(B)

$$\mathfrak{sl}_2^* = \begin{matrix} f^* \\ h^* \\ e^* \end{matrix}$$

$$\mathfrak{sl}_2^* \xrightarrow{\cong} \mathfrak{sl}_2, \qquad \begin{matrix} f^* \mapsto e \\ h^* \mapsto \tfrac{1}{2} h \\ e^* \mapsto f \end{matrix}$$

$$\Rightarrow \qquad \begin{matrix} \mathrm{Tr}(e,f) = 1 \\ \mathrm{Tr}(h,h) = 2 \end{matrix}$$

$$\begin{tikzcd} \mathfrak{sl}_2 \otimes \mathfrak{sl}_2 \arrow[r, "\text{trace form}"] & \mathbb{C} \\ \mathfrak{sl}_2^* \otimes \mathfrak{sl}_2^* \arrow[u, "\cong"] \arrow[ur, "\omega"'] & \end{tikzcd}$$

$\omega(e^*, f^*) = \omega(f^*, e^*) = 1$

$\omega(h^*, h^*) = \frac{1}{2}$ vanishes elsewhere

0 on all other basic pairings.

~~Have~~ $\mathrm{eval}: \mathfrak{sl}_2 \otimes \mathfrak{sl}_2 \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{C}}(\mathfrak{sl}_2^* \otimes \mathfrak{sl}_2^*, \mathbb{C})$

$$\Rightarrow \qquad \Omega \longleftarrow\!\shortmid\ \omega$$

$$\Omega = \tfrac{1}{2} h^2 + ef + fe.$$

We now understand how sl_2-reps behave, essentially completely: i.e. all sl_2-reps V decomp into eigenspaces for the action of $h \in sl_2(\mathbb{C})$

$$V = \bigoplus_{\mu \in \mathbb{Z}} V_\mu ,$$

are semisimple, and each simple $L(\lambda)$ is precisely structured according to some rules.

- The structure of sl_n I

For sl_2 we have the decomp $sl_2 = \mathbb{C}f \oplus \mathbb{C}h \oplus \mathbb{C}e$ which we exploit repeatedly to assess the structure of sl_2-reps.

For sl_n we have

$$sl_n = \underbrace{\mathrm{span}\{E_{ii} - E_{i+1,i+1} : 1 \le i < n\}}_{=: \mathfrak{h}} \text{ diagonal stuff}$$

$$\oplus \underbrace{\mathrm{span}\{E_{ij} : i > j\}}_{\text{strict lower } \Delta' r \;=:\; n^-} \oplus \underbrace{\mathrm{span}\{E_{ij} : i < j\}}_{\text{strict upper } \Delta' r \;=:\; n^+}$$

$$sl_n(\mathbb{C}) = n^- \oplus \mathfrak{h} \oplus n^+ .$$

Lemma 1: The bracket on $sl_n \subseteq gl_n$ is given as

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{il} E_{kj} .$$

From this one sees directly that

Lemma 2: Each subspace $n^\pm, \mathfrak{h} \subseteq sl_n$ is a Lie subalg, and $\mathfrak{h}$ is abelian (has vanishing bracket).

Lemma 3: The Lie alg sl_n is generated by the vectors $e_i = E_{i,i+1}$, $f_i = E_{i+1,i}$ [and $h_i = E_{ii} - E_{i+1,i+1}$]. Aka the only subalg which cont. these elems is all of $sl_n(\mathbb{C})$ itself.

The subalg $\mathfrak{h}$ in sl_n is called the (standard) Cartan subalg. The subalg's $n^\pm$ are the pos. and neg nilpotent sub lgs.

The subalgs $\mathfrak{b}^{\pm} := \mathfrak{h} \oplus \mathfrak{n}^{\pm}$ are called (2)
the pos. and neg. Borel subalgs.

- The structure of $\mathfrak{sl}_n$ II.

Def^{n}: For any $\mathfrak{sl}_n$-rep V, a nonzero vector $v \in V$ is said to be an eigenvector for the action of the Cartan $\mathfrak{h} \subseteq \mathfrak{sl}_n$ if for each $x \in \mathfrak{h}$ $x \cdot v \in \mathbb{C} \cdot v$.

The corresp. eigenfunction $\lambda \in \mathfrak{h}^*$ is the unique linear fun which satisfies $x \cdot v = \lambda(x) \cdot v$ at all x in $\mathfrak{h}$. Given $\lambda \in \mathfrak{h}^*$ we let V_λ denote the corresponding weight space / eigenspace. $V_\lambda = \{w \in V : x \cdot w = \lambda(x) \cdot w \text{ at all } x \in \mathfrak{h}\}$

Lemma 4: Each vector $E_{ij} \in \mathfrak{sl}_n(\mathbb{C})$, $i \neq j$ is an eigenvector for the adj action of $\mathfrak{h}$ assoc. to nonzero eigenfunction $\gamma : \mathfrak{h} \to \mathbb{C}$. Further, if $v \in \mathfrak{sl}_n(\mathbb{C})$ is an eigenvector for the Cartan then $v \in \mathfrak{h}$ or $v \in \mathbb{C} \cdot E_{ij}$ for unique i, j.

