

$$sl_2^* = \begin{matrix} f^* \\ h^* \\ e^* \end{matrix}$$

①

$$sl_2^* \xrightarrow{\cong} sl_2, \quad \begin{matrix} f^* \mapsto e \\ h^* \mapsto \frac{1}{2}h \\ e^* \mapsto f \end{matrix}$$

$$\Rightarrow \begin{matrix} \text{Tr}(e, f) = 1 \\ \text{Tr}(h, h) = 2 \end{matrix}$$

$$\begin{array}{ccc} sl_2 \otimes sl_2 & \xrightarrow{\text{trace form}} & \mathbb{C} \\ \uparrow \text{sa} & & \nearrow \kappa \\ sl_2^* \otimes sl_2^* & & \end{array}$$

$$\kappa(e^*, f^*) = \kappa(f^*, e^*) = 1$$

$$\kappa(h^*, h^*) = \frac{1}{2} \quad \text{vanishes elsewhere}$$

0 on all other basic pairs.

$$\text{Hence eval: } sl_2 \otimes sl_2 \xrightarrow{\cong} \text{Hom}_{\mathbb{C}}(sl_2^* \otimes sl_2^*, \mathbb{C})$$

$$\Rightarrow \begin{matrix} \Omega \\ \Omega = \frac{1}{2}h^2 + ef + fe \end{matrix} \longleftarrow \kappa$$

The subalgs $\mathfrak{h}^\pm := \mathfrak{h} \oplus \mathfrak{n}^\pm$ are called (2) the pos. and neg. Borel subalgs.

- The structure of $\mathfrak{sl}_n \mathbb{C}$

Def¹: For any \mathfrak{sl} -rep V , a ^{nonzero} vector $v \in V$ is said to be an eigenvector for the action of \mathfrak{h} (Cartan) $\mathfrak{h} \subseteq \mathfrak{sl}$ if for each $x \in \mathfrak{h}$ $x \cdot v \in \mathbb{C} \cdot v$.

The corresp. eigenfunction $\lambda \in \mathfrak{h}^*$ is the unique linear form which satisfies $x \cdot v = \lambda(x) \cdot v$ at all x in \mathfrak{h} . Given $\lambda \in \mathfrak{h}^*$ we let V_λ denote the corresponding weight space/eigenspace. $\{w \in V : x \cdot w = \lambda(x) \cdot w \text{ for all } x \in \mathfrak{h}\}$

Lemma 4: Each vector $E_{ij} \in \mathfrak{sl}(\mathbb{C})$, $i \neq j$ is an eigenvector for the adj action of \mathfrak{h} assoc. to nonzero eigenfunction $\gamma: \mathfrak{h} \rightarrow \mathbb{C}$. Further, if $v \in \mathfrak{sl}(\mathbb{C})$ is an eigenvector for the Cartan then $v \in \mathfrak{h}$ or $v \in \mathbb{C} \cdot E_{ij}$ for unique i, j .

Proof: The eigenvector claim is clear since for any diag matrix D , $D \cdot E_{ij}, E_{ij} \cdot D \in \mathbb{C} \cdot E_{ij}$.

For the uniqueness claim consider E_{ij}, E_{kl} with $i \neq k$. Then $[E_{ii} - E_{jj}, E_{ij}] = 2E_{ij}$ while $[E_{ii} - E_{jj}, E_{kl}] \in \mathbb{Z}_{\leq 1} \cdot E_{kl}$.

Hence the corresp. eigenfunctions for E_{ij} and E_{kl} are distinct, since they take distinct values on $E_{ii} - E_{jj} \in \mathfrak{h}$.

Lemma 5: Let $\gamma \in \mathfrak{h}^*$ be the eigenfun. for E_{ij} . Then the eigenvector for E_{ji} is $-\gamma \in \mathfrak{h}^*$.

Proof: Follows from the bracket rule given in

Lemma 1.

Def: A linear $\gamma \in \mathfrak{h}^*$ is called a root for $\mathfrak{sl}(\mathbb{C})$ if γ is nonzero and the corresp. eigenspace $\mathfrak{sl}(\mathbb{C})_\gamma$ is nonzero.

We let $\Phi = \{\text{the coll. of all roots in } \mathfrak{h}^*\}$

$\Phi^+ = \{\text{All roots } \gamma \in \Phi \text{ for which } \kappa^+ \gamma \neq 0\}$

$\Phi^- = \{\text{All roots } \gamma \in \Phi \text{ for which } \kappa^- \gamma \neq 0\}$.
 pos roots neg roots

Corollary: a) $\Phi^- = -\Phi^+$

b) $\Phi = \Phi^+ \cup \Phi^-$

c) Each weight space $\mathfrak{sl}(\mathbb{C})_\gamma$ is of dim 1.

d) $\mathfrak{sl}(\mathbb{C}) = \left(\bigoplus_{\alpha \in \Phi^-} \mathfrak{sl}(\mathbb{C})_\alpha \right) \oplus \mathfrak{h} \oplus \left(\bigoplus_{\gamma \in \Phi^+} \mathfrak{sl}(\mathbb{C})_\gamma \right)$

