KERODON REMIX PART I: ∞-CATEGORY BASICS FOR ALGEBRAISTS

CRIS NEGRON

Abstract. These are notes on ∞-categories which are (mostly) adapted from Lurie's digital text [15]. The main distinctions are the length of the document, the order of presentation, and the use of selective omission. We also deviate from [15] in that we focus on derived categories and dg categories as our primary examples of interest. A primary distinction from the related text [13] would be the complete avoidance of model structures, though this approach is already adopted in [15]. In a certain language, our approach is fundamentally analytic rather than synthetic.

We provide basic introductions to Kan complexes, ∞-categories, functors between ∞-categories, functor categories, etc. We do not cover more advanced topics like (co)limits in ∞-categories and transport for (co)cartesian fibrations. The text terminates with an extensive discussion of mapping spaces, with special attention to the dg context. We demonstrate the (well-established) fact that a functor between ∞-categories is an equivalence if and only if it is fully faithful and essentially surjective, and we compute the (pinched) mapping spaces for the dg nerves of a dg categories via the Eilenbergh-MacLane spaces of their morphism complexes.

Contents

1. Preliminary remarks 4
1.1. Claims to (un)originality 4
1.2. For algebraists 4
1.3. What's not covered 4
1.4. Why do these notes exist 4
1.5. Final disclaimer 5
2. ∞-categories (introduction) 5
2.1. What is an ∞-category, in principle 5
2.2. Preliminary words on sets and universes 5
2.3. Simplicial sets 6
2.4. Infinity categories 8
2.5. The homotopy category 9
2.6. Functor categories 12
2.7. The (simplicial) categories of categories 13
3. Basic examples 13
3.1. The nerve of a plain category 14
3.2. Nerves of dg categories 14
3.3. An integral dg example with one object 17

Date: February 20, 2024.
The author was supported by NSF Grant No. DMS-2149817, NSF CAREER Grant No. DMS-2239698, and Simons Collaboration Grant No. 999367.

3.4. More examples with one object 19
3.5. Derived categories for abelian categories 20
3.6. Derived categories for dg modules 21
3.7. Nerves of simplicial categories 22
4. The category of spaces 23
4.1. More language for simplicial sets 23
4.2. Kan complexes, Kan fibrations, and anodyne maps 25
4.3. Trivial Kan fibrations 28
4.4. Exponents for Kan complexes 28
4.5. Why maps from simplicial sets 30
4.6. Connected components and the notion of equivalence 30
4.7. Equivalences and functor spaces 31
4.8. Homotopy groups for Kan complexes 33
4.9. Homotopy groups and trivial Kan fibrations 35
4.10. Whitehead's theorem 36
4.11. A deviation on long exact sequences and homotopy groups 37
4.12. Equivalences and fibers 39
4.13. Kan complexes as spaces 40
4.14. The ∞-category of spaces 41
5. Basics for ∞-categories 42
5.1. Inner fibrations and inner anodyne maps 42
5.2. Exponentials of ∞-categories 43
5.3. Opposite categories 44
5.4. Isomorphisms in ∞-categories 45
5.5. Joins of simplicial sets 46
5.6. Overcategories and undercategories 48
5.7. Directional fibrations and under/overcategories of ∞-categories 49
5.8. Isofibrations 51
5.9. The Kan complex associated to an ∞-category 53
5.10. Equivalences and the homotopy category of ∞-categories 56
5.11. Equivalences via functor categories 57
5.12. The ∞-category of ∞-categories 58
6. Homotopy pullback and categorical pullbacks 58
6.1. Fibrant replacement 58
6.2. Introductory discussion for (homotopy) pullback 59
6.3. A functorial construction 60
6.4. Homotopy pullback squares 62
6.5. Simplicial injections, Kan fibrations, and homotopy pullback 64
6.6. Categorical pushout for simplicial sets 66
6.7. Categorical pullback square 67
7. Mapping spaces 69
7.1. Definitions 69
7.2. Natural isomorphisms 70
7.3. The mapping spaces are spaces 73
7.4. Mapping spaces as loop spaces 73
8. Fully faithful functors and equivalence 74
8.1. Fully faithful functors and equivalences 74
8.2. Equivalences are fully faithful 75

8.3. The Kan complex functor and equivalences 75
8.4. Equivalences are essentially surjective 77
8.5. Restricting fully faithful functors 77
8.6. Proof of Theorem 8.2 78
9. Composition functions for mapping spaces 80
9.1. A spacially enriched category 80
9.2. Imagining ∞-lifting 81
9.3. Comparing with endomorphism A_∞-algebras 82
10. Pinched mapping spaces 82
10.1. Pinched mapping spaces 82
10.2. Oriented products 83
10.3. Blunt joins and oriented products 84
10.4. The comparison map is a categorical equivalence 85
10.5. Slice and coslice diagonals 86
10.6. Co/slice diagonals are equivalences 87
10.7. Details for Theorem 10.14 88
10.8. Proof of Theorem 10.14 89
10.9. Comparisons of mapping spaces 89
11. Dg preparations: the Dold-Kan correspondence 91
11.1. Reminders on simplicial sets 91
11.2. Simplicial abelian groups 91
11.3. Cochains from simplicial abelian groups 92
11.4. Eilenberg-MacLane functor 93
11.5. The Dold-Kan correspondence 94
11.6. Refined Dold-Kan correspondence 95
11.7. Dold-Kan and homotopy 95
11.8. Dold-Kan and pointed homotopy 98
11.9. Proof of Theorem 11.13 99
12. Mapping spaces for dg categories 100
12.1. Main findings 100
12.2. Simplices in the pinched mapping space, explicitly 101
12.3. Identification with the Eilenbergh-MacLane construction 102
12.4. Proof of Theorem 12.3 103
12.5. A basic example: Koszul duality 104
13. Injective versus projective models for $\mathscr{D}(\mathbb{A})$ 105
13.1. The bounded setting 105
13.2. The unbounded setting 107
13.3. Uniqueness of the injective-projective transition 108
13.4. Comparison with localization 109
14. Adjoint functors 110
14.1. Adjoint functors 110
14.2. Adjoints and the homotopy category 111
14.3. Natural transformations from the dg setting 111
14.4. Adjoints from the dg setting 113
14.5. A simple example 113
References 113

1. Preliminary remarks

1.1. **Claims to (un)originality.** As the title suggests, this document is essentially a reorganization, and selective re-presentation, of materials from Lurie's Kerodon [15]–though all of the sinew materials are of my own creation. The discussions of the derived category herein also deviate from those of [14], to some degree, and the dg constructions employed in Section 13 were developed independently by myself. Outside of these few particular instances, I make not claims to originality in this work and have made copious references to the original text(s) throughout. Most importantly, the vast majority of the arguments are adapted from [15]. The reader might therefor find and reference the original texts where appropriate.

The initial aim of this document was to provide an introduction to ∞-categories which adapted perspectives from various documents in the literature. Though I began by consulting multiple texts in this regard, and do ultimately draw from a few sources other than [15], it turns out that Kerodon was essentially a perfect document for me, outside of its length and encyclopedic nature. (Of course, it is an encyclopedic text by its very nature.) So what we are left with in the end is my own "remix" of Lurie's 2024 techno-futurist journey [15].

1.2. **For algebraists.** By "for algebraists" we mean two things:

(1) We focus on dg categories and derived categories as our motivating examples.
(2) We avoid all references to (Quillen) model structures.

An interesting point in this regard is that, while the earlier text [13] is littered with references to model structures, there is not a single reference to model categories throughout the entirety of [15].

We should note, however, that we do not take the perspective that vector spaces are preferable to topological spaces (Kan complexes). Our perspective is that one might think of ∞-categories as a reformulation of the theory of dg categories in which one replaces cochains and cohomology with spaces and their homotopy groups. Additionally one avoids in this spacialization of dg categories, through some act of wizardry and black magic, the erroneous choices present in other linear approaches to derived categories such as A_∞-categories.

1.3. **What's not covered.** We cover nothing of a specialized nature, nor do we address the (very important) topics of (co)cartesian fibrations and transport, limits and colimits in ∞-categories, presentability, or stability. Monoidal structures, operads, E_n-algebras and schemes, etc. are clearly not discussed either. Some of these topics may appear in later iterations.

1.4. **Why do these notes exist.** It has never occurred to me to share any notes, lecture or otherwise, publicly. However, I hope these writings can serve some purpose within the public domain. In particular, it seems at the moment that there are not many resources on this topic which are relatively short, do not rely on advanced topological notions, which take an algebraic perspective, and which are rigorous in their treatment of the topic. I hope that this document might be consumable to the working mathematician in finite time, and might also serve as a starting point for a reader's further investigations into ∞-categories and their practical applications.

To highlight a few other references which are both rigorous and readily consumable, let me bring the reader's attention to three other fairly concise treatments:

- Land's *Introduction to infinity-categories* [11].
- Rezk's *Introduction to quasicategories* [17].
- Cisinki's *Higher categories and homotopical algebra* [3].

Of these three works, our perspective synergizes most closely with the presentation of Land. Indeed, if one replaces dg categories with simplicial categories, then Land's text is–if I may speak with some levity for a moment–a more professional and informed version of the present notes. Land also covers (co)cartesian fibrations and (co)limits in his work. We encourage the reader to check out the references above, as well as any other works on the topic which they find compelling.

1.5. **Final disclaimer.** I (the author) am a representation theorist by trade. I am not an expert in homotopy theory, nor am I an expert in ∞-categories. I've produced these notes as an outsider who is interested in leveraging homotopical methods to address specific representation theoretic problems.

2. ∞-categories (introduction)

Here we give a light introduction, and provide bare-bones definitions of simplicial sets, Kan complexes, and ∞-categories. In Section 3 we provide a number of basic examples, and subsequently begin our discussions of these topics in earnest.

2.1. **What is an ∞-category, in principle.** An infinity category $\mathscr{A}$ is, in the barest sense of things, a special type of simplicial set. We think about an infinity category as a type of "category" by considering the 0-simplices in $\mathscr{A}$ as objects, and the 1-simplices as morphisms. The 2-simplices $\zeta : \Delta^2 \to \mathscr{A}$ are *choices* of compositions $f \circ g \approx_\zeta h$, where we view ζ itself as a type of homotopy between the raw composition and the map h,

$$\zeta : \quad \begin{tikzcd} & Z \arrow[dr, "f"] & \\ X \arrow[ur, "g"] & & Y \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} & Z \arrow[dr, dotted] & \\ X \arrow[ur, dotted] \arrow[rr, "h"'] & & Y \end{tikzcd}$$

We note that without ζ this raw composition "$f \circ g$", which is simply the choice of two 1-simplices with a shared vertex, is unstructured information in $\mathscr{A}$. So, the composition *does not have a meaning* without ζ.

Note that, given our presentation above, we want $\mathscr{A}$ to admit compositions of functions. Rather, for any two 1-simplices $f, g : \Delta^1 = \{0, 1\} \to \mathscr{A}$ with matching vertices $g(1) = f(0)$ we require the existence of *some* filling $\zeta : \Delta^2 = \{0, 1, 2\} \to \mathscr{A}$ with $\zeta|_{\{0,1\}} = g$ and $\zeta|_{\{1,2\}} = f$. This choice of filling ζ shouldn't matter, up to some 3-simplex $\gamma : \Delta^3 \to \mathscr{A}$, and so on.... The notion of an infinity category is some completion of this thought.

In this section we recall the necessary (simplicial) set theoretic background, then formally define the notions ∞-categories and functors between such objects.

2.2. **Preliminary words on sets and universes.** We work with ZFCU. So, we suppose that each set X lives in a (Grothendieck) universe math $\mathbb{U}$. That is, for each set X we assume the existence of an additional set of sets $\mathbb{U}$ with X living in $\mathbb{U}$. This set $\mathbb{U}$ is assumed to contain the first infinite ordinal ω, and to be closed under all basic operations in set theory, including the formations of power sets. So, for example, Λ is in $\mathbb{U}$, and $f : \Lambda \to \mathbb{U}$, then then image $f(\Lambda) \in \mathbb{U}$, and all subsets an quotients of $f(\Lambda)$ are also in $\mathbb{U}$. $\mathbb{U}$ is also closed under unions and products. So we see that $\mathbb{U}$ is closed under colimits and limits which are indexed by sets in $\mathbb{U}$. This is all to say, we can do set theory in $\mathbb{U}$.

Given a Grothendieck universe $\mathbb{U}$, we say a set X is ($\mathbb{U}$-)small if $X \in \mathbb{U}$. So, a set X is not small if X is not an element in $\mathbb{U}$. An important feature of the universe axiom is that each universe $\mathbb{U}$ is itself a set, and hence lives in a (larger) universe, as does any set at all. So, give a set Y, we can choose a universe $\mathbb{U}'$ with $Y \cup \mathbb{U}$ in $\mathbb{U}'$. That is to say, we can always assume that a set Y is small by *enlarging* our universe. As long as our arguments are universe independent, this enlarging of the universe will cause no problems.

We note that the existence of universes is not at all implied by the usual axioms of ZFC, and requires the introduction of the new axiom "U". This axiom is equivalent to the existence of extremely large cardinals, called inaccessible cardinals.

In general, we fix a universe $\mathbb{U}$ (which we may flex slightly when needed), and a (co)complete category $\mathscr{A}$ is a category which admits all (co)limits indexed by small sets relative to the given universe. What's interesting is that this kind of category theory is essentially indistinguishable from usual category theory.

For example, if a fix a field k and a universe $\mathbb{U}$ containing k, then we may consider the category $\mathrm{Vect} = \mathrm{Vect}(\mathbb{U})$ of $\mathbb{U}$-small vector spaces. This category is closed under the formation of small limits and colimits, and so is cocomplete. In this setting all representability theorems are also perfectly viable as well.

The importance of employing universes is that we may assume that all (infinity) categories have a *set* of objects and morphisms, i.e. are special kinds of structured sets. One then can speak very clearly not only of the category of small categories

$$\mathrm{Cat}^{\mathrm{sm}} \subseteq \mathbb{U}\text{-Set},$$

but of the category of arbitrary categories

$$\mathrm{Cat} \subseteq \mathrm{Set}.$$

So our assumption that an infinity category is a type of simplicial set is not at all limiting, and allows us to pursue a rigorous analysis of the category Cat_∞ of infinity categories itself.

2.3. **Simplicial sets.** Let Δ denote the category of linearly ordered, non-empty, finite sets with weakly increasing functions. In Δ we have the objects $[n] = \{0, 1, \ldots, n\}$, with their natural ordering, which exhaust all objects up to isomorphism. A simplicial set is a functor $S : \Delta^{op} \to Set$, and we have the category of simplicial sets

$$\mathrm{sSet} = \{\text{functors } \Delta^{op} \to \mathrm{Set}, \text{ with natural transformations}\}.$$

Amongst simplicial sets one has the standard n-simplices, which are the representable functors

$$\Delta^n := \mathrm{Hom}_\Delta(-, [n]).$$

A map then $f : \Delta^n \to S$ to some simplicial set S is then a choice of elements $f(id) \in S([n])$, and we refer to those elements, or maps rather, as the n-simplices in S. We let $S[n]$ denote the set of n-simplices in S, as a shorthand.

To make this more clear, any natural transformation $f : \Delta^n \to S$ is just a collection of compatible maps between sets $f = f_J : \Delta^n(J) \to S(J)$, for all linearly ordered sets J, and for any $r \in \Delta^n(J) = \mathrm{Hom}_\Delta(J, [n])$ we have $r = r^*(id_{[n]})$ so that

$$f(r) = f(r^*(id_{[n]})) = r^*(f(id_{[n]})).$$

Hence f is determined by the value $f(id_{[n]}) \in S[n]$. Conversely, any element $x \in S[n]$ determines such a functor $f : \Delta^n \to S$, $f_J(\zeta) := \zeta_S^*(x)$. This is obviously some kind of Yoneda Tom-foolery, and Yoneda tells us directly that the map

$$\Delta \to \mathrm{sSet}, \quad J \mapsto \Delta^J$$

is fully faithful

Let us define here the boundary simplex $\partial\Delta^n \subseteq \Delta^n$ and the i-th *horn* $\Lambda_i^n \subseteq \partial\Delta^n$. We have, for each $j \leq n$, the j-th face map $\partial_j : [n-1] \to [n]$, which is the unique increasing map with j not in its image. The boundary $\partial\Delta^n$ in Δ^n is the simplicial subset consisting of all maps $r : J \to [n]$ which factor though some face $r = r'd_j$, $0 \leq j \leq n$. The i-th horn Λ_i^n in Δ^n is the simplicial subset consists of the collection of all maps $r : J \to [n]$ which factor through some face map $r = r'd_j$ with j *not* equal to i. So, for example, the i-th face map itself $d_i : [n-1] \to [n]$ lies in the boundary $\partial\Delta^n$, but does not lie in the i-th horn Λ_i^n.

The category of simplicial sets has products and coproducts, which are defined in the naïve ways

$$(K \times S)(J) = K(J) \times S(J) \quad \text{and} \quad (K \amalg S)(J) = K(J) \amalg S(J).$$

We similarly have fiber products and coproducts

$$\begin{tikzcd} K_1 \times_S K_2 \arrow[r] \arrow[d] & K_2 \arrow[d] \\ K_1 \arrow[r] & S \end{tikzcd} \qquad \begin{tikzcd} S \arrow[r] \arrow[d] & K_1 \arrow[d] \\ K_2 \arrow[r] & K_1 \amalg_S K_2, \end{tikzcd}$$

whose values on any linearly ordered set J are the fiber product and coproduct of the corresponding sets. Indeed, the category of simplicial sets is both complete and cocomplete, with

$$(\varprojlim_{i \in I} K_i)(J) = \varprojlim_{i \in I} K_i(J) \quad \text{and} \quad (\varinjlim_{i \in I} K_i)(J) = \varinjlim_{i \in I} K_i(J),$$

and any simplicial set K is reconstructible from the simplices Δ^n as the colimit

$$\varinjlim_{\Delta^n \to K} \Delta^n \xrightarrow{\sim} K.$$

Remark 2.1. This reconstruction of K from its simplices can be compared with the reconstruction of a scheme X from the category of affine schemes over X, $\mathrm{Spec}(R) \to X$, or from its Zariski site when X is separated.

Remark 2.2. We note that, for any linearly ordered set J, there is a unique isomorphism $J \cong [|J|]$ in Δ. It follows that any simplicial set $K : \Delta^{op} \to \mathrm{Set}$ is determined up to unique isomorphism by its restriction to the full subcategory $[\mathbb{Z}] \subseteq \Delta$. This is to say, any simplicial set is determined up to unique isomorphism by its collection of n-simplices $K[n]$ and the dual maps $r^* : K[n] \to K[m]$ to the collection of weakly increasing functions $r : [m] \to [n]$. So we could define a simplicial set simply as a functor $K : [\mathbb{Z}]^{op} \to \mathrm{Set}$. At times, however, it can be convenient to employ the larger category Δ of all linearly ordered sets.

2.4. **Infinity categories.** An infinity category is, at the base of it, a certain type of simplicial set. We begin with a stronger notion which encodes a simplicial version of a topological space.

Definition 2.3. A simplicial set K is a Kan complex provided that, for each positive integer n, $i \in [n]$, and map of simplicial sets $f : \Lambda^n_i \to K$, there exists an extension of f to a map $\widetilde{f} : \Delta^n \to K$.

We will refer to such an extension $\widetilde{f}$ as a "filling" for the given map $f : \Lambda^n_i \to K$. We think of such K as a type of category, with objects given by the 0-simplices K_0 and morphisms given by the 1-simplices K_1. Restricting along the inclusions $d_0, d_1 : [0] \to [1]$ gives the source and target objects for a given map $x \in K_1$. Any map $f : \Lambda^2_1 \to K$ is determined by the two 1-simplices $x \in K_1$ and $y \in K_1$ which are the images of the two maps

$$\begin{tikzcd} \Delta^1 \arrow[r, shift left, "d_2"] \arrow[r, shift right, "d_0"'] & \Lambda^2_1 \arrow[r] & K. \end{tikzcd}$$

The choice of a filling $\Delta^2 \to K$ specifies a third morphism $z \in K_1$ which provides the third face for the map $\partial\Delta^2 \to K$ and the complete filling $\Delta^2 \to K_1$ realizes an identification "$y \circ x$" $= z$:

$$\begin{tikzcd} & A_2 \arrow[dr, "y"] & \\ A_1 \arrow[ur, "x"] \arrow[rr, "z"'] \arrow[rr, phantom, shift left=4, "\widetilde{f}"] & & A_3. \end{tikzcd}$$

The higher filling axioms tell us that this filling procedure is essentially unique, in a way which is not transparent but which we leave unarticulated for the moment.

The identity morphism from an object $x \in K_0$ to itself is the image of x in K_1 along the structural map $id_? : K_0 \to K_1$ dual to the unique morphism $[1] \to [0]$. One similarly has higher identity morphisms at a given object x, which are the images of x under the structural maps $id^n_? : K_0 \to K_n$ which are again determined by the unique map $[n] \to [0]$.

Now, a Kan complex has a much more rigid structure than one needs to form this kind of conditional composition. Indeed, by considering maps $\Lambda^2_0 toK$ and $\Lambda^2_2 \to K$, one sees that all morphisms in a Kan complex are in fact invertible. So we arrive at the notion of an ∞-category.

Definition 2.4. An ∞-category is a simplicial set $\mathscr{C}$ such that, for each positive integer n, $0 < i < n$, and $f : \Lambda^n_i \to \mathscr{C}$, there exists an extension of f to a map $\widetilde{f} : \Delta^n \to \mathscr{C}$. A functor $F : \mathscr{C} \to \mathscr{D}$ between ∞-categories is simply a map of simplicial sets.

Note that we've now excluded the external horns Λ^n_0 and Λ^n_n in our lifting condition. So in particular this extension property is vacuous for 1-simplices. Note also that a Kan complex is a specific type of ∞-category.

Remark 2.5. What we call an ∞-category is also called an $(\infty, 1)$-category, and/or a weak Kan complex.

In defining functors between ∞-categories our specificity is somewhat strange. Indeed, we would imagine that one should define a functor to be some map which is compatible with the structure on the image category $\mathscr{D}$, only up to homotopy. However, one can see, for example, that a map between infinity categories $F : \mathscr{C} \to \mathscr{D}$ needn't be "compatible" with compositions of morphisms in $\mathscr{C}$ and $\mathscr{D}$. One simply needs some coherent rule which assigns for each composition "$x \circ y$" $= z$ in $\mathscr{C}$ *a* composition in $\mathscr{D}$, not *the* composition in $\mathscr{D}$, since we declined to make the mistake of choosing such a thing in the first place.

ex:Delta_infty

Example 2.6. The simplicial set Δ^n is an ∞-category. Indeed, any map $\Lambda^m_i \to \Delta^n$ from an inner horn is specified by a collection of maps $\Delta^{[m]-\{j\}} \to \Delta^n$, i.e. maps $r_j : [m] - \{j\} \to [n]$, which agree on their shared boundaries. Since the subsets $[m] - \{j\}$ cover $[m]$, when $m > 1$, the r_j glue to a unique map $r : [m] \to [n]$. To see that r is weakly increasing, note first that $m \geq 2$ in order for there to exist a inner horns for Δ^m and take any $l \in [m]$. Then l and $l+1$ are both in $[m] - \{0\}$ or $[m] - \{m\}$. In particular, there exists i so that $l, l+1 \in [m] - \{i\}$ and $r(l) = r_i(l) \leq r_i(l+1) = r(l+1)$. Since l was chosen arbitrarily it follows that r is weakly increasing.

Now, the above argument does not work when we consider maps $\Lambda^2_0 \to \Delta^n$ or $\Lambda^2_2 \to \Delta^n$ and $n \geq 1$. Indeed, consider the map $\Lambda^2_0 \to \Delta^n$ specified by the two functions $r_1 : \{0,2\} \to [n]$, $r_1(0) = r_1(2) = 0$, and $r_2 : \{0,1\} \to [n]$, $r_2(0) = 0$ and $r_2(1) = 1$. Then these r_i extend to a unique set map $r : [2] \to [n]$ given by $r(0) = 0$, $r(1) = 1$, $r(2) = 0$. This function is clearly not weakly increasing. We similarly consider the map $\Lambda^2_2 \to \Delta^n$ specified by the functions $r_0(1) = 0$, $r_0(2) = 1$, and $r_1(0) = 1$, $r_1(2) = 1$. This map extends uniquely to the function $r : [2] \to [n]$ defined by $r(0) = 1$, $r(1) = 0$, $r(2) = 1$. This function is not weakly increasing and so there exists no 2-simplex $\Delta^2 \to \Delta^n$ extending this horn. So we see that the positive dimensional standard simplices Δ^n are *not* Kan complexes.

ex:Sing_wk

Example 2.7. Let X be a topological space, and define the simplicial set $\mathrm{Sing}(X)$ by taking

$$\mathrm{Sing}(X)[n] := \mathrm{Hom}_{\mathrm{Top}}(|\Delta^n|, X),$$

where $|\Delta^n|$ is the standard topological n-simplex in $\mathbb{R}^n$. Since each inclusion of a horn horn $|\Lambda^n_i| \to |\Delta^n|$ admits a retract $|\Delta^n| \to |\Lambda^n_i|$ one sees that these singularity sets $\mathrm{Sing}(X)$ are Kan complexes.

Up to homotopy equivalence, all Kan complexes are of the form $\mathrm{Sing}(X)$. We elaborate on this point in Section 4.13.

2.5. **The homotopy category.** We construct a homotopy category $\mathrm{h}\mathscr{C}$ for any ∞-category $\mathscr{C}$, whose morphisms are certain equivalence classes of maps in $\mathscr{C}$, i.e. equivalence classes of 1-simpleces $\Delta^1 \to \mathscr{C}$. The first lemma defines the appropriate equivalence relation on maps in $\mathscr{C}$.

lem:200

Lemma 2.8. *For any two maps $\alpha, \alpha' : x \to y$ in an ∞-category $\mathscr{C}$, the following are equivalent:*

(a) *There exists a 2-simplex $s : \Delta^2 \to \mathscr{C}$ with $s|\Delta^{\{0,1\}} = \alpha$, $s|\Delta^{\{0,2\}} = \alpha'$, and $s|\Delta^{\{1,2\}} = id_y$.*
(b) *There exists a 2-simplex s with $s|\Delta^{\{0,1\}} = \alpha'$, $s|\Delta^{\{0,2\}} = \alpha$, and $s|\Delta^{\{1,2\}} = id_y$.*
(c) *There exists a 2-simplex s with $s|\Delta^{\{0,1\}} = id_x$, $s|\Delta^{\{0,2\}} = \alpha'$, and $s|\Delta^{\{1,2\}} = \alpha$.*

(d) *There exists a* 2*-simplex* s *with* $s|\Delta^{\{0,1\}} = id_x$*,* $s|\Delta^{\{0,2\}} = \alpha$*, and* $s|\Delta^{\{1,2\}} = \alpha'$*.*

Sketch proof. We prove the equivalence between (a) and (b), and (a) and (c). For (a) $\Rightarrow$ (b) consider the horn $\Lambda^3_1 \to \mathscr{C}$ which appears as

$$\begin{tikzcd} & y \arrow[dr, "id"] \arrow[dd, "id" near end] & \\ x \arrow[ur, "\alpha"] \arrow[rr, "\alpha" near start] \arrow[dr, "\alpha'"'] & & y \\ & y. \arrow[ur, "id"'] & \end{tikzcd}$$

One of the sides is given by s, and the others are given by the identity 2-simplex on y, and by expanding α along the surjective map $r : [2] \to [1]$ with $r(1) = r(2) = 1$. We fill this horn to find the simplex required by (b). One finds (b) $\Rightarrow$ (a) by swapping the roles of α and β in the above argument.

For (a) $\Rightarrow$ (c) consider the horn $\Lambda^3_2 \to \mathscr{C}$ which appears as

$$\begin{tikzcd} & x \arrow[dr, "\alpha"] \arrow[dd, "\alpha" near end] & \\ x \arrow[ur, "id"] \arrow[rr, "\alpha'" near start] \arrow[dr, "\alpha"'] & & y \\ & y. \arrow[ur, "id"'] & \end{tikzcd}$$

with sides given by s from (a), and by expanding α along the two weakly increasing surjections $r : [2] \to [1]$. We fill the horn to provide the necessary 2-simplex for (c). For the implication (c) $\Rightarrow$ (a) one fills the horn $\Lambda^3_1 \to \mathscr{C}$ which appears as above. $\square$

Let us say two maps $\alpha, \alpha' : x \to y$ are equivalent, $\alpha \sim \alpha'$, if any of the equivalent conditions of Lemma 2.8 are satisfied. After recall that we can expand any map $\alpha : x \to y$ to a 2-simplex of the form

$$\begin{tikzcd} & y \arrow[dr, "id"] & \\ x \arrow[ur, "\alpha"] \arrow[rr, "\alpha"'] & & y \end{tikzcd}$$

by restricting along the corresponding surjection $[2] \to [1]$, we see that $\alpha \sim \alpha$. This observation, along with the characterization of Lemma 2.8, assures that $\sim$ defines an equivalence relation on the set $\mathscr{C}[1]$ of edges in $\mathscr{C}$.

Let us say that a 2-simplex $s : \Delta^2 \to \mathscr{C}$ exhibits a morphism $\eta : x \to z$ as a composite or maps $\alpha : x \to y$, and $\beta : y \to z$, if

$$s|\Delta^{\{0,1\}} = \alpha, \quad s|\Delta^{\{1,2\}} = \beta, \quad \text{and} \quad s|\Delta^{\{0,2\}} = \eta.$$

Lemma 2.9. *Suppose we have two maps* $\alpha : x \to y$ *and* $\beta : y \to z$*, and* 2*-simplices* $s, s' : \Delta^2 \to \mathscr{C}$ *which exhibits two maps* $\eta, \eta' : x \to z$ *as composites of* α *and* β *in* $\mathscr{C}$*. Then* η *and* η' *are equivalent.*

Proof. The result follows by filling the inner horn $\Lambda^3_2 \to \mathscr{C}$ which appears as

$$\begin{tikzcd}
& y \arrow[dr, "\beta"] \arrow[dd, "\beta"] & \\
x \arrow[ur, "\alpha"] \arrow[rr, "\eta"] \arrow[dr, "\eta'"'] & & z \\
& z \arrow[ur, "id"'] &
\end{tikzcd} .$$

□

lem:346 **Lemma 2.10.** *Consider equivalent maps $\alpha, \alpha' : x \to y$ and $\beta, \beta' : y \to z$ in $\mathscr{C}$. Then for a map $\eta : x \to z$ the following are equivalent:*

(a) *There exists a 2-simplex $s : \Delta^2 \to \mathscr{C}$ exhibiting η as a composite of α with β.*
(b) *There exists a 2-simplex exhibiting η as a composite of α' with β.*
(c) *There exists a 2-simplex exhibiting η as a composite of α with β'.*
(d) *There exists a 2-simplex exhibiting η as a composite of α' with β'.*

Sketch proof. For (a) $\Leftrightarrow$ (b), and (b) $\Rightarrow$ (c), one fills an inner horns $\Lambda^3_i \to \mathscr{C}$ of the form

$$\begin{tikzcd}
& y \arrow[dr, "\beta"] \arrow[dd, "id"] & \\
x \arrow[ur, "\alpha"] \arrow[rr, "\eta"] \arrow[dr, "\alpha'"'] & & z \\
& y \arrow[ur, "\beta"'] &
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
& y \arrow[dr, "\beta"] \arrow[dd, "id"] & \\
x \arrow[ur, "\alpha"] \arrow[rr, "\eta"] \arrow[dr, "\alpha"'] & & z \\
& y \arrow[ur, "\beta'"'] &
\end{tikzcd} ,$$

where $i = 1$ or 2 respectively. By replacing β with β' in the argument for (a) $\Rightarrow$ (c) one finds (c) $\Leftrightarrow$ (d). □

By Lemma 2.10 we can take

$$\mathrm{Hom}_{\mathrm{h}\mathscr{C}}(x, y) := \{\alpha : \Delta^1 \to \mathscr{C} : \alpha|_0 = x \text{ and } \alpha|_1 = y\}/\sim$$

and have a well-defined composition operation

$$\circ : \mathrm{Hom}_{\mathrm{h}\mathscr{C}}(y, z) \circ \mathrm{Hom}_{\mathrm{h}\mathscr{C}}(x, y) \to \mathrm{Hom}_{\mathrm{h}\mathscr{C}}(x, z). \tag{1}$$ eq:374

This composition explicitly sends a pair of classes $[\beta] : y \to z$ and $[\alpha] : x \to y$ to the class $[\beta] \circ [\alpha] = [\eta]$ of any composite of α with β, i.e. the class of any map $\eta : x \to z$ which admits a 2-simplex exhibiting η as a composite of α with β.

lem:379 **Lemma 2.11.** *The above composition operation (1) is associative.*

Proof. Suppose we have a 2-simpleces realizing compositions $\eta = (\beta \circ \alpha)$ and $\zeta = (\gamma \circ \beta)$. Suppose we have a 2-simplex exhibiting $\vartheta : x \to a$ as a composite of α with

$\zeta = (\beta \circ \gamma)$. Fill an the inner horn $\Lambda^3_1 \to \mathscr{C}$ of the form

$$\begin{tikzcd} & y \arrow[dr, "\zeta"] \arrow[dd, "\beta"] & \\ x \arrow[ur, "\alpha"] \arrow[rr, "\vartheta"] \arrow[dr, "\eta"'] & & a \\ & z \arrow[ur, "\gamma"'] & \end{tikzcd}.$$

to find a 2-simplex exhibiting ϑ as a composite of γ with $\eta = (\beta \circ \alpha)$.

$$([\gamma] \circ [\beta]) \circ [\alpha] = [\zeta] \circ [\alpha] = [\vartheta] = [\gamma] \circ [\eta] = [\gamma] \circ ([\beta] \circ [\alpha]).$$

$\square$

Lemmas 2.8–2.11 imply that we have a well-defined category whose objects are the 0-simplices in $\mathscr{C}$ and whose morphisms are equivalence classes of 2-simplices in $\mathscr{C}$.

def:hC **Definition 2.12.** The homotopy category $\mathrm{h}\mathscr{C}$ of an ∞-category $\mathscr{C}$ is the category $\mathrm{h}\mathscr{C}$ whose objects are the 0-simplices $\mathscr{C}[0]$ and whose morphisms $\mathrm{Hom}_{\mathrm{h}\mathscr{C}}(x, y)$ are equivalence classes of vertices $\{\alpha \in \mathscr{C}[1] : \alpha|_0 = x \text{ and } \alpha|_1 = y\}$, under the equivalence relation of Lemma 2.8. Composition is defined via fillings of 2-simplices, as at (1).

Remark 2.13. In [13] the object $\mathrm{h}\mathscr{C}$ denotes two significantly different object. One is a category enriched in the homotopy category of topological spaces, which one might write as $\widetilde{\mathrm{h}\mathscr{C}}$, the other is the plain category of Definition 2.12. We have $\mathrm{h}\mathscr{C} = \pi_0(\widetilde{\mathrm{h}\mathscr{C}})$, so that these two categories are explicitly related.

In this text we avoid any reference to the to the topological category $\widetilde{\mathrm{h}\mathscr{C}}$.

sect:fun

2.6. **Functor categories.**

Definition 2.14. For simplicial sets K and S, we let $\mathrm{Fun}(K, S)$ denote the simplicial set with sections defined by maps of simplicial sets

$$\mathrm{Fun}(K, S)[n] := \mathrm{Hom}_{\mathrm{sSet}}(\Delta^n \times K, S).$$

More generally, for a linearly ordered set J we have section

$$\mathrm{Fun}(K, S)(J) := \mathrm{Hom}_{\mathrm{sSet}}(\Delta^J \times K, S).$$

Any map $a : J \to I$ in Δ defines a natural transformation of functors $a_* : \Delta^J \to \Delta^I$, and we restrict along such maps to obtain the required structural morphisms

$$a^* : \mathrm{Hom}_{\mathrm{sSet}}(\Delta^J \times K, S) \to \mathrm{Hom}_{\mathrm{sSet}}(\Delta^I \times K, S), \quad F \mapsto F|_{a_* \times id_K}.$$

To unravel things, just slightly, one sees that 0-simplices in $\mathrm{Fun}(K, S)$ are maps of simplicial sets. Any 1-simplex $F : \Delta^1 \times K \to S$ restricts to provide two maps

$$f_0 = F|_0 : K \cong K \times \{0\} \to S, \quad f_1 = F|_1 : K \to K \times \{1\} \to S,$$

and we view F as a homotopy between these two maps. Or rather, we define homotopies in this way, and higher simplices provide higher homotopies between their faces.

The following results are fundamental, and will be proved (much) later.

thm:fun **Theorem 2.15** ([13, Proposition 1.2.7.3][15, 00TN])**.** *Let K be any simplicial set.*

(i) *If $\mathscr{C}$ is an ∞-category, then the functor category* $\mathrm{Fun}(K,\mathscr{C})$ *is an ∞-category.*
(ii) *If $\mathscr{X}$ is a Kan complex, then* $\mathrm{Fun}(K,\mathscr{X})$ *is a Kan complex.*

These points are quite important, as they will be used to construct (arbitrary) limits and colimits of diagrams of (small) spaces and ∞-categories. The first point is proved at [wherever] and the second point is proved at Corollary 4.12 below.

2.7. **The (simplicial) categories of categories.** We note that the functor spaces admit natural, associative, composition maps

$$\circ : \mathrm{Fun}(K',K'') \times \mathrm{Fun}(K,K') \to \mathrm{Fun}(K,K'') \tag{2}$$

eq:comp1

which one defines in the obvious way. Namely, if we restrict to n-simplices we have

$$\circ_n : \mathrm{Fun}(K',K'')[n] \times \mathrm{Fun}(K,K')[n] \to \mathrm{Fun}(K,K'')[n],$$

$$g_n \circ_n f_n := \Delta^n \times K \overset{\delta\times id}{\longrightarrow} \Delta^n \times \Delta^n \times K \overset{id\times f_n}{\longrightarrow} \Delta^n \times K' \overset{g_n}{\to} K'',$$

where $\delta : \Delta^n \to \Delta^n \times \Delta^n$ is the diagonal map $x \mapsto (x,x)$.

The above composition maps, and functor spaces Fun, provide the category of $\mathbb{U}$-small simplicial sets $\mathrm{sSet}^{\mathbb{U}}$ with the structure of a simplicial category, where $\mathbb{U}$ is any given universe. When we restrict to the full subcategories of (small) Kan complexes and ∞-categories, this simplicial enrichment restricts to provide simplicial structures on $\mathrm{Kan}^{\mathbb{U}}$. We let $2\,\mathrm{Cat}^{\mathbb{U}}_\infty$ denote the simplicial subcategory of $\mathbb{U}$-small ∞-categories.

Lemma 2.16. *The simplicial sets* $\mathrm{Fun}(*,*)$*, and composition* (2)*, provide* $\mathrm{sSet}^{\mathbb{U}}$, $2\,\mathrm{Cat}^{\mathbb{U}}_\infty$*, and* $\mathrm{Kan}^{\mathbb{U}}$ *with natural simplicial structures.*

Notation 2.17. We let

$$\underline{\mathrm{sSet}}^{\mathbb{U}},\quad \underline{2\,\mathrm{Cat}}^{\mathbb{U}}_\infty,\quad \text{and}\quad \underline{\mathrm{Kan}}^{\mathbb{U}}$$

denote the simplicial categories of small simplicial sets, small ∞-categories, and small Kan complexes respectively.

These simplicial structures, on $\mathrm{Cat}^{\mathbb{U}}_\infty$ and $\mathrm{Kan}^{\mathbb{U}}$ in particular, provide the basis for our "meta" discussion of the homotopy theory of ∞-categories and spaces.

sect:basic_examples

3. Basic examples

We should provide some examples of ∞-categories here. Our main examples come from taking nerves of various enriched categories. In particular, we define the nerve of a plain category, a dg category, and a simplicial category. In all cases our nerve operation extends to a functor

$$\mathrm{N}_{\mathcal{E}} : \{\mathcal{E}\text{-enriched categories}\} \to \mathrm{Cat}_\infty,$$

where Cat_∞ denotes the plain (i.e. not simplicial) category of ∞-categories with ∞-functors.

3.1. **The nerve of a plain category.**

Definition 3.1. Let $\mathbb{A}$ be a plain category. The nerve $\mathrm{N}(\mathbb{A})$ of $\mathbb{A}$ is the ∞-category with n-simplices $s : \Delta^n \to \mathrm{N}(\mathbb{A})$ specified by a choice of objects $X_0, \dots, X_n$ and an $\binom{n}{2}$-tuples of maps

$$s := \{f_{ij} : X_i \to X_j : i < j,\ f_{jk}f_{ij} = f_{ik} \text{ whenever } i < j < k\}.$$

Equivalently, n-simplices are functors $s : [n] \to \mathbb{A}$. Restriction $r^* : \mathrm{N}(\mathbb{A})[n] \to \mathrm{N}(\mathbb{A})[m]$ along maps $r : [m] \to [n]$ are given by restricting functors.

So the nerve $\mathrm{N}(\mathbb{A})$ is a certain simplicial set which one assigns to a plain category. Now, an inner horn $\Lambda^n_i \to \mathrm{N}(\mathbb{A})$, for $n > 2$, is specified by a collection of maps $f_{ab} : X_a \to X_b$ for all $a < b$ with $f_{bc}f_{ab} = f_{ac}$ for all triples $a < b < c$. So, such a horn extends *uniquely* to a simplex $\Delta^n \to \mathrm{N}(\mathbb{A})$. At $n = 2$ an inner horn $\Lambda^2_1 \to \mathrm{N}(\mathbb{A})$ is a choice of two maps $f_{01} : X_0 \to X_1$ and $f_{12} : X_1 \to X_2$, which again extends uniquely to a 2-simplex $\Delta^2 \to \mathrm{N}(\mathbb{A})$. So the nerve of a plain category is an ∞-category.

In addition, for any functor $F : \mathbb{A} \to \mathsf{D}$ we obtain a map between ∞-categories $\mathsf{N}F : \mathrm{N}(\mathbb{A}) \to \mathrm{N}(\mathsf{D})$, which is simply defined by composing functors $[n] \to \mathbb{A}$ with F. These assignments

$$\mathrm{N} : \quad \mathbb{A} \mapsto \mathrm{N}(\mathbb{A}), \quad F \mapsto \mathrm{N}(F)$$

define a functors from the category of categories to the category of ∞-categories.

Proposition 3.2. *The nerve of any plain category is an ∞-category. Furthermore, any map between ∞-categories $f : \mathrm{N}(\mathbb{A}) \to \mathrm{N}(\mathsf{D})$ is of the form $f = \mathrm{N}(F)$ for a unique functor $F : \mathbb{A} \to \mathsf{D}$. This is to say, the nerve operation defines a fully faithful embedding $\mathrm{N} : \mathrm{Cat} \to \mathrm{Cat}_\infty$.*

To see that the nerve of a generic category $\mathbb{A}$ is not a weak Kan complex, a stronger notion of course, one need only consider the category $[n]$. Here we have $\mathrm{N}([n]) = \Delta^n$, and we have already seen in Example 2.6 that Δ^n is an ∞-category but not a weak Kan complex whenever $n > 0$.

We note that for any ∞-category $\mathscr{C}$ we have an obvious map of simplicial sets

$$p : \mathscr{C} \to \mathrm{N}(\mathrm{h}\mathscr{C}), \quad (s : \Delta^n \to \mathscr{C}) \mapsto (\ [s|_{\Delta^{\{i,j\}}}] : 0 \le i < j \le n),$$

which is then by definition an ∞-functor between ∞-categories. One sees immediately that p is an isomorphism whenever $\mathscr{C} = \mathrm{N}(\mathsf{C})$ for some plain category C, and we thus fund that p is an isomorphism of ∞-categories exactly when $\mathscr{C}$ is isomorphic to the nerve of some plain category.

The functor p is natural in $\mathscr{C}$, and so provides a natural endomorphism on the category of ∞-categories. Indeed, p provides the unit for an adjunction between the homotopy category functor $\mathrm{h} : \mathrm{Cat}_\infty \to \mathrm{Cat}$ and the nerve functor.

`lem:p_C` **Lemma 3.3.** *For any ∞-category we have a natural projection $p : \mathscr{C} \to \mathrm{N}(\mathrm{h}\mathscr{C})$, which is defined by taking an n-simplex s in $\mathscr{C}$ to the corresponding tuple of morphisms $(\ [s|_{\Delta^{\{i,j\}}}] : 0 \le i < j \le n)$ in the homotopy category.*

`sect:dg_nerve`

3.2. **Nerves of dg categories.** Let $\mathcal{A}$ be a dg category. We define the dg nerve $\mathrm{N}^{\mathrm{dg}}(\mathbf{A})$ to be the simplicial set with each n-simplex $\Delta^n \to \mathrm{N}^{\mathrm{dg}}(\mathbf{A})$ specified by a choice of objects $\{x_0, \dots, x_n\}$ in $\mathcal{A}$ and maps

$$f_I \in \mathrm{Hom}_{\mathbf{A}}^{-|I|+2}(x_{\min I}, x_{\max I})$$

for all subsets $I \subseteq [n]$ of order $|I| \geq 2$ which satisfy

$$d(f_I) = \sum_{t \in I - \{\min I, \max I\}} (-1)^{\mathrm{pos}(t)} (f_{I_{\geq t}} \circ f_{I_{\leq t}} - f_{I - \{t\}}). \tag{3}$$

eq:419

Here I inherits its ordering from $[n]$, so that $I_{\geq t} = \{a \in I : a \geq t\}$ for example, and $\mathrm{pos}(t) = |I_{>t}|$. Not that $\mathrm{pos}(t)$ marks the position of t relative to the *terminal* element $\max I$, not the initial element.[1]

For any weakly increasing map $r : [m] \to [n]$ the restriction

$$r^* : \mathrm{N}^{\mathrm{dg}}(\mathbf{A})[n] \to \mathrm{N}^{\mathrm{dg}}(\mathbf{A})[m], \quad \{f_I : I \subseteq [n]\} \mapsto \{f_{r,J} : J \subseteq [m]\}.$$

is defined by taking $f_{r,J} = f_{r(J)}$ if $r|_J$ is injective, $f_{r,J} = id_{x_{r(j)}}$ if $J = \{j, j'\}$ and $r(j) = r(j')$, and $f_{r,J} = 0$ otherwise. For example, pulling back $s^* : \mathrm{N}^{\mathrm{dg}}(\mathbf{A})[1] \to \mathrm{N}^{\mathrm{dg}}(\mathbf{A})[2]$ along the weakly increasing surjection $s : [2] \to [1]$ with $s(0) = s(1) = 0$ can be illustrated as

$$\begin{tikzcd} x \arrow[r, "f"] & y \end{tikzcd} \quad \overset{s^*}{\longmapsto} \quad \begin{tikzcd} & x \arrow[dr, "f"] \arrow[d, phantom, "0"] & \\ x \arrow[ur, "id"] \arrow[rr, "f"'] & & y. \end{tikzcd}$$

Lemma 3.4. *The tuple $\{f_{r,J} : J \subseteq [m]\}$ is in fact an m-simplex in $\mathrm{N}^{\mathrm{dg}}(\mathbf{A})$, and for a sequence of maps $r_2 \circ r_1 : [l] \to [m] \to [n]$ the composite $r_1^* \circ r_2^*$ equals $(r_2 \circ r_1)^*$. This is to say, $\mathrm{N}^{\mathrm{dg}}(\mathbf{A})$ is in fact a simplicial set.*

Proof. The second point follows from the fact that the composite $r_2 r_1$ has injective restriction to $K \subseteq [l]$ if and only if r_1 has injective restriction fo K and r_2 has injective restriction to $r_1(K)$.

For the first claim, fix $r : [m] \to [n]$ and take for any $J \subseteq [m]$, $f_J = f_{r,J}$. We need only establish an equality

$$0 = \sum_{t \in J - \{\min J, \max J\}} (-1)^{\mathrm{pos}(t)} (f_{J_{\geq t}} \circ f_{J \leq t} - f_{J - \{t\}}). \tag{4}$$

eq:440

when $r|_J$ has repeated indices. When $r|_J$ has more than two repeated indices, then all summands in the above expression are 0, so that the equality holds immediately. So we may suppose that $r|_J$ has precisely two repeated indices $r(j) = r(j')$, with j and j' necessarily neighbors. If $|J| = 2$ then the right above summand is empty, and thus necessarily $0 = d(id)$. So we suppose additionally that $|J| > 2$. We therefore want to establish (4) when $|J| > 2$ and $r|_J$ has precisely two repeated indices $r(j) = r(j')$.

Suppose also that $j = \min J$ or $j' = \max J$. Then the above sum has only one non-vanishing summand,

$$(-1)^{\mathrm{pos}(j)} (f_{J_{\leq j}} \circ id - f_{J - \{j\}}) = (-1)^{\mathrm{pos}(j)} (f_{J_{\leq j}} - f_{J_{\leq j}}) = 0$$

or

$$(-1)^{\mathrm{pos}(j')} (id \circ f_{J_{\geq j'}} - f_{J - \{j'\}}) = (-1)^{\mathrm{pos}(j)} (f_{J_{\geq j'}} - f_{J_{\geq j'}}) = 0$$

[1] Our particular expression of the dg nerve comes from [15], and not [13]. There is a slight difference in signs between the two resources.

So the desired equation holds. When both j and j' lie in the interior of J, this sum has precisely two summands and we have

$$\pm(f_{J_{\geq j}}\circ 0 - f_{J-\{j\}} - 0\circ f_{J_{\leq j'}} + f_{J-\{j'\}}) = 0.$$

We therefore establish (4), and see that r^* does map n-simplices to m-simplices. □

We note that there is a functor $\mathrm{dg}:\mathbb{Z}_{\geq 0}\to \mathrm{dgCat}$ so that n-simplices in $\mathrm{N}^{\mathrm{dg}}(\mathbf{A})$ are identified with dg functors $\mathrm{Hom}_{\mathrm{dgCat}}(\mathrm{dg}[n],\mathbf{A})$. So the dg nerve can be constructed in a manner similar to the nerve of a plain category. Furthermore, if we view any additive category $\mathbb{A}$ as a dg category over $\mathbb{Z}$, with all maps in degree 0, then $\mathrm{N}(\mathbb{A}) = \mathrm{N}^{\mathrm{dg}}(\mathbb{A})$.

Now, suppose we have an inner horn $a:\Lambda^n_i\to \mathrm{N}^{\mathrm{dg}}(\mathbf{A})$. Such a map of simplicial sets is specified by a tuple of morphism $(f_I : I\subseteq [n],\ [n]-\{i\}\nsubseteq I)$ which solve the equations (3), and an extension of such a to an n-simplex is an additional choice of degree $n-3$ and $n-2$ maps $f_{I-\{i\}}$ and $f_{[n]}$ respectively which have the appropriate derivatives. If we assume $f_{[n]}=0$ then the equation

$$0 = \sum_{0<t<n}(-1)^{n-t}(f_{\{t,\dots,n\}}\circ f_{[t]} - f_{[n]-\{t\}})$$

is satisfied precisely when

$$f_{[n]-\{i\}} = f_{\{i,\dots,n\}}\circ f_{[i]} + \sum_{0<t<n,\ t\neq i}(-1)^{-t+i}(f_{\{t,\dots,n\}}\circ f_{[t]} - f_{[n]-\{t\}}). \tag{5}$$

eq:469

Proposition 3.5 ([15, 00PW])**.** *For any inner horn $\Lambda^n\to \mathrm{N}^{\mathrm{dg}}(\mathbf{A})$ admits an extension to an n-simplex specified by taking $f_{[n]}=0$ and $f_{[n]-\{i\}}$ to be given by (5). In particular, the dg nerve of any dg category is an ∞-category.*

Proof. We take the differential to find

$$\begin{aligned}
d(f_{[n]-\{i\}}) \quad &= (\textstyle\sum_{t<i}(-1)^{n-t}t_{[n]-\{t,i\}}) + (\sum_{i<t}(-1)^{n-(t+1)}f_{[n]-\{i,t\}})\\
&\quad -(\textstyle\sum_{t<i}(-1)^{n-t}f_{[t,n]-\{i\}}\circ f_{[t]}) - (\sum_{i<t}(-1)^{n-t-1}f_{[t,n]}\circ f_{[t]-\{i\}})\\
&\quad + \text{other terms}\\
&= \textstyle\sum_{t\in I-\{\min I,\max I\}}(-1)^{|t|}(f_{I_{\geq t}}\circ f_{I_{\leq t}} - f_{I-\{t\}})\\
&\quad + \text{other terms.}
\end{aligned}$$

In the above expression $I=[n]-\{i\}$ and $[t,n]=\{t,\dots,n\}$, and the summands which we've written explicitly come from differentiating the term

$$\sum_{0<t<n,\ t\neq i}(-1)^{-t+i}(f_{\{t,\dots,n\}}\circ f_{[t]} - f_{[n]-\{t\}})$$

in (5). So, we want to see that these "other terms" vanish.