Proof: The eigenvector claim is clear since for any diag matrix D, $D \cdot E_{ij}, E_{ij} \cdot D \subseteq \mathbb{C} \cdot E_{ij}$. For the uniqueness claim consider E_{ij}, E_{kl} with $i \neq k$. Then $[E_{ii} - E_{jj}, E_{ij}] = 2 E_{ij}$ while $[E_{ii} - E_{jj}, E_{kl}] \in \mathbb{Z}_{\leq 1} \cdot E_{kl}$. Hence the corresp. eigenfunctions for E_{ij} and E_{kl} are distinct, since they take distinct values on $E_{ii} - E_{jj} \in \mathfrak{h}$. ∎

Lemma 5: Let $\gamma \in \mathfrak{h}^*$ be the eigenfun. for E_{ij}. Then the eigenvector for E_{ji} is $-\gamma \in \mathfrak{h}^*$.

Proof: Follows from the bracket rule given in Lemma 1. ∎

Def^n: A function $\gamma \in \mathfrak{h}^*$ is called a root for $\mathfrak{sl}_n(\mathbb{C})$ if γ is nonzero and the corresp. eigenspace $\mathfrak{sl}_n(\mathbb{C})_\gamma$ is nonzero.

We let $\Phi = \{$the coll. of all roots in $\mathfrak{h}^*\}$

pos roots $\Phi^+ = \{$All roots $\gamma \in \Phi$ for which $\mathfrak{n}^+_\gamma \neq 0\}$

neg roots $\Phi^- = \{$All roots $\gamma \in \Phi$ for which $\mathfrak{n}^-_\gamma \neq 0\}$.

Corollary 6: a) $\Phi^- = -\Phi^+$.

b) $\Phi = \Phi^+ \cup \Phi^-$.

c) Each weight space $\mathfrak{sl}_n(\mathbb{C})_\gamma$ is of dim 1.

d) $\mathfrak{sl}_n(\mathbb{C}) = \left(\bigoplus_{\mu \in \Phi^-} \mathfrak{sl}_n(\mathbb{C})_\mu\right) \oplus \mathfrak{h} \oplus \left(\bigoplus_{\gamma \in \Phi^+} \mathfrak{sl}_n(\mathbb{C})_\gamma\right)$

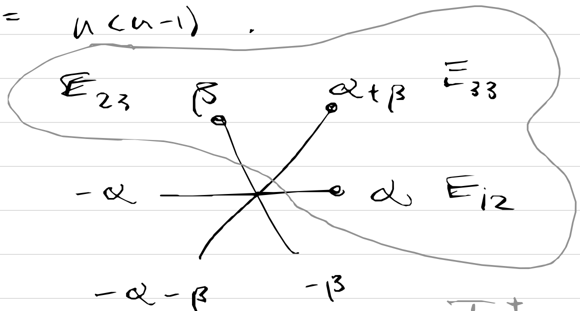
Proof: (a) Follows by Lemma 5. (b) Follows by Lemma 4. (c) Follows by Lemma 4. ∎

Rem: Note that Φ is not a subspace in $\mathfrak{h}^*$, it is a subset.

We have $|\Phi^+| = \dim \mathfrak{n}^+ = \frac{n(n-1)}{2}$

and $|\Phi| = n(n-1)$.

e.g. $\Phi(\mathfrak{sl}_3) =$



Φ^+

- Root subalgebras.

Proposition 7: For arbitrary $\gamma \in \Phi$, and nonzero $e_\gamma \in \mathfrak{sl}_n(\mathbb{C})_\gamma$, there exists a unique vector $f_\gamma \in \mathfrak{sl}_n(\mathbb{C})_{-\gamma}$ so that the unique linear map
$$i_\gamma : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{sl}_n(\mathbb{C}), \begin{cases} e \mapsto e_\gamma \\ f \mapsto f_\gamma \\ h \mapsto [e_\gamma, f_\gamma] =: h_\gamma \end{cases}$$
is an injective Lie alg homomorphism.

Furthermore, the vector h_γ is indep. of the choice (4)
of e_γ and the image $\operatorname{im}(i_\gamma) \subseteq sl_n(\mathbb{C})$
is uniquely det by γ (i.e. doesn't depend on e_γ).

Proof: First note that the triple Take E_{ij} so that $sl_n(\mathbb{C})_\gamma = \mathbb{C} \cdot E_{ij}$.

$$E_\gamma = E_{ij}, \quad F_\gamma = E_{ji}, \quad h_\gamma = E_{ii} - E_{jj}$$

def such a Lie alg embedding

$$sl_2 \to sl_n, \quad e \mapsto E_\gamma, \quad f \mapsto F_\gamma, \quad h \mapsto h_\gamma.$$

Now for any choice of e_γ we have $e_\gamma = c \cdot E_\gamma$
for unique $c \in \mathbb{C}^\times$ and for any $d \in \mathbb{C}^\times$
we have $[e_\gamma, dF_\gamma] = c \cdot d \cdot h_\gamma$ so that

$$[[e_\gamma, dF_\gamma], e_\gamma] = (2 \cdot c \cdot d) \cdot e_\gamma.$$

Hence we have the unique scaling $f_\gamma = c^{-1} \cdot F_\gamma$
so that the triple $\{e_\gamma, f_\gamma, h_\gamma\}$ specifies
such an embedding $i_\gamma : sl_2(\mathbb{C}) \hookrightarrow sl_n(\mathbb{C})$.

