

Radicals and Semisimplicity

To begin we discuss another class of rings / algebras -

~ I: Division algebras

Defn: A division ring D is a ring for which each nonzero $a \in D$ admits $a^{-1} \in D$ with $a^{-1}a = aa^{-1} = 1$.

A division algebra over a field K is an algebra which is also a division ring.

Defn: A ring R is called a domain if for each nonzero a in R an equation $a \cdot b = 0$ or $b \cdot a = 0$, at b arbitrary in R , implies $b = 0$.

Observation 1: Any division ring is a domain.

Ex: H = quaternions are divisionally over \mathbb{R} .

Lemma 2: The center of any division ring is a field.

Proof: For a in $Z(D)$ we have for a^{-1} and arbitrary b in D ,

$$a^{-1}b = a^{-1}baa^{-1} = a^{-1}aba^{-1} = ba^{-1}.$$

So $a^{-1} \in Z(D)$ as well.

Lemma 3: If \mathbb{K} is an algebraically closed field, then the only finite dimensional \mathbb{K} -algebra over \mathbb{K} is \mathbb{K} itself.

Proof: Take D a div. alg over \mathbb{K} and suppose there exists $x \in D - \mathbb{K}$. Then we have the alg. map $\phi: \mathbb{K}[X] \rightarrow D$, $X \mapsto x$. If $\ker(\phi) \neq 0$ then $\mathbb{K}[X]/\ker(\phi) \subseteq D$ is a domain, and hence $\text{mer}(\phi) = (\phi(x))$ for an irreducible poly $p(x)$. Consequently, $K[x]/(p(x))$ is a finite field extension of \mathbb{K} , $\mathbb{K} \rightarrow K[x] \subseteq D$, and has only closure $K = \mathbb{K}$. Thus $x \in \mathbb{K} \subseteq D$, a contradiction. So $\ker(\phi) = 0$ necessarily, and $\dim D \geq \dim \mathbb{K}[X] = \infty$.

As a consequence, if $\dim D < \infty$ for div. k-alg D then we must have $D = \mathbb{K}$. 

Corollary 4: The only finite division alg over \mathbb{C} , $\overline{\mathbb{Q}}$, $\overline{\mathbb{F}_p}$ etc. are $\overline{\mathbb{C}} = \overline{\mathbb{Q}}$, $\overline{\mathbb{F}_p}$ themselves, respectively.

Proposition 5: Any finitely generated module M over a div. alg D is of the form D^l for some uniq. nat. num. l , and if M 's ab. then $M = D^l$ for $l' \leq l$. Equality holds if and only if $l = l'$.

For first claim

Proof: Proceed by induction on the min. ℓ of generators, and we claim

$\ell = \ell'$ for cardinality of our minimal generating set m_1, \dots, m_{ℓ} for D .

For such a minimal generating set we have the surj morphism

$$f: \bigoplus_{i=1}^{\ell'} D_i \rightarrow M, \quad \varphi_i \mapsto m_i,$$

which we claim is injective. Summarizing $f: \bigoplus_{i=1}^{\ell'} D_i \rightarrow M$

\Rightarrow injective for $\ell \leq \ell'$ then for any $a \in D_{\ell+1}$

$$\text{we have } f(a\varphi_i) = a\varphi_{i+1} \cdot m_{i+1} = \sum_{i=1}^{\ell'} a_i \cdot m_i.$$

Since $m_i = f(\varphi_{i+1}) = \varphi_i \cdot f(\varphi_{i+1}) \in \text{Superalgebras}$

$$M_1 \rightarrow M_{\ell'},$$

a contradiction. So $f: \bigoplus_{i=1}^{\ell'} D_i \rightarrow M$ is injective as well.

