

Radicals and simplicity

To begin we discuss another class of rings/algebras -

I: Division algebras

Def¹: A division ring D is a ring for which each nonzero $a \in D$ admits $a^{-1} \in D$ with $a^{-1}a = aa^{-1} = 1$.
A division algebra over a field k is an algebra which is also a division ring.

Def²: A ring R is called a domain if for each nonzero a in R an equation $a \cdot b = 0$ or $b \cdot a = 0$, at b arbitrary in R , implies $b = 0$.

Observation 1: Any division ring is a domain.

Ex: \mathbb{H} = quaternions are a division alg over \mathbb{R} .

Lemma 2: The center of any division ring is a field.

(**Proof:** For a in $Z(D)$ we have for a^{-1} and arbitrary b in D ,

$$a^{-1}b = a^{-1}b a a^{-1} = a^{-1}a b a^{-1} = b a^{-1}.$$

So $a^{-1} \in Z(D)$ as well.

Lemma 3: If K is an algebraically closed field, then the only finite dimensional K -algebra is K itself.

Proof: Take D a div. alg over K and suppose there exists $x \in D \setminus K$. Then we have the alg. map $\phi: K[X] \rightarrow D$, $X \mapsto x$. If $\ker(\phi) \neq 0$ then $K[X]/\ker(\phi) \subseteq D$ is a domain, and hence $\ker(\phi) = (p(X))$ for an irred poly $p(X)$. Consequently $K \subseteq K[X]/\ker(\phi)$ is a finite field extension of K , $K \subseteq K \subseteq D$, and by alg closure $K = K$. Thus $x \in K \subseteq D$, a contradiction. So $\ker(\phi) = 0$ necessarily, and $\dim D \geq \dim K[X] = \infty$.

As a consequence, if $\dim D < \infty$ for div K -alg D then we must have $D = K$.

Corollary 4: No poly finite division algs over \mathbb{C} , $\overline{\mathbb{Q}}$, $\overline{\mathbb{F}_p}$ etc. are \mathbb{C} , $\overline{\mathbb{Q}}$, $\overline{\mathbb{F}_p}$ themselves, respectively.

Proposition 5: Any finitely generated module M over a div. alg D is of the form D^l for some uniq. det. l.c.s., and if $M' \subseteq M$ then $M' = D^{l'}$ for $l' \leq l$. Equality holds if and only if $l = l'$.

For first claim

Proof: Proceed by induction on the num. of generators, and we claim

$l = \{ \text{the cardinality of any minimal gen set } m_1, \dots, m_l \text{ of } R \}$

For such a minimal gen set we have the surj module map

$$f: \bigoplus_{i=1}^l D \rightarrow R, \quad e_i \mapsto m_i,$$

which we claim is injective. Assuming $f|_{\bigoplus_{i=1}^{l'} D}$

is injective for $0 \leq l' < l$ then for any $a \in D_{l'+1} \setminus \{0\}$

we have $f(ae_{l'+1}) = ae_{l'+1} \cdot m_{l'+1} = \sum_{i=1}^{l'} a_i \cdot m_i$

gives $m_{l'+1} = f(e_{l'+1}) = a_{l'+1}^{-1} \cdot f(ae_{l'+1}) \in \text{Span of } \{m_i\}_{i=1}^{l'}$

$$m_{l'+1} \rightarrow m_{l'},$$

a contradiction. So $f|_{\bigoplus_{i=1}^{l'+1} D}$ is injective as well.

By induction on l' we see f is injective, and thus an isomorphism. So

$$D^l \cong R.$$

Now, if we have an isom $f: D^{l_1} \xrightarrow{\sim} D^{l_2}$ for $l_1, l_2 > 0$ then, after precomposing w/ an aut of $G \otimes D^l$ we can assume $f|_{\text{first copy of } D} = \text{id}$. Then f induces an isomorphism.

$$D^{l_1-1} = D^{l_1} / D \xrightarrow{\sim} D^{l_2-1} = D^{l_2} / D.$$

Repeating we obtain $0 \xrightarrow{\sim} D^{l_2-l_1}$ which gives $l_2 = l_1$.