For the uniqueness claim, we always have

$$\operatorname{im}(i_\gamma) = \mathbb{C} \cdot E_\gamma \oplus \mathbb{C} H_\gamma \oplus \mathbb{C} \cdot F_\gamma$$
$$= sl_n(\mathbb{C})_\gamma \oplus \mathbb{C} \cdot h_\gamma \oplus sl_n(\mathbb{C})_{-\gamma}. \quad \#$$

Lets just collect what we've seen here:

For each positive root $\gamma \in \Phi^+$ we get a copy
of $sl_2(\mathbb{C})$ in $sl_n(\mathbb{C})$,

$$i_\gamma : sl_2(\mathbb{C}) \to sl_n(\mathbb{C}).$$

This is an <u>root subalg</u> assoc. to γ. The map i_γ
itself is not det. by γ, but its image, i.e. the assoc
subalg in $sl_n(\mathbb{C})$ is.

Further, for each root γ we have a uniquely
assoc vector $h_\gamma \in \mathfrak{h}$. There are $\frac{n(n-1)}{2}$
such vectors, and they span $\mathfrak{h}$.

- Simple root vectors

Proposition 8: There is a unique subset of positive roots $\Delta \subseteq \Phi^+$ satisfying the following

a) Δ is lin. indep in $\mathfrak{h}^*$, and in fact pov. chars.

b) $\Phi^+ \supseteq \mathbb{Z}_{\geq 0} \cdot \Delta$.

Proof: Consider

$\Delta = \{$ the weights for the superdiag. elem's $E_{i,i+1}\}$

$= \{\alpha_1, \ldots, \alpha_{n-1} : \alpha_i = \text{wt. for } E_{i,i+1}\}$.

Since for all $i < j$

$$E_{ij} = [[[E_{i,i+1}, E_{i+1,i+2}] \cdots] E_{j-1,j}]$$

and $[\mathfrak{e}_\mu, \mathfrak{e}_\nu] \in (\mathfrak{sl}_n)_{\mu+\nu}$ via Jacobi, we see that (b) holds.

Since $|\Delta| = n-1 = \dim \mathfrak{h}^*$ it suffices to show now that Δ spans $\mathfrak{h}^*$. For this it suffices to show that for each $x \in \mathfrak{h}$ we have $x = 0$ iff $\alpha(x) = 0$ at all $\alpha \in \Delta$. Write

$$x = \sum_{i=1}^{n-1} c_i h_{\alpha_i}, \quad h_{\alpha_i} = E_{ii} - E_{i+1,i+1},$$

and observe

$$\alpha_i(x) E_{i,i+1} = [x, E_{i,i+1}] = (-c_{i-1} + 2c_i - c_{i+1}) \cdot E_{i,i+1}$$

so that $\alpha_i(x) = 0$ at all i $\Leftrightarrow$

$$\mathrm{Cart}_n [c_1 \cdots c_{n-1}]^t = \vec{0} \qquad (*)$$

where

$$\mathrm{Cart}_n = \begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & 0 \\ & -1 & \ddots & -1 \\ 0 & & -1 \; 2 & -1 \\ & & -1 & 2 \end{bmatrix}.$$

We calculate by induction $\det(\mathrm{Cart}_n) = n \neq 0$ so that the eq. (*) forces $x = 0$, as desired.

We leave uniqueness as an exercise. ∎

—

Def^n: The subset $\Delta \subseteq \Phi^+$ as in Proposition 8 is called the (or a, depending) base for Φ. The elem of Δ are called the simple roots for sl_n.

Recall explicitly

$$\Delta = \{\alpha_1, \dots, \alpha_{n-1} : sl_n(\mathbb{C})_{\alpha_i} = \mathbb{C}E_{i,i+1}\}.$$

We have also $h_{\alpha_i} = E_{ii} - E_{i+1,i+1}$.

Observation/Corollary 9: For the simple roots $\alpha \in \Delta$, the corresponding vectors

$$\{h_\alpha : \alpha \in \Delta\}$$

provide a basis for the Cartan subalg $\mathfrak{h}$, and for all $\gamma \in \Phi^+$ h_γ is in the non-neg span

$$h_\gamma \in \mathbb{Z}_{\geq 0}\{h_\alpha : \alpha \in \Delta\}.$$

- Weights and dominant weights.

Def^n: A weight for $sl_n(\mathbb{C})$ is a function $\lambda \in \mathfrak{h}^*$ which takes integer values

$$\lambda(h_\gamma) \in \mathbb{Z} \text{ at all } \gamma \in \Phi^+.$$

A weight λ is called dominant if it takes nonnegative integer values

$$\lambda(h_\gamma) \in \mathbb{Z}_{\geq 0} \text{ at all } \gamma \in \Phi^+.$$

We take

$$P := \{\text{all weights in } \mathfrak{h}^*\}$$

$$P^+ := \{\text{all dominant weights in } \mathfrak{h}^*\}.$$

Lemma 10: $\lambda \in \mathfrak{h}^*$ is a wt. iff $\lambda(h_\alpha) \in \mathbb{Z}$ for all simple α, and a wt $\lambda \in P$ is dominant iff

$$\lambda(h_\alpha) \geq 0 \text{ at all simple } \alpha.$$

Proof: Immediate from Corollary 9. ∎

Lemma 11: Every root $\gamma \in \Phi$ is also a weight.

Proof: Since $\Phi = \Phi^+ \cup \Phi^-$ and $\Phi^- = -\Phi^+$, and $\Phi^+ \subseteq \mathbb{Z}_{\geq 0} \Delta$, it suffices to show that each simple root α is a weight. But we have

$$\alpha_j(h_{\alpha_i}) = \text{the coeff of } E_{\alpha_j} \text{ in } [h_{\alpha_i}, E_{\alpha_j}]$$

$$= \begin{cases} 2 & \text{if } i = j \\ -1 & \text{if } |i-j| = 1 \\ 0 & \text{else.} \end{cases}$$

We're done. ∎

Note that P is a lattice in $\mathfrak{h}^*$, i.e. a $\mathbb{Z}$-submodule w/ $\mathbb{C} \otimes_{\mathbb{Z}} P \xrightarrow{\cong} \mathfrak{h}^*$ via the natural map. We have

$Q := \mathbb{Z} \cdot \Phi \subseteq P$, another lattice.

