KERODON REMIX PART II: CARTESIAN AND COCARTESIAN FIBRATIONS, LIMITS AND COLIMITS [UNDER CONSTRUCTION]

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Abstract. These are notes on ∞-categories which are (mostly) adapted from Lurie's digital text Kerodon [3]. The main distinctions are the length of the document, the order of presentation, and selective omissions. We also deviate from [3] in that we focus on derived categories and dg categories as our primary examples of interest. A distinction from the related text [2] would be the complete avoidance of model structures, though this approach is already adopted in [3].

Following Part I, which presented the basic foundations for studies of ∞-categories, we discuss herein cartesian and cocartesian fibrations, transport functors (i.e. Grothendeick straightening and unstraightening), and limits and colimits. Specific topics include Hom functions and Yoneda embedding, calculations of limits and colimits in categories of spaces and cochains, and stable ∞-categories. While we follow the analytic, rather than synthetic, approach, we've attempted to communicate these topics in a manner which is consumable to the working mathematician.

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1. Cartesian and cocartesian fibrations

1.1. Definitions.

Definition 1.1. Consider a map of simplicial sets $q : X \to S$. We call a 1-simplex $\alpha : x \to y$ in X a q-cartesian morphism if any lifting problem

$$\begin{tikzcd} \Lambda^n_n \arrow[r, "\bar{\sigma}"] \arrow[d] & X \arrow[d, "q"] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & S \end{tikzcd} \tag{1}$$

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with $n \geq 2$ and $\bar{\sigma}|\Delta^{\{n-1,n\}} = \alpha$ admits a solution. We say $\alpha : x \to y$ is q-cocartesian if any lifting problem

$$\begin{tikzcd} \Lambda^n_0 \arrow[r, "\bar{\tau}"] \arrow[d] & X \arrow[d, "q"] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & S \end{tikzcd} \tag{2}$$

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with $n \geq 2$ and $\bar{\tau}|\Delta^{\{0,1\}} = \alpha$ admits a solution.

Though at some specific moments we will consider a case where X and S are not ∞-categories, we are primarily invested in the ∞-categorical setting.

Definition 1.2. We call a map of ∞-categories $q : \mathscr{C} \to \mathscr{D}$ a cartesian fibration if q is an inner fibration and, for any map $\bar{\alpha} : \bar{x} \to \bar{y}$ in $\mathscr{D}$ and y in $\mathscr{C}$ with $q(y) = \bar{y}$, there is a q-cartesian map $\alpha : x \to y$ in $\mathscr{C}$ with $q(\alpha) = \bar{\alpha}$.

Similarly, we call q a cocartesian fibration if it is an inner fibration and, for any map $\bar{\beta} : \bar{x} \to \bar{y}$ in $\mathscr{D}$ and x with $q(x) = \bar{x}$, there is a q-cocartesian fibration $\beta : x \to y$ with $q(\beta) = \bar{\beta}$.

The following is obvious. Recall our definitions of right and left fibrations from Definition I-4.23.

Proposition 1.3. *If $q : \mathscr{C} \to \mathscr{D}$ is a right fibration (resp. left fibration) then q is a cartesian (resp. cocartesian).*

Proof. In this case any lifting problem of the form (1), or (2) respectively, admits a solution simply by the defintion. $\square$

Obviously when $q : \mathscr{C} \to \mathscr{D}$ is a Kan fibration it is both cartesian and cocartesian.

Example 1.4. Consider a diagram $p : K \to \mathscr{C}$, with K some simplicial set. The we have the overcategory $\mathscr{C}_{/p}$ and the undercategory $\mathscr{C}_{p/}$. The two forgetful functors

$$\mathscr{C}_{/p} \to \mathscr{C} \quad \text{and} \quad \mathscr{C}_{p/} \to \mathscr{C}$$

are, respectively, a right and left fibration Proposition I-4.25. Hence these maps are respectively a cartesian and cocartesian fibration.

In the case where K is a point $x : * \to \mathscr{C}$ we recall that the fibers of the fibration $\mathscr{C}_{/x} \to \mathscr{C}$ and $\mathscr{C}_{x/} \to \mathscr{C}$ are the right and left pinched mapping spaces $\mathrm{Hom}^{\mathrm{R}}_{\mathscr{C}}(w, x)$ and $\mathrm{Hom}^{\mathrm{L}}_{\mathscr{C}}(x, y)$.

Example 1.5 ([3, 01T8])**.** Consider an ∞-category $q : \mathscr{C} \to *$. A morphism $\alpha : x \to y$ is q-cartesian if and only if α is an isomorphism. To see this consider fillings of the horns

$$\begin{tikzcd} & x \arrow[dr, "\alpha"] & \\ y \arrow[ur, dotted, "\exists\beta"] \arrow[rr, "id"'] & & y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & y \arrow[dr, "\beta"] \arrow[dd, "id"] & \\ x \arrow[ur, "\alpha"] \arrow[rr, "id"] \arrow[dr, "\alpha"'] & & x. \arrow[dl, "\alpha"] \\ & y & \end{tikzcd}$$

One similarly finds that a morphism is q-cocartesian if and only if it is an isomorphism.

Via the existence of identity morphisms the structure map $q : \mathscr{C} \to *$ is always a cartesian and cocartesian fibration. Note that this map is not a left or right fibration unless $\mathscr{C}$ is a Kan fibration.

1.2. **Imaginings: Cartesian fibrations as lax moduli.** Give a cartesian fibration $q : \mathscr{C} \to \mathscr{D}$ one might think of $\mathscr{C}$ as a lax moduli of "stuff" varying over the objects in $\mathscr{D}$. The cartesian lifts of morphisms in $\mathscr{D}$ provide transition functions between these fibers, i.e. the stuff we are parametrizing, over $\mathscr{D}$. In the case of the cartesian fibration $\mathscr{C}_{/x} \to \mathscr{C}$ the category $\mathscr{C}_{/x}$ is, in an obvious sense, the "moduli of maps to x". Let us leave the latter point about lifting maps for now, and try to make some comment on the moduli point.

Let us just consider how one classically constructs a moduli space. Here we consider the base space $\mathscr{D} = \mathrm{Sch}_k$ of schemes over k, which we can endow with some Grothendieck topology if we like, though we don't care at the moment. Then a pre-stack is a choice of a functor of plain categories $q : \mathbb{M} \to \mathrm{Sch}_k$ which makes $\mathbb{M}$ into a category fibered in groupoids over Sch_k [5, 003S]. One simply compares definitions to see that

$$\left\{ \begin{array}{c} \mathbb{M} \text{ is fibered in} \\ \text{groupoids over } \mathrm{Sch}_k \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{c} q \text{ is a cartesian fibration} \\ \text{in which all maps in } \mathbb{M} \\ \text{are } q\text{-cartesian} \end{array} \right\}.$$

In this familiar setting one can now "invert" this functor q to produce an associated 2-functor

$$q^\vee : (\mathrm{Sch}_k)^{\mathrm{op}} \to \mathrm{Groupoids} \subseteq \mathrm{Cat}, \quad Y \mapsto \mathbb{M}_Y.$$

One establishes this functor via an abuse of the axion of choice.

To elaborate a bit more, for any map of schemes $\alpha : X \to Y$ we take a lift $\alpha^* y \to y$ in $\mathbb{M}$. This lift is unique up to unique isomorphism, and via unique filling defines a functor between the fibers

$$\alpha^* : \mathbb{M}_Y \to \mathbb{M}_X, \quad y \mapsto \alpha^* y.$$

On morphisms $\xi : y_1 \to y_2$ in the fiber $\mathbb{M}_Y$, we note that the cartesian property for maps in $\mathbb{M}$ asserts the existence of a unique map $\alpha^*\xi : \alpha^* y_1 \to \alpha^* y_2$ completing the diagram

$$\begin{tikzcd} \alpha^* y_1 \arrow[r] \arrow[d, dashed, "\exists!"'] & y_1 \arrow[d, "\xi"] \\ \alpha^* y_2 \arrow[r] & y_2, \end{tikzcd}$$

where we note that uniqueness comes from filling the appropriate 3-simplex in $\mathbb{M}$. Hence α^* is well-defined on morphisms via the assignment $\xi \mapsto \alpha^*\xi$.

We note that this inversion of $q : \mathbb{M} \to \mathrm{Sch}_k$ into a functor $q^\vee : (\mathrm{Sch}_k)^{\mathrm{op}} \to \mathrm{Cat}$ does not require all maps in $\mathbb{M}$ to be cartesian. This is simply a consequence of q being a cartesian fibration between plain categories.

In the general ∞-context, we again have this inversion property for (co)cartesian fibrations. Here a cartesian fibration $q : \mathscr{C} \to \mathscr{D}$ will specify, and be specified by, a functor

$$q^\vee : \mathscr{D}^{\mathrm{op}} \to \mathscr{C}at_\infty$$

whose values over objects y in $\mathscr{D}$ are the fibers $\mathscr{C}_y$. The functors between fibers $\alpha^* : \mathscr{C}_y \to \mathscr{C}_x$ are what we've referred to as *transport* along α (following Kerodon [3]).

While such fibrations play an essentially non-existent role in plain category theory, from the perspective of the working mathematician, they play an extraordinarily important role in the development of ∞-category theory. The main point is that they tame choices in the ∞-categorical setting. While in the plain category setting we can simply make a choice, and if that choice is not unique we can simply say it's unique up to a unique isomorphism, and then if I make two of the same types of choices then any ambiguities will vanish due to sufficient uniqueness, etc. etc., such a laissez faire attitude will lead to immediate intractable problems in the ∞-context. So one generally bundles all choices of a certain "type" into a cartesian or cocartesian fibrations, and manipulates these bundles in order to make global movements between choices of different types.

1.3. **Cartesian morphisms via overcategories.**

`prop:232` **Proposition 1.6** ([3, 01TF]). *Let $q : \mathscr{C} \to \mathscr{D}$ be a map between ∞-categories. A morphism $\alpha : x \to y$ in $\mathscr{C}$ is q-cartesian if and only if the natural map*

$$\mathscr{C}_{/\alpha} \to \mathscr{C}_{/y} \times_{\mathscr{D}_{/q(y)}} \mathscr{D}_{/q(\alpha)}$$

is a trivial Kan fibration. Similarly, α is q-cocartesian if and only if the map

$$\mathscr{C}_{/\alpha} \to \mathscr{C}_{x/} \times_{\mathscr{D}_{q(x)/}} \mathscr{D}_{q(\alpha)/}$$

is a trivial Kan fibration.

For the proof we employ a specific deconstruction of the relevant horn inclusions.

`lem:joyal3.3` **Lemma 1.7** ([1, Lemma 3.3]). *For non-negative integers p and q, and $n = p+q+1$, the inclusions*

$$(\Lambda^p_0 \star \Delta^q) \coprod_{\Lambda^p_q \star \partial\Delta^q} (\Delta^p \star \partial\Delta^q) \to \Delta^p \star \Delta^q \cong \Delta^n$$

and

$$(\partial\Delta^p \star \Delta^q) \coprod_{\partial\Delta^p \star \Lambda^q_q} (\partial\Delta^p \star \Lambda^q_q) \to \Delta^p \star \Delta^q \cong \Delta^n$$

are identified with the inclusions of the extremal horns $\Lambda^n_0 \to \Delta^n$ and $\Lambda^n_n \to \Delta^n$ respectively.

One can see the text [1], or [3, 018N] for the details. We now proceed with the proof of Proposition 1.6.

Proof of Proposition 1.6. We address the cartesian situation, the cocartesian one being similar. Let $F : \mathscr{C}_{/\alpha} \to \mathscr{C}_{/y} \times_{\mathscr{D}_{/q(y)}} \mathscr{D}_{/q(\alpha)}$ denote the map under consideration. A solution to a lifting problems of the form

$$\begin{tikzcd} \partial\Delta^m \arrow[r] \arrow[d] & \mathscr{C}_{/\alpha} \arrow[d, "F"] \\ \Delta^m \arrow[r] \arrow[ur, dotted] & \mathscr{C}_{/y} \times_{\mathscr{D}_{/q(y)}} \mathscr{D}_{/q(\alpha)}, \end{tikzcd} \tag{3}$$

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with $m \geq 0$, admit a solution if and only if the equivalent lifting problem

$$\begin{tikzcd} (\partial\Delta^m \star \Delta^1) \cup (\Delta^m \star \Lambda^1_1) \arrow[r] \arrow[d] & \mathscr{C} \arrow[d, "q"] \\ \Delta^m \star \Delta^1 \arrow[r] \arrow[ur, dotted] & \mathscr{D} \end{tikzcd}$$

obtained by way of adjunction Lemma I-4.22 admits a solution. Via direct inspection the final edge $\Delta^1 \cong \emptyset \star \Delta^1 \to \mathscr{C}$ in the latter diagram is α, so that this diagram is identified, via Lemma 1.7, with a diagram of the form

$$\begin{tikzcd} \Lambda^n_n \arrow[r] \arrow[d] & \mathscr{C} \arrow[d, "q"] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & \mathscr{D} \end{tikzcd} \tag{4}$$

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in which $n \geq 2$ the edge $\Delta^{\{n-1,n\}} \to \mathscr{C}$ is α. It follows that all lifting problems of the form (3) admit a solution if and only if all lifting problems of the form (4) admit a solution, i.e. that the map F is a trivial Kan fibration if and only if the map α is q-cartesian. □

1.4. **Cartesian morphisms via mapping spaces.**

prop:cocart_maps

Proposition 1.8 ([3, 01TL]). *Consider an inner fibration $q : \mathscr{C} \to \mathscr{D}$, and a morphism $\alpha : x_1 \to x_2$ in $\mathscr{C}$ with image $\bar{\alpha} : \bar{x}_1 \to \bar{x}_2$ in $\mathscr{D}$. The morphism α is q-cartesian if and only if for each third object x_0 in $\mathscr{C}$, with corresponding triples $x : \{0,1,2\} \to \mathscr{C}$ and $\bar{x} : \{0,1,2\} \to \mathscr{D}$, the diagram*

$$\begin{tikzcd} \mathrm{Fun}(\Delta^2, \mathscr{C})_x \times_{\mathrm{Hom}_{\mathscr{C}}(x_1,x_2)} \{\alpha\} \arrow[r] \arrow[d, "q"'] & \mathrm{Hom}_{\mathscr{C}}(x_1, x_2) \arrow[d, "q"'] \\ \mathrm{Fun}(\Delta^2, \mathscr{D})_{\bar{x}} \times_{\mathrm{Hom}_{\mathscr{D}}(\bar{x}_1,\bar{x}_2)} \{\bar{\alpha}\} \arrow[r] & \mathrm{Hom}_{\mathscr{D}}(\bar{x}_1, \bar{x}_2) \end{tikzcd}$$

is a homotopy pullback diagram of Kan complexes.

We cover the proof of Proposition **??** in Section 1.5 below. Let us record now a number of examples.

sect:cocart_maps_proof

1.5. **Proof of Proposition 1.8.**

1.6. **Exponentiating cocartesian fibrations.**

Proposition 1.9 ([3, 01VG]). *If $q : X \to S$ is a cocartesian fibration, then for any simplicial set K the map $q_* : \mathrm{Fun}(K, X) \to \mathrm{Fun}(X, S)$ is a cocartesian fibration. An edge $\xi : \Delta^1 \to \mathrm{Fun}(K, X)$ is q_*-cocartesian if and only if, at each v in K, the composite $v^*\xi : \Delta^1 \to X$ is q-cocartesian in X.*

1.7. Some lifting problems.

2. Directional fibrations and Kan complexes

2.1. Exponentials for directional fibrations.

Definition 2.1. A map of simplicial sets $A \to B$ is called left anodyne (resp. right anodyne) if any lifting problem

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ B \arrow[r] \arrow[ur, dotted] & S \end{tikzcd}$$

in which f is a left (resp. right) fibration admits a solution.

One can show that the class of left anodyne maps is the saturated class generated by the horn inclusions $\Lambda^n_i \to \Delta^n$, where $0 \leq i < n$ [3, 0151]. One similarly characterizes right anodyne maps.

lem:328 **Lemma 2.2** ([3, kerodon]). *Let $i : A \to B$ and $j : K \to L$ be monomorphisms of simplicial sets. If one of i or j is left (resp. right) anodyne, then the induced map*

$$(B \times K) \coprod_{A \times K} (A \times L) \to B \times L$$

is left (resp. right) anodyne.

We refer the reader to Kerodon [3] for the proof.

prop:direct_tech **Proposition 2.3.** *Let $f : X \to S$ be a map of simplicial sets, and $j : K \to L$ be a monomorphism of simplicial sets. Consider the induced map on the functor complexes*

$$\rho : \mathrm{Fun}(L, X) \to \mathrm{Fun}(K, X) \times_{\mathrm{Fun}(K,S)} \mathrm{Fun}(L, S).$$

1. *If f is a left fibration, then ρ is a left fibration.*
2. *If f is a right fibration, then ρ is a right fibration.*
3. *If f is a left fibration and j is left anodyne, then ρ is a trivial Kan fibration.*
4. *If f is a right fibration and j is right anodyne, then ρ is a trivial Kan fibration.*

Proof. Solving a lifting problem of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \mathrm{Fun}(L, X) \arrow[d, "f"] \\ B \arrow[r] \arrow[ur, dotted] & \mathrm{Fun}(K, X) \times_{\mathrm{Fun}(K,S)} \mathrm{Fun}(L, S) \end{tikzcd}$$

is equivalent to solving the corresponding lifting problem

$$\begin{tikzcd} (B \times K) \amalg_{(A \times K)} (A \times L) \arrow[r] \arrow[d] & X \arrow[d] \\ B \times L \arrow[r] \arrow[ur, dotted] & S. \end{tikzcd}$$

So all of the claims follow from a consideration of Lemma 2.2. ∎

2.2. **Directional fibrations and Kan complexes.**

Proposition 2.4. *A cocartesian (resp. cartesian) fibration $f : X \to S$ is a left (resp. right) fibration if and only if all of the fibers X_s, at arbitrary $s : * \to S$, are Kan complexes.*

3. A deviation on $(\infty, 2)$-categories

3.1. $(\infty, 2)$**-categories.**

Definition 3.1. Let X be a simplicial set. A 2-simplex $\tau : \Delta^2 \to X$ is called thin if any horn for any $n > 2$, index $0 < i < n$, and inner horn

$$\bar{\sigma} : \Lambda^n_i \to X \quad \text{with} \quad \bar{\sigma}|\Delta^{\{i-1,i,i+1\}} = \tau,$$

the lifting problem

$$\begin{tikzcd} \Lambda^n_i \arrow[r, "\bar{\sigma}"] \arrow[d] & X \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & * \end{tikzcd}$$

admits a solution.

