KERODON REMIX PART III: A SMALL STUDY OF THE DERIVED ∞-CATEGORY

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Abstract. We employ the materials from Part II to provide a baseline analysis of the derived ∞-category of an abelian category, as a stable ∞-category. We explicitly calculate pushouts and pullbacks in the homotopy (and derived) ∞-category, directly verify stability, and explicitly realize the derived ∞-category as a localization of the homotopy ∞-category against the class of quasi-isomorphisms. We also exhibit adjunctions between left and right derived functors. We conclude the text with a discussion of ind-completion and constructions of "renormalized" derived categories. In comparing with the previous installments, Parts I and II, we are somewhat more liberal in our treatment herein, as we occasionally employ results from higher topos theory or higher algebra when needed.

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1. Derived categories, Vamos!

This document is intended to provide an "intermediate" analysis of the derived ∞-category $\mathscr{D}(\mathbb{A})$, where $\mathbb{A}$ is a (generally Grothendieck) abelian category. We assume the reader has their own motivations for coming this topic, and so won't provide our own motivations.

This document provides the third, and final, portion of our sequence of texts "Kerodon remix" Parts I, II, and III. Part I provides an introduction to the basic foundations of ∞-categories, and Part II provides an introduction to cocartesian fibrations, Hom functors, limits and colimits in ∞-categories, and the Yoneda embedding.

1.1. Contents.

The contents of this the present text proceed as follows.

In Section 2 we provide a direct, hands on analysis of limits and colimits in the homotopy ∞-category $\mathscr{K}(\mathbb{A})$. Of special interest are pullbacks and pushouts, which are both realized explicitly via a mapping cone construction. We also explain how projective and injective resolutions are realized naturally as colimits and limits of their respective finite truncations. As the derived ∞-category $\mathscr{D}(\mathbb{A})$ can be realized as a subcategory in $\mathscr{K}(\mathbb{A})$, namely as the subcategory of K-injectives or K-projectives (see Section I-12), our calculations for $\mathscr{K}(\mathbb{A})$ imply analogous calculations of limits and colimits in the derived ∞-category.

There are two general ambitions which we mean to realize in this section. The first is to simply calculate certain important limits and colimits in the homotopy and derived ∞-categories. The second is to provide a case study which demonstrates how the abstract tomfoolery from Part II, regarding limits and colimits, actually functions in concrete situation.

In Section 3 we explain how the presence of a zero object in $\mathscr{D}(\mathbb{A})$ implies that any Hom functor $H : \mathscr{D}(\mathbb{A})^{\mathrm{op}} \times \mathscr{D}(\mathbb{A}) \to \mathscr{K}an$ admits a natural pointing. This pointing is formally a lift to a $\mathscr{K}an_*$-valued functor, where $\mathscr{K}an_*$ is the ∞-category of pointed spaces, and this lift is uniquely determined by the requirement that the functor $\widetilde{H} : \mathscr{D}(\mathbb{A})^{\mathrm{op}} \times \mathscr{D}(\mathbb{A}) \to \mathscr{K}an_*$ preserves the initial/terminal object. This construction is shown the be natural in the expected sense as well. We subsequently introduce spectra, and explain how our pointed Hom functors lift further to spectra-valued Hom functors in all settings of interest.

In Section 4 we show that the derived and homotopy ∞-categories are stable. This is an immediate consequence of our calculations from Section 2. We then provide a basic overview of stable categories and special phenomena which occur in the stable setting. In short, stability provides various shortcuts which allow one to, in particular, make strong deductions for stable ∞-categories from direct investigations of their homotopy categories. We find, for example, that the homotopy and derived ∞-categories are both complete and cocomplete. We also observe that the connective derived ∞-category $\mathscr{D}(\mathbb{A})^{\leq 0}$ is cocomplete as well.

In Section 5 we return to provide a generic analysis of adjoint functors. There two main points here. The first is that any inclusion $\mathscr{C}_0 \to \mathscr{C}$ of a coreflective subcategory into an ∞-category $\mathscr{C}$ admits a left adjoint. The second is that the information of a pair of adjoint functors,

$$F : \mathscr{C} \to \mathscr{D} \text{ and } G : \mathscr{D} \to \mathscr{C},$$

can be codified in a single cartesian and cocartesian fibrations over the 1-simplex $\mathscr{E} \to \Delta^1$ whose fibers recover $\mathscr{C} = \mathscr{E}_0$ and $\mathscr{D} = \mathscr{E}_1$. This is a practical tool which allows one to both construct adjunctions, and check for the existence of adjunctions.

In Section 6 we prove that the derived category is identified, at the level of ∞-categories, as the localization $\mathscr{K}(\mathbb{A})[\mathrm{Qiso}^{-1}] = \mathscr{D}(\mathbb{A})$ relative to the class of quasi-isomorphisms in $\mathscr{K}(\mathbb{A})$. It then follows–from a particular result in [14]–that the derived ∞-category is furthermore identified with the localization of the discrete category of cochains $\mathrm{Ch}(\mathbb{A})[\mathrm{Qiso}^{-1}] = \mathscr{D}(\mathbb{A})$. This result is highly non-classical, and allows one to transfer structures directly from the abelian category of cochains to the derived setting.

In Section 7 we discuss the process of deriving functors in the ∞-categorical context. We show that the left derived functor $\mathrm{L}\,F$ of a right exact functor $F : \mathbb{A} \to \mathbb{B}$ between Grothendieck abelian categories can be defined, as in the discrete derived setting, by taking F-acyclic resolutions on the domain. Right derived functors $\mathrm{R}\,G$ can be defined universally by taking (K-)injective resolutions. Given a pair of adjoint functors $F : \mathbb{A} \leftrightarrows \mathbb{B} : G$, we show that the associated left derived functor $\mathrm{L}\,F : \mathscr{D}(\mathbb{A}) \to \mathscr{D}(\mathbb{B})$ is left adjoint to the right derived functor $\mathrm{R}\,G : \mathscr{D}(\mathbb{B}) \to \mathscr{D}(\mathbb{A})$, as expected.

In Section 8 we introduce the notion of presentability for ∞-categories. Following [14] directly, we recall that the derived ∞-category $\mathscr{D}(\mathbb{A})$ is presentable, and following [13] directly we present the adjoint functor theorem which characterizes functors between presentable ∞-categories which are left or right adjoints. This section is a bit more of a rundown of results from [13] and [14] rather than an original presentation of the topic.

in Section 9 we present the ind-completion functor and subsequent constructions of "renormalized" derived categories. These include derived categories of ind-coherent sheaves, and ind-finite representations for algebraic groups in finite characteristic.

In Appendix A we discuss idempotent complete categories, the process of idempotent completion, and $\aleph_0$-accessibility.

1.2. **Derived recollections.** As the title suggests, this is a text about the derived category. We recall the necessary fundamentals here.

Grothendieck abelian categories: In general we work with Grothendieck abelian categories. We recall that an abelian category $\mathbb{A}$ is called a Grothendieck abelian category if it admits small colimits and a generator, and if direct limits in $\mathbb{A}$ are exact. This restriction is not so important, other than the fact that it ensures the existence of enough K-injectives in the unbounded category of cochains. Such K-injectives can then be leveraged in a uniform way in the production and analysis of the (unbounded) derived ∞-category.

For those interested in finite categories, such as finite-dimensional representations of a finite group, one can move from the finite context to the Grothendieck abelian context by replacing the category of interest with its Ind-category [10, Theorem 8.6.5]. In a hands-on situation, this move to the Ind-category is achieved simply by working within the category of infinite-dimensional rather than finite-dimensional representations.

∞-categories from dg categories: From any dg category $\mathbf{A}$ we can construct an associated, or maybe *the* associated, ∞-category by applying the dg nerve $\mathscr{A} := \mathrm{N}^{\mathrm{dg}}(K\mathbf{A})$. This dg nerve construction is described in [15, 00PK], or Section I-2.2. This construction is very straightforward, and somewhat intuitive fromt he A_∞-perspective. An important point, however, is that one can equivalently construct the ∞-category $\mathscr{A}$–up to equivalence–by first factoring through the simplicial setting then apply the homotopy coherent nerve.

Specifically, we can apply the Dold-Kan functor $K : \mathrm{Ch}(\mathbb{A}) \to \mathrm{Kan}_{\mathbb{Z}}$ to morphisms to product from $\mathbf{A}$ a simplicial category $K\mathbf{A}$. This simplicial category has Hom spaces $K\,\mathrm{Hom}_{\mathbf{A}}(x, y)$ for each x and y. One then applies the homotopy coherent nerve [15, 00KM] to produce an ∞-category $\mathscr{A}' = \mathrm{N}^{\mathrm{hc}}(\mathbf{A})$ which admits an equivalence over the underlying plain category $Z^0(\mathbf{A})$,

$$\begin{tikzcd} \mathscr{A}' \arrow[rr, "\mathfrak{z}_{\mathbf{A}}"] & & \mathscr{A} \\ & Z^0(\mathbf{A}) \arrow[ul] \arrow[ur] & \end{tikzcd} .$$

See Theorem II-10.4. This simplicial construction is much more convenient to work with when considering, say, Hom functors for the associated ∞-category. See for example the materials of Section II-11.3. So, from a practical perspective, the approach via Dold-Kan and the homotopy coherent nerve is a “better” construction of this category.

We recall that in the algebraic situation, we consider an additive category $\mathbb{A}$, have its corresponding dg category $\mathbf{Ch}(\mathbb{A})$ of unbounded cochains.

Definition 1.1. For an additive category $\mathbb{A}$, the homotopy ∞-category $\mathscr{K}(\mathbb{A})$ is the associated ∞-category for the dg category of cochains over $\mathbb{A}$,

$$\mathscr{K}(\mathbb{A}) := \mathrm{N}^{\mathrm{dg}}\big(\mathbf{Ch}(\mathbb{A})\big).$$

The derived ∞-category: For a Grothendieck abelian category $\mathbb{A}$ the discrete category of cochains $\mathrm{Ch}(\mathbb{A})$ admits enough K-injectives [20], i.e. complexes I for which the cochain-valued Hom functor $\mathrm{Hom}^*_{\mathbb{A}}(-, I) : \mathrm{Ch}(\mathbb{A}) \to \mathrm{Ch}(\mathbb{Z})$ preserves acyclicity. In Parts I and II of this work we employed the following definition.

Proto-definition 1.2. For a Grothendieck abelian category $\mathbb{A}$, the derived ∞-category $\mathscr{D}(\mathbb{A})$ is the full ∞-subcategory of K-injective complexes in the homotopy ∞-category $\mathscr{K}(\mathbb{A})$.

We generally denoted this ∞-subcategory as $\mathscr{D}_{\mathrm{Inj}} = \mathscr{D}(\mathbb{A})$. It's shown in Section I-12 that one can equivalently construct the derived ∞-category via K-projective complexes, whenever such complexes are available.

From the perspective of this text, one should employ a coordinate free realization of the derived ∞-category as the localization of the homotopy ∞-category $\mathscr{D}(\mathbb{A}) = \mathscr{K}(\mathbb{A})[\mathrm{Qiso}^{-1}]$ against the class of quasi-isomorphisms in $\mathscr{K}(\mathbb{A})$. One can see in particular Theorem 6.6 below.

1.3. **Contextualization and originality.** Unlike Parts I and II, which essentially remixed and reorganized materials from Kerodon [15], the present text is a heterogenous collection of materials from Higher Topos Theory [13], Higher Algebra [14], and originally produced content. For example the majority of the contents from Sections 3 and 7 were developed independently. All of our computations from Section 2 are also original.

Additionally, in this text we regularly employ "fundamental" results from [13] and [14] without outlining or speaking to their proofs in any way. Hence we do not pursue a self-contained treatment of the topic at hand, and instead attempt to provide a treatment which is most effective in a practical sense. This deviates from the point-by-point, completionist approach taken in Parts I and II.

1.4. **Omissions et al.** In this text one finds various small omissions, such as a discussion of t-structures, and the generic omission of, say, 98% of higher algebra [14]. In any case, we expect that this point is clear to most readers. A point which might not be clear to most readers, however, is that there is some distinction between the framing of higher algebra [14] and the framing of spectral algebraic geometry [16].

From the perspective of higher algebra [14] one might locate the the derived ∞-category $\mathscr{D}(\mathbb{A})$ as a special point within the class of presentable stable ∞-categories, possibly with a t-structure. The presentation of [16] might instead emphasize the connective derived category $\mathscr{D}(\mathbb{A})^{\leq 0}$ as a special point within the class of Grothendieck prestable ∞-categories [16, Section C.1.4]. In the Grothendieck prestable setting we again have an effective algebra of categories [16, Section C.4], and such categories seemingly provide a more functional location in which to do algebraic geometry.

In any case, we'll simply alert the reader to the existence of this alternate universe of Grothendieck prestable ∞-categories, and note that the topics discussed herein are as relevant in the Grothendieck prestable context as they are in the stable presentable context. So, let us begin.

2. Limits and colimits in the homotopy ∞-category

We directly compute various limits and colimits in the homotopy ∞-category $\mathscr{K}(\mathbb{A})$, where $\mathbb{A}$ is an abelian category. Here we use the explicit descriptions of limits in $\mathscr{K}an$ provided in Section II-14.2 and the fact that limits and colimits can be detected by checking values under the application of Hom functors.

We use the calculations from this section to verify stability, and cocompleteness, of the homotopy and derived ∞-categories in Section 4 below. In considering the materials of Part II, one might also interpret the analysis herein as an exercise in which we apply the methods from Part II in a "real world" setting to compute limits and limits in a particular ∞-category of interest.

2.1. The homotopy ∞-category.

Definition 2.1. For an abelian category $\mathbb{A}$ we let $K\mathbf{Ch}(\mathbb{A})$ denote the simplicial category associated to the dg category of cochains over $\mathbb{A}$, and we take

$$\mathscr{K}(\mathbb{A})' := \mathrm{N}^{\mathrm{hc}}(K\mathbf{Ch}(\mathbb{A})).$$

We recall from Theorem II-10.4 that there is an equivalence of ∞-categories to the standard homotopy ∞-category $\mathfrak{Z} : \mathscr{K}(\mathbb{A})' \overset{\sim}{\to} \mathscr{K}(\mathbb{A})$ which is the identity on the underlying discrete categories

$$\begin{tikzcd} \mathscr{K}(\mathbb{A})' \arrow[rr, "\mathfrak{Z}"] & & \mathscr{K}(\mathbb{A}) \\ & \mathrm{Ch}(\mathbb{A}) \arrow[ul] \arrow[ur] & \end{tikzcd}$$

According to Proposition II-11.6 the functor

$$\underline{\mathrm{Hom}}_{\mathbb{A}}(V,-) := \underline{\mathrm{Hom}}_{K\mathbf{Ch}(\mathbb{A})}(V,-) : \mathscr{K}(\mathbb{A})' \to \mathscr{K}an$$

is corepresented by the given complex V. We recall that these morphism complexes in the simplicial category $\underline{\mathrm{Ch}}(\mathbb{Z})^{\leq 0}$ are explicitly given by the Eilenberg-MacLane spaces

$$\underline{\mathrm{Hom}}_{\mathbb{A}}(V,W) := K\,\mathrm{Hom}^*_{\mathbb{A}}(V,W).$$

In this particular setting Corollary II-16.17 appears as follows.

Proposition 2.2. *Given a diagram $p : K \to \mathscr{K}(\mathbb{A})'$, a given extension $\widetilde{p} : \{0\} \star K \to \mathscr{K}(\mathbb{A})'$ is a limit diagram in $\mathscr{K}(\mathbb{A})'$ if and only if, at each cochain complex V, the composite functor*

$$\underline{\mathrm{Hom}}_{\mathbb{A}}(V,-) \circ \widetilde{p} : \{0\} \star K \to \mathscr{K}an$$

is a limit diagram in $\mathscr{K}an$.

We are most interested in pullback diagrams in $\mathscr{K}(\mathbb{A})$ and/or $\mathscr{K}(\mathbb{A})'$. To get our wheels rolling in this direction, let us take a moment to recall the construction of 2-simplices in the ∞-categories $\mathscr{K}(\mathbb{A})'$ and $\mathscr{K}(\mathbb{A})$.

Directly, 2-simplices in $\underline{\mathrm{Hom}}_{\mathbb{A}}(V,W)$ are triples

$$\widetilde{h} = (h : V \to W,\ h_0 : V \to W,\ h_1 : V \to W)$$

with h of degree -1, the h_i cochain maps of degree 0, and

$$d(h) = d_W h + h d_V = h_0 - h_1.$$

The restrictions along the inclusions $\{i\} \to \Delta^2$ are as expected $\widetilde{h}|_{\{i\}} = h_i$. Now, according to the definition of the homotopy coherent nerve, a 2-simplex $\sigma : \Delta^2 \to \mathscr{K}(\mathbb{A})'$ is a not-necessarily-commuting diagram of cochain maps

$$\begin{tikzcd} & V_1 \arrow[dr, "f_{12}"] & \\ V_0 \arrow[ur, "f_{01}"] \arrow[rr, "f_{02}"'] & & V_2 \end{tikzcd}$$

and a 2-simplex $\widetilde{h}$ in $\underline{\mathrm{Hom}}_{\mathbb{A}}(V_0,V_2)$ with $\widetilde{h}|_0 = f_{12}f_{01}$ and $\widetilde{h}|_1 = f_{02}$, i.e. a choice of a degree -1 map h which establishes homotopy commutativity $d(h) = f_{12}f_{01} - f_{02}$. (See Lemma I-2.16).

To compare, a 2-simplex in the usual ∞-category $\mathscr{K}(\mathbb{A})$ is a choice of a not-necessarily-commuting diagram of cochain maps

$$\begin{tikzcd} & V_1 \arrow[dr, "f_{12}"] & \\ V_0 \arrow[ur, "f_{01}"] \arrow[rr, "f_{02}"'] & & V_2 \end{tikzcd}$$

and a degree -1 map $h : V_0 \to V_2$ satisfying $d(h) = f_{12}f_{01} - f_{02}$. These are clearly the same thing.

2.2. **Kan fibrations for simplicial abelian groups.**

Lemma 2.3. (1) *If A is a discrete simplicial abelian group, then the inclusion $0 \to A$ is a Kan fibration.*
(2) *Any surjective map of simplicial abelian groups $f : X \to Y$ is a Kan fibration.*
(3) *If A is a discrete simplicial abelian group, and $f : X \to Y$ is a surjective, then the map $[0\ f]^t : X \to A \times Y$ is a Kan fibration.*

Proof. (1) Follows from the fact that, in this case, any simplex $\Delta^n \to A$ in which a single vertex maps to 0 is of constant value 0. (2) Consider a lifting diagram

$$\begin{tikzcd} \Lambda^n_i \arrow[r, "\tau"] \arrow[d] & X \arrow[d, "f"] \\ \Delta^n \arrow[r, "\bar{\sigma}"'] & Y. \end{tikzcd}$$

We can lift $\bar{\sigma}$ arbitrarily to an n-simplex $\sigma : \Delta^n \to X$, via surjectivity of f. We can now replace $\bar{\sigma}$ with the 0 simplex and τ with $\sigma|_{\Lambda^n_i} - \tau$ to reduce to the case $Y = 0$. In this case the desired solution exists since X is a Kan complex (Proposition I-10.1). (3) In this case $[0\ f]^t$ can be identified with a product of Kan fibrations $0 \times f : 0 \times X \to A \times Y$, and is thus a Kan fibration. $\square$

Corollary 2.4. *For any map $f : V \to W$ of cochains of abelian groups in which $f^n : V^n \to W^n$ is surjective at all $n < 0$, the corresponding map $Kf : KV \to KW$ is a Kan fibration.*

Proof. We can factor f as the inclusion $V \to Z^0(W) \times V$ composed with the map $[i\ f] : Z^0(W) \times V = Z^0(W) \oplus V \to W$, where i here is the inclusion $i : Z^0(W) \to W$. Then Kf factors as the sequence

$$KV \xrightarrow{[0\ id]^t} KZ^0(W) \times KV \xrightarrow{[i\ Kf]} KW$$

in which the latter map is surjective, since the Eilenbergh-MacLane functor $K = K\tau_0$ is an equivalence on connective cochains and the map $\tau_0(Z^0(W) \times V) \to \tau_0(W)$ is surjective by construction. By Lemma 2.3 it follows that Kf is a Kan fibration. $\square$

2.3. **Claims: pullbacks in the homotopy ∞-category.**

Definition 2.5. For maps of cochains $f : V \to W$ and $f' : V' \to W$, we take

$$C(f, f') := \Sigma^{-1} \operatorname{cone}(\ [-f\ \ f'] : V \times V' \to W\ ).$$

For a single map $f : V \to W$ we take $C(f) = \Sigma^{-1} \operatorname{cone}(-f)$.

The complex $C(f, f')$ appears as

$$\left((V \times V') \oplus \Sigma^{-1} W,\ \begin{bmatrix} d_{V \times V'} & [f\ \ -f'] \\ 0 & -d_W \end{bmatrix} \right).$$

Note that we have the embedding of cochains

$$V \times_W V' = \ker([f\ \ -f']) \to C(f, f')$$

We have the degree -1 map of graded objects in $\mathbb{A}$

$$h_W = [0\ id_W] : C(f, f') \to W \tag{1}$$

with

$$d_{\mathrm{Hom}}(h_W) = \big(C(f, f') \xrightarrow{\pi} V \times V' \xrightarrow{[f\ -f']} W\big),$$

where π is the obvious projection. This homotopy defines a 2-simplex $h_W : \Delta^2 \to \mathscr{K}(\mathbb{A})$ which appears as

$$\begin{tikzcd} & V \arrow[dr, "f"] & \\ C(f, f') \arrow[ur, "p"] \arrow[rr, "f'p'"'] & & W. \end{tikzcd}$$

We append a strictly commuting diagram for V' to obtain a square $\Delta^1 \times \Delta^1 \to \mathscr{K}(\mathbb{A})$ which appears as

$$\begin{tikzcd} C(f, f') \arrow[r, "p"] \arrow[d, "p'"'] \arrow[dr, "f'p'" description] & V \arrow[d, "f"] \\ V' \arrow[r, "f'"'] & W. \end{tikzcd} \tag{2}$$

Definition 2.6. Given arbitrary morphisms $f : V \to W$ and $f' : V' \to W$ of $\mathbb{A}$-cochains, the corresponding standard pullback diagram is the diagram (2) in $\mathscr{K}(\mathbb{A})$ produced from the shifted mapping cone $C(f, f')$ and the homotopy h_Q of (1).

We prove in Proposition 2.12 below that any standard pullback diagram is in fact a limit diagram in $\mathscr{K}(\mathbb{A})$. We note that the construction of the standard pullback is natural in the sense that a strictly commuting diagram

$$\begin{tikzcd} V \arrow[r] \arrow[dr, "f"'] & X \arrow[dr, "g"] & V' \arrow[r] \arrow[dl, "f'"] & X' \arrow[dl, "g'"] \\ & W \arrow[r] & Y & \end{tikzcd}$$

extends to a diagram of the form

$$
\begin{tikzcd}
 & C(f,f') \arrow[r] \arrow[dl] \arrow[dr] & C(g,g') \arrow[dl] \arrow[dr] & \\
V' \arrow[r, "="] \arrow[dr] & X' \arrow[dr] & V \arrow[r] \arrow[dl] & X \arrow[dl] \\
 & W \arrow[r] & Y &
\end{tikzcd}
$$

in $\mathscr{K}(\mathbb{A})$.

2.4. **Pullbacks in the homotopy ∞-category.** Throughout the subsection we fix $\mathbb{A}$ an abelian category. We establish some background materials before returing to address the issue of pullbacks.

Definition 2.7. For any abelian category $\mathbb{A}$ we let

$$\tau_0 : \mathrm{Ch}(\mathbb{A}) \to \mathrm{Ch}(\mathbb{A})^{\leq 0}$$

denote the truncation functor, $\tau_0 V = \cdots \to V^{-2} \to V^{-1} \to Z^0(V) \to 0$.

If we let $\mathbf{Ch}'(\mathbb{A})$ denote the dg category of cochains with mapping complexes $\mathrm{Hom}_{\mathbb{Z}}^{\leq 0}(X,Y)$, then the functor τ_0 enhances to a dg functor

$$\tau_0 : \mathbf{Ch}'(\mathbb{A}) \to \mathbf{Ch}(\mathbb{A})^{\leq 0}.$$

In particular, τ_0 respects homotopy and homotopy equivalences. Also, since τ_0 is right adjoint to the inclusion $\mathrm{Ch}(\mathbb{A})^{\leq 0} \to \mathrm{Ch}(\mathbb{A})$ this functor commutes with limits.

Let us say that a map of $\mathbb{A}$-cochains $f : V \to W$ is termwise split surjective if, for each integer n, the map $f^n : V^n \to W^n$ is split surjective.

Lemma 2.8. *For a map of $\mathbb{A}$-cochains $f : V \to W$ the following are equivalent:*

1. *f is termwise split surjective.*
2. *f is split surjective as a map of graded objects in $\mathbb{A}$.*
3. *For each cochain X, the induced map $f_* : \mathrm{Hom}^*_{\mathbb{A}}(X,V) \to \mathrm{Hom}^*_{\mathbb{A}}(X,W)$ is surjective.*

Proof. Omitted. $\square$

The following might be seen as an algebraic analog of Corollary II-14.31.

Proposition 2.9. *Consider maps of $\mathbb{A}$-cochains $f : V \to W$ and $f' : V' \to W$, and suppose one of f or f' is termwise split surjective. Then the strictly commuting pullback diagram*

$$
\begin{tikzcd}
V \times_W V' \arrow[r] \arrow[d] & V \arrow[d] \\
V' \arrow[r] & W
\end{tikzcd}
\tag{3}
$$

is a limit diagram in $\mathscr{K}(\mathbb{A})'$.

Proof. Assume arbitrarily that f is graded split. Take

$$\mathrm{Hom}^{\bullet}_{\mathbb{A}}(X,Y) = \tau_0 \, \mathrm{Hom}^*_{\mathbb{A}}(X,Y)$$
$$= \cdots \to \mathrm{Hom}^{-2}_{\mathbb{A}}(X,Y) \to \mathrm{Hom}^{-1}_{\mathbb{A}}(X,Y) \to Z^0 \, \mathrm{Hom}^*_{\mathbb{A}}(X,Y) \to 0.$$

For each cochain complex X we have

$$\mathrm{Hom}^\bullet_{\mathbb{A}}(X, V \times_W V') = \mathrm{Hom}^\bullet_{\mathbb{A}}(X, V) \times_{\mathrm{Hom}^\bullet_{\mathbb{A}}(X,W)} \mathrm{Hom}^\bullet_{\mathbb{A}}(X, V')$$

so that the induced diagram

$$\begin{tikzcd} \mathrm{Hom}^\bullet_{\mathbb{A}}(X, V \times_W V') \arrow[r] \arrow[d] & \mathrm{Hom}^\bullet_{\mathbb{A}}(X, V) \arrow[d, "f_*"] \\ \mathrm{Hom}^\bullet_{\mathbb{A}}(X, V') \arrow[r, "f'_*"'] & \mathrm{Hom}^\bullet_{\mathbb{A}}(X, W) \end{tikzcd} \tag{4}$$

is a pullback diagram. Furthermore, by our splitting assumption, the map f_* is split in each strictly negative degrees. In particular, f_* is surjective in all strictly negative degrees.

We now apply the Eilenbergh-MacLane functor K to obtain a pullback diagram

$$\begin{tikzcd} \underline{\mathrm{Hom}}_{\mathbb{A}}(X, V \times_W V') \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}_{\mathbb{A}}(X, V) \arrow[d, "Kf_*"] \\ \underline{\mathrm{Hom}}_{\mathbb{A}}(X, V') \arrow[r, "Kf'_*"'] & \underline{\mathrm{Hom}}_{\mathbb{A}}(X, W) \end{tikzcd}$$

in which the map Kf_* is a Kan fibration by Corollary 2.4. The above diagram is therefore a pullback diagram in $\mathscr{K}an$ by Corollary II-14.31. Since X was chosen arbitrarily we apply Corollary II-16.17 to observe that the diagram (9) is a pullback diagram in $\mathscr{K}(\mathbb{A})$. $\square$

Note that the above strict pullback diagram is in the image of the inclusion $\mathrm{Ch}(\mathbb{A}) \to \mathscr{K}(\mathbb{A})'$. Since the equivalence $\mathfrak{Z} : \mathscr{K}(\mathbb{A})' \to \mathscr{K}(\mathbb{A})$ restricts to the identity on $\mathrm{Ch}(\mathbb{A})$ we see that it preserves all strict pullback diagrams. Since equivalences preserve limits, by Proposition II-13.10, Proposition 2.9 implies the following.

Corollary 2.10. *Consider maps of connective cochains $f : V \to W$ and $f' : V' \to W$, and suppose one of f or f' splits as a graded morphism. Then the strictly commuting pullback diagram*

$$\begin{tikzcd} V \times_W V' \arrow[r] \arrow[d] & V \arrow[d] \\ V' \arrow[r] & W \end{tikzcd} \tag{5}$$

is a limit diagram in $\mathscr{K}(\mathbb{A})$.

In this split setting the shifted mapping cone $C(f, f')$ one sees furthermore that the fiber product are identified.

Lemma 2.11. *Consider maps of cochains $f : V \to W$ and $f' : V' \to W$, and suppose one of f or f' splits as a map of graded objects in $\mathbb{A}$. Then the inclusion*

$$V \times_W V' \to C(f, f')$$

is a homotopy equivalence.

Proof. By replacing V with $V \oplus V'$ and V' with 0, it suffices to prove that the inclusion $\ker(f) \to C(f)$ is a homotopy equivalence in the case that $f : V \to W$ is split surjective as a graded map.

Via the splitting we can write $V \cong \Sigma L \oplus K$ with $L = \Sigma^{-1} W$ and $K = \ker(f)$. Here K is a subcomplex in V and the map $V \to W$ is just the projection onto the first factor. We may assume for simplicity that, in fact, this isomorphism is an equality of graded objects $V = \Sigma L \oplus K$.

The composite

$$\Sigma L \xrightarrow{\text{incl}} V \overset{d_V}{\to} V \xrightarrow{\text{proj}} K$$

defines a degree 1 map from ΣL, which is then a degree 0 map $g : L \to K$. This map is seen to be a cochain morphism so that

$$V = \operatorname{cone}(g) = \left(\Sigma L \oplus K, \begin{bmatrix} -d_L & g \\ 0 & d_K \end{bmatrix} \right).$$

We now have

$$C(f) = \left(\Sigma L \oplus K \oplus L, \begin{bmatrix} -d_L & g & id \\ 0 & d_K & 0 \\ 0 & 0 & d_L \end{bmatrix} \right)$$

and observe the projection

$$\pi = [0 \ id_K \ -g] : C(f) \to K.$$

We have directly $\pi \,\text{incl} = id_K : K \to K$ and the composite $\text{incl}\,\pi : C(f) \to C(f)$ is homotopic to the identity via the degree -1 map

$$h = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ id_L & 0 & 0 \end{bmatrix} : C(f) \to C(f)$$

□

Now, we have the inclusion $V \times_W V' \to C(f, f')$ and general f and f', and the homotopy h_W from (1) has trivial restriction $h_W|_{V \times_W V'}$. This inclusion therefore extends to a natural transformation of diagram

$$\begin{tikzcd}
 & V \times_W V' \arrow[r] \arrow[dl] \arrow[dr] & C(f,f) \arrow[dl] \arrow[dr] \arrow[dd, dashed] & \\
V' \arrow[r, "="] \arrow[dr] & V' \arrow[dr] & V \arrow[r, "="] \arrow[dl] & V \arrow[dl] \\
 & W \arrow[r, "="'] & W & .
\end{tikzcd} \tag{6}$$

By Lemma 2.11 this transformation is an isomorphism whenever f or f' is graded split.

Proposition 2.12. *For arbitrary maps $f : V \to W$ and $f' : V' \to W$ in $\mathscr{K}(\mathbb{A})$, the standard pullback diagram*

$$\begin{tikzcd} C(f,f') \arrow[r] \arrow[d] \arrow[dr] & V \arrow[d, "f"] \\ V' \arrow[r, "f'"'] & W \end{tikzcd} \tag{7}$$

(see Definition 2.6) is a limit diagram in $\mathscr{K}(\mathbb{A})$. In particular, the diagram (7) is isomorphic to a diagram of the form

$$\begin{tikzcd} V_0 \arrow[r] \arrow[d] & V_1 \arrow[d, "f_{12}"] \\ V_1' \arrow[r, "f_{12}'"'] & V_2 \end{tikzcd}$$

in which f_{12} is termwise split surjective and f_{12}' is injective

Proof. Take $V_2 = \mathrm{cone}(id_{V'}) \oplus W$, $V_1' = V'$, and $f_{12}' = [i\ f']^t : V' \to \mathrm{cone}(id_{V'}) \oplus W$ where $i : V' \to \mathrm{cone}(id_{V'})$ is the usual inclusion. Take now $V_1 = V \oplus C(id_{V_2})$ and $f_{12} = [\pi\ f] : V \oplus C(id_{V_2}) \to V_2$ where $\pi : C(id_{V_2}) \to V_2$ is the usual projection. The map f_{12}' is injective and g is split as a graded morphism via the identity map $V_2 to V_2 \oplus \Sigma^{-1} V_2 = C(id_{V_2})$.

Since the mapping cone of any identity morphism is contractible, the summands $C(id_{V'})$ and $C(id_{V_2})$ are contractible. The inclusion $V \to V_1$ and $V' \to V_2$ are therefore homotopy equivalence and induce an isomorphism of diagrams

$$\begin{tikzcd} & & C(f,f') \arrow[r] \arrow[dll] \arrow[dr] & C(f_{12},f_{12}') \arrow[dll] \arrow[drr] & & \\ V' \arrow[r, "="] \arrow[dr] & V_1' \arrow[drr] & & V \arrow[rr, "\mathrm{incl}"] \arrow[dll] & & V_1 \arrow[dll] \\ & W \arrow[rr, "="'] & & V_2 & & \end{tikzcd}$$

in $\mathscr{K}(\mathbb{A})$. As argued at (6) the diagram for $C(f_{12}, f_{12}')$ is furthermore isomorphic to the discrete pullback diagram

$$\begin{tikzcd} V_0 = V_1 \times_{V_2} V_1' \arrow[r] \arrow[d] & V_1 \arrow[d, "g"] \\ V_1' \arrow[r] & V_2, \end{tikzcd} \tag{8}$$

so that in total the diagram (7) is isomorphic to the diagram (8). Since the latter diagram is a pullback diagram in $\mathscr{K}(\mathbb{A})$ by Proposition 2.9, it follows by Proposition II-13.18 that the diagram (8) is a pullback diagram as well. $\square$

As a corollary we find that any diagram

$$\begin{tikzcd} & V \arrow[d] \\ V' \arrow[r] & W \end{tikzcd}$$

in $\mathscr{K}(\mathbb{A})$ admits a limit, or, in slightly more informal terms, that $\mathscr{K}(\mathbb{A})$ has pullbacks.

Corollary 2.13. *Every diagram $\Lambda^2_2 \to \mathscr{K}(\mathbb{A})$ admits a limit. That is to say, for any abelian category $\mathbb{A}$, the homotopy ∞-category $\mathscr{K}(\mathbb{A})$ admits all pullbacks.*

2.5. **Pushouts diagrams in the homotopy ∞-category.** We recall that, for a partial diagram of cochains

$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d, "g'"'] & V \\ V' \end{tikzcd}$$

the pushout is the quotient $V \amalg_{W'} V' = \operatorname{coker}(W \to V \oplus V')$.

Let us call a general morphism of $\mathbb{A}$-cochains $g : W \to V$ termwise split injective if, at each integer n, $g^n : W^n \to V^n$ is a split injective morphism in $\mathbb{A}$. We have the expected analog of Lemma 2.8.

Lemma 2.14. *For a map of $\mathbb{A}$-cochains $g : W \to V$ the following are equivalent:*

1. *g is termwise split injective.*
2. *g is split injective as a map of graded objects in $\mathbb{A}$.*
3. *For each cochain Y, the induced map $g^* : \operatorname{Hom}^*_{\mathbb{A}}(V, Y) \to \operatorname{Hom}^*_{\mathbb{A}}(W, Y)$ is surjective.*

For any abelian category $\mathbb{A}$, we apply Corollary 2.10 to the opposite category $\mathbb{B} = \mathbb{A}^{\mathrm{op}}$ to obtain the corresponding result for pushout diagrams in $\mathscr{K}(\mathbb{A})$.

Proposition 2.15. *Consider maps of $\mathbb{A}$-cochains $g : V \to W$ and $g' : V' \to W$, and suppose one of g or g' splits as a graded morphism. Then the strictly commuting pullback diagram*

$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d, "g'"'] & V \arrow[d] \\ V' \arrow[r] & V \amalg_W V' \end{tikzcd} \tag{9}$$

is a colimit diagram in $\mathscr{K}(\mathbb{A})$.

In the pushout context we return to the standard, rather than shifted, mapping cone.

Definition 2.16. Given maps $g : W \to V$ and $g' : W \to V'$ of $\mathbb{A}$-cochains we take

$$\operatorname{cone}(g, g') = \operatorname{cone}\left(\ [g \ \ -g']^t : W \to V' \oplus V\ \right).$$

Of course, in the case $V' = 0$ we have $\operatorname{cone}(g) = \operatorname{cone}(g, 0)$. We have the two inclusions from V and V' into $\operatorname{cone}(g, g')$ which provide a generally noncommuting diagram

$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d, "g'"'] & V \arrow[d] \\ V' \arrow[r] & \mathrm{cone}(g,g'). \end{tikzcd}$$

The degree -1 map $h'_W : W \to \mathrm{cone}(g,g')$ defined by the identity on W satisfies

$$d_{\mathrm{Hom}}(h'_W) = [g \ \ -g']^t : W \to V \oplus V' \subseteq \mathrm{cone}(g,g')$$

and hence produces a diagram

$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d, "g'"'] \arrow[dr] & V \arrow[d] \\ V' \arrow[r] & \mathrm{cone}(g,g'). \end{tikzcd} \tag{10}$$

in the homotopy ∞-category $\mathscr{K}(\mathbb{A})$ in which the bottom simplex is degenerate and the top simplex is exhibited via h'_W.

Definition 2.17. Given maps $g : W \to V$ and $g' : W \to V'$ in $\mathscr{K}(\mathbb{A})$, we refer to the correpsonding diagram (10) as the standard pushout diagram associated to g and g'.

Of course, we will see momentarily that standard pushout diagrams are colimit diagram in $\mathscr{K}(\mathbb{A})$. We obtain the following by applying Lemma 2.11 to the opposite category.

Lemma 2.18. *Consider maps of $\mathbb{A}$-cochains $g : V \to W$ and $g' : V' \to W$, and suppose one of g or g' splits as a graded morphism. Then the projection π : $\mathrm{cone}(g,g') \to V \amalg_W V'$ is a homotopy equivalence.*

Since the homotopy h'_W vanishes when composed with the projection $\pi : \mathrm{cone}(g,g') \to V \amalg_W V'$, this projection extends to a diagram in $\mathscr{K}(\mathbb{A})$ which appears as

$$\begin{tikzcd} & W \arrow[rr, "="] \arrow[dl] \arrow[dd, dashed] \arrow[drr] & & W \arrow[dll] \arrow[dr] & \\ V' \arrow[r, "="] \arrow[dr] & V' \arrow[d, phantom] \arrow[drr] & & V \arrow[r, "="] \arrow[dll] & V \arrow[dl] \\ & \mathrm{cone}(g,g') \arrow[rr] & & V \amalg_W V' & \end{tikzcd} .$$

According to Lemma 2.18, in the case that one of g or g' is split injective this diagram realizes an isomorphism between the two square faces, so that the standard square

$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d, "g'"'] \arrow[dr] & V \arrow[d] \\ V' \arrow[r] & \mathrm{cone}(g,g'). \end{tikzcd}$$

is observed to be a pushout square via Proposition 2.15. This realization of a colimit for such partial pushout diagrams now generalizes to arbitrary pairs of maps.

Proposition 2.19. *For arbitrary maps $g : W \to V$ and $g' : W \to V'$ in $\mathscr{K}(\mathbb{A})$, the standard pushout diagram*

$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d, "g'"'] \arrow[dr] & V \arrow[d] \\ V' \arrow[r] & \mathrm{cone}(g, g') \end{tikzcd} \tag{11}$$

is a colimit diagram in $\mathscr{K}(\mathbb{A})$. In particular, the diagram (11) is isomorphic to a diagram of the form

$$\begin{tikzcd} V_0 \arrow[r, "g_{01}"] \arrow[d, "g'_{01}"'] & V_1 \arrow[d] \\ V'_1 \arrow[r] & V_2 \end{tikzcd}$$

in which g_{01} is termwise split injective and g'_{01} is injective

Corollary 2.20. *Every diagram $\Lambda^2_0 \to \mathscr{K}(\mathbb{A})$ admits a colimit. That is to say, for any abelian category $\mathbb{A}$, the homotopy ∞-category $\mathscr{K}(\mathbb{A})$ admits all pushouts.*

2.6. **Products, coproducts, and the zero complex.**

Proposition 2.21. *For any abelian category $\mathbb{A}$, the functor* $\mathrm{Ch}(\mathbb{A}) \to \mathscr{K}(\mathbb{A})$ *preserves all small products and coproducts. In particular, the category $\mathscr{K}(\mathbb{A})$ admits all small products and coproducts.*

Proof. Since the inclusion $\mathrm{Ch}(\mathbb{A}) \to \mathscr{K}(\mathbb{A})$ factors through the equivalence $\mathfrak{Z} : \mathscr{K}(\mathbb{A})' \to \mathscr{K}(\mathbb{A})$ it suffices to show that products and coproducts in $\mathrm{Ch}(\mathbb{A})$ are products and coproducts in the simplicial construction of the homotopy ∞-category $\mathscr{K}(\mathbb{A})'$. For this we employ the Hom functor $\mathrm{K}\operatorname{Hom}^\bullet_{\mathbb{A}}$, where $\operatorname{Hom}^\bullet_{\mathbb{A}}(X, Y) = \tau_0 \operatorname{Hom}^*_{\mathbb{A}}(X, Y)$ and check that this functor turns discrete coproducts into products of spaces through the first coordinate, and discrete products into products of spaces through the secord coordinate. (Recall that products of spaces are as expected, by Example II-14.16 and Theorem II-14.25.)

Since τ_0 is a right adjoint it commutes with limits, and the Dold-Kan equivalence commutes with limits as well, it suffices to show that the functor $\operatorname{Hom}^*_{\mathbb{A}}$ sends discrete coproducts in the first coordinate to products of linear cochains, and discrete products in the second coordinate to products of linear cochains. However this follows immediately by, say, the fact that $\operatorname{Hom}^*_{\mathbb{A}}$ provide inner-Homs for the action of $\mathrm{Ch}(\mathbb{Z})$ on $\mathrm{Ch}(\mathbb{A})$. $\square$

As a corollary to Lemma II-9.18 we also see that the zero complex provides a zero object for the homotopy ∞-category.

Corollary 2.22. *The zero complex provides a simultaneous initial and terminal object in the ∞-category $\mathscr{K}(\mathbb{A})$.*

2.7. **Resolutions as (co)limits.**

Proposition 2.23. *Given a strictly commuting diagram $I(-) : \mathbb{Z}_{\leq 0} \to \mathscr{K}(\mathbb{A})$ in which each map $I(n-1) \to I(n)$ is termwise split surjective and with discrete limit*

I in $\mathrm{Ch}(\mathbb{A})$*, the discrete limit diagram*

$$\begin{tikzcd} \cdots \arrow[r] & I(m-1) \arrow[r] & I(m) \arrow[r] & I(m+1) \arrow[r] & \cdots \\ & & I \arrow[ul] \arrow[u] \arrow[ur] & & \end{tikzcd}$$

is a limit diagram in $\mathscr{K}(\mathbb{A})$*. In particular, the discrete limit is an* ∞*-categorical limit in this case.*

Proof. Since the diagram factors through $\mathrm{Ch}(\mathbb{A})$ we can work in the simplicial construction $\mathscr{K}(\mathbb{A})$. Applying the standard corepresentable functor $K\,\mathrm{Hom}^*_{\mathbb{A}}(X,-)$ to the given squence produces a sequence of Kan fibrations

$$\cdots \to K\,\mathrm{Hom}^*_{\mathbb{A}}(X, I(m-1)) \to K\,\mathrm{Hom}^*_{\mathbb{A}}(X, I(m)) \to \cdots \tag{12}$$

in Kan $\subseteq \mathscr{K}an$, by Corollary 2.4. Since K has a left adjoint, given by the normalized cochains functor, it commutes with discrete limits, so that

$$\lim_n K\,\mathrm{Hom}^*_{\mathbb{A}}(X, I(n)) = K \lim_n \mathrm{Hom}^*_{\mathbb{A}}(X, I(n)) = \mathrm{Hom}^*_{\mathbb{A}}(X, L).$$

By Proposition II-14.38 and Theorem II-14.42, in this case the discrete limit diagram for the sequence (12) in Kan provides a limit diagram in $\mathscr{K}an$. By the above formula this discrete limit diagram is the image of the discrete limit diagram for the functor $I : \mathbb{Z}_{\leq 0} \to \mathrm{Ch}(\mathbb{A}) \subseteq \mathscr{K}(\mathbb{A})$. So we see that, at each complex X, the functor $K\,\mathrm{Hom}^*_{\mathbb{A}}(X,-)$ sends the discrete limit diagram for I to a limit diagram in spaces. By Corollary II-16.17 it follows that the discrete limit diagram for I is a limit diagram in $\mathscr{K}(\mathbb{A})'$, and hence in $\mathscr{K}(\mathbb{A})$. □

By consulting the opposite category, or simply by repeating the above arguments, one obtains a statement for colimits of sequences of split injections.

Proposition 2.24. *Given a strictly commuting diagram* $P(-) : \mathbb{Z}_{\geq 0} \to \mathscr{K}(\mathbb{A})$ *in which each map* $P(n) \to P(n+1)$ *is termwise split injective and with discrete colimit* P *in* $\mathrm{Ch}(\mathbb{A})$*, the discrete colimit diagram*

$$\begin{tikzcd} & & P \arrow[dl] \arrow[d] \arrow[dr] & & \\ \cdots \arrow[r] & P(m-1) \arrow[r] & P(m) \arrow[r] & P(m+1) \arrow[r] & \cdots \end{tikzcd}$$

is a colimit diagram in $\mathscr{K}(\mathbb{A})$*. In particular, the discrete colimit is an* ∞*-categorical limit in this case.*

These propositions are relevant which expressing bounded complexes as filtered limits of bounded injectives, or filtered colimits of bounded projectives. In order to speak about this point precisely we should understand the derived ∞-category $\mathscr{D}(\mathbb{A})$ as a localization of the homotopy ∞-category $\mathscr{K}(\mathbb{A})$, i.e. as an ∞-category with some universal map $\mathscr{K}(\mathbb{A}) \to \mathscr{D}(\mathbb{A})$. This localization map will be realized, in the injective model, as a right adjoint to the inclusion $\mathscr{D}(\mathbb{A}) \to \mathscr{K}(\mathbb{A})$ and, in the projective model, as a left adjoint thereof. In particular, the localization map is seen to commute with limits in the first construction and colimits in the second construction. Via uniqueness it therefore commutes with both limits and colimits. (See Section 6.4.) So, we can simply provide the relevant statement at the homotopy level.

Example 2.25. Let $V \to I$ be an injective resolution of a bounded below complex V in $\mathrm{Ch}(\mathbb{A})$. Consider the complexes $I(n) = I/I^{\geq -n}$ and the correpsonding sequence of termwise split sujections of bounded complexes of injectives

$$\cdots \to I(-2) \to I(-1) \to I(0).$$

We have the corresponding strictly commuting diagram $I(-) : \mathbb{Z}_{\leq 0} \to \mathscr{K}(\mathbb{A})$ with discrete limit $\lim_n I(n) = I$. By Proposition 2.23 this discrete limit is limit diagram in $\mathscr{K}(\mathbb{A})$.