We call Q the <u>root lattice</u> and P the <u>weight lattice</u>.

- Δ partial ordering on the weights

Def: For weights $\mu, \lambda \in P$ we write

$$\mu \leq \lambda \quad \text{if} \quad \lambda - \mu \in \mathbb{Z}_{\geq} \cdot \Delta,$$

i.e. if λ is obtained by a positive shift via simple roots $\lambda = \mu + \sum_i d_i \cdot \alpha_i$, $d_i \in \mathbb{Z}_{\geq 0}$.

Example: Take ϑ so that $\mathfrak{sl}_n(\mathbb{C})_\vartheta$ is $\mathbb{C} \cdot E_{1n}$. Then all roots γ satisfy

$$\vartheta \geq \gamma.$$

It is the <u>longest</u> root for $\mathfrak{sl}_n(\mathbb{C})$.

– Weights for $\mathfrak{sl}_n$-representations

Proposition 12: Let V be any finite dimensional $\mathfrak{sl}_n$-representation.

a) The Cartan subalg. $\mathfrak{h}$ acts semisimply on V, i.e. V decomposes into eigenspaces for the $\mathfrak{h}$-action.

b) If the eigenspace V_λ is nonzero, for $\lambda \in \mathfrak{h}^*$, then λ lies in the weight lattice.

Proof: We have
$$\mathfrak{h} = \mathbb{C}\cdot\{h_\alpha : \alpha \text{ simple}\}$$
and the h_α provide n commuting endomorphisms on V. Hence V decomposes into generalized eigenspaces for the action of these h_α, and hence for the action of $\mathfrak{h}$, $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda^{gen}$.

For each simple α we have the root subalg $i_\alpha : \mathfrak{sl}_2 \to \mathfrak{sl}_n$, and restricting along α realizes an $\mathfrak{sl}_2$-action on V for which $h \in \mathfrak{sl}_2$ acts by h_α. Since h acts semisimply on any $\mathfrak{sl}_2$-rep, we conclude that each h_α acts semisimply on V. Hence V decomposes into its eigenspaces $V = \bigoplus_\lambda V_\lambda$.

Now, if $V_\lambda \neq 0$, then we again consider $\mathfrak{sl}_2$ acting on V_λ via i_α to see that $\lambda(\alpha) \in \mathbb{Z}$ at each simple α, via our classification of (simple) $\mathfrak{sl}_2$-representations. [Corollary 5, "Aug 28th"]. So we see that λ is in the weight lattice. ∎

Def^n: A weight vector of V is a nonzero vector which lies in some weight space V_λ, $\lambda \in P$. We call a weight vector $v \in V_\lambda$ a highest wt. vector if $e_\alpha \cdot v = 0$ for each simple root α.

Lemma 13: Any nonzero (fin.-dim) $\mathfrak{sl}_n$-rep V admits a highest wt. vector v.

Proof: Take any wt μ with V_μ nonvanishing. Since the base Δ provides a basis for $\mathfrak{h}^*$, we have for any tuple of non-neg integers $c : \Delta \to \mathbb{Z}_{\geq 0}$ and $c' : \Delta \to \mathbb{Z}_{\geq 0}$, $\mu + \sum_\alpha c_\alpha \alpha = \mu + \sum_\alpha c'_\alpha \alpha$ iff $c_\alpha = c'_\alpha$ for all α. Hence the space of wts. $\{\lambda : \lambda \geq \mu \text{ and } V_\lambda \neq 0\}$ is finite and thus contains a max elem. λ under the ordering $\geq$. Any nonzero vector $v \in V_\lambda$ provides a highest wt. vect in V. ∎

Theorem 14: If $v \in V$ is a highest wt vector, with assoc. wt. $\lambda \in P$, then λ is dominant. Furthermore the subspace

$$L(\lambda) = \mathbb{C}\cdot\{ f_{\gamma_1} \cdots f_{\gamma_t} \cdot v : t \geq 0,\ \gamma : \{1, \dots, t\} \to \Phi^+ \}$$

forms a $\mathfrak{sl}_n(\mathbb{C})$-subrep in V.

Proof: By restriction along any root subalg. $\iota_\alpha : \mathfrak{sl}_2 \to \mathfrak{sl}_n$ we realize V as a $\mathfrak{sl}_2$-rep w/ h acting by the root vector h_α and e acting by e_α. Hence v is a highest wt vector for that $\mathfrak{sl}_2$-action, and we conclude [Cor 5, Any 28] that the value $\lambda(h_\alpha)$ is a nonneg integer. Since α was chosen arbitrarily we see that λ is dominant.

