

- Abstract root systems

(1)

**Def:** A root system is a real inner-product space  $E = \{E, \langle \cdot, \cdot \rangle : E \otimes E \rightarrow \mathbb{R}\}$  equipped with a finite subset  $\Phi \subseteq E$  which satisfies the following:

i)  $\Phi$  spans  $E$ .

ii) For each  $\gamma \in \Phi$ ,  $\mathbb{R} \cdot \Phi \cap \Phi = \{\gamma, -\gamma\}$ .

iii) For each  $\gamma \in \Phi$ , the reflection

$$\sigma_\gamma : E \rightarrow E, \quad \sigma_\gamma(v) = v - \frac{2\langle v, \gamma \rangle}{\langle \gamma, \gamma \rangle} \cdot \gamma,$$

preserves  $\Phi$ ,  $\sigma_\gamma(\Phi) = \Phi$ .

iv) For each pair of elem  $\gamma, \delta \in \Phi$ ,

$$2\langle \gamma, \delta \rangle / \langle \gamma, \gamma \rangle \in \mathbb{Z}.$$

For some notation, we call the elem  $\gamma \in \Phi$  the roots in  $E$ . The automorphism  $\sigma_\gamma = ? - \frac{2\langle \cdot, \gamma \rangle}{\langle \gamma, \gamma \rangle} \gamma$  is called the reflection across  $\gamma$ .

Note that  $\sigma_\gamma$  preserves the orthogonal hyperplane  $\mathcal{H}_\gamma := \{v \in E : \langle v, \gamma \rangle = 0\} = (\mathbb{C}\gamma)^\perp$  and negates vectors in the span of  $\gamma$ .

**Def<sup>1</sup>:** Let  $\Phi \subseteq E$  be a root system, and  $\gamma, \delta \in \Phi$  be roots. The Cartan integer for  $\delta$  parallel w/  $\gamma$  is the integer

$$\langle \delta, \gamma \rangle := 2\langle \delta, \gamma \rangle / \langle \gamma, \gamma \rangle.$$

With this notation we have

$$\sigma_\gamma(\delta) = \delta - \langle \delta, \gamma \rangle \cdot \gamma.$$

Warning! : Some people take  $\langle \gamma, \delta \rangle = \frac{2\langle \delta, \gamma \rangle}{\langle \gamma, \gamma \rangle}$

So "who came first" and "who comes second" in

This notation is not fixed in the literature.

- Root systems for Lie algs.

(2)

By Thm 8.5 we understand that for any semisimple  $\mathfrak{g}$  w/ choice of Cartan  $\mathfrak{h} \subseteq \mathfrak{g}$  we have the roots  $\Phi \subseteq \mathfrak{h}^*$ , which are the nonzero wts appearing in the adj rep. We have the real span

$$\mathfrak{h}_{\mathbb{R}}^* := \mathbb{R} \cdot \Phi$$

w/ Killing form  $(-, -) : \mathfrak{h}_{\mathbb{R}}^* \otimes_{\mathbb{R}} \mathfrak{h}_{\mathbb{R}}^* \rightarrow \mathbb{R}$ .

Def<sup>1</sup>: An isomorphism of root systems  $\varphi : (\Phi, E) \xrightarrow{\sim} (\Phi', E')$  is <sup>or</sup> linear isom  $\varphi : E \rightarrow E'$  which satisfies  $\varphi(\Phi) = \Phi'$  and

$$\langle \varphi \gamma, \varphi \beta \rangle = \langle \gamma, \beta \rangle \quad (*)$$

for all pairs of simple roots.

Dem: The condition (\*) is satisfied whenever  $\varphi$  preserves the inner form "up to scaling". When  $\Phi$  is "connected", the (\*) is precisely this statement.

Thm 8.5: For any semisimple  $\mathfrak{g}$  w/ choice of Cartan  $\mathfrak{h} \subseteq \mathfrak{g}$ , the pairing  $\Phi \subseteq \mathfrak{h}_{\mathbb{R}}^*$ , along w/ the Killing form, is a root system.

Furthermore, for any other choice of Cartan  $\mathfrak{t} \subseteq \mathfrak{g}$  the two root systems are isomorphic

$$(\Phi, \mathfrak{h}_{\mathbb{R}}^*) \cong (\Psi, \mathfrak{t}_{\mathbb{R}}^*).$$

Proof: Thm 8.5 says  $(\Phi, \mathfrak{h}_{\mathbb{R}}^*)$  is a root system, as is  $(\Psi, \mathfrak{t}_{\mathbb{R}}^*)$ . Now, by uniqueness of the Cartan  $\exists$  an auto  $w : \mathfrak{g} \rightarrow \mathfrak{g}$  w/  $w(\mathfrak{h}) = \mathfrak{t}$ , and hence  $w(\mathfrak{g}/\mathfrak{h}) = \mathfrak{g}/\mathfrak{t}$  for some  $\beta$ . We have at each  $h \in \mathfrak{h}$ ,  $x \in \mathfrak{g}/\mathfrak{h}$

$$\beta(wh) \cdot w(x) = (\beta(h), w(x)) = w([h, x]) = \beta(h) \cdot w(x)$$

so that  $w^*(\mathcal{Z}) = \mathcal{Y}$ , and  $w^*(\mathcal{P}) = \mathcal{P}$ . (8)

Take now  $\varphi = (w^{-1})^*: h^* \rightarrow t^*$ . We check finally the inner form. We have

$$\begin{aligned} w^{-1} \text{ad}_{wX} w &: \mathfrak{g} \rightarrow \mathfrak{g} \\ &= \varphi \mapsto w^{-1}([wX, wY]) = [X, Y] \end{aligned}$$

$$\Rightarrow w^{-1} \text{ad}_{wX} w = \text{ad}_X.$$

$$\begin{aligned} \text{Thus } \kappa(wX, wY) &= \text{Tr}_{\mathfrak{g}}(w^{-1} \text{ad}_X \text{ad}_Y w) \\ &= \text{Tr}_{\mathfrak{g}}(\text{ad}_X \text{ad}_Y) = \kappa(X, Y). \end{aligned}$$

Hence we have

$$\begin{aligned} \kappa(wX, -) &: \mathfrak{t} \rightarrow \mathbb{C} \\ &= \kappa(X, w^{-1}(-)) : \mathfrak{t} \rightarrow \mathbb{C} \end{aligned}$$

giving a diagram

$$\begin{array}{ccc} \mathfrak{h} & \xrightarrow{w} & \mathfrak{t} \\ \downarrow f_{w\mathfrak{h}} & & \downarrow f_{w\mathfrak{t}} \\ \mathfrak{h}^* & \xrightarrow{\varphi} & \mathfrak{t}^* \end{array}$$

and from the diag

$$\begin{array}{ccc} \mathfrak{h} \otimes \mathfrak{h} & \xrightarrow{w \otimes w} & \mathfrak{t} \otimes \mathfrak{t} \\ \searrow & & \swarrow \\ & \mathbb{C} & \end{array}$$

we conclude a diag

$$\begin{array}{ccc} \mathfrak{h}^* \otimes \mathfrak{h}^* & \xrightarrow{\varphi^2} & \mathfrak{t}^* \otimes \mathfrak{t}^* \\ \searrow & & \swarrow \\ & \mathbb{C} & \end{array} \quad \checkmark (\varphi^2, \varphi).$$

So we draw that  $\varphi$  provides an  $\cong$  of root systems.

$$\varphi : (\Phi, \mathfrak{h}_{\mathbb{R}}^*) \xrightarrow{\cong} (\mathcal{P}, \mathfrak{t}_{\mathbb{R}}^*). \quad \blacksquare$$

**Def<sup>1</sup>:** For any semisimple Lie alg  $\mathfrak{g}$ , let  $\mathcal{E}(\mathfrak{g}) = (\Phi, \mathcal{E}(\mathfrak{g}))$  denote the assoc. root system, which we accept is only defined up to  $\cong$ .

