~ Roots and the root space decomp ~ ①

We have
$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\gamma \in \Phi} \mathfrak{g}_\gamma \right)$$
w/ $\mathfrak{h} = \mathfrak{g}_0$. Φ = the roots for $\mathfrak{g}$

Take for each $\gamma \in \Phi$ $t_\gamma \in \mathfrak{h}$ so that $\kappa(t_\gamma, h) = \gamma(h)$ at each $h \in \mathfrak{h}$. (Such t_γ exists by non-degen of $\kappa|_{\mathfrak{h}\times\mathfrak{h}}$.)

Structural Lemma 8.3: (a) Φ spans $\mathfrak{h}^*$.

(b) If $\gamma \in \Phi$ then $-\gamma \in \Phi$.

(c) For $\gamma \in \Phi$, $x \in \mathfrak{g}_\gamma$ and $y \in \mathfrak{g}_{-\gamma}$, we have $[x,y] = \kappa(x,y)\, t_\gamma$.

(d) $[\mathfrak{g}_\gamma, \mathfrak{g}_{-\gamma}] = \mathbb{C}\cdot t_\gamma$.

(e) $\gamma(t_\gamma) \neq 0$ at each $\gamma \in \Phi$.

(f) The normalization
$$h_\gamma = \frac{2}{\kappa(t_\gamma, t_\gamma)} \cdot t_\gamma$$
satisfies $h_{-\gamma} = -h_\gamma$ and $\gamma(h_\gamma) = 2$ at all γ.

Proof: (a) For $h \in \mathfrak{h}$ w/ $\gamma(h) = 0$ at all roots γ we have $h \cdot x = \gamma(h)\cdot x = 0$ for each root vector $x \in \mathfrak{g}_\gamma$. Hence $h \in Z(\mathfrak{g})$. By semisimp, $Z(\mathfrak{g}) = 0$ so that $h = 0$. Hence $\mathbb{C}\cdot\Phi = \mathfrak{h}^*$.

(b) Follows from the fact that $0 \neq \kappa(\mathfrak{g}_\gamma, \mathfrak{g}) = \kappa(\mathfrak{g}_\gamma, \mathfrak{g}_{-\gamma})$.

(c) At each $h \in \mathfrak{h} = \mathfrak{g}_0$ we check
$$\kappa(h, [x,y]) = \kappa([h,x], y) = \gamma(h)\, \kappa(x,y)$$
$$= \kappa(h, \kappa(x,y)\cdot t_\gamma)$$
$\Rightarrow$ $[x,y] = \kappa(x,y)\, t_\gamma$.

(2)

e) If $\gamma(t_\gamma) = 0$ then we have the subalg $\mathfrak{m} = \mathfrak{g}_\gamma \oplus \mathbb{C}\cdot t_\gamma \oplus \mathfrak{g}_{-\gamma}$ which is solvable with $[\mathfrak{m},\mathfrak{m}] = \mathbb{C}\cdot t_\gamma$. Via the adj rep we have $\mathfrak{m} \overset{\cong}{\Rightarrow} \mathrm{ad}\,\mathfrak{m} \subseteq \mathfrak{gl}(\mathfrak{g})$ w/

$$\mathrm{Tr}(\mathrm{ad}_{t_\gamma}, \mathrm{ad}_x) = \kappa(t_\gamma, x) \subseteq \mathbb{C}\cdot \gamma(t_\gamma)$$

at all $x \in \mathfrak{m}$ $\Rightarrow$ ad_{t_γ} is nilpotent! Since ad_{t_γ} is also semisimple, we now find

$\mathrm{ad}_{t_\gamma} = 0 \Rightarrow t_\gamma = 0$, which is bad!

So we see $\gamma(t_\gamma) \neq 0$ necessarily.

f) By construction $\gamma(t_\gamma) = 2$. For $x \in \mathfrak{g}_\gamma$, $y \in \mathfrak{g}_{-\gamma}$ w/ $\kappa(x,y) = 1$ we have

$t_\gamma = [x,y] = -[y,x] = -t_{-\gamma}$ so that $t_{-\gamma} = -t_\gamma$

and $h_{-\gamma} = 2t_{-\gamma}/\kappa(t_{-\gamma}, t_{-\gamma})$

$= -2t_\gamma/\kappa(t_\gamma, t_\gamma) = -h_\gamma$. ∎

We'll see in a moment that each $\mathfrak{g}_\gamma$ is 1-dim.