By induction on ℓ' we see f is injective, and thus an isomorphism. So

$$D^{\ell} \xrightarrow{\sim} M.$$

$$f: D^{\ell} \xrightarrow{\sim} D^{\ell'}$$

Now, if we have an idem $f: D^{\ell} \xrightarrow{\sim} D^{\ell}$ for $\ell, \ell > 0$ then, after precomposition w/ an aut of $\mathcal{G}(D^{\ell})$ we can assume f is the identity of D^{ℓ} . Then f induces an isomorphism.

$$(D^{\ell})^{\ell-1} = D^{\ell} / f \xrightarrow{\sim} (D^{\ell})^{\ell-1} = D^{\ell} / D.$$

Repeating we obtain $D \xrightarrow{\sim} D^{\ell-1}$, which gives $\ell_2 = \ell_1$.

For the second claim, given $M' \subseteq M$ we want to find a maximal submodule m_1, m_2, \dots, m_r for M' to be a sum of $m_i = m_1, \dots, m_r$. So let's do it to get $M' \cong D^l$ and $M \cong D^k$ with $l \leq k$ and equality holding iff $M' = M$.

Def: The rank of a D -module, for D a division ring, is the unique integer l at which we have an exact D -module $M \cong D^l$.

~ II. Matrices over division rings

On the left we see \mathbb{K}^n is simple over $M_n(\mathbb{K})$, for \mathbb{K} a field. Similarly, for any division ring D D^n is simple module over $M_n(D)$, and $M_n(D)$ admits no ideals other than 0 and $M_n(D)$ itself.

Furthermore we have

$$M_n(D) = \bigoplus_{i=1}^n D.$$

as a $M_n(D)$ module.

Here, for any ring R , $M_n(R)$ is the ring with
 $1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and multiplication $[a_{ij}] [b_{ij}] = [c_{ij}]$
 as $c_{ij} = \sum_k a_{ik} b_{kj}$.

III. Semisimple modules

Def.: A module M over an algebra A -
called semisimple if

$$M = \bigoplus_{\lambda \in \Lambda} L_\lambda$$

for simple submodules $L_\lambda \subset M$.

Equiv. M is semisimple if and only if M admits
an A -module isomorphism

$$\bigoplus_{\lambda \in \Lambda} L_\lambda \xrightarrow{\sim} M$$

for simple L_λ .

Observation 6: For a semisimple module M the
following are equivalent

a) M is finitely generated

b) M is finite length (admits a composition series)

c) For any isomorphism $\bigoplus_{\lambda \in \Lambda} L_\lambda \xrightarrow{\sim} M$

where L_λ simple, the index set Λ is finite.

Proof: Exerci.



Lemma (Schur) If L and L' are simple A -
modules, then any A -module map $f: L \rightarrow L'$ is either

ϕ , a an isomorphism.

Proof: By simplicity either $\ker(f) = 0$ or L .

In the first case f is injective with

$$0 \neq \text{im}(f) \subseteq L$$

and by simplicity again $\text{im}(f) = L$. Hence f is bijective as a map of sets, and thus invertible as a map of A -modules. In the second case, $\ker(f) = L$, we have $f = 0$.

Corollary 7: For any simple A -module L , $\text{End}_A(L)$ is a division ring.

Corollary 8: If A is a finite-dimensional algebra over an algebraically closed field K , and L is simple over A , then

$$\text{End}_A(L) = K \cdot \mathbb{D}_L \cong K.$$

Proof: Since L is cyclic we have a surjection $A \rightarrow L$ and hence $\dim_K(L) \leq \dim_K(A) < \infty$. Thus $\dim_K \text{End}_A(L) \leq \dim_K \text{End}_K(L) < \infty$.

Hence $\text{End}_A(L)$ is a finite-dimensional division algebra.

over \mathbb{C} . Since $\mathbb{C} = \bar{\mathbb{K}}$, we now have

$$\text{End}_{\mathbb{C}}(L) = \mathbb{C}$$

by Lemma 3. □

Example: For $L_{\mathbb{C}}^{(2)}$ the 2-dim simple module over $\mathbb{C} S_3$, we have

$$\text{End}_{\mathbb{C} S_3}(L_{\mathbb{C}}^{(2)}) = \mathbb{C}.$$

Also,

$$\text{End}_{\bar{\mathbb{F}}_3 S_3}(L_{\bar{\mathbb{F}}_3}^{(2)}) = \bar{\mathbb{F}}_3.$$

More generally, for any finite group G , any simple G -rep L over an alg closed field \mathbb{K} has

$$\text{End}_G(L) = \mathbb{K}.$$

Proposition 9: Let M be a finitely generated semisimple Δ -module.