**Theorem 14.2:** For semisimple Lie alg  $\mathfrak{g}$  and  $\mathfrak{g}'$ , there is an  $\cong$  of root systems  $\mathcal{E}(\mathfrak{g}) \cong \mathcal{E}(\mathfrak{g}')$  if and only if there is a Lie alg isom  $\varphi : \mathfrak{g} \xrightarrow{\cong} \mathfrak{g}'$  which lifts  $\varphi$ .

By encoding root preserving action of  $g$ , one sees that the lift  $\tilde{g}: \mathfrak{g} \rightarrow \mathfrak{g}$  is completely determined up to some  $(\mathbb{C}^*)^{x_n}$ -action.

Corollary: The associated root system  $g \mapsto E(g)$  is a simple invariant for a semisimple Lie algebra  $\mathfrak{g}$ .

This is to say it classifies semisimple Lie algs over  $\mathbb{C}$  it suffices to classify (indecomposable) root systems.

Ex: Consider  $\mathfrak{sl}_n(\mathbb{C})$ ,  $\gamma$  the root for  $E_{ij}$  & the root for  $E_{i+1, i}$ . Then

$$\langle \gamma, \gamma \rangle = \gamma(\gamma) = \text{coeff of } \gamma = E_{i+1, i} - E_{ij} \text{ acting on } E_{i+1, i}.$$

$$\begin{aligned} \gamma \cdot E_{i+1, i} &= (\delta_{i+1, i} - \delta_{ji}) - (\delta_{i+1, i} - \delta_{ji}) \\ &= \begin{cases} \pm 1 & \text{if 1 shared index} \\ \pm 2 & \text{if 2 shared indices.} \end{cases} \end{aligned}$$

Restriction to our preferred basis  $\Delta \subseteq \mathfrak{H}$ ,

$$\Delta = \{\alpha_i, \alpha_i = \text{root for } E_{i+1, i}\}$$

$$\text{we get } \langle \alpha_i, \alpha_j \rangle = \begin{cases} 2 & \text{if } i=j \\ -1 & \text{if } j=i+1 \text{ or } i=j+1. \end{cases} \quad (*)$$

Note that  $\langle \alpha_i, \alpha_j \rangle = \langle \alpha_j, \alpha_i \rangle$  so that

$$\langle -, - \rangle = \begin{cases} \text{a realisation of } (-, -) \\ \text{so that all } \langle \gamma, \gamma \rangle = 2. \end{cases}$$

Can encode (\*) in a graph (Dynkin diagram)

$$\alpha_1 \quad \alpha_2 \quad \dots \quad \alpha_{n-2} \quad \alpha_{n-1}$$

where # of edges between  $\alpha_i$  and  $\alpha_j = \langle \alpha_i, \alpha_j \rangle$ .

Will see that these numbers / this diagram completely determine the root system.

Now, back to the general setting.

~ The Weyl group

Def: Given a real vector space  $(\Phi, \langle \cdot, \cdot \rangle)$ , the associated Weyl group  $W$  is the subgroup of  $GL(\Phi)$  generated by the reflections  $\sigma_Y$ , for  $Y \in \Phi$ .

Lemma: a)  $W$  is a finite group

b) For each  $Y \in \Phi$ ,  $v, w \in \Phi$ ,  $\langle \sigma_Y v, \sigma_Y w \rangle = \langle v, w \rangle$

Proof: a) Each element is determined by its value on the finite set  $\Phi$ , so that by restricting the action  $\subseteq$  we obtain a group embedding  $W \hookrightarrow S_\Phi$ .

b) directly

$$\begin{aligned} \langle \sigma_Y v, \sigma_Y w \rangle &= \langle v, w \rangle + \frac{\langle v, Y \rangle \langle w, Y \rangle}{\langle Y, Y \rangle^2} \langle Y, Y \rangle \\ &\quad - \frac{2 \langle v, Y \rangle}{\langle Y, Y \rangle} \langle Y, w \rangle - \frac{2 \langle w, Y \rangle}{\langle Y, Y \rangle} \langle Y, v \rangle \\ &= \langle v, w \rangle. \end{aligned}$$

The following will help us to observe some relations in  $W$ .

Lemma 9.1: Let  $\sigma \in GL(\Phi)$  preserve  $\Phi$ , and for some hyperplane  $P$ , and line  $\ell = \mathbb{R}Y$  for some  $Y \in \Phi$ , then  $\sigma = \sigma_Y$ .

Proof: It suffices to show  $\sigma = \sigma_Y$  on the complexification  $\Phi_{\mathbb{C}}$ . Here we exploit the fact that any finite order linear auto on a complex vector space is semisimple. (  $\phi = \phi_s + \phi_n$ ,  $\phi^N = \phi_s^N + \underbrace{(\phi_n^N)}_{=0} = \phi_s^N$  )  
 so that  $\phi = 1$  or  $\phi_n \phi^{N-1} = 0 \Rightarrow \phi_n = 0$ .  
 So in either case  $\phi = \phi_s$ .

Consider  $\sigma \sigma_Y$  on  $\mathbb{C} \cdot Y$  we have

$\sigma \sigma_Y(Y) = iD$  and for the induced map on

$\Phi_{\mathbb{C}} / \mathbb{C} \cdot Y$  we have  $\sigma \sigma_Y$  acts as  $iD$ .

So

$$\sigma \sigma_Y = \begin{bmatrix} 1 & * \\ 0 & iD \end{bmatrix} \in GL$$

$\Rightarrow (\sigma \sigma_Y - 1)^2 = 0$  on  $\Phi_{\mathbb{C}}$ .

But since  $\sigma\sigma^{-1}$  is the identity, its value is 1. (6)

$\Phi$ , and  $\Phi$  is finite,  $\sigma\sigma^{-1}$  is surjective. Thus

$$\sigma\sigma^{-1} = \text{id} \Rightarrow \sigma = \sigma\sigma^{-1} = \sigma\sigma^{-1}.$$

**Lemma 9.2:** Let  $(\Phi, E)$  be a root system,  $\beta \in \Phi$ , and  $\sigma: E \rightarrow E$  be a linear auto with  $\sigma(\Phi) = \Phi$ . Then  $\sigma\sigma^{-1} = \text{id}$ , and  $\langle \beta, \gamma \rangle = \langle \sigma(\beta), \sigma(\gamma) \rangle$  for all pairs of roots.