Example 2.26. Let $P \to W$ be a projective resolution of a bounded above complex in $\mathrm{Ch}(\mathbb{A})$. Consider the compelxes $P(n) = P^{\geq -n}$ and the corresponding sequence of termqise split inclusions

$$P(0) \to P(1) \to P(2) \to \cdots .$$

We have the associated strictly commuting diagram in $\mathscr{K}(\mathbb{A})$, which has colimit P by Proposition 2.24.

3. Zero objects and spectral Hom functors

Given an ∞-category $\mathscr{C}$, we obtain the unique Hom functor

$$\text{“}\mathrm{Hom}_{\mathscr{C}}\text{”} = H : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{K}an$$

as the transport functor along the twisted arrows fibration (see Section II-12.1). In the event that $\mathscr{C}$ has a zero object, each space $H(x, y)$ should have a distinguished zero morphism which is preserved under composition. In this way the Hom functor should inherit a pointing, and we expect a *lifting* of H to a functor valued in the category $\mathscr{K}an_*$ of pointed spaces.

We prove that such a canonical pointing for the Hom functor exist, and prove further that, whenever $\mathscr{C}$ admits a certain symmetry around pullbacks an pushouts, this pointed Hom functor furthermore enhances to a spectra-valued Hom functor for $\mathscr{C}$.

Remark 3.1. In this text we tend not to use spectra directly, as the structure is a bit too technical for our taste at this “introductory” juncture. However, spectra will play a fundamental role in any in depth treatment of the subject, see e.g. [14]. One might keep this point in mind as they traverse the various topics, and arguments appearing in this text.

3.1. **Pointed Hom functors.**

Consider the left fibration

$$q : \mathscr{K}an_*(= \mathscr{K}an_{*/}) \to \mathscr{K}an$$

and any ∞-category $\mathscr{C}$ with a zero object. Recall that the identity morphisms $* \to *$ is simultaneously initial and terminal in the category of pointed spaces $\mathscr{K}an_*$, by Proposition II-9.15. Furthermore, any choice of a zero object $0 : * \to \mathscr{C}$ then provides an initial object

$$\vec{0} = (0, 0) : * \to \mathscr{C}^{\mathrm{op}} \times \mathscr{C}.$$

We obtain the following as a direct application of Corollary II-9.25.

Proposition 3.2. *Let $\mathscr{C}$ be an ∞-category with a zero object 0, and take $\mathscr{C}^{\mathrm{e}} = \mathscr{C}^{\mathrm{op}} \times \mathscr{C}$. The functors*

$$\mathrm{Fun}(\mathscr{C}^e, \mathscr{K}an_*) \to \mathscr{K}an_* \times_{\mathscr{K}an} \mathrm{Fun}(\mathscr{C}^e, \mathscr{K}an), \quad F \mapsto (F|_{\vec{0}}, qF), \qquad (13)$$

and

$$\mathrm{Fun}(\Delta^1 \times \mathscr{C}^{\mathrm{e}}, \mathscr{K}an_*) \to \mathscr{K}an_* \times_{\mathscr{K}an} \mathrm{Fun}(\Delta^1 \times \mathscr{C}^{\mathrm{e}}, \mathscr{K}an), \quad \zeta \mapsto (\zeta|_{(0,\vec{0})}, q\zeta),$$

are trivial Kan fibrations.

Given any Hom functor $H : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{K}an$ (Definition II-12.4), we take the fiber of the trivial Kan fibration (13) at the pairing of H with any choice of element $* \to H(0,0)$ in the contractible space $H(0,0) \cong \mathrm{Hom}_{\mathscr{C}}(0,0)$ to obtain a contractibel space of pointings for the functor H.

Corollary 3.3. *For an ∞-category $\mathscr{C}$ equipped with a choice of zero object $0 : * \to \mathscr{C}$, Hom functor $H : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{K}an$, and point $1 : * \to H(0,0)$, there is a unique functor $\widetilde{H} : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{K}an_*$ which fits into a strictly commuting diagram*

$$\begin{tikzcd} & \mathscr{K}an_* \arrow[d] \\ \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \arrow[ur, "\widetilde{H}"] \arrow[r, "H"'] & \mathscr{K}an \end{tikzcd}$$

and for which $\widetilde{H}(0,0) = (1 : * \to H(0,0))$.

One can show furthermore that there is a unique pointing of any Hom functor, in an absolute sense.

Theorem 3.4. *Let $\mathscr{C}$ be an ∞-category which admits a zero object. Then for any choice of Hom functor $H : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{K}an$, the space*

$$\mathrm{Fun}(\mathscr{C}^{\mathrm{op}} \times \mathscr{C}, \mathscr{K}an_*) \times_{\mathrm{Fun}(\mathscr{C}^{\mathrm{op}} \times \mathscr{C}, \mathscr{K}an)} \{H\}$$

of lifts, i.e. the space of functors $\widetilde{H} : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{K}an_*$ which fit into a strictly commuting diagrams

$$\begin{tikzcd} & \mathscr{K}an_* \arrow[d] \\ \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \arrow[ur, "\widetilde{H}"] \arrow[r, "H"'] & \mathscr{K}an, \end{tikzcd}$$

is a contractible Kan complex.

Proof. Take again $\mathscr{C}^{\mathrm{e}} = \mathscr{C}^{\mathrm{op}} \times \mathscr{C}$ and fix any zero object $0 : * \to \mathscr{C}$. The sequence

$$* \xrightarrow{H} \mathrm{Fun}(\mathscr{C}^{\mathrm{e}}, \mathscr{K}an) \xrightarrow{\vec{0}^*} \mathscr{K}an$$

picks out the space $H(0,0)$ in $\mathscr{K}an$, so that the fiber

$$(\mathscr{K}an_* \times_{\mathscr{K}an} \mathrm{Fun}(\mathscr{C}^{\mathrm{e}}, \mathscr{K}an)) \times_{\mathrm{Fun}(\mathscr{C}^{e}, \mathscr{K}an)} \{H\} = \mathscr{K}an_* \times_{\mathscr{K}an} \{H(0,0)\}$$

is the left pinched mapping space $\mathrm{Hom}^{\mathrm{L}}_{\mathscr{K}an}(*, H(0,0))$. Since $H(0,0)$ is contractible, it is terminal in $\mathscr{K}an$ by Lemma II-9.3. Hence this mapping space $\mathrm{Hom}^{\mathrm{L}}_{\mathscr{K}an}(*, H(0,0))$ is contractible.

Now, by the above information the fiber

$$\mathrm{Fun}(\mathscr{C}^{\mathrm{e}}, \mathscr{K}an_*) \times_{\mathrm{Fun}(\mathscr{C}^{e}, \mathscr{K}an)} \{H\}$$

$$= \mathrm{Fun}(\mathscr{C}^{\mathrm{e}}, \mathscr{K}an_*) \times_{(\mathscr{K}an_* \times_{\mathscr{K}an} \mathrm{Fun}(\mathscr{C}^{e}, \mathscr{K}an))} (\mathscr{K}an_* \times_{\mathscr{K}an} \mathrm{Fun}(\mathscr{C}^{\mathrm{e}}, \mathscr{K}an)) \times_{\mathrm{Fun}(\mathscr{C}^{e}, \mathscr{K}an)} \{H\}$$

fits into a pullback square

$$\begin{tikzcd}
\mathrm{Fun}(\mathscr{C}^e, \mathscr{K}an_*) \times_{\mathrm{Fun}(\mathscr{C}^e, \mathscr{K}an)} \{H\} \arrow[r] \arrow[d] & \mathrm{Fun}(\mathscr{C}^e, \mathscr{K}an_*) \arrow[d] \\
\mathrm{Hom}^{\mathrm{L}}_{\mathscr{C}}(0, H(0,0)) \arrow[r] & \mathscr{K}an_* \times_{\mathscr{K}an} \mathrm{Fun}(\mathscr{C}^e, \mathscr{K}an),
\end{tikzcd}$$

where the right vertical map is the trivial Kan fibration from Proposition 3.2. Hence we have a trivial Kan fibration over a contractible space

$$\mathrm{Fun}(\mathscr{C}^e, \mathscr{K}an_*) \times_{\mathrm{Fun}(\mathscr{C}^e, \mathscr{K}an)} \{H\} \to \mathrm{Hom}^{\mathrm{L}}_{\mathscr{C}}(0, H(0,0)),$$

from which we conclude that the fiber under consideration is a contractible Kan complex. $\square$

Definition 3.5. Given an ∞-category $\mathscr{C}$ which admits a zero object, a pointed Hom functor for $\mathscr{C}$ is a functor $\widetilde{H} : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{K}an_*$ whose composite

$$\mathscr{C}^{\mathrm{op}} \times \mathscr{C} \xrightarrow{\widetilde{H}} \mathscr{K}an_* \xrightarrow{forget} \mathscr{K}an$$

is a Hom functor for $\mathscr{C}$, i.e. a transport functor for the twisted arrows fibration (see Section II-12.1).

We can consider the twisted arrows fibration $\lambda : \mathscr{T}w(\mathscr{C}) \to \mathscr{C}^{\mathrm{op}} \times \mathscr{C}$, and the space of associated transport functors with a witnessing diagram $\mathscr{T}(\lambda)$ (Definition II-6.13). By Theorem II-6.14 we understand that the space $\mathscr{T}(\lambda)$ is contractible. One can rasonably define the space of pointed Hom functors, of pointed transport functors for the left fibration $\lambda : \mathscr{T}w(\mathscr{C}) \to \mathscr{C}^{\mathrm{op}} \times \mathscr{C}$, as the fiber product

$$\mathrm{Fun}(\mathscr{C}^{\mathrm{op}} \times \mathscr{C}, \mathscr{K}an_*) \times_{\mathrm{Fun}(\mathscr{C}^{\mathrm{op}} \times \mathscr{C}, \mathscr{K}an)} \mathscr{T}(\lambda).$$

Above we saw that any Hom functor for an ∞-category with zero objects admits a unique pointing. The following says that pointed Hom functors themselves are unique.

Proposition 3.6. *For any ∞-category $\mathscr{C}$ which admits a zero object, the space*

$$\mathrm{Fun}(\mathscr{C}^{\mathrm{op}} \times \mathscr{C}, \mathscr{K}an_*) \times_{\mathrm{Fun}(\mathscr{C}^{\mathrm{op}} \times \mathscr{C}, \mathscr{K}an)} \mathscr{T}(\lambda)$$

of pointed Hom functors is a contractible Kan complex.

Proof. Take $\mathscr{C}^e = \mathscr{C}^{\mathrm{op}} \times \mathscr{C}$. Since the map $\mathscr{K}an_* \to \mathscr{K}an$ is a left fibration, and in particular an isofibration, the induced map $\mathrm{Fun}(\mathscr{C}^e, \mathscr{K}an_*) \to \mathrm{Fun}(\mathscr{C}^e, \mathscr{K}an)$ is an isofibration, by Corollary I-5.14. For any choice of Hom functor H with witnessing data the corresponding map $H : * \to \mathscr{T}(\lambda)$ is a homotopy equivalence, and so the induced map on fiber products

$$\mathrm{Fun}(\mathscr{C}^e, \mathscr{K}an_*) \times_{\mathrm{Fun}(\mathscr{C}^e, \mathscr{K}an)} \{H\} \to \mathrm{Fun}(\mathscr{C}^e, \mathscr{K}an_*) \times_{\mathrm{Fun}(\mathscr{C}^e, \mathscr{K}an)} \mathscr{T}(\lambda)$$

is an equivalence by Corollay I-5.24. Since the domain space for this functor is a contractible Kan complex, by Theorem 3.4, if follows that the space of pointed Hom functors is a contractible Kan complex as well. $\square$

3.2. **Naturality for pointed Hom functors.**

Theorem 3.7. *Let $F : \mathscr{C}_0 \to \mathscr{C}_1$ be a functor between ∞-categories with zero objects, and suppose that F preserves zero objects. Take $\mathscr{C}_i^{\mathrm{e}} = \mathscr{C}_i^{\mathrm{op}} \times \mathscr{C}_i$ and*

$$\mathscr{L} = \mathrm{Fun}(\partial\Delta^1 \times \mathscr{C}_0^{\mathrm{e}}, \mathscr{K}an_*) \times_{\mathrm{Fun}(\partial\Delta^1 \times \mathscr{C}_0^{\mathrm{e}}, \mathscr{K}an)} \mathrm{Fun}(\Delta^1 \times \mathscr{C}^{\mathrm{e}}, \mathscr{K}an).$$

Let $\widetilde{H}_i : \mathscr{C}_i^{\mathrm{e}} \to \mathscr{K}an_*$ be pointed Hom functors, H_i be the underlying unpointed Hom functors, and $H_F : H_0 \to H_1 F$ be the transformation induced by F (Definition II-8.5). The space

$$\mathrm{Fun}(\Delta^1 \times \mathscr{C}_0^{\mathrm{e}}, \mathscr{K}an_*) \times_{\mathscr{L}} \{(\widetilde{H}_0, \widetilde{H}_1 F, H_F)\}$$

of of transformations lifting H_F, i.e. transformations $\widetilde{H}_F : \Delta^1 \times \mathscr{C}_0^{e} \to \mathscr{K}an_*$ which fit into a strictly commuting diagram

$$\begin{tikzcd} & \mathscr{K}an_* \arrow[d] \\ \Delta^1 \times \mathscr{C}_0 \arrow[ur, "\widetilde{H}_F"] \arrow[r, "H_F"'] & \mathscr{K}an \end{tikzcd} \tag{14}$$

and satisfy $\widetilde{H}_F|_{\{0\}} = \widetilde{H}_0$ and $\widetilde{H}_F|_{\{1\}} = \widetilde{H}_1 F$, is a contractible Kan complex.

Before giving the proof we record a useful lemma.

Lemma 3.8. *Suppose $\mathscr{E}$ is an ∞-category with an initial object $e : * \to \mathscr{E}$, and let $G : \mathscr{E} \to \mathscr{K}an$ be a functor for which the value $G(e)$ is contractible. Then the fiber product $\mathrm{Fun}(\mathscr{E}, \mathscr{K}an_*) \times_{\mathrm{Fun}(\mathscr{E}, \mathscr{K}an)} \{G\}$ is a contractible Kan complex.*

Proof. Same as the proof of Theorem 3.4. □

We now return to the matter at hand

Proof of Theorem 3.7. Given a zero object w in $\mathscr{C}$, the ∞-category $\Delta^1 \times \mathscr{C}_0^{\mathrm{e}}$ has the initial object $(0, \vec{w})$. Furthermore, since the space $H_F(0, \vec{w}) = H_0(w, w)$ is contractible, Lemma 3.8 tells us that the fiber product

$$\mathrm{Fun}(\Delta^1 \times \mathscr{C}_0^{\mathrm{e}}, \mathscr{K}an_*) \times_{\mathrm{Fun}(\Delta^1 \times \mathscr{C}_0^{\mathrm{e}}, \mathscr{K}an)} \{H_F\}$$

is a contractible Kan complex. We can rewrite this fiber product as

$$\mathrm{Fun}(\Delta^1 \times \mathscr{C}_0^{\mathrm{e}}, \mathscr{K}an_*) \times_{\mathscr{L}} (\mathscr{L} \times_{\mathrm{Fun}(\Delta^1 \times \mathscr{C}_0^{\mathrm{e}}, \mathscr{K}an)} \{H_F\})$$

and note that the fiber product

$$\mathscr{L} \times_{\mathrm{Fun}(\Delta^1 \times \mathscr{C}_0^{\mathrm{e}}, \mathscr{K}an)} \{H_F\} = \mathrm{Fun}(\partial^1\Delta^1 \times \mathscr{C}_0^{\mathrm{e}}, \mathscr{K}an_*) \times_{\mathrm{Fun}(\partial\Delta^1 \times \mathscr{C}_0^{\mathrm{e}}, \mathscr{K}an)} \{(H_0, H_1 F)\}$$

is again contractible by Lemma 3.8.

We consider the point

$$(\widetilde{H}_0, \widetilde{H}_1 F, H_F) : * \to \mathscr{L} \times_{\mathrm{Fun}(\Delta^1 \times \mathscr{C}_0^{\mathrm{e}}, \mathscr{K}an)} \{H_F\},$$

which is now a homotopy equivalence of Kan complexes. Since the map $\mathrm{Fun}(\Delta^1 \times \mathscr{C}_0^{\mathrm{e}}, \mathscr{K}an_*) \to \mathscr{L}$ is an isofibration, by Proposition I-5.13, pulling back yields an equivalence

$$\mathrm{Fun}(\Delta^1 \times \mathscr{C}_0^{\mathrm{e}}, \mathscr{K}an_*) \times_{\mathscr{L}} \{(\widetilde{H}_0, \widetilde{H}_1 F, H_F)\}$$
$$\xrightarrow{\sim} \mathrm{Fun}(\Delta^1 \times \mathscr{C}_0^{\mathrm{e}}, \mathscr{K}an_*) \times_{\mathrm{Fun}(\Delta^1 \times \mathscr{C}_0^{\mathrm{e}}, \mathscr{K}an)} \{H_F\},$$

by Corollary I-5.24. As we argued above, the target space here is contractible, so that the fiber $\mathrm{Fun}(\Delta^1 \times \mathscr{C}_0^e, \mathscr{K}an_*) \times_{\mathscr{L}} \{(\widetilde{H}_0, \widetilde{H}_1 F, H_F)\}$ is seen to be a contractible Kan complex as well. □

Definition 3.9. Let $F : \mathscr{C}_0 \to \mathscr{C}_1$ be a functor between ∞-categories with zero objects, and suppose furthermore that F preserves zero objects. Then for pointed Hom functors $\widetilde{H}_i : \mathscr{C}_i^{\mathrm{op}} \times \mathscr{C} \to \mathscr{K}an_*$, the tranformation $\widetilde{H}_F : \widetilde{H}_0 \to \widetilde{H}_1 F$ induced by F is any transformation whose composite

$$\Delta^1 \times \mathscr{C}_0^{\mathrm{op}} \times \mathscr{C}_0 \xrightarrow{\widetilde{H}_F} \mathscr{K}an_* \xrightarrow{forget} \mathscr{K}an$$

recovers the transformation $H_F : H_0 \to H_1 F$ induced by F on the underlying unpointed Hom functors (in the sense of Definition II-8.5).

Remark 3.10. We have abused language in speaking of "the" induced transformation rather than "an" induced transformation. One notes, however, that the space parametrizing such choices is contractible.

3.3. **Limits and colimits of pointed spaces.**

Lemma 3.11. *Let $\mathscr{C}$ be an ∞-category and $\mathscr{C}_{\mathrm{Term}}$ be the full ∞-subcategory of terminal objects in $\mathscr{C}$. The category $\mathscr{C}_{\mathrm{Term}}$ is complete and the inclusion $\mathscr{C}_{\mathrm{Term}} \to \mathscr{C}$ preserves limits.*

Proof. We have $\mathscr{C}_{\mathrm{Term}} \cong *$ so that it is both complete and cocomplete. To see that the inclusion $\mathscr{C}_{\mathrm{Term}} \to \mathscr{C}$ is continuous, it suffices to show that the constant diagram $\underline{t} : K \to \mathscr{C}$ at a given terminal object t is terminal in the full subcategory of constant diagrams in $\mathrm{Fun}(K, \mathscr{C})$. We have that the space

$$\mathrm{Fun}(\Delta^1, \mathscr{C}) \times_{\mathrm{Fun}(\partial\Delta^1, \mathscr{C})} \{(x,t)\} = \mathrm{Hom}_{\mathscr{C}}(x,t)$$

is contractible at each x in $\mathscr{C}$, by definition, so that the functor space

$$\begin{aligned}\mathrm{Fun}(K, \mathrm{Hom}_{\mathscr{C}}(x,t)) &= \mathrm{Fun}(\Delta^1 \times K, \mathscr{C}) \times_{\mathrm{Fun}(\partial\Delta^1 \times K, \mathscr{C})} \{(\underline{x}, \underline{t})\} \\ &= \mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{C})}(\underline{x}, \underline{t})\end{aligned}$$

is contractible as well. Hence $\underline{t}$ is terminal in $\mathrm{Fun}(K, \mathscr{C})$, and we see that the map

$$* \cong \mathrm{Hom}_{\mathscr{C}}(x,t) \to \mathrm{Hom}_{\mathrm{Fun}}(\underline{x}, \underline{t}) \overset{(id_{\underline{t}})_*}{\to} \mathrm{Hom}_{\mathrm{Fun}}(\underline{x}, \underline{t}) \cong *$$

is an equivalence at each x in $\mathscr{C}$. Therefore t is a limit for its own constant diagram. □

Proposition 3.12. (a) *The category $\mathscr{K}an_*$ is complete.*

(b) *The forgetful functor $\mathscr{K}an_* \to \mathscr{K}an$ is continuous.*

(c) *A diagram $\{0\} \star K \to \mathscr{K}an_*$ is a limit diagram if and only if its image in $\mathscr{K}an$ is a limit diagram.*

Proof. Via the coslice equivalence

$$\mathscr{K}an_* \overset{\sim}{\to} \{*\} \times^{\mathrm{or}}_{\mathscr{K}an} \mathscr{K}an$$

of Theorem I-9.14 it suffices to show that the oriented fiber product $\{*\} \times^{\mathrm{or}}_{\mathscr{K}an} \mathscr{K}an$ is complete and that the projection to $\mathscr{K}an$ both preserves and detects limits.

By Proposition II-13.29 the category $\mathrm{Fun}(\Delta^1, \mathscr{K}an)$ is complete, and a diagram $p : \{0\} \star K \to \mathrm{Fun}(\Delta^1, \mathscr{K}an)$ is a limit diagram if and only if it evaluates to a limit diagram in $\mathscr{K}an$ at both 0 and $1 : * \to \Delta^1$. Supposing that $0^* p$ takes constant value $*$, the evaluation at $*$ is already a limit diagram by Lemma 3.11. This shows that the fiber

$$\{*\} \times_{\mathrm{Fun}(\{0\},\mathscr{K}an)} \mathrm{Fun}(\Delta^1, \mathscr{K}an) = \{*\} \times^{\mathrm{or}}_{\mathscr{K}an} \mathscr{K}an$$

is a cocomplete subcategory in $\mathrm{Fun}(\Delta^1, \mathscr{K}an)$, and that the projection to $\mathscr{K}an$, i.e. the evaluation at 1 in Δ^1, both preserves and detects limits in the oriented fiber product. □

Recall that a simplicial set K is said to be weakly contractible if the terminal map $K \to *$ induces an equivalence

$$\mathscr{X} = \mathrm{Fun}(*, \mathscr{X}) \to \mathrm{Fun}(K, \mathscr{X})$$

at each Kan complex $\mathscr{X}$.

Proposition 3.13 ([15, 03PK])**.** *The class of weakly contractible simplicial sets is closed under the formation of filtered colimits in* sSet*.*

We also understand from Proposition II-9.24 that any simplicial set with an initial or terminal object is weakly contractible. We therefore observe the following.

Proposition 3.14. *The class of weakly contractible simplicial sets contains the simplicial set Δ^{op}, as well as all filtered simplicial sets.*

Proof. The category Δ^{op} has the initial object $[0]$ and is thus weakly contractible by Proposition II-9.24. For a filtered simplicial sets K, the inclusion of any subset $K' \to K$ extends to a map of simplicial sets $K' \star \{1\} \to K$. Hence K can be written as a filtered colimit of simplicial sets with a terminal object. As any simplicial set with a terminal object is contractible, again by Proposition II-9.24, we see that K is weakly contractible by Proposition 3.13. □

Proposition 3.15 ([15, 02KR])**.** *Let $q : \mathscr{E} \to \mathscr{C}$ be any left fibration of ∞-categories, K be a weakly contractible simplicial set, and suppose that $\mathscr{C}$ admits all K-indexed colimits.*

(1) *$\mathscr{E}$ admits all K-indexed colimits.*

(2) *A diagram $p : K \star \{1\} \to \mathscr{E}$ is a colimit diagram if and only if the composite $qp : K \star \{1\} \to \mathscr{C}$ is a colimit diagram.*

Outline of proof. Let $\bar{p} : K \to \mathscr{E}$ be a diagram. Take $\bar{p}_0 = q\bar{p} : K \to \mathscr{C}$. By a general result [15, 0179] the inclusion $K \to K \star \{1\}$ is left anodyne, so that any K-colimit diagram $p_0 : K \star \{1\} \to \mathscr{C}$ lifts uniquely to a diagram $p : K \star \{1\} \to \mathscr{C}$ whose restriction to K recovers $\bar{p}$. One applies [15, 02KR] (see also [15, 02KN]) to find that p is a colimit in $\mathscr{E}$.

We now have that $\mathscr{E}$ admits the proposed colimits and that q preserves colimits of the given type. Uniqueness of solutions to the the lifting problems

$$\begin{tikzcd} K \arrow[r] \arrow[d] & \mathscr{E} \arrow[d, "q"] \\ K \star \{1\} \arrow[r] \arrow[ur, dotted] & \mathscr{C} \end{tikzcd}$$

however, which we observe from [15, 0179] and Proposition I-3.11, implies that q detects such colimits in $\mathscr{E}$ as well. □

We apply this result to our favorite left fibration $\mathscr{K}an_* = \mathscr{K}an_{*/} \to \mathscr{K}an$.

Corollary 3.16. *For any weakly contractible simplicial set K, the category $\mathscr{K}an_*$ admits all K-indexed colimits, and a diagram $K \star \{1\} \to \mathscr{K}an_*$ is a colimit diagram if and only if the composite $K \star \{1\} \to \mathscr{K}an_* \to \mathscr{K}an$ is a colimit diagram in $\mathscr{K}an$.*

By Proposition 3.14 we specifically observe the following.

Corollary 3.17. *The category $\mathscr{K}an_*$ admits geometric realizations, filtered colimits, and pushouts. Furthermore, the forgetful functor $\mathscr{K}an_* \to \mathscr{K}an$ both preserves and detects such colimits in $\mathscr{K}an_*$.*

Proof. All is clear save for pushouts. However, this follows from the fact that the diagram $K = \Lambda^2_0$ admits an initial object, and is therefore weakly contractible (Proposition II-9.24). □

Remark 3.18. We see below, in Theorem 3.25 that the category $\mathscr{K}an_*$ in fact admits all small colimits, i.e. is cocomplete. It is not the case, however, that the forgetful functor $\mathscr{K}an_* \to \mathscr{K}an$ preserves general colimits. Indeed, the coproduct $* \amalg *$ in the category $\mathscr{K}an_*$ is just a point, by Lemma 3.24 below, while in $\mathscr{K}an$ is it two points.

From the above information we see that one can detect limits and colimits in a pointed ∞-category via pointed Hom functors, in addition to the unpointed Hom functors.

Proposition 3.19. *Let $\mathscr{C}$ be an ∞-category with a zero object, and $\widetilde{H} : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{K}an_*$ be a pointed Hom functor for $\mathscr{C}$. For a small diagram $p : \{0\} \star K \to \mathscr{C}$ the following are equivalent:*

(a) *p is a limit diagram in $\mathscr{C}$.*
(b) *At each object x in $\mathscr{C}$, the diagram $\widetilde{H}(x, -)p : \{0\} \star K \to \mathscr{K}an_*$ is a limit diagram in $\mathscr{K}an_*$.*

Similarly, for a diagram $q : K \star \{1\} \to \mathscr{C}$ the following are equivalent:

(a′) *q is a colimit diagram in $\mathscr{C}$.*
(b′) *At each object y in $\mathscr{C}$, the diagram $\widetilde{H}(-, y)q : \{1\} \star K^{\mathrm{op}} \to \mathscr{K}an_*$ is a limit diagram in $\mathscr{K}an_*$.*

Proof. Follows immediately from Proposition 3.12 and Corollary II-16.16. □

We similarly observe the following.

Proposition 3.20. *Let $\mathscr{C}$ be an ∞-category with a zero object, $\widetilde{H} : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{K}an_*$ be a pointed Hom functor for $\mathscr{C}$, and H be the underlying unpointed Hom functor. Suppose K is filtered, or that $K = \Delta^{\mathrm{op}}$. At a given object $x : * \to \mathscr{C}$, the functor $\widetilde{H}(x, -) : \mathscr{C} \to \mathscr{K}an_*$ preserves K-indexed colimits if and only if the functor $H(x, -) : \mathscr{C} \to \mathscr{K}an$ preserves K-indexed colimits.*

3.4. **Recasting homotopy groups for pointed spaces.** We consider the simplicial category $\underline{\mathrm{Kan}}_*$ whose objects are Kan complexes $\mathscr{X}$ with a fixed point $x : * \to \mathscr{X}$, and whose morphism complexes are the fibers

$$\mathrm{Fun}((\mathscr{X}, x), (\mathscr{Y}, y))_* := \mathrm{Fun}(\mathscr{X}, \mathscr{Y}) \times_{\mathrm{Fun}(*, \mathscr{Y})} \{y\}. \tag{15}$$

In particular, we have $\mathrm{Fun}_{*/}(*, (\mathscr{Y}, y)) = *$, and one sees that the point $*$ is both initial and terminal in the simplicial category $\underline{\mathrm{Kan}}_*$. Note also that the the functor spaces $\mathrm{Fun}(\mathscr{X},\mathscr{Y})$ are Kan complexes, and that restriction along the point x^* : $\mathrm{Fun}(\mathscr{X},\mathscr{Y}) \to \mathrm{Fun}(*,\mathscr{Y}) = \mathscr{Y}$ is a Kan fibration. Hence the fiber (15) is always a Kan complex.

Due to triviality of the mapping spaces from the one point space, one sees that any functor $\mathrm{Path}\,\Delta^n \to \underline{\mathrm{Kan}}_*$ extends uniquely to a functor

$$\mathrm{Path}(\{-1\} \star \Delta^n) \to \underline{\mathrm{Kan}}_*$$

whose value at -1 is $*$. We therefore observe a unique section

$$\mathrm{N}^{\mathrm{hc}}(\underline{\mathrm{Kan}}_*) \to \mathrm{N}^{\mathrm{hc}}(\underline{\mathrm{Kan}}_*)_{*/}$$

of the forgetful functor which is an isomorphism of simplicial sets.

Now, we have the forgetful functor $\underline{\mathrm{Kan}}_* \to \mathrm{Kan}$ which provides a map of ∞-categories

$$\mathrm{N}^{\mathrm{hc}}(\underline{\mathrm{Kan}}_*) \to \mathrm{N}^{\mathrm{hc}}(\underline{\mathrm{Kan}}) = \mathscr{K}an$$

which is furthermore an inclusion of simplicial sets. We now observe a unique lift of this map to a functor $\rho : \mathrm{N}^{\mathrm{hc}}(\underline{\mathrm{Kan}}_*) \to \mathscr{K}an_*$ which fits into a strictly commuting diagram

$$\begin{tikzcd}
\mathrm{N}^{\mathrm{hc}}(\underline{\mathrm{Kan}}_*)_{*/} \arrow[r, "forget_{*/}"] \arrow[d] & \mathscr{K}an_* \arrow[d] \\
\mathrm{N}^{\mathrm{hc}}(\underline{\mathrm{Kan}}_*) \arrow[r, "forget"'] \arrow[ur, "\rho"] & \mathscr{K}an.
\end{tikzcd}$$

As the functor to $\mathscr{K}an$ is injective, the lift ρ is injective as well. The functor ρ identifies those n-simplices in $\mathscr{K}an_*$, i.e. all maps $\{-1\} \star \Delta^n \to \mathscr{K}an$, whose restriction $\{-1\} \star \mathrm{Sk}_1(\Delta^n) \to \mathscr{K}an$ is a strictly commuting diagram.

Proposition 3.21 ([15, 0200])**.** *The inclusion $\rho : \mathrm{N}^{\mathrm{hc}}(\underline{\mathrm{Kan}}_*) \to \mathscr{K}an_*$ is an equivalence of ∞-categories.*

Let us recall now the geometric realization functor $|-| : \mathrm{sSet} \to \mathrm{Top}$, which is left adjoint to the singular complex functor $\mathrm{Sing} : \mathrm{Top} \to \mathrm{sSet}$. Being a left adjoint, the functor $|-|$ is seen to commute with colimits, and so one calculates

$$|K| = \mathrm{colim}_{n,\ \Delta^n \to K} |\Delta^n|.$$

We have the following fundamental result of Milnor.

Theorem 3.22 ([17])**.** *For any simplicial set K, the unit map $K \to \mathrm{Sing}\,|K|$ is a weak homotopy equivalence.*

Consequently, for any simplicial set K, we observe an equivalence of Kan complexes $\mathrm{Fun}(\mathrm{Sing}\,|K|,\mathscr{Y}) \to \mathrm{Fun}(K,\mathscr{Y})$ at arbitrary $\mathscr{Y}$ in $\mathscr{K}an$. In the pointed setting, restriction along the marked point provides a Kan fibration

$$x^* : \mathrm{Fun}(\mathscr{X},\mathscr{Y}) \to \mathrm{Fun}(*,\mathscr{Y}) = \mathscr{Y}$$

by Proposition I-3.11 so that taking the fiber provides, for any pointed simplicial set K, an equivalence

$$\mathrm{Fun}_{*/}(\mathrm{Sing}\,|K|,\mathscr{Y}) \to \mathrm{Fun}_{*/}(K,\mathscr{Y}).$$

As a corollary to Proposition 3.21 we now observe an identification of homotopy groups via mapping spaces in the ∞-category $\mathscr{K}an_*$.

Corollary 3.23. *Take $\mathbb{S}^n = \operatorname{Sing}|\Delta^n/\partial\Delta^n|$. For any pointed space $x : * \to \mathscr{X}$, there is a natural bijection*

$$\pi_n(\mathscr{X}, x) \overset{\cong}{\to} \pi_0 \operatorname{Hom}_{\mathscr{K}an_*}(\mathbb{S}^n, \mathscr{X}).$$

Proof. Via Theorem 3.22 restricting along the unit map $\Delta^n/\partial\Delta^n \to \mathbb{S}^n$ provides an equivalence $\operatorname{Fun}_{*/}(\mathbb{S}^n, \mathscr{X}) \overset{\sim}{\to} \operatorname{Fun}_{*/}(\Delta^n/\partial\Delta^n, \mathscr{X})$. We have also the natural equivalence

$$\operatorname{Fun}_{*/}(\mathbb{S}^n, \mathscr{X}) \overset{\sim}{\to} \operatorname{Hom}_{\mathrm{N}^{\mathrm{hc}}(\underline{\mathrm{Kan}}_*)}(\mathbb{S}^n, \mathscr{X}) \overset{\rho}{\to} \operatorname{Hom}_{\mathscr{K}an_*}(\mathbb{S}^n, \mathscr{X})$$

which provides a roof of equivalences

$$\operatorname{Fun}_{*/}(\Delta^n/\partial\Delta^n, \mathscr{X}) \leftarrow \operatorname{Fun}_{*/}(\mathbb{S}^n, \mathscr{X}) \to \operatorname{Hom}_{\mathscr{K}an_*}(\mathbb{S}^n, \mathscr{X}).$$

Taking connected components therefore provides an isomorphism

$$\begin{tikzcd}
\pi_0 \operatorname{Fun}_{*/}(\mathbb{S}^n, \mathscr{X}) \arrow[r, "\cong"] \arrow[d, "\cong"'] & \pi_0 \operatorname{Hom}_{\mathscr{K}an_*}(\mathbb{S}^n, \mathscr{X}) \\
\pi_n(\mathscr{X}) = \pi_0 \operatorname{Fun}_{*/}(\Delta^n/\partial\Delta^n, \mathscr{X}) \arrow[ur, dashed, "\cong", "\exists !"'] &
\end{tikzcd}$$

which is natural in the $\mathscr{X}$ coordinate. □

We can also use Proposition 3.21 to see that the category of pointed spaces is cocomplete. We first observe the existence of arbitrary coproducts. In the statement of the following lemma we write, for any collection of pointed spaces $\{(\mathscr{X}_\lambda, x_\lambda) : \lambda \in \Lambda\}$,

$$\mathscr{X}_\Lambda = \coprod_{\lambda\in\Lambda} \mathscr{X}_\lambda \;\text{ and }\; \mathscr{X}_\Lambda/* = \mathscr{X}_\Lambda/(\coprod_{\lambda\in\Lambda} x_\lambda).$$

Lemma 3.24. *The category $\mathscr{K}an_*$ admits all small coproducts. Specificially, for Λ a small discrete set and $\mathscr{X}_? : \Lambda \to \mathscr{K}an_*$ any functor, the maps*

$$i_\lambda = \big(\ \mathscr{X}_\lambda \to \mathscr{X}_\Lambda \to \operatorname{Sing}|\mathscr{X}_\Lambda/*|\ \big)$$

realizes tha space $\operatorname{Sing}|\mathscr{X}_\Lambda/*|$ as a colimit for the functor $\mathscr{X}_?$.

Proof. Take $\mathscr{X}_{\Lambda,*} = \operatorname{Sing}|(\coprod_\lambda \mathscr{X}_\lambda)/*|$. Via the equivalence $\mathrm{N}^{\mathrm{hc}}(\underline{\mathrm{Kan}}_*) \cong \mathscr{K}an_*$ it suffices to show that the given morphism is a coproduct in the given nerve. From Proposition II-11.6 and Corollary II-16.17 it then suffices to prove that the map

$$[i_\lambda^*; \lambda \in \Lambda]^t : \operatorname{Fun}_{*/}(\mathscr{X}_{\Lambda,*}, \mathscr{Y}) \to \prod_\lambda \operatorname{Fun}_{*/}(\mathscr{X}_\lambda, \mathscr{Y})$$

is a homotopy equivalence at all pointed spaces $\mathscr{Y}$. As the inclusion $\mathscr{X}_\Lambda/* \to \mathscr{X}_{\Lambda,*}$ is a weak homotopy equivalence [15, 0142], and the structure maps for $\mathscr{X}_{\Lambda,*}$ factor through $\mathscr{X}_\Lambda/*$, it then suffices to prove that the map

$$[\mathrm{incl}_\lambda^*; \lambda \in \Lambda]^t : \operatorname{Fun}_{*/}(\mathscr{X}_\Lambda/*, \mathscr{Y}) \to \prod_\lambda \operatorname{Fun}_{*/}(\mathscr{X}_\lambda, \mathscr{Y}) \tag{16}$$

is an equivalence.

The space $\mathrm{Fun}_{*/}(\mathscr{X}_\Lambda/*, \mathscr{Y})$ completes a pullback square

$$\begin{tikzcd} \mathrm{Fun}_{*/}(\mathscr{X}_\Lambda/*, \mathscr{Y}) \arrow[r] \arrow[d] & \mathrm{Fun}(\mathscr{X}_\Lambda/*, \mathscr{Y}) \arrow[d, "x_\Lambda^*"] \\ * \arrow[r, "y_*"'] & \mathscr{Y} \end{tikzcd}$$

and for $\mathscr{X}_\Lambda/*$ we have a pullback square

$$\begin{tikzcd} \mathrm{Fun}(\mathscr{X}_\Lambda/*, \mathscr{Y}) \arrow[r] \arrow[d] & \mathrm{Fun}(\mathscr{X}_\Lambda, \mathscr{Y}) \arrow[d, "(\amalg_\lambda x_\lambda)^*"] \\ \mathscr{Y} \arrow[r, "\mathrm{diag}"'] & \prod_\lambda \mathscr{Y}. \end{tikzcd}$$

We therefore observe the pullback square

$$\begin{tikzcd} \mathrm{Fun}_{*/}(\mathscr{X}_\Lambda/*, \mathscr{Y}) \arrow[r] \arrow[d] & \mathrm{Fun}(\mathscr{X}_\Lambda, \mathscr{Y}) \arrow[d, "(\amalg_\lambda x_\lambda)^*"] \\ * \arrow[r, "\prod_\lambda y_*"'] & \prod_\lambda \mathscr{Y}. \end{tikzcd}$$

We similarly have a pullback square

$$\begin{tikzcd} \prod_\lambda \mathrm{Fun}_{*/}(\mathscr{X}_\lambda, \mathscr{Y}) \arrow[r] \arrow[d] & \prod_\lambda \mathrm{Fun}(\mathscr{X}_\lambda, \mathscr{Y}) \arrow[d, "\prod_\lambda x_\lambda^*"] \\ * \arrow[r, "\prod_\lambda y_*"'] & \prod_\lambda \mathscr{Y} \end{tikzcd}$$

so that the unpointed isomorphism

$$[\mathrm{incl}_\lambda^*; \lambda \in \Lambda]^t : \mathrm{Fun}(\mathscr{X}_\Lambda, \mathscr{Y}) \overset{\cong}{\to} \prod_\lambda \mathrm{Fun}(\mathscr{X}_\lambda, \mathscr{Y})$$

induces an isomorphism $\mathrm{Fun}_{*/}(\mathscr{X}_\Lambda/*, \mathscr{Y}) \overset{\cong}{\to} \prod_\lambda \mathrm{Fun}_{*/}(\mathscr{X}_\lambda, \mathscr{Y})$ which is explicitly given by the map (16). □

Theorem 3.25. *The ∞-category $\mathscr{K}an_*$ is both complete and cocomplete.*

Proof. Completeness was covered in Proposition 3.12 above. For cocompleteness, we know from Corollary 3.17 and Lemma 3.24 that $\mathscr{K}an_*$ also admits pushouts and small coproducts. It follows from [13, Proposition 4.4.2.6] that $\mathscr{K}an_*$ admits all small colimit. □

3.5. **Pointed Hom functors for homotopy and derived ∞-categories.** Let $\underline{A}$ be a Kan-enriched category with a strict zero object 0, in the sense that the Hom complexes to and from 0 are just points

$$* = \underline{\mathrm{Hom}}_{\underline{A}}(x, 0) = \underline{\mathrm{Hom}}_{\underline{A}}(0, x).$$

Then the simplicial Hom functor $\underline{\mathrm{Hom}}_{\underline{A}} : \underline{A}^{\mathrm{op}} \times \underline{A} \to \underline{\mathrm{Kan}}$ admits a natural pointing via the 0 morphisms $0 : * \to \underline{\mathrm{Hom}}_{\underline{A}}(x, y)$,

$$\underline{\mathrm{Hom}}_{\underline{A}} : \underline{A}^{\mathrm{op}} \times \underline{A} \to \underline{\mathrm{Kan}}_*.$$

For $\mathscr{A} = \mathrm{N}^{\mathrm{hc}}(\underline{A})$ and $\underline{\mathrm{Hom}}_{\mathscr{A}} = \mathrm{N}^{\mathrm{hc}}\,\underline{\mathrm{Hom}}_{\underline{A}}$ we therefore obtain a functor

$$\underline{\mathrm{Hom}}_{\mathscr{A}} : \mathscr{A}^{\mathrm{op}} \times \mathscr{A} \to \mathrm{N}^{\mathrm{hc}}(\underline{\mathrm{Kan}}_*).$$

We compose with the equivalence $\rho : \mathrm{N}^{\mathrm{hc}}(\underline{\mathrm{Kan}}_*) \to \mathscr{K}an_*$ to obtain a candidate pointed Hom functor for the homotopy coherent nerve $\mathscr{A}$. We refer to this functor as the *caonnical pointed Hom functor* for $\mathscr{A} = \mathrm{N}^{\mathrm{hc}}(\underline{A})$.

Proposition 3.26. *If $\underline{A}$ is a simplicial category with a strict zero object, then the canonical pointed Hom functor*

$$\underline{\mathrm{Hom}}_{\mathscr{A}} : \mathscr{A}^{\mathrm{op}} \times \mathscr{A} \to \mathscr{K}an_*$$

constructed above is a pointed Hom functor for $\mathscr{A} = \mathrm{N}^{\mathrm{hc}}(\underline{A})$.

We consider the case of the simplicial category $K\mathbf{Ch}(\mathbb{A})$ with associated homotopy coherent nerve

$$\mathrm{N}^{\mathrm{hc}}(K\mathbf{Ch}(\mathbb{A})) \cong \mathscr{K}(\mathbb{A})$$

(see Theorem II-10.4).

Corollary 3.27. *Let $\mathbb{A}$ be an additive category. The zero morphisms $0 : * \to K\,\mathrm{Hom}^*_{\mathbb{A}}(x, y)$ endow the unpointed Hom functor $K\,\mathrm{Hom}^*_{\mathbb{A}} : \mathscr{K}(\mathbb{A})^{\mathrm{op}} \times \mathscr{K}(\mathbb{A}) \to \mathscr{K}an$ with the structure of a pointed Hom functor*

$$K\,\mathrm{Hom}^*_{\mathbb{A}} : \mathscr{K}(\mathbb{A})^{\mathrm{op}} \times \mathscr{K}(\mathbb{A}) \to \mathscr{K}an_*.$$

In the case where $\mathbb{A}$ is Grothendieck abelian, we consider the injective model for the derived ∞-category $\mathscr{D}(\mathbb{A}) = \mathscr{D}_{\mathrm{Inj}} \subseteq \mathscr{K}(\mathbb{A})$.

Corollary 3.28. *Let $\mathbb{A}$ be an Grothendieck abelian category. The zero morphisms $0 : * \to K\,\mathrm{Hom}^*_{\mathbb{A}}(x, y)$ endow the unpointed Hom functor $K\,\mathrm{Hom}^*_{\mathbb{A}} : \mathscr{D}(\mathbb{A})^{\mathrm{op}} \times \mathscr{D}(\mathbb{A}) \to \mathscr{K}an$ with the structure of a pointed Hom functor*

$$K\,\mathrm{Hom}^*_{\mathbb{A}} : \mathscr{D}(\mathbb{A})^{\mathrm{op}} \times \mathscr{D}(\mathbb{A}) \to \mathscr{K}an_*.$$

3.6. **Introduction to spectra.** Though we won't employ an in depth analysis of spectra in this work, it is convenient at times to lift from the category of pointed sets to the category of spectra. Here we provide a bare-bones introduction to spectra, in accordance with the work [14].

The category of spectra $\mathscr{S}\!p$ is, one might think, the localization of the category of pointed spaces $\mathscr{S}\!p = \mathscr{K}an_*[\Omega^{-1}]$ relative to the action of the looping functor. (See Section I-6.4 and the discussions preceding [14, Remark 1.1.2.6].) Let us define this category formally.

We first consider the full subcategory $\mathscr{K}an_*^{\mathrm{fin}}$ of $\mathscr{K}an_*$ which is generated by the one point space $*$ under finite colimits. Let us note that we have pushout diagrams

$$\begin{tikzcd} \mathbb{S}^n \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \mathbb{S}^{n+1} \end{tikzcd} \tag{17}$$

at all n, so that all spheres appear in $\mathscr{K}an_*^{\mathrm{fin}}$. (For details see Example 4.20.) We will define the category of spectra as the full subcategory of so-called reduced excisive functors from $\mathscr{K}an_*^{\mathrm{fin}}$ to $\mathscr{K}an_*$.

Definition 3.29. Let $F : \mathscr{K} \to \mathscr{C}$ be a functor between ∞-categories, and suppose that $\mathscr{K}$ admits all finite colimits and a terminal object $*$. We call F excisive if it sends pushout diagrams in $\mathscr{K}$ to pullback diagrams in $\mathscr{C}$. We call F reduced if $F(*)$ is terminal in $\mathscr{C}$.

For $\mathscr{K}$ and $\mathscr{C}$ as above, we let $\mathrm{Exc}_*(\mathscr{K}, \mathscr{C})$ denote the full ∞-subcategory in $\mathrm{Fun}(\mathscr{K}, \mathscr{C})$ spanned by those functors which are both excisive and reduced. We are particularly interested in the case $\mathscr{K} = \mathscr{K}an_*^{\mathrm{fin}}$.

Definition 3.30. For any ∞-category $\mathscr{C}$ the ∞-category of spectrum objects in $\mathscr{C}$ is defined $\mathscr{S}p(\mathscr{C}) = \mathrm{Exc}_*(\mathscr{K}an_*^{\mathrm{fin}}, \mathscr{C})$. In the particular case of $\mathscr{K}an_*$ we take

$$\mathscr{S}p := \mathscr{S}p(\mathscr{K}an_*) = \mathrm{Exc}_*(\mathscr{K}an_*^{\mathrm{fin}}, \mathscr{K}an_*).$$

For any object $\mathscr{X}_* : \mathscr{K}an_*^{\mathrm{fin}} \to \mathscr{K}an_*$ in the category of spectra, we consider the specific values of this functor on the n-spheres $\mathscr{X}_n := \mathscr{X}_*(\mathbb{S}^n)$. From the pushout diagram (17) and excisiveness of $\mathscr{X}_*$, we have identifications with the loop spaces $\mathscr{X}_n = \Omega \mathscr{X}_{n+1}$ at each nonnegative integer n.

