

# NOTES ON INFINITY CATEGORIES: PART I

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ABSTRACT. These are notes based on my own readings of [8, 9, 10]. We cover basic information for  $\infty$ -categories, and functors between  $\infty$ -categories. As a terminal point, we prove that a functor between  $\infty$ -categories is an equivalence if and only if its fully faithful and essentially surjective.

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## 1. COMMENTS AND DISCLAIMERS

Most of the arguments are adapted directly from [10]. Indeed, what follows is essentially a reorganization of (some of) the contents from [10]. Certain simplicial deviations which have content, but which are not “reusable” in our analysis of  $\infty$ -categories, are avoided. In these cases specific references are given which indicate where the required details can be found.

## 2. $\infty$ -CATEGORIES (INTRODUCTION)

Here we give a light introduction, and provide bare-bones definitions of simplicial sets, Kan complexes, and  $\infty$ -categories. In Section 3 we provide a number of basic examples, and subsequently begin our discussions of these topics in earnest.

**2.1. What is an  $\infty$ -category, in principle.** An infinity category  $\mathcal{A}$  is, in the barest sense of things, a special type of simplicial set. We think about an infinity category as a type of “category” by considering the 0-simplices in  $\mathcal{A}$  as objects, and the 1-simplices as morphisms. The 2-simplices  $\zeta : \Delta^2 \rightarrow \mathcal{A}$  are *choices* of compositions  $f \circ g \approx_\zeta h$ , where we view  $\zeta$  itself as a type of homotopy between the raw composition and the map  $h$ ,

$$\zeta : \begin{array}{ccc} & Z & \\ g \nearrow & & \searrow f \\ X & & Y \end{array} \rightsquigarrow \begin{array}{ccc} & Z & \\ \text{---} \nearrow & & \searrow \text{---} \\ X & \xrightarrow{h} & Y \end{array}$$

We note that without  $\zeta$  this raw composition “ $f \circ g$ ”, which is simply the choice of two 1-simplices with a shared vertex, is unstructured information in  $\mathcal{A}$ . So, the composition *does not have a meaning* without  $\zeta$ .

Note that, given our presentation above, we want  $\mathcal{A}$  to admit compositions of functions. Rather, for any two 1-simplices  $f, g : \Delta^1 = \{0, 1\} \rightarrow \mathcal{A}$  with matching vertices  $g(1) = f(0)$  we require the existence of *some* filling  $\zeta : \Delta^2 = \{0, 1, 2\} \rightarrow \mathcal{A}$  with  $\zeta|_{\{0,1\}} = g$  and  $\zeta|_{\{1,2\}} = f$ . This choice of filling  $\zeta$  shouldn’t matter, up to some 3-simplex  $\gamma : \Delta^3 \rightarrow \mathcal{A}$ , and so on. . . . The notion of an infinity category is some completion of this thought.

In this section we recall the necessary (simplicial) set theoretic background, then formally define the notions  $\infty$ -categories and functors between such objects.

**2.2. Preliminary words on sets and universes.** We work with ZFCU. So, we suppose that each set  $X$  lives in a (Grothendieck) universe  $\mathbb{U}$ . That is, for each set  $X$  we assume the existence of an additional set of sets  $\mathbb{U}$  with  $X$  living in  $\mathbb{U}$ . This set  $\mathbb{U}$  is assumed to contain the first infinite ordinal  $\omega$ , and to be closed under all basic operations in set theory, including the formations of power sets. So, for example,  $\Lambda$  is in  $\mathbb{U}$ , and  $f : \Lambda \rightarrow \mathbb{U}$ , then then image  $f(\Lambda) \in \mathbb{U}$ , and all subsets and quotients of  $f(\Lambda)$  are also in  $\mathbb{U}$ .  $\mathbb{U}$  is also closed under unions and products. So we see that  $\mathbb{U}$  is closed under colimits and limits which are indexed by sets in  $\mathbb{U}$ . This is all to say, we can do set theory in  $\mathbb{U}$ .

Given a Grothendieck universe  $\mathbb{U}$ , we say a set  $X$  is  $(\mathbb{U})$ -small if  $X \in \mathbb{U}$ . So, a set  $X$  is not small if  $X$  is not an element in  $\mathbb{U}$ . An important feature of the universe axiom is that each universe  $\mathbb{U}$  is itself a set, and hence lives in a (larger) universe, as does any set at all. So, give a set  $Y$ , we can choose a universe  $\mathbb{U}'$  with  $Y \cup \mathbb{U}$  in  $\mathbb{U}'$ . That is to say, we can always assume that a set  $Y$  is small by *enlarging* our universe. As long as our arguments are universe independent, this enlarging of the universe will cause no problems.

We note that the existence of universes is not at all implied by the usual axioms of ZFC, and requires the introduction of the new axiom “U”. This axiom is equivalent to the existence of extremely large cardinals, called inaccessible cardinals.

In general, we fix a universe  $\mathbb{U}$  (which we may flex slightly when needed), and a (co)complete category  $\mathcal{A}$  is a category which admits all (co)limits indexed by small sets relative to the given universe. What’s interesting is that this kind of category theory is essentially indistinguishable from usual category theory.

For example, if a fix a field  $k$  and a universe  $\mathbb{U}$  containing  $k$ , then we may consider the category  $\text{Vect} = \text{Vect}(\mathbb{U})$  of  $\mathbb{U}$ -small vector spaces. This category is closed under

the formation of small limits and colimits, and so is cocomplete. In this setting all representability theorems are also perfectly viable as well.

The importance of employing universes is that we may assume that all (infinity) categories have a *set* of objects and morphisms, i.e. are special kinds of structured sets. One then can speak very clearly not only of the category of small categories

$$\text{Cat}^{\text{sm}} \subset \mathbb{U}\text{-Set},$$

but of the category of arbitrary categories

$$\text{Cat} \subset \text{Set}.$$

So our assumption that an infinity category is a type of simplicial set is not at all limiting, and allows us to pursue a rigorous analysis of the category  $\text{Cat}_\infty$  of infinity categories itself.

**2.3. Simplicial sets.** Let  $\Delta$  denote the category of linearly ordered, non-empty, finite sets with weakly increasing functions. In  $\Delta$  we have the objects  $[n] = \{0, 1, \dots, n\}$ , with their natural ordering, which exhaust all objects up to isomorphism. A simplicial set is a functor  $S : \Delta^{\text{op}} \rightarrow \text{Set}$ , and we have the category of simplicial sets

$$\text{sSet} = \{\text{functors } \Delta^{\text{op}} \rightarrow \text{Set}, \text{ with natural transformations}\}.$$

Amongst simplicial sets one has the standard  $n$ -simplices, which are the representable functors

$$\Delta^n := \text{Hom}_\Delta(-, [n]).$$

A map then  $f : \Delta^n \rightarrow S$  to some simplicial set  $S$  is then a choice of elements  $f(id) \in S([n])$ , and we refer to those elements, or maps rather, as the  $n$ -simplices in  $S$ . We let  $S[n]$  denote the set of  $n$ -simplices in  $S$ , as a shorthand.

To make this more clear, any natural transformation  $f : \Delta^n \rightarrow S$  is just a collection of compatible maps between sets  $f = f_J : \Delta^n(J) \rightarrow S(J)$ , for all linearly ordered sets  $J$ , and for any  $r \in \Delta^n(J) = \text{Hom}_\Delta(J, [n])$  we have  $r = r^*(id_{[n]})$  so that

$$f(r) = f(r^*(id_{[n]})) = r^*(f(id_{[n]})).$$

Hence  $f$  is determined by the value  $f(id_{[n]}) \in S[n]$ . Conversely, any element  $x \in S[n]$  determines such a functor  $f : \Delta^n \rightarrow S$ ,  $f_J(\zeta) := \zeta_S^*(x)$ . This is obviously some kind of Yoneda Tom-foolery, and Yoneda tells us directly that the map

$$\Delta \rightarrow \text{sSet}, \quad J \mapsto \Delta^J$$

is fully faithful

Let us define here the boundary simplex  $\partial\Delta^n \subset \Delta^n$  and the  $i$ -th *horn*  $\Lambda_i^n \subset \partial\Delta^n$ . We have, for each  $j \leq n$ , the  $j$ -th face map  $\partial_j : [n-1] \rightarrow [n]$ , which is the unique increasing map with  $j$  not in its image. The boundary  $\partial\Delta^n$  in  $\Delta^n$  is the simplicial subset consisting of all maps  $r : J \rightarrow [n]$  which factor through some face  $r = r'd_j$ ,  $0 \leq j \leq n$ . The  $i$ -th horn  $\Lambda_i^n$  in  $\Delta^n$  is the simplicial subset consists of the collection of all maps  $r : J \rightarrow [n]$  which factor through some face map  $r = r'd_j$  with  $j$  *not* equal to  $i$ . So, for example, the  $i$ -th face map itself  $d_i : [n-1] \rightarrow [n]$  lies in the boundary  $\partial\Delta^n$ , but does not lie in the  $i$ -th horn  $\Lambda_i^n$ .

The category of simplicial sets has products and coproducts, which are defined in the naïve ways

$$(K \times S)(J) = K(J) \times S(J) \quad \text{and} \quad (K \amalg S)(J) = K(J) \amalg S(J).$$

We similarly have fiber products and coproducts

$$\begin{array}{ccc} K_1 \times_S K_2 & \longrightarrow & K_2 \\ \downarrow & & \downarrow \\ K_1 & \longrightarrow & S \end{array} \quad \begin{array}{ccc} S & \longrightarrow & K_1 \\ \downarrow & & \downarrow \\ K_2 & \longrightarrow & K_1 \amalg_S K_2, \end{array}$$

whose values on any linearly ordered set  $J$  are the fiber product and coproduct of the corresponding sets. Indeed, the category of simplicial sets is both complete and cocomplete, with

$$(\varprojlim_{i \in I} K_i)(J) = \varprojlim_{i \in I} K_i(J) \quad \text{and} \quad (\varinjlim_{i \in I} K_i)(J) = \varinjlim_{i \in I} K_i(J),$$

and any simplicial set  $K$  is reconstructible from the simplices  $\Delta^n$  as the colimit

$$\varinjlim_{\Delta^n \rightarrow K} \Delta^n \xrightarrow{\sim} K.$$

**Remark 2.1.** This reconstruction of  $K$  from its simplices can be compared with the reconstruction of a scheme  $X$  from the category of affine schemes over  $X$ ,  $\text{Spec}(R) \rightarrow X$ , or from its Zariski site when  $X$  is separated.

**Remark 2.2.** We note that, for any linearly ordered set  $J$ , there is a unique isomorphism  $J \cong [|J|]$  in  $\Delta$ . It follows that any simplicial set  $K : \Delta^{op} \rightarrow \text{Set}$  is determined up to unique isomorphism by its restriction to the full subcategory  $[\mathbb{Z}] \subset \Delta$ . This is to say, any simplicial set is determined up to unique isomorphism by its collection of  $n$ -simplices  $K[n]$  and the dual maps  $r^* : K[n] \rightarrow K[m]$  to the collection of weakly increasing functions  $r : [m] \rightarrow [n]$ . So we could define a simplicial set simply as a functor  $K : [\mathbb{Z}]^{op} \rightarrow \text{Set}$ . At times, however, it can be convenient to employ the larger category  $\Delta$  of all linearly ordered sets.

**2.4. Infinity categories.** An infinity category is, at the base of it, a certain type of simplicial set. We begin with a stronger notion which encodes a simplicial version of a topological space.

**Definition 2.3.** A simplicial set  $K$  is a Kan complex provided that, for each positive integer  $n$ ,  $i \in [n]_{\sim}$ , and map of simplicial sets  $f : \Lambda_i^n \rightarrow K$ , there exists an extension of  $f$  to a map  $\tilde{f} : \Delta^n \rightarrow K$ .

We will refer to such an extension  $\tilde{f}$  as a “filling” for the given map  $f : \Lambda_i^n \rightarrow K$ . We think of such  $K$  as a type of category, with objects given by the 0-simplices  $K_0$  and morphisms given by the 1-simplices  $K_1$ . Restricting along the inclusions  $d_0, d_1 : [0] \rightarrow [1]$  gives the source and target objects for a given map  $x \in K_1$ . Any map  $f : \Lambda_1^2 \rightarrow K$  is determined by the two 1-simplices  $x \in K_1$  and  $y \in K_1$  which are the images of the two maps

$$\Delta^1 \xrightleftharpoons[d_0]{d_2} \Lambda_1^2 \longrightarrow K.$$

The choice of a filling  $\Delta^2 \rightarrow K$  specifies a third morphism  $z \in K_1$  which provides the third face for the map  $\partial\Delta^2 \rightarrow K$  and the complete filling  $\Delta^2 \rightarrow K_1$  realizes an

identification “ $y \circ x$ ” =  $z$ :

$$\begin{array}{ccc} & A_2 & \\ x \nearrow & & \searrow y \\ A_1 & \xrightarrow{\quad z \quad} & A_3. \end{array}$$

The higher filling axioms tell us that this filling procedure is essentially unique, in a way which is not transparent but which we leave unarticulated for the moment.

The identity morphism from an object  $x \in K_0$  to itself is the image of  $x$  in  $K_1$  along the structural map  $id_? : K_0 \rightarrow K_1$  dual to the unique morphism  $[1] \rightarrow [0]$ . One similarly has higher identity morphisms at a given object  $x$ , which are the images of  $x$  under the structural maps  $id_?^n : K_0 \rightarrow K_n$  which are again determined by the unique map  $[n] \rightarrow [0]$ .

Now, a Kan complex has a much more rigid structure than one needs to form this kind of conditional composition. Indeed, by considering maps  $\Lambda_0^2 \rightarrow K$  and  $\Lambda_2^2 \rightarrow K$ , one sees that all morphisms in a Kan complex are in fact invertible. So we arrive at the notion of an  $\infty$ -category.

**Definition 2.4.** An  $\infty$ -category is a simplicial set  $\mathcal{C}$  such that, for each positive integer  $n$ ,  $0 < i < n$ , and  $f : \Lambda_i^n \rightarrow \mathcal{C}$ , there exists an extension of  $f$  to a map  $\tilde{f} : \Delta^n \rightarrow \mathcal{C}$ . A functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  between  $\infty$ -categories is simply a map of simplicial sets.

Note that we’ve now excluded the external horns  $\Lambda_0^n$  and  $\Lambda_n^n$  in our lifting condition. So in particular this extension property is vacuous for 1-simplices. Note also that a Kan complex is a specific type of  $\infty$ -category.

**Remark 2.5.** What we call an  $\infty$ -category is also called an  $(\infty, 1)$ -category, and/or a weak Kan complex.

In defining functors between  $\infty$ -categories our specificity is somewhat strange. Indeed, we would imagine that one should define a functor to be some map which is compatible with the structure on the image category  $\mathcal{D}$ , only up to homotopy. However, one can see, for example, that a map between infinity categories  $F : \mathcal{C} \rightarrow \mathcal{D}$  needn’t be “compatible” with compositions of morphisms in  $\mathcal{C}$  and  $\mathcal{D}$ . One simply needs some coherent rule which assigns for each composition “ $x \circ y$ ” =  $z$  in  $\mathcal{C}$  a composition in  $\mathcal{D}$ , not *the* composition in  $\mathcal{D}$ , since we declined to make the mistake of choosing such a thing in the first place.

ex:Delta\_infty

**Example 2.6.** The simplicial set  $\Delta^n$  is an  $\infty$ -category. Indeed, any map  $\Lambda_i^m \rightarrow \Delta^n$  from an inner horn is specified by a collection of maps  $\Delta^{[m]-\{j\}} \rightarrow \Delta^n$ , i.e. maps  $r_j : [m] - \{j\} \rightarrow [n]$ , which agree on their shared boundaries. Since the subsets  $[m] - \{j\}$  cover  $[m]$ , when  $m > 1$ , the  $r_j$  glue to a unique map  $r : [m] \rightarrow [n]$ . To see that  $r$  is weakly increasing, note first that  $m \geq 2$  in order for there to exist an inner horns for  $\Delta^m$  and take any  $l \in [m]$ . Then  $l$  and  $l + 1$  are both in  $[m] - \{0\}$  or  $[m] - \{m\}$ . In particular, there exists  $i$  so that  $l, l + 1 \in [m] - \{i\}$  and  $r(l) = r_i(l) \leq r_i(l + 1) = r(l + 1)$ . Since  $l$  was chosen arbitrarily it follows that  $r$  is weakly increasing.