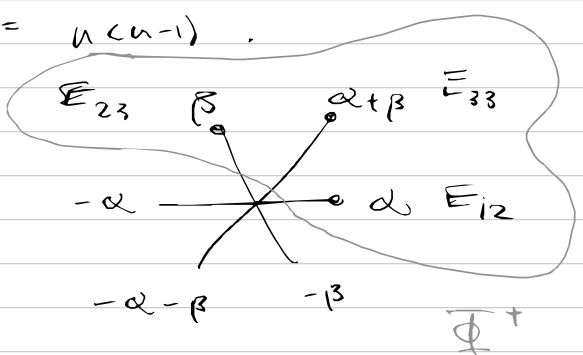
Proof: (a) Follows by Lemma 5. (b) Follows by Lemma 4. (c) Follows by Lemma 4. \square

Rem: Note that Φ is not a subspace in \mathfrak{h}^* , it is a subset.

We have $|\Phi^+| = \dim \mathfrak{h}^+ = \frac{n(n-1)}{2}$

and $|\Phi| = n(n-1)$.

e.g. $\Phi(\mathfrak{sl}_3) =$



- Root subalgebra.

Proposition 7: For arbitrary $\gamma \in \Phi$, and nonzero $e_\gamma \in \mathfrak{sl}(\mathbb{C})_\gamma$, there exists a unique vector $f_\gamma \in \mathfrak{sl}(\mathbb{C})_{-\gamma}$ so that the unique linear map $z_\gamma: \mathfrak{sl}_2(\mathbb{C}) \rightarrow \mathfrak{sl}(\mathbb{C})$, $\begin{cases} e \mapsto e_\gamma \\ f \mapsto f_\gamma \\ h \mapsto [e_\gamma, f_\gamma] = h_\gamma \end{cases}$ is an injective Lie alg homomorphism.

Furthermore, the vector h_j is indep. of the choice (4) of e_j and the image $\text{im}(\text{id}_j) \leq \mathfrak{sl}_n(\mathbb{C})$ is uniquely det. by j (i.e. doesn't depend on e_j).

Proof: Take E_{ij} so that $\mathfrak{sl}_n(\mathbb{C})|_j = \mathbb{C} \cdot E_{ij}$. First note that the triple

$$E_j = E_{ij}, F_j = E_{ji}, h_j = E_{ii} - E_{jj}$$

det. such a fix city embedding

$$\mathfrak{sl}_2 \rightarrow \mathfrak{sl}_n, e \mapsto E_j, f \mapsto F_j, h \mapsto h_j.$$

Now for any choice of e_j we have $e_j = c \cdot E_j$

for unique $c \in \mathbb{C}^\times$ and for any $d \in \mathbb{C}^\times$

we have $[e_j, dF_j] = c \cdot d \cdot h_j$ so that

$$([e_j, dF_j], e_j) = (2 \cdot c \cdot d) \cdot e_j.$$

Hence we have the unique scaling $f_j = c^{-1} \cdot F_j$

so that the triple $\{e_j, f_j, h_j\}$ specifies

such an embedding $\text{id}_j: \mathfrak{sl}_2(\mathbb{C}) \hookrightarrow \mathfrak{sl}_n(\mathbb{C})$.

For the uniqueness claim, we always have

$$\begin{aligned} \text{im}(\text{id}_j) &= \mathbb{C} \cdot E_j \oplus \mathbb{C} \cdot h_j \oplus \mathbb{C} \cdot F_j \\ &= \mathfrak{sl}_n(\mathbb{C})_j \oplus \mathbb{C} \cdot h_j \oplus \mathfrak{sl}_n(\mathbb{C})_{-j}. \end{aligned}$$

Let's just collect what we've seen here:

For each positive root $j \in \Phi^+$ we get a copy of $\mathfrak{sl}_2(\mathbb{C})$ in $\mathfrak{sl}_n(\mathbb{C})$,

$$\text{id}_j: \mathfrak{sl}_2(\mathbb{C}) \rightarrow \mathfrak{sl}_n(\mathbb{C}).$$

This is an root subalg. assoc. to j . The map id_j itself is not det. by j , but its image, i.e. the assoc. subalg in $\mathfrak{sl}_n(\mathbb{C})$ is.

Further, for each root j we have a uniquely assoc. vector $h_j \in \mathfrak{h}$. There are $\frac{n(n-1)}{2}$ such vectors, and they span \mathfrak{h} .

Proposition 7. There is a unique subset of positive roots $\Delta \subseteq \Phi^+$ satisfying the following

a) Δ is lin. indep. in \mathfrak{h}^* , and is full pos. char. s.

b) $\Phi^+ = \mathbb{Z}_{\geq 0} \cdot \Delta$.

Proof: Consider

$$\Delta = \{ \text{the weights for the simple roots } E_{i+1} \} \\ = \{ \alpha_1, \dots, \alpha_{n-1} : \alpha_i = \text{wt. for } E_{i+1} \}.$$

Since for all $i < j$

$$E_{ij} = [[E_{i+1}, E_{i+2}] \dots] E_{j-1,j}$$

and $(e_{\alpha}, e_{\beta}) \in (\mathfrak{sl}_n)_{\alpha+\beta}$ via Jacobi, we see that (b) holds.