For the second claim, given $M' \subseteq M$ we select a minimal gen set m_1, m_2, \dots, m_r for M' to a min gen set m_1, \dots, m_r for M to get $M' \cong \bigoplus_{i=1}^r R$ and $M \cong \bigoplus_{i=1}^r R$ with $\ell' \leq \ell$ and equality holding iff $M' = M$. \square

Def: The rank of a D -module, for D a division ring, is the unique integer ℓ at which we have an isomorphism of D -modules $M \cong D^\ell$.

~ II. Matrices over division rings

On the flip side over K^n is simple over $M_n(K)$, for K a field. Similarly, for any division ring D D^n is simple module over $M_n(D)$, and $M_n(D)$ admits no ideals other than 0 and $M_n(D)$ itself.

Furthermore we have

$$M_n(D) \cong \bigoplus_{i=1}^n D^n.$$

as a $M_n(D)$ -module.

Here, for any ring R , $M_n(R)$ is the ring with $1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and with $[a_{ij}][b_{ij}] = [c_{ij}]$ as $c_{ij} = \sum_k a_{ik} b_{kj}$.

III. Semisimple modules

Defⁿ: A module M over an algebra A is called semisimple if

$$M = \bigoplus_{\lambda \in \Lambda} L_{\lambda}$$

for simple submodules $L_{\lambda} \subseteq M$.

Exer^{cise}: M is semisimple if and only if M admits an A -module isomorphism

$$\bigoplus_{\lambda \in \Lambda} L_{\lambda} \xrightarrow{\sim} M$$

for simple M_{λ} .

Observation 6: For a semisimple module M the following are equivalent

a) M is finitely generated

b) M is finite length (admits a composition series)

c) For any isomorphism $\bigoplus_{\lambda \in \Lambda} L_{\lambda} \xrightarrow{\sim} M$ w/ the L_{λ} simple, the indexing set Λ is finite.


Proof: Exercise.

~~Ex~~

Lemma (Schur) If L and L' are simple A -modules, for any A -module map $f: L \rightarrow L'$ is either

0, a an isomorphism.

Proof: By simplicity either $\ker(f) = 0$ or L .
 In the first case f is injective with
 $0 \neq \operatorname{im}(f) \subseteq L$

and by simplicity again $\operatorname{im}(f) = L$. Hence f is
 bijective as a map of sets, and thus invertible as a map
 of A -modules. In the second case, $\ker(f) = L$,
 we have $f = 0$. 

Corollary 7: For any simple A -module L ,
 $\operatorname{End}_A(L)$ is a division ring.

Corollary 8: If A is a finite dimensional algebra
 over an algebraically closed field $K \subseteq \mathbb{C}$, and L
 is simple over A , then

$$\operatorname{End}_A(L) = K \cdot \operatorname{id}_L = K.$$

Proof: Since L is cyclic we have a surjection $A \rightarrow L$
 and hence $\dim_K(L) \leq \dim_K(A) < \infty$. Thus

$$\dim_K \operatorname{End}_A(L) \leq \dim_K \operatorname{End}_K(L) < \infty.$$

Thus $\operatorname{End}_A(L)$ is a finite-dimensional division algebra

over k . Since $k = \bar{k}$, we now have

$$\text{by Lemma 3.} \quad \text{End}_A(L) = k$$



Example: For $L_{\mathbb{C}}^{(2)}$ the 2-dim simple mod
over $\mathbb{C} S_3$, we have

$$\text{End}_{\mathbb{C} S_3}(L_{\mathbb{C}}^{(2)}) = \mathbb{C}.$$

Also,

$$\text{End}_{\overline{\mathbb{F}_3} S_3}(L_{\overline{\mathbb{F}_3}}^{(2)}) = \overline{\mathbb{F}_3}.$$

More generally, for any finite group G , any simple
 G -rep L over an alg closed field k has

$$\text{End}_{kG}(L) = k.$$

Proposition 9: Let M be a finitely generated
semisimple A -module.

- i) Every submodule $M' \leq M$ is semisimple.
- ii) Every quotient module M/M' is semisimple.