The subspace $L(\lambda)$ is clearly stable under the action of each f_α and h_α, for simple α, and for each e_α we have

$$e_\alpha f_{\gamma_1} \cdots f_{\gamma_t} \cdot v = [e_\alpha, f_{\gamma_1} \cdots f_{\gamma_t}] \cdot v$$

$$= \sum_i f_{\gamma_1} \cdots f_{\gamma_{i-1}} [e_\alpha, f_{\gamma_i}] \cdots f_{\gamma_t} \cdot v.$$

Each commutator $[e_\alpha, f_{\gamma_i}] \in (\mathfrak{sl}_n)_{-\gamma_i + \alpha}$ with one of three things occurring, by Proposition 8.

case I) $-\gamma_i + \alpha$ is not a root, and $[e_\alpha, f_{\gamma_i}] = 0$.

case II) $-\gamma_i + \alpha$ is a negative root, and
$$[e_\alpha, f_{\gamma_i}] = c_\alpha f_\gamma$$
for some scalar c_α and γ.

case III) $-\gamma_i + \alpha = 0$, i.e. $\gamma_i = \alpha$, and
$$[e_\alpha, f_{\gamma_i}] = h_\alpha.$$

In each case the term $f_{\gamma_1} \cdots [e_\alpha, f_{\gamma_i}] \cdots f_{\gamma_t} \cdot v$ lies in $L(\lambda)$, so that
$$e_\alpha \cdot f_{\gamma_1} \cdots f_{\gamma_t} \cdot v \in L(\lambda).$$
Since the e_α, f_α generate $\mathfrak{sl}_n(\mathbb{C})$ as a Lie alg, we see that $L(\lambda) \le V$ is a $\mathfrak{sl}$-subrep. ∎

Corollary 15: Each simple $\mathfrak{sl}_n(\mathbb{C})$-rep L admits a unique highest wt. vector $v \in L$, up to scaling.

Proof: L has some highest wt vector v. For any other highest wt vector v', Prop 14 tells us that L contains a subrep $L' \le L$ with highest wt vector w and $(L')_\mu \neq 0$ implying $\mu \le \lambda' = wt(v')$. Since $L' = L$ necessarily we have $\lambda \le \lambda'$ and similarly $\lambda' \le \lambda$, giving $\lambda' = \lambda$. Prop 14 also gives $\dim[(L')_\lambda] = 1$ so that in fact $v' = c \cdot v$ for nonzero scalar c. ∎

- Characterizing rep(sl_n(ℂ))

Main Theorem: a) For each dominant weight $\lambda \in P^+$ there is a unique simple $sl_n(\mathbb{C})$-rep $L(\lambda)$ of highest weight λ.

b) The map

$$P^+ \to \mathrm{Irrep}(sl_n(\mathbb{C}))/\cong, \quad \lambda \mapsto L(\lambda),$$

is a bijection.

c) The category $\mathrm{rep}(sl_n(\mathbb{C}))$ is semisimple.

We'll deal w/ some details of the proof later in the class. Let me sketch some details here however

First we deal w/ existence of largest wt simples / P^+.

As a starting point we again consider the standard reps.

- Fundamental wts and the standard rep < In foll. disc. we take many of simpler for granted.

Def$^{\underline{n}}$: Take $\Delta = \{\alpha_1, \ldots, \alpha_{n-1}\}$ the standard basis for sl_n, and let $h_{\alpha_i} \in \mathfrak{h}$ be the coresp vectors in the Cartan ($h_{\alpha_i} = E_{ii} - E_{i+1,i+1}$). The ith fundamental weight $\varpi_i \in P^+ \subseteq \mathfrak{h}^*$ is the unique weight satisfying

$$\varpi_i(h_{\alpha_j}) = \delta_{ij}.$$

Remark: Clearly $P = \mathbb{Z} \cdot \{\varpi_i : 1 \le i \le n-1\}$.

By tensoring, we can obtain simples of arbitrary highest wts if we can construct simples $L(\varpi_i)$ of highest wt ϖ_i at each fundamental wt.

Example: For $sl_n(\mathbb{C})$ we have the standard rep $\mathbb{V} = \mathbb{C}^n$ w/ natural action.

For $v_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow$ ith row the standard basis vector, we have

$$e_{\alpha_i} \cdot v_j = \begin{cases} v_{i-1} & \text{if } j = i+1 \\ 0 & \text{else} \end{cases}$$

$$f_{\alpha_i} \cdot v_j = \begin{cases} v_{i+1} & \text{if } j = i \\ 0 & \text{else} \end{cases}$$

$$h_{\alpha_i} \cdot v_j = \begin{cases} 1 & \text{if } j = i \\ -1 & \text{if } j = i+1 \\ 0 & \text{else} \end{cases} \qquad (*)$$

$\Rightarrow$ $\mathbb{V}$ has unique highest wt. vector v_1.

From $(*)$ we see $h_{\alpha_i} \cdot v_1 = \begin{cases} 1 & \text{if } i = 1 \\ 0 & \text{else.} \end{cases}$ so that v_1 has weight ϖ_1.

Proposition 16: The standard rep $\mathbb{V}$ over $\mathfrak{sl}_n(\mathbb{C})$ is the first fundamental simple

$$\mathbb{V} = L(\varpi_1),$$

i.e. it is a simple rep of highest wt ϖ_1.

Proof: We only note that $\mathbb{V}$ has a unique highest wt vector, up to scaling, and is hence simple by Theorem 14. We calculated the hight wt as ϖ_1 above. ∎

- Aside: Symmetric and exterior powers of representations.

One discovers directly that for any Lie alg $\mathfrak{g}$, and $\mathfrak{g}$ reps V and W, the vect. space swap

$$\tau_{V,W} : V \otimes W \to W \otimes V, \quad \tau_{V,W}(v,w) := w \otimes v,$$

is an isomorphism of $\mathfrak{g}$-representations.