Now, the above argument does not work when we consider maps  $\Lambda_0^2 \rightarrow \Delta^n$  or  $\Lambda_2^2 \rightarrow \Delta^n$  and  $n \geq 1$ . Indeed, consider the map  $\Lambda_0^2 \rightarrow \Delta^n$  specified by the two functions  $r_1 : \{0, 2\} \rightarrow [n]$ ,  $r_1(0) = r_1(2) = 0$ , and  $r_2 : \{0, 1\} \rightarrow [n]$ ,  $r_2(0) = 0$  and

$r_2(1) = 1$ . Then these  $r_i$  extend to a unique set map  $r : [2] \rightarrow [n]$  given by  $r(0) = 0$ ,  $r(1) = 1$ ,  $r(2) = 0$ . This function is clearly not weakly increasing. We similarly consider the map  $\Lambda_2^2 \rightarrow \Delta^n$  specified by the functions  $r_0(1) = 0$ ,  $r_0(2) = 1$ , and  $r_1(0) = 1$ ,  $r_1(2) = 1$ . This map extends uniquely to the function  $r : [2] \rightarrow [n]$  defined by  $r(0) = 1$ ,  $r(1) = 0$ ,  $r(2) = 1$ . This function is not weakly increasing and so there exists no 2-simplex  $\Delta^2 \rightarrow \Delta^n$  extending this horn. So we see that the positive dimensional standard simplices  $\Delta^n$  are *not* Kan complexes.

ex:Sing\_wk

**Example 2.7.** Let  $X$  be a topological space, and define the simplicial set  $\text{Sing}(X)$  by taking

$$\text{Sing}(X)[n] := \text{Hom}_{\text{Top}}(|\Delta^n|, X),$$

where  $|\Delta^n|$  is the standard topological  $n$ -simplex in  $\mathbb{R}^n$ . Since each inclusion of a horn  $|\Lambda_i^n| \rightarrow |\Delta^n|$  admits a retract  $|\Delta^n| \rightarrow |\Lambda_i^n|$  one sees that these singularity sets  $\text{Sing}(X)$  are Kan complexes.

Up to homotopy equivalence, all Kan complexes are of the form  $\text{Sing}(X)$ . We elaborate on this point in Section 4.11.

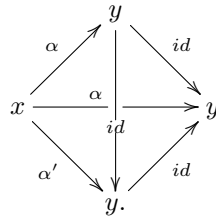
**2.5. The homotopy category.** We construct a homotopy category  $\text{h}\mathcal{C}$  for any  $\infty$ -category  $\mathcal{C}$ , whose morphisms are certain equivalence classes of maps in  $\mathcal{C}$ , i.e. equivalence classes of 1-simplices  $\Delta^1 \rightarrow \mathcal{C}$ . The first lemma defines the appropriate equivalence relation on maps in  $\mathcal{C}$ .

lem:200

**Lemma 2.8.** *For any two maps  $\alpha, \alpha' : x \rightarrow y$  in an  $\infty$ -category  $\mathcal{C}$ , the following are equivalent:*

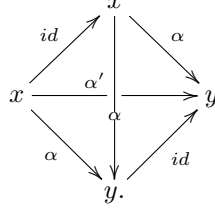
- (a) *There exists a 2-simplex  $s : \Delta^2 \rightarrow \mathcal{C}$  with  $s|_{\Delta^{\{0,1\}}} = \alpha$ ,  $s|_{\Delta^{\{0,2\}}} = \alpha'$ , and  $s|_{\Delta^{\{1,2\}}} = \text{id}_y$ .*
- (b) *There exists a 2-simplex  $s$  with  $s|_{\Delta^{\{0,1\}}} = \alpha'$ ,  $s|_{\Delta^{\{0,2\}}} = \alpha$ , and  $s|_{\Delta^{\{1,2\}}} = \text{id}_y$ .*
- (c) *There exists a 2-simplex  $s$  with  $s|_{\Delta^{\{0,1\}}} = \text{id}_x$ ,  $s|_{\Delta^{\{0,2\}}} = \alpha'$ , and  $s|_{\Delta^{\{1,2\}}} = \alpha$ .*
- (d) *There exists a 2-simplex  $s$  with  $s|_{\Delta^{\{0,1\}}} = \text{id}_x$ ,  $s|_{\Delta^{\{0,2\}}} = \alpha$ , and  $s|_{\Delta^{\{1,2\}}} = \alpha'$ .*

*Sketch proof.* We prove the equivalence between (a) and (b), and (a) and (c). For (a)  $\Rightarrow$  (b) consider the horn  $\Lambda_1^3 \rightarrow \mathcal{C}$  which appears as



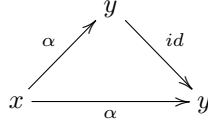
One of the sides is given by  $s$ , and the others are given by the identity 2-simplex on  $y$ , and by expanding  $\alpha$  along the surjective map  $r : [2] \rightarrow [1]$  with  $r(1) = r(2) = 1$ . We fill this horn to find the simplex required by (b). One finds (b)  $\Rightarrow$  (a) by swapping the roles of  $\alpha$  and  $\beta$  in the above argument.

For (a)  $\Rightarrow$  (c) consider the horn  $\Lambda_2^3 \rightarrow \mathcal{C}$  which appears as



with sides given by  $s$  from (a), and by expanding  $\alpha$  along the two weakly increasing surjections  $r : [2] \rightarrow [1]$ . We fill the horn to provide the necessary 2-simplex for (c). For the implication (c)  $\Rightarrow$  (a) one fills the horn  $\Lambda_1^3 \rightarrow \mathcal{C}$  which appears as above.  $\square$

Let us say two maps  $\alpha, \alpha' : x \rightarrow y$  are equivalent,  $\alpha \sim \alpha'$ , if any of the equivalent conditions of Lemma 2.8 are satisfied. After recall that we can expand any map  $\alpha : x \rightarrow y$  to a 2-simplex of the form



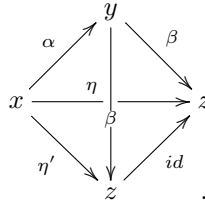
by restricting along the corresponding surjection  $[2] \rightarrow [1]$ , we see that  $\alpha \sim \alpha$ . This observation, along with the characterization of Lemma 2.8, assures that  $\sim$  defines an equivalence relation on the set  $\mathcal{C}[1]$  of edges in  $\mathcal{C}$ .

Let us say that a 2-simplex  $s : \Delta^2 \rightarrow \mathcal{C}$  exhibits a morphism  $\eta : x \rightarrow z$  as a composite of maps  $\alpha : x \rightarrow y$ , and  $\beta : y \rightarrow z$ , if

$$s|_{\Delta^{\{0,1\}}} = \alpha, \quad s|_{\Delta^{\{1,2\}}} = \beta, \quad \text{and} \quad s|_{\Delta^{\{0,2\}}} = \eta.$$

**Lemma 2.9.** *Suppose we have two maps  $\alpha : x \rightarrow y$  and  $\beta : y \rightarrow z$ , and 2-simplices  $s, s' : \Delta^2 \rightarrow \mathcal{C}$  which exhibits two maps  $\eta, \eta' : x \rightarrow z$  as composites of  $\alpha$  and  $\beta$  in  $\mathcal{C}$ . Then  $\eta$  and  $\eta'$  are equivalent.*

*Proof.* The result follows by filling the inner horn  $\Lambda_2^3 \rightarrow \mathcal{C}$  which appears as



$\square$

1em:346

**Lemma 2.10.** *Consider equivalent maps  $\alpha, \alpha' : x \rightarrow y$  and  $\beta, \beta' : y \rightarrow z$  in  $\mathcal{C}$ . Then for a map  $\eta : x \rightarrow z$  the following are equivalent:*

- (a) *There exists a 2-simplex  $s : \Delta^2 \rightarrow \mathcal{C}$  exhibiting  $\eta$  as a composite of  $\alpha$  with  $\beta$ .*
- (b) *There exists a 2-simplex exhibiting  $\eta$  as a composite of  $\alpha'$  with  $\beta$ .*
- (c) *There exists a 2-simplex exhibiting  $\eta$  as a composite of  $\alpha$  with  $\beta'$ .*
- (d) *There exists a 2-simplex exhibiting  $\eta$  as a composite of  $\alpha'$  with  $\beta'$ .*



*Sketch proof.* For (a)  $\Leftrightarrow$  (b), and (b)  $\Rightarrow$  (c), one fills an inner horns  $\Lambda_i^3 \rightarrow \mathcal{C}$  of the form



where  $i = 1$  or  $2$  respectively. By replacing  $\beta$  with  $\beta'$  in the argument for (a)  $\Rightarrow$  (c) one finds (c)  $\Leftrightarrow$  (d).  $\square$

By Lemma 2.10 we can take

$$\text{Hom}_{\mathcal{C}}(x, y) := \{\alpha : \Delta^1 \rightarrow \mathcal{C} : \alpha|_0 = x \text{ and } \alpha|_1 = y\} / \sim$$

and have a well-defined composition operation

$$\circ : \text{Hom}_{\mathcal{C}}(y, z) \circ \text{Hom}_{\mathcal{C}}(x, y) \rightarrow \text{Hom}_{\mathcal{C}}(x, z). \quad (1)$$

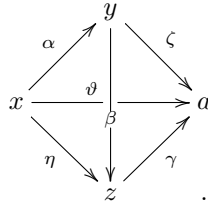
eq:374

This composition explicitly sends a pair of classes  $[\beta] : y \rightarrow z$  and  $[\alpha] : x \rightarrow y$  to the class  $[\beta] \circ [\alpha] = [\eta]$  of any composite of  $\alpha$  with  $\beta$ , i.e. the class of any map  $\eta : x \rightarrow z$  which admits a 2-simplex exhibiting  $\eta$  as a composite of  $\alpha$  with  $\beta$ .

lem:379

**Lemma 2.11.** *The above composition operation (1) is associative.*

*Proof.* Suppose we have a 2-simplex realizing compositions  $\eta = (\beta \circ \alpha)$  and  $\zeta = (\gamma \circ \beta)$ . Suppose we have a 2-simplex exhibiting  $\vartheta : x \rightarrow a$  as a composite of  $\alpha$  with  $\zeta = (\beta \circ \gamma)$ . Fill an the inner horn  $\Lambda_1^3 \rightarrow \mathcal{C}$  of the form



to find a 2-simplex exhibiting  $\vartheta$  as a composite of  $\gamma$  with  $\eta = (\beta \circ \alpha)$ .

$$([\gamma] \circ [\beta]) \circ [\alpha] = [\zeta] \circ [\alpha] = [\vartheta] = [\gamma] \circ [\eta] = [\gamma] \circ ([\beta] \circ [\alpha]).$$

$\square$

Lemmas 2.8–2.11 imply that we have a well-defined category whose objects are the 0-simplices in  $\mathcal{C}$  and whose morphisms are equivalence classes of 2-simplices in  $\mathcal{C}$ .

def:hC

**Definition 2.12.** The homotopy category  $\text{h}\mathcal{C}$  of an  $\infty$ -category  $\mathcal{C}$  is the category  $\text{h}\mathcal{C}$  whose objects are the 0-simplices  $\mathcal{C}[0]$  and whose morphisms  $\text{Hom}_{\text{h}\mathcal{C}}(x, y)$  are equivalence classes of vertices  $\{\alpha \in \mathcal{C}[1] : \alpha|_0 = x \text{ and } \alpha|_1 = y\}$ , under the equivalence relation of Lemma 2.8. Composition is defined via fillings of 2-simplices, as at (1).

**Remark 2.13.** In [8] the object  $\mathbf{h}\mathcal{C}$  denotes two significantly different object. One is a category enriched in the homotopy category of topological spaces, which one might write as  $\tilde{\mathbf{h}}\mathcal{C}$ , the other is the plain category of Definition 2.12. We have  $\mathbf{h}\mathcal{C} = \pi_0(\tilde{\mathbf{h}}\mathcal{C})$ , so that these two categories are explicitly related.

In this text we avoid any reference to the to the topological category  $\tilde{\mathbf{h}}\mathcal{C}$ .

sect:fun

## 2.6. Functor categories.

**Definition 2.14.** For simplicial sets  $K$  and  $S$ , we let  $\mathrm{Fun}(K, S)$  denote the simplicial set with sections defined by maps of simplicial sets

$$\mathrm{Fun}(K, S)[n] := \mathrm{Hom}_{\mathrm{sSet}}(\Delta^n \times K, S).$$

More generally, for a linearly ordered set  $J$  we have section

$$\mathrm{Fun}(K, S)(J) := \mathrm{Hom}_{\mathrm{sSet}}(\Delta^J \times K, S).$$

Any map  $a : J \rightarrow I$  in  $\Delta$  defines a natural transformation of functors  $a_* : \Delta^J \rightarrow \Delta^I$ , and we restrict along such maps to obtain the required structural morphisms

$$a^* : \mathrm{Hom}_{\mathrm{sSet}}(\Delta^J \times K, S) \rightarrow \mathrm{Hom}_{\mathrm{sSet}}(\Delta^I \times K, S), \quad F \mapsto F|_{a_* \times \mathrm{id}_K}.$$

To unravel things, just slightly, one sees that 0-simplices in  $\mathrm{Fun}(K, S)$  are maps of simplicial sets. Any 1-simplex  $F : \Delta^1 \times K \rightarrow S$  restricts to provide two maps

$$f_0 = F|_0 : K \cong K \times \{0\} \rightarrow S, \quad f_1 = F|_1 : K \rightarrow K \times \{1\} \rightarrow S,$$

and we view  $F$  as a homotopy between these two maps. Or rather, we define homotopies in this way, and higher simplices provide higher homotopies between their faces.

The following results are fundamental, and will be proved (much) later.

thm:fun

**Theorem 2.15** ([8, Proposition 1.2.7.3][10, 00TN]). *Let  $K$  be any simplicial set.*

- (i) *If  $\mathcal{C}$  is an  $\infty$ -category, then the functor category  $\mathrm{Fun}(K, \mathcal{C})$  is an  $\infty$ -category.*
- (ii) *If  $\mathcal{X}$  is a Kan complex, then  $\mathrm{Fun}(K, \mathcal{X})$  is a Kan complex.*

These points are quite important, as they will be used to construct (arbitrary) limits and colimits of diagrams of (small) spaces and  $\infty$ -categories. The first point is proved at [wherever] and the second point is proved at Corollary 4.12 below.

**2.7. The (simplicial) categories of categories.** We note that the functor spaces admit natural, associative, composition maps

$$\circ : \mathrm{Fun}(K', K'') \times \mathrm{Fun}(K, K') \rightarrow \mathrm{Fun}(K, K'') \tag{2}$$

eq:comp

which one defines in the obvious way. Namely, if we restrict to  $n$ -simplices we have

$$\circ_n : \mathrm{Fun}(K', K'')[n] \times \mathrm{Fun}(K, K')[n] \rightarrow \mathrm{Fun}(K, K'')[n],$$

$$g_n \circ_n f_n := \Delta^n \times K \xrightarrow{\delta \times \mathrm{id}} \Delta^n \times \Delta^n \times K \xrightarrow{\mathrm{id} \times f_n} \Delta^n \times K' \xrightarrow{g_n} K'',$$

where  $\delta : \Delta^n \rightarrow \Delta^n \times \Delta^n$  is the diagonal map  $x \mapsto (x, x)$ .

The above composition maps, and functor spaces  $\mathrm{Fun}$ , provide the category of small simplicial sets  $\mathrm{sSet}^{\mathrm{sm}}$  with the structure of a simplicial category. When we restrict to the full subcategories of (small) Kan complexes and  $\infty$ -categories, this simplicial enrichment restricts to provide simplicial structures on  $\mathrm{Kan}^{\mathrm{sm}}$  and  $\mathrm{Cat}_{\infty}^{\mathrm{sm}}$ .

**Lemma 2.16.** *The simplicial sets  $\text{Fun}(*, *)$ , and composition (2), provide  $\text{sSet}^{\text{sm}}$ ,  $\text{Cat}_{\infty}^{\text{sm}}$ , and  $\text{Kan}^{\text{sm}}$  with natural simplicial structures.*

**Notation 2.17.** We let

$$\underline{\text{sSet}}^{\text{sm}}, \underline{\text{Cat}}_{\infty}^{\text{sm}}, \text{ and } \underline{\text{Kan}}^{\text{sm}}$$

denote the simplicial categories of small simplicial sets, small  $\infty$ -categories, and small Kan complexes respectively.

These simplicial structures, on  $\text{Cat}_{\infty}^{\text{sm}}$  and  $\text{Kan}^{\text{sm}}$  in particular, provide the basis for our “meta” discussion of the homotopy theory of  $\infty$ -categories and spaces.

sect:basic\_examples

### 3. BASIC EXAMPLES

We should provide some examples of  $\infty$ -categories here. Our main examples come from taking nerves of various enriched categories. In particular, we define the nerve of a plain category, a dg category, and a simplicial category. In all cases our nerve operation extends to a functor

$$\text{N}_{\mathcal{E}} : \{\mathcal{E}\text{-enriched categories}\} \rightarrow \text{Cat}_{\infty},$$

where  $\text{Cat}_{\infty}$  denotes the plain (i.e. not simplicial) category of  $\infty$ -categories with  $\infty$ -functors.

