Errata to Aug 28:

In order for all obj. in an abelian cat $\mathcal{C}$ to have comp. series $\mathcal{C}$ must be both Artinian <u>and Noetherian</u>. So, in the statement of [Prop 4, Aug 28], which char semisimpleness via ext. of simples, we should replace "Let $\mathcal{C}$ be Artinian" with "Let $\mathcal{C}$ be Artinian and Noetherian". It is in the Art + Noeth setting that all objects have specified length and composition factors. We note that all familiar Art. cats are already Noeth. as well:

Ex 1: The cat Vect of finite-dim vect spaces is both Art. and Noeth.

Ex 2: The cat $A\text{-mod}_{fin}$ of fin-dim modules over any $\mathbb{C}$-alg A is both Art and Noeth.

Ex 3: The cat $R\text{-mod}_{fg}$ of fin. gen'd mods over an Artinian ring R is both Art. and Noeth.

Ex 4: The cat $rep(\mathfrak{g})$ of fin-dim $\mathfrak{g}$-repr, for any Lie alg $\mathfrak{g}$ is both Art. and Noeth.

Anti-Ex 5: The opposite cat $(\mathbb{C}[x]\text{-mod}_{fg})^{op}$ is Artinian but not Noeth.

These fin. constraints hold for Examples 1, 2, 4 for simple dimension reasons.

Aug 30

- $\mathfrak{sl}_2$-reps pt I: weights

$$[h,e]=2e, \quad [h,f]=-2f, \quad [e,f]=h.$$

Let V be a fin. dim $\mathfrak{sl}_2$-rep, V decomposes into generalized wt. spaces for the actn. of h

$$V = \bigoplus_{i=1}^{n} V_{\lambda_i}^{gen},$$

where each λ_i is a complex scalar and

$$V_{\lambda_i} = \ker\left((h - \lambda_i \cdot id_V)^{>>0} : V \to V \right).$$

This is clear from a consideration of the Jord. normal form of the endo

$$h|_V = \begin{bmatrix} \overbrace{\begin{bmatrix} \lambda_1 & * \\ 0 & \lambda_1 \end{bmatrix}}^{V_{\lambda_1}} & & 0 \\ & \ddots & \\ 0 & & \underbrace{\begin{bmatrix} \lambda_n & * \\ 0 & \lambda_n \end{bmatrix}}_{V_{\lambda_n}} \end{bmatrix}$$

Def: A wt vector in V is an eigenvector $v \in V$ for the actn. of h, and the assoc. wt. $\lambda = wt(v)$ is the unique scalar so that $h \cdot v = \lambda \cdot v$.

We say a wt vector $v \in V$ is a highest wt vector if $e \cdot v = 0$. Define lowest wt. vectors... $f \cdot v = 0$.

Given a scalar $\lambda \in \mathbb{C}$, the assoc. wt. space in V is the subspace $V_\lambda \subseteq V$ of all λ-eigenvectors in V, for the actn. of h.

We say V is <u>weight graded</u> if $V = \bigoplus_{i=1}^{n} V_{\lambda_i}$ for scalars λ_i.

Ex. For the adjoint rep $V = \mathfrak{sl}_2$ we have

$$V = V_{-2} \oplus V_0 \oplus V_2$$

w/ each V_i of dim 1, and $e \in V$ is the unique highest wt vector, up to scaling.

Ex: For $\mathfrak{sl}_2(\mathbb{C}) = \mathfrak{sl}_2(\mathbb{C}^2)$ we have the standard representation

$\mathbb{V} = \mathbb{C}^2$ w/ e, f, h acting w/ mat. as $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ resp. Then

$$\mathbb{V} = \mathbb{V}_{-1} \oplus \mathbb{V}_1 \quad \text{w/ highest}$$

wt vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in \mathbb{V}$.

