KERODON REMIX PART II: COCARTESIAN FIBRATIONS, TRANSPORT, AND THE YONEDA EMBEDDING

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Abstract. These are notes on ∞-categories which are (mostly) adapted from Lurie's digital text Kerodon [4]. The main distinctions are the length of the document, the order of presentation, and selective omissions. We also deviate from [4] in that we focus on derived categories and dg categories as our primary examples of interest. A distinction from the related text [2] would be the complete avoidance of model structures, though this approach is already adopted in [4].

Following Part I, which presented the basic foundations for studies of ∞-categories, we discuss herein cartesian and cocartesian fibrations, transport functors (i.e. Grothendeick straightening and unstraightening), and limits and colimits. Specific topics include Hom functions and Yoneda embedding, calculations of limits and colimits in categories of spaces and cochains, and stable ∞-categories. While we follow the analytic, rather than synthetic, approach, we've attempted to communicate these topics in a manner which is consumable to the working mathematician.

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1. Introduction to cartesian and cocartesian fibrations

1.1. Definitions.

Definition 1.1. Consider a map of simplicial sets $q : X \to S$. We call a 1-simplex $\alpha : x \to y$ in X a q-cartesian morphism if any lifting problem

$$\begin{tikzcd} \Lambda^n_n \arrow[r, "\bar{\sigma}"] \arrow[d] & X \arrow[d, "q"] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & S \end{tikzcd} \tag{1}$$

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with $n \geq 2$ and $\bar{\sigma}|\Delta^{\{n-1,n\}} = \alpha$ admits a solution. We say $\alpha : x \to y$ is q-cocartesian if any lifting problem

$$\begin{tikzcd} \Lambda^n_0 \arrow[r, "\bar{\tau}"] \arrow[d] & X \arrow[d, "q"] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & S \end{tikzcd} \tag{2}$$

eq:128

with $n \geq 2$ and $\bar{\tau}|\Delta^{\{0,1\}} = \alpha$ admits a solution.

Though at some specific moments we will consider a case where X and S are not ∞-categories, we are primarily invested in the ∞-categorical setting.

Definition 1.2. We call a map of ∞-categories $q : \mathscr{C} \to \mathscr{D}$ a cartesian fibration if q is an inner fibration and, for any map $\bar{\alpha} : \bar{x} \to \bar{y}$ in $\mathscr{D}$ and y in $\mathscr{C}$ with $q(y) = \bar{y}$, there is a q-cartesian map $\alpha : x \to y$ in $\mathscr{C}$ with $q(\alpha) = \bar{\alpha}$.

Similarly, we call q a cocartesian fibration if it is an inner fibration and, for any map $\bar{\beta} : \bar{x} \to \bar{y}$ in $\mathscr{D}$ and x with $q(x) = \bar{x}$, there is a q-cocartesian fibration $\beta : x \to y$ with $q(\beta) = \bar{\beta}$.

The following is obvious. Recall our definitions of right and left fibrations from Definition I-4.23.

Proposition 1.3. *If $q : \mathscr{C} \to \mathscr{D}$ is a right fibration (resp. left fibration) then q is a cartesian (resp. cocartesian).*

Proof. In this case any lifting problem of the form (1), or (2) respectively, admits a solution simply by the defintion. ☐

Obviously when $q : \mathscr{C} \to \mathscr{D}$ is a Kan fibration it is both cartesian and cocartesian.

Example 1.4. Consider a diagram $p : K \to \mathscr{C}$, with K some simplicial set. The we have the overcategory $\mathscr{C}_{/p}$ and the undercategory $\mathscr{C}_{p/}$. The two forgetful functors

$$\mathscr{C}_{/p} \to \mathscr{C} \quad \text{and} \quad \mathscr{C}_{p/} \to \mathscr{C}$$

are, respectively, a right and left fibration Proposition I-4.25. Hence these maps are respectively a cartesian and cocartesian fibration.

In the case where K is a point $x : * \to \mathscr{C}$ we recall that the fibers of the fibration $\mathscr{C}_{/x} \to \mathscr{C}$ and $\mathscr{C}_{x/} \to \mathscr{C}$ are the right and left pinched mapping spaces $\mathrm{Hom}^{\mathrm{R}}_{\mathscr{C}}(w, x)$ and $\mathrm{Hom}^{\mathrm{L}}_{\mathscr{C}}(x, y)$.

Example 1.5 ([4, 01T8])**.** Consider an ∞-category $q : \mathscr{C} \to *$. A morphism $\alpha : x \to y$ is q-cartesian if and only if α is an isomorphism. To see this consider fillings of the horns

$$\begin{tikzcd} & x \arrow[dr, "\alpha"] & \\ y \arrow[ur, dotted, "\exists\beta"] \arrow[rr, "id"'] & & y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & y \arrow[dr, "\beta"] \arrow[dd, "id"] & \\ x \arrow[ur, "\alpha"] \arrow[rr, "id"] \arrow[dr, "\alpha"'] & & x. \arrow[dl, "\alpha"] \\ & y & \end{tikzcd}$$

One similarly finds that a morphism is q-cocartesian if and only if it is an isomorphism.

Via the existence of identity morphisms the structure map $q : \mathscr{C} \to *$ is always a cartesian and cocartesian fibration. Note that this map is not a left or right fibration unless $\mathscr{C}$ is a Kan fibration.

1.2. **Imaginings: Cartesian fibrations as lax moduli.** Give a cartesian fibration $q : \mathscr{C} \to \mathscr{D}$ one might think of $\mathscr{C}$ as a lax moduli of "stuff" varying over the objects in $\mathscr{D}$. The cartesian lifts of morphisms in $\mathscr{D}$ provide transition functions between these fibers, i.e. the stuff we are parametrizing, over $\mathscr{D}$. In the case of the cartesian fibration $\mathscr{C}_{/x} \to \mathscr{C}$ the category $\mathscr{C}_{/x}$ is, in an obvious sense, the "moduli of maps to x". Let us leave the latter point about lifting maps for now, and try to make some comment on the moduli point.

Let us just consider how one classically constructs a moduli space. Here we consider the base space $\mathscr{D} = \mathrm{Sch}_k$ of schemes over k, which we can endow with some Grothendieck topology if we like, though we don't care at the moment. Then a pre-stack is a choice of a functor of plain categories $q : \mathbb{M} \to \mathrm{Sch}_k$ which makes $\mathbb{M}$ into a category fibered in groupoids over Sch_k [5, 003S]. One simply compares definitions to see that

$$\left\{ \begin{array}{c} \mathbb{M} \text{ is fibered in} \\ \text{groupoids over } \mathrm{Sch}_k \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{c} q \text{ is a cartesian fibration} \\ \text{in which all maps in } \mathbb{M} \\ \text{are } q\text{-cartesian} \end{array} \right\}.$$

In this familiar setting one can now "invert" this functor q to produce an associated 2-functor

$$q^\vee : (\mathrm{Sch}_k)^{\mathrm{op}} \to \mathrm{Groupoids} \subseteq \mathrm{Cat}, \quad Y \mapsto \mathbb{M}_Y.$$

One establishes this functor via an abuse of the axion of choice.

To elaborate a bit more, for any map of schemes $\alpha : X \to Y$ we take a lift $\alpha^* y \to y$ in $\mathbb{M}$. This lift is unique up to unique isomorphism, and via unique filling defines a functor between the fibers

$$\alpha^* : \mathbb{M}_Y \to \mathbb{M}_X, \quad y \mapsto \alpha^* y.$$

On morphisms $\xi : y_1 \to y_2$ in the fiber $\mathbb{M}_Y$, we note that the cartesian property for maps in $\mathbb{M}$ asserts the existence of a unique map $\alpha^*\xi : \alpha^* y_1 \to \alpha^* y_2$ completing the diagram

$$\begin{tikzcd} \alpha^* y_1 \arrow[r] \arrow[d, dashed, "\exists!"'] & y_1 \arrow[d, "\xi"] \\ \alpha^* y_2 \arrow[r] & y_2, \end{tikzcd}$$

where we note that uniqueness comes from filling the appropriate 3-simplex in $\mathbb{M}$. Hence α^* is well-defined on morphisms via the assignment $\xi \mapsto \alpha^*\xi$.

We note that this inversion of $q : \mathbb{M} \to \mathrm{Sch}_k$ into a functor $q^\vee : (\mathrm{Sch}_k)^{\mathrm{op}} \to \mathrm{Cat}$ does not require all maps in $\mathbb{M}$ to be cartesian. This is simply a consequence of q being a cartesian fibration between plain categories.

In the general ∞-context, we again have this inversion property for (co)cartesian fibrations. Here a cartesian fibration $q : \mathscr{C} \to \mathscr{D}$ will specify, and be specified by, a functor

$$q^\vee : \mathscr{D}^{\mathrm{op}} \to \mathscr{C}at_\infty$$

whose values over objects y in $\mathscr{D}$ are the fibers $\mathscr{C}_y$. The functors between fibers $\alpha^* : \mathscr{C}_y \to \mathscr{C}_x$ are what we've referred to as *transport* along α (following Kerodon [4]).

While such fibrations play an essentially non-existent role in plain category theory, from the perspective of the working mathematician, they play an extraordinarily important role in the development of ∞-category theory. The main point is that they tame choices in the ∞-categorical setting. While in the plain category setting we can simply make a choice, and if that choice is not unique we can simply say it's unique up to a unique isomorphism, and then if I make two of the same types of choices then any ambiguities will vanish due to sufficient uniqueness, etc. etc., such a laissez faire attitude will lead to immediate intractable problems in the ∞-context. So one generally bundles all choices of a certain "type" into a cartesian or cocartesian fibrations, and manipulates these bundles in order to make global movements between choices of different types.

1.3. **Discussion: Classifying functors etc.**

2. Cartesian and cocartesian fibrations

2.1. **Cartesian morphisms via overcategories.**

prop:232

Proposition 2.1 ([4, 01TF])**.** *Let $q : \mathscr{C} \to \mathscr{D}$ be a map between ∞-categories. A morphism $\alpha : x \to y$ in $\mathscr{C}$ is q-cartesian if and only if the natural map*

$$\mathscr{C}_{/\alpha} \to \mathscr{C}_{/y} \times_{\mathscr{D}_{/q(y)}} \mathscr{D}_{/q(\alpha)}$$

is a trivial Kan fibration. Similarly, α is q-cocartesian if and only if the map

$$\mathscr{C}_{/\alpha} \to \mathscr{C}_{x/} \times_{\mathscr{D}_{q(x)/}} \mathscr{D}_{q(\alpha)/}$$

is a trivial Kan fibration.

For the proof we employ a specific deconstruction of the relevant horn inclusions.

lem:joyal3.3

Lemma 2.2 ([1, Lemma 3.3])**.** *For non-negative integers p and q, and $n = p+q+1$, the inclusions*

$$(\Lambda^p_0 \star \Delta^q) \coprod_{\Lambda^p_q \star \partial\Delta^q} (\Delta^p \star \partial\Delta^q) \to \Delta^p \star \Delta^q \cong \Delta^n$$

and

$$(\partial\Delta^p \star \Delta^q) \coprod_{\partial\Delta^p \star \Lambda^q_q} (\partial\Delta^p \star \Lambda^q_q) \to \Delta^p \star \Delta^q \cong \Delta^n$$

are identified with the inclusions of the extremal horns $\Lambda^n_0 \to \Delta^n$ and $\Lambda^n_n \to \Delta^n$ respectively.

One can see the text [1], or [4, 018N] for the details. We now proceed with the proof of Proposition 2.1.

Proof of Proposition 2.1. We address the cartesian situation, the cocartesian one being similar. Let $F : \mathscr{C}_{/\alpha} \to \mathscr{C}_{/y} \times_{\mathscr{D}_{/q(y)}} \mathscr{D}_{/q(\alpha)}$ denote the map under consideration. A solution to a lifting problems of the form

$$\begin{tikzcd} \partial\Delta^m \arrow[r] \arrow[d] & \mathscr{C}_{/\alpha} \arrow[d, "F"] \\ \Delta^m \arrow[r] \arrow[ur, dotted] & \mathscr{C}_{/y} \times_{\mathscr{D}_{/q(y)}} \mathscr{D}_{/q(\alpha)}, \end{tikzcd} \tag{3}$$

eq:262

with $m \geq 0$, admit a solution if and only if the equivalent lifting problem

$$\begin{tikzcd} (\partial\Delta^m \star \Delta^1) \cup (\Delta^m \star \Lambda^1_1) \arrow[r] \arrow[d] & \mathscr{C} \arrow[d, "q"] \\ \Delta^m \star \Delta^1 \arrow[r] \arrow[ur, dotted] & \mathscr{D} \end{tikzcd}$$

obtained by way of adjunction Lemma I-4.22 admits a solution. Via direct inspection the final edge $\Delta^1 \cong \emptyset \star \Delta^1 \to \mathscr{C}$ in the latter diagram is α, so that this diagram is identified, via Lemma 2.2, with a diagram of the form

$$\begin{tikzcd} \Lambda^n_n \arrow[r] \arrow[d] & \mathscr{C} \arrow[d, "q"] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & \mathscr{D} \end{tikzcd} \tag{4}$$ eq:276

in which $n \geq 2$ the edge $\Delta^{\{n-1,n\}} \to \mathscr{C}$ is α. It follows that all lifting problems of the form (3) admit a solution if and only if all lifting problems of the form (4) admit a solution, i.e. that the map F is a trivial Kan fibration if and only if the map α is q-cartesian. □

2.2. Cartesian morphisms via mapping spaces.

prop:cocart_maps **Proposition 2.3** ([4, 01TL]). *Consider an inner fibration $q : \mathscr{C} \to \mathscr{D}$, and a morphism $\alpha : x_1 \to x_2$ in $\mathscr{C}$ with image $\bar{\alpha} : \bar{x}_1 \to \bar{x}_2$ in $\mathscr{D}$. The morphism α is q-cartesian if and only if for each third object x_0 in $\mathscr{C}$, with corresponding triples $x : \{0,1,2\} \to \mathscr{C}$ and $\bar{x} : \{0,1,2\} \to \mathscr{D}$, the diagram*

$$\begin{tikzcd} \mathrm{Fun}(\Delta^2, \mathscr{C})_x \times_{\mathrm{Hom}_{\mathscr{C}}(x_1,x_2)} \{\alpha\} \arrow[r] \arrow[d, "q"'] & \mathrm{Hom}_{\mathscr{C}}(x_1, x_2) \arrow[d, "q"'] \\ \mathrm{Fun}(\Delta^2, \mathscr{D})_{\bar{x}} \times_{\mathrm{Hom}_{\mathscr{D}}(\bar{x}_1,\bar{x}_2)} \{\bar{\alpha}\} \arrow[r] & \mathrm{Hom}_{\mathscr{D}}(\bar{x}_1, \bar{x}_2) \end{tikzcd}$$

is a homotopy pullback diagram of Kan complexes.

We cover the proof of Proposition **??** in Section 2.3 below. Let us record now a number of examples.

sect:cocart_maps_proof

2.3. Proof of Proposition 2.3.

2.4. Uniqueness for q-cocartesian lifts.

prop:cocart_uniqueness **Proposition 2.4** ([4, 01VK]). *Let $q : X \to S$ be an inner fibration of simplicial sets, and let Y be the full simplicial subset in $\mathrm{Fun}(\Delta^1, X)$ whose vertices are q-cocartesian edges in X. Let Z be the full simplicial set in $\mathrm{Fun}(\{0\}, X) \times_{\mathrm{Fun}(\{0\},S)} \mathrm{Fun}(\Delta^1, S)$ whose edges lie in the image of the composition*

$$Y \to \mathrm{Fun}(\Delta^1, X) \to \mathrm{Fun}(\{0\}, X) \times_{\mathrm{Fun}(\{0\},S)} \mathrm{Fun}(\Delta^1, S).$$

Then the induced map $Y \to Z$ is a trivial Kan fibration. The analogous statement holds when we replace Y with the full simplicial subset of q-cartesian edges in $\mathrm{Fun}(\Delta^1, X)$ as well.

Said informally, Proposition 2.4 tells us that, if a cocartesian solution to the diagram

$$\begin{tikzcd} \{0\} \arrow[r] \arrow[d] & X \arrow[d, "q"] \\ \Delta^1 \arrow[r, "\alpha"'] & S \end{tikzcd}$$

exists, then that solution is unique. We note that in the case where q itself is cocartesian, the simplicial subset Z is all of $\mathrm{Fun}(\{0\}, X) \times_{\mathrm{Fun}(\{0\},S)} \mathrm{Fun}(\Delta^1, S)$.

The proof of Proposition 2.4 employs a decomposition of a certain inclusion of simplicial sets which we record here.

lem:simpl_328 **Lemma 2.5** ([4, 00TH])**.** *At any positive integer n, the inclusion*

$$(\Delta^1 \times \partial\Delta^n) \coprod_{\{0\}\times\partial\Delta^n} (\{0\} \times \Delta^n) \to \Delta^1 \times \Delta^n$$

decomposes into a sequence of inclusions

$$(\Delta^1 \times \partial\Delta^n) \cup (\{0\} \times \Delta^n) = X(0) \to \cdots \to X(n) \to X(n+1) = \Delta^1 \times \Delta^n$$

in which each $X(i+1)$ fits into a pushout diagram

$$\begin{tikzcd} \Lambda^{n+1}_{n-i} \arrow[r] \arrow[d] & X(i) \arrow[d] \\ \Delta^{n+1} \arrow[r] & X(i+1) \end{tikzcd}$$

and the sequence

$$\Delta^{\{0,1\}} \to \Delta^{n+1} \to\to X(n+1) = \Delta^1 \times \Delta^{n+1}$$

is an isomorphism onto the edge $\Delta^1 \times \{0\}$ in $\Delta^1 \times \Delta^{n+1}$.

To be clear, our filtration is obtained by applying the opposite to the specific sequence from [4, 00TH].

Idea of proof. Consider the simplices $\sigma_i : \Delta^{n+1} \to \Delta^1 \times \Delta^n$ defined by taking $\sigma_i(j) = (0, j)$ if $j \leq n - i$ and $(1, j - 1)$ if $j > n - i$. We define sequentially $X(i + 1) = X(i) \cup \mathrm{im}(\sigma_i)$. We refer the reader to [4] for the specific details. $\square$

We now can prove our uniqueness result for cocartesian lefts.

Proof of Proposition 2.4. We deal with the case of cocartesian situation, the cartesian case following by taking opposites. We consider a lifting problem

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & Y \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & Z. \end{tikzcd} \tag{5}$$ eq:361

In the case $n = 0$, this problem admits a solution by the defintion of Z. In the case $n \geq 0$, solving this problem is equivalent to solving a lifting problem of the form

$$\begin{tikzcd} (\Delta^1 \times \partial\Delta^n) \cup (\{0\} \times \Delta^n) \arrow[r] \arrow[d, "\mathrm{incl}"'] & X \arrow[d] \\ \Delta^1 \times \Delta^n \arrow[r] \arrow[ur, dotted] & S \end{tikzcd} \tag{6}$$ eq:368

in which all of the constituent maps $\Delta^1 \times \{i\} \to X$ are q-cocartesian. In particular, the map $\Delta^1 \times \{0\} \to X$ is q-cocartesian. We decompose the map incl into a sequence of inclusion $X(i) \to X(i+1)$ as in Lemma 2.5, and produce sequential solutions to the problems

$$\begin{tikzcd} X(i) \arrow[r, "\sigma_i"] \arrow[d] & X \arrow[d, "q"] \\ X(i+1) \arrow[r] \arrow[ur, dotted, "\sigma_{i+1}"] & S \end{tikzcd}$$

for each $i < n$ since the inclusion $X(i) \to X(i+1)$ is inner anodyne in this case. For the final inclusion at $i = n$, we have the extended diagram

$$\begin{tikzcd} \Lambda^{n+1}_0 \arrow[r] \arrow[d] & X(n) \arrow[r, "\sigma_n"] \arrow[d] & X \arrow[d, "q"] \\ \Delta^{n+1} \arrow[r] \arrow[urr, dotted] & X(n+1) = \Delta^1 \times \Delta^n \arrow[r] \arrow[ur, dotted] & S \end{tikzcd}$$

and can solve the external problem since the initial edge $\Lambda^n_0 \to X$ has q-cocartesian image in X, and can therefore solve the internal lifting problem since the left-most square is a pushout diagram. We therefore obtain a solution to our original problems (5) and (6). □

2.5. Exponentiating cocartesian fibrations.

prop:397 **Proposition 2.6** ([4, 01VG]). *If $q : X \to S$ is a cocartesian fibration, then for any simplicial set K the map $q_* : \mathrm{Fun}(K, X) \to \mathrm{Fun}(K, S)$ is a cocartesian fibration. An edge $\xi : \Delta^1 \to \mathrm{Fun}(K, X)$ is q_*-cocartesian if and only if, at each v in K, the composite $v^*\xi : \Delta^1 \to X$ is q-cocartesian in X.*

The proof follows by a hands on analysis of certain lifting problems which we won't reproduce here. The reader can see the [4, 01VG & 01VM] for the details.

Let us recall that, for any inner fibration $q : X \to S$ and fixed map $\xi : A \to S$ the simplicial set $\mathrm{Fun}_S(A, X)$ is obtained as the fiber

$$\begin{tikzcd} \mathrm{Fun}_S(A, X) \arrow[r] \arrow[d] & \mathrm{Fun}(A, X) \arrow[d, "q_*"] \\ * \arrow[r, "\xi"'] & \mathrm{Fun}(A, S) \end{tikzcd}$$

Since the map q_* is an inner fibration (Corollary I-4.8) we understand that $\mathrm{Fun}_S(A, X)$ is an ∞-category. Of course, in the more restrictive case in which q is a cocartesian fibration, we have just seen that q_* is furthermore cocartesian.

thm:simp_lift **Theorem 2.7.** *Let K be any simplicial set and $q : X \to S$ be a cocartesian fibration. Any lifting problem*

$$\begin{tikzcd} \{0\} \times K \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta^1 \times K \arrow[r] \arrow[ur, dotted] & S \end{tikzcd}$$

admits a solution $\Delta^1 \times K \to X$ for which, at each vertex v in K, the composite

$$\Delta^1 \cong \Delta^1 \times \{v\} \to \Delta^1 \times K \to X$$

is a q-cocartesian edge in X. Furthermore, the full ∞-subcategory in $\mathrm{Fun}_S(\Delta^1 \times K, X)$ *spanned by such solutions is a contractible Kan complex.*

Proof. By Proposition 2.6, the map $q_* : \mathrm{Fun}(K,X) \to \mathrm{Fun}(K,S)$ is a cocartesian fibration and solutions to the above lifting problem are identified with q_*-cocartesian solutions $\widetilde{\xi} : \Delta^1 \to \mathrm{Fun}(K,X)$ to the associated lifting problem

$$\begin{tikzcd} \{0\} \arrow[r] \arrow[d] & \mathrm{Fun}(K,X) \arrow[d, "q_*"] \\ \Delta^1 \arrow[r] \arrow[ur, dotted] & \mathrm{Fun}(K,S). \end{tikzcd}$$

Existence and uniqueness of such solutions now follow by Proposition 2.4. □

In the event that $q : X \to S$ is a left fibration, all morphisms in X are cocartesian. So we see that there is a unique solution to the above lifting problem.

cor:left_lift **Corollary 2.8.** *Let K be any simplicial set and $q : X \to S$ be a left fibration. Any lifting problem*

$$\begin{tikzcd} \{0\} \times K \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta^1 \times K \arrow[r] \arrow[ur, dotted] & S \end{tikzcd}$$

admits a solution $\Delta^1 \times K \to X$, and the collection of all such solutions $\mathrm{Fun}_S(\Delta^1 \times K, X)$ *is a contractible Kan complex.*

3. Maps between fibrations

prop:318 **Proposition 3.1** ([4, 023R]). *Suppose that*

$$\begin{tikzcd} \mathscr{C} \arrow[rr, "F"] \arrow[dr, "q"'] & & \mathscr{D} \arrow[dl, "p"] \\ & S & \end{tikzcd} \qquad (7)$$ eq:320

is a map of inner fibrations, and that F is an equivalence. Then a morphism α is $\mathscr{C}$ is q-cocartesian if and only if $F\alpha$ is p-cocartesian.

Corollary 3.2. *Suppose we have a diagram (7) in which p and q are cocartesian fibrations and F is an equivalence of inner fibrations. Then F is an equivalence of cocartesian fibrations.*

4. Directional fibrations and Kan complexes

4.1. Exponentials for directional fibrations.

Definition 4.1. A map of simplicial sets $A \to B$ is called left anodyne (resp. right anodyne) if any lifting problem

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ B \arrow[r] \arrow[ur, dotted] & S \end{tikzcd}$$

in which f is a left (resp. right) fibration admits a solution.

One can show that the class of left anodyne maps is the saturated class generated by the horn inclusions $\Lambda^n_i \to \Delta^n$, where $0 \le i < n$ [4, 0151]. One similarly characterizes right anodyne maps.

lem:328

Lemma 4.2 ([4, kerodon]). *Let $i : A \to B$ and $j : K \to L$ be monomorphisms of simplicial sets. If one of i or j is left (resp. right) anodyne, then the induced map*

$$(B \times K) \coprod_{A \times K} (A \times L) \to B \times L$$

is left (resp. right) anodyne.

We refer the reader to Kerodon [4] for the proof.

prop:direct_tech

Proposition 4.3. *Let $f : X \to S$ be a map of simplicial sets, and $j : K \to L$ be a monomorphism of simplicial sets. Consider the induced map on the functor complexes*

$$\rho : \mathrm{Fun}(L, X) \to \mathrm{Fun}(K, X) \times_{\mathrm{Fun}(K,S)} \mathrm{Fun}(L, S).$$

(1) *If f is a left fibration, then ρ is a left fibration.*
(2) *If f is a right fibration, then ρ is a right fibration.*
(3) *If f is a left fibration and j is left anodyne, then ρ is a trivial Kan fibration.*
(4) *If f is a right fibration and j is right anodyne, then ρ is a trivial Kan fibration.*

Proof. Solving a lifting problem of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \mathrm{Fun}(L,X) \arrow[d, "f"] \\ B \arrow[r] \arrow[ur, dotted] & \mathrm{Fun}(K,X) \times_{\mathrm{Fun}(K,S)} \mathrm{Fun}(L,S) \end{tikzcd}$$

is equivalent to solving the corresponding lifting problem

$$\begin{tikzcd} (B \times K) \amalg_{(A \times K)} (A \times L) \arrow[r] \arrow[d] & X \arrow[d] \\ B \times L \arrow[r] \arrow[ur, dotted] & S. \end{tikzcd}$$

So all of the claims follow from a consideration of Lemma 4.2. □

4.2. Directional fibrations and Kan complexes.

Proposition 4.4. *A cocartesian (resp. cartesian) fibration $f : X \to S$ is a left (resp. right) fibration if and only if all of the fibers X_s, at arbitrary $s : * \to S$, are Kan complexes.*

5. Deviation into $(\infty, 2)$-categories

5.1. $(\infty, 2)$-categories.

Definition 5.1. Let X be a simplicial set. A 2-simplex $\tau : \Delta^2 \to X$ is called thin if any horn for any $n > 2$, index $0 < i < n$, and inner horn

$$\bar{\sigma} : \Lambda^n_i \to X \quad \text{with} \quad \bar{\sigma}|\Delta^{\{i-1,i,i+1\}} = \tau,$$

the lifting problem

$$\begin{tikzcd} \Lambda^n_i \arrow[r, "\bar{\sigma}"] \arrow[d] & X \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & * \end{tikzcd}$$

admits a solution.

One sees immediately that every 2-simplex in an ∞-category is thin, for example.

Recall our notation $s_i : [n] \to [n-1]$ for the weakly increasing surjection with $s_i(i) = s_i(i+1) = i$, for $0 \leq i \leq n-1$, and the corresponding degeneracies $s_i^* : \Delta^n \to \Delta^{n-1}$. We call an n-simplex $\sigma : \Delta^n \to X$ in a simplicial set *left degenerate* if σ factors through the degeneracy $s_0^* : \Delta^n \to \Delta^{n-1}$, and *right degenerate* if σ factors through the degeneracy $s_{n-1}^* : \Delta^n \to \Delta^{n-1}$.

def:infty2

Definition 5.2 ([4, 01W9, 01Y5])**.** A simplicial set X is called an $(\infty, 2)$-category if the following hold:

- (a) Any horn $\Lambda^2_1 \to X$ admits an extension to a thin 2-simplex.
- (b) Every degenerate 2-simplex in X is thin.
- (c.l) For $n > 2$, any horn $\bar{\sigma} : \Lambda^n_0 \to X$ in which the 2-simplex $\bar{\sigma}|\Delta^{\{0,1,n\}}$ is left degenerate admits an extension to an n-simplex in X.
- (c.r) For $n > 2$, any horn $\bar{\sigma}' : \Lambda^n_n \to X$ in which the 2-simplex $\bar{\sigma}'|\Delta^{\{0,n-1,n\}}$ is right degenerate admits an extension to an n-simplex in X.

A functor, or map, between $(\infty, 2)$-categories is a map of simplicial sets $F : X \to Y$ which preserves thin 2-simplices.

Remark 5.3. Having introduced this notion, let us recall that the term ∞-category is used interchangeably with the term $(\infty, 1)$-category.

If we consider an ∞-category $\mathscr{C}$, then in any horn $\Lambda^n_0 \to \mathscr{C}$ as in (c.l) the initial edge $\Delta^{\{0,1\}} \to \mathscr{C}$ is degenerate, and hence an isomorphism in $\mathscr{C}$. Hence we have the proposed completion to an n-simplex $\Delta^n \to \mathscr{C}$, by Proposition I-4.33. Similarly any horn $\Lambda^n_n \to \mathscr{C}$ as in (c.r) completes to an n-simplex as well. So we observe the following.

Lemma 5.4. *Any ∞-category is an $(\infty, 2)$-category. Furthermore, an $(\infty, 2)$-category X is an ∞-category if and only if every 2-simplex in X is thin.*

Recall that each simplex Δ^n is an ∞-category, and hence an $(\infty, 2)$-category.

Example 5.5. Since any degenerate 2-simplex in an $(\infty, 2)$-category is thin, any map of simplicial sets $* = \Delta^0 \to X$ is a map of $(\infty, 2)$-categories. Similarly, any map of simplicial sets $\Delta^1 \to X$ is a map of $(\infty, 2)$-categories.

One has the following practical check for maps between $(\infty, 2)$-categories.

prop:infty2_check

Proposition 5.6 ([4, 01YC])**.** *Let X and Y be $(\infty, 2)$-categories, and $F : X \to Y$ be a map of simplicial sets. Then F is a functor, i.e. preserves thin 2-simplexes, if and only if any horn $\Lambda^2_1 \to X$ can be competed to a thin simplex with thin image in Y.*

Idea of proof. The result is a consequence of stability of thin simplices under various conditions. Namely one establishes an inner-exchange property for thin simplices, which we recall below, and a 4-of-5 property which one can find at [4, 01XX]. □

5.2. **The pith of an $(\infty,2)$-category.**

Definition 5.7. Given an $(\infty,2)$-category X, the pith in X is the simplicial subset $X^{\mathrm{Pith}} \subseteq X$ whose simplices $\Delta^n \to X^{\mathrm{Pith}}$ consist of all simplices $\sigma : \Delta^n \to X$ in which each restriction along a 2-simplex

$$\Delta^2 \to \Delta^n \overset{\sigma}{\to} X$$

is thin.

Since functors between $(\infty,2)$-categories preserve thin simplices, by defintion, we see that any map $F : \mathscr{C} \to X$ from an ∞-category to an $(\infty,2)$-category factors through the pith.

lem:360 **Lemma 5.8** ([4, 01XL], Inner-exchange property)**.** *Consider a 3-simplex $\sigma : \Delta^3 \to X$ in an $(\infty,2)$-category, and suppose that the associated 2-simplices $\sigma|\Delta^{\{1,2,3\}}$ and $\sigma|\Delta^{\{0,1,2\}}$ are thin. Then the 2-simplex $\sigma|\Delta^{\{0,2,3\}}$ is thin if and only if the 2-simplex $\sigma|\Delta^{\{0,1,3\}}$ is thin.*

The proof employs certain facts about interior fibrations (see below), and is omitted. From Lemma 5.8 the proof of the following is immediate.

Proposition 5.9. *For any $(\infty,2)$-category X, the subcomplex X^{Pith} is an ∞-category.*

Proof. For any completion $\Delta^3 \to X$ of an inner horn $\Lambda^3_i \to X$ in which all of the associated face $\Delta^2 \to \Lambda^3_i \to X$ are thin, the final face $\Delta^{[3]\setminus\{i\}} \to X$ is also thin, by Lemma 5.8. This shows that the pith is stable under the completion of inner horns $\Lambda^3_i \to X^{\mathrm{Pith}}$. Stability under completion of all inner horns $\Lambda^n_i \to X^{\mathrm{Pith}}$ with $n > 3$ is immediate, since the horn Λ^n_i already contains all 2-faces in Δ^n in this case. Taken together with condition (a) of Definition 5.2, we see that any lifting problem

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & X^{\mathrm{Pith}} \arrow[d] \\ \Delta^n \arrow[r] & * \end{tikzcd}$$

with $0 < i < n$ admits a solution, as required. $\square$

5.3. $(\infty,2)$**-category via simplicial categories.** Recall that one can associate to any simplicial category $\underline{S}$ its associated homotopy coherent nerve $\mathrm{N}^{\mathrm{hc}}(\underline{S})$ Section I-2.7. The n-simplices in $\mathrm{N}^{\mathrm{hc}}(\underline{S})$ are simplicial functors from the path category $\mathrm{Path}[n]$. For $S = \mathrm{N}^{\mathrm{hc}}(\underline{S})$ we have in low dimension

$S[0] = \{$ objects in $\underline{S}$ $\}$

$S[1] = \{$ pairs of object (x_0, x_1) along with a map $f \in \underline{\mathrm{Hom}}_{\underline{S}}(x_0,x_1)[0]$ $\}$

$$S[2] = \left\{ \begin{array}{l} \text{triples of objects } (x_0,x_1,x_2)\text{, maps } f_{ij} : x_i \to x_j \text{ for each } i<j\text{, and} \\ \text{a 1-simplex } h : \Delta^1 \to \underline{\mathrm{Hom}}_{\underline{S}}(x_0,x_2) \text{ with } h|_0 = f_{02},\ h|_1 = f_{12}f_{01} \end{array} \right\}$$

Lemma I-2.16. We take the following theorem for granted.

thm:hc_infty2 **Theorem 5.10** ([4, 01YM])**.** *Let $\underline{S}$ be a simplicial category in which, at each pair of objects x and y in $\underline{S}$, the mapping complex $\underline{\mathrm{Hom}}_{\underline{S}}(x,y)$ is an ∞-category. Then the homotopy coherent nerve $\mathrm{N}^{\mathrm{hc}}(\underline{A})$ is an $(\infty,2)$-category.*

What we are most interested in here is the $(\infty,2)$-category of ∞-categories. Recall that for any ∞-categories $\mathscr{C}$ and $\mathscr{D}$ the simplicial set of functors $\mathrm{Fun}(\mathscr{C},\mathscr{D})$, whose simplicial are as expected

$$\mathrm{Fun}(\mathscr{C},\mathscr{D})[n] = \mathrm{Hom}_{\mathrm{sSet}}(\Delta^n \times \mathscr{C},\mathscr{D}),$$

form another ∞-category Corollar I-4.8. With these morphisms we obtain the simplicial category $\underline{\mathrm{Cat}}_\infty$ of ∞-categories and their functor categories. We note that $\underline{\mathrm{Cat}}_\infty$ is a full simplicial subcategory in the ambient category $\underline{\mathrm{sSet}}$ of simplicial sets.

5.4. **The $(\infty,2)$-category of ∞-categories.**

Theorem 5.11. *The homotopy coherent nerve*

$$\mathrm{Cat}_\infty := \mathrm{N}^{\mathrm{hc}}(\underline{\mathrm{Cat}}_\infty)$$

is an $(\infty,2)$-category.

According to the above analysis the 0-simplices in Cat_∞ are ∞-categories, the 1-simplices are functors between ∞-categories, and 2-simplices are triples of functors and a natural transformation

$$\begin{tikzcd} & \mathscr{C}_2 & & \mathscr{C}_2 \\ \mathscr{C}_1 \arrow[ur, "F_{12}"] & & {} \arrow[r, "\zeta"] & {} \\ & \mathscr{C}_0 \arrow[ul, "F_{01}"] & & \mathscr{C}_0. \arrow[uu, "F_{02}"] \end{tikzcd}$$

Definition 5.12. The $(\infty,2)$-category Cat_∞ is called the $(\infty,2)$-category of ∞-categories.

Remark 5.13. The $(\infty,2)$-category Cat_∞ is in our universe of "large" sets, which is strictly larger than our universe of "normal sized" set in which all other ∞-categories are assumed to live.