Since $|\Delta| = n-1 = \dim \mathfrak{h}^*$ it suffices to show now that Δ spans \mathfrak{h}^* . For this it suffices to show that for each $x \in \mathfrak{h}$ we have $x=0$ iff $\alpha(x)=0$ at all $\alpha \in \Delta$. Write

$$x = \sum_{i=1}^{n-1} c_i h_{\alpha_i}, \quad h_{\alpha_i} = E_{ii} - E_{i+1,i+1},$$

and observe

$$\alpha_i(x) E_{i+1} = [x, E_{i+1}] = (-c_{i-1} + 2c_i - c_{i+1}) E_{i+1}$$

so that $\alpha_i(x) = 0$ at all $i \Leftrightarrow$

$$\text{Cent}_n [c_1 \dots c_{n-1}]^t = \vec{0} \quad (*)$$

where $\text{Cent}_n = \begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & \ddots & -1 \\ 0 & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}.$

We calculate by induction $\det(\text{Cent}_n) = n \neq 0$ so that the eq. (*) forces $x=0$, as desired.

We have uniqueness as an exercise. \square

Def⁶: The subset $\Delta \subseteq \Phi^+$ is called

Proposition 8 is called the corollary (depending) base for Φ . The elements of Δ are called the simple roots for \mathfrak{sl}_n .

Recall explicitly

$$\Delta = \{\alpha_1, \dots, \alpha_{n-1} : \mathfrak{sl}(\mathbb{C})_{\alpha_i} = \mathbb{C} E_{i, i+1}\}.$$

We have also $h_{\alpha_i} = E_{i, i} - E_{i+1, i+1}$.

Observation/Corollary 9: For the simple roots $\alpha \in \Delta$, the corresponding vectors

$$\{h_{\alpha} : \alpha \in \Delta\}$$

provide a basis for the Cartan subalgebra \mathfrak{h} , and for

all $\gamma \in \Phi^+$ h_{γ} is in the non-neg span

$$h_{\gamma} \in \mathbb{Z}_{\geq 0} \{h_{\alpha} : \alpha \in \Delta\}.$$

- Weights and dominant weights.

Def⁷: A weight for $\mathfrak{sl}(\mathbb{C})$ is a function $\lambda \in \mathfrak{h}^*$ which takes integer values $\lambda(h_{\gamma}) \in \mathbb{Z}$ at all $\gamma \in \Phi^+$.

A weight λ is called dominant if it takes nonnegative integer values

$$\lambda(h_{\gamma}) \in \mathbb{Z}_{\geq 0} \text{ at all } \gamma \in \Phi^+.$$

We take

$$P := \{\text{all weights in } \mathfrak{h}^*\}$$

$$P^+ := \{\text{all dominant weights in } \mathfrak{h}^*\}.$$

Lemma 10: $\lambda \in \mathfrak{h}^*$ is a wt. iff $\lambda(h_{\alpha}) \in \mathbb{Z}$ for all simple α , and a wt $\lambda \in P$ is dominant iff

$$\lambda(h_{\alpha}) \geq 0 \text{ at all simple } \alpha.$$

Proof: Immediate from Corollary 9. ⑦

Lemma 11: Every root $\gamma \in \Phi$ is also a weight.

Proof: Since $\Phi = \Phi^+ \cup \Phi^-$ w/ $\Phi^- = -\Phi^+$,
and $\Phi^+ \subseteq \mathbb{Z}_{\geq 0} \Delta$, it suffices to show
that each simple root α_i is a weight. But

$$\begin{aligned} \text{we have } \alpha_j \in \mathfrak{h}_{\alpha_j} &= \text{the root of } E_{\alpha_j} \\ &\quad \text{in } [\mathfrak{h}_{\alpha_j}, E_{\alpha_j}] \\ &= \begin{cases} 2 & \text{if } i=j \\ -1 & \text{if } |i-j|=1 \\ 0 & \text{else.} \end{cases} \end{aligned}$$

We're done. ||

Note that P is a lattice in \mathfrak{h}^* , i.e. a
 \mathbb{Z} -submodule w/ $\mathbb{C} \otimes_{\mathbb{Z}} P \xrightarrow{\cong} \mathfrak{h}^*$ via the
natural map. We have

$$\mathbb{Q} = \mathbb{Z} \cdot \Phi \subseteq P, \text{ another lattice.}$$

We call \mathbb{Q} the root lattice and P the weight lattice.

- Δ partial ordering on the weights

Def¹: For weights $\mu, \lambda \in P$ we write

$$\mu \leq \lambda \text{ if } \lambda - \mu \in \mathbb{Z}_{\geq 0} \Delta,$$

i.e. if λ is obtained by a positive shift in simple

$$\text{roots } \lambda = \mu + \sum_i d_i \cdot \alpha_i, \quad d_i \in \mathbb{Z}_{\geq 0}.$$

Example: Take \mathfrak{g} so that $\mathfrak{sl}(\mathbb{C})_{\mathfrak{g}}$
is $\mathbb{C} \cdot E_{\alpha_{\max}}$. Then all roots γ satisfy
 $\gamma \leq \gamma_{\max}$.