[Rem: For $\mathfrak{sl}_n(\mathbb{C})$ we always have the standard rep $\mathbb{V} = \mathbb{C}^n$ w/ nat. action of $\mathfrak{sl}_n(\mathbb{C}) \subseteq \mathrm{End}_{\mathbb{C}}(\mathbb{C}^n)$]

- Existence of highest wt vectors $\leftarrow$ Trivial rep

Lemma 1: If V is a $\mathfrak{sl}_2$-rep and v is a wt vector of wt λ, then the following hold:

a) $e \cdot v \in V_{\lambda+2}$.

b) $h \cdot v \in V_\lambda$.

c) $f \cdot v \in V_{\lambda-2}$.

Proof: We have directly

a) $h \cdot (e \cdot v) \overset{\text{rel Jacobi}}{=} [h, e] v + e \cdot (h \cdot v)$
$= 2 e \cdot v + \lambda e \cdot v = (\lambda+2) e \cdot v$

b) $h \cdot v = \lambda \cdot v \in V_\lambda$

c) $h \cdot (f \cdot v) = [h, f] \cdot v + f \cdot (h \cdot v)$
$= -2 f \cdot v + \lambda f \cdot v = (\lambda - 2) f \cdot v$.

We're done. ∎

Proposition 2: Any nonzero $\mathfrak{sl}_2$-rep V contains a highest wt vector $v \in V$, and the subspace

$$\bigoplus_{i=1}^{n} V_{\lambda_i} \subseteq V$$

spanned by wt vectors in V is a nontrivial $\mathfrak{sl}_2$-subrep.

Proof: Let $V' = \bigoplus_{i=1}^{m} V_{\lambda_i}$ be the span of the wt vectors in V. By considering the Jord. norm form for h it is clear that V has some nonzero wt. vector $v \in V'$, since $\dim V > 0$ by hypothesis. Take λ w/ $v \in V_\lambda$.

By Lemma 1 $e^n \cdot v \in V_{\lambda+2n}$, and by fin.-dimness $V_{\lambda+2n} = 0$ for large n. So there exists some minimal n_0 w/ $e^{n_0} v \neq 0$ and $e^{n_0+1} \cdot v = 0$. $e^{n_0} \cdot v$ is therefore a highest wt. vector in V.

The latter claim, that $V' \subseteq V$ is a subrep follows by Lemma 1. ∎

Corollary 3: Every simple sl_2-representation V is weight graded,

$$V = \bigoplus_{i=1}^{n} V_{\lambda_i}.$$

— Constraining weights

Structure Thm for sl_2-reps: Let V be a simple sl_2-rep. Then

a) V has a unique highest wt. vector v, up to scaling.

b) The highest wt vector v has wt $\lambda \in \mathbb{Z}_{\geq 0}$. non-neg integral (!)

c) The nonzero wt spaces in V are precisely

$$V_{\lambda-2m} \text{ for } 0 \leq m \leq \lambda.$$

d) For each $0 \leq m \leq \lambda$, $V_{\lambda-2m}$ is 1-dim and spanned by f^m.

We decompose the proof into a seq. of Lemmas and their consequences.

Lemma 4: If $v \in V$ is a highest weight vector of weight λ then, for each $m \geq 0$,

$$e \cdot (f^m \cdot v) = m(\lambda - m + 1) f^{m-1} \cdot v$$

and

$$e^m \cdot (f^m \cdot v) = \left[\prod_{k=1}^{m} k(\lambda - k + 1)\right] \cdot v.$$

—

Proof: We have

$$
\begin{aligned}
e \cdot f^m \cdot v &= [e, f^m] \cdot v \\
&= \sum_{i=0}^{m-1} f^{m-i-1} [e,f] f^i \cdot v \\
&= \sum_{i=0}^{m-1} f^{m-i-1} h \cdot f^i \cdot v \\
&= \sum_{i=0}^{m-1} (\lambda - 2i) \cdot f^{m-1} \cdot v \\
&= \left(m\lambda - 2\left(\sum_{i=0}^{m-1} i\right)\right) \cdot f^{m-1} \cdot v \\
&= m(\lambda - m + 1) \cdot f^{m-1} \cdot v.
\end{aligned}
$$

The second eq is an immediate consequence of the first. ∎

Corollary 5: If V is a nonzero, fin-dim sl_2-rep, then for any highest wt. vector $v \in V$, $wt(v) \in \mathbb{Z}_{\geq 0}$.