- (i) Every submodule $M' \leq M$ is semisimple.
- (ii) Every quotient module M/M' is semisimple.

Prof: For (i) we claim that M' and M admit decomps into modules $L_1 \oplus \dots \oplus L_s = M'$ and $M = L_1 \oplus \dots \oplus L_p \oplus \dots \oplus L_f$ under which the inclusion

$M \rightarrow M$ is just the matrix $\begin{bmatrix} I_s \\ 0 \end{bmatrix} : \bigoplus_{i=1}^s L_i \rightarrow \bigoplus_{j=1}^t L_j$.

We proceed by induction on $\text{length}(M)$. When $\text{length}(M) = 0$ there is nothing to do. Take now $\text{length}(M) = t$ and assume the result holds for semisimple N with $\text{length}(N) < t$. Since M is of finite length [Cor 6, Fuchs] we have some simple $L_i \leq M$ and each composite

$$L \rightarrow M \rightarrow L_i$$

or either L or an isomorphism, by Schur's Lemma.

Since the inclusion $L \rightarrow M$ is nonzero we can find minimal index i_0 at which $L \rightarrow M \rightarrow L_{i_0}$ is nonzero. Then the map

$$(\bigoplus_{i < i_0} L_i) \oplus L \oplus (\bigoplus_{j > i_0} L_j) \rightarrow M$$

induced by the inclusions is an isomorphism (?) and after

replacing L_{i_0} with L and reindexing we can assume

$$\tau_0 = 1 \text{ and } \text{distrive}$$

$$\begin{array}{c} L \\ \downarrow \quad \uparrow \\ L_1 \end{array}$$

$$M' \xrightarrow{\quad M \quad} M$$

Take $L_1^\perp \leq M'$ the kernel of the sequence $M' \rightarrow M \rightarrow M / \bigoplus_{i \geq 1} L_i$. Then we have

$$M' = L_1 \oplus L_1^\perp$$

and $L_1^\perp \leq \bigoplus_{i \geq 1} L_i \leq M$. Since

$\text{length}(\oplus_{i \leq 1} L_i) = f_{-1}$ we obtain constant decomposition

$$L_1 = L_2 \oplus \dots \oplus L_s$$

$$\oplus_{i > 1} L_i = L'_2 \oplus \dots \oplus L'_t$$

and thus constant decomposition

$$M' = L_1 \oplus L'_2 \oplus \dots \oplus L'_s \quad (*)$$

$$M = L_1 \oplus L'_2 \oplus \dots \oplus L'_t -$$

(ii) Taking constant decomposition

$$M = \bigoplus_{i \leq s} L_i, \quad M = \bigoplus_{i \leq t} L_i,$$

we obtain

$$M/M' = \bigoplus_{i > s} L_i.$$

The following was already demonstrated in the proof.

Theorem 10: If M is semisimple, any exact sequence $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ splits, i.e.,

Remark: The analog of Prop 9 holds for infinite length semisimple modules as well. One can argue the point from the finite case and some Zorn's lemma argument.

Proposition 9.25: For any semisimple \mathbb{A} -module M ,
the following hold.

- i) Any submodule $M' \subseteq M$ is semisimple.
- ii) Any quotient module

Proof: Omitted. 

~ III.5 An example: Matrix rings

Example: Consider any division ring D and $M_n(D)$. We have the simple module

$$\text{standard} = D^n$$

generated by the columnar module called the decamp of the regular module

$$M_n(D) = \bigoplus_{i=1}^n \text{standard}.$$

Hence $M_n(D)$ itself is semisimple.