We have

$$\begin{aligned}
&(-1)^{n-i}\text{other terms} = \\
&(-1)^{n-i}d(f_{[i,n]})f_{[i]} - f_{[i,n]}d(f_{[i]}) \\
&+ \textstyle\sum_{0\le l<m<n,\ l,m\neq i}(-1)^{l+m}f_{[n]-\{l,m\}} + (-1)^{m+l-1}f_{[n]-\{l,m\}} \\
&- \textstyle\sum_{0\le l<m<n,\ l,m\neq i}(-1)^{l+m}f_{[n,l]-\{m\}}f_{[l]} + (-1)^{m+l-1}f_{[n,l]-\{m\}}f_{[l]} \\
&- \textstyle\sum_{0\le l<m<n,\ l,m\neq i}(-1)^{m-l+1}f_{[n,m]}f_{[m]-\{l\}} + (-1)^{-l-m}f_{[n,m]}f_{[m]-\{l\}} \\
&+ \textstyle\sum_{t<i}(-1)^{i+t}f_{[n,i]}f_{[t,i]}f_{[t]} - (-1)^{i+t}f_{[n,i]}f_{[i]-\{t\}} \\
&+ \textstyle\sum_{i<t}(-1)^{t-i-1}f_{[n,t]}f_{[i,t]}f_{[i]} - (-1)^{t+i+1}f_{[n,i]-\{t\}}f_{[i]}.
\end{aligned}$$

$$\begin{aligned}
&= -f_{[i,n]}d(f_{[i]}) + \textstyle\sum_{t<i}(-1)^{i+t}f_{[n,i]}f_{[t,i]}f_{[t]} - (-1)^{i+t}f_{[n,i]}f_{[i]-\{t\}} \\
&(-1)^{n-i}d(f_{[i,n]})f_{[i]} + \textstyle\sum_{i<t}(-1)^{t-i-1}f_{[n,t]}f_{[i,t]}f_{[i]} - (-1)^{t+i+1}f_{[n,i]-\{t\}}f_{[i]} \\
&= 0.
\end{aligned}$$

So we see that the other terms vanish, and hence that any given inner horn $\Lambda^n_i \to \mathrm{N}^{\mathrm{dg}}(\mathbf{A})$ admits an extension to an n-simplex. ☐

From a direct consideration of the equation (3) it is clear that any dg functor $\eta : \mathbf{A} \to \mathbf{B}$ defines an associated map of ∞-categories

$$\mathrm{N}^{\mathrm{dg}}(\eta) : \mathrm{N}^{\mathrm{dg}}(\mathbf{A}) \to \mathrm{N}^{\mathrm{dg}}(\mathbf{B}),\quad \begin{cases} x \mapsto F(x) & \text{for 0-simplices} \\ (f_I : I \subseteq [n]) \mapsto (\eta(f_I) : I \subseteq [n]) & \text{for } n\text{-simplices, } n > 0. \end{cases}$$

Proposition 3.6. *The dg nerve operation defines a functor* $\mathrm{dgCat} \to \mathrm{Cat}_\infty$.

`rem:dg_v_infty`

Remark 3.7. The dg nerve operation defines an equivalence from a homotopy category of dg categories, over a given base k, to the homotopy category of (certain) linear ∞-categories [4, Corollary 5.5]. In this sense, dg categories are identified with linear ∞-categories via the dg nerve functor, in some appropriate sense.

Remark 3.8. In defining the dg nerve, there are some seemingly ambiguous choices in the precise definition of the dg nerve functor. One sees, for example, in [14, Construction 1.3.1.13] an alternate construction of a "dg nerve" via Dold-Kan, which turns out to be equivalent [14, Proposition 1.3.1.17]. A construction which extends immediately to A_infty-categories is given in work on Faonte [7]. Faonte's construction can be identified with the one given above, up to some signs.

In the end, however, Antieau explains that in most reasonable cases coming from algebra, there is a unique ∞-category $\mathscr{D}$ whose homotopy category $\mathrm{h}\mathscr{D}$ is identified with the derived category of a given Grothendieck abelain category [1, Corollaries 2, 5]. So for most cases of interest to representation theorists, the ∞-category $\mathrm{N}^{\mathrm{dg}}(\mathbf{A})$ is uniquely determined, up to equivalence of ∞-categories, by its homotopy category $h\,\mathrm{N}^{\mathrm{dg}}(\mathbf{A}) = K(\mathbf{A})$. (See our discussion of the derived category below.)

`sect:dg_alg_ex`

3.3. An integral dg example with one object.

Let p be any positive integer and consider the dg category A_p over $\mathbb{Z}$ with a single object $*$ and endomorphisms

$$\mathrm{Hom}_{A_p}(*,*) = 0 \to p\mathbb{Z} \to \mathbb{Z} \to 0,$$

where the differential $p\mathbb{Z} \to \mathbb{Z}$ is just the inclusion. So, A_p is just a dg algebra which we view as a dg category. We also have the dg algebra $\mathbb{Z}/p\mathbb{Z}$ with a single object and endomorphisms $\mathbb{Z}/p\mathbb{Z}$. Let $\mathscr{A}_p$ and $\mathscr{Z}_p$ be the corresponding dg nerves of these dg categories. We describe these ∞-categories explicitly, and show that the dg algebra quasi-isomorphism $A_p \to \mathbb{Z}/p\mathbb{Z}$ admits an section $\mathscr{Z}_p \to \mathscr{A}_p$ at the level of ∞-categories, despite the fact that there are no A_∞-algebra maps $\mathbb{Z}/p\mathbb{Z} \to A_p$ at all.

An n-simplex in $a : \Delta^n \to \mathscr{Z}_p$ is a tuple of numbers $a = \{a_{ij} : i < j \in [n]\}$ with $a_{ij}a_{jk} = a_{ik}$ for all triples $i < j < k$. (This is the same as the usual nerve of $\mathbb{Z}/p\mathbb{Z}$ as a plain category.) A 1-simplex in $\mathscr{A}_p$ is a choice of element $b \in \mathbb{Z} \subseteq A_p$ and a 2-simplex $b : \Delta^2 \to \mathscr{A}_p$ is a choice of a triple of a triple of elements $\{b_{01}, b_{12}, b_{02}\} \subseteq \mathbb{Z}$ and $b_{123} \in p\mathbb{Z}$ such that

$$b_{123} = b_{02} - b_{01}b_{12}. \tag{6}$$

eq:560

In particular, b_{123} is specified uniquely by the boundary elements b_{ij}, so that a 2-simplex is simply a triple of elements satisfying $b_{01}b_{12} = b_{03} \mod p$.

For degree reasons, if we have an n-simplex $b : \Delta^n \to \mathscr{A}_p$ all functions $b_I \in A_p^{|I|-2}$ indexed by subsets $I \subseteq [n]$ of size > 3 vanish, and the equations

$$d(b_I) = \sum \pm (b_{I_{\geq t}} b_{I_{\leq t}} - b_{I - \{t\}}) \tag{7}$$

eq:570

are vacuous when $|I| > 4$, as both sides live in $A_p^{|I|-3} = 0$. We have already examined the above equation when $|I| = 3$ and $|I| = 2$ generically just says that all b_I with $|I| = 2$ are degree 0 cocycles. So we need only investigate the case $|I| = 4$. That is, we need only investigate the 3-simplices in $\mathscr{A}_p$ in order to understand its entire structure.

Consider a 3-simplex $b : \Delta^3 \to \mathscr{A}_p$. Such an object is specified by a collection of elements $\{b_I : I \subseteq [3]\}$ with all $b_{ijk} = b_{ik} - b_{ij}b_{jk}$ and $b_{0123} = 0$ for degree reasons. This first condition is equivalent to the equation (7) for I of size 3, and for the unique subset of size 4 we need to check the equation

$$0 = d(b_{0123}) = b_{023} - b_{01}b_{123} - b_{013} + b_{012}b_{23}.$$

But we simply expand, and (only) employ the 2-simplex equation (6), to find

$$\begin{aligned}
&b_{023} - b_{01}b_{123} - b_{013} + b_{012}b_{23} \\
&= b_{03} - b_{02}b_{23} - b_{01}(b_{13} - b_{12}b_{23}) - b_{03} + b_{01}b_{13} + (b_{02} - b_{01}b_{12})b_{23} \\
&= b_{03} - b_{02}b_{23} - b_{01}b_{13} + b_{01}b_{12}b_{23} - b_{03} + b_{01}b_{13} + b_{02}b_{23} - b_{01}b_{12}b_{23} \\
&= 0.
\end{aligned}$$

So we see that equation (7) at $|I| = 4$ is redundant, and thus observe a complete description of the n-simplices in $\mathscr{A}_p$,

$$\mathscr{A}_p[n] = \big\{\text{tuples } (b_{ij} : 0 \leq i < j \leq n) : b_{ij}b_{jk} = b_{ik} \mod p\big\}$$

The map of ∞-categories $\pi : \mathscr{A}_p \to \mathscr{Z}_p$ implied by the dg algebra quasi-isomorphism $A_p \to \mathbb{Z}/p\mathbb{Z}$ is defined as expected

$$\pi : \mathscr{A}_p \to \mathscr{Z}_p, \quad (b_{ij} : 0 \leq i < j \leq n) \mapsto (\bar{b}_{ij} : 0 \leq i < j \leq n).$$

For any class $a \in \mathbb{Z}/p\mathbb{Z}$ define $a' \in \mathbb{Z}$ to be the unique element in $\{0, \ldots, p-1\} \subseteq \mathbb{Z}$ such that $\bar{a}' = a$. We define a section $\pi^\vee : \mathscr{Z}_p \to \mathscr{A}_p$ on n-simplices

$$\pi_n^\vee : \mathscr{Z}_p[n] \to \mathscr{A}_p[n], \quad (a_{ij} : 0 \leq i < j \leq n) \mapsto (a'_{ij} : 0 \leq i < j \leq n).$$

One sees immediately that the composite

$$\mathscr{Z}_p \xrightarrow{\pi^\vee} \mathscr{A}_p \xrightarrow{\pi} \mathscr{Z}_p$$

is seen to be the identity. This point is remarkable, given that there aren't even any maps $\mathbb{Z}/p\mathbb{Z} \to A_p$ of $\mathbb{Z}$-modules, and highlights the content of the discussion of Remark 3.7.

We would claim furthermore that $\pi : \mathscr{A}_p \to \mathscr{Z}_p$ is an *equivalence* of ∞-categories with weak inverse $\pi^\vee$, whatever that means. This point is discussed in greater detail in Section 5.

sect:dg_alg_ex2

3.4. **More examples with one object.** We can similarly take a positive degree polynomial $q \in S = \mathbb{C}[t]$ and consider the dg algebras $A_q = 0 \to qS \to S \to 0$ and S/qS. We have the associated ∞-categories $\mathscr{A}_q$ and $\mathscr{S}_q$, and the above presentation applies verbatim to provide a complete description of the simplicial set $\mathscr{A}_q$,

$$\mathscr{A}_q[n] = \{(b_{ij} : 0 \le i < j \le n) : b_{ij}b_{jk} = b_{kl} \mod q\}.$$

We have the projection $\pi : \mathscr{A}_q \to \mathscr{S}_q$ implied by the dg algebra quasi-isomorphism $A_q \to S/qS$ and easily constructs an "inverse"

$$\pi^\vee : \mathscr{S}_q \to \mathscr{A}_q, \quad (a_{ij} : 0 \le i < j \le n) \mapsto (a'_{ij} : 0 \le i < j \le n)$$

by taking a'_{ij} to be the unique lift of $a_{ij} \in S/qS$ to a degree $< \deg(q)$ element in S.

Now that we're over $\mathbb{C}$ however, there *does* exist an A_∞ weak inverse $\iota : S/qS \to A_q$ and one can compare the construction of ι requires a choice of $\mathbb{C}$-linear section $\iota_0 : S/qS \to A_q$ and a corresponding degree -1 solution $\iota_1 : S/qS \otimes_{\mathbb{C}} S/qS \to A_q$ to a quadratic equation $d(X) = \mathrm{poly}(\iota_0)$. This quadratic equation is essentially just the equation (6), which we understand has unique solutions. So we see that in the A_∞-categorical setting we must keep track certain irrelevant information which the ∞-categorical allows us to ignore. We also note that the ∞-category map $\pi^\vee$ can be defined by any choice theoretic section $\pi_1^\vee : S/qS = \mathscr{S}_q[1] \to A_q = \mathscr{A}_q[1]$ and so disregards linearity as well.

As a last (somewhat silly) example we consider the Chevalley-Eilenberg dg algebra

$$\mathrm{CE}(\mathfrak{g}) = 0 \to \mathbb{C} \xrightarrow{0} \mathfrak{g}^* \xrightarrow{d} \mathfrak{g}^* \wedge \mathfrak{g}^* \to \cdots \to \det(\mathfrak{g}^*) \to 0.$$

Here $\mathfrak{g}$ is a Lie algebra and the differential is specified by the dual of the bracket $d^1 = [-,-]^* : \mathfrak{g}^* \to \mathfrak{g}^* \wedge \mathfrak{g}^*$. The cohomology of this dg algebra is the algebra of extensions of the trivial representation

$$H^*(\mathrm{CE}(\mathfrak{g})) = \mathrm{Ext}_{U(\mathfrak{g})}(\mathbb{C}, \mathbb{C}),$$

calculated in the category of arbitrary $U(\mathfrak{g})$-modules. We consider the associated ∞-category $\mathscr{CE}(\mathfrak{g})$.

Here there are no non-zero negative degree elements in $\mathrm{CE}(\mathfrak{g})$ to speak of, so that the unit map $\mathbb{C} \to \mathrm{CE}(\mathfrak{g})$ induces an isomorphism of ∞-categories

$$\mathrm{N}(\mathbb{C}) \xrightarrow{\cong} \mathscr{CE}(\mathfrak{g}).$$

This is despite the fact that the unit map is far from a quasi-isomorphism in general.

This example simply abuses the fact that the dg nerve functor cannot see information in a dg category $\mathbf{A}$ which is strictly contained in positive cohomological degrees. We also note that this kind of Tom-foolery will not occur if we restrict our attention to pre-triangulated dg categories [4, Definition 2.16], which provide the fibrant objects for the model structure employed in [4] anyway (cf. Remark 3.7).

3.5. **Derived categories for abelian categories.** Let $\mathbb{A}$ be an abelian category.

Definition 3.9. A complex I over $\mathbb{A}$ is called K-injective if the Hom complex functor

$$\mathrm{Hom}^*_{\mathbb{A}}(-, I) : \mathrm{Ch}(\mathbb{A})^{\mathrm{op}} \to \mathrm{Ch}(\mathbb{Z})$$

preserves acyclic complexes.

Equivalently, I is K-injective if the Hom complex functor preserves quasi-isomorphisms.

Theorem 3.10 ([18, Theorem 3.13])**.** *If $\mathbb{A}$ is a Grothendieck abelian category, then every (possibly unbounded) complex M admits a quasi-isomorphism $M \to I$ to a K-injective complex I.*

Examples of Grothendieck abelian categories include categories of arbitrary R-modules $\mathrm{Mod}(R)$ for an arbitrary ring R, categories of quasi-coherent sheaves $\mathrm{QCoh}(X)$ on an arbitrary scheme X [6], categories of representations $\mathrm{Rep}(G)$ for an algebraic group G, and categories of cohomodules $\mathrm{Comod}(C)$ for a coalgebra C.

Now, for a general abelian (or even additive) category, we can form the dg nerve $\mathrm{N}^{\mathrm{dg}}(\mathrm{Ch}(\mathbb{A}))$ of the dg category of cochains. Here we find

- $\mathrm{N}^{\mathrm{dg}}(\mathrm{Ch}(\mathbb{A}))[0] = \{$the collection of complexes over $\mathbb{A}\}$.
- $\mathrm{N}^{\mathrm{dg}}(\mathrm{Ch}(\mathbb{A}))[1] = \{$maps $f : M \to N$ of cochains$\}$.
- $\mathrm{N}^{\mathrm{dg}}(\mathrm{Ch}(\mathbb{A}))[2] =$

$$\left\{\begin{array}{c}\text{quadruples } f : L \to M,\ g : M \to N,\ h : L \to N,\ z : L \to N \\ \text{such that } f,\ g, \text{ and } h \text{ are degree 0 cocyles, i.e. maps of cochains} \\ \text{and } h = fg - d(z).\end{array}\right\}.$$

From this description of 0, 1, and 2-simplices one observes a calculation of the homotopy category.

prop:813 **Proposition 3.11.** *Let $\mathbb{A}$ be an additive category. The homotopy category of the dg nerve for $\mathrm{Ch}(\mathbb{A})$ is the usual homotopy category of dg modules for $\mathbb{A}$, i.e. the category of dg modules with homotopy classes of maps*

$$\mathrm{h}\,\mathrm{N}^{\mathrm{dg}}(\mathrm{Ch}(\mathbb{A})) = K(\mathbb{A}).$$

We refer to the dg nerve $\mathrm{N}^{\mathrm{dg}}(\mathrm{Ch}(\mathbb{A}))$ the homotopy ∞-category for $\mathbb{A}$,

$$\mathscr{K}(\mathbb{A}) := \mathrm{N}^{\mathrm{dg}}(\mathbb{A})$$

We consider inside the dg category of cochains

$$\mathrm{Ch}(\mathbb{A})_{\mathrm{Inj}} := \left\{\begin{array}{c}\text{The full dg subcategory of} \\ K\text{-injective complexes in } \mathrm{Ch}(\mathbb{A})\end{array}\right\}.$$

Definition 3.12. For a Grothendieck abelian category $\mathbb{A}$, the derived ∞-category is the dg nerve

$$\mathscr{D}(A) := \mathrm{N}^{\mathrm{dg}}(\mathrm{Ch}(\mathbb{A})_{\mathrm{Inj}}).$$

From Proposition 3.11 we have

$$\mathrm{h}\,\mathscr{D}(\mathbb{A}) = \left\{ \begin{array}{c} \text{The homotopy cat of } K\text{-} \\ \text{injective complexes over } \mathbb{A} \end{array} \right\} \cong D(\mathbb{A}),$$

so that $\mathscr{D}(\mathbb{A})$ provides an ∞-categorical lift of the usual derived category.

Now, in algebraic (as in, ring theoretic) situations one also has enough K-projectives. As one expects, these are cochains P for which the functor

$$\mathrm{Hom}^*_{\mathbb{A}}(P,-) : \mathrm{Ch}(\mathbb{A}) \to \mathrm{Ch}(\mathbb{Z})$$

preserves acyclicity. One can consider the dg category $\mathrm{Ch}(\mathbb{A})_{\mathrm{Proj}}$ of K-projective complexes in $\mathrm{Ch}(\mathbb{A})$ and, when $\mathrm{Ch}(\mathbb{A})$ has enough K-projectives, one can show that there is an equivalence of ∞-categories

$$\mathscr{D}(\mathbb{A}) = \mathrm{N}^{\mathrm{dg}}(\mathrm{Ch}(\mathbb{A})_{\mathrm{Inj}}) \overset{\sim}{\to} \mathrm{N}^{\mathrm{dg}}(\mathrm{Ch}(\mathbb{A})_{\mathrm{Proj}})$$

which is uniquely determined via some constraints. Hence one can employ either an "injective model" or a "projective model" when working with the derived ∞-category, as is traditional. We discuss this injective-projective comparison in detail in Section 13 below.

Remark 3.13. The derived ∞-category $\mathscr{D}(\mathbb{A})$ can be identified via a universal property as the localization $\mathscr{K}(\mathbb{A})[\mathrm{Qiso}^{-1}]$ at the ∞-level, or alternatively as the localization of the plain category of cochains $\mathrm{Ch}_{\mathrm{plain}}(\mathbb{A})[\mathrm{Qiso}^{-1}]$ [15, Propositions 1.3.4.5, 1.3.5.15]. The latter expression is highly non-classical.

Given some finiteness condition F for objects in $\mathbb{A}$ we take

$$\mathscr{D}_F(\mathbb{A}) = \left\{ \begin{array}{c} \text{The full } \infty\text{-subcategory of objects in} \\ \mathscr{D}(\mathbb{A}) \text{ whose cohomology have property } F \end{array} \right\}$$

For example we can consider the derived categories

$$\mathscr{D}_{fin}(G) \quad \text{and} \quad \mathscr{D}_{coh}(X)$$

of dg G-representations with finite-dimensional cohomology and of quasi-coherent dg sheaves with coherent cohomology. We define the bounded, bounded above, and bounded below derived categories

$$\mathscr{D}^b(\mathbb{A}), \quad \mathscr{D}^-(\mathbb{A}), \; \mathscr{D}^+(\mathbb{A})$$

similarly.

3.6. **Derived categories for dg modules.** Consider a dg algebra R and the category $\mathrm{dgMod}(R)$ of arbitrary dg R-modules. In this instance $\mathrm{dgMod}(R)$ still admits enough K-injectives and K-projectives [10]. So we define the derived ∞-category again as the dg nerve of the category of K-injectives

$$\mathscr{D}(R) := \mathrm{N}^{\mathrm{dg}}(\mathrm{dgMod}(R)_{\mathrm{Inj}}).$$

We again have a unique identification

$$\mathscr{D}(R) = \mathrm{N}^{\mathrm{dg}}(\mathrm{dgMod}(R)_{\mathrm{Inj}}) \overset{\sim}{\to} \mathrm{N}^{\mathrm{dg}}(\mathrm{dgMod}(R)_{\mathrm{Proj}}).$$

See Section 13.

3.7. **Nerves of simplicial categories.** First we construct a simplicial category for the simplices Δ^n [13, Definition 1.1.5.1]. We take $\mathrm{Path}\,\Delta^n$ to be the simplicial category with objects $\mathrm{obj}(\mathrm{Path}\,\Delta^n) = [n]$ and m-simplices

$$\underline{\mathrm{Hom}}_{\Delta^n}(a,b)[m] = \underline{\mathrm{Hom}}_{\mathrm{Path}\,\Delta^n}(a,b)[m] = \left\{\begin{array}{c}\text{length } m+1 \text{ sequences of subsets}\\ I_0 \subseteq \cdots \subseteq I_m \subseteq [n]\\ \text{with } a = \min I_j \text{ and } b = \max I_j\\ \text{for all } 0 \leq j \leq m\end{array}\right\}.$$

We note that each inclusion $I_j \subseteq I_{j+1}$ may be an equality in the above presentation, that these simplicial Hom sets vanishes if and only if $a > b$, and that all simplices of size $> n$ are degenerate. For any weakly increasing $r : [l] \to [m]$ the corresponding structure map is as expected,

$$r^* : \underline{\mathrm{Hom}}_{\Delta^n}(a,b)[m] \to \underline{\mathrm{Hom}}_{\Delta^n}(a,b)[l], \quad \{I_0 \cdots \subseteq I_m\} \mapsto \{I_{r(0)} \cdots \subseteq I_{r(l)}\}$$

and composition is given by taking unions

$$\{I'_0 \cdots \subseteq I'_m\} \circ \{I_0 \cdots \subseteq I_m\} = \big\{(I_0 \cup I'_0) \cdots \subseteq (I_m \cup I'_m)\big\}.$$

One similarly defines the simplicial category $\mathrm{Path}\,\Delta^J$ for any linearly ordered set J.

For any map $f : J_1 \to J_2$ in Δ we obtain a simplicial functor $f_* : \mathrm{Path}\,\Delta^{J_1} \to \mathrm{Path}\,\Delta^{J_2}$ which is defined as f on objects, and on morphisms

$$f\big(\{I_0 \cdots \subseteq I_m\}\big) := \big\{f(I_0) \cdots \subseteq f(I_m)\big\}.$$

So this path operation defines a functor $\mathrm{Path} : \Delta \to \mathrm{sCat}$.

Definition 3.14. For a simplicial category $\mathcal{A}$ we define the homotopy coherent nerve $\mathrm{N}^{\mathrm{hc}}(\mathcal{A})$ to be the simplicial set with simplices

$$\mathrm{N}^{\mathrm{hc}}(\mathcal{A})[n] = \mathrm{Fun}_{\mathrm{sCat}}(\mathrm{Path}\,\Delta^n, \mathcal{A})$$

and restriction maps $f^* := \mathrm{Fun}_{\mathrm{sCat}}(f_*, \mathcal{A})$, for each weakly increasing map $f : [m] \to [n]$.

Note that all $\geq n$-simplices in $\underline{\mathrm{Hom}}_{\Delta^n}(a,b)$ are degenerate, so that any simplicial functor $\mathrm{Path}\,\Delta^n \to \mathcal{A}$ is determined by its values on objects and on the m-simplices $\underline{\mathrm{Hom}}_{\Delta^n}(a,b)[m]$ for $m < n$. In particular, one sees

$$\mathrm{N}^{\mathrm{hc}}(\mathcal{A})[0] = \mathrm{obj}(\mathcal{A})$$

$$\mathrm{N}^{\mathrm{n}}(\mathcal{A})[1] = \{\text{pairs of objects with a specified map } f : x \to y\}.$$

lem:2simp_hc

Lemma 3.15. *For a simplicial category $\mathcal{A}$, a 2-simplex $\sigma : \Delta^2 \to \mathrm{N}^{\mathrm{hc}}(\mathcal{A})$ is specified a triple of objects (x_0, x_1, x_3), choices of maps between these objects $f_{ij} : x_i \to x_j$ for all $i < j$, and a 1-simplex $h : \Delta^1 \to \underline{\mathrm{Hom}}_{\mathcal{A}}(x_0, x_3)$ which satisfies*

$$h|_{\{0\}} = f_{13}, \quad h|_{\{1\}} = f_{23}f_{12}.$$

Proof. This simplex is, by definition, a simplicial functor $\sigma : \mathrm{Path}\,\Delta^2 \to \mathcal{A}$. The $f_{i,j}$ are the images of the unique 0-simplices $I_0 \subseteq [2]$ in $\underline{\mathrm{Hom}}_{\Delta^2}(i,j)$ with $|I_0| = 2$. The other 0-simplex in $\underline{\mathrm{Hom}}_{\delta^2}(0,2)$ is sent to $f_{12}f_{01}$ via compatibility of σ with composition. The 1-simplex h is the image of the unique non-degenerate 1-simplex $I_0 = \{0,2\} \subseteq I_1 = \{1,2,3\} \subseteq [2]$ in $\underline{\mathrm{Hom}}_{\Delta^2}(0,2)$.

Note that $\Delta^0 \cong \underline{\mathrm{Hom}}_{\Delta^2}(i, i+1)$ and that the unique 1-simplex in $\underline{\mathrm{Hom}}_{\Delta^2}(0,2)$ provides an isomorphism $\Delta^1 \cong \mathrm{Hom}_{\Delta^2}(0,2)$. Hence the functor σ is determined by the data $\{x_i : 0 \leq i \leq 1\}$, f_{ij}, h. $\square$

prop:11510

Proposition 3.16 ([13, Proposition 1.1.5.10]). *Suppose that a simplicial category $\mathcal{A}$ hat the property that, for each pair of objects in $\mathcal{A}$, the simplicial set $\underline{\mathrm{Hom}}_{\mathcal{A}}(x, y)$ is a Kan complex. Then the homotopy coherent nerve $\mathrm{N}^{\mathrm{hc}}(\mathcal{A})$ is an ∞-category.*

The proof requires an analysis of lifting properties for maps into Kan complexes which we won't recall. Our main examples of interest come from the simplicial categories of Kan complexes and ∞-categories, though we need to develop more background in order to deal with these examples in detail. (See Sections 4.14 and 5.12 below.)

We note that the description of 2-simplices in $\mathrm{N}^{\mathrm{hc}}(\mathcal{A})$, from Lemma 3.15, provides an explicit description of the homotopy category

$$\mathrm{h}\,\mathrm{N}^{\mathrm{hc}}(\mathcal{A}) = \begin{cases} \text{The plain category with objects } \mathrm{obj}(\mathcal{A}) \\ \text{and morphisms given by equiv. classes of 0-simplices} \\ f : x \to y \text{ in } \underline{\mathrm{Hom}}_{\mathcal{A}}(x, y)\text{, where } f \sim f \text{ if there} \\ \text{exists a 1-simplex } h \text{ with } h|_{\{0\}} = f \text{ and } h|_{\{1\}} = f'. \end{cases}$$

sect:spaces

4. The category of spaces

In order to speak clearly of ∞-categories, we need to understand spaces of Homs for infinity categories. From a baseline perspective, one might say that plain categories are enhanced in sets, while ∞-categories are enhanced with "spaces". So, we replace sets with spaces in our mind, and we will be on our way. But, it might be more instructive to compare with dg categories.

A dg category is, obviously, a category enriched in complexes of vector spaces. So, if one understands complexes, we see that for objects x and y in a dg category $\mathcal{A}$, one thinks of the complex of morphisms $\mathrm{Hom}_{\mathcal{A}}(x, y)$ as a somewhat dynamic object. One not only has $\mathrm{Hom}_{\mathcal{A}}(x, y)$, but one has the cohomology $H^*(\mathrm{Hom}_{\mathcal{A}}(x, y))$, the cocycles $Z^*(\mathrm{Hom}_{\mathcal{A}}(x, y))$, degree 0 cocylces $Z^0(\mathrm{Hom}_{\mathcal{A}}(x, y))$ which we view as the underlying "morphisms", boundaries which we view as homotopies, etc. So we want some fluency with the, rather dynamic, category of cochain complexes which we then use as a foundation to speak of dg categories.

The category of "spaces", i.e. Kan complexes, serves a similar purpose for ∞-categories. For objects x and y in an ∞-category $\mathscr{C}$, we will have a mapping space $\mathrm{Map}_{\mathscr{C}}(x, y)$. This space has homotopy groups, a homotopy class, components, etc. The homotopy groups in particular, can be thought of as a direct analog of cohomology groups for a complex. Indeed, one might think of ∞-categories as types of non-linear dg categories.

Just as with dg categories, in speaking of ∞-categories we want some base line understanding of these "spaces" and the category in which they live. The point of this section is to provide such a base line understanding.

4.1. **More language for simplicial sets.** A monomorphism $f : K' \to K$ between simplicial sets is a map such that, for any two morphisms $L \rightrightarrows K'$ these two maps are equal if any only if their composites $L \rightrightarrows K \to K'$ are equal. One sees that a map $f : K' \to K$ is a monomorphism if and only if each restriction $f_n : K'[n] \to K[n]$ to the n-simplices is an injective map of sets. We might also refer to a monomorphism simply as an injection.

A simplex $s : \Delta^n \to K$ in a simplicial set K is said to be non-degenerate if s admits not factorization though a simplex $\Delta^m \to K$ of lower dimension, i.e. with

$m < n$. We see that K is constructed as the colimit

$$K = \varinjlim_{\Delta/K_{\text{non-deg}}} \Delta^n, \tag{8}$$

eq:465

where $(\Delta/K)_{\text{non-deg}}$ is full "two layer" subcategory in sSet/K whose top layer consists of non-degenerate simplices $s : \Delta^n \to K$, and whose second layer consists of all simplices $\Delta^l \to K$ which admit at least one factoring $\Delta^p \xrightarrow{r} \Delta^n \xrightarrow{s} K$ through a non-degenerate simplex $s : \Delta^n \to K$ with $n > l$ and $r : \Delta^p \to \Delta^n$ a face in Δ^n. So, the category $(\Delta/K)_{\text{non-deg}}$ locally looks like

$$(\Delta/K)_{\text{non-deg}} : \quad \begin{tikzcd} & \Delta^n \arrow[ddr, dashed, "\text{non-degen}"] & \\ & \vdots & \\ \Delta^p \arrow[uur, "\text{face}"] \arrow[r] \arrow[ddr, "\text{maybe not face}"'] & \Delta^m \arrow[r, dashed] & K \\ & \vdots & \\ & \Delta^l \arrow[uur, dashed, "\text{non-degen}"'] & \end{tikzcd}$$

We say a subcomplex $K' \to K$ is obtained from K by deleting a given simplex $s : \Delta^n \to K$ if the simplices in K', $\Delta^m \to K'$ are precisely all of those simplices in K through which s admits *no* factorization $\Delta^n \to \Delta^m \to K'$. Note that if K' is obtained by deleting a *non-degenerate* n-simplex s, then $K'_{<n} = K_{<n}$, since by definition s admits no factorizations through lower dimensional simplices.

We say a non-degenerate simplex $s : \Delta^n \to K$ is an "external face" if s admits a unique factorization through a unique higher dimensional non-degenerate simplex $\Delta^n \to \Delta^{n'} \to K$, and for this factorization $n' = n + 1$. If K' is obtained from K by deleting an external face, then by the colimit expression (8) we have a pushout diagram

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & K' \arrow[d] \\ \Delta^n \arrow[r, "s"] & K. \end{tikzcd} \tag{9}$$

eq:482

Conversely, if we have such a pushout diagram then K' is obtained from K by deleting an external face.

Below, by a *lifting problem* between maps of simplicial sets $i : A \to B$ and $f : K \to S$ we mean any diagram of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & K \arrow[d, "f"] \\ B \arrow[r] & S. \end{tikzcd}$$

A solution to a lifting problem is a choice of map $\zeta : A \to K$ which produces a diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & K \arrow[d, "f"] \\ B \arrow[r] \arrow[ur, "\zeta"] & S. \end{tikzcd}$$

In order to distinguish, semantically, between a lifting problem and a generic commutative square, we generally present a lifting problem as follows:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & K \arrow[d, "f"] \\ B \arrow[r] \arrow[ur, dashed] & S. \end{tikzcd}$$

Now, if we go back in time, we see that a Kan complex is a simplicial set $\mathscr{X}$ such that all lifting problems, relative to a horn inclusion $\Lambda^n_i \to \Delta^n$ and the terminal map $\mathscr{X} \to *$, admit a solution. Similarly ∞-categories are characterized by the existence of solutions to certain lifting problems.

sect:kf_anodyne

4.2. **Kan complexes, Kan fibrations, and anodyne maps.** Let's consider some class of maps $\mathbf{Class}_p$ in sSet with a property p. (For example, we can consider the class of horn maps, or inner horn maps $\Lambda^n_i \to \Delta^n$.) Then this class generates a larger class Class_p, by closing the initial collection $\mathbf{Class}_p$ under a number of operations.

We say a class of maps Class_p is saturated [8] if it satisfies the following:

(a) Class_p contains all isomorphisms.
(b) Class_p is closed under small coproducts.
(c) Class_p is closed under pushouts: If $L \to K$ is a map in Class_p and

$$\begin{tikzcd} L \arrow[r] \arrow[d] & M \arrow[d, "f"] \\ K \arrow[r] & N \end{tikzcd}$$

is a pushout diagram, then $f : M \to N$ is in Class_p.

(d) Class_p is closed under retracts: If we have a diagram

$$\begin{tikzcd} L \arrow[r] \arrow[d, "f'"'] & M \arrow[r] \arrow[d, "f"] & L \arrow[d, "f'"] \\ K \arrow[r] & N \arrow[r] & K \end{tikzcd}$$

where the horizontal composites are the identity, and f is in Class_p, when f' is in Class_p.

(e) Class_p is closed under countable composites: If

$$K_0 \to K_1 \to K_2 \to \dots$$

is a $\mathbb{N}$-indexes sequence of morphisms in Class_p, then the structure map $K_0 \to \varinjlim_n K_n$ is in Class_p.

The minimal saturated class Class_p containing some specified collection of morphisms $\mathbf{Class}_p$ is called the saturated class generated by $\mathbf{Class}_p$. Note that condition (e) says that any saturated class of morphisms is closed under composition.

Definition 4.1. The class of anodyne morphisms in sSet is the saturated class of maps generated by the inclusions

$$\{\Lambda^n_i \to \Delta^n : \ n \geq 1, i \in [n]\}$$

We note that the class of monomorphisms is itself saturated, so that anodyne maps are in particular injective.

lem:919 **Lemma 4.2.** *A simplicial set $\mathscr{X}$ is a Kan complex if and only if for each anodyne morphism $i : A \to B$ and arbitrary map $K \to \mathscr{X}$, there is a morphism $B \to \mathscr{X}$ completing the diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & \mathscr{X} \\ B \arrow[ur, dashed, "\exists"'] & \end{tikzcd}.$$

Proof. Since the class of anodyne maps includes arbitrary horns $\Lambda^n_i \to \Delta^n$, any simplicial set with the left lifting property against anodyne maps is a Kan complex. For the converse, it suffices to show that the class LLP_{Kan} of maps $i : A \to B$ having the left lifting property to a Kan complex $\mathscr{X}$ is saturated. All of the conditions (a)–(e) are clear, save for possibly (e). Take a sequence of maps

$$A_0 \xrightarrow{i_1} A_1 \xrightarrow{i_2} A_2 \to \dots$$

is LLP_{Kan} and consider the structure map

$$\varprojlim_n \mathrm{Hom}(A_n, \mathscr{X}) = \mathrm{Hom}(\varinjlim_n A_n, \mathscr{X}) \to \mathrm{Hom}(A_0, \mathscr{X}). \tag{10}$$ eq:514

We want to say that this map is surjective.

By the left lifting property, we understand that each map $i_l^* : \mathrm{Hom}(A_l, \mathscr{X}) \to \mathrm{Hom}(A_{l+1}, \mathscr{X})$ in the sequence defining the limit in (10) is surjective. We also understand that each of the map $i_{0l}^* : \mathrm{Hom}(A_l, \mathscr{X}) \to \mathrm{Hom}(A_0, \mathscr{X})$ defining the morphism (10) is surjective, where $i_{0l} = i_l \dots i_1$. One applies Zorn's lemma to see that the corresponding map from the limit (10) is in fact surjective. So the set LLP_{Kan} is in fact saturated, and since

$$\{\Lambda^n_i \to \Delta^n : \ n \geq 1, i \in [n]\} \subseteq LLP_{\mathrm{Kan}}$$

we see that anodyne maps have the left lifting property relative to maps into Kan complexes. $\square$

For example, if we consider an extension $A' \to A$ for which A' is obtained from A via a sequence of deleting external faces in A,

$$A' \to A^{(2)} \to A^{(3)} \to \dots A^{(n)} = K,$$

then the extension $A' \to A$ is anodyne. This is due to the pushout expression (9) and the fact that anodyne maps are closed under composition.

ex:cube **Example 4.3.** Consider the n-dimensional solid cube $C^n = (\Delta^1)^n$. This simplicial set has 0-simplices $Vert^n = C^n([0])$ given by all tuples $(\delta_0, \dots, \delta_n)$, $\delta_i \in \{0, 1\}$, and m-simplices $\Delta^m \to C^n$ are given by order preserving maps

$$[m] \to Vert^n$$

where we give these vertices the dictionary ordering. We have the "sides" $F_{\delta,i} \subseteq C^n$, whose m-simplices are functions $r : [m] \to Vert^n$ with constant value δ in the i-th position, $r(j) = (\delta_0, \dots, \delta_{i-1}, \delta, \dots, \delta_n)$. We consider the "open box" $OC^n_{\delta,i} \subseteq C^n$ which is produced from C^n by deleting the side (δ, i)-side of the cube $F_{\delta,i}$ and the interior of the cube. We claim that the inclusion of the open cube $OC^n_{\delta,i} \to C^n$ is anodyne. Indeed, $OC^n_{\delta,i}$ can be obtained from C^n by a deleting external faces.

We also have a relative notion of Kan complexes.

Definition 4.4. A morphism $f : \mathscr{X} \to S$ is called a Kan fibration if any lifting problem

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \mathscr{X} \arrow[d, "f"] \\ B \arrow[r] \arrow[ur, dashed] & S \end{tikzcd}$$

in which $A \to B$ is anodyne admits a solution.

A simplicial set $\mathscr{X}$ is a Kan complex if and only if the terminal map $\mathscr{X} \to *$ is a Kan fibration. Just as in the case $S = *$, one can test Kan-ness of a morphism $\mathscr{X} \to S$ by examining the lifting property relative to the horn inclusions $\Lambda^n_i \to \Delta^n$. By the universal property of pullback, one sees that the class of Kan fibrations is closed under pullback.

lem:821 **Lemma 4.5.** *Suppose that $f : \mathscr{X} \to S$ is a Kan fibration, and*

$$\begin{tikzcd} \mathscr{X}' \arrow[r] \arrow[d, "f'"'] & \mathscr{X} \arrow[d, "f"] \\ S' \arrow[r] & S \end{tikzcd}$$

is a pullback diagram. Then f' is a Kan fibration.

By the above lemma one can view a Kan fibration as a family of Kan complexes parametrized by the base S. One can compare, for example, with notions of smoothness for varieties, and smooth morphisms. We have now defined Kan fibrations relative to anodyne maps. One can show that the converse distinction holds as well, though we do not record a proof here.

Proposition 4.6 ([9, Corollary 1.4.1])**.** *A map $i : A \to B$ is anodyne if and only if any lifting problem*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & \mathscr{X} \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & S \end{tikzcd}$$

in which $\mathscr{X} \to S$ is a Kan fibration admits a solution.

This is the definition of anodyne maps employed in [13]. The following lemma will be of use momentarily.

lem:850 **Lemma 4.7.** *Consider two monomorphisms of simplicial sets $i : A \to B$ and $j : K \to L$. If either of i or j is an anodyne morphism, then the induced map*

$$(A \times L) \coprod_{(A \times K)} (B \times K) \to (B \times L)$$

is anodyne as well.

The proof is somewhat technical and can be found in [15, 014D] or [8, Corollary 4.6].

4.3. **Trivial Kan fibrations.**

Definition 4.8. A morphism $f : \mathscr{X} \to S$ is called a trivial Kan fibration if any lifting problem

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \mathscr{X} \arrow[d, "f"] \\ B \arrow[r] \arrow[ur, dashed] & S \end{tikzcd} \tag{11}$$

eq:864

in which $A \to B$ is a monomorphism admits a solution. A simplicial set $\mathscr{X}$ is called contractible if the terminal map $\mathscr{X} \to *$ is a trivial Kan fibration.

Of course, the lifting problem here is a choice of square diagram, which produces the external square of (11), and a solution to the lifting problem is a choice of map $B \to \mathscr{X}$ which splits the diagram into two commuting squares. Clearly any trivial Kan fibration is a Kan fibrations.

Proposition 4.9. *A map $f : \mathscr{X} \to S$ is a trivial Kan fibration if and only if all lifting problems*

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d, "\mathrm{incl}"'] & \mathscr{X} \arrow[d, "f"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & S \end{tikzcd}$$

for $n \in \mathbb{Z}_{\geq 0}$, admit a solution.

Sketch proof. The point is that the saturated class of morphisms generated by the boundary inclusions $\{\partial\Delta^n \to \Delta^n : n \geq 0\}$ is the class of all monomorphisms [9, Section 1.4] [15, 0077] □

4.4. **Exponents for Kan complexes.** For simplicial sets K and S we have the mapping complex $\mathrm{Fun}(K, S)$ defined in Section 2.6. We have the evaluation map

$$ev : \mathrm{Fun}(K, S) \times K \to S \tag{12}$$

eq:eval

which is defined on simplices by

$$\mathrm{Hom}_{\mathrm{sSet}}(\Delta^n \times K, S) \times K[n] \to S[n], \quad (f, x) \mapsto f(id_{[n]}, x).$$

The fact that f is a map of simplicial sets implies that evaluation, defined as above, is also a map of simplicial sets.

Lemma 4.10 ([8, Proposition 5.1])**.** *The evaluation morphism* (12) *defines an isomorphism of (plain) sets*

$$\mathrm{Hom}_{\mathrm{sSet}}(L, \mathrm{Fun}(K, S)) \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{sSet}}(L \times K, S)$$
$$\eta \mapsto ev(\eta \times id_K)$$

which is natural in L, K, and S.

Proof. Take $\eta^! := ev(\eta \times id_K)$. The inverse to the map $\eta \mapsto \eta^!$ is given by sending a function $g : L \times K \to S$ to the maps

$$g_! : L \to \mathrm{Fun}(K, S), \quad y \mapsto [\ \Delta^n \times K \to S,\ (t, x) \mapsto g(t^* y, x)\ ].$$

Indeed, we have explicitly

$$(\eta^!)_! : L \to \mathrm{Fun}(K, S), \quad y \mapsto \{(x, t) \mapsto eta^!(x, t^* y) = ev(t^* \eta(y), x)\}$$

and $ev(t^*\eta(y),x) = t^*\eta(y)(id_{[m]},x) = \eta(y)(t,x)$. So $(\eta^!)_! = \eta$. One checks also

$$(g_!)^! : L \times K \to S, \quad (x,y) \mapsto ev(g_! \times id_K) = \left[\ (y,z) \mapsto g_!(y)(id_{[n]},z)\ \right]$$

and $g_!(y)(id_{[n]},z) = g(y,z)$, so that $(g_!)^! = g$. Naturality can be verified directly. □

The above lemma says that the functor complexes $\mathrm{Fun}(-,-)$ provide inner-Homs for the monoidal category of simplicial sets, with the symmetric monoidal structure provided by the product.

prop:tech1 **Proposition 4.11** ([8, Proposition 5.2])**.** *Suppose that $f : \mathscr{X} \to S$ is a Kan fibration and that $i : K \to L$ is a monomorphism between arbitrary simplicial sets. Then the map*

$$\mathrm{Fun}(L,\mathscr{X}) \to \mathrm{Fun}(K,\mathscr{X}) \times_{\mathrm{Fun}(K,S)} \mathrm{Fun}(L,S) \tag{13}$$ eq:903

provided by the diagram

$$\begin{tikzcd} \mathrm{Fun}(L,\mathscr{X}) \arrow[r, "i^*"] \arrow[d, "f^*"'] & \mathrm{Fun}(K,\mathscr{X}) \arrow[d, "f_*"] \\ \mathrm{Fun}(L,S) \arrow[r, "i^*"'] & \mathrm{Fun}(K,S) \end{tikzcd}$$

is a Kan fibration. When $i : K \to L$ is furthermore anodyne, or $f : \mathscr{X} \to S$ is a trivial fibration, the map (13) is a trivial Kan fibration.

Proof. First, let $A \to B$ be any anodyne map. The existence of a splitting for a given diagram

$$\begin{tikzcd} A \arrow[rr] \arrow[d] & & \mathrm{Fun}(L,\mathscr{X}) \arrow[d] \\ B \arrow[r] \arrow[urr, dashed, "\exists ?"] & \mathrm{Fun}(K,\mathscr{X}) \times_{\mathrm{Fun}(K,S)} \mathrm{Fun}(L,S) & \end{tikzcd}$$

is equivalent to the existence of a splitting for the corresponding diagram

$$\begin{tikzcd} (A \times L) \amalg_{(A\times K)} (B \times K) \arrow[r] \arrow[d] & \mathscr{X} \arrow[d, "f"] \\ (B \times L) \arrow[r] \arrow[ur, dashed, "\exists ?"] & S \end{tikzcd} \tag{14}$$ eq:925

which one obtains via adjunction for inner Homs. By Lemma 4.7 the left hand vertical map in (14) is anodyne, and the map $f : \mathscr{X} \to S$ is a Kan fibration by hypothesis, so that such a splitting is seen to exist. It follows that the map (13) is a Kan fibration. When the map $i : K \to L$ is furthermore anodyne, or $f : \mathscr{X} \to S$ is trivial, the above argument can be carried out for any monomorphism $A \to B$, from which one concludes that (13) is a trivial Kan fibration in this case. □

We find a number of interesting corollaries.

cor:Fun_kan **Corollary 4.12.** (i) *For any simplicial set L, and Kan fibration $f : \mathscr{X} \to S$, the induced map*

$$f_* : \mathrm{Fun}(L,\mathscr{X}) \to \mathrm{Fun}(L,S)$$

is a Kan fibration.

(ii) *For any monomorphism $i : K \to L$, and Kan complex $\mathscr{X}$, the induced map*
$$i^* : \mathrm{Fun}(L, \mathscr{X}) \to \mathrm{Fun}(K, \mathscr{X})$$
is a Kan fibration.

(iii) *For any anodyne morphism $i : K \to L$, and Kan complex $\mathscr{X}$, the induced map*
$$i^* : \mathrm{Fun}(L, \mathscr{X}) \to \mathrm{Fun}(K, \mathscr{X})$$
is a trivial Kan fibration.

(iv) *For any simplicial set L, and Kan complex $\mathscr{X}$, the simplicial set of functors* $\mathrm{Fun}(L, \mathscr{X})$ *is a Kan subcomplex in* $\mathrm{Fun}(K, \mathscr{X})$.

Proof. Follow by considering the cases $K = \emptyset$ and $S = *$, independently then together. □

4.5. **Why maps from simplicial sets.** In the previous subsection we have emphasized maps $\mathrm{Fun}(K, \mathscr{X})$ from some simplicial set K into a given Kan complex. Of course, a simplicial set is in general neither a Kan complex nor an ∞-category. So, one might ask: Why consider simplicial sets at all here?

The reason is rather simple. We see a map $p : K \to \mathscr{X}$ from a simplicial set as a "diagram" in $\mathscr{X}$. Then to consider the dynamics of the category $\mathrm{Fun}(K, \mathscr{X})$ is to consider the dynamics of certain diagrams in $\mathscr{X}$. This will be important in both the Kan and ∞-context when we want to speak, for example, of limits and colimits of diagrams in a given Kan complex or ∞-category. So it's actually quite important that we develop a relatively sophisticated understanding of diagrams in a generic Kan complex, or ∞-category.

4.6. **Connected components and the notion of equivalence.**

Definition 4.13. For a Kan complex $\mathscr{X}$, and 0-simplices $x, y : \Delta^0 \to \mathscr{X}$, write $x \sim y$ if there is a 2-simplex $f : \Delta^1 \to \mathscr{X}$ with $f|_0 = x$ and $f|_1 = y$. Take
$$\pi_0(\mathscr{X}) := \mathscr{X}[0]/\sim.$$
We call the set $\pi_0(\mathscr{X})$ the set of connected components for $\mathscr{X}$.

The axioms of a Kan complex say that the relation $\sim$ on $\mathscr{X}[0]$ is an equivalence relation. So the set of connected components is in fact a well-defined set. We say two 0-simplices for $\mathscr{X}$ lie in the same component of $\mathscr{X}$ if they define the have the same image in the quotient $\mathscr{X}[0] \to \pi_0(\mathscr{X})$. An interesting situation occurs when we consider functors $K \to \mathscr{X}$ into a Kan complex, and the corresponding space of functors $\mathrm{Fun}(K, \mathscr{X})$. An equivalence between functors $F, F' : K \to \mathscr{X}$ is a functor $\zeta : \Delta^1 \times K \to \mathscr{X}$ whose two restrictions
$$K \rightrightarrows \Delta^1 \times K \to \mathscr{X}$$
recovers F and F' respectively. Since $\mathrm{Fun}(K, \mathscr{X})$ is a Kan complex, according to Corollary 4.12, this relation on functors is an equivalence relation.

Definition 4.14. Let $\mathscr{X}$ be a Kan complex. We say two maps $F, F' : K \to \mathscr{X}$, from any given simplicial set, are homotopic if they lie in the same component of $\mathrm{Fun}(K, \mathscr{X})$. We say a map $f : \mathscr{X} \to \mathscr{Y}$ between Kan complexes is a (homotopy) equivalence if there exists $g : \mathscr{Y} \to \mathscr{X}$ such that fg and gf are homotopic to the identities on $\mathscr{Y}$ and $\mathscr{X}$, respectively.

One certainly anticipates the following lemma.

lem:978 **Lemma 4.15.** *Suppose that $f, f' : \mathscr{X} \to \mathscr{Y}$ are homotopic map between Kan complexes, and $F, F' : K \to \mathscr{X}$ are homotopic maps from some simplicial set. Then $f \circ F$, $f \circ F'$, $f' \circ F$, and $f' \circ F'$ are all homotopic morphisms in* $\mathrm{Fun}(K, \mathscr{Y})$.

Proof. If we choose a homotopy $\zeta : \Delta^1 \times K \to \mathscr{X}$ between F and F', then $f \circ \zeta$ provides a homotopy between $f \circ F$ and $f \circ F'$. If $z : \Delta^1 \times \mathscr{X} \to \mathscr{Y}$ is a homotopy between f and f', then $z \circ (id_{\Delta^1} \times F)$ provides a homotopy between $f \circ F$ and $f' \circ F$. Since homotopy is an equivalence relation, this suffices to show that all of the given maps are homotopic. □

Now, let us say that a morphism $g : \mathscr{Y} \to \mathscr{X}$ between Kan complexes is a left (resp. right) homotopy inverse to a map $f : \mathscr{X} \to \mathscr{Y}$ if the composite gf (resp. fg) is homotopic to the identity on $\mathscr{X}$ (resp. $\mathscr{Y}$).

Lemma 4.16. *If a map $f : \mathscr{X} \to \mathscr{Y}$ between Kan complexes admits both a left and right homotopy inverse, then f is an equivalence.*

Proof. Let g and g' be left and right homotopy inverses to f. It suffices to show that g and g' are homotopic. By Lemma 4.15 we have

$$g \sim g(fg') = (gf)g' \sim g'.$$

We are done. □

We can now reasonably define the homotopy category of Kan complexes.

Definition 4.17. The homotopy category of Kan complexes is the category

$$\mathrm{hKan} := \{\text{Kan complexes with homotopy classes of morphisms } \pi_0 \mathrm{Fun}(-,-)\}.$$

Lemma 4.15 assures us that composition is well-defined, and associative, on homotopy classes of maps so that $h\,\mathrm{Kan}$ is in fact a category. Note also that two Kan complexes are equivalent, as objects in Kan, if and only if they are isomorphic in the homotopy category hKan. If we compare with our category of "spaces", $\mathscr{S}paces = \mathrm{N_{hc}}(\underline{\mathrm{Kan}}^{\mathrm{sm}})$, we have

$$\mathrm{hKan}^{\mathrm{sm}} = \mathrm{h}\,\mathscr{S}paces^{\mathrm{sm}}.$$

4.7. **Equivalences and functor spaces.** Our main goal of the subsection is to prove the following.

prop:equiv_Fun_kan **Proposition 4.18.** *For a map $f : \mathscr{X} \to \mathscr{Y}$ between Kan complexes, the following are equivalent:*

(a) *f is an equivalence.*
(b) *For any simplicial set K, the induced map*

$$f_* : \mathrm{Fun}(K, \mathscr{X}) \to \mathrm{Fun}(K, \mathscr{Y})$$

is an equivalence of Kan complexes.
(c) *For any Kan complex $\mathscr{W}$, the induced map*

$$f_* : \mathrm{Fun}(\mathscr{W}, \mathscr{X}) \to \mathrm{Fun}(\mathscr{W}, \mathscr{Y})$$

is an equivalence.