It is the largest root for $\mathfrak{sl}(\mathbb{C})$.

- Weights for \mathfrak{sl}_n -representations

Proposition 12: Let V be any finite dimensional \mathfrak{sl}_n -representation.

- The Cartan subalgebra \mathfrak{h} acts semisimply on V , i.e. V decomposes into eigenspaces for the \mathfrak{h} -action.
- If the eigenspace V_λ is nonzero, for $\lambda \in \mathfrak{h}^*$, then λ lies in the weight lattice.

Proof: We have

$\mathfrak{h} = \mathbb{C} \cdot \{h_\alpha : \alpha \text{ simple}\}$
and the h_α provide a commuting endomorphism on V . Hence V decomposes into generalized eigenspaces for the action of these h_α , and hence for the action of \mathfrak{h} .
$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda^{\text{gen}}$$

For each simple α we have the root subalgebra $\mathfrak{sl}_\alpha: \mathfrak{sl}_\alpha \rightarrow \mathfrak{sl}_\alpha$, and restricting along α realizes as \mathfrak{sl}_α -action on V for which $h_\alpha \in \mathfrak{sl}_\alpha$ acts by h_α . Since h_α acts semisimply on any \mathfrak{sl}_α -rep, we conclude that each h_α acts semisimply on V .

Hence V decomposes into its eigenspaces $V = \bigoplus_\lambda V_\lambda$.

Now, if $V_\lambda \neq 0$, then we again consider \mathfrak{sl}_α acting on V_λ via α to see that $\lambda(\alpha) \in \mathbb{Z}$ at each simple α , via our classification of (simple) \mathfrak{sl}_α -representations. [Corollary 5, "Ans 28"].

So we see that λ is in the weight lattice. \square

Defⁿ: A weight vector $v \in V$ is a nonzero vector which lies in some weight space V_λ , $\lambda \in \mathfrak{P}$.

We call a weight vector $v \in V_\lambda$ a highest wt. vector if $e_\alpha \cdot v = 0$ for each simple root α .

Lemma 13: Any ^{nonzero} (fin. dim) sln-rep V admits ⁽⁹⁾ a highest wt. vector v .

Proof: Take any wt μ with $V_\mu \neq \emptyset$ nonempty. Since the base Δ provides a basis for \mathfrak{h}^* , we have for any tuple of non-neg integers $c: \Delta \rightarrow \mathbb{Z}_{\geq 0}$ and $c': \Delta \rightarrow \mathbb{Z}_{\geq 0}$, $\mu + \sum c_\alpha \alpha = \mu + \sum c'_\alpha \alpha$ iff $c_\alpha = c'_\alpha$ for all α . Hence the space of wts. $\{\lambda: \lambda \geq \mu \text{ and } V_\lambda \neq \emptyset\}$ is finite and thus contains a max elem. λ under the ordering \geq . Any nonzero vector $v \in V_\lambda$ provides a highest wt. vect. in V . \blacksquare

Theorem 14: If $v \in V$ is a highest wt vector, with assoc. wt. $\lambda \in \mathfrak{P}$, then λ is dominant. Furthermore the subspace

$$L(\lambda) = \mathbb{C} \cdot \{ f_{\beta_1} \cdots f_{\beta_t} \cdot v : t \geq 0, \beta_1, \dots, \beta_t \in \Phi^+ \}$$
 forms a $\mathfrak{sl}(\mathfrak{g})$ -subrep in V .

Proof: By restricting along any root subalg, $\mathfrak{sl}_\alpha: \mathfrak{sl}_\mathfrak{g} \rightarrow \mathfrak{sl}_\alpha$ we realize $V|_{\mathfrak{sl}_\alpha} = \mathfrak{sl}_\alpha$ -rep w/ h acting by the wt vector h_α and e acting by e_α . Hence v is a highest wt vector for this \mathfrak{sl}_α -action, and we conclude [Cor 5, Prop 28] that the value $\lambda(h_\alpha)$ is a nonneg integer. Since α was chosen arbitrarily we see that λ is dominant.

The subspace $L(\lambda)$ is clearly stable under the action of each f_α and h_α , for simple α , and for each e_α we have

$$\begin{aligned} e_\alpha \cdot f_{\beta_1} \cdots f_{\beta_t} \cdot v &= [e_\alpha, f_{\beta_1} \cdots f_{\beta_t}] \cdot v \\ &= \sum_i f_{\beta_1} \cdots f_{\beta_{i-1}} [e_\alpha, f_{\beta_i}] \cdots f_{\beta_t} \cdot v. \end{aligned}$$

Each commutator $[e_\alpha, f_{\beta_i}] \in (\mathfrak{sl}_\alpha)_{-\beta_i + \alpha}$ with one of three things occurring, by Proposition 8,

(10)
case I) $-\beta_i + \alpha$ is not a root, and $(e_\alpha, f_{\beta_i}) = 0$.

case II) $-\beta_i + \alpha$ is a negative root, and

$$(e_\alpha, f_{\beta_i}) = c_\alpha f_\beta$$

for some scalar c_α and f_β .

