Aug 28 (1)

Recall, we have some examples of Lie algs

$\mathfrak{gl}(V)$, $\mathfrak{gl}_n(\mathbb{C})$

$\mathfrak{sl}(V)$, $\mathfrak{sl}_n(\mathbb{C})$, $\mathfrak{sl}_2(\mathbb{C}) = \left(\begin{array}{l} \text{span}\{e, f, h\} \\ [h, e] = 2e \\ [h, f] = -2f \\ [e, f] = h. \end{array} \right.$

A $\mathfrak{g}$-representation is a vect space V equipped w/ an "action" of $\mathfrak{g}$, $\cdot : \mathfrak{g} \otimes V \to V$, which satisfies

$$[x,y] \cdot v = x \cdot y \cdot v - y \cdot x \cdot v.$$

Any rep specifies, and is specified by, its corresponding map to $\mathfrak{gl}(V)$, $\rho_V : \mathfrak{g} \to \mathfrak{gl}(V)$, $\rho_V(x) = x \cdot -$.

Ex [Adjoint rep] Any Lie alg $\mathfrak{g}$ acts on itself via the adjoint action $\cdot_{adj} : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$.

$$x \cdot_{adj} y = [x,y].$$

The requisite eq $\quad [x,y]z = x \cdot y \cdot z - y \cdot x \cdot z$

$$[[x,y]z] = [x[y,z]] - [y[x,z]]$$

is equiv to the Jacobi identity

$$[x[y,z]] = [[x,y]z] + [y[x,z]],$$

so that the adj rep $(\mathfrak{g}, \cdot_{adj})$ is seen to be a $\mathfrak{g}$-representation.

Re-Def: $\mathfrak{g}$ is simple if $\mathfrak{g}$ has no proper nonzero ideals, and $\mathfrak{g}$ is not the 1-dim abelian Lie alg.

Observation 1: If $\mathfrak{g}$ is simple, then the adj rep map $\quad ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is an inj Lie alg hom.

Proof: We already know it a Lie alg hom. Simplicity of $\mathfrak{g}$ then says $\ker(ad) = 0$ or $\ker(ad) = \mathfrak{g}$. The latter case occurs iff $\mathfrak{g}$ is abelian, which contradicts simplicity of $\mathfrak{g}$. Hence $\ker = 0$. $\blacksquare$

Short term plan:

- Provide complete analysis of rep($\mathfrak{sl}_2$). (3-4 classes)
- Discuss $\mathfrak{sl}_n$.
- Begin w/ general theory from Humphreys.

– Some category stuff

Def^n: For any finite-dimensional Lie alg $\mathfrak{g}$ we let rep($\mathfrak{g}$) denote the category of $\mathfrak{g}$-representations. The objects are $\mathfrak{g}$-reps, and morphisms are homomorphisms of $\mathfrak{g}$-representations, i.e. linear maps $\phi: V \to W$ which satisfy

$$\phi(x\cdot v) = x\cdot\phi(v) \quad \text{for all } x\in\mathfrak{g},\ v\in V.$$

A subrepresentation $V' \subseteq V$ is a linear subspace which is stable under the action of $\mathfrak{g}$.

Note that V' inherits a $\mathfrak{g}$-action, or $\mathfrak{g}$-rep structure, in this case. Call a $\mathfrak{g}$-rep <u>simple</u> if it has no proper, nonzero subrepresentations.

Example: The $\mathfrak{g}$-subreps in the adj rep are precisely the ideals $I \subseteq \mathfrak{g}$. (Hence $\mathfrak{g}$ is simple if and only if $\mathfrak{g}$ nonabelian w/ simple adjoint representation.