(d) *For any Kan complex $\mathscr{W}$, the induced map*

$$f^* : \mathrm{Fun}(\mathscr{Y}, \mathscr{W}) \to \mathrm{Fun}(\mathscr{X}, \mathscr{W})$$

is an equivalence.

We first record some lemmas, which will prove useful in any case. The following establishes the implication from (a) to (b).

lem:1195

Lemma 4.19. *Suppose that $f : \mathscr{X} \to \mathscr{Y}$ is an equivalence between Kan complexes. Then, for any simplicial set K, the induced map $f_* : \mathrm{Fun}(K, \mathscr{X}) \to \mathrm{Fun}(K, \mathscr{Y})$ is an equivalence as well.*

Proof. Let $g : \mathscr{Y} \to \mathscr{X}$ be a homotopy inverse to f. By considering the composites fg and gf, it suffices to show that for any endomorphism $F : \mathscr{X} \to \mathscr{X}$ which is homotopic to the identity $id_{\mathscr{X}}$ the induced map $F_* : \mathrm{Fun}(K, \mathscr{X}) \to \mathrm{Fun}(K, \mathscr{X})$ is homotopic to the identity. By the natural isomorphism

$$\mathrm{Hom}_{\mathrm{sSet}}(\Delta^1 \times \mathrm{Fun}(K, \mathscr{X}), \mathrm{Fun}(K\mathscr{X})) \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{sSet}}(\mathrm{Fun}(K, \mathscr{X}), \mathrm{Fun}(\Delta^1 \times K, \mathscr{X}))$$

provided by adjunction, it suffices to produce a map

$$\Theta : \mathrm{Fun}(K, \mathscr{X}) \to \mathrm{Fun}(\Delta^1 \times K, \mathscr{X})$$

whose restrictions along the two

$$-|_{\{0\}\times K},\ -|_{\{1\}\times K} : \mathrm{Fun}(\Delta^1 \times K, \mathscr{X}) \rightrightarrows \mathrm{Fun}(K, \mathscr{X})$$

recover the identity id_{Fun} and F_* respectively.

If we let $\theta : \Delta^1 \times \mathscr{X} \to \mathscr{X}$ be a homotopy between $id_{\mathscr{X}}$ and F, then we may define Θ on n-simplices by taking a function $\alpha : \Delta^n \times K \to \mathscr{X}$ to the composite $\theta \circ (\Delta^1 \times \alpha)$. Such a formula determines Θ as a map of simplicial sets and, by direct inspection, the two composites

$$\mathrm{Fun}(K, \mathscr{X}) \xrightarrow{\Theta} \mathrm{Fun}(\Delta^1 \times K, \mathscr{X}) \rightrightarrows \mathrm{Fun}(K, \mathscr{X})$$

recover id_{Fun} and F_*. Rather, Θ exhibits a homotopy between the identity and F_*, as required. □

lem:pi0_equiv

Lemma 4.20. *If $f : \mathscr{X} \to \mathscr{Y}$ is an equivalence, then the induced map on connected components $\pi_0(f) : \pi_0(\mathscr{X}) \to \pi_0(\mathscr{Y})$ is an isomorphism.*

Proof. We have a natural isomorphism

$$\pi_0(\mathscr{X}) \cong \pi_0(\mathrm{Fun}(*, \mathscr{X})) = \mathrm{Hom}_{\mathrm{hKan}}(*, \mathscr{X})$$

under which $\pi_0(f)$ is identified with procomposition

$$\bar{f}_* : \mathrm{Hom}_{\mathrm{hKan}}(*, \mathscr{X}) \to \mathrm{Hom}_{\mathrm{hKan}}(*, \mathscr{Y}).$$

Since $barf : \mathscr{X} \to \mathscr{Y}$ is an isomorphism in hKan, the corresponding map $\bar{f}_*$ is an isomorphism. □

We can now prove our proposition.

Proof of Proposition 4.18. The implication (a) $\Rightarrow$ (b) is covered by Lemma **??**, and the implication (b) $\Rightarrow$ (c) is immediate, since (c) is just some restriction of (b). Now, we claim that (c) implies (a). Suppose that (c) holds. By Lemma 4.20, the map

$$\pi_0(f_*) : \pi_0(\mathrm{Fun}(\mathscr{W}, \mathscr{X})) \to \pi_0(\mathrm{Fun}(\mathscr{W}, \mathscr{Y}))$$

is an isomorphism. But these sets, by definition, are just $\mathrm{Hom}_{\mathrm{hKan}}(\mathscr{W},\mathscr{X})$ and $\mathrm{Hom}_{\mathrm{hKan}}(\mathscr{W},\mathscr{X})$, and $\pi_0(f_*)$ is composition $\bar{f}_*$ with the homotopy class of f. So we conclude that

$$\bar{f}_* : \mathrm{Hom}_{\mathrm{hKan}}(\mathscr{W},\mathscr{X}) \to \mathrm{Hom}_{\mathrm{hKan}}(\mathscr{W},\mathscr{Y})$$

is an isomorphism at all $\mathscr{W}$ in hKan. Rather, we have a natural isomorphism

$$\bar{f}_* : \mathrm{Hom}_{\mathrm{hKan}}(-,\mathscr{X}) \overset{\cong}{\to} \mathrm{Hom}_{\mathrm{hKan}}(-,\mathscr{Y})$$

of Set valued functors. It follows by Yoneda that the map $\bar{f}$ is an isomorphism in hKan, and hence that the original map $f : \mathscr{X} \to \mathscr{Y}$ is an equivalence. So we establish (a).

One similarly argues that (d) implies (a), and so we need only establish the implication (a) $\Rightarrow$ (d). However, one can argue as in the proof of Lemma 4.19 to see directly that $f^* : \mathrm{Fun}(\mathscr{Y},\mathscr{W}) \to \mathrm{Fun}(\mathscr{X},\mathscr{W})$ is an equivalence whenever f is an equivalence. We are done. $\square$

4.8. **Homotopy groups for Kan complexes.** For any $n \geq 0$ we define the simplicial n-sphere as

$$S^n := \Delta^n \amalg_{\partial\Delta^n} *.$$

The object S^n is a simplicial set which is a quotient of the standard n-simplex. To distinguish between the simplicial and topological spheres we denote the topological n-sphere by $\mathbb{S}^n$. We have that $|S^n| = \mathbb{S}^n$, but we note that the simplicial sphere is not itself a Kan complex, and so not a "space". However, the failure of S^n to be a Kan complex is, for some specific reasons [15, 00VT], not important.

Definition 4.21. A pointed simplicial set (K, x) is a simplicial set K with a fixed map $x : * \to K$. A map between pointed simplicial sets $f : (K, x) \to (L, y)$ is a map of simplicial sets which fits into a diagram

$$\begin{tikzcd} & * \arrow[dl, "x"'] \arrow[dr, "y"] & \\ K \arrow[rr, "f"'] & & L. \end{tikzcd}$$

A pointed Kan complex is a Kan complex which is pointed as a simplicial set.

We consider the simplicial sphere S^n as a pointed simplicial set $(S^n, *)$ with distinguished point given by the unique 1-simplex $* \to S^n$. So, given a pointed simplicial set (K, x), a map of pointed simplicial sets $S^n \to (K, x)$ is a map of simplicial sets such that the unique composite $* \to S^n \to K$ recovers x. For pointed simplicial sets (K, x) and (L, y) we let

$$\mathrm{Fun}(K, L)_{\mathrm{pt}(x,y)} \subseteq \mathrm{Fun}(K, L)$$

denote the simplicial subset which is given by the pullback of the diagram

$$\begin{tikzcd} \mathrm{Fun}(K, L)_{\mathrm{pt}(x,y)} \arrow[r] \arrow[d] & \mathrm{Fun}(K, L) \arrow[d, "x^*"] \\ * \arrow[r, "y"'] & \mathrm{Fun}(*, L) = L. \end{tikzcd}$$

So a map $f : \Delta^n \times K \to L$ is an n-simplex in the pointed functor complex provided the restriction $\Delta \times * \to \Delta \times K \to L$ along the map $x : * \to K$ is of constant value y.

Lemma 4.22. *If $(\mathscr{X}, x)$ is a Kan complex, and (K, z) is an arbitrary pointed set, then the pointed mapping complex $\mathrm{Fun}(K, \mathscr{X})_{\mathrm{pt}(z,x)}$ is a Kan complex.*

Proof. By Corollary 4.12 the map $z^* : \mathrm{Fun}(K, \mathscr{X}) \to \mathrm{Fun}(*, \mathscr{X})$ is a Kan fibration, and Kan fibrations are closed under pullback by Lemma 4.5. Hence the pointed mapping complex is a Kan complex. □

Definition 4.23. Let $(\mathscr{X}, x)$ be a pointed Kan complex. The n-th homotopy group $\pi_n(\mathscr{X}, x)$ is the set of homotopy classes of maps

$$\pi_n(\mathscr{X}, x) := \pi_0\left(\mathrm{Fun}(S^n, \mathscr{X})_{\mathrm{pt}(*,x)}\right)$$

Since $\mathrm{Fun}(S^n, \mathscr{X})_{\mathrm{pt}(*,x)}$ is itself a Kan complex, it is reasonable to consider the connected components here. For $n > 0$ there is a group structure on $\pi_n(\mathscr{X}, x)$ which defined as follows: for two pointed maps $a, b : S^n \to (\mathscr{X}, x)$ which represent classes $\bar{\alpha}, \bar{\beta} \in \pi_n(\mathscr{X}, x)$, we consider the horn $w : \Lambda_1^{n+1} \to \mathscr{X}$ with restrictions

$$w|_{\Delta^{[n+1]-\{0\}}} = a, \quad w|_{\Delta^{[n+1]-\{2\}}} = b, \quad w|_{\Delta^{[n+1]-\{i\}}} = x \ \text{ when } i > 2. \tag{15}$$ eq:1088

We fill this horn to get a map $W : \Delta^{n+1} \to \mathscr{X}$, and take

$$\alpha * \beta := \overline{W|_{\Delta^{[n+1]-\{1\}}}} \quad \in \pi_n(\mathscr{X}, x).$$

lem:1097 **Theorem 4.24** ([8, Lemma 7.1, Theorem 7.2])**.** *Fix $n > 0$. For any classes $\alpha, \beta \in \pi_n(\mathscr{X}, x)$ the element $\alpha * \beta$ constructed above does note depend on the choice of representatives $a, b : S^n \to \mathscr{X}$, nor does it depend on the choice of filling for the horn (15). We thus obtain a well-defined binary operation*

$$- * - : \pi_n(\mathscr{X}, x) \times \pi_n(\mathscr{X}, x) \to \pi_n(\mathscr{X}, x). \tag{16}$$ eq:1098

The binary operation (16) provides $\pi_n(\mathscr{X}, x)$ with a group structure for which the class of the constant map $x : S^n \to \mathscr{X}$ serves as the identity. Furthermore, this group is abelian when $n > 2$.

This construction may seem somewhat mysterious, however the group structure on $\pi_n(\mathscr{X}, x)$ enjoys a certain uniqueness. Namely, this is the unique group structure for which the constant map at x provides the unit and for which an $(n+1)$-fold product $\gamma_0^{-1}\gamma_1 \dots \gamma_{n+1}^{(-1)^n}$ is the identity in $\pi_n(\mathscr{X}, x)$ if and only if the corresponding boundary map $\gamma_* : \partial\Delta^{n+1} \to \mathscr{X}$ admits a filling [15, 00VU]. These two properties are familiar from topology.

We note that the construction of the homotopy group is functorial. Namely, if we have a map of pointed Kan complexes $f : (\mathscr{X}, x) \to (\mathscr{Y}, y)$ then we have an induced map of Kan complexes

$$f_* : \mathrm{Fun}(S^n, \mathscr{X})_{\mathrm{pt}(*,x)} \to \mathrm{Fun}(S^n, \mathscr{Y})_{\mathrm{pt}(*,y)}$$

and thus an induced map on connected components $f_*(n) : \pi_n(\mathscr{X}, x) \to \pi_n(\mathscr{Y}, y)$. Since horn fillings are sent to horn fillings under f, this map on homotopy groups is in fact a map of groups. We record this observation.

Lemma 4.25. *The homotopy group constructions provides a $\mathbb{Z}_{>0}$-indexed collection of functors*

$$\pi_n(-) : \{\text{ Pointed Kan complexes with pointed maps }\} \to \mathrm{Groups}\,.$$

4.9. **Homotopy groups and trivial Kan fibrations.**

Proposition 4.26. *If $\mathscr{X}$ is a contractible Kan complex, then $\pi_0(\mathscr{X})$ has a single element and all of the homotopy groups $\pi_n(\mathscr{X}, x)$ are trivial.*

Proof. To see that $\pi_0(\mathscr{X})$ one simply notes that any map $\partial\Delta^1 \to \mathscr{X}$, which just consists of a choice of two points in $\mathscr{X}$, extends to a path $\Delta^1 \to \mathscr{X}$ via the lifting property for trivial Kan fibrations. Suppose now that $n > 0$. Consider the pushout K^n of the diagram

$$\begin{tikzcd} \partial\Delta^1 \arrow[r] \arrow[d] & \partial\Delta^1 \times S^n \arrow[d, dotted] \\ \Delta^1 \arrow[r, dotted] & K^n. \end{tikzcd}$$

The two maps $S^n \times \partial\Delta^1 \to S^n \times \Delta^1$ and $\Delta^1 \to S^n \times \Delta^1$ induce an embedding $K^n \to \Delta^1 \times \Delta^1$. Now, for any two maps $a, b : S^n \to \mathscr{X}$ there is a unique morphism $[a, b] : K^n \to \mathscr{X}$ for which $[a, b]|S^n \times \partial\Delta^1 = a \amalg b$ and $[a, b]|\Delta^1 = x$. By the lifting property there exists an extension of $[a, b]$ to a map $h : \Delta^1 \times S^n \to \mathscr{X}$. The map h provides a homotopy between a and b in $\mathrm{Fun}(S^n, \mathscr{X})_{\mathrm{pt}(*,x)}$, and hence equates the classes $\bar{a} = \bar{b} \in \pi_n(\mathscr{X}, x)$. This shows that $\pi_n(\mathscr{X}, x)$ is a singleton for all $n > 0$ as well. □

A relative version of the above proposition holds as well.

prop:1386

Proposition 4.27. *Suppose that $f : \mathscr{X} \to \mathscr{S}$ is a trivial Kan fibration, and that $\mathscr{S}$ is a Kan complex. Then the map on connected components $\pi_0 f : \pi_0(\mathscr{X}) \to \pi_0(\mathscr{S})$ is a bijection and for all $n \geq 1$ the map $\pi_n f : \pi_n(\mathscr{X}, x) \to \pi_n(\mathscr{S}, fx)$ is an isomorphism of groups.*

We first note that compositions of Kan fibrations are Kan fibrations, so that $\mathscr{X}$ itself if a Kan complex in this case

Proof. The lifting property implies directly that $\mathscr{X}[0] \to S[0]$ is surjective and that paths in $\mathscr{S}$ lift to paths in $\mathscr{X}$. So the induced map on connected components on an isomorphism. Fix now $n \geq 1$ and take, for arbitrary $x \in \mathscr{X}[0]$, $y = f(x)$. By applying the lifting property along the inclusion $* \to S^n$ we see that the map

$$f_* : \mathrm{Hom}_{\mathrm{sSet}}(S^n, \mathscr{X})_{\mathrm{pt}(*,x)} \to \mathrm{Hom}_{\mathrm{sSet}}(S^n, \mathscr{S})_{\mathrm{pt}(*,y)}$$

is surjective. Now take K^n to be the pushout

$$\begin{tikzcd} \partial\Delta^1 \arrow[r] \arrow[d] & \partial\Delta^1 \times S^n \arrow[d] \\ \Delta^1 \arrow[r] & K^n \end{tikzcd}$$

and consider the inclusion $K^n \to \Delta^1 \times S^n$. Let $\eta : \Delta^1 \times S^n \to \mathscr{S}$ be a pointed homotopy between two pointed maps $\alpha, \alpha' : S^n \to (\mathscr{S}, y)$, and consider two pointed maps $\beta, \beta' : S^n \to (\mathscr{X}, x)$ which lift α and α' respectively. Then β and β' determine a map $K^n \to \mathscr{X}$ which fits into a diagram

$$\begin{tikzcd} K^n \arrow[r] \arrow[d] & \mathscr{X} \arrow[d, "f"] \\ \Delta^1 \times S^n \arrow[r, "h"'] & \mathscr{S}. \end{tikzcd}$$

We choose a solution $\widetilde{h} : \Delta^1 \times S^n \to \mathscr{X}$ to the corresponding lifting problem to see that β and β' define the same class in $\pi_n(\mathscr{X}, x)$. This shows that the map

$$\pi_n f : \pi_n(\mathscr{X}, x) \to \pi_n(\mathscr{S}, y)$$

is bijective. $\square$

Remark 4.28. A version of Proposition 4.27 for possibly non-trivial Kan fibrations, in which the isomorphisms $\pi_n(\mathscr{X}, x) \to \pi_n(\mathscr{S}, fx)$ "spread out" into a long exact sequence of homotopy groups, is discussed in Section 4.11 below.

One can use Proposition 4.27 in place of Lemma 4.20 to see that any trivial Kan fibration is in fact an equivalence.

`prop:trivkan_equiv` **Proposition 4.29.** *Suppose that $f : \mathscr{X} \to \mathscr{S}$ is a trivial Kan fibration between Kan complexes. Then f is an equivalence.*

Proof. In this case we have that, for any Kan complex $\mathscr{W}$, the induced map

$$f_* : \mathrm{Fun}(\mathscr{W}, \mathscr{X}) \to \mathrm{Fun}(\mathscr{W}, \mathscr{X})$$

is a trivial Kan fibration, by Proposition 4.11. Then by Lemma 4.20 the induced map on connected components is an isomorphism

$$\pi_0(f_*) : \pi_0(\mathrm{Fun}(\mathscr{W}, \mathscr{X})) \to \pi_0(\mathrm{Fun}(\mathscr{W}, \mathscr{X})).$$

As in the proof of Proposition 4.27, this collection of isomorphisms, which varies naturally in $\mathscr{W}$, implies that the class of f defines an isomorphism between $\mathscr{X}$ and $\mathscr{Y}$ in the homotopy category hKan. Hence f is an equivalence. $\square$

There's actually quite a dynamic interaction between homotopy groups and Kan fibrations, trivial or otherwise. We won't cover this material here. One can see [15, 00WD, 00WL] for a more in depth discussion.

Remark 4.30. Note that the converse to Proposition 4.29 does not hold. Consider for example the 2-disk $\mathbb{D}^2$. Since this space is contractible the map $\mathrm{Sing}(\mathbb{D}^2) \to *$ for the associated singular complex is a trivial Kan fibration, and any point $x : * \to \mathrm{Sing}(\mathbb{D}^2)$ provides a homotopy inverse for this projection. However, it is clear that x is not a trivial Kan fibration since, in particular, this maps is not surjective on 0-simplices. Indeed, $\mathrm{Sing}(\mathbb{D}^2)$ has $|\mathbb{D}^2|$-many 0-simplices.

4.10. **Whitehead's theorem.**

Lemma 4.31 ([15, 00WW])**.** *Consider two maps $f, f' : K \to \mathscr{X}$ from a simplicial set K to a Kan complex $\mathscr{X}$, and suppose that for some choice of points $z \in K[0]$ and $x \in \mathscr{X}[0]$ we have $f(z) = f'(z) = x$. Then f and f' are homotopic if and only if they are pointed homotopic. This is to say, the classes of f and f' agree in $\pi_0(\mathrm{Fun}(K, \mathscr{X}))$ if and only if they agree in $\pi_0(\mathrm{Fun}(K, \mathscr{X})_{\mathrm{pt}(z,x)}$.*

The proof uses a notion of Kan replacement [15, 00UW] to reduce to the case where K is a Kan complex, and employs some technology with fundamental group(oid)s which we've not covered here. We refer the reader directly to [15] for the proof. The implication of this result is clear however.

`prop:1560` **Proposition 4.32.** *If $f : \mathscr{X} \to \mathscr{Y}$ is an equivalence between Kan complexes, and $y = f(x) \in \mathscr{Y}[0]$, then the induced map on connected components $\pi_0(f) : \mathscr{X} \to \mathscr{Y}$ is a bijection and the maps on all higher homotopy groups*

$$\pi_n f : \pi_n(\mathscr{X}, x) \to \pi_n(\mathscr{Y}, y)$$

are isomorphisms.

An amazing point is that the inverse implication holds.

thm:whitehead **Theorem 4.33** (Simplicial Whitehead theorem!)**.** *A map $f : \mathscr{X} \to \mathscr{Y}$ between Kan complexes is an equivalence if and only if the induced map on connected components $\pi_0 f : \pi_0(\mathscr{X}) \to \pi_0(\mathscr{Y})$ is a bijection and for all $x \in \mathscr{X}[0]$, $y = f(x) \in \mathscr{Y}$, the induced maps on homotopy groups*

$$\pi_n f : \pi_n(\mathscr{X}, x) \to \pi_n(\mathscr{Y}, y)$$

are all isomorphisms.

We do not provide the details here, though they are not of a completely alien nature at this point. One find a complete account in [15, 00WU] or [8, Theorem 1.10]. From this point on we simply take this beautiful result for granted.

Remark 4.34. If we compare with the category of chain complexes over some ring, then Whitehead's theorem says something like any "quasi-isomorphism" between Kan complexes is in fact a homotopy equivalence. Though we know that this statement is, in a literal sense, not true for chain complexes over $\mathbb{Z}$ for example, the example of Section 3.3 suggests that Whitehead's theorem really might be as fantastical as this analogy suggests.

sect:les

4.11. **A deviation on long exact sequences and homotopy groups.** Consider a Kan fibration $f : \mathscr{X} \to \mathscr{S}$ between Kan complexes. Furthermore, let's choose a point $x : * \to \mathscr{X}$, and $s = f \circ x : * \to \mathscr{S}$, in order to get our homotopy group machine started. We then have a pullback square

$$\begin{tikzcd} \mathscr{X}_s \arrow[r] \arrow[d] & \mathscr{X} \arrow[d, "f"] \\ * \arrow[r, "s"'] & \mathscr{S}, \end{tikzcd} \tag{17}$$ eq:1587

and a sequence of maps of homotopy groups

$$\pi_n(\mathscr{X}_s, x) \to \pi_n(\mathscr{X}, x) \xrightarrow{\pi_n f} \pi_n(\mathscr{S}, s) \tag{18}$$ eq:1594

with $\mathrm{im}\big(\pi_n(\mathscr{X}_s)\big) \subseteq \ker(f_*)$. In the case $n = 0$, we take specifically $\ker(\pi_0 f) := (\pi_n f)^{-1}([s])$.

Very coarsely, we might view a pullback diagram (17) as analogous to an exact sequence of cochain complexes. In continued analogy with the abelian setting, a diagram as in (17) in fact produces a *long exact sequence* of homotopy groups

$$\begin{aligned} \cdots \to \pi_{n+1}(\mathscr{S}, s) \xrightarrow{\partial_{n+1}} \pi_n(\mathscr{X}_s, x) \to \pi_n(\mathscr{X}, x) \to \pi_n(\mathscr{S}, s) \xrightarrow{\partial_n} \pi_{n-1}(\mathscr{X}_s, x) \to \\ \cdots \xrightarrow{\partial_1} \pi_0(\mathscr{X}_s, x) \to \pi_0(\mathscr{X}, x) \to \pi_0(\mathscr{S}, s). \end{aligned} \tag{19}$$ eq:les

which extends the sequence (18).

thm:les_pin **Theorem 4.35** ([15, 00WM])**.** *Given a Kan fibration $f : (\mathscr{X}, x) \to (\mathscr{S}, s)$ and corresponding pullback diagram as in (17), there is a connecting morphism*

$$\partial_n : \pi_n(\mathscr{S}, s) \to \pi_{n-1}(\mathscr{X}_s, x)$$

which extends the sequence (18) to a long exact sequence of homotopy groups. This connecting morphism is a morphism of groups whenever $n > 1$.

In the case where $f : \mathscr{X} \to \mathscr{S}$ is a trivial Kan fibration, so that the fiber $\mathscr{X}_s$ is contractible, the proposed long exact sequence of Theorem 4.35 recovers Proposition 4.27. Note that this theorem implicitly claims that the short sequence (18) is in fact exact [15, 00WN]. In this subsection we won't argue the exactness of the above sequence–for those details the reader should see [15]–but instead focus on the construction of the connecting morphism ∂_n and its "naturality".

For the remainder of the section we fix pointed spaces $(\mathscr{X}, x)$ and $(\mathscr{S}, s)$, a Kan fibration $f : \mathscr{X} \to \mathscr{S}$ between these pointed spaces, and the corresponding pullback diagram (17). The connecting map is specified precisely by the following information.

prop:1622 **Proposition 4.36** ([15, 00WG])**.** *There is a unique map of sets $\partial_n : \pi_n(\mathscr{S}, s) \to \pi_{n-1}(\mathscr{X}_s, x)$ for which $\pi_n([\gamma]) = [\gamma']$ precisely when the representing morphisms $\gamma : \Delta^n \to \mathscr{S}$ and $\gamma' : \Delta^{n-1} \to \mathscr{X}$ precisely when there exists a third map $\vartheta : \Delta^n \to \mathscr{X}$ with the following properties:*

(a) $f\vartheta = \gamma : \Delta^n \to \mathscr{S}$;

(b) $\vartheta|\Delta^{\{1,\ldots,n\}} = \gamma'$;

(c) $\vartheta|\Lambda^n_0 = x$.

Furthermore, ∂_n is a group map whenever $n > 1$.

As a sanity check, so to speak, let's think about conditions (a) and (b). Condition (a) requires $\vartheta(\partial\Delta^n) \in f^{-1}(s) = \mathscr{X}_s$, so that the $(n-1)$-simplex $\vartheta|\Delta^{\{1,\ldots,n\}}$ necessarily has image in $\mathscr{X}_s$. We require that this $(n-1)$-simplex agrees with γ'. Now, (c) says we simply crush the remainder of the boundary at x. So, topologically, Proposition 4.36 says $\partial([\gamma]) = [\gamma']$ if there is an map $|\vartheta| : D^n \to |\mathscr{X}|$ which lifts $|\gamma| : S^n \to |\mathscr{S}|$ and has prescribed boundary $|\vartheta||_{\mathbb{S}^{n-1}} = |\gamma'| : \mathbb{S}^{n-1} \to |\mathscr{X}_s|$. From the topological perspective the definition of the connecting morphism then becomes quite intuitive.

The explicit description of the connecting morphism provided by Proposition 4.36 allows one to witness certain naturality properties for for the long exact sequence of homotopy groups (19).

prop:2796 **Proposition 4.37.** *Suppose we have a diagram of pointed Kan complexes*

$$\begin{tikzcd} (\mathscr{X}, x) \arrow[r, "\mu"] \arrow[d, "f"'] & (\mathscr{X}, x') \arrow[d, "f'"] \\ (\mathscr{S}, s) \arrow[r, "\nu"] & (\mathscr{S}', s') \end{tikzcd}$$

in which both f and f' are Kan fibrations. Then the induced maps on homotopy groups

$$\pi_n(\mu_s) : \pi_n(\mathscr{X}_s, x) \to \pi_n(\mathscr{X}'_{s'}, x'), \quad \pi_n(\mu) : \pi_n(\mathscr{X}, x) \to \pi_n(\mathscr{X}', x'),$$
$$\pi_n(\nu) : \pi_n(\mathscr{S}, s) \to \pi_n(\mathscr{S}', s')$$

fit into a map between the corresponding long exact sequences

$$\begin{tikzcd} \cdots \arrow[r] & \pi_{n+1}(\mathscr{S}, s) \arrow[r, "\partial"] \arrow[d] & \pi_n(\mathscr{X}_s, x) \arrow[r] \arrow[d] & \pi_n(\mathscr{X}, x) \arrow[r] \arrow[d] & \pi_n(\mathscr{S}, s) \arrow[r, "\partial"] \arrow[d] & \cdots \\ \cdots \arrow[r] & \pi_{n+1}(\mathscr{S}, s) \arrow[r, "\partial"] & \pi_n(\mathscr{X}_s, x) \arrow[r] & \pi_n(\mathscr{X}, x) \arrow[r] & \pi_n(\mathscr{S}, s) \arrow[r, "\partial"] & \cdots . \end{tikzcd}$$

4.12. **Equivalences and fibers.** The following is deduced as an application of Whiteheads theorem, in conjunction with the long exact sequence for homotopy groups.

prop:weee **Proposition 4.38.** *Suppose we have a diagram of Kan complexes*

$$\begin{tikzcd} \mathscr{X} \arrow[r, "f"] \arrow[d] & \mathscr{Y} \arrow[d] \\ \mathscr{S} \arrow[r, "q"] & \mathscr{T} \end{tikzcd}$$

in which the vertical maps are Kan fibrations, and $q : \mathscr{S} \to \mathscr{T}$ is an equivalence. Then the following are equivalent:

(a) *The map $f : \mathscr{X} \to \mathscr{Y}$ is an equivalence of Kan complexes.*
(b) *For any choice of point $s : * \to \mathscr{S}$, and $t = qs : * \to \mathscr{T}$, the induced map on the fibers $f_s : \mathscr{X}_s \to \mathscr{Y}_t$ is an equivalence.*

Proof. (a) $\Rightarrow$ (b) By Whitehead's theorem it suffices to show that, for some choice of point $x : * \to \mathscr{X}_s$ and $y = fx : * \to \mathscr{Y}_t$, all of the maps on connected components and homotopy groups $\pi_n f_s : \pi_n(\mathscr{X}_s, x) \to \pi_n(\mathscr{Y}_t, y)$ are all isomorphisms.

To begin, we know that the induced maps $\pi_n f : \pi_n(\mathscr{X}, x) \to \pi_n(Y, y)$ and $\pi_n q : \pi_n(\mathscr{S}, s) \to \pi_n(\mathscr{T}, t)$ are all isomorphisms. So the five lemma and naturality of the long exact sequence on homotopy groups can be employed to see that $\pi_n f_s$ is an isomorphism whenever $n > 1$. When $n = 1$ a version of the five lemma for not-necessarily-abelian groups still holds, so that $\pi_1 f_s$ is also seen to be an isomorphism. So we need only show that $\pi_0 f_s$ is a bijection.

Note that the map $\pi_0 f_s$ does not depend on the choice of base point $x : * \to \mathscr{X}_s$, but the long exact sequence on homotopy groups does. Suppose we have a point $[z] \in \pi_0(\mathscr{Y}_t)$ and note that the image of $[z]$ under the composite

$$\pi_0(\mathscr{Y}_t) \to \pi_0(\mathscr{Y}) \to \pi_0(\mathscr{T}, t)$$

is trivial, i.e. is equal to $[t] \in \pi_0(\mathscr{T}, t)$.

We consider the image of $[z]$ in $\pi_0(\mathscr{Y})$, which we also denote $[z]$ by abuse of notation, and take the unique lift of $[z]$ to an element $[x] \in \pi_0(\mathscr{X})$ along the isomorphism $\pi_0(\mathscr{X}) \overset{\cong}{\to} \pi_0(\mathscr{Y})$. We via the diagram

$$\begin{tikzcd} \pi_0(\mathscr{X}_s) \arrow[r] \arrow[d] & \pi_0(\mathscr{X}) \arrow[r] \arrow[d, "\cong"] & \pi_0(\mathscr{S}, s) \arrow[d, "\cong"] \\ \pi_0(\mathscr{Y}_t) \arrow[r] & \pi_0(\mathscr{Y}) \arrow[r] & \pi_0(\mathscr{T}, t) \end{tikzcd} \tag{20}$$

we see that $[x]$ maps to $[s] \in \pi_0(\mathscr{S}, s)$ so that, by exactness of the rows in the above diagram we see that $[x]$ lifts to an element in $\pi_0(\mathscr{X}_s)$. Rather, we see that $[x] \in \pi_0(\mathscr{X})$ is represented by a point in $\mathscr{X}_s$, so that we may assume that x itself is a point in $\mathscr{X}_s$ and write simply $[x] \in \pi_0(\mathscr{X}_s)$ by an abuse of notation. Now, we claim that $\pi_0 f_s([x]) = [z] \in \pi_0(\mathscr{Y})$. To see this we fix $x : * \to \mathscr{X}_s$ as our base point, $y = f_s x : * \to \mathscr{Y}_t$, and consider the map between long exact sequences

$$\begin{tikzcd}
\pi_1(\mathscr{X}, x) \arrow[r] \arrow[d, "\cong"'] & \pi_1(\mathscr{S}, s) \arrow[r, "\partial"] \arrow[d, "\cong"'] & \pi_0(\mathscr{X}_s, x) \arrow[r] \arrow[d] & \pi_0(\mathscr{X}, x) \arrow[r] \arrow[d, "\cong"] & \pi_0(\mathscr{S}, s) \arrow[d, "\cong"] \\
\pi_1(\mathscr{Y}, y) \arrow[r] & \pi_1(\mathscr{T}, t) \arrow[r, "\partial"] & \pi_0(\mathscr{Y}_t, y) \arrow[r] & \pi_0(\mathscr{Y}, y) \arrow[r] & \pi_0(\mathscr{T}, t)
\end{tikzcd} \tag{21}$$

eq:2856

Via the above diagram, and the fact that $[x] \mapsto [z]$ under the map $\pi_0(\mathscr{X}) \to \pi_0(\mathscr{Y})$, we see that $[z] \mapsto [y]$ under the map $\pi_0(\mathscr{Y}_t, y) \to \pi_0(\mathscr{Y}, y)$. This is to say, $[z]$ is in the kernel of $\pi_0(\mathscr{Y}_t, y) \to \pi_0(\mathscr{Y}, y)$. Via exactness $[z] \in \pi_0(\mathscr{Y}_t, y)$ lifts to an element $[\zeta] \in \pi_1(\mathscr{T}, t)$, which then lifts uniquely to some element $[\zeta'] \in \pi_1(\mathscr{S}, s)$. The above diagram then says $\partial[\zeta'] \in \pi_0(\mathscr{X}_s, x)$ provides provides a lift of $[z] \in \pi_0(\mathscr{Y}_t, y)$ along $\pi_0 f_s$. So we see that $\pi_0 f_s : \pi_0(\mathscr{X}_s) \to \pi_0(\mathscr{Y}_t)$ is in fact surjective.

As for injectivity, we suppose $\pi_0 f_s : \pi_0(\mathscr{X}_s) \to \pi_0(\mathscr{Y}_t)$ sends two points $[x], [x'] \in \pi_0(\mathscr{X}_s)$ to the same element $[y] \in \pi_0(\mathscr{Y}_t)$, then fix $x : * \to \mathscr{X}_s$ as our base point. Via the diagram (21) we see that $[x']$ is in the kernel of the map $\pi_0(\mathscr{X}_s, x) \to \pi_0(\mathscr{X}, s)$ and hence has a preimage in $[\gamma] \in \pi_1(\mathscr{S}, s)$. The image $[\gamma'] = \pi_1 q[\gamma] \in \pi_q(\mathscr{T}, t)$ is in the kernel of the connecting morphism, and so lifts to an element $[\widetilde{\gamma}'] \in \pi_1(\mathscr{Y}, y)$. This element has a unique preimage $[\widetilde{\gamma}] \in \pi_1(\mathscr{X}, x)$ which necessarily maps to $[\gamma] \in \pi_1(\mathscr{S}, s)$, via the above diagram. So

$$[\gamma] \in \ker(\pi_1(\mathscr{S}, s) \to \pi_0(\mathscr{X}_s, x)),$$

and hence has image $[x] \in \pi_0(\mathscr{X}_s, x)$. But we chose $[\gamma]$ as a lift of $[x'] \in \pi_0(\mathscr{X}_s, x)$! We therefore have $[x] = [x']$, and conclude that $\pi_0 f_s$ is injective. This establishes bijectivity of $\pi_0 f_s$,

$$\pi_0 f_s : \pi_0(\mathscr{X}_s) \overset{\cong}{\to} \pi_0(\mathscr{Y}_t),$$

completing our proof.

(b) $\Rightarrow$ (a) This follows by a similar analysis of the long exact sequence(s) for homotopy groups, as in (21), and an application of Whitehead's theorem. $\square$

sect:kan_v_space

4.13. **Kan complexes as spaces.** Let us conclude this (long) section with a justification for the claim that Kan complexes "are" spaces.

We consider the category $\mathrm{Top} = \mathrm{Top}_{cgw}$ of compactly generated weak Hausdorff spaces. This is a certain full subcategory in the ambient category of all topological spaces which includes all locally compact Hausdorff spaces and all CW-complexes and is closed under taking limits [19, Proposition 2.30].[2] In particular, the geometric realization functor can be considered as a functor to the category of compactly generated weak Hausdorff spaces

$$|-| : \mathrm{sSet} \to \mathrm{Top}\,.$$

Now, the category Top admits a model structure for which the weak equivalences are those maps $f : X \to Y$ which induce isomorphisms on connected components and on all higher homotopy groups, $\pi_n f : \pi_n(X, x) \overset{\sim}{\to} \pi_n(Y, y)$. Under this model structure CW complexes are both fibrant and cofibrant, and the unit of the geometric realization/singular complex adjunction provides a weak equivalence

$$|\operatorname{Sing}(X)| \xrightarrow{\sim_{\mathrm{w}}} X.$$

[2]This category is cocomplete, i.e. has all small colimits, but the colimit in Top does not agree with the one in the ambient category of arbitrary topological spaces in general [19].

In particular, when X is (homotopy equivalent to) a CW complex the above map is a homotopy equivalence. This is the original version of Whitehead's theorem.

Similarly, there is a model structure on the category of simplicial sets for which the Kan complexes are the fibrant and cofibrant objects. The weak equivalences here are those maps which become weak equivalences in Top under geometric realization. When we restrict to Kan complexes, weak equivalences are simply homotopy equivalences, and the counit of the geometric realization/singular complex adjunction provides homotopy equivalences

$$\mathscr{X} \overset{\sim}{\to} \mathrm{Sing}(|\mathscr{X}|)$$

whenever $\mathscr{X}$ is a Kan complex [9, Proposition 4.6.2]. So we observe the following.

Theorem 4.39. *Geometric realization induces an equivalence on the level of homotopy categories*

$$|-| : \mathrm{hKan} \overset{\sim}{\to} \mathrm{hTop}\,. \tag{22}$$ eq:1499

In a more sophisticated statement of things, one considers the obvious simplicial structure $\underline{\mathrm{Top}}$ on the category Top [15, 00JV], and geometric realization extends to a simplicial functor

$$|-|^{\mathrm{simp}} : \underline{\mathrm{Kan}} \to \underline{\mathrm{Top}}.$$

This functor has image in the full simplicial subcategory $\underline{\mathrm{Top}}_{\mathrm{cw}}$ of spaces which are homotopic to a CW complex. If we take $\mathscr{T}op_{\mathrm{cw}}$ to be the homotopy coherent nerve of the simplicial category $\underline{\mathrm{Top}}_{\mathrm{cw}}$, and recall our notation $\mathscr{S}paces = \mathrm{N}^{\mathrm{hc}}(\underline{\mathrm{Kan}})$, then the induced map on ∞-categories

$$|-|^{\mathrm{hc}} : \mathscr{S}paces \to \mathscr{T}op_{\mathrm{cw}}. \tag{23}$$ eq:1509

This map of ∞-categories is in fact an equivalence (once we decide what this even means) which lifts the equivalence of homotopy categories presented above [15, 01Z4]. This is the sense in which the category of Kan complexes becomes a category of "spaces".

Remark 4.40. If we are more careful about set theoretic things, we should include some smallness assumptions in the above discussion. However, this smallness condition will move with our universe. So the expression (23) should be interpreted as the statement that we have a tower of equivalences which are indexed by the linearly ordered collection on universes.

sect:infty_kan

4.14. **The ∞-category of spaces.** We have already seen that the functor categories $\mathrm{Fun}(\mathscr{X}, \mathscr{Y})$ are Kan complexes whenever $\mathscr{X}$ and $\mathscr{Y}$ are Kan complexes (Corollary 4.12). Hence, for any universe $\mathbb{U}$, the simplicial category $\underline{\mathrm{Kan}}^{\mathbb{U}}$ of $\mathbb{U}$-small Kan complexes is enriched in Kan complexes. We therefore apply Proposition 3.16 to obtain the following.

Theorem 4.41. *The homotopy coherent nerve of the simplicial category of $\mathbb{U}$-small Kan complexes $\mathrm{N}^{\mathrm{hc}}(\underline{\mathrm{Kan}}^{\mathbb{U}})$ is an ∞-category.*

Definition 4.42. For any given universe $\mathbb{U}$, we take

$$\mathscr{K}an^{\mathbb{U}} := \mathrm{N}^{\mathrm{hc}}(\underline{\mathrm{Kan}}^{\mathbb{U}}).$$

We call this category the ∞-category of $\mathbb{U}$-small spaces, or $\mathbb{U}$-small spaces by an abuse of language.

When the explicit distinction of a universe is unnecessary we take

$$\mathscr{K}an := \text{The } \infty\text{-category } \mathscr{K}an^{\mathbb{U}} \text{ for some large unspecified } \mathbb{U}.$$

We take $\mathscr{K}an^{\mathrm{sm}}$ to be the ∞-category of spaces which are small for our predetermined universe of small objects.

sect:infty_cats

5. Basics for ∞-categories

5.1. **Inner fibrations and inner anodyne maps.** One should recall the notion of a saturated class of morphisms from Section 4.2.

Definition 5.1. A map of simplicial sets $i : A \to B$ is called inner anodyne if it belongs to the saturated class generated by the inclusions of inner horns $\{\Lambda_i^n \to \Delta^n : n > 0 \text{ and } 0 < i < n\}$. A map of simplicial sets $f : \mathscr{C} \to S$ is called an inner fibration if any lifting problem

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \mathscr{C} \arrow[d, "f"] \\ B \arrow[r] & S, \end{tikzcd}$$

in which i is inner anodyne, admits a solution.

As in the proof of Lemma 4.2, one sees that $\mathscr{C} \to S$ is an inner fibration if and only if one can lift morphisms morphisms to S along any inclusion of an inner horn $\Lambda_i^n \to \Delta^n$. Note also that $\mathscr{C}$ is an ∞-category if and only if the map $\mathscr{C} \to *$ is an inner fibration. Of course, the term "inner fibration" suggests an augmentation of the notion of a Kan fibration, where one replaces the role of arbitrary horn inclusions $\Lambda_i^n \to \Delta^n$ with inner horns.

We adopt the following definition directly from [15].

def:subcat

Definition 5.2. A simplicial subset $\mathscr{A} \subseteq \mathscr{C}$ in an ∞-category $\mathscr{C}$ is called an ∞-subcategory, or just a subcategory, if the inclusion map $\mathscr{A} \to \mathscr{C}$ is an inner fibration.

Note that and ∞-subcategory in a given ∞-category is an ∞-category in its own right.

Remark 5.3. Definition 5.2 is clearly somewhat restrictive. For example, if we include $\mathbb{S}^1$ into $\mathbb{S}^2$ along the equator, then the induced map $\mathrm{Sing}(\mathbb{S}^1) \to \mathrm{Sing}(\mathbb{S}^2)$ does not realize the singular complex for $\mathbb{S}^1$ as an ∞-subcategory in that of $\mathbb{S}^2$. It may be the case that the subcategory generated by the image of $\mathrm{Sing}(\mathbb{S}^1)$ in $\mathrm{Sing}(\mathbb{S}^2)$ may be equivalent to $\mathrm{Sing}(\mathbb{S}^1)$ however. The point is not clear. (See [15, 01CN].)

The following lemma is straightforward.

Lemma 5.4. *Suppose that we have a pullback diagram*

$$\begin{tikzcd} \mathscr{C}' \arrow[r] \arrow[d, "f'"'] & \mathscr{C} \arrow[d, "f"] \\ S' \arrow[r] & S \end{tikzcd}$$

in which $f : \mathscr{C} \to S$ is an inner fibration. Then f' is an inner fibration as well.

Let us say that an ∞-category $\mathscr{D}$ admits unique horn fillings if, for any inner horn $\bar{s} : \Lambda^n_i \to \mathscr{D}$, there is a *unique* n-simplex $s : \Delta^n \to \mathscr{D}$ with $s|\Lambda^n_i = \bar{s}$. An ∞-category $\mathscr{D}$ admits unique horn fillings if and only if $\mathscr{D}$ is isomorphic to the nerve of a plain category [13, Proposition 1.1.2.2].

lem:1556

Lemma 5.5. *Suppose that $\mathscr{C}$ and $\mathscr{D}$ are ∞-categories, and that $f : \mathscr{C} \to \mathscr{D}$ is a map of simplicial sets. If $\mathscr{D}$ admits unique horn fillings, then f is an inner fibration.*

Proof. Consider a diagram

$$\begin{tikzcd}
\Lambda^n_i \arrow[r, "\bar{s}"] \arrow[d] & \mathscr{C} \arrow[d, "f"] \arrow[ddr] & \\
\Delta^n \arrow[r, "t"] \arrow[drr, "pt"'] & \mathscr{D} \arrow[dr] & \\
& & *
\end{tikzcd}$$

and any lift $s : \Delta^n \to \mathscr{C}$ of pt. Then $fs : \Delta^n \to \mathscr{D}$ is a map to $\mathscr{D}$ with $fs|\Lambda^n_i = f\bar{s} = t|\Lambda^n_i$. Since $\mathscr{D}$ admits unique horn fillings, we have $fs = \bar{s}$. $\square$

As a corollary one sees that, for any ∞-category $\mathscr{C}$, the truncation $p : \mathscr{C} \to \mathrm{N}(\mathrm{h}\mathscr{C})$ is an inner fibration. One also sees that, for any ∞-category $\mathscr{D}$ with unique horn fillings, and any map $f : \mathscr{C} \to \mathscr{D}$ from an arbitrary simplicial set $\mathscr{C}$, we have that $\mathscr{C}$ is an ∞-category if and only if f is an inner fibration.

We have the following technical lemma, which is an inner variant of Lemma 4.7.

lem:1575

Lemma 5.6. *Consider two monomorphisms of simplicial sets $i : A \to B$ and $j : K \to L$. If either of i or j is an inner anodyne morphism, then the induced map*

$$(A \times L) \amalg_{(A\times K)} (B \times K) \to (B \times L)$$

is inner anodyne as well.

The proof can be found in [15, 00JB].

5.2. **Exponentials of ∞-categories.**

prop:tech2

Proposition 5.7. *Suppose that $i : K \to L$ is a monomorphism of simplicial sets, and that $f : \mathscr{C} \to S$ is an inner Kan fibration. Then the induced map on functors*

$$\mathrm{Fun}(L, \mathscr{C}) \to \mathrm{Fun}(K, \mathscr{C}) \times_{\mathrm{Fun}(K,S)} \mathrm{Fun}(L, S) \qquad (24)$$ eq:1597

is an inner fibrations. Furthermore, if $i : K \to L$ is inner anodyne, then the map (24) is a trivial Kan fibration.

Proof. One replaces Lemma 4.7 with Lemma 5.6 and proceeds exactly as in the proof of Proposition 4.11. $\square$

If one considers the inclusion $\emptyset \to K$ and, when $\mathscr{C}$ is an ∞-category, the inner fibration $\mathscr{C} \to *$, then one arrives at the following.

cor:Fun_infty

Corollary 5.8. (1) *For any simplicial set K, and ∞-category $\mathscr{C}$, the simplicial set $\mathrm{Fun}(K, \mathscr{C})$ is an ∞-category.*

(2) *For any simplicial set K, and inner fibration $f : \mathscr{C} \to S$, the morphism $f_* : \mathrm{Fun}(K, \mathscr{C}) \to \mathrm{Fun}(K, S)$ is an inner fibration.*

(3) *For any monomorphism $i : K \to L$, and ∞-category $\mathscr{C}$, the map of ∞-categories $i^* : \mathrm{Fun}(L, \mathscr{C}) \to \mathrm{Fun}(K, \mathscr{C})$ is an inner fibration.*

(4) *For any inner anodyne morphism $i : K \to L$, and ∞-category $\mathscr{C}$, the map of ∞-categories $i^* : \mathrm{Fun}(L, \mathscr{C}) \to \mathrm{Fun}(K, \mathscr{C})$ is a trivial Kan fibration.*

5.3. **Opposite categories.** For any linearly ordered set J define J^{rev} to be the set J with the reversed ordering,

$$j < j' \quad \Leftrightarrow \quad j >_{\mathrm{rev}} j'.$$

Note that for any linearly ordered sets I and J, and set map $r : I \to J$, we have that r is weakly increasing as a function from I to J if and only if r is weakly increasing when considered as a function from I^{rev} to J^{rev}. Hence the reversing operation defines an autoequivalence

$$-^{\mathrm{rev}} : \Delta \overset{\cong}{\to} \Delta.$$

Indeed, $-^{\mathrm{rev}}$ is an involution on Δ.

Clearly any linearly ordered set J admits a unique isomorphism $J \cong J^{\mathrm{rev}}$ in Δ^{op}. This isomorphism is, however, *not* natural. One can interpret this failure of naturality as an assurance that the reversal operation has some non-trivial content.

We define the opposite K^{op} of a given simplicial set K to be the simplicial set

$$K^{op} := K \circ (-^{\mathrm{rev}}).$$

So, the n-simplices $K^{op}[n]$ in K^{op} are identified with the set $K((\Delta^n)^{\mathrm{rev}})$.

Example 5.9. Let C be a plain category. Then $\mathrm{N}(C)^{op} = N(C^{op})$. As a more explicit example, one takes opposites of diagrams

$$\begin{tikzcd} & \cdot \arrow[dr] & & & & \cdot \\ \cdot \arrow[rr] & & \cdot & \overset{op}{\longmapsto} & \cdot & & \cdot \arrow[ul] \arrow[ll] \arrow[dl] \\ & \cdot \arrow[ur] & & & & \cdot \end{tikzcd}$$

Clearly a natural transformation of simplicial sets $f : K \to L$ induces a transformation $f^{op} : K^{op} \to K^{op}$, where $f^{op}_J = f_{J^{\mathrm{rev}}} : K^{op}(J) = K(J^{\mathrm{rev}}) \to L(J^{\mathrm{rev}}) = L^{op}(J)$. So the opposite operation defines an involution on sSet.

Lemma 5.10. *The opposite $\mathscr{X}^{op}$ of a Kan complex $\mathscr{X}$ is another Kan complex. The opposite $\mathscr{C}^{op}$ of an ∞-category is another ∞-category.*

Proof. This follows from the fact that the opposite of a horn inclusion $j : \Lambda^n_i \to \Delta^n$ is identified with the horn inclusion $\Lambda^n_{n-i} \to \Delta^n$, so that a lifting problem

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & \mathscr{C}^{op} \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & * \end{tikzcd}$$

admits a solution if and only if the corresponding lifting problem on opposite categories

$$\begin{tikzcd} \Lambda^n_{n-i} \arrow[r] \arrow[d] & \mathscr{C} \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & * \end{tikzcd}$$

admits a solution. $\square$

5.4. **Isomorphisms in ∞-categories.**

Definition 5.11. A morphism $f : x \to y$ in an ∞-category $\mathscr{C}$ is called an isomorphism if there exists another morphism $g : y \to x$ for which one has (not necessarily unique) 2-simplices

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "id_x"'] & & x \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & x \arrow[dr, "f"] & \\ y \arrow[ur, "g"] \arrow[rr, "id_x"'] & & y. \end{tikzcd}$$

Equivalently, f is an isomorphism in $\mathscr{C}$ if its class $\bar{f}$ in the homotopy category $\mathrm{h}\mathscr{C}$ is an isomorphism.

Remark 5.12. In [13], such a map f is simply referred to as an equivalence, while in [15] the term isomorphism is explicitly used. In familiar settings, one might use the term isomorphism to refer to a map in some category C which is literally invertible, while referring to some weaker relations as equivalences. However, one should note that in the generic ∞-categorical setting there is not stronger notion of equivalence than the one given above. This is because all 2-simplices are formally indistinguishable in $\mathscr{C}$. This is to say, there is no preferred method for "inverting" a given map f. Also, a map in the nerve $\mathrm{N}(\mathsf{C})$ of a plain category is an isomorphism if and only if the corresponding map in C is an isomorphism.

Example 5.13. A morphism $\alpha : x \to y$ in the homotopy ∞-category $\mathscr{K}(A)$ of dg modules, over some dg algebra A, is an isomorphism if and only if α is a homotopy equivalence.

`def:assoc_kan` **Definition 5.14.** Let $\mathscr{C}$ be an ∞-category. The Kan complex associated to $\mathscr{C}$ is the simplicial subset $\mathscr{C}^{\mathrm{Kan}}$ whose n-simplices $\Delta^n \to \mathscr{C}^{\mathrm{Kan}}$ consist of all n-simplices in $\mathscr{C}$, $s : \Delta^n \to \mathscr{C}$, which restrict to an isomorphism $s|\Delta^1 : \Delta^1 \to \mathscr{C}$ along each inclusion $\Delta^1 \to \Delta^n$.