**Proof:**  $\sigma\sigma^{-1}$  fixes the hyperplane  $\sigma P_\beta$  and has  $\sigma(\beta) \mapsto -\sigma(\beta)$ . So that  $\sigma\sigma^{-1} = \sigma\sigma^{-1}$  by Lemma 9.1. For the Cartan integer

$$\begin{aligned} \langle \sigma\beta, \sigma\gamma \rangle \sigma\gamma &= \sigma(\beta) - \sigma_{\sigma(\beta)}(\sigma\gamma) \\ &= \sigma(\beta - \sigma_{\sigma(\beta)}(\sigma\gamma)) \\ &= \sigma(\beta - \sigma_{\sigma(\beta)}(\sigma\gamma)) \\ &= \sigma(\langle \beta, \gamma \rangle \gamma) = \langle \beta, \gamma \rangle \sigma\gamma. \end{aligned}$$

$$\text{Hence } \langle \sigma\beta, \sigma\gamma \rangle = \langle \beta, \gamma \rangle.$$

- Roots and dual root system

**Def<sup>n</sup>:** For each  $\gamma \in \Phi$  define  $\gamma^\vee = 2\gamma / \langle \gamma, \gamma \rangle$ .

**Def<sup>n</sup>:**  $\Phi^\vee = \{ \gamma^\vee : \gamma \in \Phi \}$ . This is the (Langlands) dual root system to  $\Phi$ .

Direct calculation verifies  $\Phi^\vee$  is another root system

$$\begin{aligned} \langle \gamma^\vee, \gamma^\vee \rangle &= 4 \langle \gamma, \gamma \rangle / \langle \gamma, \gamma \rangle^2 \\ &= 4 / \langle \gamma, \gamma \rangle. \end{aligned}$$

$$\begin{aligned} \langle \beta^\vee, \gamma^\vee \rangle &= 2 \langle \beta, \gamma \rangle / \langle \beta, \beta \rangle \langle \gamma, \gamma \rangle \\ &= 2 \langle \beta, \gamma \rangle \langle \beta, \beta \rangle^{-1} \langle \gamma, \gamma \rangle^{-1} = \langle \beta, \gamma \rangle. \end{aligned}$$

**Lemma:** If  $|\beta|^2 \leq |\gamma|^2$  in  $\Phi$ , then

$$|\beta^\vee|^2 \geq |\gamma^\vee|^2 \text{ in } \Phi^\vee. \text{ Furthermore}$$

$$\langle \beta^\vee, \gamma^\vee \rangle = \langle \beta, \gamma \rangle.$$

Lemma:  $\langle \gamma, \gamma^\vee \rangle = \gamma(h_\gamma)$ . (2)

Example:  $\left[ \Phi(\mathfrak{so}_{2n}(\mathbb{C})) \right]^\vee = \Phi(\mathfrak{so}_{2n+1}(\mathbb{C}))$ .

•  $\left[ \Phi(\mathfrak{sl}_n(\mathbb{C})) \right]^\vee = \Phi(\mathfrak{sl}_n(\mathbb{C}))$ .

•  $\left[ \Phi(\mathfrak{so}_{2n}(\mathbb{C})) \right]^\vee = \Phi(\mathfrak{so}_{2n}(\mathbb{C}))$ .

Proposition:  $\sqrt{\langle E, \Phi \rangle} = \sqrt{\langle E, \Phi^\vee \rangle}$ ,

Proof: We have  $\sigma_\gamma = \sigma_{\gamma^\vee}$  since they both

fix  $P_\gamma = P_{\gamma^\vee}$  and negate  $\delta$  (and  $\gamma^\vee$ ).  $\square$

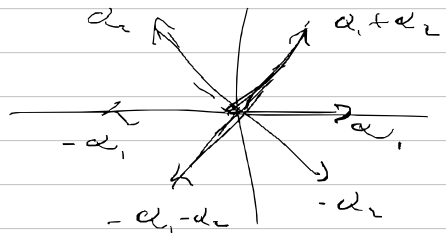
~ Constrains angles and magnitudes in a root system.

For the scaling constraint  $|R_i \gamma| \cap \Phi = \emptyset$

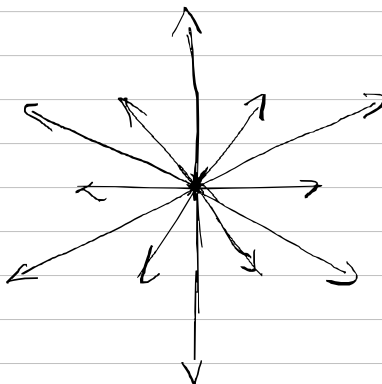
and stability under reflection, we see that  $\Phi \subseteq E$ ,

some highly symmetric star like configuration.

E.g.  $A_2 / \mathfrak{sl}_3$



$G_2$



Since we're in a

Euclidean/inner product

space we can speak

The angle is only defined mod  $\pi$ , clearly  $\delta$  angles between

so find for roots  $\gamma, \beta$

roots, and magnitudes

angle between  $\gamma, \beta$  is  $\pi$  defined by value

$$\cos(\theta).$$

We have  $|\gamma| = \sqrt{\langle \gamma, \gamma \rangle}$  and

$$\cos \theta = \frac{\langle \beta, \gamma \rangle}{|\beta| |\gamma|}.$$

$$= \frac{|\gamma|}{2|\beta|} \langle \beta, \gamma \rangle.$$

$$\Rightarrow 2|\beta||\gamma|^{-1} \cos \theta = \langle \beta, \gamma \rangle.$$

Observation 9.4A: For any pair of roots  $\beta, \gamma$ ,  $\langle \beta, \gamma \rangle \langle \gamma, \beta \rangle = \frac{1}{4} \cos^2 \theta \in \mathbb{Z}$ . (8)

But now,  $0 \leq \cos^2 \theta \leq 1$  with 1 occurring iff  $\beta = \pm \gamma$ .

Observation 9.4B: If  $\beta \neq \pm \gamma$  and  $\langle \beta, \beta \rangle \neq 0$ , then

- $\langle \beta, \gamma \rangle \in \{0, 1, 2, 3\}$ .

- $\cos \theta \in \frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}$

- If  $\langle \beta, \gamma \rangle = \langle \gamma, \beta \rangle$  then  $\langle \beta, \gamma \rangle = \langle \gamma, \beta \rangle = \pm 1$  and  $|\beta| = |\gamma|$ .

- If  $\langle \beta, \gamma \rangle \neq \langle \gamma, \beta \rangle$  then, supposing  $|\beta| > |\gamma|$ ,  $\langle \beta, \gamma \rangle = \pm 1$  and  $\langle \gamma, \beta \rangle = \pm 2, \pm 3$ .

Working things out one observes the following table

Summing  $|\beta| = |\gamma|$

$\langle \beta, \gamma \rangle$	$\langle \gamma, \beta \rangle$	$\theta$	$\frac{ \gamma ^2}{ \beta ^2}$
0	0	$\pi/2$	unk
1	1	$\pi/3$	1
-1	-1	$2\pi/3$	1
1	2	$\pi/4$	2
-1	-2	$3\pi/4$	2
1	3	$\pi/6$	3
-1	-3	$5\pi/6$	3

[Still about root strings which I'll skip]

Ex. In type  $A$ , all roots are same length, so then also angles  $\pi/3$  and  $2\pi/3$  are realized.