Proof: Take $\lambda = wt(v)$. Since V is fin dim $V_{\lambda - 2m}$ vanishes for $m \gg 0$. Hence $f^m \cdot v = 0$ for $m \gg 0$, and by Lemma 4 we see

$$k \cdot (\lambda - k + 1) = 0 \quad \text{for some } k > 0.$$

$\Rightarrow \quad \lambda = k - 1$ for some $k > 0$

$\Rightarrow \quad \lambda$ is nonnegative integral.

Proposition 6: If V is a fin-dim sl_2-rep. w/ highest wt vector v of wt $\lambda \geq 0$. Then

$$f^m \cdot v = 0 \text{ if and only if } m > \lambda,$$

and the vectors $\{v, f \cdot v, \ldots, f^{\lambda} \cdot v\}$ span a simple sl_2 subrep $L(\lambda) \subseteq V$ which has unique highest wt. vector v, up to scaling, and is of dim $\dim_{\mathbb{C}} L(\lambda) = \lambda + 1$.

Proof: By the formula from Lemma 4,

$e^{\lambda} f^{\lambda} \cdot v = \kappa \cdot v$ for a nonzero scalar κ,

so that $f^{m} \cdot v \neq 0$ whenever $m \leq \lambda$.

At $\lambda + 1$ we have

$$e \cdot f^{\lambda+1} \cdot v = (\lambda+1)(\lambda-\lambda) \cdot f^{\lambda} \cdot v = 0$$

so that either $f^{\lambda+1} \cdot v = 0$ or $f^{\lambda+1} \cdot v$

is a highest wt vector of wt

$$\lambda - 2(\lambda+1) = -\lambda - 2 < 0.$$

By Corollary 5 $\nexists$ highest wt vectors of negative wt in V, so that $f^{\lambda+1} \cdot v = 0$ and all $f^{m} \cdot v = 0$ when $m > \lambda$.

The fact that $L(\lambda)$ is a subrep, i.e. is closed under the actions of e, f, and h, is immediate from Lemma 4. For simplicity, any nonzero subrep $L \subseteq L(\lambda)$ has a highest wt vector $w \in L$, which is therefore a highest wt vector in $L(\lambda)$. But the only highest wt vector in $L(\lambda)$ is v, up to scaling, so that

$v = c \cdot w$ for some scalar c, $v \in L$, and hence $L(\lambda) = \mathrm{Span}\{f^{m} \cdot v\} \subseteq L$, so that $L = L(\lambda)$. ∎

Corollary 7: Any simple sl_2-rep L has a unique highest wt vector v, up to scaling,

$\lambda = wt(v)$ is a non-negative integer,

$$L = \mathrm{Span}_{\mathbb{C}}\{f^{m} \cdot v : 0 \leq m \leq \lambda\},$$

and

$$\dim L = \lambda + 1.$$

Def^{n}: For any simple sl_2-rep L, w/ highest wt vector of wt $\lambda \geq 0$, we say L is a simple of highest wt λ.

- Uniqueness of highest wt simples.

Proposition 8: For simple $\mathfrak{sl}_2$-reps L and L' with highest wt vectors v and v' of

$$wt(v) = \lambda = wt(v'),$$

there exists a unique isomorphism of $\mathfrak{sl}_2$-reps

$$\phi : L \longrightarrow L'$$

with $\phi(v) = v'$.

On the other hand, if L and L' have distinct highest wts then L and L' are not isom as $\mathfrak{sl}_2$-reps.

Suppose $\lambda = wt(v) = wt(v')$.