Definition 3.31. For each nonnegative integer n we take $\Omega^{\infty - n} : \mathscr{S}p \to \mathscr{K}an_*$ the evaluation functor at the n-sphere $\mathbb{S}^n$, and take in particular $\Omega^\infty := \Omega^{\infty - 0}$.

Proposition 3.32. *The category $\mathscr{S}p$ is complete and cocomplete, and the forgetful functor $\Omega^\infty : \mathscr{S}p \to \mathscr{K}an_*$ is continuous.*

Sketch proof. The category $\mathscr{K}an$ is generated by the subcategory of finite discrete sets under geometric realization. In particular, any space $\mathscr{X}$ is the colimit of its own simplicial functor $\mathscr{X} = \mathrm{colim}_{\Delta^{\mathrm{op}}} \mathscr{X}[n]$ and, under this identification, the structure map $\mathscr{X}[0] \to \mathscr{X}$ is the natural inclusion. Since the forgetful functor $\mathscr{K}an_* \to \mathscr{K}an$ preserves geometric realization we see similarly that $\mathscr{K}an_*$ is generated by the subcategory of finite pointed sets under geometric realization. In particular, the category $\mathscr{K}an_*$ is seen to be presentable. It follows that the category $\mathscr{S}p$ is presentable as well [14, Proposition 1.4.4.4], and thus complete and cocomplete, and also that the functor $\Omega^\infty : \mathscr{S}p \to \mathscr{K}an_*$ admits a left adjoint [14, Proposition 1.4.4.4]. The functor Ω^∞ is therefore continuous by Proposition II-13.24. $\square$

For ∞-categories $\mathscr{C}$ and $\mathscr{D}$ which admit finite limits, we consider the ∞-category $\mathrm{Fun}^{\mathrm{lex}}(\mathscr{C}, \mathscr{D})$ of left exact functors. By definition, this is the full subcategory in $\mathrm{Fun}(\mathscr{C}, \mathscr{D})$ whose objects are functors which preserve finite limits.

Proposition 3.33 ([14, Corollary 1.4.2.23])**.** *Suppose $\mathscr{C}$ is a pointed ∞-category which admits finite limits, and that pushout diagrams agree with pullback diagrams in $\mathscr{C}$. Then the functor*

$$\Omega^\infty_* : \mathrm{Fun}^{\mathrm{lex}}(\mathscr{C}, \mathscr{S}p) \to \mathrm{Fun}^{\mathrm{lex}}(\mathscr{C}, \mathscr{K}an_*)$$

is an equivalence of ∞-categories.

Now, one can show further that any spectrum is determined by its values on the n-spheres. To elaborate on this point, let us take $\mathscr{L}\mathscr{K}an_*$ the limit of the diagram $\omega : \mathbb{Z}_{\leq 0} \to \mathscr{C}at_\infty^{\mathrm{big}}$,

$$\mathscr{L}\mathscr{K}an_* = \lim(\cdots \to \mathscr{K}an_* \xrightarrow{\Omega} \mathscr{K}an_* \xrightarrow{\Omega} \mathscr{K}an_*).$$

Here one can take this looping functor as specifically realized by the construction of Section I-6.4 at the level of the simplicial category $\underline{\mathrm{Kan}}_*$, which then induces a functor on ∞-categories via an application of the homotopy coherent nerve and Proposition 3.21.

In terms of the explicit formulae given in Section II-14.2, this limit category is the ∞-category of sections of the weighted nerve

$$\mathscr{L}\mathscr{K}an_* = \mathrm{Fun}^{\mathrm{CCart}}_{\mathbb{Z}_{\leq 0}}(\mathbb{Z}_{\leq 0}, \mathrm{N}^{\omega}(\mathbb{Z}_{\leq 0})).$$

At each nonpositive integer i, evaluation provides a structure map

$$p_i = (ev_i)^* : \mathrm{Fun}^{\mathrm{CCart}}_{\mathbb{Z}_{\leq 0}}(\mathbb{Z}_{\leq 0}, \mathrm{N}^{\omega}(\mathbb{Z}_{\leq 0})) \to \mathrm{N}^{\omega}(\mathbb{Z}_{\leq 0})_i = \mathscr{K}an_*$$

For each object $\mathscr{X}_*$ in the limit, we take $\mathscr{X}_i = p_{-i}\mathscr{X}_*$ to see that $\mathscr{X}_*$ is defined by a sequence of pointed spaces $\{\mathscr{X}_i : i \geq 0\}$ which are equipped with homotopy equivalences $\Omega\mathscr{X}_{i+1} \overset{\sim}{\to} \mathscr{X}_i$. A morphism is a sequence of maps $f_i : \mathscr{X}_i \to \mathscr{Y}_i$ which fit into commuting diagrams

$$\begin{tikzcd} \Omega\mathscr{X}_{i+1} \arrow[r, "\Omega f_{i+1}"] \arrow[d, "\sim"'] & \Omega\mathscr{Y}_{i+1} \arrow[d, "\sim"] \\ \mathscr{X}_i \arrow[r, "f_i"'] & \mathscr{Y}_i \end{tikzcd} \tag{18}$$

in $\mathscr{K}an_*$.

Proposition 3.34. *There is a unique equivalence of ∞-categories $t : \mathscr{L}\mathscr{K}an_* \overset{\sim}{\to} \mathscr{S}p$ which fits into a 2-simplex*

$$\begin{tikzcd} & \mathscr{S}p \arrow[dr, "\Omega^{\infty-n}"] & \\ \mathscr{L}\mathscr{K}an_* \arrow[ur, "t"] \arrow[rr, "p_{-n}"'] & & \mathscr{K}an_*, \end{tikzcd}$$

at each nonnegative integer n.

Proof outline. We have the shift inclusion $-1 : \mathbb{Z}_{\leq 0} \to \mathbb{Z}_{\leq 0}$ and restricting along -1 provides a functor $\Sigma : \mathscr{L}\mathscr{K}an_* \to \mathscr{L}\mathscr{K}an_*$. This functor sends an object $\mathscr{X}_*$ to the object $\Sigma\mathscr{X}_* = \mathscr{X}_{*+1}$. We also have the looping functor $\Omega : \mathscr{L}\mathscr{K}an_* \to \mathscr{L}\mathscr{K}an_*$ which is induced by the transformation

$$\begin{tikzcd} \cdots \arrow[r] & \mathscr{K}an_* \arrow[r, "\Omega"] & \mathscr{K}an_* \arrow[r, "\Omega"] & \mathscr{K}an_* \\ \cdots \arrow[r] & \mathscr{K}an_* \arrow[r, "\Omega"'] \arrow[u, "\Omega"] & \mathscr{K}an_* \arrow[r, "\Omega"'] \arrow[u, "\Omega"] & \mathscr{K}an_*, \arrow[u, "\Omega"] \end{tikzcd}$$

and one checks that these functors are mutually inverse. It's furthermore argued in [14, Proof of Proposition 1.4.2.24] that the functor Ω is the looping functor for $\mathscr{L}\mathscr{K}an_*$ in the sense of [14, Remark 1.1.2.8]. We have now $p_{-n} = p_0\Sigma^n$.

It is argued in [14, Proof of Proposition 1.4.2.24] that we have a unique equivalence t which fits into a diagram

$$\begin{tikzcd} & \mathscr{S}p \arrow[dr, "\Omega^{\infty}"] & \\ \mathscr{L}\mathscr{K}an_* \arrow[ur, "t"] \arrow[rr, "p_0"'] & & \mathscr{K}an_*. \end{tikzcd}$$

For the delooping functor $S : \mathscr{K}an_*^{\mathrm{fin}} \to \mathscr{K}an_*^{\mathrm{fin}}$, the corresponding functor $\Sigma = S^*$ is the delooping functor on $\mathscr{S}\!p$, so that we have a commutative diagram

$$\begin{tikzcd} \mathscr{L}\mathscr{K}an_* \arrow[r, "t"] \arrow[d, "\Sigma^n"'] \arrow[dr] & \mathscr{S}\!p \arrow[d, "\Sigma^n"] \\ \mathscr{L}\mathscr{K}an_* \arrow[r, "t"'] & \mathscr{S}\!p \end{tikzcd}$$

in $\mathscr{C}at_\infty^{\mathrm{big}}$ at each integer n. This gives an isomorphism of functors $p_{-n} \cong \Omega^\infty\Sigma^n$. The latter functor evaluates a spectrum $\mathscr{X}_*$ at $S^n(\mathbb{S}^0) \cong \mathbb{S}^n$ (see Example 4.20). We therefore have an isomorphism $\Omega^\infty\Sigma^n \cong \Omega^{\infty-n}$, as desired. □

Remark 3.35. Our structural equivalences $\Omega\mathscr{X}_{n+1} \xrightarrow{\sim} \mathscr{X}_n$ are going in the "wrong direction" relative to standard practice, cf. [1, Section 2]. The direction we've employed herein is simply the direction which is implied by the description of the limit given in Section II-14.2.

3.7. **Enriching in spectra.** We begin with a helpful little lemma.

Lemma 3.36. *If a functor $F : \mathscr{D} \to \mathscr{D}'$ is an equivalence of finitely complete ∞-categories, and $\mathscr{C}$ is finitely complete as well, then the induced functor*

$$F_* : \mathrm{Fun}^{\mathrm{lex}}(\mathscr{C}, \mathscr{D}) \to \mathrm{Fun}^{\mathrm{lex}}(\mathscr{C}, \mathscr{D}') \tag{19}$$

is an equivalence.

Proof. The functor $\mathrm{Fun}(\mathscr{C}, \mathscr{D}) \to \mathrm{Fun}(\mathscr{C}, \mathscr{D}')$ is an equivalence, and equivalences preserve and detect limit diagrams. It follows that taking the fiber along the inclusion $\mathrm{incl} : \mathrm{Fun}^{\mathrm{lex}}(\mathscr{C}, \mathscr{D}') \to \mathrm{Fun}(\mathscr{C}, \mathscr{D}')$ returns the map (19). Since the class of left exact functors is stable under isomorphism, by Proposition II-13.19, we see that the map incl is an isofibration. It follows that the map (19) is an equivalence, by Corollary I-5.24 for example. □

We now observe the following.

Proposition 3.37. *Suppose $\mathscr{C}$ is a pointed ∞-category which admits finite limits, and that pushout diagrams agree with pullback diagrams in $\mathscr{C}$. Then for any pointed Hom functor $\widetilde{H} : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{K}an_*$ there is a functor $\widetilde{H}_* : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{S}\!p$ which fits into a 2-simplex*

$$\begin{tikzcd} & \mathscr{S}\!p \arrow[d] \\ \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \arrow[ur, "\widetilde{H}_*"] \arrow[r, "\widetilde{H}"'] & \mathscr{K}an_* \end{tikzcd}$$

in $\mathscr{C}at_\infty$. Furthermore, this functor $\widetilde{H}_*$ is unique up to a contractible space of choices.

Proof. Take $\mathscr{C}^{\mathrm{e}} = \mathscr{C}^{\mathrm{op}} \times \mathscr{C}$ and, for any ∞-category $\mathscr{D}$ which admits finite limits, let $\mathrm{Fun}'(\mathscr{C}^{\mathrm{e}}, \mathscr{D})$ denote the full subcategory of functors which are left exact in each factor independently. The adjunction $\mathrm{Fun}(\mathscr{C}, \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathscr{D}) \cong \mathrm{Fun}(\mathscr{C}^{\mathrm{e}}, \mathscr{D})$ then restricts to an adjunction

$$\mathrm{Fun}'(\mathscr{C}^{\mathrm{e}}, \mathscr{D}) \cong \mathrm{Fun}^{\mathrm{lex}}(\mathscr{C}, \mathrm{Fun}^{\mathrm{lex}}(\mathscr{C}^{\mathrm{op}}, \mathscr{D})),$$

where here we note that $\mathrm{Fun}^{\mathrm{lex}}(\mathscr{C}^{\mathrm{op}}, \mathscr{D})$ is closed under the formation of finite limits in the ambient category $\mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathscr{D})$ [14, Remark 1.4.2.3]. By Proposition 3.33 and Lemma 3.36 we now see that the functor Ω^∞ induces an equivalence

$$\Omega^\infty_* : \mathrm{Fun}'(\mathscr{C}^e, \mathscr{S}\!p) \overset{\sim}{\to} \mathrm{Fun}'(\mathscr{C}^e, \mathscr{K}an_*). \tag{20}$$

Now, the space of lift $\widetilde{H}_*$ which fit into a 2-simplex as proposed can be parametrized by the homotopy fiber

$$\mathrm{Fun}'(\mathscr{C}^e, \mathscr{S}\!p) \times^{\mathrm{htop}}_{\mathrm{Fun}'(\mathscr{C}^e, \mathscr{K}an_*)} \{\widetilde{H}\}.$$

Since the map (20) is an equivalence it follows, by Proposition I-5.23, that the projection

$$\mathrm{Fun}'(\mathscr{C}^e, \mathscr{S}\!p) \times^{\mathrm{htop}}_{\mathrm{Fun}'(\mathscr{C}^e, \mathscr{K}an_*)} \{\widetilde{H}\} \to *$$

is an equivalence as well. In particular, the fiber in question is a contractible Kan complex. □

The proposition says that, for such "symmetrically fibered" $\mathscr{C}$, the existence of a pointed Hom functor for $\mathscr{C}$ is more-or-less the same as a spectra-valued Hom functor for $\mathscr{C}$.

Remark 3.38. Categories satisfying the hypotheses of Proposition 3.37 are called stable ∞-categories. We see in Section 4 that the homotopy and derived ∞-categories of an abelian category are both stable. Hence the pointed Hom functors from Corollary 3.28 uniquely enhance to spectra values Hom functors.

Though this spectral perspective is commonly employed in the literature, we won't have any particular use for it here, and so are happy to work with pointed spaces throughout this work.

3.8. **Whitehead's theorem in the spectral setting.**

Proposition 3.39 ([14, Proposition 1.4.4.4])**.** *The forgetful functor $\Omega^\infty : \mathscr{S}\!p \to \mathscr{K}an_*$ admits a lift adjoint $\Sigma^\infty : \mathscr{K}an_* \to \mathscr{S}\!p$.*

Definition 3.40. The n-th sphere spectrum $\underline{\mathbb{S}}^n$ is the object $\Sigma^\infty \mathbb{S}^n$ in $\mathscr{S}\!p$.

We can define the n-th homotopy group $\pi_n(X_*)$ of a spectrum X_* as the set

$$\pi_n(X_*) = \mathrm{Hom}_{\mathrm{h}\,\mathscr{S}\!p}(\underline{\mathbb{S}}^n, X_*).$$

Via the $(\Sigma^\infty, \Omega^\infty)$-adjunction we have the natural identification

$$\pi_n(X_*) = \pi_n(X_0, x_0) \cong \pi_{n+i}(X_i, x_i),$$

where $x_k : * \to X_k$ is the implicit pointing on the k-th space. From the second perspective it is clear that we can define the homotopy groups at negative n.

Definition 3.41. For any spectrum $\mathscr{X}_*$ we define the n-th homotopy group, for $n \in \mathbb{Z}$, as the colimit

$$\pi_n(X_*) = \mathrm{colim}_{n+i \geq 0}\, \pi_{n+i}(X_i, x_i).$$

The following is a consequence of Whitehead's theorem for equivalences of Kan complexes.

Theorem 3.42 (Spectral Whitehead's theorem)**.** *For a map of spectra $f : \mathscr{X}_* \to \mathscr{Y}_*$ the following are equivalent:*

(a) *f is an isomorphism in $\mathscr{S}\!p$.*

(b) *At each integer $i \geq 0$, the induced map $f_i : \mathscr{X}_i \to \mathscr{Y}_i$ is an equivalence of Kan complexes.*

(c) *At each integer $n \in \mathbb{Z}$, the map $\pi_n f : \pi_n(\mathscr{X}_*) \to \pi_n(\mathscr{Y}_*)$ is an isomorphism.*

Proof. We identify the category of spectra with the limit category $\mathscr{L}\mathscr{K}an_*$ as in Proposition 3.34. From this perspective we have a transformation $f : \mathscr{X}_* \to \mathscr{Y}_*$ between functors $\mathscr{X}_*, \mathscr{Y}_* : \mathbb{Z}_{\leq 0} \to \mathrm{N}^\omega(\mathbb{Z}_{\leq 0})$, where ω is the diagram

$$\cdots \to \mathscr{K}an_* \overset{\Omega}{\to} \mathscr{K}an_* \overset{\Omega}{\to} \mathscr{K}an_*$$

in $\mathscr{C}at_\infty^{\mathrm{big}}$. The map $f_i : \mathscr{X}_i \to \mathscr{Y}_i$ is obtained by evaluating the transformation f at $-i \in \mathbb{Z}_{\leq 0}$. Hence the equivalence between (a) and (b) follows from the fact that a transformation between functors is an equivalence if and only if it evaluates to an equivalence at each vertex in the domain, by Proposition I-6.5. The fact that (b) implies (c) is also immediate.

For the implication from (c) to (b), consider a map of Kan complexes $g : \mathscr{S} \to \mathscr{T}$ and a point $s : * \to \mathscr{S}$ with image $t = g(s) : * \to \mathscr{T}$. Let $\mathscr{S}_0$ and $\mathscr{T}_0$ be the components of $\mathscr{S}$ and $\mathscr{T}$ containing s and t respectively. By Whitehead's theorem, Theorem I-3.37, the restriction $f|_{\mathscr{S}_0}$ is a homotopy equivalence if and only if all of the maps

$$\pi_m f : \pi_m(\mathscr{S}, s) \to \pi_m(\mathscr{T}, t)$$

is an equivalence at each integer $m \geq 0$, which then occurs if and only if the induced map on loop spaces $\Omega f : \Omega(\mathscr{S}, s) \to \Omega(\mathscr{T}, t)$ is a homotopy equivalence.

Now, at a fixed integer i, the condition that each map $\pi_n f : \pi_n(\mathscr{X}_*) \to \pi_n(\mathscr{Y}_*)$ is an isomorphism implies that $\pi_m(f_{i+1}) : \pi_m(\mathscr{X}_{i+1}) \to \pi_m(\mathscr{Y}_{i+1})$ is an isomorphism at all nonnegative integers m. Hence the induced map on loop spaces $\Omega(f_{i+1}) : \Omega(\mathscr{X}_{i+1}) \to \Omega(\mathscr{Y}_{i+1})$ is an equivalence. From the diagram (18) we conclude that each map $f_i : \mathscr{X}_i \to \mathscr{Y}_i$ is an equivalence. $\square$

4. Stability and cocompleteness of homotopy and derived ∞-categories

Given a Grothendieck abelian category $\mathbb{A}$, we prove that the homotopy and derived ∞-categories, $\mathscr{K}(\mathbb{A})$ and $\mathscr{D}(\mathbb{A})$, are both stable and cocomplete. In a very vague sense, one might view stability as a analog of abelianness in the derived/homotopical setting. From a practical perspective, stability allows one, quite often, to reduce analyses at the ∞-level to corresponding analyses at the discrete level via an application of the homotopy category functor.

4.1. **Pullbacks are puhsouts in the homotopy ∞-category.**

Lemma 4.1. *For a diagram*

$$\begin{tikzcd} V_0 \arrow[r, "g"] \arrow[d, "g'"'] & V_1 \arrow[d, "f"] \\ V_1' \arrow[r, "f'"'] & V_2 \end{tikzcd} \tag{21}$$

in $\mathrm{Ch}(\mathbb{A})$ *the following are equivalent:*

(a) *The map f is (termwise split) surjective, f' is injective, and* (21) *is a discrete pullback diagram.*

(b) *The map g is (termwise split) injective, g' is surjective, and* (21) *is a pushout diagram.*

Proof. (a) $\Rightarrow$ (b) If (21) is a pullback diagram with the prescribed properties then the map g is an inclusion which identifies V_0 as a kernel of the composite $V_1 \to V_2 \to V_2/V_1'$. This identification of V_0 with the kernel of f also tells us that g is termwise split whenever f is termwise split. Furthermore, in this case the map $g' : V_0 \to V_1'$ is simply the restriction of the projection $V_1 \to V_2$ to V_0, and hence g' is surjective as well. So we see that g is injective and g' is surjective in this case, and g is split when f is spit.

The implication (b) $\Rightarrow$ (a) is recovered by applying (a) $\Rightarrow$ (b) to the category $\mathrm{Ch}(\mathbb{A}^{\mathrm{op}}) = \mathrm{Ch}(\mathbb{A})^{\mathrm{op}}$. $\square$

Proposition 4.2. *For any abelian category $\mathbb{A}$, a diagram*

$$\begin{tikzcd} V_0 \arrow[r, "g"] \arrow[d, "g'"'] \arrow[dr] & V_1 \arrow[d, "f"] \\ V_1' \arrow[r, "f'"'] & V_2 \end{tikzcd} \tag{22}$$

in $\mathscr{K}(\mathbb{A})$ is a limit (aka pullback) diagram if and only if it is a colimit (aka pushout) diagram.

Proof. Suppose that the diagram (22) is a limit diagram. Then, according to Proposition 2.12, we can assume (22) is a strictly commuting, discrete pullback diagram in which f' is termwise split surjective. In particular, any pullback diagram in $\mathscr{K}(\mathbb{A})$ is isomorphic to such a discrete pullback diagram. In this case (22) is also a discrete pushout diagram in which g is termwise split injective, by Lemma 4.1. By Proposition 2.15 such a discrete pushout diagram is a colimit diagram in $\mathscr{K}(\mathbb{A})$, so that the diagram (22) is a colimit diagram in the homotopy ∞-category.

The converse implication is proved similarly. Namely, if (22) is a colimit diagram then we can assume it is a discrete pushout diagram with g termwise split injective, by Proposition 2.15, at which point it is seen to be a limit diagram as well by Lemma 4.1 and Proposition 2.19. $\square$

4.2. **Stable ∞-categories.**

Definition 4.3. Let $\mathscr{C}$ be an ∞-category which admits an object 0 which is both initial and terminal. A diagram of the form

$$\begin{tikzcd} x \arrow[r, "\alpha"] \arrow[d] \arrow[dr] & y \arrow[d, "\beta"] \\ 0 \arrow[r] & z \end{tikzcd}$$

is called a fiber sequence in $\mathscr{C}$ if it is a limit diagram, and a cofiber sequence if it is a colimit diagram.

Let us recall that any ∞-category with a zero object has zero morphisms. For any pair $x_0, x_1 : * \to \mathscr{C}$, this zero morphism is the unique morphism $x_0 \to x_1$ which completes a 2-simplex

$$\begin{tikzcd} & 0 \arrow[dr] & \\ x_0 \arrow[ur] \arrow[rr] & & x_1. \end{tikzcd}$$

Here we recall that the space of maps to and from 0 are contractible, by definition, so that the space of such 2-simplices is contractible.

Now, if we consider a cofiber sequence in $\mathscr{C}$, for example, one can think of z as a cokernel for the morphism $\alpha : x \to y$. Here, for the diagram $p : \Lambda_0^2 \to \mathscr{C}$ obtained from the above square by deleting z, we have an equivalence between the mapping spaces

$$\mathrm{Hom}_{\mathscr{C}}(z, w) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Fun}(\Lambda_0^2,\mathscr{C})}(p, \underline{w}) = \{p\} \times^{\mathrm{or}}_{\mathrm{Fun}(\Lambda_0^2,\mathscr{C})} \{w\},$$

simply by the definition of the colimit, and the latter space is identified with the fiber of the undercategory $\mathscr{C}_{p/}$ over w. So in total we obtain an equivalence

$$\mathrm{Hom}_{\mathscr{C}}(z, w) \xrightarrow{\sim} \mathscr{C}_{p/} \times_{\mathscr{C}} \{w\}$$

at arbitrary w in $\mathscr{C}$. The latter fiber can be viewed as the space of maps $y \to w$ whose restriction along $\alpha : x \to y$ is trivial. One can employ a similar understanding of fiber sequences.

Example 4.4. Let $\mathbb{A}$ be a discrete additive category. A diagram in $\mathbb{A}$ of the form

$$\begin{tikzcd} x \arrow[r, "\alpha"] \arrow[d] \arrow[dr] & y \arrow[d, "\beta"] \\ 0 \arrow[r] & z \end{tikzcd}$$

is a fiber sequence if and only if α is a kernel of β, and is a cofiber sequence if and only if β is a cokernel of α.

Definition 4.5. An ∞-category $\mathscr{C}$ is called stable if the following properties hold:

(a) $\mathscr{C}$ has an object 0 which is simultaneously initial and terminal.
(b) Every morphism $\alpha : x \to y$ in $\mathscr{C}$ extends to a cofiber sequence and every morphism $\beta : y \to z$ extends to a cofiber sequence.
(c) A diagram of the form

$$\begin{tikzcd} x \arrow[r, "\alpha"] \arrow[d] \arrow[dr] & y \arrow[d, "\beta"] \\ 0 \arrow[r] & z \end{tikzcd}$$

in $\mathscr{C}$ is a fiber sequence if and only if it is a cofiber sequence.

Definition 4.6. A functor between stable ∞-categories $F : \mathscr{C} \to \mathscr{D}$ is exact if F preserves initial/terminal objects and preserves fiber/cofiber sequences. A full ∞-subcategory $\mathscr{C}' \subseteq \mathscr{C}$ is called a stable subcategory if $\mathscr{C}'$ is stable and the inclusion $\mathscr{C}' \to \mathscr{C}$ is exact.

We note that $\mathscr{C}'$ is a stable ∞-subcategory in $\mathscr{C}$ if and only if, for each morphism $\alpha : x \to y$ in $\mathscr{C}'$, and pullback and pushout diagram

$$\begin{tikzcd} x \arrow[r, "\alpha"] \arrow[d] \arrow[dr] & y \arrow[d] \\ 0 \arrow[r] & \mathrm{cofib}(\alpha) \end{tikzcd} \qquad \begin{tikzcd} \mathrm{fib}(\alpha) \arrow[r] \arrow[d] \arrow[dr] & x \arrow[d] \\ 0 \arrow[r] & y \end{tikzcd}$$

in $\mathscr{C}$, there are objects z and w in $\mathscr{C}'$ which admit isomorphisms $z \cong \mathrm{cofib}(\alpha)$ and $w \cong \mathrm{fib}(\alpha)$.

We note that stability is a property, rather than a structure. For example, if $F : \mathscr{C} \to \mathscr{D}$ is an equivalence of ∞-categories, and one of $\mathscr{C}$ or $\mathscr{D}$ is stable, then both $\mathscr{C}$ and $\mathscr{D}$ are stable.

Lemma 4.7. *Suppose $F : \mathscr{C} \to \mathscr{D}$ is an equivalence of ∞-categories and that $\mathscr{C}$ is stable. Then $\mathscr{D}$ is stable.*

Proof. Left to the reader. $\square$

The following is obvious.

Lemma 4.8. *If $\mathscr{C}$ is stable then the opposite category $\mathscr{C}^{\mathrm{op}}$ is also stable.*

The quintessential stable ∞-category of is category of $\mathscr{Sp}$ of spectra.

Theorem 4.9. *The category $\mathscr{Sp}$ of spectra is stable and has zero object $0 = \Sigma^\infty *$.*

Proof. By definition, the category of spectra is the stabilization $\mathscr{Sp} = \mathscr{Sp}(\mathscr{K}an_*)$ of the ∞-category of pointed spaces. Since the category of pointed spaces complete, by Proposition 3.12, it follows that $\mathscr{Sp}$ is stable by [14, Corollary 1.4.2.17]. As for the calculation of the zero object, the left adjoint $\Sigma^\infty : \mathscr{K}an_* \to \mathscr{Sp}$ commutes with colimits, so that it preserves the colimit over the empty diagram, i.e. the initial object. By Proposition II-9.15 the object $*$ is initial in $\mathscr{K}an_*$, so that $\Sigma^\infty *$ is an initial in $\mathscr{Sp}$. $\square$

4.3. **Stability of homotopy and derived ∞-categories.**

Theorem 4.10. *For any abelian category $\mathbb{A}$, the homotopy ∞-category $\mathscr{K}(\mathbb{A})$ is stable.*

Proof. The zero complex is a zero object in $\mathscr{K}(\mathbb{A})$ by Corollary 2.22, and since $\mathscr{K}(\mathbb{A})$ admits all pullbacks and pushouts by Corollaries 2.13 and Corollary 2.20. In particular, one can complete any morphism in $\mathscr{K}(\mathbb{A})$ to both a fiber and cofiber sequence. Finally, fiber sequences and cofiber sequences in $\mathscr{K}(\mathbb{A})$ agree by Proposition 4.2. $\square$

By the description of pullbacks and pushouts in $\mathscr{K}(\mathbb{A})$ provided in Propositions 2.12 and 2.19 we see that any full subcategory $\mathscr{K} \subseteq \mathscr{K}(\mathbb{A})$ which is preserved under desuspension and the formation of mapping cones admits all pushouts an pullbacks. In particular, the inclusion $\mathscr{K} \to \mathscr{K}(\mathbb{A})$ preserves pushouts and pullbacks. Thus, under these conditions, and assuming additionally that $\mathscr{K}$ contains the zero complex, we see that $\mathscr{K}$ is a stable ∞-category as well.

Corollary 4.11. *Suppose a full ∞-subcategory $\mathscr{K} \subseteq \mathscr{K}(\mathbb{A})$ contains the zero complex, is closed under the formation of mapping cones, and is closed under the desuspension automorpihsm Σ^{-1}. Then $\mathscr{K}$ is a stable ∞-subcategory in $\mathscr{K}(\mathbb{A})$. In particular, $\mathscr{K}$ is stable.*

In the case of a Grothendieck abelian category, we have the injective construction of the (unbounded) derived ∞-category $\mathscr{D}(\mathbb{A})$, where $\mathscr{D}(\mathbb{A})$ is identified specifically as the full ∞-subcategory of K-injectives in $\mathscr{K}(\mathbb{A})$ (Definition I-2.9). As a particular instance of Corollary 4.11 we observe stability of the derived ∞-category.

Corollary 4.12. *If $\mathbb{A}$ is a Grothendieck abelian category then the derived ∞-category $\mathscr{D}(\mathbb{A})$ is a stable ∞-category.*

Proof. We construct the derived ∞-category $\mathscr{D}(\mathbb{A})$ specifically as the ∞-subcategory of K-injectives in $\mathscr{K}(\mathbb{A})$. For any map between K-injectives $f : I \to I'$, and arbitrary acyclic X, the Hom complex

$$\mathrm{Hom}^*_{\mathbb{A}}(X, \mathrm{cone}(f)) = \mathrm{cone}\left(f_* : \mathrm{Hom}^*_{\mathbb{A}}(X, I) \to \mathrm{Hom}^*_{\mathbb{A}}(X, I')\right).$$

Since both complexes in the latter cone are acyclic, the cone itself is acyclic, and we see that $\mathrm{cone}(f)$ is K-injective. By the above reasoning we now conclude that the derived ∞-category is stable. $\square$

Remark 4.13. As stated previously, stability of an ∞-category is a property not a structure. So, in the event that a Grothendieck abelian category $\mathbb{A}$ has enough projectives, one can prove stability by employing either the K-injective construction of the derived ∞-category $\mathscr{D}(\mathbb{A})$ or the K-projective construction. See Section I-12.

We similarly apply Corollary 4.11 to observe stability of the derived category under the standard bounding restrictions. We have the ∞-subcategories $\mathscr{D}^b(\mathbb{A})$, $\mathscr{D}^-(\mathbb{A})$, and $\mathscr{D}^+(\mathbb{A})$ of bounded, (cohomologically) bounded above, and (cohomologically) bounded below complexes.

Corollary 4.14. *If $\mathbb{A}$ is a Grothendieck abelian category then for $\star = b, +, -$, the appropriately bounded derived ∞-category $\mathscr{D}^\star(\mathbb{A}) \subseteq \mathscr{D}(\mathbb{A})$ is a stable ∞-subcategory. Furthermore, in the case where $\mathbb{A}$ is linear and locally finite, the full ∞-subcategory $\mathscr{D}(\mathbb{A})_{fin}$ of complexes with finite total length is a stable ∞-subcategory in $\mathscr{D}(\mathbb{A})$.*

To be clear, by a locally finite Grothendieck abelian category we mean a compactly generated abelian category all of whose compacts are of finite length, and whose morphisms between compacts $\mathrm{Hom}_{\mathbb{A}}(x, y)$ are finite-dimensional over the given base field. Such categories include representations $\mathbb{A} = \mathrm{Rep}(G)$ over an affine algebraic group G, for example.

Example 4.15 (An anti-example: The connective derived category)**.** An interesting anti-example, the connective derived ∞-category $\mathscr{D}^{\leq 0}(\mathbb{A})$ is not stable in $\mathscr{D}(\mathbb{A})$. Indeed, this ∞-category is not even stable.

We can consider explicitly a complex V which is concentrated in degree 0, and the pullback diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] \arrow[dr] & 0 \arrow[d] \\ 0 \arrow[r] & V. \end{tikzcd} \tag{23}$$

To see that this diagram is a pullback diagram we note that the truncation functor $\tau_0 : \mathscr{D}(\mathbb{A}) \to \mathscr{D}^{\leq 0}(\mathbb{A})$ (Definition 2.7) is right adjoint to the inclusion $\mathscr{D}^{\leq 0}(\mathbb{A}) \to \mathscr{D}(\mathbb{A})$ (Theorem I-13.10). Truncation therefore preserves limits by Proposition II-13.24, so that we obtain the above limit diagram by truncating the standard limit diagram in $\mathscr{D}(\mathbb{A})$

$$\begin{tikzcd} \Sigma^{-1}V \arrow[r] \arrow[d] \arrow[dr] & 0 \arrow[d] \\ 0 \arrow[r] & V \end{tikzcd}$$

provided by Proposition 2.12. However, we see simultaneously that the diagram (23) is not a pushout diagram unless $V = 0$.

4.4. **Suspension and desuspension in the general setting.** Fix $\mathscr{C}$ an ∞-category which admits a zero object. We first sketch a construction the suspension and desuspension functors [14, Section 1.1.2]: Let $\mathscr{M}^{\Sigma} = \mathscr{M}^{\Sigma}_{\mathscr{C}}$ and $\mathscr{M}^{\Omega} = \mathscr{M}^{\Omega}_{\mathscr{C}}$ denote the full subcategories

$$\mathscr{M}^{\Sigma},\ \mathscr{M}^{\Omega}\ \subseteq\ \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathscr{C})$$

whose objects are diagrams

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0' \arrow[r] & Y \end{tikzcd}$$

which are pushout diagrams, in the case of $\mathscr{M}^{\Sigma}$, and pullback diagrams, in the case of $\mathscr{M}^{\Omega}$. Here 0 and $0'$ are arbitrary zero objects in $\mathscr{C}$.

We have the evaluation functors at the initial and terminal vertices

$$ev_0 : \mathscr{M}^{\Sigma} \to \mathscr{C} \ \text{ and } \ ev_2 : \mathscr{M}^{\Omega} \to \mathscr{C}, \tag{24}$$

which are both isofibrations. The following is an application of [13, Proposition 4.3.2.15]. See the discussion preceding [14, Remark 1.1.2.6].

Proposition 4.16. *For $\mathscr{C}$ a pointed ∞-category which admits arbitrary pushouts and pullbacks, the evaluation functors (24) are both trivial Kan fibrations.*

We now can define the suspension and looping operations as endofunctors on $\mathscr{C}$.

Definition 4.17. Let $\mathscr{C}$ be a pointed ∞-category which admits all pullbacks and pushouts. The suspension functor $\Sigma : \mathscr{C} \to \mathscr{C}$ is the composite

$$\mathscr{C} \xrightarrow{s_0} \mathscr{M}^{\Sigma}_{\mathscr{C}} \xrightarrow{ev_2} \mathscr{C},$$

where s_0 is any section of the trivial fibration ev_0. Similarly, the looping functor $\Omega : \mathscr{C} \to \mathscr{C}$ is the composite

$$\mathscr{C} \xrightarrow{s_2} \mathscr{M}^{\Omega}_{\mathscr{C}} \xrightarrow{ev_0} \mathscr{C}$$

where s_2 is any section of ev_2.

At a truncated level these functors can be understood explicitly. We consider the suspension functor, as the description for the looping functor is completely similar.

In practice we can choose a natural zero object, so let us fix such a choice $0 : * \to \mathscr{C}$. For an object x in $\mathscr{C}$ let $p_x : \Lambda^2_0 \to \mathscr{C}$ denote the diagram

$$
p_x = \begin{tikzcd} x \arrow[r] \arrow[d] & 0 . \\ 0 & \end{tikzcd}
$$

The functor $\Sigma : \mathscr{C} \to \mathscr{C}$ then sends each object x to the unique object $\Sigma x = \operatorname{colim} p_x$. For any map $\alpha : x \to y$, we then have the uniquely associated transformation

$$
\begin{tikzcd}
 & x \arrow[r, "\alpha"] \arrow[dl] \arrow[dr] & y \arrow[dl] \arrow[dr] & \\
0 \arrow[r, "="'] & 0 & 0 \arrow[r, "="'] & 0
\end{tikzcd}
$$

from which we obtain a transformation $p_x \to \underline{y}$. By the definition of a colimit, this latter transformation provides a uniquely associated map $\Sigma\alpha : \Sigma x \to \Sigma y$. This describes the shift functor completely after homotopy truncation $\mathrm{h}\,\Sigma : \mathrm{h}\,\mathscr{C} \to \mathrm{h}\,\mathscr{C}$.

Example 4.18. We saw in Theorem 3.25 that the category of pointed spaces $\mathscr{K}an_*$ is both complete and cocomplete. It furthermore has the zero object $*$ provided by the one point space, by Proposition II-9.15. We therefore have the suspension and looping functors $\Sigma, \Omega : \mathscr{K}an_* \to \mathscr{K}an_*$. While the suspension functor is slightly mysterious, the looping functor on objects can be realized explicitly on pointed spaces $\mathscr{X} = (\mathscr{X}, x)$ as the assignment

$$\Omega : \mathscr{X} \mapsto \mathrm{Hom}_{\mathscr{X}}(x, x),$$

according to the materials of Section I-6.4. If we replace $\mathscr{K}an_*$ with its more rigid model $\mathrm{N}^{\mathrm{hc}}(\underline{\mathrm{Kan}}_*)$, then Ω can be described on morphisms as the assignment

$$\Omega : (F : \mathscr{X} \to \mathscr{Y}) \mapsto (F_* : \mathrm{Hom}_{\mathscr{X}}(x, x) \to \mathrm{Hom}_{\mathscr{Y}}(y, y)).$$

In general, the suspensions and loops functors are adjoint. Specifically, the suspension functor is left adjoint to the loops functor [14, Remark 1.1.2.8]. However, we are most interested in the stable categories. Here we observe the following.

Proposition 4.19. *If $\mathscr{C}$ is stable, then the endofunctors $\Sigma, \Omega : \mathscr{C} \to \mathscr{C}$ are autoequivalences which are mutually inverse.*

Proof. Since the evaluation morphisms are equivalences so are their sections s_i. Furthermore each s_i is an inverse to ev_i. So we have

$$\Sigma\Omega = \mathrm{ev}_2\, s_0\, \mathrm{ev}_0\, s_2 \cong id_{\mathscr{C}}, \quad \Omega\Sigma = \mathrm{ev}_0\, s_2\, \mathrm{ev}_2\, s_0 \cong id_{\mathscr{C}}.$$

□

In the algebriac setting the dg automorphism $\Sigma : \mathbf{Ch}(\mathbb{A}) \to \mathbf{Ch}(\mathbb{A})$ induces an equivalence of ∞-categories $\Sigma' : \mathscr{K}(\mathbb{A}) \to \mathscr{K}(\mathbb{A})$, and one can construct an explicit section s_0 for the homotopy ∞-category $\mathrm{ev}_0 : \mathscr{K}(\mathbb{A}) \to \mathscr{M}$ by sending each object V to the explicit diagram

$$
\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma X \end{tikzcd}
$$

and each n-simplex σ to the n-simplex $s_0\sigma : \Delta^n \times (\Delta^1 \times \Delta^1) \to \mathscr{C}$ which restricts to σ and $\Sigma'\sigma$ at the vertices $(0,0)$ and $(1,1)$ respectively, 0 at the vertices $(1,0)$ and $(0,1)$, and vanishes on all other non-degenerate vertices. (Here one should interpret the word vanishes correctly, based on the explicit construction of the dg nerve.) In this way we recover $\Sigma' = \mathrm{ev}_2\, s_0 = \Sigma$, and one similarly recovers $(\Sigma')^{-1} = \mathrm{ev}_0\, s_2 = \Omega$.

In general, we simply write Σ and Σ^{-1} for the suspension and looping functors on, or shift functors, on $\mathscr{K}(\mathbb{A})$ and $\mathscr{D}(\mathbb{A})$.

Example 4.20 (Suspensions of spheres)**.** Consider the n-sphere $\underline{\mathbb{S}}^n = \Sigma^\infty \mathbb{S}^n$ in $\mathscr{S}\!p$. Since the functor Σ^∞ is a left adjoint, and hence commutes with colimits, it suffices to compute the pushout

$$\begin{tikzcd} \mathbb{S}^n \arrow[r] \arrow[d] \arrow[dr] & * \arrow[d] \\ * \arrow[r] & \mathscr{X}_{\mathbb{S}^n} \end{tikzcd}$$

in the category $\mathscr{K}an_*$. By Corollary 3.17 it suffices further to compute such pushouts in the unpointed category $\mathscr{K}an$. By the explicit formula given in Section II-14.5, we have $\mathscr{X}_{\mathbb{S}^n} = Q(\mathrm{N}^p(\Lambda^0_2) = Q(\mathbb{S}^n \star \partial\Delta^1)$ where $Q(K)$ denotes any weak homotopy replacement for the given simplicial set [15, 00UV]. This weak homotopy replacement can be given as the singular complex of the geometric realization, at which point one calculates

$$\mathscr{X}_{\mathbb{S}^n} = \mathrm{Sing}\,|\mathbb{S}^n \star \partial\Delta^1| \cong \mathbb{S}^{n+1}.$$

Since the pushout is only defined up to isomorphism, we can take simply $\mathscr{X}_{\mathbb{S}^n} = \mathbb{S}^{n+1}$.

If we adopt the explicit expression given by the singular complex $\mathbb{S}^k = \mathrm{Sing}\, S^k$, then the inclusion $S^n \to S^{n+1}$ along with the two contractions onto the north and south poles provide an explicit pushout diagram

$$\begin{tikzcd} \mathrm{Sing}\, S^n \arrow[r] \arrow[d] \arrow[dr] & * \arrow[d, "\infty"] \\ * \arrow[r, "-\infty"'] & \mathrm{Sing}\, S^{n+1}. \end{tikzcd}$$

This pushout diagram determines the isomorphism $\Sigma\mathbb{S}^n \to \mathbb{S}^{n+1}$ which is unique up to a contractible space of choices.

One notes that the above analysis generalizes to provide a calculation of the suspension $\Sigma\mathscr{X}_*$ at any spectrum $\mathscr{X}_*$ which is in the image of the functor Σ^∞. As a corollary to our example we find the following.

Corollary 4.21. *At any non-negative integer n, there are isomorphisms*

$$\Sigma(\underline{\mathbb{S}}^n) \cong \mathbb{S}^{n+1},\ \Omega(\underline{\mathbb{S}}^{n+1}) \cong \mathbb{S}^n,\ \textit{and } \Sigma^n(\underline{\mathbb{S}}^0) \cong \underline{\mathbb{S}}^n.$$

4.5. **Suspension and exact functors.**

Lemma 4.22. *Let $F : \mathscr{C} \to \mathscr{D}$ be a functor between stable categories. If F respects fiber sequences then there is a uniquely associated isomorphism $F \circ \Sigma \cong \Sigma \circ F$.*

Proof. We have the induced map on the restricted functor spaces $\mathscr{M}_F : \mathscr{M}_{\mathscr{C}} \to \mathscr{M}_{\mathscr{D}}$ which fits into diagrams

$$\begin{tikzcd} \mathscr{M}_{\mathscr{C}} \arrow[r, "\mathscr{M}_F"] \arrow[d, "\mathrm{ev}_0"'] & \mathscr{M}_{\mathscr{D}} \arrow[d, "\mathrm{ev}_0"] \\ \mathscr{C} \arrow[r, "F"'] & \mathscr{D}. \end{tikzcd}$$

Since the evaluation functor is a trivial Kan fibration there is, furthermore, a unique lift $\widetilde{F}_0 : \mathscr{C} \to \mathscr{M}_{\mathscr{D}}$ of the functor $F : \mathscr{C} \to \mathscr{D}$ along ev_0. For any sections s_0 of ev_0 we then have the two lifts $\mathscr{M}_F s_0$ and $s_0 \mathscr{M}_F$, which are therefore identified, which gives

$$F\Sigma = F\,\mathrm{ev}_2\, s_0 = \mathrm{ev}_2\, \mathscr{M}_F s_0 \cong \mathrm{ev}_2\, s_0 F = \Sigma F.$$

□

One can "directly see" the enhancement of Hom functors in the stable setting to spectra, as in Proposition 3.37. Specifically, we have the following.

Proposition 4.23. *For any stable ∞-category $\mathscr{C}$, and all pairs of objects $x, y : * \to \mathscr{C}$, we have canonical isomorphisms*

$$\mathrm{can}^n_\Sigma : \Omega^n \operatorname{Hom}_{\mathscr{C}}(x,y) \xrightarrow{\sim} \operatorname{Hom}_{\mathscr{C}}(\Sigma^n x, y) \;\; \textit{and} \;\; \mathrm{can}^n_\Omega : \Omega^n \operatorname{Hom}_{\mathscr{C}}(x,y) \xrightarrow{\sim} \operatorname{Hom}_{\mathscr{C}}(x, \Omega^n y)$$

in the homotopy category $\mathrm{h}\mathscr{K}an$. Furthermore, for any exact functor $F : \mathscr{C} \to \mathscr{D}$ these isomorphisms fit into diagrams

$$\begin{tikzcd} \Omega^n \operatorname{Hom}_{\mathscr{D}}(Fx, Fy) \arrow[r, "\mathrm{can}^n"] & \operatorname{Hom}_{\mathscr{D}}(\Sigma^n Fx, Fy) \\ \Omega^n \operatorname{Hom}_{\mathscr{C}}(x,y) \arrow[r, "\mathrm{can}^n"'] \arrow[u, "F"] & \operatorname{Hom}_{\mathscr{C}}(\Sigma^n x, y) \arrow[u, "F"'] \end{tikzcd}$$

and

$$\begin{tikzcd} \Omega^n \operatorname{Hom}_{\mathscr{D}}(Fx, Fy) \arrow[r, "\mathrm{can}^n"] & \operatorname{Hom}_{\mathscr{D}}(Fx, \Omega^n Fy) \\ \Omega^n \operatorname{Hom}_{\mathscr{C}}(x,y) \arrow[r, "\mathrm{can}^n"'] \arrow[u, "F"] & \operatorname{Hom}_{\mathscr{C}}(x, \Omega^n y) \arrow[u, "F"'] \end{tikzcd}$$

which commute in $\mathrm{h}\mathscr{K}an$.

Here Ω is the space of pointed loops at the 0 map in $\operatorname{Hom}_{\mathscr{C}}(x,y)$ (Section I-6.4).

Proof. One obtains the claim about desuspension Ω from that of Σ by taking opposites. So it suffices to prove the claim about Σ. Furthermore, the claim at $n > 1$ is obtained from the claim at $n = 1$ via iteration. So we consider the case of a single application of the shift functor. We first establish existence of the claimed isomorphism, then resolve naturality.