One sees immediately that every 2-simplex in an ∞-category is thin, for example.

Recall our notation $s_i : [n] \to [n-1]$ for the weakly increasing surjection with $s_i(i) = s_i(i+1) = i$, for $0 \leq i \leq n-1$, and the corresponding degeneracies $s_i^* : \Delta^n \to \Delta^{n-1}$. We call an n-simplex $\sigma : \Delta^n \to X$ in a simplicial set *left degenerate* if σ factors through the degeneracy $s_0^* : \Delta^n \to \Delta^{n-1}$, and *right degenerate* if σ factors through the degeneracy $s_{n-1}^* : \Delta^n \to \Delta^{n-1}$.

def:infty2 **Definition 3.2** ([3, 01W9, 01Y5])**.** A simplicial set X is called an $(\infty, 2)$-category if the following hold:

(a) Any horn $\Lambda^2_1 \to X$ admits an extension to a thin 2-simplex.
(b) Every degenerate 2-simplex in X is thin.
(c.l) For $n > 2$, any horn $\bar{\sigma} : \Lambda^n_0 \to X$ in which the 2-simplex $\bar{\sigma}|\Delta^{\{0,1,n\}}$ is left degenerate admits an extension to an n-simplex in X.
(c.r) For $n > 2$, any horn $\bar{\sigma}' : \Lambda^n_n \to X$ in which the 2-simplex $\bar{\sigma}'|\Delta^{\{0,n-1,n\}}$ is right degenerate admits an extension to an n-simplex in X.

A functor, or map, between $(\infty, 2)$-categories is a map of simplicial sets $F : X \to Y$ which preserves thin 2-simplices.

Remark 3.3. Having introduced this notion, let us recall that the term ∞-category is used interchangeably with the term $(\infty, 1)$-category.

If we consider an ∞-category $\mathscr{C}$, then in any horn $\Lambda^n_0 \to \mathscr{C}$ as in (c.l) the initial edge $\Delta^{\{0,1\}} \to \mathscr{C}$ is degenerate, and hence an isomorphism in $\mathscr{C}$. Hence we have the proposed completion to an n-simplex $\Delta^n \to \mathscr{C}$, by Proposition I-4.33. Similarly any horn $\Lambda^n_n \to \mathscr{C}$ as in (c.r) completes to an n-simplex as well. So we observe the following.

Lemma 3.4. *Any ∞-category is an $(\infty, 2)$-category. Furthermore, an $(\infty, 2)$-category X is an ∞-category if and only if every 2-simplex in X is thin.*

Recall that each simplex Δ^n is an ∞-category, and hence an $(\infty, 2)$-category.

Example 3.5. Since any degenerate 2-simplex in an $(\infty,2)$-category is thin, any map of simplicial sets $*=\Delta^0\to X$ is a map of $(\infty,2)$-categories. Similarly, any map of simplicial sets $\Delta^1\to X$ is a map of $(\infty,2)$-categories.

One has the following practical check for maps between $(\infty,2)$-categories.

prop:infty2_check **Proposition 3.6** ([3, 01YC])**.** *Let X and Y be $(\infty,2)$-categories, and $F:X\to Y$ be a map of simplicial sets. Then F is a functor, i.e. preserves thin 2-simplexes, if and only if any horn $\Lambda^2_1\to X$ can be competed to a thin simplex with thin image in Y.*

Idea of proof. The result is a consequence of stability of thin simplices under various conditions. Namely one establishes an inner-exchange property for thin simplices, which we recall below, and a 4-of-5 property which one can find at [3, 01XX]. □

3.2. **The pith of an $(\infty,2)$-category.**

Definition 3.7. Given an $(\infty,2)$-category X, the pith in X is the simplicial subset $X^{\mathrm{Pith}}\subseteq X$ whose simplices $\Delta^n\to X^{\mathrm{Pith}}$ consist of all simplices $\sigma:\Delta^n\to X$ in which each restriction along a 2-simplex

$$\Delta^2\to\Delta^n\overset{\sigma}{\to}X$$

is thin.

Since functors between $(\infty,2)$-categories preserve thin simplices, by defintion, we see that any map $F:\mathscr{C}\to X$ from an ∞-category to an $(\infty,2)$-category factors through the pith.

lem:360 **Lemma 3.8** ([3, 01XL], Inner-exchange property)**.** *Consider a 3-simplex $\sigma:\Delta^3\to X$ in an $(\infty,2)$-category, and suppose that the associated 2-simplices $\sigma|\Delta^{\{1,2,3\}}$ and $\sigma|\Delta^{\{0,1,2\}}$ are thin. Then the 2-simplex $\sigma|\Delta^{\{0,2,3\}}$ is thin if and only if the 2-simplex $\sigma|\Delta^{\{0,1,3\}}$ is thin.*

The proof employs certain facts about interior fibrations (see below), and is omitted. From Lemma 3.8 the proof of the following is immediate.

Proposition 3.9. *For any $(\infty,2)$-category X, the subcomplex X^{Pith} is an ∞-category.*

Proof. For any completion $\Delta^3\to X$ of an inner horn $\Lambda^3_i\to X$ in which all of the associated face $\Delta^2\to\Lambda^3_i\to X$ are thin, the final face $\Delta^{[3]\setminus\{i\}}\to X$ is also thin, by Lemma 3.8. This shows that the pith is stable under the completion of inner horns $\Lambda^3_i\to X^{\mathrm{Pith}}$. Stability under completion of all inner horns $\Lambda^n_i\to X^{\mathrm{Pith}}$ with $n>3$ is immediate, since the horn Λ^n_i already contains all 2-faces in Δ^n in this case. Taken together with condition (a) of Definition 3.2, we see that any lifting problem

$$\begin{tikzcd}
\Lambda^n_i \arrow[r] \arrow[d] & X^{\mathrm{Pith}} \arrow[d] \\
\Delta^n \arrow[r] & *
\end{tikzcd}$$

with $0<i<n$ admits a solution, as required. □

3.3. $(\infty, 2)$**-category via simplicial categories.** Recall that one can associate to any simplicial category $\underline{S}$ its associated homotopy coherent nerve $\mathrm{N}^{\mathrm{hc}}(\underline{S})$ Section I-2.7. The n-simplices in $\mathrm{N}^{\mathrm{hc}}(\underline{S})$ are simplicial functors from the path category $\mathrm{Path}[n]$. For $S = \mathrm{N}^{\mathrm{hc}}(\underline{S})$ we have in low dimension

$$S[0] = \{ \text{ objects in } \underline{S} \ \}$$

$$S[1] = \{ \text{ pairs of object } (x_0, x_1) \text{ along with a map } f \in \underline{\mathrm{Hom}}_{\underline{S}}(x_0, x_1)[0] \ \}$$

$$S[2] = \left\{ \begin{array}{l} \text{triples of objects } (x_0, x_1, x_2), \text{ maps } f_{ij} : x_i \to x_j \text{ for each } i < j, \text{ and} \\ \text{a 1-simplex } h : \Delta^1 \to \underline{\mathrm{Hom}}_{\underline{S}}(x_0, x_2) \text{ with } h|_0 = f_{02},\ h|_1 = f_{12}f_{01} \end{array} \right\}$$

Lemma I-2.16. We take the following theorem for granted.

thm:hc_infty2

Theorem 3.10 ([3, 01YM])**.** *Let $\underline{S}$ be a simplicial category in which, at each pair of objects x and y in $\underline{S}$, the mapping complex $\underline{\mathrm{Hom}}_{\underline{S}}(x, y)$ is an ∞-category. Then the homotopy coherent nerve $\mathrm{N}^{\mathrm{hc}}(\underline{A})$ is an $(\infty, 2)$-category.*

What we are most interested in here is the $(\infty, 2)$-category of ∞-categories. Recall that for any ∞-categories $\mathscr{C}$ and $\mathscr{D}$ the simplicial set of functors $\mathrm{Fun}(\mathscr{C}, \mathscr{D})$, whose simplicial are as expected

$$\mathrm{Fun}(\mathscr{C}, \mathscr{D})[n] = \mathrm{Hom}_{\mathrm{sSet}}(\Delta^n \times \mathscr{C}, \mathscr{D}),$$

form another ∞-category Corollar I-4.8. With these morphisms we obtain the simplicial category $\underline{\mathrm{Cat}}_\infty$ of ∞-categories and their functor categories. We note that $\underline{\mathrm{Cat}}_\infty$ is a full simplicial subcategory in the ambient category $\underline{\mathrm{sSet}}$ of simplicial sets.

3.4. **The** $(\infty, 2)$**-category of** ∞**-categories.**

Theorem 3.11. *The homotopy coherent nerve*

$$\mathrm{Cat}_\infty := \mathrm{N}^{\mathrm{hc}}(\underline{\mathrm{Cat}}_\infty)$$

is an $(\infty, 2)$*-category.*

According to the above analysis the 0-simplices in Cat_∞ are ∞-categories, the 1-simplices are functors between ∞-categories, and 2-simplices are triples of functors and a natural transformation

$$\begin{tikzcd} & \mathscr{C}_2 & & & \mathscr{C}_2 \\ \mathscr{C}_1 \arrow[ur, "F_{12}"] & & \arrow[r, "\zeta"] & {} & \\ & \mathscr{C}_0 \arrow[ul, "F_{01}"] & & & \mathscr{C}_0. \arrow[uu, "F_{02}"] \end{tikzcd}$$

Definition 3.12. The $(\infty, 2)$-category Cat_∞ is called the $(\infty, 2)$-category of ∞-categories.

Remark 3.13. The $(\infty, 2)$-category Cat_∞ is in our universe of “large” sets, which is strictly larger than our universe of “normal sized” set in which all other ∞-categories are assumed to live.

We recall our ∞-category $\mathscr{C}at_\infty$ of ∞-categories, which we obtain by restricting the morphisms $\mathrm{Fun}(\mathscr{C}, \mathscr{D})$ to the associated Kan can complex $\mathrm{Fun}(\mathscr{C}, \mathscr{D})^{\mathrm{Kan}}$ then applying the simplicial nerve. The inclusions of ∞-categories

$$\mathrm{Fun}(\mathscr{C}, \mathscr{D})^{\mathrm{Kan}} \to \mathrm{Fun}(\mathscr{C}, \mathscr{D})$$

imply an inclusion of $(\infty, 2)$-categories $\mathscr{C}at_\infty \to \mathrm{Cat}_\infty$, and hence an inclusion into the pith

$$\mathscr{C}at_\infty \to (\mathrm{Cat}_\infty)^{\mathrm{Pith}}. \tag{5}$$

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By a general result one finds that this inclusion is an equality.

Proposition 3.14 ([3, 01YT])**.** *The inclusion* (5) *is an equality,* $\mathscr{C}at_\infty = (\mathrm{Cat}_\infty)^{\mathrm{Pith}}$.

3.5. **Interior fibrations.**

def:interior

Definition 3.15. A map of simplicial sets $q : X \to S$ is called an interior fibration if the following hold:

(a) At each 0-simplex x in X, the identity $id_x : x \to x$ is both q-cartesian and cocartesian.
(b) For any lifting problem

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta^n \arrow[r, "\sigma"'] \arrow[ur, dotted] & S \end{tikzcd} \tag{6}$$

eq:400

in which $0 < i < n$ and $\sigma|\Delta^{\{i-1,i,i\}}$ is thin in S, (6) admits a solution.

It is clear that if $f : S' \to S$ is a map of simplicial sets which preserves thin 2-simplices, and the diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "q'"'] & X \arrow[d, "q"] \\ S' \arrow[r, "f"] & S \end{tikzcd}$$

is a pullback diagram of simplicial sets in which q is an interior fibration, then the map $q' : X' \to S'$ is an interior fibration as well.

One also observes the following.

lem:495

Lemma 3.16. *If S is an $(\infty, 2)$-category, and $q : X \to S$ is an interior fibration, then X is also an $(\infty, 2)$-category and q is a functor between $(\infty, 2)$-categories.*

Proof. One sees via the lifting property for q that any 2-simplex $\Delta^2 \to X$ which has thin image in S is thin in X. From this we see that any horn $\Lambda^2_1 \to X$ can be completed to a thin 2-simplex in X. One obtains this thin completion by lifting a thin completion $\Delta^2 \to S$. We are left to prove that any appropriate degenerate horn $\Lambda^n_0 \to X$ or $\Lambda^n_n \to X$, at $n > 2$, completes to an n-simplex. However this follows from the fact the fact that identity maps in X are both q-cartesian and cocartesian, and the fact that the corresponding horns in S admit completions. We now see that X is an $(\infty, 2)$-category. One sees that q is a functor, i.e. preserves thin 2-simplices, by applying Proposition 3.6. □

As we see in the above proof, given an interior fibration $q : X \to S$ over an $(\infty, 2)$-category, one can detect thin simplices in X by considering their images in S along q.

lem:505

Lemma 3.17. *If $q : X \to S$ is an interior fibration then a 2-simplex in X is thin if and only if its image in S is thin.*

cor:interior_pullback **Corollary 3.18.** *Consider a pullback diagram*

$$\begin{tikzcd} Z \arrow[r, "p_2"] \arrow[d, "p_1"'] & X \arrow[d, "q"] \\ Y \arrow[r, "F"'] & S \end{tikzcd}$$

in which q is an interior fibration and F is a map between $(\infty,2)$-categories. Then Z is an $(\infty,2)$-category, p_1 is an interior fibration, and p_2 is a map of $(\infty,2)$-categories.

Proof. The fact that Z is an $(\infty,2)$-category and p_1 is a map of $(\infty,2)$-categories follows by Lemma 3.16. As for p_1, we consider a thin 2-simplex $\Delta^2 \to Z$, and note that its image in Y is thin. Hence its image in S is thin, and so its image in X is thin by Lemma 3.17. It follows that p_2 is a map of $(\infty,2)$-categories, by definition. □

We are especially interested in the fiberings of interior fibrations over ∞-categories.

lem:502 **Lemma 3.19.** *Let $\mathscr{C}$ be an ∞-category. A map of simplicial sets $q : X \to \mathscr{C}$ is an interior fibration if and only if it is an inner fibration.*

Proof. If q is an interior fibration then it is an inner fibration since all 2-simplices in $\mathscr{C}$ are thin. Conversely, if q is an inner fibration then X is an ∞-category and q is therefore an inner fibration between ∞-categories. Condition (a) of Definition 3.15 now follows from the fact that the identity in an ∞-category is an isomorphism, and an application of Proposition I-4.33. □

One combines Lemma 3.19 with the above discussion of fiber products to obtain the following corollary.

Corollary 3.20. *Consider an interior fibration $q : X \to S$ over an $(\infty,2)$-category S.*

(a) *For any ∞-category $\mathscr{C}$, and any functor of $(\infty,2)$-categories $\mathscr{C} \to S$, the fiber product $X \times_S \mathscr{C}$ is an ∞-category. Furthermore, the projection $X \times_S \mathscr{C} \to \mathscr{C}$ is an inner fibration.*
(b) *At each point $s : * \to S$ the fiber X_s is an ∞-category.*

cor:interior_pith **Corollary 3.21.** *If $q : X \to S$ is an interior fibration over an $(\infty,2)$-category S then the diagram*

$$\begin{tikzcd} X^{\mathrm{Pith}} \arrow[r] \arrow[d] & X \arrow[d] \\ S^{\mathrm{Pith}} \arrow[r] & S \end{tikzcd}$$

is a pullback diagram, and the map $X^{\mathrm{Pith}} \to S^{\mathrm{Pith}}$ is an interior fibration.

Proof. In this case the pullback $X \times_S S^{\mathrm{Pith}}$ is an ∞-category and the projection to X is a map of $(\infty,2)$-categories. So the identification

$$X^{\mathrm{Pith}} = X \times_S S^{\mathrm{Pith}}$$

follows via an application of the universal property for the pullback and the universal property for the pith. □

3.6. **Undercategories and overcategories and pointed ∞-categories.** In the $(\infty,2)$-setting we can define overcategories and undercategories exactly as in the ∞-setting. Namely, for a map of simplicial sets $p : K \to X$ the overcategory $X_{p/}$ is the simplicial set with n-simplices provided by the join

$$X_{p/}[n] := \mathrm{Hom}_{\mathrm{sSet}}(K \star \Delta^n, X)_p,$$

and similarly for the undercategory

$$X_{/p}[n] := \mathrm{Hom}_{\mathrm{sSet}}(\Delta^n \star K, X)_p$$

Section I-4.6. In the case in which X is an ∞-category, we saw that the forgetful functors

$$\begin{tikzcd} X_{p/} \arrow[dr] & & X_{/p} \arrow[dl] \\ & X & \end{tikzcd}$$

obtained by restricting along the inclusions $\Delta^n \to \Delta^n \star K$ and $\Delta^n \to K \star \Delta^n$ are directional fibrations, and in particular isofibrations. We have a similar result in the 2-categorical context.

prop:504 **Proposition 3.22** ([3, 01WU])**.** *Let X be an $(\infty,2)$-category and $p : K \to X$ be a map of simplicial sets. The the forgetful maps*

$$X_{p/} \to X \quad \textit{and} \quad X_{/p} \to X$$

are both interior fibrations.

At this point we'll begin to leave many of the details unaccounted for. In particular, we direct the reader to the original text [3] for the details on Proposition 3.22. In any case, we record some corollaries.