Now, since any finitely generated $M_n(D)$ -module M admits a surjection

$$\bigoplus_{i=1}^m M_n(D) = \bigoplus_{i=1}^m \text{standard} \rightarrow M$$

Prop 9 (ii) tells us that M is also semisimple, and [Prop 9, Fin dim] forces (as expected)

$$M = \bigoplus_{i=1}^t \text{standard}.$$

Corollary 11: For any division ring D , any
finitely generated $M_n(D)$ -module is semisimple, and

The standard column module is the only simple $M(\mathbb{C})$ module, up to isomorphism.

IV Radicals for modules

M_0 is said a proper submodule of M if maximal if any larger module $M_0 \subset M_1 \subset M$ has either $M_1 = M_0$ or $M_1 = M$.

Lemma 11: A proper submodule $M_0 \subset M$ is maximal if and only if M/M_0 is simple.

(Proof: Consider the surjection $\pi: M \rightarrow N$, with $N = M/M_0$. We have the bijection:

$$\begin{cases} \text{Submodules } M_1 \\ \text{with } M_0 \subset M_1 \subset M \end{cases} \xrightarrow{\quad} \begin{cases} \text{Submodules } N_1 \\ N_1 \subset N \end{cases}$$

$$M_1 \mapsto \pi(M_1)$$

$$\pi^{-1}(N_1) \leftarrow N_1.$$

Indeed, for $m \in M$ with $\pi(m) = \pi(m')$ for m' in M_1 , we have $m - m' \in \ker(\pi) = M_0$ so that $m = m' + (m - m') \in M_1$. This gives $\pi^{-1}(\pi(M_1)) = M_1$.

The equality $\pi(\pi^{-1}(N_1)) = N_1$, at any submodule $N_1 \subset N$ follows by surjectivity of π . Hence

The only submodules between M_0 and M are M_0 and M itself if and only if the only submodules in N are $\{0\}$ and N itself. Rather, M_0 is maximal if and only if $N = M/M_0$ is simple. ■

Def: Given a ring A and a nonzero A -ideal M , the radical is \sqrt{M} is the intersection

$$\text{Rad}(M) = \bigcap_{M_0 \subseteq M, M_0 \neq M} M_0.$$

Example: For an infinite field K (e.g. \mathbb{Q} , \mathbb{C} , or $\overline{\mathbb{F}_p}$) and any $\alpha \in K$ we have the max ideal / submodule $(x-\alpha)$ in the regular module $K[x]$, and the quotient

$$(K[x])/(x-\alpha) \xrightarrow{\sim} K(\alpha)$$

is the 1-dimensional $K[x]$ -module K on which $x \cdot f_\alpha = \alpha f_\alpha$. Now, for any poly

$$p(x) \in \text{Rad}(K[x])$$

we have $p(x) \cdot f_\alpha = p(\alpha) \cdot f_\alpha = 0$ at all $\alpha \in K$.

Hence $p(x)$ has as many zeros, and we conclude $p(x) = 0$. So

$$\text{Rad}(K[x]) = 0.$$

Example: At each $i=1, \dots, n$, and \mathbb{k} a field,
we have the surjection

$$\pi_i: M_n(\mathbb{k}) \rightarrow L_{\text{standard}} = \mathbb{k}^n$$

with kernel $\ker(\pi_i) = \left\{ \begin{bmatrix} \text{stuff} & | & 0 \\ & | & \text{stuff} \end{bmatrix} \right\}$
ith column

Since L_{standard} is simple this gives

$$\text{Dual}(M_n(\mathbb{k})) \subseteq \bigcap_i \ker(\pi_i) = 0.$$

Similarly, for any division ring D , $\text{Dual}(M_n(D)) = 0$.

Example: For $M = \mathbb{k}^{(r \times s)} / (x^{\infty})$

as a module over $\mathbb{k}^{(r \times s)}$, any simple quotient

$$\pi: M \rightarrow L$$

we have $x_-: L \rightarrow L$ an endomorphism. By

schur either $x_- = 0$ or an isomorphism. Since

$$(x_-)^{\infty} = x^{\infty} \cdot - = 0$$

we conclude $x_- = 0$. Rather, x annihilates L .