Since there are no inclusions of Δ^1 into Δ^0, $\mathscr{C}^{\mathrm{Kan}}$ has the same objects as $\mathscr{C}$, and the morphisms in $\mathscr{C}^{\mathrm{Kan}}$ are precisely the isomorphisms in $\mathscr{C}$. Clearly $\mathscr{C}^{\mathrm{Kan}}$ is the largest simplicial subset in $\mathscr{C}$ with this property.

Lemma 5.15. *For any ∞-category $\mathscr{C}$, the associated Kan complex $\mathscr{C}^{K}$ is an ∞-category, and the inclusion $\mathscr{C}^{\mathrm{Kan}} \to \mathscr{C}$ is an inner fibration.*

Proof. It suffices to prove that the inclusion $i : \mathscr{C}^{\mathrm{Kan}} \to \mathscr{C}$ is an inner fibration. Consider a lifting problem

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & \mathscr{C}^{\mathrm{Kan}} \arrow[d, "i"] \\ \Delta^n \arrow[r, "s"'] \arrow[ur, dashed] & \mathscr{C} \end{tikzcd}$$

with $0 < i < n$. When $n = 2$, we simply note that isomorphisms in the homotopy category $\mathrm{h}\mathscr{C}$ have the 2-of-3 property. So if $s : \Delta^2 \to \mathscr{C}$ is a 2-simplex whose restriction to a horn $\Lambda^2_i \to \Delta^2 \to \mathscr{C}$ factors $\Lambda^n_i \to \mathscr{C}^{\mathrm{Kan}} \to \mathscr{C}$, then s itself factors through $\mathscr{C}^{\mathrm{Kan}}$. For all higher simplices, $n > 2$, we note that all 1-simplices $\Delta^1 \to \Delta^n$ all factor uniquely through the horn $\Lambda^n_i \to \Delta^n$ so see that the simplex

s is actually a simplex in $\mathscr{C}^{\mathrm{Kan}}$. We therefore have a unique solution to the given lifting problem. □

One of the aims of this section is to prove that $\mathscr{C}^{\mathrm{Kan}}$ is not only an ∞-category, but a Kan complex. In order to prove this result we need to develop a number of notions which are quite important in their own right. We discuss overcategories, undercategories, and isofibrations then return to the topic of the associated Kan complex in Section 5.9.

Remark 5.16. Lurie refers to the subcategory $\mathscr{C}^{\mathrm{Kan}}$ as the *core* of $\mathscr{C}$, and denotes this subcategory $\mathscr{C}^{\simeq}$.

5.5. **Joins of simplicial sets.** For a linearly ordered set I let's take $P_{\pm}(I)$ to be the collection of all partitions $I = I_- \amalg I_+$ such that $i < j$ for each $i \in I_-$ and $j \in I_+$. Note that $P_{\pm}(I)$ is identified with the collection of weakly increasing maps $I \to [1]$, $P_{\pm}(I) = \mathrm{Hom}_{\Delta}(I, [2])$.

Definition 5.17. Let A and B be simplicial sets. The join of A and B, denoted $A \star B$, is the simplicial set with I-simplices

$$(A \star B)(I) := \coprod_{(I_-, I_+) \in P_{\pm}(I)} A(I_-) \times B(I_+),$$

where one takes formally $A(\emptyset) = B(\emptyset) = *$. Given any weakly decreasing map $r : I \to J$, restriction

$$r^* : (A \star B)(J) \to (A \star B)(I)$$

sends a pair $(s, t) \in A(J_-) \times B(J_+)$ to $(r^*s, r^*t) \in A(r^{-1}J_-) \times B(r^{-1}J_+)$.

One should note that we have canonical inclusions $A \to A \star B$ and $B \to A \star B$. We note also that the join forms a bifunctor

$$\star : \mathrm{sSet} \times \mathrm{sSet} \to \mathrm{sSet}\,.$$

Given two maps $f : A \to A'$ and $g : B \to B'$ the join $f \star g : A \star B \to A' \star B'$ sends each pair of simplices (s, t) in $A \star B$ to the corresponding pair (fs, gt) in $A' \star B'$. In terms of bifunctoriality, the two inclusions of the original simplicial sets into $A \star B$ are deduced as the composites

$$A \cong A \star \emptyset \to A \star B \quad \text{and} \quad B \cong \emptyset \star B \to A \star B,$$

where the maps $\emptyset \star B \to A \star B$ and $A \star \emptyset \to A \star B$ are the joins of the unique morphisms from the empty set with the respective identities.

ex:delta_join

Example 5.18. There are isomorphisms $\Delta^0 \star \Delta^n \cong \Delta^{n+1}$ and $\Delta^n \star \Delta^0 \cong \Delta^{n+1}$. In the first case we take a pair of maps $s_- : I_- \to \{0\}$ and $s_+ : I_+ \to [n]$ to the unique map $s : I \to [n+1]$ with $s|I_- = s_-$ and $s(i) = s_+(i-1)$ for all $i \in I_+$. The second isomorphism is defined similarly. Indeed, we can argue as above to obtain an isomorphism

$$\Delta^m \star \Delta^n \overset{\cong}{\to} \Delta^{m+n+1}$$

We note that, since Δ^{m+n+1} admits no nontrivial automorphism, the above isomorphism is *unique*.

If we take $\Delta^{-1} = \emptyset$ we can describe n-simplices in the join $s : \Delta^n \to A \star B$ as a choice of pair of integers $m, m' \geq -1$ such that $m + m' = n - 1$, and a choice of simplices $s_- : \Delta^m \to A$ and $s_+ : \Delta^{m'} \to B$. We reconstruct the original simplex as the composite of the unique isomorphism $\Delta^n \cong \Delta^m \star \Delta^{m'}$ with the join $s_- \star s_+ : \Delta^m \star \Delta^{m'} \to A \star B$,

$$s = \big(\ \Delta^n \cong \Delta^m \star \Delta^{m'} \xrightarrow{s_- \star s_+} A \star B\ \big).$$

One finds s_- and s_+ as the maximal initial and terminal faces in Δ^n which have images in the subcomplexes $A \subseteq A \star B$ and $B \subseteq A \star B$ respectively.

prop:1713 **Proposition 5.19.** *Suppose that $f : \mathscr{C} \to S$ and $g : \mathscr{D} \to T$ are inner fibrations. Then the join*

$$f \star g : \mathscr{C} \star \mathscr{D} \to S \star T$$

is another inner fibration.

Proof. The important point to keep in mind here is that an l-simplex $t : \Delta^l \to \mathscr{C} \star \mathscr{D}$ has image in $\mathscr{C}$ if and only if its composite $\Delta^l \to \mathscr{C} \star \mathscr{D}$ along $f \star g$ has image in S. Similarly, t has image in $\mathscr{D}$ if and only if its composite along $f \star g$ has image in T. So, one should keep this basic point in mind.

Consider a lifting problem

$$\begin{tikzcd} \Lambda^n_i \arrow[r, "\bar{s}"] \arrow[d] & \mathscr{C} \star \mathscr{D} \arrow[d, "f \star g"] \\ \Delta^n \arrow[r, "s"'] \arrow[ur, dashed] & S \star T. \end{tikzcd}$$

where $\Lambda^n_i \to \Delta^n$ is an inner horn. As was discussed above, we can decompose s uniquely into the join of simplices $s_- : \Delta^m \to S$ and $s_+ : \Delta^{m'} \to T$, where $m, m' \geq -1$ and $m + m' = n - 1$. If m or m' is less that 0, then s factors through S and $\bar{s}$ factors through $\mathscr{C}$, or s factors through T and $\bar{s}$ factors through $\mathscr{D}$. In either case, one can resolve this lifting problem. So let us suppose $m, m' \geq 0$.

We have the faces $\Delta^m \to \Delta^m \star \Delta^{m'}$ and $\Delta^{m'} \to \Delta^m \star \Delta^{m'}$ and, by the fact that our horn is inner, both of these faces factor through the horn

$$\Delta^m,\ \Delta^n \to \Lambda^n_i \to \Delta^m \star \Delta^{m'}.$$

We therefore have unique lifts $s'_- : \Delta^m \to \mathscr{C} \star \mathscr{D}$ and $s'_+ : \Delta^{m'} \to \mathscr{C} \star \mathscr{D}$. Since the images of s'_- and s'_+ in $S \star T$ lie in the simplicial subsets S and T respectively, we see that s'_- and s'_+ themselves factor uniquely through $\mathscr{C}$ and $\mathscr{D}$ respectively. So we consider the n-simplex

$$s' = s'_- \star s'_+ : \Delta^n \to \mathscr{C} \star \mathscr{D}.$$

We have $(f \star g)s' = (fs'_-) \star (gs'_+) = s_- \star s_+ = s$. We claim that $s'|\Lambda^n_i = \bar{s}$, and hence that s' solves our lifting problem.

To establish the equality $s'|\Lambda^n_i = \bar{s}$ it suffices to establish equalities $s'|\Delta^{n-1} = \bar{s}|\Delta^{n-1}$ for all of the $(n-1)$-faces $\Delta^{n-1} \to \Lambda^n_i$. But here we can factor each composite $s'|\Delta^{n-1} : \Delta^{n-1} \to \mathscr{C} \star \mathscr{D}$ as

$$\Delta^{n-1} = \Delta^l \star \Delta^{l'} \xrightarrow{i \star i'} \Delta^m \star \Delta^{m'} \xrightarrow{s'_- \star s'_+} \mathscr{C} \star \mathscr{D},$$

where $i : \Delta^l \to \Delta^m$ and $i' : \Delta^{l'} \to \Delta^{m'}$ are face inclusions. By construction $s'|\Delta^l = \bar{s}|\Delta^l$ and $s'|\Delta^{l'} = \bar{s}|\Delta^{l'}$. So $\bar{s}$ maps Δ^l into $\mathscr{C}$, and maps $\Delta^{l'}$ into $\mathscr{D}$, and we therefore conclude

$$\bar{s}|\Delta^{n-1} = (\bar{s}|\Delta^l) \star (\bar{s}|\Delta^{l'}) = (s'_-|\Delta^l) \star (s'_+|\Delta^{l'}) = s'|\Delta^{n-1},$$

as required. So we see $s'|\Lambda^n_i = \bar{s}$, and hence that the original lifting problem admits a solution. $\square$

Corollary 5.20. *If $\mathscr{C}$ and $\mathscr{D}$ are ∞-categories, then the join $\mathscr{C} \star \mathscr{D}$ is an ∞-category.*

Proof. The map $\mathscr{C} \star \mathscr{D} \to \Delta^0 \star \Delta^0 = \Delta^1$ is an inner fibration by Proposition 5.19. Since $\Delta^1 = \mathrm{N}(\{0 < 1\})$ has admits unique horn fillings, Lemma 5.5 now implies that $\mathscr{C} \star \mathscr{D}$ is an ∞-category. $\square$

5.6. **Overcategories and undercategories.** Let $p : K \to \mathscr{C}$ be a map from a simplicial set K to another simplicial set $\mathscr{C}$. We will generally refer to p as a "diagram" in $\mathscr{C}$. (The case where $\mathscr{C}$ is an ∞-category is most important for us, but it well be helpful to just consider simplicial sets for now.)

We define the overcategory $\mathscr{C}/p$ to be the simplicial set with simplices

$$\mathscr{C}_{/p}[n] = \mathrm{Hom}_{\mathrm{sSet}}(\Delta^n \star K, \mathscr{C})_p,$$

where the subscript p indicates that we consider maps of simplicial sets $p' : \Delta^n \star K \to \mathscr{C}$ for which $p'|K = p$. Restriction in $\mathscr{C}_{/p}$ along some weakly increasing function $[m] \to [n]$ is defined by restricting along the corresponding map of simplicial sets $\Delta^m \to \Delta^n$. We similarly define the undercategory $\mathscr{C}_{p/}$ as the simplicial set with simplices

$$\mathscr{C}_{p/} = \mathrm{Hom}_{\mathrm{sSet}}(K \star \Delta^n, \mathscr{C})_p.$$

By construction there are identifications

$$\mathrm{Hom}_{\mathrm{sSet}}(\Delta^n, \mathscr{C}_{/p}) \overset{\cong}{\to} \mathrm{Hom}_{\mathrm{sSet}}(\Delta^n \star K, \mathscr{C})_p, \quad f \mapsto f(id_{[n]}) \tag{25}$$ eq:1784

and

$$\mathrm{Hom}_{\mathrm{sSet}}(\Delta^n, \mathscr{C}_{p/}) \overset{\cong}{\to} \mathrm{Hom}_{\mathrm{sSet}}(K \star \Delta^n, \mathscr{C})_p, \quad g \mapsto g(id_{[n]}). \tag{26}$$ eq:1788

Since the join functor is seen to commute with colimits the above isomorphism extend to adjunctions which are natural in all factors.

Lemma 5.21 ([15, 017Z])**.** *For any diagram $p : K \to \mathscr{C}$, which can similarly be considered as a diagram to the opposite category $p : K^{op} \to \mathscr{C}^{op}$, there is an identification of simplicial sets $(\mathscr{C}_{/p})^{op} \cong (\mathscr{C}^{op})_{p/}$.*

Proof. Follows from the identification $(\Delta^n \star K)^{op} = K^{op} \star (\Delta^n)^{op} \cong K^{op} \star \Delta^n$. $\square$

lem:1798 **Lemma 5.22** ([15, 0189])**.** *Consider an arbitrary diagram $p : K \to \mathscr{C}$ in a simplicial set $\mathscr{C}$. For any simplicial set L, there are unique isomorphisms*

$$\mathrm{Hom}_{\mathrm{sSet}}(L, \mathscr{C}_{/p}) \overset{\cong}{\to} \mathrm{Hom}_{\mathrm{sSet}}(L \star K, \mathscr{C})_p \tag{27}$$ eq:1795

and

$$\mathrm{Hom}_{\mathrm{sSet}}(L, \mathscr{C}_{p/}) \overset{\cong}{\to} \mathrm{Hom}_{\mathrm{sSet}}(K \star L, \mathscr{C})_p$$

which are natural in L and recover the isomorphisms (25, 26) when $L = \Delta^n$.

Construction. We construct the isomorphism for $\mathscr{C}_{/p}$. The construction for $\mathscr{C}_{p/}$ is similar, or can be recovered by replacing $\mathscr{C}$ with its opposite category. First we consider the map

$$ev : \mathscr{C}_{/p} \star K \to \mathscr{C}, \quad (s : \Delta^l \to \mathscr{C}_{/p},\ t : \Delta^m \to K) \mapsto s(id_{[l]}, t).$$

(Here we've abused notation to identify the function $s : \Delta^l \to \mathscr{C}_{/p}$ with its value at $id_{[l]}$.) Then the isomorphism (27) sends a function $f : L \to \mathscr{C}_{/p}$ to the composite

$ev \circ (f \star id_K)$. The inverse takes a map $\eta : L \star K \to \mathscr{C}$ with appropriate restriction to the map

$$\eta' : L \to \mathscr{C}_{/p}, \quad (s : \Delta^n \to L) \mapsto \eta \circ (s \star id_K).$$

□

We note that the overcategory/undercategory construction does enjoy some functoriality. For example, if we have a map of simplicial sets $i : A \to K$ and an arbitrary diagram $p : K \to \mathscr{C}$, then restricting along i defines maps

$$i^*_{\text{over}} : \mathscr{C}_{/p} \to \mathscr{C}_{/pi} \quad \text{and} \quad i^*_{\text{under}} : \mathscr{C}_{p/} \to \mathscr{C}_{pi/}.$$

Similarly, if we have a map of simplicial sets $f : \mathscr{C} \to \mathscr{D}$ then composing with f defines a maps

$$f^{\text{over}}_* : \mathscr{C}_{/p} \to \mathscr{D}_{/fp} \quad \text{and} \quad f^{\text{under}}_* : \mathscr{C}_{p/} \to \mathscr{D}_{fp/}.$$

If we consider such induced morphisms, the isomorphism of Lemma 5.22 will be natural in the K and $\mathscr{C}$ variables as well.

5.7. Directional fibrations and under/overcategories of ∞-categories.

def:lr_anodyne

Definition 5.23. The class of left (resp. right) anodyne maps $i : A \to B$ is the saturated class of morphisms in sSet generated by the collection of horn inclusions $\Lambda^n_i \to \Delta^n$, where n is arbitrary and $0 \leq i < n$ (resp. $0 < i \leq n$).

A map of simplicial sets $f : \mathscr{C} \to S$ is called a left fibration (resp. right fibration) if any lifting problem

$$\begin{array}{ccc} A & \longrightarrow & \mathscr{C} \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\ B & \longrightarrow & S \end{array}$$

in which i is left anodyne (resp. right anodyne) admits a solution.

So, it is *easier* to be a right/left anodyne map, and hence *harder* for a map of simplicial sets to be a left/right fibration. We have the join analog of Lemma 4.7, which is slightly more robust.

lem:1806

Lemma 5.24 ([15, 018J])**.** *For monomorphisms $i : A \to B$ and $j : K \to L$, the corresponding map from the pushout*

$$(A \star L) \coprod_{(A \star K)} (B \star K) \to B \star L$$

is inner anodyne whenever i is right anodyne or j is left anodyne.

One employs Lemma 5.24 and follows the arguments of Proposition 4.11 to obtain the following. We repeat these arguments, since we haven't seen them in a while, and we miss them.

prop:tech3

Proposition 5.25. *Let $f : \mathscr{C} \to S$ be an inner fibration, and $p : K \to \mathscr{C}$ be any diagram in $\mathscr{C}$. Then for any monomorphism $j : K_0 \to K$, and corresponding diagram $\pi = pj : K_0 \to \mathscr{C}$, the induced map on diagram overcategories*

$$\mathscr{C}_{/p} \to \mathscr{C}_{/\pi} \times_{S_{/\pi}} S_{/p}$$

is a right fibration, and the induced map on undercategories

$$\mathscr{C}_{p/} \to \mathscr{C}_{\pi/} \times_{S_{\pi/}} S_{p/}$$

is a left fibration.

Clearly we have abused notation here and written $S_{/p}$ instead of $S_{/pf}$ and $S_{/\pi}$ instead of $S_{/\pi f}$, for example.

Proof. The claim about undercategories follows from the claim about overcategories, after applying the opposite involution on sSet. So, we deal with the overcategories.

Any lifting problem

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & \mathscr{C}_{/p} \arrow[d] \\ B \arrow[r] & \mathscr{C}_{/\pi} \times_{S_{/\pi}} S_{/p} \end{tikzcd}$$

defines a corresponding lifting problem

$$\begin{tikzcd} (A \star K) \coprod_{(A\star K_0)} (B \star K_0) \arrow[r] \arrow[d] & \mathscr{C} \arrow[d, "f"] \\ B \star K \arrow[r] \arrow[ur, dashed] & S \end{tikzcd}$$

via adjunction, where the left vertical map is defined by the maps $i \star id_K$ and $id_B \star j$. If we suppose that $i : A \to B$ is right anodyne, then Lemma 5.24, and the fact that f is an inner fibration, assures us that the second lifting problem admits a solution. Hence the first lifting admits a solution. $\square$

The same argument can be used to establish the following.

prop:tech3.5

Proposition 5.26. *Suppose that we are in the setting of Proposition 5.25. If the inclusion $j : K_0 \to K$ is left anodyne, then the induced map on overcategories*

$$\mathscr{C}_{/p} \to \mathscr{C}_{/\pi} \times_{S_{/\pi}} S_{/p}$$

is a trivial Kan fibration. Similarly, if $j : K_0 \to K$ is right anodyne, the induced map

$$\mathscr{C}_{p/} \to \mathscr{C}_{\pi/} \times_{S_{\pi/}} S_{p/}$$

is a trivial Kan fibration.

If we consider the case $S = *$ and $K_0 = \emptyset$, then we have identifications $\mathscr{C}_{/\pi} = \mathscr{C}$ and $S_{/p} = S_{/\pi} = *$, essentially by the adjunction of Lemma 5.22. Similarly $\mathscr{C}_{\pi/} = \mathscr{C}$ and $S_{p/} = S_{\pi/} = *$ in this case. So we obtain the following.

cor:1947

Corollary 5.27. *Let $\mathscr{C}$ be an ∞-category and $p : K \to \mathscr{C}$ be an arbitrary diagram. Then restricting along the inclusions $\Delta^n \to \Delta^n \star K$ defines a map of simplicial sets*

$$\mathscr{C}_{/p} \to \mathscr{C}.$$

This map is an right fibration. Similarly, restriction defines a map

$$\mathscr{C}_{p/} \to \mathscr{C}$$

which is a left fibration. In particular, both the overcategory $\mathscr{C}_{/p}$ and the undercategory $\mathscr{C}_{p/}$ are ∞-categories.

Let's take a moment to explain clearly what this map $\mathscr{C}_{/p} \to \mathscr{C}$ actually "looks like". An n-simplex in $\mathscr{C}$ can be thought of as a Δ^n-shaped diagram in $\mathscr{C}$. Directly, an n-simplex in $\mathscr{C}$ is identified with a map of simplicial sets $s : \Delta^n \to \mathscr{C}$. To compare, an n-simplex in $\mathscr{C}_{/p}$ is a $\Delta^n \star K$-shaped diagram in $\mathscr{C}$, i.e. a map $s' : \Delta^n \star K \to \mathscr{C}$.

Now, the simplicial set $\Delta^n \star K$ can be visualized as a copy of Δ^n floating in space, a copy of K floating in space, and a bunch of connective simplices from Δ^n to K. So a $\Delta^n \star K$-shaped diagram in $\mathscr{C}$ appears as follows:

$$\text{an } n\text{-simplex in } \mathscr{C}_{/p} \quad = \quad \begin{tikzcd} & x_2 \arrow[dr, dash, dotted] \arrow[ddrr, bend left] & & \\ x_1 \arrow[ur, dash] \arrow[rr, dash, dotted] \arrow[drrr, bend right] & & x_n \arrow[dr] & \\ & & & {\color{red} p(K)}. \end{tikzcd} \quad \subseteq \quad \mathscr{C}$$

The map $\mathscr{C}_{/p} \to \mathscr{C}$ simply keeps the uncolored portion of the above diagram and forgets about the colored portions,

$$\mathscr{C}_{/p} \qquad \to \qquad \mathscr{C}$$

$$\begin{tikzcd} & x_2 \arrow[dr, dash, dotted] \arrow[ddr, bend left] & \\ x_1 \arrow[ur, dash] \arrow[rr, dash, dotted] \arrow[ddr, bend right] & & x_n \arrow[d] \\ & & {\color{red} p(K)} \end{tikzcd} \quad \overset{\text{forget}}{\rightsquigarrow} \quad \begin{tikzcd} & x_2 \arrow[dr, dash, dotted] & \\ x_1 \arrow[ur, dash] \arrow[rr, dash, dotted] & & x_n. \end{tikzcd}$$

Similarly, the map $\mathscr{C}_{/p} \to \mathscr{C}_{/\pi}$ forgets *some* portion of the colored diagram, while remembering others. One can provide a similar describe the map $\mathscr{C}_{p/} \to \mathscr{C}$.

We record a final corollary to Proposition 5.26.

cor:1998

Corollary 5.28. *Let $\mathscr{C}$ be an ∞-category and $p : K \to \mathscr{C}$ be an arbitrary diagram. Adopt the notation from Proposition 5.25. If the inclusion $j : K_0 \to K$ is left anodyne then the induced map*

$$\mathscr{C}_{/p} \to \mathscr{C}_{/\pi}$$

is a trivial Kan fibration. If the inclusion $j : K_0 \to K$ is right anodyne, then the induced map

$$\mathscr{C}_{p/} \to \mathscr{C}_{\pi/}$$

is a trivial Kan fibration.

5.8. **Isofibrations.**

def:isofib

Definition 5.29. A map of ∞-categories $F : \mathscr{C} \to \mathscr{D}$ is called an isofibrations if

(a) F is an inner fibration.

(b) For every object x in $\mathscr{C}$, and isomorphism $\gamma' : y' \to F(x)$ in $\mathscr{D}$, there exists an object y in $\mathscr{C}$ and an isomorphism $\gamma : y \to x$ with $F(\gamma) = \gamma'$.

(b') For every object y in $\mathscr{C}$, and isomorphism $\theta' : F(y) \to x'$ in $\mathscr{D}$, there exists an object x in $\mathscr{C}$ and an isomorphism $\theta : y \to x$ with $F(\theta) = \theta'$.

Colloquially, an isofibration is an inner fibration along which one can lift isomorphisms.

lem:2034 **Lemma 5.30.** *Let $F : \mathscr{C} \to \mathscr{D}$ be an inner fibration between ∞-categories. Then F has satisfies property* (b) *from Definition 5.29 if and only if F satisfies property* (b'). *Rather, these two lifting properties are equivalent for any inner fibration between ∞-categories.*

Proof. Suppose F satisfies property (b), and consider a pairing of an object y in $\mathscr{C}$ and an isomorphism $\theta' : F(y) \to x'$ in $\mathscr{D}$. Take $y' = F(y)$ and consider a homotopy inverse $\zeta' : x' \to y'$ in $\mathscr{D}$. We can lift this inverse to an isomorphism $\zeta : x \to y$ in $\mathscr{C}$, by hypothesis. In particular, we can lift x' to the object x in $\mathscr{C}$. Consider an inverse $\vartheta : y \to x$ to ζ in $\mathscr{C}$, write $\theta'' = F(\vartheta) : y' \to x'$. Note that θ'' provides an inverse to ζ'. By uniqueness of inverses (up to 2-simplices) we have a 2-simplex $s : \Delta^2 \to \mathscr{D}$ of the form

$$\begin{tikzcd} & x' \arrow[dr, "id"] & \\ y' \arrow[ur, "\theta''"] \arrow[rr, "\theta'"'] & & x' \end{tikzcd}$$

of which the inner horn $\Lambda^2_1 \to \Delta^2 \to \mathscr{D}$ lifts to an inner horn in $\mathscr{C}$, $\bar{s} : \Lambda^2_1 \to \mathscr{C}$, which appears as

$$\begin{tikzcd} & x \arrow[dr, "id"] & \\ y \arrow[ur, "\vartheta"] & & x. \end{tikzcd}$$

So we have a lifting problem

$$\begin{tikzcd} \Lambda^2_1 \arrow[r, "\bar{s}"] \arrow[d] & \mathscr{C} \arrow[d, "F"] \\ \Delta^2 \arrow[r, "s"'] \arrow[ur, dashed] & \mathscr{D}. \end{tikzcd}$$

Since F is an inner fibration, by hypothesis, there exists a solution to this lifting problem $s' : \Delta^2 \to \mathscr{C}$,

$$s' = \begin{tikzcd} & x \arrow[dr, "id"] & \\ y \arrow[ur, "\vartheta"] \arrow[rr, "\theta"'] & & x, \end{tikzcd}$$

and the map $\theta := s'|\Delta^{\{1,3\}} : y \to x$ is such that $F(\theta) = \theta : y' \to x'$. By the two-of-three property for isomorphisms in $\mathscr{C}$, we see that $\theta : y \to x$ is in fact an isomorphism, and hence provides the desired lift of θ' to an isomorphism in $\mathscr{C}$. This shows that condition (b) of Definition 5.29 implies condition (b'). The opposite implication follows by applying the finding (b) $\Rightarrow$ (b') to the opposite map $F^{op} : \mathscr{C}^{op} \to \mathscr{D}^{op}$. $\square$

lem:2071 **Lemma 5.31.** *If $\mathscr{C}$ and $\mathscr{D}$ are ∞-categories, and $F : \mathscr{C} \to \mathscr{D}$ is a left or right fibration, then F is an isofibration. Furthermore, if F is a left or right fibration then a given morphism γ in $\mathscr{C}$ is an isomorphism if and only if its image in $\mathscr{D}$ is an isomorphism.*

The proof just follows by fiddling with diagrams. We will give a detailed account in any account.

Proof. Suppose that F is a right fibration, for example. Then we can lift any map $\gamma' : y' \to F(x)$ in $\mathscr{D}$ to a map $\gamma : y \to x$ in $\mathscr{C}$. This just follows by solving the corresponding lifting problem

$$\begin{tikzcd} \Lambda^1_0 \arrow[r] \arrow[d] & \mathscr{C} \arrow[d, "F"] \\ \Delta^1 \arrow[r] \arrow[ur, dashed] & \mathscr{D}. \end{tikzcd}$$

One chooses an inverse ζ' to γ' in $\mathscr{D}$ and lifts the appropriate diagram to find a diagram

$$\begin{tikzcd} & y \arrow[dr, "\gamma"] & \\ x \arrow[ur, dashed, "\exists\zeta"] \arrow[rr, "id"'] & & x \end{tikzcd}$$

in $\mathscr{C}$ to see that γ admits a right inverse ζ. A similar lifting of a 2-simplex implies that ζ admits a left inverse as well, and hence that ζ is an isomorphism in $\mathscr{C}$. We see subsequently that γ is an isomorphism as well. So F is an inner fibration which satisfies condition (b) of Definition 5.29, and by Lemma 5.30 we see that F is an isofibration.

Now suppose $\gamma : x \to y$ is an map in $\mathscr{C}$ with $F(\gamma) : Fx \to Fy$ an isomorphism. By lifting a diagram for the right inverse to $F(\gamma)$ we see that γ admits a right inverse ζ, and we lift to find a diagram

$$\begin{tikzcd} & x \arrow[dr, "\zeta"] & \\ y \arrow[ur, dashed, "\exists"] \arrow[rr, "id"'] & & y \end{tikzcd}$$

which shows that ζ itself admits a right inverse. So ζ is invertible and γ is invertible as well.

The proof in the case that F is a left fibration follows by a similar lifting argument, or by considering opposite categories. $\square$

sect:assoc_kan

5.9. **The Kan complex associated to an ∞-category.** Let us recall the associated Kan complex functor, which associates to any ∞-category $\mathscr{C}$ the ∞-subcategory $\mathscr{C}^{\mathrm{Kan}} \subseteq \mathscr{C}$ which is the maximal ∞-subcategory whose morphisms are the isomorphisms in $\mathscr{C}$. So, an n-simplex $s : \Delta^n \to \mathscr{C}$ lies in $\mathscr{C}^{\mathrm{Kan}}$ if and only if each restriction $s|\Delta^{\{i,j\}} \in \mathscr{C}[1]$ is an isomorphism in $\mathscr{C}$. Since the image of an isomorphism under any map of ∞-categories remains an isomorphism, we simply restrict to observe functoriality of this assignment

$$F : \mathscr{C} \to \mathscr{D} \quad \rightsquigarrow \quad F^{\mathrm{Kan}} : \mathscr{C}^{\mathrm{Kan}} \to \mathscr{D}^{\mathrm{Kan}}.$$

thm:assoc_kan

Theorem 5.32. *For any ∞-category $\mathscr{C}$, the associated Kan complex $\mathscr{C}^{\mathrm{Kan}}$ is in fact a Kan complex.*

This theorem will essentially follow from the next proposition. We provide this proposition, a number of immediate corollaries, then give the proof of Theorem 5.32

prop:2112 **Proposition 5.33.** *Let $F : \mathscr{C} \to \mathscr{D}$ be an inner fibration between ∞-categories. Suppose $n \geq 2$ and consider a lifting problem of the form*

$$\textit{Form A)} \quad \begin{tikzcd} \Lambda^n_n \arrow[r, "s"] \arrow[d] & \mathscr{C} \arrow[d, "F"] \\ \Delta^n \arrow[r, "\sigma"'] & \mathscr{D} \end{tikzcd} \quad \textit{or} \quad \textit{Form B)} \quad \begin{tikzcd} \Lambda^n_0 \arrow[r, "s"] \arrow[d] & \mathscr{C} \arrow[d, "F"] \\ \Delta^n \arrow[r, "\sigma"'] & \mathscr{D}. \end{tikzcd} \tag{28}$$
eq:2105

A lifting problem of Form A admits a solution provided $s|\Delta^{\{n-1,n\}}$ is an isomorphism in $\mathscr{C}$. A lifting problem of Form B admits a solution provided $s|\Delta^{\{0,1\}}$ is an isomorphism in $\mathscr{C}$.

In the case of a diagram of Form A, the proof leverages of overcategories as a means of decomposing the Λ^n_n-shaped and Δ^n-shaped diagrams in $\mathscr{C}$ and $\Delta^n \to \mathscr{D}$. For a diagrams of Form B one employs undercategories in an analogous manner.

Proof. We assume $s|\Delta^{\{n-1,n\}}$ is an isomorphism in $\mathscr{C}$, and consider a lifting problem of Form A. The Form B case follows by considering opposite categories.

We consider the inclusion

$$\Delta^{n-1} = \Delta^{n-2} \star \emptyset \to \Delta^{n-2} \star \Delta^1 = \Delta^n$$

This inclusion factors through the horn Λ^n_n and, in terms of the decomposition $\Delta^n = \Delta^{n-2} \star \Delta^1$, this horn is the union

$$\Lambda^n_n = (\Delta^{n-2} \star \{1\}) \cup (\partial\Delta^{n-2} \star \Delta^1).$$

The external horn $s : \Lambda^n_n \to \mathscr{C}$ restricts along the inclusion $\Delta^{n-2} \to \Lambda^n_n$ to define a simplex $f := s|\Delta^{n-2} : \Delta^{n-2} \to \mathscr{C}$, and we can restrict further to the boundary to get $g = f|\partial\Delta^{n-2} \to \mathscr{C}$,

$$\begin{tikzcd} & \partial\Delta^{n-2} \arrow[dl, "\mathrm{incl}"'] \arrow[dr, "g"] & \\ \Delta^{n-2} \arrow[rr, "f"'] & & \mathscr{C}. \end{tikzcd}$$

We have the various undercategories $\mathscr{C}_{f/}$, $\mathscr{C}_{g/}$, $\mathscr{D}_{f/}$ and $\mathscr{D}_{g/}$, where we have abused notation and written simply f and g for the maps to $\mathscr{D}$ given by composing with F. The functor F induces a map to the fiber product

$$\Theta_F : \mathscr{C}_{f/} \to \mathscr{C}_{g/} \times_{\mathscr{D}_{g/}} \mathscr{D}_{f/}. \tag{29}$$
eq:2142

We will construct the desired solution $s' : \Delta^n \to \mathscr{C}$ to our lifting problem via an analysis of the map (29).

Let us take $\mathscr{E} = \mathscr{C}_{g/} \times_{\mathscr{D}_{g/}} \mathscr{D}_{f/}$, and for convenience identify Δ^1 in the formula $\Delta^{n-2} \star \Delta^1 = \Delta^n$ with $\Delta^{\{-1,1\}}$. Note that $\mathscr{E}$ is an ∞-category, by Corollary 5.27.

The restrictions

$$s|_{\text{subthing}} : \partial\Delta^{n-2} \star \{\pm 1\} \to \mathscr{C} \quad \text{and} \quad \sigma|_{\text{subthing}} : \Delta^{n-2} \star \{\pm 1\} \to \mathscr{D}$$

specify objects $\bar{x}_-$ and $\bar{x}_+$ in $\mathscr{E}$, and the corresponding restrictions

$$s|_{\text{subthing}} : \partial\Delta^{n-2} \star \Delta^1 \to \mathscr{C} \quad \text{and} \quad \sigma : \Delta^{n-2} \star \Delta^1 = \Delta^n \to \mathscr{D}$$

specify a morphism $\bar{\gamma} : \bar{x}_- \to \bar{x}_+$ in $\mathscr{E}$. The restriction

$$s|_{\text{subthing}} : \Delta^{n-2} \star \{1\} \to \mathscr{C}$$

Specifies an object x_+ in $\mathscr{C}_{f/}$ with $\Theta_F(x_+) = \bar{x}_+$.

Now, a lifting of $\bar{\gamma}$ to a(n object x_- with a specified) morphism $\gamma : x_- \to x_+$ in $\mathscr{C}_{f/}$ is the information of a simplex

$$s' : \Delta^{n-2} \star \Delta^1 = \Delta^n \to \mathscr{C}$$

with

$$s'|\Delta^{n-2}\star\{1\} = s|\Delta^{n-2}\star\{1\}, \text{ and } F(s') = \sigma, \text{ and } s'|\partial\Delta^{n-2}\star\Delta^1 = s|\partial\Delta^{n-2}\star\Delta^1.$$

The first and third equalities say precisely that s' restricts s along the inclusion $\Lambda^n_n \to \Delta^n$. So, there exists a solution to the lifting problem

$$\begin{tikzcd} \Delta^0 = \Lambda^1_1 \arrow[r, "x_+"] \arrow[d, "1"'] & \mathscr{C}_{f/} \arrow[d, "\Theta_F"] \\ \Delta^1 \arrow[r, "\bar{\gamma}"'] \arrow[ur, dashed] & \mathscr{E} \end{tikzcd}$$

if and only if there exists a solution to the lifting problem (29). But now, we have already seen that Θ_F is a left fibration, at Proposition 5.25, and hence an isofibration by Lemma 5.31. So such a lifting $\gamma : \Delta^1 \to \mathscr{C}_{f/}$ exists, provided $\bar{\gamma}$ is an isomorphism in $\mathscr{E}$.

To see that $\bar{\gamma}$ is an isomorphism, we consider the functor $w : \mathscr{E} \to \mathscr{C}$ which is obtained by composing the structural projection $p_1 : \mathscr{E} \to \mathscr{C}_{g/}$ with the forgetful functor $\mathscr{C}_{g/} \to \mathscr{C}$. Both of the maps in this composite are left fibrations, so that $w : \mathscr{E} \to \mathscr{C}$ is a left fibration. We have

$$w(\bar{\gamma}) = (s|\partial\Delta^{n-2} \star \Delta^1)|\Delta^1 = s|\Delta^{\{n-1,n\}},$$

which is an isomorphism in $\mathscr{C}$ by hypothesis. It follows that $\bar{\gamma}$ is an isomorphism, by Lemma 5.31. Since Θ_F is an isofibration we can then find our desired lift $\gamma : x_- \to x_+$ in $\mathscr{C}_{f/}$, and hence find a solution to the lifting problem (28). □

cor:2209 **Corollary 5.34.** *If $F : \mathscr{C} \to \mathscr{D}$ is an isofibration between ∞-categories, then the associated map $F^{\mathrm{Kan}} : \mathscr{C}^{\mathrm{Kan}} \to \mathscr{D}^{\mathrm{Kan}}$ is a Kan fibration.*

Proof. Consider a lifting problem

$$\begin{tikzcd} \Delta^n_i \arrow[r] \arrow[d] & \mathscr{C}^{\mathrm{Kan}} \arrow[d, "F^{\mathrm{Kan}}"] \\ \Delta^1 \arrow[r] \arrow[ur, dashed] & \mathscr{D}^{\mathrm{Kan}} \end{tikzcd} \tag{30}$$ eq:2215

with $0 \leq i \leq n$. When $n = 1$ there exists solutions since F is an isofibration. When $n > 1$ we compose with the inclusions to $\mathscr{C}$ and $\mathscr{D}$ and find a solution $s : \Delta^n \to \mathscr{C}$ to the corresponding lifting problem along for the original $F : \mathscr{C} \to \mathscr{D}$. Now, since $n > 1$ all 2-simplices in Δ^n lie in the horn Λ^n_i, and we conclude that all of the maps $s|\Delta^{\{i,j\}} \in \mathscr{C}[1]$ are isomorphisms. So s has image in $\mathscr{C}^{\mathrm{Kan}}$, and thus provides a solution to the lifting problem (30). We are done. □

The proof of Theorem 5.32 is now transparent.

Proof of Theorem 5.32. Follows by Corollary 5.34, where we consider the case $\mathscr{D} = *$. □

We now can see this "Kan complex functor" in the appropriate light.

Definition 5.35. The (associated) Kan complex functor

$$-^{\mathrm{Kan}} : \mathrm{Cat}_\infty \to \mathrm{Kan}$$

is defined by taking any ∞-category $\mathscr{C}$ to its maximal subcategory of isomorphisms $\mathscr{C}^{\mathrm{Kan}}$, and by restriction of functors to these maximal subcategories.

Evidently, any map $\mathscr{X} \to \mathscr{C}$ from a Kan complex to an ∞-category factors through $\mathscr{C}^{\mathrm{Kan}}$. Hence the Kan complex functor provides a right adjoint to the inclusion $\mathrm{incl}_{\mathrm{Kan}} : \mathrm{Kan} \to \mathrm{Cat}_\infty$. If we label everything explicitly, we have a bifunctorial identification

$$\mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathrm{incl}_{\mathrm{Kan}}\,\mathscr{X}, \mathscr{C}) = \mathrm{Hom}_{\mathrm{Kan}}(\mathscr{X}, \mathscr{C}^{\mathrm{Kan}}). \tag{31}$$ eq:2258

sect:infty_equiv

5.10. **Equivalences and the homotopy category of ∞-categories.** Given Theorem 5.32 and Corollary 5.8, we see that for any ∞-category $\mathscr{C}$ and simplicial set K we can consider the mapping space

$$\mathrm{Fun}(K, \mathscr{C})^{\mathrm{Kan}} \subseteq \mathrm{Fun}(K, \mathscr{C}).$$

This is the Kan complex parametrizing maps of simplicial set $f : K \to \mathscr{C}$ with natural isomorphisms. Since $\mathrm{Fun}(K, \mathscr{C})^{\mathrm{Kan}}$ is a Kan complex, and products of Kan complexes are Kan complexes, the adjunction (31) implies that composition for the simplicial category $\underline{\mathrm{Cat}_\infty}$ restricts to provide composition maps

$$\circ : \mathrm{Fun}(\mathscr{D}, \mathscr{E})^{\mathrm{Kan}} \times \mathrm{Fun}(\mathscr{C}, \mathscr{D})^{\mathrm{Kan}} \to \mathrm{Fun}(\mathscr{C}, \mathscr{E})^{\mathrm{Kan}}.$$

We take connected components to obtain a well-defined, associative composition on the sets of connected components

$$\mathrm{Hom}_{\mathrm{hCat}_\infty}(-,-) := \pi_0\big(\mathrm{Fun}(\mathscr{D}, \mathscr{E})^{\mathrm{Kan}}\big), \tag{32}$$ eq:2274

and a well-defined category.

Definition 5.36. The homotopy category hCat_∞ of ∞-categories is the category whose objects are ∞-categories and whose morphisms are homotopy classes of maps, as defined at (32).

Definition 5.37. A functor $F : \mathscr{C} \to \mathscr{D}$ between ∞-categories is called an equivalence if there exists a functor $G : \mathscr{D} \to \mathscr{C}$ such that FG and GF are isomorphic to $id_{\mathscr{D}}$ and $id_{\mathscr{C}}$ in $\mathrm{Fun}(\mathscr{D}, \mathscr{D})$ and $\mathrm{Fun}(\mathscr{C}, \mathscr{C})$ respectively.

The following lemma is transparent.

lem:equiv_htop

Lemma 5.38. *A map $F : \mathscr{C} \to \mathscr{D}$ between ∞-categories is an equivalence if and only if its class $\bar{F}$ in the homotopy category hCat_∞ is an isomorphism.*

We end with some simple lemmas about equivalences of functors.

lem:2293

Lemma 5.39. *Let $v : \Delta^1 \times \mathscr{C} \to \mathscr{D}$ is an isomorphism between functors F_0 and F_1 in $\mathrm{Fun}(\mathscr{C}, \mathscr{D})$. Then for any $x : \Delta^0 \to \mathscr{C}$ the restriction $v_x : \Delta^1 = \Delta^1 \times \Delta^0 \to \Delta^1 \times \mathscr{C} \to \mathscr{D}$ is an isomorphism in $\mathscr{D}$, $v_x : F_0(x) \to F_1(x)$.*

Proof. One chooses 2-simplices $V, V' : \Delta^2 \times \mathscr{C} \to \mathscr{D}$ with

$$V|\Delta^{\{0,2\}} = id_{F_0}, \quad V'|\Delta^{\{0,2\}} = id_{F_1}, \quad V|\Delta^{\{0,1\}} = v, \quad V|\Delta^{\{1,2\}} = v.$$

Evaluating these 2-simplex at $x : \Delta^0 \to \mathscr{C}$ exhibit right and left inverses to v_x. $\square$

lem:2305 **Lemma 5.40.** *If $F_0, F_1 : \mathscr{C} \to \mathscr{D}$ are isomorphic functors between ∞-categories, then their restrictions $F_0^{\mathrm{Kan}}, F_1^{\mathrm{Kan}} : \mathscr{C}^{\mathrm{Kan}} \to \mathscr{D}^{\mathrm{Kan}}$ are isomorphic as well. In particular, any isomorphism between the F_i restricts to an isomorphism between the F_i^{Kan}.*

Proof. It suffices to show that the restriction of $v : \Delta^1 \times \mathscr{C}^{\mathrm{Kan}} \to \mathscr{D}$ of an isomorphism $\widetilde{v} : F_0 \xrightarrow{\sim} F_1$ has image in $\mathscr{D}^{\mathrm{Kan}}$. The 1-simplices in $\Delta^1 \times \mathscr{C}^{\mathrm{Kan}}$ are of the form $\mathrm{vert}_x = \Delta^1 \times \{x\}$, $\alpha^0 : \Delta^{\{0\}} \times \alpha$, $\alpha^1 : \Delta^{\{1\}} \times \alpha$, and

$$\Delta^1 \xrightarrow{\delta} \Delta^1 \times \Delta^1 \xrightarrow{id \times \alpha} \Delta^1 \times \mathscr{C}, \tag{33}$$ eq:2311

where δ is the diagonal map $0 \mapsto (0,0)$, $1 \mapsto (1,1)$. By Lemma 5.39 vert_x maps to an isomorphism v_x in $\mathscr{D}$, and the fact that the functors $F_0 = v|_0$ and $F_1 = v|_1$ preserve isomorphisms says that α^0 and α^1 map to isomorphisms in $\mathscr{D}$. We need to deal with the map (33).

Write $\mathrm{diag}_\alpha : (0,x) \to (1,y)$ for the map (33) in $\Delta^1 \times \mathscr{C}$. We have the 2-simplex $\Delta^2 \to \Delta^1 \times \Delta^1$ defined by the function $[2] \to [1] \times [1]$, $0 \mapsto (0,0)$, $1 \mapsto (1,0)$, $2 \mapsto (1,1)$, and the composite

$$\Delta^2 \to \Delta^1 \times \Delta^1 \xrightarrow{id \times \alpha} \Delta^1 \times \mathscr{C}$$

defines a 2-simplex s in the product with faces $s|\Delta^{\{0,1\}} = \mathrm{vert}_x$, $s|\Delta^{\{1,2\}} = \alpha^1$, and $s|\Delta^{\{0,2\}} = \mathrm{diag}_\alpha$. The image of s in $\mathscr{D}$ is a 2-simplex exhibiting $v(\mathrm{diag}_\alpha)$ as a composite of v_x with $F_1(\alpha)$. Since both of these maps are isomorphisms in $\mathscr{D}$, $v(\mathrm{diag}_\alpha)$ is seen to be an isomorphism in $\mathscr{D}$ as well. It follows that v has image in $\mathscr{D}^{\mathrm{Kan}}$, as desired. □

5.11. **Equivalences via functor categories.** The point of this final subsection is to prove the following characterization of equivalences between ∞-categories via spaces of functors.

thm:Fun_equiv_infty **Theorem 5.41.** *Let $F : \mathscr{C} \to \mathscr{D}$ be a functor between ∞-categories. The following are equivalent:*

(a) *F is an equivalence.*

(b) *For any simplicial set K, the map $F_* : \mathrm{Fun}(K, \mathscr{C}) \to \mathrm{Fun}(K, \mathscr{D})$ is an equivalence.*

(c) *For any ∞-category $\mathscr{A}$, the map $F_* : \mathrm{Fun}(\mathscr{A}, \mathscr{C}) \to \mathrm{Fun}(\mathscr{A}, \mathscr{D})$ is an equivalence of ∞-categories.*

(d) *For any ∞-category $\mathscr{A}$, the map $F^* : \mathrm{Fun}(\mathscr{D}, \mathscr{A}) \to \mathrm{Fun}(\mathscr{C}, \mathscr{A})$ is an equivalence of ∞-categories.*

(e) *For any ∞-category $\mathscr{A}$, the map $F_* : \mathrm{Fun}(\mathscr{A}, \mathscr{C})^{\mathrm{Kan}} \to \mathrm{Fun}(\mathscr{A}, \mathscr{D})^{\mathrm{Kan}}$ is an equivalence of Kan complexes.*

(f) *For any ∞-category $\mathscr{A}$, the map $F^* : \mathrm{Fun}(\mathscr{D}, \mathscr{A})^{\mathrm{Kan}} \to \mathrm{Fun}(\mathscr{C}, \mathscr{A})^{\mathrm{Kan}}$ is an equivalence of Kan complexes.*

Proof. One employs homotopies directly, as in the proof of Lemma 4.19, to see that (a) implies (b), and the implication (b) $\Rightarrow$ (c) follows since ∞-categories are simplicial sets. Lemma 5.40 shows that (c) implies (e). By applying π_0 to (e) and considering Yoneda's Lemma we see that (e) implies $\mathscr{C}$ and $\mathscr{D}$ are isomorphic in the homotopy category of ∞-categories. By Lemma 5.38 it follows that $F : \mathscr{C} \to \mathscr{D}$

is an equivalence. So (e) $\Rightarrow$ (a). We thus fund an equivalence between (a), (b), (c), and (e).

We similarly see that (a) $\Rightarrow$ (d) $\Rightarrow$ (f) $\Rightarrow$ (a), and hence that (a), (d), and (f) are equivalent. We are done. $\square$

sect:infty_infty

5.12. **The ∞-category of ∞-categories.** Consider ∞-categories $\mathscr{C}$ and $\mathscr{D}$. From Corollary 5.8 and Theorem 5.32 we understand that the simplicial subset $\mathrm{Fun}(\mathscr{C},\mathscr{D})^{\mathrm{Kan}}$ of functors and natural isomorphisms in $\mathrm{Fun}(\mathscr{C},\mathscr{D})$ is a Kan complex. As observed in Section 5.10 above, we now have the simplicial subcategory

$$\underline{Cat}_\infty^{\mathbb{U}} \subseteq \underline{2\,\mathrm{Cat}}_\infty^{\mathbb{U}}$$

in the ambient category of $\mathbb{U}$-small ∞-category whose morphisms are the Kan complexes

$$\mathrm{Hom}_{\underline{Cat}_\infty^{\mathbb{U}}}(\mathscr{C},\mathscr{D}) := \mathrm{Fun}(\mathscr{C},\mathscr{D})^{\mathrm{Kan}}$$

As in the Kan setting, we apply Proposition 3.16 to obtain the following.

Proposition 5.42. *For any universe $\mathbb{U}$, the homotopy coherent nerve $\mathrm{N}^{\mathrm{hc}}(\underline{Cat}_\infty^{\mathbb{U}})$ is an ∞-category.*

Definition 5.43. For a given universe $\mathbb{U}$, we take

$$\mathscr{C}\!at_\infty^{\mathbb{U}} := \mathrm{N}^{\mathrm{hc}}(\underline{Cat}_\infty^{\mathbb{U}}).$$

We call this the ∞-category of $\mathbb{U}$-small ∞-categories.

When the specification of a universe is immaterial we take

$$\mathscr{C}\!at_\infty := \text{The } \infty\text{-category } \mathscr{C}\!at_\infty^{\mathbb{U}} \text{ for some large unspecified } \mathbb{U}.$$

We let $\mathscr{C}\!at_\infty^{\mathrm{sm}}$ denote the ∞-category of ∞-categories in our pre-determined universe of small sets. As one expects, the homotopy category of $\mathscr{C}\!at_\infty$ recovers the homotopy category of ∞-categories introduced in Section 5.10 above,

$$\mathrm{h}\,\mathscr{C}\!at_\infty = \mathrm{h}\,\mathrm{Cat}_\infty\,.$$

sect:htop_pullback

6. Homotopy pullback and categorical pullbacks

6.1. **Fibrant replacement.** While we are not especially interested in introducing the general notion of a model structure, let us provide some basic structuring of the category of Kan complexes.

Lemma 6.1. *If $f:\mathscr{X}\to\mathscr{X}'$ is and anodyne morphism between Kan complexes then f is a homotopy equivalence.*

Proof. Follows by Corollary 4.12, Proposition 4.29, and Proposition 4.18. $\square$

Proposition 6.2. *Any map of Kan complexes $f:\mathscr{X}\to\mathscr{Y}$ admits a factorization*

$$\mathscr{X}\overset{t}{\to}\mathscr{X}'\overset{f'}{\to}\mathscr{Y}$$

with t an anodyne equivalence and f' a Kan fibration.

We do not cover the proof, and refer directly to the text [15, 00UU] for the details. We only note that, most immediately, [15] provides a factorization as above where $\mathscr{X}'$ is only assumed to be a simplicial set. However, since $\mathscr{X}'\to\mathscr{Y}$ is a Kan fibration and $\mathscr{Y}$ is itself a Kan complex, it follows that $\mathscr{X}'$ is a Kan complex as well.

There are points at which the fact that t is an anodyne map will be important. However, we will often only use the fact that any map factors $\mathscr{X} \to \mathscr{X}' \to \mathscr{Y}$ where the second map is a Kan fibration and the first map is simply an equivalence. In such a situation we refer to $\mathscr{X}'$ as a "fibrant replacement" for $\mathscr{X}$ in the category of Kan complexes over $\mathscr{Y}$.