#### 3.1. The nerve of a plain category.

**Definition 3.1.** Let  $\mathbf{C}$  be a plain category. The nerve  $\text{N}(\mathbf{C})$  of  $\mathbf{C}$  is the  $\infty$ -category with  $n$ -simplices  $s : \Delta^n \rightarrow \text{N}(\mathbf{C})$  specified by a choice of objects  $X_0, \dots, X_n$  and an  $\binom{n}{2}$ -tuples of maps

$$s := \{f_{ij} : X_i \rightarrow X_j : i < j, f_{jk}f_{ij} = f_{ik} \text{ whenever } i < j < k\}.$$

Equivalently,  $n$ -simplices are functors  $s : [n] \rightarrow \mathbf{C}$ . Restriction  $r^* : \text{N}(\mathbf{C})[n] \rightarrow \text{N}(\mathbf{C})[m]$  along maps  $r : [m] \rightarrow [n]$  are given by restricting functors.

So the nerve  $\text{N}(\mathbf{C})$  is a certain simplicial set which one assigns to a plain category. Now, an inner horn  $\Lambda_i^n \rightarrow \text{N}(\mathbf{C})$ , for  $n > 2$ , is specified by a collection of maps  $f_{ab} : X_a \rightarrow X_b$  for all  $a < b$  with  $f_{bc}f_{ab} = f_{ac}$  for all triples  $a < b < c$ . So, such a horn extends *uniquely* to a simplex  $\Delta^n \rightarrow \text{N}(\mathbf{C})$ . At  $n = 2$  an inner horn  $\Lambda_1^2 \rightarrow \text{N}(\mathbf{C})$  is a choice of two maps  $f_{01} : X_0 \rightarrow X_1$  and  $f_{12} : X_1 \rightarrow X_2$ , which again extends uniquely to a 2-simplex  $\Delta^2 \rightarrow \text{N}(\mathbf{C})$ . So the nerve of a plain category is an  $\infty$ -category.

In addition, for any functor  $F : \mathbf{C} \rightarrow \mathbf{D}$  we obtain a map between  $\infty$ -categories  $\text{N}F : \text{N}(\mathbf{C}) \rightarrow \text{N}(\mathbf{D})$ , which is simply defined by composing functors  $[n] \rightarrow \mathbf{C}$  with  $F$ . These assignments

$$\text{N} : \mathbf{C} \mapsto \text{N}(\mathbf{C}), \quad F \mapsto \text{N}(F)$$

define a functors from the category of categories to the category of  $\infty$ -categories.

**Proposition 3.2.** *The nerve of any plain category is an  $\infty$ -category. Furthermore, any map between  $\infty$ -categories  $f : \text{N}(\mathbf{C}) \rightarrow \text{N}(\mathbf{D})$  is of the form  $f = \text{N}(F)$  for a unique functor  $F : \mathbf{C} \rightarrow \mathbf{D}$ . This is to say, the nerve operation defines a fully faithful embedding  $\text{N} : \text{Cat} \rightarrow \text{Cat}_{\infty}$ .*

To see that the nerve of a generic category  $\mathbf{C}$  is not a weak Kan complex, a stronger notion of course, one need only consider the category  $[n]$ . Here we have  $N([n]) = \Delta^n$ , and we have already seen in Example 2.6 that  $\Delta^n$  is an  $\infty$ -category but not a weak Kan complex whenever  $n > 0$ .

We note that for any  $\infty$ -category  $\mathcal{C}$  we have an obvious map of simplicial sets

$$p : \mathcal{C} \rightarrow N(\mathbf{h}\mathcal{C}), \quad (s : \Delta^n \rightarrow \mathcal{C}) \mapsto ([s|_{\Delta^{\{i,j\}}}] : 0 \leq i < j \leq n),$$

which is then by definition an  $\infty$ -functor between  $\infty$ -categories. One sees immediately that  $p$  is an isomorphism whenever  $\mathcal{C} = N(\mathbf{C})$  for some plain category  $\mathbf{C}$ , and we thus find that  $p$  is an isomorphism of  $\infty$ -categories exactly when  $\mathcal{C}$  is isomorphic to the nerve of some plain category.

The functor  $p$  is natural in  $\mathcal{C}$ , and so provides a natural endomorphism on the category of  $\infty$ -categories. Indeed,  $p$  provides the unit for an adjunction between the homotopy category functor  $\mathbf{h} : \mathbf{Cat}_\infty \rightarrow \mathbf{Cat}$  and the nerve functor.

lem:p\_C

**Lemma 3.3.** *For any  $\infty$ -category we have a natural projection  $p : \mathcal{C} \rightarrow N(\mathbf{h}\mathcal{C})$ , which is defined by taking an  $n$ -simplex  $s$  in  $\mathcal{C}$  to the corresponding tuple of morphisms  $([s|_{\Delta^{\{i,j\}}}] : 0 \leq i < j \leq n)$  in the homotopy category.*

**3.2. Nerves of dg categories.** Let  $\mathcal{A}$  be a dg category. We define the dg nerve  $N^{\mathrm{dg}}(\mathcal{A})$  to be the simplicial set with each  $n$ -simplex  $\Delta^n \rightarrow N^{\mathrm{dg}}(\mathcal{A})$  specified by a choice of objects  $\{x_0, \dots, x_n\}$  in  $\mathcal{A}$  and maps

$$f_I \in \mathrm{Hom}_{\mathcal{A}}^{-|I|+2}(x_{\min I}, x_{\max I})$$

for all subsets  $I \subset [n]$  of order  $|I| \geq 2$  which satisfy

$$d(f_I) = \sum_{t \in I - \{\min I, \max I\}} (-1)^{\mathrm{pos}(t)} (f_{I_{\geq t}} \circ f_{I_{\leq t}} - f_{I - \{t\}}). \quad (3) \quad \text{eq:419}$$

Here  $I$  inherits its ordering from  $[n]$ , so that  $I_{\geq t} = \{a \in I : a \geq t\}$  for example, and  $\mathrm{pos}(t) = |I_{>t}|$ . Note that  $\mathrm{pos}(t)$  marks the position of  $t$  relative to the *terminal* element  $\max I$ , not the initial element.<sup>1</sup>

For any weakly increasing map  $r : [m] \rightarrow [n]$  the restriction

$$r^* : N^{\mathrm{dg}}(\mathcal{A})[n] \rightarrow N^{\mathrm{dg}}(\mathcal{A})[m], \quad \{f_I : I \subset [n]\} \mapsto \{f_{r,J} : J \subset [m]\}.$$

is defined by taking  $f_{r,J} = f_{r(J)}$  if  $r|_J$  is injective,  $f_{r,J} = \mathrm{id}_{x_{r(j)}}$  if  $J = \{j, j'\}$  and  $r(j) = r(j')$ , and  $f_{r,J} = 0$  otherwise. For example, pulling back  $s^* : N^{\mathrm{dg}}(\mathcal{A})[1] \rightarrow N^{\mathrm{dg}}(\mathcal{A})[2]$  along the weakly increasing surjection  $s : [2] \rightarrow [1]$  with  $s(0) = s(1) = 0$  can be illustrated as

$$\begin{array}{ccc} & & \xrightarrow{s^*} \\ & & \begin{array}{ccc} & x & \\ \mathrm{id} \nearrow & & \searrow f \\ x & 0 & y \\ & \xrightarrow{f} & \end{array} \\ x & \xrightarrow{f} & y \end{array}$$

**Lemma 3.4.** *The tuple  $\{f_{r,J} : J \subset [m]\}$  is in fact an  $m$ -simplex in  $N^{\mathrm{dg}}(\mathcal{A})$ , and for a sequence of maps  $r_2 \circ r_1 : [l] \rightarrow [m] \rightarrow [n]$  the composite  $r_1^* \circ r_2^*$  equals  $(r_2 \circ r_1)^*$ . This is to say,  $N^{\mathrm{dg}}(\mathcal{A})$  is in fact a simplicial set.*

<sup>1</sup>Our particular expression of the dg nerve comes from [10], and not [8]. There is a slight difference in signs between the two resources.

*Proof.* The second point follows from the fact that the composite  $r_2 r_1$  has injective restriction to  $K \subset [l]$  if and only if  $r_1$  has injective restriction to  $K$  and  $r_2$  has injective restriction to  $r_1(K)$ .

For the first claim, fix  $r : [m] \rightarrow [n]$  and take for any  $J \subset [m]$ ,  $f_J = f_{r,J}$ . We need only establish an equality

$$0 = \sum_{t \in J - \{\min J, \max J\}} (-1)^{\text{pos}(t)} (f_{J_{\geq t}} \circ f_{J_{\leq t}} - f_{J - \{t\}}). \quad (4) \quad \boxed{\text{eq:440}}$$

when  $r|_J$  has repeated indices. When  $r|_J$  has more than two repeated indices, then all summands in the above expression are 0, so that the equality holds immediately. So we may suppose that  $r|_J$  has precisely two repeated indices  $r(j) = r(j')$ , with  $j$  and  $j'$  necessarily neighbors. If  $|J| = 2$  then the right above summand is empty, and thus necessarily  $0 = d(\text{id})$ . So we suppose additionally that  $|J| > 2$ . We therefore want to establish (4) when  $|J| > 2$  and  $r|_J$  has precisely two repeated indices  $r(j) = r(j')$ .

Suppose also that  $j = \min J$  or  $j' = \max J$ . Then the above sum has only one non-vanishing summand,

$$(-1)^{\text{pos}(j)} (f_{J_{\leq j}} \circ \text{id} - f_{J - \{j\}}) = (-1)^{\text{pos}(j)} (f_{J_{\leq j}} - f_{J_{\leq j}}) = 0$$

or

$$(-1)^{\text{pos}(j')} (\text{id} \circ f_{J_{\geq j'}} - f_{J - \{j'\}}) = (-1)^{\text{pos}(j)} (f_{J_{\geq j'}} - f_{J_{\geq j'}}) = 0$$

So the desired equation holds. When both  $j$  and  $j'$  lie in the interior of  $J$ , this sum has precisely two summands and we have

$$\pm (f_{J_{\geq j}} \circ 0 - f_{J - \{j\}} - 0 \circ f_{J_{\leq j'}} + f_{J - \{j'\}}) = 0.$$

We therefore establish (4), and see that  $r^*$  does map  $n$ -simplices to  $m$ -simplices.  $\square$

We note that there is a functor  $\text{dg} : \mathbb{Z}_{\geq 0} \rightarrow \text{dgCat}$  so that  $n$ -simplices in  $\text{N}^{\text{dg}}(\mathcal{A})$  are identified with dg functors  $\text{Hom}_{\text{dgCat}}(\text{dg}[n], \mathcal{A})$ . So the dg nerve can be constructed in a manner similar to the nerve of a plain category. Furthermore, if we view any additive category  $\mathcal{C}$  as a dg category over  $\mathbb{Z}$ , with all maps in degree 0, then  $\text{N}(\mathcal{C}) = \text{N}^{\text{dg}}(\mathcal{C})$ .

Now, suppose we have an inner horn  $a : \Lambda_i^n \rightarrow \text{N}^{\text{dg}}(\mathcal{A})$ . Such a map of simplicial sets is specified by a tuple of morphism  $(f_I : I \subset [n], [n] - \{i\} \not\subset I)$  which solve the equations (3), and an extension of such  $a$  to an  $n$ -simplex is an additional choice of degree  $n-3$  and  $n-2$  maps  $f_{I - \{i\}}$  and  $f_{[n]}$  respectively which have the appropriate derivatives. If we assume  $f_{[n]} = 0$  then the equation

$$0 = \sum_{0 < t < n} (-1)^{n-t} (f_{\{t, \dots, n\}} \circ f_{[t]} - f_{[n] - \{t\}})$$

is satisfied precisely when

$$f_{[n] - \{i\}} = f_{\{i, \dots, n\}} \circ f_{[i]} + \sum_{0 < t < n, t \neq i} (-1)^{-t+i} (f_{\{t, \dots, n\}} \circ f_{[t]} - f_{[n] - \{t\}}). \quad (5) \quad \boxed{\text{eq:469}}$$

**Proposition 3.5** ([10, 00PW]). *For any inner horn  $\Lambda^n \rightarrow \text{N}^{\text{dg}}(\mathcal{A})$  admits an extension to an  $n$ -simplex specified by taking  $f_{[n]} = 0$  and  $f_{[n] - \{i\}}$  to be given by (5). In particular, the dg nerve of any dg category is an  $\infty$ -category.*

*Proof.* We take the differential to find

$$\begin{aligned}
d(f_{[n]-\{i\}}) &= (\sum_{t < i} (-1)^{n-t} t_{[n]-\{t,i\}}) + (\sum_{i < t} (-1)^{n-(t+1)} f_{[n]-\{i,t\}}) \\
&\quad - (\sum_{t < i} (-1)^{n-t} f_{[t,n]-\{i\}} \circ f_{[t]}) - (\sum_{i < t} (-1)^{n-t-1} f_{[t,n]} \circ f_{[t]-\{i\}}) \\
&\quad + \text{other terms} \\
&= \sum_{t \in I - \{\min I, \max I\}} (-1)^{|t|} (f_{I \geq t} \circ f_{I \leq t} - f_{I - \{t\}}) \\
&\quad + \text{other terms.}
\end{aligned}$$

In the above expression  $I = [n] - \{i\}$  and  $[t, n] = \{t, \dots, n\}$ , and the summands which we've written explicitly come from differentiating the term

$$\sum_{0 < t < n, t \neq i} (-1)^{-t+i} (f_{\{t, \dots, n\}} \circ f_{[t]} - f_{[n]-\{t\}})$$

in (5). So, we want to see that these “other terms” vanish.

We have

$$\begin{aligned}
&(-1)^{n-i} \text{other terms} = \\
&(-1)^{n-i} d(f_{[i,n]}) f_{[i]} - f_{[i,n]} d(f_{[i]}) \\
&+ \sum_{0 \leq l < m < n, l, m \neq i} (-1)^{l+m} f_{[n]-\{l,m\}} + (-1)^{m+l-1} f_{[n]-\{l,m\}} \\
&- \sum_{0 \leq l < m < n, l, m \neq i} (-1)^{l+m} f_{[n,l]-\{m\}} f_{[l]} + (-1)^{m+l-1} f_{[n,l]-\{m\}} f_{[l]} \\
&- \sum_{0 \leq l < m < n, l, m \neq i} (-1)^{m-l+1} f_{[n,m]} f_{[m]-\{l\}} + (-1)^{-l-m} f_{[n,m]} f_{[m]-\{l\}} \\
&+ \sum_{t < i} (-1)^{i+t} f_{[n,i]} f_{[t,i]} f_{[t]} - (-1)^{i+t} f_{[n,i]} f_{[i]-\{t\}} \\
&+ \sum_{i < t} (-1)^{t-i-1} f_{[n,t]} f_{[i,t]} f_{[i]} - (-1)^{t+i+1} f_{[n,i]-\{t\}} f_{[i]}. \\
&= -f_{[i,n]} d(f_{[i]}) + \sum_{t < i} (-1)^{i+t} f_{[n,i]} f_{[t,i]} f_{[t]} - (-1)^{i+t} f_{[n,i]} f_{[i]-\{t\}} \\
&(-1)^{n-i} d(f_{[i,n]}) f_{[i]} + \sum_{i < t} (-1)^{t-i-1} f_{[n,t]} f_{[i,t]} f_{[i]} - (-1)^{t+i+1} f_{[n,i]-\{t\}} f_{[i]} \\
&= 0.
\end{aligned}$$

So we see that the other terms vanish, and hence that any given inner horn  $\Lambda_i^n \rightarrow \mathcal{N}^{\text{dg}}(\mathcal{A})$  admits an extension to an  $n$ -simplex.  $\square$

From a direct consideration of the equation (3) it is clear that any dg functor  $\eta : \mathcal{A} \rightarrow \mathcal{B}$  defines an associated map of  $\infty$ -categories

$$\mathcal{N}^{\text{dg}}(\eta) : \mathcal{N}^{\text{dg}}(\mathcal{A}) \rightarrow \mathcal{N}^{\text{dg}}(\mathcal{B}), \quad \begin{cases} x \mapsto F(x) & \text{for 0-simplices} \\ (f_I : I \subset [n]) \mapsto (\eta(f_I) : I \subset [n]) & \text{for } n\text{-simplices, } n > 0. \end{cases}$$

**Proposition 3.6.** *The dg nerve operation defines a functor  $\text{dgCat} \rightarrow \text{Cat}_\infty$ .*

rem:dg\_v\_infty

**Remark 3.7.** The dg nerve operation defines an equivalence from a homotopy category of dg categories, over a given base  $k$ , to the homotopy category of (certain) linear  $\infty$ -categories [2, Corollary 5.5]. In this sense, dg categories are identified with linear  $\infty$ -categories via the dg nerve functor, in some appropriate sense.