Both angles realized already in  $A_2$ .

- In type  $C$  and  $B$ , rd root lengths of 2 occur, and hence more acute angles occur.

- In type  $G_2$ , rd root length of 3 occur.



Lemma 9.4: Suppose  $\beta \neq \pm \gamma$ . Then (9)

•  $\langle \gamma, \beta \rangle > 0$  Then  $\gamma - \beta$  is a root.

•  $\langle \gamma, \beta \rangle < 0$  Then  $\gamma + \beta$  is a root.

Proof: Suppose  $\langle \gamma, \beta \rangle > 0$  then one of  $\langle \gamma, \beta \rangle$  or  $\langle \beta, \gamma \rangle = 1$ . Suppose arbitrarily that  $\langle \beta, \gamma \rangle = 1$ , we get

$$\sigma_{\beta}(\beta) = \beta - \langle \beta, \gamma \rangle \gamma = \beta - \gamma \text{ a root,}$$

and thus  $\gamma - \beta$  a root as well by stability under

negation. If  $\langle \beta, \gamma \rangle < 0$  then one of  $\langle \beta, \gamma \rangle$

or  $\langle \gamma, \beta \rangle = -1$ . Suppose arbitrarily  $\langle \beta, \gamma \rangle = -1$

we get  $\sigma_{\gamma}(\beta) = \beta + \gamma \in \Phi$ . ~~1~~

~ Basis for root system.

Fix a root system  $(E, \Phi)$ . We seek now that which has been promised to us by the exercise:

(1)  $\Delta$  good basis of char  $\Delta \subseteq \Phi$  for  $E$ .

(2)  $\Delta$  splitting  $\Phi = \Phi^+ \cup \Phi^-$  w/  $\Phi^- = -\Phi^+$  and  $\Delta \in \Phi^+$ .

Def<sup>1</sup>: A base for a root system  $(E, \Phi)$  is a choice of subset  $\Delta \subseteq \Phi$  which satisfies the following:

a)  $\Delta$  is a basis for  $E$ .

b) For each root  $\gamma$ ,  $\gamma = \sum_{\alpha \in \Delta} c_{\alpha} \alpha$  with all  $c_{\alpha} \in \mathbb{Z}_{\geq 0}$ , or all  $c_{\alpha} \in \mathbb{Z}_{\leq 0}$ .

Ex:  $\alpha_2 = \text{root for } E_{\text{int}}$  form a base

$$\{\alpha_1, \dots, \alpha_{n-1}\}$$

for  $\Phi(E_n)$ .

(10)

Def<sup>h</sup>: For a root system  $(E, \Phi)$  w/ chosen base  $\Delta$ , we define

$$\Phi^+ = \left\{ \gamma \in \Phi : \gamma = \sum_{\alpha \in \Delta} c_\alpha \alpha \text{ w/ all } c_\alpha \geq 0 \right\}$$

$$\Phi^- = \left\{ \gamma \in \Phi : \gamma = - \sum_{\alpha \in \Delta} c_\alpha \alpha \text{ w/ all } c_\alpha \geq 0 \right\}$$

Clearly  $\Phi^- = -\Phi^+$ , and by def of a base

$$\Phi = \Phi^+ \sqcup \Phi^-$$

Ex: For  $\Phi(\mathfrak{sl}_n)$  w/ chosen base  $\Delta = \{\alpha_i : 1 \leq i \leq n-1\}$ ,  $\Phi^+ = \{\text{roots for upper } \Delta \text{ elem } E_{ij}\}$ .

Lemma 10.1: If  $\alpha, \beta \in \Delta$  are distinct, then  $(\alpha, \beta) \leq 0$ .

Proof:  $\alpha - \beta$  is not a root, by def of base. Then the constraint  $c_{\alpha}, c_{\beta} \leq 0$  occurs by Lemma 9.4.  $\square$

$\sim$  Existence of a base.

Consider  $(E, \Phi)$  and the collection of hyperplanes  $\{P_\gamma : \gamma \in \Phi\}$ ,  $P_\gamma = \{v \in E : (v, \gamma) = 0\}$ .

The union  $\bigcup_{\gamma \in \Phi} P_\gamma \subseteq E$  is not an equality, by linear algebra, so that  $\bigcup_{\gamma \in \Phi} P_\gamma$  is a non-trivial closed alg. subspace in  $E$ , and the complement

$$E - \bigcup_{\gamma \in \Phi} P_\gamma$$

is an open alg. subspace, and hence dense in  $E$ . Anyway, we can find elem  $v$  in the complement.

Def<sup>n</sup>: Call  $v \in E$  regular if  $v \in E - \bigcup_{\gamma \in \Phi} P_\gamma$ , and singular otherwise. For regular  $v$  we write

$$\Phi(v)^+ = \{\gamma \in \Phi : (v, \gamma) > 0\}$$

$$\Phi(v)^- = \{\gamma \in \Phi : (v, \gamma) < 0\}$$

Clearly each  $\beta \in \Phi$  is either in  $\Phi(v)^+$  or  $\Phi(v)^-$ , but not both, and  $\Phi(v)^- = -\Phi(v)^+$ . (11)

**Def<sup>1</sup>:** Call  $\alpha \in \Phi(v)^+$  indecomposable if  $\alpha \neq \alpha_1 + \alpha_2$  for any  $\alpha_i \in \Phi(v)^+$ .

Note that such indecomp roots exist, by choosing  $\alpha \in \Phi(v)^+$  w/ minimal value  $(\alpha, v) > 0$ , for example.

**Theorem 0.1:** For any regular  $v \in E$ , the set  $\Delta(v) = \{ \alpha \in \Phi(v)^+ : \alpha \text{ indecomposable} \}$

provides a basis for  $(E, \Phi)$ . Furthermore, all bases occur in this way.

**Proof:** **Step 1:** Note that the collection

$$\{ (v, \beta) : \beta \in \Phi(v)^+ \} = \{ m_1 < \dots < m_r \}$$

is a finite and naturally linearly ordered. If  $\alpha$  is such that  $(\alpha, v)$  is minimal in it then  $\alpha$  is indecomp.

Now suppose  $\beta \in \Phi^+(v)$  is such that each  $\gamma \in \Phi^+$  w/  $(v, \gamma) < (v, \beta)$  is expressible as sum of neg linear combos of elem in  $\Delta(v)$ . If

$\beta \in \Delta(v)$  then  $\beta \in \mathbb{Z}_{\geq 0} \cdot \Delta(v)$  and otherwise  $\beta = \beta_1 + \beta_2$  w/  $\beta_i \in \Phi^+$  and wce.  $(v, \beta_i) < (v, \beta)$  so that both  $\beta_i \in \mathbb{Z}_{\geq 0} \Delta(v)$ , giving  $\beta = \beta_1 + \beta_2 \in \mathbb{Z}_{\geq 0} \Delta(v)$ . So we

see  $\Phi^+(v) \supseteq \mathbb{Z}_{\geq 0} \cdot \Delta(v)$ . By negation

$$\Phi^-(v) \supseteq \mathbb{Z}_{\leq 0} \cdot \Delta(v) \text{ as well.}$$

**Step 2:** For distinct  $\alpha, \beta \in \Delta(v)$  we

have  $\alpha - \beta$  not a root. Otherwise, up to swapping

$\alpha$  and  $\beta$ ,  $\alpha - \beta \in \Phi^+(v)$  and we decompose  $\alpha$

$$\alpha = \beta + c(\alpha - \beta), \text{ which is conv.} \quad (12)$$

So  $\alpha - \beta$  is not a root, and we conclude

$c(\alpha, \beta) \leq 0$  whenever  $\alpha, \beta \in \Delta(w)$  are distinct by Lemma 9.4.