We recall our ∞-category $\mathscr{C}at_\infty$ of ∞-categories, which we obtain by restricting the morphisms $\mathrm{Fun}(\mathscr{C},\mathscr{D})$ to the associated Kan can complex $\mathrm{Fun}(\mathscr{C},\mathscr{D})^{\mathrm{Kan}}$ then applying the simplicial nerve. The inclusions of ∞-categories

$$\mathrm{Fun}(\mathscr{C},\mathscr{D})^{\mathrm{Kan}} \to \mathrm{Fun}(\mathscr{C},\mathscr{D})$$

imply an inclusion of $(\infty,2)$-categories $\mathscr{C}at_\infty \to \mathrm{Cat}_\infty$, and hence an inclusion into the pith

$$\mathscr{C}at_\infty \to (\mathrm{Cat}_\infty)^{\mathrm{Pith}}. \tag{8}$$ eq:444

By a general result one finds that this inclusion is an equality.

Proposition 5.14 ([4, 01YT])**.** *The inclusion (8) is an equality, $\mathscr{C}at_\infty = (\mathrm{Cat}_\infty)^{\mathrm{Pith}}$.*

5.5. **Interior fibrations.**

def:interior **Definition 5.15.** A map of simplicial sets $q : X \to S$ is called an interior fibration if the following hold:

(a) At each 0-simplex x in X, the identity $id_x : x \to x$ is both q-cartesian and cocartesian.

(b) For any lifting problem

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta^n \arrow[r, "\sigma"'] \arrow[ur, dotted] & S \end{tikzcd} \tag{9}$$

eq:400

in which $0 < i < n$ and $\sigma|\Delta^{\{i-1,i,i\}}$ is thin in S, (9) admits a solution.

It is clear that if $f : S' \to S$ is a map of simplicial sets which preserves thin 2-simplices, and the diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "q'"'] & X \arrow[d, "q"] \\ S' \arrow[r, "f"] & S \end{tikzcd}$$

is a pullback diagram of simplicial sets in which q is an interior fibration, then the map $q' : X' \to S'$ is an interior fibration as well.

One also observes the following.

lem:495

Lemma 5.16. *If S is an $(\infty,2)$-category, and $q : X \to S$ is an interior fibration, then X is also an $(\infty,2)$-category and q is a functor between $(\infty,2)$-categories.*

Proof. One sees via the lifting property for q that any 2-simplex $\Delta^2 \to X$ which has thin image in S is thin in X. From this we see that any horn $\Lambda^2_1 \to X$ can be completed to a thin 2-simplex in X. One obtains this thin completion by lifting a thin completion $\Delta^2 \to S$. We are left to prove that any appropriate degenerate horn $\Lambda^n_0 \to X$ or $\Lambda^n_n \to X$, at $n > 2$, completes to an n-simplex. However this follows from the fact the fact that identity maps in X are both q-cartesian and cocartesian, and the fact that the corresponding horns in S admit completions. We now see that X is an $(\infty,2)$-category. One sees that q is a functor, i.e. preserves thin 2-simplices, by applying Proposition 5.6. □

As we see in the above proof, given an interior fibration $q : X \to S$ over an $(\infty,2)$-category, one can detect thin simplices in X by considering their images in S along q.

lem:505

Lemma 5.17. *If $q : X \to S$ is an interior fibration then a 2-simplex in X is thin if and only if its image in S is thin.*

cor:interior_pullback

Corollary 5.18. *Consider a pullback diagram*

$$\begin{tikzcd} Z \arrow[r, "p_2"] \arrow[d, "p_1"'] & X \arrow[d, "q"] \\ Y \arrow[r, "F"'] & S \end{tikzcd}$$

in which q is an interior fibration and F is a map between $(\infty,2)$-categories. Then Z is an $(\infty,2)$-category, p_1 is an interior fibration, and p_2 is a map of $(\infty,2)$-categories.

Proof. The fact that Z is an $(\infty,2)$-category and p_1 is a map of $(\infty,2)$-categories follows by Lemma 5.16. As for p_1, we consider a thin 2-simplex $\Delta^2 \to Z$, and note that its image in Y is thin. Hence its image in S is thin, and so its image in X is thin by Lemma 5.17. It follows that p_2 is a map of $(\infty, 2)$-categories, by definition. $\square$

We are especially interested in the fiberings of interior fibrations over ∞-categories.

lem:502

Lemma 5.19. *Let $\mathscr{C}$ be an ∞-category. A map of simplicial sets $q : X \to \mathscr{C}$ is an interior fibration if and only if it is an inner fibration.*

Proof. If q is an interior fibration then it is an inner fibration since all 2-simplices in $\mathscr{C}$ are thin. Conversely, if q is an inner fibration then X is an ∞-category and q is therefore an inner fibration between ∞-categories. Condition (a) of Definition 5.15 now follows from the fact that the identity in an ∞-category is an isomorphism, and an application of Proposition I-4.33. $\square$

One combines Lemma 5.19 with the above discussion of fiber products to obtain the following corollary.

Corollary 5.20. *Consider an interior fibration $q : X \to S$ over an $(\infty, 2)$-category S.*

(a) *For any ∞-category $\mathscr{C}$, and any functor of $(\infty, 2)$-categories $\mathscr{C} \to S$, the fiber product $X \times_S \mathscr{C}$ is an ∞-category. Furthermore, the projection $X \times_S \mathscr{C} \to \mathscr{C}$ is an inner fibration.*
(b) *At each point $s : * \to S$ the fiber X_s is an ∞-category.*

cor:interior_pith

Corollary 5.21. *If $q : X \to S$ is an interior fibration over an $(\infty, 2)$-category S then the diagram*

$$\begin{tikzcd} X^{\mathrm{Pith}} \arrow[r] \arrow[d] & X \arrow[d] \\ S^{\mathrm{Pith}} \arrow[r] & S \end{tikzcd}$$

is a pullback diagram, and the map $X^{\mathrm{Pith}} \to S^{\mathrm{Pith}}$ is an interior fibration.

Proof. In this case the pullback $X \times_S S^{\mathrm{Pith}}$ is an ∞-category and the projection to X is a map of $(\infty, 2)$-categories. So the identification

$$X^{\mathrm{Pith}} = X \times_S S^{\mathrm{Pith}}$$

follows via an application of the universal property for the pullback and the universal property for the pith. $\square$

5.6. **Undercategories and overcategories and pointed ∞-categories.** In the $(\infty, 2)$-setting we can define overcategories and undercategories exactly as in the ∞-setting. Namely, for a map of simplicial sets $p : K \to X$ the overcategory $X_{p/}$ is the simplicial set with n-simplices provided by the join

$$X_{p/}[n] := \mathrm{Hom}_{\mathrm{sSet}}(K \star \Delta^n, X)_p,$$

and similarly for the undercategory

$$X_{/p}[n] := \mathrm{Hom}_{\mathrm{sSet}}(\Delta^n \star K, X)_p$$

Section I-4.6. In the case in which X is an ∞-category, we saw that the forgetful functors

$$\begin{tikzcd} X_{p/} \arrow[dr] & & X_{/p} \arrow[dl] \\ & X & \end{tikzcd}$$

obtained by restricting along the inclusions $\Delta^n \to \Delta^n \star K$ and $\Delta^n \to K \star \Delta^n$ are directional fibrations, and in particular isofibrations. We have a similar result in the 2-categorical context.

prop:504 **Proposition 5.22** ([4, 01WU]). *Let X be an $(\infty, 2)$-category and $p : K \to X$ be a map of simplicial sets. The the forgetful maps*

$$X_{p/} \to X \quad \textit{and} \quad X_{/p} \to X$$

are both interior fibrations.

At this point we'll begin to leave many of the details unaccounted for. In particular, we direct the reader to the original text [4] for the details on Proposition 5.22. In any case, we record some corollaries.

Corollary 5.23. *For an $(\infty, 2)$-category X and a diagram $p : K \to X$, the simplical sets $X_{p/}$ and $X_{/p}$ are $(\infty, 2)$-category and the forgetful maps are both functors between $(\infty, 2)$-categories.*

cor:601 **Corollary 5.24.** *Let $p : K \to X$ be a map from a simplicial set into an $(\infty, 2)$-category. At any point $x : * \to X$ the fibers $(X_{p/})_x$ and $(X_{/p})_x$ are both ∞-categories.*

We apply this corollary in the case where the diagram p is a point $x : * \to X$ to obtain mapping categories for any $(\infty, 2)$-category X.

Definition 5.25. For any $(\infty, 2)$-category X, and objects $x, y : * \to X$, the left pinched mapping ∞-category is the fiber

$$\mathrm{Hom}^{\mathrm{L}}_X(x, y) := (X_{x/}) \times_X \{y\}.$$

Similarly,the right pinched mapping ∞-category is the fiber

$$\mathrm{Hom}^{\mathrm{R}}_X(x, y) = \{x\} \times_X (X_{/y}).$$

As with any interior fibration, we can restrict the forgetful functor to the piths to obtain inner fibrations of ∞-categories

$$(X_{p/})^{\mathrm{Pith}} \to X^{\mathrm{Pith}} \quad \text{and} \quad (X_{/p})^{\mathrm{Pith}} \to X^{\mathrm{Pith}}.$$

In this particular instance one can observe a stronger characterization of these functors.

prop:over_cartesian **Proposition 5.26** ([4, 01YE]). *For X and $p : K \to X$ as above, the restrictions of the forgetful functors*

$$(X_{p/})^{\mathrm{Pith}} \to X^{\mathrm{Pith}} \quad \textit{and} \quad (X_{/p})^{\mathrm{Pith}} \to X^{\mathrm{Pith}}.$$

are, respectively, a cocartesian fibration and a cartesian fibration.

One might view this result in analogy with the ∞-setting, where the forgetful functors were observed to be right and left fibrations Corollary I-4.27.

5.7. **Mapping categories in the homotopy coherent nerve.** Let $\underline{S}$ be a simplicial category whose morphism complexes are weak Kan complexes, and let S be the homotopy coherent nerve, $S = \mathrm{N}^{\mathrm{hc}}(\underline{S})$. We recall that S is an $(\infty, 2)$-category in this case. By an abuse of notation take

$$\underline{\mathrm{Hom}}_S(x, y) = \underline{\mathrm{Hom}}_{\underline{S}}(x, y)$$

for any given pair of objects in S. We construct a map of simplicial sets

$$\theta : \underline{\mathrm{Hom}}_S(x,y) \to \mathrm{Hom}^{\mathrm{L}}_S(x,y)$$

[4, 01LD] which is subsequently found to be an equivalence of ∞-categories.

To begin, for any simplicial set K we consider the simplicial category $E(K)$ with objects x_- and x_+ and morphisms

$$\mathrm{Hom}_{E(K)}(x_-,x_-) = \mathrm{Hom}_{E(K)}(x_+,x_+) = * \quad\text{and}\quad \mathrm{Hom}_{E(K)}(x_-,x_+) = K.$$

We consider the $(n+1)$-simplex $\{-1\}\star\Delta^n \cong \Delta^{n+1}$ and the simplicial path category $\mathrm{Path}(\{-1\}\star\Delta^n)$ whose morphisms are given by the nerves

$$\mathrm{Hom}_{\mathrm{Path}(\{-1\}\star\Delta^n)}(l,m) = \mathrm{N}(\mathrm{Subsets}^{\mathrm{op}}_{l,m})$$

where $\mathrm{Subsets}_{l,m}$ is the partially ordered set of subsets $S \subseteq [n]$ with $\min S = l$ and $\max S = m$, ordered by inclusion.

At each integer n we have a simiplicial functor

$$\theta^*_n : \mathrm{Path}(\{-1\}\star\Delta^n) \to E\Delta^n$$

which is define on objects by taking $\theta^*_n(-1) = x_-$, and $\theta^*_n(i) = x_+$ for all $i \geq 0$, and defined on morphisms by the simplicial map

$$\theta^*_n : \mathrm{Hom}_{\mathrm{Path}(\{-1\}\star\Delta^n)}(l,m) = \mathrm{N}(\mathrm{Subsets}^{\mathrm{op}}_{l,m}) \to \mathrm{Hom}_E(x_-,x_+) = \Delta^n = \mathrm{N}([n])$$

associated to the functor $\mathrm{Subsets}^{\mathrm{op}}_{l,m} \to [n]$ which sends each subset $S = \{l < s_1 < \dots s_r < m\}$ to s_1 and each inclusion $S' \supseteq S$ to the inequality $s'_1 \leq s_1$.

For objects x and y in S, n-simplices in $\underline{\mathrm{Hom}}_S(x,y)$ are identified with simplicial functors $\mathrm{Fun}_{\mathrm{sCat}}(E\Delta^n, \underline{S})$ in the fiber over (x,y) in $\mathrm{Fun}(E\emptyset, \underline{S})$. Each such functor now defined an $(n+1)$-simplex in S via a consideration of the identification

$$S[n+1] = \mathrm{Fun}_{\mathrm{sCat}}(\mathrm{Path}(\{-1\}\star\Delta^n), \underline{S})$$

and restricting along θ^*_n. One sees, by the definiton of θ^*_n that this associated $(n+1)$-simplex has initial vertex x and all other vertices y, and restricts trivially to $\Delta^n \subseteq \{-1\}\star\Delta^n$. So we obtain a map of sets

$$\theta_n : \underline{\mathrm{Hom}}_S(x,y)[n] \to (S_{x/}) \times_S \{y\} = \mathrm{Hom}^{\mathrm{L}}_S(x,y)[n],$$
$$(f : E\Delta^n \to \underline{S}) \mapsto (f\theta^*_n : \mathrm{Path}\{-1\}\star\Delta^n \to \underline{S}).$$

One observes directly that any increasing function $t : [n] \to [n']$ produces a commutative diagram

$$\begin{tikzcd}
\mathrm{Path}(\{-1\}\star\Delta^n) \arrow[r, "\theta_n"] \arrow[d, "t_*"'] & E\Delta^n \arrow[d, "t_*"] \\
\mathrm{Path}(\{-1\}\star\Delta^{n'}) \arrow[r, "\theta_{n'}"'] & E\Delta^{n'},
\end{tikzcd}$$

from which we see that the θ_n assemble into a map of simplicial sets, or a map of ∞-categories,

$$\theta : \underline{\mathrm{Hom}}_S(x,y) \to \mathrm{Hom}^{\mathrm{L}}_S(x,y).$$

thm:pinched_simplicial

Theorem 5.27 ([4, 01LG])**.** *Let $\underline{S}$ be a simplicial category whose morphism complexes are ∞-categories. Take $S = \mathrm{N}^{\mathrm{hc}}(\underline{S})$. For any objects $x, y : * \to S$ there is a natural equivalence of ∞-categories*

$$\theta : \underline{\mathrm{Hom}}_S(x,y) \to \mathrm{Hom}^{\mathrm{L}}_S(x,y).$$

We do not cover the details, and refer the reader to the text [4].

cor:simplicial_pullback

Corollary 5.28. *Take $\underline{S}$ and S as above. For any pair of points $x, y : * \to S$ there is a categorical pullback diagram*

$$\begin{tikzcd} \underline{\mathrm{Hom}}_S(x,y) \arrow[r, "\theta"] \arrow[d] & (S_{x/})^{\mathrm{Pith}} \arrow[d] \\ * \arrow[r, "y"'] & S^{\mathrm{Pith}}. \end{tikzcd} \tag{10}$$

eq:704

Proof. By Corollary 5.18 and Proposition 5.22 the projection map $\mathrm{Hom}^{\mathrm{L}}_S(x,y) \to S_{x/}$ has image in the Pith $(S_{x/})^{\mathrm{Pith}}$. Applying this fact in conjunction with Corollary 5.21, we observe a pullback diagram of ∞-categories

$$\begin{tikzcd} \mathrm{Hom}^{\mathrm{L}}_S(x,y) \arrow[r, "\theta"] \arrow[d] & (S_{x/})^{\mathrm{Pith}} \arrow[d] \\ * \arrow[r, "y"] & S^{\mathrm{Pith}}. \end{tikzcd}$$

in which the right-hand map is an inner fibration. This diagram is additionally a categorical pullback square by Corollary I-5.22 and Proposition 5.26. Since $\theta : \underline{\mathrm{Hom}}_S(x,y) \to \mathrm{Hom}^{\mathrm{L}}_S(x,y)$ is an equivalence of ∞-categories it follows that the corresponding diagram (10) is a categorical pullback square as well (see Proposition I-5.23). $\square$

sect:transport

6. Transport I: Classifying functors

6.1. **Preliminary discussion.** In analogy with the plain categorical setting, we claim that cocartesian fibrations $q : \mathscr{E} \to \mathscr{C}$ over a given ∞-category are "the same thing" as functors into the ∞-category of ∞-categories $F : \mathscr{C} \to \mathscr{C}at_\infty$. In our imaginations, the functor F should evaluate as the fibers $F(x) \cong \mathscr{E}_x$ and the image of a given map $\alpha : x \to y$ should be some kind of pushforward functor $\alpha_* : \mathscr{E}_x \to \mathscr{E}_y$ which "moves along" cartesian lifts $\widetilde{\alpha} : \widetilde{x} \to \widetilde{y}$, so that $\alpha_*(\widetilde{x}) \cong \widetilde{y}$.

Of course one can not simply construct the desired functor $F : \mathscr{C} \to \mathscr{C}at_\infty$ by hand. We (or rather, Lurie) instead proceed(s) by establishing a universal cocartesian fibration over ∞-categories

$$U : \mathscr{Z} \to \mathscr{C}at_\infty.$$

It is then shown that each cocartesian fibration is realized as a (categorical) pullback along U,

$$\begin{tikzcd} \mathscr{E} \arrow[r] \arrow[d, "q"'] & \mathscr{Z} \arrow[d] \\ \mathscr{C} \arrow[r, "F"'] & \mathscr{C}at_\infty, \end{tikzcd}$$

and furthermore that the space of such pullback diagrams assembles into a contractible space. We refer to this uniquely determined functor F as the covariant transport functor along q, or as the functor which classifies q. One obtains a completely similar analysis of cartesian fibrations and classification via an applications of the opposite involution.

In this section we outline the above construction. Unlike at other points in this text we are not especially concerned with (all of) the technical details, and seek only to provide a coherent narrative which explains clearly what's going on and how this stuff works.

We begin with a detour into $(\infty, 2)$-categories. We then construct the universal cocartesian fibration $\mathscr{X}$ via a certain "category of objects", and explain how each fiber $\mathscr{X}_{\mathscr{E}}$ over a given ∞-category $\mathscr{E} : * \to \mathscr{C}at_\infty$ reproduces $\mathscr{E}$ itself, up to equivalence. We define the space $\mathscr{T}(q)$ of classifying diagrams and recall the contractibility of this space from [4]. The section concludes with a description of the pushforward functors α_* appearing the transport F.

sect:univ_fib

6.2. **Categories with objects and the universal cocartesian fibration.** From the $(\infty, 2)$-category Cat_∞ we can produce the simplicial of pointed ∞-categories

$$(\mathrm{Cat}_\infty)_* := (\mathrm{Cat}_\infty)_{\Delta^0/}.$$

By Proposition 5.22 the simplicial set $(\mathrm{Cat}_\infty)_*$ is an $(\infty, 2)$-category and the forgetful functor $(\mathrm{Cat}_\infty)_* \to \mathrm{Cat}_\infty$ is a interior fibration.

Definition 6.1. The ∞-category of ∞-categories with a distinguished object is the pith of the $(\infty, 2)$-category of pointed ∞-categories,

$$\mathscr{P}.\mathscr{C}at_\infty := \big((\mathrm{Cat}_\infty)_*\big)^{\mathrm{Pith}}.$$

Remark 6.2. The $\mathscr{P}$ suffix stands for "pointed", though we heed the warning from [4, 020W] and do not label this ∞-category as such.

Remark 6.3. There is a comparison functor $\mathscr{P}.\mathscr{C}at_\infty \to (\mathscr{C}at_\infty)_{*/}$ which is, apparently, bijective on objects. However this map is not bijective on 1-morphisms so that it is not an isomorphism [4, 020Z].

Via an application of Corollary 5.21 we see that the forgetful functor restricts to provide a pullback diagram

$$\begin{tikzcd} \mathscr{P}.\mathscr{C}at_\infty \arrow[r] \arrow[d] & (\mathrm{Cat}_\infty)_{*/} \arrow[d] \\ \mathscr{C}at_\infty \arrow[r] & \mathrm{Cat}_\infty . \end{tikzcd}$$

The forgetful functor $\mathscr{P}.\mathscr{C}at_\infty \to \mathscr{C}at_\infty$ is furthermore seen to be a cocartesian fibration in this case, via an application of Proposition 5.26. We record this result.

Proposition 6.4 ([4, 0213])**.** *The forgetful functor $\mathscr{P}.\mathscr{C}at_\infty \to \mathscr{C}at_\infty$ is a cocartesian fibration.*

We call the above forgetful functor the *universal cocartesian fibration*, for reasons which will be apparent shortly.

Definition 6.5. We let $\mathrm{univ} : \mathscr{P}.\mathscr{C}at_\infty \to \mathscr{C}at_\infty$ denote the cocartesian fibration induced by the forgetful functor $(\mathrm{Cat}_\infty)_* \to \mathrm{Cat}_\infty$, as considered above.

At a baseline, objects in the ∞-category $\mathscr{P}.\mathscr{C}at_\infty$ are simply pointed ∞-categories $x : * \to \mathscr{C}$. A map in $\mathscr{P}.\mathscr{C}at_\infty$ between two objects $(\mathscr{C}, x)$ and $(\mathscr{D}, y)$ consists of a functor $F : \mathscr{C} \to \mathscr{D}$ along with a map

$$\alpha : \Delta^1 \to \mathscr{D}$$

with $\alpha(0) = F(x)$ and $\alpha(1) = y$, i.e. a choice of a morphism $\alpha : F(x) \to y$ in $\mathscr{D}$. A 2-simplex in $\mathscr{P}.\mathscr{C}at_\infty$ consists of a choice of the following data:

- Three functors with moprhisms $F_{ij} : \mathscr{C}_i \to \mathscr{C}_j$ for $1 \le i < j \le 3$ and a natural isomorphism $\beta : F_{23}F_{12} \to F_{13}$,
- Morphisms $\alpha_{ij} : F_{ij}(x_i) \to x_j$ in $\mathscr{C}_j$.
- Morphisms $\alpha^1_{123} : F_{32}F_{12}(x_1) \to F_{13}(x_1)$, $\alpha^2_{123} : F_{32}F_{12}(x_2) \to F_{23}(x_2)$, and $\alpha^3_{123} : F_{23}F_{12}(x_1) \to x_3$ in $\mathscr{C}_3$ with $\alpha^1 = \beta(x_1)$.
- Two 2-simplices $\sigma_k : \Delta^2 \to \mathscr{C}_3$ with $\sigma_k|_{\Delta^{\{0,1\}}} = \alpha^k_{123}$, $\sigma_k|_{\Delta^{\{1,2\}}} = \alpha_{k3}$, and $\sigma_k|_{\Delta^{\{0,3\}}} = \alpha^3_{123}$.

We have the following characterization of univ-cocartesian edges.

prop:univ_cocart

Proposition 6.6 ([4, 026X,01YE]). *A morphism $(F,\alpha) : (\mathscr{C}, x) \to (\mathscr{D}, y)$ in $\mathscr{P}.\mathscr{C}at_\infty$ is cocartesian for the universal fibration* univ : $\mathscr{P}.\mathscr{C}at_\infty$ *if and only if the underlying map $\alpha : F(x) \to y$ is an isomoprhism in $\mathscr{D}$.*

6.3. **A remark on notation.** Our $(\infty, 2)$-category Cat_∞ is the $(\infty, 2)$-category denoted by a bold $\mathcal{QC}$ in [4, 020K]. Our $(\mathrm{Cat}_\infty)_{*/}$ is the $(\infty, 2)$-category denoted by a bold $\mathcal{QC}_{\mathrm{Obj}}$ in [4, 0210]. The associated piths, which we've denoted $\mathscr{C}at_\infty$ and $\mathscr{P}.\mathscr{C}at_\infty$ respectively, are the non-bolded ∞-categories $\mathcal{QC}$ and $\mathcal{QC}_{\mathrm{Obj}}$ in [4].

6.4. **Fibers of the map** univ : $\mathscr{P}.\mathscr{C}at_\infty \to \mathscr{C}at_\infty$**.** In considering the universal cocartesian fibration $\mathscr{P}.\mathscr{C}at_\infty \to \mathscr{C}at$, any point $e : \Delta^0 \to \mathscr{C}at$ corresponds to an ∞-category $\mathscr{E} = e(0)$ and we have the pullback

$$\mathscr{P}.\mathscr{C}at_\infty \times_{\mathscr{C}at_\infty} \{e\}$$

which is some other ∞-category. Now, objects in this fiber are simply maps of ∞-categories $* \to \mathscr{E}$, and hence are identified with objects in $\mathscr{E}$. Similarly, 1-simplices in the fiber are identified 2-simplices in the ∞-category of ∞-categories

$$\begin{tikzcd} & * \arrow[dl, "x"'] \arrow[dr, "y"] & \\ \mathscr{E} \arrow[rr, "="'] & & \mathscr{E}. \end{tikzcd}$$

These are, by definition, natural transformations $\alpha \in \mathrm{Hom}_{\mathrm{sSet}}(\Delta^1, \mathscr{E})$ with $\alpha|_0 = x$ and $\alpha|_1 = y$, i.e. 1-simplices $\alpha : x \to y$. So we observe an identification of 1-skeleta

$$\mathscr{E}[\le 1] \;=\; \mathscr{P}.\mathscr{C}at_\infty \times_{\mathscr{C}at_\infty} \{e\}[\le 1].$$

As an application of Theorem 5.27 and Corollary 5.28, we see that this direct identification of simplices in low-dimension expands to an equivalence of ∞-categories which calculates the fiber.

prop:univ_fibs

Proposition 6.7. *For any ∞-category $\mathscr{E}$, which we can understand as a point $\mathscr{E} : * \to \mathscr{C}at_\infty$, we have a categorical pullback square*

$$\begin{tikzcd} \mathscr{E} \arrow[r] \arrow[d] & \mathscr{P}.\mathscr{C}at_\infty \arrow[d] \\ * \arrow[r, "\mathscr{E}"'] & \mathscr{C}at_\infty, \end{tikzcd}$$

and a corresponding equivalence between ∞-categories $\theta : \mathscr{E} \xrightarrow{\sim} \mathscr{P}.\mathscr{C}at_\infty \times_{\mathscr{C}at_\infty} \{\mathscr{E}\}$.

6.5. **Covariant transport: classifying cocartesian fibrations.**

Definition 6.8. Let $q : \mathscr{E} \to \mathscr{C}$ be a cocartesian fibration of ∞-categories. We say a functor $F : \mathscr{C} \to \mathscr{C}at_\infty$ classifies the cocartesian fibration q if the functors q and F fit into a categorical pullback diagram

$$\begin{tikzcd} \mathscr{E} \arrow[r, "\widetilde{F}"] \arrow[d, "q"'] & \mathscr{P}.\mathscr{C}at_\infty \arrow[d, "\mathrm{univ}"] \\ \mathscr{C} \arrow[r, "F"'] & \mathscr{C}at_\infty. \end{tikzcd} \tag{11}$$

eq:672

In this case we say the above diagram witnesses F as a (covariant) transport functor along q.

Remark 6.9. The term "classifies" is used with some frequency in the works [2, 3]. However, [4] seems to prefer the term "transport representation" for a functor F as above. We will usually just refer to F as a, or the, transport functor for q.

We have an alternate characterization of transport functors via the fiber product $\mathscr{C} \times_{\mathscr{C}at_\infty} \mathscr{P}.\mathscr{C}at_\infty$.

Lemma 6.10. *Let $q : \mathscr{E} \to \mathscr{C}$ be a cocartesian fibration. A diagram (14) witnesses $F : \mathscr{C} \to \mathscr{C}at_\infty$ as transport along q if and only if the induced map to the fiber product*

$$\mathscr{E} \to \mathscr{C} \times_{\mathscr{C}at_\infty} \mathscr{P}.\mathscr{C}at_\infty$$

is an equivalence of cocartesian fibrations over $\mathscr{C}$.

Proof. The fact that the map to the fiber is an equivalence follows by Proposition I-5.23. It follows by Proposition **??** and the diagram

$$\begin{tikzcd} \mathscr{E} \arrow[rr, "\sim"] \arrow[dr, "q"'] & & \mathscr{C} \times_{\mathscr{C}at_\infty} \mathscr{P}.\mathscr{C}at_\infty \arrow[dl] \\ & \mathscr{C} & \end{tikzcd}$$

that the equivalence in question is an equivalence of cocartesian fibrations. $\square$

The fiber product considered above is often denoted

$$\int_{\mathscr{C}} F := \mathscr{C} \times_{\mathscr{C}at_\infty} \mathscr{P}.\mathscr{C}at_\infty,$$

so that any functors $F : \mathscr{C} \to \mathscr{C}at_\infty$ determines an associated cocartesian fibration $\int_{\mathscr{C}} F \to \mathscr{C}$. By definition, F is a transport functor for $q : \mathscr{E} \to \mathscr{C}$ if there is an equivalence

$$\mathscr{E} \overset{\sim}{\to} \int_{\mathscr{C}} F \ \text{ in the overcategory } \mathrm{sSet}_{/\mathscr{C}}.$$

As a corollary to Proposition 6.6 we can characterize cocartesian edges in the fibration $\int_{\mathscr{C}} F$.

cor:c_univ_cocart **Corollary 6.11.** *Consider a functor of ∞-categories $F : \mathscr{C} \to \mathscr{C}at_\infty$ and $q : \int_{\mathscr{C}} F \to \mathscr{C}$ be the corresponding cocartesian fibration. An edge*

$$(\zeta, \alpha) : (x, a : * \to F(x)) \to (y, b : * \to F(y))$$

in $\int_{\mathscr{C}} F$ is q-cocartesian if and only if the underlying morphism $\alpha : F(\zeta)(a) \to b$ is an isomorphism in the ∞-category $F(b)$.

One observes that transport functors are stable under restriction.

lem:687 **Lemma 6.12.** *Suppose we have a pullback square*

$$\begin{tikzcd} \mathscr{E}' \arrow[r, "g"] \arrow[d, "q'"'] & \mathscr{E} \arrow[d, "q"] \\ \mathscr{C}' \arrow[r, "f"'] & \mathscr{C} \end{tikzcd} \tag{12}$$ eq:689

in which q and q' are are cocartesian fibrations between ∞-categories, and consider a diagram of the form (14) which witnesses a functor $F : \mathscr{C} \to \mathscr{C}at_\infty$ as transport along q. Then for $\widetilde{F}' = \widetilde{F}g$ and $F' = Ff$, the diagram

$$\begin{tikzcd} \mathscr{E}' \arrow[r, "\widetilde{F}'"] \arrow[d, "q'"'] & \mathscr{P}.\mathscr{C}at_\infty \arrow[d] \\ \mathscr{C}' \arrow[r, "F'"'] & \mathscr{C}at_\infty. \end{tikzcd}$$

witnesses F' as a transport functor along q'.

Proof. Since q is a cocartesian fibration, and in particular an isofibration, the diagram (12) is a categorical pullback square Corollary I-5.22. The result now follows from the fact that categorical pullback squares are stable under composition [4, 033J]. $\square$

Given a cocartesian fibration we can now consider the simplicial subset in the functor category

$$\mathrm{Fun}(\mathscr{C}, \mathscr{C}at_\infty) \times_{\mathrm{Fun}(\mathscr{E}, \mathscr{C}at_\infty)} \mathrm{Fun}(\mathscr{E}, \mathscr{P}.\mathscr{C}at_\infty) \tag{13}$$ eq:710

which consists of diagrams witnessing transport for a given cocartesian fibration $q : \mathscr{E} \to \mathscr{C}$.

Definition 6.13. For a given cocartesian fibration $q : \mathscr{E} \to \mathscr{C}$, we let $\mathscr{T}(q)$ denote the simplicial subset in the fiber product (13) whose simplices correspond to diagrams

$$\begin{tikzcd} \Delta^n \times \mathscr{E} \arrow[r, "\widetilde{F}"] \arrow[d, "\Delta^n \times q"'] & \mathscr{P}.\mathscr{C}at_\infty \arrow[d] \\ \Delta^n \times \mathscr{C} \arrow[r, "F"'] & \mathscr{C}at_\infty. \end{tikzcd}$$

which witness F as a covariant transport functor along $\Delta^n \times q$.

Stability of such diagrams under restriction (Lemma 6.12) assures us that $\mathscr{T}(q)$ is in fact a simplicial subset in the given fiber product.

thm:transport **Theorem 6.14** (Universality theorem [4, 02SC])**.** *For any cocartesian fibration $q : \mathscr{E} \to \mathscr{C}$, the terminal map $\mathscr{T}(q) \to *$ is a trivial Kan fibration.*

This result says that any cocartesian fibration q admits a covariant transport functor $F : \mathscr{C} \to \mathscr{C}at_\infty$, and that this functor is uniquely determined up to a contractible space of choices.

6.6. **Transport for cartesian fibrations.** Given any cartesian fibration $p : \mathscr{E} \to \mathscr{C}$, we have the associated cocartesian fibration $p^{\mathrm{op}} : \mathscr{E}^{\mathrm{op}} \to \mathscr{C}^{\mathrm{op}}$. So our analysis of classifying functors for cocartesian fibrations dualizes in the obvious ways to provide an analysis of classifying functors for cartesian fibrations.

Definition 6.15. Let $p : \mathscr{E} \to \mathscr{C}$ be a cartesian fibration of ∞-categories. We say a functor $F : \mathscr{C}^{\mathrm{op}} \to \mathscr{C}at_\infty$ classifies the cartesian fibration q if the functors p^{op} and F fit into a categorical pullback diagram

$$\begin{tikzcd} \mathscr{E}^{\mathrm{op}} \arrow[r, "\widetilde{F}"] \arrow[d, "q"'] & \mathscr{P}.\mathscr{C}at_\infty \arrow[d, "\mathrm{univ}"] \\ \mathscr{C}^{\mathrm{op}} \arrow[r, "F"'] & \mathscr{C}at_\infty . \end{tikzcd} \tag{14}$$ eq:672

In this case we say the above diagram witnesses F as a (contravariant) transport functor along p.

Gives a cartesian fibration $p : \mathscr{E} \to \mathscr{C}$, we define the space of transport functors with witness in the obvious way $\mathscr{T}(p) := \mathscr{T}(p^{\mathrm{op}})$. Theorem 6.14 implies contractibility of this space immediately.

thm:uniq_transp **Theorem 6.16** (Contravariant universality)**.** *For any cartesian fibration $p : \mathscr{E} \to \mathscr{C}$, the terminal map $\mathscr{T}(p) \to *$ from the space of transport functors along p is a trivial Kan fibration.*

Again this establishes both the existence and uniqueness of contravariant transport.

6.7. **Classification of left and right fibrations.** We have the simplicial subcategory $\underline{\mathrm{Kan}} \to \underline{\mathrm{Cat}}_\infty$ and subsequent simplicial subset $\mathscr{K}an \subseteq \mathrm{Cat}_\infty$. This simplicial subset is the full $(\infty, 2)$-subcategory whose objects are precisely those ∞-categories which are Kan complexes, and so the inclusion preserves thin 2-simplices. We now have the full $(\infty, 2)$-subcategory $\mathscr{K}an_{*/} \to (\mathrm{Cat}_\infty)_{*/}$ of pointed Kan complexes and the pullback diagram

$$\begin{tikzcd} \mathscr{K}an_{*/} \arrow[r] \arrow[d] & (\mathrm{Cat}_\infty)_{*/} \arrow[d] \\ \mathscr{K}an \arrow[r] & \mathrm{Cat}_\infty \end{tikzcd}$$

which restricts to a pullback diagram into the piths

$$\begin{tikzcd} \mathscr{K}an_{*/} \arrow[r] \arrow[d] & (\mathrm{Cat}_\infty)_{*/} \arrow[d] \\ \mathscr{K}an \arrow[r] & \mathrm{Cat}_\infty \end{tikzcd} .$$

We recall that the map $\mathscr{K}an_{*/} \to \mathscr{K}an$ is a left fibration, by Corollary I-**??**.

prop:kan_transp **Proposition 6.17.** *A cocartesian fibration $q : \mathscr{E} \to \mathscr{C}$ is a left fibration if and only if the corresponding transport functor $F : \mathscr{C} \to \mathscr{C}at_\infty$ has image in $\mathscr{K}an$. Similarly, a cartesian fibration $p : \mathscr{E} \to \mathscr{C}$ is a right fibration if and only if the corresponding transport functor $G : \mathscr{C}^{\mathrm{op}} \to \mathscr{C}at_\infty$ has image in $\mathscr{K}an$.*

Proof. By Proposition **??**, a cocartesian (resp. cartesian) fibration $\mathscr{E} \to \mathscr{C}$ is a left (resp. right) fibration if and only if its fibers over object sin $\mathscr{C}$ are Kan complexes. So the result follows by the calculation of the fibers of the pullback fibration $\int_{\mathscr{C}} F \to \mathscr{C}$ provided in Proposition 6.7. □

This proposition tells us that any left fibration $q : \mathscr{E} \to \mathscr{C}$ fits into a categorical pullback square

$$\begin{tikzcd} \mathscr{E} \arrow[r] \arrow[d, "q"'] & \mathscr{K}an_{*/} \arrow[d] \\ \mathscr{C} \arrow[r, "F"'] & \mathscr{K}an \end{tikzcd}$$

for some functor F, and that F, considered as a functor into $\mathscr{C}at_\infty$, is the transport functor for q. In this way left fibrations are classified by maps into the ∞-category of Kan complexes. One obtains similar statements for right fibrations by applying the opposite functor.

sect:weight_nerv

6.8. **Weighted nerves.** In the following two subsections we provide a complete and explicit description of the fibration $\int_{\mathbb{A}} F \to \mathbb{A}$ in the case where $\mathbb{A}$ is a discrete category and F comes from a simplicial functor. We subsequently obtain an explicit classification of cocartesian fibrations over discrete categories, up to equivalence.

