Lie Cartan

– An aside: Derivations

Def^{n}: A derivation of a Lie alg $\mathfrak{g}$ is a linear endo $d: \mathfrak{g} \to \mathfrak{g}$ which satisfies

$$d([x,y]) = [dx, y] + [x, dy].$$

We take $\mathrm{Der}(\mathfrak{g}) \subseteq \mathrm{End}_{\mathbb{C}}(\mathfrak{g})$ the subspace consisting of Lie alg derivations on $\mathfrak{g}$.

Ex: For any $x \in \mathfrak{g}$, $\mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g}$ is a derivation of $\mathfrak{g}$.

Lemma: a) $\mathrm{Der}(\mathfrak{g})$ is a Lie subalg in $\mathfrak{gl}(\mathfrak{g}) = \mathrm{End}_{\mathbb{C}}(V)^{Lie}$.

b) The adj rep $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ has image in the Lie subalg $\mathrm{Der}(\mathfrak{g}) \subseteq \mathfrak{gl}(V)$.

Proof: (b) Trivial. (a) For der d and d' we have

$$(dd' - d'd)[x,y] = [[d,d']x, y] + [x, [d,d']y] + [dx, d'y] + [d'x, dy] - [d'x, dy] - [dx, d'y]$$

$$= [[d,d']x, y] + [x, [d,d']y].$$

Theorem 5.3: For any semisimple Lie alg $\mathfrak{g}$ the map $\rho_{adj} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ restricts to provide an $\cong$ of Lie algs $\rho_{adj} : \mathfrak{g} \xrightarrow{\cong} \mathrm{Der}(\mathfrak{g})$.

Proof: Given any derivation $d : \mathfrak{g} \to \mathfrak{g}$ we obtain the extension of the adj rep

$$0 \to \mathfrak{g} \xrightarrow{i} \mathfrak{g}_d \xrightarrow{\pi} \mathbb{C} \to 0$$

where $\mathfrak{g}_d = \mathfrak{g} \oplus \mathbb{C}\cdot d$ w/ action

$$x \cdot (y + c d) = [x,y] + c \cdot d(x).$$

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The map i is the obvious inclusion and π is the projection $\pi(x + c\cdot d) = c$. Since rep$(\mathfrak{g})$ is semisimple this extension is split by a map $\tau: \mathbb{C} \to \mathfrak{g}_d$, $\tau(1) = x_d + d$, and $\mathfrak{g}$-invariance demands

$$0 = [x, x_d] + d(x) = -ad_{x_d}(x) + d(x)$$

at all x. Hence $d = ad_{x_d}$. We now find that $\rho_{adj}: \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$ is surjective, and semisimplicity implies ρ_{adj} is injective. ∎

— The universal Jordan decomposition

Theorem 5.4: For any semisimple Lie alg $\mathfrak{g}$, and $x \in \mathfrak{g}$, there exist $x_s, x_n \in \mathfrak{g}$ such that $x = x_s + x_n$, $[x_s, x_n] = 0$, and for each $\mathfrak{g}$-rep

$$\rho: \mathfrak{g} \to gl(V)$$

the decomp

$$\rho(x) = \underbrace{\rho(x_s)}_{\text{semisimple endo of } V} + \underbrace{\rho(x_n)}_{\text{nilpotent endo of } V}$$

is the Jordan decomp for $\rho(x)$.

Proof: (Uniqueness) The uniqueness follows when we consider any faithful rep, such as $V = $ adj rep, and uniqueness of usual J-C decomp in $gl(V)$.

(Existence) By considering the decomp into simples

$$\mathfrak{g} = \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_t$$

it suffices to establish such a decomp $x_i = x_s + x_n$ in the case of simple $\mathfrak{g}$.

Suppose $\mathfrak{g}$ simple and V any $\mathfrak{g}$-rep. If V is trivial then $\rho(x) = 0$ at all $x \in \mathfrak{g}$, so that any decomp $x = x_s + x_n$ has the prescribed property at V. If V not trivial then V is

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faithful over $\mathfrak{g}$.