6.2. **Introductory discussion for (homotopy) pullback.** In general, for a pullback diagram

$$\begin{tikzcd} \mathscr{Z} \arrow[r] \arrow[d, "q'"'] & \mathscr{Y} \arrow[d, "q"] \\ \mathscr{X} \arrow[r] & \mathscr{S} \end{tikzcd}$$

the fact that q is a homotopy equivalence does not imply that q' is a homotopy equivalence. Consider for example the case where $q : \mathscr{Y} \to \mathscr{S}$ is a map between two contractible spaces which is not surjective on points. In this case we can consider a point $x : * \to \mathscr{S}$ which does not lie in the image of $\mathscr{Y}$ to obtain a pullback diagram

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d, "q'"'] & \mathscr{Y} \arrow[d, "p"] \\ * \arrow[r] & \mathscr{S} \end{tikzcd} \tag{34}$$ eq:3224

in which the map q' is, obviously, not a equivalence.

In comparison with the dg setting, for dg schemes (or affine dg schemes if one likes), the preservation of equivalences under base change might be understood as a kind of flatness condition on morphisms. The Kan condition provides such a condition.

prop:kan_basechange **Proposition 6.3** ([15, 0109])**.** *Suppose we have a diagram*

$$\begin{tikzcd} \mathscr{X} \arrow[d, "t"'] \arrow[dr, "f"] & & \mathscr{Y} \arrow[dl] \arrow[d, "u"] \\ \mathscr{X}' \arrow[dr, "f'"'] & \mathscr{S} \arrow[d, "v"] & \mathscr{Y}' \arrow[dl] \\ & \mathscr{S}' & \end{tikzcd}$$

in which the maps f and f' are Kan fibrations, and all vertical maps are homotopy equivalences. Then the induced map on fiber products $\mathscr{X} \times_{\mathscr{S}} \mathscr{Y} \to \mathscr{X}' \times_{\mathscr{S}'} \mathscr{Y}'$ is an equivalence.

Proof. We consider the diagram

$$\begin{tikzcd} \mathscr{X} \times_{\mathscr{S}} \mathscr{Y} \arrow[r] \arrow[d] & \mathscr{X}' \times_{\mathscr{S}'} \mathscr{Y}' \arrow[d] \\ \mathscr{Y} \arrow[r, "u"] & \mathscr{Y}' \end{tikzcd}$$

in which the vertical maps are Kan fibrations, and the bottom map is a homotopy equivalence. Hence, by Proposition 4.38, the map on fiber products is an equivalence if and only if we have an equivalence

$$\mathscr{X}_s = (\mathscr{X} \times_{\mathscr{S}} \mathscr{Y})_y \to (\mathscr{X}' \times_{\mathscr{S}'} \mathscr{Y}')_y = \mathscr{X}'_s$$

at all points $y : * \to \mathscr{Y}$ with corresponding point $s : * \to \mathscr{S}$. However, the above map is the fiber of the equivalence t over s, which we know to be an equivalence by Proposition 4.38 applied to the diagram

$$\begin{tikzcd} \mathscr{X} \arrow[r, "t"] \arrow[d, "f"'] & \mathscr{X}' \arrow[d, "f'"] \\ \mathscr{S} \arrow[r, "v"] & \mathscr{S}'. \end{tikzcd}$$

□

As a particular application of Proposition 6.3 we see that the base change $\mathscr{X} \times_{\mathscr{S}} \mathscr{Y} \to \mathscr{X} \times_{\mathscr{S}} \mathscr{Y}$ of an equivalence $\mathscr{Y} \to \mathscr{Y}'$, for spaces over $\mathscr{S}$, remains an equivalence provided $\mathscr{X}$ is a Kan fibration over $\mathscr{S}$.

In vague analogy to the dg setting, one might define the homotopy (aka "derived") pullback of spaces by first taking a fibrant replacement

$$\mathscr{X} \to \mathscr{S} \quad \rightsquigarrow \quad \mathscr{X} \overset{\sim}{\dashrightarrow} \mathscr{X}' \to \mathscr{S}$$

and replacing the usual fiber product $\mathscr{X} \times_{\mathscr{S}} \mathscr{Y}$ with the product $\mathscr{X}' \times_{\mathscr{S}} \mathscr{Y}$, at least up to some first order. As one might understand from experience, this construction has the drawback of being ambiguous and non-functorial.

In some sense, one of the points of working in the ∞, rather that dg, setting is to provide explicit control on such ambiguities. However, let us leave this issue for now and simply proceed with our presentation.

6.3. **A functorial construction.**

Definition 6.4. Given maps of Kan complexes $\mathscr{X} \to \mathscr{S}$ and $\mathscr{Y} \to \mathscr{S}$ we define the homotopy pullback as

$$\mathscr{X} \times^{\mathrm{htop}}_{\mathscr{S}} \mathscr{Y} := \mathscr{X} \times_{\mathrm{Fun}(\{0\},\mathscr{S})} \mathrm{Fun}(\Delta^1, \mathscr{S}) \times_{\mathrm{Fun}(\{1\},\mathscr{S})} \mathscr{Y}.$$

Note that we can write, alternatively,

$$\mathscr{X} \times^{\mathrm{htop}}_{\mathscr{S}} \mathscr{Y} = \mathrm{Fun}(\Delta^1, \mathscr{S}) \times_{\mathrm{Fun}(\partial\Delta^1,\mathscr{S})} (\mathscr{X} \times \mathscr{Y}).$$

Let us be clear that the homotopy pullback $\mathscr{X} \times^{\mathrm{htop}}_{\mathscr{S}} \mathscr{Y}$ does not fit into a diagram over $\mathscr{X}$ and $\mathscr{Y}$ in general. Instead we have a diagram

$$\begin{tikzcd} \mathscr{X} \times^{\mathrm{htop}}_{\mathscr{S}} \mathscr{Y} \arrow[r] \arrow[d] & \mathscr{Y} \arrow[d] \\ \mathscr{X} \arrow[r] & \mathscr{S} \end{tikzcd}$$

in the ∞-category of spaces, where the necessary homotopy between the two maps to $\mathscr{S}$ is given by evaluation

$$\Delta^1 \times \left(\mathrm{Fun}(\Delta^1, \mathscr{S}) \times_{\mathrm{Fun}(\partial\Delta^1,\mathscr{S})} (\mathscr{X} \times \mathscr{Y})\right) \overset{\text{project}}{\to} \Delta^1 \times \mathrm{Fun}(\Delta^1, \mathscr{S}) \overset{\text{eval}}{\to} \mathscr{S}$$

Example 6.5. Consider two distinct points in a contractible space $x, y : * \to \mathscr{S}$. The homotopy fiber product in this case is the mapping space

$$\{x\} \times^{\text{htop}}_{\mathscr{S}} \{y\} = \mathrm{Hom}_{\mathscr{S}}(x, y).$$

Since the terminal map $\mathscr{S} \to *$ is an equivalence, it is in particular fully faithful. So the induced map $\mathrm{Hom}_{\mathscr{S}}(x, y) \to \mathrm{Hom}_*(*, *) = *$ is an equivalence. In particular, the homotopy fiber product is a non-empty contractible space.

More generally, when $\mathscr{S}$ is non-contractible the homotopy fiber product $x \times^{\text{htop}}_{\mathscr{S}} y$ is empty if and only if the two points live in different components in $\mathscr{S}$. In the event that x and y are in the same component we claim, without proof, that the homotopy fiber product is equivalent to the loop space for $\mathscr{S}$ at either x or y (cf. Section 7.4).

One sees that restriction along the map $\Delta^1 \to *$ provides a binatural embedding $\mathscr{X} \times \mathscr{Y} \to \mathscr{X} \times^{\text{htop}}_{\mathscr{S}} \mathscr{Y}$. We refer to this embedding informally as the "comparison map".

prop:3319 **Proposition 6.6** ([15, 0329])**.** *For any partial diagram*

$$\begin{tikzcd} \mathscr{X} \arrow[dr, "f"'] & & \mathscr{Y} \arrow[dl, "g"] \\ & \mathscr{S} & \end{tikzcd}$$

in which either f or g is a Kan fibration the comparison map $\mathscr{X} \times_{\mathscr{S}} \mathscr{Y} \to \mathscr{X} \times^{\text{htop}}_{\mathscr{S}} \mathscr{Y}$ is an equivalence.

Proof. Suppose that $\mathscr{X} \to \mathscr{S}$ is a Kan fibration. In this case the base changed map $\mathscr{X} \times_{\mathscr{S}} \mathrm{Fun}(\Delta^1, \mathscr{S}) \to \mathrm{Fun}(\Delta^1, \mathscr{S})$ is also a Kan fibration. Furthermore, since the inclusion $1 : \Delta^0 \to \Delta^1$ is anodyne the induced map $\mathrm{Fun}(\Delta^1, \mathscr{S}) \to \mathscr{S}$ is a trivial Kan fibration, by Corollary 4.12, and in particular an equivalence by Proposition 4.29. We consider also the map

$$h = \left(\mathscr{Y} \overset{g}{\to} \mathscr{S} \overset{\text{const}}{\to} \mathrm{Fun}(\Delta^1, \mathscr{S}) \right).$$

We now have a diagram

$$\begin{tikzcd} \mathscr{X} \times_{\mathscr{S}} \mathrm{Fun}(\Delta^1, \mathscr{S}) \arrow[r] \arrow[d, "="'] & \mathrm{Fun}(\Delta^1, \mathscr{S}) \arrow[d, "1^*"] & \mathscr{Y} \arrow[l, "h"'] \arrow[d, "="] \\ \mathscr{X} \times_{\mathscr{S}} \mathrm{Fun}(\Delta^1, \mathscr{S}) \arrow[r] & \mathscr{S} & \mathscr{Y} \arrow[l, "g"] \end{tikzcd}$$

in which the vertical maps are equivalences (in particular trivial Kan fibrations) and the maps from $\mathscr{X} \times_{\mathscr{S}} \mathrm{Fun}(\Delta^1, \mathscr{S})$ are Kan fibrations. So, by Proposition 6.3, the induced map

$$\mathscr{X} \times_{\mathscr{S}} \mathscr{Y} \cong (\mathscr{X} \times_{\mathscr{S}} \mathrm{Fun}(\Delta^1, \mathscr{S})) \times_{\mathrm{Fun}(\Delta^1, \mathscr{S})} \mathscr{Y}$$
$$\to \mathscr{X} \times_{\mathscr{S}} \mathrm{Fun}(\Delta^1, \mathscr{S}) \times_{\mathscr{S}} \mathscr{Y} = \mathscr{X} \times^{\text{htop}}_{\mathscr{S}} \mathscr{Y}$$

is a homotopy equivalence. The case where the g is a Kan fibration is dealt with similarly. □

As one imagines at this point, we have a homotopy analog of Proposition 6.3 in which the Kan condition is now obviated.

prop:htop_basechange

Proposition 6.7 ([15, 032B])**.** *Suppose we have a diagram*

$$\begin{tikzcd} \mathscr{X} \arrow[d, "t"'] \arrow[dr] & & \mathscr{Y} \arrow[dl] \arrow[d, "u"] \\ \mathscr{X}' \arrow[dr] & \mathscr{S} \arrow[d, "v"] & \mathscr{Y}' \arrow[dl] \\ & \mathscr{S}' & \end{tikzcd}$$

in which all vertical maps are homotopy equivalences. Then the induced map on homotopy fiber products

$$\mathscr{X} \times^{\mathrm{htop}}_{\mathscr{S}} \mathscr{Y} \to \mathscr{X}' \times^{\mathrm{htop}}_{\mathscr{S}'} \mathscr{Y}'$$

is a homotopy equivalence.

Proof. In this case the map on functor spaces $v^* : \mathrm{Fun}(K, \mathscr{S}) \to \mathrm{Fun}(K, \mathscr{S}')$ is an equivalence at all simplicial sets K, by Proposition 4.18. Furthermore restricting along any inclusion $L \to K$ produces a Kan fibration $\mathrm{Fun}(K, \mathscr{S}) \to \mathrm{Fun}(L, \mathscr{S})$ by Corollary 4.12. So the result follows by applying Proposition 6.3 to the diagram

$$\begin{tikzcd} \mathrm{Fun}(\Delta^1, \mathscr{S}) \arrow[r] \arrow[d] & \mathrm{Fun}(\partial\Delta^1, \mathscr{S}) \arrow[d] & \mathscr{X} \times \mathscr{Y} \arrow[l] \arrow[d] \\ \mathrm{Fun}(\Delta^1, \mathscr{S}') \arrow[r] & \mathrm{Fun}(\partial\Delta^1, \mathscr{S}') & \mathscr{X}' \times \mathscr{Y}'. \arrow[l] \end{tikzcd}$$

□

6.4. **Homotopy pullback squares.**

Definition 6.8. A commutative diagram of Kan complexes

$$\begin{tikzcd} \mathscr{Z} \arrow[r] \arrow[d] & \mathscr{Y} \arrow[d] \\ \mathscr{X} \arrow[r] & \mathscr{S} \end{tikzcd}$$

is called a homotopy pullback square if the induced map

$$\mathscr{Z} \to \mathscr{X} \times_{\mathscr{S}} \mathscr{Y} \to \mathscr{X} \times^{\mathrm{htop}}_{\mathscr{S}} \mathscr{Y}$$

is a homotopy equivalence.

We make no claim that any partial diagram $\mathscr{X} \to \mathscr{S} \leftarrow \mathscr{Y}$ can be completed to a homotopy pullback square. Proposition 6.6 says that the usual pullback provides a homotopy pullback square whenever either of the maps $\mathscr{X} \to \mathscr{S}$ or $\mathscr{Y} \to \mathscr{S}$ is a Kan fibration.

We have the following interpretation of homotopy pullback squares via fibrant replacements.

Proposition 6.9. *Consider a diagram of Kan complexes*

$$\begin{tikzcd} \mathscr{Z} \arrow[r] \arrow[d] & \mathscr{Y} \arrow[d, "g"] \\ \mathscr{X} \arrow[r, "f"'] & \mathscr{S}. \end{tikzcd}$$

The following are equivalent:

(a) *The above diagram is a homotopy pullback diagram.*

(b) *For each factorization $\mathscr{X} \xrightarrow{t} \mathscr{X}' \xrightarrow{f'} \mathscr{S}$ in which t is a homotopy equivalence and f' is a Kan fibration, the induced map $\mathscr{Z} \to \mathscr{X}' \times_{\mathscr{S}} \mathscr{Y}$ is an equivalence.*

(c) *For each factorization $\mathscr{Y} \xrightarrow{u} \mathscr{Y}' \xrightarrow{g'} \mathscr{S}$ in which u is a homotopy equivalence and g' is a Kan fibration, the induced map $\mathscr{Z} \to \mathscr{X} \times_{\mathscr{S}} \mathscr{Y}'$ is an equivalence.*

(d) *for any two factorizations as in (b) and (c) the induced map $\mathscr{Z} \to \mathscr{X}' \times_{\mathscr{S}} \mathscr{Y}'$ is an equivalence.*

Proof. The result follows by a consideration of the diagram of equivalences

$$\begin{tikzcd}
\mathscr{X}' \times_{\mathscr{S}} \mathscr{Y} \arrow[r] \arrow[d] & \mathscr{X}' \times_{\mathscr{S}} \mathscr{Y}' \arrow[d] & \mathscr{X} \times_{\mathscr{S}} \mathscr{Y}' \arrow[l] \arrow[d] \\
\mathscr{X}' \times^{\mathrm{htop}}_{\mathscr{S}} \mathscr{Y} \arrow[r] & \mathscr{X}' \times^{\mathrm{htop}}_{\mathscr{S}} \mathscr{Y}' & \mathscr{X} \times^{\mathrm{htop}}_{\mathscr{S}} \mathscr{Y}' \arrow[l] \\
& \mathscr{X} \times^{\mathrm{htop}}_{\mathscr{S}} \mathscr{Y} \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}.$$

We leave it to the interested reader to fill in the details. □

As with the functorial construction of the homotopy pullback, homotopy pullback squares enjoy invariance under homotopy equivalence.

prop:htpy_invariance

Proposition 6.10 (Homotopy invariance, [15, 0111])**.** *Consider a diagram of Kan complexes*

$$\begin{tikzcd}
\mathscr{Z} \arrow[rr] \arrow[dd] \arrow[dr, "t"] & & \mathscr{Y} \arrow[dd] \arrow[dr, "t_2"] & \\
& \mathscr{Z}' \arrow[rr] \arrow[dd] & & \mathscr{Y}' \arrow[dd] \\
\mathscr{X} \arrow[rr] \arrow[dr, "t_1"] & & \mathscr{S} \arrow[dr, "t_0"] & \\
& \mathscr{X}' \arrow[rr] & & \mathscr{S}'
\end{tikzcd} \tag{35}$$

eq:34321

in which all of the t_i is a equivalence. Then the following are equivalent:

(a) *Both the back and front faces in (35) are homotopy pullback squares.*

(b) *Either the back face or the front face in (35) is a homotopy pullback square, and the map t is an equivalence.*

Proof. Follows by homotopy invariance of the homotopy fiber product, by Proposition 6.7, and the diagram

$$\begin{tikzcd}
\mathscr{X} \times^{\mathrm{htop}}_{\mathscr{S}} \mathscr{Y} \arrow[r, "\sim"] & \mathscr{X}' \times^{\mathrm{htop}}_{\mathscr{S}'} \mathscr{Y}' \\
\mathscr{Z} \arrow[r, "t"] \arrow[u] & \mathscr{Z}' \arrow[u]
\end{tikzcd}.$$

□

6.5. **Simplicial injections, Kan fibrations, and homotopy pullback.** We claim that homotopy pullback squares appear naturally when one applies the functor $\mathrm{Fun}(-,\mathscr{C})^{\mathrm{Kan}}$ to from certain pushout squares for simplicial sets. We begin our analysis with a technical lifting result.

lem:3465 **Lemma 6.11** ([15, 01NX])**.** *Consider a lifting problem*

$$\begin{tikzcd} (\Delta^1\times A)\coprod_{(\{1\}\times A)}(\{1\}\times B) \arrow[r] \arrow[d] & \mathscr{C} \arrow[d] \\ \Delta^1\times B \arrow[r, "\nu"'] \arrow[ur, dashed] & \mathscr{D} \end{tikzcd}$$

associated to an inner fibration of ∞-categories $\mathscr{C}\to\mathscr{D}$ and injective map of simplicial sets $i: A\to B$. Suppose that for every simplex $s:\Delta^n\to B$ which is not contained in the image of i the associated map in $\mathscr{C}$

$$\nu(id_{\Delta^1}\times s(n))\Delta^1 = \Delta^1\times * \to \mathscr{C}$$

is an isomorphism, and that $i(A)$ contains all vertices in B. (Here $n:*\to\Delta^n$ is the terminal point in Δ^n and $s(n):*\to B$ is the composite map to B.) Then the above lifting problem has a solution.

We only relay the basic idea of the proof, and refer the reader to the source [15] for any details.

Idea of proof. One reduces to the case where $A=\partial\Delta^n$, $B=\Delta^n$, and $i:A\to B$ is the inclusion. One then factors this inclusion into a sequence

$$(\Delta^1\times\partial\Delta^n)\cup(\{0\}\times\Delta^n)\subseteq X_1\subseteq\cdots\subseteq X_n=\Delta^1\times\Delta^n$$

in which each X_{i+1} is obtained from X_i via a pushout diagram of the form

$$\begin{tikzcd} \Lambda^{n+1}_i \arrow[r] \arrow[d] & X_i \arrow[d] \\ \Delta^{n+1}_i \arrow[r, "s_i"'] & X_{i+1}. \end{tikzcd}$$

One can furthermore assume that the final simplex we adjoin $s_{n+1}:\Delta^{n+1}\to X_{n+1}=\Delta^1\times\Delta^n$ we adjoin has $s_{n+1}(\Delta^{\{n,n+1\}})=\Delta^1\times\{n\}$ [15, Proof of 00TH]. We therefore reduce the lifting problem for to the inclusion $(\Delta^1\times\partial\Delta^n)\cup(\{0\}\times\Delta^n)\to\Delta^1\times\Delta^n$ to a sequence of lifting problems of the form

$$\begin{tikzcd} \Lambda^{n+1}_i \arrow[r] \arrow[d] & \mathscr{C} \arrow[d] \\ \Delta^{n+1} \arrow[r, "\tau_i"'] & \mathscr{D}, \end{tikzcd}$$

with $0<i\leq n+1$ and τ_n sending the final edge in Δ^{n+1} to an isomorphism in $\mathscr{D}$. We can solve all such lifting problems via the weak Kan condition and Proposition 5.33. □

Now, for any inclusion of simplicial sets $i : A \to B$ we can expand i to an inclusion $i' : A' \to B$ for which all vertices in B lie in the image of i' by simply adjoining the 0 skeleton of B to A. By considering such minimal extensions one obtains the following adjacent lifting property from that of Lemma 6.11.

cor:3506 **Corollary 6.12** ([15, 01NY]). *Consider a lifting problem*

$$\begin{tikzcd}
(\Delta^1 \times A)\coprod_{(\{1\}\times A)}(\{1\}\times B) \arrow[r] \arrow[d] & \mathscr{C} \arrow[d] \\
\Delta^1 \times B \arrow[r, "\nu"'] \arrow[ur, dashed] & \mathscr{D}
\end{tikzcd}$$

associated to an isofibration of ∞-categories $\mathscr{C} \to \mathscr{D}$ and an arbitrary injective map of simplicial sets $i : A \to B$. The above lifting problem admits a solution provided the following conditions hold:

(a) *For each vertex $a : * \to A$ the corresponding edge $\Delta^1 \times \{a\} \to \mathscr{C}$ is an isomorphism in $\mathscr{C}$.*
(b) *For each vertex $b : * \to B$ the corresponding edge $\Delta^1 \times \{b\} \to \mathscr{D}$ is an isomorphism in $\mathscr{D}$.*

Furthermore, when the above hypotheses are satisfied, there exists a solution $\eta : \Delta \times B \to \mathscr{C}$ for which each edge $\Delta^1 \times \{b\} \to \mathscr{C}$ is an isomorphism in $\mathscr{C}$.

prop:inj_isofib **Proposition 6.13.** *Let $L \to K$ be an injective map of simplicial sets and $\mathscr{C} \to \mathscr{D}$ be an isofibration of ∞-categories. Then the induced map*

$$\theta : \mathrm{Fun}(K,\mathscr{C}) \to \mathrm{Fun}(L,\mathscr{C}) \times_{\mathrm{Fun}(L,\mathscr{D})} \mathrm{Fun}(K,\mathscr{D})$$

is an isofibration of ∞-categories.

Proof. First note that the restriction map $\mathrm{Fun}(K,\mathscr{D}) \to \mathrm{Fun}(L,\mathscr{D})$ is an inner fibration by Corollary 5.8, so that the projection $\mathrm{Fun}(L,\mathscr{C}) \times_{\mathrm{Fun}(L,\mathscr{D})} \mathrm{Fun}(K,\mathscr{D}) \to \mathrm{Fun}(L,\mathscr{C})$ is an inner fibration as well. It follows that the fiber product is in fact an ∞-category. We also know that θ is an inner fibration by Proposition 5.7. So we need only know that for any object ξ in $\mathrm{Fun}(K,\mathscr{C})$ and isomorphism $a : \eta \to \theta(\xi)$ in the fiber product, we can lift a to an isomorphism $a' : \eta' \to \xi$ in $\mathrm{Fun}(K,\mathscr{C})$.

The map a in the fiber product is precisely the information of a diagram

$$\begin{tikzcd}
(\Delta^1 \times L)\coprod_{(\{1\}\times L)}(\{1\}\times K) \arrow[r] \arrow[d] & \mathscr{C} \arrow[d] \\
\Delta^1 \times K \arrow[r, "\nu"'] \arrow[ur, dashed] & \mathscr{D}
\end{tikzcd}$$

and the desired lift $\widetilde{a}$ is precisely the information of a solution to the corresponding lifting problem. Furthermore, since a is a natural isomorphism the maps $\Delta^1 \times L \to \mathscr{C}$ and $\Delta^1 \times K \to \mathscr{D}$ both restrict to isomorphisms on all edges of the form $\Delta^1 \times \{x\}$. So we see that the desired solution $\widetilde{a}$ exists by Corollary 6.12. $\square$

We note that an analog of Proposition 6.13 in the case where the map $\mathscr{C} \to \mathscr{D}$ is simply an inner fibration and $L \to K$ is an injection which contains all vertices in K in its image. The following corollary is immediate.

cor:inj_isofib

Corollary 6.14. (1) *For any injection $i : L \to K$, and any ∞-category $\mathscr{C}$, the map*

$$i^* : \mathrm{Fun}(K, \mathscr{C}) \to \mathrm{Fun}(L, \mathscr{C})$$

is an isofibration.

(2) *For any simplicial set K, and any isofibration $F : \mathscr{C} \to \mathscr{D}$, the map*

$$F_* : \mathrm{Fun}(K, \mathscr{C}) \to \mathrm{Fun}(K, \mathscr{D})$$

is an isofibration.

We recall that any isofibration produces a Kan fibration on associated Kan complexes, by Corollary 5.34. So we have the following.

cor:inj_kan

Corollary 6.15. (1) *For any injection $i : L \to K$, and any ∞-category $\mathscr{C}$, the map*

$$i^* : \mathrm{Fun}(K, \mathscr{C})^{\mathrm{Kan}} \to \mathrm{Fun}(L, \mathscr{C})^{\mathrm{Kan}}$$

is a Kan fibration.

(2) *For any simplicial set K, and any isofibration $F : \mathscr{C} \to \mathscr{D}$, the map*

$$F_* : \mathrm{Fun}(K, \mathscr{C})^{\mathrm{Kan}} \to \mathrm{Fun}(K, \mathscr{D})^{\mathrm{Kan}}$$

is a Kan fibration.

cor:3681

Corollary 6.16. *Suppose we have a pushout diagram of simplicial sets*

$$\begin{tikzcd} T \arrow[r] \arrow[d, "\mu"'] & A \arrow[d] \\ B \arrow[r] & K \end{tikzcd}$$

in which the map μ is injective. Then the corresponding diagram

$$\begin{tikzcd} \mathrm{Fun}(K, \mathscr{C})^{\mathrm{Kan}} \arrow[r] \arrow[d] & \mathrm{Fun}(A, \mathscr{C})^{\mathrm{Kan}} \arrow[d] \\ \mathrm{Fun}(B, \mathscr{C})^{\mathrm{Kan}} \arrow[r, "\mu^*"'] & \mathrm{Fun}(T, \mathscr{C})^{\mathrm{Kan}} \end{tikzcd}$$

is a homotopy pullback diagram.

Proof. Follows from the fact that μ^* is a Kan fibration, by Corollary 6.15, and Proposition 6.6. $\square$

6.6. Categorical pushout for simplicial sets.

Definition 6.17 ([15, 01F7])**.** We call a diagram of simplicial sets

$$\begin{tikzcd} T \arrow[r] \arrow[d] & A \arrow[d] \\ B \arrow[r] & K \end{tikzcd}$$

a categorical pushout square if, for any ∞-category $\mathscr{C}$, the corresponding diagram of Kan complexes

$$\begin{tikzcd} \mathrm{Fun}(K,\mathscr{D})^{\mathrm{Kan}} \arrow[r] \arrow[d] & \mathrm{Fun}(A,\mathscr{D})^{\mathrm{Kan}} \arrow[d] \\ \mathrm{Fun}(B,\mathscr{D})^{\mathrm{Kan}} \arrow[r] & \mathrm{Fun}(T,\mathscr{D})^{\mathrm{Kan}} \end{tikzcd}$$

is a homotopy pullback square.

Corollary 6.16 can now be rephrased as follows.

cor:3659 **Corollary 6.18.** *A pushout diagram of simplicial sets*

$$\begin{tikzcd} T \arrow[r] \arrow[d, "\mu"'] & A \arrow[d] \\ B \arrow[r] & K \end{tikzcd}$$

in which μ is injective is a categorical pushout diagram.

6.7. **Categorical pullback square.** Categorical pullback squares are not of immediate interest for us here. However, they are of use in further analyses of mapping spaces. So we record the appropriate foundations in this section.

Definition 6.19. We call a diagram of ∞-categories

$$\begin{tikzcd} \mathscr{K} \arrow[r] \arrow[d] & \mathscr{C} \arrow[d] \\ \mathscr{D} \arrow[r] & \mathscr{T} \end{tikzcd}$$

is a categorical pullback square if, for any ∞-category $\mathscr{A}$, the corresponding diagram of Kan complexes

$$\begin{tikzcd} \mathrm{Fun}(\mathscr{A},\mathscr{K})^{\mathrm{Kan}} \arrow[r] \arrow[d] & \mathrm{Fun}(\mathscr{A},\mathscr{C})^{\mathrm{Kan}} \arrow[d] \\ \mathrm{Fun}(\mathscr{A},\mathscr{D})^{\mathrm{Kan}} \arrow[r] & \mathrm{Fun}(\mathscr{A},\mathscr{D})^{\mathrm{Kan}} \end{tikzcd}$$

is a homotopy pullback square.

Given a partial diagram of ∞-categories

$$\begin{tikzcd} & \mathscr{C} \arrow[d] \\ \mathscr{D} \arrow[r] & \mathscr{T} \end{tikzcd}$$

we extend the definition of the homotopy pullback square as follows

$$\mathscr{C} \times^{\mathrm{htop}}_{\mathscr{T}} \mathscr{D} := \mathrm{Isom}(\mathscr{T}) \times_{\mathrm{Fun}(\partial\Delta^1,\mathscr{T})} (\mathscr{C} \times \mathscr{D}),$$

where $\mathrm{Isom}(\mathscr{T})$ is the full ∞-subcategory of $\mathrm{Fun}(\Delta^1,\mathscr{T})$ spanned by isomorphisms in $\mathscr{T}$. We note that the restriction map $\mathrm{Fun}(\Delta^1,\mathscr{T}) \to \mathrm{Fun}(\partial\Delta^1,\mathscr{T})$ is an inner fibration, by Corollary 5.8, and hence the composite

$$\mathrm{Isom}(\mathscr{T}) \to \mathrm{Fun}(\Delta^1,\mathscr{T}) \to \mathrm{Fun}(\partial\Delta^1,\mathscr{T})$$

is also an inner fibration. So the homotopy pullback of ∞-categories is in fact an ∞-category.

Lemma 6.20. *For any ∞-category $\mathscr{T}$, the inclusion $\mathscr{T}^{\mathrm{Kan}} \to \mathscr{T}$ induces an isomorphism of simplicial sets*

$$\mathrm{Fun}(\Delta^1, \mathscr{T}^{\mathrm{Kan}}) \overset{\cong}{\to} \mathrm{Isom}(\mathscr{T})^{\mathrm{Kan}} \tag{36}$$ eq:3701

Proof. By Theorem 7.5 a map $\eta : \Delta^1 \to \mathrm{Fun}(\Delta^1, \mathscr{T})$ is an isomorphism if and only if $\eta(0)$ and $\eta(1)$ are isomorphisms in $\mathscr{T}$. This implies that $\mathrm{Isom}(\mathscr{T})^{\mathrm{Kan}}$ is identified with the simplicial subset in $\mathrm{Fun}(\Delta^1, \mathscr{T})$ whose n-simplices are precisely those n-simplices

$$\Delta^n \times \Delta^1 \to \mathscr{T}$$

whose restrictions to all points $\{x\} \times \Delta^1$ and $\Delta^{\{i,j\}} \times \{y\}$ are isomorphisms in $\mathscr{T}$. This implies that this n-simplex has image in $\mathscr{T}^{\mathrm{Kan}}$, so that we have an inclusion

$$\mathrm{Isom}(\mathscr{T})^{\mathrm{Kan}} \subseteq \mathrm{Hom}(\Delta^1, \mathscr{T}^{\mathrm{Kan}})$$

occurring in $\mathrm{Fun}(\Delta^1, \mathscr{T})$. The converse inclusion is clear, so that the two simplicial subsets are identified in $\mathrm{Fun}(\Delta^1, \mathscr{T})$, and we subsequently conclude that the map (36) is an isomorphism. $\square$

One now sees via a basic manipulation that, for any ∞-category $\mathscr{A}$, we have a canonical isomorphism of simplicial sets

$$\mathrm{Fun}(\mathscr{A}, \mathscr{C} \times^{\mathrm{htop}}_{\mathscr{T}} \mathscr{D}) \cong \mathrm{Fun}(\mathscr{A}, \mathscr{C}) \times^{\mathrm{htop}}_{\mathrm{Fun}(\mathscr{A}, \mathscr{T})} \mathrm{Fun}(\mathscr{A}, \mathscr{D})$$

and in particular an equivalence of ∞-categories. We apply the Kan complex functor to obtain a natural isomorphism of Kan complexes

$$\mathrm{Fun}(\mathscr{A}, \mathscr{C} \times^{\mathrm{htop}}_{\mathscr{T}} \mathscr{D})^{\mathrm{Kan}} \cong \mathrm{Fun}(\mathscr{A}, \mathscr{C})^{\mathrm{Kan}} \times^{\mathrm{htop}}_{\mathrm{Fun}(\mathscr{A}, \mathscr{T})^{\mathrm{Kan}}} \mathrm{Fun}(\mathscr{A}, \mathscr{D})^{\mathrm{Kan}}.$$

One uses this identification to obtain the following, simply from the definition of a homotopy pullback square and Theorem 5.41.

prop:3708 **Proposition 6.21.** *A diagram ∞-categories*

$$\begin{tikzcd} \mathscr{K} \arrow[r] \arrow[d] & \mathscr{C} \arrow[d] \\ \mathscr{D} \arrow[r] & \mathscr{T} \end{tikzcd}$$

is a categorical pullback diagram if and only if the natural map $\mathscr{K} \to \mathscr{C} \times^{\mathrm{htop}}_{\mathscr{T}} \mathscr{D}$ is an equivalence of ∞-categories.

We apply Proposition 6.21 and Corollary 6.15 to see the following.

cor:3721 **Corollary 6.22.** *A standard pullback diagram*

$$\begin{tikzcd} \mathscr{C} \times_{\mathscr{T}} \mathscr{D} \arrow[r] \arrow[d] & \mathscr{C} \arrow[d, "F"] \\ \mathscr{D} \arrow[r, "G"'] & \mathscr{T} \end{tikzcd}$$

of ∞-categories is a categorical pullback diagram provided one of F or G is an isofibration.

The following invariance result now follows from homotopy invariance of the homotopy pullback and Proposition 6.10.

prop:equiv_invariance **Proposition 6.23.** *Consider a diagram of ∞-categories*

$$\begin{tikzcd} \mathscr{K} \arrow[rr] \arrow[dd] \arrow[dr, "F"] & & \mathscr{C} \arrow[dd] \arrow[dr, "F_2"] & \\ & \mathscr{K}' \arrow[rr] \arrow[dd] & & \mathscr{C}' \arrow[dd] \\ \mathscr{D} \arrow[rr] \arrow[dr, "F_1"'] & & \mathscr{T} \arrow[dr, "F_0"] & \\ & \mathscr{D}' \arrow[rr] & & \mathscr{T}' \end{tikzcd} \tag{37}$$ eq:3432

in which all of the F_i is a equivalence. Then the following are equivalent:

(a) *Both the back and front faces in (37) are categorical pullback squares.*
(b) *Either the back face or the front face in (37) is a categorical pullback square, and the map F is an equivalence.*

cor:3776 **Corollary 6.24.** *Suppose we have a diagram*

$$\begin{tikzcd} \mathscr{C} \arrow[d] \arrow[drr, "F"] & & & & \mathscr{D} \arrow[d] \arrow[dll] \\ \mathscr{C}' \arrow[drr, "F'"'] & & \mathscr{T} \arrow[d] & & \mathscr{D}' \arrow[dll] \\ & & \mathscr{T}' & & \end{tikzcd}$$

in which F and F' are isofibrations, and all vertical maps are equivalences. Then the induced map $\mathscr{C} \times_{\mathscr{T}} \mathscr{D} \to \mathscr{C}' \times_{\mathscr{T}'} \mathscr{D}'$ is an equivalence of ∞-categories.

Proof. Apply Corollary 6.22 and Proposition 6.23. □

7. Mapping spaces

7.1. Definitions.

Definition 7.1. Let x and y be objects in an ∞-category $\mathscr{C}$. The mapping space $\mathrm{Hom}_{\mathscr{C}}(x, y)$ is the simplicial subset in $\mathrm{Fun}(\Delta^1, \mathscr{C})$ whose n-simplices are all maps $f : \Delta^n \times \Delta^1 \to \mathscr{C}$ with

$$f|\Delta^n \times \{0\} = id_x \quad \text{and} \quad f|\Delta^n \times \{1\} = id_y.$$

We've abused notation to write, for any object z in $\mathscr{C}$, $id_z : \Delta^n \to \mathscr{C}$ for the composite of the terminal map $\Delta^n \to *$ with the map $z : * \to \mathscr{C}$. We can reproduce $\mathrm{Hom}_{\mathscr{C}}(x, y)$ as a pullback

$$\begin{tikzcd} \mathrm{Hom}_{\mathscr{C}}(x, y) \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^1, \mathscr{C}) \arrow[d, "\mathrm{res}"] \\ * \arrow[r, "x \amalg y"'] & \mathrm{Fun}(\partial \Delta^1, \mathscr{C}). \end{tikzcd} \tag{38}$$ eq:2459

Lemma 7.2. *The mapping space* $\mathrm{Hom}_{\mathscr{C}}(x,y)$ *is an* ∞*-category.*

Proof. By Proposition 5.25 the restriction map from $\mathrm{Fun}(\Delta^1,\mathscr{C})$ is an inner fibration. So the terminal map $\mathrm{Hom}_{\mathscr{C}}(x,y)\to *$ is an inner fibration, and hence $\mathrm{Hom}_{\mathscr{C}}(x,y)$ is an ∞-category. $\square$

We will see momentarily that these morphism spaces $\mathrm{Hom}_{\mathscr{C}}(x,y)$ are Kan complexes.

Now, for any functor $F:\mathscr{C}\to\mathscr{D}$ the induced map $F_*:\mathrm{Fun}(\Delta^1,\mathscr{C})\to\mathrm{Fun}(\Delta^1,\mathscr{D})$ restricts to provide a map

$$F_*:\mathrm{Hom}_{\mathscr{C}}(x,y)\to\mathrm{Hom}_{\mathscr{D}}(Fx,Fy).$$

Remark 7.3. These Hom spaces do not admit a natural composition operation, at least before taking some homotopy truncation. So we are *not* providing an object and morphism description of our ∞-category $\mathscr{C}$.

Definition 7.4. A functor between ∞-categories $F:\mathscr{C}\to\mathscr{D}$ is called fully faithful if, for each pair of objects x and y in $\mathscr{C}$, the induced map on Hom spaces $F_*:\mathrm{Hom}_{\mathscr{C}}(x,y)\to\mathrm{Hom}_{\mathscr{D}}(Fx,Fy)$ is an equivalence.

7.2. **Natural isomorphisms.**

thm:natty_isom **Theorem 7.5.** *Let* $f,f':K\to\mathscr{C}$ *be two functors maps from a simplicial set to an* ∞*-category* $\mathscr{C}$*, and consider a natural transformation* $u:f\to f'$*, i.e. a map from* f *to* f' *in* $\mathrm{Fun}(K,\mathscr{C})$*. Then* t *is an isomorphism if and only if, for all vertices* x *in* K*, the map* $u_z:f(z)\to f(z)$ *is an isomorphism in* $\mathscr{C}$*.*

The proof relies on certain technical results about simplicial sets, which we recall here.

lem:2504 **Lemma 7.6** ([15, 01DN])**.** *For any integers* $m\geq 0$ *and* $n\geq 2$*, there is a sequence of simplicial sets*

$$(\Delta^m\times\Lambda^n_0\cup\partial\Delta^m\times\Delta^n)=X(0)\subseteq X(1)\subseteq\cdots\subseteq X(l)=\Delta^m\times\Delta^n$$

such that, for each positive integer $k\leq l$*, there are integers* $2\leq q$ *and* $p<q$ *which admit a pushout diagram*

$$\begin{tikzcd}
\Lambda^q_p \arrow[r] \arrow[d] & X(k-1) \arrow[d, "\mathrm{incl}"] \\
\Delta^q \arrow[r, "s"] & X(k).
\end{tikzcd}$$

Furthermore, if $p=0$ *then the map* s *can be chosen so that* $s(0)=(0,0)$ *and* $s(1)=(0,1)$*.*

In the expressions for $s(0)$ and $s(1)$, we identify the map $s:\Delta^p\to X(k)\subseteq\Delta^m\times\Delta^n$ with a choice of function $[p]\to[m]\times[n]$. We refer the reader directly to [15] for the proof.

lem:2521 **Lemma 7.7.** *Let* $Y\to S$ *be an inner fibration and* $\bar{F}:B\to S$ *be any map of simplicial sets. Consider any simplicial subset* $A\subseteq B$ *and integer* $n\geq 2$*, and let* $B\times\Delta^n\to S$ *be the composite of the projection* $B\times\Delta^n\to B$ *composed with* $\bar{F}$*.*

Suspect that we have a lifting problem

$$\begin{tikzcd} (A \times \Delta^n) \amalg_{A \times \Lambda^n_0} (B \times \Lambda^n_0) \arrow[r, "F"] \arrow[d] & Y \arrow[d] \\ B \times \Delta^n \arrow[r] \arrow[ur, dashed] & S \end{tikzcd} \tag{39}$$

eq:2523

for which, at every choice of vertex b in B, the corresponding edge

$$\Delta^1 \cong \{b\} \times \Delta^{\{0,1\}} \to \{b\} \times \Lambda^n_0 \to \{Fb\} \times_S Y = Y_b$$

is an isomorphism in the ∞-category Y_b. Then the problem (39) admits a solution.

Proof. We can replace B by any intermediate complex $A \subseteq K \subseteq B$, and consider the corresponding lifting problem L_K obtained by restricting F and $\bar{F}$ to $(A \times \Delta^n) \amalg_{A \times \Lambda^n_0} (K \times \Lambda^n_0)$ and K respectively. We consider the collection $\mathscr{P}$ of pairs (K, F_K) consisting of a choice of an intermediate complexes K and a map $F_K : K \times \Delta^n \to Y$ which solves the lifting problem $\mathbb{L}_K$. This collection $\mathscr{P}$ admits a natural partially ordered, where we take $(K, F_K) \leq (L, F_L)$ if and only if $K \subseteq L$ and $F_K = F_L|(K \times \Delta^n)$. By taking unions we see that any chain in $\mathscr{P}$ admits an upper bound, so we apply Zorns lemma to see that $\mathscr{P}$ admits a maximal element $(K_{\max}, F_{\max})$, and we claim that $K_{\max} = B$, so that the original lifting problem (39) admits a solution.

Suppose, by way of contradiction, that $K_{\max}$ is not B, and choose a non-degenerate simplex $z : \Delta^m \to B$ of minimal dimension which does not factor through $K_{\max}$. The minimality condition tells us that the restriction $z|\partial\Delta^m$ factors through $K_{\max}$, and we take $L = (K_{\max} \cup z) \subseteq B$. We therefore have the pushout diagram

$$\begin{tikzcd} \partial\Delta^m \arrow[r] \arrow[d] & K_{\max} \arrow[d] \\ \Delta^m \arrow[r] & L \end{tikzcd}$$

and consider the lifting problem

$$\begin{tikzcd} (K_{\max} \times \Delta^n) \amalg_{K_{\max} \times \Lambda^n_0} (L \times \Lambda^n_0) \arrow[r] \arrow[d] & Y \arrow[d] \\ L \times \Delta^n \arrow[r] \arrow[ur, dashed] & S \end{tikzcd} \tag{40}$$

eq:2548

where top map is obtained from $F_{\max} : K_{\max} \times \Delta^n \to Y$ and $F|(L \times \Lambda^n_0)$, and the bottom map is $\bar{F}|_L \circ \mathrm{proj}_1$. Via the above pushout diagram we can solve the lifting problem (40) by solving a corresponding pushout diagram

$$\begin{tikzcd} (\partial\Delta^m \times \Delta^n) \amalg_{\partial\Delta^m \times \Lambda^n_0} (\Delta^m \times \Lambda^n_0) \arrow[r, "f"] \arrow[d] & \Delta^m \times_S Y \arrow[d] \\ \Delta^m \times \Delta^n \arrow[r, "\bar{f}"'] \arrow[ur, dashed] & \Delta^m \end{tikzcd} \tag{41}$$

eq:2555

We note that at this point the base category Δ^m is an ∞-category, as is the fiber product $\Delta^m \times_S Y$. This is our main goal.

We break the inclusion

$$(\partial\Delta^m \times \Delta^n) \amalg_{\partial\Delta^m\times\Lambda^n_0} (\Delta^m \times \Lambda^n_0) = (\partial\Delta^m \times \Delta^n) \cup (\Delta^m \times \Lambda^n_0) \to \Delta^m \times \Delta^n$$

into a sequence of inclusion

$$(\partial\Delta^m \times \Delta^n) \cup (\Delta^m \times \Lambda^n_0) = X(0) \subseteq X(1) \subseteq \cdots \subseteq X(l) = \Delta^m \times \Delta^n$$

which satisfy the conclusions of Lemma 7.6. We have a map $f_0 = f : X(0) \to \Delta^m \times_S Y$ and we claim that for each $0 \leq k \leq l$ there is a map $f_k : X(k) \to \Delta^m \times_S Y$ which solve the lifting problem

$$\begin{tikzcd} X(0) \arrow[r, "f"] \arrow[d] & \Delta^m \times_S Y \arrow[d] \\ X(k) \arrow[r, "\bar{f}|X(k)"'] \arrow[ur, dashed, "f_k"] & \Delta^m. \end{tikzcd}$$

We prove this claim by induction.

Suppose that we have the desired map $f_{k-1} : X(k-1) \to \Delta^m \times_S Y$, for some $k > 0$. By hypotheses, we have a pushout diagram

$$\begin{tikzcd} \Lambda^q_p \arrow[r] \arrow[d] & X(k-1) \arrow[d] \\ \Delta^q \arrow[r, "s"] & X(k), \end{tikzcd}$$

where $p < q$. Supposing $0 < p$, the fact that the map $\Delta^m \times_S Y \to \Delta^m$ is an inner fibration tells us that we can lift $\bar{f} : X(k) \to \Delta^m$ to a map $f_k : X(k) \to \Delta^m \times_S Y$ with $f_k|X(k-1) = f_{k-1}$. Supposing $p = 0$, then we can find a lift $f_k : X(k) \to \Delta^m \times_S Y$ by our hypotheses that $s(0) = (0,0)$, $s(1) = (0,1)$, and the hypotheses that $f_{k-1} : \Delta^{\{0,1\}} \to \{\bar{f}0\} \times_S Y$ is an isomorphism, and a subsequent application of Proposition 5.33.

In any case, we can always extend the map f_{k-1} to f_k, as desired, and the map $f_l : X(l) = \Delta^m \times \Delta^n \to \Delta^m \times_S Y$ solves the lifting problem (41). We can therefore solve the original lifting problem (40), which contradicts the maximality of the pair $(K_{\max}, F_{\max})$. So we have necessarily $K_{\max} = B$, and $F_{\max} : B \times \Delta^n \to Y$ solves our lifting problem (39). $\square$

Theorem 7.5 is now obtained as a special case of the following.

prop:2597 **Proposition 7.8.** *Consider an inner fibration $\mathscr{C} \to S$ and an arbitrary map of simplicial sets $a : K \to S$, and let $u : F \to F'$ be a map in the ∞-category $\mathrm{Fun}_{/S}(K, \mathscr{C})$. Then u is an isomorphism if and only if, at every vertex z in K, the map $u_z : F(z) \to F'(z)$ is an isomorphism in $\mathscr{C}_z = \{a(z)\} \times_S \mathscr{C}$.*

To be clear, the space $\mathrm{Fun}_{/S}(K, \mathscr{C})$ is the fiber product

$$\mathrm{Fun}_{/S}(K, \mathscr{C}) = \mathrm{Fun}(K, \mathscr{C}) \times_{\mathrm{Fun}(K,S)} \{a\}$$

Since $\mathscr{C} \to S$ is an inner fibration, the map $\mathrm{Fun}(K, \mathscr{C}) \to \mathrm{Fun}(K, S)$ is an inner fibration by Corollary 5.8. So the fiber $\mathrm{Fun}_{/S}(K, \mathscr{C})$ is in fact an ∞-category.

Proof. If u is an isomorphism, then one can evaluate any choice of inverse map $v : F' \to F$ at the vertices in K to produce inverses to the u_z. For the converse claim, let's suppose that u_z is an isomorphism at all $z \in B[0]$. We consider the diagrams

$$\begin{tikzcd} K \times \Lambda^2_0 \arrow[r] \arrow[d] & \mathscr{C} \arrow[d] \\ K \times \Delta^2 \arrow[r] & S \end{tikzcd} \tag{42}$$ eq:2609

where the bottom map is the projection $K \times \Delta^2 \to K$ composed with a, and the top map has restrictions F, F', and F to $K \times \Delta^{\{0\}}$, $K \times \Delta^{\{1\}}$, and $K \times \Delta^{\{2\}}$ respectively. The restrictions to $K \times \Delta^{\{0,1\}}$ and $K \times \Delta^{\{0,2\}}$ are u and id_F respectively. By the hypothesis that u_z is an isomorphism at all z, we may apply Lemma 7.7 to find a map $\eta : K \times \Delta^2 \to \mathscr{C}$ with solves the lifting problem corresponding to (42). For $v = \eta | K \times \Delta^{\{1,2\}} : F' \to F$ we therefore have $v \circ u \sim id_F$. So u admits a left inverse in $\mathrm{Fun}_{/S}(K, \mathscr{C})$. We consider opposite categories to similarly find that the opposite map $u^{op} : (F')^{op} \to F^{op}$ admits a left inverse w^{op}, and hence that u admits a right inverse w. Hence u is an isomorphism. □

Proof of Theorem 7.5. Apply Proposition 7.8 in the case $S = *$. □

7.3. **The mapping spaces are spaces.**

Theorem 7.9. *For any objects x, y in an ∞-category $\mathscr{C}$, the ∞-category of maps $\mathrm{Hom}_{\mathscr{C}}(x, y)$ is a Kan complex.*

Proof. It suffices to show that all morphisms in $\mathrm{Hom}_{\mathscr{C}}(x, y)$ are isomorphisms. Objects in $\mathrm{Hom}_{\mathscr{C}}(x, y)$ are maps $f : \Delta^1 \to \mathscr{C}$ with $f(0) = x$ and $f(1) = y$, and a morphism $f \to f'$ in $\mathrm{Hom}_{\mathscr{C}}(x, y)$ is a natural transformation $u : f \to f'$ with $u_0 = id_x : f(0) \to f(0)$ and $u_1 = id_y : f(1) \to f(1)$. So we just apply Theorem 7.5 directly to see that any map in $\mathrm{Hom}_{\mathscr{C}}(x, y)$ is an isomorphism. □

sect:loop

7.4. **Mapping spaces as loop spaces.**

We consider an example of such a fiber diagram (17) which relates mapping spaces and for a Kan complex $\mathscr{X}$ to the underlying space $\mathscr{X}$ itself, and in particular allows us to access the (higher) homotopy group for $\mathscr{X}$ via the homotopy groups of its mapping spaces.

Consider any Kan complex $\mathscr{X}$, and the inclusion $i : \Delta^{\{0\}} \to \Delta^1$. Since i is anodyne the corresponding restriction map $i^* : \mathrm{Fun}(\Delta^1, \mathscr{X}) \to \mathscr{X}$ is a trivial Kan fibration, by Corollary 4.12. Taking the fiber along any point $x : * \to \mathscr{X}$ provides another trivial Kan fibration

$$\{x\} \times_{\mathrm{Fun}(\Delta^{\{0\}}, \mathscr{X})} \mathrm{Fun}(\Delta^1, \mathscr{X}) \to *,$$

so that the domain is seen to be contractible.

Now, restriction along the inclusion $j : \partial\Delta^1 \to \Delta^1$ provides a Kan fibration $j^* : \mathrm{Fun}(\Delta^1, \mathscr{X}) \to \mathrm{Fun}(\partial\Delta^1, \mathscr{X})$ and we have the pullback diagram

$$\begin{tikzcd} \mathscr{X} \arrow[r] \arrow[d] & * \arrow[d, "x"] \\ \mathrm{Fun}(\partial\Delta^1, \mathscr{X}) \arrow[r] & \mathrm{Fun}(\Delta^{\{0\}}, \mathscr{X}) \cong \mathscr{X}, \end{tikzcd}$$

identifying $\mathscr{X}$ with the subcomplex of maps $\partial\Delta^1 \to \mathscr{X}$ which take constant value x on $\Delta^{\{0\}}$. Pulling back j^* along the inclusion $\mathscr{X} \to \mathrm{Fun}(\partial\Delta^1, \mathscr{X})$ then yields the map

$$\{x\} \times_{\mathrm{Fun}(\Delta^{\{0\}},\mathscr{X})} \mathrm{Fun}(\Delta^1,\mathscr{X}) \to \mathscr{X}, \quad (s,t) \mapsto t|_{\Delta^{\{1\}}},$$

which we now conclude is a Kan fibration as well. Let us call this map f.