Corollary 3.23. *For an $(\infty,2)$-category X and a diagram $p : K \to X$, the simplical sets $X_{p/}$ and $X_{/p}$ are $(\infty,2)$-category and the forgetful maps are both functors between $(\infty,2)$-categories.*

cor:601 **Corollary 3.24.** *Let $p : K \to X$ be a map from a simplicial set into an $(\infty,2)$-category. At any point $x : * \to X$ the fibers $(X_{p/})_x$ and $(X_{/p})_x$ are both ∞-categories.*

We apply this corollary in the case where the diagram p is a point $x : * \to X$ to obtain mapping categories for any $(\infty,2)$-category X.

Definition 3.25. For any $(\infty,2)$-category X, and objects $x, y : * \to X$, the left pinched mapping ∞-category is the fiber

$$\mathrm{Hom}^{\mathrm{L}}_X(x,y) := (X_{x/}) \times_X \{y\}.$$

Similarly,the right pinched mapping ∞-category is the fiber

$$\mathrm{Hom}^{\mathrm{R}}_X(x,y) = \{x\} \times_X (X_{/y}).$$

As with any interior fibration, we can restrict the forgetful functor to the piths to obtain inner fibrations of ∞-categories

$$(X_{p/})^{\mathrm{Pith}} \to X^{\mathrm{Pith}} \;\text{ and }\; (X_{/p})^{\mathrm{Pith}} \to X^{\mathrm{Pith}}.$$

In this particular instance one can observe a stronger characterization of these functors.

prop:over_cartesian

Proposition 3.26 ([3, 01YE]). *For X and $p : K \to X$ as above, the restrictions of the forgetful functors*

$$(X_{p/})^{\mathrm{Pith}} \to X^{\mathrm{Pith}} \quad \textit{and} \quad (X_{/p})^{\mathrm{Pith}} \to X^{\mathrm{Pith}}.$$

are, respectively, a cocartesian fibration and a cartesian fibration.

One might view this result in analogy with the ∞-setting, where the forgetful functors were observed to be right and left fibrations Corollary I-4.27.

3.7. **Mapping categories in the homotopy coherent nerve.** Let $\underline{S}$ be a simplicial category whose morphism complexes are weak Kan complexes, and let S be the homotopy coherent nerve, $S = \mathrm{N}^{\mathrm{hc}}(\underline{S})$. We recall that S is an $(\infty, 2)$-category in this case. By an abuse of notation take

$$\underline{\mathrm{Hom}}_S(x, y) = \underline{\mathrm{Hom}}_{\underline{S}}(x, y)$$

for any given pair of objects in S. We construct a map of simplicial sets

$$\theta : \underline{\mathrm{Hom}}_S(x, y) \to \mathrm{Hom}^{\mathrm{L}}_S(x, y)$$

[3, 01LD] which is subsequently found to be an equivalence of ∞-categories.

To begin, for any simplicial set K we consider the simplicial category $E(K)$ with objects x_- and x_+ and morphisms

$$\mathrm{Hom}_{E(K)}(x_-, x_-) = \mathrm{Hom}_{E(K)}(x_+, x_+) = * \quad \text{and} \quad \mathrm{Hom}_{E(K)}(x_-, x_+) = K.$$

We consider the $(n+1)$-simplex $\{-1\}\star\Delta^n \cong \Delta^{n+1}$ and the simplicial path category $\mathrm{Path}(\{-1\} \star \Delta^n)$ whose morphisms are given by the nerves

$$\mathrm{Hom}_{\mathrm{Path}(\{-1\}\star\Delta^n)}(l, m) = \mathrm{N}(\mathrm{Subsets}^{\mathrm{op}}_{l,m})$$

where $\mathrm{Subsets}_{l,m}$ is the partially ordered set of subsets $S \subseteq [n]$ with $\min S = l$ and $\max S = m$, ordered by inclusion.

At each integer n we have a simiplicial functor

$$\theta^*_n : \mathrm{Path}(\{-1\} \star \Delta^n) \to E\Delta^n$$

which is define on objects by taking $\theta^*_n(-1) = x_-$, and $\theta^*_n(i) = x_+$ for all $i \geq 0$, and defined on morphisms by the simplicial map

$$\theta^*_n : \mathrm{Hom}_{\mathrm{Path}(\{-1\}\star\Delta^n)}(l, m) = \mathrm{N}(\mathrm{Subsets}^{\mathrm{op}}_{l,m}) \to \mathrm{Hom}_E(x_-, x_+) = \Delta^n = \mathrm{N}([n])$$

associated to the functor $\mathrm{Subsets}^{\mathrm{op}}_{l,m} \to [n]$ which sends each subset $S = \{l < s_1 < \ldots s_r < m\}$ to s_1 and each inclusion $S' \supseteq S$ to the inequality $s'_1 \leq s_1$.

For objects x and y in S, n-simplices in $\underline{\mathrm{Hom}}_S(x, y)$ are identified with simplicial functors $\mathrm{Fun}_{\mathrm{sCat}}(E\Delta^n, \underline{S})$ in the fiber over (x, y) in $\mathrm{Fun}(E\emptyset, \underline{S})$. Each such functor now defined an $(n + 1)$-simplex in S via a consideration of the identification

$$S[n + 1] = \mathrm{Fun}_{\mathrm{sCat}}(\mathrm{Path}(\{-1\} \star \Delta^n), \underline{S})$$

and restricting along θ^*_n. One sees, by the definiton of θ^*_n that this associated $(n + 1)$-simplex has initial vertex x and all other vertices y, and restricts trivially to $\Delta^n \subseteq \{-1\} \star \Delta^n$. So we obtain a map of sets

$$\theta_n : \underline{\mathrm{Hom}}_S(x, y)[n] \to (S_{x/}) \times_S \{y\} = \mathrm{Hom}^{\mathrm{L}}_S(x, y)[n],$$

$$(f : E\Delta^n \to \underline{S}) \mapsto (f\theta^*_n : \mathrm{Path}\{-1\} \star \Delta^n \to \underline{S}).$$

One observes directly that any increasing function $t : [n] \to [n']$ produces a commutative diagram

$$\begin{tikzcd} \mathrm{Path}(\{-1\} \star \Delta^n) \arrow[r, "\theta_n"] \arrow[d, "t_*"'] & E\Delta^n \arrow[d, "t_*"] \\ \mathrm{Path}(\{-1\} \star \Delta^{n'}) \arrow[r, "\theta_{n'}"'] & E\Delta^{n'}, \end{tikzcd}$$

from which we see that the θ_n assemble into a map of simplicial sets, or a map of ∞-categories,

$$\theta : \underline{\mathrm{Hom}}_S(x, y) \to \mathrm{Hom}^{\mathrm{L}}_S(x, y).$$

thm:pinched_simplicial

Theorem 3.27 ([3, 01LG])**.** *Let $\underline{S}$ be a simplicial category whose morphism complexes are ∞-categories. Take $S = \mathrm{N}^{\mathrm{hc}}(\underline{S})$. For any objects $x, y : * \to S$ there is a natural equivalence of ∞-categories*

$$\theta : \underline{\mathrm{Hom}}_S(x, y) \to \mathrm{Hom}^{\mathrm{L}}_S(x, y).$$

We do not cover the details, and refer the reader to the text [3].

cor:simplicial_pullback

Corollary 3.28. *Take $\underline{S}$ and S as above. For any pair of points $x, y : * \to S$ there is a categorical pullback diagram*

$$\begin{tikzcd} \underline{\mathrm{Hom}}_S(x, y) \arrow[r, "\theta"] \arrow[d] & (S_{x/})^{\mathrm{Pith}} \arrow[d] \\ * \arrow[r, "y"'] & S^{\mathrm{Pith}}. \end{tikzcd} \tag{7}$$

eq:704

Proof. By Corollary 3.18 and Proposition 3.22 the projection map $\mathrm{Hom}^{\mathrm{L}}_S(x, y) \to S_{x/}$ has image in the Pith $(S_{x/})^{\mathrm{Pith}}$. Applying this fact in conjunction with Corollary 3.21, we observe a pullback diagram of ∞-categories

$$\begin{tikzcd} \mathrm{Hom}^{\mathrm{L}}_S(x, y) \arrow[r, "\theta"] \arrow[d] & (S_{x/})^{\mathrm{Pith}} \arrow[d] \\ * \arrow[r, "y"] & S^{\mathrm{Pith}}. \end{tikzcd}$$

in which the right-hand map is an inner fibration. This diagram is additionally a categorical pullback square by Corollary I-5.22 and Proposition 3.26. Since $\theta : \underline{\mathrm{Hom}}_S(x, y) \to \mathrm{Hom}^{\mathrm{L}}_S(x, y)$ is an equivalence of ∞-categories it follows that the corresponding diagram (7) is a categorical pullback square as well (see Proposition I-5.23). □

sect:transport

4. Transport functors

4.1. **Preliminary discussion.** In analogy with the plain categorical setting, we claim that cocartesian fibrations $q : \mathscr{E} \to \mathscr{C}$ over a given ∞-category are "the same thing" as functors into the ∞-category of ∞-categories $F : \mathscr{C} \to \mathscr{C}at_\infty$. In our imaginations, the functor F should evaluate as the fibers $F(x) \cong \mathscr{E}_x$ and the image of a given map $\alpha : x \to y$ should be some kind of pushforward functor $\alpha_* : \mathscr{E}_x \to \mathscr{E}_y$ which "moves along" cartesian lifts $\widetilde{\alpha} : \widetilde{x} \to \widetilde{y}$, so that $\alpha_*(\widetilde{x}) \cong \widetilde{y}$.

Of course one can not simply construct the desired functor $F : \mathscr{C} \to \mathscr{C}at_\infty$ by hand. We (or rather, Lurie) instead proceed(s) by establishing a universal cocartesian fibration over ∞-categories

$$U : \mathscr{Z} \to \mathscr{C}at_\infty.$$

It is then shown that each cocartesian fibration is realized as a (categorical) pullback along U,

$$\begin{tikzcd} \mathscr{E} \arrow[r] \arrow[d, "q"'] & \mathscr{Z} \arrow[d] \\ \mathscr{C} \arrow[r, "F"'] & \mathscr{C}at_\infty, \end{tikzcd}$$

and furthermore that the space of such pullback diagrams assembles into a contractible space. We refer to this uniquely determined functor F as the covariant transport functor along q, or as the functor which classifies q. One obtains a completely similar analysis of cartesian fibrations and classification via an applications of the opposite involution.

In this section we outline the above construction. Unlike at other points in this text we are not especially concerned with (all of) the technical details, and seek only to provide a coherent narrative which explains clearly what's going on and how this stuff works.

We begin with a detour into $(\infty, 2)$-categories. We then construct the universal cocartesian fibration $\mathscr{Z}$ via a certain "category of objects", and explain how each fiber $\mathscr{Z}_\mathscr{E}$ over a given ∞-category $\mathscr{E} : * \to \mathscr{C}at_\infty$ reproduces $\mathscr{E}$ itself, up to equivalence. We define the space $\mathscr{T}(q)$ of classifying diagrams and recall the contractibility of this space from [3]. The section concludes with a description of the pushforward functors α_* appearing the transport F.

4.2. **Categories with objects and the universal cocartesian fibration.** From the $(\infty, 2)$-category Cat_∞ we can produce the simplicial of pointed ∞-categories

$$(\mathrm{Cat}_\infty)_* := (\mathrm{Cat}_\infty)_{\Delta^0/}.$$

By Proposition 3.22 the simplicial set $(\mathrm{Cat}_\infty)_*$ is an $(\infty, 2)$-category and the forgetful functor $(\mathrm{Cat}_\infty)_* \to \mathrm{Cat}_\infty$ is a interior fibration.

Definition 4.1. The ∞-category of ∞-categories with a distinguished object is the pith of the $(\infty, 2)$-category of pointed ∞-categories,

$$\mathscr{P}.\mathscr{C}at_\infty := \big((\mathrm{Cat}_\infty)_*\big)^{\mathrm{Pith}}.$$

Remark 4.2. The $\mathscr{P}$ suffix stands for "pointed", though we heed the warning from [3, 020W] and do not label this ∞-category as such.

Remark 4.3. There is a comparison functor $\mathscr{P}.\mathscr{C}at_\infty \to (\mathscr{C}at_\infty)_{*/}$ which is, apparently, bijective on objects. However this map is not bijective on 1-morphisms so that it is not an isomorphism [3, 020Z].

Via an application of Corollary 3.21 we see that the forgetful functor restricts to provide a pullback diagram

$$\begin{tikzcd} \mathscr{P}.\mathscr{C}at_\infty \arrow[r] \arrow[d] & (\mathrm{Cat}_\infty)_{*/} \arrow[d] \\ \mathscr{C}at_\infty \arrow[r] & \mathrm{Cat}_\infty\,. \end{tikzcd}$$

The forgetful functor $\mathscr{P}.\mathscr{C}at_\infty \to \mathscr{C}at_\infty$ is furthermore seen to be a cocartesian fibration in this case, via an application of Proposition 3.26. We record this result.

Proposition 4.4 ([3, 0213])**.** *The forgetful functor $\mathscr{P}.\mathscr{C}at_\infty \to \mathscr{C}at_\infty$ is a cocartesian fibration.*

We call the above forgetful functor the *universal cocartesian fibration*, for reasons which will be apparent shortly.

Definition 4.5. We let univ : $\mathscr{P}.\mathscr{C}at_\infty \to \mathscr{C}at_\infty$ denote the cocartesian fibration induced by the forgetful functor $(\mathrm{Cat}_\infty)_* \to \mathrm{Cat}_\infty$, as considered above.

4.3. **A remark on notation.** Our $(\infty,2)$-category Cat_∞ is the $(\infty,2)$-category denoted by a bold $\mathcal{QC}$ in [3, 020K]. Our $(\mathrm{Cat}_\infty)_{*/}$ is the $(\infty,2)$-category denoted by a bold $\mathcal{QC}_{\mathrm{Obj}}$ in [3, 0210]. The associated piths, which we've denoted $\mathscr{C}at_\infty$ and $\mathscr{P}.\mathscr{C}at_\infty$ respectively, are the non-bolded ∞-categories $\mathcal{QC}$ and $\mathcal{QC}_{\mathrm{Obj}}$ in [3].

4.4. **Fibers of the universal map $\mathscr{P}.\mathscr{C}at_\infty \to \mathscr{C}at_\infty$.** In considering the universal cocartesian fibration $\mathscr{P}.\mathscr{C}at_\infty \to \mathscr{C}at$, any point $e : \Delta^0 \to \mathscr{C}at$ corresponds to an ∞-category $\mathscr{E} = e(0)$ and we have the pullback

$$\mathscr{P}.\mathscr{C}at_\infty \times_{\mathscr{C}at_\infty} \{e\}$$

which is some other ∞-category. Now, objects in this fiber are simply maps of ∞-categories $* \to \mathscr{E}$, and hence are identified with objects in $\mathscr{E}$. Similarly, 1-simplices in the fiber are identified 2-simplices in the ∞-category of ∞-categories

$$\begin{tikzcd} & * \arrow[dl, "x"'] \arrow[dr, "y"] & \\ \mathscr{E} \arrow[rr, "="'] & & \mathscr{E}. \end{tikzcd}$$

These are, by definition, natural transformations $\alpha \in \mathrm{Hom}_{\mathrm{sSet}}(\Delta^1, \mathscr{E})$ with $\alpha|_0 = x$ and $\alpha|_1 = y$, i.e. 1-simplices $\alpha : x \to y$. So we observe an identification of 1-skeleta

$$\mathscr{E}[\leq 1] \ = \ \mathscr{P}.\mathscr{C}at_\infty \times_{\mathscr{C}at_\infty} \{e\}[\leq 1].$$

As an application of Theorem 3.27 and Corollary 3.28, we see that this direct identification of simplices in low-dimension expands to an equivalence of ∞-categories which calculates the fiber.

prop:univ_fibs **Proposition 4.6.** *For any ∞-category $\mathscr{E}$, which we can understand as a point $\mathscr{E} : * \to \mathscr{C}at_\infty$, we have a categorical pullback square*

$$\begin{tikzcd} \mathscr{E} \arrow[r] \arrow[d] & \mathscr{P}.\mathscr{C}at_\infty \arrow[d] \\ * \arrow[r, "\mathscr{E}"'] & \mathscr{C}at_\infty, \end{tikzcd}$$

and a corresponding equivalence between ∞-categories $\theta : \mathscr{E} \xrightarrow{\sim} \mathscr{P}.\mathscr{C}at_\infty \times_{\mathscr{C}at_\infty} \{\mathscr{E}\}$.

4.5. **Covariant transport: classifying cocartesian fibrations.**

Definition 4.7. Let $q : \mathscr{E} \to \mathscr{C}$ be a cocartesian fibration of ∞-categories. We say a functor $F : \mathscr{C} \to \mathscr{C}at_\infty$ classifies the cocartesian fibration q if the functors q and F fit into a categorical pullback diagram

$$\begin{tikzcd} \mathscr{E} \arrow[r, "\widetilde{F}"] \arrow[d, "q"'] & \mathscr{P}.\mathscr{C}at_\infty \arrow[d, "\mathrm{univ}"] \\ \mathscr{C} \arrow[r, "F"'] & \mathscr{C}at_\infty . \end{tikzcd} \tag{8}$$

eq:672

In this case we say the above diagram witnesses F as a (covariant) transport functor along q.

Remark 4.8. The term "classifies" is used with some frequency in the works [2, **?**]. However, [3] seems to prefer the term "transport representation" for a functor F as above. We will usually just refer to F as a, or the, transport functor for q.

We have an alternate characterization of transport functors via the fiber product $\mathscr{C} \times_{\mathscr{C}at_\infty} \mathscr{P}.\mathscr{C}at_\infty$.

Lemma 4.9. *Let $q : \mathscr{E} \to \mathscr{C}$ be a cocartesian fibration. A diagram (11) witnesses $F : \mathscr{C} \to \mathscr{C}at_\infty$ as transport along q if and only if the induced map to the fiber product*

$$\mathscr{E} \to \mathscr{C} \times_{\mathscr{C}at_\infty} \mathscr{P}.\mathscr{C}at_\infty$$

is an equivalence of cocartesian fibrations over $\mathscr{C}$.