Suppose V is faithful and, by identifying $\mathfrak{g}$ w/ $\rho(\mathfrak{g}) \subseteq \mathfrak{gl}(V)$, we may assume $\mathfrak{g} \subseteq \mathfrak{gl}(V)$.
We have the internal decomp

$$x = s + n \quad w/ \; s, n \in \mathfrak{gl}(V)$$

and $\mathrm{ad}_x = \mathrm{ad}_s + \mathrm{ad}_n$ is the JC decomp of the adj operator [Lemma 4.2 A]. Hence

$$\mathrm{ad}_s(\mathfrak{g}),\ \mathrm{ad}_n(\mathfrak{g}) \subseteq \mathfrak{g}$$

by JH (c). In particular,

$$\mathrm{ad}_s,\ \mathrm{ad}_n \in \mathrm{Der}(\mathfrak{g}),$$

and we can find $x_s(V), x_n(V) \in \mathfrak{g}$ w/

$$\mathrm{ad}_{x_s(V)} = \mathrm{ad}_s \qquad \mathrm{ad}_{x_n(V)} = \mathrm{ad}_n .$$

We claim now $x_s(V) = x_s(V')$, $x_n(V) = x_n(V')$ for any pair of repr V, V'. Indeed, for

$$W = V \oplus V'$$ we consider $x_s(W), x_n(W)$

and have

$$x_s(W)\big|_V + x_n(W)\big|_V = \text{JC decomp for } x|_V$$

$$x_s(W)\big|_{V'} + x_s(W)\big|_{V'} = \text{JC decomp for } x|_{V'}$$

via uniqueness. Hence

$$x_s(W) = x_s(V) = x_s(V')$$

and

$$x_n(W) = x_n(V) = x_n(V').$$

We therefore obtain the proposed universal decomp.

$$x = x_s + x_n$$ ▨

Def$^{\underline{n}}$: For $\mathfrak{g}$ semisimple, $x \in \mathfrak{g}$, call the decomp $x = x_s + x_n$ as in Thm 5.4 the internal/univ JH decomp for x. Call an element $x \in \mathfrak{g}$ semisimple if $x = x_s$, and call x nilpotent if $x = x_n$.

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Observe, x is semisimple $\Leftrightarrow$ x acts semisimply on all $\mathfrak{g}$-reps V $\Leftrightarrow$ ad_x is semisimple.

x nilpotent $\Leftrightarrow$ x acts nilpotently on every $\mathfrak{g}$-rep V $\Leftrightarrow$ ad_x is nilpotent.

Example: For $sl_n(\mathbb{C})$, V = standard rep, we see x acts semisimply on V then x acts semisimply on all $sl_n(\mathbb{C})$-reps.

– Cartan Subalgs

Def^n: Let $\mathfrak{g}$ be semisimple. A Cartan subalg in $\mathfrak{g}$ is a maximal subalg $\mathfrak{h} \subseteq \mathfrak{g}$ satisfying the following:

a) Every elem $h \in \mathfrak{h}$ is semisimple.

Observation 8.1: If any subalg $\mathfrak{t} \leq \mathfrak{g}$ satisfies (a) then $\mathfrak{t}$ is already abelian.

Proof: We have $\mathfrak{t}$ acting semisimply on $\mathfrak{g}$ via adj rep and stabilizing $\mathfrak{t}$, hence $\mathfrak{t}$ acts semisimp on itself. For any eigenvector $y \in \mathfrak{t}$ for the adj action of $\mathfrak{t}$ on itself we have at all $x \in \mathfrak{t}$

$$[x, y] = \lambda(x) \cdot y$$

$$\Rightarrow 0 = [y, [y, x]] = ad_y^2(x)$$ so that x is a gen eigenvector for the action of y, and hence

$$-\lambda(x) \cdot y = ad_y(x) = 0$$

via semisimplicity. Thus $\lambda(x) = 0$ at all x, and we see $\mathfrak{t} = \mathfrak{t}_0$, i.e. $\mathfrak{t}$ is abelian. ∎

Ex: In $\mathfrak{sl}_n(\mathbb{C})$, HW 2, tells us that (5)
all Cartans in $\mathfrak{sl}_n(\mathbb{C})$ are conjugate (to the diag subalg),
under the adj action of $SL_n(\mathbb{C})$.