For a discrete category $\mathbb{A}$, which we view as an ∞-category with unique horn fillings, we let $\mathrm{Plain}(\mathbb{A})$ denote its corresponding "classical category", which we might view as a simplicial category with completely degenerate mapping complexes.

Definition 6.18. Let $F : \mathrm{Plain}(\mathbb{A}) \to \underline{\mathrm{Cat}}_\infty^+$ be a functor from a discrete category to the simplicial category of ∞-categories. We define the weighted nerve $\mathrm{N}^F(\mathbb{A})$ to be the simplicial set with n-simplices

$$\mathrm{N}^F(\mathbb{A})[n] = (\sigma : \Delta^n \to \mathbb{A}, \tau_i : \Delta^i \to F(a_i) : 0 \le i \le n).$$

Here a_i is the object $\sigma(i)$ in $\mathbb{A}$, and the τ_i are required to fit into diagrams

$$\begin{tikzcd} \Delta^i \arrow[r] \arrow[d, "\tau_i"'] & \Delta^{i+1} \arrow[d, "\tau_{i+1}"] \\ F(a_i) \arrow[r, "F(i<i+1)"'] & F(a_{i+1}) \end{tikzcd}$$

at all $i < n$, where $\Delta^i \to \Delta^{i+1}$ is induced by the inclusion $[i] \to [i+1]$.

Below we may write τ for the tuple $\tau = \{\tau_i : \Delta^i \to F(a_i) : 0 \le i \le n\}$, so that an n-simplex in the weighted nerve appears as a pair (σ, τ).

As for the restriction maps, given a non-decreasing function $t : [m] \to [n]$ the restriction function t^* sends a pair (σ, τ) to the pair of the m-simplex σt^* in $\mathbb{A}$ with the tuple of simplices

$$\tau'_j : \Delta^j \to \Delta^{t(j)} \to F(a_{t(j)})$$

where the first map is simply given by restricting t to a function $t : [j] \to [t(j)]$. We have the obvious forgetful map

$$\mathrm{N}^F(\mathbb{A}) \to \mathbb{A}, \quad (\sigma, \tau) \mapsto \sigma.$$

At the lowest levels, one sees that vertices $* \to \mathrm{N}^F(\mathbb{A})$ consist of a choice of object $\bar{x}$ in $\mathbb{A}$ and an object $x : * \to F(\bar{x})$ in the ∞-category over $\bar{x}$. An edge

$\Delta^1 \to \mathrm{N}^F(\mathbb{A})$ over a morphism $\alpha : \bar{x} \to \bar{y}$ in $\mathbb{A}$ consists of a choice of objects $x : * \to F(\bar{x})$ and $y : * \to F(y)$, along with with a morphism $\xi : F(\alpha)(x) \to y$ in the ∞-category $F(y)$.

Remark 6.19. We note that any functor from a discrete category $\mathrm{Plain}(\mathbb{A})$ into a simplicial category $\underline{S}$ is simply a functor into the underlying discrete category $\underline{S}[0]$. So our functor $F : \mathrm{Plain}(\mathbb{A}) \to \underline{\mathrm{Cat}}_\infty^+$ simply specifies an $\mathbb{A}$-shaped diagram of ∞-categories.

ex:point_nerv

Example 6.20 ([4, 025Y])**.** Let $\mathscr{C} : * \to \underline{\mathrm{Cat}}_\infty$ be the function which just picks an ∞-category $\mathscr{C}$. Then the first term in an n-simplex (σ, τ) in $\mathrm{N}^{\mathscr{C}}(*)$ is trivial, and the second term determines an increasing sequence of simplices in $\mathscr{C}$,

$$\begin{tikzcd}
\Delta^0 \arrow[r] \arrow[drr, "\tau_0"'] & \Delta^1 \arrow[r] \arrow[dr, "\tau_1"] & \cdots \arrow[r] & \Delta^{n-1} \arrow[r] \arrow[dl, "\tau_{n-1}"'] & \Delta^n \arrow[dll, "\tau_n"] \\
& & \mathscr{C} & &
\end{tikzcd}$$

So we see that the functions $\phi[n] : \mathrm{N}^{\mathscr{C}}(*)[n] \to \mathscr{C}[n]$, $(\sigma, \tau) \mapsto \tau_n$ determine a isomorphism of simplicial sets $\mathrm{N}^{\mathscr{C}}(*) \overset{\cong}{\to} \mathscr{C}$.

Example 6.21. For the constant functor $* : \mathrm{Plain}(\mathbb{A}) \to \underline{\mathrm{Cat}}_\infty^+$, which sends all object to the terminal space $*$, the second factor in any n-simplex (σ, τ) in the weighted nerve is completely determined, so that we obtain an isomorphism $\mathrm{N}^*(\mathbb{A}) \cong \mathbb{A}$.

We note that the weighted nerve construction is functorial in the obvious ways. Namely, if $F : \mathrm{Plain}(\mathbb{A}) \to \underline{\mathrm{Cat}}_\infty^+$ is a functor and $\phi : \mathbb{B} \to \mathbb{A}$ is a functor between ∞-categories, then we have a map of simplicial sets

$$\mathrm{N}(\phi) : \mathrm{N}^{F\phi}(\mathbb{B}) \to \mathrm{N}^F(\mathbb{A})$$

which just composes simplices in the first factor $\sigma \mapsto \sigma\phi$, and is the identity in the second factor $\tau \mapsto \tau$. The following calculation is immediate.

lem:1277

Lemma 6.22. *Given a functor between discrete categories $\phi : \mathbb{B} \to \mathbb{A}$ and a functor $F : \mathrm{Plain}(\mathbb{A}) \to \underline{\mathrm{Cat}}_\infty^+$, the weighted nerves fit into a pullback diagram*

$$\begin{tikzcd}
\mathrm{N}^{F\phi}(\mathbb{B}) \arrow[r, "\mathrm{N}(\phi)"] \arrow[d] & \mathrm{N}^F(\mathbb{A}) \arrow[d] \\
\mathbb{B} \arrow[r, "\phi"'] & \mathbb{A}.
\end{tikzcd}$$

Our main objective for the remainder of the subsection is to prove the following result.

prop:cocart_relnev

Proposition 6.23. *Let $F, G : \mathrm{Plain}(\mathbb{A}) \to \underline{\mathrm{Cat}}_\infty^+$ be functors, and $\xi : F \to G$ be a natural transformation. Suppose that at each object a in $\mathbb{A}$ the morphism $\xi(a) : F(a) \to G(a)$ is a cocartesian fibration, and that for any morphism $t : a \to b$ the map $F(t) : F(a) \to F(b)$ preserves ξ-cocartesian maps. Then the following hold:*

1. *The induced map $\mathrm{N}^\xi : \mathrm{N}^F(\mathbb{A}) \to \mathrm{N}^G(\mathbb{A})$ is a cocartesian fibration.*
2. *An edge $\lambda : F(\alpha)(x) \to y$ in $\mathrm{N}^F(\mathbb{A})$, over an edge $t : a \to b$ in $\mathbb{A}$, is N^ξ-cocartesian if and only if the underlying map $\lambda : \Delta^1 \to F(b)$ is $\xi(b)$-cocartesian.*

Here N^ξ is the obvious map, i.e. the map which sends an n-simplex $(\sigma, \tau_i : 0 \leq i \leq n)$ in $\mathrm{N}^F(\mathbb{A})$ to $\sigma : \Delta^n \to \mathbb{A}$ paired with the composites $\xi(a_i)\tau_i : \Delta^i \to F(a_i) \to G(a_i)$. In the case of the constant functor $G : \mathrm{Plain}(\mathbb{A}) \to \underline{\mathrm{Cat}}^+_\infty$, with $G(a) = *$ at all a in $\mathbb{A}$, Proposition 6.23 appears as follows.

Corollary 6.24. *The forgetful functor $q : \mathrm{N}^F(\mathbb{A}) \to \mathbb{A}$ is a cocartesian fibration, and for any map $t : a \to b$ in $\mathbb{A}$, a morphism $\lambda : F(\alpha)(x) \to y$ over t in $\mathrm{N}^F(\mathbb{A})$ is q-cocartesian if and only if λ is an isomorphism in $F(b)$.*

For the proof of Proposition 6.23 we employ a relative join construction $\mathscr{C} \star_{\mathscr{T}} \mathscr{D}$. Here we consider the ∞-categories over $\mathscr{T}$, we have the unique map

$$\Delta^1 \times \mathscr{T} \to \mathscr{T} \star \mathscr{T}$$

with $\{0\} \times \mathscr{T}$ mapping identically to $\mathscr{T} \star \emptyset = \mathscr{T}$ and $\{1\} \times \mathscr{T}$ identically to $\emptyset \star \mathscr{T} = \mathscr{T}$, and we consider the fiber product

$$\mathscr{C} \star_{\mathscr{T}} \mathscr{D} := (\mathscr{C} \star \mathscr{D}) \times_{\mathscr{T} \star \mathscr{T}} (\Delta^1 \times \mathscr{T}).$$

lem:1307 **Lemma 6.25.** *Let $\mathscr{C} \to \mathscr{T}$ and $\mathscr{D} \to \mathscr{T}$ be maps of ∞-categories. The map $\mathscr{C} \star_{\mathscr{T}} \mathscr{D} \to \mathscr{C} \star \mathscr{D}$ is an inner fibration.*

Proof. It suffices to show that the map $\Delta^1 \times \mathscr{T} \to \mathscr{T} \star \mathscr{T}$ is an inner fibration. This follows from the fact that the composite

$$\Delta^1 \times \mathscr{T} \to \mathscr{T} \star \mathscr{T} \to \Delta^0 \star \Delta^0 = \Delta^1,$$

which one sees is just the projection onto the first factor, is an inner fibration. Indeed, for a given lifting problem for an inner horn

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & \Delta^1 \times \mathscr{T} \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & \mathscr{T} \star \mathscr{T} \end{tikzcd} \tag{15}$$ eq:1317

either Δ^n has image in one of the two $\mathscr{T}$ factors in $\mathscr{T} \star \mathscr{T}$, in which case the problem has a unique solution, or else a solution to the associated lifting problem

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & \Delta^1 \times \mathscr{T} \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & \mathscr{T} \star \mathscr{T} \end{tikzcd}$$

solves (15). $\square$

It is relatively easy to see that the map from the usual join $\mathscr{C} \star \mathscr{D} \to \Delta^0 \star \Delta^0 = \Delta^1$ is an inner fibration, so that $\mathscr{C} \star \mathscr{D}$ is in particular an ∞-category. It follows from Lemma 6.25 that the relative join is an ∞-category as well.

Corollary 6.26. *For ∞-categories $\mathscr{C}$ and $\mathscr{D}$ over another ∞-category $\mathscr{T}$, the relative join $\mathscr{C} \star_{\mathscr{T}} \mathscr{D}$ is also an ∞-category.*

To describe the relative join, we have the projection

$$\mathscr{C} \star_{\mathscr{T}} \mathscr{D} \to \mathscr{C} \star \mathscr{D} \to \Delta^1$$

and one the fibers over $\{0\}$ and $\{1\}$ are copies of $\mathscr{C}$ and $\mathscr{D}$ respectively. Furthermore, for any map $\alpha : \Delta^1 \to \mathscr{C} \star_{\mathscr{T}} \mathscr{D}$ with $\alpha|_{\{0\}} = x$ in $\mathscr{C}$ and $\alpha|_{\{1\}} = y$ in $\mathscr{D}$ one calculates

$$\mathrm{Hom}_{\mathscr{C}\star_{\mathscr{T}}\mathscr{D}}(x, y) = \mathrm{Hom}_{\Delta^1\times\mathscr{T}}((0, \bar{x}), (1, \bar{y})),$$

where $\bar{x}$ and $\bar{y}$ are the images of x and y in $\mathscr{T}$, respectively. This latter mapping space is the product

$$\mathrm{Hom}_{\Delta^1}(0, 1) \times \mathrm{Hom}_{\mathscr{T}}(\bar{x}, \bar{y}) = \{*\} \times \mathrm{Hom}_{\mathscr{T}}(\bar{x}, \bar{y}).$$

So in total we calculate

$$\mathrm{Hom}_{\mathscr{C}\star_{\mathscr{T}}\mathscr{D}}(x, y) = \begin{cases} \mathrm{Hom}_{\mathscr{C}}(x, y) & \text{if } x \text{ and } y \text{ are in } \mathscr{C} \\ \mathrm{Hom}_{\mathscr{D}}(x, y) & \text{if } x \text{ and } y \text{ are in } \mathscr{D} \\ \mathrm{Hom}_{\mathscr{T}}(\bar{x}, \bar{y}) & \text{if } x \text{ is in } \mathscr{C} \text{ and } y \text{ in } \mathscr{D}. \end{cases}$$

The mapping spaces vanish when x is in $\mathscr{D}$ and y is in $\mathscr{C}$, as there are simply no 1-simplices which begin at x and end in y by the definition of the join.

The proof of Proposition 6.23 relies on the following generic observation.

lem:reljoin_cocart

Lemma 6.27 ([4, 02RH]). *Consider a diagram of ∞-categories*

$$\begin{tikzcd} \mathscr{C} \arrow[r, "F"] \arrow[d, "q"'] & \mathscr{T} \arrow[d, "p"] \\ \mathscr{D} \arrow[r] & \mathscr{V} \end{tikzcd}$$

in which q and p are cocartesian fibrations and F preserved cocartesian morphisms. The the induced map $z : \mathscr{C} \star_{\mathscr{T}} \mathscr{T} \to \mathscr{D} \star_{\mathscr{V}} \mathscr{V}$ is also a cocartesian fibration. Furthermore, a map $\alpha : x \to y$ in $\mathscr{C} \star_{\mathscr{T}} \mathscr{T}$ is z-cocartesian if and only if one of the following two conditions holds:

(a) *Both x and y are in $\mathscr{C}$, and α is q-cocartesian.*
(b) *One of x or y is not in $\mathscr{C}$, and the image of α along the map $\mathscr{C} \star_{\mathscr{T}} \mathscr{T} \to \Delta^1 \times \mathscr{T} \to \mathscr{T}$ is p-cocartesian.*

While the proof is not extraordinarily complications, but requires one to touch on various points regarding the relative join. We refer the reader to [4], and in particular [4, 0241], for the details.

We now provide our argument for Proposition 6.23.

Proof of Proposition 6.23. Via Lemma 6.22 we reduce to the case of an n-simplex $\mathbb{A} = \Delta^n$, and we proceed by induction to observe the conclusions of Proposition 6.23 over such a base. In the case $n = 0$, the functors $F : G : * \to \underline{\mathrm{Cat}}^+_\infty$ just choose ∞-categories and the transformation ξ is just a map of ∞-categories $\xi : F(*) \to G(*)$ which is specifically a cocartesian fibration. Under the identifications

$$\mathrm{N}^F(*) = F(*) \quad \text{and} \quad \mathrm{N}^G(*) = G(*)$$

of Example 6.20 we have $\mathrm{N}^\xi = \xi$. Since ξ is a cocartesian fibration by hypothesis we obtain condition (1). Condition (2) demands that a map $\alpha : x \to y$ in $F(*)$ is N^ξ-cocartesian if and only if it is ξ-cocartesian, which is a tautology and in particular is true.

Suppose now that the result holds over $\mathbb{A}_0 = \Delta^{n-1}$, and consider $\mathbb{A} = \Delta^n = \mathbb{A}_0 \star \{n\}$. We take two functors

$$F, G : \mathrm{Plain}(\mathbb{A}) \to \underline{\mathrm{Cat}}^+_\infty$$

and a transformation $\xi : F \to G$ as prescribed. We have natural decompositions

$$\mathrm{N}^{F_0}(\mathbb{A}) \cong \mathrm{N}^F(\mathbb{A}_0) \star_{F(n)} F(n) \quad \text{and} \quad \mathrm{N}^{G_0}(\mathbb{A}) \cong \mathrm{N}^G(\mathbb{A}_0) \star_{G(n)} G(n)$$

where the map $\mathrm{N}^F(\mathbb{A}) \to F(n)$ and $\mathrm{N}^G(\mathbb{A}) \to G(n)$ are provided by the structural morphisms $F(m \leq n) : F(m) \to F(n)$ and $G(m \leq n) : G(m) \to G(n)$. Here also F_0 and G_0 are the obvious restrictions.

The map $N^{\xi_0} : \mathrm{N}^{F_0}(\mathbb{A}_0) \to \mathrm{N}^{G_0}(\mathbb{A}_0)$ is a cocartesian fibration by our induction hypothesis, and $\xi(n) : F(n) \to G(n)$ is a cocartesian fibration by assumption. Also by assumption the map $N^{F_0}(\mathbb{A}_0) \to F(n)$ preserves cocartesian edges. It follows that the map in question

$$\mathrm{N}^\xi = \mathrm{N}^\xi \star_{\xi(n)} \xi(n) : \mathrm{N}^F(\mathbb{A}_0) \star_{F(n)} F(n) \to \mathrm{N}^G(\mathbb{A}_0) \star_{G(n)} G(n)$$

is a cocartesian fibration by Lemma 6.27. Lemma 6.27 also verifies the proposed description of cocartesian morphisms in $\mathrm{N}^F(\mathbb{A})$. ☐

sect:weighted_fib

6.9. **Fibrations over discrete categories via weighted nerves.**

thm:weight_nerv_univ

Theorem 6.28 ([4, 027J]). *Let $\mathbb{A}$ be a discrete category and $F : \mathrm{Plain}(\mathbb{A}) \to \underline{\mathrm{Cat}}_\infty^+$ be a functor between simplicial categories. Then for the associated functor between ∞-categories $\mathrm{N}^{\mathrm{hc}}(F) : \mathbb{A} \to \mathscr{C}at_\infty$ we have an equivalence of cocartesian fibrations*

$$\begin{tikzcd} \mathrm{N}^F(\mathbb{A}) \arrow[rr, "\sim", "\mu"'] \arrow[dr] & & \int_{\mathbb{A}} \mathrm{N}^{\mathrm{hc}}(F) \arrow[dl] \\ & \mathbb{A} & \end{tikzcd} .$$

We describe the functor $\mu : \mathrm{N}^F(\mathbb{A}) \to \int_{\mathbb{A}} \mathrm{N}^{\mathrm{hc}}(F)$, but leave the verification that it is an equivalence to the text [4]. Our job here is simple–given an n-simplex in the weighted nerve we need to produce an n-simplex in the ∞-category $\int_{\mathbb{A}} \mathrm{N}^{\mathrm{hc}}(F)$.

Let us take $f = \mathrm{N}^{\mathrm{hc}}(F)$ for simplicity

$$f = \mathrm{N}^{\mathrm{hc}}(F) : \mathbb{A} \to \mathscr{C}at_\infty.$$

We consider an n-simplex $\omega : \Delta^n \to \mathrm{N}^F(\mathbb{A})$, which is specified by a pair $(\sigma : \Delta^n \to \mathbb{A}, \tau_i : \Delta^i \to F(a_i))$, where $a_i = \sigma(i)$. From ω we produce a diagram

$$\omega' : \{-1\} \star \Delta^n \to \mathrm{N}^{\mathrm{hc}}(\underline{\mathrm{Cat}}_\infty) = \mathrm{Cat}_\infty$$

with $\omega'|_{\Delta^n} = f(\sigma)$ and $\omega'(-1) = *$. Such a diagram corresponds to an n-simplex in the undercategory $(\mathrm{Cat}_\infty)_{*/}$ so that we have a diagram

$$\begin{tikzcd} \Delta^n \arrow[r, "\omega'"] \arrow[d] & (\mathrm{Cat}_\infty)_{*/} \arrow[d] \\ \mathbb{A} \arrow[r, "\mathrm{N}^{\mathrm{hc}}(F)"'] & \mathrm{Cat}_\infty \end{tikzcd} .$$

Since $\mathbb{A}$ has image in the pith $\mathscr{C}at_\infty = (\mathrm{Cat}_\infty)^{\mathrm{Pith}}$, any such lift ω' also has image in the pith $\mathscr{P}.\mathscr{C}at_\infty = (\mathrm{Cat}_\infty)_{*/})^{\mathrm{Pith}}$ by Corollary 5.21 and Proposition 5.22. Hence ω' defines a map into the fiber product $\Delta^n \to \int_{\mathbb{A}} \mathrm{N}^{\mathrm{hc}}(F)$, and we denote this n-simplex simply ω' by an abuse of notation.

Let us proceed with the construction of the associated simplex $\omega' : \{-1\} \star \Delta^n \to \mathrm{Cat}_\infty$. We define, following [4, 027A], ω' as the unique simplicial functor

$$\omega' : \mathrm{Path}(\{-1\} \star \Delta^n) \to \underline{\mathrm{Cat}}_\infty$$

with

$$\omega'|_{\mathrm{Path}(\Delta^n)} = F \circ \sigma : \mathrm{Path}(\Delta^n) \to \mathrm{Plain}(\mathbb{A}) \to \underline{\mathrm{Cat}}_\infty$$

and $\omega'(-1) = *$ and each map

$$\underline{\mathrm{Hom}}(-1, i) = \mathrm{N}(\mathrm{Subsets}^{\mathrm{op}}_{-1,i}) \to \mathrm{Fun}(*, F(a_i)) = F(a_i)$$

given as the composite

$$\mathrm{N}(\mathrm{Subsets}^{\mathrm{op}}_{-1,i}) \xrightarrow{\rho} \Delta^i \xrightarrow{\tau_i} F(a_i),$$

where ρ is induced by the map of partially ordered sets

$$\mathrm{Subsets}^{\mathrm{op}}_{-1,i} \to \Delta^i, \quad S \mapsto \min(S - \{-1\}).$$

One can check that ω' is in fact a well-defined simplicial functor, and given a diagram

$$\begin{tikzcd}
\Delta^n \arrow[r, "\omega"] & \mathrm{N}^F(\mathbb{A}) \\
\Delta^m \arrow[u] \arrow[r, "\nu"'] & \mathrm{N}^F(\mathbb{A}) \arrow[u, "="']
\end{tikzcd}$$

one checks directly from the definition the corresponding diagram

$$\begin{tikzcd}
\mathrm{Path}\{-1\} \star \Delta^n \arrow[r, "\omega'"] & \underline{\mathrm{Cat}}_\infty \\
\mathrm{Path}\{-1\} \star \Delta^m \arrow[u] \arrow[r, "\nu'"'] & \underline{\mathrm{Cat}}_\infty. \arrow[u, "="']
\end{tikzcd}$$

In this way we obtain maps $\mathrm{N}^F(\mathbb{A})[n] \to (\int_{\mathbb{A}} f)[n]$ which assemble into a map of simplicial sets $\mu : \mathrm{N}^F(\mathbb{A}) \to \int_A f$.

Finally, it is argued in [4, Proof of] that the map μ is an equivalence by first noting that the induced maps on each fiber

$$\mu_a : F(a) \cong \mathrm{N}^F(\mathbb{A})_a \to (\int_{\mathbb{A}} f)_a$$

is an equivalence, which is certainly expected from the fiber calculations of 6.7, and als that μ preserved cocartesian edges. This latter point is determined via the explicit descriptions of cocartesian edges in each ∞-category provided in Proposition 6.23 and Corollary 6.11.

lem:simplicial_fun

Lemma 6.29. *Let $\mathbb{A}$ be a discrete category. Any functor $F' : \mathbb{A} \to \mathscr{C}at_\infty$ is isomorphic to one of the form $\mathrm{N}^{\mathrm{hc}}(F)$ for F a functor of simplicial categories $F : \mathrm{Plain}(\mathbb{A}) \to \underline{\mathrm{Cat}}^+_\infty$.*

Proof. Let $\mathrm{Cat}_{\mathrm{Kan}}$ be the category of simplicial categories which are enriched in Kan complexes. By [2, Theorem 2.2.5.1] the homotopy coherent nerve provides an equivalence of homotopy categories

$$\mathrm{h}\,\mathrm{N}^{\mathrm{hc}} : \mathrm{h}\,\mathrm{Cat}_{\mathrm{Kan}} \xrightarrow{\sim} \mathrm{h}\,\mathscr{C}at_\infty.$$

In particular, every functor between homotopy coherent nerves $F' : \mathrm{N}^{\mathrm{hc}}(\underline{A}) \to \mathrm{N}^{\mathrm{hc}}(\underline{B})$ is, up to natural isomorphism, identified with $\mathrm{N}^{\mathrm{hc}}(F)$ for some simplicial functor F. □

From Theorem 6.28 in conjunction with a result from Section 8 below, Theorem 8.19 and Proposition 8.23, we obtain a classification of all cocartesian fibrations over a discrete category.

`thm:discrete_cocart`

Theorem 6.30. *Let $\mathbb{A}$ be a discrete category and $q : \mathscr{E} \to \mathbb{A}$ be an arbitrary cocartesian fibration.*

`sect:htf`

7. Transport II: Homotopy representations

7.1. **Homotopy transport representations.** Consider a cocartesian fibration $q : X \to S$ and an edge $\alpha : s \to t$ in S. We then have the fibers X_s and X_t over these respective points, both of which are ∞-categories.

We consider the diagram

$$\begin{tikzcd} \{0\} \times X_s \arrow[r] \arrow[d] & X \arrow[d, "q"] \\ \Delta^1 \times X_s \arrow[r] & S \end{tikzcd}$$

where the top arrow is the inclusion and the bottom arrow is the composite of the projection with α,

$$\Delta^1 \times X_s \to \Delta^1 \times \{s\} \overset{ev_\alpha}{\to} S.$$

By Theorem 2.7 the above diagram is split by a transformation

$$\xi_\alpha : \Delta^1 \times X_s \to X$$

which has $\alpha_!|_{\{0\}\times X_s}$ equal to the inclusion and has

$$\xi_\alpha|_{\Delta^1 \times \{s'\}} : \Delta^1 \times \{s'\} \to X$$

a q-cocartesian morphism over α. In particular, the restriction at 1 produces a functor

$$\alpha_! := \xi_\alpha|_{\{1\}\times X_s} : X_s \to X_t.$$

Furthermore, this transformation ξ_α is uniquely determined up to a contractible space of choices, so that $\alpha_!$ is similarly uniquely determined up to a contractible space as well.

Definition 7.1. Given a cocartesian fibration $q : X \to S$ and any edge $\alpha : s \to t$ in the base, we let $\alpha_! : X_s \to X_t$ denote the uniquely determined functor which comes equipped with a cocartesian transformation ξ_α over α, as above. We call $\alpha_! : X_s \to X_t$ the homotopy transport functor over α.

`prop:1297`

Proposition 7.2. *Let $X \to S$ be a cocartesian fibration. Suppose that we have a 2-simplex $A : \Delta^2 \to S$ and take $\alpha_{ij} = A|_{\Delta^{\{i,j\}}} : s_i \to s_j$. Then there is an isomorphism*

$$(\alpha_{02})_! \cong (\alpha_{12})_!(\alpha_{01})_!$$

in $\mathrm{Fun}(X_{s_0}, X_{s_2})$.

Proof. Take $X_i = X_{s_i}$. Consider a diagram $\widetilde{A} : \Delta^1 \times \Delta^1 \to S$ which appears as

$$\begin{tikzcd}
s_2 \arrow[r, "id"] & s_2 \\
s_0 \arrow[u, "\alpha_{02}"] \arrow[ur, "\alpha_{02}"] \arrow[r, "\alpha_{01}"'] & s_1, \arrow[u, "\alpha_{12}"']
\end{tikzcd}$$

which we might obtain by expanding A for example. Then we have a lifting problem

$$\begin{tikzcd}
\{0\} \times \Delta^1 \times X_0 \arrow[r, "\xi_{\alpha_{02}}"] \arrow[d] & X \arrow[d] \\
\Delta^1 \times \Delta^1 \times X_0 \arrow[r, "\widetilde{A} \times \{s_0\}"'] \arrow[ur, dotted] & S
\end{tikzcd}$$

and can take $\xi_A : \Delta^1 \times \Delta^1 \times X_0 \to X$ to be the unique cocartesian solution. Restricting ξ_A to $\Delta^1 \times \{0\} \times X_0$ provides a cocartesian lift of α_{01} and so is identified with $\xi_{\alpha_{01}}$, and similarly restricting ξ_A to the edge $\Delta^1 \times \{1\} \times X_0$ is identified with $id_{\alpha_{02}}$. By the 2-of-3 property for q-cocartesian maps, we also see that the restriction of ξ_A to the diagonal $\Delta^1 \times X_0$ is a cocartesian lift of α_{02} and so is identified with $\xi_{\alpha_{02}}$ as well.

We have only the edge $\{1\} \times \Delta^1 \times X_0 \to X$ to be identified. By the 2-of-3 property again we see that F_A provides a cocartesian solution to the diagram

$$\begin{tikzcd}
\{1\} \times \{0\} \times X_0 \arrow[r, "(\alpha_{01})_!"'] \arrow[d] \arrow[rr, bend left, "\xi_A|_{\ldots}"] & \{0\} \times X_1 \arrow[r] \arrow[d, dashed] & X \arrow[d] \\
\{1\} \times \Delta^1 \times X_0 \arrow[r, "id \times (\alpha_{01})_!"'] \arrow[urr, "\xi_A|_{\ldots}"'] & \Delta^1 \times X_1 \arrow[r] & S.
\end{tikzcd}$$

However we have the alternate cocartesian lift

$$\begin{tikzcd}
\{1\} \times \{0\} \times X_0 \arrow[r, "(\alpha_{01})_!"'] \arrow[d] & \{0\} \times X_1 \arrow[r] \arrow[d, dashed] & X \arrow[d] \\
\{1\} \times \Delta^1 \times X_0 \arrow[r, "id \times (\alpha_{01})_!"'] & \Delta^1 \times X_1 \arrow[r] \arrow[ur, "\xi_{\alpha_{12}}"] & S
\end{tikzcd}$$

so that there is a unique isomorphism

$$\xi_A|_{\{1\} \times \Delta^1 \times X_0} \cong \xi_{\alpha_{12}}(id \times (\alpha_{01})_!).$$

So we restrict further to $\{1\} \times \{1\} \times X_0$ to get

$$(\alpha_{02})_! = \xi_{\alpha_{02}}|_{\{1\}} \cong \xi_A|_{\{1\} \times \{1\}} \cong \xi_{\alpha_{12}}(id \times (\alpha_{01})_!)|_{\{1\}} = (\alpha_{12})_!(\alpha_{01})_!.$$

$\square$

Corollary 7.3. *Let $q : \mathscr{E} \to \mathscr{C}$ be a cocartesian fibration over an ∞-category $\mathscr{C}$. The functors $\alpha_! : X_s \to X_t$ assemble into a functor into the homotopy category of ∞-categories $\bar{q}_! : \mathrm{h}\mathscr{C} \to \mathrm{h}\mathscr{C}at_\infty$.*

Definition 7.4. Let $q : \mathscr{E} \to \mathscr{C}$ be a cocartesian fibration over an ∞-category $\mathscr{C}$. The homotopy transport representation for q is the functor on homotopy categories

$$\bar{q}_! : \mathrm{h}\mathscr{C} \to \mathrm{h}\mathscr{C}at_\infty$$

is the functor whose value at each object $x : * \to \mathscr{C}$ is the fiber $\bar{q}_!(x) = \mathscr{E}_x$, and whose value at any morphism $\alpha : x \to y$ in $\mathscr{C}$ is the associated homotopy transport functor $\alpha_! : \mathscr{E}_x \to \mathscr{E}_y$.

More generally, we call any functor $F : \mathrm{h}\mathscr{C} \to \mathrm{h}\mathscr{C}at_\infty$ which comes equipped with a natural isomorphism $\zeta : F \overset{\sim}{\to} \bar{q}_!$ a homotopy transport representation for q.

One observes that the homotopy transport representation is natural in diagrams of cocartesian fibrations.

lem:1366 **Lemma 7.5.** *Let*

$$\begin{tikzcd} Y \arrow[r, "F"] \arrow[d] & X \arrow[d] \\ T \arrow[r, "f"'] & S \end{tikzcd}$$

be a diagram of cocartesian fibrations, i.e. a diagram in which F preserves cocartesian maps. Then for any edge $\alpha : t \to t'$ in T the homotopy transport functors fit into a diagram

$$\begin{tikzcd} Y_t \arrow[r, "\alpha_!"] \arrow[d, "F"'] & Y_{t'} \arrow[d, "F"] \\ X_{f(t)} \arrow[r, "f(\alpha)_!"'] & X_{f(t')} \end{tikzcd}$$

in $\mathscr{C}at_\infty$.

Proof. Follows from the fact that both $F\xi_\alpha$ and $\xi_{f(\alpha)}f$ provide cocartesian lifts for the diagram

$$\begin{tikzcd} \{0\} \times Y_t \arrow[r] \arrow[d] & \{0\} \times X_{f(t)} \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta^1 \times Y_t \arrow[r] & \Delta^1 \times X_{f(t)} \arrow[r] & S. \end{tikzcd}$$

□

7.2. **The $\mathrm{h}\mathscr{K}an$-enriched category of ∞-categories.** Let $\underline{A}$ be a simplicial category whose morphism complexes are all Kan complexes. Via the functor to the homotopy category $\pi : \mathrm{Kan} \to \mathrm{h}\mathscr{K}an$ we obtain a new category $\pi\underline{A}$ which is enriched in $\mathrm{h}\mathscr{K}an$. (Here we note that the usual product of Kan complexes endows $\mathrm{h}\mathscr{K}an$ with a unique symmetric monoidal structure under which the projection $\pi : \mathrm{Kan} \to \mathrm{h}\mathscr{K}an$ is symmetric monoidal.) We compare this Kan enriched category to the Kan enriched category $\pi\,\mathrm{N}^{\mathrm{hc}}(\underline{A})$ obtained via the mapping spaces in the homotopy coherent nerve and their associated composition functions of Section I-8.

prop:pi_hcnerve **Proposition 7.6** ([4, 02LN])**.** *Let $\underline{A}$ be a simplicial category whose morphism compelxes are Kan complexes, and let $\mathscr{A} = \mathrm{N}^{\mathrm{hc}}(\underline{A})$ denote its associated ∞-category. Then the natural equivalences*

$$\underline{\mathrm{Hom}}_{\underline{A}}(x,y) \overset{\sim}{\to} \mathrm{Hom}^{\mathrm{L}}_{\mathscr{A}}(x,y) \overset{\sim}{\to} \mathrm{Hom}_{\mathscr{A}}(x,y)$$

supplied by Theorems 5.27 and I-9.3 define an equivalence of $\mathrm{h}\mathscr{K}an$*-enriched categories* $\pi\underline{A} \to \pi\mathscr{A}$*.*

`sect:ehtf` 7.3. **Enriched homotopy transport.** Given a simplicial set S and vertices $s, t : * \to S$ we take $\mathrm{Hom}_S(s,t) = \mathrm{Fun}(\Delta^1, S) \times_{S\times S} \{(s,t)\}$. For any cocartesian fibration $q : X \to S$ we consider the evaluation map

$$\Delta^1 \times \mathrm{Hom}_S(s,t) \times X_s \to \Delta^1 \times \mathrm{Fun}(\Delta^1, S) \times \{s\} \overset{ev}{\to} S$$

and the diagram

$$\begin{tikzcd}
\{0\} \times \mathrm{Hom}_S(s,t) \times X_s \arrow[r] \arrow[d] & X \arrow[d] \\
\Delta^1 \times \mathrm{Hom}_S(s,t) \times X_s \arrow[r] & S.
\end{tikzcd}$$

The top map is the composite of the projection to X_s with the inclusion $X_s \to X$. By Theorem 2.7 there is a unique functor

$$\xi : \Delta^1 \times \mathrm{Hom}_S(s,t) \times X_s \to X$$

which splits the above diagram and sends each edge $\Delta^1 \times \{(\alpha, x)\}$ to a q-cocartesian morphism in X. The uniqueness claim of Theorem 2.7 tells us that, and each α in $\mathrm{Hom}_S(s,t)$, ξ restricts to the transformation ξ_α appearing in the defintion of the homotopy transport functor $\alpha_!$. So the map

$$\xi|_{\{1\}} : \mathrm{Hom}_S(s,t) \times X_s \to X_t$$

provides a paramterized family of morphisms whose fibers are the homotopy transport functors $\alpha_!$.

`def:pht` **Definition 7.7.** Given a cocartesian fibration $q : X \to S$, we call the functor $\xi|_{\{1\}} : \mathrm{Hom}_S(s,t) \times X_s \to X_t$ constructed above the parametrized homotopy transport functor for q.