Consider a pointed Hom functor $H_{\mathscr{C}}$ for $\mathscr{C}$. As $H_{\mathscr{C}}$ is left exact in both coordinates, we apply the functor $H_{\mathscr{C}}(-,y)$ to the (co)fiber diagram

$$\begin{tikzcd} x \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma x \end{tikzcd}$$

to obtain a pullback diagram

$$\begin{tikzcd}
H_{\mathscr{C}}(\Sigma x, y) \arrow[r] \arrow[d] & H_{\mathscr{C}}(0, y) \arrow[d] \\
H_{\mathscr{C}}(0, y) \arrow[r] & H_{\mathscr{C}}(x, y)
\end{tikzcd}$$

Since the point $*$ is initial in $\mathscr{K}an_*$, and the space $H_{\mathscr{C}}(0, y)$ is contractible, we have a uniquely associated isomorphism of diagrams

$$\begin{tikzcd}
* \arrow[r, "\sim"] \arrow[dr] & H_{\mathscr{C}}(0, y) \arrow[dr] & * \arrow[r, "\sim"] \arrow[dl] & H_{\mathscr{C}}(0, y) \arrow[dl] \\
& H_{\mathscr{C}}(x, y) \arrow[r, "id"'] & H_{\mathscr{C}}(x, y) &
\end{tikzcd} \tag{25}$$

in $\mathrm{h}\,\mathrm{Fun}(\Lambda^2_2, \mathscr{K}an_*)$ which provided a uniquely associated morphism $\mathrm{can} : \Omega H_{\mathscr{C}}(x, y) \to H_{\mathscr{C}}(\Sigma x, y)$ in $\mathrm{h}\,\mathscr{K}an$, by the definition of the limit. Since the transformation (25) is an isomorphism, the map can is an isomorphism as well (Proposition II-13.17).

For the naturality claim, for any pointed Hom functor $H_{\mathscr{D}}$ for $\mathscr{D}$ we have the transformation $H_F : H_{\mathscr{C}} \to H_{\mathscr{D}}$ induced by F. Let us define the diagram $p_{\mathscr{C}}(x, y) : \Lambda^2_2 \to \mathscr{K}an_*$ as

$$\begin{tikzcd}
* \arrow[dr] & & * \arrow[dl] \\
& H_{\mathscr{C}}(x, y) &
\end{tikzcd}$$

and $p'_{\mathscr{C}}(x, y)$ as

$$\begin{tikzcd}
H_{\mathscr{C}}(0, y) \arrow[dr] & & H_{\mathscr{C}}(0, y) \arrow[dl] \\
& H_{\mathscr{C}}(x, y) & .
\end{tikzcd}$$

Define the diagrams $p_{\mathscr{D}}(Fx, Fy), p'_{\mathscr{D}}(Fx, Fy) : \Lambda^2_2 \to \mathscr{K}an_*$ similarly.

We have the unique isomorphisms

$$p_{\mathscr{C}}(x, y) \overset{\sim}{\to} p'_{\mathscr{C}}(x, y) \quad \text{and} \quad p_{\mathscr{D}}(Fx, Fy) \overset{\sim}{\to} p'_{\mathscr{D}}(Fx, Fy)$$

defined as in (25), and the unique map $\theta : p_{\mathscr{C}}(x, y) \to p'_{\mathscr{D}}(x, y)$ which appears as

$$\begin{tikzcd}
* \arrow[r] \arrow[dr] & H_{\mathscr{D}}(0, Fy) \arrow[dr] & * \arrow[r] \arrow[dl] & H_{\mathscr{D}}(0, Fy) \arrow[dl] \\
& H_{\mathscr{C}}(x, y) \arrow[r, "H_F"'] & H_{\mathscr{D}}(Fx, Fy) & .
\end{tikzcd}$$

Now, the tranformation H_F induces maps

$$p_{\mathscr{C}}(x, y) \overset{\sim}{\to} p_{\mathscr{D}}(Fx, Fy) \quad \text{and} \quad p'_{\mathscr{C}}(x, y) \overset{\sim}{\to} p'_{\mathscr{D}}(Fx, Fy)$$

which now complete a diagram

$$\begin{tikzcd}
p'_{\mathscr{C}}(x, y) \arrow[r, "H_F"] & p'_{\mathscr{D}}(Fx, Fy) \\
p_{\mathscr{C}}(x, y) \arrow[u] \arrow[r, "H_F"'] \arrow[ur, "\theta"] & p_{\mathscr{D}}(Fx, Fy) \arrow[u]
\end{tikzcd}$$

in $\mathrm{h}\,\mathrm{Fun}(\Lambda^2_2, \mathscr{K}an_*)$. From this diagram one observes that the uniquely associated morphisms in $\mathrm{h}\,\mathscr{K}an_*$ obtained via applications of the universal property on the limits, fit into a diagram

$$\begin{tikzcd} \Omega H_{\mathscr{D}}(Fx, Fy) \arrow[r, "\mathrm{can}"] & H_{\mathscr{D}}(\Sigma Fx, Fy) \\ \Omega H_{\mathscr{C}}(x, y) \arrow[r, "\mathrm{can}"'] \arrow[u, "\Omega H_F"] & H_{\mathscr{C}}(\Sigma x, y). \arrow[u, "H_F"'] \end{tikzcd}$$

We finally note the natural isomorphisms $\mathrm{Hom}_{\mathscr{C}} \cong H_{\mathscr{C}}$ and $\mathrm{Hom}_{\mathscr{D}} \cong H_{\mathscr{D}}$ of $\mathrm{h}\,\mathscr{K}an$-valued functors, from Corollary II-7.12, to obtain the claimed result. □

Remark 4.24. Let us be clear that Proposition 4.23 is completely redundant for those who wish to employ spectra directly in their analysis of stable categories. It provides, however, a means of *circumventing* spectra for those who may be less familiar with the subject, and provides simultaneously an indication of the kinds of information which spectra encode in the stable setting.

4.6. **Overview: Limits and colimits in stable ∞-categories.** Our main aims of this section are two-fold: First, we want to prove homotopy and derived ∞-categories are stable. Second, we want to explain how one observes cocompleteness of the derived ∞-category. We hope this provides the reader with the proper entrance point need to begin to apply some of the essential findings from [14]. For the moment, we simply record some of the basic properties of stable categories, with appropriate referenceing to [14].

Proposition 4.25 ([14, Lemma 1.1.2.9])**.** *Any stable ∞-category $\mathscr{C}$ admits finite products and coproducts, and for any pair of object $x, y : * \to \mathscr{C}$ the canonical map*

$$\begin{bmatrix} id_x & 0 \\ 0 & id_y \end{bmatrix} : x \times y \to x \amalg y$$

is an isomorphism in $\mathscr{C}$.

Proposition 4.26 ([14, Propositions 1.1.3.4 & 1.1.4.1])**.** *Any stable ∞-category $\mathscr{C}$ admits all finite limits and colimits. Furthermore, for a functor between ∞-categories $F : \mathscr{C} \to \mathscr{D}$ the following are equivalent:*

(a) *F is exact, i.e. preserves fiber sequences.*
(b) *F preserves all finite limits.*
(c) *F preserves all finite colimits.*

Proposition 4.27 ([14, Proposition 1.1.3.4])**.** *A diagram*

$$\begin{tikzcd} x \arrow[r] \arrow[d] \arrow[dr] & y \arrow[d] \\ y' \arrow[r] & z \end{tikzcd}$$

in a stable ∞-category $\mathscr{C}$ is a pullback diagram if and only if it is a pushout diagram.

We are especially interested in cocompleteness and cocontinuity for stable ∞-categories. As in the abelian setting, existence of all small coproducts in the stable setting ensures the existence of all small colimits.

Proposition 4.28 ([14, Proposition 1.4.4.1])**.** *Let $\mathscr{C}$ be a stable ∞-category. The following are equivalent:*

(1) *$\mathscr{C}$ admits all small coproducts.*
(2) *$\mathscr{C}$ is cocomplete, i.e. admits all small colimits.*

Furthermore, if $F : \mathscr{C} \to \mathscr{D}$ is functor between stable ∞-categories, and $\mathscr{C}$ admits all small coproducts, then F is cocontinuous if and only if F is exact and preserves small coproducts.

By considering opposite categories we similarly find that a stable category $\mathscr{C}$ is complete if and only if $\mathscr{C}$ admits all small products.

Proposition 4.29. *Consider a full ∞-subcategory $\mathscr{C}'$ in a cocomplete stable ∞-category $\mathscr{C}$. If $\mathscr{C}'$ contains the zero object, and is closed under small coproducts and the formation of cofiber sequences in $\mathscr{C}$, then $\mathscr{C}'$ is cocomplete and the inclusion $\mathscr{C}' \to \mathscr{C}$ is cocontinuous.*

Sketch proof. By [13, Proposition 4.4.3.2] and ∞-category is cocomplete if and only if it admits small coproducts and coequalizaers. As $\mathscr{C}'$ is closed under the formation of small coproducts in $\mathscr{C}$, it suffices to prove that $\mathscr{C}'$ is similarly closed under the formation of coequalizers.

As argued in the proof of [14, Proposition 1.1.3.1], the coequalizer c of a pair of maps $\alpha, \alpha' : x \to y$ is the cofiber of the difference $c = \mathrm{cofib}(\alpha - \alpha')$. Indeed, we have the standard pushout diagram

$$\begin{tikzcd} x \amalg x \arrow[r, "{[\alpha\ \alpha']}"] \arrow[d, "\mathrm{diag}"'] & y \arrow[d] \\ x \arrow[r] & c \end{tikzcd}$$

which then extends to a concatenation of pushout diagrams

$$\begin{tikzcd} x \arrow[r, "{[1\ -1]^t}"] \arrow[d] & x \times x \arrow[r] \arrow[d] & y \arrow[d] \\ 0 \arrow[r] & x \arrow[r] & c \end{tikzcd}$$

and the outer square recovers the claimed fiber sequence

$$\begin{tikzcd} x \arrow[r, "(\alpha-\alpha')"] \arrow[d] & y \arrow[d] \\ 0 \arrow[r] & c. \end{tikzcd}$$

So we see that stability under the formation of cofibers implies the existence of coequalizers in any full ∞-subcategory, and we conclude that $\mathscr{C}'$ is is fact cocomplete.

As for the claim that the inclusion $\mathscr{C}' \to \mathscr{C}$ is cocontinuous, for object $x, y, y' : * \to \mathscr{C}'$ the pushout z of a diagram

$$\begin{tikzcd} x \arrow[r, "\alpha"] \arrow[d, "\alpha'"'] & y \\ y' & \end{tikzcd}$$

in $\mathscr{C}$ is the coequalizer of the maps $\alpha, \alpha' : x \to y \amalg y'$. This follows by [13, Proposition 4.4.3.1]. Since $\mathscr{C}'$ is stable under the formation of coequalizers, we now find that $\mathscr{C}'$ is stable under the formation of pushouts in $\mathscr{C}$ as well. Cocontinuity of the inclusion follows by [13, Proposition 4.4.2.7]. $\square$

Remark 4.30. Clearly the argument for Proposition 4.29 is simpler if we just assume that $\mathscr{C}'$ is stable under the formation of coproducts and pushouts in $\mathscr{C}$. In the only example we are interested in, that of connective cochains, such pushout stability is clear.

As in the usual discrete setting, one has the notion of compact objects in an ∞-category.

Definition 4.31. Let $\mathscr{C}$ be a cocomplete ∞-category. An object x in an $\mathscr{C}$ is called compact if there exists a functor $h^x : \mathscr{C} \to \mathscr{K}an$ which is represented by x and which preserves small filtered colimits [15, 02PB].

Of course, since all functors which are represented by x are isomorphic, there exists such a functor h^x which preserves filtered colimits if and only if all functors which are represented by x preserve small filtered colimits.

Proposition 4.32 ([14, Proposition 1.4.4.1])**.** *Let $\mathscr{C}$ be a cocomplete, stable ∞-category. An object $x : * \to \mathscr{C}$ is compact if and only if, for each small coproduct $\amalg_{\lambda\in\Lambda} y_\lambda$ and map $\alpha : x \to \coprod_\lambda y_\lambda$, there exists a finite subset $\{\lambda_0, \dots, \lambda_m\} \subseteq \Lambda$ for which α factors as a composite*

$$x \to (y_{\lambda_0} \amalg \cdots \amalg y_{\lambda_m}) \to \coprod_{\lambda\in\Lambda} y_\lambda.$$

4.7. **Overview: The homotopy category under stability.** We record the following for the sake of completeness.

Theorem 4.33 ([14, Theorem 1.1.2.14])**.** *For any stable ∞-category $\mathscr{C}$, the homotopy category $\mathrm{h}\mathscr{C}$ inherits a natural triangulated structure in which the exact triangles*

$$x \to y \to z$$

are exactly the images of fiber sequences in $\mathscr{C}$. The connecting morphism $\delta : z \to \Sigma x$ for such a triangle is provided by the universal property of the pushout applied to the diagram

$$\begin{tikzcd} x \arrow[r] \arrow[d] \arrow[dr] & y \arrow[d, "0"] \\ 0 \arrow[r] & \Sigma x. \end{tikzcd}$$

One can check directly that this "natural" triangulated structure on $\mathrm{h}\mathscr{C}$ recovers the standard triangulated structure on the discrete homotopy category $K(\mathbb{A}) = \mathrm{h}\mathscr{K}(\mathbb{A})$ and discrete derived category $D(\mathbb{A}) = \mathrm{h}\mathscr{D}(\mathbb{A})$. (See for example [23, Sections 10.2 & 10.4].) We leave this as an exercise for the interested reader.

Corollary 4.34. *Any exact functor between stable ∞-categories $F : \mathscr{C} \to \mathscr{D}$ induces an exact functor of triangulated categories $\mathrm{h}F : \mathrm{h}\mathscr{C} \to \mathrm{h}\mathscr{D}$.*

Additionally, Proposition 4.32 tells us that compactness of objects in a stable ∞-category can be checked at the level of the homotopy category. We recall that an object x in a triangulated category $\mathbb{C}$ which admits small sums is called compact if the functor $\mathrm{Hom}_{\mathbb{C}}(x, -)$ commutes with small sums. The following is now simply a repackaging of Proposition 4.32.

Corollary 4.35. *An object x in a stable ∞-category $\mathscr{C}$ is compact if and only if its image is compact in the homotopy category $\mathrm{h}\,\mathscr{C}$.*

4.8. **Fully faithfulness for exact functors.** Of course, it is not the case, generally speaking, that one can detect equivalences between ∞-categories at the level of the homotopy category. Consider, for example, the inclusion $0 : * \to \mathrm{Sing}(S^2)$ of a point into the circle. We have $\mathrm{h}\,\mathrm{Sing}(S^2) = *$, so that the map 0 induces an equivalence on homotopy categories. However, this map is not an equivalence since, using PropositionI-6.10, we have

$$\pi_1 \mathrm{Hom}_{\mathrm{Sing}(S^2)}(0, 0) \cong \pi_2 S^2 = \mathbb{Z}.$$

In the stable setting such phenomena never occurs, as all of the higher homotopy groups in the mapping spaces $\mathrm{Hom}_{\mathscr{C}}(x, y)$ are realized as the 0-th homotopy group of a shifted space $\mathrm{Hom}_{\mathscr{C}}(\Sigma^n x, y)$. We have the following fundamental fact.

Proposition 4.36 ([2, Proposition 5.10])**.** *For an exact functor between stable ∞-categories $F : \mathscr{C} \to \mathscr{D}$, the following are equivalent:*

(a) *F is fully faithful (resp. an equivalence).*

(b) *$\mathrm{h}\,F : \mathrm{h}\,\mathscr{C} \to \mathrm{h}\,\mathscr{D}$ is fully faithful (resp. an equivalence).*

One can approach this result directly via Whitehead's theorem and the spectral enrichments of $\mathscr{C}$ and $\mathscr{D}$. Alternatively, we can employ pointed Hom functors and use the equivalences of Proposition 4.23 directly in place of this spectral structure. We opt for the latter approach and leave the spectral analog of this argument to the interested reader.

Proof of Proposition 4.36. It is clear that if F is fully faithful, or an equivalence, then the map on homotopy categories $\mathrm{h}\,F$ is also fully faithful, or an equivalence. So the implication (a) $\Rightarrow$ (b) is clear. For the converse claim (b) $\Rightarrow$ (a), essential surjectivity can be checked at the level of the homotopy category. So we need only deal with fully faithfulness.

Suppose that $\mathrm{h}\,F$ is fully faithful. Then the maps

$$\pi_0 F : \pi_0 \mathrm{Hom}_{\mathscr{C}}(x, y) \to \pi_0 \mathrm{Hom}_{\mathscr{D}}(Fx, Fy)$$

are isomorphisms at all x and y in $\mathscr{C}$, where be base our mapping spaces at 0. We recall that there are natural isomorphisms $\pi_n(\mathscr{X}) = \pi_0 \Omega^n(\mathscr{X})$, at a general pointed space $\mathscr{X}$. (See for example Section I-6.4.) By applying shifts, and considering Proposition 4.23, we therefore find that the maps

$$\pi_n F : \pi_n \mathrm{Hom}_{\mathscr{C}}(x, y) \to \pi_n \mathrm{Hom}_{\mathscr{D}}(Fx, Fy)$$

are isomorphisms at all $n \geq 0$, where we again base at 0. As in the proof of Whitehead's theorem in the spectral setting, Theorem 3.42, we employ the natural isomorphisms of Proposition 4.23 to see that each map $\mathrm{Hom}_{\mathscr{C}}(x, y) \to \mathrm{Hom}_{\mathscr{D}}(Fx, Fy)$ is in fact a homotopy equivalence. Hence F is fully faithful. $\square$

Corollary 4.37. *An exact functor between stable ∞-categories $F : \mathscr{C} \to \mathscr{D}$ is an equivalence if and only if the induced map on homotopy categories $\mathrm{h}\,F : \mathrm{h}\,\mathscr{C} \to \mathrm{h}\,\mathscr{D}$ is an equivalence.*

Proof. By definition F is essentially surjective if and only if $\mathrm{h}\,F$ is essentially surjective, and F is an equivalence if and only if F is essentially surjective and fully faithful by Theorem I-7.2. So the result follows by Proposition 4.36. $\square$

4.9. **(Co)completeness of the derived ∞-category.** We consider again a Grothendieck abelian category $\mathbb{A}$. As explained in the proof of Corollary 4.12, the class of K-injectives in $\mathrm{Ch}(\mathbb{A})$ is stable under the formation of mapping cones. It is also clearly stable under suspension and desuspension. So, as was already argued implicitly, the full ∞-subcategory of K-injectives $\mathscr{D}(\mathbb{A})$ in $\mathscr{K}(\mathbb{A})$ is stable under the formation of both pullbacks and pushouts. This follows by the explicit constructions provided in Propositions 2.12 and 2.19 above.

Proposition 4.38. *Let $\mathbb{A}$ be a Grothendieck abelian category. The derived ∞-category $\mathscr{D}(\mathbb{A})$ admits all pullbacks and pushouts, and the inclusion $\mathscr{D}(\mathbb{A}) \to \mathscr{K}(\mathbb{A})$ preserves all pullback and pushout diagrams.*

Since products of acyclic complexes are acyclic, it is clear that products of K-injectives are K-injective. It follows tha te derived ∞-category $\mathscr{D}(\mathbb{A})$ is stable under products in $\mathscr{K}(\mathbb{A})$, and in particular admits all products. One can show that this ∞-category admits all coproducts as well.

Proposition 4.39. *For any Grothendieck abelian category $\mathbb{A}$, the derived ∞-category $\mathscr{D}(\mathbb{A})$ is admits all small products and coproducts.*

In the case that $\mathbb{A}$ has enough projectives, one can simply employ the construction of the derived ∞-category via K-projectives, and note that arbitrary sums of K-projectives remain K-projective. In a case like $\mathbb{A} = \mathrm{Rep}\,G$ for a smooth affine algebraic group of positive dimension, for example, we have no such projectives. As we demonstrate in the proof, in this case we form sums of K-injectives by taking the ordinary sum, then taking a K-projective replacement.

Proof of Proposition 4.39. The situation with products is as explained above. Consider now any collection of K-injectives $I_- : \Lambda \to \mathscr{D}(\mathbb{A})$ indexed over a small discrete set Λ. Take any injective resolution of the resulting coproduct $k : \oplus_{\lambda\in\Lambda} I_\lambda \to I$ and for each index λ let $k_\lambda : I_\lambda \to I$ be the composition of the structural map $I_\lambda \to \oplus_{\lambda\in\Lambda} I_\lambda$ with k.

Take $\mathscr{D}(\mathbb{A})'$ the simplicial construction of $\mathscr{D}(\mathbb{A})$, with natural equivalence $\mathfrak{Z} : \mathscr{D}(\mathbb{A})' \to \mathscr{D}(\mathbb{A})$ as in Theorem II-10.4. It suffices to show that the extension diagram $\{0\} \star \Lambda \to \mathscr{D}(\mathbb{A})'$, with cone point I and maps given by the k_λ, is a limit diagram.

We apply the representable functor

$$\underline{\mathrm{Hom}}_{\mathbb{A}}(-, J) = K\,\mathrm{Hom}^*_{\mathbb{A}}(-, J) : (\mathscr{D}(\mathbb{A})')^{\mathrm{op}} \subseteq (\mathscr{K}(\mathbb{A})')^{\mathrm{op}} \to \mathscr{K}an$$

to obtain an equivalence

$$\underline{\mathrm{Hom}}_{\mathbb{A}}(I, J) \overset{\sim}{\to} \underline{\mathrm{Hom}}_{\mathbb{A}}(\oplus_\lambda I_\lambda, J) = \prod_{\lambda\in\Lambda} \underline{\mathrm{Hom}}_{\mathbb{A}}(I_\lambda, J).$$

By considering a corresponding diagram in $\mathrm{Ch}(\mathbb{A})$, this equivalence is seen to extend to a strictly commuting diagram $\Delta^1 \times (\{0\} \star \Lambda) \to \mathscr{K}an$ which realizes an isomorphism between the standard product diagram

$$\begin{tikzcd} & & \underline{\mathrm{Hom}}_{\mathbb{A}}(I,J) \arrow[dl, "(k_{\mu'})^*"'] \arrow[d, "(k_\mu)^*"] \arrow[dr, "(i_{\mu''})^*"] & & \\ \cdots & \underline{\mathrm{Hom}}_{\mathbb{A}}(I_{\mu'},J) & \underline{\mathrm{Hom}}_{\mathbb{A}}(I_{\mu},J) & \underline{\mathrm{Hom}}_{\mathbb{A}}(I_{\mu''},J) & \cdots \end{tikzcd} \tag{26}$$

and the standard product diagram

$$\begin{tikzcd} & & \prod_\lambda \underline{\mathrm{Hom}}_{\mathbb{A}}(I_\lambda,J) \arrow[dl] \arrow[d] \arrow[dr] & & \\ \cdots & \underline{\mathrm{Hom}}_{\mathbb{A}}(I_{\mu'},J) & \underline{\mathrm{Hom}}_{\mathbb{A}}(I_{\mu},J) & \underline{\mathrm{Hom}}_{\mathbb{A}}(I_{\mu''},J) & \cdots. \end{tikzcd} \tag{27}$$

Since products in $\mathscr{K}an$ are realized via the discrete product in Kan, the diagram (27) is a limit diagram at all J, and it follows from Proposition II-13.18 that the diagram (26) is a limit diagram at all J. By Corollaries II-16.17 and II-11.13, it follows that I provides a coproduct for the diagram $I_{-}:\Lambda\to\mathscr{D}(\mathbb{A})'$, and hence for I is a coproduct for the original diagram in $\mathscr{D}(\mathbb{A})$. □

Corollary 4.40. *For any Grothendieck abelian category $\mathbb{A}$, the homotopy ∞-category $\mathscr{K}(\mathbb{A})$ is both complete and cocomplete, as is the derived ∞-category $\mathscr{D}(\mathbb{A})$.*

We also observe cocompleteness, and also completeness, of the connective derived category.

Corollary 4.41. *Consider the connective derived category $\mathscr{D}^{\leq 0}(\mathbb{A})$ for $\mathbb{A}$ Grothendieck abelian. The following hold:*

(1) *The category $\mathscr{D}^{\leq 0}(\mathbb{A})$ is both complete and cocomplete.*

(2) *The inclusion $\mathscr{D}^{\leq 0}(\mathbb{A})\to\mathscr{D}(\mathbb{A})$ is cocontinuous.*

Proof. For completeness, the inclusion $i:\mathscr{D}^{\leq 0}(\mathbb{A})\to\mathscr{D}(\mathbb{A})$ has a right adjoint which is given by the truncation $\tau_0:\mathscr{D}(\mathbb{A})\to\mathscr{D}^{\leq 0}(\mathbb{A})$. This follows as an application of Theorem I-13.10 for example. By Proposition II-13.24 the functor τ_0 is continuous, and hence any diagram $p:K\to\mathscr{D}^{\leq 0}(\mathbb{A})$ has a limit which can be computed as

$$\lim(p)=\tau_0(\lim(ip)).$$

So we see that the connective derived category of complete.

To obtain cocompleteness, and cocontinuity of the inclusion i, we observe directly from the formula for pushouts in $\mathscr{D}(\mathbb{A})$ given in Proposition 2.19 that $\mathscr{D}^{\leq 0}(\mathbb{A})$ is stable under the formation of pushouts in $\mathscr{D}(\mathbb{A})$. Similarly, from the formula for the coproduct given in the proof of Proposition 4.39, we also see that $\mathscr{D}^{\leq 0}(\mathbb{A})$ is stable under the formation of small coproducts in $\mathscr{D}(\mathbb{A})$. We now apply [13, Propositions 4.4.2.6 & 4.4.2.7] to see that $\mathscr{D}^{\leq 0}(\mathbb{A})$ is cocomplete and that the inclusion i : $\mathscr{D}^{\leq 0}(\mathbb{A})\to\mathscr{D}(\mathbb{A})$. □

4.10. **(Co)completeness via the homotopy category.** If one accepts that the discrete derived category $\mathrm{D}(\mathbb{A})$ admits both small coproducts and products, then we can alternatively approach the proofs of Proposition 4.39 and Corollary 4.40 via the homotopy category.

Proposition 4.42. *A stable ∞-category $\mathscr{C}$ is cocomplete (resp. complete) if and only if its homotopy category $\mathrm{h}\mathscr{C}$ admits all small coproducts (resp. products).*

Proof. We address cocompleteness. Completeness follows by considering the opposite category.

Let $\{x_\lambda : \lambda \in \Lambda\}$ be a small collection of objects in $\mathscr{C}$, and let x_Λ be a coproduct for this collection in $\mathrm{h}\mathscr{C}$ along with the structure maps $i_\lambda : x_\lambda \to x_\Lambda$. Then at each y in $\mathscr{C}$ the i_λ induce isomorphisms

$$\pi_0[i_\lambda^* : \lambda \in \Lambda]^t : \pi_0 \operatorname{Hom}_{\mathscr{C}}(x_\Lambda, y) \to \prod_{\lambda\in\Lambda} \pi_0 \operatorname{Hom}_{\mathscr{C}}(y, x_\lambda) = \pi_0(\prod_{\lambda\in\Lambda} \operatorname{Hom}_{\mathscr{C}}(y, x_\lambda)).$$

Via the identifications of Proposition 4.23 it follows that the induced maps on all higher homotopy groups

$$\pi_n[i_\lambda^* : \lambda \in \Lambda]^t : \pi_n \operatorname{Hom}_{\mathscr{C}}(x_\Lambda, y) \to \prod_{\lambda\in\Lambda} \pi_n \operatorname{Hom}_{\mathscr{C}}(y, x_\lambda) = \pi_n(\prod_{\lambda\in\Lambda} \operatorname{Hom}_{\mathscr{C}}(y, x_\lambda))$$

all based at 0, are isomorphisms as well. Now, by applying an additive shift by any map $\alpha : x_\Lambda \to y$ we observe that all of the maps

$$\pi_n([i_\lambda^* : \lambda \in \Lambda]^t, \alpha)$$

based now at α are also isomorphisms. Hence the map $[i_\lambda^* : \lambda \in \Lambda]^t$ is a homotopy equivalence, and we see that $\mathscr{C}$ admits coproducts. It follows that $\mathscr{C}$ is cocomplete via Proposition 4.28. ☐

5. Adjoints, again

Our next goal is to realize the derived ∞-category $\mathscr{D}(\mathbb{A})$ as a localization of the homotopy ∞-category $\mathscr{K}(\mathbb{A})$ relative to the class of quasi-isomorphisms. The localization functor in this case is obtained as a left adjoint $\mathrm{loc} : \mathscr{K}(\mathbb{A}) \to \mathscr{D}(\mathbb{A})$ to the inclusion $\mathscr{D}(\mathbb{A}) = \mathscr{D}_{\mathrm{Inj}} \to \mathscr{K}(\mathbb{A})$.

In order to facilitate such an analysis we return to the topic of adjunctions, in the general setting. We prove that adjoint pairs of functors can be identified with simultaneous cartesian and cocartesian fibrations over the edge Δ^1, and determine when the inclusion $\mathscr{C}' \to \mathscr{C}$ of a full ∞-subcategory admits a left (or right) adjoint. Our findings are not only applied to address the localization problem discussed above, but also in our discussions of derived functors, indization of small ∞-categories, and idempotent completion in the appendix.

5.1. **Reflexive subcategories.**

Definition 5.1 ([15, 02F6])**.** Let $\mathscr{C}' \subseteq \mathscr{C}$ be a full ∞-subcategory. Given an object x in $\mathscr{C}$, a morphism $f : x \to y$ with y in $\mathscr{C}'$ is said to exhibit y as a $\mathscr{C}'$-reflection of x if, for each third object z in $\mathscr{C}'$, the precomposition function

$$f^* : \operatorname{Hom}_{\mathscr{C}}(y, z) \to \operatorname{Hom}_{\mathscr{C}}(x, z)$$

is an isomorphism in $\mathrm{h}\mathscr{K}an$. Similarly, a morphism $g : y \to x$ with y in $\mathscr{C}'$, is said to exhibit y as a $\mathscr{C}'$-coreflection of x if, for each third object z in $\mathscr{C}'$, the composition function

$$g_* : \operatorname{Hom}_{\mathscr{C}}(z, y) \to \operatorname{Hom}_{\mathscr{C}}(z, x)$$

is an isomorphism in $\mathrm{h}\mathscr{K}an$.

We say $\mathscr{C}'$ itself is a reflective (resp. coreflective) subcategory in $\mathscr{C}$ if every object x in $\mathscr{C}$ admits a morphism $x \to y$ (resp. $y \to x$) which exhibits y as a $\mathscr{C}'$-reflection (resp. $\mathscr{C}'$-coreflection) of x.

One sees directly that taking opposites $\mathscr{C} \mapsto \mathscr{C}^{\mathrm{op}}$ exchanges reflections and coreflections, and exchanges reflective and coreflective subcategories as well. So, throughout the section, we may prove a result only for reflections with the understanding that the analogous result for coreflections is obtained by applying opposites.

Example 5.2 (K-injectives)**.** Let $\mathbb{A}$ be a Grothendiek abelian category, and let $\mathscr{D}_{\mathrm{Inj}} \subseteq \mathscr{K}(\mathbb{A})$ denote the full subcategory of K-injective complexes. Let $\mathscr{D}'_{\mathrm{Inj}} \subseteq \mathscr{K}(\mathbb{A})'$ be the corresponding full subcategory in the simplicial construction of the homotopy ∞-category.

Every complex V in $\mathrm{Ch}(\mathbb{A})$ admits a K-injective resolution $f : V \to I_V$. This map induces a quasi-isomorphism

$$f^* : \mathrm{Hom}^*_{\mathbb{A}}(I_V, J) \to \mathrm{Hom}^*_{\mathbb{A}}(V, J),$$

which then induces a homotopy equivalence

$$f^* : K\,\mathrm{Hom}^*_{\mathbb{A}}(I_V, J) \to K\,\mathrm{Hom}^*_{\mathbb{A}}(V, J)$$

so that f is a $\mathscr{D}'_{\mathrm{Inj}}$-reflection in $\mathscr{K}(\mathbb{A})'$, by Proposition II-**??**. It follows via the natural equivalence $\mathfrak{Z}_?$ of Theorem II-10.4 that $f : V \to I_V$ is also a $\mathscr{D}_{\mathrm{Inj}}$-reflection in $\mathscr{K}(\mathbb{A})$. So we conclude that $\mathscr{D}_{\mathrm{Inj}}$ is a reflective subcategory in $\mathscr{K}(\mathbb{A})$.

Example 5.3 (K-projectives)**.** Suppose that an abelian category $\mathbb{A}$ has enough projectives, and let $\mathscr{D}_{\mathrm{Proj}}$ be the full ∞-subcategory of K-projectives in $\mathscr{K}(\mathbb{A})$. Each complex V admits a K-projective resolution $g : P_V \to V$. One argues as in Example 5.2 to see that g is a $\mathscr{D}_{\mathrm{Proj}}$-coflection, and hence to see that $\mathscr{D}_{\mathrm{Proj}}$ is a coreflective subcategory in $\mathscr{K}(\mathbb{A})$.

Lemma 5.4. *Given any morphism $f : x \to y$, the forgetful functor $\mathscr{C}_{f/} \to \mathscr{C}_{y/}$ is a trivial Kan fibration.*

Proof. By [9, Lemma 3.3] the apparent map

$$(\Delta^1 \star \partial\Delta^n) \coprod_{(\{1\}\star\partial\Delta^n)} (\{1\} \star \Delta^n) \to \Delta^1 \star \Delta^n \cong \Delta^{n+2}$$

is an isomorphism onto the inner horn $\Lambda^{n+2}_1 \subseteq \Delta^{n+2}$. Hence solving a lifting problem of the form

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & \mathscr{C}_{f/} \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & \mathscr{C}_{y/} \end{tikzcd}$$

is equivalent to solving a lifting problem of the form

$$\begin{tikzcd} \Lambda^{n+2}_1 \arrow[r] \arrow[d] & \mathscr{C} \arrow[d] \\ \Delta^{n+2} \arrow[r] \arrow[ur, dotted] & *. \end{tikzcd}$$

Since $\mathscr{C}$ is an ∞-category, the latter problem always admits a solution. $\square$

Lemma 5.5 ([15, 02LL])**.** *For any morphism $f : x \to y$ and object $z : * \to \mathscr{C}$, we have a commuting diagram*

$$\begin{tikzcd}
\mathscr{C}_{y/} \times_{\mathscr{C}} \{z\} \arrow[d, "\cong"'] & \mathscr{C}_{f/} \times_{\mathscr{C}} \{z\} \arrow[l, "\cong"'] \arrow[r] & \mathscr{C}_{x/} \times_{\mathscr{C}} \{z\} \arrow[d, "\cong"] \\
\mathrm{Hom}_{\mathscr{C}}(y, z) \arrow[rr, "f^*"'] & & \mathrm{Hom}_{\mathscr{C}}(x, z).
\end{tikzcd}$$

in $\mathrm{h}\mathscr{K}an$*.*

Here the vertical maps specifically those induced by the coslice diagonal equivalences, where we observe that the fibers of this equivalence remain equivalences by Corollary I-5.22 and Proposition I-5.23.

Idea of proof. One produces a morphism

$$i : \mathscr{C}_{f/} \times_{\mathscr{C}} \{z\} \to \{f\} \times_{\mathrm{Hom}_{\mathscr{C}}(x,y)} \mathrm{Fun}(\Delta^2, \mathscr{C})_{\vec{x}}$$

which bisects the diagram

$$\begin{tikzcd}
\mathscr{C}_{y/} \times_{\mathscr{C}} \{z\} \arrow[d, "\cong"'] & \mathscr{C}_{f/} \times_{\mathscr{C}} \{z\} \arrow[l, "\cong"'] \arrow[r] \arrow[d] & \mathscr{C}_{x/} \times_{\mathscr{C}} \{z\} \arrow[d, "\cong"] \\
\mathrm{Hom}_{\mathscr{C}}(y, z) & \{f\} \times_{\mathrm{Hom}_{\mathscr{C}}(x,y)} \mathrm{Fun}(\Delta^2, \mathscr{C})_{\vec{x}} \arrow[l] \arrow[r] & \mathrm{Hom}_{\mathscr{C}}(x, z).
\end{tikzcd}$$

at the level of the discrete category Kan. □

Proposition 5.6. *Let $\mathscr{C}' \subseteq \mathscr{C}$ be a full ∞-subcategory. For a fixed morphism $f : x \to y$, with y in $\mathscr{C}'$, the following are equivalent:*

(a) *f exhibits y as a $\mathscr{C}'$-reflection of x.*
(b) *At each object z in $\mathscr{C}'$ the forgetful functor $\mathscr{C}_{f/} \times_{\mathscr{C}} \{z\} \to \mathscr{C}_{x/} \times_{\mathscr{C}} \{z\}$ is an equivalence.*
(c) *The map $\mathscr{C}_{f/} \times_{\mathscr{C}} \mathscr{C}' \to \mathscr{C}_{f/} \times_{\mathscr{C}} \mathscr{C}'$ is a trivial Kan fibration.*
(d) *Each lifting problem*

$$\begin{tikzcd}
\Lambda^n_0 \arrow[r, "\tau"] \arrow[d] & \mathscr{C} \arrow[d] \\
\Delta^n \arrow[r] & *
\end{tikzcd}$$

with $n \geq 2$ and $\tau|_{\Delta^{\{1,\ldots,n\}}}$ having imagine in $\mathscr{C}'$, admits a solution.

Proof. The equivalence (a) ⇔ (b) is a consequence of Lemma 5.5. For (b) ⇔ (c), we recall that the map

$$\mathscr{C}_{f/} = (\mathscr{C}_{x/})_{y/} \to \mathscr{C}_{x/}$$

is a left fibration by Corollary I-4.27, and hence its base change $\mathscr{C}_{f/} \times_{\mathscr{C}} \mathscr{C}' \to \mathscr{C}_{f/} \times_{\mathscr{C}} \mathscr{C}'$ is a left fibration as well. This left fibration furthermore fits into a diagram of left fibrations

$$\begin{tikzcd}
\mathscr{C}_{f/} \times_{\mathscr{C}} \mathscr{C}' \arrow[rr] \arrow[dr] & & \mathscr{C}_{f/} \times_{\mathscr{C}} \mathscr{C}' \arrow[dl] \\
& \mathscr{C}' &
\end{tikzcd}$$

.

Theorem II-3.8 and Corollary II-9.8 together now imply that the forgetful functor $\mathscr{C}_{f/} \times_{\mathscr{C}} \mathscr{C}' \to \mathscr{C}_{x/} \times_{\mathscr{C}} \mathscr{C}'$ is a trivial Kan fibration if and only if at each object $z : * \to \mathscr{C}'$ the fiber

$$\mathscr{C}_{f/} \times_{\mathscr{C}} \{z\} \to \mathscr{C}_{x/} \times_{\mathscr{C}} \{z\}$$

is an equivalence. Statement (d) is identified with (c) via the identification of lifting problems

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & \mathscr{C}_{f/} \times_{\mathscr{C}} \mathscr{C}' \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & \mathscr{C}_{x/} \times_{\mathscr{C}} \mathscr{C}' \end{tikzcd} \quad = \quad \begin{tikzcd} \Lambda^{n+2}_0 \arrow[r] \arrow[d] & \mathscr{C} \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & * \end{tikzcd}$$

which is implied by Lemma II-9.10. □

5.2. **Reflexivity and adjoints.**

Proposition 5.7. *Let $\mathscr{C}' \subseteq \mathscr{C}$ be a full ∞-subcategory and $i : \mathscr{C}' \to \mathscr{C}$ be the inclusion. The subcategory $\mathscr{C}'$ is reflective if and only if there is a functor $L : \mathscr{C} \to \mathscr{C}'$ and a transformation $u : id_{\mathscr{C}} \to iL$ for which, at each x in $\mathscr{C}$, the map $u_x : x \to L(x)$ exhibits $L(x)$ as a $\mathscr{C}'$-reflection of x.*

Proof. Let $\mathscr{E} \subseteq \mathscr{C} \times \Delta^1$ be the full ∞-subcategory whose objects $\mathscr{E}[0]$ are the union $(\mathscr{C}[0] \times \{0\}) \cup (\mathscr{C}'[0] \times \{1\})$. By Proposition 5.6 the projection

$$q : \mathscr{E} \to \Delta^1$$

is a cocartesian fibration, and a map $f : (x, 0) \to (y, 1)$ in $\mathscr{E}$ is q-cocartesian if and only if the underlying map $f : x \to y$ in $\mathscr{C}$ exhibits y as a $\mathscr{C}$-reflection of x.

Now, by Theorem II-2.7 there exists a unique functor

$$U : \Delta^1 \times \mathscr{C} \to \mathscr{E}$$

which splits the diagram

$$\begin{tikzcd} \{0\} \times \mathscr{C} \arrow[r, "\mathrm{incl}"] \arrow[d] & \mathscr{E} \arrow[d, "q"] \\ \Delta^1 \times \mathscr{C} \arrow[r, "p_1"'] \arrow[ur, "U"] & \Delta^1 \end{tikzcd}$$

and sends each map $\Delta^1 \times \{x\}$ to a q-cocartesian morphism in $\mathscr{E}$. For $L : \mathscr{C} \to \mathscr{C}' \subseteq \mathscr{C}$ defined as the composite

$$\mathscr{C} \cong \{1\} \times \mathscr{C} \to \Delta^1 \times \mathscr{C} \to \mathscr{E} \xrightarrow{p_1} \mathscr{C},$$

the transformation $u = p_1 U : \Delta^1 \times \mathscr{C} \to \mathscr{C}$ has the prescribed property. □

Lemma 5.8 ([15, 02DK])**.** *Let $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ be functors between ∞-categories, and $u : id_{\mathscr{C}} \to GF$ be a transformation. Suppose that the induced transformations*

$$Fu : F \to F(GF) \quad \textit{and} \quad uG : G \to (GF)G$$

are isomorphisms in $\mathrm{Fun}(\mathscr{C}, \mathscr{D})$ *and* $\mathrm{Fun}(\mathscr{D}, \mathscr{C})$ *respectively, and that G is fully faithful. Then u is the unit of an adjunction, and the counit $\epsilon : FG \to id_{\mathscr{D}}$ is a natural isomorphism.*

Sketch proof. Since G is fully faithful the induced map $G_* : \mathrm{Fun}(\mathscr{C}, \mathscr{D}) \to \mathrm{Fun}(\mathscr{C}, \mathscr{C})$ is fully faithful. Hence there is a unique transformation $\epsilon : FG \to id_{\mathscr{D}}$ which lifts the isomorphism $(uG)^{-1} : GFG \to G$. Fully faithfulness implies that ϵ is an isomorphism as well.

We can replace the functor categories $\mathrm{Fun}(\mathscr{A}, \mathscr{B})$ with their homotopy categories $\mathrm{h}\,\mathrm{Fun}(\mathscr{A}, \mathscr{B})$ and work with the corresponding 2-category Cat^2_∞ obtained from the simplicial category $\underline{\mathrm{Cat}}_\infty$. At this level we consider the composites

$$F \overset{Fu}{\to} FGF \overset{\epsilon F}{\to} F \quad \text{and} \quad G \overset{uG}{\to} GFG \overset{G\epsilon}{\to} G.$$

The latter composite is the identity by the definition of ϵ. We consider now the composite $\beta = (\epsilon F)(Fu)$. By our assumptions, Fu is an isomorphism, so that β is an isomorphism as well. One now argues, using [15, 02CX], that β also satisfies $\beta^2 = \beta$ and hence $\beta = id_F$ necessarily. □

Remark 5.9. If a transformation $u : id_{\mathscr{C}} \to GF$ admits some transformation $FG \to id_{\mathscr{D}}$ which exhibits F as left adjoint to G, then this transformation is fixed up to the action of $\mathrm{Aut}_{\mathrm{Fun}(\mathscr{D},\mathscr{C})}(G)$ [15, 02D7]. In particular, any transformation $\epsilon' : FG \to id_{\mathscr{D}}$ which pairs with u to realize F as left adjoint to G, in the situation of Lemma 5.8, must be a natural isomorphism.

Proposition 5.10. *Let $i : \mathscr{C}' \to \mathscr{C}$ be the inclusion of a full ∞-subcategory into an ∞-category $\mathscr{C}$. Consider any functor $L : \mathscr{C} \to \mathscr{C}'$ and transformation $u : id_{\mathscr{C}} \to iL$. The following are equivalent:*

(a) *The transformation u is part of an adjunction which exhibits L as left adjoint to the inclusion $i : \mathscr{C}' \to \mathscr{C}$.*

(b) *At each object x in $\mathscr{C}$, the map $u_x : x \to L(x)$ exhibits $L(x)$ as a $\mathscr{C}'$-reflection of x.*

(c) *At each x in $\mathscr{C}$ the map $L(u_x) : L(x) \to LL(x)$ is an isomorphism and, at each y in $\mathscr{C}'$, the map $u_y : y \to L(y)$ is an isomorphism in $\mathscr{C}'$.*

Furthermore, in this case, any transformation $\epsilon : Li \to id_{\mathscr{C}'}$ which pairs with u to realize L as left adjoint to i is a natural isomorphism.

Proof. Supposing (a) and (c) hold, the claim about ϵ follows by Lemma 5.8 and Remark 5.9. Now, suppose (a) holds. Then by Corollary I-13.4 the transformation u realizes the $\mathrm{h}\,\mathscr{K}an$-enriched functor $\pi L : \pi\mathscr{C} \to \pi\mathscr{C}'$ as left adjoint to the enriched embedding $\pi i : \pi\mathscr{C}' \to \pi\mathscr{C}$. Hence, at each z in $\mathscr{C}'$, u induces isomorphisms

$$u^* : \mathrm{Hom}_{\mathscr{C}}(L(x), z) \to \mathrm{Hom}_{\mathscr{C}}(x, z)$$

in $\mathrm{h}\,\mathscr{K}an$. Thus (b) holds.

Supposing (b) holds. When y is in $\mathscr{C}'$ applying u yields an isomorphism of sets

$$u^* : \pi_0\,\mathrm{Hom}_{\mathscr{C}'}(L(y), z) \to \pi_0\,\mathrm{Hom}_{\mathscr{C}'}(y, z)$$

which shows, via Yoneda, that $u_y : y \to L(y)$ is an isomorphism in $\mathrm{h}\,\mathscr{C}'$. By definition this implies that u_y is an isomorphism in $\mathscr{C}'$.

As for the transformation $L(u_x) : L(x) \to LL(x)$ at general x, we have the diagram

$$\begin{tikzcd}
x \arrow[r, "u_x"] \arrow[d, "u_x"'] & L(x) \arrow[d, "u_{L(x)}"] \\
L(x) \arrow[r, "L(u_x)"'] & LL(x)
\end{tikzcd}$$

in $\mathrm{h}\mathscr{C}$ and apply $\mathrm{Hom}_{\mathrm{h}\mathscr{C}}(-, z)$ at arbitrary z in $\mathscr{C}'$ to obtain a diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{h}\mathscr{C}}(L(x), z) \arrow[r, "\cong"] \arrow[d, "\cong"'] & \mathrm{Hom}_{\mathrm{h}\mathscr{C}'}(x, z) \arrow[d, "\cong"] \\
\mathrm{Hom}_{\mathrm{h}\mathscr{C}'}(LL(x), z) \arrow[r, "L(u)^*"'] & \mathrm{Hom}_{\mathrm{h}\mathscr{C}'}(L(x), z).
\end{tikzcd}$$

From this we conclude that $L(u_x)^*$ is an isomorphism at all z, and hence that $L(u_x)$ is an isomorphism in $\mathrm{h}\mathscr{C}$. It follows that $L(u_x)$ is an isomorphism in $\mathscr{C}$.

Finally, Lemma 5.8 tells us directly that (c) implies (a). This completes the proof. □

One combines Propositions 5.7 and 5.10 to obtain the following.

Theorem 5.11. *Consider a full ∞-subcategory $\mathscr{C}' \subseteq \mathscr{C}$ in an arbitrary ∞-category $\mathscr{C}$. Then $\mathscr{C}'$ is reflective in $\mathscr{C}$ if and only if the inclusion $i : \mathscr{C}' \to \mathscr{C}$ admits a left adjoint $L : \mathscr{C} \to \mathscr{C}'$ whose unit and counit transformations have the properties outlined in Proposition 5.10 above. Similarly, $\mathscr{C}'$ is coreflective if and only if the inclusion admits a right adjoint $R : \mathscr{C} \to \mathscr{C}'$.*

Proof. The claim about reflective subcategories is clear, on the claim about coreflective subcategories is obtain by applying opposites. □

Example 5.12. As an application of Theorem 5.11 to Example 5.2 we observe, for Grothendieck abelian $\mathbb{A}$, the existence of a left adjoint $L : \mathscr{K}(\mathbb{A}) \to \mathscr{D}_{\mathrm{Inj}} = \mathscr{D}(\mathbb{A})$ to the inclusion $i : \mathscr{D}(\mathbb{A}) \to \mathscr{K}(\mathbb{A})$. This left adjoint exhibits $\mathscr{D}(\mathbb{A})$ as a localization $\mathscr{K}(\mathbb{A})[\mathrm{Qiso}^{-1}]$ of the homotopy ∞-category at the class of quasi-isomorphisms. We return to this topic in Section 6 below.