Proof: For (i) we claim that M' and M admit
decomp into simples $L_1 \oplus \dots \oplus L_s = M'$ and $M =$
 $L_1 \oplus \dots \oplus L_r \oplus \dots \oplus L_t$ under which the inclusion

$M \rightarrow M$ is just the matrix $\begin{bmatrix} I_s \\ 0 \end{bmatrix} : \bigoplus_{i=1}^s L_i \rightarrow \bigoplus_{j=1}^t L_j$.

We proceed by induction on $\text{length}(M)$. When $\text{length}(M) = 0$ there's nothing to do. Take now $\text{length}(M) = t$ and assume the result holds for semisimple N with $\text{length}(N) < t$. Since M' is of finite length [Cor 6, Fuchs] we have some simple $L \leq M'$ and each composite

$$L \rightarrow M \rightarrow L_i$$

is either 0 or an isomorphism, by Schur's Lemma.

Since the inclusion $L \rightarrow M$ is nonzero we can find minimal index i_0 at which $L \rightarrow M \rightarrow L_{i_0}$ is nonzero. Then the map

$$(\bigoplus_{i < i_0} L_i) \oplus L \oplus (\bigoplus_{j > i_0} L_j) \rightarrow M$$

induced by the inclusions is an isomorphism(?) and after replacing L_{i_0} with L and reindexing we can assume $i_0 = 1$ and denote L_1

$$\begin{array}{ccc} & \swarrow & \searrow \\ & M' & \longrightarrow M \end{array}$$

Take $L_1^\perp \leq M'$ the kernel of the sequence $M' \rightarrow M \rightarrow M / \bigoplus_{i>1} L_i$. Then

we have

$$M' = L_1 \oplus L_1^\perp$$

and $L_1^\perp \leq \bigoplus_{i>1} L_i \leq M$. Since

$\text{length}(\bigoplus_{i=1}^t L_i) = t-1$ we obtain consistent decomposition

$$L_1 = L'_1 \oplus \dots \oplus L'_s$$

$$\bigoplus_{i=1}^t L_i = L'_1 \oplus \dots \oplus L'_t$$

and thus consistent decomposition

$$M' = L'_1 \oplus L'_2 \oplus \dots \oplus L'_s \quad (*)$$

$$M = L_1 \oplus L'_2 \oplus \dots \oplus L'_t$$

(ii) Taking consistent decomposition

$$M = \bigoplus_{i=1}^s L_i, \quad M' = \bigoplus_{i=1}^t L_i$$

we obtain

$$M/M' = \bigoplus_{i>s} L_i$$

The following was already demonstrated in the proof.

Theorem 10: If M is semisimple, any exact sequence $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ splits, i.e.,

Remark: The analog of Prop 9 holds for infinite length semisimple modules as well. One can argue the point from the finite case and some Zorn's lemma argument.

Proposition 9.2: For any semisimple A -module M , the following hold.

- i) Any submodule $M' \subseteq M$ is semisimple.
- ii) Any quotient module

Proof: Omitted. ■

~ III is An example: Matrix rings

Example: Consider any division ring D and $M_n(D)$. We have the simple module

$$L_{\text{standard}} = D^n$$

provided by the columnar module under the decomp. of the regular module

$$M_n(D) = \bigoplus_{i=1}^n L_{\text{standard}}.$$

Since $M_n(D)$ itself is semisimple.

Now, since any fin gen $M_n(D)$ -module M admits a surjection

$$\bigoplus_{i=1}^m M_n(D) = \bigoplus_{i=1}^{n \cdot m} L_{\text{standard}} \rightarrow M$$

Prop 9 (ii) tells us that M is also semisimple, and [Prop 9, Fuchs] shows (as expected)

$$M = \bigoplus_{i=1}^t L_{\text{standard}}.$$

Corollary 11: For any division ring D , any finitely generated $M_n(D)$ -module is semisimple, and

The standard column module is the only simple $M_n(D)$ module, up to isomorphism.

IV Quotients for modules

$\mathcal{M} \supseteq \mathcal{M}_0 \supsetneq$ proper submodule $\mathcal{M}_0 \subsetneq \mathcal{M}$ maximal if any larger module $\mathcal{M}_0 \subseteq \mathcal{M}_1 \subseteq \mathcal{M}$ has either $\mathcal{M}_1 = \mathcal{M}_0$ or $\mathcal{M}_1 = \mathcal{M}$.