**Remark 3.8.** In defining the dg nerve, there are some seemingly ambiguous choices in the precise definition of the dg nerve functor. One sees, for example, in [9, Construction 1.3.1.13] an alternate construction of a “dg nerve” via Dold-Kan, which turns out to be equivalent [9, Proposition 1.3.1.17]. A construction which

extends immediately to *A<sub>i</sub>nfty*-categories is given in work on Faonte [4]. Faonte's construction can be identified with the one given above, up to some signs.

In the end, however, Antieau explains that in most reasonable cases coming from algebra, there is a unique  $\infty$ -category  $\mathcal{D}$  whose homotopy category  $h\mathcal{D}$  is identified with the derived category of a given Grothendieck abelian category [1, Corollaries 2, 5]. So for most cases of interest to representation theorists, the  $\infty$ -category  $N^{\text{dg}}(\mathcal{A})$  is uniquely determined, up to equivalence of  $\infty$ -categories, by its homotopy category  $hN^{\text{dg}}(\mathcal{A}) = K(\mathcal{A})$ . (See our discussion of the derived category below.)

sect:dg\_alg\_ex

**3.3. An integral dg example with one object.** Let  $p$  be any positive integer and consider the dg category  $A_p$  over  $\mathbb{Z}$  with a single object  $*$  and endomorphisms

$$\text{Hom}_{A_p}(*, *) = 0 \rightarrow p\mathbb{Z} \rightarrow \mathbb{Z} \rightarrow 0,$$

where the differential  $p\mathbb{Z} \rightarrow \mathbb{Z}$  is just the inclusion. So,  $A_p$  is just a dg algebra which we view as a dg category. We also have the dg algebra  $\mathbb{Z}/p\mathbb{Z}$  with a single object and endomorphisms  $\mathbb{Z}/p\mathbb{Z}$ . Let  $\mathcal{A}_p$  and  $\mathcal{Z}_p$  be the corresponding dg nerves of these dg categories. We describe these  $\infty$ -categories explicitly, and show that the dg algebra quasi-isomorphism  $A_p \rightarrow \mathbb{Z}/p\mathbb{Z}$  admits an section  $\mathcal{Z}_p \rightarrow \mathcal{A}_p$  at the level of  $\infty$ -categories, despite the fact that there are no  $A_\infty$ -algebra maps  $\mathbb{Z}/p\mathbb{Z} \rightarrow A_p$  at all.

An  $n$ -simplex in  $a : \Delta^n \rightarrow \mathcal{Z}_p$  is a tuple of numbers  $a = \{a_{ij} : i < j \in [n]\}$  with  $a_{ij}a_{jk} = a_{ik}$  for all triples  $i < j < k$ . (This is the same as the usual nerve of  $\mathbb{Z}/p\mathbb{Z}$  as a plain category.) A 1-simplex in  $\mathcal{A}_p$  is a choice of element  $b \in \mathbb{Z} \subset A_p$  and a 2-simplex  $b : \Delta^2 \rightarrow \mathcal{A}_p$  is a choice of a triple of elements  $\{b_{01}, b_{12}, b_{02}\} \subset \mathbb{Z}$  and  $b_{123} \in p\mathbb{Z}$  such that

$$b_{123} = b_{02} - b_{01}b_{12}. \quad (6)$$

eq:560

In particular,  $b_{123}$  is specified uniquely by the boundary elements  $b_{ij}$ , so that a 2-simplex is simply a triple of elements satisfying  $b_{01}b_{12} = b_{03} \pmod{p}$ .

For degree reasons, if we have an  $n$ -simplex  $b : \Delta^n \rightarrow \mathcal{A}_p$  all functions  $b_I \in A_p^{|I|-2}$  indexed by subsets  $I \subset [n]$  of size  $> 3$  vanish, and the equations

$$d(b_I) = \sum \pm (b_{I_{\geq t}} b_{I_{\leq t}} - b_{I - \{t\}}) \quad (7)$$

eq:570

are vacuous when  $|I| > 4$ , as both sides live in  $A_p^{|I|-3} = 0$ . We have already examined the above equation when  $|I| = 3$  and  $|I| = 2$  generically just says that all  $b_I$  with  $|I| = 2$  are degree 0 cocycles. So we need only investigate the case  $|I| = 4$ . That is, we need only investigate the 3-simplices in  $\mathcal{A}_p$  in order to understand its entire structure.

Consider a 3-simplex  $b : \Delta^3 \rightarrow \mathcal{A}_p$ . Such an object is specified by a collection of elements  $\{b_I : I \subset [3]\}$  with all  $b_{ijk} = b_{ik} - b_{ij}b_{jk}$  and  $b_{0123} = 0$  for degree reasons. This first condition is equivalent to the equation (7) for  $I$  of size 3, and for the unique subset of size 4 we need to check the equation

$$0 = d(b_{0123}) = b_{023} - b_{01}b_{123} - b_{013} + b_{012}b_{23}.$$

But we simply expand, and (only) employ the 2-simplex equation (6), to find

$$\begin{aligned} & b_{023} - b_{01}b_{123} - b_{013} + b_{012}b_{23} \\ &= b_{03} - b_{02}b_{23} - b_{01}(b_{13} - b_{12}b_{23}) - b_{03} + b_{01}b_{13} + (b_{02} - b_{01}b_{12})b_{23} \\ &= b_{03} - b_{02}b_{23} - b_{01}b_{13} + b_{01}b_{12}b_{23} - b_{03} + b_{01}b_{13} + b_{02}b_{23} - b_{01}b_{12}b_{23} \\ &= 0. \end{aligned}$$

So we see that equation (7) at  $|I| = 4$  is redundant, and thus observe a complete description of the  $n$ -simplices in  $\mathcal{A}_p$ ,

$$\mathcal{A}_p[n] = \{\text{tuples } (b_{ij} : 0 \leq i < j \leq n) : b_{ij}b_{jk} = b_{ik} \pmod{p}\}$$

The map of  $\infty$ -categories  $\pi : \mathcal{A}_p \rightarrow \mathcal{Z}_p$  implied by the dg algebra quasi-isomorphism  $A_p \rightarrow \mathbb{Z}/p\mathbb{Z}$  is defined as expected

$$\pi : \mathcal{A}_p \rightarrow \mathcal{Z}_p, \quad (b_{ij} : 0 \leq i < j \leq n) \mapsto (\bar{b}_{ij} : 0 \leq i < j \leq n).$$

For any class  $a \in \mathbb{Z}/p\mathbb{Z}$  define  $a' \in \mathbb{Z}$  to be the unique element in  $\{0, \dots, p-1\} \subset \mathbb{Z}$  such that  $\bar{a}' = a$ . We define a section  $\pi^\vee : \mathcal{Z}_p \rightarrow \mathcal{A}_p$  on  $n$ -simplices

$$\pi_n^\vee : \mathcal{Z}_p[n] \rightarrow \mathcal{A}_p[n], \quad (a_{ij} : 0 \leq i < j \leq n) \mapsto (a'_{ij} : 0 \leq i < j \leq n).$$

One sees immediately that the composite

$$\mathcal{Z}_p \xrightarrow{\pi^\vee} \mathcal{A}_p \xrightarrow{\pi} \mathcal{Z}_p$$

is seen to be the identity. This point is remarkable, given that there aren't even any maps  $\mathbb{Z}/p\mathbb{Z} \rightarrow A_p$  of  $\mathbb{Z}$ -modules, and highlights the content of the discussion of Remark 3.7.

We would claim furthermore that  $\pi : \mathcal{A}_p \rightarrow \mathcal{Z}_p$  is an *equivalence* of  $\infty$ -categories with weak inverse  $\pi^\vee$ , whatever that means. This point is discussed in greater detail in Section 5.

sect:dg\_alg\_ex2

**3.4. More examples with one object.** We can similarly take a positive degree polynomial  $q \in S = \mathbb{C}[t]$  and consider the dg algebras  $A_q = 0 \rightarrow qS \rightarrow S \rightarrow 0$  and  $S/qS$ . We have the associated  $\infty$ -categories  $\mathcal{A}_q$  and  $\mathcal{S}_q$ , and the above presentation applies verbatim to provide a complete description of the simplicial set  $\mathcal{A}_q$ ,

$$\mathcal{A}_q[n] = \{(b_{ij} : 0 \leq i < j \leq n) : b_{ij}b_{jk} = b_{kl} \pmod{q}\}.$$

We have the projection  $\pi : \mathcal{A}_q \rightarrow \mathcal{S}_q$  implied by the dg algebra quasi-isomorphism  $A_q \rightarrow S/qS$  and easily constructs an “inverse”

$$\pi^\vee : \mathcal{S}_q \rightarrow \mathcal{A}_q, \quad (a_{ij} : 0 \leq i < j \leq n) \mapsto (a'_{ij} : 0 \leq i < j \leq n)$$

by taking  $a'_{ij}$  to be the unique lift of  $a_{ij} \in S/qS$  to a degree  $< \deg(q)$  element in  $S$ .

Now that we're over  $\mathbb{C}$  however, there *does* exist an  $A_\infty$  weak inverse  $\iota : S/qS \rightarrow A_q$  and one can compare the construction of  $\iota$  requires a choice of  $\mathbb{C}$ -linear section  $\iota_0 : S/qS \rightarrow A_q$  and a corresponding degree  $-1$  solution  $\iota_1 : S/qS \otimes_{\mathbb{C}} S/qS \rightarrow A_q$  to a quadratic equation  $d(X) = \text{poly}(\iota_0)$ . This quadratic equation is essentially just the equation (6), which we understand has unique solutions. So we see that in the  $A_\infty$ -categorical setting we must keep track certain irrelevant information which the  $\infty$ -categorical allows us to ignore. We also note that the  $\infty$ -category map  $\pi^\vee$  can be defined by any choice theoretic section  $\pi_1^\vee : S/qS = \mathcal{S}_q[1] \rightarrow A_q = \mathcal{A}_q[1]$  and so disregards linearity as well.

As a last (somewhat silly) example we consider the Chevalley-Eilenberg dg algebra

$$\text{CE}(\mathfrak{g}) = 0 \rightarrow \mathbb{C} \xrightarrow{0} \mathfrak{g}^* \xrightarrow{d} \mathfrak{g}^* \wedge \mathfrak{g}^* \rightarrow \dots \rightarrow \det(\mathfrak{g}^*) \rightarrow 0.$$

Here  $\mathfrak{g}$  is a Lie algebra and the differential is specified by the dual of the bracket  $d^1 = [-, -]^* : \mathfrak{g}^* \rightarrow \mathfrak{g}^* \wedge \mathfrak{g}^*$ . The cohomology of this dg algebra is the algebra of extensions of the trivial representation

$$H^*(\text{CE}(\mathfrak{g})) = \text{Ext}_{U(\mathfrak{g})}(\mathbb{C}, \mathbb{C}),$$



calculated in the category of arbitrary  $U(\mathfrak{g})$ -modules. We consider the associated  $\infty$ -category  $\mathcal{CE}(\mathfrak{g})$ .

Here there are no non-zero negative degree elements in  $\mathbf{CE}(\mathfrak{g})$  to speak of, so that the unit map  $\mathbb{C} \rightarrow \mathbf{CE}(\mathfrak{g})$  induces an isomorphism of  $\infty$ -categories

$$N(\mathbb{C}) \xrightarrow{\cong} \mathcal{CE}(\mathfrak{g}).$$

This is despite the fact that the unit map is far from a quasi-isomorphism in general.

This example simply abuses the fact that the dg nerve functor cannot see information in a dg category  $\mathcal{A}$  which is strictly contained in positive cohomological degrees. We also note that this kind of Tom-foolery will not occur if we restrict our attention to pre-triangulated dg categories [2, Definition 2.16], which provide the fibrant objects for the model structure employed in [2] anyway (cf. Remark 3.7).

**3.5. Derived categories of dg modules.** We discuss the construction of the derived  $\infty$ -category for a dg algebra. This is really just a small taste of a topic which lies at the center of this text.

Let  $A$  be a dg algebra and consider the dg category  $A\text{-dgmod}$  or arbitrary dg modules, i.e. dg modules with no finiteness assumptions. So, Hom-complexes are given by

$$\mathrm{Hom}_A^*(M, N) = \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_A^i(M, N), \quad d_{\mathrm{Hom}}(f) = d_N f - (-1)^{|f|} f d_M.$$

Here  $\mathrm{Hom}_A^i(M, N)$  denotes the collection of degree  $i$  maps of graded  $A$ -modules  $f : M \rightarrow N$ . We can form the dg nerve  $N_{dg}(A\text{-dgmod})$  in which

- $N^{\mathrm{dg}}(A\text{-dgmod})_0 = \{\text{the collection of dg-modules}\}.$
- $N^{\mathrm{dg}}(A\text{-dgmod})_1 = \{\text{maps } f : M \rightarrow N \text{ of dg-modules}\}.$
- $N^{\mathrm{dg}}(A\text{-dgmod})_2 = \left\{ \begin{array}{l} \text{quadruples } f : L \rightarrow M, g : M \rightarrow N, h : L \rightarrow N, z : L \rightarrow N \\ \text{such that } f, g, \text{ and } h \text{ are degree 0 cocycles, i.e. dg module maps} \\ \text{and } h = fg - d(z). \end{array} \right\}.$

From this description of 0, 1, and 2-simplices one observes a calculation of the homotopy category.

**Proposition 3.9.** *The homotopy category of the dg nerve for  $A\text{-dgmod}$  is the usual homotopy category of dg modules for  $A$ , i.e. the category of dg modules with homotopy classes of maps*

$$h N^{\mathrm{dg}}(A\text{-dgmod}) = K(A\text{-dgmod}).$$

We refer to the dg nerve  $N^{\mathrm{dg}}(A\text{-dgmod})$  the homotopy  $\infty$ -category for  $A$ ,

$$\mathcal{K}(A) := N^{\mathrm{dg}}(A\text{-dgmod})$$

**Definition 3.10.** A dg module  $M$  over  $A$  is called K-projective (resp. K-injective) if the mapping complex functor  $\mathrm{Hom}_A^*(M, -)$  (resp.  $\mathrm{Hom}_A^*(-, M)$ ) preserves quasi-isomorphisms.

We recall that any dg module  $M$  admits a quasi-isomorphism  $q : M' \rightarrow M$  from a K-projective dg module  $M'$ . See for example [7, Theorem 3.1] or [3, Lemma 13.3].

We let  $K\text{-proj} \subset A\text{-dgmod}$  denote the full subcategory of  $K$ -projective dg modules,  $\text{Acyc} \subset A\text{-dgmod}$  denote the thick subcategory of acyclic objects, and recall that the sequence

$$K(K\text{-proj}) \hookrightarrow K(A\text{-dgmod}) \rightarrow K(A\text{-dgmod})/\text{Acyc} = D(A)$$

provides an equivalence onto the unbounded derived category  $K(K\text{-proj}) \simeq D(A)$ .

**Definition 3.11.** For any dg algebra  $A$ , the derived  $\infty$ -category is the dg nerve

$$\mathcal{D}(A) := N^{\text{dg}}(K\text{-proj}).$$

When  $A$  satisfies the appropriate bounding conditions we can reasonably define the bounded above, bounded below, and totally bounded derived  $\infty$ -categories

$$\mathcal{D}^-(A), \mathcal{D}^+(A), \mathcal{D}^b(A)$$

as the full  $\infty$ -subcategories in  $\mathcal{D}(A)$  consisting of  $K$ -projective complexes with bounded (above, below, etc.) cohomology.

Of course, it is more reasonably at times to employ an injective construction of the derived category. We could define instead

$$\mathcal{D}(A) = N^{\text{dg}}(K\text{-inj}).$$

While we won't verify this at the moment, these two constructions of the derived category are in fact equivalent. Indeed, they are both  $\infty$ -categorical localizations of  $\mathcal{K}(A)$  at the class of quasi-isomorphisms [9, Theorems 1.3.4.20, 1.3.5.13].

We will discuss derived categories for Grothendieck abelian categories, which is the appropriate setting for sheaves on schemes and stacks, in later parts of the text.