5.3. **Adjoints via simultaneous fibrations.**

Lemma 5.13. *Let $q : \mathscr{E} \to \Delta^1$ be an inner fibration of ∞-categories, with fibers $\mathscr{E}_i = \mathscr{E} \times_{\Delta^1} \{i\}$. The following hold:*

1. *The subcategory $\mathscr{E}_1$ is a reflective subcategory in $\mathscr{E}$ if and only if q is a cocartesian fibration, and in this case a map $f : x \to y$ over $0 < 1$ is q-cocartesian if and only if it exhibits y as a $\mathscr{E}_1$-reflection of x.*
2. *The subcategory $\mathscr{E}_0$ is a coreflective subcategory in $\mathscr{E}$ if and only if q is a cartesian fibration, and in this case a map $g : y \to x$ over $0 < 1$ is q-cartesian if and only if it exhibits y as a $\mathscr{E}_0$-coreflection of x.*

Proof. Follows by Proposition 5.6 (c). □

Proposition 5.14. *Let $q : \mathscr{E} \to \Delta^1$ be a cocartesian fibration, and $F : \mathscr{E}_0 \to \mathscr{E}_1$ be the functor given by covariant transport along q (Definiiton II-7.1). The functor F*

admits a right adjoint $G : \mathscr{E}_1 \to \mathscr{E}_0$ *if and only if* q *is a cartesian fibration as well, and in this case* G *is given by contravariant transport along* q.

Our claim that G is "given by contravariant transport" should be interpreted in a strict sense. Namely, we claim that when F admits a right adjoint G, there is a cartesian transformation $\Delta^1 \times \mathscr{E}_1 \to \mathscr{E}$, i.e. a cartesian solution to the lifting problem

$$\begin{tikzcd} \{1\} \times \mathscr{E}_1 \arrow[r] \arrow[d] & \mathscr{E} \arrow[d, "q"] \\ \Delta^1 \times \mathscr{E}_1 \arrow[r] \arrow[ur, dotted] & \Delta^1 \end{tikzcd}$$

(see Proposition II-2.6) whose restriction to $\{0\} \times \mathscr{E}_1$ recovers G.

Proof. First suppose that F admits such a right adjoint $G : \mathscr{E}_1 \to \mathscr{E}_0$, and consider the unit and counit transformations

$$u : id_{\mathscr{E}_0} \to GF \quad \text{and} \quad \epsilon : FG \to id_{\mathscr{E}_1}$$

respectively. By Proposition 5.6 (c) we understand that q is cartesian if and only if the subcategory $\mathscr{E}_0$ is coreflective in $\mathscr{E}$. So we seek to demonstrate $\mathscr{E}_0$-coreflections $g_x : y \to x$ at each x in $\mathscr{E}$. When x is in $\mathscr{E}_0$ we can just take $g_x = id_x$, so that we can assume here that x is in the fiber $\mathscr{E}_1$.

First, let us consider the extended fibration $q^l : \mathscr{E}^l \to \Delta^1$ where $\mathscr{E}^l \subseteq \mathscr{E} \times \Delta^1$ is the full ∞-subcategory with objects $(\mathscr{E}[0] \times \{0\}) \cup (\mathscr{E}_1[0] \times \{1\})$. Here q^l is specifically the projection onto the second factor. As in the proof of Proposition 5.7, we see that q^l is a cocartesian fibration and one observes directly the inclusion of cocartesian fibrations $\mathscr{E} \to \mathscr{E}^l$ provided by restricting the image of the product map $[id_{\mathscr{E}}\ q]^t : \mathscr{E} \to \mathscr{E} \times \Delta^1$.

The transport functor $L : \mathscr{E} \to \mathscr{E}_1$ along q^l comes equipped with a transformation $\eta : id_{\mathscr{E}} \to L$ which evaluates to a q^l-cocartesian morphism at each x in $\mathscr{E}$, by definition. By uniqueness of transport functors and the fact that the restriction functor

$$\mathrm{Fun}(\Delta^1 \times \mathscr{E}, \mathscr{E}) \to \mathrm{Fun}(\Delta^1 \times \mathscr{E}_0, \mathscr{E})$$

is an isofibration (Corollary I-5.14), we can choose L so that $F = L|_{\mathscr{E}_0}$. Note also that η exhibits L as left adjoint to the inclusion $i_1 : \mathscr{E}_1 \to \mathscr{E}$, by Lemma 5.8.

At each x in $\mathscr{E}_1$ define the map $g_x : G(x) \to x$ as a composite

$$G(x) \xrightarrow{\eta} LG(x) = FG(x) \xrightarrow{\epsilon} x.$$

We then have at each z in $\mathscr{E}_0$, in the homotopy category of spaces, the sequence of maps

$$\mathrm{Hom}_{\mathscr{E}_0}(z, G(x)) \xrightarrow{F} \mathrm{Hom}_{\mathscr{E}_1}(F(z), FG(x)) \xrightarrow{\epsilon_*} \mathrm{Hom}_{\mathscr{E}_1}(F(z), x) \xrightarrow{\eta^*} \mathrm{Hom}_{\mathscr{E}}(z, x).$$

As the first two maps compose to an isomorpihsm, and the third map is also an isomorphism, this composite is an isomorphism. By commutativity of the operations $\epsilon_* = \epsilon \circ -$ and $\eta^* = - \circ \eta$, i.e. by associativity of composition in $\mathrm{h}\mathscr{K}an$, and naturality of η (Lemma I-13.3), the above composite is equal to the map

$$(g_x)_* : \mathrm{Hom}_{\mathscr{E}_0}(z, G(x)) \to \mathrm{Hom}_{\mathscr{E}}(z, x),$$

which we conclude is an isomorphism. So each $g_x : G(x) \to x$ exhibits $G(x)$ as a $\mathscr{E}_0$-coreflecton, $\mathscr{E}_0$ is seen to be coreflective in $\mathscr{E}$, and q is therefore a cartesian fibration.

Suppose now that $q : \mathscr{E} \to \Delta^1$ is cartesian. Then the subcategory $\mathscr{E}_0$ is coreflective in $\mathscr{E}$ by Proposition 5.6 and we have the associated cartesian fibration $q^r : \mathscr{E}^r \to \Delta^1$ obtained by considering the full ∞-subcategory $\mathscr{E}^r \subseteq \mathscr{E} \times \Delta^1$ whose objects are provided by the union $(\mathscr{E}_0[0] \times \{0\}) \cup (\mathscr{E}[0] \times \{1\})$. We have the contravariant transport functor $R : \mathscr{E} \to \mathscr{E}_0$ for q^r along with the transformation $\pi : i_0 R \to id_{\mathscr{E}}$ which evaluates to a q^r-cartesian morphism at each x in $\mathscr{E}$. By Lemma 5.8 the transformation π exhibits R as right adjoint to the inclusion $\mathscr{E}_0 \to \mathscr{E}$.

Define $G : \mathscr{E}_1 \to \mathscr{E}_0$ as the composite $Ri_1 : \mathscr{E}_1 \to \mathscr{E} \to \mathscr{E}_0$. The inclusion $\mathscr{E}_1 \to \mathscr{E}$ has left adjoint $L : \mathscr{E} \to \mathscr{E}_1$ which restricts to F on $\mathscr{E}_0$, and we see that G is a composite of right adjoints. Hence G itself is right adjoint to the functor $Li_0 = F$, as desired (see [15, 02DT]). We see also that G is a contravariant trasport functor for $q : \mathscr{E} \to \Delta^1$ since restricting along the inclusion of cartesian fibrations $\mathscr{E} \to \mathscr{E}^r$ recovers transport functors for q from transport functors for q^r.

As for the claim that any right adjoint to F is given by contravariant transport, consider $G : \mathscr{E}_1 \to \mathscr{E}_0$ a right adjoint to F with counit transformation $\epsilon : FG \to id_{\mathscr{E}_1}$. Consider the transformation $\widetilde{\epsilon} : \Delta^1 \times \mathscr{E}_1 \to \mathscr{E}$ which sends an n-simplex $(\alpha, \sigma) : \Delta^n \to \Delta^1 \times \mathscr{E}_1$ to the triple

$$(\alpha,\ G\sigma|_{\Delta^{\alpha^{-1}(0)}},\ \sigma) : \Delta^n \to \mathscr{E} = \mathrm{N}^F(\Delta^1).$$

At each $y : * \to \mathscr{E}_1$, $\widetilde{\epsilon}$ sends the edge $\Delta^1 \times \{y\}$ to the edge $(G(y), \epsilon_y : FG(y) \to y)$ in $\mathscr{E}$. Let us denote this edge $\widetilde{\epsilon}_y : \Delta^1 \to \mathscr{E}$.

By Corollary I-13.5, ϵ realizes the $\mathrm{h}\mathscr{K}an$-enriched functor πF and left adjoint to πG, so that the composite

$$\mathrm{Hom}_{\mathscr{E}_0}(z, G(y)) \xrightarrow{F} \mathrm{Hom}_{\mathscr{E}_1}(F(z), FG(y)) \xrightarrow{\epsilon_*} \mathrm{Hom}_{\mathscr{E}_1}(F(z), y)$$

is an isomorphism at arbitrary z in $\mathscr{C}$ and y in $\mathscr{D}$. One checks that this sequence recovers composition $(\widetilde{\epsilon}_y)_* : \mathrm{Hom}_{\mathscr{E}}(z, G(y)) \to \mathrm{Hom}_{\mathscr{E}}(z, y)$ in $\pi\mathscr{E}$ to see that $\widetilde{\epsilon}_y$ is an $\mathscr{E}_0$-coreflection in $\mathscr{E}$. By Lemma 5.13 we conclude that each $\widetilde{\epsilon}_y$ is q-cartesian in $\mathscr{E}$, and hence that $\widetilde{\epsilon}$ is a cartesian solution to the lifting problem

$$\begin{tikzcd} \{1\} \times \mathscr{E}_1 \arrow[r] \arrow[d] & \mathscr{E} \arrow[d, "q"] \\ \Delta^1 \times \mathscr{E}_1 \arrow[r] \arrow[ur, dotted] & \Delta^1. \end{tikzcd}$$

By construction $\widetilde{\epsilon}|_{\{0\} \times \mathscr{E}_1} = G$. $\square$

We now obtain a characterization of adjunctions via fibrations over the 1-simplex.

Theorem 5.15. *Given a pair of functor $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$, the following are equivalent:*

(a) *The functors F and G admit transformations which exhibit F as left adjoint to G.*

(b) *There is a simultaneous cartesian and cocartesian fibration $q : \mathscr{E} \to \Delta^1$ with fixed isomorphisms at the fibers $\mathscr{C} \cong \mathscr{E}_0$ and $\mathscr{D} \cong \mathscr{E}_1$, and for which F*

and G are recovered respectivley as covariant and contravariant transport along q.

Proof. Note that F defines a functor $F : \Delta^1 \to \mathscr{C}at_\infty$ and consider the weighted nerve $q : \mathscr{E} = \mathrm{N}^F(\Delta^1) \to \Delta^1$. The fact that F is recovered by covariant transport along q is implicit in the claim that there is an isomorphism of fibrations $\mathscr{E} \cong \int_{\Delta^1} F$ (Theorem II-6.28). However, we can just observe this fact directly.

For an n-simplex $\sigma = (\sigma', \sigma'') : \Delta^n \to \Delta^1 \times \mathscr{C} = \mathscr{E}_0$ take $\Delta^{n_0} = (\sigma')^{-1}(0)$. Define now

$$\widetilde{\sigma} : \Delta^n \to \mathscr{E}$$

as the pairing of the n-simplex $\sigma' : \Delta^n \to \Delta^1$ with the pair of n-simplices $\sigma''|_{\Delta^{n_0}} : \Delta^{n_0} \to \mathscr{C}$ and $F\sigma'' : \Delta^n \to \mathscr{D}$. The assignment $\sigma \mapsto \widetilde{\sigma}$ defines a functor cocartesian lift of the inclusion $\{0\} \times \mathscr{C} \to \mathscr{E}$, and so recovers $F : \mathscr{C} \to \mathscr{E}$ as covariant transport along q.

In any case, we recover the claimed equivalence of (a) and (b) by applying Proposition 5.14 to the weighted nerve for F. □

5.4. **Local criterion for adjunction.**

Theorem 5.16. *A functor between ∞-categories $F : \mathscr{C} \to \mathscr{D}$ admits a right adjoint if and only if, for each y in $\mathscr{D}$, there exists a morphism $g_y : F(x) \to y$ from some object x in $\mathscr{C}$ such that, at any other z in $\mathscr{C}$, the sequence*

$$\mathrm{Hom}_{\mathscr{C}}(z, x) \xrightarrow{F} \mathrm{Hom}_{\mathscr{D}}(F(z), F(x)) \xrightarrow{(g_y)_*} \mathrm{Hom}_{\mathscr{D}}(F(z), y) \tag{28}$$

is an isomorphism in $\mathrm{h}\mathscr{K}an$.

Proof. If there exists an adjoint G then we can take $x = G(y)$ and g the counit morphism. Conversely, suppose we can always find such a g_y at each y in $\mathscr{D}$. Then for the weighted nerve $q : \mathscr{E} = \mathrm{N}^F(\Delta^1) \to \Delta^1$ covariant trasport along q recovers the functor F, as was argued in the proof of Theorem 5.15.

In the weighted nerve

$$\mathrm{Hom}_{\mathscr{E}}(z, x) = \mathrm{Hom}_{\mathscr{C}}(z, x) \quad \text{and} \quad \mathrm{Hom}_{\mathscr{E}}(z, y) = \mathrm{Hom}_{\mathscr{E}}(F(z), y),$$

and one can check directly that the composition function

$$\mathrm{Hom}_{\mathscr{E}}(x, y) \times \mathrm{Hom}_{\mathscr{E}}(z, x) \to \mathrm{Hom}_{\mathscr{E}}(z, y)$$

is obtained by applying $F : \mathrm{Hom}_{\mathscr{C}}(z, x) \to \mathrm{Hom}_{\mathscr{D}}(F(z), F(x))$ then composing in $\mathscr{D}$. Hence if we view g_y as a morphism in $\mathscr{E}$, $g_y : x \to y$, then the operation

$$(g_y)_* : \mathrm{Hom}_{\mathscr{E}}(z, x) \to \mathrm{Hom}_{\mathscr{E}}(z, y)$$

is identified with the sequence (28). So, we conclude that the sequence (28) is an isomorphism at some g_y, for each y, if and only if the fiber $\mathscr{D} = \mathscr{E}_1$ is coreflective in $\mathscr{E}$, which then occurs if and only if $q : \mathscr{E} \to \Delta^1$ is a cartesian fibration by Proposition 5.6 (c). We apply Proposition 5.14 to see that F has a right adjoint in this case. □

Taking opposites, we observe the analagous local criterion for the existence of left adjoints.

Theorem 5.17. *A functor between ∞-categories $G : \mathscr{D} \to \mathscr{C}$ admits a left adjoint if and only if, for each x in $\mathscr{C}$, there exists a morphism $f_x : x \to G(y)$ from some object y in $\mathscr{D}$ such that, at any other z in $\mathscr{D}$, the sequence*

$$\mathrm{Hom}_{\mathscr{D}}(y, z) \xrightarrow{G} \mathrm{Hom}_{\mathscr{C}}(G(y), G(z)) \xrightarrow{f_x^*} \mathrm{Hom}_{\mathscr{C}}(x, G(z))$$

is an isomoprhism in $\mathrm{h}\mathscr{K}an$.

5.5. Going halfsies on adjoints.

Proposition 5.18. *For functors $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ between ∞-categories the following are equivalent:*

(a) *The transformation F is left adjoint to G.*

(b) *There is a tranformation $\epsilon : FG \to id_{\mathscr{D}}$ for which, at each z in $\mathscr{C}$ and y in $\mathscr{D}$, the composite*

$$\mathrm{Hom}_{\mathscr{C}}(z, G(y)) \xrightarrow{F} \mathrm{Hom}_{\mathscr{D}}(F(z), FG(y)) \xrightarrow{(\epsilon_y)_*} \mathrm{Hom}_{\mathscr{D}}(F(z), y)$$

is an isomorphism in $\mathrm{h}\mathscr{K}an$.

(c) *There is a transformation $\eta : id_{\mathscr{C}} \to GF$ for which, at each x in $\mathscr{C}$ and z in $\mathscr{D}$, the composite*

$$\mathrm{Hom}_{\mathscr{D}}(F(x), z) \xrightarrow{G} \mathrm{Hom}_{\mathscr{C}}(GF(x), G(z)) \xrightarrow{\eta_x^*} \mathrm{Hom}_{\mathscr{C}}(x, G(z))$$

is an isomorphism in $\mathrm{h}\mathscr{K}an$.

Furthermore, in the cases of (b)*, the transformation ϵ is the counit of an adjunction between F and G, and in the case of* (c)*, the transformation η is the unit of such an adjunction.*

Proof. The implications (a) $\Rightarrow$ (b) and (a) $\Rightarrow$ (c) are immediate. For the implication (b) $\Rightarrow$ (a), suppose we have a tranformation $\epsilon : FG \to id_{\mathscr{D}}$ as in (b). Consider the weighted nerve $q : \mathscr{E} = \mathrm{N}^F(\Delta^1) \to \Delta^1$. As was argued in the proof of Theorem 5.16, $\mathscr{D} = \mathscr{E}_1$ is coreflective in $\mathscr{E}$ and by Lemma 5.13 the fibration q is both cartesian and cocartesian. Furthermore Lemma 5.13 tells us that, at each y in $\mathscr{D}$, the morphism $(G(y),\ \epsilon_y : FG(y) \to y)$ is q-cartesian in $\mathscr{E}$.

Define the functor $\widetilde{\epsilon} : \Delta^1 \times \mathscr{D} = \Delta^1 \times \mathscr{E}_1 \to \mathscr{E}$ which takes a simplex $\widetilde{\sigma} = (\alpha, \sigma) : \Delta^n \to \Delta^1 \times \mathscr{D}$ to the triple

$$\big\{\alpha : \Delta^n \to \Delta^1,\ G\sigma|_{\Delta^{\alpha^{-1}(0)}} : \Delta^{\alpha^{-1}(0)} \to \mathscr{C},\ \epsilon\widetilde{\sigma} : \Delta^n \to \mathscr{D}\big\}.$$

By direct inspection $\widetilde{\epsilon}$ fits into a diagram

$$\begin{tikzcd} \{1\} \times \mathscr{E}_1 \arrow[rr, "\mathrm{incl}"] \arrow[d] & & \mathscr{E} \arrow[d, "q"] \\ \Delta^1 \times \mathscr{D} = \Delta^1 \times \mathscr{E}_1 \arrow[urr, "\widetilde{\epsilon}"] \arrow[rr, "\mathrm{proj}"'] & & \Delta^1 \end{tikzcd} \tag{29}$$

and at each y in $\mathscr{D}$ the edge $\widetilde{\epsilon}|_{\Delta^1 \times \{y\}} = (G(y),\ \epsilon_y : FG(y) \to y) : \Delta^1 \to \mathscr{E}$ is q-cartesian, as explained above. So $\widetilde{\epsilon}$ is the unique cartesian transformation which splits the above square (see Corollary II-2.8), and by Theorem 5.15 the functor $G = \widetilde{\epsilon}|_{\{0\} \times \mathscr{E}_1}$ is seen to be right adjoint to F.

We claim now that $\epsilon : FG \to id_{\mathscr{D}}$ is the counit transformation in a pair of transformation which (ϵ, ϵ) which realize G as right adjoint to F. However, this follows by uniqueness of the cartesian transformation which splits the diagram (29). Specifically, the counit $\epsilon' : FG \to id_{\mathscr{D}}$ can be used to define a cocartesian transformation $\widetilde{\epsilon}' : \Delta^1 \times \mathscr{D} \to \mathscr{E}$ exactly as above, so that we obtain an isomorphism $\widetilde{\epsilon} \cong \widetilde{\epsilon}'$ in $\mathrm{Fun}(\Delta^1 \times \mathscr{D}, \mathscr{E})$.

We have the transformation $k : F \to id_{\mathscr{D}}$ in $\mathrm{Fun}(\Delta^1, \mathscr{C}at_\infty)$ which one simply observes by the existence of the strictly commuting diagram

$$\begin{tikzcd} \mathscr{C} \arrow[r, "F"] \arrow[d, "F"'] & \mathscr{D} \arrow[d, "id_{\mathscr{D}}"] \\ \mathscr{D} \arrow[r, "id_{\mathscr{D}}"'] & \mathscr{D}. \end{tikzcd}$$

This tranformation defines a functor $\mathrm{N}^k : \mathscr{E} = \mathrm{N}^F(\Delta^1) \to \mathrm{N}^{id_{\mathscr{D}}}(\Delta^1) = \Delta^1 \times \mathscr{D}$, and we compose with the projection $\Delta^1 \times \mathscr{D} \to \mathscr{D}$ to obtain a functor

$$\pi : \mathscr{E} \to \mathscr{D}.$$

The compositions recover our original functors $\pi\widetilde{\epsilon} = \epsilon$ and $\pi\widetilde{\epsilon}' = \epsilon'$ so that we obtain an isomorphism $\epsilon \cong \epsilon'$ in $\mathrm{Fun}(\Delta^1 \times \mathscr{D}, \mathscr{D})$. Since compositions of morphisms are stable under isomorphisms of morphisms, in any ∞-category, the fact that ϵ' can be paired with a transformation $\eta : id_{\mathscr{C}} \to GF$ with witnesses G as right adjoint to F implies that the pair (ϵ, η) also witnesses G as right adjoint to F.

We now obtain the converse implication (b) $\Rightarrow$ (a), and hence the equivalence (a) $\Leftrightarrow$ (b). The equivalence (a) $\Rightarrow$ (c) follows by taking opposites. $\square$

5.6. **Universal properties of adjoints.**

Proposition 5.19. *Suppose a functor $F : \mathscr{C} \to \mathscr{D}$ admits a right adjoint $G : \mathscr{D} \to \mathscr{C}$, and let $\epsilon : FG \to id_{\mathscr{D}}$ be the counit transformation for this adjunction. Suppose that we have another functor $G' : \mathscr{D} \to \mathscr{C}$ and a transformation $\epsilon' : FG' \to id_{\mathscr{D}}$. Then the following hold:*

(1) *There exists a transformation $\zeta : G' \to G$ and an identification of ϵ' as a composite $\epsilon' = \epsilon(F\zeta)$ in $\mathrm{Fun}(\Delta^1 \times \mathscr{D}, \mathscr{D})$.*

(2) *The transformation ζ from (1) is an isomorphism if and only if ϵ' realizes G' as a(nother) right adjoint F.*

Proof. (1) We have the cartesian (and cocartesian) fibration $q : \mathscr{E} = \mathrm{N}^F(\Delta^1) \to \Delta^1$ and the cocartesian lift

$$\begin{tikzcd} \{1\} \arrow[r, "\mathrm{incl}"] \arrow[d] & \mathrm{Fun}(\mathscr{D}, \mathscr{E}) \arrow[d, "q_*"] \\ \Delta^1 \arrow[r, "\mathrm{proj}"'] \arrow[ur, "\widetilde{\epsilon}"] & \mathrm{Fun}(\mathscr{D}, \Delta^1) \end{tikzcd}$$

as in the proof of Proposition 5.18. (Here we recall that q_* is a cartesian fibration, by Proposition II-2.6.) The map $\widetilde{\epsilon} : \Delta^1 \times \mathscr{D} \to \mathscr{E}$ has restrictions

$$\widetilde{\epsilon}|_0 = (\mathscr{D} \overset{G}{\to} \mathscr{C} = \mathscr{E}_0 \to \mathscr{E}) \quad \text{and} \quad \widetilde{\epsilon}|_1 = (\mathscr{D} = \mathscr{E}_1 \to \mathscr{E}).$$

Let us denote these maps i_0G and i_1 respectively.

Pulling back, we obtain a cocartesian (and cocartesian) fibration $\mathscr{F} \to \Delta^1$ and a cartesian lift $\widetilde{\epsilon}$ of the morphism $0 < 1$. From ϵ' we obtain a note-necessarily-cartesian lift $\widetilde{\epsilon}' : \Delta^1 \to \mathscr{F}$ which is defined in exactly the same manner.

By Lemma 5.13 the map $\widetilde{\epsilon}$ provides an equivalence

$$\widetilde{\epsilon}_* : \mathrm{Hom}_{\mathrm{Fun}(\mathscr{D},\mathscr{C})}(G', G) \to \mathrm{Hom}_{\mathrm{Fun}(\mathscr{D},\mathscr{E})}(i_0 G', i_1)$$

in $\mathrm{h}\mathscr{K}an$, so that the transformation $\widetilde{\epsilon}' : i_0 G' \to i_1$ lifts uniquely to a map $\zeta : G' \to G$ with $\widetilde{\epsilon}\zeta = \widetilde{\epsilon}'$ in $\mathrm{Fun}(\Delta^1, \mathrm{Fun}(\mathscr{D},\mathscr{E})) = \mathrm{Fun}(\Delta^1 \times \mathscr{D}, \mathscr{E})$.

Again we consider the transformation $k : \Delta^1 \to \mathrm{Fun}(\Delta^1, \mathscr{C}at_\infty)$ between F and $id_{\mathscr{D}}$ provided by the strictly commuting diagram

$$\begin{tikzcd} \mathscr{C} \arrow[r, "F"] \arrow[d, "F"'] & \mathscr{D} \arrow[d, "id"] \\ \mathscr{D} \arrow[r, "id"'] & \mathscr{D} \end{tikzcd}$$

to obtain a projection $\pi : \mathscr{E} \to \mathscr{D}$ via the composite

$$\mathscr{E} = \mathrm{N}^F(\Delta^1) \xrightarrow{\mathrm{N}^k} \mathrm{N}^{id_{\mathscr{D}}}(\Delta^1) = \Delta^1 \times \mathscr{D} \xrightarrow{p_2} \mathscr{D}.$$

Composing with π provides the claimed identification $\epsilon(F\zeta) = \pi\widetilde{\epsilon}\zeta = \pi\widetilde{\epsilon}' = \epsilon'$. The proof of (2) is left to the interested reader. $\square$

Of course we have the analogous universal property for the left adjoint, which one obtains by taking opposites.

Proposition 5.20. *Suppose a functor $G : \mathscr{D} \to \mathscr{C}$ admits a left adjoint $F : \mathscr{C} \to \mathscr{D}$, and let $\eta : id_{\mathscr{C}} \to GF$ be the unit transformation for this adjunction. Suppose that we have another functor $F' : \mathscr{D} \to \mathscr{C}$ and a transformation $\eta' : id_{\mathscr{C}} \to GF'$. Then the following hold:*

(1) *There exists a transformation $\zeta : F \to F'$ and an identification of η' as a composite $\eta' = (G\zeta)\eta$ in $\mathrm{Fun}(\Delta^1 \times \mathscr{C}, \mathscr{C})$.*

(2) *The transformation ζ from (1) is an isomorphism if and only if η' realizes F' as a(nother) left adjoint F.*

6. The derived ∞-category via localization

We show that the derived ∞-category $\mathscr{D}(\mathbb{A})$ is obtainable as a localization $\mathscr{D}(\mathbb{A}) = \mathscr{K}(\mathbb{A})[\mathrm{Qiso}^{-1}]$. After applying some fundamental results from higher algebra [14], we observe a further identification of the derived ∞-category as a localization of the *discrete* category of cochains

$$\mathscr{D}(\mathbb{A}) = \mathrm{Ch}(\mathbb{A})[\mathrm{Qiso}^{-1}]. \tag{30}$$

This latter characterization is, at least from the perspective of the author, mind blowing, as it is not premeditated by anything observed in classical representation theory or geometry.

Though it is outside of the scope of the present document, the characterization from (30) can be used to transfer various structures, e.g. symmetric or braided monoidal structures, from the abelian setting to the derived setting.

6.1. **The setup.** Throughout this section $\mathbb{A}$ is a Grothendieck abelian category. We recall that $\mathbb{A}$ admits enough injectives in this case, and that every complex V admits a quasi-isomorphism $V \to I$ to a K-injectives [20, Theorem 3.13]. From this we conclude that the full subcategory

$$\mathscr{D}_{\mathrm{Inj}} := \mathscr{K}(\mathbb{A})_{\mathrm{Inj}} = \left\{ \begin{array}{c} \text{The full } \infty\text{-subcategory of } K\text{-injective} \\ \text{complexes in } \mathscr{K}(\mathbb{A}). \end{array} \right.$$

is reflective in $\mathscr{K}(\mathbb{A})$ (Example 5.2). Of course, the ∞-category $\mathscr{D}_{\mathrm{Inj}}$ is one of our standard models for the derived ∞-category of $\mathbb{A}$ (Section I-2.5). However, the point of this section is to argue that the derived category should be characterized, or defined, via a universal property. So we emphasize throughout the section the particular nature of $\mathscr{D}_{\mathrm{Inj}}$ as the ∞-subcategory of K-injectives in $\mathscr{K}(\mathbb{A})$.

We are also interested in the cases where $\mathbb{A}$ admits enough projectives. For examples, one might consider:

- $\mathbb{A} = A\text{-Mod}$ for a ring A.
- $\mathbb{A} = \mathrm{QCoh}(\mathfrak{X})$ where $\mathfrak{X} = [X/G]$ is the quotient stack of an affine scheme by the action of a reductive algebraic group G, in characteristic 0. In this case the projective objects are identified with equivariant vector bundles on X, under the pullback equivalence between $\mathrm{QCoh}(\mathfrak{X})$ and equivariant vector bundles on X.
- $\mathbb{A} = \mathrm{Rep}_q(G)$ the category of quantum group representations for a semisimple algebraic group G at an arbitrary complex parameter $q \in \mathbb{C}$.

In this case the full subcategory of K-projectives

$$\mathscr{D}_{\mathrm{Proj}} := \mathscr{K}(\mathbb{A})_{\mathrm{Proj}} = \left\{ \begin{array}{c} \text{The full } \infty\text{-subcategory of } K\text{-injective} \\ \text{complexes in } \mathscr{K}(\mathbb{A}) \end{array} \right.$$

forms a coreflective subcategory in $\mathscr{K}(\mathbb{A})$ (Example 5.3), and we have a canonical equivalence

$$\mathscr{D}_{\mathrm{Proj}} \cong \mathscr{D}_{\mathrm{Inj}}$$

by Theorem I-12.5. So in this case we have reflective a coreflective subcategories

$$\mathscr{D}_{\mathrm{Proj}},\ \mathscr{D}_{\mathrm{Inj}}\ \subseteq\ \mathscr{K}(\mathbb{A})$$

which provide two distinct, but categorically indestinguishable, "models" for the derived ∞-category.

6.2. **Reflections and coreflections as reoslutions.**

Lemma 6.1. *For a morphism $f : V \to X$ in $\mathscr{K}(\mathbb{A})$, the following are equivalent:*

(a) *The object X is K-injective and f is a quasi-isomorpihsm.*
(b) *The morphism f is a $\mathscr{D}_{\mathrm{Inj}}$-reflection.*

Proof. The implication (a) $\Rightarrow$ (b) follows from the fact that maps into any K-injective $\mathrm{Hom}^*_{\mathbb{A}}(-, Z)$ preserve quasi-isomoprphisms. Hence the functor

$$\mathrm{Hom}_{\mathscr{K}(\mathbb{A})}(-, Z) \cong K\,\mathrm{Hom}^*_{\mathbb{A}}(-, Z) : \mathscr{K}(\mathbb{A}) \to \mathrm{h}\,\mathscr{K}an$$

sends quasi-isomorphisms to isomorphisms.

For (b) $\Rightarrow$ (a), suppose f is a $\mathscr{D}_{\mathrm{Inj}}$-reflection. Then X is K-injective, by definition. Suppose, by way of contradiction, that f is not a quasi-isomorphism. Then the mapping cone $\mathrm{cone}(f)$ is not acyclic, and there is some integer i so that

$$H^i(\mathrm{cone}(f)) \neq 0.$$

Let $\alpha'' : H^i(\mathrm{cone}(f)) \to I^0$ be an inclusion into an injective object (which exists since $\mathbb{A}$ has enough injectives), $\alpha' : Z^0(\mathrm{cone}(f)) \to I^0$ be the restriction along the projection from the cocycles, and $\alpha : (\mathrm{cone}(f))^i \to I^0$ be an arbitrary lift to degree i cochains. We note that such a lift exists via injectivity of I^0.

Take now $I = \Sigma^{-i} I^0$, considered as a complex. The map α now defines a map of cochains

$$\alpha : \mathrm{cone}(f) \to I$$

which recovers α'' on cohomology. In particular, α is not homotopically trivial, and hence realizes a nonzero class in cohomology

$$\bar{\alpha} \in H^0 \operatorname{Hom}^*_{\mathbb{A}}(\mathrm{cone}(f), I) \cong H^0 \left(\Sigma^{-1} \mathrm{cone}(f^*)\right).$$

It follows that the induced map $f^* : \operatorname{Hom}^*_{\mathbb{A}}(X, I) \to \operatorname{Hom}^*_{\mathbb{A}}(V, I)$ is not a quasi-isomorphism.

We have in particular

$$\operatorname{gr} H^0 \left(\Sigma^{-1} \mathrm{cone}(f^*)\right) = \begin{array}{c} \ker\left(H^0 \operatorname{Hom}^*_{\mathbb{A}}(X, I) \to H^0 \operatorname{Hom}^*_{\mathbb{A}}(V, I)\right) \\ \oplus \\ \operatorname{coker}\left(H^{-1} \operatorname{Hom}^*_{\mathbb{A}}(X, I) \to H^{-1} \operatorname{Hom}^*_{\mathbb{A}}(V, I)\right) \end{array}$$

under the apparent filtation on the mapping cone so that the above arguments show that at least one of the maps

$$H^{\varepsilon}(f^*) : H^{\varepsilon} \operatorname{Hom}^*_{\mathbb{A}}(X, I) \to H^{\varepsilon} \operatorname{Hom}^*_{\mathbb{A}}(V, I)$$

at $\varepsilon = 0, -1$, is not an isomorphism. It follows that the induced map of simplicial abelian groups

$$f^* : \operatorname{Hom}_{\mathscr{K}(\mathbb{A})}(X, I) \cong K \operatorname{Hom}^*_{\mathbb{A}}(X, I) \to K \operatorname{Hom}^*_{\mathbb{A}}(V, I) \cong \operatorname{Hom}_{\mathscr{K}(\mathbb{A})}(V, I)$$

is not a isomorphism in $\mathrm{h}\mathscr{K}an$ by Theorem I-10.13. Since the complex I is K-injective, this contradicts the assumption that f is a $\mathscr{D}_{\mathrm{Inj}}$-reflection, and we conclude that reflection-ness of f forces f to be a quasi-isomorphism. $\square$

Completely similar arguments apply in the projective situation.

Lemma 6.2. *Suppose that $\mathbb{A}$ has enough projectives. Then for a morphism $g : X \to V$ in $\mathscr{K}(\mathbb{A})$ the following are equivalent:*

(a) *The object X is K-projective and g is a quasi-isomorphism.*
(b) *The morphism g is a $\mathscr{D}_{\mathrm{Proj}}$-coreflection.*

6.3. **Precomposition and natural isomorphisms in functor categories.**

Lemma 6.3. *Let $\zeta : F_0 \to F_1$ be a natural transformation between functors $F_i : \mathscr{K} \to \mathscr{K}'$.*

(1) *For each ∞-category $\mathscr{C}$ the functors ζ induces a natural transformation $\zeta^* : F_0^* \to F_1^*$ between the corresponding functors $F_i^* : \operatorname{Fun}(\mathscr{K}', \mathscr{C}) \to \operatorname{Fun}(\mathscr{K}, \mathscr{C})$.*
(2) *If ζ is an isomorphism, then ζ^* is an isomorphism as well.*

Construction 6.3. The transformation ζ is a 2-simplex $\zeta : \Delta^1 \to \operatorname{Fun}(\mathscr{K}, \mathscr{K}')$ which restricts to F_i at $\{i\}$, for $i = 0, 1$. So composition in the simplicial category $\underline{\mathrm{Cat}}_{\infty}$ provides us with a map

$$\zeta^* : \operatorname{Fun}(\mathscr{K}', \mathscr{C}) \times \Delta^1 \to \operatorname{Fun}(\mathscr{K}', \mathscr{C}) \times \operatorname{Fun}(\mathscr{K}, \mathscr{K}') \xrightarrow{\circ} \operatorname{Fun}(\mathscr{K}, \mathscr{C})$$

whose restrictions to $\{i\} \subseteq \Delta^1$ recover the maps F_i^*. So ζ^* is a transformation $\zeta^* : F_0^* \to F_1^*$.

Similarly, if we have an n-simplex $\sigma : \Delta^n \to \mathrm{Fun}(\mathscr{K}, \mathscr{K}')$ with vertices $G_i : \mathscr{K} \to \mathscr{K}'$ we get an n-simplex $\sigma^* : \mathrm{Fun}(\mathscr{K}', \mathscr{C}) \times \Delta^n \to \mathrm{Fun}(\mathscr{K}, \mathscr{C})$ with vertices G_i^*, and one sees that σ^* is degenerate whenever σ is degenerate. Hence diagrams of the form

$$\begin{tikzcd} & F_1 \arrow[dr, "\eta"] & \\ F_0 \arrow[ur, "\zeta"] \arrow[rr, "id_{F_0}"'] & & F_0 \end{tikzcd} \qquad \begin{tikzcd} & F_0 \arrow[dr, "\zeta"] & \\ F_1 \arrow[ur, "\eta"] \arrow[rr, "id_{F_1}"'] & & F_1 \end{tikzcd}$$

in $\mathrm{Fun}(\mathscr{K}, \mathscr{K}')$ imply diagrams of the form

$$\begin{tikzcd} & F_1^* \arrow[dr, "\eta^*"] & \\ F_0^* \arrow[ur, "\zeta^*"] \arrow[rr, "id_{F_0^*}"'] & & F_0^* \end{tikzcd} \qquad \begin{tikzcd} & F_0^* \arrow[dr, "\zeta^*"] & \\ F_1^* \arrow[ur, "\eta^*"] \arrow[rr, "id_{F_1^*}"'] & & F_1^* \end{tikzcd}$$

in $\mathrm{Fun}(\mathrm{Fun}(\mathscr{K}', \mathscr{C}), \mathrm{Fun}(\mathscr{K}, \mathscr{C}))$. So we see directly that ζ^* is isomorphism whenever ζ is an isomorphism. □

6.4. **Localizating the homotopy ∞-category against quasi-isomorphisms.**

Proposition 6.4. *Let $L : \mathscr{K}(\mathbb{A}) \to \mathscr{D}_{\mathrm{Inj}}$ be the left adjoint to the inclusion $i : \mathscr{D}_{\mathrm{Inj}} \to \mathscr{K}(\mathbb{A})$, along with the unit and counit transformations*

$$u : id_{\mathscr{K}(\mathbb{A})} \to iL \quad \textit{and} \quad \epsilon : Li \to id_{\mathscr{D}_{\mathrm{Inj}}}$$

as in Proposition 5.10.

(1) *At each x in $\mathscr{K}(\mathbb{A})$ the unit transformation $u_x : x \to L(x)$ is a quasi-isomorphism.*

(2) *A map $\alpha : x \to y$ in $\mathscr{K}(\mathbb{A})$ is a quasi-isomorphism if and only if $L(\alpha) : L(x) \to L(y)$ is an isomoprhism in $\mathscr{D}_{\mathrm{Inj}}$.*

(3) *The counit transformation is a natural isomoprhism from the composite*

$$\mathscr{D}_{\mathrm{Inj}} \xrightarrow{i} \mathscr{K}(\mathbb{A}) \xrightarrow{L} \mathscr{D}_{\mathrm{Inj}}$$

to the identity $id_{\mathscr{D}_{\mathrm{Inj}}}$, and so defines a 2-simplex in the mapping complex $\epsilon : \Delta^2 \to \mathrm{Fun}(\mathscr{D}_{\mathrm{Inj}}, \mathscr{D}_{\mathrm{Inj}})^{\mathrm{Kan}}$.

Proof. Statement (1) follows from the characterization of $\mathscr{D}_{\mathrm{Inj}}$-reflections provided in Lemma 6.1 and Theorem 5.11. For (2), naturality of u implies, for each morphism $\alpha : x \to y$, the existence of a diagram

$$\begin{tikzcd} L(x) \arrow[r, "L(\alpha)"] & L(y) \\ x \arrow[u, "u_x"] \arrow[r, "\alpha"'] & y \arrow[u, "u_y"'] \end{tikzcd}$$

in the discrete homotopy category $K(\mathbb{A}) = \mathrm{h}\,\mathscr{K}(\mathbb{A})$. The vertical maps in this diagram are quasi-isomorpihsms by (1), so that α is a quasi-isomorphism if and only if $L(\alpha)$ is a quasi-isomorphism. However, a map between K-injectives is a quasi-isomorphism if and only if it is a homotopy equivalence, i.e. an isomoprhism in $\mathscr{D}_{\mathrm{Inj}}$. So we conclude that α is a quasi-isomorphism if and only if $L(\alpha)$ is an isomorphism in $\mathscr{D}_{\mathrm{Inj}}$. Statement (3) is implied directly by the generic description of ϵ given in Proposition 5.10 □

Let us recall that the localization $\mathscr{C}[W^{-1}]$ of an ∞-category at a class of morphisms $W \subseteq \mathscr{C}[1]$, with all degenerate 1-simplices in W, is any ∞-category $\mathscr{D}$ equipped with a functor $F : \mathscr{C} \to \mathscr{D}$ which induces, at all ∞-categories $\mathscr{E}$, a fully faithful functor

$$F^* : \mathrm{Fun}(\mathscr{D}, \mathscr{E}) \to \mathrm{Fun}(\mathscr{C}, \mathscr{E})$$

whose image is the full ∞-subcategory spanned by all functors $\mathscr{C} \to \mathscr{E}$ which sends all maps in W to isomorphisms in $\mathscr{E}$ (Definition II-14.17). In this case we write, somewhat ambiguously, $\mathscr{D} = \mathscr{C}[W^{-1}]$.

Remark 6.5. One can think of localization fairly clearly from the perspective of marked simplicial sets. A functor $F : \mathscr{C} \to \mathscr{D}$ is a localization of $\mathscr{C}$ relative to a class of maps W if and only if it induces an equivalence between the category of marked functors

$$\mathrm{Fun}(\mathscr{C}, \mathscr{D}) = \mathrm{Fun}^\flat((\mathscr{C}, \mathrm{Isom}), (\mathscr{D}, \mathrm{Isom})),$$

at general $\mathscr{E}$, and the category of marked functors $\mathrm{Fun}^\flat((\mathscr{C}, W), (\mathscr{E}, \mathrm{Isom}))$ [13, Section 3.1.3].

Theorem 6.6. *Let $\mathbb{A}$ be a Grothendieck abelian category.*

(1) *Given any ∞-category $\mathscr{E}$, restriction along the left adjoint $L : \mathscr{K}(\mathbb{A}) \to \mathscr{D}_{\mathrm{Inj}}$ to the inclusion provides a fully faithful functor*

$$L^* : \mathrm{Fun}(\mathscr{D}_{\mathrm{Inj}}, \mathscr{E}) \to \mathrm{Fun}(\mathscr{K}(\mathbb{A}), \mathscr{E})$$

which is an equivalence onto the full subcategory spanned by fnuctors which sends all quasi-isomoprihsms in $\mathscr{K}(\mathbb{A})$ to isomoprhisms in $\mathscr{E}$.

(2) *For $\mathrm{Fun}(\mathscr{K}(\mathbb{A}), \mathscr{E})^{\mathrm{Qiso}}$ the full subcategory spanned by all functors which send quasi-isomorphisms to isomorphisms in $\mathscr{E}$, the inverse to the equivalence L^* is provided by restiction $i^* : \mathrm{Fun}(\mathscr{K}(\mathbb{A}), \mathscr{E})^{\mathrm{Qiso}} \to \mathrm{Fun}(\mathscr{D}_{\mathrm{inj}}, \mathscr{E})$ along the inclusion $i : \mathscr{D}_{\mathrm{Inj}} \to \mathscr{K}(\mathbb{A})$.*

(3) *the functor $L : \mathscr{K}(\mathbb{A}) \to \mathscr{D}_{\mathrm{Inj}}$ exhibits $\mathscr{D}_{\mathrm{Inj}}$ as a localization $\mathscr{D}_{\mathrm{Inj}} = \mathscr{K}(\mathbb{A})[\mathrm{Qiso}^{-1}]$.*

Proof. (3) Follows from (1), simply by the definition of a localization. We prove (1) and (2). Take $\mathscr{K} = \mathscr{K}(\mathbb{A})$ and $\mathscr{D} = \mathscr{D}_{\mathrm{Inj}}$, and let $\mathscr{E}$ be arbitrary. Take $\mathrm{Fun}(\mathscr{K}, \mathscr{E})^{\mathrm{Qiso}}$ the full ∞-subcategory of functors $T : \mathscr{K} \to \mathscr{E}$ which send quasi-isomoprhisms in $\mathscr{K}$ to isomorpsms in $\mathscr{E}$. Then, by Proposition 6.4 (2), the functor L^* has image in $\mathrm{Fun}(\mathscr{K}, \mathscr{E})^{\mathrm{Qiso}}$ and so restricts to a map

$$L^* : \mathrm{Fun}(\mathscr{D}, \mathscr{E}) \to \mathrm{Fun}(\mathscr{K}, \mathscr{E})^{\mathrm{Qiso}}.$$

We have the functor $i^* : \mathrm{Fun}(\mathscr{K}, \mathscr{E})^{\mathrm{Qiso}} \to \mathrm{Fun}(\mathscr{D}, \mathscr{E})$ provided by restricting along the inclusion $i^* : \mathscr{D} \to \mathscr{K}$. We claim that these functors are mutually inverse, and so realize the claimed equivalence. More prescisely, we claim that the counit and unit transformations ϵ and u induce isomorphisms

$$\epsilon^* : i^* L^* \to id_{\mathrm{Fun}(\mathscr{D}, \mathscr{E})} \quad \text{and} \quad u^* : id_{\mathrm{Fun}(\mathscr{K}, \mathscr{E})^{\mathrm{Qiso}}} \to L^* i^*.$$

The fact that ϵ^* is an isomorphism just follows from the fact that ϵ itself is a isomorphism. See Proposition 6.4 and Lemma 6.3. So we need only address the transformation u^*.

First note that

$$u^* : \mathrm{Fun}(\mathscr{K}, \mathscr{E}) \times \Delta^1 \to \mathrm{Fun}(\mathscr{K}, \mathscr{E})$$

sends each object in the subcategory $\mathrm{Fun}(\mathscr{K}, \mathscr{E})^{\mathrm{Qiso}} \times \Delta^1$ to an object in $\mathrm{Fun}(\mathscr{K}, \mathscr{E})^{\mathrm{Qiso}}$, since u^* is a transformation between $id_{\mathrm{Fun}(\mathscr{K},\mathscr{E})}$ and L^*i^* and these endofunctors preserve the subcategory $\mathrm{Fun}(\mathscr{K}, \mathscr{E})^{\mathrm{Qiso}}$. Since $\mathrm{Fun}(\mathscr{K}, \mathscr{E})^{\mathrm{Qiso}}$ is full in $\mathrm{Fun}(\mathscr{K}, \mathscr{E})$ it follows that u^* does in fact restrict to a transformation

$$u^* : \mathrm{Fun}(\mathscr{K}, \mathscr{E})^{\mathrm{Qiso}} \times \Delta^1 \to \mathrm{Fun}(\mathscr{K}, \mathscr{E})^{\mathrm{Qiso}}$$

between the identity and L^*i^*.