Proof: We have $L = \mathrm{Span}\{v, fv, \ldots, f^\lambda v\}$ and $L' = \mathrm{Span}\{v', fv', \ldots, f^\lambda \cdot v'\}$ so that, by the action formulas of Lemma 4, the unique linear map

$$\phi : L \to L' \text{ w/ } \phi(f^n \cdot v) = f^m \cdot v'$$

provides the desired isomorphism. ∎

Example (adj rep): The adjoint rep

$$V_{adj} = V_{-2} \oplus V_0 \oplus V_2$$

is the unique simple of highest wt 2, $V_{adj} = L(2)$

Example (standard rep): The standard rep

$$\mathbb{V} = \mathbb{V}_{-1} \oplus \mathbb{V}_1$$

is the unique simple of highest wt 1, $\mathbb{V} = L(1)$.

Example (trivial rep): The trivial rep $\mathbb{C}$ is the unique simple of highest wt. 0, $\mathbb{C} = L(0)$.

- Aside: Tensor products of $\mathfrak{g}$-reps.

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Lemma 9: Let $\mathfrak{g}$ be an arbitrary Lie algebra. For any two $\mathfrak{g}$-repr V and W the tensor product $V \otimes W = V \otimes_{\mathbb{C}} W$ admits a unique $\mathfrak{g}$-rep structure under the action

$$x \cdot (v \otimes w) := (x \cdot v) \otimes w + v \otimes (x \cdot w).$$

Proof: For each $x \in \mathfrak{g}$ we have the endos $x_V : V \to V$ and $x_W : W \to W$ so that we have the assoc. linear endo

$$x_V \otimes id_W + id_V \otimes x_W : V \otimes_{\mathbb{C}} W \to V \otimes_{\mathbb{C}} W,$$

via naturality of the tensor product. We claim that the assoc. linear map

$$\phi_{V \otimes W} : \mathfrak{g} \longrightarrow \mathfrak{gl}(V \otimes_{\mathbb{C}} W), \quad \phi_{V \otimes W} = \phi_V \otimes id + id \otimes \phi_W,$$

defines a $\mathfrak{g}$-rep structure on the tensor product.

We check relative Jacobi directly on monomials in $V \otimes_{\mathbb{C}} W$,

$$[x,y] \cdot (v \otimes w) = [x,y] \cdot v \otimes w + v \otimes [x,y] \cdot w$$
$$= xyv \otimes w + v \otimes xyw - yx \cdot v \otimes w - v \otimes yxw$$
$$= xyv \otimes w + xv \otimes yw + yv \otimes xw + v \otimes xyw$$
$$- yxv \otimes w - yv \otimes xw - xv \otimes yw - v \otimes yxw$$
$$= x \cdot y \cdot (v \otimes w) - yx(v \otimes w).$$

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Example: Let L and L' be simple $\mathfrak{sl}_2$-reps of highest wts λ and λ' resp. Let $v \in L$ and $v' \in L'$ be highest wt vectors.

Then $v \otimes v'$ is a highest wt vector in $L \otimes L'$ and

$$h \cdot (v \otimes v') = hv \otimes v' + v \otimes h \cdot v'$$
$$= (\lambda + \lambda')(v \otimes v').$$

So $L \otimes L'$ contains a highest wt vector of wt $\lambda + \lambda'$.

- Existence and uniqueness for simple $\mathfrak{sl}_2$-reps

Theorem 10: For each $\lambda \geq 0$, there exists a unique simple $\mathfrak{sl}_2$-representation $L(\lambda)$ of highest wt. λ. Furthermore, for any highest wt v, we have $f^m \cdot v \neq 0$ for all $m \leq \lambda$ and

$$L(\lambda) = \mathrm{span}_{\mathbb{C}}\{v, f v, \ldots, f^{\lambda} v\}. \quad (*)$$

Proof: Uniqueness was covered in Proposition 8, and the structure $(*)$ follows by Corollary 7. So we need only establish existence.