Theorem 8.3: For semisimple $\mathfrak{g}$, $\gamma \in \Phi$, and nonzero $e_\gamma \in \mathfrak{g}_\gamma$, there exists unique $f_\gamma \in \mathfrak{g}_{-\gamma}$ for which the linear map

$$i_\gamma : \mathfrak{sl}_2(\mathbb{C}) \longrightarrow \mathfrak{g}, \quad \left\{ \begin{array}{l} e \mapsto e_\gamma \\ h \mapsto h_\gamma \\ f \mapsto f_\gamma \end{array} \right.$$

is an embedding of Lie algs.

We'll see later that Φ also splits as

$\Phi = \Phi^+ \amalg \Phi^-$.

At positive $\gamma \in \Phi^+$ the various

$i_\gamma : \mathfrak{sl}_2(\mathbb{C}) \hookrightarrow \mathfrak{g}$

provide our root subalg's in $\mathfrak{g}$, from which we observe our analysis of $\mathfrak{g}$-reps via highest wts. Hold...

~ An analysis of the root decomp "at a root".

Fix $\mathfrak{g}$ semisimple and $\gamma \in \Phi$. Then we have the copy of $\mathfrak{sl}_2$

$$i_\gamma : \mathfrak{sl}_2(\mathbb{C}) \hookrightarrow \mathfrak{g}$$

given by choosing any nonzero $e_\gamma \in \mathfrak{g}_\gamma$. Take

$$V \subseteq \mathfrak{g}, \quad V = \left(\sum_{c \in \mathbb{C}^*} \mathfrak{g}_{c\gamma}\right) \oplus \mathfrak{h}.$$

Then V is a $\mathfrak{sl}_2(\mathbb{C})$ subrep, where $\mathfrak{sl}_2$ acts via i_γ. Since $c \cdot \gamma(h_\gamma) = 2 \cdot c$ at all complex c, we have for the 0th wt space $\mathfrak{h} = V_0$.

For $\ker(\gamma) \subseteq \mathfrak{h}$ there's a complement subspace which is a $\mathfrak{sl}_2$-subrep, and we split V as

$$V = \ker(\gamma) \oplus i_\gamma(i\gamma) \oplus V'$$

with $(V')_0 = 0$ implying $\mathfrak{g}_{2\gamma} \subseteq V_4 = V'_4 = 0$.

One now sees, by way of contradiction, that $\mathfrak{g}_{\frac{1}{2}\gamma} = 0$.

Now, $\mathfrak{g}_{c\gamma} = V_{2c}$ at all complex $\mathbb{C}$ so that

$$\sum_c \mathfrak{g}_{c\gamma} = \bigoplus_{n \in \mathbb{Z}_{\neq 0}} \mathfrak{g}_{n\gamma/2}$$

w/ each $\mathfrak{g}_{n\gamma/2} = V_n$. So we see

$$V'_1 = 0 \text{ and } V'_0 = 0 \Rightarrow V' = 0.$$

So we conclude

$$V = \mathfrak{h} \oplus \mathbb{C}e_\gamma \oplus \mathbb{C}f_\gamma.$$

Since $\mathfrak{g}_{\pm\gamma} \subseteq V$ we conclude

$$\mathfrak{g}_\gamma = \mathbb{C} \cdot e_\gamma \text{ and } \mathfrak{g}_{-\gamma} = \mathbb{C} \cdot f_\gamma.$$

Structural Lemma 8.4A: Let $\mathfrak{g}$ be semisimple, and $\gamma \in \Phi$. Then (a) $\mathfrak{g}_\gamma$ is 1-dim'l.

(b) $\mathbb{Z} \cdot \gamma \cap \Phi = \{\gamma, -\gamma\}$.

(c) For any $\eta \in \Phi$, $\eta(h_\gamma) \in \mathbb{Z}$ and

Let us now consider the sl_2-subrep
$$V(\eta;\gamma) = \bigoplus_{m\in\mathbb{Z}} \mathfrak{g}_{\eta+m\gamma}$$
at η distinct from γ. We have
$$(\eta+m\gamma)(h_\gamma) = \eta(h_\gamma) + 2m$$
so that each wt. space in $V(\eta;\gamma)$ is 1-dim. Hence $V(\eta;\gamma)$ is simple over $sl_2(\mathbb{C})$ and we have

$$V(\eta;\gamma) = \mathfrak{g}_{\eta+q\gamma} \oplus \mathfrak{g}_{\eta+(q-1)\gamma} \oplus \cdots \oplus \mathfrak{g}_{\eta-r\gamma}$$
w/ each $\mathfrak{g}_{\eta+m\gamma} \neq 0$ for $q \geq m \geq -r$, and
$$\eta(h_\gamma) - 2r = -(\eta(h_\gamma) + 2q)$$
$$\Rightarrow \quad \eta(h_\gamma) = r-q.$$
$$\Rightarrow \quad \mathfrak{g}_{\eta-\eta(h_\gamma)\cdot\gamma} \text{ is nonzero.}$$