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23

Lemma 17: For any maps of $\mathfrak{g}$-reps $\phi: V_0 \to V_1$, $\psi: W_0 \to W_1$, the map $\phi \otimes \psi$ is a map of $\mathfrak{g}$-reps and the diagram

$$\begin{tikzcd} V_0 \otimes W_0 \arrow[r, "\tau"] \arrow[d, "\phi\otimes\psi"'] & W_0 \otimes V_0 \arrow[d, "\psi\otimes\phi"] \\ V_1 \otimes W_1 \arrow[r, "\tau"] & W_1 \otimes V_1 \end{tikzcd}$$

commutes.

Proof: Clear by inspection. ∎

For any $\mathfrak{g}$-rep V and $n \geq 1$, the automorphisms

$$\tau_i := \mathrm{id}^{\otimes i-1} \otimes \tau_{V,V} \otimes \mathrm{id}^{\otimes n-i-1} : V^{\otimes n} \circlearrowleft$$

satisfy the relations

$$(*) \quad \begin{cases} \tau_i^2 = \mathrm{id} \\ \tau_i \tau_j = \tau_j \tau_i & \text{when } |i-j|>1 \\ \tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1} \end{cases}$$

and hence define an action of

$$S_n = \langle \tau_1, \dots, \tau_{n-1} \mid \text{rels } (*) \rangle$$

via $\mathfrak{g}$-automorphisms,

$$S_n \longrightarrow \mathrm{Aut}_{\mathfrak{g}}(V^{\otimes n}).$$

Def^{n}: Given any $\mathfrak{g}$-representation V, the n-th symmetric power is the $\mathfrak{g}$-representation

$$S^n(V) := (V^{\otimes n})^{S_n} \quad \text{or } S^n\text{-invariants}$$

and the n-th exterior power

$$\Lambda^n(V) := \left[(V^{\otimes n}) \otimes \mathrm{sgn}\right]_{S^n} \quad \text{or } S^n\text{-coinvariants.}$$

Explicitly, $S^n(V) = \{ w \in V^{\otimes n} : w = \tau_i w \text{ at all } i = 1, \dots, n-1 \}$

$$\overset{\text{or}}{=} \mathrm{Hom}_{S^n}(\mathbb{C}, V^{\otimes n})$$

$$\overset{\text{or}}{=} \text{equilizer}\left(V^{\otimes n} \underset{\tau_i}{\overset{\mathrm{id}}{\rightrightarrows}} V^{\otimes n} : i = 1, \dots, n-1 \right)$$

$$\Lambda^n(V) = V^{\otimes n} / (w - \tau_i w : w \in V^{\otimes n} \text{ and } i = 1, \dots, n-1)$$

$$\overset{\text{or}}{=} (V^{\otimes n}) \otimes_{\mathbb{C} S^n} \mathrm{sgn}$$

$$\overset{\text{or}}{=} \text{coequalizer}\left(V^{\otimes n} \underset{-\tau_i}{\overset{\mathrm{id}}{\rightrightarrows}} V^{\otimes n} : i = 1, \dots, n-1 \right).$$

Exercise: Verify directly that $S^n(V)$ is a $\mathfrak{g}$-subrep in $V^{\otimes n}$ and that $\Lambda^n(V)$ is a quotient $\mathfrak{g}$-rep of $V^{\otimes n}$.

Example: For $\mathbb{V}$ $= L(1)$ the standard rep for $\mathfrak{sl}_2(\mathbb{C})$, $S^n(L(1))$ has highest wt. vector $v_1^{\otimes n} = \begin{bmatrix}1\\0\end{bmatrix}^{\otimes n}$ which is of wt $n \cdot 1 = n$. We have

$$\dim S^n(L(1))$$

$$= \dim \mathbb{C}\cdot\left\{ \frac{1}{n!}\left(\sum_{\sigma\in S_n} \sigma\right)\cdot v_1^{\otimes n-m} \otimes v_{-1}^{\otimes m} \,\middle|\, 0\le m\le n\right\}$$

$$= n+1.$$

For wt. reasons now $L(n) \subseteq S^n(L(1))$ and for dim reasons this inclusion is an isomorphism;

$$S^n(L(1)) \cong L(n) \text{ at all } n.$$

For the exterior powers,

$$\Lambda^1(L(1)) = L(1), \quad \Lambda^2(L(2)) = \mathbb{C}\cdot\{v_1\wedge v_{-1}\} \cong L(0),$$

$$\Lambda^{\geq 3}(L(2)) = 0.$$

- Exterior powers of the standard rep

Consider the n-dimensional standard rep

$$\mathbb{V} = \mathbb{C}^n,$$

we have the highest wt. vector

$$\eta_k := v_1\wedge v_2\wedge\cdots\wedge v_k \in \Lambda^k(\mathbb{V}),$$

whenever $k \le n$. < we exploit basis $\eta(\vec{\imath}) = v_{i_1}\wedge\cdots\wedge v_{i_k}$ ind by...

Lemma 18: η_k is of weight ϖ_k, and is the unique highest wt. vector in $\Lambda^k(\mathbb{V})$.

Proof: We have for $i<k$ directly

$$h_{\alpha_i}\cdot\eta_k = v_1\wedge\cdots\wedge h_{\alpha_i}v_i\wedge\cdots\wedge v_k$$

$$+\ v_1\wedge\cdots\wedge h_{\alpha_i}v_{i+1}\wedge\cdots\wedge v_k$$

$$= \eta_k - \eta_k = 0$$

and for $k<i$ $\quad h_{\alpha_i}\cdot\eta_k = 0$ as well. At $i=k$

$$h_{\alpha_k}\cdot\eta_k = v_1\wedge\cdots\wedge h_{\alpha_k}v_k = \eta_k.$$

Then $h \cdot \eta_k = w_k(h) \cdot \eta_k$ at all $h \in \mathfrak{h}$, giving η_k of weight w_k.