We need to show that u^* is a natural isomorphism. By Proposition I-6.8 it suffices to show that u^* evaluates to an isomorphism in $\mathrm{Fun}(\mathscr{K}, \mathscr{E})^{\mathrm{Qiso}}$ at each functor $T : \mathscr{K} \to \mathscr{E}$ in $\mathrm{Fun}(\mathscr{K}, \mathscr{E})^{\mathrm{Qiso}}$. By the definition of u^* from Construction 6.3 we have

$$u^*_T : \Delta^1 \overset{u}{\to} \mathrm{Fun}(\mathscr{K}, \mathscr{K}) \overset{T_*}{\to} \mathrm{Fun}(\mathscr{K}, \mathscr{E})^{\mathrm{Qiso}} \subseteq \mathrm{Fun}(\mathscr{K}, \mathscr{E}),$$

and to see that u^*_T is an isomoprhism it again suffices to show that u^*_T evaluates to an isomorphism at each x in $\mathscr{K}$. At any such x we have

$$(u^*_T)_x = T(u_x) : T(x) \to TL(x).$$

By Proposition 6.4 (1) each map u_x is a quasi-isomorphism, and since T sends quasi-isomorphisms to isomorphism we have that $(u^*_T)_x$ is an isomoprhism in $\mathscr{E}$, as desired. So we see that u^* itself is a natural isomorpihsm.

We now have natural isomorphisms $\epsilon^* : i^*L^* \to id_{\mathrm{Fun}(\mathscr{D},\mathscr{E})}$ and $(u^*)^{-1} : L^*i^* \to id_{\mathrm{Fun}(\mathscr{K},\mathscr{E})^{\mathrm{Qiso}}}$. By the definition of $\mathscr{C}at_\infty$ as the homotopy coherent nerve of the simplicial category $\underline{\mathrm{Cat}}^+_\infty$, these natural isomorphsms provide 2-simplices

$$\begin{tikzcd} & \mathrm{Fun}(\mathscr{K}, \mathscr{E})^{\mathrm{Qiso}} \arrow[dr, "i^*"] & \\ \mathrm{Fun}(\mathscr{D}, \mathscr{E}) \arrow[ur, "L^*"] \arrow[rr, "id_{\mathrm{Fun}(\mathscr{K},\mathscr{D})}"'] & & \mathrm{Fun}(\mathscr{D}, \mathscr{E}) \end{tikzcd}$$

and

$$\begin{tikzcd} & \mathrm{Fun}(\mathscr{D}, \mathscr{E}) \arrow[dr, "L^*"] & \\ \mathrm{Fun}(\mathscr{K}, \mathscr{E})^{\mathrm{Qiso}} \arrow[ur, "i^*"] \arrow[rr, "id_{\mathrm{Fun}(\mathscr{K},\mathscr{E})^{\mathrm{Qiso}}}"'] & & \mathrm{Fun}(\mathscr{K}, \mathscr{E})^{\mathrm{Qiso}} \end{tikzcd}$$

in $\mathscr{C}at_\infty$ which realize L^* and i^* as mutually inverse. $\square$

In the event that $\mathbb{A}$ has enough projectives, we can consider the right adjoint $R : \mathscr{K}(\mathbb{A}) \to \mathscr{D}_{\mathrm{Proj}}$ along with its unit and counit transformations $u : id_{\mathscr{D}_{\mathrm{Proj}}} \to Ri$ and $\epsilon : iR \to id_{\mathscr{K}(\mathbb{A})}$. We have that u is a natural isomorphism, that ϵ evaluates to a quasi-isomorphism $\epsilon_x : R(x) \to x$ at each x in $\mathscr{K}(\mathbb{A})$, and that a map $\alpha : x \to y$ in $\mathscr{K}(\mathbb{A})$ is a quasi-isomorphism if and only if $R(\alpha) : R(x) \to R(y)$ is an isomorphism. To observe these properties one argues exactly as in the proof of Proposition 6.4.

We can therefore argue as in the proof of Theorem 6.6 to realize the projective construction of the derived ∞-category as a localization.

Theorem 6.7. *Let $\mathbb{A}$ be a Grothendieck abelian category, and suppose that $\mathbb{A}$ has enough projectives.*

(1) *For any ∞-category $\mathscr{E}$, restriction along the right adjoint $R : \mathscr{K}(\mathbb{A}) \to \mathscr{D}_{\mathrm{Proj}}$ to the inclusion provides a fully faithful functor*
$$R^* : \mathrm{Fun}(\mathscr{D}_{\mathrm{Proj}}, \mathscr{E}) \to \mathrm{Fun}(\mathscr{K}(\mathbb{A}), \mathscr{E})$$
which is an equivalence onto the full ∞-subcategory spanned by those functors $T : \mathscr{K}(\mathbb{A}) \to \mathscr{E}$ which send quasi-isomorphisms in $\mathscr{K}(\mathbb{A})$ to isomorphisms in $\mathscr{E}$.

(2) *For $\mathrm{Fun}(\mathscr{K}(\mathbb{A}), \mathscr{E})^{\mathrm{Qiso}}$ the full subcategory spanned by all functors which send quasi-isomorphisms to isomorphisms in $\mathscr{E}$, the inverse to R^* is given by restriction $i^* : \mathrm{Fun}(\mathscr{K}(\mathbb{A}), \mathscr{E})^{\mathrm{Qiso}} \to \mathrm{Fun}(\mathscr{D}_{\mathrm{Proj}}, \mathscr{E})$ along the inclusion $i : \mathscr{D}_{\mathrm{Proj}} \to \mathscr{K}(\mathbb{A})$.*

(3) *The functor $R : \mathscr{K}(\mathbb{A}) \to \mathscr{D}_{\mathrm{Proj}}$ realizes $\mathscr{D}_{\mathrm{Proj}}$ as a localization $\mathscr{D}_{\mathrm{Proj}} = \mathscr{K}(\mathbb{A})[\mathrm{Qiso}^{-1}]$.*

As in the case of $\mathscr{D}_{\mathrm{Inj}}$, one sees that the inverse to R^* is given by restricting along the inclusion $\mathscr{D}_{\mathrm{Proj}} \to \mathscr{K}(\mathbb{A})$.

Corollary 6.8. *For any Grothendieck abelian category $\mathbb{A}$ which has enough projectives, there is a unique equivalence $\psi : \mathscr{D}_{\mathrm{Inj}} \xrightarrow{\sim} \mathscr{D}_{\mathrm{Proj}}$ which fits into a diagram*
$$\begin{tikzcd} & \mathscr{K}(\mathbb{A}) \arrow[dl, "L^*"'] \arrow[dr, "R^*"] & \\ \mathscr{D}_{\mathrm{Inj}} \arrow[rr, "\sim", "\psi"'] & & \mathscr{D}_{\mathrm{Proj}} \end{tikzcd}$$
in $\mathscr{C}at_\infty$.

We leave the following exercise to the interested reader.

Exercise 6.9. Provie that the equivalence $\psi : \mathscr{D}_{\mathrm{Inj}} \to \mathscr{D}_{\mathrm{Proj}}$ is precisely the equivalence realized previously in Section I-12. (Here of course we accept that the functor ψ is only defined up to equivalence.)

6.5. **Re-definining the derived ∞-category.**

Definition 6.10. Given a Grothendieck abelian category $\mathbb{A}$, the derived ∞-category $\mathscr{D}(\mathbb{A})$ is the localization of the homotopy ∞-category relaztive to the class of quasi-isomorphisms
$$\mathscr{D}(\mathbb{A}) := \mathscr{K}(\mathbb{A})[\mathrm{Qiso}^{-1}].$$

Theorems 6.6 and 6.7 say that we can construct the derived ∞-category via K-injectives in $\mathrm{Ch}(\mathbb{A})$, or via K-projectives when they exist. Up to equivalence, and relative specifically to the universal property of localization, it's all the same.

6.6. **Stable localization.** In the triangulated setting, for a triangulated subcategory T in a triangulated category C we have the Verdier localization

$$C/T := C[W_T^{-1}],$$

where W_T is the collection of all morphisms whose mapping cone lies in T. We have the canonical functor $C \to C/T$ whose kernel is T and which is universal amongst all exact functors which annihilate T.

Definition 6.11. Given a stable subcategory $\mathscr{T}$ in a stable ∞-category $\mathscr{C}$, let $W_{\mathscr{T}}$ be the collection of all morphisms $\alpha : x \to y$ in $\mathscr{C}$ whose cofiber $\mathrm{cofib}(\alpha)$ is isomorphic to an object in $\mathscr{T}$. (Note that identity morphisms are all $\mathscr{T}$-acyclic since $\mathscr{T}$ contains the zero object in $\mathscr{C}$.) The Verdier quotient of $\mathscr{C}$ by $\mathscr{T}$ is the localization

$$\mathscr{C}/\mathscr{T} := \mathscr{C}[W_{\mathscr{T}}^{-1}].$$

Given a discrete category $\mathbb{A}$ and a class of morphisms S in $\mathbb{A}$, which we assume contains all identity maps, we can consider the discrete localization $\mathbb{A}[S^{-1}]_{\mathrm{disc}}$. This is any discrete category equipped with a functor $\mathbb{A} \to \mathbb{A}[S^{-1}]_{\mathrm{disc}}$ which is universal amongst all functors to a discrete target which turn morphisms in S into isomorphisms. One can construct such a discrete localization by taking the homotopy category of the ∞-categorical (Dwyer-Kan) localization, for example.

Lemma 6.12. *Consider an ∞-category $\mathscr{C}$ and class of morphisms $W \subseteq \mathscr{C}[1]$ which contains all degenerate edges. Let $F : \mathscr{C} \to \mathscr{C}[W^{-1}]$ be the localization functor. Then the unique map*

$$\bar{F} : (\mathrm{h}\,\mathscr{C})[W^{-1}]_{\mathrm{disc}} \to \mathrm{h}(\mathscr{C}[W^{-1}])$$

induced by $\mathrm{h}\,F$ is an equivalence of categories.

Proof. Take $\mathbb{C} = \mathrm{h}\,\mathscr{C}$ and consider the discrete localization $\mathbb{C}[W^{-1}]_{\mathrm{disc}}$. For any discrete category $\mathbb{E}$ let $\mathrm{Fun}(\mathbb{C}, \mathbb{E})^W$ and $\mathrm{Fun}(\mathscr{C}, \mathbb{E})^W$ denote the full subcategories of functors which send maps in W to isomorphisms in $\mathbb{E}$. If follows that, for any discrete category $\mathbb{E}$, restriction along the localization map $\mathbb{C} \to \mathbb{C}[W^{-1}]$, which exists and is produced via a calculus of localization [23, Theorem 10.3.7], provides an equivalence

$$\mathrm{Fun}(\mathbb{C}[W^{-1}]_{\mathrm{disc}}, \mathbb{E}) \xrightarrow{\sim} \mathrm{Fun}(\mathbb{C}, \mathbb{E})^W \xrightarrow{\sim} \mathrm{Fun}(\mathscr{C}, \mathbb{E})^W.$$

The latter equivalence is ensured since the homotopy category functor is left adjoint to the inclusion $\mathrm{Cat} \to \mathrm{Cat}_\infty$.

Similarly, we have an equivalence

$$\mathrm{Fun}(\mathrm{h}(\mathscr{C}[W^{-1}]), \mathbb{E}) \xrightarrow{\sim} \mathrm{Fun}(\mathscr{C}[W^{-1}], \mathbb{E}) \xrightarrow{\sim} \mathrm{Fun}(\mathscr{C}, \mathbb{E})^W.$$

These two equivalences fit into a diagram

$$\begin{tikzcd} \mathrm{Fun}(\mathscr{C}, \mathbb{E})^W & \\ \mathrm{Fun}(\mathbb{C}[W^{-1}]_{\mathrm{disc}}, \mathbb{E}) \arrow[u, "\sim"] & \mathrm{Fun}(\mathscr{C}[W^{-1}], \mathbb{E}), \arrow[l, "\bar{F}^*"'] \arrow[ul, "\sim"'] \end{tikzcd}$$

at arbitrary discrete $\mathbb{E}$, from which we conclude that $\bar{F}^*$ is an equivalence at all $\mathbb{E}$. This implies that $\bar{F}$ itself is an equivalence. $\square$

As a particular example, we find that the homotopy category of the Verdier localization $\mathscr{C}/\mathscr{T}$, where $\mathscr{C}$ is stable and $\mathscr{T}$ is thick in $\mathscr{C}$, recovers the discrete Verdier localization of the homotopy category,

$$\mathrm{h}\,\mathscr{C}/\mathrm{h}\,\mathscr{T} \overset{\sim}{\to} \mathrm{h}(\mathscr{C}/\mathscr{T}).$$

See for example [12, Lemma 4.6.1, Proposition 4.6.2].

From this observation one sees that the homotopy category of the localization carries a unique triangulated structure so that the localization map $F : \mathscr{C} \to \mathscr{C}/\mathscr{T}$ induces an exact functor $\mathrm{h}\,F : \mathrm{h}\,\mathscr{C} \to \mathrm{h}\,\mathscr{C}/\mathscr{T}$. One can show that this discrete triangulated structure lifts, in the most advantageous fashion, to the ∞-categorical level.

Theorem 6.13 ([19, Theorem I.3.3]). *Given a thick subcategory $\mathscr{T}$ in a stable ∞-category $\mathscr{C}$, the following hold:*

1. *The Verdier quotient $\mathscr{C}/\mathscr{T}$ is stable.*
2. *The localization functor $l : \mathscr{C} \to \mathscr{C}/\mathscr{T}$ is exact.*
3. *The induced map on the homotopy category $\bar{l} : \mathrm{h}\,\mathscr{C}/\mathrm{h}\,\mathscr{T} \to \mathrm{h}(\mathscr{C}/\mathscr{T})$ is an equivalence of triangulated categories.*
4. *For any stable ∞-category $\mathscr{D}$, and exact functor $\mathscr{C} \to \mathscr{D}$ which sends $\mathscr{T}$ to the subcategory $\mathscr{D}_{\mathrm{Zero}}$ of zero objects, the induced map $F' : \mathscr{C}/\mathscr{T} \to \mathscr{D}$ is also exact.*

Proof. All is covered in [19] save for (4). For (4), since F and l preserve zero objects, F' preserves zero objects as well. So we need only show that F' preserves cofiber sequences. Any morphism $\alpha' : x \to y'$ in $\mathscr{C}/\mathscr{T}$ fits into a diagram

$$\begin{tikzcd} x \arrow[r, "\alpha"] \arrow[d, "="'] \arrow[dr] & y \arrow[d, "\sim"] \\ x \arrow[r, "\alpha'"'] & y' \end{tikzcd}$$

where α is in the image of the localization map. This just follows from the fact that morphisms in the homotopy category $\mathrm{h}\,\mathscr{C}/\mathrm{h}\,\mathscr{T}$ are determined by a calculus of fractions.

Taking cofibers we obtain an isomorphism of pushout diagrams

$$\begin{tikzcd} & y \arrow[dr] & \\ x \arrow[ur, "\alpha"] \arrow[rr] \arrow[dr] & & \mathrm{cofib}(\alpha) \\ & 0 \arrow[ur] & \end{tikzcd} \quad \overset{\sim}{\to} \quad \begin{tikzcd} & y' \arrow[dr] & \\ x \arrow[ur, "\alpha'"] \arrow[rr] \arrow[dr] & & \mathrm{cofib}(\alpha') \\ & 0 \arrow[ur] & \end{tikzcd}$$

by Proposition II-13.20. Applying F we obtain an another isomorphism of pushout diagrams

$$\begin{tikzcd} & Fy \arrow[dr] & \\ Fx \arrow[ur, "\alpha"] \arrow[rr] \arrow[dr] & & F\,\mathrm{cofib}(\alpha) \\ & F0 \arrow[ur] & \end{tikzcd} \quad \overset{\sim}{\to} \quad \begin{tikzcd} & F'y' \arrow[dr] & \\ F'x \arrow[ur, "\alpha'"] \arrow[rr] \arrow[dr] & & F'\,\mathrm{cofib}(\alpha') \\ & F'0 \arrow[ur] & \end{tikzcd}$$

in $\mathscr{D}$. Since F is exact the left hand diagram is a cofiber sequence and we conclude that the right hand diagram is a cofiber diagram as well, by Proposition II-13.18. □

6.7. **Small derived categories via localization.**

Proposition 6.14. *Let $\mathbb{A}$ be a Grothendieck abelian category, $\mathscr{K} \subseteq \mathscr{K}(\mathbb{A})$ be a full stable subcategory in the homotopy ∞-category, and take $\mathrm{K} = \mathrm{h}\mathscr{K}$. Let* Acyc *and* $\mathrm{Acyc}_{\mathrm{K}}$ *be the triangulated subcategories of acyclic complexes in* $\mathrm{K}(\mathbb{A})$ *and* K*, respectively, and suppose that the induced map on discrete Veridier quotients*

$$\mathrm{K}\,/\,\mathrm{Acyc}_{\mathrm{K}} \to \mathrm{K}(\mathbb{A})/\,\mathrm{Acyc} = \mathrm{D}(\mathbb{A})$$

is fully faithful. Then for $\mathscr{D} \subseteq \mathscr{D}(\mathbb{A})$ the full subcategory spanned by the image of $\mathscr{K}$ under the localization map, the functor

$$\mathscr{K}[\mathrm{Qiso}^{-1}] \to \mathscr{D}$$

induced by the sequence $\mathscr{K} \to \mathscr{K}(\mathbb{A}) \to \mathscr{D}(\mathbb{A})$ is an equivalence.

In each of the following examples one employs standard arguments to prove that the relevant map $\mathrm{K}\,/\,\mathrm{Acyc}_{\mathrm{K}} \to \mathrm{D}(\mathbb{A})$ is fully faithful. Since I cannot find a reference for such arguments, we record all of the details for (only) the first example.

Example 6.15 (The bounded derived category)**.** Let $\mathbb{A}$ be any Grothendieck abelian category and $\mathscr{K}^b(\mathbb{A})$ be the homotopy ∞-category of bounded complexes in $\mathbb{A}$ and $\mathscr{D}^b(\mathbb{A})$ be the full subcategory in $\mathscr{D}(\mathbb{A})$ spanned by complexes with bounded cohomology. We also consider the ∞-categories $\mathscr{K}^-(\mathbb{A})$ of bounded above complexes. We claim that the functor $\mathscr{K}^b(\mathbb{A})[\mathrm{Qiso}^{-1}] \to \mathscr{D}^b(\mathbb{A})$ is an equivalence. For this it suffices to show, via Proposition 6.14, that the functor $\mathrm{K}(\mathbb{A})/\,\mathrm{Acyc}^b \to D(\mathbb{A})$ is fully faithful. We express within this example morphisms in $D(\mathbb{A})$ by either left or right fractions, following [12, Section 3].

Since any cochain complex X which has bounded above cohomology admits a quasi-isomorphism $X' \to X$ from a bounded above complex, we see that every morphism $V \leftarrow X \to W$ between bounded above complexes in $D(\mathbb{A})$ is equivalent to a morphism $V \leftarrow X' \to W$ which only involves bounded above complexes. This shows, from the perspective of the calculus of left fractions, that the functor $K^-(\mathbb{A})/\,\mathrm{Acyc}^- \to D(\mathbb{A})$ is full. This same fact, and a direct consideration of the equivalence relation on morphisms $V \leftarrow X \to W$ in the calculus of left fractions, also tells us that the given functor is full. We conclude that the functor $K^-(\mathbb{A})/\,\mathrm{Acyc}^- \to D(\mathbb{A})$ is fully faithful.

It now suffices to show that the functor $K^b(\mathbb{A})/\,\mathrm{Acyc}^b \to K^-(\mathbb{A})/\,\mathrm{Acyc}^-$ is fully faithful. For this one notes that any bounded above complex X which has bounded cohomology admits a quasi-isomorphism $X \to X''$ onto a bounded complex. From this fact, and a direct consideration of the calculus of right fractions which describes morphisms in the latter category, we see that the given functor is in fact fully faithful. In total, this recovers the well-known fact that the map on discrete categories

$$\mathrm{K}^b(\mathbb{A})/\,\mathrm{Acyc}^b \to \mathrm{D}(\mathbb{A})$$

is fully faithful, and hence that the restriction $\mathscr{K}^b(\mathbb{A}) \to \mathscr{D}^b(\mathbb{A})$ of the localization functor $L : \mathscr{K}(\mathbb{A}) \to \mathscr{D}(\mathbb{A})$ induces an equivalence of ∞-categories

$$\mathscr{K}^b(\mathbb{A})[\mathrm{Qiso}^{-1}] \overset{\sim}{\to} \mathscr{D}^b(\mathbb{A}).$$

Example 6.16 (The finite-dimensional derived category)**.** Let G be an algebraic group in finite characteristic, say, and consider the representation category $\mathrm{Rep}(G)$. Take $\mathscr{K}(\mathbb{A})_{fin}$ the homotopy ∞-category of bounded complexes of finite-dimensional representations. Take $\mathscr{D}(G)_{fin}$ the full subcategory in $\mathscr{D}(G) = \mathscr{D}(\mathrm{Rep}(G))$ consisting of all complexes with finite-dimensional cohomology. By Proposition 6.14, the functor $L : \mathscr{K}(G)_{fin} \to \mathscr{D}(G)_{fin}$ given by restricting the localization functor on $\mathscr{K}(G)$ induces an equivalence of ∞-categories

$$\mathscr{K}(G)_{fin}[\mathrm{Qiso}^{-1}] \to \mathscr{D}(G)_{fin}.$$

Example 6.17 (The coherent derived category)**.** Let X be a Noetherian scheme, for example a quasi-projective scheme over a Noetherian ring k, and let $\mathscr{K}(X)_{coh}$ denote the homotopy ∞-category of bounded complexes of coherent sheaves on X. Let $\mathscr{D}(X)_{coh}$ be the full subcategory of complexes in $\mathscr{D}(X) = \mathscr{D}(\mathrm{QCoh}(X))$ with coherent total cohomology. Then the restriction of the localization functor $L : \mathscr{K}(X)_{coh} \to \mathscr{D}(X)_{coh}$ induces an equivalence

$$\mathscr{K}(X)_{coh}[\mathrm{Qiso}^{-1}] \overset{\sim}{\to} \mathscr{D}(X)_{coh}$$

Example 6.18 (The derived category via flat sheaves)**.** Let X be a quasi-compact quasi-separated scheme and $\mathscr{K}(X)_{flat}$ be the homotopy ∞-category of K-flat quasi-coherent sheaves on X. Then the restriction of the localization functor $L : \mathscr{K}(X)_{flat} \to \mathscr{D}(X)$ induces an equivalence

$$\mathscr{K}(X)_{flat}[\mathrm{Qiso}^{-1}] \overset{\sim}{\to} \mathscr{D}(X)$$

[21, Lemma 3.3].

Example 6.19 (The perfect derived category)**.** Let X be any Noetherian algebraic stack with the resolution property, i.e. for which every coherent sheaf M admits a surjection $V \to M$ from a finite rank vector bundle. Let $mscD(X)_{perf}$ be the full ∞-subcategory of perfect sheaves in $\mathscr{D}(X) = \mathscr{D}(\mathrm{QCoh}(X))$, i.e. sheaves whose image in $D(X)$ is dualizable for the product $\otimes^{\mathrm{L}}_{\mathscr{O}_X}$, and $\mathscr{K}(X)_{vec}$ be the homotopy ∞-category of bounded complexes of finite rank vector bundles on X. Then the restriction of the localization functor $L : \mathscr{K}(X)_{vec} \to \mathscr{D}(X)_{perf}$ induces an equivalence

$$\mathscr{K}(X)_{vec}[\mathrm{Qiso}^{-1}] \overset{\sim}{\to} \mathscr{D}(X)_{perf}.$$

6.8. **Localizing directly from cochains.**

Theorem 6.20 ([14, Proposition 1.3.4.5])**.** *Consider an additive category $\mathbb{A}$ and any full subcategory $\mathrm{Ch}^\star(\mathbb{A}) \subseteq \mathrm{Ch}(\mathbb{A})$ which is closed under finite sums and the formation of mapping cones. Then the inclusion $\mathrm{Ch}^\star(\mathbb{A}) \to \mathscr{K}(\mathbb{A})$ induces an fully faithful functor from the localization of $\mathrm{Ch}^\star(\mathbb{A})$ against the class of homotopy equivalences*

$$\mathrm{Ch}^\star(\mathbb{A})[\mathrm{Htop}^{-1}] \overset{\sim}{\to} \mathscr{K}(\mathbb{A}).$$

which is an equivalence onto the full subcategory in $\mathscr{K}(\mathbb{A})$ spanned by the complexes in $\mathrm{Ch}^\star(\mathbb{A})$.

Corollary 6.21. *Let $\mathbb{A}$ be a Grothendieck abelian category and $\mathrm{Ch}^\star(\mathbb{A}) \subseteq \mathrm{Ch}(\mathbb{A})$ be a full subcategory which is stable under suspension, desuspension, and the formation of mapping cones. Suppose also that the, for the corresponding homotopy ∞-category $\mathscr{K}^\star(\mathbb{A})$, the induced map $\mathrm{h}\,\mathscr{K}^\star(\mathbb{A})/\,\mathrm{Acyc} \to \mathrm{D}(\mathbb{A})$ is fully faithful. Then the inclusion of simplicial sets $\mathrm{Ch}^\star(\mathbb{A}) \to \mathscr{D}(\mathbb{A})$ induces a fully faithful functor*

$$\mathrm{Ch}^\star(\mathbb{A})[\mathrm{Qiso}^{-1}] \to \mathscr{D}(\mathbb{A})$$

which is an equivalence onto the full subcategory in $\mathscr{D}(\mathbb{A})$ spanned by the (images of the) complexes in $\mathrm{Ch}^\star(\mathbb{A})$.

Of course, in the absolute setting we obtain an equivalence

$$\mathrm{Ch}(\mathbb{A})[\mathrm{Qiso}^{-1}] \xrightarrow{\sim} \mathscr{D}(\mathbb{A}).$$

Corollary 6.21 applies to all of the examples discussed in Section 6.7 above. We employ the notations from Examples 6.15–6.19 and recall few of these instances here.

Example 6.22 (The bounded derived category)**.** For $\mathbb{A}$ Grothendieck abelian, the canonical functor

$$\mathrm{Ch}^b(\mathbb{A})[\mathrm{Qiso}^{-1}] \to \mathscr{D}^b(\mathbb{A})$$

is an equivalence.

Example 6.23 (The finite-dimensional derived category)**.** For any algebraic group G, the canonical functor

$$\mathrm{Ch}(G)_{fin}[\mathrm{Qiso}^{-1}] \to \mathscr{D}(G)_{fin}$$

is an equivalence.

Example 6.24 (The perfect derived category)**.** For any Noetherian algebraic stack X with the resolution property, the canonical functor

$$\mathrm{Ch}(X)_{vec}[\mathrm{Qiso}^{-1}] \to \mathscr{D}(X)_{perf}$$

is an equivalence.

7. Left and right derived functors

We define the left and right derived functors

$$\mathrm{L}\,F : \mathscr{D}(\mathbb{A}) \to \mathscr{D}(\mathbb{B}) \quad \text{and} \quad \mathrm{R}\,G : \mathscr{D}(\mathbb{B}) \to \mathscr{D}(\mathbb{A})$$

for pairs of adjoint functors which exist at the level of the homotopy ∞-category. We show, in particular, that the left derived functor can be computed by taking F-acyclic resolutions in the domain. This approach mirrors directly the approach taken in the discrete derived setting.

7.1. **Derived functors in ideal situations.**

Consider Grothendieck abelian $\mathbb{A}$ with enough projectives, and $\mathbb{B}$ Grothendieck abelian. Let

$$\bar{F} : \mathbf{Ch}(\mathbb{A}) \to \mathbf{Ch}(\mathbb{B}) \quad \text{and} \quad \bar{G} : \mathbf{Ch}(\mathbb{B}) \to \mathbf{Ch}(\mathbb{A})$$

be dg functors with $\bar{F}$ left adjoint to $\bar{G}$. For example, we can consider the case where $\bar{F}$ and $\bar{G}$ are induced by adjoint functors between $\mathbb{A}$ and $\mathbb{G}$. In this situation, by Theorem I-13.10, the induced functors on homotopy ∞-categories

$$F : \mathscr{K}(\mathbb{A}) \to \mathscr{K}(\mathbb{B}) \quad \text{and} \quad G : \mathscr{K}(\mathbb{B}) \to \mathscr{K}(\mathbb{A})$$

are such that F is left adjoint to G.

In this situation, we define the left and right derived functors in the expected ways.

Definition 7.1. Let $\mathbb{A}$ and $\mathbb{B}$ be Grothendieck abelian categories. Take $\mathrm{R} : \mathscr{D}(\mathbb{B}) \to \mathscr{K}(\mathbb{B})$ the right adjoint to the localization functor for $\mathbb{B}$.

For any continuous functor $G : \mathscr{K}(\mathbb{B}) \to \mathscr{K}(\mathbb{A})$ we define the right derived functor $\mathrm{R}\,G : \mathscr{D}(\mathbb{B}) \to \mathscr{D}(\mathbb{A})$ as the composite

$$\mathrm{R}\,G : \mathscr{D}(\mathbb{B}) \xrightarrow{\mathrm{R}} \mathscr{K}(\mathbb{B}) \xrightarrow{G} \mathscr{K}(\mathbb{A}) \xrightarrow{\mathrm{loc}} \mathscr{D}(\mathbb{A}).$$

We note that G is continuous whenever it admits a left adjoint $F : \mathscr{D}(\mathbb{A}) \to \mathscr{D}(\mathbb{B})$.

Definition 7.2. Let $\mathbb{A}$ and $\mathbb{B}$ be Grothendieck abelian categories, and suppose the localization functor $\mathscr{K}(\mathbb{A}) \to \mathscr{D}(\mathbb{A})$ has a left adjoint $\mathrm{L} : \mathscr{D}(\mathbb{A}) \to \mathscr{K}(\mathbb{A})$.

For any cocontinuous functor $F : \mathscr{K}(\mathbb{A}) \to \mathscr{K}(\mathbb{B})$ We define the left derived functor $\mathrm{L}\,F : \mathscr{D}(\mathbb{A}) \to \mathscr{D}(\mathbb{B})$ as the composite

$$\mathrm{L}\,F : \mathscr{D}(\mathbb{A}) \xrightarrow{\mathrm{L}} \mathscr{K}(\mathbb{A}) \xrightarrow{F} \mathscr{K}(\mathbb{B}) \xrightarrow{\mathrm{loc}} \mathscr{D}(\mathbb{B}).$$

Proposition 7.3. *Take $\mathbb{A}$ and $\mathbb{B}$ Grothendieck abelian and let $F : \mathscr{K}(\mathbb{A}) \to \mathscr{K}(\mathbb{B})$ be left adjoint to a functor $G : \mathscr{K}(\mathbb{B}) \to \mathscr{K}(\mathbb{A})$. Suppose that the localization functor $\mathscr{K}(\mathbb{A}) \to \mathscr{D}(\mathbb{A})$ admits a left adjoint. Then the functor $\mathrm{L}\,F : \mathscr{K}(\mathbb{A}) \to \mathscr{K}(\mathbb{B})$ is left adjoint to $\mathrm{R}\,G : \mathscr{D}(\mathbb{B}) \to \mathscr{D}(\mathbb{A})$.*

Proof. This just follows from the fact that a composite of left adjoints is left adjoint to the respective composite of right adjoints (Proposition **??**). □

7.2. **The candidate left derived functor.** In the general setting we do not have enough projective in $\mathbb{A}$, so that we cannot define the left derived functor as in the ideal setting discussed above. (One might consider for example the case of sheaves $\mathbb{A} = \mathrm{QCoh}(X)$ on a projective variety X.) However, we can also approach the sitation via relatively acyclic complexes.

Definition 7.4. Let $\mathbb{A}$ and $\mathbb{B}$ be Grothendieck abelian categories, and $F : \mathscr{K}(\mathbb{A}) \to \mathscr{K}(\mathbb{B})$ be any cocontinuous functor. We say $\mathscr{K}(\mathbb{A})$ has enough F-acyclic objects if there is a stable subcategory $\mathscr{K}(\mathbb{A})_{F\text{-ac}}$ in $\mathscr{K}(\mathbb{A})$ which satisfies the following:

(a) $\mathscr{K}(\mathbb{A})_{F\text{-ac}}$ admits small colimits and the inclusion $\mathscr{K}(\mathbb{A})_{F\text{-ac}} \to \mathscr{K}(\mathbb{A})$ is cocontinuous.
(b) Every object V in $\mathscr{K}(\mathbb{A})$ admits a quasi-isomorphism $W \to V$ from an object W in $\mathscr{K}(\mathbb{A})_{F\text{-ac}}$.
(c) If an object W in $\mathscr{K}(\mathbb{A})_{F\text{-ac}}$ is acyclic then the image $F(W)$ in $\mathscr{K}(\mathbb{B})$ is acyclic as well.

The following is more-or-less apparent.

Lemma 7.5. *Let $\mathbb{A}$ and $\mathbb{B}$ be Grothendieck abelian, $F : \mathscr{K}(\mathbb{A}) \to \mathscr{K}(\mathbb{B})$ be a cocontinuous functor, and suppose that $\mathscr{K}(\mathbb{A})$ has enough F-acyclic objects. Let $\mathscr{K}(\mathbb{A})_{F\text{-}ac} \subseteq \mathscr{K}(\mathbb{A})$ be a full subcategory of F-acyclics which satisfies the constraints of Definition 7.4.*

There is a unique functor $\mathrm{L}\,F : \mathscr{D}(\mathbb{A}) \to \mathscr{D}(\mathbb{B})$ which fits into a 2-simplex

$$\begin{tikzcd}
\mathscr{K}(\mathbb{A})_{F\text{-}ac} \arrow[r, "F"] \arrow[d, "\mathrm{loc}"'] \arrow[dr] & \mathscr{K}(\mathbb{B}) \arrow[d, "\mathrm{loc}"] \\
\mathscr{D}(\mathbb{A}) \arrow[r, "\mathrm{L}\,F"'] & \mathscr{D}(\mathbb{B})
\end{tikzcd}$$

in $\mathscr{C}at_\infty$.

Proof. As in all of the examples from Section 6.7, one observes that the sequence

$$\mathscr{K}(\mathbb{A})_{F\text{-ac}} \xrightarrow{\mathrm{incl}} \mathscr{K}(\mathbb{A}) \xrightarrow{\mathrm{loc}} \mathscr{D}(\mathbb{A})$$

identifies the derived ∞-category with the localization $\mathscr{K}(\mathbb{A})_{F\text{-ac}}[\mathrm{Qiso}^{-1}] = \mathscr{D}(\mathbb{A})$. Furthermore, by our assumption that F preserves acyclics in $\mathscr{K}(\mathbb{A})_{F\text{-ac}}$ exactness implies that F preserves quasi-isomorphisms. Hence the restriction functor

$$\mathrm{Fun}(\mathscr{D}(\mathbb{A}), \mathscr{D}(\mathbb{B})) \to \mathrm{Fun}(\mathscr{K}(\mathbb{A})_{F\text{-ac}}, \mathscr{D}(\mathbb{B}))$$

is an equivalence. Existence and uniqueness of the functor $\mathrm{L}\,F$ now follows by contractibility of the homotopy fiber

$$\mathrm{Fun}(\mathscr{D}(\mathbb{A}), \mathscr{D}(\mathbb{B})) \times^{\mathrm{htop}}_{\mathrm{Fun}(\mathscr{K}(\mathbb{A})_{F\text{-ac}}, \mathscr{D}(\mathbb{B}))} \{\mathrm{loc}\, F|_{F\text{-ac}}\}.$$

□

7.3. **Checking adjoints at the homotopy level.**

Proposition 7.6 ([15, 02EY])**.** *Suppose $G : \mathscr{D} \to \mathscr{C}$ is a functor between ∞-categories which admits a left adjoint. Let $F : \mathscr{C} \to \mathscr{D}$ be any functor which we pair with a tranformation $\eta : id_{\mathscr{C}} \to GF$. Then the following are equivalent:*

(1) *The transformation η exhibits F as left adjoint to G.*

(2) *The induced transformation $\mathrm{h}\,\eta : id_{\mathrm{h}\,\mathscr{C}} \to \mathrm{h}\,G\,\mathrm{h}\,F$ exhibits $\mathrm{h}\,F$ as left adjoint to $\mathrm{h}\,G$.*

Proof. Let $F' : \mathscr{C} \to \mathscr{D}$ be left adjoint to G with unit transformation $\eta' : id \to GF$. Then by Proposition 5.20 there is a transformation $\zeta : F' \to F$ with $\eta = (G\zeta)\eta'$, and η is a unit transformation which realizes F as left adjoint to G if and only if ζ is a natural isomorphism. Taking the homotopy categories, we see that $\mathrm{h}\,\zeta : \mathrm{h}\,F' \to \mathrm{h}\,F$ is an isomorphism since $\mathrm{h}\,F$ is left adjoint to $\mathrm{h}\,G$.

We have that ζ is a natural isomorphism if and only if, at each x in $\mathscr{C}$, the map $\zeta_x : F'(x) \to F(x)$ is an isomorphism in $\mathscr{C}$. See Theorem I-6.5. This property can be checked at the level of the homotopy category. So we see that ζ is in fact a natural isomorphism since $\mathrm{h}\,\zeta$ is a natural isomorphism. □

The obvious analog of Proposition 7.6 holds for right adjoints, simply by taking opposites. In the stable setting we can forgo the presupposition that a left adjoint to G exists.

Proposition 7.7. *Let $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ be exact functors between stable ∞-categories, and consider a transformation $\eta : id_{\mathscr{C}} \to GF$. Then the following are equivalent:*

(a) *The transformation η exhibits F as left adjoint to G.*

(b) *The induced transformation $\mathrm{h}\,\eta$ exhibits $\mathrm{h}\,F$ as left adjoint to $\mathrm{h}\,G$.*

Proof. We consider the composite

$$\mathrm{Hom}_{\mathscr{D}}(Fx, y') \overset{G}{\to} \mathrm{Hom}_{\mathscr{C}}(GFx, Gy') \overset{\eta^*}{\to} \mathrm{Hom}_{\mathscr{C}}(x, Gy')$$

in $\mathrm{h}\,\mathscr{K}an_*$. By exactness of G and Proposition 4.23, the fact that the composite

$$\pi_0\,\mathrm{Hom}_{\mathscr{D}}(Fx, y') \overset{G}{\to} \pi_0\,\mathrm{Hom}_{\mathscr{C}}(GFx, Gy') \overset{\eta^*}{\to} \pi_0\,\mathrm{Hom}_{\mathscr{C}}(x, Gy')$$

is an isomorphism at arbitrary y' implies that at all higher homotopy groups, based at 0, the composite

$$\pi_n\,\mathrm{Hom}_{\mathscr{D}}(Fx, y') \overset{G}{\to} \pi_n\,\mathrm{Hom}_{\mathscr{C}}(GFx, Gy') \overset{\eta^*}{\to} \pi_n\,\mathrm{Hom}_{\mathscr{C}}(x, Gy')$$

are isomorphisms. Hence the induced map on loop spaces $\Omega\operatorname{Hom}_{\mathscr{D}}(Fx, y') \to \Omega\operatorname{Hom}_{\mathscr{C}}(x, Gy')$ is an equivalence. We considering the case $y' = \Sigma y$, and apply the isomorphism Proposition 4.23, so see that the composite map $\operatorname{Hom}_{\mathscr{D}}(Fx, y) \to \operatorname{Hom}_{\mathscr{C}}(x, Gy)$ is an equivalence at all objects $x, y : * \to \mathscr{D}$. By Proposition 5.18 it follows that η realizes F as left adjoint to G. □

7.4. **Left adjoints to right derived functors.**

Proposition 7.8. *Let $\mathbb{A}$ and $\mathbb{B}$ be Grothendieck abelian categories and $F : \mathscr{K}(\mathbb{A}) \to \mathscr{K}(\mathbb{B})$ be left adjoint to a functor $G : \mathscr{K}(\mathbb{B}) \to \mathscr{K}(\mathbb{A})$. Suppose that $\mathscr{K}(\mathbb{A})$ has enough F-acyclic objects, and let $\mathrm{L}\,F : \mathscr{D}(\mathbb{A}) \to \mathscr{D}(\mathbb{B})$ be as in the statement of Lemma 7.5. Then $\mathrm{L}\,F$ is left adjoint to the right derived functor $\mathrm{R}\,G : \mathscr{D}(\mathbb{B}) \to \mathscr{D}(\mathbb{A})$.*

Before giving the proof let us consider the situation at the level of homotopy categories. We have the left adjoint $i : \mathrm{D}(\mathbb{B}) \to \mathrm{K}(\mathbb{B})$ to the localization functor, and recall that the composite $i\,\mathrm{loc} : \mathrm{K}(\mathbb{B}) \to \mathrm{K}(\mathbb{B})$, along with the transformation $u_V : V \to i\,\mathrm{loc}(V)$, takes K-injective resolutions of complexes. We now have the commuting diagram

$$\begin{tikzcd}
\mathrm{K}(\mathbb{A})_{F\text{-ac}} \arrow[r, "F"] \arrow[d, "\mathrm{loc}"'] & \mathrm{K}(\mathbb{B}) \arrow[r, "Gi\,\mathrm{loc}"] \arrow[d, "\mathrm{loc}"'] & \mathrm{K}(\mathbb{B}) \arrow[d, "\mathrm{loc}"'] \\
\mathrm{D}(\mathbb{A}) \arrow[r, "\mathrm{L}\,F"'] & \mathrm{D}(\mathbb{B}) \arrow[r, "\mathrm{R}\,G"'] & \mathrm{D}(\mathbb{A})
\end{tikzcd}$$

and the unit transformation $id_{\mathrm{K}(\mathbb{A})} \to GF$ induces a unique transformation

$$\mathrm{loc}\,|_{F\text{-ac}} \to \mathrm{loc}\,GF|_{F\text{-ac}} \xrightarrow{u} \mathrm{loc}\,Gi\,\mathrm{loc}\,F|_{F\text{-ac}} \cong \mathrm{R}\,G\,\mathrm{loc}\,F|_{F\text{-ac}} \cong \mathrm{R}\,G\,\mathrm{L}\,F\,\mathrm{loc}\,|_{F\text{-ac}}.$$

(Here we have omitted the h from the notations, and taken for example $F = \mathrm{h}\,F$, to ease notation.) The above transformation induces a unique transformation for the discrete derived category

$$\eta : id_{\mathrm{D}(\mathbb{A})} \to \mathrm{R}\,G\,\mathrm{L}\,F. \tag{31}$$

Lemma 7.9. *The transformation $\eta : id_{\mathrm{D}(\mathbb{A})} \to \mathrm{R}\,G\,\mathrm{L}\,F$ realizes $\mathrm{L}\,F$ as left adjoint to $\mathrm{R}\,G$ at the level of the discrete derived category.*

Proof. At the discrete level the derived category is realizable via a calculus of fractions, so that the localization functor $\mathrm{loc} : \mathrm{K}(\mathbb{A}) \to \mathrm{D}(\mathbb{A})$ essentially does nothing. Let $u : id_{\mathrm{K}(\mathbb{B})} \to i(= i\,\mathrm{loc})$ denote the unit transformation for the (loc, i)-adjunction and $\bar{\eta} : id_{\mathrm{K}(\mathbb{B})} \to GF$ denote the unit of the (F, G)-adjunction.

For F-acyclic M and K-injective N we have the diagram

$$\begin{tikzcd}
\operatorname{Hom}_{\mathrm{K}(\mathbb{B})}(FM, N) \arrow[r, "i"] & \operatorname{Hom}_{\mathrm{K}(\mathbb{B})}(iFM, iN) \arrow[r, "G"] \arrow[d, "u^*"'] & \operatorname{Hom}_{\mathrm{K}(\mathbb{A})}(GiFM, GiN) \arrow[d, "Gu^*"] \\
 & \operatorname{Hom}_{\mathrm{K}(\mathbb{B})}(FM, iN) \arrow[r, "G"'] & \operatorname{Hom}_{\mathrm{K}(\mathbb{A})}(GFM, GiN) \arrow[d, "\bar{\eta}^*"] \\
 & & \operatorname{Hom}_{\mathrm{K}(\mathbb{A})}(M, GiN)
\end{tikzcd}$$

in which the maps i and u^* are isomorphisms, since N is K-injective. Hence the top sequence is an isomorphism if and only if the composite

$$\mathrm{Hom}_{\mathrm{K}(\mathbb{B})}(FM, iN) \xrightarrow{G} \mathrm{Hom}_{\mathrm{K}(\mathbb{A})}(GFM, GiN) \xrightarrow{\bar{\eta}^*} \mathrm{Hom}_{\mathrm{K}(\mathbb{A})}(M, GiN)$$

is an isomorphism. However, this holds since $\bar{\eta}$ is the unit of the relevant adjunction.

We apply the localization functor to obtain a diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{K}(\mathbb{B})}(FM, N) \arrow[r] \arrow[d, "\mathrm{loc}"', "\cong"] & \mathrm{Hom}_{\mathrm{K}(\mathbb{A})}(GiFM, GiN) \arrow[r, "\bar{\eta}^*"] \arrow[d, "\mathrm{loc}"'] & \mathrm{Hom}_{\mathrm{K}(\mathbb{A})}(M, GiN) \arrow[d, "\mathrm{loc}"'] \\
\mathrm{Hom}_{\mathrm{D}(\mathbb{B})}(\mathrm{L}\,FM, N) \arrow[r] & \mathrm{Hom}_{\mathrm{D}(\mathbb{A})}(\mathrm{R}\,G\,\mathrm{L}\,FM, \mathrm{R}\,GN) \arrow[r, "\eta^*"'] & \mathrm{Hom}_{\mathrm{D}(\mathbb{A})}(M, \mathrm{R}\,GN).
\end{tikzcd}$$

To see that the bottom sequence is an isomorphism it now suffices to prove that the localization functor induces an isomorphism

$$\mathrm{loc} : \mathrm{Hom}_{\mathrm{K}(\mathbb{A})}(M, GN') \to \mathrm{Hom}_{\mathrm{D}(\mathbb{A})}(M, GN') \tag{32}$$

whenever N' is K-injective and M is F-acyclic.

Since $\mathrm{K}(\mathbb{A})$ has enough F-acyclics we can write maps in the derived category as the colimit

$$\mathrm{colim}_\alpha\, \mathrm{Hom}_{\mathrm{K}(\mathbb{A})}(M_\alpha, GN') = \mathrm{Hom}_{\mathrm{D}(\mathbb{A})}(M, GN')$$

over quasi-isomorphisms $\alpha : M_\alpha \to M$ between F-acyclics, and under this identification the localization map

$$\mathrm{loc} : \mathrm{Hom}_{\mathrm{K}(\mathbb{A})}(M, GN') \to \mathrm{colim}_\alpha\, \mathrm{Hom}_{\mathrm{K}(\mathbb{A})}(M_\alpha, GN')$$

is just the structure map at $\alpha = id_M$. But now, via (F, G)-adjunction and F-acyclicity each transition function

$$\alpha^* : \mathrm{Hom}_{\mathrm{K}(\mathbb{A})}(M, GN') \to \mathrm{Hom}_{\mathrm{K}(\mathbb{A})}(M_\alpha, GN')$$

is an isomorphism, so that this colimit diagram is essentially constant. Hence the map (32) is seen to be an isomorphism, and that the composite

$$\mathrm{Hom}_{\mathrm{D}(\mathbb{B})}(FM, N) \xrightarrow{\mathrm{R}\,G} \mathrm{Hom}_{\mathrm{D}(\mathbb{A})}(\mathrm{R}\,G\,\mathrm{L}\,FM, \mathrm{R}\,GN) \xrightarrow{\eta^*} \mathrm{Hom}_{\mathrm{D}(\mathbb{A})}(M, \mathrm{R}\,GN) \tag{33}$$

is an isomorphism at all F-acyclic M and K-injective N. Since we have enough F-acyclics and enough K-injectives, it follows that the sequence (33) is an isomorphism at all M and N, and hence that η realizes the claimed adjunction. $\square$

We now provide the proof of Proposition 7.8.