**3.6. Nerves of simplicial categories.** First we construct a simplicial category for the simplices  $\Delta^n$  [8, Definition 1.1.5.1]. We take  $\text{Path } \Delta^n$  to be the simplicial category with objects  $\text{obj}(\text{Path } \Delta^n) = [n]$  and  $m$ -simplices

$$\underline{\text{Hom}}_{\Delta^n}(a, b)[m] = \underline{\text{Hom}}_{\text{Path } \Delta^n}(a, b)[m] = \left\{ \begin{array}{l} \text{length } m+1 \text{ sequences of subsets} \\ I_0 \subseteq \cdots \subseteq I_m \subseteq [n] \\ \text{with } a = \min I_j \text{ and } b = \max I_j \\ \text{for all } 0 \leq j \leq m \end{array} \right\}.$$

We note that each inclusion  $I_j \subset I_{j+1}$  may be an equality in the above presentation, that these simplicial Hom sets vanishes if and only if  $a > b$ , and that all simplices of size  $> n$  are degenerate. For any weakly increasing  $r : [l] \rightarrow [m]$  the corresponding structure map is as expected,

$$r^* : \underline{\text{Hom}}_{\Delta^n}(a, b)[m] \rightarrow \underline{\text{Hom}}_{\Delta^n}(a, b)[l], \quad \{I_0 \cdots \subseteq I_m\} \mapsto \{I_{r(0)} \cdots \subseteq I_{r(l)}\}$$

and composition is given by taking unions

$$\{I'_0 \cdots \subseteq I'_m\} \circ \{I_0 \cdots \subseteq I_m\} = \{(I_0 \cup I'_0) \cdots \subseteq (I_m \cup I'_m)\}.$$

One similarly defines the simplicial category  $\text{Path } \Delta^J$  for any linearly ordered set  $J$ .

For any map  $f : J_1 \rightarrow J_2$  in  $\Delta$  we obtain a simplicial functor  $f_* : \text{Path } \Delta^{J_1} \rightarrow \text{Path } \Delta^{J_2}$  which is defined as  $f$  on objects, and on morphisms

$$f(\{I_0 \cdots \subseteq I_m\}) := \{f(I_0) \cdots \subseteq f(I_m)\}.$$

So this path operation defines a functor  $\text{Path} : \Delta \rightarrow \text{sCat}$ .

**Definition 3.12.** For a simplicial category  $\mathcal{S}$  we define the homotopy coherent nerve  $N^{\text{hc}}(\mathcal{S})$  to be the simplicial set with simplices

$$N^{\text{hc}}(\mathcal{S})[n] = \text{Fun}_{\text{sCat}}(\text{Path } \Delta^n, \mathcal{S})$$

and restriction maps  $f^* := \text{Fun}_{\text{sCat}}(f_*, \mathcal{S})$ , for each weakly increasing map  $f : [m] \rightarrow [n]$ .

prop:11510

**Proposition 3.13** ([8, Proposition 1.1.5.10]). *Suppose that a simplicial category  $\mathcal{S}$  has the property that, for each pair of objects in  $\mathcal{S}$ , the simplicial set  $\underline{\text{Hom}}_{\mathcal{S}}(x, y)$  is a Kan complex. Then the homotopy coherent nerve  $N^{\text{hc}}(\mathcal{S})$  is an  $\infty$ -category.*

The proof requires an analysis of lifting properties for maps into Kan complexes. We will cover this later. However, let us list a corollary. If we accept both Theorem 2.15 and Proposition 3.13 then we find

**Corollary 3.14.** *The homotopy coherent nerves*

$$N^{\text{hc}}(\underline{\text{Kan}}^{\text{sm}}) \text{ and } N^{\text{hc}}(\underline{\text{Cat}}_{\infty}^{\text{sm}})$$

*are  $\infty$ -category.*

It is this homotopy coherent nerve which we refer to as the category of (small) spaces

$$\mathcal{S}paces^{\text{sm}} := N^{\text{hc}}(\underline{\text{Kan}}^{\text{sm}}).$$

We return to consider this category in Section 4. We also have the  $\infty$ -category of small  $\infty$ -categories

$$\mathcal{C}at_{\infty}^{\text{sm}} := N^{\text{hc}}(\underline{\text{Cat}}_{\infty}^{\text{sm}}).$$

### 3.7. Low dimensional simplices in the homotopy coherent nerve.

## 4. THE CATEGORY OF SPACES

sect:spaces

In order to speak clearly of  $\infty$ -categories, we need to understand spaces of Homs for infinity categories. From a baseline perspective, one might say that plain categories are enhanced in sets, while  $\infty$ -categories are enhanced with “spaces”. So, we replace sets with spaces in our mind, and we will be on our way. But, it might be more instructive to compare with dg categories.

A dg category is, obviously, a category enriched in complexes of vector spaces. So, if one understands complexes, we see that for objects  $x$  and  $y$  in a dg category  $\mathcal{A}$ , one thinks of the complex of morphisms  $\text{Hom}_{\mathcal{A}}(x, y)$  as a somewhat dynamic object. One not only has  $\text{Hom}_{\mathcal{A}}(x, y)$ , but one has the cohomology  $H^*(\text{Hom}_{\mathcal{A}}(x, y))$ , the cocycles  $Z^*(\text{Hom}_{\mathcal{A}}(x, y))$ , degree 0 cocycles  $Z^0(\text{Hom}_{\mathcal{A}}(x, y))$  which we view as the underlying “morphisms”, boundaries which we view as homotopies, etc. So we want some fluency with the, rather dynamic, category of cochain complexes which we then use as a foundation to speak of dg categories.

The category of “spaces”, i.e. Kan complexes, serves a similar purpose for  $\infty$ -categories. For objects  $x$  and  $y$  in an  $\infty$ -category  $\mathcal{C}$ , we will have a mapping space  $\text{Map}_{\mathcal{C}}(x, y)$ . This space has homotopy groups, a homotopy class, components, etc. The homotopy groups in particular, can be thought of as a direct analog of cohomology groups for a complex. Indeed, one might think of  $\infty$ -categories as types of non-linear dg categories.

Just as with dg categories, in speaking of  $\infty$ -categories we want some base line understanding of these “spaces” and the category in which they live. The point of this section is to provide such a base line understanding.

**4.1. More language for simplicial sets.** A monomorphism  $f : K' \rightarrow K$  between simplicial sets is a map such that, for any two morphisms  $L \rightrightarrows K'$  these two maps are equal if and only if their composites  $L \rightrightarrows K \rightarrow K'$  are equal. One sees that a map  $f : K' \rightarrow K$  is a monomorphism if and only if each restriction  $f_n : K'[n] \rightarrow K[n]$  to the  $n$ -simplices is an injective map of sets. We might also refer to a monomorphism simply as an injection.

A simplex  $s : \Delta^n \rightarrow K$  in a simplicial set  $K$  is said to be non-degenerate if  $s$  admits not factorization through a simplex  $\Delta^m \rightarrow K$  of lower dimension, i.e. with  $m < n$ . We see that  $K$  is constructed as the colimit

$$K = \varinjlim_{\Delta/K_{\text{non-deg}}} \Delta^n, \quad (8) \quad \boxed{\text{eq:465}}$$

where  $(\Delta/K)_{\text{non-deg}}$  is full “two layer” subcategory in  $\mathbf{sSet}/K$  whose top layer consists of non-degenerate simplices  $s : \Delta^n \rightarrow K$ , and whose second layer consists of all simplices  $\Delta^l \rightarrow K$  which admit at least one factoring  $\Delta^p \xrightarrow{r} \Delta^n \xrightarrow{s} K$  through a non-degenerate simplex  $s : \Delta^n \rightarrow K$  with  $n > l$  and  $r : \Delta^p \rightarrow \Delta^n$  a face in  $\Delta^n$ . So, the category  $(\Delta/K)_{\text{non-deg}}$  locally looks like

$$(\Delta/K)_{\text{non-deg}} :$$

We say a subcomplex  $K' \rightarrow K$  is obtained from  $K$  by deleting a given simplex  $s : \Delta^n \rightarrow K$  if the simplices in  $K'$ ,  $\Delta^m \rightarrow K'$  are precisely all of those simplices in  $K$  through which  $s$  admits *no* factorization  $\Delta^n \rightarrow \Delta^m \rightarrow K'$ . Note that if  $K'$  is obtained by deleting a *non-degenerate*  $n$ -simplex  $s$ , then  $K'_{<n} = K_{<n}$ , since by definition  $s$  admits no factorizations through lower dimensional simplices.

We say a non-degenerate simplex  $s : \Delta^n \rightarrow K$  is an “external face” if  $s$  admits a unique factorization through a unique higher dimensional non-degenerate simplex  $\Delta^n \rightarrow \Delta^{n'} \rightarrow K$ , and for this factorization  $n' = n + 1$ . If  $K'$  is obtained from  $K$  by deleting an external face, then by the colimit expression (8) we have a pushout diagram

$$\begin{array}{ccc} \Delta_i^n & \longrightarrow & K' \\ \downarrow & & \downarrow \\ \Delta^n & \xrightarrow{s} & K. \end{array} \quad (9) \quad \boxed{\text{eq:482}}$$

Conversely, if we have such a pushout diagram then  $K'$  is obtained from  $K$  by deleting an external face.

Below, by a *lifting problem* between maps of simplicial sets  $i : A \rightarrow B$  and  $f : K \rightarrow S$  we mean any diagram of the form

$$\begin{array}{ccc} A & \longrightarrow & K \\ i \downarrow & & \downarrow f \\ B & \longrightarrow & S. \end{array}$$

A solution to a lifting problem is a choice of map  $\zeta : A \rightarrow K$  which produces a diagram

$$\begin{array}{ccc} A & \longrightarrow & K \\ i \downarrow & \nearrow \zeta & \downarrow f \\ B & \longrightarrow & S. \end{array}$$

In order to distinguish, semantically, between a lifting problem and a generic commutative square, we generally present a lifting problem as follows:

$$\begin{array}{ccc} A & \longrightarrow & K \\ i \downarrow & \nearrow \text{---} & \downarrow f \\ B & \longrightarrow & S. \end{array}$$

Now, if we go back in time, we see that a Kan complex is a simplicial set  $\mathcal{X}$  such that all lifting problems, relative to a horn inclusion  $\Lambda_i^n \rightarrow \Delta^n$  and the terminal map  $\mathcal{X} \rightarrow *$ , admit a solution. Similarly  $\infty$ -categories are characterized by the existence of solutions to certain lifting problems.

sect:kf\_anodyne

**4.2. Kan complexes, Kan fibrations, and anodyne maps.** Let's consider some class of maps  $\mathbf{Class}_p$  in  $\mathbf{sSet}$  with a property  $p$ . (For example, we can consider the class of horn maps, or inner horn maps  $\Lambda_i^n \rightarrow \Delta^n$ .) Then this class generates a larger class  $\mathbf{Class}_p$ , by closing the initial collection  $\mathbf{Class}_p$  under a number of operations.

We say a class of maps  $\mathbf{Class}_p$  is saturated [5] if it satisfies the following:

- (a)  $\mathbf{Class}_p$  contains all isomorphisms.
- (b)  $\mathbf{Class}_p$  is closed under small coproducts.
- (c)  $\mathbf{Class}_p$  is closed under pushouts: If  $L \rightarrow K$  is a map in  $\mathbf{Class}_p$  and

$$\begin{array}{ccc} L & \longrightarrow & M \\ \downarrow & & \downarrow f \\ K & \longrightarrow & N \end{array}$$

is a pushout diagram, then  $f : M \rightarrow N$  is in  $\mathbf{Class}_p$ .

- (d)  $\mathbf{Class}_p$  is closed under retracts: If we have a diagram

$$\begin{array}{ccccc} L & \longrightarrow & M & \longrightarrow & L \\ f' \downarrow & & \downarrow f & & \downarrow f' \\ K & \longrightarrow & N & \longrightarrow & K \end{array}$$

where the horizontal composites are the identity, and  $f$  is in  $\mathbf{Class}_p$ , when  $f'$  is in  $\mathbf{Class}_p$ .

- (e)  $\mathbf{Class}_p$  is closed under countable composites: If

$$K_0 \rightarrow K_1 \rightarrow K_2 \rightarrow \dots$$

is a  $\mathbb{N}$ -indexes sequence of morphisms in  $\mathbf{Class}_p$ , then the structure map  $K_0 \rightarrow \varinjlim_n K_n$  is in  $\mathbf{Class}_p$ .

The minimal saturated class  $\mathbf{Class}_p$  containing some specified collection of morphisms  $\mathbf{Class}_p$  is called the saturated class generated by  $\mathbf{Class}_p$ . Note that condition (e) says that any saturated class of morphisms is closed under composition.

**Definition 4.1.** The class of anodyne morphisms in  $\mathbf{sSet}$  is the saturated class of maps generated by the inclusions

$$\{\Lambda_i^n \rightarrow \Delta^n : n \geq 1, i \in [n]\}$$

We note that the class of monomorphisms is itself saturated, so that anodyne maps are in particular injective.

lem:919

**Lemma 4.2.** *A simplicial set  $\mathcal{X}$  is a Kan complex if and only if for each anodyne morphism  $i : A \rightarrow B$  and arbitrary map  $K \rightarrow \mathcal{X}$ , there is a morphism  $B \rightarrow \mathcal{X}$  completing the diagram*

$$\begin{array}{ccc} A & \longrightarrow & \mathcal{X} \\ i \downarrow & \nearrow \exists & \\ B & & . \end{array}$$

*Proof.* Since the class of anodyne maps includes arbitrary horns  $\Lambda_i^n \rightarrow \Delta^n$ , any simplicial set with the left lifting property against anodyne maps is a Kan complex. For the converse, it suffices to show that the class  $LLP_{\text{Kan}}$  of maps  $i : A \rightarrow B$  having the left lifting property to a Kan complex  $\mathcal{X}$  is saturated. All of the conditions (a)–(e) are clear, save for possibly (e). Take a sequence of maps

$$A_0 \xrightarrow{i_1} A_1 \xrightarrow{i_2} A_2 \rightarrow \dots$$

is  $LLP_{\text{Kan}}$  and consider the structure map

$$\varprojlim_n \text{Hom}(A_n, \mathcal{X}) = \text{Hom}(\varinjlim_n A_n, \mathcal{X}) \rightarrow \text{Hom}(A_0, \mathcal{X}). \quad (10)$$

eq:514

We want to say that this map is surjective.

By the left lifting property, we understand that each map  $i_l^* : \text{Hom}(A_l, \mathcal{X}) \rightarrow \text{Hom}(A_{l+1}, \mathcal{X})$  in the sequence defining the limit in (10) is surjective. We also understand that each of the map  $i_{0l}^* : \text{Hom}(A_l, \mathcal{X}) \rightarrow \text{Hom}(A_0, \mathcal{X})$  defining the morphism (10) is surjective, where  $i_{0l} = i_l \dots i_1$ . One applies Zorn's lemma to see that the corresponding map from the limit (10) is in fact surjective. So the set  $LLP_{\text{Kan}}$  is in fact saturated, and since

$$\{\Lambda_i^n \rightarrow \Delta^n : n \geq 1, i \in [n]\} \subset LLP_{\text{Kan}}$$

we see that anodyne maps have the left lifting property relative to maps into Kan complexes.  $\square$

For example, if we consider an extension  $A' \rightarrow A$  for which  $A'$  is obtained from  $A$  via a sequence of deleting external faces in  $A$ ,

$$A' \rightarrow A^{(2)} \rightarrow A^{(3)} \rightarrow \dots A^{(n)} = K,$$

then the extension  $A' \rightarrow A$  is anodyne. This is due to the pushout expression (9) and the fact that anodyne maps are closed under composition.

ex:cube

**Example 4.3.** Consider the  $n$ -dimensional solid cube  $C^n = (\Delta^1)^n$ . This simplicial set has 0-simplices  $Vert^n = C^n([0])$  given by all tuples  $(\delta_0, \dots, \delta_n)$ ,  $\delta_i \in \{0, 1\}$ , and  $m$ -simplices  $\Delta^m \rightarrow C^n$  are given by order preserving maps

$$[m] \rightarrow Vert^n$$

where we give these vertices the dictionary ordering. We have the “sides”  $F_{\delta,i} \subset C^n$ , whose  $m$ -simplices are functions  $r : [m] \rightarrow Vert^n$  with constant value  $\delta$  in the  $i$ -th

position,  $r(j) = (\delta_0, \dots, \delta_{i-1}, \delta, \dots, \delta_n)$ . We consider the “open box”  $OC_{\delta,i}^n \subset C^n$  which is produced from  $C^n$  by deleting the side  $(\delta, i)$ -side of the cube  $F_{\delta,i}$  and the interior of the cube. We claim that the inclusion of the open cube  $OC_{\delta,i}^n \rightarrow C^n$  is anodyne. Indeed,  $OC_{\delta,i}^n$  can be obtained from  $C^n$  by a deleting external faces.