For a general nonzero vector $\eta \in \Lambda^k(\mathbb{V})$ take $\eta(i) = v_{i_1} \wedge \cdots \wedge v_{i_k}$ for i an increasing function $i : \{1, .., k\} \to \{1, .., n\}$, and order such fns via the dictionary ordering. Then

$$\eta = c_i \cdot \eta(i) + \sum'_{j<i} c_j \cdot \eta(j)$$

with c_i nonzero. Supposing $\eta \neq \eta_k$ we have a final index $i_\ell \in \{i_1, .., i_k\}$ with $i_\ell - i_{\ell-1} > 1$, where we take formally $i_0 = 0$, and for

$$i' = \{i_1, \ldots, i_\ell - 1, i_{\ell+1}, .., i_k\}$$

we have

$$e_{\alpha_{i_\ell - 1}} \cdot \eta = c_i \, \eta(i') + \sum'_{j'<i'} d_{j'} \cdot \eta(j').$$

In particular $e_{\alpha_{i_\ell - 1}} \cdot \eta \neq 0$, and η is not a highest wt. vector. ∎

Theorem 19: For each integer $k = 1, .., n-1$ the exterior power $\Lambda^k(\mathbb{V})$ is a simple $\mathfrak{sl}_n(\mathbb{C})$-rep. of highest wt. w_k,

$$\Lambda^k(\mathbb{V}) = L(w_k).$$

Proof: By Theorem 14 any nonzero $\mathfrak{sl}_n(\mathbb{C})$-subrep $L \subseteq \Lambda^k(\mathbb{V})$ contains a highest wt vector, and hence contains η_k by Lemma 18.

We claim now that

$$\Lambda^k(\mathbb{V}) = \mathbb{C}\{ f_{\alpha_{m_1}} \cdots f_{\alpha_{m_t}} \cdot \eta_k : t \geq 0, \alpha_m : \{1, .., t\} \to \Delta \}, \quad (*)$$

so that any subrep containing η_k must be all of $\Lambda^k(\mathbb{V})$.

For this consider again the basis vectors

$$\{ \eta(i) = v_{i_1} \wedge \cdots \wedge v_{i_k} : \text{increasing fns } i : \{1, .., k\} \to \{1, .., n\} \}$$

→.

We claim that each basic vector $\xi(\vec{i})$ is in the span $(*)$, so that any subrep which contains $\xi_k = \xi(\vec{i}_{min}) = \xi(1,2,\to k)$ is necessarily $=$ to $\Lambda^k(\mathbb{V})$. We proceed by ind. under the dictionary ordering on the set of increasing funs $\vec{i}: \{1,\to k\} \to \{1,\to n\}$.

For the min index $\vec{i}_{min} = (1,\to k)$ we have $\xi_k = \xi(\vec{i}_{min}) \in \text{span}(*)$, and suppose now $\vec{i} > \vec{i}_{min}$ with $\xi(\vec{j}) \in \text{span}(*)$ for all $\vec{j} < \vec{i}$.

Since $\vec{i} > \vec{i}_{min}$ there is a first index

$$i_\ell \in \{i_1, \dots, i_k\}$$

at which $i_\ell - i_{\ell-1} > 1$, where we take $i_0 = 0$. Then for the index $\vec{i}' < \vec{i}$ def by

$$\vec{i}' = (i_1, \dots, i_\ell - 1, \dots, i_k)$$

we have $\xi(\vec{i}') \in \text{span}(*)$ and

$$\xi(\vec{i}) = f_{\alpha_{i_\ell - 1}} \cdot \xi(\vec{i}')$$

giving $\xi(\vec{i}) \in \text{span}(*)$ as well. Hence all basis vectors $\xi(\vec{i}) \in \text{span}(*)$ by induction.

Consequently, any subrep $W \leq \Lambda^k(\mathbb{V})$ which contains the highest wt. vector ξ_k is equal to $\Lambda^k(\mathbb{V})$ and we conclude our arbitrary subrep L is in fact all of $\Lambda^k(\mathbb{V})$. This establishes simplicity. ∎

Conclusion: For each fundamental wt ϖ_k, $k = 1, \dots, n-1$,

$$\varpi_k(h_{\alpha_i}) = \delta_{ik},$$

the k-th exterior power of the standard rep $\mathbb{V}$ realizes a simple $\mathfrak{sl}_n(\mathbb{C})$-rep of highest wt ϖ_k.

In particular, such highest wt. simples exist.

- Existence of highest wt. simples over P^+

Proposition 20: For each dominant weight $\lambda \in P^+$, there is a simple representation $L(\lambda)$ with a single highest wt. vector of wt. λ.

Proof: Since λ is dominant we have unique nonnegative integers $m_1, \ldots, m_{n-1}$ so that

$$\lambda = m_1 \varpi_1 + \cdots + m_{n-1} \varpi_{n-1}.$$

By Thm 19 there exist simple sl_n-reps $L(\varpi_k)$ of highest wt ϖ_k of each fundamental wt. ϖ_k.