Step 3. By Step 1

$$\begin{aligned} R \cdot \Delta(w) &\supseteq R \cdot \Phi^+(w) + R \cdot \Phi^-(w) \\ &= R \cdot \Phi = E, \end{aligned}$$

so that  $\Delta(w)$  spans  $E$ . By step 1 again

we see now  $\Delta(w)$  is a base with its linear indep.

For this consider an expression

$$0 = \sum_{\alpha} r_{\alpha} \alpha \text{ w/ all } r_{\alpha} \text{ real.}$$

Then for  $\Delta' = \{\alpha \text{ w/ } r_{\alpha} > 0\}$  we have

$$0 = \sum_{\alpha \in \Delta'} r_{\alpha} \alpha = \sum_{\beta \notin \Delta'} s_{\beta} \beta$$


w/ all  $r_{\alpha}, s_{\beta} \geq 0$ . Then

$$(0, 0) = \sum_{\alpha, \beta} r_{\alpha} s_{\beta} (\alpha, \beta) \leq 0$$

$\Rightarrow 0 = 0$ . So we reduce to the case of a sum

$$0 = \sum_{\alpha} r_{\alpha} \alpha \text{ w/ all } r_{\alpha} \geq 0. \text{ In this case}$$

$$0 = (0, \sum_{\alpha} r_{\alpha} \alpha) = \sum_{\alpha} r_{\alpha} (0, \alpha) \Rightarrow \text{all } r_{\alpha} = 0.$$

So  $\Delta(w)$  is in fact a lin. indep. set. 

Weyl Chamber  $\text{Root} \cong \text{Exterior}$

Def<sup>1</sup>: The Weyl Chambers  $C \subseteq E$  are the components of the complement  $E - \bigcup_{\alpha \in \Delta} \alpha^{\perp}$ .

Since each  $C$  is path connected (in fact convex)

we see that  $C$  is determined by any chosen regular  $v \in C$ , and for any  $v' \in C$  we have

$$\Delta(v) = \Delta(v').$$

Given a base  $\Delta = \Delta(w)$  we write  $C^{\pm}(\Delta) := \left\{ \begin{array}{l} \text{the unique} \\ \text{chamber} \\ \text{containing } v. \end{array} \right.$

(13)

$\mathcal{W}$  = non-trivial bijection

$$\{B_{\text{case}} \text{ for } (E, \Phi)\} \leftrightarrow \left\{ \begin{array}{l} \text{Components of} \\ E - V/P_V \end{array} \right\}$$

= Weyl chambers.

~ Some next-o bits and bobs

Fix fundamental  $(\Phi, E)$  root system with specified base.

$$\Delta \subseteq \Phi.$$

Def<sup>t</sup>:  $\mathcal{W}$  call a reflection  $\sigma_\alpha \in \mathcal{W}$  a simple reflection if  $\alpha \in \Delta$ .

We'll see in a moment that  $\mathcal{W}$  is generated by simple reflections.

Lemma 10.2 A: For positive non-simple  $\gamma \in \Phi^+$ , there exists  $\alpha \in \Delta$  with  $\gamma - \alpha \in \Phi^+$ .

Proof: Write  $\gamma = \sum_{\beta} c_\beta \beta$  for non-neg  $c_\beta$ . Then  $0 < (\gamma, \gamma) = \sum_{\beta} c_\beta (\beta, \beta)$  so that  $(\beta, \alpha) > 0$  at some  $\alpha$ . In this case

$\gamma - \alpha \in \Phi$  by Lemma 9.7. Since  $\gamma \neq \alpha$  by hypothesis, and  $\gamma - \alpha \in \Phi^+$  or  $\Phi^-$  necessarily, the expression  $\gamma - \alpha = (c_\alpha - 1)\alpha + \sum_{\beta \neq \alpha} c_\beta \beta$

forces  $\gamma - \alpha \in \Phi^+$ .  $\square$

Corollary 10.2 A: Each  $\gamma \in \Phi^+$  admits an expression  $\gamma = \sum_{i=1}^k \alpha_i$  with all  $\alpha_i$  simple and each sub-sum  $\sum_{i=1}^{k-m} \alpha_i \in \Phi^+$  as well.

Proof: Immediate  $\square$

Lemma 10.2 B: For  $\alpha$  simple,

$$\sigma_\alpha(\Phi^+ - \{\alpha\}) = \Phi^+ - \{\alpha\}.$$

Proof: For  $\gamma \in \Phi^+ - \{\alpha\}$  write  $\gamma = \sum_{\beta} c_\beta \beta$

to get  $\sigma_\alpha(\gamma) = \sum_{\beta} c_\beta \beta - (\sum_{\beta} c_\beta (\beta, \alpha))\alpha$ .

— Since  $\gamma \neq \alpha$  we have  $c_\beta \neq 0$  at some  $\beta \neq \alpha$

giving  $\sigma_a(\gamma) \notin \Phi^-$ . Hence  $\sigma_a(\gamma) \in \Phi^+$ . (14)

Corollary 10.2B: For the half-sum of the positive roots  $\rho = \frac{1}{2} \sum_{\gamma \in \Phi^+} \gamma$  we have

a)  $\langle \rho, \alpha \rangle = 1$  at all simple  $\alpha$ .

b)  $\sigma_a(\rho) = \rho - \alpha$ .

Proof: (a) Follows from (b). For (b) we have

$$\rho = \left( \frac{1}{2} \sum_{\gamma \neq \alpha} \gamma \right) + \frac{1}{2} \alpha$$

$$\begin{aligned} \Rightarrow \sigma_a(\rho) &= \left( \frac{1}{2} \sum_{\gamma \neq \alpha} \sigma_a(\gamma) \right) - \frac{1}{2} \alpha \\ &= \left( \frac{1}{2} \sum_{\gamma \neq \alpha} \gamma \right) - \frac{1}{2} \alpha \\ &= \rho - \alpha. \end{aligned}$$

Lemma 10.2C: Let  $\alpha_1, \dots, \alpha_t \in \Delta$  be simple, and  $\sigma_t = \sigma_{\alpha_t}$  at all  $i$ . If

$$\sigma_1 \dots \sigma_{t-1}(\alpha) \in \Phi^-$$

then there exists an index  $1 \leq s < t$  for which

$$\sigma_1 \dots \sigma_t = \sigma_1 \dots \sigma_{s-1} \sigma_{s+1} \dots \sigma_{t-1}.$$

Proof: Let  $s$  be maximal so that

$$\sigma_s \sigma_{s+1} \dots \sigma_{t-1}(\alpha) \text{ is negative.}$$

Then  $\beta = \sigma_{s+1} \dots \sigma_{t-1}(\alpha)$  is positive, and further

$$\alpha_s = \beta. \text{ For } \sigma = \sigma_{s+1} \dots \sigma_{t-1} \text{ we now have}$$

$$\sigma_s = \sigma_{\sigma(\alpha)} = \sigma \sigma_{\alpha} \sigma^{-1}$$

$$\begin{aligned} \text{so that } \sigma_1 \dots \sigma_s \dots \sigma_{t-1} \sigma_t &= \sigma_1 \dots \sigma \sigma_{\alpha} \sigma^{-1} \sigma_t \\ &= \sigma_1 \dots \sigma_{s-1} \sigma_{s+1} \dots \sigma_{t-1} \end{aligned}$$

as claimed.