Note that we can view $\xi|_{\{1\}}$ as a functor

$$q_{s,t} : \mathrm{Hom}_S(s,t) \to \mathrm{Fun}(X_s, X_t)$$

via adjunction. In the case that S is an ∞-category, we note that $\widetilde{q}_{s,t}$ is a functor between ∞-categories.

Consider now a cocartesian fibration $\mathscr{E} \to \mathscr{C}$ over an ∞-category $\mathscr{C}$. Recall our notation $\pi\mathscr{C}$ for the $\mathrm{h}\mathscr{K}an$-enriched category with By similar arguments to those employed in our analysis of the homotopy transport functors $\alpha_! : X_s \to X_t$, one sees that these maps assemble into a functor of $\mathrm{h}\mathscr{K}an$-enriched categories

$$\pi\mathscr{C} \to \pi\underline{\mathrm{Cat}}_\infty^+$$

which lifts the homotopy transport functor $\bar{q}_!$ of Section 7.1.

Definition 7.8. Given a cocartesian fibration $q : \mathscr{E} \to \mathscr{C}$ over an ∞-category $\mathscr{C}$, we let

$$q_! : \pi\mathscr{C} \to \pi\mathscr{C}at_\infty$$

denote the $\mathrm{h}\mathscr{K}an$-enriched functor whose value at any object $x : * \to \mathscr{C}$ is the fiber $\mathscr{E}_x$, and whose values on morphisms

$$q_! : \mathrm{Hom}_{\mathscr{C}}(x,y) \to \mathrm{Fun}(\mathscr{E}_x, \mathscr{E}_y)^{\mathrm{Kan}} \cong \mathrm{Hom}_{\mathscr{C}at_\infty}(\mathscr{E}_x, \mathscr{E}_y)$$

are the functors induced by parametrized homotopy transport. We call $q_!$ the enriched homotopy transport representation associated to q.

We generally, we call any enriched functor $F : \pi\mathscr{C} \to \pi\mathscr{C}at_\infty$ which comes equipped with a natural isomorphism $\zeta : F \xrightarrow{\sim} q_!$ a homotopy transport representation for q.

Note that here we've employed the natural identification $\pi\underline{\mathrm{Cat}}^+_\infty \cong \mathscr{C}at_\infty$ provided by Proposition 7.6 here when we replace the functor categories with the Hom-spaces for $\mathscr{C}at_\infty$.

One again sees that the enriched homotopy transport representation is natural

lem:enriched_pullback **Lemma 7.9.** *Consider a diagram of ∞-categories*

$$\begin{tikzcd} \mathscr{E} \arrow[r, "G"] \arrow[d, "q"'] & \mathscr{K} \arrow[d, "p"] \\ \mathscr{C} \arrow[r, "F"'] & \mathscr{D} \end{tikzcd} \tag{16}$$ eq:1464

in which p and q are cocartesian fibrations and G preserves cocartesian maps. Suppose also that $\mathscr{C}$ and $\mathscr{D}$ are ∞-categories. Then the maps $G|_{\mathscr{E}_x} : \mathscr{E}_x \to \mathscr{K}_{F(x)}$ define a natural transformation between the enriched homotopy transport representations

$$G_! : q_! \to p_! F.$$

The proof is the same as that of Lemma 7.5. We note that when the diagram (16) is a categorical pullback diagram then the maps $G|_{\mathscr{E}_x} : \mathscr{E}_x \to \mathscr{K}_{F(x)}$ are isomorphisms in $\mathrm{h}\,\mathscr{K}an$, so that $\widetilde{G}_!$ is a natural isomorphism of $\mathrm{h}\,\mathscr{K}an$-enriched functors.

Lemma 7.10. *If a diagram (16) is a pullback diagram of cocartesian fibrations, then the composite functor*

$$\pi\mathscr{C} \xrightarrow{F} \pi\mathscr{D} \xrightarrow{p_!} \pi\mathscr{C}at_\infty$$

is an enriched homotopy transport functor for q.

7.4. **Transport functors induce homotopy transport.** We have the following fundamental result concerning homotopy transport.

thm:transport_v_transport **Theorem 7.11** ([4, 02S5])**.** *Consider the universal cocartesian fibration $U : \mathscr{P}.\mathscr{C}at_\infty \to \mathscr{C}at_\infty$. The equivalences*

$$\theta : \mathscr{C} \to (\mathscr{P}.\mathscr{C}at_\infty)_\mathscr{C}$$

from Corollary 5.28 define a natural isomorphism $id_{\pi\mathscr{C}at_\infty} \xrightarrow{\sim} U_!$. This isomorphism realizes the identity functor $id : \pi\mathscr{C}at_\infty \to \pi\mathscr{C}at_\infty$ as enriched homotopy transport for the universal fibration.

The proof proceeds by a more general analysis of homotopy transport for cocartesian fibrations of the form

$$\big(\mathrm{N}^{\mathrm{hc}}(\underline{A})_{x/}\big)^{\mathrm{Pith}} \to \mathrm{N}^{\mathrm{hc}}(\underline{A})^{\mathrm{Pith}},$$

where $\underline{A}$ a simplicial category which is enriched in ∞-categories [4, 02RZ]. We omit the details and refer the reader instead to the text [4].

Now, given an arbitrary cocartesian fibration $q : \mathscr{E} \to \mathscr{C}$ we have a categorical pullback diagram

eq:1525
$$\begin{tikzcd} \mathscr{E} \arrow[r] \arrow[d, "q"'] & \mathscr{P}.\mathscr{C}at_\infty \arrow[d] \\ \mathscr{C} \arrow[r, "F"'] & \mathscr{C}at_\infty \end{tikzcd} \tag{17}$$

which identifies a given functor F as the transport functor for q. Since homotopy transport is preserved under categorical pullback, we conclude that transport functors always induce enriched homotopy transport at the level of the enriched homotopy category.

cor:transport_v_transport
Corollary 7.12. *For any cocartesian fibration $q : \mathscr{E} \to \mathscr{C}$, and classifying functor $F : \mathscr{C} \to \mathscr{C}at_\infty$, the induced map on enriched homotopy categories*

$$\pi F : \pi\mathscr{C} \to \pi\mathscr{C}at_\infty$$

is an enriched homotopy transport functor for q. More specifically, the isomorphisms $\mathscr{E}_x \to F(x)$ in $\mathrm{h}\mathscr{K}an$ provided by the diagram (17) and Theorem 5.27 define an isomorphism of enriched functors $\pi F \overset{\sim}{\to} q_!$.

7.5. **Homotopy transport and enriched Hom functors.** For an ∞-category $\mathscr{C}$ and an object $x : * \to \mathscr{C}$, we recall the oriented fiber product $\{x\} \times^{\mathrm{or}}_{\mathscr{C}} \mathscr{C}$, which is the explicitly the isofibration

$$\{x\} \times_{\mathrm{Fun}(\{0\},\mathscr{C})} \mathrm{Fun}(\Delta^1, \mathscr{C}) \to \mathrm{Fun}(\{1\}, \mathscr{C}) = \mathscr{C}.$$

We have the equivalence of isofibrations

$$\begin{tikzcd} \mathscr{C}_{x/} \arrow[rr, "\sim"] \arrow[dr] & & \{x\} \times^{\mathrm{or}}_{\mathscr{C}} \mathscr{C} \arrow[dl] \\ & \mathscr{C} & \end{tikzcd}$$

of Theorem I-9.14, from which we conclude that the projection $\{x\} \times_{\mathscr{C}} \mathscr{C} \to \mathscr{C}$ is in fact a left fibration.

We assess homotopy transport $\mathscr{C}_{x/} \to \mathscr{C}$ by considering this equivalent fibration. We note that the fibers of $\{x\} \times^{\mathrm{or}}_{\mathscr{C}} \mathscr{C}$ over $\mathscr{C}$ are simply the mapping spaces $\mathrm{Hom}_{\mathscr{C}}(x, y)$.

prop:comp_transp
Proposition 7.13. *Let $\mathscr{C}$ be an ∞-category and $x_0 : * \to \mathscr{C}$ be any object. The composition functions*

$$\circ : \mathrm{Hom}_{\mathscr{C}}(x_1, x_2) \times \mathrm{Hom}_{\mathscr{C}}(x, x_1) \to \mathrm{Hom}_{\mathscr{C}}(x, x_2)$$

from Section I-8.1 are parametrized homotopy transport for the left fibration $\{x\} \times^{\mathrm{or}}_{\mathscr{C}} \mathscr{C} \to \mathscr{C}$.

Proof. Take $x_0 = x$ and $\vec{x} = (x_0, x_1, x_2)$. A morphism $\Delta^1 \times \mathrm{Hom}(x_1, x_2) \times \mathrm{Hom}(x_0, x_1) \to \{x_0\} \times^{\mathrm{or}}_{\mathscr{C}} \mathscr{C}$ is equivalent to a choice of a morphism

$$\Delta^1 \times \Delta^1 \times \mathrm{Hom}(x_1, x_2) \times \mathrm{Hom}(x_0, x_1) \to \mathscr{C}$$

whose restriction to $\{0\}$ in the first argument is of constant value x_0. Let $h : \Delta^1 \times \Delta^1 \to \Delta^2$ be the map which sends $(0, j)$ to 0 and $(1, j)$ to $j + 1$ and let

$$\omega : \mathrm{Hom}(x_1, x_2) \times \mathrm{Hom}(x_0, x_1) \to \mathrm{Fun}(\Delta^2, \mathscr{C})_{\vec{x}}$$

be any section of the trivial Kan fibration

$$\mathrm{Fun}(\Delta^2, \mathscr{C})_{\vec{x}} \to \mathrm{Fun}(\Lambda^2_1, \mathscr{C})_{\vec{x}} = \mathrm{Hom}(x_1, x_2) \times \mathrm{Hom}(x_0, x_1).$$

We consider the composite

$$\begin{aligned}\Delta^1 \times \Delta^1 \times \mathrm{Hom}(x_1, x_2) \times \mathrm{Hom}(x_0, x_1) &\overset{h\times id}{\to} \Delta^2 \times \mathrm{Hom}(x_1, x_2) \times \mathrm{Hom}(x_0, x_1) \\ &\overset{id\times\omega}{\to} \Delta^2 \times \mathrm{Fun}(\Delta^2, \mathscr{C})_{\vec{x}} \overset{ev}{\to} \mathscr{C}.\end{aligned} \tag{18}$$

eq:1582

One sees directly that this composite is of constant value x_0 when restricted to $\{0\}$ in the first argument, and the restriction to $\{1\}$ in the first argument yields the map

$$\Delta^1 \times \mathrm{Hom}(x_1, x_2) \times \mathrm{Hom}(x_0, x_1) \to \Delta^1 \times \mathrm{Hom}(x_1, x_2) \overset{ev}{\to} \mathscr{C}$$

since ω is a section of the aforementioned fibration. This implies commutativity of the diagram

$$\begin{tikzcd}
\{0\} \times \mathrm{Hom}(x_1, x_2) \times \mathrm{Hom}(x_0, x_1) \arrow[r] \arrow[d] & \{x\} \times^{\mathrm{or}}_{\mathscr{C}} \mathscr{C} \arrow[d] \\
\Delta^1 \times \mathrm{Hom}(x_1, x_2) \times \mathrm{Hom}(x_0, x_1) \arrow[r] \arrow[ur, "\xi"] & \mathscr{C},
\end{tikzcd}$$

where ξ is adjoint to the composite (18). So therefore realize the restriction

$$\xi|_{\{1\}} : \Delta^1 \times \mathrm{Hom}(x_1, x_2) \times \mathrm{Hom}(x_0, x_1) \to \mathrm{Hom}(x_0, x_2)$$

as enriched transport for the given fibration, which one checks directly is simply the composition function for $\mathrm{Hom}_{\mathscr{C}}$, i.e. the uniquely determined composite

$$\mathrm{Hom}(x_1, x_2) \times \mathrm{Hom}(x_0, x_1) \to \mathrm{Fun}(\Delta^2, \mathscr{C})_{\vec{x}} \to \mathrm{Fun}(\Delta^{\{0,1\}}, \mathscr{C})_{(x_0, x_2)} = \mathrm{Hom}(x_0, x_2)$$

in $\mathrm{h}\mathscr{K}an$. $\square$

We now find that the Hom-functor

$$\mathrm{Hom}_{\mathscr{C}}(x, -) : \pi\mathscr{C} \to \pi\mathscr{K}an$$

is the enriched homotopy transport representation for the oriented fiber product $\{x\} \times^{\mathrm{or}} \times^{\mathrm{or}}_{\mathscr{C}} \mathscr{C} \to \mathscr{C}$, and hence also for the fibration $\mathscr{C}_{x/} \to \mathscr{C}$.

Corollary 7.14. *Let $\mathscr{C}$ be any ∞-category and $x : * \to \mathscr{C}$ be any object. The* Hom-*functor*

$$\mathrm{Hom}_{\mathscr{C}}(x, -) : \pi\mathscr{C} \to \pi\mathscr{K}an$$

is an enriched homotopy transport functor for the left fibration $\mathscr{C}_{x/} \to \mathscr{C}$.

sect:straight_unstraight

8. Transport III: Perspective via straightening and unstraightening

In this section we explain how the assignment which sends a functor $F : \mathscr{C} \to \mathscr{C}at_\infty$ to the corresponding cocartesian fibration

$$\int_{\mathscr{C}} F = \mathscr{C} \times_{\mathscr{C}at_\infty} \mathscr{P}.\mathscr{C}at_\infty \to \mathscr{C}$$

extends to an equivalence of ∞-categories

$$\mathrm{Un} : \mathrm{Fun}(\mathscr{C}, \mathscr{C}at_\infty) \overset{\sim}{\to} \mathrm{Cocart}(\mathscr{C}),$$

where $\mathrm{Cocart}(\mathscr{C})$ is an ∞-category of cocartesian fibrations over $\mathscr{C}$. Similarly, we have have an equivalence for cartesian fibrations

$$\mathrm{Un} : \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathscr{C}at_\infty) \overset{\sim}{\to} \mathrm{Cart}(\mathscr{C}),$$

which one obtains simply by applying the opposite involution. These functors are referred to as the unstraightening functors [2].

Our investigation centers not the unstraightening functor per-se, but its discrete inverse, the aptly named straightening functor. In our discussion we generalize our base to a simplicial set S rather than an ∞-category.

8.1. **Marked simplicial sets.** A marked simplicial set is a pair (K, W) consisting of a simplicial set K and a choice of 1-simplices $W \subseteq K[1]$ which contains all degenerate 1-simplices. A map between marked simplicial sets $f : (K, W) \to (K', W')$ is a map of simplicial sets which sends W into W'. In this way we obtain the category sSet^+ of marked simplicial sets.

The forgetful functor $\mathrm{sSet}^+ \to \mathrm{sSet}$ has a right adjoint $-^\# : \mathrm{sSet} \to \mathrm{sSet}^+$, which sends a simplicial set K to the simplicial set K with all 1-simplices marked. The category sSet^+ also admits products, with $(K, W) \times (K', W') = (K \times K', W \times W')$, and the right adjoint $-^\#$ is seen to be symmetric monoidal. Via this monoidal functor we obtain an action of sSet on sSet^+.

As a notational point, from this point on we often omit markings from our notation, and simply say K is a marked simplicial set to indicate that K is a simplicial set equipped with some specified marking (K, W).

Lemma 8.1. *The* sSet*-module category* sSet^+ *admits inner-Homs* $\underline{\mathrm{Hom}}_{\mathrm{sSet}^+}(K, K')$. *The n-simplices in the underlying simplicial set maps*

$$\underline{\mathrm{Hom}}_{\mathrm{sSet}^+}(K, K')[n] := \mathrm{Hom}_{\mathrm{sSet}^+}((\Delta^n)^\# \times K, K').$$

Note that this simplicial set is a simplicial subset in the usual inner-Homs for simplicial sets.

lem:1074 **Lemma 8.2.** *Let (K, W) and (K', W') be marked simplicial sets and suppose that the marked vertices in K' are stable under compositions, i.e. that for any simplex*

$$s : \Delta^2 \to K'$$

in which edges $s|\Delta^{\{i,i+1\}}$ are marked, the edge $s|\Delta^{\{0,2\}}$ is marked as well. A map of unmarked simplicial sets $F : \Delta^n \times K \to K'$ is an n-simplex in $\underline{\mathrm{Hom}}_{\mathrm{sSet}^+}(K, K')$ if and only if the following hold:

(a) *The restrict to each vertex $F|_{\Delta^{\{i\}}} : K \to K'$ is a map of marked simplicial sets.*
(b) *At each vertex $x : * \to K$, and each $0 \le i < j \le n$, the edge $F|\Delta^{\{i,j\}} \times \{x\} \to K$ is marked in K'.*

Proof. The marked edges in $(\Delta^n)^\# \times K$ are all pairs (α_{ij}, w) where $\alpha_{ij} : [1] \to [n]$ is the unique increasing map with image $\{i, j\}$ and $w : x \to y$ is any marked edge in K. Since all degenerate 1-simplices are marked, it is clear that any marked map F must satisfy (a) and (b). So let us suppose now that F satisfies (a) and (b), and consider a marked edge (α_{ij}, w) with $i \le j$. Such an edge appears as the $\{0, 3\}$ edge in a 2-simplex

$$t : \Delta^2 \to \Delta^n \times K \text{ with } t|\Delta^{\{0,1\}} = (\alpha_{i,j}, x) \text{ and } t|\Delta^{\{1,2\}} = (\alpha_{i,j}, w).$$

and $Ft : \Delta^2 \to K'$ is now a 2-simplex in K' with both edges $Ft|\Delta^{\{i,i+1\}}$ marked. (In the case $i = j$ the simplex $Ft|\Delta^{\{i,j\}}$ is marked simply because it is degenerate, otherwise this follows by (a).) It follows by stability under composition that $Ft|\Delta^{\{0,2\}} = F(\alpha_{ij}, w)$ is marked as well. So F preserves markings. $\square$

The following is an alternate phrasing of this lemma

lem:1098 **Lemma 8.3.** *The mapping complex $\underline{\mathrm{Hom}}_{\mathrm{sSet}^+}(K, K')$ is a subcomplex in the complex of simplicial maps $\mathrm{Fun}(K, K')$. An n-simplex $F : \Delta^n \to \mathrm{Fun}(K, K')$ lies in $\underline{\mathrm{Hom}}_{\mathrm{sSet}^+}(K, K')$ if and only if each functor $F_i : \Delta^{\{i\}} \to \mathrm{Fun}(K, K')$ preserved markings and, at each $x : * \to K'$ and $i < j$, the image*

$$\Delta^1 \cong \Delta^{\{i,j\}} \overset{F_{ij}}{\to} \mathrm{Fun}(K, K') \overset{x^*}{\to} Fun(*, K') = K'$$

is a marked map in K'.

We are also interested in the nature of 2-simplices in $\underline{\mathrm{Hom}}_{\mathrm{sSet}^+}(K, K')$. The following is obtained from Lemma 8.3, essentially emediately.

cor:1108 **Corollary 8.4.** *Suppose that K and K' are marked simplicial sets, and that the markings in K' satisfy the 2-of-3 property. A 2-simplex $\Delta^2 \to \mathrm{Fun}(K, K')$ lies in $\underline{\mathrm{Hom}}_{\mathrm{sSet}^+}(K, K')$ if and only if each constituent map $\Delta^{\{i\}} \to \mathrm{Fun}(K, K')$ is marked, and 2 of the 3 edges $\Delta^{\{i<j\}} \to \mathrm{Fun}(K, K')$ is marked.*

Given a simplicial set S we let $\underline{\mathrm{sSet}}^+_{/S}$ denote the simplicial category whose objects are morphisms of marked simplicial sets $p : K \to S^\#$, whose morphsism complexes $\underline{\mathrm{Hom}}_S(K, K')$ are the fiber products

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_S(K, K') \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}_{\mathrm{sSet}^+}(K, K') \arrow[d, "(p')_*"] \\
* \arrow[r, "p"'] & \underline{\mathrm{Hom}}_{\mathrm{sSet}^+}(K, S^\#) = \mathrm{Fun}(K, S)
\end{tikzcd}$$

prop:cocart_kan **Proposition 8.5.** *Let $q : X \to S$ be a cocartesian fibration, and consider the associated map of marked simplicial sets $X^q \to S^\#$ where X^q is X paired with the collection of all q-cocartesian maps. Then for any marked simplicial set K over $S^\#$, the mapping complex $\underline{\mathrm{Hom}}_S(K, X^q)$ is a Kan complex.*

Proof. By I4.8 the map $\mathrm{Fun}(K, X) \to \mathrm{Fun}(K, S)$ is an inner fibration, so that the fiber $\mathrm{Fun}_S(K, X)$ over the diagram p is an ∞-category. We have now that $\underline{\mathrm{Hom}}_S(K, X^q)$ is a subcomplex in the ∞-category $\mathrm{Fun}_S(K, X)$ whose n-simplices are precisely those n-simplices in $\mathrm{Fun}_S(K, X)$ which satisfy the conditions specified in Lemma 8.2 (a) and (b), since q-cocartesian maps in X are stable under composition. This stability under composition in X also implies that any completion $\Delta^2 \times K \to X$ of an inner horn $\Lambda^2_1 \times K \to X$ which lies in $\underline{\mathrm{Hom}}_S(K, X^q)$ also lies in $\underline{\mathrm{Hom}}_S(K, X^q)$. Since the subcomplex $\underline{\mathrm{Hom}}_S(K, X^q)$ in $\mathrm{Fun}(K, X)$ is characterized by a restriction on the 1-simplices, it follows that all higher dimensional horns in $\underline{\mathrm{Hom}}_S(K, X^q)$ complete to siplices in $\underline{\mathrm{Hom}}_S(K, X^q)$. So $\underline{\mathrm{Hom}}_S(K, X^q)$ is an ∞-subcategory in $\mathrm{Fun}_S(K, X)$.

Now, for any 1-simplex $\zeta : \Delta^1 \to \underline{\mathrm{Hom}}_S(K, X^q)$ and vertex $x : * \to K$ with image s in S, the composite

$$\Delta^1 \to \underline{\mathrm{Hom}}_S(K, X^q) \overset{x^*}{\to} X$$

has q-cocartesian image in the ∞-category X_s, and in particular is an isomorphism in X_s. This implies that ζ is an isomorphism in the ambient category $\mathrm{Fun}_S(K, X)$, by Proposition I-6.8. From the 2 of 3 property for q-cocartesian morphisms, and Corollary 8.4, it follows that any inverse $\zeta^{-1} : \Delta^1 \to \mathrm{Fun}_S(K, X)$ to ζ is also in

$\underline{\mathrm{Hom}}_S(K, X^q)$. So we see that every morphism in $\underline{\mathrm{Hom}}_S(K, X^q)$ is an isomorphism, and so this complex is a Kan complex. $\square$

8.2. **The ∞-category of cocartesian fibrations over a base.** We consider each cocartesian fibration $q : X \to S$ as a morphism in sSet^+ by applying the q-cocartesian marking X^q on X and the maximal marking $S^\#$ on S, and we have a the full simplicial subcategory $\underline{\mathrm{Cocart}}(S) \subseteq \underline{\mathrm{sSet}}^+_{/S}$ of cocartesian fibrations over S. By Proposition 8.5 this simpilicial subcategory is enriched in Kan complexes, so that the homotopy coherent nerve is an ∞-category.

Definition 8.6. Given a simplicial set S, the ∞-category of cocartesian fibrations over S is the homotopy coherent nerve

$$\mathrm{Cocart}(S) := \mathrm{N}^{\mathrm{hc}}(\underline{\mathrm{Cocart}}(S))$$

Example 8.7. When $S = *$ a cocartesian fibration over $*$ is an ∞-category $\mathscr{C}$. The associated cocartesian marking marks equivalences in $\mathscr{C}$. Since all functors between ∞-categories preserve equivalences we have

$$\underline{\mathrm{Hom}}_*(\mathscr{C}, \mathscr{D}) = \mathrm{Fun}(\mathscr{C}, \mathscr{D}) \quad \text{and} \quad \underline{\mathrm{Hom}}_{\mathrm{Cocart}(*)}(\mathscr{C}, \mathscr{D}) = \mathrm{Fun}(\mathscr{C}, \mathscr{D})^{\mathrm{Kan}}.$$

Hence $\mathrm{Cocart}(*) = \mathscr{C}at_\infty$.

Remark 8.8. Of course, we have an $(\infty, 2)$-category of cocartesian fibrations, which we obtain by applying the homotopy coherent nerve to the simplicial category $\underline{\mathrm{Cocart}}(S)'$. We have no intentions of using this $(\infty, 2)$-category in this work, and so disregard it.

To state things clearly, objects in $\mathrm{Cocart}(S)$ are cocartesian fibrations $X \to S$ and morphisms are functors $F : X \to Y$ which preserve cocartesian maps and fit into a diagram over the base

$$\begin{tikzcd} X \arrow[rr, "F"] \arrow[dr] & & Y \arrow[dl] \\ & S & . \end{tikzcd}$$

A 2-simplex $\Delta^2 \to \mathrm{Cocart}(S)$ is a pair of partial diagrams of cocartesian fibrations

$$\begin{tikzcd} X_0 \arrow[r, "F_{01}"] \arrow[dr, "q_0"'] & X_1 \arrow[r, "F_{12}"] \arrow[d, "q_1"] & X_2 \arrow[dl, "q_2"] \\ & S & \end{tikzcd} \qquad \begin{tikzcd} X_0 \arrow[rr, "F_{02}"] \arrow[dr, "q_0"'] & & X_2 \arrow[dl, "q_2"] \\ & S & \end{tikzcd}$$

and a map

$$\begin{tikzcd} \Delta^1 \times X_0 \arrow[rr, "F"] \arrow[dr, "q_0 p_2"'] & & X_2 \arrow[dl, "q_2"] \\ & S & \end{tikzcd}$$

which satisfies $F|_0 = F_{12}F_{01}$ and $F|_1 = F_{02}$, and for which $F|_{\Delta_1 \times \{x\}}$ is an isomorphism in the ∞-category $X_2 \times_S \{s\}$.

8.3. **Simplicial enrichment for simplicial functors.** For any simplicial set K, and any simplicial functor $F : \underline{A} \to \underline{\mathrm{sSet}}^+$ we define $K \times F : \underline{A} \to \underline{\mathrm{sSet}}^+$ to be the composite

$$\underline{A} \xrightarrow{F} \underline{\mathrm{sSet}}^+ \xrightarrow{K\times -} \underline{\mathrm{sSet}}^+.$$

One sees that the action of K is in fact a simplicial functor via the symmetry $K \times \Delta^n \times - \cong \Delta^n \times K \times -$.

For two functors $F, F' : \underline{A} \to \underline{\mathrm{sSet}}^+$ we let $\mathrm{Nat}(F, F')$ denote the simplicial set with n-simplices

$$\mathrm{Nat}(F, F')[n] := \{\text{Natural transformations } \Delta^n \times F \to F'\}.$$

The composition operation

$$\mathrm{Nat}(F', F'') \times \mathrm{Nat}(F, F') \to \mathrm{Nat}(F, F'')$$

takes transformations $\Delta^n \times F' \to F''$ and $\Delta^n \times F \to F'$ to the transformation

$$\Delta^n \times F \xrightarrow{\delta\times 1} (\Delta \times \Delta) \times F = \Delta^n \times (\Delta^n \times F) \to \Delta^n \times F' \to F''.$$

Definition 8.9. For any simplicial category $\underline{A}$, we let $\mathrm{Fun}(\underline{A}, \underline{\mathrm{sSet}}^+)$ denote the simplicial category of simplicial functors, with morphism complexes $\mathrm{Nat}(F, F')$.

We have the evaluation functor

$$ev : \mathrm{Fun}(\underline{A}, \underline{\mathrm{sSet}}^+) \times \underline{A} \to \underline{\mathrm{sSet}}^+.$$

This functor sends a pair (F, a) of a functor and an object in $\underline{A}$ to $F(a)$, and on morphisms the map of simplicial sets

$$\mathrm{Nat}(F, F') \times \underline{\mathrm{Hom}}_{\underline{A}}(a, a') \to \underline{\mathrm{Hom}}^+_{\mathrm{sSet}}(Fa, F'a')$$

sends a pair

$$\left(\Delta^n \times F \to F', \sigma : \Delta^n \to \underline{\mathrm{Hom}}_{\underline{A}}(a, a')\right)$$

to the composite

$$\Delta^n \times Fa \xrightarrow{\delta\times 1} \Delta^n \times \Delta^n \times Fa \xrightarrow{1\times F\sigma} \Delta^n \times Fa' \to F'a'.$$

Lemma 8.10. *There is a natural isomorphism*

$$\mathrm{Hom}_{\mathrm{sCat}}(\underline{A}', \mathrm{Fun}(\underline{A}, \underline{\mathrm{sSet}}^+)) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{sCat}}(\underline{A}' \times \underline{A}, \underline{\mathrm{sSet}}^+)$$

$$G \mapsto ev(G \times id_{\underline{A}}).$$

Proof. The inverse sends a functor $\Theta : A' \times A \to \underline{\mathrm{sSet}}^+$ to the functor

$$\theta : \underline{A}' \to \mathrm{Fun}(\underline{A}, \underline{\mathrm{sSet}}^+),\ \theta(a) = \Theta(a, -)$$

$$\theta_{ab} : \underline{\mathrm{Hom}}_{\underline{A}'}(a, b) \to \mathrm{Nat}(\Theta(a, -), \Theta(b, -)),$$

where θ_{ab} sends an n-simplex σ to the transformation which evaluates at each x in $\underline{A}$ as

$$\Theta(\sigma, x) : \Delta^n \times \Theta(a, x) \to \Theta(b, x).$$

□

8.4. **Simplicial functors as functor categories.** As the category of simplicial categories is cocomplete [4, 00K3], one sees that the path category construction admits a unique extension from the class of simplices to the entire category of simplicial sets.

Lemma 8.11 ([4, 00L4])**.** *The association* $\Delta^n \mapsto \mathrm{Path}\,\Delta^n$ *extends to a functor* $\mathrm{Path} : \mathrm{sSet} \to \mathrm{sCat}$ *which provides a left adjoint to the homotopy coherent nerve,*

$$\mathrm{Hom}_{\mathrm{sSet}}(-, \mathrm{N}^{\mathrm{hc}}\, -) \cong \mathrm{Hom}_{\mathrm{sCat}}(\mathrm{Path}\, -, -).$$

The product of the unit map $S \to \mathrm{N}^{\mathrm{hc}}\,\mathrm{Path}\,S$, and commutativity of the homotopy coherent nerve with products, provide natural maps

$$\mathrm{Path}(S' \times S) \to \mathrm{Path}(S') \times \mathrm{Path}(S)$$

from which the path category functor becomes op-lax monoidal. Via this op-lax structure we obtain a map

$$\mathrm{Hom}_{\mathrm{sCat}}(\mathrm{Path}\,\Delta^n, \mathrm{Fun}(\mathrm{Path}\,S, \underline{\mathrm{sSet}}^+)) \overset{\sim}{\to} \mathrm{Hom}_{\mathrm{sCat}}(\mathrm{Path}\,\Delta^n \times \mathrm{Path}\,S, \underline{\mathrm{sSet}}^+)$$

$$\to \mathrm{Hom}_{\mathrm{sCat}}(\mathrm{Path}(\Delta^n \times S), \underline{\mathrm{sSet}}^+) \cong \mathrm{Hom}_{\mathrm{sSet}}(\Delta^n \times S, \mathrm{N}^{\mathrm{hc}}\,\underline{\mathrm{sSet}}^+).$$

Taking these maps collectively across various n provides a map of simplicial sets

$$\mathrm{N}^{\mathrm{hc}}\,\mathrm{Fun}(\mathrm{Path}\,S, \underline{\mathrm{sSet}}^+) \to \mathrm{Fun}(S, \mathrm{N}^{\mathrm{hc}}\,\underline{\mathrm{sSet}}^+).$$

We restrict to consider those functors which land in the non-full simplicial subcategory of ∞-categories, with Kanified morphism complexes, and consider the resultant map

$$\mathrm{comp} : \mathrm{N}^{\mathrm{hc}}\,\mathrm{Fun}(\mathrm{Path}\,S, \underline{\mathrm{Cat}}^+_\infty) \to \mathrm{Fun}(S, \mathscr{C}at_\infty). \qquad (19)$$ eq:comp_NN

prop:nerv_to_nerv **Proposition 8.12.** *The simplicial category* $\mathrm{Fun}(\mathrm{Path}\,S, \underline{\mathrm{Cat}}^+_\infty)$ *is enriched in Kan complexes, and the comparison functor* (19) *is an equivalence of* ∞*-categories.*

We outline how this result occurs, according to the logic of [2]. So the proof is not so much a proof as an "authentication ticket" which the reader might verify for themselves.

Proof. The categories $\underline{\mathrm{sSet}}^+$ and $\mathrm{Fun}(\mathrm{Path}\,S, \underline{\mathrm{sSet}}^+)$ admit combinatorial simplicial model structures under which the subcategories of fibrant-cofibrant objects are precisely te subcategories

$$\underline{\mathrm{Cat}}^+_\infty \subseteq \underline{\mathrm{sSet}}^+ \quad \text{and} \quad \mathrm{Fun}(\mathrm{Path}\,S, \underline{\mathrm{Cat}}^+_\infty) \subseteq \mathrm{Fun}(\mathrm{Path}\,S, \underline{\mathrm{sSet}}^+)$$

[2, Proposition 3.1.3.7, Corollary 3.1.4.4, Proposition A.3.3.2, Remark A.3.3.4]. It follows that the simplicial category $\mathrm{Fun}(\mathrm{Path}\,S, \underline{\mathrm{Cat}}^+_\infty)$ is enriched in Kan complexes [2, Remark A.3.1.8], and also that $\underline{\mathrm{sSet}}^+$ provides a $\mathrm{Path}\,S$-chunk [2, Definition A.3.4.9] of itself [2, Example A.3.4.4]. We now see from [2, Proposition 4.2.4.4] that the comparison functor. □

Remark 8.13. The specific claim of Proposition 8.12 seems not to appear explicitly in [2], though it may be implicit. We found this particular claim in [**?**, Remark 3.7].

8.5. **Non-enriched Straightening and unstraightening.** Let $p : K \to S$ be a cocartesian fibration from a marked simplicial set (K, W), and consider the pushout S_p of the following diagram

$$\begin{tikzcd} K \arrow[r, "i"] \arrow[d, "p"'] & \{*\} \star K \arrow[d] \\ S \arrow[r, "i_p"'] & S_p. \end{tikzcd}$$

We now have the functor of plain categories

$$\mathrm{St}[0] : \underline{\mathrm{sSet}}^+_{/S}[0] \to \mathrm{Fun}(\mathrm{Path}\, S, \underline{\mathrm{sSet}}^+_\infty)[0]$$

which sends each object $q : K \to S$ to the representable functor $\underline{\mathrm{Hom}}_{\mathrm{Path}\, S_p}(*, -)$, where each value $\underline{\mathrm{Hom}}_{\mathrm{Path}\, S_p}(*, s)$ is equipped with a natural marking W_s. (Here we are writing s for $i_p(s)$ by an abuse of notation.) The marked edges are all those which appear as follows:

- Take a marked edge $\alpha : x' \to y'$ in K, and consider the extension to a 2-simplex
$$\mathrm{Ext}(\alpha) := \{*\} \star \alpha : \Delta^2 \to \{*\} \star K$$
which then has image $\mathrm{Ext}'(\alpha) : \Delta^2 \to S_p$. This 2-simplex appears as
$$\mathrm{Ext}'(\alpha) : \quad \begin{tikzcd} & px \arrow[dr, "p\alpha"] & \\ * \arrow[ur] \arrow[rr] & & py. \end{tikzcd}$$
which then determines a 1-simplex $\mathrm{E}(\alpha) : \Delta^1 \to \underline{\mathrm{Hom}}_{\mathrm{Path}\, S_p}(*, py)$.
- Take any 1-simplex $B : \Delta^1 \to \underline{\mathrm{Hom}}_{\mathrm{Path}\, S}(py, s)$.
- Consider the composite $B\, \mathrm{E}(\alpha) : \Delta^1 \to \underline{Hom}_{\mathrm{Path}\, S_p}(*, s)$.

The marked edges in W_s in $\underline{\mathrm{Hom}}_{\mathrm{Path}\, S_p}(*, s)$ are precisely those edges which appear as above.

Definition 8.14. The functor $\mathrm{St}[0] : \underline{\mathrm{sSet}}^+_{/S}[0] \to \mathrm{Fun}(\mathrm{Path}\, S, \underline{\mathrm{sSet}}^+_\infty)[0]$ constructed above is called the non-enriched straightening functor.

It can be shown that the straightening fnuctor is cocontinuous [2, Proposition 3.2.1.4] and hence admits a right adjoint.