We see now $x \in \ker(\pi)$ at all such π , giving

$$\bar{x} \in \text{Dual}(M) \Rightarrow \mathbb{k}^{(r \times s)} \bar{x} = x \cdot M \subseteq \text{Dual}(M).$$

Since the quotient $M / x \cdot M \cong \mathbb{k}$ is simple,
we get $\text{Dual}(M) = x \cdot M$.

$\sim \text{IV } \frac{1}{2}$, Radicals and semisimplicity

Theorem 12: For any finite length module M , $M/\text{Rad}(M)$ is semisimple. Furthermore, for any surjective A -module map $\pi: M \rightarrow N$, N is semisimple if and only if $\text{Rad}(M) \subseteq \ker(\pi)$.

Before we prove the result, we note the following

Lemma 13: For any surjective module map $\pi: M \rightarrow N$ to nonzero N , $\text{Rad}(\text{col}) \subseteq \pi^{-1}(\text{Rad}(N))$.

Proof: For maximal $N_i \subseteq N$, $\pi^{-1}(N_i)$ is maximal in M . Hence

$$\pi^{-1}(\text{Rad}(N)) = \bigcap_{N_i \text{ max}} \pi^{-1}(N_i)$$

$$= \bigcap_{N_i \text{ max}} \text{col}(\pi^{-1}(N_i)) \supseteq \bigcap_{N_i \text{ max}} \text{col}(\text{col}) = \text{Rad}(\text{col}).$$

Lemma 14: If N is finite length and semisimple, then $\text{Rad}(N) = 0$.

Proof: Given an expression $N = \bigoplus_{i=1}^r L_i$ with the L_i simple, the kernel $(L_i)^{\perp}$ is N or each

projection $p_i: N \rightarrow L_i$ satisfying

$$\bigcap_{i=1}^r L_i = \ker(p_1 \cdots p_r)^t: N \rightarrow \bigoplus_{i=1}^r L_i.$$

But $(p_1 \cdots p_r)^t$ just recovers the identity on N , and hence $\text{Rad}(N) \subseteq \bigcap_{i=1}^r L_i = 0 \Rightarrow \text{Rad}(N) = 0$.

We now prove our main theorem.

Proof of Theorem 2: We have - via duality - up to the quotient $M/\text{Rad}(M)$ the existence of finitely many maximal submodules $L_1, \dots, L_r \subseteq M$ for which $\text{Rad}(M) = L_1 \cap \dots \cap L_r$. Hence,

$\text{Rad}(M)$ is the kernel of the map

$$M \rightarrow \bigoplus_{i=1}^r L_i, \quad L_i \mapsto M/L_i,$$

induced by the individual quotients $M \rightarrow L_i$, giving

$$M/\text{Rad}(M) \cong \bigoplus_{i=1}^r L_i \text{ as semisimple.}$$

By Proposition 9 we conclude $M/\text{Rad}(M)$ is semisimple.

As for the second claim, for semisimple M we have $\text{Rad}(N) = 0$ by Lemma 14 and hence

$$\text{Rad}(M) \subseteq \ker(\pi) = \pi^{-1}\text{Rad}(N) \text{ by Lemma 13.}$$

Conversely, if $\pi: M \rightarrow N$ is surj with $\text{Rad}(M) \subseteq \ker(\pi)$ then $\sqrt{\pi}$ is the quotient of the semisimple module $M/\text{Rad}(M)$.

By Proposition 9 we conclude that N is semisimple. \blacksquare

IV Socles for modules

Recor. 15: For a Noetherian module M (e.g. a finitely generated module over a domain A), the sum $\text{soc}(M) = \sum_{i \in \Delta} L_i \subseteq M$ (4)

over the collection Δ of all simple submodules $L_i \leq M$ is a finite length semisimple submodule in M .

Furthermore, for any semisimple module N , any module map $f: N \rightarrow M$ has image in $\text{soc}(M)$.