Lemma 11: A proper submodule $\mathcal{M}_0 \subsetneq \mathcal{M}$ is maximal if and only if $\mathcal{M}/\mathcal{M}_0$ is simple.

(Proof: Consider the surjection $\pi: \mathcal{M} \rightarrow \mathcal{N}$, with $\mathcal{N} = \mathcal{M}/\mathcal{M}_0$. We have the bijection

$$\begin{aligned} \left\{ \begin{array}{l} \text{Submodules } \mathcal{M}_1 \\ \text{with } \mathcal{M}_0 \subseteq \mathcal{M}_1 \subseteq \mathcal{M} \end{array} \right\} &\longleftrightarrow \left\{ \begin{array}{l} \text{Submodules} \\ \mathcal{N}_1 \subseteq \mathcal{N} \end{array} \right\} \\ \mathcal{M}_1 &\longmapsto \pi(\mathcal{M}_1) \\ \pi^{-1}(\mathcal{N}_1) &\longleftarrow \mathcal{N}_1. \end{aligned}$$

Indeed, for $m \in \mathcal{M}$ with $\pi(m) = \pi(m')$ for m' in \mathcal{M}_1 , we have $m - m' \in \ker(\pi) = \mathcal{M}_0$ so that $m = m' + (m - m') \in \mathcal{M}_1$. This gives

$$\pi^{-1}(\pi(\mathcal{M}_1)) = \mathcal{M}_1.$$

The equality $\pi(\pi^{-1}(\mathcal{N}_1)) = \mathcal{N}_1$, at any submodule $\mathcal{N}_1 \subseteq \mathcal{N}$ follows by surjectivity of π . Hence

The only submodules⁰ between M_0 and M are M_0 and M itself if and only if the only submodules in N are 0 and N itself. Rather, M_0 is maximal if and only if $N = M/M_0$ is simple. 1

Def¹: Given a ring A and a nonzero A -module M , the radical in M is the intersection

$$\text{Rad}(M) = \bigcap_{M_0 \in \text{Max}} M_0.$$

Example: For an infinite field K (e.g. \mathbb{Q} , \mathbb{C} , or $\overline{\mathbb{F}_p}$) and any $\alpha \in K$ we have the maximal / submodule $(x - \alpha)$ in the regular module $K[x]$, and the quotient

$$K[x] / (x - \alpha) \xrightarrow{\sim} K(\alpha)$$

is the 1-dimensional $K[x]$ -module $K(\alpha)$ which

$$x \cdot \mathbf{1}_\alpha = \alpha \mathbf{1}_\alpha. \quad \text{Now, for any poly}$$

$$p(x) \in \text{Rad}(K[x])$$

we have $p(x) \cdot \mathbf{1}_\alpha = p(\alpha) \cdot \mathbf{1}_\alpha = 0$ at all $\alpha \in K$.

Hence $p(x)$ has as many zeros, and we conclude

$$p(x) = 0.$$

So

$$\text{Rad}(K[x]) = 0.$$

Example: At each $i=1, \dots, n$, and k a field, we have the surjection

$$\pi_i: M_n(k) \rightarrow L_{\text{standard}} = k^n$$

with kernel $\ker(\pi_i) = \left\{ \begin{bmatrix} \text{stuff} & 0 & \text{stuff} \end{bmatrix} \right\}$
↑
its column

Since L_{standard} is simple this gives

$$\text{Rad}(M_n(k)) \subseteq \bigcap_i \ker(\pi_i) = 0.$$

Similarly, for any division ring D , $\text{Rad}(M_n(D)) = 0$.

Example: For $M = k[x] / (x^n)$

as a module over $k[x]$, any simple quotient

$$\pi: M \rightarrow L$$

we have $x \cdot -: L \rightarrow L$ an endomorphism. By Schur either $x \cdot -$ is 0 or an isomorphism. Since

$$(x \cdot -)^{n-1} = x^{n-1} \cdot - = 0$$

we conclude $x \cdot - = 0$. Rather, x annihilates L .

We see now $x \in \ker(\pi)$ at each such π , giving

$$\bar{x} \in \text{Rad}(M) \Rightarrow k[x] \cdot \bar{x} = x \cdot M \subseteq \text{Rad}(M).$$

Since the quotient $M / x \cdot M \cong k$ is simple, we get $\text{Rad}(M) = x \cdot M$.