Corollary 10.2C: For  $\sigma = \sigma_1 \dots \sigma_t \in W$  with the  $\sigma_i$  simple reflections  $\sigma_i = \sigma_{\alpha_i}$ , and  $t$  minimal so that  $\sigma$  admits such an expression, we have  $\sigma(\alpha_t)$  a negative root.

Proof: Otherwise  $\sigma\sigma_t(\alpha_t)$  is negative (15)

implying  $\sigma = \sigma_1 \dots \sigma_{s-1} \sigma_{s+1} \dots \sigma_{t-1}$ .

Lemma 10.2.D: If  $\sigma = \sigma_1 \dots \sigma_t$  is a minimal such exp. in terms of simple reflections, then

$$t = |\{ \gamma \in \Phi^+ : \sigma(\gamma) \in \Phi^- \}|.$$

Proof: Proceed by ind. on  $t$ , the  $t=1$  case follows by Lemma 10.2.B. Define now for

$$s \in \Delta \quad \sigma \in W \quad \langle \sigma\alpha : \alpha \in \Delta \rangle \subseteq W$$

$l(\sigma) = \min \{ t : \sigma = \sigma_1 \dots \sigma_t \text{ for some simple } \sigma_i \}$  and suppose the result holds for all  $\tau$  w/  $l(\tau) \leq t$ , and take  $\sigma$  s.t.  $l(\sigma) = t$ .

Then  $\sigma_1 \dots \sigma_{t-1}$  is necessarily a min length exp, so that  $|\{ \gamma : \sigma(\gamma) \text{ negative} \}| = t-1$ .

If  $\sigma(\alpha_t)$  negative then  $\sigma = \tau\sigma_t$  has a length  $t-2$  exp by Lemma 10.2.C, a contradiction.

So  $\sigma(\alpha_t)$  is positive, and  $\sigma(\alpha_t)$  is negative and  $\{ \gamma : \sigma(\gamma) \text{ negative} \}$

$$= \{ \alpha_t \} \cup \{ \sigma_t(\gamma) : \gamma \in \Phi^+ \text{ and } \sigma(\gamma) < 0 \}$$

$$\Rightarrow |\{ \gamma : \sigma(\gamma) \text{ negative} \}| = t, \text{ as desired.}$$

~ Generating the Weyl group.

Theorem 10.3: Let  $\Delta$  be a base for  $(\Phi, E)$ .

i) If  $\lambda \in E$  is regular then there exists  $\sigma \in W$  so that  $(\sigma(\lambda), \alpha) > 0$  at all  $\alpha \in \Delta$ .

ii) For any other base  $\Delta'$  there exists  $\sigma \in W$  with  $\sigma(\Delta') = \Delta$ .

iii) For arbitrary  $\gamma \in \Phi$  there exists  $\sigma \in W$  such that  $\sigma(\gamma) \in \Delta$ .

iv)  $\mathcal{W}$  is generated by the  $\sigma_a: a \in \Delta$ . (16)

v)  $\sigma(\Delta) = \Delta$  iff  $\sigma = id$ .

Proof: We prove (i)-(iv) for  $\mathcal{W}' = \langle \sigma_a: a \in \Delta \rangle$  which will imply (iv).

(i) For  $\rho = \frac{1}{2} \sum_{\gamma \in \Phi^+} \gamma$  take  $\sigma \in \mathcal{W}'$  so that  $(\sigma(\lambda), \rho)$  is maximal. Then for each simple root  $\alpha$  we have

$$(\sigma \alpha \sigma(\lambda), \rho) = (\sigma(\lambda), \sigma \alpha(\rho)) = (\sigma(\lambda), \rho) - (\sigma(\lambda), \alpha).$$

$$u/ \quad (\sigma \alpha \sigma(\lambda), \rho) \leq (\sigma(\lambda), \rho). \quad \text{Hence}$$

$(\sigma(\lambda), \alpha) > 0$  by regularity of  $\lambda$ , and hence  $\sigma$  of  $\sigma(\lambda)$ .

(ii) Write  $\Delta' = \Delta(\lambda)$  and choose  $\sigma \in \mathcal{W}'$  so that  $(\sigma(\lambda), \alpha) > 0$  at all simple  $\alpha \Rightarrow$

$$\sigma(\Delta') = \Delta(\sigma\lambda) = \Delta.$$

(iii) Take first  $\lambda' \in P_f$  with  $\lambda' \notin \bigcup_{\beta \neq \gamma} P_\beta$ .

Such  $\lambda'$  exists since the  $P_\beta$  intersect  $P_f$  at a lower dimension space and  $(\bigcup_{\beta \neq \gamma} P_\beta) \cap P_f = \bigcap_{\beta \neq \gamma} (P_\beta \cap P_f)$ .

Now we can choose  $\lambda = \lambda' + \epsilon \gamma$  with  $\epsilon$  small enough so that  $(\gamma, \lambda) = \epsilon(\gamma, \gamma) > 0$  and

$$(\gamma, \lambda) < |(\beta, \lambda)| \quad \text{at all } \beta \in \Phi - \{\gamma\}.$$

Then  $\gamma \in \Delta(\lambda)$  by minimality of  $(\gamma, \lambda)$ , and for  $\sigma$  as in (i) we get  $\sigma(\gamma) \in \Delta$ .

(iv) For any  $\delta \in \Phi$  choose  $\sigma \in \mathcal{W}'$  so that  $\sigma(\delta) \in \Delta$ . Then  $\sigma \sigma_\gamma \sigma^{-1} = \sigma(\sigma\gamma) \in \mathcal{W}'$

and hence  $\sigma_\gamma = \sigma^{-1} \sigma(\sigma\gamma) \sigma \in \mathcal{W}'$ . If

follows that  $\mathcal{W} = \langle \sigma_\gamma: \gamma \in \Phi \rangle = \mathcal{W}'$ .



ex) If  $\sigma(\Delta) = \Delta$  then  $\sigma(\Phi^+) = \Phi^+$ . (17)

BZ Corollary 10.2 C we have  $\sigma = \text{id}$  necessarily.

— Lengths of elem in the Weyl group.