Let's take

$$V(\lambda) := L(\varpi_1)^{\otimes m_1} \otimes \cdots \otimes L(\varpi_k)^{\otimes m_k}.$$

Note that $\dim V(\lambda)_\lambda = 1$ and by Thm 14 applied to the $L(\varpi_k)$ we see

$$V(\lambda)_\mu \neq 0 \Rightarrow \mu < \lambda. \qquad (*)$$

Thus, for any composition series

$$0 = V_t \leq V_{t-1} \leq \cdots \leq V_0 = V(\lambda)$$

there exists a unique index i at which the (simple) composition factor

$$L = V_i / V_{i+1}$$

has $\dim_{\mathbb{C}} L_\lambda = 1$ and $L_\mu \neq 0 \Rightarrow \mu < \lambda$.

It follows that $L = L(\lambda)$ is of highest wt λ. ∎

- Sketch proof of Uniqueness

Thm 21: Given $sl_n(\mathbb{C})$-simples L and L' of highest wt. λ, with highest wt. vectors v and v', there exists a unique isom $\phi: L \to L'$ over $sl_n(\mathbb{C})$ with $\phi(v) = v'$.

-

We'll be more careful about the proof when we deal w/ the general case. Let me sketch the details however.

Sketch Proof: We have the univ. env. alg. $\mathcal{U}(\mathfrak{sl}_2)$ and the subalg. env. alg. for the Borel $\mathcal{U}(\mathfrak{b}^+)$, $\mathfrak{b}^+ = \mathfrak{h} \oplus \mathfrak{n}^+$. We have the 1-dim simple $\mathbb{C}_\lambda$ of wt. λ over $\mathfrak{b}^+$ and define the Verma module $M(\lambda) := \mathcal{U}(\mathfrak{sl}_2) \otimes_{\mathcal{U}(\mathfrak{b}^+)} \mathbb{C}_\lambda$. This is a highest weight, ∞-dim, weight graded, $\mathfrak{sl}_2$-rep and restriction along the inclusion $\mathbb{C}_\lambda \to M(\lambda)$ provides a linear $\cong$

$$\mathrm{Hom}_{\mathfrak{sl}_2}(M(\lambda), V) \xrightarrow{\sim} \mathbb{C}\cdot\left\{ \begin{matrix} \text{highest wt. vectors} \\ v \in V \text{ of wt. } \lambda \end{matrix} \right\}.$$
$$f \longmapsto f(1_\lambda).$$

For grading reasons, there is a unique simple quotient $\pi : M(\lambda) \to \bar{M}(\lambda)$ $(= L(\lambda))$, and hence the surjective $\mathfrak{sl}_2$-maps

$$f : M(\lambda) \to L, \quad f' : M(\lambda) \to L'$$
$$f(1_\lambda) = v_\lambda \qquad f'(1_\lambda) = v'_\lambda$$

induce isoms

$$\begin{tikzcd} & \bar{M}(\lambda) \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & \\ L \arrow[rr, dashed, "\exists!"', "\phi"] & & L' \end{tikzcd}.$$

The induced $\cong$ $\phi : L \to L'$ completing the above diagram does the desired job. ∎

Corollary 22: The assignment
$$P^+ \to \mathrm{Irrep}(\mathfrak{sl}_2(\mathbb{C})), \quad \lambda \mapsto L(\lambda)$$
is a bijection, i.e. classifies all irreducible $\mathfrak{sl}_2(\mathbb{C})$-reps up to isom.

—

— Semisimplicity for rep($\mathfrak{sl}_n(\mathbb{C})$).

Prop 22: For simple $\mathfrak{sl}_n(\mathbb{C})$-reps $L(\lambda)$ and $L(\mu)$, and extension

$$0 \to L(\mu) \to V \to L(\lambda) \to 0$$

is split.

Again, we'll cover the details more clearly in the general setting. We again sketch the details.

Proof: The adjoint rep $\mathfrak{sl}_n$ is again self dual, so that the trace form on $\mathfrak{sl}_n$ induces a symm non-deg $\mathfrak{sl}_n$-invariant form

$$\kappa: \mathfrak{sl}_n^* \otimes \mathfrak{sl}_n^* \to \mathbb{C},$$

and under the natural $\mathfrak{sl}_n$ iso

$$\mathfrak{sl}_n \otimes \mathfrak{sl}_n \cong (\mathfrak{sl}_n^* \otimes \mathfrak{sl}_n^*)^*$$

κ determines an elem $\Omega \in \mathfrak{sl}_n \otimes \mathfrak{sl}_n$,

$$\Omega = \Big(\sum_{\gamma \in \Phi^+} c_\gamma (f_\gamma \cdot e_\gamma + e_\gamma f_\gamma)\Big) + \Big(\sum_i \text{Cartan terms}\Big),$$

w/ $x \cdot \Omega = 0$ at all $x \in \mathfrak{sl}_n(\mathbb{C})$, aka Ω is an invariant elem in $\mathfrak{sl}_n^{\otimes 2}$.

The elem Ω therefore acts by $\mathfrak{sl}_n$-linear endos on all reps $\Omega_V = \Omega \cdot - : V \to V$, and we can use Ω to split extensions of simples, just as in the case of $\mathfrak{sl}_2$. [Exercise] ∎

Using [Prop 6, Aug 28] we now see that the category rep($\mathfrak{sl}_n(\mathbb{C})$) is semisimple.

Theorem 23: The cat rep($\mathfrak{sl}_n(\mathbb{C})$) is semisimple, and the simple $\mathfrak{sl}_n(\mathbb{C})$-reps are in bij correspondence with dominant wts, $\lambda \mapsto L(\lambda)$.

— We've now recovered our "Main Theorem" for $\mathfrak{sl}_n(\mathbb{C})$.

End.