We also have a relative notion of Kan complexes.

**Definition 4.4.** A morphism  $f : \mathcal{X} \rightarrow S$  is called a Kan fibration if any lifting problem

$$\begin{array}{ccc} A & \longrightarrow & \mathcal{X} \\ \downarrow & \nearrow & \downarrow f \\ B & \longrightarrow & S \end{array}$$

in which  $A \rightarrow B$  is anodyne admits a solution.

A simplicial set  $\mathcal{X}$  is a Kan complex if and only if the terminal map  $\mathcal{X} \rightarrow *$  is a Kan fibration. Just as in the case  $S = *$ , one can test Kan-ness of a morphism  $\mathcal{X} \rightarrow S$  by examining the lifting property relative to the horn inclusions  $\Lambda_i^n \rightarrow \Delta^n$ . By the universal property of pullback, one sees that the class of Kan fibrations is closed under pullback.

1em:821

**Lemma 4.5.** Suppose that  $f : \mathcal{X} \rightarrow S$  is a Kan fibration, and

$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & \mathcal{X} \\ f' \downarrow & & \downarrow f \\ S' & \longrightarrow & S \end{array}$$

is a pullback diagram. Then  $f'$  is a Kan fibration.

By the above lemma one can view a Kan fibration as a family of Kan complexes parametrized by the base  $S$ . One can compare, for example, with notions of smoothness for varieties, and smooth morphisms. We have now defined Kan fibrations relative to anodyne maps. One can show that the converse distinction holds as well, though we do not record a proof here.

**Proposition 4.6** ([6, Corollary 1.4.1]). A map  $i : A \rightarrow B$  is anodyne if and only if any lifting problem

$$\begin{array}{ccc} A & \longrightarrow & \mathcal{X} \\ i \downarrow & \nearrow & \downarrow \\ B & \longrightarrow & S \end{array}$$

in which  $\mathcal{X} \rightarrow S$  is a Kan fibration admits a solution.

This is the definition of anodyne maps employed in [8]. The following lemma will be of use momentarily.

1em:850

**Lemma 4.7.** Consider two monomorphisms of simplicial sets  $i : A \rightarrow B$  and  $j : K \rightarrow L$ . If either of  $i$  or  $j$  is an anodyne morphism, then the induced map

$$(A \times L) \coprod_{(A \times K)} (B \times K) \rightarrow (B \times L)$$

is anodyne as well.

The proof is somewhat technical and can be found in [10, 014D] or [5, Corollary 4.6].

### 4.3. Trivial Kan fibrations.

**Definition 4.8.** A morphism  $f : \mathcal{X} \rightarrow S$  is called a trivial Kan fibration if any lifting problem

$$\begin{array}{ccc} A & \longrightarrow & \mathcal{X} \\ \downarrow & \nearrow & \downarrow f \\ B & \longrightarrow & S \end{array} \quad (11) \quad \boxed{\text{eq:864}}$$

in which  $A \rightarrow B$  is a monomorphism admits a solution. A simplicial set  $\mathcal{X}$  is called contractible if the terminal map  $\mathcal{X} \rightarrow *$  is a trivial Kan fibration.

Of course, the lifting problem here is a choice of square diagram, which produces the external square of (11), and a solution to the lifting problem is a choice of map  $B \rightarrow \mathcal{X}$  which splits the diagram into two commuting squares. Clearly any trivial Kan fibration is a Kan fibrations.

**Proposition 4.9.** A map  $f : \mathcal{X} \rightarrow S$  is a trivial Kan fibration if and only if all lifting problems

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & \mathcal{X} \\ \text{incl} \downarrow & \nearrow & \downarrow f \\ \Delta^n & \longrightarrow & S \end{array}$$

for  $n \in \mathbb{Z}_{\geq 0}$ , admit a solution.

*Sketch proof.* The point is that the saturated class of morphisms generated by the boundary inclusions  $\{\partial\Delta^n \rightarrow \Delta^n : n \geq 0\}$  is the class of all monomorphisms [6, Section 1.4] [10, 0077]  $\square$

**4.4. Exponents for Kan complexes.** For simplicial sets  $K$  and  $S$  we have the mapping complex  $\text{Fun}(K, S)$  defined in Section 2.6. We have the evaluation map

$$ev : \text{Fun}(K, S) \times K \rightarrow S \quad (12) \quad \boxed{\text{eq:eval}}$$

which is defined on simplices by

$$\text{Hom}_{\text{sSet}}(\Delta^n \times K, S) \times K[n] \rightarrow S[n], \quad (f, x) \mapsto f(id_{[n]}, x).$$

The fact that  $f$  is a map of simplicial sets implies that evaluation, defined as above, is also a map of simplicial sets.

**Lemma 4.10** ([5, Proposition 5.1]). *The evaluation morphism (12) defines an isomorphism of (plain) sets*

$$\begin{aligned} \text{Hom}_{\text{sSet}}(L, \text{Fun}(K, S)) &\xrightarrow{\cong} \text{Hom}_{\text{sSet}}(L \times K, S) \\ \eta &\mapsto ev(\eta \times id_K) \end{aligned}$$

which is natural in  $L$ ,  $K$ , and  $S$ .

*Proof.* Take  $\eta^! := ev(\eta \times id_K)$ . The inverse to the map  $\eta \mapsto \eta^!$  is given by sending a function  $g : L \times K \rightarrow S$  to the maps

$$g_! : L \rightarrow \text{Fun}(K, S), \quad y \mapsto [ \Delta^n \times K \rightarrow S, (t, x) \mapsto g(t^*y, x) ].$$



Indeed, we have explicitly

$$(\eta^!)_! : L \rightarrow \text{Fun}(K, S), \quad y \mapsto \{(x, t) \mapsto \text{eta}^!(x, t^*y) = \text{ev}(t^*\eta(y), x)\}$$

and  $\text{ev}(t^*\eta(y), x) = t^*\eta(y)(\text{id}_{[m]}, x) = \eta(y)(t, x)$ . So  $(\eta^!)_! = \eta$ . One checks also

$$(g_!)^! : L \times K \rightarrow S, \quad (x, y) \mapsto \text{ev}(g_! \times \text{id}_K) = [(y, z) \mapsto g_!(y)(\text{id}_{[n]}, z)]$$

and  $g_!(y)(\text{id}_{[n]}, z) = g(y, z)$ , so that  $(g_!)^! = g$ . Naturality can be verified directly.  $\square$

The above lemma says that the functor complexes  $\text{Fun}(-, -)$  provide inner-Homs for the monoidal category of simplicial sets, with the symmetric monoidal structure provided by the product.

**prop:tech1**

**Proposition 4.11** ([5, Proposition 5.2]). *Suppose that  $f : \mathcal{X} \rightarrow S$  is a Kan fibration and that  $i : K \rightarrow L$  is a monomorphism between arbitrary simplicial sets. Then the map*

$$\text{Fun}(L, \mathcal{X}) \rightarrow \text{Fun}(K, \mathcal{X}) \times_{\text{Fun}(K, S)} \text{Fun}(L, S) \quad (13)$$

**eq:903**

*provided by the diagram*

$$\begin{array}{ccc} \text{Fun}(L, \mathcal{X}) & \xrightarrow{i^*} & \text{Fun}(K, \mathcal{X}) \\ f^* \downarrow & & \downarrow f_* \\ \text{Fun}(L, S) & \xrightarrow{i_*} & \text{Fun}(K, S) \end{array}$$

*is a Kan fibration. When  $i : K \rightarrow L$  is furthermore anodyne, or  $f : \mathcal{X} \rightarrow S$  is a trivial fibration, the map (13) is a trivial Kan fibration.*

*Proof.* First, let  $A \rightarrow B$  be any anodyne map. The existence of a splitting for a given diagram

$$\begin{array}{ccc} A & \xrightarrow{\quad} & \text{Fun}(L, \mathcal{X}) \\ \downarrow & \dashrightarrow^{\exists?} & \downarrow \\ B & \xrightarrow{\quad} & \text{Fun}(K, \mathcal{X}) \times_{\text{Fun}(K, S)} \text{Fun}(L, S) \end{array}$$

is equivalent to the existence of a splitting for the corresponding diagram

$$\begin{array}{ccc} (A \times L) \amalg_{(A \times K)} (B \times K) & \xrightarrow{\quad} & \mathcal{X} \\ \downarrow & \dashrightarrow^{\exists?} & \downarrow f \\ (B \times L) & \xrightarrow{\quad} & S \end{array} \quad (14)$$

**eq:925**

which one obtains via adjunction for inner Homs. By Lemma 4.7 the left hand vertical map in (14) is anodyne, and the map  $f : \mathcal{X} \rightarrow S$  is a Kan fibration by hypothesis, so that such a splitting is seen to exist. It follows that the map (13) is a Kan fibration. When the map  $i : K \rightarrow L$  is furthermore anodyne, or  $f : \mathcal{X} \rightarrow S$  is trivial, the above argument can be carried out for any monomorphism  $A \rightarrow B$ , from which one concludes that (13) is a trivial Kan fibration in this case.  $\square$

We find a number of interesting corollaries.

cor:Fun\_kan

**Corollary 4.12.** (i) For any simplicial set  $L$ , and Kan fibration  $f : \mathcal{X} \rightarrow S$ , the induced map

$$f_* : \text{Fun}(L, \mathcal{X}) \rightarrow \text{Fun}(L, S)$$

is a Kan fibration.

(ii) For any monomorphism  $i : K \rightarrow L$ , and Kan complex  $\mathcal{X}$ , the induced map

$$i^* : \text{Fun}(L, \mathcal{X}) \rightarrow \text{Fun}(K, \mathcal{X})$$

is a Kan fibration.

(iii) For any anodyne morphism  $i : K \rightarrow L$ , and Kan complex  $\mathcal{X}$ , the induced map

$$i^* : \text{Fun}(L, \mathcal{X}) \rightarrow \text{Fun}(K, \mathcal{X})$$

is a trivial Kan fibration.

(iv) For any simplicial set  $L$ , and Kan complex  $\mathcal{X}$ , the simplicial set of functors  $\text{Fun}(L, \mathcal{X})$  is a Kan subcomplex in  $\text{Fun}(K, \mathcal{X})$ .

*Proof.* Follow by considering the cases  $K = \emptyset$  and  $S = *$ , independently then together.  $\square$

**4.5. Why maps from simplicial sets.** In the previous subsection we have emphasized maps  $\text{Fun}(K, \mathcal{X})$  from some simplicial set  $K$  into a given Kan complex. Of course, a simplicial set is in general neither a Kan complex nor an  $\infty$ -category. So, one might ask: Why consider simplicial sets at all here?

The reason is rather simple. We see a map  $p : K \rightarrow \mathcal{X}$  from a simplicial set as a “diagram” in  $\mathcal{X}$ . Then to consider the dynamics of the category  $\text{Fun}(K, \mathcal{X})$  is to consider the dynamics of certain diagrams in  $\mathcal{X}$ . This will be important in both the Kan and  $\infty$ -context when we want to speak, for example, of limits and colimits of diagrams in a given Kan complex or  $\infty$ -category. So it’s actually quite important that we develop a relatively sophisticated understanding of digrams in a generic Kan complex, or  $\infty$ -category.

#### 4.6. Connected components and the notion of equivalence.

**Definition 4.13.** For a Kan complex  $\mathcal{X}$ , and 0-simplices  $x, y : \Delta^0 \rightarrow \mathcal{X}$ , write  $x \sim y$  if there is a 2-simplex  $f : \Delta^1 \rightarrow \mathcal{X}$  with  $f|_0 = x$  and  $f|_1 = y$ . Take

$$\pi_0(\mathcal{X}) := \mathcal{X}[0] / \sim.$$

We call the set  $\pi_0(\mathcal{X})$  the set of connected components for  $\mathcal{X}$ .

The axioms of a Kan complex say that the relation  $\sim$  on  $\mathcal{X}[0]$  is an equivalence relation. So the set of connected components is in fact a well-defined set. We say two 0-simplices for  $\mathcal{X}$  lie in the same component of  $\mathcal{X}$  if they define the have the same image in the quotient  $\mathcal{X}[0] \rightarrow \pi_0(\mathcal{X})$ . An interesting situation occurs when we consider functors  $K \rightarrow \mathcal{X}$  into a Kan complex, and the corresponding space of functors  $\text{Fun}(K, \mathcal{X})$ . An equivalence between functors  $F, F' : K \rightarrow \mathcal{X}$  is a functor  $\zeta : \Delta^1 \times K \rightarrow \mathcal{X}$  whose two restrictions

$$K \rightrightarrows \Delta^1 \times K \rightarrow \mathcal{X}$$

recoves  $F$  and  $F'$  respectively. Since  $\text{Fun}(K, \mathcal{X})$  is a Kan complex, according to Corollary 4.12, this relation on functors is an equivalence relation.

**Definition 4.14.** Let  $\mathcal{X}$  be a Kan complex. We say two maps  $F, F' : K \rightarrow \mathcal{X}$ , from any given simplicial set, are homotopic if they lie in the same component of  $\text{Fun}(K, \mathcal{X})$ . We say a map  $f : \mathcal{X} \rightarrow \mathcal{Y}$  between Kan complexes is a (homotopy) equivalence if there exists  $g : \mathcal{Y} \rightarrow \mathcal{X}$  such that  $fg$  and  $gf$  are homotopic to the identities on  $\mathcal{Y}$  and  $\mathcal{X}$ , respectively.

One certainly anticipates the following lemma.

lem:978

**Lemma 4.15.** *Suppose that  $f, f' : \mathcal{X} \rightarrow \mathcal{Y}$  are homotopic map between Kan complexes, and  $F, F' : K \rightarrow \mathcal{X}$  are homotopic maps from some simplicial set. Then  $f \circ F, f \circ F', f' \circ F$ , and  $f' \circ F'$  are all homotopic morphisms in  $\text{Fun}(K, \mathcal{Y})$ .*

*Proof.* If we choose a homotopy  $\zeta : \Delta^1 \times K \rightarrow \mathcal{X}$  between  $F$  and  $F'$ , then  $f \circ \zeta$  provides a homotopy between  $f \circ F$  and  $f \circ F'$ . If  $z : \Delta^1 \times \mathcal{X} \rightarrow \mathcal{Y}$  is a homotopy between  $f$  and  $f'$ , then  $z \circ (id_{\Delta^1} \times F)$  provides a homotopy between  $f \circ F$  and  $f' \circ F$ . Since homotopy is an equivalence relation, this suffices to show that all of the given maps are homotopic.  $\square$

Now, let us say that a morphism  $g : \mathcal{Y} \rightarrow \mathcal{X}$  between Kan complexes is a left (resp. right) homotopy inverse to a map  $f : \mathcal{X} \rightarrow \mathcal{Y}$  if the composite  $gf$  (resp.  $fg$ ) is homotopic to the identity on  $\mathcal{X}$  (resp.  $\mathcal{Y}$ ).

**Lemma 4.16.** *If a map  $f : \mathcal{X} \rightarrow \mathcal{Y}$  between Kan complexes admits both a left and right homotopy inverse, then  $f$  is an equivalence.*

*Proof.* Let  $g$  and  $g'$  be left and right homotopy inverses to  $f$ . It suffices to show that  $g$  and  $g'$  are homotopic. By Lemma 4.15 we have

$$g \sim g(fg') = (gf)g' \sim g'.$$

We are done.  $\square$

We can now reasonably define the homotopy category of Kan complexes.

**Definition 4.17.** The homotopy category of Kan complexes is the category

$$\text{hKan} := \{\text{Kan complexes with homotopy classes of morphisms } \pi_0 \text{Fun}(-, -)\}.$$

Lemma 4.15 assures us that composition is well-defined, and associative, on homotopy classes of maps so that  $\text{hKan}$  is in fact a category. Note also that two Kan complexes are equivalent, as objects in  $\text{Kan}$ , if and only if they are isomorphic in the homotopy category  $\text{hKan}$ . If we compare with our category of “spaces”,  $\mathcal{S}paces = N_{\text{hc}}(\underline{\text{Kan}}^{\text{sm}})$ , we have

$$\text{hKan}^{\text{sm}} = \text{h}\mathcal{S}paces^{\text{sm}}.$$

**4.7. Equivalences and functor spaces.** Our main goal of the subsection is to prove the following.

prop:equiv\_Fun\_kan

**Proposition 4.18.** *For a map  $f : \mathcal{X} \rightarrow \mathcal{Y}$  between Kan complexes, the following are equivalent:*

- (a)  *$f$  is an equivalence.*
- (b) *For any simplicial set  $K$ , the induced map*

$$f_* : \text{Fun}(K, \mathcal{X}) \rightarrow \text{Fun}(K, \mathcal{Y})$$

*is an equivalence of Kan complexes.*

(c) For any Kan complex  $\mathcal{W}$ , the induced map

$$f_* : \text{Fun}(\mathcal{W}, \mathcal{X}) \rightarrow \text{Fun}(\mathcal{W}, \mathcal{Y})$$

is an equivalence.