Proposition 8.15 ([2, Corollary 3.2.1.5])**.** *The functor* $\mathrm{St}[0]$ *admits a right adjoint*

$$\mathrm{Un}[0] : \mathrm{Fun}(\mathrm{Path}\, S, \underline{\mathrm{sSet}}^+_\infty)[0] \to \underline{\mathrm{sSet}}^+_{/S}[0].$$

8.6. **Enriched unstraightening.** At each simplicial set L and map $p : K \to S$ from a marked simplicial set, we have the new map

$$Lp : L^\# \times K \to K \to S$$

and hence a new object in $\mathrm{sSet}^+_{/S}$. This product construction endows $\mathrm{sSet}^+_{/S}$ with a module category structure over $\mathrm{sSet}^+_{/S}$ whose inner-Homs recover the simpicial mapping complexes for the enhancement $\underline{\mathrm{sSet}}^+_{/S}$.

We have the natural map of simplicial sets

$$\mathrm{St}[0](L^\# \times K) \to L^\# \times \mathrm{St}[0](K)$$

[2, Corollary 3.2.1.15] which endows the straightening functor with a op-lax module category structure. The op-lax structure on St[0] endow the adjoint Un[0] with a lax module category structure, and this lax structure provides a canonical enrichment on the unstraightening functor to the simplicial setting. Specifically, we have the natural maps

$$\mathrm{Hom}_{\mathrm{sSet}}(L, \underline{\mathrm{Hom}}(F,G)) = \mathrm{Hom}_{\mathrm{Fun}}(L^{\#}\times F, G) \to \mathrm{Hom}_{\mathrm{sSet}^+_{/S}}(\mathrm{Un}[0](L^{\#}\times F)$$

$$\to \mathrm{Hom}_{\mathrm{sSet}^+_{/S}}(L^{\#}\times \mathrm{Un}[0]F, \mathrm{Un}[0]G) = \mathrm{Hom}_{\mathrm{sSet}}(L, \underline{\mathrm{Hom}}(\mathrm{Un}[0]F, \mathrm{Un}[0]G))$$

across all simplicial sets L. Via Yoneda this lifts the application of the unstraightening functor on morphism sets to the simplicial level,

$$\mathrm{Un} : \underline{\mathrm{Hom}}_{\mathrm{Fun}(\mathrm{Path}\, S, \underline{\mathrm{sSet}}^+)}(F,G) \to \underline{\mathrm{Hom}}_{\mathrm{sSet}^+_{/S}}(\mathrm{Un}[0]F, \mathrm{Un}[0]G),$$

and so enriches the non-simplicial unstraightening functor.

Definition 8.16. The unstraightening functor

$$\underline{\mathrm{Un}} : \mathrm{Fun}(\mathrm{Path}\, S, \underline{\mathrm{sSet}}^+) \to \underline{\mathrm{sSet}}^+_{/S}$$

is the simplicial enrichment of the non-enriched unstraightening functor, as constructed above.

8.7. The straightening and unstraightening equivalences.

thm:1382 **Theorem 8.17.** *The unstraightening functor restricts to an equivalence of simplicial categories*

$$\underline{\mathrm{Un}} : \mathrm{Fun}(\mathrm{Path}\, S, \underline{\mathrm{Cat}}^+_\infty) \to \underline{\mathrm{Cocart}}(S).$$

Proof. Follows by [2, Lemma 3.2.4.1] and [2, Theorem 3.2.0.1]. □

In the statement of Theorem 8.17 by an equivalence of simplicial categories we mean an equivalence specifically in the sense of [2, Definition A.3.2.1]. Equivalently, we are saying that the induced functor on homotopy categories is an equivalence and that the maps on morphism complexes are equivalences of Kan complexes. We refer to this latter property as fully faithfulness.

Lemma 8.18. *If $\Theta : \underline{A} \to \underline{B}$ is an equivalence of Kan-enriched simplicial categories, then the induced functor on homotopy coherent nerves*

$$\mathrm{N}^{\mathrm{hc}}\,\Theta : \mathrm{N}^{\mathrm{hc}}\,\underline{A} \to \mathrm{N}^{\mathrm{hc}}\,\underline{B}$$

is an equivalence of ∞-categories.

Proof. Since the induced functor on homotopy categories is an equivalence, $\mathrm{N}^{\mathrm{hc}}\,\Theta$ is essentially surjective. Fully faithfulness follows from fully faithfulness of Θ and the calculation of the (left pinched) mapping spaces in the homotopy coherent nerve via the mapping complexes in the original categories (see Theorem 5.27). □

We now conclude that unstraightening induces an equivalence on the associated ∞-categories

$$\mathrm{N}^{\mathrm{hc}}\,\underline{\mathrm{Un}} : \mathrm{Fun}(\mathrm{Path}\, S, \underline{\mathrm{Cat}}^+_\infty) \to \mathrm{N}^{\mathrm{hc}}\,\underline{\mathrm{Cocart}}(S) = \mathrm{Cocart}(S)$$

is an equivalence of ∞-categories. Finally, we compose with the inverse comparison equivalence to obtain an equivalence of ∞-categories

$$\mathrm{Fun}(S, \mathscr{C}at_\infty) \overset{\sim}{\to} \mathrm{Cocart}(S).$$

We record this finding.

thm:unstrt_equiv **Theorem 8.19.** *The functor*

$$\mathrm{Un} := \mathrm{comp}^{-1}\,\mathrm{N}^{\mathrm{hc}}\,\mathrm{Un} : \mathrm{Fun}(S, \mathscr{C}at_\infty) \xrightarrow{\sim} \mathrm{Cocart}(S).$$

is an equivalence of ∞-categories.

Definition 8.20. The unstraightening equivalence is the equivalence of Theorem 8.19. The straightening equivalence is the inverse functor

$$\mathrm{St} : \mathrm{Cocart}(S) \xrightarrow{\sim} \mathrm{Fun}(S, \mathscr{C}at_\infty).$$

8.8. **A remark on uniqueness.**

8.9. **Recovering transport via straightening.** For any map of simplicial sets $f : S \to S'$, we have the enriched pullback functor

$$f^* : \underline{\mathrm{Cocart}}(S') \to \underline{\mathrm{Cocart}}(S), \quad (K \to S') \mapsto (K \times_{S'} S \to S).$$

lem:1434 **Lemma 8.21** ([?, Observation 2.13])**.** *For any map of simplicial sets $f : S \to S'$, there is a commutative diagram at the level of homotopy categories*

$$\begin{tikzcd}
\mathrm{h}\,\mathrm{Cocart}(S) & \mathrm{h}\,\mathrm{Fun}(S, \mathscr{C}at_\infty) \arrow[l, "\mathrm{Un}"'] \\
\mathrm{h}\,\mathrm{Cocart}(S') \arrow[u, "f^*"] & \mathrm{h}\,\mathrm{Fun}(S', \mathscr{C}at_\infty) \arrow[l, "\mathrm{Un}"] \arrow[u, "f^*"']
\end{tikzcd}$$

Proof. The functor $f^* : \mathrm{Fun}(\mathrm{Path}\,S', \underline{\mathrm{sSet}}^+) \to \mathrm{Fun}(\mathrm{Path}\,S, \underline{\mathrm{sSet}}^+)$ has a left adjoint $f_!$ and we have a natural isomorphism $\mathrm{St}_f[0] \cong f_!\,\mathrm{St}[0]$ [2, Proposition 3.2.1.4], where the functor St_f is as in [2]. This implies and identification of non-enriched functors $\mathrm{Un}_f[0] \cong \mathrm{Un}[0]f^*$. But, by construction, $\mathrm{St}_f[0] = \mathrm{St}[0]f$, where

$$f : \underline{\mathrm{sSet}}^+_{/S} \to \underline{\mathrm{sSet}}^+_{/S'}$$

just composes maps over S with f. Now one can see directly that the pullback functor on marked simplicial sets over S' is right adjoint to this composition functor f, so that we have $\mathrm{Un}_f[0] = f^*\,\mathrm{Un}[0]$. So we have finally $f^*\,\mathrm{Un}[0] \cong \mathrm{Un}[0]f^*$, and this identification of non-enriched functors implies a corresponding identification of the ∞-functors at the level of homotopy categories. □

Remark 8.22. We expect that all of the identifications employed in the proof are compatible with (op)-lax module category structures, so that the identification $f^*\,\mathrm{Un}[0] \cong \mathrm{Un}[0]f^*$ enriches to an identification at the simplicial level, and hence at the ∞-level $f^*\,\mathrm{Un} \cong \mathrm{Un}\,f^*$. This homotopy-level identification suffices for our purposes however.

We can now consider the universal cocartesian fibration, i.e. the cocartesian fibration over $\mathscr{C}at_\infty$ which is associated to the identity functor

$$\mathscr{Z} := \mathrm{Un}(id_{\mathscr{C}at_\infty}).$$

By the materials of Section 6 we understand that there is an equivalence of fibrations

$$\begin{tikzcd}
\mathscr{Z} \arrow[rr, "\sim"] \arrow[dr] & & \mathscr{P}.\mathscr{C}at_\infty \arrow[dl] \\
& \mathscr{C}at_\infty &
\end{tikzcd}$$

.

Now the diagram of Lemma 8.21 implies the existence of an equivalence of cocartesian fibrations

$$\mathrm{Un}(F) \cong S \times_{\mathscr{C}at_\infty} \mathscr{Z} =: \int_S F$$

at any functor F in $\mathrm{Fun}(S, \mathscr{C}at_\infty)$.

prop:strt_transport

Proposition 8.23 ([2, §3.3.2])**.** *For any cocartesian fibration $q : X \to S$ there is an equivalence of cocartesian fibrations*

$$\begin{tikzcd} X \arrow[rr, "\sim"] \arrow[dr, "q"'] & & \int_S \mathrm{St}(q) \arrow[dl] \\ & S & \end{tikzcd}.$$

In particular, when the base S is an ∞-category the straightening functor $\mathrm{St}(q) : S \to \mathscr{C}_\infty$ is a transport functor for $q : X \to S$.

Proof. We have $q \cong \mathrm{Un}\,\mathrm{St}(q) \cong \int_S \mathrm{St}(q)$. □

We can consider now the ∞-category of left fibrations.

Definition 8.24. Let $\mathrm{LFib}(S)$ denote the full ∞-subcategory in $\mathrm{Cocart}(S)$ whose objects are precisely the left fibrations $q : X \to S$.

8.10. **Straightening and unstraightening for left fibrations.** Since the straightening functor $\mathrm{St}(q)$ produces a transport functor for any cocartesian fibration $q : X \to S$, and the unstraightening functor pulls back along the universal fibrations, Proposition **??** and the fibrer calculation of Proposition 6.7 now imply that unstraightening and straightening restrict to equivalences between left fibrations over S and functors into the ∞-subcategory of spaces $\mathscr{K}an \subseteq \mathscr{C}at_\infty$.

cor:strt_kan

Corollary 8.25. *The straightening and unstraightening equivalences restrict to inverse equivalences*

$$\mathrm{St} : \mathrm{LFib}(S) \overset{\sim}{\to} \mathrm{Fun}(S, \mathscr{K}an) \quad \textit{and} \quad \mathrm{Un} : \mathrm{Fun}(S, \mathscr{K}an) \overset{\sim}{\to} \mathrm{LFib}(S).$$

8.11. **Some naturality over the base.**

prop:small_nat

Proposition 8.26. *Any diagram of cocartesian fibrations*

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "p"'] & X \arrow[d, "q"] \\ T \arrow[r, "\xi"'] & S \end{tikzcd}$$

determines a uniquely associated transformation $\mathrm{St}(p) \to \mathrm{St}(q)\xi$ in the ∞-category $\mathrm{Fun}(T, \mathscr{C}at_\infty)$.

In the case where the bases are ∞-categories, this transformation is visualized as a diagram

$$\begin{tikzcd} T \arrow[rr] \arrow[dr, "\mathrm{St}(p)"'] & & S \arrow[dl, "\mathrm{St}(q)"] \\ & \mathscr{C}at_\infty & \end{tikzcd}$$

in the $(\infty, 2)$-category of (big) ∞-categories.

Construction. Take $F = \mathrm{St}(q)$, $G = \mathrm{St}(p)$. The composite $F\xi$ is a transport functor for the fibration $q' : X \times_S T \to T$, which then determines an isomorphism $F\xi \cong \mathrm{St}(q')$ which is unique up to a contractible space of choices, by Theorem 6.16 (or rather its generalized form [4, 02SC]). Additionally the given diagram specifies, and is specified by, a morphism

$$\begin{tikzcd} Y \arrow[rr, "l"] \arrow[dr, "p"'] & & X \times_S T \arrow[dl, "q'"] \\ & T & \end{tikzcd}$$

in $\mathrm{Cocart}(T)$. Via straightening this morphism determines a map

$$\mathrm{St}(l) : G \to \mathrm{St}(q') \cong F\xi.$$

□

8.12. **Straightening for cartesian fibrations.** As with the cocartesian case, one can show that the full simplicial subcategory $\underline{\mathrm{Cart}}(S) \subseteq \underline{\mathrm{sSet}}_{/S}$ consisting of cartesian fibrations over S is enriched in Kan complexes. We can therefore consider the ∞-category

$$\mathrm{Cart}(S) = \mathrm{N}^{\mathrm{hc}}\, \underline{\mathrm{Cart}}(S).$$

An application of the opposite functor provides an isomorphism of ∞-categories $\mathrm{Cart}(S) \cong \mathrm{Cocart}(S^{\mathrm{op}})$. We therefore obtain the following results for cartesian fibrations:

There are mutually inverse equivalences $\mathrm{St} : \mathrm{Cart}(S) \overset{\sim}{\to} \mathrm{Fun}(S^{\mathrm{op}}, \mathscr{C}at_\infty)$ and $\mathrm{Un} : \mathrm{Fun}(S, \mathscr{C}at_\infty) \overset{\sim}{\to} \mathrm{Cart}(S)$ (Theorem 8.19). These equivalences restrict to equivalences

$$\mathrm{St} : \mathrm{RFib}(S) \overset{\sim}{\to} \mathrm{Fun}(S^{\mathrm{op}}, \mathscr{K}an) \quad \text{and} \quad \mathrm{Un} : \mathrm{Fun}(S^{\mathrm{op}}, \mathscr{K}an) \overset{\sim}{\to} \mathrm{RFib}(S)$$

(Corollary 8.25). The value $\mathrm{St}(q) : S^{\mathrm{op}} \to \mathscr{C}at_\infty$ at any cartesian fibration $q : X \to S$ is a transport functor for q (Proposition 8.23). Finally, any diagram of cocartesian fibrations

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "p"'] & X \arrow[d, "q"] \\ T \arrow[r, "\xi"'] & S \end{tikzcd}$$

determines a uniquely associated transformation between the transport functors $\mathrm{St}(p) \to \mathrm{St}(q)\xi$ (Proposition 8.26).

9. Initial and terminal objects

Before beginning with our study in earnest, with the introduction of Hom functors and the Yoneda embedding for ∞-categories, we discuss the notions of initial and terminal objects in an ∞-category.

9.1. **Initial and terminal basics.**

Definition 9.1. Let $\mathscr{C}$ be an ∞-category. An object x in $\mathscr{C}$ is called initial if, for each object z in $\mathscr{C}$, the mapping space $\mathrm{Hom}_{\mathscr{C}}(x, z)$ is contractible. An object z in $\mathscr{C}$ is called terminal if, for each object z in $\mathscr{C}$, the space $\mathrm{Hom}_{\mathscr{C}}(z, y)$ is is contractible.

One sees that an object x is initial (resp. terminal) in $\mathscr{C}$ if and only if x is terminal (resp. initial) in the opposite category $\mathscr{C}^{\mathrm{op}}$. So we can freely translate between results for initial versus terminal objects. Note also that we can replace the mapping space $\mathrm{Hom}_{\mathscr{C}}(x, y)$ with either the left or right pinched spaces when evaluating initial-ness or terminal-ness of objects.

lem:init_unique

Lemma 9.2. *Let $\mathscr{C}$ be an ∞-category, and let $\mathscr{C}_{\mathrm{Init}}$ and $\mathscr{C}_{\mathrm{Term}}$ denote the full ∞-subcategories whose objects are the initial and terminal objects in $\mathscr{C}$, respectively. Then each of the categories $\mathscr{C}_{\mathrm{Init}}$ and $\mathscr{C}_{\mathrm{Term}}$ is either empty or a contractible Kan complex.*

This is to say, the initial (or terminal) object in an ∞-category $\mathscr{C}$ is unique, provided any such object exists.

Proof. We only consider the case of $\mathscr{C}_{\mathrm{Init}}$. Let us suppose that this subcategory is nonempty. Via contractibility of the mapping spaces we conclude that the functor $\mathscr{C}_{\mathrm{Init}} \to *$ is fully faithful and essentially surjective, and hence an equivalence of ∞-categories. So $\mathscr{C}_{\mathrm{Init}}$ is a contractible Kan complex. $\square$

lem:2181

Lemma 9.3. *If x is initial (resp. terminal) in $\mathscr{C}$, then another object x' is initial (resp. terminal) in $\mathscr{C}$ if and only if x' is isomorphic to x.*

Proof. For any isomorphism $\alpha : x \to x'$ the induced maps

$$\alpha^* : \mathrm{Hom}_{\mathscr{C}}(x, y) \to \mathrm{Hom}_{\mathscr{C}}(x', y) \quad \text{and} \quad \alpha_* : \mathrm{Hom}_{\mathscr{C}}(y, x) \to \mathrm{Hom}_{\mathscr{C}}(y, x')$$

are isomorphisms in $\mathrm{h}\,\mathscr{K}an$, at all y in $\mathscr{C}$. So contractibility of the left-hand spaces implies contratibility of the right-hand spaces. $\square$

One also sees that equivalences of ∞-categories preserve initial and terminal objects.

lem:equiv_initial

Lemma 9.4. *If $F : \mathscr{C} \to \mathscr{D}$ is an equivalence between ∞-categories, and x is initial (resp. terminal) in $\mathscr{C}$, then $F(x)$ is initial (resp. terminal) in $\mathscr{D}$*

Proof. Suppose that x is initial in $\mathscr{C}$. First note that any isomorphism $\beta : y \to y'$ in $\mathscr{D}$ induces isomorphisms

$$\beta_* : \mathrm{Hom}_{\mathscr{D}}(z, y) \overset{\sim}{\to} \mathrm{Hom}_{\mathscr{D}}(z, y')$$

in the homotopy category of Kan complexes. So an object z in $\mathscr{D}$ is initial if and only if the relevant mapping spaces are contractible at a dense collection of objects in $\mathscr{D}$. (By a dense collection we mean a collection which contains a representative for every isoclass in $\mathscr{D}$.) Since any equivalence is both fully faithful and essentially surjective, we have that the mapping spaces $\mathrm{Hom}_{\mathscr{D}}(Fx, y)$ are contractible at all y in the image of $\mathscr{C}$, and hence at all y in $\mathscr{D}$. So $F(x)$ is initial in $\mathscr{D}$. The case where x is terminal is proved similarly. $\square$

Warning 9.5. Initial and terminal objects are not well-behaved under fibering. Consider for example the cone $C = \{x^2 + y^2 = z : x, y, z \in \mathbb{R}\}$ and its projection onto the z-axis line $R_z \cong \mathbb{R}$. The projection $\mathrm{Sing}(C) \to \mathrm{Sing}(R_z)$ is a Kan fibration and the objects $\vec{1} = (1,1,1)$ and 1 are both initial and terminal in $\mathrm{Sing}(C)$ and $\mathrm{Sing}(R_z)$ respectively, since these spaces are contractible. However, $\vec{1}$ is not initial or terminal in the fiber $\mathrm{Sing}(C)_1 = \mathrm{Sing}(S^1)$. In fact, this fiber admits no such objects.

9.2. **Aside: trivial fibrations via the fibers.** For the analysis that follows, it is convenient to have a characterization of trivial Kan fibrations which can be checked on the fibers.

`prop:triv_fibs` **Proposition 9.6.** *A map of simplicial sets $f : \mathscr{C} \to S$ is a trivial Kan fibration if and only if f is a left (or right) fibration and, at each point $s : * \to S$, the fiber $\mathscr{C}_s$ is a contractible Kan complex.*

Sketch proof. If f is a trivial Kan fibration then it is both a left and right fibration, and all of its fibers are contractible. As for the other direction, assume now that f is a left fibration and that all of its fibers are contractible. (The case of a right fibrations is then obtained by taking opposites.)

We must show that each lifting problem of the form

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & \mathscr{C} \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & S \end{tikzcd}$$

admits a solution. In the case that $n = 0$, such a solution exists since the fibers $\mathscr{C}_s$ are all non-empty. So we assume $n > 0$. By replacing S with Δ^n, and $\mathscr{C}$ with $\Delta^n \times_S \mathscr{C}$, we may assume also that both S and $\mathscr{C}$ are ∞-categories. For fun we can finally replace this lifting problem with the related lifting problem

$$\begin{tikzcd} \{0\} \times \partial\Delta^n \arrow[r, "\bar{\sigma}"] \arrow[d] & \mathscr{C} \arrow[d] \\ \Delta^1 \times \Delta^n \arrow[r, "\sigma"'] \arrow[ur, dotted] & S, \end{tikzcd} \tag{20}$$ `eq:1722`

which is obtained by restricting along the projections

$$p : \Delta^1 \times \Delta^n \to \Delta^n, \quad p(0,i) = i, \; p(1,i) = n,$$

and which recovers our original problem after restricting to $\{0\} \times \Delta^n$. It suffices to solve this second problem.

By [4, 0153] the class of left anodyne maps is stable under the cartesian action of sSet on itself, so that the inclusion $\{0\} \times \partial\Delta^n \to \Delta^1 \times \partial\Delta^n$ is left anodyne. Since the map f is left anodyne, it follows that the lifting problem (20) extends to a problem

$$\begin{tikzcd} \Delta^1 \times \partial\Delta^n \arrow[r, "\bar{\sigma}"] \arrow[d] & \mathscr{C} \arrow[d] \\ \Delta^1 \times \Delta^n \arrow[r, "\sigma"'] \arrow[ur, dotted] & S, \end{tikzcd}$$

Since the fibers of f are trivial Kan fibrations, the above problem extends further to a problem of the form

$$\begin{tikzcd} Y(0) \arrow[r, "\bar{\sigma}"] \arrow[d] & \mathscr{C} \arrow[d] \\ \Delta^1 \times \Delta^n \arrow[r, "\sigma"'] \arrow[ur, dotted] & S, \end{tikzcd}$$

where $Y(0)$ is the pushout

$$Y(0) = (\Delta^1 \times \partial\Delta^n) \coprod_{\{1\}\times\partial\Delta^n} (\{1\} \times \Delta^n).$$

By [4, Proof of 00TH] the inclusion $Y(0) \to \Delta^1 \times \Delta^n$ can be factored into a sequence $Y(0) \to Y(1) \to \cdots \to Y(n+1) = \Delta^1 \times \Delta^n$ with each $Y(i+1)$ fitting into a pushout diagram

$$\begin{tikzcd} \Lambda^{n+1}_{i+1} \arrow[r] \arrow[d] & Y(i) \arrow[d] \\ \Delta^{n+1} \arrow[r] & Y(i+1). \end{tikzcd}$$

Furthermore, this sequence can be constructed so that at $Y(n) = \Delta^1 \times \Delta^n$ the sequence

$$\Delta^1 \cong \Delta^{\{n,n+1\}} \to \Delta^{n+1} \to \Delta^1 \times \Delta^n$$

recovers the edge $\Delta^1 \times \{n\}$.

Now, since f is inner anodyne we can solve, in order, the lifting problems

$$\begin{tikzcd} Y(i-1) \arrow[r] \arrow[d] & \mathscr{C} \arrow[d] \\ Y(i) \arrow[r] \arrow[ur, dotted] & S \end{tikzcd}$$

at all $0 < i \leq n$. For the final lifting problem, along the inclusion $Y(n) \to Y(n+1)$, we need to solve a lifting problem of the form

$$\begin{tikzcd} \Lambda^{n+1}_{n+1} \arrow[r] \arrow[d] & \mathscr{C} \arrow[d] \\ \Delta^{n+1} \arrow[r] \arrow[ur, dotted] & S \end{tikzcd}$$

in which $\sigma|\Delta^{\{n,n+1\}}$ is of a constant value s in S. Since the fiber $\mathscr{C}_s$ is a Kan complex, the morphism $\Delta^{\{n,n+1\}} \to \Lambda^{n+1}_{n+1} \to \mathscr{C}$ is an isomorphism in $\mathscr{C}$. We can therefore solve this final lifting problem, by Proposition I-4.33, and hence obtain the desired solution to the problem (20). □

When applied to the case of a Kan fibration we have the following, which can also be deduced from Propositions I-3.30 and I-3.42.

Corollary 9.7. *A map between Kan complexes $f : \mathscr{X} \to \mathscr{Y}$ is a trivial Kan fibration if and only if it is a Kan fibration and, at each point $y : * \to \mathscr{Y}$, the fiber $\mathscr{X}_y$ is contractible.*

9.3. **Initial objects and undercategories.**

`prop:terminal_over` **Proposition 9.8.** *An object in an ∞-category $x : * \to \mathscr{C}$ is initial if and only if the forgetful functor $\mathscr{C}_{x/} \to \mathscr{C}$ is a trivial Kan fibration. Dually, an object $y : * \to \mathscr{C}$ is terminal if and only if the functor $\mathscr{C}_{/y} \to \mathscr{C}$ is a trivial Kan fibration.*

Proof. If x is initial then all of the left pinched mapping spaces are contractible, so that all of the fibers of the left fibration $\mathscr{C}_{x/} \to \mathscr{C}$ are contractible. It follows that this map is a trivial Kan fibration. For the converse, we simply note that trivial Kan fibrations are stable under pullback. The arguments in the terminal case are similar. $\square$

Let us now give a technical lemma.

`lem:1699` **Lemma 9.9.** *For each positive integer n, the map*

$$(\Delta^1 \star \partial\Delta^n) \coprod_{(\{0\}\star\partial\Delta^n)} \{0\} \star \Delta^n \to \Delta^1 \star \Delta^n \cong \Delta^{n+2}$$

induced by the respective inclusions is an isomorphism onto the horn Λ^{n+2}_0.

See [4] for the proof. We have the following characterization of isomorphisms via initial and terminal objects.

`prop:isom_initial` **Proposition 9.10.** *For a map $\alpha : x \to y$ in an ∞-category $\mathscr{C}$, the following are equivalent:*

(a) *α is an isomorphism.*
(b) *α is initial when considered as an object in the undercategory $\mathscr{C}_{x/}$.*
(c) *α is terminal when considered as an object in the overcategory.*

Proof. We prove the equivalence between (a) and (b). The equivalence between (a) and (c) is obtained by taking opposites. The implication (b) $\Rightarrow$ (a) just follows by considering maps between α and id_x in the undercategory. So suppose that α is an isomorphism. By Proposition 9.8, α is initial in $\mathscr{C}_{x/}$ if and only if the forgetful functor

$$\mathscr{C}_{\alpha/} \cong (\mathscr{C}_{x/})_{\alpha/} \to \mathscr{C}_{x/}$$

is a trivial Kan fibration. Now, via a consideration of the identification from Lemma 9.9, solving a lifting problem of the form

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & \mathscr{C}_{\alpha/} \arrow[d] \\ \Delta^n \arrow[r] & \mathscr{C}_{/x} \end{tikzcd}$$

is equivalent to solving the corresponding lifting problem

$$\begin{tikzcd} \Lambda^{n+2}_0 \arrow[r] \arrow[d] & \mathscr{C} \arrow[d] \\ \Delta^{n+2} \arrow[r] & * \end{tikzcd}$$

in which the initial edge $\Delta^{\{0,1\}} \to \mathscr{C}$ is α. Such a problem admits a solution by Proposition I-4.33, so that the forgetful functor is seen to be a trivial Kan fibration. $\square$

We take a moment to discuss some examples before returning to the theoretical foundations of this topic.

9.4. **Initial and terminal objects in simplicial nerves.**

Definition 9.11. An object x in a simplicial category $\underline{A}$ is called initial (resp. terminal) if, for each y in $\underline{A}$, the mapping complex $\underline{\mathrm{Hom}}_{\underline{A}}(x,y)$ (resp. $\underline{\mathrm{Hom}}_{\underline{A}}(y,x)$) is a contractible Kan complexes.

The easiest way for this to occur is if the relevant mapping complexes are just points. For example, one sees immediately that $\emptyset$ and $*$ are initial and terminal in $\underline{\mathrm{Kan}}$, respectively.

For $\underline{A}$ enriched in Kan complexes, and $\mathscr{A} = \mathrm{N}^{\mathrm{hc}}(\underline{A})$, the equivalence

$$\underline{\mathrm{Hom}}_{\underline{A}}(x,y) \overset{\sim}{\to} \mathrm{Hom}^{\mathrm{L}}_{\mathscr{A}}(x,y)$$

of Theorem 5.27 tells us that an object x is initial (resp. terminal) in $\underline{A}$ if and only if x is initial (resp. terminal) when considered as an object in the ∞-category $\mathscr{A}$. The analogous claim is seen to hold for terminal objects via a consideration of the opposite categories.

Lemma 9.12. *Let $\underline{A}$ be a simplicial category which is enriched in Kan complexes. Then an object x is initial (resp. terminal) in $\underline{A}$ if and only if the corresponding object x is initial (resp. terminal) in $\mathrm{N}^{\mathrm{hc}}(\underline{A})$.*

The following corollary is not an immediate consequence of triviality of the mapping categories $\mathrm{Fun}(\emptyset, \mathscr{C})$ and $\mathrm{Fun}(\mathscr{C}, *)$, when $\mathscr{C}$ is an ∞-category.

cor:1761 **Corollary 9.13.** *The empty set $\emptyset$ is initial in both $\mathscr{K}an$ and $\mathscr{C}at_\infty$. The point $*$ is terminal in both $\mathscr{K}an$ and $\mathscr{C}at_\infty$.*

9.5. **Zero objects in pointed spaces.** Though we will not use the term explicitly, a zero object in an ∞-category is an object which is simultaneously initial and terminal. Such objects are familiar from our studies of abelian categories. In the ∞-setting, the theory of abelian categories is, to some extent and in an indirect manner, reflected in the theory of stable categories. In the stable setting one again demands the existence of a zero object.

Proposition 9.14. *If x is terminal in an ∞-category $\mathscr{C}$, then x is both initial and terminal in the category $\mathscr{C}_{x/}$.*

By x in $\mathscr{C}_{x/}$ we mean any morphism $x \to x$. Since x is terminal, this lift of x to an object in $\mathscr{C}_{x/}$ is uniquely determined up to a contractible space. Practically speaking, we can just take this lift to be $id_x : x \to x$.

Proof. The fact that x is initial in $\mathscr{C}_{x/}$ follows by Proposition 9.10. For terminality, we consider the forgetful functor

$$\mathscr{C}_{x//x} \to \mathscr{C},$$

where $\mathscr{C}_{x//x} = (\mathscr{C}_{x/})_{/x} = (\mathscr{C}_{/x})_{x/}$. For any inclusion of simplicial sets $A \to B$, the existence of a solution to a lifting problem

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & \mathscr{C}_{x//x} \arrow[d] \\
B \arrow[r] & \mathscr{C}_{x/}
\end{tikzcd}$$

is equivalent to the existence of a solution to the corresponding lifting problem

$$\begin{tikzcd} \{x\} \star A \arrow[r] \arrow[d] & \mathscr{C}_{/x} \arrow[d] \\ \{x\} \star B \arrow[r] & \mathscr{C}. \end{tikzcd}$$

By Proposition 9.8 a solution to the latter problem exists, since x is terminal in $\mathscr{C}$. It follows that the map $\mathscr{C}_{x//x} \to \mathscr{C}_{x/}$ is a trivial Kan fibration, and hence that x is terminal in $\mathscr{C}_{x/}$, by Proposition 9.8. □

Recall form Corollary 9.13 that the 1-point space $*$ is terminal in $\mathscr{K}an$.

Corollary 9.15. *The* 1*-point space* $*$ *is both initial and terminal in the* ∞*-category* $\mathscr{K}an_{*/}$ *of pointed Kan complexes.*

9.6. **Zero objects in derived categories.**

Definition 9.16. An object x in a dg category $\mathbf{A}$ is said to be initial (resp. terminal) if, at each y in $\mathbf{A}$, the Hom complex $\mathrm{Hom}^*_{\mathbf{A}}(x,y)$ (resp. $\mathrm{Hom}^*_{\mathbf{A}}(y,x)$) is acyclic.

Recall our calculation of the mapping spaces in the dg nerve $\mathscr{A} = \mathrm{N}^{\mathrm{dg}}(\mathbf{A})$ via the Hom complexes in $\mathbf{A}$,

$$\mathrm{Hom}^{\mathrm{L}}_{\mathscr{A}}(x,y) \overset{\sim}{\to} K(\mathrm{Hom}^*_{\mathbf{A}}(x,y))$$

(Proposition I-11.7). By Theorem I-10.13, the above calculation tells us that the mapping $\mathrm{Hom}^{\mathrm{L}}_{\mathscr{A}}(x,y)$ are contractible whenever the complex $\mathrm{Hom}^*_{\mathbf{A}}(x,y)$ is acyclic. So we observe the following.

lem:init_dg **Lemma 9.17.** *Let* $\mathbf{A}$ *be a dg category and take* $\mathscr{A} = \mathrm{N}^{\mathrm{dg}}(\mathbf{A})$*. If an object* x *is initial (resp. terminal) in* $\mathbf{A}$*, then the corresponding object* x *is initial (resp. terminal) in* $\mathscr{A}$*.*

Remark 9.18. The converse to Lemma 9.17 holds if we assume that our dg category $\mathbf{A}$ has a good shift functor (see Section 11.1).

For any abelian category $\mathbb{A}$, the object 0 is both initial and terminal in the dg category $\mathrm{Ch}(\mathbb{A})$ of cochains over $\mathbb{A}$, and hence also in the subcategories of K-projective and K-injective complexes. We recall that the derived ∞-category $\mathscr{D}(\mathbb{A})$ is defined by taking the dg nerve of the dg category of K-injective objects in $\mathrm{Ch}(\mathbb{A})$ when we have enough such objects, or K-projectives when we have enough such objects (see Section I-12).

Corollary 9.19. *For any Grothendieck abelian category* $\mathbb{A}$*, the zero complex* 0 *is both initial and terminal in the derived* ∞*-category* $\mathscr{D}(\mathbb{A})$*.*

9.7. **Initial objects and weak contractibility.** We phrase all results below in terms of initial objects. The corresponding results hold for terminal objects via duality.

Lemma 9.20. *A Kan complex* $\mathscr{X}$ *admits an initial object if and only if* $\mathscr{X}$ *is contractible.*

Proof. If x is initial in $\mathscr{X}$, then every object in $\mathscr{X}$ admits a morphism from $\mathscr{X}$, and hence is isomorphic to x (since $\mathscr{X}$ is a Kan complex). Since any object which is isomorphic to an initial object is initial, we conclude that $\mathscr{X}$ consists entirely of initial objects. We conclude that $\mathscr{X}$ is contractible by Lemma 9.2. $\square$

In the case of an ∞-category $\mathscr{C}$ we do not gain such a precise understanding of $\mathscr{C}$ via the existence of an initial object. This is clear from the examples discussed above. We can, however, constrain certain relative phenomena between ∞-categories via the preservation of initial objects. The remainder of this section is dedicated to an elaboration on this, somewhat criptic, point.

lem:1936 **Lemma 9.21.** *An object x in $\mathscr{C}$ is initial if and only if the forgetful functor $\mathscr{C}_{x/} \to \mathscr{C}$ admits a section $F : \mathscr{C} \to \mathscr{C}_{x/}$ with $F(x) = id_x$.*

Proof. If x is initial then the forgetful functor is a trivial Kan fibration, by Proposition 9.8. It follows that the lifting problem

$$\begin{tikzcd} * \arrow[r, "id_x"] \arrow[d, "x"'] & \mathscr{C}_{x/} \arrow[d] \\ \mathscr{C} \arrow[r, "id"'] \arrow[ur, dotted] & \mathscr{C} \end{tikzcd}$$

admits a solution $s : \mathscr{C} \to \mathscr{C}_{x/}$. This solution provides the desired section. Conversely, if we have such a section F then for each y in $\mathscr{C}$ we can split the identity on the mapping space as

$$\mathrm{Hom}_{\mathscr{C}}(x,y) \overset{F}{\to} \mathrm{Hom}_{\mathscr{C}_{x/}}(id_x, F(y)) \overset{forget}{\to} \mathrm{Hom}_{\mathscr{C}}(x,y).$$

Since id_x is initial in $\mathscr{C}_{x/}$, by Proposition 9.10, each mapping space $\mathrm{Hom}_{\mathscr{C}_{x/}}(id_x, F(y))$ is contractible. Thus each mapping space $\mathrm{Hom}_{\mathscr{C}}(x,y)$ is a retract of a contractible space, and hence contractible itself. $\square$

We record a little lemma.

lem:1957 **Lemma 9.22** ([4, 0196])**.** *If $i : A \to B$ is an inclusion of simplicial sets, then the induced map*

$$\{*\} \star i : \{*\} \star A \to \{*\} \star B$$

is left anodyne.