At low wts we have

$\mathcal{L}(0)$ = trivial rep, $\mathcal{L}(1)$ = standard rep

$\mathcal{L}(2)$ = adjoint rep.

Now for all $\lambda \geq 1$ we note that

$$\mathcal{L}(1)^{\otimes \lambda}$$

has a highest wt vector $v = \begin{bmatrix}1\\0\end{bmatrix} \otimes \cdots \otimes \begin{bmatrix}1\\0\end{bmatrix}$ of wt. λ, so that $\mathcal{L}(1)^{\otimes\lambda}$ contains a, and hence the, simple sl_2-rep $\mathcal{L}(\lambda) \subseteq \mathcal{L}(1)^{\otimes \lambda}$ of highest wt. λ by Proposition 6. ∎

We've now classified simple sl_2-representations:

$$\begin{aligned} \mathbb{Z}_{\geq 0} &\xrightarrow{\ \cong\ } \{\text{simple } sl_2\text{-reps}\}/\cong \\ \lambda &\longmapsto \mathcal{L}(\lambda). \end{aligned}$$

$$\mathcal{L}(\lambda) = \begin{tikzcd} v_\lambda \arrow[d, bend left, "f"] \arrow[loop right, "h"] \\ v_{\lambda-2} \arrow[u, bend left, "e"] \arrow[loop right, "h"] \\ \vdots \\ v_{-\lambda+2} \arrow[d, bend left, "f"] \arrow[loop right, "h"] \\ v_{-\lambda} \arrow[u, bend left, "e"] \arrow[loop right, "h"] \end{tikzcd}$$

- Next: semisimplicity of $\mathrm{rep}(sl_2(\mathbb{C}))$.

We know, from [Prop 4, Aug 28], that $\mathrm{rep}(\mathfrak{g})$ is semisimple iff each extension of simples

$$0 \to \mathcal{L}(\mu) \to V \to \mathcal{L}(\lambda) \to 0$$

split. Since we know so much about simples, one can observe such splittings directly. However, let us take an approach which mirrors the higher rank setting.

- The Casimir element.

For each $sl_2(\mathbb{C})$-rep V define

$\Omega_V : V \to V$ as:

$$\Omega_V := \tfrac{1}{2} h^2 + ef + fe \in \mathrm{End}_{\mathbb{C}}(V)$$

Lemma 11: a) For each map $\xi : V \to W$ of $\mathfrak{sl}_2$-reps, the diagram

$$\begin{tikzcd} V \arrow[r, "\xi"] \arrow[d, "\Omega_V"'] & W \arrow[d, "\Omega_W"] \\ V \arrow[r, "\xi"'] & W \end{tikzcd}$$

commutes.

b) Each linear endo Ω_V is in fact an $\mathfrak{sl}_2$-linear endo of V.

c) For each simple rep $L(\lambda)$, $\lambda \in \mathbb{Z}_{\geq 0}$,
$\Omega_{L(\lambda)} = \frac{1}{2}\lambda(\lambda+2)\cdot \mathrm{id}_{L(\lambda)}$.

Proof: a) Is clear as at each $v \in V$ we have

$$\xi\left[\left(\tfrac{1}{2}h.h + e.f + f.e\right).v\right)$$
$$= \left(\tfrac{1}{2}h^2 + e.f + f.e\right)\cdot \xi(v),$$

via $\mathfrak{sl}_2$-linearity of ξ. (b) We want to show $x\cdot\Omega_V = \Omega_V x$ for each $x \in \mathfrak{sl}_2$, i.e. $[x, \Omega_V] = 0$ in $\mathfrak{gl}(V) = \mathrm{End}(V)^{\mathrm{Lie}}$.