Proof of Proposition 7.8. Let $\eta_{\mathscr{K}} : id_{\mathscr{K}(\mathbb{A})} \to GF$ denote the unit of the adjunction. We obtain a transformation $\eta' : \Delta^1 \times \mathscr{K}(\mathbb{A}) \to \mathscr{D}(\mathbb{A})$ as the composite

$$\mathrm{loc}\,|_{F\text{-ac}} \xrightarrow{\eta_{\mathscr{K}}} \mathrm{loc}\, GF|_{F\text{-ac}} \xrightarrow{u} \mathrm{loc}\, Gi\, \mathrm{loc}\, F|_{F\text{-ac}} \cong \mathrm{R}\,G\, \mathrm{loc}\, F|_{F\text{-ac}} \cong \mathrm{R}\,G\,\mathrm{L}\,F\,\mathrm{loc}\,|_{F\text{-ac}}, \tag{34}$$

where $i : \mathscr{D}(\mathbb{B}) \to \mathscr{K}(\mathbb{B})$ is the right adjoint to localization. (Recall that the composite $i\,\mathrm{loc}$ takes functorial injective resolutions of objects.) We claim that this transformation induces a unique transformation at the level of derived categories $\eta : id_{\mathscr{D}(\mathbb{A})} \to \mathrm{R}\, G\mathbb{L}F$ which fits into a diagram

$$\begin{tikzcd} \Delta^1 \times \mathscr{K}(\mathbb{A}) \arrow[d, "\mathrm{loc}"'] \arrow[drr, "\eta'"] & & \\ \Delta^1 \times \mathscr{D}(\mathbb{A}) \arrow[rr, "\eta"'] & & \mathscr{D}(\mathbb{A}) \end{tikzcd} \tag{35}$$

in $\mathscr{C}at_\infty$.

Take $\mathscr{D} = \mathscr{D}(\mathbb{A})$ and $\mathscr{K} = \mathscr{K}(\mathbb{A})$. First note that the restriction functor

$$\mathrm{Fun}(\Delta^1 \times \mathscr{D}, \mathscr{D}) \to \mathrm{Fun}(\Delta^1 \times \mathscr{K}, \mathscr{D})^{\mathrm{Qiso}}$$

is an equivalence, so that we can find some functor $\eta'' : \Delta^1 \times \mathscr{D} \to \mathscr{D}$ which fits into a 2-simplex as in (35). On the boundary $\partial\Delta^1 \times \mathscr{D}$ we obtain unique functors F_i (up to a contractible space of choices) which fit into diagrams

$$\begin{tikzcd} & \mathscr{K} \arrow[dl, "\mathrm{loc}"'] \arrow[d, "\mathrm{loc}"] \\ \mathscr{D} & \mathscr{D} \arrow[l, "F_0"] \end{tikzcd} \qquad \begin{tikzcd} \mathscr{K} \arrow[d, "\mathrm{loc}"'] \arrow[dr, "\mathrm{R}\, f_*\mathbb{L}f^*\, \mathrm{loc}"] & \\ \mathscr{D} \arrow[r, "F_1"'] & \mathscr{D} \end{tikzcd} \tag{36}$$

in $\mathscr{C}at_\infty$. Via uniqueness we have $F_0 \cong id_{\mathscr{D}}$ and $F_1 \cong \mathrm{R}\, G\mathbb{L}F$.

Since the restriction functor

$$\mathrm{Fun}(\Delta^1 \times \mathscr{D}, \mathscr{D}) \to \mathrm{Fun}(\partial\Delta^1 \times \mathscr{D}, \mathscr{D})$$

is an isofibration (Corollary I-5.14) we can replace η'' with an isomorphic map $\eta : \Delta^1 \times \mathscr{D} \to \mathscr{D}$ which completes the proposed diagram (35) and has the required restrictions $\eta|_{\{0\}\times\mathscr{D}} = id_{\mathscr{D}}$ and $\eta|_{\{1\}\times\mathscr{D}} = \mathrm{R}\, G\mathbb{L}F$. By considering the isofibration

$$\mathrm{Fun}(\Delta^1 \times \Delta^1 \times \mathscr{K}, \mathscr{D}) \to \mathrm{Fun}(\Delta^1 \times \partial\Delta^1 \times \mathscr{K}, \mathscr{D})$$

we can also assume that the diagram (35) restricts to simplices on the boundary in which the right simplex in (36) is degenerate and the left simplex is given as the composite (34).

Finally, Proposition 7.7 tells us that the transformation $\eta : id_{\mathscr{D}} \to \mathrm{R}\, G\mathbb{L}F$ is the unit of an adjunction between $\mathbb{L}F$ and $\mathrm{R}\, G$ if and only if the induced transformation on the homotopy category exhibits $\mathrm{h}\,\mathbb{L}F$ as left adjoint to $\mathrm{h}\,\mathrm{R}\, G$. However, this was already argued in Lemma 7.9, so that the transformation η is seen to realize $\mathbb{L}F : \mathscr{D}(\mathbb{A}) \to \mathscr{D}(\mathbb{B})$ the left adjoint to $\mathrm{R}\, G$. □

7.5. **Left derived functors via acyclic resolutions.**

Definition 7.10. Let $\mathbb{A}$ and $\mathbb{B}$ be Grothendieck abelian categories. Suppose that an exact functor $F : \mathscr{K}(\mathbb{A}) \to \mathscr{K}(\mathbb{B})$ is left adjoint to an exact functor $G : \mathscr{K}(\mathbb{B}) \to \mathscr{K}(\mathbb{A})$. Suppose also that $\mathscr{K}(\mathbb{A})$ admits enough F-acyclic complexes.

In this setting, the left derived functor $\mathrm{L}\, F : \mathscr{D}(\mathbb{A}) \to \mathscr{D}(\mathbb{B})$ for F is defined as the left adjoint to the right derived functor $\mathrm{R}\, G : \mathscr{D}(\mathbb{B}) \to \mathscr{D}(\mathbb{A})$.

We note that, by Proposition 7.8, such a left adjoint exists. Furthermore, by uniqueness of adjoints, the functor $\mathrm{L}\, F$ can be computed via any sufficiently large collection of F-acyclic complexes $\mathscr{K}(\mathbb{A})_{F\text{-ac}} \subseteq \mathscr{K}(\mathbb{A})$, as in Lemma 7.5.

Example 7.11 (The push-pull adjunction)**.** Let $f : X \to Y$ be a map of quasi-compact and quasi-separated schemes. Then we have the push-pull adjunction on quasi-coherent sheaves, which implies an adjunction between functors

$$f^* : \mathscr{K}(\mathrm{QCoh}(Y)) \to \mathscr{K}(\mathrm{QCoh}(X)) \quad \text{and} \quad f_* : \mathscr{K}(\mathrm{QCoh}(X)) \to \mathscr{K}(\mathrm{QCoh}(Y))$$

at the level of homotopy ∞-categories. This follows by Theorem I-13.10, for example.

The category $\mathscr{K}(\mathrm{QCoh}(Y))$ admits enough K-flat complexes, so that we can define the derived pullback functor

$$\mathrm{L}\, f^* : \mathscr{D}(\mathrm{QCoh}(Y)) \to \mathscr{D}(\mathrm{QCoh}(X)),$$

loosely speaking, by taking K-flat resolutions over Y and applying the underived pullback functor. This is precisely what one expects from standard algebraic practices, as outlined in [23] for example.

This example generalizes to the setting where X and Y are geometric stacks. Here we note that the categories of quasi-coherent sheaves remain Grothendieck abelian [22, Corollary 5.10] and that the categories of complexes of quasi-coherent sheaves admit enough K-flat objects by [7, Theorem 3.5.5].

8. Presentability of the derived ∞-category

We provide a very concise discussion of presentability and the adjoint functor theorem. The main deliverable here is that the derived ∞-category of a Grothendieck abelian category $\mathscr{D}(\mathbb{A})$ is presentable.

Presentability is a kind of lax compact generation condition which places an ∞-category in a functional algebraic context, where the term algebraic here indicates a kind of "algebra of categories". In this context we can also begin to speak of linear categories [16, Section D.1.2], sheaves of categories [16, Section 10.2], etc. One can see for example results from higher algebra [14, section 4.8], which are pre-mediated by earlier results of Kelley in the discrete setting [11, Section 6.5] [4].

Our main ambitions of the section are simply to define presentability and introduce the adjoint functor theorem. The explicit and implicit points appearing in this section are then relevant for our analysis of the indization functor, and renormalization of derived categories as presented in Section 9 below.

8.1. **Filtering and compactness at a regular cardinal.**

By a regular cardinal we mean, formally, a cardinal κ for which satisfies the following: If Λ is an indexing set of cardinality $|\Lambda| < \kappa$, and $\{S_\lambda : \lambda \in \Lambda\}$ is a collection of sets with $|S_\lambda| < \kappa$ at all λ, then the union $\cup_{\lambda\in\Lambda} S_\lambda$ also has cardinality less that κ. For some examples, $\aleph_0$, which characterizes countably infinite sets, is regular. Under the continuum hypothesis, $2^{\aleph_0}$ is also regular, and in general any successor cardinal is regular. Informally, regular cardinals just provide a mechanism through which, in some instances, to impose functional size constraints.

We say a simplicial set K is κ-small, for a regular cardinal κ, if the collection of non-degenerate simplices in $\amalg_{n\geq 0} K[n]$ has cardinality $< \kappa$.

Definition 8.1. An ∞-category $\mathscr{K}$ is called κ-filtered if, for any κ-small simplicial set K and map $i : A \to \mathscr{K}$, there a map $i^+ : A \star \{*\} \to \mathscr{K}$ with $i^+|_A = i$. We call $\mathscr{K}$ filtered if it is $\aleph_0$-filtered.

We note that, via functoriality of the join, all maps $i : A \to \mathscr{K}$ from κ-small K admit an extension as proposed if and only if all injective maps from κ-small K admit such an extension. Hence $\mathscr{K}$ is filtered if and only if each finite simplicial subset in $\mathscr{K}$ admits a cone point in $\mathscr{K}$. In particular, we recover the familiar notion of a filtered category in the discrete setting.

Remark 8.2. Though we won't use the notion here, in [13, Remark 5.3.1.11] Lurie defines a simplicial set S to be κ-filtered if it admits a categorical equivalence $S \to \mathscr{C}$ to a κ-filtered ∞-category $\mathscr{C}$.

We say an ∞-category $\mathscr{C}$ admits κ-filtered colimits if each diagram $p : \mathscr{K} \to \mathscr{C}$ from a κ-filtered ∞-category admits a colimit in $\mathscr{C}$.

Definition 8.3. Let $\mathscr{C}$ be an ∞-category which admits κ-filtered colimits. We call a functor $F : \mathscr{C} \to \mathscr{D}$ κ-cocontinuous if F preserves κ-filtered limits. We call an object x in $\mathscr{C}$ κ-compact if each functor $h^x : \mathscr{C} \to \mathscr{K}an$ which is corepresented by x is κ-cocontinuous. We call x compact if it is $\aleph_0$-compact.

Proposition 8.4. *For an ∞-category $\mathscr{C}$, the following are equivalent:*

(a) *$\mathscr{C}$ is κ-cocomplete.*

(b) *For each κ-filtered partially ordered set A, and diagram $p : A \to \mathscr{C}$, p admits a colimit in $\mathscr{C}$.*

Supposing $\mathscr{C}$ is κ-cocomplete, and $F : \mathscr{C} \to \mathscr{D}$ is any functor, the following are equivalent:

(a′) *F is κ-cocontinuous.*

(b′) *For each κ-filtered partially ordered set A, F preserves A-filtered colimits.*

Outline. [15, 02QA, 02NU]. $\square$

8.2. **Compact generation and presentability of derived categories.**

Definition 8.5. A cocomplete ∞-category $\mathscr{C}$ is called compactly generated if $\mathscr{C}$ is generated under small colimits by an essentially small subcategory of compact objects.

Lemma 8.6. *A locally small stable ∞-category $\mathscr{C}$ is compactly generated if and only if its homotopy category $\mathrm{h}\mathscr{C}$ is compactly generated. More precisely, if $\mathbb{C}_0 \subseteq \mathrm{h}\mathscr{C}$ is a small subcategory which generates $\mathrm{h}\mathscr{C}$, and $\mathscr{C}_0$ is the full ∞-subcategory in $\mathscr{C}$ spanned by the objects of $\mathbb{C}_0$, then*

(1) *$\mathbb{C}_0$ consists of compacts in $\mathrm{h}\mathscr{C}$ if and only if $\mathscr{C}_0$ consists of compacts in $\mathscr{C}$, and*

(2) *$\mathbb{C}_0$ generates $\mathrm{h}\mathscr{C}$ under small coproducts and the formation of mapping cones if and only if $\mathscr{C}_0$ generates $\mathscr{C}$ under small colimits.*

Proof. Let $\mathrm{h}\mathscr{C}_0 \subseteq \mathrm{h}\mathscr{C}$ be any full subcategory, and $\mathscr{C}_0$ be the full subcategory in $\mathscr{C}$ spanned by the objects in $\mathrm{h}\mathscr{C}_0$. Since $\mathscr{C}$ is locally small, we understand that $\mathscr{C}_0$ is essentially small if and only if $\mathrm{h}\mathscr{C}_0$ is essentially small. Also, by Corollary 4.35, $\mathscr{C}_0$ consists entirely of compact objects in $\mathscr{C}$ if and only if $\mathrm{h}\mathscr{C}_0$ consists entirely of compact objects in $\mathrm{h}\mathscr{C}$. Let us suppose that $\mathscr{C}_0$ is in fact essentially small and consists of compact objects.

Take now $\mathrm{h}\,\mathrm{Loc}(\mathscr{C}_0)$ the smallest subcategory in $\mathrm{h}\mathscr{C}$ which is stable under the formation of triangles and small sums, as well as isomorphisms, and $\mathrm{Loc}(\mathscr{C}_0)$ the corresponding lift to a full ∞-subcategory in $\mathscr{C}$. Then $\mathrm{Loc}(\mathscr{C}_0)$ is the smallest full subcategory in $\mathscr{C}$ which is stable under sums, the formation of cofibers, and isomorphisms in $\mathscr{C}$. It follows by Proposition 4.28 that $\mathrm{Loc}(\mathscr{C}_0)$ is the smallest ∞-subcategory in $\mathscr{C}$ which is stable under the formation of small colimits, contains $\mathscr{C}_0$, and is closed under isomorphism. Clearly we have

$$\mathrm{h}\,\mathrm{Loc}(\mathscr{C}_0) = \mathrm{h}\,\mathscr{C} \;\Leftrightarrow\; \mathrm{Loc}(\mathscr{C}_0) = \mathscr{C}.$$

By definition, $\mathrm{h}\,\mathscr{C}$ is compactly generated if we can find such an essentially small subcategory of compacts $\mathscr{C}_0$ with $\mathrm{h}\,\mathrm{Loc}(\mathscr{C}_0) = \mathrm{h}\,\mathscr{C}$. By the information above we also see that $\mathscr{C}$ is compactly generated if and only if we can find such $\mathscr{C}_0$ with $\mathrm{Loc}(\mathscr{C}_0) = \mathscr{C}$. So we observe that $\mathrm{h}\,\mathscr{C}$ is compactly generated if and only if $\mathscr{C}$ is compactly generated. □

We apply Lemma 8.6 to observe a number of familiar examples.

Example 8.7. Let R be any ring and take $\mathscr{D}(R) = \mathscr{D}(R\text{-Mod})$. All bounded complexes of projectives are compact in the discrete derived category $\mathrm{D}(R)$, and $\mathrm{D}(R)$ is generated by this subcategory under small sums and the formation of cones. Hence $\mathrm{D}(R)$ is compactly generated, and we conclude that $\mathscr{D}(R)$ is compactly generated.

Example 8.8. If X is a quasi-projective scheme, or more generally quasi-compact and separated, then the discrete derived category $\mathrm{D}(X) = \mathrm{D}(\mathrm{QCoh}(X))$ is compactly generated [3, Theorems 3.1.1, 3.1.3]. It is, in particular, generated by the full subcategory $\mathrm{D}(X)_{perf}$ of perfect sheaves. It follows that the derived ∞-category $\mathscr{D}(X)$ is generated by the essentially small subcategory of compacts $\mathscr{D}(X)_{perf}$, and in particular is compactly generated.

We consider the following generalization of compact generation via κ-filtered simplicial sets and κ-compact objects.

Definition 8.9 ([13, Theorem 5.5.1.1])**.** An ∞-category $\mathscr{C}$ is called presentable if it satisfies the following:

(a) $\mathscr{C}$ is cocomplete.

(b) For some fixed regular cardinal κ, $\mathscr{C}$ is generated under colimits by a small, finitely cocomplete subcategory $\mathscr{C}_0$ of κ-small objects.

In "tame" situations, the derived category $\mathscr{D}(\mathbb{A})$ is simply compactly generated. However, there are very reasonable settings where this is not the case. For example, if we take $\mathbb{A} = \mathrm{Rep}\,G$ for a reductive algebraic group G in finite characteristic, then the discrete derived category $\mathrm{D}(\mathrm{Rep}\,G)$ is not compactly generated [8, Theorem 1.1] and so the derived ∞-category $\mathscr{D}(\mathrm{Rep}\,G)$ is not compactly generated either. We do, however, have the following consolation prize.

Theorem 8.10 ([14, Proposition 1.3.5.21])**.** *For any Grothendieck abelian category $\mathbb{A}$, the derived category $\mathscr{D}(\mathbb{A})$ is presentable.*

8.3. **Presentability and the adjoint functor theorem.** As an example application of presentability, we record below the adjoint functor theorem.

Definition 8.11. Let $\mathscr{C}$ be a cocomplete ∞-category. A functor $F : \mathscr{C} \to \mathscr{D}$ is called accessible if there exists a regular cardinal for which F preserves all κ-filtered colimits.

Remark 8.12. One generally defines accessibility of a functor $F : \mathscr{C} \to \mathscr{D}$ in the situation where $\mathscr{C}$ is only accessible [13, Definition 5.4.2.5], not necessarily cocomplete. As all of our categories in this text are cocomplete, and we don't want to introduce this additional notion, we stick with the more restrictive setting of Definition 8.11.

The easiest way for a functor to be accessible if for it to be cocomplete. We have the following fundamental characterization of left and right adjoints.

Theorem 8.13 (Adjoint functor theorem, [13, Corollary 5.5.2.9])**.** *Let $F : \mathscr{C} \to \mathscr{D}$ be a functor between presentable categories. The following hold:*

(1) *F admits a right adjoint if and only if it is cocontinuous.*
(2) *F admits a left adjoint if and only if it is accessible and continuous.*

8.4. **Derived functors and the adjoint functor theorem.** As an immediate consequence of Theorem 8.13 and Proposition 7.8 we obtain the following.

Corollary 8.14. *In the situation of Lemma 7.5, the left derived functor $\mathrm{L}\,F : \mathscr{D}(\mathbb{A}) \to \mathscr{D}(\mathbb{B})$ is cocontinuous and the right derived functor $\mathrm{R}\,G : \mathscr{D}(\mathbb{B}) \to \mathscr{D}(\mathbb{A})$ is accessible.*

It is relatively easy to check that the functor $\mathrm{L}\,F$ is cocontinuous by hand. Accessibility of the right derived functor $\mathrm{R}\,G$ is not clear to us in general, however.

Exercise 8.15. Prove accessibility of the right derived functor $\mathrm{R}\,G$ without reference to Proposition 7.8.

9. Ind-completion and renormalized derived categories

In concluding, we discuss the ind-completion functor $\mathrm{Ind} : \mathscr{C}at_\infty^{\mathrm{sm}} \to \mathscr{C}at_\infty$ and its relatively recent appearances in studies of geometry and representation theory. The main point here is fairly benign; Namely, ind-completion provides an alternate means of "compactifying" small derived categories of interest. This compactifying process produces, from any essentially small ∞-category $\mathscr{C}$ with all finite colimits, a tautological cocompletion $\mathrm{Ind}(\mathscr{C})$ of $\mathscr{C}$ which is presentable and recovers $\mathscr{C}$ as its subcategory of compact objects.

Such alternate compactifications, or renormalizations of derived categories, have actually appeared constantly in the discrete derived setting, though we've not recognized them as such. For example, Koszul duality provides an equivalence between the discrete derived category $\mathrm{D}(\Lambda)_{fin}$ of finite-dimensional representations over the exterior algebra and the derived category $\mathrm{D}(S)_{coh}$ of coherent dg modules over the polynomial ring. This equivalence, however, does not extend to the unbounded setting as there is a disagreement between the compact objects in $\mathrm{D}(\Lambda)$ and in $\mathrm{D}(S)$. Instead, one sees clearly from the indization perspective that Koszul duality calculates the ind-completion of $\mathscr{D}(\Lambda)_{fin}$ as the unbounded category of dg modules over S,

$$\mathrm{Ind}\,\mathscr{D}(\Lambda)_{fin} \cong \mathscr{D}(S).$$

This example is discussed, along with a few others, in the final subsection below.

9.1. Ind-completion of small ∞-categories.

Notation 9.1. For ∞-categories $\mathscr{A}$ and $\mathscr{B}$ which admit small ($\aleph_0$-)filtered colimits, we let $\mathrm{Fun}^{\aleph_0}(\mathscr{A},\mathscr{B})$ denote the full ∞-subcategory of functors in $\mathrm{Fun}(\mathscr{A},\mathscr{B})$ which preserve small filtered colimits.

For $\mathscr{A}$ admitting small filtered colimits, we recall that an object $a : * \to \mathscr{A}$ compact if any functor $h^a : \mathscr{A} \to \mathscr{K}an$ which is corepresented by a commutes with all small filtered colimits.

Theorem 9.2 ([13, Corollary 5.3.5.4 & Proposition 5.3.5.11])**.** *For any essentially small ∞-category $\mathscr{C}$, there is an ∞-category $\mathscr{C}'$ and a functor $i : \mathscr{C} \to \mathscr{C}'$ which has the following properties:*

(1) *$\mathscr{C}'$ admits all small filtered colimits.*
(2) *i is fully faithful, and for each x in $\mathscr{C}$ the image $F(x)$ is compact in $\mathscr{C}'$.*
(3) *Every object z in $\mathscr{C}'$ admits a filtered diagram $p : \mathscr{K} \to \mathscr{C}$ for which z is a colimit of the corresponding diagram $ip : \mathscr{K} \to \mathscr{C}'$.*

Furthermore, the category $\mathscr{C}'$ is uniquely determined, as an object in the undercategory $(\mathscr{C}at_\infty)_{\mathscr{C}/}$, up to equivalence.

Let us outline how this theorem works, according to the fundamentals laid out in [13].

Outline proof. Define the full subcategory $\mathscr{C}' = \mathrm{Ind}(\mathscr{C}) \subseteq \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathscr{K}an)$ spanned by those functors $\xi : \mathscr{C}^{\mathrm{op}} \to \mathscr{K}an$ which classify a cartesian fibration $\mathscr{E} \to \mathscr{C}$ for which $\mathscr{E}$ is filtered as an ∞-category. We note that all representable functors lie in $\mathrm{Ind}(\mathscr{C})$ since each fibration $\mathscr{C}_{/x} \to \mathscr{C}$, for x in $\mathscr{C}$, has terminal object $id_x : x \to x$ (Proposition II-9.11). Hence the Yoneda embedding $i : \mathscr{C} \to \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathscr{K}an)$ has image in $\mathrm{Ind}(\mathscr{C})$. Furthermore, i is fully faithful since the Yoneda embedding is fully faithful, by Theorem II-16.1.

The fact that $\mathrm{Ind}(\mathscr{C})$ is stable under taking filtered limits in $\mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathscr{K}an)$ is covered in [13, Proposition 5.3.5.3], and the fact that $\mathrm{Ind}(\mathscr{C})$ is generated by the image of $\mathscr{C}$ under small filtered limits is covered in [13, Corollary 5.3.5.4]. Compactness of the image $i(x)$ for each x in $\mathscr{C}$ is covered in [13, Proposition 5.3.5.5].

As for the uniqueness claim, for any ∞-category $\mathscr{A}$ which admits small filtered limits, and any functor $f : \mathscr{C} \to \mathscr{A}$ with compact image, f admits a unique extension to a $\aleph_0$-cocontinuous functor $F : \mathrm{Ind}(\mathscr{C}) \to \mathscr{A}$ which is obtained via left Kan extension [13, Lemma 5.3.5.8]. This functor is an equivalence if and only if f is fully faithful and $\mathscr{A}$ is generated by the image of $\mathscr{C}$ under filtered limits [13, Proposition 5.3.5.11]. □

Remark 9.3. We note that essential smallness of $\mathscr{C}$ is required to ensure the existence of the Kan extension $F : \mathrm{Ind}(\mathscr{C}) \to \mathscr{A}$ employed in the proof of Theorem 9.2.

Definition 9.4. Given an essentially small ∞-category $\mathscr{C}$, any $\aleph_0$-cocomplete ∞-category $\mathscr{C}'$ equipped with a fully faithful functor $i : \mathscr{C} \to \mathscr{C}'$ as in Theorem 9.2 is called an ind-completion of $\mathscr{C}$.

We have the following universal property for the ind-completion.

Theorem 9.5 ([13, Proposition 5.3.5.10])**.** *Let $\mathscr{C}$ be a small ∞-category and $i : \mathscr{C} \to \mathscr{C}'$ be any ind-completion of $\mathscr{C}$. Then for any ∞-category $\mathscr{A}$ which admits small filtered colimits, restriction along i provides an equivalence*

$$i^* : \mathrm{Fun}^{\aleph_0}(\mathscr{C}', \mathscr{A}) \xrightarrow{\sim} \mathrm{Fun}(\mathscr{C}, \mathscr{A}).$$

We can be more precise about our uniqueness claim from Theorem 9.2.

Theorem 9.6 ([13, Proposition 5.3.5.11])**.** *Let $\mathscr{A}$ be an ∞-category which admits all small filtered colimits, and $f : \mathscr{C} \to \mathscr{A}$ be a functor from an essentially small ∞-category. Consider any ind-completion $i : \mathscr{C} \to \mathscr{C}'$ and the unique extension $F : \mathscr{C}' \to \mathscr{A}$ of f to an $\aleph_0$-cocontinuous map from $\mathscr{C}'$. Then*

1. *F is fully faithful provided f is fully faithful and has compact image.*
2. *F is an equivalence if and only if f is fully faithful with compact image, and $\mathscr{A}$ is generated by the image of $\mathscr{C}$ under small filtered colimits.*

We consider some examples. Below we take for granted that, for any ring R, the compact objects in the discrete derived category $\mathrm{D}(R) = \mathscr{D}(R\text{-mod})$ are precisely those objects which are quasi-isomorphic to a bounded complex of projectives, and for any quasi-compact quasi-separated scheme X the compacts in the discrete derived category $\mathrm{D}(X) = \mathscr{D}(\mathrm{QCoh}(X))$ are the perfect complexes, i.e. those complexes which are isomorphic over any affine open to a bounded complex of locally free sheaves.

Example 9.7 (Ind-finite representations I)**.** Let R be a finite-dimensional algebra and $\mathscr{D}(R)$ be the derived ∞-category. Let $\mathscr{D}(R)_{fin}$ be the full subcategory of complexes with finite-dimensional cohomology. As $\mathscr{D}(R)$ is cocomplete it admits all small filtered colimits, and by Corollary 4.35 the compact objects in $\mathscr{D}(R)$ are precisely those complexes of finite projective dimension, i.e. which are quasi-isomorphic to bounded complexes of projectives. Hence, if R is of finite global dimension, then all objects in $\mathscr{D}(R)_{fin}$ are compact in $\mathscr{D}(R)$ and the inclusion $\mathscr{D}(R)_{fin} \to \mathscr{D}(R)$ identifies $\mathscr{D}(R)$ as an ind-completion of $\mathscr{D}(R)_{fin}$.

Example 9.8 (Ind-coherent sheaves I)**.** Let X be a complex variety with a singular point $x : \mathrm{Spec}(\mathbb{C}) \to X$. Then the residue field $k(x)$ is a coherent sheaf on X which is non-compact in the derived ∞-category $\mathscr{D}(X) = \mathscr{D}(\mathrm{QCoh}(X))$. Hence, for $\mathscr{D}(X)_{coh}$ the full subcategory of complexes with coherent cohomology, the inclusion $\mathscr{D}(X)_{coh} \to \mathscr{D}(X)$ does *not* identify the derived ∞-category as an ind-completion of $\mathscr{D}(X)_{coh}$.

We note, however, that in this case we can still consider the ind-completion $\mathrm{Ind}\,\mathscr{D}(X)_{coh}$ and we have the unique extension $F : \mathrm{Ind}\,\mathscr{D}(X)_{coh} \to \mathscr{D}(X)$ of the inclusion. To say that $\mathscr{D}(X)$ is not an ind-completion of $\mathscr{D}(X)$ is, more precisely, to say that the functor F is not an equivalence.

Example 9.9 (Ind-coherent sheaves II)**.** For X a smooth Noetherian scheme, $\mathscr{D}(X)_{perf} = \mathscr{D}(X)_{coh}$, and $\mathscr{D}(X)$ is an ind-completion of $\mathscr{D}(X)_{coh}$.

Example 9.10 (Ind-finite representations II)**.** Let G be a finite group with p dividing the order of G, for some prime p. Let k be a field of characteristic p. Then the trivial representation k is not compact in $\mathscr{D}(G) = \mathscr{D}(\mathrm{Rep}_k(G))$, and the unique extension

$$F : \mathrm{Ind}\,\mathscr{D}(G)_{fin} \to \mathscr{D}(G)$$

of the inclusion $\mathscr{D}(G)_{fin} \to \mathscr{D}(G)$ is not an equivalence.

9.2. **Ind-completion as a functor.** Let $\mathscr{C}at_\infty^{\aleph_0}$ denote the non-full ∞-subcategory whose objects are ∞-categories which admit all small filtered colimits, and whose maps are those functors $F : \mathscr{A} \to \mathscr{B}$ which commute with small filtered colimits. Consider the non-full subcategory

$$\mathscr{M}_{\mathrm{Ind}} \subseteq \mathscr{C}at_\infty \times \Delta^1$$

whose objects are the union

$$\mathscr{M}_{\mathrm{Ind}}[0] = (\mathscr{C}at_\infty^{\mathrm{sm}}[0] \times \{0\}) \cup \left(\mathscr{C}at_\infty^{\aleph_0}[0] \times \{1\}\right).$$

The morphisms (edges) over $\{0\}$ and over $0 < 1$, under the projection $\mathscr{C}at_\infty \times \Delta^1 \to \Delta^1$, are arbitrary maps between essentially small ∞-categories and from essentially small ∞-categories to $\aleph_0$-cocomplete categories. The morphisms over 1 are $\aleph_0$-cocontinuous functors. The n-simplices $\sigma : \Delta^n \to \mathscr{M}_{\mathrm{Ind}}$ are precisely those n-simplices in $\mathscr{C}at_\infty \times \Delta^1$ whose vertices $\sigma|_{\Delta^{\{i\}}}$ and edges $\sigma|_{\Delta^{\{i,i+1\}}}$ satisfy the above restrictions.

One sees that $\mathscr{M}_{\mathrm{Ind}}$ is an ∞-category since composites of κ-cocomplete functors are κ-cocomplete. The fibers over Δ^1 under the projection $q : \mathscr{M}_{\mathrm{Ind}} \to \Delta^1$ are precisely

$$(\mathscr{M}_{\mathrm{Ind}})_0 = \mathscr{C}at_\infty^{\mathrm{sm}} \quad \text{and} \quad (\mathscr{M}_{\mathrm{Ind}})_1 = \mathscr{C}at_\infty^{\aleph_0}.$$

One sees that this projection is an inner fibration, as it is a composite of the inner fibrations provided by the inclusion $\mathscr{M}_{\mathrm{Ind}} \to \mathscr{C}at_\infty \times \Delta^1$ and the projection $\mathscr{C}at_\infty \times \Delta^1 \to \Delta^1$.

Lemma 9.11. *The projection $q : \mathscr{M}_{\mathrm{Ind}} \to \Delta^1$ is a cocartesian fibration, and for any ∞-category $\mathscr{C}$ over 0, i.e. any essentially small ∞-category, an edge $i : \mathscr{C} \to \mathscr{C}'$ over $0 < 1$ is q-cocartesian if and only if i realizes $\mathscr{C}'$ as an ind-completion of $\mathscr{C}$.*

Proof. The category $\mathscr{C}at_\infty \times \Delta^1$ is the homotopy coherent nerve of the simplicial category $\underline{\mathrm{Cat}}_\infty^+ \times \{0 < 1\}$, and $\mathscr{M}_{\mathrm{Ind}}$ is the homotopy coherent nerve of the simplicial uncategory $\underline{\mathrm{M}}$ with prescribed objects and morphisms given by

$$\underline{\mathrm{Hom}}_{\underline{\mathrm{M}}}(\mathscr{C}, \mathscr{A}) = \begin{cases} \mathrm{Fun}(\mathscr{C}, \mathscr{A}) & \text{if } \mathscr{C} \text{ lies over } 0 \\ \mathrm{Fun}^{\aleph_0}(\mathscr{C}, \mathscr{A}) & \text{if } \mathscr{C} \text{ and } \mathscr{A} \text{ lie over } 1 \\ \emptyset & \text{otherwise.} \end{cases}$$

We therefore identify the $\mathrm{h}\mathscr{K}an$-enriched category $\pi\mathscr{M}_{\mathrm{Ind}}$ with the category $\pi\underline{\mathrm{M}}$ obtained by applying the symmetric monoidal functor $\pi : \mathscr{K}an \to \mathrm{h}\mathscr{K}an$ to the morphisms, by Proposition II-7.6. By Theorem 9.5 we therefore conclude that the subcategory

$$\mathscr{C}at_\infty^{\aleph_0} = (\mathscr{M}_{\mathrm{Ind}})_1 \ \subseteq \ \mathscr{M}_{\mathrm{Ind}}$$

is reflective in $\mathscr{M}_{\mathrm{Ind}}$. By Lemma 5.13 it follows that the projection $q : \mathscr{M}_{\mathrm{Ind}} \to \Delta^1$ is a cocartesian fibration, and a map $i : \mathscr{C} \to \mathscr{C}'$ is a q-cocartesian edge in $\mathscr{M}_{\mathrm{Ind}}$, over $0 < 1$, if and only if i realizes $\mathscr{C}'$ as an ind-completion of $\mathscr{C}$. $\square$

We now have the unique cocartesian transformation $I : \Delta^1 \times \mathscr{C}at_\infty^{\mathrm{sm}} \to \mathscr{M}_{\mathrm{Ind}}$ which solves the lifting problem

$$\begin{tikzcd} \{0\} \times \mathscr{C}at_\infty^{\mathrm{sm}} \arrow[r] \arrow[d] & \mathscr{M}_{\mathrm{Ind}} \arrow[d, "q"] \\ \Delta^1 \times \mathscr{C}at_\infty^{\mathrm{sm}} \arrow[r] \arrow[ur, dotted, "I"] & \Delta^1, \end{tikzcd} \tag{37}$$

i.e. the unique functor which solves the lifting problem and evaluates to a cocartesian functor in $\mathscr{M}_{\mathrm{Ind}}$ at each edge $\Delta^1 \times \{\mathscr{C}\}$. By Lemma 9.11 this cocartesian edge is an ind-completion $i : \mathscr{C} \to \mathscr{C}'$ of $\mathscr{C}$.

We now consider the other projection

$$p : \mathscr{M}_{\mathrm{Ind}} \subseteq \mathscr{C}at_\infty \times \Delta^1 \to \mathscr{C}at_\infty$$

and the composition $\widetilde{\mathrm{Ind}} := pI : \Delta^1 \times \mathscr{C}at_\infty^{\mathrm{sm}} \to \mathscr{C}at_\infty$.

Definition 9.12. The indization functor is the pairing $(\mathrm{Ind}, \widetilde{\mathrm{Ind}})$ of the restriction

$$\mathrm{Ind} := \widetilde{\mathrm{Ind}}|_{\{1\}\times \mathscr{C}at_\infty^{\mathrm{sm}}} : \mathscr{C}at_\infty^{\mathrm{sm}} \to \mathscr{C}at_\infty^{\aleph_0}$$

along with the transformation $\widetilde{\mathrm{Ind}} : \Delta^1 \times \mathscr{C}at_\infty^{\mathrm{sm}} \to \mathscr{C}at_\infty$ constructed above.

We note that the cocartesian solution from (37) is unique up to a contractible space of choices, by Theorem II-2.7. Hence the indization functor is determined up to a contractible space of choices.

Remark 9.13. By an abuse of notation we often refer to the functor $\mathrm{Ind} : \mathscr{C}at_\infty^{\mathrm{sm}} \to \mathscr{C}at_\infty$ itself as the indization functor. We also generally let i denote the structural transformation $i = \widetilde{\mathrm{Ind}}$ in order to ease notation.

Remark 9.14. Despite the apparent differences in the constructions, our indization functor is the same as the one introduced in [13, Proposition 5.4.2.19]. This follows by Proposition A.21 below.

To be very clear, the functor $\widetilde{\mathrm{Ind}}$ picks, at each object $(1, \mathscr{C})$ in $\Delta^1 \times \mathscr{C}at_\infty^{\mathrm{sm}}$, an ind-completion $\mathrm{Ind}(\mathscr{C})$ and $\widetilde{\mathrm{Ind}}$ itself is a tranformation between the inclusion $\mathscr{C}at_\infty^{\mathrm{sm}} \to \mathscr{C}at_\infty$ and the functor $\mathrm{Ind} : \mathscr{C}at_\infty^{\mathrm{sm}} \to \mathscr{C}at_\infty^{\aleph_0} \subseteq \mathscr{C}at_\infty$ which evaluates at each essentially small ∞-category $\mathscr{C}$ to a functor $i : \mathscr{C} \to \mathrm{Ind}(\mathscr{C})$ which specifically witnesses $\mathrm{Ind}(\mathscr{C})$ as an ind-completion of $\mathscr{C}$.

9.3. **Ind-completion and products.**

Proposition 9.15. *The indization functor* $\mathrm{Ind} : \mathscr{C}at_\infty^{\mathrm{sm}} \to \mathscr{C}at_\infty$ *commutes with arbitrary products.*

In the proof we employ the notations and constructions introduced in Appendix A.

Proof. By [13, Proposition 5.4.7.3] the category $\mathrm{Acc}_{\aleph_0}$ admits all small limits and the inclusion $\mathrm{Acc}_{\aleph_0} \to \mathscr{C}at_\infty$ is continuous. Since the restriction to idempotent complete categories

$$\mathrm{Ind} : (\mathscr{C}at_\infty^{\mathrm{sm}})_{\mathrm{split}} \overset{\sim}{\to} \mathrm{Acc}_{\aleph_0}$$

is an equivalence by Corollary A.20, we conclude that the category of idempotent split ∞-categories is complete as well and that indization from $(\mathscr{C}at_\infty^{\mathrm{sm}})_{\mathrm{split}}$. From the factorization of Proposition A.21 it suffices to provide that the idempotent completion functor $(-)^\vee : \mathscr{C}at_\infty^{\mathrm{sm}} \to (\mathscr{C}at_\infty^{\mathrm{sm}})_{\mathrm{split}}$ preserves products.

For any simplicial set K and small collection of categories $\{\mathscr{C}_\lambda : \lambda \in \Lambda\}$ we have the natural isomorphism

$$\mathrm{Fun}(K, \prod_\lambda \mathscr{C}_\lambda) \overset{\sim}{\to} \prod_\lambda \mathrm{Fun}(K, \mathscr{C}_\lambda)$$

so that, by the characterization of idempotent complete categories provided in Proposition A.5, $(\mathscr{C}at_\infty^{\mathrm{sm}})_{\mathrm{split}}$ is stable under small products in $\mathscr{C}at_\infty$, and hence products in the complete category $(\mathscr{C}at_\infty^{\mathrm{sm}})_{\mathrm{split}}$ are the usual cartesian product.

We consider a small collection $\{\mathscr{C}_\lambda : \lambda \in \Lambda\}$ of arbitrary essentially small ∞-categories and consider the idempotent completions $i_\lambda : \mathscr{C}_\lambda \to \mathscr{C}_\lambda^\vee$. Consider also the idempotent completion of the product $i : \prod_\lambda \mathscr{C}_\lambda \to (\prod_\lambda \mathscr{C}_\lambda)^\vee$. Via naturality of the transformation $i : id_{\mathscr{C}at_\infty} \to (-)^\vee$, each projection $p_\mu^\vee : (\prod_\lambda \mathscr{C}_\lambda)^\vee \to \mathscr{C}_\mu^\vee$ fits into a diagram

$$\begin{tikzcd}
(\prod_\lambda \mathscr{C}_\lambda)^\vee \arrow[r, "p_\mu^\vee"] & \mathscr{C}_\mu^\vee \\
\prod_\lambda \mathscr{C}_\lambda \arrow[u, "i"] \arrow[r, "p_\mu"'] & \mathscr{C}_\mu^\vee. \arrow[u, "i_\mu"']
\end{tikzcd}$$

Hence the induced map to the product fits into a diagram

$$\begin{tikzcd}
(\prod_\lambda \mathscr{C}_\lambda)^\vee \arrow[rr, "\mathrm{comp}"] & & \prod_\lambda \mathscr{C}_\mu^\vee \\
& \prod_\lambda \mathscr{C}_\lambda. \arrow[ul, "i"] \arrow[ur, "\prod i_\lambda"'] &
\end{tikzcd}$$

To show that this comparison map is an equivalence it suffices to show that the product $\prod_\lambda \mathscr{C}_\lambda^\vee$ is an idempotent completion of $\prod_\lambda \mathscr{C}_\lambda$.

For this final point, since each functor $i_\lambda : \mathscr{C} \to \mathscr{C}_\lambda$ is fully faithful, the product map $\prod_\lambda i_\lambda$ is also fully faithful. Furthermore, for a tuple of objects $y = (y_\lambda : \lambda \in \Lambda)$ in $\prod_\lambda \mathscr{C}_\lambda^\vee$ any collection of retract diagrams $r_\lambda : \mathrm{Ret} \to \mathscr{C}_\lambda^\vee$ which realize each y_λ as a retract of some x_λ in $\mathscr{C}_\lambda$, the uniquely associated diagram

$$r = [r_\lambda : \lambda \in \Lambda]^t : \mathrm{Ret} \to \prod_\lambda \mathscr{C}_\lambda^\vee$$

expresses y as a retract of the object $(x_\lambda : \lambda \in \Lambda)$ in $\prod_\lambda \mathscr{C}_\lambda$. Therefore, by definition, the product map

$$\prod_\lambda i_\lambda : \prod_\lambda \mathscr{C}_\lambda \to \prod_\lambda \mathscr{C}_\lambda^\vee$$

exhibits the target category as an idempotent completion of the product $\prod_\lambda \mathscr{C}_\lambda$. as desired. $\square$

Remark 9.16. This preservation of products is important when considering the transfer of monoidal structures, or actions, through the ind-completion functor. See for example [14, Corollary 2.4.1.8].

9.4. Ind-completion in the $\aleph_0$-cocomplete and stable settings.

Theorem 9.17. *Consider an essentially small ∞-category $\mathscr{C}$. If $\mathscr{C}$ admits all finite colimits then the following hold:*

(1) *The ind-completion* $\mathrm{Ind}(\mathscr{C})$ *is presentable and the structure map* $\mathscr{C} \to \mathrm{Ind}(\mathscr{C})$ *preserves finite colimits.*

(2) *For any cocomplete ∞-category, and functor $\mathscr{C} \to \mathscr{A}$ which commutes with finite colimits, the induced functor* $\mathrm{Ind}(\mathscr{C}) \to \mathscr{A}$ *commutes with all small colimits.*

We first offer a supporting lemma regarding idempotent completion. (See Appendix A.)

Lemma 9.18. *If $i : \mathscr{C} \to \mathscr{D}$ is an idempotent completion in which $\mathscr{C}$ and $\mathscr{D}$ both admit κ-small limits (resp. colimits), then i commutes with κ-small limits (resp. colimits).*

Proof. We argue the point for limits. The case of colimits follows by taking opposites.

By identifying $\mathscr{C}$ with its essential image in $\mathscr{C}^\vee$ we may assume that i is an inclusion of simplicial sets. For any diagram $p : K \to \mathscr{C}$ the corresponding inclusion $\mathscr{C}_{/p} \to \mathscr{D}_{/p}$ is an idempotent completion of $\mathscr{C}_{/p}$ by [13, Lemma 5.1.4.4]. So it suffices to prove that idempotent completion preserves terminal objects. So, let us suppose that an object z is terminal in $\mathscr{C}$. For any y in $\mathscr{D}$ express y as a retract $y \to x \to y$ of an object x in $\mathscr{C}$ to obtain a retract diagram

$$\begin{tikzcd} & \mathrm{Hom}_{\mathscr{C}}(x,z) \arrow[dr] & \\ \mathrm{Hom}_{\mathscr{D}}(y,z) \arrow[ur] \arrow[rr, "id"'] & & \mathrm{Hom}_{\mathscr{D}}(y,z) \end{tikzcd}$$

in $\mathrm{h}\mathscr{K}an$. Contractibility of $\mathrm{Hom}_{\mathscr{C}}(x,z)$ now implies contractibility of $\mathrm{Hom}_{\mathscr{D}}(y,z)$. Since y was chosen arbitrarily we see that, by definition, z is terminal in $\mathscr{D}$. $\square$

As we explain in Remark 9.19 below, it is in fact the case that the idempotent completion $\mathscr{C}^\vee$ admits κ-small limits (resp. colimits) whenever $\mathscr{C}$ admits such limits. So the proper definitive statement is that any idempotent completion admits κ-small limits (colimits) whenever $\mathscr{C}$ does, and that the structure map $i : \mathscr{C} \to \mathscr{D}$ always respects such limits (colimits). In any case, we can now record the proof of Theorem 9.17.

Proof of Theorem 9.17. (1) By [13, Theorem 5.5.1.1] the category $\mathrm{Ind}(\mathscr{C})$ is presentable and by [13, Corollary 5.3.4.15] the subcategory of compacts $\mathrm{Ind}(\mathscr{C})^c$ is stable under finite colimits in $\mathrm{Ind}(\mathscr{C})$. By Lemma A.16 the fully faithful functor $\mathscr{C} \to \mathrm{Idem}(\mathscr{C})^c$ exhibits $\mathrm{Idem}(\mathscr{C})^c$ as an idempotent completion of $\mathscr{C}$. By Lemma 9.18 the map $\mathscr{C} \to \mathrm{Idem}(\mathscr{C})$ preserves finite colimits, so that the composite

$$\mathscr{C} \to \mathrm{Ind}(\mathscr{C})^c \subseteq \mathrm{Ind}(\mathscr{C})$$

preserves finite colimits as well. Claim (2) follows by [13, Proposition 5.5.1.9]. $\square$

Remark 9.19. Since $\mathrm{Ind}(\mathscr{C})^c$ is an idempotent completion of $\mathscr{C}$, one can employ [13, Corollary 5.3.4.15] and Lemma 9.18 to see that the idempotent completion $\mathscr{C}^\vee$ of any finitely cocomplete ∞-category $\mathscr{C}$ admits all finite colimits, and the structure map $i : \mathscr{C} \to \mathscr{C}^\vee$ commutes with finite colimits. One can similarly argue, by replacing $\mathrm{Ind} = \mathrm{Ind}_{\aleph_0}$ with Ind_κ at a generic regular cardinal, that the idempotent completion $\mathscr{C}^\vee$ admits κ-small colimits whenever $\mathscr{C}$ admits κ-small colimits and that the structure map $i : \mathscr{C} \to \mathscr{C}^\vee$ respects such colimits. By taking opposite categories the analogous results are seen to hold for ($\kappa-$)small limits as well.

Remark 9.20. As in the accessible situation, represented in Corollary A.20, one observes that the indization functor Ind provides an equivalence between the category of idempotent complete, essentially small, and finitely cocomplete ∞-categories and a certain category $\mathrm{Pr}_{\aleph_0}$ of compactly generated presentable ∞-categories. See [14, Lemma 5.3.2.9].

As one expects, stability is preserved under ind-completion as well.