case III) $-\beta_i + \alpha = 0$, i.e. $\beta_i = \alpha$, and

$$(e_\alpha, f_{\beta_i}) = h_\alpha.$$

In each case the term $f_{\beta_i} \dots (e_\alpha, f_{\beta_i}) \dots f_{\beta_t} v$ lies in $L(\lambda)$, so that

$$c_\alpha f_{\beta_i} \dots f_{\beta_t} v \in L(\lambda).$$

Since the e_α, f_α generate $\mathfrak{sl}(\mathbb{C})$ as a Lie alg, we see that $L(\lambda) \subseteq V$ is a \mathfrak{sl} -subrep. \square

Corollary 15: Each simple $\mathfrak{sl}_n(\mathbb{C})$ -rep L admits a unique highest wt. vector $v \in L$, up to scaling.

Proof: L has some highest wt vector v . For any other highest wt vector v' , Prop 14 tells us that L contains a subrep $L' \subseteq L$ with highest wt vector w and $(L')_\mu \neq 0$ implying,

$\mu \leq \lambda' = \text{wt}(v')$. Since $L' = L$ necessarily we

have $\lambda \leq \lambda'$ and similarly $\lambda' \leq \lambda$, giving $\lambda' = \lambda$.

Prop 14 also gives $\dim(L')_{\lambda'} = 1$ so that

in fact $v' = c \cdot v$ for nonzero scalar c . \square

- Characterizing $\text{rep}(\mathfrak{sl}_n(\mathbb{C}))$

6

Main Theorem: a) For each dominant weight $\lambda \in P^+$ there is a unique simple $\mathfrak{sl}_n(\mathbb{C})$ -rep $L(\lambda)$ of highest weight λ .

b) The map

$$P^+ \rightarrow \text{Irr}(\mathfrak{sl}_n(\mathbb{C})) / \cong, \quad \lambda \mapsto L(\lambda),$$

is a bijection.

c) The category $\text{rep}(\mathfrak{sl}_n(\mathbb{C}))$ is semisimple.

We'll deal w/ some details of the proof later in the class. Let me sketch some details here however

First we deal w/ existence of highest wt samples / P^+ .

As a starting point we again consider the standard rep.

- Fundamental wts and the standard rep < In foll. disc. we take more samples

Def⁶: Take $\Delta = \{\alpha_1, \dots, \alpha_{n-1}\}$ the standard basis for \mathfrak{sl}_n , and let $h_{\alpha_i} \in \mathfrak{h}$ be the corresp vectors in the Cartan ($h_{\alpha_i} \in E_{\alpha_i} - E_{-\alpha_i}$).

The i -th fundamental weight $w_i \in P^+ \subseteq \mathfrak{h}^*$ is the unique weight satisfying

$$w_i(h_{\alpha_j}) = \delta_{ij}.$$

Remark: Clearly $P = \mathbb{Z} \cdot \{w_i : 1 \leq i \leq n-1\}$.

By Lemma 1, we can observe samples of arbitrary highest wt if we can construct samples $L(w_i)$ of highest wt w_i at each fundamental wt.

Example: For $\mathfrak{sl}_n(\mathbb{C})$ we have the standard rep $V = \mathbb{C}^n$ w/ natural action.

For $v_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$ ← i th row the standard (12)

base vector, we have

$$e_{\alpha_i} \cdot v_j = \begin{cases} v_{i-1} & \text{if } j=i+1 \\ 0 & \text{else} \end{cases}$$

$$f_{\alpha_i} \cdot v_j = \begin{cases} v_{i+1} & \text{if } j=i \\ 0 & \text{else} \end{cases}$$

$$h_{\alpha_i} \cdot v_j = \begin{cases} 1 & \text{if } j=i \\ -1 & \text{if } j=i+1 \\ 0 & \text{else} \end{cases} \quad (*)$$


$\Rightarrow \mathbb{V}$ has unique highest wt. vector v_1 .

From $(*)$ we see $h_{\alpha_i} \cdot v_1 = \begin{cases} 1 & \text{if } i=1 \\ 0 & \text{else} \end{cases}$
so that v_1 has weight w_1 .

Proposition 16: The standard rep \mathbb{V} on $\mathfrak{sl}_n(\mathbb{C})$ is the first fundamental simple

$$\mathbb{V} = L(w_1),$$

i.e. it is a simple rep of highest wt w_1 .

(Prob: We only note that \mathbb{V} has a unique highest wt vector, up to scaling, and is hence simple by Theorem 14. We calculated the highest wt as w_1 above. 

- Defn: Symmetric and exterior powers of representations.

One shows directly that for any \mathfrak{g} -alg \mathfrak{g} , and \mathfrak{g} reps V and W , the vect. space isom

$$T_{V,W}: V \otimes W \rightarrow W \otimes V, \quad T_{V,W}(v,w) := w \otimes v,$$

is an isomorphism of \mathfrak{g} -representations.