Proof. The fact that the map to the fiber is an equivalence follows by Proposition I-5.23. It follows by Proposition **??** and the diagram

$$\begin{tikzcd} \mathscr{E} \arrow[rr, "\sim"] \arrow[dr, "q"'] & & \mathscr{C} \times_{\mathscr{C}at_\infty} \mathscr{P}.\mathscr{C}at_\infty \arrow[dl] \\ & \mathscr{C} & \end{tikzcd}$$

that the equivalence in question is an equivalence of cocartesian fibrations. $\square$

The fiber product considered above is often denoted

$$\int_{\mathscr{C}} F := \mathscr{C} \times_{\mathscr{C}at_\infty} \mathscr{P}.\mathscr{C}at_\infty,$$

so that any functors $F : \mathscr{C} \to \mathscr{C}at_\infty$ determines an associated cocartesian fibration $\int_{\mathscr{C}} F \to \mathscr{C}$. We find that F is a transport functor for $q : \mathscr{E} \to \mathscr{C}$ if there is an equivalence

$$\mathscr{E} \xrightarrow{\sim} \int_{\mathscr{C}} F \quad \text{in the overcategory } \mathrm{sSet}_{/\mathscr{C}} .$$

We also notes that transport functors are stable under restriction.

lem:687 **Lemma 4.10.** *Suppose we have a pullback square*

$$\begin{tikzcd} \mathscr{E}' \arrow[r, "g"] \arrow[d, "q'"'] & \mathscr{E} \arrow[d, "q"] \\ \mathscr{C}' \arrow[r, "f"'] & \mathscr{C} \end{tikzcd} \tag{9}$$ eq:689

in which q and q' are are cocartesian fibrations between ∞-categories, and consider a diagram of the form (11) which witnesses a functor $F : \mathscr{C} \to \mathscr{C}at_\infty$ as transport along q. Then for $\widetilde{F}' = \widetilde{F}g$ and $F' = Ff$, the diagram

$$\begin{tikzcd} \mathscr{E}' \arrow[r, "\widetilde{F}'"] \arrow[d, "q'"'] & \mathscr{P}.\mathscr{C}at_\infty \arrow[d] \\ \mathscr{C}' \arrow[r, "F'"'] & \mathscr{C}at_\infty. \end{tikzcd}$$

witnesses F' as a transport functor along q'.

Proof. Since q is a cocartesian fibration, and in particular an isofibration, the diagram (9) is a categorical pullback square Corollary I-5.22. The result now follows from the fact that categorical pullback squares are stable under composition [3, 033J]. ☐

Given a cocartesian fibration we can now consider the simplicial subset in the functor category

$$\mathrm{Fun}(\mathscr{C}, \mathscr{C}at_\infty) \times_{\mathrm{Fun}(\mathscr{E}, \mathscr{C}at_\infty)} \mathrm{Fun}(\mathscr{E}, \mathscr{P}.\mathscr{C}at_\infty) \tag{10}$$ eq:710

which consists of diagrams witnessing transport for a given cocartesian fibration $q : \mathscr{E} \to \mathscr{C}$.

Definition 4.11. For a given cocartesian fibration $q : \mathscr{E} \to \mathscr{C}$, we let $\mathscr{T}(q)$ denote the simplicial subset in the fiber product (10) whose simplices correspond to diagrams

$$\begin{tikzcd} \Delta^n \times \mathscr{E} \arrow[r, "\widetilde{F}"] \arrow[d, "\Delta^n \times q"'] & \mathscr{P}.\mathscr{C}at_\infty \arrow[d] \\ \Delta^n \times \mathscr{C} \arrow[r, "F"'] & \mathscr{C}at_\infty. \end{tikzcd}$$

which witness F as a covariant transport functor along $\Delta^n \times q$.

Stability of such diagrams under restriction (Lemma 4.10) assures us that $\mathscr{T}(q)$ is in fact a simplicial subset in the given fiber product.

thm:transport **Theorem 4.12** (Universality theorem [3, 02SC])**.** *For any cocartesian fibration $q : \mathscr{E} \to \mathscr{C}$, the terminal map $\mathscr{T}(q) \to *$ is a trivial Kan fibration.*

This result says that any cocartesian fibration q admits a covariant transport functor $F : \mathscr{C} \to \mathscr{C}at_\infty$, and that this functor is uniquely determined up to a contractible space of choices.

4.6. **Transport for cartesian fibrations.** Given any cartesian fibration $p : \mathscr{E} \to \mathscr{C}$, we have the associated cocartesian fibration $p^{\mathrm{op}} : \mathscr{E}^{\mathrm{op}} \to \mathscr{C}^{\mathrm{op}}$. So our analysis of classifying functors for cocartesian fibrations dualizes in the obvious ways to provide an analysis of classifying functors for cartesian fibrations.

Definition 4.13. Let $p : \mathscr{E} \to \mathscr{C}$ be a cartesian fibration of ∞-categories. We say a functor $F : \mathscr{C}^{\mathrm{op}} \to \mathscr{C}at_\infty$ classifies the cartesian fibration q if the functors p^{op} and F fit into a categorical pullback diagram

$$\begin{tikzcd} \mathscr{E}^{\mathrm{op}} \arrow[r, "\widetilde{F}"] \arrow[d, "q"'] & \mathscr{P}.\mathscr{C}at_\infty \arrow[d, "\mathrm{univ}"] \\ \mathscr{C}^{\mathrm{op}} \arrow[r, "F"'] & \mathscr{C}at_\infty. \end{tikzcd} \tag{11}$$

eq:672

In this case we say the above diagram witnesses F as a (contravariant) transport functor along p.

Gives a cartesian fibration $p : \mathscr{E} \to \mathscr{C}$, we define the space of transport functors with witness in the obvious way $\mathscr{T}(p) := \mathscr{T}(p^{\mathrm{op}})$. Theorem 4.12 implies contractibility of this space immediately.

thm:uniq_transp **Theorem 4.14** (Contravariant universality)**.** *For any cartesian fibration $p : \mathscr{E} \to \mathscr{C}$, the terminal map $\mathscr{T}(p) \to *$ from the space of transport functors along p is a trivial Kan fibration.*

Again this establishes both the existence and uniqueness of contravariant transport.

4.7. **Transport along edges.** Let $q : \mathscr{E} \to \Delta^1$ be a cocartesian fibration, and $f : \Delta^1 \to \mathscr{C}at_\infty$ be a transport functor for q. The functor f specifies a pair of ∞-categories and a functor $F' : \mathscr{E}'_0 \to \mathscr{E}'_1$. We then have partial diagrams

$$\begin{tikzcd} \mathscr{E}_i \arrow[dr] & & \mathscr{E}'_i \arrow[dl, "\theta"] \\ & \{i\} \times_{\Delta^1} \left(\int_{\Delta^1} f\right), & \end{tikzcd}$$

where as before $\int_{\Delta^1} f$ is the fiber product $\Delta^1 \times_{\mathscr{C}at_\infty} \mathscr{P}.\mathscr{C}at_\infty$, θ is the equivalence of Section **??**, and $\mathscr{E}_i \to \int_{\Delta^1} f$ is the equivalence witnessing f as a transport functor. We choose equivalences $\xi_i : \mathscr{E}_i \to \mathscr{E}'_i$ which complete a 2-simplex

$$\begin{tikzcd} \mathscr{E}_i \arrow[rr, "\xi_i"] \arrow[dr] & & \mathscr{E}'_i \arrow[dl, "\theta"] \\ & \{i\} \times_{\Delta^1} \left(\int_{\Delta^1} f\right), & \end{tikzcd}$$

in $\mathscr{C}at_\infty$. By pulling back along the ξ_i, the defining functor $F' : \mathscr{E}_0 \to \mathscr{E}_1$ is identified with a functor between the $\mathscr{E}_i$. The following determines this functor via implicit criterion.

prop:1_transport

Proposition 4.15. *Fix $q : \mathscr{E} \to \Delta^1$ and $f : \Delta^1 \to \mathscr{C}at_\infty$ as above. There is a functor $F : \mathscr{E}_1 \to \mathscr{E}_2$ and a natural transformation*

$$\widetilde{F} : \Delta^1 \times \mathscr{E}_0 \to \mathscr{E}$$

from the inclusion $\mathscr{E}_1 \to \mathscr{E}$ to the composite of F with the inclusion $\mathscr{E}_1 \to \mathscr{E}$ for which, at each object $s : * \to \mathscr{E}_0$, the corresponding map

$$\widetilde{F}_s : \Delta^1 = \Delta^1 \times \{s\} \to \mathscr{E}$$

is q-cocartesian. This functor is unique up to equivalence, and fits into a diagram

$$\begin{tikzcd} \mathscr{E}_0 \arrow[r, "F"] \arrow[d, "\xi_0"'] \arrow[dr] & \mathscr{E}_1 \arrow[d, "\xi_1"] \\ \mathscr{E}'_0 \arrow[r, "F'"'] & \mathscr{E}'_1, \end{tikzcd}$$

in the ∞-category $\mathscr{C}at_\infty$, where the ξ_i are as above.

Proof. Combine [3, 01VS] and [3, 027K]. □

Now consider the global situation, where we have some cocartesian fibration $q : \mathscr{E} \to \mathscr{C}$ and classifying functor $\mathscr{C} \to \mathscr{C}at_\infty$. At any morphism $\alpha : x \to y$ in $\mathscr{C}$, transport restricts to a functor $F' : \mathscr{E}'_x \to \mathscr{E}'_y$. According to Proposition 4.15, this functor is uniquely determined by a functorial choice of q-cocartesian maps between the fiber categories $\mathscr{E}_x$ and $\mathscr{E}_y$. There is, in particular, a unique functor $F : \mathscr{E}_x \to \mathscr{E}_y$ and a transformation which fits into a diagram

$$\begin{tikzcd} \Delta^1 \times \mathscr{E}_x \arrow[r] \arrow[d] & \mathscr{E} \arrow[d] \\ \Delta^1 \arrow[r, "\alpha"'] & \mathscr{C} \end{tikzcd}$$

and sends each edge $\Delta^1 \times \{s\}$ in $\Delta^1 \times \mathscr{E}_x$ to a q-cartesian morphism in $\mathscr{E}$ which lies over α. This functor fits into a diagram

$$\begin{tikzcd} \mathscr{E}_x \arrow[r, "F"] \arrow[d, "\sim"'] \arrow[dr] & \mathscr{E}_y \arrow[d, "\sim"] \\ \mathscr{E}'_x \arrow[r, "F'"'] & \mathscr{E}'_y. \end{tikzcd}$$

4.8. **Classification of left and right fibrations.** We have the simplicial subcategory $\underline{\mathrm{Kan}} \to \underline{\mathrm{Cat}}_\infty$ and subsequent simplicial subset $\mathscr{K}an \subseteq \mathrm{Cat}_\infty$. This simplicial subset is the full $(\infty, 2)$-subcategory whose objects are precisely those ∞-categories which are Kan complexes, and so the inclusion preserves thin 2-simplices. We now have the full $(\infty, 2)$-subcategory $\mathscr{K}an_{*/} \to (\mathrm{Cat}_\infty)_{*/}$ of pointed Kan complexes and the pullback diagram

$$\begin{tikzcd} \mathscr{K}an_{*/} \arrow[r] \arrow[d] & (\mathrm{Cat}_\infty)_{*/} \arrow[d] \\ \mathscr{K}an \arrow[r] & \mathrm{Cat}_\infty \end{tikzcd}$$

which restricts to a pullback diagram into the piths

$$\begin{tikzcd} \mathscr{K}an_{*/} \arrow[r] \arrow[d] & (\mathrm{Cat}_\infty)_{*/} \arrow[d] \\ \mathscr{K}an \arrow[r] & \mathrm{Cat}_\infty . \end{tikzcd}$$

We recall that the map $\mathscr{K}an_{*/} \to \mathscr{K}an$ is a left fibration, by Corollary I-**??**.

prop:kan_transp

Proposition 4.16. *A cocartesian fibration $q : \mathscr{E} \to \mathscr{C}$ is a left fibration if and only if the corresponding transport functor $F : \mathscr{C} \to \mathscr{C}at_\infty$ has image in $\mathscr{K}an$. Similarly, a cartesian fibration $p : \mathscr{E} \to \mathscr{C}$ is a right fibration if and only if the corresponding transport functor $G : \mathscr{C}^{\mathrm{op}} \to \mathscr{C}at_\infty$ has image in $\mathscr{K}an$.*

Proof. By Proposition **??**, a cocartesian (resp. cartesian) fibration $\mathscr{E} \to \mathscr{C}$ is a left (resp. right) fibration if and only if its fibers over object sin $\mathscr{C}$ are Kan complexes. So the result follows by the calculation of the fibers of the pullback fibration $\int_{\mathscr{C}} F \to \mathscr{C}$ provided in Proposition 4.6. □

This proposition tells us that any left fibration $q : \mathscr{E} \to \mathscr{C}$ fits into a categorical pullback square

$$\begin{tikzcd} \mathscr{E} \arrow[r] \arrow[d, "q"'] & \mathscr{K}an_{*/} \arrow[d] \\ \mathscr{C} \arrow[r, "F"'] & \mathscr{K}an \end{tikzcd}$$

for some functor F, and that F, considered as a functor into $\mathscr{C}at_\infty$, is the transport functor for q. In this way left fibrations are classified by maps into the ∞-category of Kan complexes. One obtains similar statements for right fibrations by applying the opposite functor.

5. Naturality of transport: straightening and unstraightening

In this section we explain how the assignment which sends a functor $F : \mathscr{C} \to \mathscr{C}at_\infty$ to the corresponding cocartesian fibration

$$\int_{\mathscr{C}} F = \mathscr{C} \times_{\mathscr{C}at_\infty} \mathscr{P}.\mathscr{C}at_\infty \to \mathscr{C}$$

extends to an equivalence of ∞-categories

$$\mathrm{Un} : \mathrm{Fun}(\mathscr{C}, \mathscr{C}at_\infty) \overset{\sim}{\to} \mathrm{Cocart}(\mathscr{C}),$$

where $\mathrm{Cocart}(\mathscr{C})$ is an ∞-category of cocartesian fibrations over $\mathscr{C}$. Similarly, we have have an equivalence for cartesian fibrations

$$\mathrm{Un} : \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathscr{C}at_\infty) \overset{\sim}{\to} \mathrm{Cart}(\mathscr{C}),$$

which one obtains simply by applying the opposite involution. These functors are referred to as the unstraightening functors [2].

Our investigation centers not the unstraightening functor per-se, but its discrete inverse, the aptly named straightening functor. In our discussion we generalize our base to a simplicial set S rather than an ∞-category.

5.1. **Marked simplicial sets.** A marked simplicial set is a pair (K, W) consisting of a simplicial set K and a choice of 1-simplices $W \subseteq K[1]$ which contains all degenerate 1-simplices. A map between marked simplicial sets $f : (K, W) \to (K', W')$ is a map of simplicial sets which sends W into W'. In this way we obtain the category sSet^+ of marked simplicial sets.

The forgetful functor $\mathrm{sSet}^+ \to \mathrm{sSet}$ has a right adjoint $-^{\#} : \mathrm{sSet} \to \mathrm{sSet}^+$, which sends a simplicial set K to the simplicial set K with all 1-simplices marked. The category sSet^+ also admits products, with $(K, W) \times (K', W') = (K \times K', W \times W')$, and the right adjoint $-^{\#}$ is seen to be symmetric monoidal. Via this monoidal functor we obtain an action of sSet on sSet^+.

As a notational point, from this point on we often omit markings from our notation, and simply say K is a marked simplicial set to indicate that K is a simplicial set equipped with some specified marking (K, W).

Lemma 5.1. *The* sSet*-module category* sSet^+ *admits inner-Homs* $\underline{\mathrm{Hom}}_{\mathrm{sSet}^+}(K, K')$*. The* n*-simplices in the underlying simplicial set maps*

$$\underline{\mathrm{Hom}}_{\mathrm{sSet}^+}(K, K')[n] := \mathrm{Hom}_{\mathrm{sSet}^+}((\Delta^n)^{\#} \times K, K').$$

Note that this simplicial set is a simplicial subset in the usual inner-Homs for simplicial sets.

lem:1074 **Lemma 5.2.** *Let* (K, W) *and* (K', W') *be marked simplicial sets and suppose that the marked vertices in* K' *are stable under compositions, i.e. that for any simplex*

$$s : \Delta^2 \to K'$$

in which edges $s|\Delta^{\{i,i+1\}}$ *are marked, the edge* $s|\Delta^{\{0,2\}}$ *is marked as well. A map of unmarked simplicial sets* $F : \Delta^n \times K \to K'$ *is an* n*-simplex in* $\underline{\mathrm{Hom}}_{\mathrm{sSet}^+}(K, K')$ *if and only if the following hold:*

(a) *The restrict to each vertex* $F|_{\Delta^{\{i\}}} : K \to K'$ *is a map of marked simplicial sets.*
(b) *At each vertex* $x : * \to K$*, and each* $0 \leq i < j \leq n$*, the edge* $F|\Delta^{\{i,j\}} \times \{x\} \to K$ *is marked in* K'*.*

Proof. The marked edges in $(\Delta^n)^{\#} \times K$ are all pairs (α_{ij}, w) where $\alpha_{ij} : [1] \to [n]$ is the unique increasing map with image $\{i, j\}$ and $w : x \to y$ is any marked edge in K. Since all degenerate 1-simplices are marked, it is clear that any marked map F must satisfy (a) and (b). So let us suppose now that F satisfies (a) and (b), and consider a marked edge (α_{ij}, w) with $i \leq j$. Such an edge appears as the $\{0, 3\}$ edge in a 2-simplex

$$t : \Delta^2 \to \Delta^n \times K \;\text{ with }\; t|\Delta^{\{0,1\}} = (\alpha_{i,j}, x) \text{ and } t|\Delta^{\{1,2\}} = (\alpha_{i,j}, w).$$

and $Ft : \Delta^2 \to K'$ is now a 2-simplex in K' with both edges $Ft|\Delta^{\{i,i+1\}}$ marked. (In the case $i = j$ the simplex $Ft|\Delta^{\{i,j\}}$ is marked simply because it is degenerate, otherwise this follows by (a).) It follows by stability under composition that $Ft|\Delta^{\{0,2\}} = F(\alpha_{ij}, w)$ is marked as well. So F preserves markings. □

The following is an alternate phrasing of this lemma

lem:1098 **Lemma 5.3.** *The mapping complex* $\underline{\mathrm{Hom}}_{\mathrm{sSet}^+}(K, K')$ *is a subcomplex in the complex of simplicial maps* $\mathrm{Fun}(K, K')$*. An* n*-simplex* $F : \Delta^n \to \mathrm{Fun}(K, K')$ *lies in*

$\underline{\mathrm{Hom}}_{\mathrm{sSet}^+}(K,K')$ if and only if each functor $F_i:\Delta^{\{i\}}\to \mathrm{Fun}(K,K')$ preserved markings and, at each $x:*\to K'$ and $i<j$, the image

$$\Delta^1\cong\Delta^{\{i,j\}}\overset{F_{ij}}{\to}\mathrm{Fun}(K,K')\overset{x^*}{\to}Fun(*,K')=K'$$

is a marked map in K'.