We now, finally, have a pullback diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathscr{X}}(x,x) \arrow[r] \arrow[d] & \{x\} \times_{\mathrm{Fun}(\Delta^{\{0\}},\mathscr{X})} \mathrm{Fun}(\Delta^1,\mathscr{X}) \arrow[d, "f"] \\ * \arrow[r, "x"'] & \mathscr{X} \end{tikzcd} \tag{43}$$

eq:maps_loop

with f a Kan fibration from a contractible domain. The long exact sequence on homotopy groups now provides the following.

prop:based

Proposition 7.10. *For any Kan complex $\mathscr{X}$, point $x : * \to \mathscr{X}$, and integer $n \geq 0$, there is a natural isomorphisms of homotopy groups*

$$\partial_{n+1} : \pi_{n+1}(\mathscr{X}, x) \overset{\sim}{\to} \pi_n(\mathrm{Hom}_{\mathscr{X}}(x,x), id_x).$$

We only note that naturality (with respect to maps of Kan complexes $\mathscr{X} \to \mathscr{Y}$) comes from naturality of the above constructions in conjunction with Proposition 4.37.

Remark 7.11 ([15, 01JE])**.** The diagram (43) identifies the mapping space $\mathrm{Hom}_{\mathscr{X}}(x,x)$ as the based loop space for $\mathscr{X}$, in the homotopy category of spaces.

Via the naturality claim from Proposition 7.10 and Whitehead's Theorem we observe that any map of Kan complexes which is essentially surjective and fully faithful is also an equivalence.

cor:ffes_kan

Corollary 7.12. *If $f : \mathscr{X} \to \mathscr{Y}$ is a map of Kan complexes which is fully faithful and essentially surjective, then f is an equivalence.*

sect:mapping_spaces

8. Fully faithful functors and equivalence

8.1. Fully faithful functors and equivalences.

Definition 8.1. We say a functor between ∞-categories $F : \mathscr{C} \to \mathscr{D}$ is fully-faithful if the induced maps

$$F_* : \mathrm{Hom}_{\mathscr{C}}(x,y) \to \mathrm{Hom}_{\mathscr{D}}(Fx, Fy)$$

are equivalences of Kan complexes, at all x, y in $\mathscr{C}$. We say F is essentially surjective if F induces a surjection

$$\pi_0(F^{\mathrm{Kan}}) : \pi_0(\mathscr{C}^{\mathrm{Kan}}) \to \pi_0(\mathscr{D}^{\mathrm{Kan}})$$

on isoclasses of objects.

The main result of the section is to prove the following essential result.

thm:ffes_equiv

Theorem 8.2. *A functor $F : \mathscr{C} \to \mathscr{D}$ between ∞-categories is an equivalence if and only if F is fully faithful and essentially surjective.*

While any equivalence F is easily seen to be essentially surjective, the fact that the operation $\mathrm{Hom}_{\mathscr{C}}(x,-) : \mathscr{C}[1] \to \mathrm{Kan}$ is not a priori defined on morphisms $\mathscr{C}[2]$ makes fully faithfulness slightly unclear. So, we are interested in both of the claims in Theorem 8.2. We first deal with the "easier" implication

$$F \text{ is an equivalence } \Rightarrow F \text{ is fully faithful and essentially surjective},$$

which still requires quite a significant analysis.

8.2. **Equivalences are fully faithful.** Let us recall that the restriction functor

$$\mathrm{Fun}(\Delta^1, \mathscr{C}) \to \mathrm{Fun}(\partial\Delta^1, \mathscr{C})$$

by Proposition 6.13, and hence the map on associated Kan complexes is a Kan fibration by Corollary 5.34. We now find ourselves in a particularly advantageous situation. Since $\mathrm{Hom}_{\mathscr{C}}(x, y)$ is itself a Kan complex we see that the inclusion $\mathrm{Hom}_{\mathscr{C}}(x, y) \to \mathrm{Fun}(\Delta^1, \mathscr{C})$ factors through the associated Kan complex, and we therefore realize $\mathrm{Hom}_{\mathscr{C}}(x, y)$ as the pullback of a point along a Kan fibration

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{C}}(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\textit{pullback sq}"] & \mathrm{Fun}(\Delta^1, \mathscr{C})^{\mathrm{Kan}} \arrow[d, "\textit{Kan fib}"] \\
* \arrow[r] & \mathrm{Fun}(\partial\Delta^1, \mathscr{C})^{\mathrm{Kan}}
\end{tikzcd}$$

With this framing in mind, we now deduce fully faithfulness for equivalences of ∞-categories.

`cor:equiv_ff` **Corollary 8.3.** *Any equivalence of ∞-categories $F : \mathscr{C} \to \mathscr{D}$ is fully faithful.*

Proof. At any pair of objects $x, y : * \to \mathscr{C}$ we have the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{C}}(x,y) \arrow[rr] \arrow[d] \arrow[dr] & & \mathrm{Hom}_{\mathscr{D}}(Fx,Fy) \arrow[d] \arrow[dr] & \\
* \arrow[dr] & \mathrm{Fun}(\Delta^1, \mathscr{C})^{\mathrm{Kan}} \arrow[rr, "\sim"] \arrow[d] & * \arrow[dr] & \mathrm{Fun}(\Delta^1, \mathscr{D})^{\mathrm{Kan}} \arrow[d] \\
& \mathrm{Fun}(\partial\Delta^1, \mathscr{C})^{\mathrm{Kan}} \arrow[rr, "\sim"] & & \mathrm{Fun}(\partial\Delta^1, \mathscr{D})^{\mathrm{Kan}}.
\end{tikzcd} \tag{44}$$ `eq:2914`

Here the horizontal (red) maps are induced by F and those maps labeled $\sim$ are equivalences of ∞-categories, by Theorem 5.41, and the (white) squares involving the point $*$ are pullback diagrams. Hence the map

$$F_* : \mathrm{Hom}_{\mathscr{C}}(x, y) \to \mathrm{Hom}_{\mathscr{D}}(Fx, Fy)$$

is an equivalence, by Proposition 4.38. □

8.3. **The Kan complex functor and equivalences.** The aim of this subsection is to prove the following.

`prop:2881` **Proposition 8.4.** *Suppose $F : \mathscr{C} \to \mathscr{D}$ is an equivalence between ∞-categories. Then the induced map $F^{\mathrm{Kan}} : \mathscr{C}^{\mathrm{Kan}} \to \mathscr{D}^{\mathrm{Kan}}$ is also an equivalence.*

Proposition 8.4 is used in our proof of the fact that equivalences of ∞-categories are essentially surjective. The proof is relatively straightforward, but requires some technical details. We provide the necessary details then return to provide the proof of Proposition 8.4.

`lem:2822` **Lemma 8.5.** *Consider ∞-categories $\mathscr{A}$ and $\mathscr{B}$, and morphisms $\alpha : a_1 \to a_2$ and $\beta : b_1 \to b_2$ in $\mathscr{A}$ and $\mathscr{B}$ respectively. Then the product map $[\alpha, \beta] : \Delta^1 \to \mathscr{A} \times \mathscr{B}$ is a composition $[\alpha, \beta] = [a_1, \beta] \circ [\alpha, b_1]$ in $\mathscr{A} \times \mathscr{B}$*

Proof. We need to construct a 2-simplex $c : \Delta^2 \to \mathscr{A} \times \mathscr{B}$ with the three maps above appearing as the boundary morphisms for c. By definition, any such c is specified by two 2-simplices $c_1 : \Delta^2 \to \mathscr{A}$ and $c_2 : \Delta^2 \to \mathscr{B}$. We take

$$c_1 = \Delta^2 \overset{d_1}{\to} \Delta^1 \overset{\alpha}{\to} \mathscr{A}, \quad c_2 = \Delta^2 \overset{d_0}{\to} \Delta^1 \overset{\beta}{\to} \mathscr{B},$$

to produce our 2-simplex $c = [c_1, c_2]$ with

$$c|\Delta^{\{0,1\}} = [\alpha, id_{b_1}],\ c|\Delta^{\{1,2\}} = [id_{a_2}, \beta],\ \text{and}\ c|\Delta^{\{0,3\}} = [\alpha, \beta],$$

as desired. □

Recall that for any Kan complexes $\mathscr{X}$ and $\mathscr{Y}$ the functor space $\mathrm{Fun}(\mathscr{X}, \mathscr{Y})$ is a Kan complex. Hence for any Kan complex $\mathscr{X}$ and ∞-category $\mathscr{E}$ the inclusion $\mathscr{E}^{\mathrm{Kan}} \to \mathscr{E}$ induces a natural map of Kan complexes

$$\mathrm{Fun}(\mathscr{X}, \mathscr{E}^{\mathrm{Kan}}) \to \mathrm{Fun}(\mathscr{X}, \mathscr{E})^{\mathrm{Kan}}. \tag{45}$$ eq:2841

lem:2845 **Lemma 8.6.** *For any Kan complex $\mathscr{X}$, and ∞-category $\mathscr{E}$, the inclusion (45) is a isomorphism of Kan complexes.*

Proof. Take any n-simplex $w : \Delta^n \times \mathscr{X} \to \mathscr{E}$ in $\mathrm{Fun}(\mathscr{X}, \mathscr{E})^{\mathrm{Kan}}$. We want to show that w has image in $\mathscr{E}^{\mathrm{Kan}}$, so that w factors uniquely through an n-simplex $\widetilde{w} : \Delta^n \times \mathscr{X} \to \mathscr{E}^{\mathrm{Kan}}$. We just need to show that the image $w(\xi)$ of every morphism ξ in the ∞-category $\Delta^n \times \mathscr{X}$ is an isomorphism in $\mathscr{E}$.

Now, morphisms $\xi : \Delta^1 \to \Delta^n \times \mathscr{X}$ are of three flavors. In the first case $\xi = [\xi_1, x]$ for some point $x : \Delta^0 \to \mathscr{X}$, and

$$w(\xi) : \xi_1|_{\Delta^{\{0\}}}(x) \to \xi_1|_{\Delta^{\{1\}}}(x)$$

is the evaluation of a natural transformation $\xi|\Delta^1 \times \mathscr{X}$ between the functors $\xi_1|_{\Delta^{\{0\}}}$ and $\xi_1|_{\Delta^{\{1\}}}$ at x. Since ξ is a 2-simplex in $\mathrm{Fun}(\mathscr{X}, \mathscr{E})^{\mathrm{Kan}}$, this natural transformation is in particular a natural isomorphism. By Proposition 7.8 it follows that $w(\xi)$ is an isomorphism. In the second case $\xi = [i, \xi_2]$ for a morphism $\xi_2 : \Delta^2 \to \mathscr{X}$, so that $w(\xi)$ is the image of a morphism in $\mathscr{X}$ under the composite functor

$$\mathscr{X} \overset{\cong}{\to} \Delta^{\{i\}} \times \mathscr{X} \overset{w_i}{\to} \mathscr{E}.$$

Since functors between ∞-categories preserve isomorphisms, we see that $w(\xi)$ is an isomorphism. Lastly, when $\xi = [\xi_1, \xi_2]$ for morphisms ξ_1 in Δ^1 and ξ_2 in $\mathscr{X}$, we realize $w(\xi)$ as a composite

$$w(\xi) = w([i, \xi_2]) \circ w([\xi_1, x])$$

by Lemma 8.5. We have just argued that $w([\xi_1, x])$ and $w([i, \xi_2])$ are isomorphisms in $\mathscr{E}$, so that $w(\xi)$ is again an isomorphism in $\mathscr{E}$.

We now see any n-simplex $w : \Delta^n \times \mathscr{X} \to \mathscr{E}$ has image in $\mathscr{E}$. This show that all n-simplices in $\mathrm{Fun}(\mathscr{X}, \mathscr{E})^{\mathrm{Kan}}$ lie in the subcomplex $\mathrm{Fun}(\mathscr{X}, \mathscr{E}^{\mathrm{Kan}})$, and hence that these two complexes are identified. □

The above Proposition tells us that the associated Kan complex functor $-^{\mathrm{Kan}}$ lifts to a functor between simplicial categories, and subsequently between ∞-categories

$$-^{\mathrm{Kan}} : \mathscr{C}at_\infty \to \mathscr{K}an = \mathscr{S}paces.$$

We then get an induced map on homotopy categories

$$\mathrm{h}(-^{\mathrm{Kan}}) : \mathrm{h}\mathscr{C}at_\infty \to \mathrm{h}\mathscr{S}paces.$$

This functor necessarily sends isomorphisms to isomorphisms, which implies Proposition 8.4. We record a more direct proof in any case.

Proof of Proposition 8.4. Consider an inverse $F' : \mathscr{D} \to \mathscr{C}$ and natural isomorphisms

$$\xi : \Delta^1 \to \mathrm{Fun}(\mathscr{C}, \mathscr{C})^{\mathrm{Kan}}, \;\; \eta : \Delta^1 \to \mathrm{Fun}(\mathscr{D}, \mathscr{D})^{\mathrm{Kan}}$$

between $F'F$ and the identity on $\mathscr{C}$, and FF' and the identity on $\mathscr{D}$, respectively. We restrict the domains to produce natural isomorphisms

$$\bar{\xi} : \Delta^1 \to \mathrm{Fun}(\mathscr{C}^{\mathrm{Kan}}, \mathscr{C})^{\mathrm{Kan}} = \mathrm{Fun}(\mathscr{C}^{\mathrm{Kan}}, \mathscr{C}^{\mathrm{Kan}})$$

and

$$\bar{\eta} : \Delta^1 \to \mathrm{Fun}(\mathscr{D}^{\mathrm{Kan}}, \mathscr{D})^{\mathrm{Kan}} = \mathrm{Fun}(\mathscr{D}^{\mathrm{Kan}}, \mathscr{D}^{\mathrm{Kan}})$$

which realize $(F')^{\mathrm{Kan}}$ as an inverse to F^{Kan}. □

8.4. **Equivalences are essentially surjective.**

prop:equiv_ffes **Proposition 8.7.** *An equivalence between ∞-categories $F : \mathscr{C} \to \mathscr{D}$ is essentially surjective.*

Proof. The induced map on Kan complexes $F^{\mathrm{Kan}} : \mathscr{C}^{\mathrm{Kan}} \to \mathscr{D}^{\mathrm{Kan}}$ is also an equivalence, by Proposition 8.4. By Whitehead's theorem, the induced map on connected components $\pi_0 F^{\mathrm{Kan}} : \pi_0(\mathscr{C}^{\mathrm{Kan}}) \to \pi_0(\mathscr{D}^{\mathrm{Kan}})$ is an isomorphism, so that F is essentially surjective. □

We've now established the "easier" direction for Theorem 8.2,

$$\text{equivalence} \;\Rightarrow\; \text{fully faithful} + \text{essentially surjective}.$$

8.5. **Restricting fully faithful functors.** We now want to establish the implication

$$\text{fully faithful} + \text{essentially surjective} \;\Rightarrow\; \text{equivalence}.$$

In the most basic terms, this implication follows via a reduction argument from the ∞-category setting to the Kan complex setting. We first prove that any fully faithful functor between ∞-categories restricts to a fully faithful functor on the associated Kan complexes.

lem:3002 **Lemma 8.8.** *Let $F : \mathscr{X} \to \mathscr{Y}$ be a equivalence between Kan complexes and let $\mathscr{X}' \subseteq \mathscr{X}$ and $\mathscr{Y}' \subseteq \mathscr{Y}$ be full subcategories. Suppose that F restricts to a map $F' : \mathscr{X}' \to \mathscr{Y}'$ and that the induced map on connected components $\pi_0(\mathscr{X}') \to \pi_0(\mathscr{Y}')$ is a bijection. Then the restriction F' is an equivalence of Kan complexes.*

Proof. For any $x \in \mathscr{X}'$ and $y \in \mathscr{Y}'$ the respective inclusions induce equalities

$$\pi_n(\mathscr{X}', x) = \pi_n(\mathscr{X}, x) \;\text{ and }\; \pi_n(\mathscr{Y}', y) = \pi(\mathscr{Y}, y)$$

for all positive integers n. Via Whitehead's theorem we understand that F induces an isomorphism on all homotopy groups for $\mathscr{X}$ and $\mathscr{Y}$, and therefore that F' induces isomorphisms on all homotopy groups for $\mathscr{X}'$ and $\mathscr{Y}'$. Apply Whitehead again to observe that F' is an equivalence. □

lem:3082 **Lemma 8.9.** *Suppose that a functor between ∞-categories $F : \mathscr{C} \to \mathscr{D}$ is fully faithful. Then the induced map $F^{\mathrm{Kan}} : \mathscr{C}^{\mathrm{Kan}} \to \mathscr{D}^{\mathrm{Kan}}$ is fully faithful as well.*

Proof. For objects x and x' in $\mathscr{C}$, with images y and y' in $\mathscr{D}$, we consider the map of Kan complexes

$$\mathrm{Hom}_{\mathscr{C}^{\mathrm{Kan}}}(x,x') \to \mathrm{Hom}_{\mathscr{D}^{\mathrm{Kan}}}(y,y').$$

These Kan complexes sit as fully subcategories in the ambient categories $\mathrm{Hom}_{\mathscr{C}}(x,x')$ and $\mathrm{Hom}_{\mathscr{D}}(y,y')$, and we have a diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathscr{C}^{\mathrm{Kan}}}(x,x') \arrow[r, "F^{\mathrm{Kan}}"] & \mathrm{Hom}_{\mathscr{D}^{\mathrm{Kan}}}(y,y') \\ \mathrm{Hom}_{\mathscr{C}}(x,x') \arrow[u] \arrow[r, "F"] & \mathrm{Hom}_{\mathscr{D}}(y,y'). \arrow[u] \end{tikzcd}$$

So it suffices to show that F^{Kan} induces a bijection on connected components, by Lemma 8.8. For this it suffices to prove that every equivalence $\beta : y \to y'$ lifts to an equivalence $\alpha : x \to x'$ in $\mathscr{C}$. Let us choose an arbitrary map $\alpha : x \to x'$ with $F\alpha \simeq \beta$. We claim that α is an equivalence.

Take any lift $\alpha' : x' \to x$ of an inverse $\beta^{-1} : y' \to y$, and consider a composites $\alpha'\alpha$ and $\alpha\alpha'$. We have $F(\alpha'\alpha) \simeq id_y$ and since F indues an equivalence on morphism spaces we have $\alpha'\alpha \simeq id_x$, and similarly find $\alpha\alpha' \simeq id_y$. So α is in fact an equivalence, i.e. a map in $\mathscr{C}^{\mathrm{Kan}}$, as desired. □

As we saw in the proof, for a given fully faithful functor $F : \mathscr{C} \to \mathscr{D}$ we find that a given map α in $\mathscr{C}$ is an equivalence if and only if $F\alpha$ is an equivalence.

Definition 8.10. A functor between ∞-categories $F : \mathscr{C} \to \mathscr{D}$ is called conservative if a map $\alpha : x \to x'$ in $\mathscr{C}$ is an equivalence if and only if $F\alpha : Fx \to Fx'$ is an equivalence.

We have the following.

Lemma 8.11 ([15, 01JN])**.** *Any fully faithful functor between ∞-categories is conservative.*

8.6. **Proof of Theorem 8.2.** We begin with a refinement of Proposition 8.4.

lem:3123 **Lemma 8.12.** *Suppose $F : \mathscr{C} \to \mathscr{D}$ is a trivial Kan fibration between ∞-categories. Then the map $F^{\mathrm{Kan}} : \mathscr{C}^{\mathrm{Kan}} \to \mathscr{D}^{\mathrm{Kan}}$ is a trivial Kan fibration as well.*

Proof. The lifting property applies to the inclusions $\Lambda^2_0 \to \Delta^2$ and Λ^2_2 imply that F is conservative. Hence the lifting property for F immediately restricts to a lifting property for F^{Kan}. □

The following result allows us to descend from the Kan setting to the ∞-setting.

thm:OK **Theorem 8.13** ([15, 01HG])**.** *A map between ∞-categories $F : \mathscr{C} \to \mathscr{D}$ is an equivalence if and only if the associated map of Kan complexes*

$$\mathrm{Fun}(\Delta^1, F)^{\mathrm{Kan}} : \mathrm{Fun}(\Delta^1, \mathscr{C})^{\mathrm{Kan}} \to \mathrm{Fun}(\Delta^1, \mathscr{D})^{\mathrm{Kan}}$$

is an equivalence.

In the proof we only outline some of the points.

Sketch proof. Take

$$F^K = \mathrm{Fun}(K, F)^{\mathrm{Kan}} : \mathrm{Fun}(K, \mathscr{C})^{\mathrm{Kan}} \to \mathrm{Fun}(K, \mathscr{D})^{\mathrm{Kan}}$$

at a given simplicial set K. By Theorem 5.41, F is an equivalence if and only if F^K is an equivalence at all K. In particular, if F is an equivalence then F^{Δ^1} is an equivalence.

For the converse claim, suppose that F^{Δ^1} is an equivalence and let Υ denote the class of all simplicial sets K at which F^K is an equivalence. One can show the following: (O1) If we have a directed sequence $K_0 \to K_1 \to \cdots$ is maps of simplicial sets, and all F^{K_i} are equivalences, then the map F^K is an equivalence for $K = \varinjlim_n K_n$. (O2) If we have a retract $L \to K \to L$, and F^K is an equivalence, then F^L is an equivalence. (O3) If $\{K_\lambda\}_\lambda$ is a set of simplicial sets for which all F^{K_λ} are equivalences, then the map $F^{\amalg_\lambda K_\lambda}$ is an equivalence. (04) If $L \to K$ is inner anodyne, then F^L is an equivalence if and only if F^K is an equivalence (Corollary 5.8 and Lemma 8.12). (O5) If

$$\begin{tikzcd} L_0 \arrow[r, "\mu_1"] \arrow[d, "\mu_2"'] & L_1 \arrow[d] \\ L_2 \arrow[r] & K \end{tikzcd}$$

is a categorical pushout square and all of the F^{L_i} are equivalences, then F^K is an equivalence (Proposition 6.10). This applies, in particular, to the case of a standard pushout square in which one of the μ_i is injective (Corollary 6.18).

We now see that Υ is a class of simplicial sets which contains Δ^1 and which is stable under the operations (O1)–(O5). Since the 0-simplex is a retract of Δ^1, we see that Υ contains Δ^0. Now, supposing all Δ^m are in Υ for $m < n$, and $n \geq 2$, one sees that all horns Λ^{n+1}_i are in Υ by considering the appropriate pushout diagram, and subsequently sees that Δ^n is in Υ by considering the inner anodyne morphism $\Lambda^n_1 \to \Delta^n$ and applying (O4). It follows by induction that all standard simplices are in Υ, and subsequently that Υ contains all simplicial sets via applications of (O5) and (O1). □

We now provide a proof of Theorem 8.2.

Proof. Let $F : \mathscr{C} \to \mathscr{D}$ be a functor between ∞-categories. If F is an equivalence, the F is fully faithful and essentially surjective by Corollary 8.3 and Proposition 8.7. Suppose conversely that F is fully faithful and essentially surjective.

Let us consider the diagram

$$\begin{tikzcd} \mathrm{Fun}(\Delta^1, \mathscr{C})^{\mathrm{Kan}} \arrow[r, "F_*"] \arrow[d] & \mathrm{Fun}(\Delta^1, \mathscr{D})^{\mathrm{Kan}} \arrow[d] \\ \mathrm{Fun}(\partial\Delta^1, \mathscr{C})^{\mathrm{Kan}} \arrow[r, "F_*"] & \mathrm{Fun}(\partial\Delta^1, \mathscr{D})^{\mathrm{Kan}}. \end{tikzcd}$$

We know that the bottom map is an fully faithful and essentially surjective, by Lemma 8.9, and hence an equivalence by Corollary 7.12. Also by Corollary 6.15 the vertical maps are both Kan fibrations.

Since the fibers of the vertical maps over points in $\mathrm{Fun}(\partial\Delta^1, \mathscr{C})$ and $\mathrm{Fun}(\partial\Delta^1, \mathscr{D})$ are the respective mapping spaces, and F_* restricts the the induced morphism

$$F_{x,y} : \mathrm{Hom}_{\mathscr{C}}(x, y) \to \mathrm{Hom}_{\mathscr{D}}(Fx, Fy)$$

on these fibers. So, via fully faithfulness of F, we apply Proposition 4.38 to see that $F_* : \mathrm{Fun}(\Delta^1, \mathscr{C})^{\mathrm{Kan}} \to \mathrm{Fun}(\Delta^1, \mathscr{D})^{\mathrm{Kan}}$ is an equivalence. By Theorem 8.13 if follows that F is an equivalence. We are done. □

9. Composition functions for mapping spaces

We conclude with an indication, so to speak, of a composition function for mapping spaces in an ∞-category.

9.1. A spacially enriched category. Take

$$P^n = \Delta^{\{0,1\}} \coprod_{\{1\}} \Delta^{\{1,2\}} \cdots \coprod_{\{n-1\}} \Delta^{\{n-1,n\}}.$$

Since the functor $\mathrm{Fun}(K, -)$ is right adjoint to the functor $K \times -$, and the product $- \times L$ is a left adjoint, we see that the functor $\mathrm{Fun}(-, \mathscr{C})$ sends colimits to limits. Hence we have an identification

$$\mathrm{Fun}(P^n, \mathscr{C}) = \mathrm{Fun}(\Delta^{\{n-1,n\}}, \mathscr{C}) \times_{\mathrm{Fun}(\{n-1\}, \mathscr{C})} \cdots \times_{\mathrm{Fun}(\{1\}, \mathscr{C})} \mathrm{Fun}(\Delta^{\{0,1\}}, \mathscr{C})$$

at each ∞-category $\mathscr{C}$.

We have the inclusion $P^n \to \Delta^n$. One can show the following.

Lemma 9.1 ([15, 00JA])**.** *For all $n \geq 2$, the inclusion $P^n \to \Delta^n$ is inner anodyne.*

We now find, via Corollary 5.8 that the functor

$$\mathrm{Fun}(\Delta^n, \mathscr{C}) \to \mathrm{Fun}(\Delta^{\{n-1,n\}}, \mathscr{C}) \times_{\mathrm{Fun}(\{n-1\}, \mathscr{C})} \cdots \times_{\mathrm{Fun}(\{1\}S, \mathscr{C})} \mathrm{Fun}(\Delta^{\{0,1\}}, \mathscr{C})$$

obtained via restriction along the inclusion $P^n \to \Delta^n$, is a trivial Kan fibration. In particular, it is an equivalence. We record an overdue lemma in this regard.

Lemma 9.2. *A trivial Kan fibration between ∞-categories is an equivalence.*

Proof. For such a trivial Kan fibration $F : \mathscr{C} \to \mathscr{D}$, one solves the relevant lifting problem to see that the induced map $F_* : \mathrm{Fun}(K, \mathscr{C}) \to \mathrm{Fun}(K, \mathscr{D})$ is a trivial Kan fibration at any simplicial set K (cf. proof of Proposition 5.7). It follows that the induced map on Kan complexes F_*^{Kan} is an equivalence, and hence that F is an equivalence by Theorem 5.41. □

We now consider the inclusion $\Delta^{\{0,n\}} \to \Delta^n$ and defined the "nth composition functions" for $\mathscr{C}$ as the diagram

$$\begin{tikzcd}
\mathrm{Fun}(\Delta^n, \mathscr{C}) \arrow[d, "\text{triv Kan}"'] \arrow[dr] & \\
\mathrm{Fun}(\Delta^{\{n-1,n\}}, \mathscr{C}) \times_{\mathrm{Fun}(\{n-1\}, \mathscr{C})} \cdots \times_{\mathrm{Fun}(\{1\}S, \mathscr{C})} \mathrm{Fun}(\Delta^{\{0,1\}}, \mathscr{C}) & \mathrm{Fun}(\Delta^{\{0,n\}}, \mathscr{C}).
\end{tikzcd}$$

Taking the fiber at an n-tuple of points $\vec{x} : \amalg_{i=1}^n \{i\} \to \mathscr{C}$ provides an "n-th composition function" for the mapping spaces

$$\begin{tikzcd}
\mathrm{Fun}(\Delta^n, \mathscr{C})_{\vec{x}} \arrow[d, "\text{triv Kan}"'] \arrow[dr] & \\
\mathrm{Hom}_{\mathscr{C}}(x_{n-1}, x_n) \times \cdots \times \mathrm{Hom}_{\mathscr{C}}(x_0, x_1) & \mathrm{Hom}_{\mathscr{C}}(x_0, x_n).
\end{tikzcd} \tag{46}$$ eq:comp

Note that the fact that the product on the lower left is a Kan complex, and that the map from $\mathrm{Fun}(\Delta^n, \mathscr{C})_{\vec{x}}$ is a trivial Kan fibration, implies that this fiber is a Kan complex as well.

By considering the appropriate diagrams one sees that the above composition functions are sufficiently associative. In particular, for any n-tuple of objects $\vec{x}$ in $\mathscr{C}$, we obtain a contravariant functor

$$\mathrm{sSet}^{\mathrm{op}}_{/\Delta^n} \to \mathrm{Fun}(-, \mathscr{C})_{\vec{x}}.$$

Proposition 9.3 ([15, 01PS, 01PT]). *The diagram* (46) *defines an associative and unital composition operation*

$$\circ : \mathrm{Hom}_{\mathscr{C}}(y, z) \times \mathrm{Hom}_{\mathscr{C}}(x, y) \to \mathrm{Hom}_{\mathscr{C}}(x, z)$$

in the homotopy category of spaces.

We now find that the pairing of the objects $\mathscr{C}[0]$ with the mapping spaces $\mathrm{Hom}_{\mathscr{C}}(x, y)$ define a category enriched in the homotopy category of spaces.

Proposition 9.4. *For any ∞-category $\mathscr{C}$, the objects $\mathscr{C}[0]$, mapping spaces $\mathrm{Hom}_{\mathscr{C}}(x, y)$, and composition operations given above define a category $\widetilde{\mathrm{h}\mathscr{C}})$ enriched in the homotopy category of spaces. Applying the functor*

$$\pi_0 : \mathrm{h}\,\mathscr{S}paces \to \mathrm{Set}$$

recovers the usual homotopy category $\pi_0\widetilde{\mathrm{h}\mathscr{C}}) = \mathrm{h}\,\mathscr{C}$.

We note that the constriction $\mathscr{C} \rightsquigarrow \widetilde{\mathrm{h}\mathscr{C}}$ is functorial. In particular, functors between ∞-categories define functors between their associated spacially enriched categories.

9.2. **Imagining ∞-lifting.** It is clear that we can collect the objects $\mathscr{C}[0]$ and generalized mapping spaces $\mathrm{Hom}(\Delta^n, \mathscr{C})_{\vec{x}}$ to produce some intricate algebraic object which functions in a completely coherent manner at the level of the ∞-category of Kan complexes, rather than in the homotopy category.

Let us consider a slightly more restrained proposition. Let's fix a single object x and the associated "endomorphism spaces"

$$\mathrm{End}_{\mathscr{C}}(x)_n := \mathrm{Fun}(\Delta^n, \mathscr{C})_{(x,\dots,x)}.$$

These endomorphism spaces are the values of the associated functors

$$\underline{\mathrm{End}}_{\mathscr{C}}(x)_n : \mathrm{sSet}^{\mathrm{op}}_{/\Delta^n} \to \mathscr{C}$$

on the initial object Δ^n, and the structural maps

$$p_i : \mathrm{End}_{\mathscr{C}}(x)_n \to \mathrm{End}_{\mathscr{C}}(x)_1$$

dual to the inclusions $\Delta^1 \cong \Delta^{\{i-1,i\}} \to \Delta^n$ induce an isomorphism onto the product

$$\mathrm{End}_{\mathscr{C}}(x)_n \xrightarrow{\sim} \mathrm{End}_{\mathscr{C}}(x)_1 \times \cdots \times \mathrm{End}_{\mathscr{C}}(x)_1$$

in the ∞-category of spaces $\mathscr{S}paces = \mathrm{N}^{\mathrm{hc}}(\underline{\mathrm{Kan}})$. (Here we indulge ourselves and imagine/assume that the ∞-category of spaces admits products, and that these products are realized as the usual products of spaces.)

The object $\mathrm{End}(x)_*$ is now seen to define an algebra object in the category of spaces, considered along with its cartesian monoidal structure [12, §1.2] [14,

§2.4.1]. If we stretch our imaginations again, slightly, we can see the objects $\mathscr{C}[0]$ and morphism spaces

$$\mathrm{Hom}_{\mathscr{C}}(\vec{x}:\Delta^n[0]\to\mathscr{C}) := \mathrm{Fun}(\Delta^n,\mathscr{C})_{\vec{x}}$$

as defining an algebra with many objects in the ∞-category of spaces, i.e. as a "category enriched in the ∞-category of spaces".

9.3. **Comparing with endomorphism A_∞-algebras.** Given an complex x of objects in a linear abelian category $\mathbf{C}$, we have the extension algebra $\mathrm{Ext}_{\mathbf{C}}(x,x)$ and a lift of this algebra to an ∞-algebra $\mathcal{E}xt_{\mathbf{C}}(x,x)$. This A_∞-algebra is completely non-canonical in general, and can be defined by choosing a linear section section

$$\mathrm{Ext}_{\mathbf{C}}(x,x) = H^*(\mathrm{End}_{\mathbf{C}}(P_x)) \to \mathrm{End}_{\mathbf{C}}(P_x),$$

where P_x is some choice of (sufficiently) projective resolution $P_x \to x$.

The structure of this A_∞-algebra is, in practice, completely non-canonical, completely non-uniform, and completely uncontrollable. (This is, at least, the author's experience/opinion.) For example, if we have a functor $F:\mathbf{C}\to\mathbf{D}$ the corresponding map of A_∞-algebras $\mathrm{Ext}_{\mathbf{C}}(x,x)\to\mathrm{Ext}_{\mathbf{C}}(x,x)$ is not defined in any natural way. (Such an A_∞-algebra map can be defined by making some explicit comparisons between various choices.) Similarly, the A_∞-lift of the derived category $D(\mathbf{C})$ is a rather intractable object as well.

Although the tame nature of the construction $\mathrm{End}_{\mathscr{C}}(x)_*$ may be misleading, it is clear that this algebra is naturally constructed from the ∞-category $\mathscr{C}$, and is functional in a way that the A_∞-algebra $\mathcal{E}xt_{\mathbf{C}}(x,x)$ simply is not.

Remark 9.5. One can also compare with working at the dg level. Here endomorphism algebras are naturally defined and function very well. However, the categories of dg algebras and dg categories seem to lack the kinds of flexibility enjoyed by the category of spaces. To recall one simple issue, quasi-isomorphisms of dg things do not admit dg inverses in general.

10. Pinched mapping spaces

10.1. **Pinched mapping spaces.** Recall that, for any diagram $p:K\to\mathscr{C}$, we have the forgetful functors $\mathscr{C}_{p/}\to\mathscr{C}$ and $\mathscr{C}_{/p}\to\mathscr{C}$ which are dual to the functorial embeddings $L\to K\star L$ and $L\to L\star K$, at varied L.

Definition 10.1. Let $x,y:*\to\mathscr{C}$ be two objects in an ∞-category. We define the left and right pinched mapping spaces as

$$\mathrm{Hom}^{\mathrm{L}}_{\mathscr{C}}(x,y) := \mathscr{C}_{x/}\times_{\mathscr{C}}\{y\} \quad\text{and}\quad \mathrm{Hom}^{\mathrm{R}}_{\mathscr{C}}(x,y) := \{x\}\times_{\mathscr{C}}\mathscr{C}_{/y},$$

respectively.

We note that there is an identification

$$\mathrm{Hom}^{\mathrm{R}}_{\mathscr{C}}(x,y) = \mathrm{Hom}^{\mathrm{L}}_{\mathscr{C}^{\mathrm{op}}}(y,x)^{\mathrm{op}}.$$

So we are free to focus on the lift pinched space in our analysis. Note also that any functor between ∞-categories $F:\mathscr{C}\to\mathscr{D}$ fits into a diagram

$$\begin{tikzcd} \mathscr{C}_{x/} \arrow[r, "F_{x/}"] \arrow[d] & \mathscr{D}_{Fx/} \arrow[d] \\ \mathscr{C} \arrow[r, "F"] & \mathscr{D}. \end{tikzcd}$$

Taking the fiber of this diagram over a given point $y : * \to \mathscr{C}$ provides an induced map on the pinched spaces

$$F : \mathrm{Hom}^{\mathrm{L}}_{\mathscr{C}}(x, y) \to \mathrm{Hom}^{\mathrm{L}}_{\mathscr{D}}(Fx, Fy).$$

Our first aim is to prove that the pinched mapping spaces are in fact spaces.

Lemma 10.2. *At any pair of objects in an ∞-category $\mathscr{C}$, the pinched mapping spaces $\mathrm{Hom}^{\mathrm{L}}_{\mathscr{C}}(x, y)$ and $\mathrm{Hom}^{\mathrm{R}}_{\mathscr{C}}(x, y)$ are Kan complexes.*

Proof. By Corollary 5.27 the forgetful functor $\mathscr{C}_{x/} \to \mathscr{C}$ is a left fibration. It follows that the fiber over $y : * \to \mathscr{C}$ is a left fibration

$$\mathrm{Hom}^{\mathrm{L}}_{\mathscr{C}}(x, y)$$

and in particular is conservative by Lemma 5.31. Hence every map in $\mathscr{C}$ is an isomorphism, and $\mathscr{C}$ is a therefore a Kan complex by Theorem 5.32. The proof for $\mathrm{Hom}^{\mathrm{R}}_{\mathscr{C}}(x, y)$ is similar. □

Our second aim is to proved that the pinched mapping space is naturally equivalent to the "standard mapping space" studied in Section 8 above.

thm:pre_left_right **Theorem 10.3.** *At any pair of objects $x, y : * \to \mathscr{C}$ there are natural homotopy equivalences*

$$\begin{tikzcd}
\mathrm{Hom}^{\mathrm{L}}_{\mathscr{C}}(x, y) \arrow[dr, "\sim"'] & & \mathrm{Hom}^{\mathrm{R}}_{\mathscr{C}}(x, y) \arrow[dl, "\sim"] \\
& \mathrm{Hom}_{\mathscr{C}}(x, y). &
\end{tikzcd}$$

A more refined statement of this result, as well as its proof, appears in Section 10.9 below.

We warn the reader that the proof of Theorem 10.3 is a rather intricate deviation into rather specific constructions for simplicial sets, which one then analyzes mostly through the analyses of categorical pushouts and pullbacks provided in Section 6. So one might only peruse the details on a first reading, then return to the topic after considering its applications to the simplicial and dg settings.

10.2. **Oriented products.**

Definition 10.4. Given a partial diagram of ∞-categories

$$\begin{tikzcd}
\mathscr{C} \arrow[dr] & & \mathscr{D} \arrow[dl] \\
& \mathscr{T} &
\end{tikzcd}$$

the oriented fiber product is the ∞-category

$$\mathscr{C} \times^{\mathrm{or}}_{\mathscr{T}} \mathscr{D} := \mathscr{C} \times_{\mathrm{Fun}(\Delta^{\{0\}}, \mathscr{T})} \mathrm{Fun}(\Delta^1, \mathscr{T}) \times_{\mathrm{Fun}(\Delta^{\{1\}}, \mathscr{T})} \mathscr{D}.$$

Equivalently, we may take $\mathscr{C} \times^{\mathrm{or}}_{\mathscr{T}} \mathscr{D} = \mathrm{Fun}(\Delta^1, \mathscr{T}) \times_{\mathrm{Fun}(\partial\Delta^1, \mathscr{T})} (\mathscr{C} \times \mathscr{D})$.

Since the restriction functor $\mathrm{Fun}(\Delta^1, \mathscr{T}) \to \mathrm{Fun}(\partial\Delta^1, \mathscr{T})$ is an isofibration (Corollary 6.14) the projection

$$\mathscr{C} \times^{\mathrm{or}}_{\mathscr{T}} \mathscr{D} \to \mathscr{C} \times \mathscr{D} \qquad (47)$$ eq:3998

is an isofibration as well. This shows, in particular, that the oriented fiber product is an ∞-category, and that for each pair of points (x, y) in $\mathscr{C} \times \mathscr{D}$ the fiber ${}_x(\mathscr{C} \times^{\mathrm{or}}_{\mathscr{T}} \mathscr{D})_y$ is a Kan complex.

Example 10.5. If $\mathscr{T}$ is a Kan complex then $\mathrm{Fun}(\Delta^1, \mathscr{T}) = \mathrm{Isom}(\mathscr{T})$ and the oriented fiber product $\mathscr{C} \times^{\mathrm{or}}_{\mathscr{T}} \mathscr{D}$ agrees with the categorical pullback $\mathscr{C} \times^{\mathrm{htop}}_{\mathscr{T}} \mathscr{D}$.

ex:mapping_sp **Example 10.6.** For objects $x, y : * \to \mathscr{C}$ the mapping space is obtained as the fiber

$$\mathrm{Hom}_{\mathscr{C}}(x,y) = \mathrm{Fun}(\Delta^1, \mathscr{C}) \times_{\mathrm{Fun}(\partial\Delta^1, \mathscr{C})} \{(x,y)\} = {}_x(\mathscr{C} \times^{\mathrm{or}}_{\mathscr{T}} \mathscr{C})_y$$

Of course, we are particularly interested in the latter example.

10.3. **Blunt joins and oriented products.**

Definition 10.7 ([15, 01HR])**.** For simplicial sets K and L the blunt join $K \diamond L$ is the pushout

$$\begin{tikzcd} K \times \partial\Delta^1 \times L \arrow[r] \arrow[d] & K \times \Delta^1 \times L \arrow[d] \\ (K \times \{0\}) \amalg (\{1\} \times L) \arrow[r] & K \diamond L. \end{tikzcd}$$

For the left-hand map, we have the identification

$$K \times \partial\Delta^1 \times L = (K \times \{0\} \times L) \amalg (K \times \{1\} \times L),$$

and project onto the $K \times \{0\}$ and $\{1\} \times L$ factors, respectively. Hence a map out of $K \diamond L$ is a map out of the product

$$K \times \Delta^1 \times L \to B$$

whose restriction to $K \times \{0\} \times L$ is constant along L and whose restriction to $K \times \{1\} \times L$ is constant along K.

lem:4038 **Lemma 10.8.** *For simplicial sets K and L, and an ∞-category $\mathscr{C}$, we have a natural identification*

$$\mathrm{Fun}(K \diamond L, \mathscr{C}) = \mathrm{Fun}(K, \mathscr{C}) \times^{\mathrm{or}}_{\mathrm{Fun}(K \times L, \mathscr{C})} \mathrm{Fun}(L, \mathscr{C}).$$

Proof. Commutativity of $\mathrm{Fun}(-, \mathscr{C})$ with colimits identifies $\mathrm{Fun}(K \diamond L, \mathscr{C})$ with the fiber product

$$\mathrm{Fun}(\Delta^1, \mathrm{Fun}(K \times L, \mathscr{C})) \times_{\mathrm{Fun}(\partial\Delta^1, \mathrm{Fun}(K \times L, \mathscr{C}))} (\mathrm{Fun}(K, \mathscr{C}) \times \mathrm{Fun}(L, \mathscr{C})).$$

$\square$

We claim that, in the case of the join, there is a canonical map $K \times \Delta^1 \times L \to K \star L$ with the appropriate constancy conditions so that we obtain a map from the blunt join $K \diamond L$. Explicitly, any map $\sigma : \Delta^n \to \Delta^1$ splits Δ^n as

$$\Delta^n = \Delta^{\sigma_0} \star \Delta^{\sigma_1} \quad \text{where} \quad \Delta^{\sigma_0} := \Delta^{\sigma^{-1}(0)} \text{ and } \Delta^{\sigma_1} = \Delta^{\sigma^{-1}(1)}.$$

This is the unique splitting so that σ now appears as the join of the projections $\Delta^{\sigma_i} \to \Delta^0$.

For any simplex $\Sigma : \Delta^n \to K \times \Delta^1 \times L$, with associated triple of n-simplices $(\sigma_K, \sigma, \sigma_L)$, we associate the n-simplex

$$\Sigma' : \Delta^n \cong \Delta^{\sigma_0} \star \Delta^{\sigma_1} \overset{\sigma_K \star \sigma_L}{\to} K \star L.$$

(Here we have abused notation and written σ_K and σ_L for the restrictions of these n-simplices to the appropriate sub-simplices Δ^{σ_i}.) The association $\Sigma \to \Sigma'$ determines a map of simplicial sets

$$\widetilde{c} : K \times \Delta^1 \times L \to K \star L$$

whose restrictions to $K \times \{0\} \times L$ and $K \times \{1\} \times L$ are constant along L and K respectively. Hence we get an induced map from the blunt join

$$c_{K,L} : K \diamond L \to K \star L.$$

We call $c_{K,L}$ the *comparison map*, and note that this map is natural in both K and L. Hence we obtain a natural transformation of bifunctors $c : - \diamond - \to - \star -$.

The following can be checked directly.

Lemma 10.9. *The structural maps $K = K \times \{0\} \to K \diamond L$ and $L \cong \{1\} \times L \to K \diamond L$ fit into diagrams*

$$\begin{tikzcd} & K \arrow[dl] \arrow[dr] & \\ K \diamond L \arrow[rr, "c"] & & K \star L \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & L \arrow[dl] \arrow[dr] & \\ K \diamond L \arrow[rr, "c"] & & K \star L. \end{tikzcd}$$

We claim that, at least in some sense, the comparison map is an "equivalence" of simplicial sets.

10.4. **The comparison map is a categorical equivalence.**

Definition 10.10. A map of simplicial sets $f : K \to L$ is called a categorical equivalence if, for every ∞-category $\mathscr{C}$, the induced map on Kan complexes

$$\mathrm{Fun}(L, \mathscr{C})^{\mathrm{Kan}} \to \mathrm{Fun}(K, \mathscr{C})^{\mathrm{Kan}}$$

is a homotopy equivalence.

This notion of categorical equivalence is somewhat flexible, and has a number of equivalent expressions.

Lemma 10.11 ([15, 01EF])**.** *For a map of simplicial sets $f : K \to L$, the following are equivalent:*

(a) *For any ∞-category $\mathscr{C}$ the induced map*

$$\mathrm{Fun}(L, \mathscr{C})^{\mathrm{Kan}} \to \mathrm{Fun}(K, \mathscr{C})^{\mathrm{Kan}}$$

is an equivalence of Kan complexes (i.e. f is a categorical equivalence).

(b) *For any ∞-category $\mathscr{C}$ the induced map*

$$\mathrm{Fun}(L, \mathscr{C}) \to \mathrm{Fun}(K, \mathscr{C})$$

is an equivalence of ∞-categories.

(c) *For any ∞-category $\mathscr{C}$ the induced map*

$$\pi_0\left(\mathrm{Fun}(L, \mathscr{C})^{\mathrm{Kan}}\right) \to \pi_0\left(\mathrm{Fun}(K, \mathscr{C})^{\mathrm{Kan}}\right)$$

is a bijection of sets.

Proof. The equivalence (a) $\Leftrightarrow$ (b) follows from a consideration of the ∞-category $\mathrm{Fun}(\Delta^1, \mathscr{C})$, the adjunction

$$\mathrm{Fun}(\Delta^1, \mathrm{Fun}(-, \mathscr{C}))^{\mathrm{Kan}} \cong \mathrm{Fun}(-, \mathrm{Fun}(\Delta^1, \mathscr{C}))^{\mathrm{Kan}},$$

and Theorem 8.13. The implication (a) $\Rightarrow$ (c) is clear. We deal finally with the implication (c) $\Rightarrow$ (a). Via adjunction (c) implies that f induces a bijection

$$\pi_0\left(\mathrm{Fun}(\mathscr{X}, \mathrm{Fun}(L, \mathscr{C}))^{\mathrm{Kan}}\right) \to \pi_0\left(\mathrm{Fun}(\mathscr{X}, \mathrm{Fun}(K, \mathscr{C}))^{\mathrm{Kan}}\right)$$

at each pair of an ∞-categories $\mathscr{C}$ and Kan complex $\mathscr{X}$. But now the natural map

$$\mathrm{Fun}(\mathscr{X}, \mathrm{Fun}(L, \mathscr{C})^{\mathrm{Kan}}) \to \mathrm{Fun}(\mathscr{X}, \mathrm{Fun}(L, \mathscr{C}))^{\mathrm{Kan}}$$

is an isomorphism at an arbitrary simplicial set A, by Lemma 8.6. Hence (c) implies that the induced map

$$\pi_0\left(\mathrm{Fun}(\mathscr{X}, \mathrm{Fun}(L, \mathscr{C})^{\mathrm{Kan}})\right) \to \pi_0\left(\mathrm{Fun}(\mathscr{X}, \mathrm{Fun}(K, \mathscr{C})^{\mathrm{Kan}})\right).$$

So $f^* : \mathrm{Fun}(L, \mathscr{C})^{\mathrm{Kan}} \to \mathrm{Fun}(K, \mathscr{C})^{\mathrm{Kan}}$ is an equivalence in the homotopy category of spaces, and thus a homotopy equivalence, and we therefore observe the implication (c) $\Rightarrow$ (a). □

Our primary claim is the following.

prop:comp_map

Proposition 10.12 ([15, 01HV])**.** *At an arbitrary pair of simplicial sets K and L, the comparison map $c_{K,L} : K \diamond L \to K \star L$ is a categorical equivalence.*

We are claiming, equivalently, that the diagram of simplicial sets

$$\begin{tikzcd}
K \times \partial\Delta^1 \times L \arrow[r] \arrow[d] & K \times \Delta^1 \times L \arrow[d, "\tilde{c}"] \\
(K \times \{0\}) \amalg (\{1\} \times L) \arrow[r] & K \star L.
\end{tikzcd}$$

is a categorical pushout square. The proof, which we do not cover, follows by a reduction to the case where $K = \Delta^1$ [15, 01HX]. This reduction is similar to the one outlined in the proof of Theorem 8.13. One then deals with the special case where L is a point [15, 01HZ] and follows a relatively intricate argument with categorical pushouts and anodyne maps. See [15, 01HV] and the surrounding commentary for details.

sect:slice_diag

10.5. **Slice and coslice diagonals.** If we fix a diagram $p : K \to \mathscr{C}$, then at arbitrary L we have

$$\mathrm{Fun}(K \diamond L, \mathscr{C})_p = \{p\} \times^{\mathrm{or}}_{\mathrm{Fun}(K \times L, \mathscr{C})} \mathrm{Fun}(L, \mathscr{C}).$$

We expand the right hand side to the expression

$$\begin{aligned}
&\mathrm{Fun}(\Delta^1, \mathrm{Fun}(K \times L, \mathscr{C})) \times_{\mathrm{Fun}(\partial\Delta^1, \mathrm{Fun}(K \times L, \mathscr{C}))} \mathrm{Fun}(L, \mathscr{C}) \\
&= \mathrm{Fun}(L, \mathrm{Fun}(\Delta^1, \mathrm{Fun}(K, \mathscr{C}))) \times_{\mathrm{Fun}(L, \mathrm{Fun}(\partial\Delta^1, \mathrm{Fun}(K, \mathscr{C})))} \mathrm{Fun}(L, \mathscr{C}) \\
&= \mathrm{Fun}\left(L, \{p\} \times^{\mathrm{or}}_{\mathrm{Fun}(K, \mathscr{C})} \mathscr{C}\right).
\end{aligned}$$

So in total we have a "restricted adjunction"

$$\mathrm{Fun}(K \diamond L, \mathscr{C})_p \cong \mathrm{Fun}\left(L, \{p\} \times^{\mathrm{or}}_{\mathrm{Fun}(K, \mathscr{C})} \mathscr{C}\right), \tag{48}$$ eq:4109

and similarly obtain an adjunction

$$\mathrm{Fun}(K \diamond L, \mathscr{C})_p \cong \mathrm{Fun}\left(L, \mathscr{C} \times^{\mathrm{or}}_{\mathrm{Fun}(L, \mathscr{C})} \{q\}\right) \tag{49}$$ eq:4113

over any diagram $q : L \to \mathscr{C}$ [15, 01KN].

Now, at such diagrams q and l, we have the evaluation maps $ev_p : K \star \mathscr{C}_{p/} \to \mathscr{C}$ and $ev_q : \mathscr{C}_{q/} \star L \to \mathscr{C}$ which restrict to p and q on K and L respectively. We compose with the comparison maps $K \diamond \mathscr{C}_{p/} \to K \star \mathscr{C}_{p/}$ and $\mathscr{C}_{/q} \diamond L \to \mathscr{C}_{/q} \star L$ to obtain corresponding functions

$$\delta'_{p/} : K \diamond \mathscr{C}_{p/} \to \mathscr{C} \quad \text{and} \quad \delta'_{/q} : \mathscr{C}_{/q} \diamond L \to \mathscr{C}.$$

Via the restricted adjunctions (48) and (49) we obtain finally maps

$$\delta_{p/} : \mathscr{C}_{p/} \to \{p\} \times^{\mathrm{or}}_{\mathrm{Fun}(K,\mathscr{C})} \mathscr{C} \quad \text{and} \quad \delta_{/q} : \mathscr{C}_{/q} \to \mathscr{C} \times^{\mathrm{or}}_{\mathrm{Fun}(L,\mathscr{C})} \{q\}.$$

We refer to these maps as the coslice and slice diagonals, respectively [15, 02GH].

lem:4197

Lemma 10.13. *Consider diagrams $p : K \to \mathscr{C}$ and $q : L \to \mathscr{C}$. Composition with the co/slice diagonal fits into a diagram of natural transformations*

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{sSet}}(-, \mathscr{C}_{p/}) \arrow[r, "\delta_*"] \arrow[d, "\cong"'] & \mathrm{Hom}_{\mathrm{sSet}}(-, \{p\} \times^{\mathrm{or}}_{\mathrm{Fun}(K,\mathscr{C})} \mathscr{C}) \arrow[d, "\cong"] \\
\mathrm{Hom}_{\mathrm{sSet}}(K \star -, \mathscr{C})_p \arrow[r, "c^*_{K,-}"] & \mathrm{Hom}_{\mathrm{sSet}}(K \diamond -, \mathscr{C})_p
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{sSet}}(-, \mathscr{C}_{q/}) \arrow[r, "\delta_*"] \arrow[d, "\cong"'] & \mathrm{Hom}_{\mathrm{sSet}}(-, \mathscr{C} \times^{\mathrm{or}}_{\mathrm{Fun}(K,\mathscr{C})} \{q\}) \arrow[d, "\cong"] \\
\mathrm{Hom}_{\mathrm{sSet}}(- \star L, \mathscr{C})_q \arrow[r, "c^*_{-,L}"] & \mathrm{Hom}_{\mathrm{sSet}}(- \diamond L, \mathscr{C})_q
\end{tikzcd}$$

Proof. Follows by Yoneda's lemma. □

By considering the above diagrams one sees that the co/slice diagonals fit into diagrams

$$\begin{tikzcd}
\mathscr{C}_{p/} \arrow[rr, "\delta"] \arrow[dr, "\text{forget}"'] & & \{p\} \times^{\mathrm{or}}_{\mathrm{Fun}(K,\mathscr{C})} \mathscr{C} \arrow[dl, "p_2"] \\
& \mathscr{C} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{C}_{/q} \arrow[rr, "\delta"] \arrow[dr, "\text{forget}"'] & & \mathscr{C} \times^{\mathrm{or}}_{\mathrm{Fun}(L,\mathscr{C})} \{q\} \arrow[dl, "p_1"] \\
& \mathscr{C} &
\end{tikzcd}.$$

10.6. **Co/slice diagonals are equivalences.**

thm:slice_equiv

Theorem 10.14. *At an arbitrary diagram $p : K \to \mathscr{C}$, the slice an coslice diagonals*

$$\delta_{/p} : \mathscr{C}_{/p} \to \mathscr{C} \times^{\mathrm{or}}_{\mathrm{Fun}(K,\mathscr{C})} \{p\} \quad \textit{and} \quad \delta_{p/} : \mathscr{C}_{p/} \to \{p\} \times^{\mathrm{or}}_{\mathrm{Fun}(K,\mathscr{C})} \mathscr{C}$$

are equivalences of ∞-categories.