— Cartan subalgs are self centralizing.

OK, we consider a Cartan subalg $\mathfrak{h} \leq \mathfrak{g}$,
w/ $\mathfrak{g}$ semisimple. Then we obtain a decomp of
$\mathfrak{g}$ into weight spaces for the adj action of $\mathfrak{h}$,

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\gamma \in \Phi} \mathfrak{g}^{\gamma},$$

where $\Phi = \{\text{nonzero fns } \gamma \in \mathfrak{h}^* \text{ w/ } \mathfrak{g}^\gamma \neq 0\}$.
Here $\mathfrak{g}_0 = C_{\mathfrak{g}}(\mathfrak{h}) \; [= Z_{\mathfrak{g}}(\mathfrak{h})]$
$= \{x \in \mathfrak{g} : [x, \mathfrak{h}] = 0\}$.

Lemma 8.1. a) $[\mathfrak{g}_\gamma, \mathfrak{g}_\mu] \leq \mathfrak{g}_{\gamma+\mu}$

b) $\kappa(\mathfrak{g}_\gamma, \mathfrak{g}_\mu) \neq 0$ if and only if $\mu = -\gamma$.

c) κ restricts to a nondegenerate form on $\mathfrak{g}_0 = C_{\mathfrak{g}}(\mathfrak{h})$.

Proof: (a) Trivial. (b) Just like on HW. (c)
Follows from (b). ∎

Lemma 8.2: If $x, y \in \mathfrak{g}$ with $[x,y] = 0$
and y nilpotent, then $\mathrm{ad}_x \mathrm{ad}_y = 0$.

Proof: In this case $[\mathrm{ad}_x, \mathrm{ad}_y] = \mathrm{ad}_{[x,y]} = 0$ w/
ad_y nilpotent. Hence $(\mathrm{ad}_x \mathrm{ad}_y)^{>>0} = \mathrm{ad}_x^{>>0} \mathrm{ad}_y^{>>0} = 0$. ∎

Proposition 8.2: If $\mathfrak{h} \leq \mathfrak{g}$ is a Cartan subalg
then $\mathfrak{h} = C_{\mathfrak{g}}(\mathfrak{h}) (= \mathfrak{g}_0)$.

Sketch Proof: One observes for $x \in C_{\mathfrak{g}}(\mathfrak{h})$,
$x = x_s + x_n$ with $x_s, x_n \in C_{\mathfrak{g}}(\mathfrak{h})$, by TE, and
hence $x_s \in \mathfrak{h}$. So need only show $x_n = 0$.

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By Lemma 8.1 we see that, for each $z \in h$, (6)

$$\kappa(z, C_{g}(h)) = \kappa(z, \text{semisimple parts of elements}) = \kappa(z, h)$$

so that $\kappa \big|_{h \times h}$ is non-degenerate.

Using this information one argues first that $C_g(h)$ is nilpotent, one then finds $[C_g(h), C_g(h)] = 0$ so that $C_g(h)$ is abelian, then finds all nilpotent parts x_n must vanish since we can find

$$\kappa(x_n, C_g(h)) = 0 \Rightarrow x_n = 0$$

by Lemma 8.2. ∎

We now have, for any choice of Cartan, the expected decomposition

$$g = h \oplus \bigoplus_{\gamma \in \Phi} g_\gamma$$

- Errata

Lemma 6.4: Let $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ be a semisimple Lie alg and $x \in \mathfrak{g}$ be arbitrary. Then for the Jord decomp $x = x_s + x_n$ in $\mathfrak{gl}(V)$, we have $x_s, x_n \in \mathfrak{g}$.