We are also interested in the nature of 2-simplices in $\underline{\mathrm{Hom}}_{\mathrm{sSet}^+}(K,K')$. The following is obtained from Lemma 5.3, essentially emediately.

cor:1108 **Corollary 5.4.** *Suppose that K and K' are marked simplicial sets, and that the markings in K' satisfy the 2-of-3 property. A 2-simplex $\Delta^2\to\mathrm{Fun}(K,K')$ lies in $\underline{\mathrm{Hom}}_{\mathrm{sSet}^+}(K,K')$ if and only if each constituent map $\Delta^{\{i\}}\to\mathrm{Fun}(K,K')$ is marked, and 2 of the 3 edges $\Delta^{\{i<j\}}\to\mathrm{Fun}(K,K')$ is marked.*

Given a simplicial set S we let $\underline{\mathrm{sSet}}^+_{/S}$ denote the simplicial category whose objects are morphisms of marked simplicial sets $p:K\to S^\#$, whose morphsism complexes $\underline{\mathrm{Hom}}_S(K,K')$ are the fiber products

$$\begin{array}{ccc}\underline{\mathrm{Hom}}_S(K,K') & \longrightarrow & \underline{\mathrm{Hom}}_{\mathrm{sSet}^+}(K,K')\\ \downarrow & & \downarrow{\scriptstyle (p')_*}\\ * & \xrightarrow[p]{} & \underline{\mathrm{Hom}}_{\mathrm{sSet}^+}(K,S^\#)=\mathrm{Fun}(K,S)\end{array}$$

prop:cocart_kan **Proposition 5.5.** *Let $q:X\to S$ be a cocartesian fibration, and consider the associated map of marked simplicial sets $X^q\to S^\#$ where X^q is X paired with the collection of all q-cocartesian maps. Then for any marked simplicial set K over $S^\#$, the mapping complex $\underline{\mathrm{Hom}}_S(K,X^q)$ is a Kan complex.*

Proof. By I4.8 the map $\mathrm{Fun}(K,X)\to\mathrm{Fun}(K,S)$ is an inner fibration, so that the fiber $\mathrm{Fun}_S(K,X)$ over the diagram p is an ∞-category. We have now that $\underline{\mathrm{Hom}}_S(K,X^q)$ is a subcomplex in the ∞-category $\mathrm{Fun}_S(K,X)$ whose n-simplices are precisely those n-simplices in $\mathrm{Fun}_S(K,X)$ which satisfy the conditions specified in Lemma 5.2 (a) and (b), since q-cocartesian maps in X are stable under composition. This stability under composition in X also implies that any completion $\Delta^2\times K\to X$ of an inner horn $\Lambda^2_1\times K\to X$ which lies in $\underline{\mathrm{Hom}}_S(K,X^q)$ also lies in $\underline{\mathrm{Hom}}_S(K,X^q)$. Since the subcomplex $\underline{\mathrm{Hom}}_S(K,X^q)$ in $\mathrm{Fun}(K,X)$ is characterized by a restriction on the 1-simplices, it follows that all higher dimensional horns in $\underline{\mathrm{Hom}}_S(K,X^q)$ complete to siplices in $\underline{\mathrm{Hom}}_S(K,X^q)$. So $\underline{\mathrm{Hom}}_S(K,X^q)$ is an ∞-subcategory in $\mathrm{Fun}_S(K,X)$.

Now, for any 1-simplex $\zeta:\Delta^1\to\underline{\mathrm{Hom}}_S(K,X^q)$ and vertex $x:*\to K$ with image s in S, the composite

$$\Delta^1\to\underline{\mathrm{Hom}}_S(K,X^q)\overset{x^*}{\to}X$$

has q-cocartesian image in the ∞-category X_s, and in particular is an isomorphism in X_s. This implies that ζ is an isomorphism in the ambient category $\mathrm{Fun}_S(K,X)$, by Proposition I-6.8. From the 2 of 3 property for q-cocartesian morphisms, and Corollary 5.4, it follows that any inverse $\zeta^{-1}:\Delta^1\to\mathrm{Fun}_S(K,X)$ to ζ is also in $\underline{\mathrm{Hom}}_S(K,X^q)$. So we see that every morphism in $\underline{\mathrm{Hom}}_S(K,X^q)$ is an isomorphism, and so this complex is a Kan complex. $\square$

5.2. **The ∞-category of cocartesian fibrations over a base.** We consider each cocartesian fibration $q : X \to S$ as a morphism in sSet^+ by applying the q-cocartesian marking X^q on X and the maximal marking $S^\#$ on S, and we have a the full simplicial subcategory $\underline{\mathrm{Cocart}}(S) \subseteq \underline{\mathrm{sSet}}^+_{/S}$ of cocartesian fibrations over S. By Proposition 5.5 this simpilicial subcategory is enriched in Kan complexes, so that the homotopy coherent nerve is an ∞-category.

Definition 5.6. Given a simplicial set S, the ∞-category of cocartesian fibrations over S is the homotopy coherent nerve

$$\mathrm{Cocart}(S) := \mathrm{N}^{\mathrm{hc}}(\underline{\mathrm{Cocart}}(S))$$

Example 5.7. When $S = *$ a cocartesian fibration over $*$ is an ∞-category $\mathscr{C}$. The associated cocartesian marking marks equivalences in $\mathscr{C}$. Since all functors between ∞-categories preserve equivalences we have

$$\underline{\mathrm{Hom}}_*(\mathscr{C},\mathscr{D}) = \mathrm{Fun}(\mathscr{C},\mathscr{D}) \quad \text{and} \quad \underline{\mathrm{Hom}}_{\mathrm{Cocart}(*)}(\mathscr{C},\mathscr{D}) = \mathrm{Fun}(\mathscr{C},\mathscr{D})^{\mathrm{Kan}}.$$

Hence $\mathrm{Cocart}(*) = \mathscr{C}at_\infty$.

Remark 5.8. Of course, we have an $(\infty, 2)$-category of cocartesian fibrations, which we obtain by applying the homotopy coherent nerve to the simplicial category $\underline{\mathrm{Cocart}}(S)'$. We have no intentions of using this $(\infty, 2)$-category in this work, and so disregard it.

To state things clearly, objects in $\mathrm{Cocart}(S)$ are cocartesian fibrations $X \to S$ and morphisms are functors $F : X \to Y$ which preserve cocartesian maps and fit into a diagram over the base

$$\begin{tikzcd} X \arrow[rr, "F"] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}.$$

A 2-simplex $\Delta^2 \to \mathrm{Cocart}(S)$ is a pair of partial diagrams of cocartesian fibrations

$$\begin{tikzcd} X_0 \arrow[r, "F_{01}"] \arrow[dr, "q_0"'] & X_1 \arrow[r, "F_{12}"] \arrow[d, "q_1"] & X_2 \arrow[dl, "q_2"] \\ & S & \end{tikzcd} \qquad \begin{tikzcd} X_0 \arrow[rr, "F_{02}"] \arrow[dr, "q_0"'] & & X_2 \arrow[dl, "q_2"] \\ & S & \end{tikzcd}$$

and a map

$$\begin{tikzcd} \Delta^1 \times X_0 \arrow[rr, "F"] \arrow[dr, "q_0 p_2"'] & & X_2 \arrow[dl, "q_2"] \\ & S & \end{tikzcd}$$

which satisfies $F|_0 = F_{12}F_{01}$ and $F|_1 = F_{02}$, and for which $F|_{\Delta_1 \times \{x\}}$ is an isomorphism in the ∞-category $X_2 \times_S \{s\}$.

5.3. **Simplicial enrichment for simplicial functors.** For any simplicial set K, and any simplicial functor $F : \underline{A} \to \underline{\mathrm{sSet}}^+$ we define $K \times F : \underline{A} \to \underline{\mathrm{sSet}}^+$ to be the composite

$$\underline{A} \overset{F}{\to} \underline{\mathrm{sSet}}^+ \overset{K\times-}{\to} \underline{\mathrm{sSet}}^+.$$

One sees that the action of K is in fact a simplicial functor via the symmetry $K \times \Delta^n \times - \cong \Delta^n \times K \times -$.

For two functors $F, F' : \underline{A} \to \underline{\mathrm{sSet}}^+$ we let $\mathrm{Nat}(F, F')$ denote the simplicial set with n-simplices

$$\mathrm{Nat}(F, F')[n] := \{\text{Natural transformations } \Delta^n \times F \to F'\}.$$

The composition operation

$$\mathrm{Nat}(F', F'') \times \mathrm{Nat}(F, F') \to \mathrm{Nat}(F, F'')$$

takes transformations $\Delta^n \times F' \to F''$ and $\Delta^n \times F \to F'$ to the transformation

$$\Delta^n \times F \overset{\delta\times 1}{\to} (\Delta \times \Delta) \times F = \Delta^n \times (\Delta^n \times F) \to \Delta^n \times F' \to F''.$$

Definition 5.9. For any simplicial category $\underline{A}$, we let $\mathrm{Fun}(\underline{A}, \underline{\mathrm{sSet}}^+)$ denote the simplicial category of simplicial functors, with morphism complexes $\mathrm{Nat}(F, F')$.

We have the evaluation functor

$$ev : \mathrm{Fun}(\underline{A}, \underline{\mathrm{sSet}}^+) \times \underline{A} \to \underline{\mathrm{sSet}}^+.$$

This functor sends a pair (F, a) of a functor and an object in $\underline{A}$ to $F(a)$, and on morphisms the map of simplicial sets

$$\mathrm{Nat}(F, F') \times \underline{\mathrm{Hom}}_{\underline{A}}(a, a') \to \underline{\mathrm{Hom}}^+_{\mathrm{sSet}}(Fa, F'a')$$

sends a pair

$$\left(\Delta^n \times F \to F', \sigma : \Delta^n \to \underline{\mathrm{Hom}}_{\underline{A}}(a, a')\right)$$

to the composite

$$\Delta^n \times Fa \overset{\delta\times 1}{\to} \Delta^n \times \Delta^n \times Fa \overset{1\times F\sigma}{\longrightarrow} \Delta^n \times Fa' \to F'a'.$$

Lemma 5.10. *There is a natural isomorphism*

$$\mathrm{Hom}_{\mathrm{sCat}}(\underline{A}', \mathrm{Fun}(\underline{A}, \underline{\mathrm{sSet}}^+)) \overset{\sim}{\to} \mathrm{Hom}_{\mathrm{sCat}}(\underline{A}' \times \underline{A}, \underline{\mathrm{sSet}}^+)$$

$$G \mapsto ev(G \times id_{\underline{A}}).$$

Proof. The inverse sends a functor $\Theta : A' \times A \to \underline{\mathrm{sSet}}^+$ to the functor

$$\theta : \underline{A}' \to \mathrm{Fun}(\underline{A}, \underline{\mathrm{sSet}}^+),\ \theta(a) = \Theta(a, -)$$

$$\theta_{ab} : \underline{\mathrm{Hom}}_{\underline{A}'}(a, b) \to \mathrm{Nat}(\Theta(a, -), \Theta(b, -)),$$

where θ_{ab} sends an n-simplex σ to the transformation which evaluates at each x in $\underline{A}$ as

$$\Theta(\sigma, x) : \Delta^n \times \Theta(a, x) \to \Theta(b, x).$$

$\square$

5.4. **Simplicial functors as functor categories.** As the category of simplicial categories is cocomplete [3, 00K3], one sees that the path category construction admits a unique extension from the class of simplices to the entire category of simplicial sets.

Lemma 5.11 ([3, 00L4])**.** *The association $\Delta^n \mapsto \mathrm{Path}\,\Delta^n$ extends to a functor* $\mathrm{Path} : \mathrm{sSet} \to \mathrm{sCat}$ *which provides a left adjoint to the homotopy coherent nerve,*

$$\mathrm{Hom}_{\mathrm{sSet}}(-, \mathrm{N}^{\mathrm{hc}}\, -) \cong \mathrm{Hom}_{\mathrm{sCat}}(\mathrm{Path}\, -, -).$$

The product of the unit map $S \to \mathrm{N}^{\mathrm{hc}}\,\mathrm{Path}\, S$, and commutativity of the homotopy coherent nerve with products, provide natural maps

$$\mathrm{Path}(S' \times S) \to \mathrm{Path}(S') \times \mathrm{Path}(S)$$

from which the path category functor becomes op-lax monoidal. Via this op-lax structure we obtain a map

$$\mathrm{Hom}_{\mathrm{sCat}}(\mathrm{Path}\,\Delta^n, \mathrm{Fun}(\mathrm{Path}\, S, \underline{\mathrm{sSet}}^+)) \overset{\sim}{\to} \mathrm{Hom}_{\mathrm{sCat}}(\mathrm{Path}\,\Delta^n \times \mathrm{Path}\, S, \underline{\mathrm{sSet}}^+)$$

$$\to \mathrm{Hom}_{\mathrm{sCat}}(\mathrm{Path}(\Delta^n \times S), \underline{\mathrm{sSet}}^+) \cong \mathrm{Hom}_{\mathrm{sSet}}(\Delta^n \times S, \mathrm{N}^{\mathrm{hc}}\, \underline{\mathrm{sSet}}^+).$$

Taking these maps collectively across various n provides a map of simplicial sets

$$\mathrm{N}^{\mathrm{hc}}\,\mathrm{Fun}(\mathrm{Path}\, S, \underline{\mathrm{sSet}}^+) \to \mathrm{Fun}(S, \mathrm{N}^{\mathrm{hc}}\, \underline{\mathrm{sSet}}^+).$$

We restrict to consider those functors which land in the non-full simplicial subcategory of ∞-categories, with Kanified morphism complexes, and consider the resultant map

$$\mathrm{comp} : \mathrm{N}^{\mathrm{hc}}\,\mathrm{Fun}(\mathrm{Path}\, S, \underline{\mathrm{Cat}}^+_\infty) \to \mathrm{Fun}(S, \mathscr{C}at_\infty). \tag{12}$$ eq:comp_NN

prop:nerv_to_nerv **Proposition 5.12.** *The simplicial category* $\mathrm{Fun}(\mathrm{Path}\, S, \underline{\mathrm{Cat}}^+_\infty)$ *is enriched in Kan complexes, and the comparison functor* (12) *is an equivalence of ∞-categories.*

We outline how this result occurs, according to the logic of [2]. So the proof is not so much a proof as an "authentication ticket" which the reader might verify for themselves.

Proof. The categories $\underline{\mathrm{sSet}}^+$ and $\mathrm{Fun}(\mathrm{Path}\, S, \underline{\mathrm{sSet}}^+)$ admit combinatorial simplicial model structures under which the subcategories of fibrant-cofibrant objects are precisely te subcategories

$$\underline{\mathrm{Cat}}^+_\infty \subseteq \underline{\mathrm{sSet}}^+ \quad \text{and} \quad \mathrm{Fun}(\mathrm{Path}\, S, \underline{\mathrm{Cat}}^+_\infty) \subseteq \mathrm{Fun}(\mathrm{Path}\, S, \underline{\mathrm{sSet}}^+)$$

[2, Proposition 3.1.3.7, Corollary 3.1.4.4, Proposition A.3.3.2, Remark A.3.3.4]. It follows that the simplicial category $\mathrm{Fun}(\mathrm{Path}\, S, \underline{\mathrm{Cat}}^+_\infty)$ is enriched in Kan complexes [2, Remark A.3.1.8], and also that $\underline{\mathrm{sSet}}^+$ provides a $\mathrm{Path}\, S$-chunk [2, Definition A.3.4.9] of itself [2, Example A.3.4.4]. We now see from [2, Proposition 4.2.4.4] that the comparison functor. □

Remark 5.13. The specific claim of Proposition 5.12 seems not to appear explicitly in [2], though it may be implicit. We found this particular claim in [**?**, Remark 3.7].