Def<sup>1</sup>: For  $\sigma \in W = \langle \sigma_{\alpha_i} : \alpha_i \text{ simple} \rangle$ ,  
the length of  $\sigma$  is

$$l(\sigma) = |\{ \gamma \in \Phi^+ : \sigma(\gamma) \in \Phi^- \}|$$

Lemma 10.20  $\{ f : \sigma \text{ admit an expression } \sigma = \sigma_1 \dots \sigma_f \text{ with the } \sigma_i \text{ simple refls} \}$   
= min

We call an expression  $\sigma = \sigma_1 \dots \sigma_f$  reduced if  
 $f = l(\sigma)$

Lemma (the longest word): There is a unique element  
 $w \in W$  with  $l(w) = |\Phi^+|$ .

Furthermore, for all other  $\sigma \in W$   $l(\sigma) < l(w)$ .

Proof: For  $\Delta$  the given base we have  $-\Delta$   
another base. Hence, by Thm 10.3 there is an element  
 $w \in W$  with  $w(\Delta) = -\Delta$  and thus

$$w(\Phi^+) = \Phi^-.$$

So  $l(w) = |\Phi^+|$  is maximal. For any  
other  $w' \neq w$   $l(w') < |\Phi^+|$  we have  $w'(\Phi^-) = \Phi^+$   
so that  $w'w^{-1}(\Delta) = w'(-\Delta) = \Delta$ . Then by  
Thm 10.3 (v)  $w'w^{-1} = 1 \Rightarrow w' = w$ .

For any other  $\sigma \neq w$  we have  $l(\sigma) < l(w)$   
by uniqueness of  $w$  and the def. of length.  $\square$

- Irreducible root systems

Def<sup>n</sup>: Call a root system  $(E, \Phi)$  reducible if there exist orthogonal subspaces  $E_1, E_2$  for which

$$E = E_1 + E_2 (= E_1 \oplus E_2)$$

and  $\Phi = \Phi_1 \cup \Phi_2$ , where  $\Phi_i = E_i \cap \Phi$ .

Call  $(E, \Phi)$  irreducible if it is not reducible.

Lemma 10.48: A root system  $(E, \Phi)$  is irreducible if and only if  $E$  is an irreducible  $W$ -representation.

Proof: If  $(E, \Phi)$  reduces as

$$E = E_1 + E_2, \quad \Phi = \Phi_1 \cup \Phi_2,$$

then the  $E_i$  are both  $W$ -subreps. Hence  $E$  is not an irreducible  $W$ -rep. Suppose now that  $E$  is <sup>not</sup> irreducible over  $W$ , and choose any <sup>proper nonzero</sup>  $W$ -subrep  $E_1 \subset E$ . We

have the linear splitting  $E = E_1 \oplus E_1^\perp$  and we claim

$$\Phi = \Phi_1 \cup \Phi_1^\perp, \quad \Phi_i = E_i \cap \Phi.$$

For any <sup>nonzero</sup>  $v \notin E_1$ ,  $E_1^\perp$  we have  $\langle v, v \rangle > 0$

in  $E_1$  with  $\langle v, v \rangle > 0$  and hence

$$\sigma_v(w) = w - \frac{2\langle w, v \rangle}{\langle v, v \rangle} v \notin E_1.$$

From this, and stability of  $E_1$  under the  $W$ -action,

we see all roots lie in  $E_1$  or  $E_1^\perp$ . Since  $\Phi$

spans all of  $E$  we now find  $\Phi_1 \neq \emptyset$  and  $\Phi_1^\perp \neq \emptyset$ .

Thus we decompose  $E$  as a root system

$$(E, \Phi) = (E_1, \Phi_1) \oplus (E_1^\perp, \Phi_1^\perp).$$

Lemma 2.4 C: If  $(E, \Phi)$  is irreducible then

at most two root lengths occur in  $\Phi$ , and both the

short roots and long roots in  $\Phi$  consist of a single  $W$ -orbit

— (independently).

Proof: Let  $\alpha, \gamma \in \Phi$  be of <sup>same length</sup>, and take  $\sigma \in W$  with  $\langle \sigma\alpha, \gamma \rangle \neq 0$ . (We know such  $\sigma$  exists by irreducibility of  $\Phi$  and  $W$ , Lemma 10.4B.) Since  $\alpha$  and  $\gamma$  have same length we have  $\langle \sigma\alpha, \gamma \rangle = \langle \gamma, \sigma\alpha \rangle \Rightarrow \langle \sigma\alpha, \gamma \rangle = \langle \gamma, \sigma\alpha \rangle = \pm 1$ , and by replacing  $\alpha$  w/  $-\alpha = \sigma_\alpha(\alpha)$  if necessary we can assume  $\langle \sigma\alpha, \gamma \rangle = 1$ .

Now we calculate, for  $\beta = \sigma\alpha$ ,  

$$\sigma_\gamma \sigma_\beta \sigma_\gamma(\sigma\alpha) = \sigma_\gamma \sigma_\beta \sigma_\gamma(\beta) = \sigma_\gamma \sigma_\beta(\beta - \gamma)$$

$$= \sigma_\gamma(-\beta - \gamma + \beta) = \gamma.$$

So  $\gamma$  and  $\alpha$  are in the same  $W$ -orbit. (same length)

As for the two root lengths being <sup>supposing</sup> <sup>occurs</sup>  $\neq 1$ , for  $\alpha$  of shortest length, and two longer roots  $\gamma_1, \gamma_2$

we have  $\sigma_i \in W$  so that  $\langle \sigma_i \alpha, \gamma_i \rangle \neq 0$ , and

hence  $\frac{|\gamma_i|^2}{|\sigma_i \alpha|^2} = \frac{|\gamma_i|^2}{|\alpha|^2} \in 1, 2, 3$  by Table 1.9.4,

and we can also find  $\tau$  w/  $\langle \tau \gamma_1, \gamma_2 \rangle \neq 0$  so that

$$\frac{|\gamma_1|^2}{|\gamma_2|^2} = \frac{|\gamma_1|^2}{|\alpha|^2} \cdot \frac{|\alpha|^2}{|\gamma_2|^2} \in 1, 2, 3.$$

This forces  $\frac{|\gamma_1|^2}{|\alpha|^2} = \frac{|\gamma_2|^2}{|\alpha|^2} \Rightarrow |\gamma_1|^2 = |\gamma_2|^2$ .

So our possible length ratios are

1, 1 and 2, or 1 and 3.

— Irreducibility vs. simplicity

Recall: A semisimple Lie algebra  $\mathfrak{g}$  is simple if and only if the associated root system  $(\Phi_{\mathfrak{g}}, \Phi_{\mathfrak{g}})$  is irreducible.

Proof: If  $\mathfrak{g} = \mathfrak{g}_1 \times \mathfrak{g}_2$  w/  $\dim \mathfrak{g}_i > 0$ , the roots decompose as  $\Phi_1 = \text{roots for } \mathfrak{g}_1$  and  $\Phi_2 = \text{roots for } \mathfrak{g}_2$ , and these roots are orthogonal for

The Killing form, as

$$\kappa(y_1, y_2) = \gamma_i(h_{y_2}) = 0$$

whenever  $\gamma_i$  is not for  $\sigma_i$ . Hence, for

$$E_i = \mathbb{R}_{\sigma_i} \oplus \mathbb{R}_{\sigma_i}$$

we have  $\mathfrak{h}_{\mathbb{R}} = E_1 \oplus E_2$ ,  $\mathfrak{F} = \mathfrak{F}_1 \perp \mathfrak{F}_2$ .