Proof. The class of i at which $\{*\} \star i$ is left anodyne is saturated. We we need only show that it contains the inclusions $\partial\Delta^n \to \Delta^n$. But in this case the inclusion in question is identified with the left anodyne map $\Lambda^{n+1}_0 \to \Delta^{n+1}$. $\square$

prop:init_lanodyne **Proposition 9.23.** *An object x in $\mathscr{C}$ is initial if and only if the inclusion $x : * \to \mathscr{C}$ is left anodyne. If y is terminal in $\mathscr{C}$, then the inclusion $y : * \to \mathscr{C}$ is right anodyne.*

Proof. We deal with the initial claim. If $x : * \to \mathscr{C}$ is left anodyne then we can solve the lifting problem

$$\begin{tikzcd} * \arrow[r, "id_x"] \arrow[d, "x"'] & \mathscr{C}_{x/} \arrow[d] \\ \mathscr{C} \arrow[r, "id"'] \arrow[ur, dotted] & \mathscr{C} \end{tikzcd}$$

and hence obtain a section $F : \mathscr{C} \to \mathscr{C}_{x/}$ as in Lemma 9.21. We conclude that x is initial in $\mathscr{C}$.

Conversely, if x is initial then the section $F : \mathscr{C} \to \mathscr{C}_{x/}$ of Lemma 9.21 provides a map $F' : \{*\} \star \mathscr{C} \to \mathscr{C}$ with $F'|_{\mathscr{C}} = id_{\mathscr{C}}$, $F'(*) = x$, and $F'(* \to x) = id_x$. In particular, F' is defined on each simplex outside of $\mathscr{C}$ by taking

$$F'(\{*\} \star \Delta^m) = F(\Delta^m).$$

This map F' gives a diagram

$$\begin{tikzcd} * \arrow[r, "x"] \arrow[d, "x"'] & \{*\} \star \{x\} \arrow[r] \arrow[d] & * \arrow[d, "x"] \\ \mathscr{C} \arrow[r] & \{*\} \star \mathscr{C} \arrow[r, "F"] & \mathscr{C} \end{tikzcd}$$

so that the inclusion $\{x\} \to \mathscr{C}$ is a retract of the inclusion $\{*\} \star \{x\} \to \{*\} \star \mathscr{C}$. Since this latter inclusion is left anodyne, by Lemma 9.22, we conclude that the inclusion $\{x\} \to \mathscr{C}$ is left anodyne as well. □

As a consequence of Proposition 9.23 we observe a kind of relative triviality for $\mathscr{C}$.

cor:initial_eval **Corollary 9.24.** *Suppose $f : X \to S$ is a left fibration, and that $\mathscr{C}$ admits an initial object $x : * \to \mathscr{C}$. Then the map*

$$\mathrm{Fun}(\mathscr{C}, X) \to X \times_S \mathrm{Fun}(\mathscr{C}, S), \quad F \mapsto (F|_x, fF). \tag{21}$$ eq:2002

is a trivial Kan fibration.

Proof. Immediate from Propositions 9.23 and 4.3. □

Let's consider what Corollary 9.24 is telling, in semi-human terms. In the extreme case where $\mathscr{X} \to *$ is a Kan complex, the right hand side of (21) is just $\mathscr{X}$ and we obtain a trivial Kan fibration

$$ev_x : \mathrm{Fun}(\mathscr{C}, \mathscr{X}) \to \mathscr{X}$$

which just evaluates a functor F at $x : * \to \mathscr{C}$. This says that for any choice of a point $z : * \to \mathscr{X}$ there is a unique functor $F_z : \mathscr{C} \to \mathscr{X}$ which evaluates as $F_z(x) = z$. Indeed, we can just take the fiber of ev_x at z to obtain a space $\mathrm{Fun}(\mathscr{C}, \mathscr{X})_z$ which parametrizes such functors, and observe that this space is contractible. In this way $\mathscr{C}$ "looks like a point" relative to any Kan complex.

In the relative setting, we consider a left fibration $f : X \to S$ and see that for any choice of a functor $\bar{F} : \mathscr{C} \to S$, and a point z in X which lifts $F(x)$, there is a unique lift of $\bar{F}$ to a functor $F : \mathscr{C} \to X$ with $F(x) = z$. Rather, we observe that any lifting problem of the form

$$\begin{tikzcd} * \arrow[r] \arrow[d, "x"'] & X \arrow[d, "f"] \\ \mathscr{C} \arrow[r] \arrow[ur, dotted] & S \end{tikzcd}$$

admits a unique solution.

9.8. **Equivalences of fibrations via initial objects.**

prop:equiv_initial **Proposition 9.25.** *Let*

$$\begin{tikzcd} \mathscr{C} \arrow[rr, "F"] \arrow[dr] & & \mathscr{D} \arrow[dl] \\ & \mathscr{T} & \end{tikzcd}$$

be a diagram of ∞-categories in which both of the maps to $\mathscr{T}$ are left fibrations. If $\mathscr{C}$ admits an initial object x, then F is an equivalence if and only if $F(x)$ is initial in $\mathscr{D}$.

Proof. If F is an equivalence then it preserves initial objects, by Lemma 9.4. Suppose conversely that F is such a map, that x is initial in $\mathscr{C}$, and that the image $F(x)$ is initial in $\mathscr{D}$. Consider the lifting problem

$$\begin{tikzcd} * \arrow[r, "x"] \arrow[d, "F(x)"'] & \mathscr{C} \arrow[d] \\ \mathscr{D} \arrow[r] \arrow[ur, dotted] & \mathscr{T}. \end{tikzcd}$$

By Corollary 9.24 there exists a solution to this problem, and hence there exists a functor $G : \mathscr{D} \to \mathscr{C}$ over $\mathscr{T}$ with $GF(x) = x$. By Corollary 9.24 again the composition GF is also seen to be isomorphic to the identity in $\mathrm{Fun}_{\mathscr{T}}(\mathscr{C}, \mathscr{C})$. We also have $FG(Fx) = F(x)$ so that FG is isomorphic to the identity on $\mathscr{D}$. Thus F is an equivalence. □

Again, one observes a corresponding statement for right fibrations and terminal objects, via the opposite duality.

At first glance this proposition seems ridiculous. Indeed, it suggests that if $f : \mathscr{E} \to \mathscr{C}$ is a left fibration of ∞-categories, and e is an object in $\mathscr{E}$ with image x in $\mathscr{C}$, then the induced map on undercategories $F : \mathscr{E}_{e/} \to \mathscr{C}_{x/}$ is an equivalence. This is because F fits into a diagram

$$\begin{tikzcd} \mathscr{E}_{e/} \arrow[rr] \arrow[dr] & & \mathscr{C}_{x/} \arrow[dl] \\ & \mathscr{C} & \end{tikzcd}$$

and which both maps to $\mathscr{E}$ are left fibrations, and F is seen to send the initial object id_e to the initial object id_x. However, one sees that this is as bad as it gets.

cor:2491 **Corollary 9.26.** *Let $\mathscr{C} \to \mathscr{T}$ be a left fibration, suppose that $\mathscr{C}$ admits an initial object x, and let t denote the image of x in $\mathscr{T}$. Then there is an equivalence $F : \mathscr{C} \to \mathscr{T}_{t/}$ of left fibrations over $\mathscr{T}$ which sends x to id_t.*

So Proposition 9.25, said another way, classifies left fibrations up to equivalence via isoclasses of objects in $\mathscr{T}$.

Proof. By Proposition 9.25 the map $\mathscr{C}_{x/} \to \mathscr{T}_{t/}$ is an equivalence of left fibrations which sends id_x to id_t. The proposed equivalence $\mathscr{C} \to \mathscr{T}_{t/}$ is obtained by composing the equivalence $\mathscr{C}_{x/} \to \mathscr{T}_{t/}$ with a section $F : \mathscr{C} \to \mathscr{C}_{x/}$ as in Lemma 9.21. □

10. Aside: Simplicialification for dg categories

10.1. **Simplicial categories for dg categories.** Let k be a commutative ring and consider the category sSet_k of simplicial k-modules. This category is symmetric monoidal with the expected product $M \otimes_k N$, where

$$(M \otimes_k N)[r] := M[r] \otimes_k N[r].$$

This product admits inner-Homs, so that sSet_k is naturally enriched in itself, and we obtain the corresponding simplicial category $\underline{\mathrm{sSet}_k}$. Explicitly the morphism complexes $\mathrm{Fun}_k(M, N)$ are the complexes with n-simplices

$$\mathrm{Fun}_k(M, N)[n] = \mathrm{Hom}_{\mathrm{sSet}_k}(k\Delta^n \otimes_k M, N).$$

Since all simplicial k-modules are Kan complexes, the simplicial category $\underline{\mathrm{sSet}_k}$ is Kan-enriched and we have the forgetful simplicial functor $\underline{\mathrm{sSet}_k} \to \underline{Kan}$.

Definition 10.1. We take $\mathscr{K}an_k := \mathrm{N}^{\mathrm{hc}}(\underline{\mathrm{sSet}_k})$.

Via faithfulness of the functor $\underline{\mathrm{sSet}_k} \to \underline{Kan}$ we observe that the induced functor on ∞-categories $\mathscr{K}an_k \to \mathscr{K}an$ is an inclusion of simplicial sets.

Recall that we have the Dold-Kan functor

$$K : \mathrm{Ch}(k) \to \mathrm{sSet}_k$$

which restricts to an equivalence $K^{\leq 0}$ from connective cochains. In particular, K factors through the truncation

$$\mathrm{Ch}(k) \to \mathrm{Ch}(k)^{\leq 0}, \quad X \mapsto (\cdots \to X^{-1} \to Z^0 X \to 0)$$

and is identified with the composite of this truncation and the equivalence $K^{\leq 0}$. We let $\mathbf{Ch}(k)$ denote the usual dg category of k-cochains.

prop:K_lax

Proposition 10.2 ([4, 00SD])**.** *The functor K admits a lax-monoidal structure $m_{V,W} : K(V) \otimes_k K(W) \to K(V \otimes_k W)$. Furthermore, at each pair of objects this morphism $m_{V,W}$ is a (non-linear) homotopy equivalence.*

Proof. This lax monoidal structure is adjoint to the monoidal structure on the normalized cochain functor provided by the Alexander-Whitney map [4, 00S6]. Since the Alexander-Whitney maps are quasi-isomorphism [4, 00SB], we conclude that each $m_{V,W}$ is a homotopy equivalence. In particular, $m_{V,W}$ is obtained as the composite

$$K(V)\otimes_k K(W) \overset{\sim}{\to} KN(K(V)\otimes_k K(W)) \overset{K(\mathrm{AW})}{\to} K(NK(V)\otimes_k NK(W)) \overset{\sim}{\to} K(V\otimes_k W).$$

∎

In the case that one of V or W is concentrated in degree 0 one observes natural identifications $K(V) \otimes_k K(W) \cong K(V \otimes W)$ and the aforementioned lax monoidal structure extends these identifications to arbitrary complexes. In the case where V is concentrated in degree 0, for example, we have $K(V)[n] = V$ at all n and the aforementioned identification is explicitly the map

$$V \otimes_k \mathrm{Hom}^*_k(N\Delta^n, W) \to \mathrm{Hom}^*_k(N\Delta^n, V \otimes_k W), \quad v \otimes f \mapsto (x \mapsto v \otimes f(x)).$$

Corollary 10.3. *For any dg category $\mathbf{A}$ there is an associated simplicial category $K\mathbf{A}$ obtained by applying the lax monoidal functor K to the morphism complexes.*

The identifications $K(V)[0] = V^0$ induce an identification of homotopy categories

$$\mathrm{h}\,\mathrm{N}^{\mathrm{hc}}(K\mathbf{A}) = \mathrm{h}\,K\mathbf{A} = H^0\mathbf{A} = \mathrm{h}\,\mathrm{N}^{\mathrm{dg}}(\mathbf{A}).$$

It's shown in [3, 4] that this equivalence lifts to the ∞-categorical level.

thm:dk_compare

Theorem 10.4 ([4, 00SV])**.** *For any dg category $\mathbf{A}$, there is a natural equivalence of ∞-categories*

$$\mathfrak{Z}_{\mathbf{A}} : \mathrm{N}^{\mathrm{hc}}(K\mathbf{A}) \overset{\sim}{\to} \mathrm{N}^{\mathrm{dg}}(\mathbf{A})$$

which is furthermore a trivial Kan fibration and lifts the identification on homotopy categories $\mathrm{h}\,K\mathbf{A} = H^0\mathbf{A}$.

By naturality, we mean that any dg functor F fits into a diagram

$$\begin{tikzcd}
\mathrm{N}^{\mathrm{dg}}(\mathbf{A}) \arrow[r, "\mathrm{N}^{\mathrm{dg}}F"] & \mathrm{N}^{\mathrm{dg}}(B) \\
\mathrm{N}^{\mathrm{hc}}(K\mathbf{A}) \arrow[u, "\mathfrak{Z}"] \arrow[r, "N^{\mathrm{hc}}KF"] & \mathrm{N}^{\mathrm{hc}}(K\mathbf{B}). \arrow[u, "\mathfrak{Z}"']
\end{tikzcd}$$

10.2. **Derived Dold-Kan for vector spaces.** If one considers dg categories as, loosely, categories enriched in the ∞-category of vector

Lax monoidality tells us that the Dold-Kan functor K enriches to a simplicial functor $K\,\mathrm{Ch}(k)^* \to \underline{\mathrm{sSet}}_k$ which just sends a cochain V to the object KV and which is defined on morphisms via the unique map

$$K\,\mathrm{Hom}_k(V, W) \to \underline{\mathrm{Hom}}_{\mathrm{sSet}_k}(KV, KW)$$

which is compatible with evaluation.

thm:enriched_dk

Theorem 10.5. *Let $\mathbf{Ch}(k)$ denote the dg category of cochains. If k is a field, the simplicial functor*

$$K : K\mathbf{Ch}(k)^* \to \underline{\mathrm{sSet}}_k$$

restricts to an equivalence on $K\mathbf{Ch}(k)^{\leq 0}$.

Proof. In this case a map in $\mathrm{Ch}(k)^{\leq 0}$ is a homotopy equivalence if and only if it is a quasi-isomorphism. It follows via the Dold-Kan equivalence, Theorems I-10.12 & I-10.13 and Proposition I-10.16, that a map in sSet_k is is linear homotopy equivalence if and only if it is a homotopy equivalence. In particular, the comparison map $m_{V,W} : K(V) \otimes_k K(W) \to K(V \otimes_k W)$ is a linear homotopy equivalence. It follows that K induces a monoidal euqivalence on homotopy categories

$$\mathrm{h}\,K : D(k)^{\leq 0} \overset{\sim}{\to} \mathrm{h}\,\mathscr{K}an_k.$$

Now, both of the categories $D(k)$ and $\mathrm{h}\,\mathscr{K}an_k$ admit inner-Homs, which are just given by the inner-Homs at the pre-homotopical level Hom_k^* and $\underline{\mathrm{Hom}}_{\mathrm{sSet}_k}$. Since $\mathrm{h}\,K$ is an equivalence we now obtain a unique isomorphism of inner-Homs

$$K\,\mathrm{Hom}_k^*(V, W) \overset{\sim}{\to} \underline{\mathrm{Hom}}_{\mathrm{sSet}_k}(KV, KW)$$

in $\mathrm{h}\,\mathscr{K}an_k$ which is compatible with evaluation. This unique isomorphism is the image of the correpsonding map at the pre-homotopical level, from which we conclude that the original morphism is a homotopy equivalence. This implies that K is fully faithful, and essential surjectivity just follows form the fact that the non-enriched Dold-Kan functor is essentially surjective. $\square$

We refer to the enriched equivalence of Theorem 10.5 as the enriched Dold-Kan equivalence.

cor:infty_dk

Corollary 10.6. *Suppose k is a field. Then enriched Dold-Kan provides a functor between ∞-categories $K : \mathscr{V}ect_k \to \mathscr{K}an_k$ which restricts to an equivalence of ∞-categories*

$$K : \mathscr{V}ect_k^{\leq 0} \xrightarrow{\sim} \mathscr{K}an_k.$$

Here $\mathscr{V}ect_k := \mathrm{N}^{\mathrm{dg}}(\mathbf{Ch}(k))$ denotes the ∞-category of connective cochains, and we've written simply K for the composite of equivalences

$$\mathscr{V}ect_k \xrightarrow{\sim} \mathrm{N}^{\mathrm{hc}}(K\mathbf{Ch}(k)) \xrightarrow{\mathrm{N}^{\mathrm{hc}}K} \mathscr{K}an_k$$

by an abuse of notaiton.

10.3. **Derived Dold-Kan for abelian categories with projectives.** Let $\mathbb{A}$ be an abelian category with enough projectives. We recall the following strengthening of the usual Dold-Kan equivalence.

Theorem 10.7 (Dold-Kan, [3, Theorem 1.2.3.7])**.** *Let $\mathbb{A}$ be any abelian category and $\mathbb{A}'$ be any additive subcategory in $\mathbb{A}$ which is closed under taking summands. Then the Dold-Kan functor $K : \mathrm{Ch}(\mathbb{A}') \to \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathbb{A}')$ restricts to an equivalence*

$$K : \mathrm{Ch}(\mathbb{A}')^{\leq 0} \xrightarrow{\sim} \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathbb{A}')$$

with inverse given by the normalized cochains functor.

Though our notation is a bit cumbersome, we recall that the category $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathbb{A}')$ is simply the category of simplicial "sets" T whose simplicices $T[n]$ have the structure of objects in $\mathbb{A}'$, and whose structure maps are all maps in $\mathbb{A}'$. In the case where $\mathbb{A}'$ is the full subcategory $\mathrm{Proj}_{\mathbb{A}}$ of projectives in $\mathbb{A}$ we obtain an equivalence

$$\mathrm{Proj}_{\mathbb{A}}^{\leq 0} \xrightarrow{\sim} \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Proj}_{\mathbb{A}}).$$

Definition 10.8. Let k be a field and Vect be the category of finite-dimensional vector spaces. A k-linear category $\mathbb{A}$ is a Vect-module category for which the action map $\mathrm{Vect} \times \mathbb{A} \to \mathbb{A}$ commutes with all colimits which exist in either factor.

Remark 10.9. When $\mathbb{A}$ is locally finite, i.e. has all objects of finite length and finite dimensional morphisms, one should replace Vect with the subcategory Vect_{fin} of finite dimensional vector spaces in the above definition. Equivalently, one can consider such $\mathbb{A}$ along with a Vect-action on its Ind-category.

Note that when $\mathbb{A}$ is k-linear then the category sSet_k acts on simplicial objects in $\mathbb{A}$ via the expected formula

$$V \otimes_k M[n] := V[n] \otimes_k M[n].$$

Via the same arguments as employed in [4, 00RF, kerodon] one produces the Eilenberg-Zilber morphisms, Alexander-Whitney morphisms, and (associative and unital) adjoint morphism

$$m_{V,M} : K(V) \otimes_k K(M) \to K(V \otimes_k M)$$

which compares the Vect-action on $\mathrm{Ch}(\mathbb{A})$ with the sSet_k-action on $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathbb{A})$.

Lemma 10.10. *Suppose $\mathbb{A}$ is a k-linear abelian category. Then for an arbitrary complex of vector spaces V and complex of objects M in $\mathbb{A}$, the map $m_{V,M} : K(V) \otimes_k K(M) \to K(V \otimes_k M)$ is a quasi-isomorphism.*

Proof. One argues exactly as in [4] and Proposition 10.2. However, when necessary, on should replace the simplicial abelian group $\mathbb{Z}[\Delta^n]$ with $T[\Delta^n]$ in arguments from [4], where T is an arbitrary object in $\mathbb{A}$. □

Using the lax structure maps m we can enrich the Dold-Kan equivalence to a simplicial functor

$$K : K\mathbf{Ch}(\mathbb{A}) \to \underline{\mathrm{Fun}}(\Delta^{\mathrm{op}}, \mathbb{A}),$$

where the k-linear simplicial mapping complexes on $\underline{\mathrm{Fun}}(\Delta^{\mathrm{op}}, \mathbb{A})$ are the inner-Homs relative to the sSet_k-action. On objects we simply M take the associated space KM and the maps

$$K\,\mathrm{Hom}^*_{\mathbb{A}}(M, N) \to \underline{\mathrm{Hom}}_{\mathrm{Fun}}(KM, KN)$$

are adjoint to the evaluation morphisms

$$K\,\mathrm{Hom}^*_{\mathbb{A}}(M, N) \otimes_k KM \overset{m}{\to} K(\mathrm{Hom}^*_{\mathbb{A}}(M, N) \otimes_k M) \overset{Kev}{\to} KN.$$

Definition 10.11. For a k-linear abelian category $\mathbb{A}$ with enough projectives, take

$$\mathscr{K}an_{\mathbb{A}} := \mathrm{N}^{\mathrm{hc}}\,\underline{\mathrm{Fun}}(\Delta^{\mathrm{op}}, \mathrm{Proj}_{\mathbb{A}}).$$

thm:enriched_d2

Theorem 10.12. *Let $\mathbb{A}$ be a k-linear abelian category with enough projectives, and take $\mathbf{Proj}^{\leq 0}_{\mathbb{A}}$ the dg category of connective cochains of projectives. The enriched Dold-Kan functor*

$$K : K\mathbf{Proj}^{\leq 0}_{\mathbb{A}} \to \underline{\mathrm{Fun}}(\Delta^{\mathrm{op}}, \mathrm{Proj}_{\mathbb{A}})$$

is an equivalence of simplicial categories.

Proof. Same as the proof of Theorem 10.5. □

cor:derived_dk_A

Corollary 10.13. *For any k-linear abelian category $\mathbb{A}$ which has enough projectives, there is an equivalence of ∞-categories*

$$K : \mathscr{D}^{\leq 0}(\mathbb{A}) \overset{\sim}{\to} \mathscr{K}an_{\mathbb{A}}.$$

11. Representable and corepresentable functors

Definition 11.1. Let $\mathscr{C}$ be an ∞-category. A functor $F : \mathscr{C} \to \mathscr{K}an$ is corepresented by an object x in $\mathscr{C}$ if F is a transport functor for the left fibration $\mathscr{C}_{x/} \to \mathscr{C}$. We say F is corepresentable if it is corepresented by some object in $\mathscr{C}$.

We say a functor $G : \mathscr{C}^{\mathrm{op}} \to \mathscr{K}an$ is represented by an object y in $\mathscr{C}$ if it is corepresented by y when considered as an object in $\mathscr{C}^{\mathrm{op}}$, i.e. if it is a contravariant transport functor for the right fibration $\mathscr{C}_{/y} \to \mathscr{C}$. A corepresentable functor is a fnuctor which is corepresented by some object in $\mathscr{C}$.

We note that if F and F' are corepresented by an object x in $\mathscr{C}$, then F and F' are isomorphic, via the uniqueness of transport functors. We also see, by Corollary 9.26, that a functor $F : \mathscr{C} \to \mathscr{K}an$ is representable if the corresponding ∞-category $\mathrm{Un}(F) \cong \int_{\mathscr{C}} F$ has an initial object. It is clear from Corollary 9.26 that h is representable if, in some sense, it has an "initial object" in some fiber $F(x)$.

def:initial_fun

Definition 11.2. Given a functor $F : \mathscr{C} \to \mathscr{K}an$, we say an object $1_x : * \to F(x)$ is an initial object for F, over x, if at each y in $\mathscr{C}$ the composite

$$\mathrm{Hom}_{\mathscr{C}}(x, y) \overset{F}{\to} \mathrm{Hom}_{\mathscr{K}an}(F(x), F(y)) \overset{\theta^{-1}}{\to} \mathrm{Fun}_{\mathscr{K}an}(F(x), F(y)) \overset{1_x^*}{\to} F(y) \tag{22}$$ eq:initial_fun

is an isomorphism in $\mathrm{h}\,\mathscr{K}an$.

Note that this condition is really a restriction on the induced functor on enriched categories $\pi F : \pi\mathscr{C} \to \pi\mathscr{K}an$. Since the isomorpihsms θ of Theorem 5.27 is seen to preserve identity morphisms, we see that the above composite at x,

$$\mathrm{Hom}_{\mathscr{C}}(x,x) \to F(x)$$

sends the identity $id_x : x \to x$ to 1_x. Since the θ^{-1} assemble into an equivalence of H$\,\mathscr{K}an$-enriched categories $\pi\mathscr{K}an \xrightarrow{\sim} \pi\underline{\mathrm{Kan}}$ (Proposition 7.6), we also observes at any choice of $1_x : * \to F(x)$ a diagram

$$\begin{tikzcd} F(x) \arrow[r, "F(\alpha)"] & F(y) \\ \mathrm{Hom}_{\mathscr{C}}(x,x) \arrow[u] \arrow[r, "\alpha_*"'] & \mathrm{Hom}_{\mathscr{C}}(x,y). \arrow[u] \end{tikzcd}$$

Our first aim is to prove the following.

thm:rep_funs **Theorem 11.3.** *A functor $F : \mathscr{C} \to \mathscr{K}an$ is corepresented by an object x in $\mathscr{C}$ if and only if F admits an initial object which lies over x, $1_x : * \to F(x)$.*

11.1. **Left fibrations with initial transport.** Consider a left fibration $q : \mathscr{E} \to \mathscr{C}$ with transport functor $F : \mathscr{C} \to \mathscr{K}an$. The induced functor $\pi F : \pi\mathscr{C} \to \pi\mathscr{K}an$ is determined by paramtrized homotopy transport, according to Corollary 7.12. Hence the composite of (22) at any object $1_x : * \to F(x) \cong \mathscr{E}_x$ is identified with the composite

$$\mathrm{Hom}_{\mathscr{C}}(x,y) \times \{1_x\} \to \mathrm{Hom}_{\mathscr{C}}(x,y) \times \mathscr{E}_x \to \mathscr{E}_y, \qquad (23)$$ eq:2540

where the final map is given by parametrized homotopy transport (Definition 7.7). More precisely, the adjunction

$$\mathrm{Adj} : \mathrm{Hom}_{\mathrm{Kan}}(\mathrm{Hom}(x,y), \mathrm{Fun}(*, \mathscr{E}_y)) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Kan}}(\mathrm{Hom}(x,y) \times \{*\}, \mathscr{E}_y)$$

the map (22) is identified with the map (23). One sees directly that any map $\gamma : \mathrm{Hom}_{\mathscr{C}}(x,y) \to \mathrm{Fun}(*, \mathscr{E}_y) \cong \mathscr{E}_y$ fits into a diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathscr{C}}(x,y) \times \{*\} \arrow[r, "\mathrm{Adj}(\gamma)"] & \mathscr{E}_y \\ \mathrm{Hom}_{\mathscr{C}}(x,y) \arrow[u, "\cong"] \arrow[r, "\gamma"'] & \mathrm{Fun}(*, \mathscr{E}_y), \arrow[u, "\cong"'] \end{tikzcd}$$

so that γ is an euqivalence if and only if $\mathrm{Adj}(\gamma)$ is an equivalence. So we see that an object $1_x : * \to F(x)$ is initial if and only if the maps (23) are all equivalences.

prop:2653 **Proposition 11.4.** *Let $F : \mathscr{C} \to \mathscr{K}an$ be any functor and take $q : \mathscr{E} = \int_{\mathscr{C}} F \to \mathscr{C}$ the corresponding left fibration. For any choice object x in $\mathscr{C}$, any object $1_x : * \to F(x)$, and let $\widetilde{x}$ be the image of 1_x in $\mathscr{E}_x$ under the equivalence $\theta : F(x) \to \mathscr{E}_x$. Then $\widetilde{x}$ is initial in $\mathscr{E}$ if and only if 1_x is an initial object for F.*

Proof. Via the equivalence $\mathscr{C}_{x/} \xrightarrow{\sim} \{x\}\widetilde{\times}_{\mathscr{C}}\mathscr{C}$ we see that the oriented fiber product admits an initial object. By the specific expression of this equivalence give in Section I-(9.5) one sees that this equivalence sends id_x to id_x, so that the identity specifically is seen to be initial in $\{x\} \times^{\mathrm{or}}_{\mathscr{C}} \mathscr{C}$.

By Corollary 9.24 there is a unique map of left fibrations $t : \{x\} \times^{\mathrm{or}}_{\mathscr{C}} \mathscr{C} \to \mathscr{E}$ which sends id_x to $\widetilde{x}$. This map is an equivalence if and only if all of the induced maps on the fibers

$$t : \mathrm{Hom}_{\mathscr{C}}(x, y) \to \mathscr{E}_y$$

are isomorphisms.

Since enriched homotopy transport for the oriented fiber product is given by composition on $\mathrm{Hom}_{\mathscr{C}}$ (Proposition **??**), the diagrams

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{C}}(x,y) \times \mathscr{E}_x \arrow[r] & \mathscr{E}_y \\
\mathrm{Hom}_{\mathscr{C}}(x,y) \times \mathrm{Hom}_{\mathscr{C}}(x,x) \arrow[u, "id\times t"] \arrow[r, "\circ"] & \mathrm{Hom}_{\mathscr{C}}(x,y) \arrow[u, "t"']
\end{tikzcd}$$

commute at all y in $\mathscr{C}$, where the top map is enriched homotopy transport for $\mathscr{E}$. Restricting to id_x in the second factor produces a diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{C}}(x,y) \times \{\widetilde{x}\} \arrow[r] & \mathscr{E}_y \\
\mathrm{Hom}_{\mathscr{C}}(x,y) \times \{id_x\} \arrow[u, "id"] \arrow[r, "id"'] & \mathrm{Hom}_{\mathscr{C}}(x,y) \arrow[u, "t_y"']
\end{tikzcd}$$

from which we see that t_y is an equivalence at all y if and only if the maps (23) are all isomorphisms in $\mathrm{h}\,\mathscr{K}an$. Via the equivalence of enriched functors $\pi F \overset{\sim}{\to} q_!$ we observe that these maps are all equivalences if and only if 1_x is an initial object for F. □

Now, for an arbitrary left fibration $q : \mathscr{E} \to \mathscr{C}$, with transport functor $F : \mathscr{C} \to \mathscr{K}an$, we have the equivalence of left fibrations $\mathscr{E} \overset{\sim}{\to} \int_{\mathscr{C}} F$ which is implicit in the assertion that F is a transport functor. We therefore see that $\mathscr{E}$ admits an initial object if and only if $\int_{\mathscr{C}} F$ admits an initial object. So Proposition 11.4 implies the following.

Corollary 11.5. *Let $q : \mathscr{E} \to \mathscr{C}$ be a left fibration between ∞-categories. Then $\mathscr{E}$ admits an initial object $\widetilde{x}$ over a point x in $\mathscr{C}$ if and only if the corresponding transport functor F admits an initial object $1_x : * \to F(x)$.*

We now observe the proof of Theorem 11.3.

Proof of Theorem 11.3. By Proposition 11.4 a functor F admits an initial object over x if and only if the corresponding left fibration $\int_{\mathscr{C}} F$ admits an initial object in the fiber over x. This occurs if and only if there is an equivalence of left fibrations

$$\mathscr{C}_{x/} \overset{\sim}{\to} \int_{\mathscr{C}} F$$

over $\mathscr{C}$. The existence of such an equivalence, by definition, characterizes F as a transport functor for the fibration $\mathscr{C}_{x/} \to \mathscr{C}$. □

11.2. **Corepresentable functors for simplicial categories.** Let $\underline{A}$ be a simplicial category which is enriched in Kan complexes. At any choice of an object x in $\underline{A}$ we have the simplicial functor

$$\underline{\mathrm{Hom}}_{\underline{A}}(x, -) : \underline{A} \to \underline{\mathrm{Kan}}.$$

Taking homotopy coherent nerves then provides a functor

$$\underline{\mathrm{Hom}}_{\mathrm{N}^{\mathrm{hc}}(\underline{A})}(x,-):\mathrm{N}^{\mathrm{hc}}(\underline{A})\to\mathscr{K}an.$$

prop:simplicial_corep

Proposition 11.6. *Let $\underline{A}$ be a* Kan*-enriched simplicial category and take $\mathscr{A}=\mathrm{N}^{\mathrm{hc}}(\underline{A})$. At any object x in $\mathscr{A}$ the functor $\underline{\mathrm{Hom}}_{\mathscr{A}}(x,-):\mathscr{A}\to\mathscr{K}an$ is corepresented by x.*

Proof. Since the map $\pi\underline{A}\to\pi\mathscr{A}$ induced by the equivalences of Theorem 5.27 is an equivalence of ∞-categories, we have an identification of $\mathrm{h}\,\mathscr{K}an$-enriched functors

$$\underline{\mathrm{Hom}}_{\mathscr{A}}(x,-)\cong\mathrm{Hom}_{\mathscr{A}}(x,-).$$

Since the functor $\mathrm{Hom}_{\mathscr{A}}(x,-)$ admits an initial object over x, so does $\underline{\mathrm{Hom}}_{\mathscr{A}}(x,-)$. It follows that $\underline{\mathrm{Hom}}_{\mathscr{A}}(x,-)$ is corepresented by x. □

Corollary 11.7. *For $\underline{A}$ and $\mathscr{A}$ as in Proposition 11.6, a functor $F:\mathscr{A}\to\mathscr{K}an$ is corepresentable if and only if F admits an isomorphism*

$$\underline{\mathrm{Hom}}_{\mathscr{A}}(x,-)\overset{\sim}{\to}F$$

at some x in $\underline{A}$.

11.3. **Corepresentable functors for dg categories.** Let $\mathbf{A}$ be a dg category, and $\mathscr{A}=\mathrm{N}^{\mathrm{dg}}(\mathbf{A})$. At any object x in $\mathbf{A}$ we have the dg functor $\mathrm{Hom}^*_{\mathbf{A}}(x,-):\mathbf{A}\to\mathrm{Ch}(k)$.

lem:2851

Lemma 11.8. *Let $\mathbf{A}$ be a dg category. At any object V in $\mathbf{A}$ the simplicial functor*

$$K\circ\mathrm{Hom}^*_{\mathbf{A}}(V,-):K\mathbf{A}\to K\mathbf{Ch}(k)\overset{K}{\to}\underline{\mathrm{sSet}}_k$$

is equal to the functor $\underline{\mathrm{Hom}}_{KA}(V,-):K\mathbf{A}\to\underline{\mathrm{sSet}}_k$.

Proof. Take $h_V=\mathrm{Hom}^*_{\mathbf{A}}(V,-)$ and $\underline{h}_V=\underline{\mathrm{Hom}}_{K\mathbf{A}}(V,-)$. On objects these functors are the same. For the composite, the original map

$$h_V:\mathrm{Hom}^*_{\mathbf{A}}(W,W')\to\mathrm{Hom}^*_k(h_VW,h_VW')$$

fits into, and is specified by, a diagram

$$\begin{tikzcd}
\mathrm{Hom}^*_{\mathbf{A}}(W,W')\otimes_k h_V(W) \arrow[d, "h_V\otimes_k id"'] \arrow[dr, "\circ"] & \\
\mathrm{Hom}^*_k(h_VW,h_VW')\otimes h_V(W) \arrow[r, "ev"'] & h_V(W').
\end{tikzcd}$$

Hence Kh_V fits into a diagram

$$\begin{tikzcd}
K\,\mathrm{Hom}^*_{\mathbf{A}}(W,W')\otimes_k Kh_V(W) \arrow[d, "Kh_V\otimes_k id"'] \arrow[dr, "K\circ"] & \\
K\,\mathrm{Hom}^*_k(h_VW,h_VW')\otimes_k Kh_V(W) \arrow[r, "ev"'] & Kh_V(W').
\end{tikzcd}$$

Taking inner-Homs for sSet_k now gives a diagram

$$\begin{tikzcd}
K\,\mathrm{Hom}^*_{\mathbf{A}}(W,W') \arrow[d, "Kh_V"'] \arrow[dr, "\underline{h}_V"] & \\
K\,\mathrm{Hom}^*_k(h_VW,h_VW') \arrow[r, "K"'] & \underline{\mathrm{Hom}}_{\underline{\mathrm{sSet}}_k}(Kh_VW,Kh_VW').
\end{tikzcd}$$

This implies an equality between these two functors on morphism complexes as well. ◻

lem:2889

Lemma 11.9. *Given a diagram in $\mathscr{C}at_\infty$*

$$\begin{tikzcd} \mathscr{C} \arrow[d, "\xi"'] \arrow[dr, "F"] & \\ \mathscr{D} \arrow[r, "G"'] & \mathscr{K}an \end{tikzcd}$$

in which ξ is an equivalence, then F is corepresentable by an object x if and only if G is corepresentable by $\xi(x)$.

Proof. We have an isomorphism of functors $F \cong G \circ \xi$. Since representability is stable under isomorphism, we may assume $F = G \circ \xi$. Fix arbitrary points $x, y : * \to \mathscr{C}$ and take $x' = \xi(x)$, $y' = \xi(y)$.

