Aug 28.

– Lie algs from Lie groups.

Given any manifold M the vector fields $\{x : M \to TM\}$ have their natural bracket

$\mathrm{Vect}(M)$ $[x,y] : M \to TM$

which is given in coordinates by the expected formula. For $M = G$ a Lie group, or alg grp, we have the Lie subalg $\mathrm{Lie}(G) \subseteq \mathrm{Vect}(G)$ consisting of vector fields satisfying $g^{-1} x \cdot g = x$. Evaluating such vect fields at 1 gives

$$\mathfrak{g} \xrightarrow{\cong} T_1 G.$$

Def$^{\underline{n}}$: For an algebraic group G, a (rational, or algebraic) G representation is a vect space V equipped w/ a group action for which the action map

$$G \times V \to V$$

is a map of schemes/algebraic varieties.

Any G-representation inherits an "infinitesimal action" of the associated Lie alg $\mathfrak{g} = \mathrm{Lie}(G)$, which defines a functor $\mathrm{rep}(G) \to \mathrm{rep}(\mathfrak{g})$.

Theorem: For G a semisimple (connected) affine alg group $/\mathbb{C}$, then taking infinitesimal action defines a fully faithful functor

$$\mathrm{inf} : \mathrm{rep}(G) \longrightarrow \mathrm{rep}(\mathfrak{g}),$$

where $\mathfrak{g} = \mathrm{Lie}(G)$. When G is simply connected, this functor is an <u>equivalence</u>.

- Classical simple (alg.) groups and their Lie alg.

Type A_n) $SL_n(\mathbb{C})$ = det 1 $n\times n$ matrices

we have the det fun. on $GL_n(\mathbb{C})$

$$\det(I_n + [x_{ij}]) = 1 + (\textstyle\sum_i x_{ii}) + \text{higher deg.}$$

so that the infinitesimal part of the det fun is the trace fun $Tr: \mathfrak{gl}_n(\mathbb{C}) \to \mathbb{C}$

Hence $Lie(SL_n(\mathbb{C})) = \mathfrak{sl}_n(\mathbb{C})$.

Type $B_n + D_n$) The orthogonal group is the matrix group

$$O(m, \mathbb{C}) = \{A \in GL_m(\mathbb{C}) : A \cdot A^T = I_m\}$$

This group is not conn., the conn component of the id is

$$SO(m, \mathbb{C}) = \{A \in O(m, \mathbb{C}) : \det A = 1\}.$$

The type B_n alg group is

$$SO(2n+1, \mathbb{C})$$

type D_n is

$$SO(2n, \mathbb{C})$$

The corresp Lie alg's are

$$\mathfrak{so}(m) := \{x \in \mathfrak{gl}_m(\mathbb{C}) : x = -x^t\}$$

for m odd and even respectively.

Type C_n) Consider the vector space $\mathbb{C}^{2n}$ w/ its unique (up to $\cong$) symplectic form

$$\langle e_i, e_j \rangle := \begin{cases} 1 & \text{if } j = n+i \\ -1 & \text{if } i = n+j \\ 0 & \text{else} \end{cases}$$

Take $Sp(2n, \mathbb{C}) := \{A \in GL_{2n} : \langle v, w\rangle = \langle A\cdot v, A\cdot w\rangle$ for all $v, w\}$

= matrices which preserve the unique symplectic form on $\mathbb{C}^{2n}$.

This group is (almost-) simple w/ Lie algebra

$$\mathfrak{sp}(2n, \mathbb{C}) := \{x \in \mathfrak{gl}_{2n} : \langle x\cdot v, w\rangle + \langle v, x\cdot w\rangle = 0 \text{ for all } v, w.\}$$

There are also "exceptional" (almost-)simple groups and Lie algebras of types E_6, E_7, E_8), F_4 and G_2.