5.5. **Non-enriched Straightening and unstraightening.** Let $p : K \to S$ be a cocartesian fibration from a marked simplicial set (K, W), and consider the pushout S_p of the following diagram

$$\begin{tikzcd} K \arrow[r, "i"] \arrow[d, "p"'] & \{*\} \star K \arrow[d] \\ S \arrow[r, "i_p"'] & S_p. \end{tikzcd}$$

We now have the functor of plain categories

$$\mathrm{St}[0] : \underline{\mathrm{sSet}}^+_{/S}[0] \to \mathrm{Fun}(\mathrm{Path}\, S, \underline{\mathrm{sSet}}^+_\infty)[0]$$

which sends each object $q : K \to S$ to the representable functor $\underline{\mathrm{Hom}}_{\mathrm{Path}\, S_p}(*, -)$, where each value $\underline{\mathrm{Hom}}_{\mathrm{Path}\, S_p}(*, s)$ is equipped with a natural marking W_s. (Here we are writing s for $i_p(s)$ by an abuse of notation.) The marked edges are all those which appear as follows:

- Take a marked edge $\alpha : x' \to y'$ in K, and consider the extension to a 2-simplex
$$\mathrm{Ext}(\alpha) := \{*\} \star \alpha : \Delta^2 \to \{*\} \star K$$
which then has image $\mathrm{Ext}'(\alpha) : \Delta^2 \to S_p$. This 2-simplex appears as
$$\mathrm{Ext}'(\alpha) : \quad \begin{tikzcd} & px \arrow[dr, "p\alpha"] & \\ * \arrow[ur] \arrow[rr] & & py. \end{tikzcd}$$
which then determines a 1-simplex $\mathrm{E}(\alpha) : \Delta^1 \to \underline{\mathrm{Hom}}_{\mathrm{Path}\, S_p}(*, py)$.
- Take any 1-simplex $B : \Delta^1 \to \underline{\mathrm{Hom}}_{\mathrm{Path}\, S}(py, s)$.
- Consider the composite $B\,\mathrm{E}(\alpha) : \Delta^1 \to \underline{Hom}_{\mathrm{Path}\, S_p}(*, s)$.

The marked edges in W_s in $\underline{\mathrm{Hom}}_{\mathrm{Path}\, S_p}(*, s)$ are precisely those edges which appear as above.

Definition 5.14. The functor $\mathrm{St}[0] : \underline{\mathrm{sSet}}^+_{/S}[0] \to \mathrm{Fun}(\mathrm{Path}\, S, \underline{\mathrm{sSet}}^+_\infty)[0]$ constructed above is called the non-enriched straightening functor.

It can be shown that the straightening fnuctor is cocontinuous [2, Proposition 3.2.1.4] and hence admits a right adjoint.

Proposition 5.15 ([2, Corollary 3.2.1.5])**.** *The functor* $\mathrm{St}[0]$ *admits a right adjoint*

$$\mathrm{Un}[0] : \mathrm{Fun}(\mathrm{Path}\, S, \underline{\mathrm{sSet}}^+_\infty)[0] \to \underline{\mathrm{sSet}}^+_{/S}[0].$$

5.6. **Enriched unstraightening.** At each simplicial set L and map $p : K \to S$ from a marked simplicial set, we have the new map

$$Lp : L^\# \times K \to K \to S$$

and hence a new object in $\mathrm{sSet}^+_{/S}$. This product construction endows $\mathrm{sSet}^+_{/S}$ with a module category structure over $\mathrm{sSet}^+_{/S}$ whose inner-Homs recover the simpicial mapping complexes for the enhancement $\underline{\mathrm{sSet}}^+_{/S}$.

We have the natural map of simplicial sets

$$\mathrm{St}[0](L^\# \times K) \to L^\# \times \mathrm{St}[0](K)$$

[2, Corollary 3.2.1.15] which endows the straightening functor with a op-lax module category structure. The op-lax structure on St[0] endow the adjoint Un[0] with a lax module category structure, and this lax structure provides a canonical enrichment on the unstraightening functor to the simplicial setting. Specifically, we have the natural maps

$$\mathrm{Hom}_{\mathrm{sSet}}(L, \underline{\mathrm{Hom}}(F,G)) = \mathrm{Hom}_{\mathrm{Fun}}(L^{\#} \times F, G) \to \mathrm{Hom}_{\mathrm{sSet}^+_{/S}}(\mathrm{Un}[0](L^{\#} \times F)$$

$$\to \mathrm{Hom}_{\mathrm{sSet}^+_{/S}}(L^{\#} \times \mathrm{Un}[0]F, \mathrm{Un}[0]G) = \mathrm{Hom}_{\mathrm{sSet}}(L, \underline{\mathrm{Hom}}(\mathrm{Un}[0]F, \mathrm{Un}[0]G))$$

across all simplicial sets L. Via Yoneda this lifts the application of the unstraightening functor on morphism sets to the simplicial level,

$$\mathrm{Un} : \underline{\mathrm{Hom}}_{\mathrm{Fun}(\mathrm{Path}\, S, \underline{\mathrm{sSet}}^+)}(F,G) \to \underline{\mathrm{Hom}}_{\mathrm{sSet}^+_{/S}}(\mathrm{Un}[0]F, \mathrm{Un}[0]G),$$

and so enriches the non-simplicial unstraightening functor.

Definition 5.16. The unstraightening functor

$$\underline{\mathrm{Un}} : \mathrm{Fun}(\mathrm{Path}\, S, \underline{\mathrm{sSet}}^+) \to \underline{\mathrm{sSet}}^+_{/S}$$

is the simplicial enrichment of the non-enriched unstraightening functor, as constructed above.

5.7. **The straightening and unstraightening equivalences.**

thm:1382 **Theorem 5.17.** *The unstraightening functor restricts to an equivalence of simplicial categories*

$$\underline{\mathrm{Un}} : \mathrm{Fun}(\mathrm{Path}\, S, \underline{\mathrm{Cat}}^+_\infty) \to \underline{\mathrm{Cocart}}(S).$$

Proof. Follows by [2, Lemma 3.2.4.1] and [2, Theorem 3.2.0.1]. □

In the statement of Theorem 5.17 by an equivalence of simplicial categories we mean an equivalence specifically in the sense of [2, Definition A.3.2.1]. Equivalently, we are saying that the induced functor on homotopy categories is an equivalence and that the maps on morphism complexes are equivalences of Kan complexes. We refer to this latter property as fully faithfulness.

Lemma 5.18. *If $\Theta : \underline{A} \to \underline{B}$ is an equivalence of Kan-enriched simplicial categories, then the induced functor on homotopy coherent nerves*

$$\mathrm{N}^{\mathrm{hc}}\, \Theta : \mathrm{N}^{\mathrm{hc}}\, \underline{A} \to \mathrm{N}^{\mathrm{hc}}\, \underline{B}$$

is an equivalence of ∞-categories.

Proof. Since the induced functor on homotopy categories is an equivalence, $\mathrm{N}^{\mathrm{hc}}\,\Theta$ is essentially surjective. Fully faithfulness follows from fully faithfulness of Θ and the calculation of the (left pinched) mapping spaces in the homotopy coherent nerve via the mapping complexes in the original categories (see Theorem 3.27). □

We now conclude that unstraightening induces an equivalence on the associated ∞-categories

$$\mathrm{N}^{\mathrm{hc}}\, \underline{\mathrm{Un}} : \mathrm{Fun}(\mathrm{Path}\, S, \underline{\mathrm{Cat}}^+_\infty) \to \mathrm{N}^{\mathrm{hc}}\, \underline{\mathrm{Cocart}}(S) = \mathrm{Cocart}(S)$$

is an equivalence of ∞-categories. Finally, we compose with the inverse comparison equivalence to obtain an equivalence of ∞-categories

$$\mathrm{Fun}(S, \mathscr{C}at_\infty) \xrightarrow{\sim} \mathrm{Cocart}(S).$$

We record this finding.

thm:unstrt_equiv **Theorem 5.19.** *The functor*

$$\mathrm{Un} := \mathrm{comp}^{-1}\,\mathrm{N}^{\mathrm{hc}}\,\mathrm{Un} : \mathrm{Fun}(S, \mathscr{C}at_\infty) \xrightarrow{\sim} \mathrm{Cocart}(S).$$

is an equivalence of ∞-categories.

Definition 5.20. The unstraightening equivalence is the equivalence of Theorem 5.19. The straightening equivalence is the inverse functor

$$\mathrm{St} : \mathrm{Cocart}(S) \xrightarrow{\sim} \mathrm{Fun}(S, \mathscr{C}at_\infty).$$

5.8. **A remark on uniqueness.**

5.9. **Recovering transport via straightening.** For any map of simplicial sets $f : S \to S'$, we have the enriched pullback functor

$$f^* : \underline{\mathrm{Cocart}}(S') \to \underline{\mathrm{Cocart}}(S), \quad (K \to S') \mapsto (K \times_{S'} S \to S).$$

lem:1434 **Lemma 5.21** ([?, Observation 2.13])**.** *For any map of simplicial sets $f : S \to S'$, there is a commutative diagram at the level of homotopy categories*

$$\begin{tikzcd}
\mathrm{h}\,\mathrm{Cocart}(S) & \mathrm{h}\,\mathrm{Fun}(S, \mathscr{C}at_\infty) \arrow[l, "\mathrm{Un}"'] \\
\mathrm{h}\,\mathrm{Cocart}(S') \arrow[u, "f^*"] & \mathrm{h}\,\mathrm{Fun}(S', \mathscr{C}at_\infty) \arrow[l, "\mathrm{Un}"] \arrow[u, "f^*"']
\end{tikzcd}$$

Proof. The functor $f^* : \mathrm{Fun}(\mathrm{Path}\, S', \underline{\mathrm{sSet}}^+) \to \mathrm{Fun}(\mathrm{Path}\, S, \underline{\mathrm{sSet}}^+)$ has a left adjoint $f_!$ and we have a natural isomorphism $\mathrm{St}_f[0] \cong f_!\,\mathrm{St}[0]$ [2, Proposition 3.2.1.4], where the functor St_f is as in [2]. This implies and identification of non-enriched functors $\mathrm{Un}_f[0] \cong \mathrm{Un}[0]f^*$. But, by construction, $\mathrm{St}_f[0] = \mathrm{St}[0]f$, where

$$f : \underline{\mathrm{sSet}}^+_{/S} \to \underline{\mathrm{sSet}}^+_{/S'}$$

just composes maps over S with f. Now one can see directly that the pullback functor on marked simplicial sets over S' is right adjoint to this composition functor f, so that we have $\mathrm{Un}_f[0] = f^*\,\mathrm{Un}[0]$. So we have finally $f^*\,\mathrm{Un}[0] \cong \mathrm{Un}[0]f^*$, and this identification of non-enriched functors implies a corresponding identification of the ∞-functors at the level of homotopy categories. □

Remark 5.22. We expect that all of the identifications employed in the proof are compatible with (op)-lax module category structures, so that the identification $f^*\,\mathrm{Un}[0] \cong \mathrm{Un}[0]f^*$ enriches to an identification at the simplicial level, and hence at the ∞-level $f^*\,\mathrm{Un} \cong \mathrm{Un}\, f^*$. This homotopy-level identification suffices for our purposes however.

We can now consider the universal cocartesian fibration, i.e. the cocartesian fibration over $\mathscr{C}at_\infty$ which is associated to the identity functor

$$\mathscr{Z} := \mathrm{Un}(id_{\mathscr{C}at_\infty}).$$

By the materials of Section 4 we understand that there is an equivalence of fibrations

$$\begin{tikzcd}
\mathscr{Z} \arrow[rr, "\sim"] \arrow[dr] & & \mathscr{P}.\mathscr{C}at_\infty \arrow[dl] \\
& \mathscr{C}at_\infty &
\end{tikzcd}.$$

Now the diagram of Lemma 5.21 implies the existence of an equivalence of cocartesian fibrations

$$\mathrm{Un}(F) \cong S \times_{\mathscr{C}at_\infty} \mathscr{Z} =: \int_S F$$

at any functor F in $\mathrm{Fun}(S, \mathscr{C}at_\infty)$.

prop:strt_transport

Proposition 5.23 ([2, §3.3.2])**.** *For any cocartesian fibration $q : X \to S$ there is an equivalence of cocartesian fibrations*

$$\begin{tikzcd} X \arrow[rr, "\sim"] \arrow[dr, "q"'] & & \int_S \mathrm{St}(q) \arrow[dl] \\ & S & \end{tikzcd}.$$

In particular, when the base S is an ∞-category the straightening functor $\mathrm{St}(q) : S \to \mathscr{C}_\infty$ is a transport functor for $q : X \to S$.

Proof. We have $q \cong \mathrm{Un}\,\mathrm{St}(q) \cong \int_S \mathrm{St}(q)$. □

We can consider now the ∞-category of left fibrations.

Definition 5.24. Let $\mathrm{LFib}(S)$ denote the full ∞-subcategory in $\mathrm{Cocart}(S)$ whose objects are precisely the left fibrations $q : X \to S$.

5.10. **Straightening and unstraightening for left fibrations.** Since the straightening functor $\mathrm{St}(q)$ produces a transport functor for any cocartesian fibration $q : X \to S$, and the unstraightening functor pulls back along the universal fibrations, Proposition **??** and the fibrer calculation of Proposition 4.6 now imply that unstraightening and straightening restrict to equivalences between left fibrations over S and functors into the ∞-subcategory of spaces $\mathscr{K}an \subseteq \mathscr{C}at_\infty$.

cor:strt_kan

Corollary 5.25. *The straightening and unstraightening equivalences restrict to inverse equivalences*

$$\mathrm{St} : \mathrm{LFib}(S) \xrightarrow{\sim} \mathrm{Fun}(S, \mathscr{K}an) \quad \textit{and} \quad \mathrm{Un} : \mathrm{Fun}(S, \mathscr{K}an) \xrightarrow{\sim} \mathrm{LFib}(S).$$

5.11. **Some naturality over the base.**

prop:small_nat

Proposition 5.26. *Any diagram of cocartesian fibrations*

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "p"'] & X \arrow[d, "q"] \\ T \arrow[r, "\xi"'] & S \end{tikzcd}$$

determines a uniquely associated transformation $\mathrm{St}(p) \to \mathrm{St}(q)\xi$ in the ∞-category $\mathrm{Fun}(T, \mathscr{C}at_\infty)$.

In the case where the bases are ∞-categories, this transformation is visualized as a diagram

$$\begin{tikzcd} T \arrow[rr] \arrow[dr, "\mathrm{St}(p)"'] & & S \arrow[dl, "\mathrm{St}(q)"] \\ & \mathscr{C}at_\infty & \end{tikzcd}$$

in the $(\infty, 2)$-category of (big) ∞-categories.

Construction. Take $F = \mathrm{St}(q)$, $G = \mathrm{St}(p)$. The composite $F\xi$ is a transport functor for the fibration $q' : X \times_S T \to T$, which then determines an isomorphism $F\xi \cong \mathrm{St}(q')$ which is unique up to a contractible space of choices, by Theorem 4.14 (or rather its generalized form [3, 02SC]). Additionally the given diagram specifies, and is specified by, a morphism

$$\begin{tikzcd} Y \arrow[rr, "l"] \arrow[dr, "p"'] & & X \times_S T \arrow[dl, "q'"] \\ & T & \end{tikzcd}$$

in $\mathrm{Cocart}(T)$. Via straightening this morphism determines a map

$$\mathrm{St}(l) : G \to \mathrm{St}(q') \cong F\xi.$$

□

5.12. **Straightening for cartesian fibrations.** As with the cocartesian case, one can show that the full simplicial subcategory $\underline{\mathrm{Cart}}(S) \subseteq \underline{\mathrm{sSet}}_{/S}$ consisting of cartesian fibrations over S is enriched in Kan complexes. We can therefore consider the ∞-category

$$\mathrm{Cart}(S) = \mathrm{N}^{\mathrm{hc}}\,\underline{\mathrm{Cart}}(S).$$

An application of the opposite functor provides an isomorphism of ∞-categories $\mathrm{Cart}(S) \cong \mathrm{Cocart}(S^{\mathrm{op}})$. We therefore obtain the following results for cartesian fibrations:

There are mutually inverse equivalences $\mathrm{St} : \mathrm{Cart}(S) \xrightarrow{\sim} \mathrm{Fun}(S^{\mathrm{op}}, \mathscr{C}at_\infty)$ and $\mathrm{Un} : \mathrm{Fun}(S, \mathscr{C}at_\infty) \xrightarrow{\sim} \mathrm{Cart}(S)$ (Theorem 5.19). These equivalences restrict to equivalences

$$\mathrm{St} : \mathrm{RFib}(S) \xrightarrow{\sim} \mathrm{Fun}(S^{\mathrm{op}}, \mathscr{K}an) \quad \text{and} \quad \mathrm{Un} : \mathrm{Fun}(S^{\mathrm{op}}, \mathscr{K}an) \xrightarrow{\sim} \mathrm{RFib}(S)$$

(Corollary 5.25). The value $\mathrm{St}(q) : S^{\mathrm{op}} \to \mathscr{C}at_\infty$ at any cartesian fibration $q : X \to S$ is a transport functor for q (Proposition 5.23). Finally, any diagram of cocartesian fibrations

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "p"'] & X \arrow[d, "q"] \\ T \arrow[r, "\xi"'] & S \end{tikzcd}$$

determines a uniquely associated transformation between the transport functors $\mathrm{St}(p) \to \mathrm{St}(q)\xi$ (Proposition 5.26).

6. Initial and terminal objects

Before beginning with our study in earnest, with the introduction of Hom functors and the Yoneda embedding for ∞-categories, we discuss the notions of initial and terminal objects in an ∞-category.

6.1. **Initial and terminal basics.**

Definition 6.1. Let $\mathscr{C}$ be an ∞-category. An object x in $\mathscr{C}$ is called initial if, for each object z in $\mathscr{C}$, the mapping space $\mathrm{Hom}_{\mathscr{C}}(x,z)$ is contractible. An object z in $\mathscr{C}$ is called terminal if, for each object z in $\mathscr{C}$, the space $\mathrm{Hom}_{\mathscr{C}}(z,y)$ is is contractible.

One sees that an object x is initial (resp. terminal) in $\mathscr{C}$ if and only if x is terminal (resp. initial) in the opposite category $\mathscr{C}^{\mathrm{op}}$. So we can freely translate between results for initial versus terminal objects. Note also that we can replace the mapping space $\mathrm{Hom}_{\mathscr{C}}(x,y)$ with either the left or right pinched spaces when evaluating initial-ness or terminal-ness of objects.

lem:init_unique

Lemma 6.2. *Let $\mathscr{C}$ be an ∞-category, and let $\mathscr{C}_{\mathrm{Init}}$ and $\mathscr{C}_{\mathrm{Term}}$ denote the full ∞-subcategories whose objects are the initial and terminal objects in $\mathscr{C}$, respectively. Then each of the categories $\mathscr{C}_{\mathrm{Init}}$ and $\mathscr{C}_{\mathrm{Term}}$ is either empty or a contractible Kan complex.*

This is to say, the initial (or terminal) object in an ∞-category $\mathscr{C}$ is unique, provided any such object exists.