(d) For any Kan complex  $\mathcal{W}$ , the induced map

$$f^* : \text{Fun}(\mathcal{Y}, \mathcal{W}) \rightarrow \text{Fun}(\mathcal{X}, \mathcal{W})$$

is an equivalence.

We first record some lemmas, which will prove useful in any case. The following establishes the implication from (a) to (b).

**lem:1195**

**Lemma 4.19.** *Suppose that  $f : \mathcal{X} \rightarrow \mathcal{Y}$  is an equivalence between Kan complexes. Then, for any simplicial set  $K$ , the induced map  $f_* : \text{Fun}(K, \mathcal{X}) \rightarrow \text{Fun}(K, \mathcal{Y})$  is an equivalence as well.*

*Proof.* Let  $g : \mathcal{Y} \rightarrow \mathcal{X}$  be a homotopy inverse to  $f$ . By considering the composites  $fg$  and  $gf$ , it suffices to show that for any endomorphism  $F : \mathcal{X} \rightarrow \mathcal{X}$  which is homotopic to the identity  $id_{\mathcal{X}}$  the induced map  $F_* : \text{Fun}(K, \mathcal{X}) \rightarrow \text{Fun}(K, \mathcal{X})$  is homotopic to the identity. By the natural isomorphism

$$\text{Hom}_{\text{sSet}}(\Delta^1 \times \text{Fun}(K, \mathcal{X}), \text{Fun}(K, \mathcal{X})) \xrightarrow{\cong} \text{Hom}_{\text{sSet}}(\text{Fun}(K, \mathcal{X}), \text{Fun}(\Delta^1 \times K, \mathcal{X}))$$

provided by adjunction, it suffices to produce a map

$$\Theta : \text{Fun}(K, \mathcal{X}) \rightarrow \text{Fun}(\Delta^1 \times K, \mathcal{X})$$

whose restrictions along the two

$$-|_{\{0\} \times K}, -|_{\{1\} \times K} : \text{Fun}(\Delta^1 \times K, \mathcal{X}) \rightrightarrows \text{Fun}(K, \mathcal{X})$$

recover the identity  $id_{\text{Fun}}$  and  $F_*$  respectively.

If we let  $\theta : \Delta^1 \times \mathcal{X} \rightarrow \mathcal{X}$  be a homotopy between  $id_{\mathcal{X}}$  and  $F$ , then we may define  $\Theta$  on  $n$ -simplices by taking a function  $\alpha : \Delta^n \times K \rightarrow \mathcal{X}$  to the composite  $\theta \circ (\Delta^1 \times \alpha)$ . Such a formula determines  $\Theta$  as a map of simplicial sets and, by direct inspection, the two composites

$$\text{Fun}(K, \mathcal{X}) \xrightarrow{\Theta} \text{Fun}(\Delta^1 \times K, \mathcal{X}) \rightrightarrows \text{Fun}(K, \mathcal{X})$$

recover  $id_{\text{Fun}}$  and  $F_*$ . Rather,  $\Theta$  exhibits a homotopy between the identity and  $F_*$ , as required.  $\square$

**lem:pi0\_equiv**

**Lemma 4.20.** *If  $f : \mathcal{X} \rightarrow \mathcal{Y}$  is an equivalence, then the induced map on connected components  $\pi_0(f) : \pi_0(\mathcal{X}) \rightarrow \pi_0(\mathcal{Y})$  is an isomorphism.*

*Proof.* We have a natural isomorphism

$$\pi_0(\mathcal{X}) \cong \pi_0(\text{Fun}(*, \mathcal{X})) = \text{Hom}_{\text{hKan}}(*, \mathcal{X})$$

under which  $\pi_0(f)$  is identified with procomposition

$$\bar{f}_* : \text{Hom}_{\text{hKan}}(*, \mathcal{X}) \rightarrow \text{Hom}_{\text{hKan}}(*, \mathcal{Y}).$$

Since  $\text{bar } f : \mathcal{X} \rightarrow \mathcal{Y}$  is an isomorphism in  $\text{hKan}$ , the corresponding map  $\bar{f}_*$  is an isomorphism.  $\square$

We can now prove our proposition.

*Proof of Proposition 4.18.* The implication (a)  $\Rightarrow$  (b) is covered by Lemma ??, and the implication (b)  $\Rightarrow$  (c) is immediate, since (c) is just some restriction of (b). Now, we claim that (c) implies (a). Suppose that (c) holds. By Lemma 4.20, the map

$$\pi_0(f_*) : \pi_0(\text{Fun}(\mathcal{W}, \mathcal{X})) \rightarrow \pi_0(\text{Fun}(\mathcal{W}, \mathcal{Y}))$$

is an isomorphism. But these sets, by definition, are just  $\text{Hom}_{\text{hKan}}(\mathcal{W}, \mathcal{X})$  and  $\text{Hom}_{\text{hKan}}(\mathcal{W}, \mathcal{Y})$ , and  $\pi_0(f_*)$  is composition  $\bar{f}_*$  with the homotopy class of  $f$ . So we conclude that

$$\bar{f}_* : \text{Hom}_{\text{hKan}}(\mathcal{W}, \mathcal{X}) \rightarrow \text{Hom}_{\text{hKan}}(\mathcal{W}, \mathcal{Y})$$

is an isomorphism at all  $\mathcal{W}$  in  $\text{hKan}$ . Rather, we have a natural isomorphism

$$\bar{f}_* : \text{Hom}_{\text{hKan}}(-, \mathcal{X}) \xrightarrow{\cong} \text{Hom}_{\text{hKan}}(-, \mathcal{Y})$$

of Set valued functors. It follows by Yoneda that the map  $\bar{f}$  is an isomorphism in  $\text{hKan}$ , and hence that the original map  $f : \mathcal{X} \rightarrow \mathcal{Y}$  is an equivalence. So we establish (a).

One similarly argues that (d) implies (a), and so we need only establish the implication (a)  $\Rightarrow$  (d). However, one can argue as in the proof of Lemma 4.19 to see directly that  $f^* : \text{Fun}(\mathcal{Y}, \mathcal{W}) \rightarrow \text{Fun}(\mathcal{X}, \mathcal{W})$  is an equivalence whenever  $f$  is an equivalence. We are done.  $\square$

**4.8. Homotopy groups for Kan complexes.** For any  $n \geq 0$  we define the simplicial  $n$ -sphere as

$$S^n := \Delta^n \amalg_{\partial \Delta^n} *$$

The object  $S^n$  is a simplicial set which is a quotient of the standard  $n$ -simplex. To distinguish between the simplicial and topological spheres we denote the topological  $n$ -sphere by  $\mathbb{S}^n$ . We have that  $|S^n| = \mathbb{S}^n$ , but we note that the simplicial sphere is not itself a Kan complex, and so not a “space”. However, the failure of  $S^n$  to be a Kan complex is, for some specific reasons [10, 00VT], not important.

**Definition 4.21.** A pointed simplicial set  $(K, x)$  is a simplicial set  $K$  with a fixed map  $x : * \rightarrow K$ . A map between pointed simplicial sets  $f : (K, x) \rightarrow (L, y)$  is a map of simplicial sets which fits into a diagram

$$\begin{array}{ccc} & * & \\ x \swarrow & & \searrow y \\ K & \xrightarrow{f} & L. \end{array}$$

A pointed Kan complex is a Kan complex which is pointed as a simplicial set.

We consider the simplicial sphere  $S^n$  as a pointed simplicial set  $(S^n, *)$  with distinguished point given by the unique 1-simplex  $* \rightarrow S^n$ . So, given a pointed simplicial set  $(K, x)$ , a map of pointed simplicial sets  $S^n \rightarrow (K, x)$  is a map of simplicial sets such that the unique composite  $* \rightarrow S^n \rightarrow K$  recovers  $x$ . For pointed simplicial sets  $(K, x)$  and  $(L, y)$  we let

$$\text{Fun}(K, L)_{\text{pt}(x, y)} \subset \text{Fun}(K, L)$$

denote the simplicial subset which is given by the pullback of the diagram

$$\begin{array}{ccc} \mathrm{Fun}(K, L)_{\mathrm{pt}(x, y)} & \longrightarrow & \mathrm{Fun}(K, L) \\ \downarrow & & \downarrow x^* \\ * & \xrightarrow{y} & \mathrm{Fun}(*, L) = L. \end{array}$$

So a map  $f : \Delta^n \times K \rightarrow L$  is an  $n$ -simplex in the pointed functor complex provided the restriction  $\Delta \times * \rightarrow \Delta \times K \rightarrow L$  along the map  $x : * \rightarrow K$  is of constant value  $y$ .

**Lemma 4.22.** *If  $(\mathcal{X}, x)$  is a Kan complex, and  $(K, z)$  is an arbitrary pointed set, then the pointed mapping complex  $\mathrm{Fun}(K, \mathcal{X})_{\mathrm{pt}(z, x)}$  is a Kan complex.*

*Proof.* By Corollary 4.12 the map  $z^* : \mathrm{Fun}(K, \mathcal{X}) \rightarrow \mathrm{Fun}(*, \mathcal{X})$  is a Kan fibration, and Kan fibrations are closed under pullback by Lemma 4.5. Hence the pointed mapping complex is a Kan complex.  $\square$

**Definition 4.23.** Let  $(\mathcal{X}, x)$  be a pointed Kan complex. The  $n$ -th homotopy group  $\pi_n(\mathcal{X}, x)$  is the set of homotopy classes of maps

$$\pi_n(\mathcal{X}, x) := \pi_0(\mathrm{Fun}(S^n, \mathcal{X})_{\mathrm{pt}(*, x)})$$

Since  $\mathrm{Fun}(S^n, \mathcal{X})_{\mathrm{pt}(*, x)}$  is itself a Kan complex, it is reasonable to consider the connected components here. For  $n > 0$  there is a group structure on  $\pi_n(\mathcal{X}, x)$  which defined as follows: for two pointed maps  $a, b : S^n \rightarrow (\mathcal{X}, x)$  which represent classes  $\bar{\alpha}, \bar{\beta} \in \pi_n(\mathcal{X}, x)$ , we consider the horn  $w : \Lambda_1^{n+1} \rightarrow \mathcal{X}$  with restrictions

$$w|_{\Delta^{[n+1]-\{0\}}} = a, \quad w|_{\Delta^{[n+1]-\{2\}}} = b, \quad w|_{\Delta^{[n+1]-\{i\}}} = x \quad \text{when } i > 2. \quad (15)$$

eq:1088

We fill this horn to get a map  $W : \Delta^{n+1} \rightarrow \mathcal{X}$ , and take

$$\alpha * \beta := \overline{W|_{\Delta^{[n+1]-\{1\}}}} \in \pi_n(\mathcal{X}, x).$$

lem:1097

**Theorem 4.24** ([5, Lemma 7.1, Theorem 7.2]). *Fix  $n > 0$ . For any classes  $\alpha, \beta \in \pi_n(\mathcal{X}, x)$  the element  $\alpha * \beta$  constructed above does not depend on the choice of representatives  $a, b : S^n \rightarrow \mathcal{X}$ , nor does it depend on the choice of filling for the horn (15). We thus obtain a well-defined binary operation*

$$- * - : \pi_n(\mathcal{X}, x) \times \pi_n(\mathcal{X}, x) \rightarrow \pi_n(\mathcal{X}, x). \quad (16)$$

eq:1098

*The binary operation (16) provides  $\pi_n(\mathcal{X}, x)$  with a group structure for which the class of the constant map  $x : S^n \rightarrow \mathcal{X}$  serves as the identity. Furthermore, this group is abelian when  $n > 2$ .*

This construction may seem somewhat mysterious, however the group structure on  $\pi_n(\mathcal{X}, x)$  enjoys a certain uniqueness. Namely, this is the unique group structure for which the constant map at  $x$  provides the unit and for which an  $(n+1)$ -fold product  $\gamma_0^{-1} \gamma_1 \dots \gamma_{n+1}^{(-1)^n}$  is the identity in  $\pi_n(\mathcal{X}, x)$  if and only if the corresponding boundary map  $\gamma_* : \partial \Delta^{n+1} \rightarrow \mathcal{X}$  admits a filling [10, 00VU]. These two properties are familiar from topology.

We note that the construction of the homotopy group is functorial. Namely, if we have a map of pointed Kan complexes  $f : (\mathcal{X}, x) \rightarrow (\mathcal{Y}, y)$  then we have an induced map of Kan complexes

$$f_* : \mathrm{Fun}(S^n, \mathcal{X})_{\mathrm{pt}(*, x)} \rightarrow \mathrm{Fun}(S^n, \mathcal{Y})_{\mathrm{pt}(*, y)}$$

and thus an induced map on connected components  $f_*(n) : \pi_n(\mathcal{X}, x) \rightarrow \pi_n(\mathcal{Y}, y)$ . Since horn fillings are sent to horn fillings under  $f$ , this map on homotopy groups is in fact a map of groups. We record this observation.

**Lemma 4.25.** *The homotopy group constructions provides a  $\mathbb{Z}_{>0}$ -indexed collection of functors*

$$\pi_n(-) : \{ \text{Pointed Kan complexes with pointed maps} \} \rightarrow \text{Groups}.$$

#### 4.9. Homotopy groups and trivial Kan fibrations.

**Proposition 4.26.** *If  $\mathcal{X}$  is a contractible Kan complex, then  $\pi_0(\mathcal{X})$  has a single element and all of the homotopy groups  $\pi_n(\mathcal{X}, x)$  are trivial.*

*Proof.* To see that  $\pi_0(\mathcal{X})$  one simply notes that any map  $\partial\Delta^1 \rightarrow \mathcal{X}$ , which just consists of a choice of two points in  $\mathcal{X}$ , extends to a path  $\Delta^1 \rightarrow \mathcal{X}$  via the lifting property for trivial Kan fibrations. Suppose now that  $n > 0$ . Consider the pushout  $K^n$  of the diagram

$$\begin{array}{ccc} \partial\Delta^1 & \longrightarrow & \partial\Delta^1 \times S^n \\ \downarrow & & \downarrow \text{dotted} \\ \Delta^1 & \xrightarrow{\text{dotted}} & K^n. \end{array}$$

The two maps  $S^n \times \partial\Delta^1 \rightarrow S^n \times \Delta^1$  and  $\Delta^1 \rightarrow S^n \times \Delta^1$  induce an embedding  $K^n \rightarrow \Delta^1 \times \Delta^1$ . Now, for any two maps  $a, b : S^n \rightarrow \mathcal{X}$  there is a unique morphism  $[a, b] : K^n \rightarrow \mathcal{X}$  for which  $[a, b]|_{S^n \times \partial\Delta^1} = a \amalg b$  and  $[a, b]|_{\Delta^1} = x$ . By the lifting property there exists an extension of  $[a, b]$  to a map  $h : \Delta^1 \times S^n \rightarrow \mathcal{X}$ . The map  $h$  provides a homotopy between  $a$  and  $b$  in  $\text{Fun}(S^n, \mathcal{X})_{\text{pt}(*, x)}$ , and hence equates the classes  $\bar{a} = \bar{b} \in \pi_n(\mathcal{X}, x)$ . This shows that  $\pi_n(\mathcal{X}, x)$  is a singleton for all  $n > 0$  as well.  $\square$

A relative version of the above proposition holds as well.

prop:1386

**Proposition 4.27.** *Suppose that  $f : \mathcal{X} \rightarrow \mathcal{S}$  is a trivial Kan fibration, and that  $\mathcal{S}$  is a Kan complex. Then the map on connected components  $\pi_0 f : \pi_0(\mathcal{X}) \rightarrow \pi_0(\mathcal{S})$  is a bijection and for all  $n \geq 1$  the map  $\pi_n f : \pi_n(\mathcal{X}, x) \rightarrow \pi_n(\mathcal{S}, f(x))$  is an isomorphism of groups.*

We first note that compositions of Kan fibrations are Kan fibrations, so that  $\mathcal{X}$  itself is a Kan complex in this case

*Proof.* The lifting property implies directly that  $\mathcal{X}[0] \rightarrow \mathcal{S}[0]$  is surjective and that paths in  $\mathcal{S}$  lift to paths in  $\mathcal{X}$ . So the induced map on connected components is an isomorphism. Fix now  $n \geq 1$  and take, for arbitrary  $x \in \mathcal{X}[0]$ ,  $y = f(x)$ . By applying the lifting property along the inclusion  $* \rightarrow S^n$  we see that the map

$$f_* : \text{Hom}_{\text{sSet}}(S^n, \mathcal{X})_{\text{pt}(*, x)} \rightarrow \text{Hom}_{\text{sSet}}(S^n, \mathcal{S})_{\text{pt}(*, y)}$$

is surjective. Now take  $K^n$  to be the pushout

$$\begin{array}{ccc} \partial\Delta^1 & \longrightarrow & \partial\Delta^1 \times S^n \\ \downarrow & & \downarrow \\ \Delta^1 & \longrightarrow & K^n \end{array}$$

and consider the inclusion  $K^n \rightarrow \Delta^1 \times S^n$ . Let  $\eta : \Delta^1 \times S^n \rightarrow \mathcal{S}$  be a pointed homotopy between two pointed maps  $\alpha, \alpha' : S^n \rightarrow (\mathcal{S}, y)$ , and consider two pointed maps  $\beta, \beta' : S^n \rightarrow (\mathcal{X}, x)$  which lift  $\alpha$  and  $\alpha'$  respectively. Then  $\beta$  and  $\beta'$  determine a map  $K^n \rightarrow \mathcal{X}$  which fits into a diagram

$$\begin{array}{ccc} K^n & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow f \\ \Delta^1 \times S^n & \xrightarrow{h} & \mathcal{S}. \end{array}$$

We choose a solution  $\tilde{h} : \Delta^1 \times S^n \rightarrow \mathcal{X}$  to the corresponding lifting problem to see that  $\beta$  and  $\beta'$  define the same class in  $\pi_n(\mathcal{X}, x)$ . This shows that the map

$$\pi_n f : \pi_n(\mathcal{X}, x) \rightarrow \pi_n(\mathcal{S}, y)$$

is bijective. □

**Remark 4.28.** A version of Proposition 4.27 for possibly non-trivial Kan fibrations, in which the isomorphisms  $\pi_n(\mathcal{X}, x) \rightarrow \pi_n(\mathcal{S}, fx)$  “spread out” into a long exact sequence of homotopy groups, is discussed in Section 6.6 below.