~ IV 2 Radicals and Semisimplicity

Theorem 12: For any finite length module M , $M/\text{Rad}(M)$ is semisimple. Furthermore, for any surjective A -module map $\pi: M \rightarrow N$, N is semisimple if and only if $\text{Rad}(M) \subseteq \ker(\pi)$.

Before we prove the result, we note the following

Lemma 13: For any surjective module map $\pi: M \rightarrow N$ to nonzero N , $\text{Rad}(M) \subseteq \pi^{-1}(\text{Rad}(N))$.

Proof: For maximal $N_0 \subseteq N$, $\pi^{-1}(N_0)$ is max in M . Hence

$$\begin{aligned} \pi^{-1}(\text{Rad}(N)) &= \pi^{-1}\left(\bigcap_{N_0 \text{ max}} N_0\right) \\ &= \bigcap_{N_0 \text{ max}} \pi^{-1}(N_0) \subseteq \bigcap_{N_0 \text{ max}} M_0 = \text{Rad}(M). \end{aligned}$$

Lemma 14: If N is finite length and semisimple, then $\text{Rad}(N) = 0$.

Proof: Given an expression $N = \bigoplus_{i=1}^r L_i$ with the L_i simple, the kernel $(R_i) \subseteq N$ of each

projection $p_i: \mathcal{N} \rightarrow \mathcal{L}_i$ satisfying

$$\bigcap_{i=1}^r \mathcal{K}_i = \ker([p_1 \dots p_r]^t: \mathcal{N} \rightarrow \bigoplus_{i=1}^r \mathcal{L}_i).$$

But $[p_1 \dots p_r]^t$ just recovers the identity on \mathcal{N} , and hence $\text{Rad}(\mathcal{N}) \subseteq \bigcap_{i=1}^r \mathcal{K}_i = 0 \Rightarrow \text{Rad}(\mathcal{N}) = 0.$

We now prove our main theorem.

Proof of Theorem 12: We have, via Artinian-ness of the quotient $\mathcal{M}/\text{Rad}(\mathcal{M})$ the existence of finitely many maximal submodules $\mathcal{K}_1, \dots, \mathcal{K}_r \subseteq \mathcal{M}$ for which $\text{Rad}(\mathcal{M}) = \mathcal{K}_1 \cap \dots \cap \mathcal{K}_r$. Hence, $\text{Rad}(\mathcal{M})$ is the kernel of the map

$$\mathcal{M} \rightarrow \bigoplus_{i=1}^r \mathcal{L}_i, \quad \mathcal{L}_i := \mathcal{M}/\mathcal{K}_i,$$


induced by the individual quotients $\mathcal{M} \rightarrow \mathcal{L}_i$, giving $\mathcal{M}/\text{Rad}(\mathcal{M}) \subseteq \bigoplus_{i=1}^r \mathcal{L}_i \cong \text{semisimple}.$

By Proposition 9 we conclude $\mathcal{M}/\text{Rad}(\mathcal{M})$ is semisimple.

As for the second claim, for semisimple \mathcal{N} we have $\text{Rad}(\mathcal{N}) = 0$ by Lemma 14 and hence

$$\text{Rad}(\mathcal{M}) \subseteq \ker(\pi) = \pi^{-1}(\text{Rad}(\mathcal{N})) \text{ by Lemma 13.}$$

Conversely, if $\pi: \mathcal{M} \rightarrow \mathcal{N}$ is surj with $\text{Rad}(\mathcal{M}) \subseteq \ker \pi$ then \mathcal{N} is the quotient of the semisimple module $\mathcal{M}/\text{Rad}(\mathcal{M})$.

By Proposition 9 we conclude that N is semisimple. 

~ V Socles for modules

Theorem 15: For a Noetherian module M (e.g. a finitely generated module over $R/\dim A$), the sum $\text{soc}(M) = \sum_{L \in \Delta} L$ is a finite length semisimple submodule of M (e.g. $\text{soc}(M)$)

over the collection Δ of all simple submodules $L \subseteq M$ is a finite length semisimple submodule in M .

Furthermore, for any semisimple module N , any module map $f: N \rightarrow M$ has image in $\text{soc}(M)$.