Lemma 2: If $\phi: V\to W$ is a homomorph of $\mathfrak{g}$-reps then

a) The kernel $\ker(\phi)\subseteq V$ is a subrepresentation in V,

b) The image $\phi(V)\subseteq W$ is a subrep in W,

c) The quotient $W/\phi(V)$ inherits a unique $\mathfrak{g}$-rep structure so that the quotient map $\pi: W\to W/\phi(V)$ is a map of $\mathfrak{g}$-reps.

d) ϕ is an isomorphism iff $\ker(\phi)=0$ and $\phi(V)=W$

Proof: The proof just follows by standard observations. For example, (a) if $v \in \ker(\phi)$ then $\phi(x\cdot v) = x\cdot\phi(v) = x\cdot 0 = 0$. Hence the kernel is stable under the action of $\mathfrak{g}$, and thus a $\mathfrak{g}$-subrep. For (c) we have the right exact seq of vect spaces $V \to W \to W' \to 0$ for $W' = W/\phi(V)$ and apply the right exact fun $\mathfrak{g}\otimes -$ to get

$$\mathfrak{g}\otimes V \to \mathfrak{g}\otimes W \to \mathfrak{g}\otimes W' \to 0$$

and by univ prop of cokernel $\exists$ unique action $\mathfrak{g}\otimes W' \to W'$ which completes the diag

$$\begin{tikzcd}
\mathfrak{g}\otimes V \arrow[r] \arrow[d, "\cdot"] & \mathfrak{g}\otimes W \arrow[r] \arrow[d, "\cdot"] & \mathfrak{g}\otimes W' \arrow[r] \arrow[d, dashed, "\exists!"] & 0 \\
V \arrow[r] & W \arrow[r] & W' \arrow[r] & 0
\end{tikzcd}$$

This action is given on classes by $x\cdot\bar{w} := \overline{x\cdot w}$, and inherits the identities $[x,y]\cdot\bar{w} = x\cdot y\cdot\bar{w} - y\cdot x\cdot\bar{w}$ from the corresp. id on W.

(d) For the linear inverse ϕ^{-1} we have

$$\phi^{-1}(x\cdot w) = \phi^{-1}(\phi(x\cdot\phi^{-1}(w))) = x\cdot\phi^{-1}(w),$$

so that ϕ^{-1} seen to be a map of $\mathfrak{g}$-reps. ∎

Also easy to check the following:

- A $\mathbb{C}$-scaling $c\cdot\phi$ of a $\mathfrak{g}$-rep hom $\phi: V\to W$ is again a map of $\mathfrak{g}$-reps, as is any sum $\phi+\phi'$ of $\mathfrak{g}$-rep homs. Hence

$$\mathrm{Hom}_{\mathfrak{g}}(V,W) := \mathrm{Hom}_{\mathrm{rep}(\mathfrak{g})}(V,W)$$

is a vector subspace in $\mathrm{Hom}_{\mathbb{C}}(V,W)$.

- The sum $V_1\oplus V_2$ inherits a unique $\mathfrak{g}$-rep structure so that the two inclusions $V_i \to V_1\oplus V_2$ are — maps of $\mathfrak{g}$-reps. Furthermore, this sum is both a product and coproduct in $\mathrm{rep}(\mathfrak{g})$ (look it up!

Taken together we conclude that

$\mathrm{rep}(\mathfrak{g})$ is a $\mathbb{C}$-linear abelian category.

can take linear combos of morphisms

has kernels and cokernels

Def^{n}: Call an abelian cat $\mathcal{C}$ Artinian if every seq of subobjects $V_0 \supseteq V_1 \supseteq V_2 \supseteq \cdots$ stabilizes.

Call $\mathcal{C}$ semisimple if every exact sequence

$$0 \to V \xrightarrow{\phi} W \xrightarrow{\phi'} V' \to 0$$

splits, i.e. if there exists $\psi : W \to V$ satisfying $\psi\phi = \mathrm{id}_V$ or $\psi' : V' \to W$ w/ $\phi'\psi' = \mathrm{id}_{V'}$.

Observe that $\mathcal{C} = \mathrm{rep}(\mathfrak{g})$ is Artinian.