Proposition 9.21 ([14, Proposition 1.1.3.6])**.** *If $\mathscr{C}$ is essentially small and stable, then the ind-completion $\mathrm{Ind}(\mathscr{C})$ is presentable and stable. Furthermore, the structure map $i : \mathscr{C} \to \mathrm{Ind}(\mathscr{C})$ is exact by Theorem 9.17 (2).*

Proof. Stability is covered in [14], and exactness of i follows by □

9.5. **Renormalized derived categories.** One can employ ind-completion to produce presentable and stable alternatives to the unbounded derived category. The fundamental point here is the following.

Proposition 9.22. *If $\mathscr{C}$ is an idempotent complete essentially small ∞-category, then the structure map $i : \mathscr{C} \to \mathrm{Ind}(\mathscr{C})$ restricts to an equivalence*

$$\mathscr{C} \overset{\sim}{\to} \mathrm{Ind}(\mathscr{C})^c.$$

Proof. This is Lemma A.17 (2). □

Hence, for $\mathscr{D}(\mathbb{A})_{fav}$ some essentially small stable subcategory of objects in $\mathscr{D}(\mathbb{A})$ with a "favored" property, the ind-completion $\mathrm{Ind}\,\mathscr{D}(\mathbb{A})_{fav}$ is a stable completion of $\mathscr{D}(\mathbb{A})_{fav}$ which is freely generated by $\mathscr{D}(\mathbb{A})_{fav}$ and from which one recovers $\mathscr{D}(\mathbb{A})_{fav}$ as the subcategory of compacts. Taking the homotopy category, we obtain a triangulated category $\mathrm{D} = \mathrm{h}\,\mathrm{Ind}\,\mathscr{D}(\mathbb{A})_{fav}$ which contains the discrtete derived category $\mathrm{D}(\mathbb{A})_{fav}$, and with

$$\mathrm{Loc}\,\mathrm{D}(\mathbb{A})_{fav} = \mathrm{D}\,.$$

We note furthermore that we have, via the universal property of ind-completion, a unique cocontinuous and exact functor $\mathrm{Ind}\,\mathscr{D}(\mathbb{A})_{fin} \to \mathscr{D}(\mathbb{A})$ which fits into a diagram

$$\begin{tikzcd} \mathrm{Ind}\,\mathscr{D}(\mathbb{A})_{fin} \arrow[rr] & & \mathscr{D}(\mathbb{A}) \\ & \mathscr{D}(\mathbb{A})_{fav} \arrow[ul] \arrow[ur] & \end{tikzcd} .$$

One can view the construction $\mathrm{Ind}\,\mathscr{D}(\mathbb{A})_{fin}$ as a kind of "renormalization" of the derived ∞-category.

We do not attempt to convince the reader that this kind of construction is of any particular relevance to them, but cover two examples which already appear in the literature.

Example 9.23 (Ind-coherent sheaves)**.** Let X be a reasonable scheme or stack, and $\mathscr{D}(X)_{\mathrm{coh}}$ be the derived category of dg quasi-coherent sheaves with (bounded) coherent cohomology. Take

$$\mathbf{IndCoh}(X) := \mathrm{Ind}\,\mathscr{D}(X)_{coh},$$

where the derived-ness is implicit in the left-hand notation. This is the category of so-called ind-coherent sheaves on X.

Ind-coherent sheaves play an essential role in the geometric Langlands program [5, 6].

Example 9.24 (Ind-finite representations)**.** Let $\mathbb{G}$ be an affine algebraic group in finite characteristic. Take $\mathscr{D}(\mathbb{G})_{fin}$ the derived category of dg $\mathbb{G}$-representations with (bounded) finite dimensional cohomology, and take also

$$\mathbf{Rep}(\mathbb{G}) = \operatorname{Ind}\mathscr{D}(\mathbb{G})_{fin}.$$

This is a compactly generated stable ∞-category with compacts $\mathbf{Rep}(\mathbb{G})_{fin} := \mathscr{D}(\mathbb{G})_{fin}$.

In comparing with the derived ∞-category, we have that $\mathscr{D}(\mathbb{G})$ is not even compactly generated except under very specific circumstances [8]. Furthermore, if we consider $\mathscr{D}(\mathbb{G})$ as a symmetric monoidal ∞-category, it has dualizable objects $\mathscr{D}(\mathbb{G})_{fin}$, so that its dualizable and compact objects disagree horribly. In comparing with ind-coherent sheaves, we have in this case $\mathbf{Rep}(\mathbb{G}) = \mathbf{IndCoh}(*/\mathbb{G})$ where here the quotient is specifically the quotient stack.

The following example demonstrates that ind-constructions appear very naturally, and should be expected to appear generically, when considering equivalences of derived categories.

Example 9.25 (Koszul duality)**.** Let V be a finite dimensional vector space. Take $A = \wedge^*(V)$ and $S = k[\Sigma^{-1}V^*]$, where we consider S as a dg algebra generated in degree 1. Koszul duality provides an equivalence

$$\mathrm{Kos} = \mathrm{RHom}_A(k, -) : \mathscr{D}(A)_{fin} \to \mathscr{D}(S)_{coh}.$$

To be clear, $\mathscr{D}(S)$ is obtained as the dg nerve of the dg category S-dgmod of arbitrary dg S-modules, and $\mathscr{D}(S)_{coh}$ is the full subcategory of complexes with finitely generated cohomology.

The equivalence Kos sends the trivial representation k to $\mathrm{Kos}(k) = S$, and we note that the trivial representation is non-compact in $\mathscr{D}(A)$, since A is Frobenius. Specifically, an A-module M is compact in $\mathscr{D}(A)$ if and only if M is projective. So we see that Kos does *not* extend to an equivalence on unbounded derived categories, from $\mathscr{D}(A)$ to $\mathscr{D}(S)$. However, trivially, Kos does extend to an equivalence

$$\operatorname{Ind}\mathrm{Kos} : \operatorname{Ind}\mathscr{D}(A)_{fin} \to \operatorname{Ind}\mathscr{D}(S)_{coh} \cong \mathscr{D}(S).$$

So this Ind-category $\operatorname{Ind}\mathscr{D}(A)_{fin}$ is seen to be the natural cocompletion of the finite derived category which is implied by Koszul duality.

Appendix A. Idempotents and idempotent completion

A.1. **Idempotents and retracts.** By a *linear equivalence relation* on an ordered set I we mean an equivalence relation $\sim$ which satisfies the following:

$$\text{if } a \sim c \text{ and } a \leq b \leq c, \text{ then } a \sim b \sim c.$$

Take Idem the simplicial set with n-simplices

$$\mathrm{Idem}[n] = \big\{\text{the set of linear equivalence relations} \sim \text{on } [n]\big\}$$

The structure maps are the apparent ones. Note that Idem contains a single nondegenerate simplex in each dimension, which is given specifically by the equivalence relation with $a \sim b$ if and only if $a = b$. Furthermore, each face map $d_i^* : \mathrm{Idem}[n] \to \mathrm{Idem}[n-1]$ sends the unique nondegenerate simplex in dimension n to the nondegenerate simplex in dimension $n-1$.

A diagram in an ∞-category $F : \mathrm{Idem} \to \mathscr{C}$ can be seen as an "infinitely coherent idempotent" in $\mathscr{C}$, and consists of a choice of object x, endomorphism $e : x \to x$, 2-simplex

$$\begin{tikzcd} & x \arrow[dr, "e"] & \\ x \arrow[ur, "e"] \arrow[rr, "e"'] & & x \end{tikzcd}$$

and furthermore an n-simplex $F_n : \Delta^n \to \mathscr{C}$ at all $n \geq 0$ all of whose faces are equal in $\mathscr{C}$.

Definition A.1. An idempotent in and ∞-category $\mathscr{C}$ is a diagram $e : \mathrm{Idem} \to \mathscr{C}$.

We consider also the simplicial set $\mathrm{Ret} = \Delta^2/\Delta^{\{0,2\}}$ whose diagrams $\mathrm{Ret} \to \mathscr{C}$ classify retracts in $\mathscr{C}$. Explicitly, a diagram $\mathrm{Ret} \to \mathscr{C}$ is a choice of a 2-simplex of the form

$$\begin{tikzcd} & x \arrow[dr, "\pi"] & \\ y \arrow[ur, "l"] \arrow[rr, "id_y"'] & & y. \end{tikzcd}$$

We claim that, as in the discrete setting, any retract diagram $r : \mathrm{Ret} \to \mathscr{C}$ determines a uniquely associated idempotent $e : \mathrm{Idem} \to \mathscr{C}$. In establishing this relation we factor through an intermediate construction Idem^+.

We define the complex Idem^+ whose n-simplices are

$$\mathrm{Idem}^+[n] = \big\{\text{subsets } I \subseteq [n] \text{ equipped with a linear equivalence relation } \sim\big\}.$$

Given a map $\alpha : [m] \to [n]$ the function $\alpha^* : \mathrm{Idem}^+[n] \to \mathrm{Idem}^+[m]$ sends the pair $(I, \sim)$ to $\alpha^{-1}I \subseteq [m]$ paired with the relation $\sim_\alpha$ where $a \sim_\alpha b$ in $\alpha^{-1}I$ if and only if $\alpha(a) \sim \alpha(b)$.

We have the inclusion $\mathrm{Idem} \to \mathrm{Idem}^+$ and in low dimensions we have non-degenerate vertices

$$\mathrm{Idem}^+[0] = \{[0], \emptyset\}, \quad \mathrm{Idem}^+[1]_{\mathrm{nd}} = \big\{[1], \{0\}, \{1\}\big\},$$

$$\mathrm{Idem}^+[2]_{\mathrm{nd}} = \big\{[2], \{0,1\}, \{0,2\}, \{1,2\}, \{1\}\big\}$$

where in each case above we give the set I the equivalence relation $a \sim b$ if and only if $a = b$. We have additionally the inclusion $j : \mathrm{Ret} = \Delta^2/\Delta^{\{0,2\}} \to \mathrm{Idem}^+$ which has

$$j(0) = j(2) = \emptyset,\ j(1) = [1],\ j(0 < 1) = \{1\},\ j(1 < 2) = \{0\},\ j(0 < 2) = \emptyset$$

and $j(0 < 1 < 2) = \{1\}$. The union $\mathrm{Idem} \cup \mathrm{Ret}$ covers all 2-simplices save for the [1] and [2].

We see now that a diagram $e^+ : \mathrm{Idem}^+ \to \mathscr{C}$ specifies an idempotent $e : \mathrm{Idem} \to \mathscr{C}$ and a retract $r : \mathrm{Ret} \to \mathscr{C}$, which one might display as above, along with additional simplices which validate the equation "$e = \pi l : x \to y \to x$" up to all higher levels of compatibility. For example e^+ specifies diagrams of the form

$$e^+(\{0,1\} \subseteq [2]) = \begin{tikzcd} & x \arrow[dr, "\pi"] & \\ x \arrow[ur, "e"] \arrow[rr, "\pi"'] & & y \end{tikzcd} \qquad e^+(\{0,2\} \subseteq [2]) = \begin{tikzcd} & y \arrow[dr, "l"] & \\ x \arrow[ur, "\pi"] \arrow[rr, "e"'] & & x \end{tikzcd}$$

$$e^+(\{1,2\} \subseteq [2]) = \begin{tikzcd} & x \arrow[dr, "e"] & \\ y \arrow[ur, "l"] \arrow[rr, "l"'] & & x \end{tikzcd}$$

in $\mathscr{C}$, in dimension 2, and also diagrams of the form

$$e^+(\{0,1,2\} \subseteq [3]) = \begin{tikzcd} & x \arrow[d, "e"] \arrow[ddl, "e"'] \arrow[ddr, "\pi"] & \\ & x \arrow[dr, "\pi"] & \\ x \arrow[ur, "e"] \arrow[rr, "\pi"'] & & y \end{tikzcd} \qquad e^+(\{1,2,3\} \subseteq [3]) = \begin{tikzcd} & y \arrow[d, "l"] \arrow[ddl, "l"'] \arrow[ddr, "l"] & \\ & x \arrow[dr, "e"] & \\ x \arrow[ur, "e"] \arrow[rr, "e"'] & & x \end{tikzcd}$$

$$e^+(\{0,1,3\} \subseteq [3]) = \begin{tikzcd} & x \arrow[d, "\pi"] \arrow[ddl, "e"'] \arrow[ddr, "e"] & \\ & y \arrow[dr, "l"] & \\ x \arrow[ur, "e"] \arrow[rr, "e"'] & & x \end{tikzcd} \qquad e^+(\{0,2,3\} \subseteq [3]) = \begin{tikzcd} & x \arrow[d, "e"] \arrow[ddl, "e"'] \arrow[ddr, "\pi"] & \\ & x \arrow[dr, "e"] & \\ y \arrow[ur, "l"] \arrow[rr, "l"'] & & x \end{tikzcd}$$

in dimension 3.

Proposition A.2 ([13, Proposition 4.4.5.6])**.** *The inclusion* $\mathrm{Ret} \to \mathrm{Idem}^+$ *is inner anodyne.*

We can now reasonably speak of split idempotents.

Corollary A.3. *At any* ∞*-category* $\mathscr{C}$ *there is a functor* $\mathrm{split} : \mathrm{Fun}(\mathrm{Ret}, \mathscr{C}) \to \mathrm{Fun}(\mathrm{Idem}, \mathscr{C})$ *which into a* 2*-simplex*

$$\begin{tikzcd} & \mathrm{Fun}(\mathrm{Idem}^+, \mathscr{C}) \arrow[dl, "\mathrm{restrict}"'] \arrow[dr, "\mathrm{restrict}"] & \\ \mathrm{Fun}(\mathrm{Ret}, \mathscr{C}) \arrow[rr, "\mathrm{split}"'] & & \mathrm{Fun}(\mathrm{Idem}, \mathscr{C}). \end{tikzcd}$$

in $\mathscr{C}at_\infty$*. Furthermore, this functor* split *is uniquely determined up to a contractible space of choices.*

Proof. Since $\mathrm{Ret} \to \mathrm{Idem}^+$ is inner anodyne, the map $\mathrm{Fun}(\mathrm{Idem}^+, \mathscr{C}) \to \mathrm{Fun}(\mathrm{Ret}, \mathscr{C})$ is a trivial Kan fibration, by Proposition I-4.7. Hence the functor

$$\mathrm{Fun}(\mathrm{Fun}(\mathrm{Ret}, \mathscr{C}), \mathrm{Fun}(\mathrm{Idem}, \mathscr{C})) \to \mathrm{Fun}(\mathrm{Fun}(\mathrm{Idem}^+, \mathscr{C}), \mathrm{Fun}(\mathrm{Idem}, \mathscr{C}))$$

is an equivalence, and the space

$$\{\mathrm{restrict}\} \times^{\mathrm{htop}}_{\mathrm{Fun}(\mathrm{Fun}(\mathrm{Idem}^+, \mathscr{C}), \mathrm{Fun}(\mathrm{Idem}, \mathscr{C}))} \mathrm{Fun}(\mathrm{Fun}(\mathrm{Ret}, \mathscr{C}), \mathrm{Fun}(\mathrm{Idem}, \mathscr{C}))$$

of functors completing the given diagram is contractible. $\square$

Definition A.4. An idempotent $e : \mathrm{Idem} \to \mathscr{C}$ is called split if e is in the essential image of the functor $\mathrm{split} : \mathrm{Fun}(\mathrm{Ret}, \mathscr{C}) \to \mathrm{Fun}(\mathrm{Idem}, \mathscr{C})$.

Proposition A.5 ([13, Corollary 4.4.5.14])**.** *For a given* ∞*-category* $\mathscr{C}$*, the following are equivalent:*

(a) *Every idempotent in $\mathscr{C}$ is split.*

(b) *The restriction functor* $\mathrm{Fun}(\mathrm{Idem}^+, \mathscr{C}) \to \mathrm{Fun}(\mathrm{Idem}, \mathscr{C})$ *is a trivial Kan fibration.*

(c) *The functor* $\mathrm{split} : \mathrm{Fun}(\mathrm{Ret}, \mathscr{C}) \to \mathrm{Fun}(\mathrm{Idem}, \mathscr{C})$ *is an equivalence.*

Proof. The equivalence between (a) and (b) is covered in [13, Corollary 4.4.5.14]. For the equivalence between (b) and (c), the diagram from Corollary A.3 and the fact that the restriction $\mathrm{Fun}(\mathrm{Idem}^+, \mathscr{C}) \to \mathrm{Fun}(\mathrm{Ret}, \mathscr{C})$ is an equivalence tells us that the map split is an equivalence if and only if the restriction functor $\mathrm{Fun}(\mathrm{Idem}^+, \mathscr{C}) \to \mathrm{Fun}(\mathrm{Idem}, \mathscr{C})$ is an equivalence.

Since the map $\mathrm{Idem} \to \mathrm{Idem}^+$ is injective, the restriction functor

$$\mathrm{Fun}(\mathrm{Idem}^+, \mathscr{C}) \to \mathrm{Fun}(\mathrm{Idem}, \mathscr{C})$$

is an isofibration (Corollary I-5.14). To close we recall that, by Proposition II-16.12, an isofibration between ∞-categories is a trivial Kan fibration if and only if it is an equivalence. □

Definition A.6. We call an ∞-category $\mathscr{C}$ idempotent complete if any of the equivalent conditions from Proposition A.5 hold.

The following tells us that any cocomplete ∞-category is idempotent complete.

Proposition A.7 ([13, Corollary 4.4.5.16])**.** *Let κ be a regular cardinal. If $\mathscr{A}$ is a κ-cocomplete ∞-category then $\mathscr{A}$ is idempotent complete.*

A.2. **Idempotent completions.**

Definition A.8. An idempotent completion of an ∞-category $\mathscr{C}$ is a functor $i : \mathscr{C} \to \mathscr{C}^\vee$ to an ∞-category $\mathscr{C}^\vee$ for which:

(a) The functor i is fully faithful.

(b) $\mathscr{C}^\vee$ is idempotent complete.

(c) Every object in $\mathscr{C}^\vee$ is a retract of an object in the image of $\mathscr{C}$.

By a retract of an object x in $\mathscr{C}^\vee$ we mean, of course, an object y which admits a retract diagram $r : \mathrm{Ret} \to \mathscr{C}^\vee$ with $r(0) = y$ and $r(1) = x$. The following tells us that the collection of retracts for a class of objects $O \subset \mathscr{C}^\vee[0]$ is stable under all expected operations.

Lemma A.9. *Let $\mathscr{C}$ be any ∞-category, and $x, x', x'' : * \to \mathscr{C}$ be arbitrary objects. If x' is a retract of x, and x'' is a retract of x', then x'' is a retract of x'.*

Proof. Follows by a consideration of the diagram

$$\begin{tikzcd}
 & & x \arrow[dr] & & \\
 & x' \arrow[ur] \arrow[rr, "id"] \arrow[dr] & & x' \arrow[dr] & \\
x'' \arrow[ur] \arrow[rr, "id"'] & & x'' \arrow[ur] \arrow[rr, "id"'] & & x''
\end{tikzcd}$$

in $\mathrm{h}\mathscr{C}$. □

We note that idempotent completions exist in complete generality.

Lemma A.10. (1) *For any ∞-category $\mathscr{C}$, an idempotent completion $i : \mathscr{C} \to \mathscr{C}^\vee$ exists.*

(2) *If $\mathscr{C}$ is essentially small, then there is an idempotent completion $i : \mathscr{C} \to \mathscr{C}^\vee$ for which $\mathscr{C}^\vee$ is also essentially small.*

Proof of Lemma A.10. We will prove that any essentially small ∞-category $\mathscr{C}$ admits an essentially small idempotent completion. For (1) one just repeats the same arguments without regard for size constraints.

Suppose that $\mathscr{C}$ is essentially small. Since the category $\mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathscr{K}an^{\mathrm{sm}})$ is cocomplete it is idempotent split. Let $\mathscr{C}' \subseteq \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathscr{K}an^{\mathrm{sm}})$ denote the essential image of the Yoneda embedding $\mathscr{C} \to \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathscr{K}an^{\mathrm{sm}})$. Now, the collection of idempotents $\mathrm{Fun}(\mathrm{Idem}, \mathscr{C}')$ is also essentially small, as the simplicial set Idem is idempotent complete.

Let us take

$$\mathscr{Z} = \mathrm{Fun}(\mathrm{Idem}^+ \times \mathscr{C}^{\mathrm{op}}, \mathscr{K}an^{\mathrm{sm}}) \times_{\mathrm{Fun}(\mathrm{Idem} \times \mathscr{C}^{\mathrm{op}}, \mathscr{K}an^{\mathrm{sm}})} \mathrm{Fun}(\mathrm{Idem}, \mathscr{C}').$$

As the restriction functor $\mathrm{Fun}(\mathrm{Idem}^+ \times \mathscr{C}^{\mathrm{op}}, \mathscr{K}an^{\mathrm{sm}}) \to \mathrm{Fun}(\mathrm{Idem} \times \mathscr{C}^{\mathrm{op}}, \mathscr{K}an^{\mathrm{sm}})$ is a trivial Kan fibration [13, Corollary 4.4.5.16.], the projection $p : \mathscr{Z} \to \mathrm{Fun}(\mathrm{Idem}, \mathscr{C}')$ is a trivial Kan fibration as well.

We note that $\mathscr{Z}$ is a fully subcategory in $\mathrm{Fun}(\mathrm{Idem}^+ \times \mathscr{C}^{\mathrm{op}}, \mathscr{K}an^{\mathrm{sm}})$. The restriction functor $\mathscr{Z} \to \mathrm{Fun}(\mathrm{Ret} \times \mathscr{C}^{\mathrm{op}}, \mathscr{K}an^{\mathrm{sm}})$ is a trivial Kan fibration onto its image as well [13, Proposition 4.4.5.6]. (Here we mean strict image, not essential image.) This image, call it $\mathscr{Y}$, is precisely the collection of diagrams $F : \mathrm{Ret} \to \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathscr{K}an^{\mathrm{sm}})$ with $F(1)$ in $\mathscr{C}'$, and we deduce an equivalence $\mathscr{Y} \overset{\sim}{\to} \mathrm{Fun}(\mathrm{Idem}, \mathscr{C}')$. In particular $\mathscr{Y}$ is essentially small.

Evaluating at $0 = 2$ we obtain a functor

$$ev_0 : \mathscr{Y} \to \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathscr{K}an^{\mathrm{sm}})$$

and let $\mathscr{C}^\vee \subseteq \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathscr{K}an^{\mathrm{sm}})$ denote the full ∞-subcategory spanned by the image of $\mathscr{Y}$. Since $\mathscr{Y}$ is essentially small and $\mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathscr{K}an^{\mathrm{sm}})$ is locally small, we conclude that $\mathscr{C}^\vee$ is essentially small. From a consideration of degenerate diagram

$$\mathrm{Ret} \to * \xrightarrow{x} \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathscr{K}an^{\mathrm{sm}})$$

at any x in $\mathscr{C}'$ it is clear that $\mathscr{C}^\vee$ contains $\mathscr{C}'$, and we see directly that $\mathscr{C}^\vee$ is spanned by all objects which are retracts of objects in $\mathscr{C}'$. Hence the Yoneda embedding provides a fully faithful functor $i : \mathscr{C} \to \mathscr{C}^\vee$ for at which all objects in $\mathscr{C}^\vee$ are obtained by taking retracts of objects in the image of i. Furthermore, by Proposition A.7 the functor category $\mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathscr{K}an^{\mathrm{sm}})$ is idempotent split, and hence $\mathscr{C}^\vee$ is idempotent complete by Lemma A.9. So the Yoneda embedding realizes $\mathscr{C}^\vee$ as an idempotent completion of $\mathscr{C}$. □

We have the following universal property for idempotent completion.

Proposition A.11 ([13, Proposition 5.1.4.9])**.** *If $i : \mathscr{C} \to \mathscr{C}^\vee$ is an idempotent completion of an ∞-category $\mathscr{C}$, then for any idempotent complete ∞-category $\mathscr{A}$ restricting along i provides an equivalence of ∞-categories*

$$i^* : \mathrm{Fun}(\mathscr{C}^\vee, \mathscr{A}) \to \mathrm{Fun}(\mathscr{C}, \mathscr{A}).$$

Corollary A.12. *For any ∞-category $\mathscr{C}$, the idempotent completion $i : \mathscr{C} \to \mathscr{C}^\vee$ is uniquely determined up to isomorphism in the undercategory $(\mathscr{C}at_\infty)_{\mathscr{C}/}$.*

We combine uniqueness with the smallness assertion form Lemma A.10 to observe that idempotent completion preserves essentially small ∞-categories.

Corollary A.13. *If $\mathscr{C}$ is essentially small, then any idempotent completion $\mathscr{C}^\vee$ of $\mathscr{C}$ is also essentially small.*

Using Proposition A.11 one can also establish the existence of a unique idempotent completion functor

$$(-)^\vee : \mathscr{C}at_\infty \to \mathscr{C}at_\infty$$

which comes equipped with a transformation $i : id_{\mathscr{C}at_\infty} \to (-)^\vee$ which evaluates to an idempotent completion $i : \mathscr{C} \to \mathscr{C}^\vee$ at each ∞-category $\mathscr{C}$. To see this one constructs a cocartesian fibration $q : \mathscr{M}_{\mathrm{Idem}} \to \Delta^1$ for idempotent completions and proceeds exactly as in Section 9.2, where we construct the indization functor.

Below we let $(\mathscr{C}at_\infty)_{\mathrm{split}}$ denote the full subcategory of idempotent split ∞-categories in $\mathscr{C}at_\infty$. We leave the proof of the following as an exercise for the interested reader.

Proposition A.14. *There is a functor $(-)^\vee : \mathscr{C}at_\infty \to (\mathscr{C}at_\infty)_{\mathrm{split}}$ is right adjoint to the inclusion $(\mathscr{C}at_\infty)_{\mathrm{split}} \to \mathscr{C}at_\infty$, and the unit transformation $i : id_{\mathscr{C}at_\infty} \to (-)^\vee$ evaluates to an idempotent completion $i : \mathscr{C} \to \mathscr{C}^\vee$ at each ∞-category $\mathscr{C}$.*

A.3. $\aleph_0$-accessibility and idempotent completion.

Definition A.15. An ∞-category $\mathscr{A}$ is called $\aleph_0$-accessible if $\mathscr{A}$ admits a functor $i : \mathscr{C} \to \mathscr{A}$ from an essentially small ∞-category $\mathscr{C}$ which exhibits $\mathscr{A}$ as an ind-completion of $\mathscr{C}$ (Definition 9.4).

The category $\mathrm{Acc}_{\aleph_0}$ of $\aleph_0$-accessible categories is the non-full subcategory in $\mathscr{C}at_\infty$ whose n-simplices $\sigma : \Delta^n \to \mathrm{Acc}_{\aleph_0}$ are those simplices $\sigma : \Delta^n \to \mathscr{C}at_\infty$ for which each vertex $\mathscr{A}_i = \sigma(\{i\})$ is $\aleph_0$-accessible, and for which each edge $F_{ij} = \sigma|_{\Delta^{\{i,j\}}} : \mathscr{A}_i \to \mathscr{A}_j$ is $\aleph_0$-cocontinuous and preserves ($\aleph_0-$)compact objects.

We can construct the category $\mathrm{Acc}_{\aleph_0}$ as the homotopy coherent nerve of the simplicial subcategory $\underline{\mathrm{Acc}}_{\aleph_0} \subseteq \underline{\mathrm{Cat}}^+_\infty$ whose objects are $\aleph_0$-accessible categories and whose mapping complexes are the full subcategories

$$\mathrm{Fun}_{\aleph_0}(\mathscr{A}, \mathscr{B}) \ \subseteq \ \mathrm{Fun}(\mathscr{A}, \mathscr{B})^{\mathrm{Kan}}$$

spanned by accessible functors.

Taking compacts provides a simplicial functor $\underline{\mathrm{Acc}}_{\aleph_0} \to \underline{\mathrm{Cat}}^+_\infty$ and we apply the homotopy coherent nerve to obtain a functor $(-)^c : \mathrm{Acc}_{\aleph_0} \to \mathscr{C}at_\infty$. The following gives essential information about the image of the compacts functor.

Lemma A.16 ([13, Lemma 5.4.2.4])**.** *If $\mathscr{C}$ is essentially small, then the subcategory $\mathrm{Ind}(\mathscr{C})^c$ of compact objects in $\mathrm{Ind}(\mathscr{C})$ is an idempotent completion of $\mathscr{C}$.*

Applying Corollary A.13, we see that the full subcategory of compacts $\mathscr{A}^c$ in any accessible ∞-category form an essentially small ∞-category.

Lemma A.17. (1) *If $\mathscr{A}$ is an $\aleph_0$-accessible ∞-category then the full subcategory of compacts $\mathscr{A}^c$ is essentially small, and the inclusion $\mathscr{A}^c \to \mathscr{A}$ induces an equivalence $\mathrm{Ind}(\mathscr{A}^c) \to \mathscr{A}$.*

(2) *If $\mathscr{C}$ is essentially small and idempotent complete, then the inclusion $i : \mathscr{C} \to \mathrm{Ind}(\mathscr{C})^c$ is an equivalence.*

Proof. Statement (2) is clear, by uniqueness of idempotent completion. For (1) we have an equivalence $\mathrm{Ind}(\mathscr{C}) \xrightarrow{\sim} \mathscr{A}$ for some essentially small $\mathscr{C}$, so that the inclusion $\mathscr{C} \to \mathrm{Ind}(\mathscr{C}) \to \mathscr{A}^c$ exhibits $\mathscr{A}^c$ as an idempotent completion of $\mathscr{C}$. By Corollary A.13 it follows that $\mathscr{A}^c$ is essentially small. Furthermore, since $\mathscr{A}$ is generated by the image of $\mathscr{C}$ under filtered colimits, we see that $\mathscr{A}$ is generated by $\mathscr{A}^c$ under filtered colimits. Hence the induced map $\mathrm{Ind}(\mathscr{A}^c) \to \mathscr{A}$ is an equivalence by Theorem 9.6. $\square$

We note that, via the structural transformation $i : \mathrm{incl} \to \mathrm{Ind}$ we have a diagram

$$\begin{tikzcd}
\mathrm{Ind}(\mathscr{C}) \arrow[r, "\mathrm{Ind}(f)"] & \mathrm{Ind}(\mathscr{C}') \\
\mathscr{C} \arrow[u, "i"] \arrow[r, "f"'] \arrow[ur] & \mathscr{C}' \arrow[u, "i"']
\end{tikzcd}$$

at any map $f : \mathscr{C} \to \mathscr{C}'$ between essentially small ∞-categories. It follows that the functor $\mathrm{Ind}(f)$ preserves compact objects, and is therefore $\aleph_0$-accesible. We cen therefore restrict the codomain to obtain a functor $\mathrm{Ind} : \mathscr{C}at_\infty^{\mathrm{sm}} \to \mathrm{Acc}_{\aleph_0}$.

Let us take

$(\mathscr{C}at_\infty^{\mathrm{sm}})_{\mathrm{split}}$:= the full subcategory of idempotent complete ∞-categories in $\mathscr{C}at_\infty^{\mathrm{sm}}$.

We have functors

$$(-)^c : \mathrm{Acc}_{\aleph_0} \to (\mathscr{C}at_\infty^{\mathrm{sm}})_{\mathrm{split}} \quad \text{and} \quad \mathrm{Ind} : (\mathscr{C}at_\infty^{\mathrm{sm}})_{\mathrm{split}} \to \mathrm{Acc}_{\aleph_0},$$

and by Lemma A.17 the functor $(-)^c$, taken now to have image in idempotent complete categories, is essentially surjective. Furthermore, for the simplicial category $\underline{\mathrm{Acc}}_{\aleph_0}$, Lemma A.17 and Theorem 9.5 tell us that restricting to the compacts provides a fully faithful functor

$$\mathrm{Fun}_{\aleph_0}(\mathscr{A}, \mathscr{B}) \to \mathrm{Fun}(\mathscr{A}^c, \mathscr{B})$$

whose essential image is precisely the subcategory $\mathrm{Fun}(\mathscr{A}^c, \mathscr{B}^c)$. So, at the level of simplicial categories, the functor given by restricting to the compacts

$$\underline{\mathrm{Acc}}_{\aleph_0} \to (\underline{\mathrm{Cat}}_\infty^{\mathrm{sm}})_{\mathrm{split}}$$

is fully faithful. By Proposition II-7.6 it follows that the functor $(-)^c : \mathrm{Acc}_{\aleph_0} \to (\mathscr{C}at_\infty^{\mathrm{sm}})_{\mathrm{split}}$ is fully faithful as well.

Theorem A.18. *The functor* $(-)^c : \mathrm{Acc}_{\aleph_0} \to (\mathscr{C}at_\infty^{\mathrm{sm}})_{\mathrm{split}}$ *is an equivalence.*

Proof. By the information above, this functor is fully faithful and essentially surjective, and hence an equivalence by Theorem I-7.2. $\square$

We claim finally that the inverse to $(-)^c$ is provided by the indization functor. To prove this it suffices to prove that indization is left adjoint to the compact objects functor.

Proposition A.19. *The functor* $(-)^c : \mathrm{Acc}_{\aleph_0} \to \mathscr{C}at_\infty^{\mathrm{sm}}$ *is right adjoint to the indization functor* $\mathrm{Ind} : \mathscr{C}at_\infty^{\mathrm{sm}} \to \mathrm{Acc}_{\aleph_0}$.

Proof. Let $\mathscr{M}_{\aleph_0} \subseteq \Delta^1 \times \mathscr{C}at_\infty$ denote the non-full subcategory whose objects are

$$\mathscr{M}_{\aleph_0}[0] = (\{0\} \times \mathscr{C}at_\infty^{\mathrm{sm}}[0]) \amalg (\{1\} \times \mathrm{Acc}_{\aleph_0}[0])$$

and whose edges $(i, \mathscr{C}) \to (j, \mathscr{A})$ are arbitrary maps of essentially small ∞-categories when $i = j = 0$, maps in $\mathrm{Acc}_{\aleph_0}$ when $i = j = 1$, and maps $\mathscr{C} \to \mathscr{A}$ which have image in $\mathscr{A}^c$ when $i < j$. We place no other restrictions on n-simplices in $\mathscr{M}_{\aleph_0}$. One checks directly that the inclusion $\mathscr{M}_{\aleph_0} \to \Delta^1 \times \mathscr{C}at_\infty$ is an inner fibration, so that the composte $q : \mathscr{M}_{\aleph_0} \to \Delta^1$ of the inclusion with the projection $\Delta^1 \times \mathscr{C}at_\infty \to \Delta^1$ is also an inner fibration. One argues exactly as in the proof of Lemma **??** to see that the projection $q : \mathscr{M}_{\aleph_0} \to \Delta^1$ is a cocartesian fibration with q-cocartesian edges over $0 < 1$ provided by those functors $\mathscr{C} \to \mathscr{A}$ which realize $\mathscr{A}$ as an ind-completion of $\mathscr{C}$.

(To recall, we can identify $\mathscr{M}_{\aleph_0}$ with the homotopy coherent nerve of the apparent simplicial subcategory $\mathrm{M}_{\aleph_0} \subseteq \{0 < 1\} \times \underline{\mathrm{Cat}}_\infty^{\mathrm{sm}}$, and use this simplicial construction along with Proposition II-7.6 to check reflexivity of $\mathrm{Acc}_{\aleph_0}$ in $\mathscr{M}_{\aleph_0}$ directly via the functor categories and the isomorphism

$$i^* : \mathrm{Fun}_{\aleph_0}(\mathscr{C}', \mathscr{A}) \to \mathrm{Fun}_c(\mathscr{C}, \mathscr{A})$$

given by restricting along any ind-completion $i : \mathscr{C} \to \mathscr{C}'$ at essentially small $\mathscr{C}$, where $\mathrm{Fun}_c(\mathscr{C}, \mathscr{A})$ here denotes the full subcategory of functors with image in $\mathscr{A}^c$.)

From this characterization of q-cocartesian edges we see that the indization functor, as defined in Section 9.2, provides the unique cocartesian solution to the lifting problem

$$\begin{tikzcd}
\{0\} \times \mathscr{C}at_\infty^{\mathrm{sm}} \arrow[r] \arrow[d] & \mathscr{M}_{\aleph_0} \arrow[d, "q"] \\
\Delta^1 \times \mathscr{C}at_\infty^{\mathrm{sm}} \arrow[r] \arrow[ur, "I"] & \Delta^1,
\end{tikzcd}$$

and hence that the functor $\mathrm{Ind} : \mathscr{C}at_\infty^{\mathrm{sm}} \to \mathrm{Acc}_{\aleph_0}$ is realized as homotopy transport for along q.

We claim now that $q : \mathscr{M}_{\aleph_0} \to \Delta^1$ is also cartesian. For this it suffices to show that $\mathscr{C}at_\infty^{\mathrm{sm}} \to \mathscr{M}_{\aleph_0}$ is coreflexive. However, this is clear since for any $\aleph_0$-accessible ∞-category $\mathscr{A}$ the inclusion $k : \mathscr{A}^c \to \mathscr{A}$ provides an equivalence $k_* : \mathrm{Hom}_{\mathscr{M}}(\mathscr{C}, \mathscr{A}^c) \to \mathrm{Hom}_{\mathscr{M}}(\mathscr{C}, \mathscr{A})$. One observes this fact precisely by considering the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{M}}(\mathscr{C}, \mathscr{A}^c) \arrow[r, "k_*"] & \mathrm{Hom}_{\mathscr{M}}(\mathscr{C}, \mathscr{A}) \\
\mathrm{Fun}(\mathscr{C}, \mathscr{A}^c) \arrow[r, "\sim", "k_*"'] \arrow[u, "\sim"] & \mathrm{Fun}_c(\mathscr{C}, \mathscr{A}) \arrow[u, "\sim"']
\end{tikzcd}$$

in $\mathrm{h}\mathscr{K}an$, which exists by Proposition II-7.6.

Using Lemma 5.13 we furthermore characterize q-cartesian edges over $0 < 1$ as those map $k : \mathscr{A}' \to \mathscr{A}$ which are an equivalence onto the compacts in $\mathscr{A}$. At the level of simplicial categories, the inclusions $\mathscr{A}^c \to \mathscr{A}$ provide a transformation $(-)^c \to \mathrm{incl}$ between the simplicial functor $(-)^c : \underline{\mathrm{Acc}}_{\aleph_0} \to \underline{\mathrm{Cat}}_\infty$ and the inclusion $\mathrm{incl} : \underline{\mathrm{Acc}}_{\aleph_0} \to \underline{\mathrm{Cat}}_\infty^+$, and we obtain an induced transformation $k : (-)^c \to \mathrm{incl}$ between objects in $\mathrm{Fun}(\mathrm{Acc}_\lambda, \mathscr{C}at_\infty)$. This transformation provides the unique cartesian solution to the lifting problem

$$\begin{tikzcd} \{1\} \times \mathrm{Acc}_{\aleph_0} \arrow[r] \arrow[d] & \mathscr{M}_{\aleph_0} \arrow[d, "q"] \\ \Delta^1 \times \mathrm{Acc}_{\aleph_0} \arrow[r] \arrow[ur, "\mathrm{K}"] & \Delta^1, \end{tikzcd}$$

which is given explicitly by the sequence

$$\Delta^1 \times \mathrm{Acc}_{\aleph_0} \xrightarrow{\delta\times 1} \Delta^1 \times \Delta^1 \times \mathrm{Acc}_{\aleph_0} \xrightarrow{1\times k} \mathscr{M}_{\aleph_0} \subseteq \Delta^1 \times \mathscr{C}at_\infty.$$

Evaluating at 0 recovers the functor which takes compacts

$$\mathrm{K}\,|_{\{0\}\times \mathrm{Acc}_{\aleph_0}} = (-)^c : \mathrm{Acc}_{\aleph_0} \to \mathscr{C}at_\infty^{\mathrm{sm}}$$

so that this functor is realized as homotopy contraviant transport along q. It follows by Theorem 5.15 that $(-)^c$ is right adjoint to the indization functor. □

Since taking the compacts in $\mathrm{Acc}_{\aleph_0}$ has image in $(\mathscr{C}at_\infty^{\mathrm{sm}})_{\mathrm{split}}$, we see that the restricted functor

$$\mathrm{Ind} : (\mathscr{C}at_\infty^{\mathrm{sm}})_{\mathrm{split}} \to \mathrm{Acc}_{\aleph_0}$$

remains left adjoint to the functor $(-)^c$. By uniqueness of adjoints, we conclude that Ind is in fact the inverse functor to the equivalence $(-)^c$.

Corollary A.20. *The functors*

$$\mathrm{Ind} : (\mathscr{C}at_\infty^{\mathrm{sm}})_{\mathrm{split}} \to \mathrm{Acc}_{\aleph_0} \quad \textit{and} \quad (-)^c : \mathrm{Acc}_{\aleph_0} \to (\mathscr{C}at_\infty^{\mathrm{sm}})_{\mathrm{split}}$$

are mutually inverse equivalences.

The following identifies our indization functor with the corresponding functor appearing in [13, Proposition 5.4.2.19].

Proposition A.21. *In* $\mathrm{Fun}(\mathscr{C}at_\infty^{\mathrm{sm}}, \mathscr{C}at_\infty)$*, there is a natural isomorphism of functors*

$$\mathrm{Ind} \cong (\mathrm{Ind}\,|_{(\mathscr{C}at_\infty^{\mathrm{sm}})_{\mathrm{split}}}) \circ (-)^\vee.$$

Proof. If F_i is left adjoint to G_i then the composite F_1F_0 is left adjoint to G_0G_1. Hence, by Proposition A.14 and Corollary A.20. Hence the composite $(\mathrm{Ind}\,|_{(\mathscr{C}at_\infty^{\mathrm{sm}})_{\mathrm{split}}})\circ (-)^\vee$ is left adjoint to the composite

$$(-)^c = \mathrm{incl} \circ (-)^c : \mathrm{Acc}_{\aleph_0} \to (\mathscr{C}at_\infty^{\mathrm{sm}})_{\mathrm{split}} \to \mathscr{C}at_\infty^{\mathrm{sm}}.$$

By uniqueness of left adjoints and Proposition A.19. □

A.4. **Idempotent splitting in the stable setting.**

Proposition A.22. *A stable ∞-category $\mathscr{C}$ is idempotent complete if and only if its homotopy category $\mathrm{h}\mathscr{C}$ is idempotent complete.*

Proof. By enlarging our universe if necessary, we may assume $\mathscr{C}$ is essentially small. Taking $\mathscr{A} = \mathrm{Ind}(\mathscr{C})$, we have the exact embedding $\mathscr{C} \to \mathscr{A}$ into a presentable stable idempotent complete ∞-category, by Proposition 9.21. As $\mathscr{A}^c$ is an idempotent completion of $\mathscr{C}$, we understand that $\mathscr{C}$ is idempotent complete if and only if the map $\mathscr{C} \to \mathscr{A}^c$ is an equivalence.

Note that $\mathrm{h}\mathscr{A}$ is idempotent complete, as it is triangulated with all small coproducts [18, Proposition 1.6.8]. So we have a bijection between isoclasses of retract diagrams in $\mathrm{h}\mathscr{A}$ and idempotents in $\mathrm{h}\mathscr{A}$. Since every retract diagram in $\mathrm{h}\mathscr{A}$

lifts to a retract diagram in $\mathscr{A}$, we obtain a bijection between isoclasses of retract diagrams in $\mathscr{A}$ and idempotents in $\mathrm{h}\,\mathscr{A}$. This bijection simply sends a retract $r = (y \to x \to y)$ to the idempotent $e_r : x \to y \to x$ in $\mathrm{h}\,\mathscr{A}$.

Since $\mathrm{h}\,\mathscr{A}$ is idempotent complete, we have that the subcategory of compacts $(\mathrm{h}\,\mathscr{A})^c$ is idempotent complete. Now, at the level of the homotopy category we have $\mathrm{h}(\mathscr{A}^c) = (\mathrm{h}\,\mathscr{A})^c$, by Corollary 4.35. So if $\mathscr{C}$ is idempotent complete we have that the homotopy category $\mathrm{h}\,\mathscr{C} \cong (\mathrm{h}\,\mathscr{A})^c$ is idempotent complete as well.

Suppose conversely that $\mathrm{h}\,\mathscr{C}$ is idempotent complete. By the definition of idempotent completion we have that every object in $\mathrm{h}(\mathscr{A}^c) = (\mathrm{h}\,\mathscr{A})^c$ is a retract of an object in the image of $\mathrm{h}\,\mathscr{C}$. By idempotent completeness of $\mathrm{h}(\mathscr{A}^c)$ it follows that every object is obtained by splitting an idempotent in the image of $\mathrm{h}\,\mathscr{C}$. In the case that $\mathrm{h}\,\mathscr{C}$ is idempotent split, it follows that every object in $\mathrm{h}\,\mathscr{A})^c$ is in the essential image of $\mathrm{h}\,\mathscr{C}$, and hence that the map $\mathrm{h}\,\mathscr{C} \to \mathrm{h}(\mathscr{A}^c)$ is an equivalence. So we see that $\mathscr{C}$ is idempotent complete. □

References

[1] J. F. Adams. *Stable homotopy and generalised homology.* University of Chicago press, 1974.

[2] A. J. Blumberg, D. Gepner, and G. Tabuada. A universal characterization of higher algebraic k-theory. *Geom. Topol.*, 17(2):733–838, 2013.

[3] A. Bondal and M. V. den Bergh. Generators and representability of functors in commutative and noncommutative geometry. *Mosc. Math. J.*, 3(1):1–36, 2003.

[4] A. Chirvasitu and T. Johnson-Freyd. The fundamental pro-groupoid of an affine 2-scheme. *Appl. Categor. Struct.*, 21(5):469–522, 2013.

[5] D. Gaitsgory. Ind-coherent sheaves. *preprint arXiv:1105.4857.*

[6] D. Gaitsgory and N. Rozenblyum. *A study in derived algebraic geometry: Volume I: correspondences and duality*, volume 221. American Mathematical Society, 2019.

[7] P. Gross. *Vector bundles as generators on schemes and stacks.* PhD thesis, Düsseldorf, Univ., Diss., 2010, 2010.

[8] J. Hall, A. Neeman, and D. Rydh. One positive and two negative results for derived categories of algebraic stacks. *J. Inst. Math. Jussieu*, 18(5):1087–1111, 2019.

[9] A. Joyal. Quasi-categories and kan complexes. *J. Pure Appl. Algebra*, 175(1-3):207–222, 2002.

[10] M. Kashiwara and P. Schapira. *Categories and Sheaves.* Springer Berlin Heidelberg, 2006.

[11] G. M. Kelly. *Basic concepts of enriched category theory*, volume 64. CUP Archive, 1982.

[12] H. Krause. *Localization theory for triangulated categories*, pages 161–235. London Mathematical Society Lecture Note Series. Cambridge University Press, 2010.

[13] J. Lurie. *Higher topos theory.* Number 170. Princeton University Press, Princeton, New Jersey 08540, 2009.

[14] J. Lurie. Higher algebra. *www.math.ias.edu/ lurie/papers/HA.pdf*, 2017.

[15] J. Lurie. Kerodon. *https://kerodon.net*, 2024.

[16] J. Lurie. Spectral algebraic geometry. *https://www.math.ias.edu/ lurie/papers/SAG-rootfile.pdf*, updated 2018.

[17] J. Milnor. The geometric realization of a semi-simplicial complex. *Ann. of Math.*, 65(2):357–362, 1957.

[18] A. Neeman. *Triangulated categories.* Number 148. Princeton University Press, 2001.

[19] T. Nikolaus and P. Scholze. On topological cyclic homology. *Acta Math.*, 221(2):203 – 409, 2018.

[20] C. Serpé. Resolution of unbounded complexes in grothendieck categories. *J. Pure Appl. Algebra*, 177(1):103–112, 2003.

[21] L. A. Tarrío, A. J. López, M. P. Rodríguez, and M. J. V. Gonsalves. The derived category of quasi-coherent sheaves and axiomatic stable homotopy. *Adv. Math.*, 218(4):1224–1252, 2008.

[22] L. A. Tarrío, A. J. López, M. P. Rodríguez, and M. J. V. Gonsalves. A functorial formalism for quasi-coherent sheaves on a geometric stack. *Expo. Math.*, 33(4):452–501, 2015.

[23] C. A. Weibel. *An introduction to homological algebra*, volume 38. Cambridge university press, 1994.

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