Lemma 17: For any maps of \mathfrak{g} -reps

$\phi: V_0 \rightarrow V_1$, $\psi: W_0 \rightarrow W_1$, the map $\phi \otimes \psi$ is a map of \mathfrak{g} -reps and for diagram

$$\begin{array}{ccc} V_0 \otimes W_0 & \xrightarrow{\tau} & W_0 \otimes V_0 \\ \phi \otimes \psi \downarrow & & \downarrow \psi \otimes \phi \\ V_1 \otimes W_1 & \xrightarrow{\tau} & W_1 \otimes V_1 \end{array}$$

commutes.

Proof: Clear by inspection. \blacksquare

For any \mathfrak{g} -rep V and $n \geq 1$, the automorphisms

$$\tau_i = id^{\otimes i-1} \otimes \tau_{V,V} \otimes id^{\otimes n-i-1} : V^{\otimes n} \rightarrow V^{\otimes n}$$

satisfy the relations

$$(*) \quad \begin{cases} \tau_i^2 = id \\ \tau_i \tau_j = \tau_j \tau_i \quad \text{when } |i-j| > 1 \\ \tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1} \end{cases}$$

and hence define an action of

$$S_n = \langle \tau_1, \dots, \tau_{n-1} \mid \text{reps } (*) \rangle$$

v.i. \mathfrak{g} -automorphisms,

$$S_n \rightarrow \text{Aut}_{\mathfrak{g}}(V^{\otimes n}).$$

Defⁿ: Given any \mathfrak{g} -representation V , the n -th symmetric power is the \mathfrak{g} -representation

$$S^n(V) := (V^{\otimes n})^{S_n} \quad \text{or } S^n\text{-invariants}$$

and the n -th exterior power

$$\Lambda^n(V) := [(V^{\otimes n})^{\otimes \text{sgn}}]^{S_n} \quad S^n\text{-coinvariants.}$$

$$\text{Explicitly, } S^n(V) = \left\{ w \in V^{\otimes n} : w = \tau_i w \text{ at all } i=1, \dots, n-1 \right\}$$

$$= \text{Hom}_{S_n}(\mathbb{C}, V^{\otimes n})$$

$$= \text{equiv.} \left(V^{\otimes n} \xrightleftharpoons[\tau_i]{id} V^{\otimes n} : i=1, \dots, n-1 \right)$$

$$\Lambda^n(V) = V^{\otimes n} / (w - \epsilon_i w : w \in V^{\otimes n} \text{ and } i=1, \dots, n-1)$$

$$= (V^{\otimes n})^{\otimes \text{sgn}}_{S_n}$$

$$= \text{coequiv.} \left(V^{\otimes n} \xrightleftharpoons[\tau_i]{id} V^{\otimes n} : i=1, \dots, n-1 \right)$$

Exercise: Verify ^{directly} that $S^n(V)$ is a of subrep in $V^{\otimes n}$ and that $\tilde{\Lambda}^n(V)$ is a quotient of rep of $V^{\otimes n}$. (14)

Example: For $\mathbb{V} = L(1)$ the standard rep for $sl_2(\mathbb{C})$, $S^n(L(1))$ has highest wt. vector $v_1^{\otimes n} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^{\otimes n}$ which is of wt $n \cdot 1 = n$. We have

$$\begin{aligned} \dim S^n(L(1)) &= \dim \mathbb{C} \cdot \left\{ \frac{1}{n!} \left(\sum_{\sigma \in S^n} \sigma \right) \cdot v_1^{\otimes n-m} \otimes v_{-1}^{\otimes m} \mid 0 \leq m \leq n \right\} \\ &= n+1. \end{aligned}$$

For wt. reasons now $L(n) \subseteq S^n(L(1))$ and for dim reasons this inclusion is an isomorphism;

$$S^n(L(1)) \cong L(n) \text{ at all } n.$$

For the exterior powers,

$$\begin{aligned} \tilde{\Lambda}^1(L(1)) &= L(1), \quad \tilde{\Lambda}^2(L(2)) = \mathbb{C} \cdot \{v_1 \wedge v_2\} \cong L(2), \\ \tilde{\Lambda}^{23}(L(2)) &= 0. \end{aligned}$$

- Exterior power of the standard rep

Consider the n -dimensional standard rep

$$\mathbb{V} = \mathbb{C}^n,$$

we have the highest wt. vector

$$z_k := v_1 \wedge v_2 \wedge \dots \wedge v_k \in \tilde{\Lambda}^k(\mathbb{V}),$$

whenever $k \leq n$. \leftarrow we exploit basis $z_k(i) = v_{i_1} \wedge \dots \wedge v_{i_k}$ with $i_1 < \dots < i_k$

Lemma 18: z_k is of weight w_k , and is the unique highest wt. vector in $\tilde{\Lambda}^k(\mathbb{V})$.

Proof: We have for $i < k$ directly

$$\begin{aligned} h_{\alpha_i} \cdot z_k &= v_1 \wedge \dots \wedge h_{\alpha_i} v_i \wedge \dots \wedge v_k \\ &\quad + v_1 \wedge \dots \wedge h_{\alpha_i} v_{i+1} \wedge \dots \wedge v_k \\ &= z_k - z_k = 0 \end{aligned}$$

and for $k < i$ $h_{\alpha_i} \cdot z_k = 0$ as well. At $i=k$

$$- h_{\alpha_k} \cdot z_k = v_1 \wedge \dots \wedge h_{\alpha_k} v_k = z_k.$$

Then $h \cdot z_k = w_k(h) \cdot z_k$ at all $h \in \mathfrak{h}$, giving z_k a weight w_k .