Proof: Since N is Noetherian, the sum in (4) is finite. $\sum_i L_i = L_1 + \dots + L_r$ for some simple $L_i \leq N$. Then \sum_i is the image of the map

$$\bigoplus_{i=1}^r L_i \rightarrow M$$

induced by the inclusions, giving $\text{soc}(M)$ as a quotient of a finite length semisimple module. Hence $\text{soc}(M)$ is finite length semisimple, by Proposition 9. For the second claim, write $N = \bigoplus_{i \in I} V_i$ for simple V_i to get $\text{soc}(N) = \sum_{i \in I} \text{soc}(V_i)$ with each $\text{soc}(V_i)$

either simple or zero, by simplicity. Hence $m(N) \leq$
502 (all).

Def^t: The sum of all simple submodules is a Noetherian module M is called the socle in M .

Ex: For $K[x]/(x^n)$, considered as a module over $K[x]$, any simple submodule $M \subseteq K[x]/(x^n)$ is annihilated by x^n , and thus simplicity forces $x \cdot M = 0$. This gives

$$\text{soc}(K[x]/(x^n)) = x \cdot \overline{x^{n-1}} \stackrel{?}{=} \{0\}.$$

The I -chain simple on which K acts as 0.

HW

1. For the regular A -module A , prove that there is a canonical ring isomorphism

$$A^{\text{op}} \xrightarrow{\sim} \text{End}_A(A).$$

2. Let G be a group and suppose G acts a K -algebra A by algebra automorphisms. Suppose K is a field, and take $A \rtimes G := A \otimes_{\mathbb{K}} K[G]$ with the unique bilinear map $\cdot : A \rtimes G \times A \rtimes G \rightarrow A \rtimes G$ satisfying
- $$(a \otimes g) \cdot (b \otimes h) = a(g \ast h) \otimes gh$$
- as demanded. Prove that $A \rtimes G$ is a K -algebra.

3. Let $\mathbb{Z}/n\mathbb{Z}$ act on \mathbb{C}^{\times} via the automorphism $m \cdot \varphi(x) = \varphi(g \cdot x)$ for $g \in \mathbb{C}^{\times}$ an n -th root of unity. Take $A_g = \mathbb{C}(x) \rtimes \mathbb{Z}/n\mathbb{Z}$.

- a) Prove that $M_r = A_g / A_g \cdot x^r$ is a free module over $\mathbb{C}[\mathbb{Z}/n\mathbb{Z}]$ of rank r . Give a basis for M_r over $\mathbb{C}[\mathbb{Z}/n\mathbb{Z}]$.

- b) Calculate $\text{Res}_L(M_r)$, $\text{det}_L(\text{Res}_L(M_r))$, and $\text{soc}(\text{Res}_L(M_r))$.

2. a) For $\mathbb{Z}/n\mathbb{Z}$. Prove that $\mathbb{Z}/n\mathbb{Z}$ has precisely n non-isomorphic (simple) 1-dimensional representations L_i over \mathbb{C} .

b) Prove that each simple 1-dimensional representation L_i over $\mathbb{Z}/n\mathbb{Z}$ admits an injective $\mathbb{C}\mathbb{Z}/n\mathbb{Z}$ -module map $L_i \rightarrow \mathbb{C}\mathbb{Z}/n\mathbb{Z}$, and that that map is unique up to rescaling.

c) Provide an isomorphism of $\mathbb{C}\mathbb{Z}/n\mathbb{Z}$ -modules

$$\bigoplus_{i=1}^n L_i \xrightarrow{\sim} \mathbb{C}\mathbb{Z}/n\mathbb{Z}.$$

d) Prove that every finite-dimensional $\mathbb{Z}/n\mathbb{Z}$ -module M over \mathbb{C} decomposes as

$$M \cong \bigoplus_{i=1}^n m_i \cdot L_i$$

where each $m_i = [L_i : M]$.

3. For any finite length semi-simple module M over a ring A , prove that there are divisors m_1, D_1, \dots, D_t and integers n_1, \dots, n_t for which

$$\text{End}_A(M) \cong \bigcap_{i=1}^t \text{End}_{A/(D_i)}(D_i)$$

are true.