Proof: Since M is Noetherian, the sum in (*) is finite. $\sum_{i=1}^r L_i = L_1 + \dots + L_r$ for some simple $L_i \subseteq M$. The sum is the image of the map

$$\bigoplus_{i=1}^r L_i \rightarrow M$$

induced by the inclusions, giving $\text{soc}(M)$ as a quotient of a finite length semisimple module. Hence $\text{soc}(M)$ is finite length semisimple, by Proposition 9. For the second claim, write $N = \bigoplus_{\alpha} V_{\alpha}$ for simple V_{α} to get $\text{im}(N) = \sum_{\alpha} \text{im}(V_{\alpha})$ with each $\text{im}(V_{\alpha})$

either simple or zero, by simplicity. Hence $\text{soc}(M) \subseteq \text{soc}(A)$.

Def¹: The sum of all simple submodules is a Noether A -module. M is called the socle in M .

Ex: For $K[X]/(X^n)$, considered as a module over $K[X]$, any simple submodule $M \subseteq K[X]/(X^n)$ is annihilated by X^n , and has simplicity forced $X \cdot M = 0$. This gives

$$\text{soc}(K[X]/(X^n)) = K \cdot \bar{X}^{n-1} = \sum_{i=0}^{n-1} K \bar{X}^i.$$

the 1-dim simple on which K acts as 0.

HW

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1. For the regular A -module A , prove that there is a canonical isomorphism

$$A^* \xrightarrow{\sim} \text{End}_A(A).$$

2. Let G be a group and suppose G act on a k -algebra A by algebra automorphisms. Suppose k is a field, and take

$$A \rtimes G := A \otimes_k k[G] \text{ with the unique bilinear map}$$

$$\text{map } \cdot : A \rtimes G \times A \rtimes G \rightarrow A \rtimes G \text{ satisfying}$$

$$(a \otimes g) \cdot (b \otimes h) = a(g \cdot b) \otimes gh$$

is monomial. Prove that $A \rtimes G$ is a k -algebra.

3. Let $\mathbb{Z}/n\mathbb{Z}$ act on $\mathbb{C}[x]$ via the automorphism $m \cdot p(x) = p(g \cdot x)$ for $g \in \mathbb{C}^\times$ an n -th root of unity. Take $A_g = \mathbb{C}[x] \rtimes \mathbb{Z}/n\mathbb{Z}$.

a) Prove that $M_r = A_g / A_g \cdot x^r$ is a free module over $\mathbb{C}\mathbb{Z}/n\mathbb{Z}$ of rank r . Give a basis for M_r over $\mathbb{C}\mathbb{Z}/n\mathbb{Z}$.

b) Calculate $\text{Rank}(M_r)$, $M_r / \text{Rad}(M_r)$, and $\text{soc}(M_r)$.

2. a) For $\mathbb{Z}/n\mathbb{Z}$. Prove that $\mathbb{Z}/n\mathbb{Z}$ has precisely n non-isomorphic (simple) 1-dimensional representations L_i over \mathbb{C} .

b) Prove that each simple 1-dimensional representation L_i over $\mathbb{Z}/n\mathbb{Z}$ admits an injective $\mathbb{C}\mathbb{Z}/n\mathbb{Z}$ -module map $L_i \rightarrow \mathbb{C}\mathbb{Z}/n\mathbb{Z}$, and that this map is unique up to scaling.

c) Provide an isomorphism of $\mathbb{C}\mathbb{Z}/n\mathbb{Z}$ -modules

$$\bigoplus_{i=1}^n L_i \xrightarrow{\sim} \mathbb{C}\mathbb{Z}/n\mathbb{Z}.$$

d) Prove that every finite-dimensional $\mathbb{Z}/n\mathbb{Z}$ -module M over \mathbb{C} decomposes as

$$M \cong \bigoplus_{i=1}^n m_i \cdot L_i$$

where each $m_i = [L_i : M]$.

5. For any finite length semisimple module M over a ring A , prove that there are division rings D_1, \dots, D_k and integers n_1, \dots, n_k for which

$$\text{End}_A(M) \cong \prod_{i=1}^k M_{n_i}(D_i)$$

as rings.