For a general nonzero vector $z \in \Lambda^k(\mathbb{V})$ take $z(i) = v_{i_1} \wedge \dots \wedge v_{i_k}$ for i an increasing function $i: \{1, \dots, k\} \rightarrow \{1, \dots, n\}$, and order such i via the dictionary ordering. Then

$$z = c_i \cdot z(i) + \sum_{j < i} c_j \cdot z(j)$$

with c_i nonzero. Supposing $z \neq z_k$ we have a final index $i_\ell \in \{i_1, \dots, i_k\}$ with $i_\ell - i_{\ell-1} > 1$, where we take formally $i_0 = 0$, and for

$$i' = \{i_1, \dots, i_{\ell-1}, i_{\ell+1}, \dots, i_k\}$$

we have
$$e_{\alpha_{i_\ell-1}} \cdot z = c_i \cdot z(i') + \sum_{j < i} d_j \cdot z(j).$$

In particular $e_{\alpha_{i_\ell-1}} \cdot z \neq 0$, and z is not a highest wt. vector. ✎

Theorem 19: For each integer $k=1, \dots, n-1$ the exterior power $\Lambda^k(\mathbb{V})$ is a simple $\mathfrak{sl}_n(\mathbb{C})$ -rep. of highest wt. w_k ,

$$\Lambda^k(\mathbb{V}) = L(w_k).$$

Proof: By Theorem 14, every ^{nonzero} $\mathfrak{sl}_n(\mathbb{C})$ -subrep $L \subseteq \Lambda^k(\mathbb{V})$ contains a highest wt vector, and hence contains z_k by Lemma 17.

We claim now that

$$\Lambda^k(\mathbb{V}) = \mathbb{C} \cdot \{ f_{\alpha_1} \dots f_{\alpha_k} \cdot z_k : t \geq 0, \alpha_i: \{1, \dots, k\} \rightarrow \Delta \}, \tag{*}$$

so that any subrep containing z_k must be all of $\Lambda^k(\mathbb{V})$.

For this consider again the basis vectors

$$\{ z(i) = v_{i_1} \wedge \dots \wedge v_{i_k} : \text{increasing fun } i: \{1, \dots, k\} \rightarrow \{1, \dots, n\} \}$$

We claim that each basic vector $z(i)$ is in the span $(*)$, so that any subrep which contains $z_k = z(i_{\min}) = z(1, 2, \dots, k)$ is necessarily = to $\Lambda^k(V)$. We proceed by ind. under the lexicographic order on the set of increasing sums $i: \{1, \dots, k\} \rightarrow \{1, \dots, n\}$.

For the min index $i_{\min} = (1, \dots, k)$ we have $z_k = z(i_{\min}) \in \text{span } (*)$, and suppose now $i > i_{\min}$ with $z(j) \in \text{span } (*)$ for all $j < i$.

Since $i > i_{\min}$ there is a first index

$$i_\ell \in \{i_1, \dots, i_k\}$$

at which $i_\ell - i_{\ell-1} > 1$, where we take $i_0 = 0$.

Then for the index $i' < i$ def by


$$i' = (i_1, \dots, i_{\ell-1}, \dots, i_k)$$

we have $z(i') \in \text{span } (*)$ and

$$z(i) = f_{e_{i_\ell-1}} \cdot z(i')$$

giving $z(i) \in \text{span } (*)$ as well. Hence

all basis vectors $z(i) \in \text{span } (*)$ by induction.

Consequently, any subrep $W \subseteq \Lambda^k(V)$ which contains the highest wt. vector z_k is equal to $\Lambda^k(V)$ and we conclude our arbitrary subrep L is in fact all of $\Lambda^k(V)$. This establishes simplicity. 

Conclusion: For each fundamental wt

$$\omega_k, \quad k=1, \dots, k-1,$$

$$\omega_k(\alpha_{e_i}) = \delta_{ik},$$

The k -th exterior power of the standard rep V realizes a simple $\mathfrak{sl}_n(\mathbb{C})$ -rep of highest wt ω_k .

— In particular, such highest wt. examples exist.

- Existence of highest wt. simples over P^+

Proposition 20: For each dominant weight $\lambda \in P^+$, there is a simple representation $L(\lambda)$ with a unique highest wt. vector of wt. λ .

Proof: Since λ is dominant we have unique nonnegative integers m_1, \dots, m_n so that

$$\lambda = m_1 \alpha_1 + \dots + m_n \alpha_n.$$

By Thm 14 there exist simple sub-reps $L(\alpha_k)$ of highest wt α_k at each fundamental wt. α_k .

Let's take

$$V(\lambda) := L(\alpha_1)^{\otimes m_1} \otimes \dots \otimes L(\alpha_n)^{\otimes m_n}.$$

Note that $\dim V(\lambda)_\lambda = 1$ and by Thm 14 applied to the $L(\alpha_k)$ we see

$$V(\lambda)_\mu \neq 0 \Rightarrow \mu \leq \lambda. \quad (*)$$

Thus, for any composition series

$$0 = V_t \subset V_{t-1} \subset \dots \subset V_0 = V(\lambda)$$

there exists a unique index i at which the simple

$$\text{composition factor } L = V_i / V_{i+1}$$

has $\dim L_\lambda = 1$ and $L_\mu = 0 \Rightarrow \mu < \lambda$.