We understand at this point that the comparison map $K \diamond L \to K \star L$ is a categorical equivalence. Hence we have an equivalence of ∞-categories

$$c^* : \mathrm{Fun}(K \star L, \mathscr{C}) \xrightarrow{\sim} \mathrm{Fun}(K \diamond L, \mathscr{C})$$

at arbitrary $\mathscr{C}$. We take the fiber at any diagram $p : K \to \mathscr{C}$ to (presumably) deduce an equivalence

$$c^* : \mathrm{Fun}(K \star L, \mathscr{C})_p \xrightarrow{\sim} \mathrm{Fun}(K \diamond L, \mathscr{C})_p \cong \mathrm{Fun}\left(L, \{p\} \times^{\mathrm{or}}_{\mathrm{Fun}(K,\mathscr{C})} \mathscr{C}\right).$$

Now, in our dreams we apply an adjunction to obtain a diagram

$$\begin{tikzcd}
\mathrm{Fun}(K \star L, \mathscr{C})_p \arrow[r, "\sim"] \arrow[d, dotted, "\text{fantasy adjoint}"'] & \mathrm{Fun}(K \diamond L, \mathscr{C})_p \arrow[d, "\sim"] \\
\mathrm{Fun}(L, \mathscr{C}_{p/}) \arrow[r, "\delta_*"] & \mathrm{Fun}\left(L, \{p\} \times^{\mathrm{or}}_{\mathrm{Fun}(K,\mathscr{C})} \mathscr{C}\right)
\end{tikzcd} \tag{50}$$

eq:4229

and so deduce that δ_* is an equivalence at arbitrary J, from which we conclude that δ is itself an equivalence.

Of course, this fantasy argument does not apply since the fantasy adjunction for the join is not valid. However, we do have such a diagram at the level of 0-simplices (objects), via Lemma 10.13. Faced with this depressing situation, we will provide a more intricate argument to find that the existence of such a diagram on 0-simplices (and connected components), at varied L, still forces δ to be an equivalence.

10.7. **Details for Theorem 10.14.**

lem:4260 **Lemma 10.15** ([15, 01KV])**.** *Fix a diagram $p : K \to \mathscr{C}$. The adjunction $\mathrm{Hom}_{\mathrm{sSet}}(K\star -, \mathscr{C})_p \cong \mathrm{Hom}_{\mathrm{sSet}}(-, \mathscr{C}_{p/})$ induces a unique natural isomorphism on connected components*

$$\begin{tikzcd} \mathrm{Hom}_{\mathrm{sSet}}(K\star -, \mathscr{C})_p \arrow[r, "\cong"] \arrow[d] & \mathrm{Hom}_{\mathrm{sSet}}(-, \mathscr{C}_{p/}) \arrow[d] \\ \pi_0\left(\mathrm{Fun}(K\star -, \mathscr{C})_p^{\mathrm{Kan}}\right) \arrow[r, "\cong"] & \pi_0\left(\mathrm{Fun}(-, \mathscr{C}_{p/})^{\mathrm{Kan}}\right). \end{tikzcd} \tag{51}$$ eq:4262

The analogous result holds for the adjunction $\mathrm{Hom}_{\mathrm{sSet}}(-\star K, \mathscr{C})_p \cong \mathrm{Hom}_{\mathrm{sSet}}(-, \mathscr{C}_{/p})$.

The proof does not follow a direct argument, and involves the use of categorical mapping cones for simplicial sets. We avoid these details, but make a few remarks.

Remarks on the proof of Lemma 10.15. Let us only deal with the adjunction for $K\star -$. In [15, 01KV] it's shown that at any simplicial set L two maps $f_0, f_1 : K\star L \to \mathscr{C}$ are homotopic if and only if the corresponding maps under adjunction $f_0', f_1' : L \to \mathscr{C}_{p/}$ are homotopic. Equivalently, the adjunction

$$\mathrm{Hom}_{\mathrm{sSet}}(K\star L, \mathscr{C})_p \overset{\cong}{\to} \mathrm{Hom}_{\mathrm{sSet}}(L, \mathscr{C}_{p/}) \tag{52}$$ eq:4275

induces a bijection on isoclasses of objects in $\mathrm{Fun}(K\star L, \mathscr{C})$ and $\mathrm{Fun}(L, \mathscr{C}_{p/})$. This final statement is equivalent to the existence of an isomorphism on connected components in the associated Kan complex which completes the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathrm{sSet}}(K\star L, \mathscr{C})_p \arrow[r, "\cong"] \arrow[d] & \mathrm{Hom}_{\mathrm{sSet}}(L, \mathscr{C}_{p/}) \arrow[d] \\ \pi_0\left(\mathrm{Fun}(K\star L, \mathscr{C})_p^{\mathrm{Kan}}\right) \arrow[r, dashed, "\exists!"] & \pi_0\left(\mathrm{Fun}(L, \mathscr{C}_{p/})^{\mathrm{Kan}}\right). \end{tikzcd}$$

Since the adjunction (52) is natural in both L and $\mathscr{C}$ the induced isomorphism on connected components is also natural in both L and $\mathscr{C}$. □

Proposition 10.16. *A map of ∞-categories $F : \mathscr{C} \to \mathscr{D}$ is an equivalence if and only if the induced map*

$$F_* : \pi_0\left(\mathrm{Fun}(L, \mathscr{C})^{\mathrm{Kan}}\right) \to \pi_0\left(\mathrm{Fun}(L, \mathscr{D})^{\mathrm{Kan}}\right) \tag{53}$$ eq:4290

is an isomorphism at each simplicial set L.

Proof. Suppose F is an equivalence. By Theorem 5.41 it follows that the induced map on functor categories is an equivalence, and hence that the induced map

$$F_* : \mathrm{Fun}(L, \mathscr{C})^{\mathrm{Kan}} \to \mathrm{Fun}(L, \mathscr{D})^{\mathrm{Kan}}$$

is an equivalence, by Proposition 8.4. The induced map on connected components is therefore a bijection by Lemma 4.20.

Conversely, suppose the induced map (53) is an equivalence at all simplicial sets. Then via adjunction we find that the induced map

$$F_* : \pi_0\left(\mathrm{Fun}(\mathscr{X}, \mathrm{Fun}(L, \mathscr{C}))^{\mathrm{Kan}}\right) \to \pi_0\left(\mathrm{Fun}(\mathscr{X}, \mathrm{Fun}(L, \mathscr{D}))^{\mathrm{Kan}}\right)$$

is an equivalence at all Kan complexes $\mathscr{X}$. Applying the natural identification $\mathrm{Fun}(\mathscr{X}, \mathscr{E})^{\mathrm{Kan}} = \mathrm{Fun}(\mathscr{X}, \mathscr{E}^{\mathrm{Kan}})$ of Lemma 8.6, at an arbitrary ∞-category $\mathscr{E}$, we find that the map

$$F_* : \pi_0\left(\mathrm{Fun}(\mathscr{X}, \mathrm{Fun}(L, \mathscr{C})^{\mathrm{Kan}})\right) \to \pi_0\left(\mathrm{Fun}(\mathscr{X}, \mathrm{Fun}(L, \mathscr{D})^{\mathrm{Kan}})\right)$$

is an isomorphism at all L and $\mathscr{X}$. So the induced map $F_* : \mathrm{Fun}(L, \mathscr{C})^{\mathrm{Kan}} \to \mathrm{Fun}(L, \mathscr{D})^{\mathrm{Kan}}$ is an equivalence in the homotopy category of Kan complexes at arbitrary L, i.e. a homotopy equivalence, and hence F is an equivalence by Theorem 5.41. □

10.8. **Proof of Theorem 10.14.** We can now prove our main intermediate result.

Proof of Theorem 10.14. We prove that $\delta_{p/}$ is an equivalence. The case of $\delta_{/p}$ is completely similar.

By Proposition 10.12 the comparison map $c_{K,L} : K \diamond L \to K \star L$ is an equivalence. Hence the map on functor spaces

$$c^* : \mathrm{Fun}(K \star L, \mathscr{C})^{\mathrm{Kan}} \to \mathrm{Fun}(K \diamond L, \mathscr{C})^{\mathrm{Kan}}$$

is an equivalence at an arbitrary ∞-category $\mathscr{C}$. Restricting along the inclusions $K \to K \star L$ and $K \to K \diamond L$ provide a diagram

$$\begin{tikzcd} \mathrm{Fun}(K \star L, \mathscr{C})^{\mathrm{Kan}} \arrow[rr] \arrow[dr] & & \mathrm{Fun}(K \diamond L, \mathscr{C}) \arrow[dl] \\ & \mathrm{Fun}(K, \mathscr{C})^{\mathrm{Kan}} & \end{tikzcd}$$

in which the two maps to $\mathrm{Fun}(K, \mathscr{C})^{\mathrm{Kan}}$ are Kan fibrations by Corollary 6.15. Hence the induced maps on fibers is an equivalence

$$\mathrm{Fun}(K \star L, \mathscr{C})^{\mathrm{Kan}}_p \overset{\sim}{\to} \mathrm{Fun}(K \diamond L, \mathscr{C})^{\mathrm{Kan}}_p$$

over any given diagram $p : K \to \mathscr{C}$, by Proposition 4.38, and the maps on connected components are subsequently an equivalence as well. We apply Lemma 10.15 to find that the induced map on connected components is an equivalence

$$\delta_* : \pi_0\left(\mathrm{Fun}(L, \mathscr{C}_{p/})\right) \overset{\sim}{\to} \pi_0(\mathrm{Fun}(L, \{p\} \times^{\mathrm{or}}_{\mathrm{Fun}(K,\mathscr{C})} \mathscr{C}))$$

at arbitrary L. Hence the map $\delta_{p/} : \mathscr{C}_{p/} \to \{p\} \times^{\mathrm{or}}_{\mathrm{Fun}(K,\mathscr{C})} \mathscr{C}$ is an equivalence. □

sect:left_right

10.9. **Comparisons of mapping spaces.** S

Fix objects x and y in an ∞-category $\mathscr{C}$. We interpret x as a diagram $x : \Delta^0 \to \mathscr{C}$, and consider the coslice diagonal

$$\delta : \mathscr{C}_{x/} \to \{x\} \times^{\mathrm{or}}_{\mathscr{C}} \mathscr{C},$$

which we now understand is an equivalence by Theorem 10.14. Via the explicit construction from Section 10.5 we also have a diagram of functors

$$\begin{tikzcd}
\mathscr{C}_{x/} \arrow[rr, "\delta"] \arrow[dr, "\text{forget}"'] & & \{x\} \times^{\mathrm{or}}_{\mathscr{C}} \mathscr{C} \arrow[dl, "\text{projection (47)}"] \\
& \mathscr{C} = \{x\} \times \mathscr{C} & .
\end{tikzcd}$$

We take the fiber over $y : * \to \mathscr{C}$ to obtain a map of Kan complexes

$$\delta^{\mathrm{L}}_{x,y} : \mathrm{Hom}^{\mathrm{L}}_{\mathscr{C}}(x,y) = \mathscr{C}_{x/} \times_{\mathscr{C}} \{y\} \to \{x\} \times^{\mathrm{or}}_{\mathscr{C}} \{y\} = \mathrm{Hom}_{\mathscr{C}}(x,y).$$

One similarly takes the fiber of the slice diagonal $\mathscr{C}_{/y} \to \mathscr{C} \times^{\mathrm{or}}_{\mathscr{C}} \{y\}$ over x to obtain a map

$$\delta^{\mathrm{R}}_{x,y} : \mathrm{Hom}^{\mathrm{R}}_{\mathscr{C}}(x,y) \to \mathrm{Hom}_{\mathscr{C}}(x,y).$$

We refer to these maps as the left and right comparison maps respectively, via a slight abuse of language.

One sees directly that the co/slice diagonals are natural in functors between ∞-categories $F : \mathscr{C} \to \mathscr{D}$ so that the left and right comparison maps are natural in ∞-category functors as well. Explicitly, at any pair of objects in $\mathscr{C}$, and any functor F, we have a diagram

$$\begin{tikzcd}
& & \mathrm{Hom}_{\mathscr{D}}(Fx,Fy) & & \\
\mathrm{Hom}^{\mathrm{L}}_{\mathscr{D}}(Fx,Fy) \arrow[urr, "\delta^{\mathrm{L}}_{Fx,Fy}"] & & \mathrm{Hom}_{\mathscr{C}}(x,y) \arrow[u, "F"] & & \mathrm{Hom}^{\mathrm{L}}_{\mathscr{D}}(Fx,Fy) \arrow[ull, "\delta^{\mathrm{R}}_{Fx,Fy}"'] \\
\mathrm{Hom}^{\mathrm{L}}_{\mathscr{C}}(x,y) \arrow[u, "F"] \arrow[urr, "\delta^{\mathrm{L}}_{x,y}"'] & & & & \mathrm{Hom}^{\mathrm{L}}_{\mathscr{C}}(x,y). \arrow[u, "F"'] \arrow[ull, "\delta^{\mathrm{R}}_{x,y}"]
\end{tikzcd}$$

thm:left_right **Theorem 10.17.** *The left and right comparison maps*

$$\delta^{\mathrm{L}}_{x,y}\, \mathrm{Hom}^{\mathrm{L}}_{\mathscr{C}}(x,y) \to \mathrm{Hom}_{\mathscr{C}}(x,y) \quad \textit{and} \quad \delta^{\mathrm{R}}_{x,y} : \mathrm{Hom}^{\mathrm{R}}_{\mathscr{C}}(x,y) \to \mathrm{Hom}_{\mathscr{C}}(x,y)$$

are homotopy equivalences.

Proof. The forgetful functor $\mathscr{C}_{x/} \to \mathscr{C}$ is a left fibration by Corollary 5.27, and hence an isofibration by Lemma 5.31. The projection $\{x\} \times^{\mathrm{or}}_{\mathscr{C}} \mathscr{C} \to \mathscr{C}$ is also an isofibration, by Corollary 6.14 if one likes. We now have a diagram

$$\begin{tikzcd}
\mathscr{C}_{x/} \arrow[rr, "\delta"] \arrow[dr] & & \{x\} \times^{\mathrm{or}}_{\mathscr{C}} \mathscr{C} \arrow[dl] \\
& \mathscr{C} = \{x\} \times \mathscr{C} &
\end{tikzcd}$$

in which the maps to $\mathscr{C}$ are isofibrations and the map δ is an equivalence (Theorem 10.14). We apply Corollary **??** to conclude that the induced map on fibers

$$\delta^{\mathrm{L}}_{x,y} : \mathrm{Hom}^{\mathrm{R}}_{\mathscr{C}}(x,y) \to \mathrm{Hom}_{\mathscr{C}}(x,y)$$

is an equivalence. One follows a completely similar argument to see that the map $\delta^{\mathrm{L}}_{x,y}$ is an equivalence as well. $\square$

11. Dg preparations: the Dold-Kan correspondence

We want to calculate the mapping spaces for dg categories. The claim is ultimately that, given a dg category $\mathbf{A}$, the pinched mapping spaces for dg nerve $\mathscr{A} = N^{\mathrm{dg}}(\mathbf{A})$ are identified with the Eilenbergh-MacLane construction for the mapping complexes

$$\mathrm{Hom}^{\mathrm{L}}_{\mathscr{A}}(x,y) = K(\mathrm{Hom}^*_{\mathbf{A}}(x,y)).$$

With a sufficiently strong understanding of this construction, we can then hope to compare (quasi-)fully faithfulness for dg functors to fully faithfulness of functors in the ∞-setting.

This Eilenbergh-MacLane construction is specifically a functor

$$K : \{\text{cochains over } \mathbb{Z}\} \to \{\text{simplicial abelian groups}\} \subseteq \mathrm{Kan}$$

which provides one-half of the so-called Dold-Kan correspondence. In this section we provide a relatively detailed discussion of the Dold-Kan correspondence, and a proof a proof of the fact that the functor K transforms cohomology groups of cochains into homotopy groups of spaces, at least in non-positive degrees,

$$\pi_n(K(X)) = H^{-n}(X).$$

11.1. Reminders on simplicial sets.

In this section we approach simplicial sets from a more combinatorial perspective. A simplicial set K can be described as a $\mathbb{Z}_{\geq 0}$-collection of sets $\{K[n]\}_{n\geq 0}$ equipped with maps

$$\partial_i^* : K[n] \to K[n-1] \quad \text{and} \quad s_j^* : K[n-1] \to K[n]$$

for all n, $0 \leq i \leq n$, and $0 \leq j \leq n-1$, which satisfy the relations

- $d_i^* d_j^* = d_{j-1}^* d_i^*$ if $i < j$.
- $s_i^* s_j^* = s_{j+1}^* s_i^*$ if $i \leq j$.
- $d_i^* s_j^* = s_{j-1}^* d_i^*$ if $i < j$.
- $d_i^* s_j^* = s_j^* d_{i-1}^*$ if $i > j+1$.
- $d_i^* s_j^* = id$ if $i = j$ or $j+1$.

As one might surmise, these maps d_i^* and s_j^* are dual to the unique increasing inclusion $d_i : [n-1] \to [n]$ which does not contain i in its image, and the unique weakly increasing surjection $s_j : [n] \to [n-1]$ with $s_j(j) = s_j(j+1) = j$.

11.2. Simplicial abelian groups.

We recall that a simplicial abelian group is a functor $A : \Delta^{\mathrm{op}} \to \mathrm{Ab}$. Let sAb denote the category of simplicial abelian groups. We have the forgetful functor $\mathrm{sAb} \to \mathrm{sSet}$ and note that that the free simplicial abelian group functor provides a right adjoint to this functor

$$\mathbb{Z}- : \mathrm{sSet} \to \mathrm{sAb}\,.$$

Explicitly, for any simplicial set K, $(\mathbb{Z}K)[n] = \mathbb{Z}(K[n])$ and the structure maps are extended linearly from the structure maps for K.

prop:ab_kan

Proposition 11.1 ([16, 08N])**.** *Any simplicial abelian group is a Kan complex.*

Proof. Suppose we have a horn $\Lambda^n_i \to A$. Such a horn is specified by a tuple of simplices $\sigma_j : \Delta^{[n]\setminus\{j\}} \to A$, for $j \neq i$, and we seek an n-simplex $\sigma : \Delta^n \to A$ which satisfies $d_j^*\sigma = \sigma_k$ at all k.

We proceed via two induction processes. For the first inductive argument we claim that there exists simplices x_k in $A[n]$ for each $0 \leq k < i$ for which $d_j^* x_k = \sigma_j$

whenever $j \leq k$. This claim is trivially satisfied when $i = 0$, and otherwise we begin by taking $x_0 = s_0^*\sigma_0$. Now given x_{k-1} as desired we define x_k as

$$x_k = x_{k-1} - s_k^* d_k^*(x_{k-1}) + s_k^*(\sigma_k),$$

and find directly that x_k has the claimed property. Via induction we obtain an element $x = x_{i-1}$ for which $d_j^*(x) = \sigma_j$ whenever $j = i$.

For our second argument we claim the existence of n-simplices x'_m for $0 \leq m \leq n-i$ for which $d_j(x'_m) = \sigma_j$ for all j with $0 \leq j < i$ or $n-m < j \leq n$. We begin by taking $x'_0 = x$, and given x'_{m-1} as desired we define

$$x'_m = x'_{m-1} - s_{n-m}^* d_{n-m+1}^*(x'_{m-1}) + s_{n-m}^*(\sigma_{n-m}).$$

One checks directly that x'_m has the desired property, and by induction we obtain all x'_m as claimed. Take finally $\sigma = x'_{n-i}$. $\square$

11.3. **Cochains from simplicial abelian groups.** We eventually consider the normalized Moore complex functor

$$N^* : \mathrm{sAb} \to \mathrm{Ch}(\mathbb{Z})$$

from the category of simplicial abelian groups to the category of cochains of abelian groups. This construction begins with a consideration of the standard Moore complex functor.

For any complex of abelian groups A take $C^*(A)$ to be the complex with

$$C^{-n}(A) = A[n] \ \text{ and differential } \ d^{-n}(\sigma) = \sum_{i=0}^{n} (-1)^i \partial_i^*(\sigma),$$

where $\partial_i^* : A[n] \to A[n-1]$ is the i-th face map. One directly verifies that this differential does in fact square to 0, and we note that this construction is clearly functorial, and so provides a functor

$$C^* : \mathrm{sAb} \to \mathrm{Ch}(\mathbb{Z}).$$

We have the subcomplex of degenerate simplices $D^*(A)$ with $D^{-n}(A) = \sum_i s_i^*(A[n-1])$. This subcomplex is also functorial in A.

Lemma 11.2 ([20, Theorem 8.3.8])**.** *The subcomplex $D^*(A)$ of degenerate simplices in $C^*(A)$ is acyclic.*

Idea of proof. One applies some filtration on the on $D^*(A)$ induced by the p-th face operators, then sees that the E_2-page of the associated spectral sequence already vanishes, i.e. that the associated graded complex $\mathrm{gr}\, D^*(A)$ is acyclic. $\square$

One now reduces by this degenerate complex to produce a quasi-isomorphic complex.

Definition 11.3. We define the normalized (Moore) cochain complex functor

$$N^* : \mathrm{sAb} \to \mathrm{Ch}(\mathbb{Z})$$

by taking $N^*(A) := C^*(A)/D^*(A)$.

Lemma 11.4. *For any simplicial abelian group A, the reduction map $C^*(A) \to N^*(A)$ is a quasi-isomorphism.*

For a general simplicial set K, we can take $\mathbb{Z}K$ the associated free simplicial abelian group. This is the simplicial abelian group whose n-simplices are the free abelian group generated by the n-simplices in K, $K[n]$, and whose face maps are the unique linear maps which extend the face maps $f^* : K[n] \to K[m]$.

Definition 11.5. Define the functor

$$N^*(-, \mathbb{Z}) : \mathrm{sSet} \to \mathrm{Ch}(\mathbb{Z})$$

by taking $N^*(K, \mathbb{Z}) := N^*(\mathbb{Z}K)$.

11.4. **Eilenberg-MacLane functor.** For any cochain complex X we define the functor

$$K(X) : \Delta \to \mathrm{sAb}, \quad [n] \mapsto \mathrm{Hom}_{\mathrm{Ch}(\mathbb{Z})}(N^*(\Delta^n, \mathbb{Z}), X).$$

Explicitly, an n-simplex $\sigma : \Delta^n \to K(X)$ consists of the following data: For each subset $J \subseteq [n]$, with its inherited ordering $J = \{j_0, \dots, j_t\}$, σ associates a cochain x_J of degree $-|J|$ in X. These cochains are required to satisfy

$$d(x_J) = \sum_{i=0}^{|J|-1} (-1)^i x_{J \setminus \{j_i\}} \quad \text{and} \quad d(x_J) = 0 \text{ when } |J| = 1.$$

The abelian group structure on $K(X)[n]$ is the obvious one,

$$\{x_J : J \subseteq [n]\} \pm \{y_J : J \subseteq [n]\} = \{x_J \pm y_J : J \subseteq [n]\}$$

Given a map $f : [n] \to [m]$ the corresponding map $K(X)[n] \to K(X)[m]$ sends a tuple $\{x_J\}$ to the tuple $\{x_{f^{-1}K} : K \subseteq [m]\}$.

lem:4566 **Lemma 11.6.** *For each positive n, any cochain $x \in X^{-n}$ appears as the leading term $x = x_{[n]}$ for an n-simplex $\sigma : \Delta^n \to K(X)$. A degree 0 cochain $x \in X^0$ appears as the leading term in a 0-simplex if and only if x is a cocycle.*

Proof. When $n > 0$ take σ specified by the unique tuple with $x_{[n]} = x$, $x_{\{1,\dots,n\}} = d(x)$, and all other $x_J = 0$. The result at $n = 0$ just follows from the definition of $K(X)[0]$ as the collection of sets of a single element $\{x\}$ with $x \in X^0$ and $d(x) = 0$. $\square$

For a function $f : [n] \to [m]$ the map $f^* : K(X)[m] \to K(X)[n]$ sends each tuple $\{x_K : K \subseteq [m]\}$ to the tuple $\{x_J : J \subseteq [n]\}$ with

$$x_J = \begin{cases} x_{f(J)} & \text{if } f|_J \text{ is injective,} \\ 0 & \text{otherwise.} \end{cases}$$

lem:4583 **Lemma 11.7.** *Consider an n-simplex $\sigma : \Delta^n \to K(X)$ and corresponding tuple of cochains $\sigma = \{x_J : J \subseteq [n]\}$. The simplex σ is a sum of degenerate simplices, i.e. in $D^n K(X) = \sum_i s_i^* K(X)[n-1]$, if and only if the minimal element $x_{[n]} \in X^{-n}$ vanishes.*

Proof. Say σ is of type m if $x_J = 0$ whenever $|J| > m$. Note that the only simplex of type 0 is the zero simplex. Consider σ of type m with $m \leq n$, and suppose σ is not of type $m-1$. Order the cochains x_J with $|J| = m$ in the dictionary order

$$\{x_{J_0}, \dots, x_{J_t}\},$$

and let I be the minimal size m subset for which x_I is nonzero. Let i be the maximal element in $[n]$ with $i \notin I$ and consider the endomorphism $f : [n] \to [n]$

which is a bijection on $[n]-\{i\}$ and sends i to $i-1$. Then $f(I)=I$ and for any J we have $f(J)\leq J$ whenever $f|_J$ is injective. So the simplex $f^*(\sigma)$ is of type m and specified by a tuple $\{x'_J\}$ with

$$x'_I = x_I \ \text{ and } \ x'_J = 0 \ \text{ whenever } |J| = m \text{ and } J < I.$$

We note that f factors through $[n-1]$ so that the simplex $f(\sigma)$ is degenerate, that $f(\sigma)$ is of type m, and that

$$\sigma - f(\sigma) = \{x''_J : J \subseteq [n]\} \ \text{ with } x''_J = 0 \ \text{ whenever } |J| = m \text{ and } J \leq I.$$

In this way we can eliminate all nonzero cochains x_J with $|J|=m$, in order, by successively adding degenerate simplices, and in totality we observe the existence of a sum of degenerate simplices σ' for which

$$\sigma - \sigma' \ \text{ is of type } m-1.$$

It follows, by induction if one likes, that there exists a sum of degenerate simplices σ'' so that $\sigma - \sigma'' = 0$ whenever σ is of type n. Equivalently, any $\sigma : \Delta^n \to K(X)$ is a sum of degenerate simplices if and only if the leading term $x_{[n]}$ vanishes. □

In the statement below we let $\tau_n X$ denote the truncated subcomplex

$$\tau_n X = \cdots \to X^{-2} \to X^{-1} \to Z^0(X) \to 0$$

in any given cochain complex X.

cor:4617 **Corollary 11.8.** *There is a natural map*

$$\epsilon_X : N^*K(X) \to X$$

which sends the class of each generator $\{x_J : J \subseteq [n]\}$ in $N^{-n}K(X)$ to its leading term $x_{[n]}$ in X^{-n}. This natural map is an isomorphism onto the subcomplex $\tau_0 X$ in X.

The proof only requires a check of the differential, which we omit.

11.5. **The Dold-Kan correspondence.**

Lemma 11.9. *The functor $N^* : \mathrm{sAb} \to \mathrm{Ch}(\mathbb{Z})$ commutes with colimits.*

Proof. One observes directly that N commutes with direct sums and exact sequences $A' \to A \to A'' \to 0$. So N commutes with colimits as well. It follows that the composite $N^*(-,\mathbb{Z})$ commutes with colimits. □

We note that any simplicial abelian group can be placed in an exact sequence

$$\oplus_\mu N^*(\Delta^{n(\mu)},\mathbb{Z}) \to \oplus_\lambda N^*(\Delta^{n(\lambda)},\mathbb{Z}) \to A \to 0$$

to now observe that the natural identification

$$\mathrm{Hom}_{\mathrm{sSet}}(\Delta^n, K(X)) = \mathrm{Hom}_{\mathrm{Ch}(\mathbb{Z})}(N(\Delta^n,\mathbb{Z}), X)$$

extends to an adjunction between N^* and K.

Proposition 11.10. *There is a unique adjunction*

$$\mathrm{Hom}_{\mathrm{Ch}(\mathbb{Z})}(N^*-,-) \cong \mathrm{Hom}(-,K-)$$

which extends the identification

$$\mathrm{Hom}_{\mathrm{sAb}}(N(\Delta^n,\mathbb{Z}), K(X)) = \mathrm{Hom}_{\mathrm{Ch}(\mathbb{Z})}(N^*(\Delta^n,\mathbb{Z}), X).$$

The unit of this adjunction $A \to KN^*(A)$ sends a simplex $\sigma : \Delta^n \to A$ to the simplex $\sigma' = N^*(\sigma) : \Delta^n \to KN^*(A)$. One can additionally check that the counit of this adjunction is is precisely the natural map $\epsilon_X : N^*K(X) \to X$ from Corollary 11.8. We recall that this map is counit map is an isomorphism whenever X is concentrated in nonpositive degrees.

prop:4760

Proposition 11.11. *For any simplicial abelian group A, the unit map $u_A : A \to KN^*(A)$ is an isomorphism.*

We omit the proof, as it is somewhat intricate, and refer the reader instead to the presentation of Weibel [20, Section 8.4] or Lurie [15, 00QQ]. We now observe the Dold-Kan correspondence as a consequence of Corollary 11.8 and Proposition 11.11.

Theorem 11.12 (Dold-Kan)**.** *The functors $N^* : \mathrm{sAb} \to \mathrm{Ch}(\mathbb{Z})$ and $K : \mathrm{Ch}(\mathbb{Z}) \to \mathrm{sAb}$ restrict to mutually inverse equivalences of abelian categories*

$$\mathrm{sAb} \overset{\sim}{\to} \mathrm{Ch}^{\leq 0}(\mathbb{Z}) \quad \textit{and} \quad \mathrm{Ch}^{\leq 0}(\mathbb{Z}) \overset{\sim}{\to} \mathrm{sAb}\,.$$

Proof. Corollary 11.8 and Proposition 11.11 tell us that, after restricting to $\mathrm{Ch}^{\leq 0}(\mathbb{Z})$, the two composites KN^* and N^*K are naturally isomorphic to the identity. Since N and K are additive functors, they are furthermore equivalences of abelian categories. □

11.6. **Refined Dold-Kan correspondence.** An important aspect of the Dold-Kan equivalence is that it transforms homotopy groups for simplicial abelian groups into cohomology groups for cochain complexes.

thm:dk_htopy_cohom

Theorem 11.13 (Dold-Kan II)**.** *Let A be a simplicial abelian group and X be an integral cochain complex. The Dold-Kan equivalence admits natural isomorphisms*

$$\pi_n(A, 0) \cong H^{-n}(N^*A) \quad \textit{and} \quad H^{-n}(X) \cong \pi_n(K(X), 0) \tag{54}$$ eq:4661

at all non-negative integers n.

For a basic outline of the proof, we obtain the fundamental group $\pi_n(A)$ as the collection of (pointed) homotopy classes of maps

$$\pi_n(A, 0) = \{\text{pointed maps } S^n \to A\}/\sim_{\text{pt htop}}.$$

One calculates $N^*(S^n, \mathbb{Z}) = \mathbb{Z}[n] \oplus \mathbb{Z}[0]$, where $\mathbb{Z}[i]$ is a free abelian group concentrated in degree $-i$. Hence $H^{-n}(X)$ is identified with homotopy classes of maps from $N^*(S^n, \mathbb{Z})$ which vanish on $\mathbb{Z}[0]$. We might think of such maps as pointed homotopy classes of pointed maps to obtain

$$H^{-n}(X) = \{\text{pointed maps } N^*(S^n, \mathbb{Z}) \to X\}/\sim_{\text{pt htop}}.$$

So we expect to obtain the identifications (54) from a homotopy sensitive articulation of the Dold-Kan correspondence. We take a moment to explain how Dold-Kan interacts with homotopy, and subsequently how the identifications (54) are extracted out of the Dold-Kan equivalence.

11.7. **Dold-Kan and homotopy.** Recall that any simplicial abelian group is a Kan complex. We can therefore speak of homotopy equivalences between maps between simplicial abelian groups in the usual way, i.e. as maps $A \times \Delta^1 \to B$ with prescribed value on $A \times \partial\Delta^1$.

lem:4708 **Lemma 11.14** ([20, Theorem 8.3.12]). *A homotopy $h : K \times \Delta^1 \to L$ between two maps of simplicial abelian groups $f, f' : K \to L$ is equivalent to the following data: At each $n \geq 0$ we have a tuple of maps $h_i[n] : K[n] \to B[n+1]$, indexed $0 \leq i \leq n$, which satisfy*

$$\partial_0^* h_0 = f \quad \text{and} \quad \partial_{n+1}^* h_n[n] = f'[n]$$

as well as the intermediate formulae

$$\partial_i^* h_j = \begin{cases} h_{j-1}\partial_i^* & \text{if } i < j \\ \partial_i^* h_{i-1} & \text{if } i = j \neq 0 \\ h_j \partial_{i-1}^* & \text{if } i < j+1 \end{cases} \quad \text{and} \quad s_i^* h_j = \begin{cases} h_{j+1}s_i^* & \text{if } i \leq j \\ h_j s_{i-1}^* & \text{if } i > j. \end{cases} \tag{55}$$ eq:4725

Given a homotopy $h : K \times \Delta^1 \to L$ we refer to the corresponding maps $\{h_i[n] : n, 0 \leq i \leq n\}$ as the *simplicial data* for h. We recall the construction of this bijection directly from [20].

Construction. Fix an integer n. For each $-1 \leq i \leq n$ let $\alpha_i : [n] \to [1]$ be the unique increasing map with $\alpha_i^{-1}(0) = \{0, \ldots, i\}$. To be clear, α_{-1} takes constant value 1. We now have

$$(K \times \Delta^1)[n] = \amalg_{i=-1}^n K[n] \times \{\alpha_i\} \cong \amalg_{i=-1}^n K[n]$$

Given a homotopy $h : K \times \Delta^1 \to L$ between maps f and f' define

$$h_i[n] := (h|A[n+1] \times \{\alpha_i\})s_i^* : K[n] \to K[n+1]$$

Conversely, given data $h_i[n]$ as above, let $h : K \times \Delta^1 \to L$ be the unique simplicial map with

$$h[n]|_{K[n]\times\{\alpha_i\}} = \begin{cases} f'[n] & \text{when } i = -1 \\ d_{i+1}^* h_i[n] & \text{when } 0 \leq i < n \\ f[n] & \text{when } i = n. \end{cases} \tag{56}$$ eq:4751

□

Definition 11.15. Suppose A and B are simplicial abelian groups. A homotopy $h : A \times \Delta^1 \to B$ between maps of simplicial abelian groups is said to be an additive homotopy, or $\mathbb{Z}$-homotopy, if all of the maps $h_i[n] : A[n] \to B[n+1]$ in the corresponding simplicial data are maps of abelian groups.

Clearly one can add and subtract additive simplicial data so that additive homotopy classes are stable under linear combinations. In particular we have an identification of quotients

$$\mathrm{Hom}_{\mathrm{sAb}}(A, B)/\mathbb{Z}\text{-htopy} = \mathrm{Hom}_{\mathrm{sAb}}(A, B)/\{f : f \text{ htopic to } 0\},$$

and we see that the quotient inherits an additive group structure from that of $\mathrm{Hom}_{\mathrm{sAb}}(A, B)$.

By extending linearly we furthermore see that the adjunction

$$\mathrm{Hom}_{\mathrm{sSet}}(K, B) \overset{\sim}{\to} \mathrm{Hom}_{\mathrm{sAb}}(\mathbb{Z}K, B),$$

at any simplicial set K and simplicial abelian group B, identifies homotopy classes of maps from K to B with additive homotopy classes of maps from $\mathbb{Z}K$ to B. So we have an induced adjunction at the homotopy level

$$\mathrm{Hom}_{\mathrm{sSet}}(K, B)/\text{ htopy} \overset{\sim}{\to} \mathrm{Hom}_{\mathrm{sAb}}(\mathbb{Z}K, B)/\mathbb{Z}\text{-htopy}. \tag{57}$$ eq:4769

prop:dk_htopy **Proposition 11.16.** *The isomorphism*

$$N^* : \mathrm{Hom}_{\mathrm{sAb}}(A,B) \overset{\sim}{\to} \mathrm{Hom}_{\mathrm{Ch}(\mathbb{Z})}(N^*A, N^*B)$$

identifies additive homotopy equivalence classes of simplicial maps with homotopy equivalence classes of chain maps, and hence induces a natural isomorphism on the quotients

$$N^* : \mathrm{Hom}_{\mathrm{sAb}}(A,B)/\mathbb{Z}\text{-htopy} \overset{\sim}{\to} \mathrm{Hom}_{\mathrm{K}(\mathbb{Z})}(N^*A, N^*B).$$

Note the appearance of the homotopy category $\mathrm{K}(\mathbb{Z})$ in the second expression above. We only indicate the main idea of the proof.

Sketch proof. One shows that any additive homotopy $\{h_i[n] : A[n] \to B[n+1]\}_{i,n}$ between maps $f, f' : A \to B$ defines a homotopy ξ_h between the associated maps on complexes $N^*(f), N^*(f') : N^*(A) \to N^*(B)$. Specifically we can take ξ_h with each $\xi_h^n = \sum_i (-1)^i h_i[n]$ [20, Lemma 8.3.13].

Conversely, consider two maps between nonpositively graded cochain complexes $g, g' : X \to Y$ which are homotopic via some cochain homotopy ξ. Consider at each n the natural inclusions

$$X^{-n} \to K(X)[n], \quad x \mapsto \{x_J : J \subseteq [n]\}$$

$$\text{where } x_{[n]} = x,\ x_{[n-1]} = (-1)^n d(x),\ x_J = 0 \text{ otherwise.}$$

We note that for any additive homotopy h between $K(g)$ and $K(g')$, h is determined uniquely by its values on the subspaces $X^{-n} \subseteq K(X)[n]$. Indeed, $X^0 = K(X)[0]$, at all positive n

$$K(X)[n] = X^{-n} + \text{degenerate simplices},$$

and we see from the constraints h from (55) that the values of $h_*[n]$ on degenerate simplices are determined completely by previous map $h_*[n-1]$.

One shows finally that ξ determines a (unique) additive homotopy h^ξ between $K(g), K(g') : KN^*(A) \to KN^*(B)$ for which the simplicial data satisfies

$$h_i^\xi[n]|_{X^{-n}} = \begin{cases} s_i^* g & \text{if } i < n-1 \\ s_{n-1}^* g - s_n^* \xi^{-n+1} d & \text{if } i = n-1 \\ s_n^*(g - \xi^{-n+1} d) - \xi^{-n} & \text{if } i = n \end{cases}$$

at each n and $0 \le i \le n$. See [20, pg 273–274]. □

We recall that any simplicial abelian group B is a Kan complex (Proposition 11.1). Hence the simplicial set $\mathrm{Fun}(K,B)$ is a Kan complex, at any simplicial set K, and we have

$$\pi_0(\mathrm{Fun}(K,B)) = \mathrm{Hom}_{\mathrm{sSet}}(K,B)/\,\mathrm{htopy}\,.$$

We now obtain the following corollary via the identification (57) and Proposition 11.16.

cor:dk_htopy **Corollary 11.17.** *Let K be a simplicial set and B be a simplicial abelian group. The isomorphism*

$$N^* : \mathrm{Hom}_{\mathrm{sSet}}(K,B) \overset{\sim}{\to} \mathrm{Hom}_{\mathrm{Ch}(\mathbb{Z})}(N^*(K,\mathbb{Z}), N^*B)$$

reduces to a natural isomorphism

$$N^* : \pi_0(\mathrm{Fun}(K,B)) \overset{\sim}{\to} \mathrm{Hom}_{\mathrm{K}(\mathbb{Z})}(N^*A, N^*B).$$

11.8. **Dold-Kan and pointed homotopy.** Let $x : * \to K$ be a pointed simplicial set, and consider any abelian group B. Note that B is naturally pointed via the additive unit $0 : * \to B$. We then have the additive subgroup of pointed maps

$$\mathrm{Hom}_{\mathrm{sSet}}(K,B)_{\mathrm{pt}} = \mathrm{Hom}_{\mathrm{sSet}}(K,B)_{\mathrm{pt}(x,0)}$$

in $\mathrm{Hom}_{\mathrm{sSet}}(K,B)$.

Definition 11.18. The category of pointed cochains is the undercategory $\mathrm{Ch}(\mathbb{Z})_{\mathbb{Z}/}$, and a pointed homotopy between pointed maps $f, f : X \to Y$ is a homotopy $h : X \to \Sigma Y$ whose restriction along the structure map $\mathbb{Z} \to X$ is identically 0.

As with simplicial abelian groups, we always have the 0-pointing, which gives an embedding $\mathrm{Ch}(\mathbb{Z}) \to \mathrm{Ch}(\mathbb{Z})_{\mathbb{Z}/}$. This embedding is left adjoint to the forgetful functor.

Since $N^*(*, \mathbb{Z}) = \mathbb{Z}$, we see that N^* restricts to an equivalence between the categories of pointed simplicial abelian groups, i.e. simplicial abelian groups with a fixed map $* \to A$, and pointed nonnegatively graded cochains. In particular, we have the binatural isomorphism

$$N^* : \mathrm{Hom}_{\mathrm{sSet}}(K,B)_{\mathrm{pt}} \overset{\sim}{\to} \mathrm{Hom}_{\mathrm{Ch}(\mathbb{Z})}(N^*(K,\mathbb{Z}), N^*B)_{\mathrm{pt}}.$$

Though we don't define pointed additive homotopy for pointed simplicial abelian groups in general, for maps $f, f' : (A,a) \to (B,0)$ between pointed simplicial sets in which B has the 0 pointing, we say f and f' are pointed additively homotopic if there is an additive homotopy $\{h_i[n]\}_{n,i}$ for which all $h_i[n]|_{a[n]} = 0$.

lem:4853 **Lemma 11.19.** *Let $a : * \to A$ and $b : * \to B$ be pointed simplicial abelian groups, and suppose B has trivial pointing $b = 0$. Then the embedding*

$$\mathrm{Hom}_{\mathrm{sAb}}(A/\mathbb{Z}a, B) \to \mathrm{Hom}_{\mathrm{sAb}}(A,B)$$

is an isomorphism onto $\mathrm{Hom}_{\mathrm{sAb}}(A,B)_{\mathrm{pt}}$, and reduces to an isomorphism

$$\mathrm{Hom}_{\mathrm{sAb}}(A/\mathbb{Z}a, B)/\{\mathbb{Z}\text{-htopy}\} \overset{\sim}{\to} \mathrm{Hom}_{\mathrm{sAb}}(A,B)_{\mathrm{pt}}/\{\text{pointed } \mathbb{Z}\text{-htopy}\}.$$

A similar result holds for cochains. Namely, we have a natural isomorphism between maps in the homotopy category $\mathrm{Hom}_{\mathrm{K}(\mathbb{Z})}(X/\mathbb{Z}x, Y)$ and pointed homotopy classes of maps in $\mathrm{Hom}_{\mathrm{Ch}(\mathbb{Z})}(X,Y)_{\mathrm{pt}}$, whenever Y has the 0 pointing.

prop:4870 **Proposition 11.20.** *Let $x : * \to K$ be a pointed simplicial set and B be a simplicial abelian group. Give B its 0 pointing. Then the isomorphism*

$$N^* : \mathrm{Hom}_{\mathrm{sSet}}(K,B)_{\mathrm{pt}} \overset{\sim}{\to} \mathrm{Hom}_{\mathrm{Ch}(\mathbb{Z})}(N^*(K,\mathbb{Z}), N^*B)_{\mathrm{pt}}$$

reduces to an isomorphism

$$\pi_0(\mathrm{Fun}(K,B)_{\mathrm{pt}}) \overset{\sim}{\to} \mathrm{Hom}_{\mathrm{Ch}(\mathbb{Z})}(N^*(K,\mathbb{Z}), N^*B)_{\mathrm{pt}}/\{\text{pointed htopy}\}.$$

Let us recall that $\mathrm{Fun}(K,B)_{\mathrm{pt}}$ is the fiber space

$$\begin{tikzcd} \mathrm{Fun}(K,B)_{\mathrm{pt}} \arrow[r] \arrow[d] & \mathrm{Fun}(K,B) \arrow[d] \\ * \arrow[r, "0"'] & \mathrm{Fun}(x,B). \end{tikzcd}$$

Proof. We have

$$\begin{aligned}\pi_0(\mathrm{Fun}(K,B)_{\mathrm{pt}}) &= \mathrm{Hom}_{\mathrm{sSet}}(K,B)_{\mathrm{pt}}/\{\text{pointed htopy}\}\\ &\cong \mathrm{Hom}_{\mathrm{sAb}}(\mathbb{Z}K,B)_{\mathrm{pt}}/\{\text{pointed }\mathbb{Z}\text{-htopy}\}.\end{aligned}$$

So the result follows from the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{sAb}}(\mathbb{Z}K/\mathbb{Z}x,B) \arrow[r, "\sim"] \arrow[d, "\sim"'] & \mathrm{Hom}_{\mathrm{Ch}(\mathbb{Z})}(N^*(K,\mathbb{Z})/\mathbb{Z}x,N^*B) \arrow[d, "\sim"] \\
\mathrm{Hom}_{\mathrm{sAb}}(\mathbb{Z}K,B)_{\mathrm{pt}} \arrow[r, "\sim"] & \mathrm{Hom}_{\mathrm{Ch}(\mathbb{Z})}(N^*(K,\mathbb{Z}),N^*B)_{\mathrm{pt}},
\end{tikzcd}$$

which reduced to a diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{sAb}}(\mathbb{Z}K/\mathbb{Z}x,B)/\sim \arrow[r, "\sim"] \arrow[d, "\sim"'] & \mathrm{Hom}_{\mathrm{K}(\mathbb{Z})}(N^*(K,\mathbb{Z})/\mathbb{Z}x,N^*B) \arrow[d, "\sim"] \\
\pi_0(\mathrm{Fun}(K,B)_{\mathrm{pt}}) \arrow[r, dashed, "\exists!\sim"] & \mathrm{Hom}_{\mathrm{Ch}(\mathbb{Z})}(N^*(K,\mathbb{Z}),N^*B)_{\mathrm{pt}}/\sim,
\end{tikzcd} \tag{58}$$

eq:49700

via Corollary 11.17 and Lemma 11.19. By considering a sufficiently large diagram, which connects the two squares above, we see that the completing isomorphism in (58) is necessarily induced by the isomorphism

$$N^* : \mathrm{Hom}_{\mathrm{sSet}}(K,B)_{\mathrm{pt}} \overset{\sim}{\to} \mathrm{Hom}_{\mathrm{Ch}(\mathbb{Z})}(N^*(K,\mathbb{Z}),N^*B)_{\mathrm{pt}}$$

□

11.9. **Proof of Theorem 11.13.** For the simplicial n-sphere S^n we have

$$N^*(S^n,\mathbb{Z}) = \cdots \to 0 \to \cdots 0 \to \mathbb{Z}\sigma_n \to 0 \cdots \to \mathbb{Z}\sigma_0 \to 0$$

with the unique n-face sitting in cohomological degree $-n$ and the unique 0-face sitting in cohomological degree 0. When $n > 1$ the differential on this complex is 0, for degree reasons, and at $n = 1$ the differential is still 0 since $d(\sigma_1) = \sigma_0 - \sigma_0 = 0$ directly. Hence

$$\mathrm{Hom}_{\mathrm{Ch}(\mathbb{Z})}(N^*(S^n,\mathbb{Z}),X) = Z^{-n}(X) \oplus Z^0(X)$$

for any cochain complex X. Now, if we give X its 0 pointing, and give $N^*(S^n,\mathbb{Z})$ its unique pointing induced by the pointing on S^n, then we have

$$\mathrm{Hom}_{\mathrm{Ch}(\mathbb{Z})}(N^*(S^n,\mathbb{Z}),X)_{\mathrm{pt}} \cong Z^{-n}(X)$$

and one observes further a natural identification

$$\mathrm{Hom}_{\mathrm{Ch}(\mathbb{Z})}(N^*(S^n,\mathbb{Z}),X)_{\mathrm{pt}}/\{\text{pointed htopy}\} \cong H^{-n}(X).$$

Let us recall that, for any simplicial abelian group, we have

$$\pi_n(B,0) = \pi_0(\mathrm{Fun}(S^n,B)_{\mathrm{pt}})$$

by definition. So we obtain Theorem 11.13 as an consequence of Proposition 11.20.

Proof of Theorem 11.13. The result for A is a consequence of Proposition 11.20, as explained above. The result for X follows by by the natural isomorphism $N^*K(X) \cong \tau_0 X$, where $\tau_0 X \subseteq X$, and the identifications $H^{-n}(\tau_0 X) = H^{-n}(X)$ at all positive n. □

Remark 11.21. Note that we've not claimed that the isomorphisms of Theorem 11.13 are isomorphisms of groups, under the natural group structure on $\pi_n(A, 0)$. They are however natural isomorphisms of groups under a group structure on $\pi_n(A, 0)$ induced by the group structure on A.

However, it can be shown that the usual group structure on the homotopy groups is identified with the one induced by the additive structure on A [8, Discussion preceding Corollary 2.7]. From this we conclude that the isomorphisms of Theorem 11.13 do in fact respect the group structures.

12. Mapping spaces for dg categories

We adopt the following notion of fully faithfulness for dg functors.

Definition 12.1. Call a dg functor $f : \mathbf{A} \to \mathbf{B}$ fully faithful if the induced maps on Hom complexes

$$f : \mathrm{Hom}^*_{\mathbf{A}}(x, y) \to \mathrm{Hom}^*_{\mathbf{B}}(fx, fy)$$

are all quasi-isomorphisms.

Here we investigate relationship between fully faithfulness for a given dg functor and fully faithfulness for the associated map on the dg nerves. As a fundamental point, we identify the mapping spaces for the dg nerve $\mathrm{N}^{\mathrm{dg}}(\mathbf{A})$ with the Eilenbergh-MacLane spaces for the Hom complexes $\mathrm{Hom}^*_{\mathbf{A}}(x, y)$.

12.1. Main findings.

prop:4410 **Proposition 12.2.** *Consider a dg category $\mathbf{A}$, and let K denote the Eilenbergh-MacLane construction. Take $\mathscr{A} = \mathrm{N}_{\mathrm{dg}}(\mathbf{A})$. For each pair of objects x and y in $\mathbf{A}$ there is a natural identification of Kan complexes*

$$\mathrm{Hom}^{\mathrm{L}}_{\mathscr{A}}(x, y) \cong \mathrm{K}(\mathrm{Hom}^*_{\mathbf{A}}(x, y)).$$

By natural here we mean natural in maps between dg categories. As we recall below, we have natural isomorphisms which identify the homotopy groups of the Eilenbergh-MacLane construction with the cohomology of the incoming complex,

$$\pi_n(\mathrm{K}(X)) \cong H^{-n}(X) \quad \text{for all } n \geq 0.$$

Essentially as a corollary to Proposition 12.2 we observe the following.

thm:fullyfaith_dg **Theorem 12.3.** *Suppose $f : \mathbf{A} \to \mathbf{B}$ be a functor between dg categories and let $F = \mathrm{N}_{\mathrm{dg}}(f) : \mathscr{A} \to \mathscr{B}$ be the corresponding functor between dg nerves. Suppose furthermore that $\mathbf{A}$ and $\mathbf{B}$ admit shift functors and that f commutes with shift functors on $\mathbf{A}$ and $\mathbf{B}$. Then f is fully faithful if and only if F is fully faithful.*

By a shift functor we mean an automorphism Σ on a dg category $\mathbf{A}$ which represents the shift functor on Hom complexes

$$\mathrm{Hom}^*_{\mathbf{A}}(X, \Sigma -) \cong \Sigma\,\mathrm{Hom}^*_{\mathbf{A}}(X, -),$$

and by fully faithful we mean that f induces quasi-isomorphisms on Hom complexes. Via the sequence of maps

$$\begin{aligned} \mathrm{Hom}^*_{\mathbf{A}}(\Sigma X, \Sigma X) &\cong \Sigma\,\mathrm{Hom}^*_{\mathbf{A}}(\Sigma X, X) \\ &\xrightarrow{f} \Sigma\,\mathrm{Hom}^*_{\mathbf{B}}(f(\Sigma X), fX) \cong \mathrm{Hom}^*_{\mathbf{B}}(f(\Sigma X), \Sigma f(X)) \end{aligned}$$

we see that there is a natural map $f(\Sigma X) \to \Sigma f(X)$ at all X in $\mathbf{A}$, namely the image of the identity under the above sequence. We say that f commutes with shifts if this map is an isomorphism at all X. We note that this situation occurs for essentially all functors which arise in both representation theoretic and algebro-geometric settings.

