Midterm 1 (Math 410, Fall 2023) Name:__________________

- There are four questions. Use *back* of sheets, and/or page 5, for scratch work. Write your final answer directly below the statement of the question.

- Justify all steps in your proofs. If you use a result from class, or from the text, provide a generic reference. E.g. "By [Artin, Ch 2], it follows that" or "The above equation follows by [Lectures on orders of elements]".

1. [20 pt] Prove the following theorem from Artin:[1] A group homomorphism $f : G \to G'$ is injective if and only if $\ker(f)$ is trivial.

Suppose f is injective. We have $f(e) = e$, since f is a group homomorphism. Hence, via injectivity $f(g) = e$ implies $g = e$. So $\ker(f) = \{e\}$.

Conversely, if $\ker(f) = \{e\}$, then ~~the~~ equality

$$f(g) = f(g') \quad \text{implies} \quad e = f(g)^{-1} f(g') = f(g^{-1} g'),$$

so that $g^{-1} g' \in \ker(f)$. Triviality now gives

$$g^{-1} g' = e \implies g' = g.$$

So f is seen to be injective.

[1] Obviously, you can't simply reference "a result from [Artin]" for this. You must provide a direct proof for full credit.

2. [30 pt] (a) What is the order of a cycle in S_n of the form $\sigma = (a_1 \ldots a_m)$? Provide a few sentences to justify your answer. [Hint: Where does σ^r send a_j?]

(b) Suppose that $\sigma_1, \ldots, \sigma_l$ are disjoint cycles in S_n, with each σ_i appearing as

$$\sigma_i = (a(i)_1 \ldots a(i)_{m_i}).$$

What is the order of the product $\sigma_1 \cdots \sigma_l$? Provide a few sentences to justify your answers.

(c) Let H be the subgroup in S_6 generated by the permutation $\omega = (135)(26)$. How many distinct H-cosets are there in S_6?

(a) $\operatorname{ord}(\sigma) = m$. [Reasoning].

(b) $\operatorname{ord}(\sigma_1 \cdots \sigma_l) = \operatorname{lcm}(m_1, \ldots, m_l)$. [Reasoning]

(c) $|S_6/H| = |S_6|/|H| = 6!/\operatorname{lcm}(3,2)$

$= 6!/6 = 6 \cdot 5 \cdot 4 = 120.$

$[= 5 \cdot 4 \cdot 3 \cdot 2 = 120]$.

Note

3. [20 pt] Consider groups G_1 and G_2, and normal subgroups $K_i \subseteq G_i$. ~~Prove~~ that $K_1 \times K_2$ is a normal subgroup in $G_1 \times G_2$ ~~and~~ that there is an isomorphism of groups

• Prove

$$f : (G_1 \times G_2)/(K_1 \times K_2) \xrightarrow{\sim} (G_1/K_1) \times (G_2/K_2).$$

[Hint: Begin by considering a group homomorphism from $G_1 \times G_2$ to $(G_1/K_1) \times (G_2/K_2)$.]

The two projection $p_i : G_i \to G_i/K_i$ give a ~~project~~ surjective group homomorphism

$$p : G_1 \times G_2 \longrightarrow (G_1/K_1) \times (G_2/K_2)$$

defined by $p(x,y) := (p_1(x), p_2(y))$. We have

$$p(x,y) \cdot p(x',y') = (p_1(x), p_2(y)) \cdot (p_1(x'), p_2(y'))$$

$$= (p_1(x \cdot x'), p_2(y \cdot y')) \quad [\text{since the } p_i \text{ are hom.}]$$

$$= p(x \cdot x', y \cdot y')$$

$$= p((x,y) \cdot (x',y')).$$

So p is in fact a group homomorphism. The kernel of p is $K_1 \times K_2 \subseteq G_1 \times G_2$, since $p(x,y) = e$ if and only if $x \in \ker(p_1)$ and $y \in \ker(p_2)$. So by the first isomorphism theorem we have an induced isom. from the quotient

$$\bar{p} : (G_1 \times G_2)/(K_1 \times K_2) \xrightarrow{\sim} (G_1/K_1) \times (G_2/K_2).$$

4. [25 pt] (a) Consider the subset $\mathrm{O}_n(\mathbb{R}) := \{A \in \mathrm{GL}_n(\mathbb{R}) : A^t = A^{-1}\}$. Prove that $\mathrm{O}_n(\mathbb{R})$ is a subgroup in $\mathrm{GL}_n(\mathbb{R})$.

(b) Consider the subset $\mathrm{P}_n(\mathbb{R}) := \{A \in \mathrm{GL}_n(\mathbb{R}) : A^t = A\}$. Prove, by way of example, that $\mathrm{P}_n(\mathbb{R})$ is *not* a subgroup in $\mathrm{GL}_n(\mathbb{R})$ whenever $n \geq 2$. [Hint: Consider the case $n = 2$.]

(a) We have $I_n^t = I_n = I_n^{-1}$, so that $I_n \in O_n(\mathbb{R})$, and also

$$(A^{-1})^t \cdot A^t = (A \cdot A^{-1})^t = I_n^t = I_n$$

$$A^t \cdot (A^{-1})^t = (A^{-1} A)^t = I_n \quad \text{for each } A \in O_n.$$

So O_n is nonempty and stable under inversion. Finally, we have for $A, B \in O_n(\mathbb{R})$,

$$(A \cdot B)^t = B^t \cdot A^t = B^{-1} \cdot A^{-1} = (A \cdot B)^{-1}.$$

So $A \cdot B \in O_n(\mathbb{R})$. Hence $O_n(\mathbb{R})$ is a subgroup in $GL_n(\mathbb{R})$.

(b) We have

$$\underbrace{\begin{bmatrix} 0 & 2 \\ 2 & 1 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}}_{B} = \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix}.$$

Since $A, B \in P_2(\mathbb{R})$, while $A \cdot B \notin P_2(\mathbb{R})$, we see $P_2(\mathbb{R})$ is not a subgroup in $GL_2(\mathbb{R})$. More generally,

for $A_n = \begin{bmatrix} 0 & 2 & & & \\ 2 & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}$ and $B_n = \begin{bmatrix} 1 & 1 & & & \\ 1 & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}$ we have $A_n \cdot B_n \notin P_n(\mathbb{R})$

So $P_n(\mathbb{R})$ is not a subgroup for any $n \geq 2$