Structural Lemma 8.4.18: For $\gamma, \eta \in \Phi$

(a) The function $\eta - \eta(h_\gamma)\cdot\gamma$
$$= \eta - {}^{2(\eta,\gamma)}\!/\!_{(\gamma,\gamma)} \cdot \gamma$$

is also a root.

(b) Supposing $\eta \neq \pm\gamma$, $[\mathfrak{g}_\gamma, \mathfrak{g}_\eta] = \mathfrak{g}_{\gamma+\eta}$ iff $\gamma+\eta$ is a root.

(c) Supposing $\eta \neq \pm\gamma$, q, r the max integers so that $\eta+q\gamma$ and $\eta-r\gamma$ are roots, we have
$$\eta + i\gamma \in \Phi \text{ at all } q \geq i \geq -r,$$
and $\dfrac{2(\eta,\gamma)}{(\gamma,\gamma)} = r-q.$

– The Killing form on $\mathfrak{h}^*$

We have the nondegen restriction $\kappa|_{\mathfrak{h}\times\mathfrak{h}}$ providing a linear $\cong$, $\mathfrak{h} \to \mathfrak{h}^*$, $t_\gamma \mapsto \gamma$. Via this iso we observe a non-degen form on $\mathfrak{h}^*$, which we also denote simply by $(-,-) : \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{C}$.

At each pair of roots γ, ξ we have

$$(\gamma, \xi) = \kappa(t_\gamma, t_\xi)$$

– Cartan integers

Def^n: Given roots $\xi, \gamma \in \Phi$, the assoc. Cartan integer is the integer

$$\xi(h_\gamma) = 2(\xi, \gamma)/(\gamma, \gamma) \in \mathbb{Z}.$$

– Δ root system for any semisimple $\mathfrak{g}$.

We consider $\mathfrak{h}^*$ along w/ its induc. (Killing) form $(-,-): \mathfrak{h}^* \times \mathfrak{h}^*$.

Theorem 8.5A: For any choice of subset $\Delta \subseteq \Phi$ which prov. a basis for $\mathfrak{h}^*$, and γ in Φ, there exist $c_i \in \mathbb{Q}$ for which $\gamma = \sum_{i=1}^n c_i \alpha_i$.

$\Delta = \{\alpha_1, \ldots, \alpha_n\}$

Corollary 8.5A: For $\mathfrak{h}^*_F = F \cdot \Phi$ we have

$$\dim_F(\mathfrak{h}^*_F) = \dim_{\mathbb{C}}(\mathfrak{h}^*),$$

and the natural map

$$\mathbb{C} \otimes_F \mathfrak{h}^*_F \longrightarrow \mathfrak{h}^*$$

is a linear isomorphism.

Any intermediate field $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$

Proof of Proposition 8.5: Write (uniquely) $\gamma = \sum_j c_j \alpha_j$ w/ the $c_j \in \mathbb{C}$.

Then we have

$$(\alpha_i, \gamma) = \sum_j c_j (\alpha_i, \alpha_j)$$

$$\Rightarrow 2(\alpha_i, \gamma)/(\alpha_i, \alpha_i) = \sum_j 2c_j (\alpha_i, \alpha_j)/(\alpha_i, \alpha_i).$$

w/ all $2(\alpha_i, \beta)/(\alpha_i, \alpha_i)$ integers by Lemma 8.4B.