Remark 12.4. In a more thoughtful analysis, the shift functor is defined as the pushout of a particular diagram over 0. So this "commutation" with the shift would follow, for example, from cocontinuity of the functor F, or exactness of F.

One has a strongly related statement which forgoes any reference to shifting.

thm:ffes_dg **Theorem 12.5.** *Suppose a dg functor $f : \mathbf{A} \to \mathbf{B}$ is fully faithful (resp. fully faithful and essentially surjective). Then the corresponding functor on ∞-categories $F : \mathscr{A} \to \mathscr{B}$ is fully faithful (resp. an equivalence).*

We provide a proof which is contingent on Proposition 12.2.

Proof. We recall that the Eilenbergh-MacLane construction naturally identifies homotopy groups with cohomology (Theorem 11.13). So, from the identification between the pinched and standard mapping spaces of Theorem 10.17, we see that F is fully faithful whenever f is fully faithful. It now follows that F is essentially surjective and fully faithful whenever f is, and hence F is an equivalence whenever f is fully faithful and essentially surjective by Theorem 8.2. $\square$

Let us now establish Proposition 12.2, then return to prove Theorem 12.3.

sect:simplices_exp

12.2. **Simplices in the pinched mapping space, explicitly.** Take $\mathscr{A} = \mathrm{N}^{\mathrm{dg}}(\mathbf{A})$, for a dg category $\mathbf{A}$. An n-simplex $\Delta^n \to \mathrm{Hom}^{\mathrm{L}}_{\mathscr{A}}(x, y)$ is simply an $(n+1)$-simplex $\Delta^{n+1} \to \mathscr{A}$ whose value on $\Delta^{\{0\}}$ is x, and whose value on the 0-th face $\Delta^{\{1,\dots,n+1\}}$ is of constant value y.

Let us recall that an m-simplex in such a dg nerve $\mathscr{A}$, or $m \geq 1$, is a choice of m objects x_i in $\mathbf{A}$ along with a tuple of maps $\{f_I\}$ indexed by subsets $I \subseteq [m]$ with $|I| \geq 2$,

$$f_I \in \mathrm{Hom}_{\mathbf{A}}^{-|I|+2}(x_{\min I}, x_{\max I}),$$

for which satisfy

$$d(f_I) = \sum_{t \in I \setminus \{\min I, \max I\}} (-1)^{|I_{>t}|}(f_{I_{\geq t}} \circ f_{I_{\leq t}} - f_{I-\{t\}}). \tag{59}$$ eq:4970

Here $I_{\geq t} = \{i \in I : i \geq t\}$, $I_{\leq t} = \{i \in I : i \leq t\}$ etc. The constant n-simplex at y is the simplex specified by the maps $f_I : y \to y$ with

$$f_I = \begin{cases} id_y & \text{if } |I| = 2 \\ 0 & \text{else.} \end{cases}$$

Given a subset $J \subseteq [m]$, restricting such a simplex as above to the corresponding face $s^* : \Delta^J \to \Delta^m$ produces the $(|J| - 1)$-simplex specified by the data

$$s^*\{f_I\}_I = \{f_I : I \subseteq [m],\ |I| \geq 2,\ I \subseteq J\}.$$

More generally, for any map $\xi : [k] \to [m]$ we have

$$\xi^*\{f_I\}_I = \{f_{\xi,J} : J \subseteq [k],\ |J| \geq 2\}$$

where

$$f_{\xi,J} = \begin{cases} id_{\xi(J)} & \text{if } |J| = 2 \text{ and } |\xi(J)| = 1 \\ f_{\xi(J)} & \text{if } \xi|_J \text{ is injective} \\ 0 & \text{else.} \end{cases}$$

So, returning to the issue at hand, an n-simplex in $\mathrm{Hom}^{\mathrm{L}}_{\mathscr{A}}(x,y)$ is a tuple of maps $\{f_I\}_I$ indexed by subsets $I \subseteq [n+1]$ of size ≥ 2 with

$$f_I = 0 \text{ whenever } 0 \notin I \text{ and } |I| > 2,\ x_0 = x,\ x_i = y \text{ whenever } i > 0,$$
$$\text{and } f_{\{i,j\}} = id_y \text{ whenever } i, j > 0.$$

The differential constraint (59) is vacuous when $0 \notin I$, and when $0 \in I$ it reduces to give

$$d(f_I) = -f_{I\setminus\{\max I\}} - \sum_{t \in I\setminus\{\min I, \max I\}} (-1)^{|I_{>t}|} f_{I\setminus\{t\}} = -\sum_{t\in I, t>0} (-1)^{|I_{>t}|} f_{I\setminus\{t\}}.$$

By deleting 0 from these $I \subseteq [n+1]$ with $0 \in I$ we obtain the following explicit description of n-simplices in the pinched mapping space.

lem:5012

Lemma 12.6 ([15, 01L9])**.** *For* $\mathbf{A}$ *a dg category and* $\mathscr{A} = \mathrm{N}^{\mathrm{dg}}(\mathbf{A})$, *an* n*-simplex in* $\mathrm{Hom}^{\mathrm{L}}_{\mathscr{A}}(x,y)$ *is specified by a tuple of maps* $f_J \in \mathrm{Hom}^{-|J|+1}_{\mathbf{A}}(x,y)$ *indexed by nonempty subsets of* $J \subseteq [n]$ *which satisfy the constraint*

$$d(f_J) = -\sum_{t\in J} (-1)^{|J_{>t}|} f_{J\setminus\{t\}}.$$

Given $\xi : [m] \to [n]$ *the restricted simplex* $\xi^*\{f_J\}_J$ *is specified by maps* $f_{\xi,H}$, $H \subseteq [n]$, *with*

$$f_{\xi,H} = \begin{cases} f_{\xi(J)} & \text{if } \xi|_J \text{ is injective} \\ 0 & \text{otherwise.} \end{cases}$$

12.3. Identification with the Eilenbergh-MacLane construction.

prop:5029

Proposition 12.7 ([15, 01L9])**.** *Take* $\mathscr{A} = \mathrm{N}^{\mathrm{dg}}(\mathbf{A})$, *for a dg category* $\mathbf{A}$. *Suppose a tuple of maps*

$$\{f_J \in \mathrm{Hom}^{|J|+1}_{\mathbf{A}}(x,y) : J \subseteq [n]\}$$

specifies an n*-simplex in* $\mathrm{Hom}^{\mathrm{L}}_{\mathbf{A}}(x,y)$, *in the manner outlined in Section 12.2. Then the tuple* $\{f'_J : J \subseteq [n]\}$,

$$f'_J = (-1)^{|J|(|J|-1)/2} f_J,$$

specified an n*-simplex in* $K(\mathrm{Hom}^*_{\mathbf{A}}(x,y))$. *Furthermore, the assignment*

$$\vartheta_{x,y} : \mathrm{Hom}^{\mathrm{L}}_{\mathscr{A}}(x,y) \to K(\mathrm{Hom}^*_{\mathbf{A}}(x,y)),\quad \{f_J\}_J \mapsto \{f'_J\}_J,$$

is an isomorphism of Kan complexes.

Proof. We simply check the constraint for $\{f'_J\}_J$. Fix a nonempty subset $J \subseteq [n]$ and take $m = |J|$. When $|J| = 1$ we have $d(f_J) = d(f'_J) = 0$ and there is nothing to check. So we assume $m \geq 2$.

We note that at any $t \in J$ we have

$$|J_{>t}| = m - |J_{\leq t}| \ \text{ and thus }\ |J_{>t}| + \frac{m(m-1)}{2} = \frac{m(m+1)}{2} - |J_{\leq t}|.$$

Additionally,

$$\frac{m(m+1)}{2} - \frac{(m-1)(m-2)}{2} = (m-1) + m = 2m - 1 \equiv 1 \mod 2.$$

Hence

$$\begin{aligned} d(f'_J) &= -\textstyle\sum_t (-1)^{|J_{\leq t}| + m(m+1)/2} f_{J\setminus\{t\}} \\ &= -\textstyle\sum_t -(-1)^{|J_{\leq t}|} f'_{J\setminus\{t\}} \\ &= \textstyle\sum_t (-1)^{|J_{\leq t}|} f'_{J\setminus\{t\}}. \end{aligned}$$

This is precisely the differential constraint for simplices in $K(\mathrm{Hom}^*_{\mathbf{A}}(x,y))$. We therefore obtain a well-defined map

$$\vartheta_{x,y}[n] : \mathrm{Hom}^{\mathrm{L}}_{\mathscr{A}}(x,y)[n] \to K(\mathrm{Hom}^*_{\mathbf{A}}(x,y))[n], \quad \{f_J\}_J \mapsto \{f'_J\}_J$$

on n-simplices, at arbitrary n.

Via the same scaling one produces the inverse to $\vartheta_{x,y}[n]$, so that each $\vartheta_{x,y}[n]$ is seen to be a bijection. Compatibility with the structure maps follows by a direct check, via Lemma 12.6 and the definition of the restriction maps on the Eilenbergh-MacLane space. So we see the $\vartheta_{x,y}[n]$ assemble to provide the claimed isomorphism $\vartheta_{x,y}$ of simplicial sets. □

For the proof of Proposition 12.2 we need only deal with naturality under dg functors.

Proof of Proposition 12.2. Let $\tau : \mathbf{A} \to \mathbf{B}$ be a dg functor with corresponding ∞-functor $\mathrm{T} : \mathscr{A} \to \mathscr{B}$. For an $(n+1)$-simplex $\{f_I\}_I$ in $\mathscr{A}$ we have

$$\mathrm{T}\{f_I\}_I = \{\tau f_I\}_I.$$

Hence for an n-simplex $\{f_J\}_J$ in the pinched mapping space we have $\mathrm{T}\{f_J\}_J = \{\tau f_J\}_J$. To compare, for an n-simplex $\{f'_J\}_J$ in the Eilenbergh-MacLane we have

$$K(\tau)\{f'_J\}_J = \{\tau f'_J\}_J.$$

Since τ is $\mathbb{Z}$-linear on morphisms we have $\tau(\pm f_J) = \pm\tau(f_J)$. Such $\mathbb{Z}$-linearity implies commutativity of the required diagram

$$\begin{tikzcd}
\mathrm{Hom}^{\mathrm{L}}_{\mathscr{A}}(x,y) \arrow[r, "\vartheta_{x,y}"] \arrow[d, "\mathrm{T}"'] & K(\mathrm{Hom}^*_{\mathbf{A}}(x,y)) \arrow[d, "K(\tau)"] \\
\mathrm{Hom}^{\mathrm{L}}_{\mathscr{B}}(\tau x,\tau y) \arrow[r, "\vartheta_{\tau x,\tau y}"] & K(\mathrm{Hom}^*_{\mathbf{B}}(\tau x,\tau y)),
\end{tikzcd}$$

where θ is the isomorphism from Proposition 12.7. □

12.4. **Proof of Theorem 12.3.**

Proof of Theorem 12.3. Let us adopt a slightly different notation and take $\tau : \mathbf{A} \to \mathbf{B}$ a dg functor with corresponding functor between ∞-categories $\mathrm{T} : \mathscr{A} \to \mathscr{B}$. In this case, commutation with shifts implies that τ is fully faithful if and only if τ induces quasi-isomorphisms on the truncated complexes

$$\cdots \to \mathrm{Hom}^{-2}(u,v) \to \mathrm{Hom}^{-1}(u,v) \to Z^0(\mathrm{Hom}(u,v)) \to 0.$$

This is to say, τ is fully faithful if and only if the induced maps on cohomology groups

$$H^{-n}(\mathrm{Hom}^*_{\mathbf{A}}(x,y)) \to H^{-n}(\mathrm{Hom}^*_{\mathbf{B}}(\mathrm{T}\,x,\mathrm{T}\,y))$$

are isomorphisms at all $n \geq 0$.

We have the diagram

$$\begin{tikzcd}
\pi_n K(\mathrm{Hom}^*_{\mathbf{A}}(x,y)) \arrow[r, "\pi_n K(\tau)"] \arrow[d, "\cong"'] & \pi_n K(\mathrm{Hom}^*_{\mathbf{B}}(\tau x,\tau y)) \arrow[d, "\cong"] \\
H^{-n}(\mathrm{Hom}^*_{\mathbf{A}}(x,y)) \arrow[r, "H^{-n}\tau"] & H^{-n}(\mathrm{Hom}^*_{\mathbf{B}}(\tau x,\tau y)).
\end{tikzcd}$$

and from Proposition 12.2 deduce a diagram

$$\begin{tikzcd}
\pi_n \operatorname{Hom}^{\mathrm{L}}_{\mathscr{A}}(x,y) \arrow[r, "\pi_n \mathrm{T}"] \arrow[d, "\cong"'] & \pi_n \operatorname{Hom}^{\mathrm{L}}_{\mathscr{B}}(\mathrm{T}\,x, \mathrm{T}\,y) \arrow[d, "\cong"] \\
\pi_n K(\operatorname{Hom}^*_{\mathbf{A}}(x,y)) \arrow[r, "\pi_n K(\tau)"] & \pi_n K(\operatorname{Hom}^*_{\mathbf{B}}(\tau x, \tau y))
\end{tikzcd}$$

These two diagrams together imply that $H^{-n}(\tau)$ is an isomorphism at all nonnegative n if and only if $\pi_n(\mathrm{T})$ is an isomorphism at all nonnegative n. The above information, and the fact that the left pinched mapping spaces are naturally identified with the standard mapping spaces (Theorem 10.17), tells us that $\tau : \mathbf{A} \to \mathbf{B}$ is fully faithful if and only if the associated map on ∞-categories $\mathrm{T} : \mathscr{A} \to \mathscr{B}$ is fully faithful. □

So, all that is left is to deal with the proof of Proposition 12.2. We first recall the contents of the Dold-Kan correspondence, then leverage these materials in our pursuit of this fundamental identification.

12.5. **A basic example: Koszul duality.** In most instances one should approach derived functors via localization [14, Propositions 1.3.4.5, 1.3.5.15] and stability [14, Proposition 1.3.5.9] [2, Corollary 5.11], however in some instances one can deal directly with the explicit injective or model for the derived ∞-category.

Let's consider the case of Koszul duality. Take $S = k[x_1, \dots, x_n]$ with the x_i in degree 1 and $\Lambda = \wedge_k(y_1, \dots, y_n)$ with the y_i in degree 0. Take $\mathbf{A}$ the dg category of K-projective dg Λ-modules with finite-dimensional cohomology and $\mathbf{B}$ the dg category of K-projective dg S-modules with coherent cohomology. We have

$$\mathscr{D}_{fin}(\Lambda) \text{ and } \mathscr{D}_{coh}(S),$$

the derived ∞-categories of finite-dimensional dg Λ-modules and coherent dg S-modules, which we construct explicitly via the dg nerves of the corresponding dg categories of K-projective dg modules.

Consider the Koszul resolution

$$Kos = (\Lambda \otimes S^*,\ d = \sum_i y_i \otimes x_i),$$

with its natural dg (Λ, S)-bimodule structure, and consider the dg function

$$r = \operatorname{Hom}^*_\Lambda(Kos, -) : \operatorname{dgMod}(\Lambda)_{\mathrm{Proj}} \to \operatorname{dgMod}(S)_{\mathrm{Proj}}.$$

This functor commutes with shifts. We have the associated functor on derived ∞-categories, restricted to the finite ∞-subcategories, which we simply denote

$$R : \mathscr{D}_{fin}(\Lambda) \to \mathscr{D}_{fin}(S).$$

It is well-known that the functor $\operatorname{Hom}^*_\Lambda(Kos, -)$ induces an equivalence on the corresponding homotopy categories $D_{fin}(\Lambda) \to D_{fin}(S)$, and hence that the maps on morphism complexes

$$r_{M,N} : \operatorname{Hom}^*_\Lambda(M, N) \to \operatorname{Hom}^*_S(rM, rN)$$

are quasi-isomorphisms at all M and N. (Here one uses the shift to move from 0-th cohomology to all cohomology.) It follows by Theorem 12.3 that the associated functor on ∞-categories $R : \mathscr{D}_{fin}(\Lambda) \to \mathscr{D}_{coh}(S)$ is fully faithful. Essential surjectivity follows from, and is equivalent to, essential surjectivity of the map on homotopy categories. So we see that the functor R is in fact and equivalence of ∞-categories, by Theorem 8.2. In this way Koszul duality lifts to an equivalence of ∞-categories.

sect:inj_proj

13. Injective versus projective models for $\mathscr{D}(\mathbb{A})$

Throughout this section we fix $\mathbb{A}$ an Grothendieck abelian category with enough projectives. The most basic example would be the category of arbitrary modules over some ring R.

In this setting one has the "algebraists model" for the derived category $D(\mathbb{A})$, which employs projectives and projective resolutions. This model is convenient for deriving tensor products, for example, and can also be used to derived Hom. (The injective model is "bad" for tensor product functors.) While this model is generally bad for the algebro-geometric setting, one might imagine reducing to the affine setting where projectives can again be effectively be employed.

Given $\mathbb{A}$ as described, we provide below a unique identification

$$\mathscr{D}(\mathbb{A}) = \{\text{the } \infty\text{-category of sufficiently injective complexes}\}$$

$$\cong \{\text{the } \infty\text{-category of sufficiently projective complexes}\}.$$

We begin in the bounded setting, where the existence of projective and injective resolutions is more familiar, then address the unbounded setting.

sect:D_bounded

13.1. **The bounded setting.**

Let $\mathbb{A}$ be an abelian category with enough projectives and injectives, and take

$$\mathbf{D}^b_{\mathrm{Proj}} := \left\{ \begin{array}{c} \text{The dg category of bounded} \\ \text{above complexes of projectives} \\ P \text{ with bounded cohomology} \end{array} \right\}, \ \mathbf{D}^b_{\mathrm{Inj}} := \left\{ \begin{array}{c} \text{The dg category of bounded} \\ \text{below complexes of injectives} \\ I \text{ with bounded cohomology} \end{array} \right\}.$$

We claim that there is a canonical identification of the associated dg nerves

$$\mathscr{D}^b_{\square} := \mathrm{N}^{\mathrm{dg}}(\mathbf{D}^b_{\square}), \quad \mathscr{D}^b(\mathbb{A}) \text{``} = \text{''} \mathscr{D}^b_{\mathrm{Proj}} \simeq \mathscr{D}^b_{\mathrm{Inj}}.$$

Let's define a third dg category $\mathbf{D}^b_{\mathrm{Bal}}$ whose objects are triples

$$M_\alpha = (P, I, \alpha : P \to I),$$

where P is in $\mathbf{D}^b_{\mathrm{Proj}}$, I is in $\mathbf{D}^b_{\mathrm{Inj}}$, and α is a quasi-isomorphism of cochains over $\mathbb{A}$. (Really the quasi-isomorphism α already specifies the triple.) As graded spaces, the mapping complexes

$$\mathrm{Hom}^*_{\mathbf{D}^b_{\mathrm{Bal}}}(M_\alpha, M_\beta) =$$

are the upper triangular matrices

$$\mathrm{Hom}^*_{\mathbf{D}^b_{\mathrm{Bal}}}(M_\alpha, M_\beta) = \begin{bmatrix} \mathrm{Hom}^*_{\mathbb{A}}(I, I') & \Sigma^{-1}\mathrm{Hom}^*_{\mathbb{A}}(P, I') \\ 0 & \mathrm{Hom}^*_{\mathbb{A}}(P, P') \end{bmatrix}$$

with composition

$$[f_{ij}] \cdot [g_{ij}] = f_{11}g_{11} + f_{22}g_{22} + (-1)^{|f_{11}|} f_{11}g_{12} + f_{12}g22.$$

The differential is the modified differential

$$d([f_{ij}]) = d(f_{11}) + d(f_{22}) - d(f_{12}) + \beta f_{22} - f_{11}\alpha.$$

To make sense of this construction one can consider the following situation: Consider a dg (S,R)-bimodule M over dg algebras S and R. Then we have the shifted dg bimodule ΣM, which has negated differential and new actions

$$s \cdot_{\text{shifted}} m = (-1)^{|s|} s \cdot m \quad \text{and} \quad m \cdot_{\text{shifted}} r = m \cdot r$$

We then have the upper triangular matrix algebra

$$\mathrm{UMat}(\Sigma M) = \begin{bmatrix} S & \Sigma M \\ 0 & R \end{bmatrix}$$

with natural dg structure induced by the dg algebra/bimodule structures on the entries. Now, any degree 0 cocyle α in M produces a degree 1 solution to the Maurer-Cartan equation $d(\alpha) + \alpha^2 = 0$, and we twist by this solution to obtain the new dg algebra

$$\mathrm{UMat}(\Sigma M)_\alpha = \mathrm{UMat}(\Sigma M) \text{ with new differential } d + [\alpha, -].$$

In the case $R = \mathrm{End}^*_{\mathbb{A}}(I)$, $S = \mathrm{End}^*_{\mathbb{A}}(P)$, $M = \mathrm{Hom}^*_{\mathbb{A}}(P, I)$ this matrix algebra, with α-twisted differential, reproduces the endomorphisms $\mathrm{End}^*_{\mathbf{D}^b_{\mathrm{Bal}}}(M_\alpha)$.

Lemma 13.1. *The dg category $\mathbf{D}^b_{\mathrm{Bal}}$ is in fact a dg category.*

Proof. All of the calculations are similar to the calculations which show that $\mathrm{UMat}(\Sigma M)_\alpha$ is a dg algebra. We omit the check of associativity.

For the square of the differential, we have

$$d = d' + (\beta \cdot -) - (- \cdot \alpha)$$

with $d'([g_{ij}]) = [\pm d(g_{ij})]$. Since α and β in M_α and M_β are cocycles we have

$$-d(\beta \cdot g_{22}) = -\beta \cdot d(g_{22}) \quad \text{and} \quad -d(g_{11} \cdot \alpha) = -d(g_{11}) \cdot \alpha.$$

Hence

$$d^2([g_{ij}]) = (d')^2([g_{ij}]) - d(\beta \cdot g_{22}) + d(g_{11} \cdot \alpha) + \beta \cdot d(g_{22}) - d(g_{11}) \cdot \alpha = 0.$$

For compatibility with the product, given $f = [f_{ij}]$ in $\mathrm{Hom}^*(M_\beta, M_\gamma)$ and $g = [g_{ij}]$ in $\mathrm{Hom}^*(M_\alpha, M_\beta)$ one checks directly

$$d'(f \cdot g) = d'(f) \cdot g + (-1)^{|f|} f \cdot d'(g)$$

so that

$$\begin{gathered} d(f \cdot g) - d(f) \cdot g - (-1)^{|f|} f \cdot d(g) \\ = \gamma f_{22} g_{22} - f_{11} g_{11} \alpha - \gamma f_{22} g_{22} + f_{11} \beta g_{22} - (-1)^{|f|+|f|} f_{11} \beta g_{22} + (-1)^{|f|+|f|} f_{11} g_{11} \alpha \\ = 0. \end{gathered}$$

□

We consider the dg projections on Hom-complexes

$$\begin{tikzcd} & \mathrm{Hom}^*_{\mathbf{D}^b_{\mathrm{Bal}}}(M_\alpha, M_\beta) \arrow[dl, "\pi_{11}"'] \arrow[dr, "\pi_{22}"] & \\ \mathrm{Hom}_{\mathbb{A}}(I, I') & & \mathrm{Hom}^*_{\mathbb{A}}(P, P'). \end{tikzcd}$$

which define dg functors $\pi_I : \mathbf{D}^b_{\mathrm{Bal}} \to \mathbf{D}^b_{\mathrm{Inj}}$ and $\pi_P : \mathbf{D}^b_{\mathrm{Bal}} \to \mathbf{D}^b_{\mathrm{Proj}}$.

prop:bounded **Proposition 13.2.** *The two projections π_I and π_P are fully faithful and essentially surjective.*

Proof. Since any complex with bounded cohomology admits a resolution by a bounded above complex of projectives, and a resolution by a bounded below complex of injectives, the projections π_X are essentially surjective.

As for the claim that these functors induce isomorphisms on Hom-complexes, we have in the injective instance an exact sequence of cochains

$$0 \to \Sigma^{-1}\operatorname{cone}(\beta_* : \operatorname{Hom}(P, P') \to \operatorname{Hom}(P, I')) \to \operatorname{Hom}_{\mathbf{D}^b_{\mathrm{Bal}}}(M_\alpha, M_\beta) \xrightarrow{\pi_I} \operatorname{Hom}(I, I') \to 0. \tag{60}$$

eq:5221

Since β is a quasi-isomorphism and $\operatorname{Hom}^*_{\mathbb{A}}(P, -)$ preserves quasi-isomorphisms the corresponding map $\beta \cdot - = \beta_*$ is a quasi-isomorphism, and the mapping cone appearing in (60) in acyclic. From exactness of the sequence (60) we now conclude that the projection to $\operatorname{Hom}^*_{\mathbb{A}}(I, I')$ is in fact a quasi-isomorphism, and hence that π_I is fully faithful. The argument for π_P is completely similar. □

We now consider the dg nerves $\mathscr{D}^b_\square = \mathrm{N}^{\mathrm{dg}}(\mathbf{D}^b_\square)$, and the corresponding functors between ∞-categories $\Pi_\square : \mathscr{D}^b_{\mathrm{Bal}} \to \mathscr{D}^b_\square$.

We now apply Theorem 12.5 to immediately observe an equivalence between the projective and injective models for the ∞-derived category.

thm:boundedD_bal **Theorem 13.3.** *The two functors*

$$\begin{tikzcd} & \mathscr{D}^b_{\mathrm{Bal}} \arrow[dl, "\Pi_I"'] \arrow[dr, "\Pi_P"] & \\ \mathscr{D}^b_{\mathrm{Inj}} & & \mathscr{D}^b_{\mathrm{Proj}} \end{tikzcd}$$

are equivalences of ∞-categories. In particular, completing the diagram yields equivalences of ∞-categories

$$(\mathscr{D}^b(\mathbb{A}) :=)\mathscr{D}^b_{\mathrm{Inj}} \xrightarrow{\sim} \mathscr{D}^b_{\mathrm{Proj}} \quad \textit{and} \quad \mathscr{D}^b_{\mathrm{Proj}} \xrightarrow{\sim} \mathscr{D}^b_{\mathrm{Inj}}$$

which are unique up to an isomorphism of ∞-functors.

sect:D_unbounded

13.2. **The unbounded setting.** There is nothing special about the bounded situation here. It's simply the most familiar local to work with for such an argument. Let us recall that a complex P (resp. I) in the dg category $\mathrm{Ch}(\mathbb{A})$ is called K-projective (resp. K-injective) if the functor

$$\operatorname{Hom}^*_{\mathbb{A}}(P, -) : \mathrm{Ch}(\mathbb{A}) \to \mathrm{Ch}(\mathbb{Z})$$

$$\big(\text{resp.}\ \operatorname{Hom}^*_{\mathbb{A}}(-, I) : \mathrm{Ch}(\mathbb{A})^{\mathrm{op}} \to \mathrm{Ch}(\mathbb{Z})\big)$$

preserves acyclicity. By considering mapping cones, one sees that a complex is K-projective, or K-injective, if and only if the corresponding Hom-complex functor preserves quasi-isomorphisms.

For general $\mathbb{A}$ with enough injectives and projectives one considers $\mathbf{D}_{\mathrm{Proj}}$ and $\mathbf{D}_{\mathrm{Inj}}$ as the dg categories of K-projective and K-injective complexes to find again, in the unbounded setting, an identification between injective and projective models for the derived ∞-category.

prop:unbounded **Proposition 13.4.** *Let $\mathbb{A}$ be a Grothendieck abelian category with enough projectives, and let $\mathbf{D}_{\mathrm{Proj}}$ and $\mathbf{D}_{\mathrm{Inj}}$ be the dg categories of K-projective and K-injective cochains over $\mathbb{A}$, respectively. Consider the balanced dg category $\mathbf{D}_{\mathrm{Bal}}$, which we* *constructs exactly as in the bounded setting (see Section 13.1). Then the two projections*

$$\begin{tikzcd} & \mathbf{D}_{\mathrm{Bal}} \arrow[dl, "\pi_I"'] \arrow[dr, "\pi_P"] & \\ \mathbf{D}_{\mathrm{Inj}} & & \mathbf{D}_{\mathrm{Proj}}. \end{tikzcd}$$

are fully faithful and essentially surjective.

Proof. Under our Grothendieck restriction, any complex M admits a K-injective resolution $M \to I$ [18, Theorem 3.13]. Since $\mathbb{A}$ has enough projectives, one can also construct K-projective resolutions $P \to M$ at arbitrary M. (One constructs such a resolution via a cell attaching process, as in [5, Theorem III.2.10].) Hence the two functors $\pi_\square$ are essentially surjective. Fully faithfulness is established exactly as in the proof of Proposition 13.2. □

We take associated ∞-categories, via the dg nerve, to observe a balancing result for the derived ∞-category. As usual, we take $\mathscr{D}_\square = \mathrm{N}^{\mathrm{dg}}(\mathbf{D}_\square)$.

thm:D_bal **Theorem 13.5.** *Let $\mathbb{A}$ be a Grothendieck abelian category which has enough projectives. There is an equivalence of ∞-categories $\mathscr{D}(\mathbb{A}) = \mathscr{D}_{\mathrm{Inj}} \overset{\sim}{\to} \mathscr{D}_{\mathrm{Proj}}$ which completes a 2-simplex*

$$\begin{tikzcd} & \mathscr{D}_{\mathrm{Bal}} \arrow[dl, "\Pi_I"'] \arrow[dr, "\Pi_P"] & \\ (\mathscr{D}(\mathbb{A}) =)\mathscr{D}_{\mathrm{Inj}} \arrow[rr, "\sim", "\exists"'] & & \mathscr{D}_{\mathrm{Proj}}. \end{tikzcd} \tag{61}$$ eq:5280

in the ∞-category $\mathscr{C}at_\infty$. This equivalence is uniquely determined by the above diagram, up to natural isomorphism.

Proof. Again follows by Theorem 12.5 and Proposition 13.4. □

Remark 13.6. The obvious analog of Theorem 13.5 holds when we replace $\mathrm{Ch}(\mathbb{A})$ with the dg category of dg modules R-dgmod for a dg algebra R. The proof is exactly the same.

13.3. **Uniqueness of the injective-projective transition.** We have claimed that the equivalence $\mathscr{D}_{\mathrm{Inj}} \overset{\sim}{\to} \mathscr{D}_{\mathrm{Proj}}$ appearing in Theorem 13.5 is unique up to an isomorphism of ∞-functors. This notion of uniqueness is weaker, however, than a complete uniqueness claim. We would propose that the "space of choices" for morphisms completing the diagram (61) is contractible. Let us first decide what this space of fillings is, then address its (non-)triviality.

Consider the ∞-category $\mathscr{C}at_\infty$ of ∞-categories which are small for some sufficiently large universe. A map completing the given diagram is an object in the mapping space

$$\mathrm{Hom}_{(\mathscr{C}at_\infty)_{\mathscr{D}_{\mathrm{Bal}}/}}(\Pi_I, \Pi_P)$$

for the undercategory $(\mathscr{C}at_\infty)_{\mathscr{D}_{\mathrm{Bal}}/}$. We can replace this mapping space with the left pinched space, by Theorem 10.17. This pinched space is explicitly the fiber of the double undercategory

$$\big((\mathscr{C}at_\infty)_{\mathscr{D}_{\mathrm{Bal}}/}\big)_{\Pi_I/}$$

over the point Π_P in $(\mathscr{C}at_\infty)_{\mathscr{D}_{\mathrm{Bal}}/}$. This double undercategory is directly identified with the undercategory

$$(\mathscr{C}at_\infty)_{\Pi_I/}$$

via associativity of the join if one likes. Hence we obtain an identification, at least up to homotopy,

$$\{\text{the space of fillings (61)}\} \simeq \{\Pi_P\} \times_{(\mathscr{C}at_\infty)_{\mathscr{D}_{\mathrm{Bal}}/}} (\mathscr{C}at_\infty)_{\Pi_I/}.$$

prop:unique **Proposition 13.7.** *The ∞-category*

$$\{\Pi_P\} \times_{(\mathscr{C}at_\infty)_{\mathscr{D}_{\mathrm{Bal}}/}} (\mathscr{C}at_\infty)_{\Pi_I/}$$

of functors $\mathscr{D}_{\mathrm{Inj}} \to \mathscr{D}_{\mathrm{Proj}}$ completing the diagram (61) is a contractible Kan complex.

We won't provide all of the details for this uniqueness result, however the argument is fairly straightforward. Since the functor Π_I is an equivalence it follows that the forgetful functor on undercategories

$$(\mathscr{C}at_\infty)_{\Pi_I/} \to (\mathscr{C}at_\infty)_{\mathscr{D}_{\mathrm{Bal}}/}$$

is a trivial Kan fibration [15, 02J2]. Hence the fiber of this forgetful functor along any point is a contractible space. In particular, the fiber along the point

$$\Pi_P : * \to (\mathscr{C}at_\infty)_{\mathscr{D}_{\mathrm{Bal}}/}$$

is contractible, as claimed.

13.4. **Comparison with localization.** This final subsection is a more casual discussion concerning the derived category.

At this point it is apparent that we don't actually want to define the derived ∞-category $\mathscr{D}(\mathbb{A})$ in a way which makes any explicit reference to some specific dg model of injective, projective, flat, etc. complexes. Though the description still leaves something to be desired, one can in fact obtain $\mathscr{D}(\mathbb{A})$ as an ∞-categorical localization of the plain category of cochains over $\mathbb{A}$

$$\mathscr{D}(\mathbb{A}) := \mathrm{Ch}_{\mathrm{plain}}(\mathbb{A})[\mathrm{Qiso}^{-1}] \tag{62}$$ eq:5343

relative to the class of quasi-isomorphism [14, Propositions 1.3.4.5, 1.3.5.15]. This localization happened to be identified with the localization of the homotopy ∞-category $\mathscr{K}(\mathbb{A})$ at quasi-isomorphisms as well [14, Propositions 1.3.4.5].

Remark 13.8. One should note the absurdity of the localization claim (62), when considered from the classical algebraic perspective, as it circumvents the homotopy category completely.

Remark 13.9. The reference to chain complexes at all still references a kind of model for the derived category. In particular, the move from the category $\mathbb{A}$ to cochains over $\mathbb{A}$ has is not premeditated by any generic principle, and so cannot be rationalized via any such principle. One might instead approach the issue via stability and stabilization [14, Section 1.4.2], though we've see no treatments which follow this line of reasoning.

Now, the α appearing in the objects $\alpha : P \to I$ in the balanced category $\mathbf{D}_{\mathrm{Bal}}$ define a dg transformation which fills the 2-diagram

$$\begin{tikzcd}
 & \mathbf{D}_{\mathrm{Bal}} \arrow[dl, "\pi"'] \arrow[dr, "\pi"] & \\
\mathbf{D}_{\mathrm{Inj}} \arrow[dr, "\mathrm{incl}"'] & & \mathbf{D}_{\mathrm{Proj}} \arrow[dl, "\mathrm{incl}"] \\
 & \mathrm{Ch}(\mathbb{A}) &
\end{tikzcd}$$

Taking ∞-categories provides a corresponding diagram in $\mathscr{C}at_\infty$, i.e. a morphism between functors in $\mathrm{Fun}(\mathscr{D}_{\mathrm{Bal}}, \mathscr{K}(\mathbb{A}))$, and this extends to a diagram

$$\begin{tikzcd}
 & \mathscr{D}_{\mathrm{Bal}} \arrow[dl, "\pi"'] \arrow[dr, "\pi"] & \\
\mathscr{D}_{\mathrm{Inj}} \arrow[r, dashed] \arrow[dr, "\sim"'] & \mathscr{K}(\mathbb{A}) \arrow[d, dashed] & \mathscr{D}_{\mathrm{Proj}} \arrow[l, dashed] \arrow[dl, "\sim"] \\
 & \mathrm{Ch}_{\mathrm{plain}}(\mathbb{A})[\mathrm{Qiso}^{-1}] \cong \mathscr{K}(\mathbb{A})[\mathrm{Qiso}^{-1}] &
\end{tikzcd}$$

in $\mathscr{C}at_\infty$ (Proposition 14.4). From this one can show that any morphism $\mathscr{D}_{\mathrm{Inj}} \to \mathscr{D}_{\mathrm{Proj}}$ which completes a diagram under $\mathscr{D}_{\mathrm{Bal}}$ simultaneously completes a diagram over the localization $\mathrm{Ch}_{\mathrm{plain}}(\mathbb{A})[\mathrm{Qiso}^{-1}]$ (Proposition 5.33). This is to say, the unique equivalence

$$\mathrm{IvP} : \mathscr{D}_{\mathrm{Inj}} \overset{\sim}{\to} \mathscr{D}_{\mathrm{Proj}}$$

appearing in Theorem 13.5 is simultaneously the unique equivalence completing the diagram

$$\begin{tikzcd}
\mathscr{D}_{\mathrm{Inj}} \arrow[rr, "\exists! \, = \, \mathrm{IvP}"] \arrow[dr, "\sim"'] & & \mathscr{D}_{\mathrm{Proj}} \arrow[dl, "\sim"] \\
 & \mathscr{D}(\mathbb{A}) \underset{\text{better}}{:=} \mathrm{Ch}_{\mathrm{plain}}(\mathbb{A})[\mathrm{Qiso}^{-1}] &
\end{tikzcd}.$$

Hence our equivalence from Theorem 13.5 is the correct one from the perspective of localization as well.

14. Adjoint functors

A discussion of adjoint functors is in some sense premature, as this discussion is more thoroughly developed through the lenses of presentability and localization. However, we can a make a few comments here about adjoint functors in the ∞-setting and their dg counterparts.

14.1. Adjoint functors.

def:adjoints **Definition 14.1** ([15, 02EL])**.** Given a pair of functors $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$, we say F is left adjoint to G, or equivalently G is right adjoint to F, if there are natural transformations $\eta : id_{\mathscr{C}} \to GF$ and $\epsilon : FG \to id_{\mathscr{D}}$ for which have 2-simplices

$$\begin{tikzcd} & FGF \arrow[dr, "\epsilon"] & \\ F \arrow[ur, "F(\eta)"] \arrow[rr, "id_F"'] & & F \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & GFG \arrow[dr, "G(\epsilon)"] & \\ G \arrow[ur, "\eta"] \arrow[rr, "id_G"'] & & G \end{tikzcd}$$

in $\mathrm{Fun}(\mathscr{C}, \mathscr{D})$ and $\mathrm{Fun}(\mathscr{D}, \mathscr{C})$ respectively.

Given natural transformations η and ϵ which exhibit F as a left adjoint to G we refer to $\eta : id_{\mathscr{C}} \to GF$ as the unit of the adjunction and $\epsilon : FG \to id_{\mathscr{D}}$ as the counit of the adjunction.

Remark 14.2. It is interesting to compare the above definition to the initial definition from [13, Definition 5.2.2.1]. While the two definitions are equivalent [13, Proposition 5.2.2.8] one sees a distinction in the presentations of [13] and [15] which is somewhat indicative of the comparative natures of these two documents. While the definition from [13] is very efficient, and technically malleable, the definition from [15] is highly legible and agrees with out preexisting understanding of adjoint functors from usual category theory.

14.2. **Adjoints and the homotopy category.**

Proposition 14.3. *Suppose a functor $F : \mathscr{C} \to \mathscr{D}$ is left adjoint to a functor $G : \mathscr{D} \to \mathscr{C}$. Then the induced map on homotopy categories $\mathrm{h}F : \mathrm{h}\mathscr{C} \to \mathrm{h}\mathscr{D}$ is left adjoint to $\mathrm{h}G : \mathrm{h}\mathscr{D} \to \mathrm{h}\mathscr{C}$. Furthermore, any transformations $id \to GF$ and $FG \to id$ which reduce to unit and counit transformations at the level of homotopy categories are already unit and counit transformations at the level of ∞-categories.*

Proof. Suppose $\mathscr{E}$ is an ∞-category and $\theta, \theta' : m \to n$ are isomorphic natural transformations in $\mathrm{Fun}(\mathscr{E}, \mathscr{E})$. Then at each object x in $\mathscr{E}$ we have a diagram

$$\begin{tikzcd} & n(x) \arrow[dr, "id"] & \\ m(x) \arrow[ur, "\theta"] \arrow[rr, "\theta'"'] & & n(x). \end{tikzcd}$$

Hence in the homotopy category the natural transformation θ and θ' are equal. This implies that the composite transformations

$$F \to FGF \to F \text{ and } G \to GFG \to G$$

become the identity in the homotopy category. □

14.3. **Natural transformations from the dg setting.**

By a natural transformation between dg functors $t, t' : \mathbf{A} \to \mathbf{B}$ we mean a degree collection of degree 0 cocyles $\theta_x : t(x) \to t'(x)$ in $\mathrm{Hom}^*_{\mathbf{B}}(t(x), t'(x))$ which satisfy

$$\theta_y t(\xi) = t'(\xi)\theta_x \quad \text{at each} \quad \xi \in \mathrm{Hom}^*_{\mathbf{A}}(x, y).$$

prop:5112 **Proposition 14.4.** *Let $t, t' : \mathbf{A} \to \mathbf{B}$ be two dg functors between dg categories, and $\theta : t \to t'$ be any natural transformation. Then there is an explicitly defined natural* *transformation* $\Theta : \Delta^1 \times \mathscr{A} \to \mathscr{B}$ *between the associated functors* $T, T' : \mathscr{A} \to \mathscr{B}$ *on the dg nerves which satisfies*

$$\Theta|_{\Delta^1 \times \{x\}} = \theta_x : t(x) \to t'(x)$$

Construction 14.4. We have already described the values of Θ on 0-simplices in the product. Now suppose $n \geq 1$.

An n-simplex in the product $\Delta^1 \times \mathscr{A}$ is a pair of n-simplices $\{\alpha, \sigma\}$ in $\Delta^1[n] \times \mathscr{A}[n]$. Here α is a map $\alpha : [n] \to [1]$, and this map is determined by a splitting $[n] = [n]_- \amalg [n]_+$, with $[n]_- = \alpha^{-1}(0)$ and $[n]_+ = \alpha^{-1}(1)$. Additionally σ is determined by a tuple of maps $\{f_J : J \subseteq [n]\}$ with $|J| \geq 2$. Let us take $x_i = \sigma|_{\Delta^{\{i\}}}$.

Define $\Theta(\alpha, \sigma) = \{g_J : J \subseteq [n]\}$ where

$$g_J = \begin{cases} t(f_J) & \text{when } J \subseteq [n]_- \\ t'(f_J) & \text{when } J \subseteq [n]_+ \\ t'(f_J)\theta_{x_{\min(J)}} = \theta_{x_{\max(J)}} t(f_J) & \text{otherwise,} \end{cases}$$

The differential constraints on these g_J follow from naturality of θ, and so one sees that the tuple $\{g_J : J \subseteq [n]\}$ defines an n-simplex in $\mathscr{B}$.

We now have well defined maps

$$\Theta[n] : \Delta^1[n] \times \mathscr{A}[n] \to \mathscr{B}[n]$$

at all n, and a direct check verifies that the $\Theta[n]$ assemble into a map of simplicial sets. By construction $\Theta|_{\{0\} \times \mathscr{A}} = T$, $\Theta|_{\{1\} \times \mathscr{A}} = T'$, and $\Theta|_{\Delta^1 \times \{x\}} = \theta_x$. $\square$

Given a dg transformation $\theta : t \to t'$ between dg functors $t, t' : \mathbf{A} \to \mathbf{B}$, and dg functors $f : \mathbf{Z} \to \mathbf{A}$, $g : \mathbf{B} \to \mathbf{C}$, we let $\theta_f : tf \to t'f$ and $\theta^g : gt \to gt'$ denote the natural transformations with

$$(\theta_f)_z = \theta_{f(z)} \quad \text{and} \quad (\theta^g)_x = g(\theta_x)$$

at each z in $\mathbf{Z}$ and x in $\mathbf{A}$. We adopt a similar notation for natural transformations between functors on ∞-categories.

lem:5166 **Lemma 14.5.** *Suppose we are in the situation of Proposition 14.4, and consider dg functors* $f : \mathbf{Z} \to \mathbf{A}$ *and* $g : \mathbf{B} \to \mathbf{C}$. *Let* $F : \mathscr{Z} \to \mathscr{A}$ *and* $G : \mathscr{B} \to \mathscr{C}$ *be the associated functors on* ∞*-categories. Then* $\Theta_F : TF \to T'F$ *and* $\Theta^G : GT \to GT'$ *are the natural transformations associated to the dg transformations* θ_f *and* θ^g*, respectively.*

Proof. The transformations Θ_F and Θ^G are explicitly the composites

$$\Delta^1 \times \mathscr{Z} \overset{[id\ F]}{\to} \Delta^1 \times \mathscr{A} \overset{\Theta}{\to} \mathscr{B} \quad \text{and} \quad \Delta^1 \times \mathscr{A} \overset{\Theta}{\to} \mathscr{B} \overset{G}{\to} \mathscr{C},$$

respectively. One simply checks, using Construction 14.4 directly, that these functors are the natural transformations associated to θ_f and θ^g respectively. $\square$

We finally consider composites of dg transformations and their ∞-counterparts.

lem:5180 **Lemma 14.6.** *Suppose* $t, t', t'' : \mathbf{A} \to \mathbf{B}$ *are dg functors with dg transformations* $\theta : t \to t'$ *and* $\theta' : t' \to t''$. *Take* $\theta'' = \theta'\theta$, *and let* $\Theta, \Theta', \Theta'' : \Delta^1 \times \mathscr{A} \to \mathscr{B}$ *be the associated* ∞*-categorical transformations. There exists a 2-simplex in the mapping category* $\mathrm{M} : \Delta^2 \times \mathscr{A} \to \mathscr{B}$ *with*

$$\mathrm{M}|_{\Delta^{\{0,1\}} \times \mathscr{A}} = \Theta, \ \mathrm{M}|_{\Delta^{\{1,2\}} \times \mathscr{A}} = \Theta', \ \textit{and} \ \mathrm{M}|_{\Delta^{\{0,2\}} \times \mathscr{A}} = \Theta''.$$

This is to say, Θ'' *is a composition of* Θ *and* Θ' *in the* ∞*-category* $\mathrm{Fun}(\mathscr{A}, \mathscr{B})$.

The construction of M is similar to Construction 14.4, and is omitted.

14.4. **Adjoints from the dg setting.** By an adjoint pair of dg functors we mean a pair of dg functors

$$f : \mathbf{A} \to \mathbf{B} \quad \text{and} \quad g : \mathbf{B} \to \mathbf{A}$$

with corresponding dg transformations $u : id_{\mathbf{A}} \to gf$ and $\epsilon : gf \to id_{\mathbf{B}}$ for which the composites

$$F \to FGF \to F \quad \text{and} \quad G \to GFG \to G$$

are both the identity.

thm:dg_adjoints **Theorem 14.7.** *Suppose $f : \mathbf{A} \to \mathbf{B}$ is left adjoint to $g : \mathbf{B} \to \mathbf{A}$, and let $u : id_{\mathbf{A}} \to gf$ and $c : gf \to id_{\mathbf{B}}$ be the dg transformations which exhibit this adjunction. Let $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{A}$ be the induced functors on dg nerves.*

The transformations $\eta : id_{\mathscr{A}} \to GF$ and $\epsilon : GF \to id_{\mathscr{B}}$ which are associated to u and c, as in Construction 14.4, exhibit F as left adjoint to G.

Proof. By Lemmas 14.5 and 14.6, the identity transformations $F \to F$ and $G \to G$ are composites of $\eta_F : F \to FGF$ with $\epsilon^F : FGF \to F$, and $\eta_G : G \to GFG$ with $\epsilon^G : FGF \to G$, respectively. Hence F is left adjoint to G, in the precise sense of Definition 14.1. □

14.5. **A simple example.** Let's consider a basic example. Consider algebras S and R over a field k. Let M be a bounded complex of (S, R)-bimodule and consider the functors

$$M \otimes_R - : \mathrm{Ch}^b(R) \to \mathrm{Ch}^b(S) \quad \text{and} \quad \mathrm{Hom}^*_S(M, -) : \mathrm{Ch}^b(S) \to \mathrm{Ch}^b(R).$$

Let $M' \to M$ be a bounded above resolution of M by projective $S \otimes_k R^{\mathrm{op}}$-modules. Then we have the induced functor on dg categories

$$M' \otimes_R - : \mathrm{Proj}^b(R) \to \mathrm{Proj}^b(S) \quad \text{and} \quad \mathrm{Hom}^*_S(M', -) : \mathrm{Proj}^b(S) \to \mathrm{Proj}^b(R),$$

where the b here indicated bounded above complexes with bounded cohomology. Take dg nerves to get induced functors on the bounded derived ∞-categories

$$M \otimes^{\mathrm{L}}_R - : \mathscr{D}^b(R) \to \mathscr{D}^b(S) \quad \text{and} \quad \mathrm{RHom}_S(M, -) : \mathscr{D}^b(S) \to \mathscr{D}^b(R).$$

The usual unit and counit transformations

$$u : id_R \to \mathrm{Hom}^*_S(M', M' \otimes_R -) \quad \text{and} \quad c : M' \otimes_R \mathrm{Hom}^*_S(M', -) \to id_S$$

now induce natural transformations at the level of ∞-categories

$$\eta : id_R \to \mathrm{RHom}_S(M, M \otimes^{\mathrm{L}}_R -) \quad \text{and} \quad \epsilon : M \otimes^{\mathrm{L}}_R \mathrm{RHom}_S(M, -) \to id_S$$

which exhibit $M \otimes^{\mathrm{L}}_R -$ as left adjoint to RHom_S, by Theorem 14.7.

References

antieau [1] B. Antieau. On the uniqueness of infinity-categorical enhancements of triangulated categories. *arXiv:1812.01526*.

blumberggepnertabuada13 [2] A. J. Blumberg, D. Gepner, and G. Tabuada. A universal characterization of higher algebraic k-theory. *Geom. Topol.*, 17(2):733–838, 2013.

cisinski19 [3] D.-C. Cisinski. *Higher categories and homotopical algebra*, volume 180. Cambridge University Press, 2019.

cohn [4] L. Cohn. Differential graded categories are k-linear stable infinity categories. *arXiv:1308.2587*.

elmendorfetal97 [5] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May. *Rings, modules, and algebras in stable homotopy theory*. Number 47. American Mathematical Soc., 1997.

enochsestrada05 [6] E. Enochs and S. Estrada. Relative homological algebra in the category of quasi-coherent sheaves. *Adv. Math.*, 194(2):284–295, 2005.

faonte17 [7] G. Faonte. Simplicial nerve of an A_∞-category. *Theory Appl. Categ*, 32(31-52):3607208, 2017.

goerssjardine [8] P. G. Goerss and J. F. Jardine. *Simplicial homotopy theory*. Springer Science & Business Media, 2009.

joyaltierney [9] A. Joyal and M. Tierney. Notes on simplicial homotopy theory. *online at ncat-lab.org/nlab/files/JoyalTierneySimplicialHomotopyTheory.pdf*, 2008.

keller94 [10] B. Keller. Deriving DG categories. In *Ann. scient. Éc. Norm. Sup*, volume 27, pages 63–102, 1994.

land21 [11] M. Land. *Introduction to Infinity-categories*. Springer Nature, 2021.

lurie_dagII [12] J. Lurie. Derived algebraic geometry ii: Noncommutative algebra. *online at arXiv:0702299*, version 5.

htt [13] J. Lurie. *Higher topos theory*. Number 170. Princeton University Press, Princeton, New Jersey 08540, 2009.

ha [14] J. Lurie. Higher algebra. *www.math.ias.edu/ lurie/papers/HA.pdf*, 2017.

kerodon [15] J. Lurie. Kerodon. *https://kerodon.net*, 2024.

stacks [16] S. project authors. Stacks project. *stacks.math.columbia.edu*.

rezk [17] C. Rezk. Introduction to quasicategories. *rezk.web.illinois.edu/quasicats.pdf*.

serpe03 [18] C. Serpé. Resolution of unbounded complexes in grothendieck categories. *J. Pure Appl. Algebra*, 177(1):103–112, 2003.

strickland [19] N. Strickland. The category of CGWH spaces, 2009.

weibel94 [20] C. A. Weibel. *An introduction to homological algebra*, volume 38. Cambridge university press, 1994.

Department of Mathematics, University of Southern California, Los Angeles, CA 90007

Email address: cnegron@usc.edu