Proof. We only consider the case of $\mathscr{C}_{\mathrm{Init}}$. Let us suppose that this subcategory is nonempty. Via contractibility of the mapping spaces we conclude that the functor $\mathscr{C}_{\mathrm{Init}} \to *$ is fully faithful and essentially surjective, and hence an equivalence of ∞-categories. So $\mathscr{C}_{\mathrm{Init}}$ is a contractible Kan complex. $\square$

Lemma 6.3. *If x is initial (resp. terminal) in $\mathscr{C}$, then another object x' is initial (resp. terminal) in $\mathscr{C}$ if and only if x' is isomorphic to x.*

Proof. For any isomorphism $\alpha : x \to x'$ the induced maps

$$\alpha^* : \mathrm{Hom}_{\mathscr{C}}(x,y) \to \mathrm{Hom}_{\mathscr{C}}(x',y) \quad \text{and} \quad \alpha_* : \mathrm{Hom}_{\mathscr{C}}(y,x) \to \mathrm{Hom}_{\mathscr{C}}(y,x')$$

are isomorphisms in $\mathrm{h}\,\mathscr{K}an$, at all y in $\mathscr{C}$. So contractibility of the left-hand spaces implies contratibility of the right-hand spaces. $\square$

One also sees that equivalences of ∞-categories preserve initial and terminal objects.

lem:equiv_initial

Lemma 6.4. *If $F : \mathscr{C} \to \mathscr{D}$ is an equivalence between ∞-categories, and x is initial (resp. terminal) in $\mathscr{C}$, then $F(x)$ is initial (resp. terminal) in $\mathscr{D}$*

Proof. Suppose that x is initial in $\mathscr{C}$. First note that any isomorphism $\beta : y \to y'$ in $\mathscr{D}$ induces isomorphisms

$$\beta_* : \mathrm{Hom}_{\mathscr{D}}(z,y) \overset{\sim}{\to} \mathrm{Hom}_{\mathscr{D}}(z,y')$$

in the homotopy category of Kan complexes. So an object z in $\mathscr{D}$ is initial if and only if the relevant mapping spaces are contractible at a dense collection of objects in $\mathscr{D}$. (By a dense collection we mean a collection which contains a representative for every isoclass in $\mathscr{D}$.) Since any equivalence is both fully faithful and essentially surjective, we have that the mapping spaces $\mathrm{Hom}_{\mathscr{D}}(Fx,y)$ are contractible at all y in the image of $\mathscr{C}$, and hence at all y in $\mathscr{D}$. So $F(x)$ is initial in $\mathscr{D}$. The case where x is terminal is proved similarly. $\square$

Warning 6.5. Initial and terminal objects are not well-behaved under fibering. Consider for example the cone $C = \{x^2 + y^2 = z : x, y, z \in \mathbb{R}\}$ and its projection onto the z-axis line $R_z \cong \mathbb{R}$. The projection $\mathrm{Sing}(C) \to \mathrm{Sing}(R_z)$ is a Kan fibration and the objects $\vec{1} = (1,1,1)$ and 1 are both initial and terminal in $\mathrm{Sing}(C)$ and $\mathrm{Sing}(R_z)$ respectively, since these spaces are contractible. However, $\vec{1}$ is not initial or terminal in the fiber $\mathrm{Sing}(C)_1 = \mathrm{Sing}(S^1)$. In fact, this fiber admits no such objects.

6.2. **Aside: trivial fibrations via the fibers.** For the analysis that follows, it is convenient to have a characterization of trivial Kan fibrations which can be checked on the fibers.

prop:triv_fibs **Proposition 6.6.** *A map of simplicial sets $f : \mathscr{C} \to S$ is a trivial Kan fibration if and only if f is a left (or right) fibration and, at each point $s : * \to S$, the fiber $\mathscr{C}_s$ is a contractible Kan complex.*

Sketch proof. If f is a trivial Kan fibration then it is both a left and right fibration, and all of its fibers are contractible. As for the other direction, assume now that f is a left fibration and that all of its fibers are contractible. (The case of a right fibrations is then obtained by taking opposites.)

We must show that each lifting problem of the form

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & \mathscr{C} \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & S \end{tikzcd}$$

admits a solution. In the case that $n = 0$, such a solution exists since the fibers $\mathscr{C}_s$ are all non-empty. So we assume $n > 0$. By replacing S with Δ^n, and $\mathscr{C}$ with $\Delta^n \times_S \mathscr{C}$, we may assume also that both S and $\mathscr{C}$ are ∞-categories. For fun we can finally replace this lifting problem with the related lifting problem

$$\begin{tikzcd} \{0\} \times \partial\Delta^n \arrow[r, "\bar{\sigma}"] \arrow[d] & \mathscr{C} \arrow[d] \\ \Delta^1 \times \Delta^n \arrow[r, "\sigma"'] \arrow[ur, dotted] & S, \end{tikzcd} \tag{13}$$ eq:1722

which is obtained by restricting along the projections

$$p : \Delta^1 \times \Delta^n \to \Delta^n, \quad p(0, i) = i, \; p(1, i) = n,$$

and which recovers our original problem after restricting to $\{0\} \times \Delta^n$. It suffices to solve this second problem.

By [3, 0153] the class of left anodyne maps is stable under the cartesian action of sSet on itself, so that the inclusion $\{0\} \times \partial\Delta^n \to \Delta^1 \times \partial\Delta^n$ is left anodyne. Since the map f is left anodyne, it follows that the lifting problem (13) extends to a problem

$$\begin{tikzcd} \Delta^1 \times \partial\Delta^n \arrow[r, "\bar{\sigma}"] \arrow[d] & \mathscr{C} \arrow[d] \\ \Delta^1 \times \Delta^n \arrow[r, "\sigma"'] \arrow[ur, dotted] & S, \end{tikzcd}$$

Since the fibers of f are trivial Kan fibrations, the above problem extends further to a problem of the form

$$\begin{tikzcd} Y(0) \arrow[r, "\bar{\sigma}"] \arrow[d] & \mathscr{C} \arrow[d] \\ \Delta^1 \times \Delta^n \arrow[r, "\sigma"'] \arrow[ur, dotted] & S, \end{tikzcd}$$

where $Y(0)$ is the pushout

$$Y(0) = (\Delta^1 \times \partial\Delta^n) \coprod_{\{1\}\times\partial\Delta^n} (\{1\} \times \Delta^n).$$

By [3, Proof of 00TH] the inclusion $Y(0) \to \Delta^1 \times \Delta^n$ can be factored into a sequence $Y(0) \to Y(1) \to \cdots \to Y(n+1) = \Delta^1 \times \Delta^n$ with each $Y(i+1)$ obtained from $Y(i)$ as a pushout for a partial diagram

$$\begin{tikzcd} \Lambda^{n+1}_{i+1} \arrow[r] \arrow[d] & Y(i) \\ \Delta^{n+1} & . \end{tikzcd}$$

Furthermore, this sequence can be constructed so that at $Y(n) = \Delta^1 \times \Delta^n$ the sequence

$$\Delta^1 \cong \Delta^{\{n,n+1\}} \to \Delta^{n+1} \to \Delta^1 \times \Delta^n$$

recovers the edge $\Delta^1 \times \{n\}$.

Now, since f is inner anodyne we can solve, in order, the lifting problems

$$\begin{tikzcd} Y(i-1) \arrow[r] \arrow[d] & \mathscr{C} \arrow[d] \\ Y(i) \arrow[r] \arrow[ur, dotted] & S \end{tikzcd}$$

at all $0 < i \leq n$. For the final lifting problem, along the inclusion $Y(n) \to Y(n+1)$, we need to solve a lifting problem of the form

$$\begin{tikzcd} \Lambda^{n+1}_{n+1} \arrow[r] \arrow[d] & \mathscr{C} \arrow[d] \\ \Delta^{n+1} \arrow[r] \arrow[ur, dotted] & S \end{tikzcd}$$

in which $\sigma|\Delta^{\{n,n+1\}}$ is of a constant value s in S. Since the fiber $\mathscr{C}_s$ is a Kan complex, the morphism $\Delta^{\{n,n+1\}} \to \Lambda^{n+1}_{n+1} \to \mathscr{C}$ is an isomorphism in $\mathscr{C}$. We can therefore solve this final lifting problem, by Proposition I-4.33, and hence obtain the desired solution to the problem (13). $\square$

When applied to the case of a Kan fibration we have the following, which can also be deduced from Propositions I-3.30 and I-3.42.

Corollary 6.7. *A map between Kan complexes $f : \mathscr{X} \to \mathscr{Y}$ is a trivial Kan fibration if and only if it is a Kan fibration and, at each point $y : * \to \mathscr{Y}$, the fiber $\mathscr{X}_y$ is contractible.*

6.3. **Initial objects and undercategories.**

prop:terminal_over **Proposition 6.8.** *An object in an ∞-category $x : * \to \mathscr{C}$ is initial if and only if the forgetful functor $\mathscr{C}_{x/} \to \mathscr{C}$ is a trivial Kan fibration. Dually, an object $y : * \to \mathscr{C}$ is terminal if and only if the functor $\mathscr{C}_{/y} \to \mathscr{C}$ is a trivial Kan fibration.*

Proof. If x is initial then all of the left pinched mapping spaces are contractible, so that all of the fibers of the left fibration $\mathscr{C}_{x/} \to \mathscr{C}$ are contractible. It follows that this map is a trivial Kan fibration. For the converse, we simply note that trivial Kan fibrations are stable under pullback. The arguments in the terminal case are similar. $\square$

Let us now give a technical lemma.

lem:1699 **Lemma 6.9.** *For each positive integer n, the map*

$$(\Delta^1 \star \partial\Delta^n) \coprod_{(\{0\}\star\partial\Delta^n)} \{0\} \star \Delta^n \to \Delta^1 \star \Delta^n \cong \Delta^{n+2}$$

induced by the respective inclusions is an isomorphism onto the horn Λ_0^{n+2}.

See [3] for the proof. We have the following characterization of isomorphisms via initial and terminal objects.

prop:isom_initial **Proposition 6.10.** *For a map $\alpha : x \to y$ in an ∞-category $\mathscr{C}$, the following are equivalent:*

(a) *α is an isomorphism.*
(b) *α is initial when considered as an object in the undercategory $\mathscr{C}_{x/}$.*
(c) *α is terminal when considered as an object in the overcategory.*

Proof. We prove the equivalence between (a) and (b). The equivalence between (a) and (c) is obtained by taking opposites. The implication (b) $\Rightarrow$ (a) just follows by considering maps between α and id_x in the undercategory. So suppose that α is an isomorphism. By Proposition 6.8, α is initial in $\mathscr{C}_{x/}$ if and only if the forgetful functor

$$\mathscr{C}_{\alpha/} \cong (\mathscr{C}_{x/})_{\alpha/} \to \mathscr{C}_{x/}$$

is a trivial Kan fibration. Now, via a consideration of the identification from Lemma 6.9, solving a lifting problem of the form

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & \mathscr{C}_{\alpha/} \arrow[d] \\ \Delta^n \arrow[r] & \mathscr{C}_{/x} \end{tikzcd}$$

is equivalent to solving the corresponding lifting problem

$$\begin{tikzcd} \Lambda_0^{n+2} \arrow[r] \arrow[d] & \mathscr{C} \arrow[d] \\ \Delta^{n+2} \arrow[r] & * \end{tikzcd}$$

in which the initial edge $\Delta^{\{0,1\}} \to \mathscr{C}$ is α. Such a problem admits a solution by Proposition I-4.33, so that the forgetful functor is seen to be a trivial Kan fibration. $\square$

We take a moment to discuss some examples before returning to the theoretical foundations of this topic.

6.4. **Initial and terminal objects in simplicial nerves.**

Definition 6.11. An object x in a simplicial category $\underline{A}$ is called initial (resp. terminal) if, for each y in $\underline{A}$, the mapping complex $\underline{\mathrm{Hom}}_{\underline{A}}(x,y)$ (resp. $\underline{\mathrm{Hom}}_{\underline{A}}(y,x)$) is a contractible Kan complexes.

The easiest way for this to occur is if the relevant mapping complexes are just points. For example, one sees immediately that $\emptyset$ and $*$ are initial and terminal in $\underline{\mathrm{Kan}}$, respectively.

For $\underline{A}$ enriched in Kan complexes, and $\mathscr{A} = \mathrm{N}^{\mathrm{hc}}(\underline{A})$, the equivalence

$$\underline{\mathrm{Hom}}_{\underline{A}}(x,y) \overset{\sim}{\to} \mathrm{Hom}^{\mathrm{L}}_{\mathscr{A}}(x,y)$$

of Theorem 3.27 tells us that an object x is initial (resp. terminal) in $\underline{A}$ if and only if x is initial (resp. terminal) when considered as an object in the ∞-category $\mathscr{A}$. The analogous claim is seen to hold for terminal objects via a consideration of the opposite categories.

Lemma 6.12. *Let $\underline{A}$ be a simplicial category which is enriched in Kan complexes. Then an object x is initial (resp. terminal) in $\underline{A}$ if and only if the corresponding object x is initial (resp. terminal) in $\mathrm{N}^{\mathrm{hc}}(\underline{A})$.*

The following corollary is not an immediate consequence of triviality of the mapping categories $\mathrm{Fun}(\emptyset, \mathscr{C})$ and $\mathrm{Fun}(\mathscr{C}, *)$, when $\mathscr{C}$ is an ∞-category.

cor:1761 **Corollary 6.13.** *The empty set $\emptyset$ is initial in both $\mathscr{K}an$ and $\mathscr{C}at_\infty$. The point $*$ is terminal in both $\mathscr{K}an$ and $\mathscr{C}at_\infty$.*

6.5. **Zero objects in pointed spaces.** Though we will not use the term explicitly, a zero object in an ∞-category is an object which is simultaneously initial and terminal. Such objects are familiar from our studies of abelian categories. In the ∞-setting, the theory of abelian categories is, to some extent and in an indirect manner, reflected in the theory of stable categories. In the stable setting one again demands the existence of a zero object.

Proposition 6.14. *If x is terminal in an ∞-category $\mathscr{C}$, then x is both initial and terminal in the category $\mathscr{C}_{x/}$.*

By x in $\mathscr{C}_{x/}$ we mean any morphism $x \to x$. Since x is terminal, this lift of x to an object in $\mathscr{C}_{x/}$ is uniquely determined up to a contractible space. Practically speaking, we can just take this lift to be $id_x : x \to x$.

Proof. The fact that x is initial in $\mathscr{C}_{x/}$ follows by Proposition 6.10. For terminality, we consider the forgetful functor

$$\mathscr{C}_{x//x} \to \mathscr{C},$$

where $\mathscr{C}_{x//x} = (\mathscr{C}_{x/})_{/x} = (\mathscr{C}_{/x})_{x/}$. For any inclusion of simplicial sets $A \to B$, the existence of a solution to a lifting problem

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \mathscr{C}_{x//x} \arrow[d] \\ B \arrow[r] & \mathscr{C}_{x/} \end{tikzcd}$$

is equivalent to the existence of a solution to the corresponding lifting problem

$$\begin{tikzcd} \{x\} \star A \arrow[r] \arrow[d] & \mathscr{C}_{/x} \arrow[d] \\ \{x\} \star B \arrow[r] & \mathscr{C}. \end{tikzcd}$$

By Proposition 6.8 a solution to the latter problem exists, since x is terminal in $\mathscr{C}$. It follows that the map $\mathscr{C}_{x//x} \to \mathscr{C}_{x/}$ is a trivial Kan fibration, and hence that x is terminal in $\mathscr{C}_{x/}$, by Proposition 6.8. □

Recall form Corollary 6.13 that the 1-point space $*$ is terminal in $\mathscr{K}an$.

Corollary 6.15. *The* 1*-point space* $*$ *is both initial and terminal in the* ∞*-category* $\mathscr{K}an_{*/}$ *of pointed Kan complexes.*

6.6. **Zero objects in derived categories.**

Definition 6.16. An object x in a dg category $\mathbf{A}$ is said to be initial (resp. terminal) if, at each y in $\mathbf{A}$, the Hom complex $\mathrm{Hom}^*_{\mathbf{A}}(x,y)$ (resp. $\mathrm{Hom}^*_{\mathbf{A}}(y,x)$) is acyclic.

Recall our calculation of the mapping spaces in the dg nerve $\mathscr{A} = \mathrm{N}^{\mathrm{dg}}(\mathbf{A})$ via the Hom complexes in $\mathbf{A}$,

$$\mathrm{Hom}^{\mathrm{L}}_{\mathscr{A}}(x,y) \overset{\sim}{\to} K(\mathrm{Hom}^*_{\mathbf{A}}(x,y))$$

(Proposition I-11.7). By Theorem I-10.13, the above calculation tells us that the mapping $\mathrm{Hom}^{\mathrm{L}}_{\mathscr{A}}(x,y)$ are contractible whenever the complex $\mathrm{Hom}^*_{\mathbf{A}}(x,y)$ is acyclic. So we observe the following.

lem:init_dg **Lemma 6.17.** *Let* $\mathbf{A}$ *be a dg category and take* $\mathscr{A} = \mathrm{N}^{\mathrm{dg}}(\mathbf{A})$*. If an object* x *is initial (resp. terminal) in* $\mathbf{A}$*, then the corresponding object* x *is initial (resp. terminal) in* $\mathscr{A}$*.*

Remark 6.18. The converse to Lemma 6.17 holds if we assume that our dg category $\mathbf{A}$ has a good shift functor (see Section 11.1).

For any abelian category $\mathbb{A}$, the object 0 is both initial and terminal in the dg category $\mathrm{Ch}(\mathbb{A})$ of cochains over $\mathbb{A}$, and hence also in the subcategories of K-projective and K-injective complexes. We recall that the derived ∞-category $\mathscr{D}(\mathbb{A})$ is defined by taking the dg nerve of the dg category of K-injective objects in $\mathrm{Ch}(\mathbb{A})$ when we have enough such objects, or K-projectives when we have enough such objects (see Section I-12).