Proof: We have $ad_x(\mathfrak{g}) \subseteq \mathfrak{g}$ so that $ad_{x_s}(\mathfrak{g}), ad_{x_n}(\mathfrak{g}) \subseteq \mathfrak{g}$ as well, by Jord-Chev Thm. Hence x_s and x_n are in the normalizer $N_{\mathfrak{gl}}(\mathfrak{g})$. Further for any $\mathfrak{g}$-subrep $W \leq V$ we have

$$x_s(W), x_n(W) \subseteq W$$

and $Tr_W(x_s) = Tr_W(x_n) = Tr_x(x) = 0$ by Lemma 6.3 [Thm]. Take now $\mathfrak{g}' \subseteq \mathfrak{gl}(V)$ def by

$$\mathfrak{g}' = \left\{ y \in \mathfrak{gl}(V) : \begin{array}{l} y \text{ normalizes } \mathfrak{g} \\ y \text{ stabilizes all } \mathfrak{g}\text{-subreps in } V \\ Tr_W(y) = 0 \text{ for all } \mathfrak{g}\text{-subrep } W \leq V \end{array} \right\}$$

Note $\mathfrak{g} \subseteq \mathfrak{g}'$ and $x_s, x_n \in \mathfrak{g}'$. I claim $\mathfrak{g} = \mathfrak{g}'$.

For this, we have that $\mathfrak{g}'$ is a Lie subalg in $\mathfrak{gl}(V)$ and $\mathfrak{g}$-subrep under the adj actn. By semisimplicity we now have a decomp as $\mathfrak{g}$-rep

$$\mathfrak{g}' = \mathfrak{g} \oplus \mathfrak{m}.$$

Since all of $\mathfrak{g}' \subseteq N_{\mathfrak{gl}}(\mathfrak{g})$ we have

$$[\mathfrak{g}, \mathfrak{m}] \subseteq (\mathfrak{g} \cap \mathfrak{m}) = 0$$

and hence all $y \in \mathfrak{m}$ act on each simple subrep $W \leq V$ as a $\mathfrak{g}$-lin endo, and thus a scalar. Vanishing of the trace $Tr_W(y) = 0$ then forces $y|_W = 0$. Since V decomp into simple $\mathfrak{g}$-reps via semisimplicity we find $y = 0$. The

$n = 0$ and we have $\mathfrak{g}' = \mathfrak{g}$. This gives $x_s, x_n \in \mathfrak{g}$. ∎

Thm 6.4: For $\mathfrak{g}$ semisimple, $x \in \mathfrak{g}$, there exists a unique decomp $x = x_s + x_n$ in $\mathfrak{g}$ such that

$[x_s, x_n] = 0$

and for each $\mathfrak{g}$-rep $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$

$\rho(x) = \rho(x_s) + \rho(x_n)$

is the JC decomp of $\rho(x)$. (x_s acts semisimply on all V and x_n acts nilpotently on all V.)

Proof: Reduce to the simple case and faithful V, as before. Then we have, by Lemma 6.?, commuting $x_s^V, x_n^V \in \mathfrak{g}$ w/ x_s^V and x_n^V acting semisimply, and nilpotently on V. We claim now that $x_s^V = x_s^W$ and $x_n^V = x_n^W$ at each pair of non-trivial $\mathfrak{g}$-reps V and W.

For that consider the sum $V \oplus W$ and $x_s^{V \oplus W}$, $x_n^{V \oplus W}$.

Then $x_s^{V \oplus W}$ and $x_n^{V \oplus W}$ act as commuting semis/nilpotent endos on the subreps $V, W \subseteq V \oplus W$, implied by Jord Chev $x_s^V = x_s^{V \oplus W} = x_s^W$ and $x_n^V = x_n^{V \oplus W} = x_n^W$. ∎