Hence for the rational matrix $A = [2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)]$

and $\vec{c} = [c_1 \ldots c_n]$, $\vec{v}_\gamma = [2(\alpha_i, \gamma)/(\alpha_i, \alpha_i)]^t$, We have

$$\vec{v_\gamma} = A \cdot \vec{c}.$$

For an arbitrary function $\sum_{\gamma} \xi_j \cdot \alpha_j$ we have some $\overset{\text{"}}{f}$

i at which

$$0 \neq (f, \alpha_i) \Rightarrow 2(f, \alpha_i)/(\alpha_i, \alpha_i) \neq 0$$

$$\Rightarrow A \cdot [\xi_1 \cdots \xi_n]^t \neq 0.$$

So A has no kernel, and is thus invertible over $\mathbb{Q}$. Hence we have

$$\vec{c} = A^{-1} \cdot \vec{v_\gamma} \in \mathbb{Q}^n,$$

so that $\gamma \in \mathbb{Q} \cdot \Delta$. ∎

Lemma 8.5B: For any $v, w \in \mathfrak{h}^*_{\mathbb{R}}$,

$$(v, w) = \sum{}'_{\gamma \in \Phi} (\gamma, v) \cdot (\gamma, w).$$

Proof: For $t = t_v$ and $t' = t_w$ we have

$$(v, w) = \kappa(t, t') = \mathrm{Tr}_{\mathfrak{g}}(\mathrm{ad}_t\, \mathrm{ad}_{t'})$$

$$= \sum{}'_{\gamma} \gamma(t) \cdot \gamma(t')$$

$$= \sum{}'_{\gamma} \kappa(t, t_\gamma) \cdot \kappa(t', t_\gamma)$$

$$= \sum_{\gamma} (v, \gamma) \cdot (w, \gamma).$$ ∎

Lemma 8.5C: For $v, w \in \mathfrak{h}^*_{F}$,

$$(v, w) \in F \subseteq \mathbb{C}.$$

Rather, the form $(-,-)$ is F-valued on $\mathfrak{h}^*_F$.

Proof: Suffices to show $(\xi_1, \xi_2) \in \mathbb{Q}$ at all $\xi_i \in \Phi$.

We have $(\xi, \xi) = \sum{}'_{\gamma} (\xi, \gamma)^2$

$$\Rightarrow 4(\xi, \xi)^{-1} = \sum{}'_{\gamma} 4(\xi, \gamma)^2/(\xi, \xi)^2 \in \mathbb{Z}$$

$$\Rightarrow (\xi, \xi) \in \mathbb{Q}. \quad \text{at all } \xi \in \Phi$$

Since $2(\xi_1, \xi_2)/(\xi_1, \xi_1) \in \mathbb{Z}$ we conclude now $(\xi_1, \xi_2) \in \mathbb{Q}$ as well. ∎

Corollary 8.5C: For any nonzero $v \in \mathfrak{h}_{\mathbb{R}}$, $(v, v) > 0$.

Proof: We have $(v,v) = \sum'_{\gamma \in \Phi} (\gamma, v)^2$ with all (γ, v) real, by Lemma 8.5.C. By non-degeneracy we also have $(\gamma, v) \neq 0$ at some γ, so that $(v, v) \in \mathbb{R}_{>0}$. ∎

~ The root system associated to semisimple $\mathfrak{g}$

Theorem D: For any semisimple $\mathfrak{g}$ w/ choice of Cartan $\mathfrak{h}$. The Killing form provides a real, symmetric, positive definite form

$$(-,-) : \mathfrak{h}^*_{\mathbb{R}} \otimes \mathfrak{h}^*_{\mathbb{R}} \to \mathbb{R}$$

on the real span of the roots $\mathfrak{h}^*_{\mathbb{R}} = \mathbb{R} \cdot \Phi$.

Furthermore, the roots $\Phi \subseteq \mathfrak{h}^*_{\mathbb{R}}$ satisfy the following:

i) Φ is finite and spans $\mathfrak{h}^*_{\mathbb{R}}$.

ii) For $\gamma \in \Phi$, $\mathbb{R} \cdot \gamma \cap \Phi = \{\gamma, -\gamma\}$.

iii) For any $\gamma \in \Phi$, the reflection across $\gamma^{\perp}$

$$\sigma_\gamma : \mathfrak{h}^*_{\mathbb{R}} \to \mathfrak{h}^*_{\mathbb{R}}, \quad \sigma_\gamma(v) = v - \frac{2(\gamma, v)}{(\gamma, \gamma)} \gamma,$$

preserves Υ.

iv) For each $\gamma, \xi \in \Phi$, $2(\gamma, \xi)/(\gamma, \gamma)$ is an integer.

Def^{n}: A root system is the pairing (E, Φ) of a Euclidean $\mathbb{R}$-vector space w/ a finite set Φ which satisfies conditions (i) – (iv) from Thm D.