Corollary 6.19. *For any Grothendieck abelian category* $\mathbb{A}$*, the zero complex* 0 *is both initial and terminal in the derived* ∞*-category* $\mathscr{D}(\mathbb{A})$*.*

6.7. **Initial objects and weak contractibility.** We phrase all results below in terms of initial objects. The corresponding results hold for terminal objects via duality.

Lemma 6.20. *A Kan complex* $\mathscr{X}$ *admits an initial object if and only if* $\mathscr{X}$ *is contractible.*

Proof. If x is initial in $\mathscr{X}$, then every object in $\mathscr{X}$ admits a morphism from $\mathscr{X}$, and hence is isomorphic to x (since $\mathscr{X}$ is a Kan complex). Since any object which is isomorphic to an initial object is initial, we conclude that $\mathscr{X}$ consists entirely of initial objects. We conclude that $\mathscr{X}$ is contractible by Lemma 6.2. □

In the case of an ∞-category $\mathscr{C}$ we do not gain such a precise understanding of $\mathscr{C}$ via the existence of an initial object. This is clear from the examples discussed above. We can, however, constrain certain relative phenomena between ∞-categories via the preservation of initial objects. The remainder of this section is dedicated to an elaboration on this, somewhat criptic, point.

lem:1936

Lemma 6.21. *An object x in $\mathscr{C}$ is initial if and only if the forgetful functor $\mathscr{C}_{x/} \to \mathscr{C}$ admits a section $F : \mathscr{C} \to \mathscr{C}_{x/}$ with $F(x) = id_x$.*

Proof. If x is initial then the forgetful functor is a trivial Kan fibration, by Proposition 6.8. It follows that the lifting problem

$$\begin{tikzcd} * \arrow[r, "id_x"] \arrow[d, "x"'] & \mathscr{C}_{x/} \arrow[d] \\ \mathscr{C} \arrow[r, "id"'] \arrow[ur, dotted] & \mathscr{C} \end{tikzcd}$$

admits a solution $s : \mathscr{C} \to \mathscr{C}_{x/}$. This solution provides the desired section. Conversely, if we have such a section F then for each y in $\mathscr{C}$ we can split the identity on the mapping space as

$$\mathrm{Hom}_{\mathscr{C}}(x, y) \xrightarrow{F} \mathrm{Hom}_{\mathscr{C}_{x/}}(id_x, F(y)) \xrightarrow{forget} \mathrm{Hom}_{\mathscr{C}}(x, y).$$

Since id_x is initial in $\mathscr{C}_{x/}$, by Proposition 6.10, each mapping space $\mathrm{Hom}_{\mathscr{C}_{x/}}(id_x, F(y))$ is contractible. Thus each mapping space $\mathrm{Hom}_{\mathscr{C}}(x, y)$ is a retract of a contractible space, and hence contractible itself. □

We record a little lemma.

lem:1957

Lemma 6.22 ([3, 0196])**.** *If $i : A \to B$ is an inclusion of simplicial sets, then the induced map*

$$\{*\} \star i : \{*\} \star A \to \{*\} \star B$$

is left anodyne.

Proof. The class of i at which $\{*\} \star i$ is left anodyne is saturated. We we need only show that it contains the inclusions $\partial\Delta^n \to \Delta^n$. But in this case the inclusion in question is identified with the left anodyne map $\Lambda^{n+1}_0 \to \Delta^{n+1}$. □

prop:init_lanodyne

Proposition 6.23. *An object x in $\mathscr{C}$ is initial if and only if the inclusion $x : * \to \mathscr{C}$ is left anodyne. If y is terminal in $\mathscr{C}$, then the inclusion $y : * \to \mathscr{C}$ is right anodyne.*

Proof. We deal with the initial claim. If $x : * \to \mathscr{C}$ is left anodyne then we can solve the lifting problem

$$\begin{tikzcd} * \arrow[r, "id_x"] \arrow[d, "x"'] & \mathscr{C}_{x/} \arrow[d] \\ \mathscr{C} \arrow[r, "id"'] \arrow[ur, dotted] & \mathscr{C} \end{tikzcd}$$

and hence obtain a section $F : \mathscr{C} \to \mathscr{C}_{x/}$ as in Lemma 6.21. We conclude that x is initial in $\mathscr{C}$.

Conversely, if x is initial then the section $F : \mathscr{C} \to \mathscr{C}_{x/}$ of Lemma 6.21 provides a map $F' : \{*\} \star \mathscr{C} \to \mathscr{C}$ with $F'|_{\mathscr{C}} = id_{\mathscr{C}}$, $F'(*) = x$, and $F'(* \to x) = id_x$. In particular, F' is defined on each simplex outside of $\mathscr{C}$ by taking

$$F'(\{*\} \star \Delta^m) = F(\Delta^m).$$

This map F' gives a diagram

$$\begin{tikzcd} * \arrow[r, "x"] \arrow[d, "x"'] & \{*\} \star \{x\} \arrow[r] \arrow[d] & * \arrow[d, "x"] \\ \mathscr{C} \arrow[r] & \{*\} \star \mathscr{C} \arrow[r, "F"] & \mathscr{C} \end{tikzcd}$$

so that the inclusion $\{x\} \to \mathscr{C}$ is a retract of the inclusion $\{*\} \star \{x\} \to \{*\} \star \mathscr{C}$. Since this latter inclusion is left anodyne, by Lemma 6.22, we conclude that the inclusion $\{x\} \to \mathscr{C}$ is left anodyne as well. □

As a consequence of Proposition 6.23 we observe a kind of relative triviality for $\mathscr{C}$.

cor:initial_eval **Corollary 6.24.** *Suppose $f : X \to S$ is a left fibration, and that $\mathscr{C}$ admits an initial object $x : * \to \mathscr{C}$. Then the map*

$$\mathrm{Fun}(\mathscr{C}, X) \to X \times_S \mathrm{Fun}(\mathscr{C}, S), \quad F \mapsto (F|_x, fF). \tag{14}$$ eq:2002

is a trivial Kan fibration.

Proof. Immediate from Propositions 6.23 and 2.3. □

Let's consider what Corollary 6.24 is telling, in semi-human terms. In the extreme case where $\mathscr{X} \to *$ is a Kan complex, the right hand side of (14) is just $\mathscr{X}$ and we obtain a trivial Kan fibration

$$ev_x : \mathrm{Fun}(\mathscr{C}, \mathscr{X}) \to \mathscr{X}$$

which just evaluates a functor F at $x : * \to \mathscr{C}$. This says that for any choice of a point $z : * \to \mathscr{X}$ there is a unique functor $F_z : \mathscr{C} \to \mathscr{X}$ which evaluates as $F_z(x) = z$. Indeed, we can just take the fiber of ev_x at z to obtain a space $\mathrm{Fun}(\mathscr{C}, \mathscr{X})_z$ which parametrizes such functors, and observe that this space is contractible. In this way $\mathscr{C}$ "looks like a point" relative to any Kan complex.

In the relative setting, we consider a left fibration $f : X \to S$ and see that for any choice of a functor $\bar{F} : \mathscr{C} \to S$, and a point z in X which lifts $F(x)$, there is a unique lift of $\bar{F}$ to a functor $F : \mathscr{C} \to X$ with $F(x) = z$. Rather, we observe that any lifting problem of the form

$$\begin{tikzcd} * \arrow[r] \arrow[d, "x"'] & X \arrow[d, "f"] \\ \mathscr{C} \arrow[r] \arrow[ur, dotted] & S \end{tikzcd}$$

admits a unique solution.

6.8. **Equivalences of fibrations via initial objects.**

prop:equiv_initial

Proposition 6.25. *Let*

$$\begin{tikzcd} \mathscr{C} \arrow[rr, "F"] \arrow[dr] & & \mathscr{D} \arrow[dl] \\ & \mathscr{T} & \end{tikzcd}$$

be a diagram of ∞-categories in which both of the maps to $\mathscr{T}$ are left fibrations. If $\mathscr{C}$ admits an initial object x, then F is an equivalence if and only if $F(x)$ is initial in $\mathscr{D}$.

Proof. If F is an equivalence then it preserves initial objects, by Lemma 6.4. Suppose conversely that F is such a map, that x is initial in $\mathscr{C}$, and that the image $F(x)$ is initial in $\mathscr{D}$. Consider the lifting problem

$$\begin{tikzcd} * \arrow[r, "x"] \arrow[d, "F(x)"'] & \mathscr{C} \arrow[d] \\ \mathscr{D} \arrow[r] \arrow[ur, dotted] & \mathscr{T}. \end{tikzcd}$$

By Corollary 6.24 there exists a solution to this problem, and hence there exists a functor $G : \mathscr{D} \to \mathscr{C}$ over $\mathscr{T}$ with $GF(x) = x$. By Corollary 6.24 again the composition GF is also seen to be isomorphic to the identity in $\mathrm{Fun}_{\mathscr{T}}(\mathscr{C}, \mathscr{C})$. We also have $FG(Fx) = F(x)$ so that FG is isomorphic to the identity on $\mathscr{D}$. Thus F is an equivalence. □

Again, one observes a corresponding statement for right fibrations and terminal objects, via the opposite duality.

At first glance this proposition seems ridiculous. Indeed, it suggests that if $f : \mathscr{E} \to \mathscr{C}$ is a left fibration of ∞-categories, and e is an object in $\mathscr{E}$ with image x in $\mathscr{C}$, then the induced map on undercategories $F : \mathscr{E}_{e/} \to \mathscr{C}_{x/}$ is an equivalence. This is because F fits into a diagram

$$\begin{tikzcd} \mathscr{E}_{e/} \arrow[rr] \arrow[dr] & & \mathscr{C}_{x/} \arrow[dl] \\ & \mathscr{C} & \end{tikzcd}$$

and which both maps to $\mathscr{E}$ are left fibrations, and F is seen to send the initial object id_e to the initial object id_x. However, one sees that this is as bad as it gets.

Corollary 6.26. *Let $\mathscr{C} \to \mathscr{T}$ be a left fibration, suppose that $\mathscr{C}$ admits an initial object x, and let t denote the image of x in $\mathscr{T}$. Then there is an equivalence $F : \mathscr{C} \to \mathscr{T}_{t/}$ of left fibrations over $\mathscr{T}$ which sends x to id_t.*

So Proposition 6.25, said another way, classifies left fibrations up to equivalence via isoclasses of objects in $\mathscr{T}$.

Proof. By Proposition 6.25 the map $\mathscr{C}_{x/} \to \mathscr{T}_{t/}$ is an equivalence of left fibrations which sends id_x to id_t. The proposed equivalence $\mathscr{C} \to \mathscr{T}_{t/}$ is obtained by composing the equivalence $\mathscr{C}_{x/} \to \mathscr{T}_{t/}$ with a section $F : \mathscr{C} \to \mathscr{C}_{x/}$ as in Lemma 6.21. □

7. Hom functors for ∞-categories

7.1. **The twisted arrows category.**

Definition 7.1. Given a simplicial set $\mathscr{C}$, we define the twisted arrow category $\mathscr{T}w(\mathscr{C})$ as the simplicial set whose n-simplices are

$$\mathscr{T}w(\mathscr{C})[n] := \mathrm{Hom}_{\mathrm{sSet}}((\Delta^n)^{\mathrm{op}} \star \Delta^n, \mathscr{C}).$$

Restricting along the inclusions

$$(\Delta^n)^{\mathrm{op}} \to (\Delta^n)^{\mathrm{op}} \star \Delta^n \quad \text{and} \quad \Delta^n \to (\Delta^n)^{\mathrm{op}} \star \Delta^n$$

provides a natural map to the product

$$\lambda : \mathscr{T}w(\mathscr{C}) \to \mathscr{C}^{\mathrm{op}} \times \mathscr{C}. \tag{15}$$ eq:lambda

To get our heads on straight here, let's observe directly that an object in $\mathscr{T}w(\mathscr{C})$ is a choice of a morphism $\alpha : x \to y$ in $\mathscr{C}$. A morphism from an objects $\alpha : x \to y$ to some other $\alpha' : x' \to y'$ is a diagram of the form

$$\begin{tikzcd} x \arrow[r, "\alpha"] \arrow[dr] & y \arrow[d] \\ x' \arrow[u] \arrow[r, "\alpha'"'] \arrow[ur] & y'. \end{tikzcd}$$

If we consider the fiber $\{(x,y)\} \times_{(\mathscr{C}^{\mathrm{op}} \times \mathscr{C})} \mathscr{T}w(\mathscr{C})$, a simplex in this fiber can be visualized as some directed diagram from x to y which is "completely filled in",

$$\begin{tikzcd} x \arrow[r, bend left, dotted] \arrow[r, dotted] \arrow[r, bend right, dotted] & y. \end{tikzcd}$$

We prove below that these fibers are a type of bifunctorial Hom space for $\mathscr{C}$, where bifunctoriality simply refers to the fact that one has two variables to tune in the base.

We note that the join $(\Delta^n)^{\mathrm{op}} \star \Delta^n$ is identified with Δ^{2n+1} via the bijection

$$[2n+1] \to [n] \amalg [n], \quad i \mapsto \begin{cases} n - i \text{ in the first set if } i \leq n \\ i - n \text{ in the second set if } i \geq n. \end{cases}$$

prop:tw_inner **Proposition 7.2** ([3, 03JT]). *The restriction map $\lambda : \mathscr{T}w(\mathscr{C}) \to \mathscr{C}^{\mathrm{op}} \times \mathscr{C}$ is a left fibration. More generally, if $\mathscr{C} \to S$ is an inner fibration of simplicial sets, then the restriction map $\mathscr{T}w(\mathscr{C}) \to (\mathscr{C}^{\mathrm{op}} \times \mathscr{C}) \times_{(S^{\mathrm{op}} \times S)} \mathscr{T}w(S)$ is a left fibration.*

We only outline the main points of the proof. The reader can find details in the cited text.

Proof outline. We wish to show that any lifting problem of the form

$$\begin{tikzcd} \Lambda^n_{n-i} \arrow[r] \arrow[d] & \mathscr{T}w(\mathscr{C}) \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & (\mathscr{C}^{\mathrm{op}} \times \mathscr{C}) \times_{(S^{\mathrm{op}} \times S)} \mathscr{T}w(S), \end{tikzcd}$$

with $i \leq n$, admits a solution. Such a lifting problem admits a solution if an only if the corresponding problem

$$\begin{tikzcd} K_0 \arrow[r] \arrow[d] & \mathscr{C} \arrow[d] \\ \Delta^{2n+1} \arrow[r] \arrow[ur, dotted] & S \end{tikzcd} \tag{16}$$

eq:1641

admits a solution, where $K_0 \subseteq \Delta^{2n+1}$ is some subcomplex which we descirbe below.

Given a subset $J \subseteq [2n+1]$, the non-degenerate simplex $\Delta^J \subseteq \Delta^{2n+1}$ lies in K_0 if and only if J is contained in one of $[n]$ or $[2n+1] - [n]$, or J is contained in a subset $[2n+1] - \{j, 2n+1-j\}$ with $j \neq i$. It is argued in [3] that the inclusion $K_0 \to \Delta^{2n+1}$ is in fact anodyne, by factoring this map into a sequence of inclusions

$$K_0 \to K_1 \to \cdots \to K_m = \Delta^{2n+1}$$

in which each K_{i+1} is obtained from K_i by attaching a single non-degenerate simplex. Each such inclusion $K_i \to K_{i+1}$ is shown to be anodyne, so that the composition $K_0 \to \Delta^{2n+1}$ is in fact anodyne, and we find that the problem (16) admits a solution, as desired. $\square$

Since $\mathscr{C}^{\mathrm{op}} \times \mathscr{C}$ is itself an ∞-category whenever $\mathscr{C}$ is an ∞-category, we find that the twisted arrow category $\mathscr{T}w(\mathscr{C})$ is also an ∞-category in this case.

Corollary 7.3. *If $\mathscr{C}$ is an ∞-category, the twisted arrow category $\mathscr{T}w(\mathscr{C})$ is also an ∞-category.*

Via Proposition 7.2, and the general phenomena of transport for left fibrations (Proposition 4.16), we understand that the left fibration $\lambda : \mathscr{T}w(\mathscr{C}) \to \mathscr{C}^{\mathrm{op}} \times \mathscr{C}$ identifies a associated transport functor

$$H : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{K}an.$$

This transport functor is uniquely determined, up to a contractible space of choices, by the assertion that H fits into a categorical pullback diagram

$$\begin{tikzcd} \mathscr{T}w(\mathscr{C}) \arrow[r] \arrow[d] & \mathscr{K}an_{*/} \arrow[d] \\ \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \arrow[r, "H"'] & \mathscr{K}an. \end{tikzcd}$$

Definition 7.4. A Hom-functor for and ∞-category $\mathscr{C}$ is a transport functor

$$H : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{K}an$$

for the left fibration $\lambda : \mathscr{T}w(\mathscr{C}) \to \mathscr{C}^{\mathrm{op}} \times \mathscr{C}$.

The first aims of this section are to provide a calculation of the fibers of the twisted arrow fibration sufficient conditions which allow us to identify a Hom functor when we see one. Of interest are Hom functors for nerves of dg and simplicial categories (e.g. Hom functors for derived categories).

Let us note, as a bit of foreshadowing, that any Hom functor determines maps into the functor categories

$$H_* : \mathscr{C} \to \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathscr{K}an) \quad \text{and} \quad H^* : \mathscr{C}^{\mathrm{op}} \to \mathrm{Fun}(\mathscr{C}, \mathscr{K}an).$$

We will eventually find that these functors are both fully faithful embeddings.

7.2. Fibers of the twisted arrows fibration.

7.3. Lol.

8. Hom functors for dg categories

9. Limits and colimits

10. Coproducts and cofibers

11. Homotopy limits and colimits Kan complexes

12. Limits and colimits in $\mathscr{K}an$

13. Stability and cocompleteness

14. Limits via mapping spaces

15. Cocompletion and presentability

16. Exactness and adjoints

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