We consider the corresponding maps on π-enriched categories to observe a diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathscr{C}}(x,y) \times F(x) \arrow[r] \arrow[d, "\xi \times id"'] & \mathrm{Hom}_{\mathscr{K}an}(F(x), F(y)) \times F(x) \arrow[r, "ev"] \arrow[d, "="] & F(y) \arrow[d, "="] \\ \mathrm{Hom}_{\mathscr{D}}(x', y') \times G(\xi x) \arrow[r] & \mathrm{Hom}_{\mathscr{K}an}(G(x'), G(y')) \times G(x') \arrow[r, "ev"'] & G(y') \end{tikzcd}$$

in $\mathrm{h}\,\mathscr{K}an$ in which all vertical maps are isomorphisms. From this diagram we see that F admits an initial object if and only if G admits an initial object. Hence F is representable if and only if G is representable. ◻

We understand that, at any dg category, the functor

$$\underline{\mathrm{Hom}}_{K\mathbf{A}}(V, -) : \mathrm{N}^{\mathrm{hc}}(K\mathbf{A}) \to \mathscr{K}an_k \subseteq \mathscr{K}an$$

is corepresented by the given object V. This follows by Proposition 11.6. Naturality of the equivalence $\mathfrak{Z}$ from Theorem 10.4, in conjunction with Lemmas 11.8 and 11.9 above, tell us that the Hom-complexes for dg categories also provide representable functors, in the only way that makes sense.

prop:dg_corep

Proposition 11.10. *Let $\mathbf{A}$ be any dg category with associated ∞-category $\mathscr{A} = N^{\mathrm{dg}}(A)$. At any object V in $\mathbf{A}$ the functor*

$$K\,\mathrm{Hom}^*_{\mathbf{A}}(V, -) : \mathscr{A} \to \mathscr{V}ect \to \mathscr{K}an$$

is corepresented by V.

Corollary 11.11. *Let $\mathbf{A}$ be any dg category with associated ∞-category $\mathscr{A} = N^{\mathrm{dg}}(A)$. A functor $F : \mathscr{A} \to \mathscr{K}an$ is corepresented by an object V in $\mathscr{A}$ if and only if F is isomorphic to the functor $K\,\mathrm{Hom}^*_{\mathbf{A}}(V, -)$.*

11.4. **Representable functors for simplicial and dg categories.**

Lemma 11.12. *Let $\underline{A}$ be a simplicial category which is enriched in Kan complexes. Then we have $\mathrm{N}^{\mathrm{hc}}(\underline{A}^{\mathrm{op}}) = \mathrm{N}^{\mathrm{hc}}(\underline{A})^{\mathrm{op}}$.*

Here the opposite category $\underline{A}^{\mathrm{op}}$ is simply the category obtained by applying the symmetry on sSet to the morphisms. The identification of opposites follows from the identification

$$\mathrm{Fun}_{\mathrm{sCat}}(\mathrm{Path}\,\Delta^n, \underline{A}^{\mathrm{op}}) = \mathrm{Fun}((\mathrm{Path}\,\Delta^n)^{\mathrm{op}}, \underline{A}) = \mathrm{Fun}(\mathrm{Path}((\Delta^n)^{op}), \underline{A})$$

We now consider the functor $\underline{\mathrm{Hom}}_{\underline{A}}(-,y)$, for y in $\underline{A}$ as a functor from the opposite category

$$\underline{\mathrm{Hom}}_{\underline{A}}(-,y) : \underline{A}^{\mathrm{op}} \to \underline{\mathrm{sSet}}.$$

As a corollary to Proposition 11.6 we observe the following.

Corollary 11.13. *Let $\underline{A}$ be a simplicial category which is enriched in Kan complexes. Then at any object y in $\underline{A}$, the functor*

$$\underline{\mathrm{Hom}}_{\underline{A}}(-,y) : \mathrm{N}^{\mathrm{hc}}(\underline{A}) \to \mathscr{K}an$$

is a representable functor which is represented by y.

In the dg setting we also have the opposite category $\mathbf{A}^{\mathrm{op}}$. Here we employ the Koszul sign rule in the symmetric on $\mathrm{Ch}(k)$, so that composition for $\mathbf{A}^{\mathrm{op}}$ inherits a sign

$$f \cdot_{\mathrm{op}} g := (-1)^{\deg(f)\deg(g)} gf.$$

One can check the following.

lem:2965 **Lemma 11.14.** *For any dg category $\mathbf{A}$, the maps on n-simplices*

$$\mathrm{N}^{dg}(\mathbf{A})^{\mathrm{op}}[n] \to \mathrm{N}^{\mathrm{dg}}(\mathbf{A}^{\mathrm{op}})[n], \quad \{f_I : I \subseteq [n]\} \mapsto \{(-1)^{|I|(|I|-1)/2} f_I : I \subseteq [n]\},$$

define an isomorphism of ∞-categories $\mathrm{N}^{dg}(\mathbf{A})^{\mathrm{op}} \overset{\sim}{\to} \mathrm{N}^{\mathrm{dg}}(\mathbf{A}^{\mathrm{op}})$.

From Proposition 11.10 and Lemma 11.9 we now observe the following.

Proposition 11.15. *For any dg category $\mathbf{A}$, and any object W in $\mathbf{A}$, the functor*

$$K\,\mathrm{Hom}^*_{\mathbf{A}}(-,W) : \mathrm{N}^{\mathrm{dg}}(\mathbf{A})^{\mathrm{op}} \to \mathscr{K}an$$

is a representable functor which is represented by W.

Here of course we have abused notation to view the functor $\mathrm{Hom}^*_{\mathbf{A}}(-,W) : \mathrm{N}^{\mathrm{hc}}(\mathbf{A}^{\mathrm{op}}) \to \mathscr{V}ect$ as a functor from $\mathrm{N}^{\mathrm{dg}}(\mathbf{A})^{\mathrm{op}}$, via the identification of Lemma 11.14.

12. Twisted arrows and bifunctorial Homs

12.1. The twisted arrows category.

Definition 12.1. Given a simplicial set $\mathscr{C}$, we define the twisted arrow category $\mathscr{T}w(\mathscr{C})$ as the simplicial set whose n-simplices are

$$\mathscr{T}w(\mathscr{C})[n] := \mathrm{Hom}_{\mathrm{sSet}}((\Delta^n)^{\mathrm{op}} \star \Delta^n, \mathscr{C}).$$

Restricting along the inclusions

$$(\Delta^n)^{\mathrm{op}} \to (\Delta^n)^{\mathrm{op}} \star \Delta^n \quad \text{and} \quad \Delta^n \to (\Delta^n)^{\mathrm{op}} \star \Delta^n$$

provides a natural map to the product

$$\lambda : \mathscr{T}w(\mathscr{C}) \to \mathscr{C}^{\mathrm{op}} \times \mathscr{C}. \tag{24}$$ eq:lambda

To get our heads on straight here, let's observe directly that an object in $\mathscr{T}w(\mathscr{C})$ is a choice of a morphism $\alpha : x \to y$ in $\mathscr{C}$. A morphism from an objects $\alpha : x \to y$ to some other $\alpha' : x' \to y'$ is a diagram of the form

$$\begin{tikzcd} x \arrow[r, "\alpha"] \arrow[dr] & y \arrow[d] \\ x' \arrow[u] \arrow[r, "\alpha'"'] \arrow[ur] & y'. \end{tikzcd}$$

If we consider the fiber $\{(x,y)\}\times_{(\mathscr{C}^{\mathrm{op}}\times\mathscr{C})}\mathscr{T}w(\mathscr{C})$, a simplex in this fiber can be visualized as some directed diagram from x to y which is "completely filled in",

$$\begin{tikzcd} x \arrow[r, dotted] \arrow[r, bend left, dotted] \arrow[r, bend right, dotted] & y. \end{tikzcd}$$

We prove below that these fibers are a type of bifunctorial Hom space for $\mathscr{C}$, where bifunctoriality simply refers to the fact that one has two variables to tune in the base.

We note that the join $(\Delta^n)^{\mathrm{op}}\star\Delta^n$ is identified with Δ^{2n+1} via the bijection

$$[2n+1]\to[n]\amalg[n],\quad i\mapsto\begin{cases} n-i \text{ in the first set if } i\leq n\\ i-n \text{ in the second set if } i\geq n.\end{cases}$$

prop:tw_inner

Proposition 12.2 ([4, 03JT])**.** *The restriction map $\lambda:\mathscr{T}w(\mathscr{C})\to\mathscr{C}^{\mathrm{op}}\times\mathscr{C}$ is a left fibration. More generally, if $\mathscr{C}\to S$ is an inner fibration of simplicial sets, then the restriction map $\mathscr{T}w(\mathscr{C})\to(\mathscr{C}^{\mathrm{op}}\times\mathscr{C})\times_{(S^{\mathrm{op}}\times S)}\mathscr{T}w(S)$ is a left fibration.*

We only outline the main points of the proof. The reader can find details in the cited text.

Proof outline. We wish to show that any lifting problem of the form

$$\begin{tikzcd} \Lambda^n_{n-i} \arrow[r] \arrow[d] & \mathscr{T}w(\mathscr{C}) \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & (\mathscr{C}^{\mathrm{op}}\times\mathscr{C})\times_{(S^{\mathrm{op}}\times S)}\mathscr{T}w(S), \end{tikzcd}$$

with $i\leq n$, admits a solution. Such a lifting problem admits a solution if an only if the corresponding problem

$$\begin{tikzcd} K_0 \arrow[r] \arrow[d] & \mathscr{C} \arrow[d] \\ \Delta^{2n+1} \arrow[r] \arrow[ur, dotted] & S \end{tikzcd} \tag{25}$$

eq:1641

admits a solution, where $K_0\subseteq\Delta^{2n+1}$ is some subcomplex which we descirbe below.

Given a subset $J\subseteq[2n+1]$, the non-degenerate simplex $\Delta^J\subseteq\Delta^{2n+1}$ lies in K_0 if and only if J is contained in one of $[n]$ or $[2n+1]-[n]$, or J is contained in a subset $[2n+1]-\{j,2n+1-j\}$ with $j\neq i$. It is argued in [4] that the inclusion $K_0\to\Delta^{2n+1}$ is in fact anodyne, by factoring this map into a sequence of inclusions

$$K_0\to K_1\to\cdots\to K_m=\Delta^{2n+1}$$

in which each K_{i+1} is obtained from K_i by attaching a single non-degenerate simplex. Each such inclusion $K_i\to K_{i+1}$ is shown to be anodyne, so that the composition $K_0\to\Delta^{2n+1}$ is in fact anodyne, and we find that the problem (25) admits a solution, as desired. $\square$

Since $\mathscr{C}^{\mathrm{op}}\times\mathscr{C}$ is itself an ∞-category whenever $\mathscr{C}$ is an ∞-category, we find that the twisted arrow category $\mathscr{T}w(\mathscr{C})$ is also an ∞-category in this case.

Corollary 12.3. *If $\mathscr{C}$ is an ∞-category, the twisted arrow category $\mathscr{T}w(\mathscr{C})$ is also an ∞-category.*

Via Proposition 12.2, and the general phenomena of transport for left fibrations (Proposition 6.17), we understand that the left fibration $\lambda : \mathscr{T}w(\mathscr{C}) \to \mathscr{C}^{\mathrm{op}} \times \mathscr{C}$ identifies a associated transport functor

$$H : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{K}an.$$

This transport functor is uniquely determined, up to a contractible space of choices, by the assertion that H fits into a categorical pullback diagram

$$\begin{tikzcd} \mathscr{T}w(\mathscr{C}) \arrow[r] \arrow[d] & \mathscr{K}an_{*/} \arrow[d] \\ \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \arrow[r, "H"'] & \mathscr{K}an. \end{tikzcd}$$

Definition 12.4. A Hom-functor for and ∞-category $\mathscr{C}$ is a transport functor

$$H : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{K}an$$

for the left fibration $\lambda : \mathscr{T}w(\mathscr{C}) \to \mathscr{C}^{\mathrm{op}} \times \mathscr{C}$.

The first aims of this section are to provide a calculation of the fibers of the twisted arrow fibration sufficient conditions which allow us to identify a Hom functor when we see one. Of interest are Hom functors for nerves of dg and simplicial categories (e.g. Hom functors for derived categories).

Let us note, as a bit of foreshadowing, that any Hom functor determines maps into the functor categories

$$H_* : \mathscr{C} \to \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathscr{K}an) \quad \text{and} \quad H^* : \mathscr{C}^{\mathrm{op}} \to \mathrm{Fun}(\mathscr{C}, \mathscr{K}an).$$

We will eventually find that these functors are both fully faithful embeddings.

12.2. **Fibers of the twisted arrows fibration.** At any points in an ∞-category $x : * \to \mathscr{C}$ we can restrict along the projection $(\Delta^n)^{\mathrm{op}} \to *$ to obtain an inclusion into the twisted arrows category

$$\mathscr{C}_{x/} \to \{x\} \times_{\mathscr{C}^{\mathrm{op}}} \mathscr{T}w(\mathscr{C}). \tag{26}$$

eq:2199

This map fits into a diagram over $\mathscr{C}$.

Lemma 12.5 ([4, 03JW])**.** *At any object $x : * \to \mathscr{C}^{\mathrm{op}}$, and isomorphism $\alpha : x \to x'$, the map α is an initial object in the fiber $\{x\} \times_{\mathscr{C}^{\mathrm{op}}} \mathscr{T}w(\mathscr{C})$.*

We note that we can take $x' = x$ and $\alpha = id_x$. In particular, we observe that the fiber of the twisted arrow category over any point in $\mathscr{C}^{\mathrm{op}}$ admits an initial object.

Outline of proof. We want to show that the forgetful map

$$(\{x\} \times_{\mathscr{C}^{\mathrm{op}}} \mathscr{T}w(\mathscr{C}))_{\alpha/} \to \{x\} \times_{\mathscr{C}^{\mathrm{op}}} \mathscr{T}w(\mathscr{C})$$

is a trivial Kan fibration. We note that solving the relevant lifting problem along an inclusion $\partial\Delta^{n-1} \to \Delta^{n-1}$ is equivalent to extending the boundary of an n-simplex $\bar{\sigma} : \partial\Delta^n \to \mathscr{T}w(\mathscr{C})$, with $n > 0$ and $\bar{\sigma}|_{\{0\}} = \alpha$, to an n-simplex $\sigma : \Delta^n \to \mathscr{T}w(\mathscr{C})$. This problem, in turn, is equivalent to solving a lifting problem of the form

$$\begin{tikzcd} K \arrow[r, "\tau_0"] \arrow[d] & \mathscr{C} \arrow[d] \\ \Delta^{2n+1} \arrow[r] \arrow[ur, dotted] & * \end{tikzcd} \tag{27}$$

eq:2209

with K the subcomplex in Δ^{2n+1} which is the union of the J-simpleces $\Delta^J \to \Delta^{2n+1}$, where $J \subseteq [2n+1]$ is any subset which is either contained in $[n]$ or $[2n+1]-[n]$ or $[2n+1]-\{i, 2n+1-i\}$ for some i. The assumption that $\bar{\sigma}$ lands in the fiber $\{x\} \times_{\mathscr{C}^{\mathrm{op}}} \mathscr{T}w(\mathscr{C})$ forces $\tau_0|_{\Delta^n}$ to be of constant value x, and the assmuption that $\bar{\sigma}|_{\{0\}} = \alpha$ forces $\tau_0|\Delta^{\{n,n+1\}} = \alpha$

Now, one argues that there is a factoring of the inclusion $K \to \Delta^{2n+1}$ into a sequence of inclusions

$$K = K_0 \to K_1 \to \cdots \to K_m = \Delta^{2n+1}$$

with each K_{i+1} fitting into a pushout square

$$\begin{tikzcd} \Lambda^{d_i}_{k_i} \arrow[r] \arrow[d] & K_i \arrow[d] \\ \Delta^{d_i} \arrow[r] & K_{i+1} \end{tikzcd}$$

with each $k_i < d_i$, or $k_i = 0$, $d_i > 1$, and $\Delta^{\{0,1\}} \to \Lambda^{d_i}_0 \to \Delta^{2n+1}$ landing in the 1-skeleton $\mathrm{Sk}_1\,\Delta^{n+1} \subseteq K$. This final condition implies that, in the case $k_i = 0$, and map $\tau_i : K_i \to \mathscr{C}$ extending $\tau_0 : K \to \mathscr{C}$ sends the initial vertex

$$\Delta^{\{0,1\}} \to \Lambda^{d_i}_0 \to K_i \overset{\tau_i}{\to} \mathscr{C}$$

to an isomorphism in $\mathscr{C}$.

Using the above information, and Proposition I-4.33, we can produce sequential solutions to the lifting problems

$$\begin{tikzcd} K_i \arrow[r, "\tau_i"] \arrow[d] & \mathscr{C} \arrow[d] \\ K_{i+1} \arrow[r] \arrow[ur, dotted, "\tau_{i+1}"] & * \end{tikzcd}$$

in order to produce the desired solution $\tau = \tau_m : \Delta^{2n+1} \to \mathscr{C}$ to the problem (27). $\square$

prop:2273

Proposition 12.6. *The maps (26) define equivalences of left and right fibrations*

$$\begin{tikzcd} \mathscr{C}_{x/} \arrow[rr, "\sim"] \arrow[dr] & & \{x\} \times_{\mathscr{C}^{\mathrm{op}}} \mathscr{T}w(\mathscr{C}) \arrow[dl] \\ & \mathscr{C} & \end{tikzcd} .$$

Proof. The map $F : \mathscr{C}_{x/} \to \{x\} \times_{\mathscr{C}^{\mathrm{op}}} \mathscr{T}w(\mathscr{C})$ preserves the initial object id_x, and so is an equivalence of ∞-categories by Proposition 9.25. It follows that F is an equivalence of cocartesian fibrations, by Proposition 3.1. $\square$

We recall that the fibers of any equivalence of isofibrations are again equivalences (Corollary I-5.24). So Proposition 12.6 tells us that the fibers of the twisted arrows category $\mathscr{T}w(\mathscr{C})$ are identified with the mapping spaces for $\mathscr{C}$.

cor:tw_fibers

Corollary 12.7. *Let $\mathscr{C}$ be an ∞-category. At any pair of points $x, y : * \to \mathscr{C}$ the natural map*

$$\mathrm{Hom}^{\mathrm{L}}_{\mathscr{C}}(x,y) = \mathscr{C}_{x/} \times_{\mathscr{C}} \{y\} \to \{x\} \times_{\mathscr{C}^{\mathrm{op}}} \mathscr{T}w(\mathscr{C}) \times_{\mathscr{C}} \{y\}$$

is an equivalence of Kan complexes.

Corollary 12.8. *For any Hom functor $H : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{K}an$, i.e. classifying functor for the twisted arrows fibration, the restriction*

$$\mathscr{C} \overset{x \times id}{\to} \mathscr{C} \times \mathscr{C} \overset{H}{\to} \mathscr{K}an$$

at any point $x : * \to \mathscr{C}$ is a classifying functor for the forgetful functor $\mathscr{C}_{x/} \to \mathscr{C}$.

12.3. Recognition for Hom functors.

12.4. Hom functors for simplicial and dg categories.

13. Limits and colimits in ∞-categories

Let $\mathscr{C}$ be an ∞-category and K be a simplicial set. We consider the embedding

$$\mathscr{C} \cong \mathrm{Fun}(*, \mathscr{C}) \to \mathrm{Fun}(K, \mathscr{C})$$

induced by the terminal map $K \to *$ and, for any object x in $\mathscr{C}$, let $\underline{x}$ denote the corresponding image in $\mathrm{Fun}(K, \mathscr{C})$. This map $\underline{x}$ is just the constant function at x, i.e. the composite

$$K \to * \overset{x}{\to} \mathscr{C}.$$

def:lim_colim

Definition 13.1. Let $\mathscr{C}$ be an ∞-category and $p : K \to \mathscr{C}$ be an arbitrary map from a simplicial set. A transformation $l : \underline{y} \to p$ is said to exhibit y as a limit of p if, for each object x in $\mathscr{C}$, the composite

$$\mathrm{Hom}_{\mathscr{C}}(z, y) \to \mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{C})}(\underline{z}, \underline{y}) \overset{l_*}{\to} \mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{C})}(\underline{z}, p)$$

is an isomorphism in $\mathrm{h}\,\mathscr{K}an$. A transformation $c : p \to \underline{x}$ is said to exhibit x as a colimit of p if at each z in $\mathscr{C}$ the composite

$$\mathrm{Hom}_{\mathscr{C}}(x, z) \to \mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{C})}(\underline{x}, \underline{z}) \overset{c^*}{\to} \mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{C})}(p, \underline{z})$$

is an isomorphism in $\mathrm{h}\,\mathscr{K}an$.

Definition 13.2. Let $p : K \to \mathscr{C}$ be a diagram in an ∞-category $\mathscr{C}$. We say an object y is a limit (resp. colimit) for p if there is a transformation $l : \underline{y} \to p$ (resp. $c : p \to \underline{y}$) which exhibits y as a limit (resp. colimit) for p.

sect:char_lim

13.1. **Characterizations of limits.** Having fixed diagram $p : K \to \mathscr{C}$, we recall the oriented fiber product

$$\mathscr{C} \times^{\mathrm{or}}_{\mathrm{Fun}(K,\mathscr{C})} \{p\} := \mathrm{Fun}(\Delta^1 \times K, \mathscr{C}) \times_{\mathrm{Fun}(\partial\Delta^1 \times K, \mathscr{C})} (\mathscr{C} \times \{p\}).$$

Here $\mathscr{C}$ maps to $\mathrm{Fun}(\{0\} \times K, \mathscr{C})$ via the constant functions and p maps to $\mathrm{Fun}(\{1\} \times K, \mathscr{C})$ as the given diagram. The objects in this oriented product are transformations $\underline{z} \to p$ in $\mathrm{Fun}(K, \mathscr{C})$.

prop:3309

Proposition 13.3. *Let $\mathscr{C}$ be an ∞-category and $p : K \to \mathscr{C}$ be an arbitrary diagram. For a transformation $l : \underline{y} \to p$ in $\mathrm{Fun}(K, \mathscr{C})$, the following are equivalent:*

(a) *l exhibits y as a limit of p in $\mathscr{C}$.*
(b) *l is terminal in the category $\mathscr{C} \times^{\mathrm{or}}_{\mathrm{Fun}(K,\mathscr{C})} \{p\}$.*

Proof. We have the pullback diagram of right fibrations

$$\begin{tikzcd} \mathscr{C} \times^{\mathrm{or}}_{\mathrm{Fun}(K,\mathscr{C})} \{p\} \arrow[r] \arrow[d, "f'"'] & \mathrm{Fun}(K,\mathscr{C}) \times^{\mathrm{or}}_{\mathrm{Fun}(K,\mathscr{C})} \{p\} \arrow[d, "f"] \\ \mathscr{C} \arrow[r, "\mathrm{const}"'] & \mathrm{Fun}(K,\mathscr{C}) \end{tikzcd}$$

so that enriched homotopy transport for the fibration f' is the composite of the constant functor with the transport functor for f,

$$f'_! = f_!\,\mathrm{const} : \pi\mathscr{C}^{\mathrm{op}} \to \pi\mathscr{K}an,$$

by Lemma 7.9. Additionally the fiber of $\mathscr{C} \times^{\mathrm{or}}_{\mathrm{Fun}(K,\mathscr{C})} \{p\}$ over a given point z is the fiber of the fibration $\mathrm{Fun}(K,\mathscr{C}) \times^{\mathrm{or}}_{\mathrm{Fun}(K,\mathscr{C})} \{p\}$ at the constant diagram $\underline{z}$, i.e. the space $\mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{C})}(\underline{z}, p)$. Hence enriched transport for f' is the composite

$$\mathrm{Hom}_{\mathscr{C}}(z,y) \times \mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{C})}(\underline{y}, p) \overset{\mathrm{const}\times 1}{\to} \mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{C})}(\underline{z}, \underline{y}) \times \mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{C})}(\underline{y}, p)$$
$$\overset{\circ^{\mathrm{op}}}{\to} \mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{C})}(\underline{z}, p),$$

by Proposition 7.13. Hence a transformation l in $\mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{C})}(\underline{y}, p)$ exhibits y as a limit for p if and only if l provides a terminal object for the transport functor associated to the right fibration $\mathscr{C} \times^{\mathrm{or}}_{\mathrm{Fun}(K,\mathscr{C})} \{p\} \to \mathscr{C}$. Here by a terminal object for a "contravariant" functor to $\mathscr{K}an$ we simply mean an initial object for the functor $\mathscr{C}^{\mathrm{op}} \to \mathscr{K}an$, in the sense of Definition 11.2.

As in the proof of Proposition 11.4, we see that l is terminal for the transport functor associated to the given fibration if and only if l is terminal as an object in the oriented fiber product $\mathscr{C} \times^{\mathrm{or}}_{\mathrm{Fun}(K,\mathscr{C})} \{p\}$. □

As in the case of a single object $K = *$, we have at general K the equivalence of right fibrations

$$\begin{tikzcd} \mathscr{C}_{/p} \arrow[rr, "\sim"] \arrow[dr] & & \mathscr{C} \times^{\mathrm{or}}_{\mathrm{Fun}(K,\mathscr{C})} \{p\} \arrow[dl] \\ & \mathscr{C} & \end{tikzcd}$$

from Theorem I-9.14. On objects this equivalence sends a diagram $t : \Delta^0 \star K \to \mathscr{C}$ to the diagram $\Delta^1 \times K \to \mathscr{C}$ obtained by restricting along the comparison map

$$\Delta^1 \times K \to \Delta^0 \star K$$

of Section I-9.3, which sends the the subcomplex $\{0\} \times K$ to the cone point $\{0\}$ and sends $\{1\} \times K$ identically to the subcomplex K.

cor:lim_cone

Corollary 13.4. *There exists a transformation $l : y \to p$ which exhibits an object y as a limit of a diagram $p : K \to \mathscr{C}$ if and only if there is a diagram $l' : \Delta^0 \star K \to \mathscr{C}$ for which the corresponding object $l' : * \to \mathscr{C}_{/p}$ is terminal and which has $l(0) = y$.*

Furthermore, if l exhibits y as a limit of p, then a diagram $l' : \Delta^0 \star K \to \mathscr{C}$ is terminal in $\mathscr{C}_{/p}$ if and only if its image in $\mathscr{C} \times^{\mathrm{or}}_{\mathrm{Fun}(K,\mathscr{C})} \{p\}$ is isomorphic to l.

Proof. Follows from Proposition 13.3, the fact that terminal objects are preserved under equivalence (Lemma 9.4), and the fact that terminal objects are unique up to isomorphism (Lemma 9.2). □

cor:isom_lim

Corollary 13.5. *Consider any diagram $p : K \to \mathscr{C}$, and suppose there is an isomorphism $\alpha : y \xrightarrow{\sim} y'$ in $\mathscr{C}$. Then y is a limit for p if and only if y' is a limit for p.*

Proof. The forgetful functor $\mathscr{C}_{/p} \to \mathscr{C}$ is a right fibration, and hence an isofibration by Lemma I-4.31. So if $\widetilde{y} : * \to \mathscr{C}_{/p}$ is any lift of y to the overcategory, there is an isomorphism $\widetilde{\alpha} : \widetilde{y} \to \widetilde{y}'$ in $\mathscr{C}_{/p}$ which lifts the given isomorphism α. Hence $\widetilde{y}$ is initial if and only if $\widetilde{y}'$ is initial, by Lemma 9.3. It follows by Corollary 13.4 that y is a limit of p if and only if y' is a limit of p. $\square$

cor:lim_isom

Corollary 13.6. *If y and y' are limits for a diagram $p : K \to \mathscr{C}$, then y and y' are isomorphic in $\mathscr{C}$.*

Proof. By Lemma 9.2 the space of terminal objects in $\mathscr{C}_{/p}$ is contractible. In particular, any two initial lifts $\widetilde{y}$ and $\widetilde{y}'$ of y and y' are necessarily isomorphic. $\square$

sect:char_colim

13.2. **Characterizations of colimits.** The obvious analogs of the arguments provided in Section 13.1 provide the following characterization of colimits in a given ∞-category $\mathscr{C}$.

prop:char_colim

Proposition 13.7. *Let $p : K \to \mathscr{C}$ be an arbitrary diagram in an ∞-category $\mathscr{C}$. The following are equivalent:*

(a) *There exists a morphism $c : p \to \underline{x}$ in $\mathrm{Fun}(K, \mathscr{C})$ which exhibits x as a colimit of p.*
(b) *There is an initial object c in $\{p\} \times^{\mathrm{or}}_{\mathrm{Fun}(K,\mathscr{C})} \mathscr{C}$ which lies over x in $\mathscr{C}$.*
(c) *There exists an initial object $\widetilde{x}$ in $\mathscr{C}_{p/}$ which maps to x in $\mathscr{C}$.*

We can furthermore compare objects in the undercategory $\mathscr{C}_{p/}$ and tranformations in $\mathrm{Fun}(K, \mathscr{C})$ via the equivalence of left fibrations

$$\begin{tikzcd} \mathscr{C}_{p/} \arrow[rr, "\sim"] \arrow[dr] & & \{p\} \times^{\mathrm{or}}_{\mathrm{Fun}(K,\mathscr{C})} \mathscr{C} \arrow[dl] \\ & \mathscr{C} & \end{tikzcd}$$

from Theorem I-9.14. In particular, an object $\widetilde{x}$ in $\mathscr{C}_{p/}$ is initial if and only if the image $c : p \to \underline{x}$ in $\{p\} \times^{\mathrm{or}}_{\mathrm{Fun}(K,\mathscr{C})} \mathscr{C}$ is initial, by Lemma 9.4. Finally c is initial in the oriented fiber product if and only if c exhibits x as a colimit of p, in the sense of Definition 13.1, by similar arguments to those employed in the proof of Proposition 13.3.

Corollary 13.8. *Given a digram $p : K \to \mathscr{C}$, and isomorphic objects $x, x' : * \to \mathscr{C}$, x is a colimit of p if and only if x' is a colimit of p. Furthermore, any two colimits of p are isomorphic in $\mathscr{C}$.*

Proof. Follows from the fact that the forgetful functor $\mathscr{C}_{p/} \to \mathscr{C}$ is an isofibration and stability of initial objects under isomorphism. $\square$

13.3. **Limit and colimit diagrams.** Given the conclusions of Corollary 13.4 and Proposition 13.7, the following definition now makes sense.

Definition 13.9. Let $p : K \to \mathscr{C}$ be a diagram in an ∞-category. We call a diagram $l' : \Delta^0 \star K \to \mathscr{C}$ a limit diagram for p if $l'|_K = p$ and l' is terminal when considered as an object in the overcategory $\mathscr{C}_{/p}$. We call a diagram $c' : K \star \Delta^0 \to \mathscr{C}$

a colimit diagram if $c'|_K = p$ and c' is initial when considered as an object in the undercategory $\mathscr{C}_{p/}$

We say a diagram $l : * \to \mathscr{C}_{/p}$ exhibits an object y as a limit of p if l is terminal and $l(0) = y$. Similarly, we say $c : * \to \mathscr{C}_{p/}$ exhibits an object x as a colimit of p if c is initial and $c(0) = x$.

13.4. **Some results for change of diagrams.**

Definition 13.10 ([4, 01E8])**.** A map of simplicial sets $i : K \to L$ is said to be a categorical equivalence if, for each ∞-category $\mathscr{C}$, the induced map

$$i_* : \pi_0(\mathrm{Fun}(L, \mathscr{C})^{\mathrm{Kan}}) \to \pi_0(\mathrm{Fun}(K, \mathscr{C})^{\mathrm{Kan}})$$

is a bijection of sets.

Lemma 13.11. *A map of simplicial sets $i : K \to L$ is a categorical equivalence if and only if, for each ∞-category $\mathscr{C}$, the induced map*

$$i_* : \mathrm{Fun}(L, \mathscr{C}) \to \mathrm{Fun}(K, \mathscr{C})$$

is an equivalence of ∞-categories.

Proof. At any ∞-category $\mathscr{D}$ we have the adjunction

$$\mathrm{Fun}(\mathscr{D}, \mathrm{Fun}(-, \mathscr{C})) \cong \mathrm{Fun}(-, \mathrm{Fun}(\mathscr{D}, \mathscr{C})).$$

It follows that the induced map

$$\pi_0(\mathrm{Fun}(\mathscr{D}, \mathrm{Fun}(L, \mathscr{C}))^{\mathrm{Kan}}) \to \pi_0(\mathrm{Fun}(\mathscr{D}, \mathrm{Fun}(K, \mathscr{C}))^{\mathrm{Kan}})$$

is a bijection at any ∞-category $\mathscr{D}$, and hence that $i_* : \mathrm{Fun}(L, \mathscr{C}) \to \mathrm{Fun}(K, \mathscr{C})$ is an isomorphism in the homotopy category $\mathrm{h}\,\mathscr{C}at_\infty$. Equivalently, i_* is an equivalence of ∞-categories. □

Example 13.12. If $i : K \to L$ is inner anodyne, then i is a categorical equivalence by Corollary I-4.8.

Example 13.13. If $F : \mathbb{A} \to \mathbb{B}$ is an equivalence of discrete categories–or more generally ∞-categories–then F is a categorical equivalence.

prop:lim_catequiv **Proposition 13.14.** *Suppose that a map $i : L \to K$ is a categorical equivalence, and let $p : K \to \mathscr{C}$ be a diagram in an ∞-category. A transformation $l : \underline{y} \to p$ exhibits an object y as a limit if p if and only if the corresponding transformation $l|_L : \underline{y} \to p|_L$ exhibits y as a limit of $p|_L$. Dually, a transformation $c : p \to \underline{x}$ exhibits x as a colimit of p if and only if $c|_L : p|_L \to \underline{x}$ exhibits x as a colimit of $p|_L$.*

Proof. We have the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{C}}(z, y) \arrow[r] \arrow[dr] & \mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{C})}(\underline{z}, \underline{y}) \arrow[r, "l^*"] \arrow[d, "\sim"] & \mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{C})}(\underline{z}, p) \arrow[d, "\sim"] \\
 & \mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{C})}(\underline{z}, \underline{y}) \arrow[r, "(li)^*"] & \mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{C})}(\underline{z}, p)
\end{tikzcd}$$

in $\mathrm{h}\,\mathscr{K}an$, from which we conclude that the top composite is an equivalence if and only if the bottom composite is an equivalence. Hence l is a limit transformation if and only if li is a limit transformation. The case of c versus ci is similar. □

Corollary 13.15. *Let K be a category, or more generally an ∞-category. If K has an initial object $a : * \to K$, then for any diagram $p : K \to \mathscr{C}$ the value $p(a)$ is a limit for p. Similarly, if K has a terminal object $z : * \to K$, then for any diagram $p : K \to \mathscr{C}$ the value $p(z)$ is a colimit for p.*

Proof. If a is initial or terminal then the inclusion $a : * \to \mathbb{K}$ is inner anodyne, by Proposition 9.23. □

Corollary 13.16. *Suppose $i : L \to K$ is a categorical equivalence and let $p : K \to \mathscr{C}$ be a diagram. Take $p' = pi : L \to \mathscr{C}$.*

1. *For any object $l : * \to \mathscr{C}_{/p}$, l is a limit diagram if and only if the image of l under the forgetful functor $\mathscr{C}_{/p} \to \mathscr{C}_{/p'}$ is a limit diagram.*
2. *For an object $c : * \to \mathscr{C}_{p/}$, c is a colimit diagram if and only if the image of c under the forgetful functor $\mathscr{C}_{p/} \to \mathscr{C}_{p'/}$ is a colimit diagram.*

Proof. We prove (1). Consider the diagram

$$\begin{tikzcd} \mathscr{C}_{/p} \arrow[r, "\sim"] \arrow[d] & \mathscr{C} \times^{\mathrm{or}}_{\mathrm{Fun}(K,\mathscr{C})} \{p\} \arrow[d] \\ \mathscr{C}_{/p'} \arrow[r, "\sim"] & \mathscr{C} \times^{\mathrm{or}}_{\mathrm{Fun}(L,\mathscr{C})} \{p'\}. \end{tikzcd}$$

By Propositions 13.3 and 13.14 we see that an object in $\mathscr{C} \times^{\mathrm{or}}_{\mathrm{Fun}(K,\mathscr{C})} \{p\}$ is terminal if and only if its image in $\mathscr{C} \times^{\mathrm{or}}_{\mathrm{Fun}(L,\mathscr{C})} \{p'\}$. By preservation of terminal objects under equivalence, Lemma 9.4, we therefore find that a diagram l is initial, i.e. a limit diagram, if and only if its image in $\mathscr{C}_{/p'}$ is initial. □

`prop:isom_replace`

Proposition 13.17. *Suppose that $p, p' : K \to \mathscr{C}$ are diagrams in an ∞-category, and that $\xi : p \to p'$ is a morphism in $\mathrm{Fun}(K, \mathscr{C})$ for which, at each vertex $x : * \to K$, the map $\xi(x) : p(x) \to p'(x)$ is an isomorphism in $\mathscr{C}$.*

1. *A transformation $l : \underline{y} \to p$ exhibits y as a limit of p if and only if the transformation ξl exhibits y as a limit of p'.*
2. *A transformation $c : p' \to \underline{x}$ exhibits x as a colimit of p' if and only if $c\xi : p \to \underline{x}$ exhibits x as a colimit of p.*

Proof. By Proposition I-6.8 the map ξ is an isomorphism in the ∞-category $\mathrm{Fun}(K, \mathscr{C})$. So (1) follows from a consideration of the diagram

$$\begin{tikzcd} & \mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{C})}(\underline{z}, \underline{y}) \arrow[dl, "l_*"'] \arrow[dr, "\xi_* l_*"] & \\ \mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{C})}(\underline{z}, p) \arrow[rr, "\xi_*"] & & \mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{C})}(\underline{z}, p') \end{tikzcd}$$

in $\mathrm{h}\mathscr{K}an$. One similarly observes (2). □

13.5. **Limits and colimits under adjunctions.** As in the discrete setting, one can show that any left adjoint respects colimits and any right adjoint respects limits. We begin this discussion with a result concerning exponentiation of natural transformations. For each functor $F : \mathscr{C} \to \mathscr{D}$ between ∞-categories, and simplicial set K, we have the induced functor

$$F_* : \mathrm{Fun}(K, \mathscr{C}) \to \mathrm{Fun}(K, \mathscr{D}).$$

This assignment $F \mapsto F_*$ extends to a map of simplicial sets.

prop:exp_nat

Proposition 13.18. *Let $\mathscr{C}_i$ and $\mathscr{D}$ be ∞-categories, and K be a simplicial set. Take $\mathscr{E}^K = \mathrm{Fun}(K, \mathscr{E})$ for any ∞-category $\mathscr{E}$. There is a map of ∞-categories*

$$(-)_* : \mathrm{Fun}(\mathscr{C}, \mathscr{D}) \to \mathrm{Fun}(\mathscr{C}^K, \mathscr{D}^K)$$

which is natural in K, which is the apparent isomorphism when $K = *$, and which on objects sends a functor F to the induced map F_*. Furthermore, for any triple of ∞-categories the diagram

$$\begin{tikzcd} \mathrm{Fun}(\mathscr{C}_1, \mathscr{C}_2) \times \mathrm{Fun}(\mathscr{C}_0, \mathscr{C}_1) \arrow[r, "(-)_* \times (-)_*"] \arrow[d, "\circ"'] & \mathrm{Fun}(\mathscr{C}_1^K, \mathscr{C}_2^K) \times \mathrm{Fun}(\mathscr{C}_0^K, \mathscr{C}_1^K) \arrow[d, "\circ"] \\ \mathrm{Fun}(\mathscr{C}_0, \mathscr{C}_2) \arrow[r, "(-)_*"'] & \mathrm{Fun}(\mathscr{C}_0^K, \mathscr{C}_2^K) \end{tikzcd}$$

commutes.