Conversely, if  $\mathfrak{F}$  has an orthogonal decomp

$\mathfrak{F} = \mathfrak{F}_1 \perp \mathfrak{F}_2$  then for  $\gamma_i \in \mathfrak{F}_i$  we

have  $\gamma_1 + \gamma_2 \notin \mathfrak{F}$  as  $(\gamma_i, \gamma_i + \gamma_2) \neq 0$


at both  $i=1, 2$ . Thus

$$[\sigma \gamma_1, \sigma \gamma_2] = 0 \text{ whenever } \gamma_1 \in \mathfrak{F}_1, \gamma_2 \in \mathfrak{F}_2$$

and hence

$$\sigma \mathfrak{F} = \text{span}\{h_{\gamma}; \gamma \in \mathfrak{F}_1\} \oplus \left( \bigoplus_{\gamma \in \mathfrak{F}_1} \sigma \gamma \right) \\ \times \\ \text{span}\{h_{\gamma}; \gamma \in \mathfrak{F}_2\} \oplus \left( \bigoplus_{\gamma \in \mathfrak{F}_2} \sigma \gamma \right)$$

is a decomp of  $\sigma \mathfrak{F}$  into orthogonal ideals. So  $\sigma \mathfrak{F}$  is

not simple. 

Conclg:  $\mathcal{D}_{\mathbb{R}}$  bipedini

$$\{\text{semisimple Lie algs}\} / \cong \rightarrow \{\text{rat systems}\} / \cong$$

restricts to a bipedini

$$\{\text{simple Lie algs}\} / \cong \rightarrow \{\text{indecomposable rat systems}\} / \cong$$

Main Goal: Classify indecomposable rat systems!

- No Classification of indecomposable rat systems/  
simple Lie algs.

Def<sup>n</sup>: The Dynkin diagram for  $(E, \Phi)$   
is defined as the directed, unrooted graph w/  
vertices

$$\text{vert Dynkin}(E, \Phi) = \Delta, \text{ where } \Delta \text{ is any choice of base}$$

(21)

# of edges between  $\alpha, \beta \in \Delta$

$$= \max \{ \langle \alpha, \beta \rangle, \langle \beta, \alpha \rangle \}.$$

When # edges  $> 1$  we draw the edges w/ a greater than sign towards the larger root

e.g.  $\alpha \rightrightarrows \beta$  means  $|\alpha|^2/|\beta|^2 = 3$  eqval.  
 $\langle \alpha, \beta \rangle = 3.$

Ex: The Dynkin diagrams for the Classical type algebras

Type  $A_n, \mathfrak{sl}_{n+1}(\mathbb{C})$ :  $\alpha_1 \text{---} \alpha_2 \text{---} \alpha_3 \text{---} \dots \text{---} \alpha_n$

Type  $B_n, \mathfrak{so}_{2n+1}(\mathbb{C})$ :

$\alpha_1 \text{---} \alpha_2 \text{---} \dots \text{---} \alpha_{n-1} \xrightarrow{1 \quad 2} \alpha_n$  (the unique short root)

Type  $C_n, \mathfrak{sp}_{2n}(\mathbb{C})$ :

$\alpha_1 \text{---} \alpha_2 \text{---} \alpha \text{---} \alpha_{n-1} \xleftarrow{1 \quad 2} \alpha_n$  (the unique long root).

Type  $D_n, \mathfrak{so}_{2n}(\mathbb{C})$ :

$\alpha_1 \text{---} \alpha_2 \text{---} \dots \text{---} \alpha_{n-2} \text{---} \alpha_{n-1} \text{---} \alpha_n$

The information of the Dynkin diagram is alternatively contained in the Coxeter matrix

$$\text{Cox}(E, \Phi) = [\langle \alpha_i, \alpha_j \rangle]$$

where  $\{\alpha_1, \dots, \alpha_n\} = \Delta$

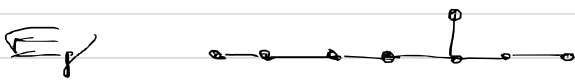
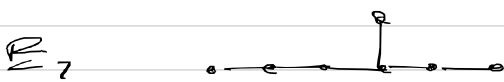
Example:  $\text{Cox}(C_n) = \begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & \ddots & -1 \\ & & -1 & 2 & -2 \\ 0 & & & -1 & 2 \end{bmatrix}$

Characterization (Humphreys 11.4): For

$(E, \Phi)$  an irreducible root system,  $\text{Dynkin}(E, \Phi)$  is, up to isomorphism, one of the following

Classical Types  $\begin{cases} A_n & C_n & n \in \mathbb{Z}_{>0} \\ B_n & D_n \end{cases}$

Exceptional types



We're done

Corollary 11.4: A simple Lie algebra  $\mathfrak{g}$  is one of the following

Classical  $\{ \mathfrak{sl}_n(\mathbb{C}), \mathfrak{sp}_{2n}(\mathbb{C}), \mathfrak{so}_n(\mathbb{C}) \}$

Exceptional  $\{ \mathfrak{g}(E_6), \mathfrak{g}(E_7), \mathfrak{g}(E_8), \mathfrak{g}(F_4), \mathfrak{g}(G_2) \}$ .

Each of these algebras are described via Serre's presentation

— Serre relations

Theorem (Serre): Let  $\mathfrak{g}$  be a semisimple Lie algebra w/ Cartan matrix  $A = [a_{ij}]$ ,  $a_{ij} = \langle \alpha_i, \alpha_j \rangle$ .

(Here  $A = [\alpha_1, \dots, \alpha_n]$  is some choice of base for Cartan  $\mathfrak{h} \subseteq \mathfrak{g}$ .) Then  $\mathfrak{g}$  admits a presentation

via generators  $e_i \rightarrow \alpha_i, f_i \rightarrow -\alpha_i, h_i \rightarrow \alpha_i$

and relations

$$(S) \begin{cases} \cdot [h_i, h_j] = 0 & \cdot [h_i, e_j] = a_{ji} e_j \\ \cdot [h_i, f_j] = -a_{ji} f_j & \cdot [e_i, f_j] = 0 \text{ if } i \neq j \\ \cdot \operatorname{ad}_{e_i}^{-a_{ji}+1}(e_j) = \operatorname{ad}_{f_i}^{-a_{ji}+1}(f_j) = 0 \text{ if } i \neq j \end{cases}$$

Furthermore, if  $(E, \mathfrak{F})$  is any root system w/ Cartan matrix  $A = [a_{ij}]$ , then the Lie algebra

$$\mathfrak{g}(E, \mathfrak{F}) := \left\{ \begin{array}{l} \text{Lie alg w/ gens } e_1, \dots, e_n, \\ f_1, \dots, f_n, h_1, \dots, h_n, \text{ and rels (S)} \end{array} \right\}$$

is a semisimple Lie algebra.