However this follows by the calculations

$$[h, \tfrac{1}{2}h^2 + ef + fe] = \underbrace{2ef + (-2)ef + (-2)fe + 2fe}_{=0}$$
$$[e, \tfrac{1}{2}h^2 + ef + fe] = \underbrace{-eh - he + eh + he}_{=0}$$
$$[f, \tfrac{1}{2}h^2 + ef + fe] = \underbrace{fh + hf - hf - fh}_{=0}.$$

c) By Schur's Lemma $\mathrm{End}_{\mathfrak{sl}_2}(L(\lambda)) = \mathbb{C}$, so that $\Omega_{L(\lambda)} = c\cdot \mathrm{id}$ for some scalar c.

We can find the scalar c by evaluating on the highest wt. vector $v \in L(\lambda)_\lambda$. We have

$$\left(\tfrac{1}{2}h^2 + ef + fe\right).v = \tfrac{1}{2}\lambda^2 v + ef.v$$
$$= \tfrac{1}{2}\lambda^2 v + [e,f].v$$
$$= \tfrac{1}{2}\lambda^2 v + \lambda v$$
$$= \tfrac{1}{2}\lambda(\lambda+2).v. \quad \blacksquare$$

Remark: Ω_V is the action of the element $\Omega = \frac{1}{2}h^2 + ef + fe$ in $\mathcal{U}(sl_2)$ on the given $sl_2(\mathbb{C})$-rep V. This element

$$\Omega \in \mathcal{U}(sl_2)$$

is central, by (b). It is called the Casimir element.

— Splitting extensions

Proposition 12: Any extension of simple sl_2-reps

$$0 \to L(\mu) \to V \to L(\lambda) \to 0 \qquad (*)$$

is split.

Proof: If $\lambda = \mu$ then $V(\lambda) = \mathbb{C}w \oplus \mathbb{C}w'$ where w is the image of the highest wt. vector $v \in L(\lambda)$ under the given inclusion and w' maps to v under the projection $V \to L(\lambda)$. By Proposition 6 we have two simple subreps

$$L, L' \subseteq V, \quad L, L' \cong L(\lambda),$$

with highest wt. vectors w and w' respectively.

The map $L(\lambda) \to V$ is therefore an $\cong$ onto L and the map $V \to L(\lambda)$ restricts to an isomorphism $L \to V \to L(\lambda)$. The inverse morphism $L(\lambda) \to L \hookrightarrow V$ provides the desired splitting.

If $\mu \neq \lambda$ then $\frac{1}{2}\mu(\mu+2) \neq \frac{1}{2}\lambda(\lambda+2)$. By Lemma 11 the operator $\Omega_V : V \to V$ has eigenvalues $\frac{1}{2}\mu(\mu+1)$ and $\frac{1}{2}\lambda(\lambda+1)$ and the resp. generalized eigenspaces $V(\mu)$ and $V(\lambda)$ are nonvanishing subreps in V w/

$$V(\mu) \oplus V(\lambda) = V.$$

Since $\mathrm{Length}(V) = 2$ we have

$$V(\mu) \simeq L(\lambda)$$

and the composite $V(\lambda) \to V \to L(\lambda)$ is

an isomorphism of sl_2-repr. The inverse

$$L(\lambda) \xrightarrow{\cong} V(\lambda) \hookrightarrow V$$

then provides the required splitting. ∎

Theorem [semisimplicity of $\mathrm{rep}(sl_2)$]:

a) The category $\mathrm{rep}(sl_2(\mathbb{C}))$ is semisimple.

b) The simples in $\mathrm{rep}(sl_2(\mathbb{C}))$ are classified by their highest wts,

$$\mathbb{Z}_{\geq 0} \xrightarrow{\cong} \{\text{simple } sl_2(\mathbb{C})\text{-reps}\}/\cong.$$

c) Every fin-dim $sl_2(\mathbb{C})$-rep V decomposes uniquely into a sum

$$V = \bigoplus_{i=1}^{n} m(\lambda_i) \cdot L(\lambda_i)$$

with $m(\lambda_i) = \dim \mathrm{Hom}_{sl_2(\mathbb{C})}(L(\lambda_i), V)$.

Proof: Immediate from Prop 12 and [Prop 4, Aug 28]. ∎