Construction 13.18. Given an n-simplex $F : \Delta^n \times \mathscr{C} \to \mathscr{D}$ define the n-simplex $F_* : \Delta^n \times \mathscr{C}^K \to \mathscr{D}^K$ to be the map of simplicial sets which sends an m-simplex $(\tau, \sigma) : \Delta^m \to \Delta^n \times \mathscr{C}^K$ to the m-simplex

$$\Delta^m \times K \overset{\delta \times 1}{\to} \Delta^m \times \Delta^m \times K \overset{\tau \times \sigma}{\to} \Delta^n \times \mathscr{C} \overset{F}{\to} \mathscr{D}.$$

The assignment $F \mapsto F_*$ defines the proposed map of simplicial sets. Given a map of simplices $s : \Delta^{n'} \to \Delta^n$ the equality $F_*(s \times id) = (F(s \times id))_*$ follows by a direct comparison, so that the assignment $F \mapsto F_*$ defines a map of simplicial sets

$$(-)_* : \mathrm{Fun}(\mathscr{C}, \mathscr{D}) \to \mathrm{Fun}(\mathscr{C}^K, \mathscr{D}^K).$$

Naturality in K is apparent.

As for commutativity of the given diagram, proceeding along the bottom takes a pair of n-simplices (F_1, F_0) to the n-simplex $(F_1F_0)_*$. Proceeding along the top produces a map $(F_1)_*(F_0)_* : \Delta^n \times \mathscr{C}_0^K \to \mathscr{C}_2^K$ which is the composite

$$\Delta^n \times \mathscr{C}_0^K \overset{\delta \times 1}{\to} \Delta^n \times \Delta^n \times \mathscr{C}_0^K \overset{1 \times (F_0)_*}{\to} \Delta^n \times \mathscr{C}_1^K \overset{(F_1)_*}{\to} \mathscr{C}_2^K.$$

This map explicitly takes an m-simplex $(\tau, \sigma) : \Delta^m \to \Delta^n \times \mathscr{C}_0^K$ to the m-simplex $(\tau, (F_0)_*(\tau, \sigma))$ in $\Delta^1 \times \mathscr{C}_1^K$, which then maps to the m-simplex

$$\Delta^m \times K \overset{\delta \times 1}{\to} \Delta^m \times \Delta^m \times K \overset{\tau \times (F_0)_*(\tau, \sigma)}{\to} \Delta^n \times \mathscr{C}_1 \overset{F_1}{\to} \mathscr{C}_2.$$

This composite expands to the composite

$$\Delta^m \times K \overset{\delta \times 1}{\to} \Delta^m \times \Delta^m \times K \overset{1 \times \delta \times 1}{\to} \Delta^m \times \Delta^m \times \Delta^m \times K$$
$$\overset{\tau \times \tau \times \sigma}{\to} \Delta^n \times \Delta^n \times \mathscr{C}_0 \overset{1 \times F_0}{\to} \Delta^n \times \mathscr{C}_1 \overset{F_1}{\to} \mathscr{C}_2.$$

Via coassociativity and naturality of the diagonal map, this composite is equal to the composite

$$\Delta^m \times K \overset{\delta \times 1}{\to} \Delta^m \times \Delta^m \times K \overset{\tau \times \sigma}{\to} \Delta^n \times \mathscr{C}_0 \overset{\delta \times 1}{\to} \Delta^n \times \Delta^n \times \mathscr{C}_0 \overset{F_1(1 \times F_0)}{\to} \mathscr{C}_2.$$

This m-simplex is precisely $(F_1F_0)_*(\tau, \sigma)$, verifying the desired equality

$$(F_1F_0)_* = (F_1)_*(F_0)_*.$$

$\square$

We note that the composite of a natural transformation $\zeta : \Delta^1 \times \mathscr{C}_0 \to \mathscr{C}_1$ with a functor $F : \mathscr{C}_1 \to \mathscr{C}_2$ is recovered as the composite of 1-simplices

$$F\zeta = (s_0^* F)\eta$$

where $s_0 : \Delta^1 \to \Delta^0$ is the degeneracy map. Similarly the composition of ζ with a functor $G : \mathscr{C}_{-1} \to \mathscr{C}_0$ is identified with a composite of 1-simplices

$$\zeta(id_{\Delta^1} \times G) = \zeta(s_0^* G).$$

It follows that the functor from Proposition 13.18 respects these compositions between natural transformations and functors. We therefore observe preservation of adjoints under exponentiation.

cor:exp_adj **Corollary 13.19.** *Suppose a functor $F : \mathscr{C} \to \mathscr{D}$ is left adjoint to a functor $G : \mathscr{D} \to \mathscr{C}$. Let $\eta : id_{\mathscr{C}} \to GF$ and $\epsilon : FG \to id_{\mathscr{D}}$ be the associated unit and counit transformations. Then for any simplicial set K, the transformations η_* and ϵ_* exhibit the functor $F_* : \mathrm{Fun}(K, \mathscr{C}) \to \mathrm{Fun}(K, \mathscr{D})$ as left adjoint to the functor $G_* : \mathrm{Fun}(K, \mathscr{D}) \to \mathrm{Fun}(K, \mathscr{C})$.*

We use the above information to prove that left adjoints preserve colimits.

prop:left_cocont **Proposition 13.20.** *Suppose a functor $F : \mathscr{C} \to \mathscr{D}$ admits a right adjoint, and let $p : K \to \mathscr{C}$ be a diagram in $\mathscr{C}$. If a given transformation $c : p \to \underline{x}$ exhibits an object x as a colimit to p in $\mathscr{C}$, then the transformation $Fc : Fp \to \underline{F(x)}$ exhibits $F(x)$ as a colimit to the diagram $Fp : K \to \mathscr{D}$.*

Proof. By Corollary 13.19 the functor $F_* : \mathrm{Fun}(K, \mathscr{C}) \to \mathrm{Fun}(K, \mathscr{D})$ admits some right adjoint G_* which is induced by a right adjoint $G : \mathscr{D} \to \mathscr{C}$ to F. We therefore have, at each z in $\mathscr{D}$, a diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{D}}(F(x), z) \arrow[r] \arrow[d, "\cong"'] & \mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{D})}(F_*\underline{x}, \underline{z}) \arrow[r, "(F_*c)^*"] \arrow[d, "\cong"'] & \mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{D})}(F_*p, \underline{z}) \arrow[d, "\cong"'] \\
\mathrm{Hom}_{\mathscr{C}}(x, G(z)) \arrow[r] & \mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{C})}(\underline{x}, G_*\underline{z}) \arrow[r, "c^*"'] & \mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{C})}(p, G_*\underline{z}),
\end{tikzcd}$$

in $\mathrm{h}\mathscr{K}an$ by Corollary I-13.5. It follows that the top row is an isomorphism in $\mathrm{h}\mathscr{K}an$ since the bottom row is an isomorphism in $\mathrm{h}\mathscr{K}an$ by hypothesis. $\square$

Similar arguments establish the analogous result for limits and right adjoints.

prop:right_cont **Proposition 13.21.** *Suppose a functor $G : \mathscr{D} \to \mathscr{C}$ admits a left adjoint, and let $q : K \to \mathscr{D}$ be a diagram in $\mathscr{D}$. If a given transformation $l : \underline{y} \to q$ exhibits an object y as a limit to q in $\mathscr{D}$, then the transformation $Gl : \underline{G(y)} \to Gq$ exhibits $G(y)$ as a limit to the diagram $Gq : K \to \mathscr{C}$.*

13.6. **Co/completeness and co/continuity.**

Definition 13.22. We say an ∞-category $\mathscr{C}$ is complete (resp. cocomplete) if, for each diagram $p : K \to \mathscr{C}$ from a small simplicial set, p admits a limit (resp. colimit) in $\mathscr{C}$. A functor $F : \mathscr{C} \to \mathscr{D}$ from a complete (resp. cocomplete) category is called continuous (resp. cocontinuous) if F commutes with limits (resp. colimits).

From Propositions 13.20 and 13.21 we observe the following.

Proposition 13.23. *Suppose $\mathscr{C}$ and $\mathscr{D}$ are both complete and cocomplete, and let $F : \mathscr{C} \to \mathscr{D}$ be any functor. If F admits a right adjoint then F is cocontinuous. If F admits a left adjoint then F is continuous.*

14. Limits and colimits in $\mathscr{C}at_\infty$ and $\mathscr{K}an$

14.1. **Limits in ∞-categories.** Consider an arbitrary diagram $p : K \to \mathscr{C}at_\infty$ and the corresponding cocartesian fibration $\mathscr{E} \to K$. One can take explicitly $\mathscr{E} = \int_K p$ here. A limit diagram for p is a particular diagram from the join $\widetilde{p} : \Delta^0 \star K \to \mathscr{C}at_\infty$, which is then specified by the corresponding fibration $\mathscr{E}' \to \Delta^0 \star K$, which fits into a pullback diagram

$$\begin{tikzcd} \mathscr{E} \arrow[r] \arrow[d] & \mathscr{E}' \arrow[d] \\ \mathscr{C} \arrow[r] & \Delta^0 \star K. \end{tikzcd}$$

Definition 14.1. For any cocartesian fibrations $X \to S$ and $Y \to S$, take $\mathrm{Fun}_S^{\mathrm{CCart}}(X, Y)$ to be the full ∞-subcategory in $\mathrm{Fun}_S(X, Y)$ spanned by those functors which preserve cocartesian edges.

Remark 14.2. Note that $\mathrm{Fun}_S^{\mathrm{CCart}}(X, Y)$ is generally larger than the inner-morphisms for the simplicial category of cocartesian fibrations $\underline{\mathrm{Hom}}_{\mathrm{Cocart}(S)}(X, Y)$. This is because natural transformations in the latter ∞-category are restricted according to the conditions outlined in Lemma 8.3.

Notation 14.3. For any simplicial set K, we take $K^< := \Delta^0 \star K$. We refer to the vertex $\{0\}$ in $K^<$ as the cone point in $K^<$.

We have the following general result.

Lemma 14.4 ([4, 018Q]). *Consider inclusions of simplicial sets $f : A \to A'$ and $g : B \to B'$, and the corresponding inclusion*

$$\mu : (A \star B') \coprod_{(A \star B)} (A' \star B) \to A' \star B'.$$

If f is anodyne, then μ is left anodyne. If g is anodyne, then μ is right anodyne.

In considering the extreme cases where $f = id_A$ and $B = \emptyset$ we find the following.

cor:cone_lano **Corollary 14.5.** *The inclusions $\{0\} \to K^<$ is left anodyne.*

We also have the following basic result, whose proof we omit.

prop:3558 **Proposition 14.6** ([4, 035S]). *For any cocartesian fibration $q : X \to S$, left anodyne morphisms of simplicial sets $S_0 \to S$, and $X_0 = X \times_S S_0$, the restriction functor*

$$\mathrm{Fun}_S^{\mathrm{CCart}}(S, X) \to \mathrm{Fun}_{S_0}^{\mathrm{CCart}}(S_0, X_0)$$

is a trivial Kan fibration.

Note that in the case of a cocartesian fibration over a point $\mathscr{C} \to *$, i.e. an ∞-category, we have $\mathrm{Fun}_*^{\mathrm{CCart}}(*, \mathscr{C}) = \mathrm{Fun}(*, \mathscr{C}) \cong \mathscr{C}$.

cor:conical_sections **Corollary 14.7.** *Let K be any simplicial set. For any cocartesian fibration $\mathscr{E} \to K^<$ the restriction functor*

$$\mathrm{Fun}_{K^<}^{\mathrm{CCart}}(K^<, \mathscr{E}) \to \mathrm{Fun}_{\{0\}}^{\mathrm{CCart}}(\{0\}, \mathscr{E}_0) = \mathscr{E}_0$$

is a trivial Kan fibration.

Proof. By Corollary 14.5 the inclusion $\{0\} \to K^<$ is left anodyne, and hence the result follows by Proposition 14.6. $\square$

`thm:diff_crit` **Theorem 14.8** (Diffraction criterion, [4, 02T8])**.** *Given any diagram $p : K \to \mathscr{C}at_\infty$, an extension $p' : K^< \to \mathscr{C}at_\infty$ is a limit diagram if and only if, for the corresponding cocartesian fibrations $\mathscr{E} = \int_K p$ and $\mathscr{E}' = \int_{K^<} p'$, the restriction functor*

$$\mathrm{Fun}^{\mathrm{CCart}}_{K^<}(K^<, \mathscr{E}') \to \mathrm{Fun}^{\mathrm{CCart}}_K(K, \mathscr{E})$$

is an equivalence of ∞-categories.

Lurie then shows that such extensions p' always exist [4, 02TG], so that we obtain completeness of the ∞-category $\mathscr{C}at_\infty$.

`thm:infty_lim` **Theorem 14.9.** *Any diagram $p : K \to \mathscr{C}at_\infty$ admits a limit. Furthermore, for the associated cocartesian fibration $\mathscr{E} = \int_K p$, the ∞-category $\mathrm{Fun}^{\mathrm{CCart}}_K(K, \mathscr{E})$ is a limit for the diagram p.*

Proof. Existence of the limit follows by the Diffraction Criterion of Theorem 14.8, and for any limit diagram $p' : K^< \to \mathscr{C}at_\infty$ with corresponding cocartesian fibration $\mathscr{E}'$ we have equivalences

$$p'(0) \xrightarrow{\sim} \mathscr{E}'_0 \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{CCart}}_{K^<}(K^<, \mathscr{E}') \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{CCart}}_K(K, \mathscr{E}).$$

Here the first equivalence follows by the fiber calculation of Theorem 5.27, the second equivalence is from Corollary 14.7, and the third equivalence follows by the Diffraction Criterion. By definition $p'(0)$ is a limit for p, and by Corollary 13.5 we find that the ∞-category $\mathrm{Fun}^{\mathrm{CCart}}_K(K, \mathscr{E})$ is also a limit for p. □

We now restrict our attention to small diagrams $K \to \mathscr{C}at_\infty$ to observe completeness.

`cor:infty_complete` **Corollary 14.10.** *The ∞-category of ∞-categories $\mathscr{C}at_\infty$ is complete.*

14.2. **Describing limits in $\mathscr{C}at_\infty$.** From the description of the cocartesian fibration $\int_K F \to K$ provided in Section 6.2 we can understand objects in the ∞-category $\mathrm{Fun}^{\mathrm{CCart}}_K(K, \int_K F)$, to some minimal extent. A section $t : K \to \int_K F$ specifies, at least, a choice of an object $t_x : * \to F(x)$ over each vertex x in K, and over each edge $\alpha : x \to y$ we have a morphism $t_\alpha : F(\alpha)(t_x) \to t_y$. Since every edge in K is cocartesian, the section t lies in $\mathrm{Fun}^{\mathrm{CCart}}_K(K, \int_F K)$ if and only if each map t_α is an isomorphism in $F(y)$. Apply t to 2-simplices in K provide compatibilities for the t_α.

In the special case where K is a discrete category, this relatively shallow description of objects in the space of sections can be filled out completely. First, by Lemma 6.29 we may assume that $F : K \to \mathscr{C}at_\infty$ is a strictly commuting diagram of ∞-categories, i.e. is the nerve of a simplicial functor from the underlying plain category. The ∞-category $\int_K F$ can then be replaced with the weighted nerve $\mathrm{N}^F(K)$ by Theorem 6.28. A section $t : K \to \mathrm{N}^F(K)$ in $\mathrm{Fun}^{\mathrm{CCart}}_K(K, \mathrm{N}^F(K))$ specifies for each compatible collection of morphisms $\alpha_{ij} : x_i \to x_j$ in K for $0 \le i < j \le n$, i.e. each n-simplex $\alpha : \Delta^n \to K$, an expanding sequence of simplices

$$\begin{tikzcd}
\Delta^i \arrow[r] \arrow[d, "\tau_i(\alpha)"'] & \Delta^j \arrow[d, "\tau_j(\alpha)"] \\
F(x_i) \arrow[r, "F(\alpha_{ij})"'] & F(x_j)
\end{tikzcd}$$

in which each constituent morphism $\tau_i(\alpha)|_{\Delta^{\{a,b\}}} : \Delta^1 \to F(x_i)$ is an isomorphism in the ∞-category $F(x_i)$. Said imprecisely, an object in $\mathrm{Fun}_K^{\mathrm{CCart}}(K, \mathrm{N}^F(K))$ is just a choice of objects $t_x : * \to F(x)$ for each x in K, a choice of an isomorphism $t_\alpha : F(\alpha)(t_x) \to t_y$ over each morphism $\alpha : x \to y$ in K which enjoy arbitrarily high levels of compatibility under composition.

14.3. **Colimits in ∞-categories.** We again outline a construction of colimits in the ∞-category of ∞-categories. We provide the basic rationalle, but omit arguments for some of the technical pinch points. Of course, one can find all details in the original text [4]. We begin with a brisk discussion of localization. Recall that a marked simplicial set (E, W) is a simplicial set E with a prescribed collection $W \subseteq E[1]$ which contains all degenerate edges.

Definition 14.11. For a marked simplicial set (E, W) a (Dwyer-Kan) localization of E relative to W is a map of simplicial sets $F : E \to \mathscr{D}$ into an ∞-category $\mathscr{D}$ for which, at each ∞-category $\mathscr{C}$, the restriction functor

$$F^* : * : \mathrm{Fun}(\mathscr{D}, \mathscr{C}) \to \mathrm{Fun}(E, \mathscr{C})$$

is an equivalence onto the full subcategory of all maps $E \to \mathscr{C}$ which send all edges in W to equivalences in $\mathscr{C}$.

It is the case that such localizations always exist and are unique.

Lemma 14.12. *For any marked simplicial set (E, W), a localization functor θ : $E \to \mathscr{D}$ exists. Furthermore, any two localizations $F : E \to \mathscr{D}$ and $F' : E \to \mathscr{D}'$ admit an equivalence $\vartheta : \mathscr{D} \to \mathscr{D}'$ which fits into homotopy commuting diagram*

$$\begin{tikzcd} \mathscr{D} \arrow[rr, "\vartheta"] & & \mathscr{D}' \\ & E \arrow[ul, "F"] \arrow[ur, "F'"'] & \end{tikzcd} .$$

Proof. Take $\mathscr{D}$ a fibrant replacement for (E, W) in the model category of marked simplicial sets, with respect to the cartesian model structure [2]. Uniqueness follows by the expected nonsense argument. $\square$

In the event that we do not have a specific realization of the lecalization in mind, we write $\mathscr{E}[W^{-1}]$ for the localization of a simplicial set E relative to a collection of edges W.

Given a pullback diagram of cocartesian fibrations

$$\begin{tikzcd} \mathscr{E} \arrow[r, "i"] \arrow[d, "q"'] & \widetilde{\mathscr{E}} \arrow[d, "\widetilde{q}"] \\ K \arrow[r] & K \star \{1\} \end{tikzcd} \tag{28}$$ eq:3939

we have the unique solution the lifting problem

$$\begin{tikzcd} \{0\} \times \mathscr{E} \arrow[r, "i"] \arrow[d] & \widetilde{\mathscr{E}} \arrow[d, "\widetilde{q}"] \\ \Delta^1 \times \mathscr{E} \arrow[r] \arrow[ur, dotted] & K \star \{1\} \end{tikzcd} \tag{29}$$ eq:3946

where the bottom map is the unique map of simplicial sets which is q on $\{0\} \times \mathscr{E}$ and the terminal map $\mathscr{E} \to \{1\}$ on $\{1\} \times \mathscr{E}$ (see Section 6.8). We recall that there is a unique cocartesian solution $s : \Delta^1 \times \mathscr{E} \to \widetilde{\mathscr{E}}$ to this problem by Theorem 2.7, i.e. one which sends each edge $\Delta^1 \times \{e\}$ to a cocartesian morphism in $\widetilde{\mathscr{E}}$.

Definition 14.13. Given a pullback diagram of cocartesian fibrations (28), and an equivalence $\widetilde{\mathscr{E}} \times_{(K\star\{1\})} \{1\} \to \mathscr{E}_1$, a refraction diagram for $\widetilde{\mathscr{E}}$ is a map of simplicial sets $\mathrm{Rf} : \mathscr{E} \to \mathscr{E}_1$ which appears as the restriction $\mathrm{Rf} = s|_{\{1\}\times\mathscr{E}}$ of a cocartesian solution $s : \Delta^1 \times \mathscr{E} \to \widetilde{\mathscr{E}}$ to the lifting problem (30).

By uniqueness of s, we see that the refraction diagram $\mathrm{Rf} : \mathscr{E} \to \mathscr{E}_1$ is uniquely specified by the fibration $\widetilde{q} : \widetilde{\mathscr{E}} \to K \star \{1\}$ and the choice of identification $\widetilde{\mathscr{E}}_1 \cong \mathscr{E}_1$.

thm:refract **Theorem 14.14** (Refraction Criterion, [4, 02UU])**.** *Let $F : K \to \mathscr{C}at_\infty$ be a diagram of ∞-categories, and $q : \mathscr{E} = \int_K F \to K$ the corresponding cocartesian fibration. An ∞-category $\mathscr{E}_1$ is a colimit for the diagram F if and only if there is a pullback diagram of cocartesian fibrations*

$$\begin{tikzcd} \mathscr{E} \arrow[r] \arrow[d, "q"'] & \widetilde{\mathscr{E}} \arrow[d, "\widetilde{q}"] \\ K \arrow[r] & K \star \{1\} \end{tikzcd}$$

for which the fiber $\widetilde{\mathscr{E}}_1$ admits an equivalence $\widetilde{\mathscr{E}}_q \xrightarrow{\sim} \mathscr{E}_1$, and for which the refraction diagram $\mathrm{Rf} : \mathscr{E} \to \mathscr{E}_1$ exhibits $\mathscr{E}_1$ as a localization of $\mathscr{E}$ relative to the class W of q-cocartesian morphisms.

We note that the transport functor $\widetilde{F} : K \star \{1\} \to \mathscr{C}at_\infty$ associated to such a cocartesian fibration $\widetilde{\mathscr{E}} \to K\star\{1\}$ has $\widetilde{F}|_K \cong F$ and has $\widetilde{F}(1)$ equivalent to $\mathscr{E}_1$. Since the forgetful functor $(\mathscr{C}at_\infty)_{\widetilde{F}|_K/} \to \mathscr{C}at_\infty$ we see that the diagram $\widetilde{F}$ determines, up to isomorphism, an object $e_1 : * \to \mathscr{C}_{\widetilde{F}|_K/}$ over $\mathscr{E}_1$. Now, by Proposition 13.17, e_1 realizes $\mathscr{E}_1$ as a colimit of $\widetilde{F}|_K$ if and only if $\mathscr{E}_1$ is a colimit of F. So the assertion of Theorem 14.14 at least makes sense. Furthermore, by considering refraction diagrams one can intuit the nature of the refraction criterion and the necessity of the proposed localization.

We refer the reader directly to [4] for a precise accounting of Theorem 14.14 and its proof.

prop:refract_exist **Proposition 14.15.** *Given any cocartesian fibration $q : \mathscr{E} \to K$ over a simplicial set K, there exists a cocartesian fibration $\widetilde{q} : \widetilde{\mathscr{E}} \to K \star \{1\}$ for which the refraction diagram $\mathrm{Rf} : \mathscr{E} \to \mathscr{E}_1$ exhibits $\mathscr{E}_1$ as a localization of $\mathscr{E}$.*

Proof. Take any localization $F : \mathscr{E} \to \mathscr{E}_1$ and corresponding cocartesian fibration $\mathscr{E}[W^{-1}] \to \{1\}$. Then we have the diagram

$$\begin{tikzcd} \mathscr{E} \arrow[r, "F"] \arrow[d, "q"'] & \mathscr{E}[W^{-1}] \arrow[d] \\ K \arrow[r] & \{1\} \end{tikzcd}$$

where the top map F preserves cocartesian edges. By Lemma 6.27 we then obtain a cocartesian fibration

$$\widetilde{q} : \widetilde{\mathscr{E}} := \mathscr{E} \star_{\mathscr{E}_1} \mathscr{E}_1 \to K \star_{\{1\}} \{1\}.$$

We have $K \star_{\{1\}} \{1\} \cong K \star \{1\}$, and identifications of the fibers

$$\mathscr{E} \overset{\cong}{\to} \widetilde{\mathscr{E}} \times_{(K\star\{1\})} K, \quad \mathscr{E}_1 \overset{\cong}{\to} \widetilde{\mathscr{E}} \times_{(K\star\{1\})} \{1\}.$$

We have the unique map $\Delta^1 \times \mathscr{E} \to \mathscr{E} \star \mathscr{E}_1$ which is the identity $\mathscr{E} \to \mathscr{E} \subseteq \mathscr{E} \star \mathscr{E}_1$ at $\{0\}$ and the localization $F : \mathscr{E} \to \mathscr{E}_1 \subseteq \mathscr{E} \star \mathscr{E}_1$ at $\{1\}$. We also have the map $id \times F : \Delta^1 \times \mathscr{E} \to \Delta^1 \times \mathscr{E}_1$. These two morphisms fit into the appropriate diagram

$$\begin{tikzcd}
\Delta^1 \times \mathscr{E} \arrow[r] \arrow[d] & \Delta^1 \times \mathscr{E}_1 \arrow[d] \\
\mathscr{E} \star \mathscr{E}_1 \arrow[r] & \mathscr{E}_1 \star \mathscr{E}_1
\end{tikzcd}$$

and so define a morphism to the relative join $s : \Delta^1 \times \mathscr{E} \to \widetilde{\mathscr{E}}$ which sends each edge $\Delta^1 \times \{e\}$ to the pairing of the unique edge $e \to F(e)$ in $\mathscr{E} \star \mathscr{E}_1$ with the edge $\Delta^1 \times \{F(e)\}$ in $\mathscr{E}_1$. By the characterization of $\widetilde{q}$-cocartesian edges in $\widetilde{\mathscr{E}}$ provided in Lemma **??** we see that s provides a cocartesian solution to the lifting problem

$$\begin{tikzcd}
\{0\} \times \mathscr{E} \arrow[r, "i"] \arrow[d] & \widetilde{\mathscr{E}} \arrow[d, "\widetilde{q}"] \\
\Delta^1 \times \mathscr{E} \arrow[r] \arrow[ur, dotted, "s"] & K \star \{1\}.
\end{tikzcd} \tag{30}$$

eq:3946

Since $s|_{\{1\}\times\mathscr{E}} = F$, we see that F is realized as the refraction diagram for $\widetilde{q}$. □

We combine Theorem 14.14 with Proposition 14.15 to see that $\mathscr{C}at_\infty$ arbitrary colimits.

cor:infty_co_complete

Corollary 14.16. *The ∞-category $\mathscr{C}at_\infty$ is both complete and cocomplete.*

Proof. Follows by Corollary 14.10, Theorem 14.14, and Proposition 14.15. □

Remark 14.17. It is a bit odd that we've, apparently, shown that $\mathscr{C}at_\infty$ admits limits and colimits indexed by *arbitrary* simplicial sets, rather than all small sets. However we recall our sizing conventions. All simplicial sets, ∞-categories, and Kan complexes, are in our universe of "medium sized" sets, while we make special exemptions for the ∞-categories $\mathscr{C}at_\infty$ and $\mathscr{K}an$, which lift in our universe of "large" sets. So all simplicial sets are small relative to our large categories $\mathscr{C}at_\infty$ and $\mathscr{K}an$, and the claim that $\mathscr{C}at_\infty$ admits all colimits indexed by medium sized simplicial sets poses no philosophical or material error. In any case, we are only concerned with the existence of small limits and colimits in $\mathscr{C}at_\infty$.

14.4. **Limits and colimits in spaces.** Our ultimate conclusion here is that $\mathscr{K}an$ is complete and cocomplete, and is in fact closed under both limits and colimits in the ambient category $\mathscr{C}at_\infty$.

Theorem 14.18. *The ∞-category $\mathscr{K}an$ is complete and the inclusion $\mathscr{K}an \to \mathscr{C}at_\infty$ is continuous.*

Proof. Given a diagram $p : K \to \mathscr{K}an$ we have the inclusion $\mathscr{K}an_{/p} \to (\mathscr{C}at_\infty)_{/p}$ which identifies $\mathscr{K}an_{/p}$ with a full ∞-subcategory in $(\mathscr{C}at_\infty)_{/p}$. Furthermore, $\mathscr{K}an_{/p}$ is seen to fit into a pullback diagram

$$\begin{tikzcd} \mathscr{K}an_{/p} \arrow[r] \arrow[d] & (\mathscr{C}at_\infty)_{/p} \arrow[d] \\ \mathscr{K}an \arrow[r] & \mathscr{C}at_\infty. \end{tikzcd}$$

It follows that the limit diagram $l : * \to (\mathscr{C}at_\infty)_{/p}$ is a limit in $\mathscr{K}an_{/p}$ if and only if the cone point $\lim(p)$ in $\mathscr{C}at_\infty$ is a Kan complex.

Take $\mathscr{E} = \int_K p \to K$ the left fibration associated to p. By the calculation of Theorem 14.9 we must show that the functor category $\mathrm{Fun}_K^{\mathrm{CCart}}(K, \mathscr{E})$ is a Kan complex, and for this it suffices to show that the ∞-category $\mathrm{Fun}_K(K, \mathscr{E})$ is a Kan complex. Since the fibers of $\mathscr{E}_x$ over K are all Kan complexes, it follows that for any natural transformation $\zeta : \Delta^1 \times K \to \mathscr{E}$ between functors F and G the maps $\zeta(x) : F(x) \to G(x)$ are isomorphisms in $\mathscr{E}_x$. Hence each transformation in $\mathrm{Fun}_K(K, \mathscr{E})$ is an isomorphism by Proposition I-6.8, we see that $\mathrm{Fun}_K(K, \mathscr{E})$ is a Kan complex, and since $\mathrm{Fun}_K^{\mathrm{CCart}}(K, \mathscr{E})$ is a full ∞-subcategory in $\mathrm{Fun}_K(K, \mathscr{E})$ we conclude that $\mathrm{Fun}_K^{\mathrm{CCart}}(K, \mathscr{E})$ is a Kan complex. □

Theorem 14.19. *The ∞-category $\mathscr{K}an$ is cocomplete and the inclusion $\mathscr{K}an \to \mathscr{C}at_\infty$ is cocontinuous.*

Proof. Fix a diagram $p : K \to \mathscr{K}an$. Since $\mathscr{C}at_\infty$ is cocomplete by Corollary 14.16, it suffices to show that the colimit $\mathscr{C} = \mathrm{colim}(p)$ is a Kan complex.

Let $i : \mathscr{K}an \to \mathscr{C}at_\infty$ be the inclusion, and recall from Proposition I-13.16 that this functor is left adjoint to the associated Kan complex functor. The counit of this adjunction is provided by a transformation $id_{\mathscr{C}at_\infty} \to i(-)^{\mathrm{Kan}}$ which is just the inclusion $\mathscr{C}^{\mathrm{Kan}} \to \mathscr{C}$ on objects. This counit transformation realizes these functors as adjoints at the level of the enriched homotopy categories as well, by Corollary **??**, and we have an induced adjunction for the functors

$$i_* : \mathrm{Fun}(K, \mathscr{K}an) \to \mathrm{Fun}(K, \mathscr{C}at_\infty) \quad \text{and} \quad (-)^{\mathrm{Kan}}_* : \mathrm{Fun}(K, \mathscr{C}at_\infty) \to \mathrm{Fun}(K, \mathscr{K}an)$$

by Proposition 13.18. Here i_* is just the inclusion.

Take $c : i_* p \to \underline{\mathscr{C}}$ a transformation which exhibits an ∞-category $\mathscr{C}$ as a colimit for $i_* p$ in $\mathscr{C}at_\infty$ and take $\mathscr{X} = \mathscr{C}^{\mathrm{Kan}}$. Let $\epsilon : \mathscr{X} \to \mathscr{C}$ be the inclusion. Then we have the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathscr{C}at_\infty}(\mathscr{C}, \mathscr{X}) \arrow[r] \arrow[d, "\epsilon_*"] & \mathrm{Hom}_{\mathrm{Fun}(K, \mathscr{C}at_\infty)}(\underline{\mathscr{C}}, \underline{\mathscr{X}}) \arrow[r, "c^*"] \arrow[d, "\underline{\epsilon}_*"] & \mathrm{Hom}_{\mathrm{Fun}(K, \mathscr{K}an)}(p, \underline{\mathscr{X}}) \arrow[d, "\underline{\epsilon}_*", "\cong"'] \\ \mathrm{Hom}_{\mathscr{C}at_\infty}(\mathscr{C}, \mathscr{C}) \arrow[r] & \mathrm{Hom}_{\mathrm{Fun}(K, \mathscr{C}at_\infty)}(\underline{\mathscr{C}}, \underline{\mathscr{C}}) \arrow[r, "c^*"'] & \mathrm{Hom}_{\mathrm{Fun}(K, \mathscr{C}at_\infty)}(i_* p, \underline{\mathscr{C}}) \end{tikzcd}$$

Since c exhibits $\mathscr{C}$ as a colimit to p both the top and bottom composites are isomorphisms. It follows that the map

$$\epsilon_* : \mathrm{Hom}_{\mathscr{C}at_\infty}(\mathscr{C}, \mathscr{X}) \to \mathrm{Hom}_{\mathscr{C}at_\infty}(\mathscr{C}, \mathscr{C})$$

is an isomorphism. Take $f : \mathscr{C} \to \mathscr{X}$ a homotopy lift of the identity on $\mathscr{C}$, so that we have a 2-simplex

$$\begin{tikzcd} & \mathscr{X} \arrow[dr, "\epsilon"] & \\ \mathscr{C} \arrow[ur, "f"] \arrow[rr, "id"'] & & \mathscr{C}. \end{tikzcd}$$

in $\mathscr{C}at_\infty$.

By definition, this 2-simplex is a simplicial map

$$\mathrm{Path}(\Delta^2) \to \underline{\mathrm{Cat}}_\infty$$

with the appropriate restrictions, which is then a choice of an isomorphism $\epsilon f \overset{\sim}{\to} id_{\mathscr{C}}$ in $\mathrm{Fun}(\mathscr{C}, \mathscr{C})$. Via the identification

$$\epsilon_* : \mathrm{Fun}(\mathscr{X}, \mathscr{X}) \overset{\sim}{\to} \mathrm{Fun}(\mathscr{X}, \mathscr{C})^{\mathrm{Kan}}$$

from Lemma I-7.6 this natural isomorphism restricts to a natural isomorphism

$$f\epsilon \overset{\sim}{\to} id_{\mathscr{X}}.$$

So, by definition, the inclusion $\mathscr{X} \to \mathscr{C}$ is an equivalence with inverse $f : \mathscr{C} \to \mathscr{X}$. Consequently, $\mathscr{C}$ is a Kan complex and so is in fact equal to $\mathscr{X}$. $\square$

14.5. **Example: pullback vs. homotopy pullback.**

14.6. **Other examples of complete and cocomplete categories.**

15. Kan extension

16. Yoneda embedding

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