It follows that $L = L(\lambda)$ is of highest wt λ .

- Sketch proof of uniqueness

Thm 21: Given simple L and L' of highest wt. λ , with highest wt. vectors v and v' , there exists a unique isom $\phi: L \rightarrow L'$ over $\text{sh}(\mathbb{C})$ with $\phi(v) = v'$.

We'll be more careful about the proof when we deal w/ the general case. Let us sketch the details however

Sketch Proof: We have the univ. env. alg. $\mathcal{U}(\mathfrak{sl}_n)$ and the subalg. env. alg. for the Borel $\mathcal{U}(\mathfrak{b}^+)$, $\mathfrak{b}^+ = \mathfrak{h} \oplus \mathfrak{n}^+$. We have the 1-dim simple \mathbb{C}_λ of wt. λ over \mathfrak{b}^+ and define the Verma module $\mathcal{M}(\lambda) = \mathcal{U}(\mathfrak{sl}_n) \otimes_{\mathcal{U}(\mathfrak{b}^+)} \mathbb{C}_\lambda$. This is a highest weight, co-finite, weight graded, \mathfrak{sl}_n -rep and restriction along the inclusion $\mathbb{C}_\lambda \rightarrow \mathcal{M}(\lambda)$ provides a linear \cong

$$\text{Hom}_{\mathfrak{sl}_n}(\mathcal{M}(\lambda), V) \xrightarrow{\cong} \mathbb{C} \cdot \left\{ \begin{array}{l} \text{highest wt. vectors} \\ v \in V \text{ of wt. } \lambda \end{array} \right\}.$$

$$f \longmapsto f(\mathfrak{t}_\lambda).$$

For grading reasons, there is a unique simple quotient $\pi: \mathcal{M}(\lambda) \rightarrow \overline{\mathcal{M}}(\lambda)$, $(= \mathcal{L}(\lambda))$, and hence the unique \mathfrak{sl}_n -maps

$$f: \mathcal{M}(\lambda) \rightarrow L, \quad f': \mathcal{M}(\lambda) \rightarrow L'$$

$$f(\mathfrak{t}_\lambda) = v_\lambda, \quad f'(\mathfrak{t}_\lambda) = v'_\lambda$$

induce isms

$$L \begin{array}{ccc} & \overline{\mathcal{M}}(\lambda) & \\ \swarrow & \xrightarrow{\pi} & \searrow \\ & L' & \end{array}$$

The induced $\cong \varphi: L \rightarrow L'$ completes the above diagram doing the desired job. \square

Corollary 21: The assignment

$$\mathfrak{p}^+ \mapsto \text{Im}(\mathfrak{sl}_n(\mathbb{C})), \quad \lambda \mapsto \mathcal{L}(\lambda)$$

is a bijection, i.e. classifies all irreducible $\mathfrak{sl}_n(\mathbb{C})$ -reps up to isom.

- Semisimplicity for $\text{rep}(sl_n(\mathbb{C}))$.

Prop 22: For simple $sl_n(\mathbb{C})$ -reps $L(\lambda)$ and $L(\mu)$, and extension

$$0 \rightarrow L(\mu) \rightarrow V \rightarrow L(\lambda) \rightarrow 0$$

is split.

Again, we'll cover the details more slowly in the general setting. We again sketch the details.

Proof: The adjoint rep sl_n is again self dual, so that the trace form on sl_n induces a symmetric non-deg sl_n -invariant form

$$\kappa: sl_n^* \otimes sl_n^* \rightarrow \mathbb{C},$$


and under the natural sl_n action

$$sl_n \otimes sl_n \cong (sl_n^* \otimes sl_n^*)^*$$

κ determines an element $\Omega \in sl_n \otimes sl_n$,

$$\Omega = \left(\sum_{i,j \in \mathbb{Z}^+} c_{ij} (e_i \cdot e_j + e_j \cdot e_i) \right) + \left(\sum_i \text{Cartan terms} \right),$$

st $x \cdot \Omega = 0$ at all $x \in sl_n(\mathbb{C})$, also Ω is an invariant element in $sl_n^{\otimes 2}$.

The element Ω therefore acts by sl_n -linear endomorphisms on all reps $\Omega_V = \Omega \cdot - : V \rightarrow V$, and we can use Ω to split extensions of simples, just as in the case of sl_2 . (Exercise) 

Using [Prop 6, Aug 28] we now see that the category $\text{rep}(sl_n(\mathbb{C}))$ is semisimple.

Theorem 23: The cat $\text{rep}(sl_n(\mathbb{C}))$ is semisimple, and the simple $sl_n(\mathbb{C})$ -reps are in bij correspondence with dominant wts, $\lambda \mapsto L(\lambda)$.

— We've now recovered our "Main Theorem" for $sl_n(\mathbb{C})$.

End.