Indeed, since each obj is fin. dim / $\mathbb{C}$ and desc. seq of subreps must stabilize ~~for~~ dim reasons.

Goal: $\mathrm{rep}(\mathfrak{sl}_2)$ is semisimple.

- Aside: lengths and JH series.

Let $\mathcal{C}$ be an Artinian cat, and V be an object. A Jordan-Hölder series for V is a seq of proper submodules

$$0 = V_n \subsetneq V_{n-1} \subsetneq \cdots \subsetneq V_0 = V \qquad (*)$$

for which each quotient V_i / V_{i+1} is a nonzero simple object in $\mathcal{C}$. (Here simple means cont. no proper nonzero subobj.) The length of such a series $(*)$ is n.

Theorem 3 (JH series) For any two JH series

$$0 = V'_m \subsetneq V'_{m-1} \subsetneq \cdots \subseteq V'_0 = V$$

$$0 = V_n \subsetneq V_{n-1} \subseteq \cdots \subseteq V_0 = V$$

we have $n = m$, and for some permutation $\sigma \in S_n$ there are isoms

$$V_i / V_{i+1} \cong V_{\sigma(i)} / V_{\sigma(i)+1}$$

in $\mathcal{C}$.

Proof: Exercise. ∎

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Def^{n}: For any object V in an Artinian cat $\mathcal{C}$, the length of V is the length n of any JH series

$$0 = V_n \subseteq V_{n-1} \subseteq \cdots \subseteq V_0 = V.$$

The composition factors are, up to isomorphism, the simples which appear in the collection $\{ V_i / V_{i+1} : 0 \le i \le n-1 \}$.

Proposition 4: For an Artinian category $\mathcal{C}$ the following are equivalent.

a) $\mathcal{C}$ is semisimple.

c) Every obj V decomp. as a sum of simples $V = \bigoplus_{i=1}^{m} L_i$.

b) Any extension $0 \to V \to W \to V' \to 0$ in which V and V' are simple splits.

Sketch Proof: (a) $\Rightarrow$ (b) is trivial. Assume now that (b) holds. Then any seq

$$0 \to V \to W \to V' \to 0 \qquad (*)$$

in which

$$\text{length}(W) = \text{length}(V) + \text{length}(V') \le 2$$

is split. Suppose now that a seq $(*)$ is s.t. $\text{length}(W) = n+1$ and that all seq w/ middle term of length $\le n$ split. We can assume $n > 2$, so that one of $\text{length}(V)$ or $\text{length}(V') > 1$. Assume first that $\text{length}(V') > 1$, and consider an exact sequence

$$0 \to V_1' \to V' \to V_0' \to 0$$

with V_0' simple.

By taking fiber products we obtain an exact seq

$$0 \to V \to W_1 = W \times_{V'} V_1' \to V_1' \to 0,$$

which is split since $\text{length}(W_1) = \text{length}(V) + \text{length}(V_1') = n$.

So we have a splitting

$$W_1 \cong V \oplus V_1'.$$

Take now $W_0 = W/\mathrm{im}\, V_1'$, with the splitting map $V_1' \xrightarrow{s} W_1 \hookrightarrow W$, and note the exact seq $0 \to V \to W_0 \to V_0' \to 0$ and a diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
V \arrow[r, "="] \arrow[d] & V \arrow[d] \\
W \arrow[r] \arrow[d] & W_0 \arrow[d] \\
V' \arrow[r] \arrow[d] & V_0' \arrow[d] \\
0 & 0
\end{tikzcd}$$

and the induced map to the fibre product

$$W \to V' \times_{V_0} W_0$$

is an isomorphism. So we see that the projection $W \to V'$ is split if the projection $W_0 \to V_0'$ is split. However, the latter splitting occurs by our inductive hypothesis, so that the seq $0 \to V \to W \to V' \to 0$ is in fact split.

The argument in the case $\mathrm{length}(V) > 1$ is similar. ∎