One can use Proposition 4.27 in place of Lemma 4.20 to see that any trivial Kan fibration is in fact an equivalence.

`prop:trivkan_equiv`

**Proposition 4.29.** *Suppose that  $f : \mathcal{X} \rightarrow \mathcal{S}$  is a trivial Kan fibration between Kan complexes. Then  $f$  is an equivalence.*

*Proof.* In this case we have that, for any Kan complex  $\mathcal{W}$ , the induced map

$$f_* : \text{Fun}(\mathcal{W}, \mathcal{X}) \rightarrow \text{Fun}(\mathcal{W}, \mathcal{S})$$

is a trivial Kan fibration, by Proposition 4.11. Then by Lemma 4.20 the induced map on connected components is an isomorphism

$$\pi_0(f_*) : \pi_0(\text{Fun}(\mathcal{W}, \mathcal{X})) \rightarrow \pi_0(\text{Fun}(\mathcal{W}, \mathcal{S})).$$

As in the proof of Proposition 4.27, this collection of isomorphisms, which varies naturally in  $\mathcal{W}$ , implies that the class of  $f$  defines an isomorphism between  $\mathcal{X}$  and  $\mathcal{S}$  in the homotopy category  $\text{hKan}$ . Hence  $f$  is an equivalence. □

There’s actually quite a dynamic interaction between homotopy groups and Kan fibrations, trivial or otherwise. We won’t cover this material here. One can see [10, 00WD, 00WL] for a more in depth discussion.

**Remark 4.30.** Note that the converse to Proposition 4.29 does not hold. Consider for example the 2-disk  $\mathbb{D}^2$ . Since this space is contractible the map  $\text{Sing}(\mathbb{D}^2) \rightarrow *$  for the associated singular complex is a trivial Kan fibration, and any point  $x : * \rightarrow \text{Sing}(\mathbb{D}^2)$  provides a homotopy inverse for this projection. However, it is clear that  $x$  is not a trivial Kan fibration since, in particular, this maps is not surjective on 0-simplices. Indeed,  $\text{Sing}(\mathbb{D}^2)$  has  $|\mathbb{D}^2|$ -many 0-simplices.



#### 4.10. Whitehead's theorem.

**Lemma 4.31** ([10, 00WW]). *Consider two maps  $f, f' : K \rightarrow \mathcal{X}$  from a simplicial set  $K$  to a Kan complex  $\mathcal{X}$ , and suppose that for some choice of points  $z \in K[0]$  and  $x \in \mathcal{X}[0]$  we have  $f(z) = f'(z) = x$ . Then  $f$  and  $f'$  are homotopic if and only if they are pointed homotopic. This is to say, the classes of  $f$  and  $f'$  agree in  $\pi_0(\text{Fun}(K, \mathcal{X}))$  if and only if they agree in  $\pi_0(\text{Fun}(K, \mathcal{X})_{\text{pt}(z,x)})$ .*

The proof uses a notion of Kan replacement [10, 00UW] to reduce to the case where  $K$  is a Kan complex, and employs some technology with fundamental group(oid)s which we've not covered here. We refer the reader directly to [10] for the proof. The implication of this result is clear however.

prop:1560

**Proposition 4.32.** *If  $f : \mathcal{X} \rightarrow \mathcal{Y}$  is an equivalence between Kan complexes, and  $y = f(x) \in \mathcal{Y}[0]$ , then the induced map on connected components  $\pi_0(f) : \mathcal{X} \rightarrow \mathcal{Y}$  is a bijection and the maps on all higher homotopy groups*

$$\pi_n f : \pi_n(\mathcal{X}, x) \rightarrow \pi_n(\mathcal{Y}, y)$$

*are isomorphisms.*

An amazing point is that the inverse implication holds.

thm:whitehead

**Theorem 4.33** (Simplicial Whitehead theorem!). *A map  $f : \mathcal{X} \rightarrow \mathcal{Y}$  between Kan complexes is an equivalence if and only if the induced map on connected components  $\pi_0 f : \pi_0(\mathcal{X}) \rightarrow \pi_0(\mathcal{Y})$  is a bijection and for all  $x \in \mathcal{X}[0]$ ,  $y = f(x) \in \mathcal{Y}$ , the induced maps on homotopy groups*

$$\pi_n f : \pi_n(\mathcal{X}, x) \rightarrow \pi_n(\mathcal{Y}, y)$$

*are all isomorphisms.*

We do not provide the details here, though they are not of a completely alien nature at this point. One find a complete account in [10, 00WU] or [5, Theorem 1.10]. From this point on we simply take this beautiful result for granted.

**Remark 4.34.** If we compare with the category of chain complexes over some ring, then Whitehead's theorem says something like any “quasi-isomorphism” between Kan complexes is in fact a homotopy equivalence. Though we know that this statement is, in a literal sense, not true for chain complexes over  $\mathbb{Z}$  for example, the example of Section 3.3 suggests that Whitehead's theorem really might be as fantastical as this analogy suggests.

sect:kan\_v\_space

**4.11. Kan complexes as spaces.** Let us conclude this (long) section with a justification for the claim that Kan complexes “are” spaces.

We consider the category  $\text{Top} = \text{Top}_{cgu}$  of compactly generated weak Hausdorff spaces. This is a certain full subcategory in the ambient category of all topological spaces which includes all locally compact Hausdorff spaces and all CW-complexes and is closed under taking limits [11, Proposition 2.30].<sup>2</sup> In particular, the geometric realization functor can be considered as a functor to the category of compactly generated weak Hausdorff spaces

$$|-| : \text{sSet} \rightarrow \text{Top}.$$

<sup>2</sup>This category is cocomplete, i.e. has all small colimits, but the colimit in  $\text{Top}$  does not agree with the one in the ambient category of arbitrary topological spaces in general [11].

Now, the category  $\mathbf{Top}$  admits a model structure for which the weak equivalences are those maps  $f : X \rightarrow Y$  which induce isomorphisms on connected components and on all higher homotopy groups,  $\pi_n f : \pi_n(X, x) \xrightarrow{\sim} \pi_n(Y, y)$ . Under this model structure CW complexes are both fibrant and cofibrant, and the unit of the geometric realization/singular complex adjunction provides a weak equivalence

$$|\mathrm{Sing}(X)| \xrightarrow{\sim_w} X.$$

In particular, when  $X$  is (homotopy equivalent to) a CW complex the above map is a homotopy equivalence. This is the original version of Whitehead’s theorem.

Similarly, there is a model structure on the category of simplicial sets for which the Kan complexes are the fibrant and cofibrant objects. The weak equivalences here are those maps which become weak equivalences in  $\mathbf{Top}$  under geometric realization. When we restrict to Kan complexes, weak equivalences are simply homotopy equivalences, and the counit of the geometric realization/singular complex adjunction provides homotopy equivalences

$$\mathcal{X} \xrightarrow{\sim} \mathrm{Sing}(|\mathcal{X}|)$$

whenever  $\mathcal{X}$  is a Kan complex [6, Proposition 4.6.2]. So we observe the following.

**Theorem 4.35.** *Geometric realization induces an equivalence on the level of homotopy categories*

$$|-| : \mathbf{hKan} \xrightarrow{\sim} \mathbf{hTop}. \quad (17)$$

eq:1499

In a more sophisticated statement of things, one considers the obvious simplicial structure  $\underline{\mathbf{Top}}$  on the category  $\mathbf{Top}$  [10, 00JV], and geometric realization extends to a simplicial functor

$$|-|^{\mathrm{simp}} : \underline{\mathbf{Kan}} \rightarrow \underline{\mathbf{Top}}.$$

This functor has image in the full simplicial subcategory  $\underline{\mathbf{Top}}_{\mathrm{cw}}$  of spaces which are homotopic to a CW complex. If we take  $\mathcal{T}op_{\mathrm{cw}}$  to be the homotopy coherent nerve of the simplicial category  $\underline{\mathbf{Top}}_{\mathrm{cw}}$ , and recall our notation  $\mathcal{S}paces = \mathbf{N}^{\mathrm{hc}}(\underline{\mathbf{Kan}})$ , then the induced map on  $\infty$ -categories

$$|-|^{\mathrm{hc}} : \mathcal{S}paces \rightarrow \mathcal{T}op_{\mathrm{cw}}. \quad (18)$$

eq:1509

This map of  $\infty$ -categories is in fact an equivalence (once we decide what this even means) which lifts the equivalence of homotopy categories presented above [10, 01Z4]. This is the sense in which the category of Kan complexes becomes a category of “spaces”.

**Remark 4.36.** If we are more careful about set theoretic things, we should include some smallness assumptions in the above discussion. However, this smallness condition will move with our universe. So the expression (18) should be interpreted as the statement that we have a tower of equivalences which are indexed by the linearly ordered collection on universes.

sect:infty\_cats

## 5. BASIC NOTIONS FOR $\infty$ -CATEGORIES

**5.1. Inner fibrations and inner anodyne maps.** One should recall the notion of a saturated class of morphisms from Section 4.2.

**Definition 5.1.** A map of simplicial sets  $i : A \rightarrow B$  is called inner anodyne if it belongs to the saturated class generated by the inclusions of inner horns  $\{\Lambda_i^n \rightarrow \Delta^n : n > 0 \text{ and } 0 < i < n\}$ . A map of simplicial sets  $f : \mathcal{C} \rightarrow S$  is called an inner fibration if any lifting problem

$$\begin{array}{ccc} A & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow f \\ B & \longrightarrow & S, \end{array}$$

in which  $i$  is inner anodyne, admits a solution.

As in the proof of Lemma 4.2, one sees that  $\mathcal{C} \rightarrow S$  is an inner fibration if and only if one can lift morphisms to  $S$  along any inclusion of an inner horn  $\Lambda_i^n \rightarrow \Delta^n$ . Note also that  $\mathcal{C}$  is an  $\infty$ -category if and only if the map  $\mathcal{C} \rightarrow *$  is an inner fibration. Of course, the term “inner fibration” suggests an augmentation of the notion of a Kan fibration, where one replaces the role of arbitrary horn inclusions  $\Lambda_i^n \rightarrow \Delta^n$  with inner horns.

We adopt the following definition directly from [10].

**def:subcat**

**Definition 5.2.** A simplicial subset  $\mathcal{A} \subseteq \mathcal{C}$  in an  $\infty$ -category  $\mathcal{C}$  is called an  $\infty$ -subcategory, or just a subcategory, if the inclusion map  $\mathcal{A} \rightarrow \mathcal{C}$  is an inner fibration.

Note that an  $\infty$ -subcategory in a given  $\infty$ -category is an  $\infty$ -category in its own right.

**Remark 5.3.** Definition 5.2 is clearly somewhat restrictive. For example, if we include  $\mathbb{S}^1$  into  $\mathbb{S}^2$  along the equator, then the induced map  $\text{Sing}(\mathbb{S}^1) \rightarrow \text{Sing}(\mathbb{S}^2)$  does not realize the singular complex for  $\mathbb{S}^1$  as an  $\infty$ -subcategory in that of  $\mathbb{S}^2$ . It may be the case that the subcategory generated by the image of  $\text{Sing}(\mathbb{S}^1)$  in  $\text{Sing}(\mathbb{S}^2)$  may be equivalent to  $\text{Sing}(\mathbb{S}^1)$  however. The point is not clear. (See [10, 01CN].)

The following lemma is straightforward.

**Lemma 5.4.** *Suppose that we have a pullback diagram*

$$\begin{array}{ccc} \mathcal{C}' & \longrightarrow & \mathcal{C} \\ f' \downarrow & & \downarrow f \\ S' & \longrightarrow & S \end{array}$$

*in which  $f : \mathcal{C} \rightarrow S$  is an inner fibration. Then  $f'$  is an inner fibration as well.*

Let us say that an  $\infty$ -category  $\mathcal{D}$  admits unique horn fillings if, for any inner horn  $\bar{s} : \Lambda_i^n \rightarrow \mathcal{D}$ , there is a *unique*  $n$ -simplex  $s : \Delta^n \rightarrow \mathcal{D}$  with  $s|_{\Lambda_i^n} = \bar{s}$ . An  $\infty$ -category  $\mathcal{D}$  admits unique horn fillings if and only if  $\mathcal{D}$  is isomorphic to the nerve of a plain category [8, Proposition 1.1.2.2].

**lem:1556**

**Lemma 5.5.** *Suppose that  $\mathcal{C}$  and  $\mathcal{D}$  are  $\infty$ -categories, and that  $f : \mathcal{C} \rightarrow \mathcal{D}$  is a map of simplicial sets. If  $\mathcal{D}$  admits unique horn fillings, then  $f$  is an inner fibration.*

*Proof.* Consider a diagram

$$\begin{array}{ccc}
 \Lambda_i^n & \xrightarrow{\bar{s}} & \mathcal{C} \\
 \downarrow & & \downarrow f \\
 \Delta^n & \xrightarrow{t} & \mathcal{D} \\
 & \searrow pt & \searrow \\
 & & *
 \end{array}$$

and any lift  $s : \Delta^n \rightarrow \mathcal{C}$  of  $pt$ . Then  $fs : \Delta^n \rightarrow \mathcal{D}$  is a map to  $\mathcal{D}$  with  $fs|_{\Lambda_i^n} = f\bar{s} = t|_{\Lambda_i^n}$ . Since  $\mathcal{D}$  admits unique horn fillings, we have  $fs = \bar{s}$ .  $\square$

As a corollary one sees that, for any  $\infty$ -category  $\mathcal{C}$ , the truncation  $p : \mathcal{C} \rightarrow \mathbf{N}(\mathbf{h}\mathcal{C})$  is an inner fibration. One also sees that, for any  $\infty$ -category  $\mathcal{D}$  with unique horn fillings, and any map  $f : \mathcal{C} \rightarrow \mathcal{D}$  from an arbitrary simplicial set  $\mathcal{C}$ , we have that  $\mathcal{C}$  is an  $\infty$ -category if and only if  $f$  is an inner fibration.

We have the following technical lemma, which is an inner variant of Lemma 4.7.

lem:1575

**Lemma 5.6.** *Consider two monomorphisms of simplicial sets  $i : A \rightarrow B$  and  $j : K \rightarrow L$ . If either of  $i$  or  $j$  is an inner anodyne morphism, then the induced map*

$$(A \times L) \amalg_{(A \times K)} (B \times K) \rightarrow (B \times L)$$

*is inner anodyne as well.*

The proof can be found in [10, 00JB].

## 5.2. Exponentials of $\infty$ -categories.

prop:tech2

**Proposition 5.7.** *Suppose that  $i : K \rightarrow L$  is a monomorphism of simplicial sets, and that  $f : \mathcal{C} \rightarrow S$  is an inner Kan fibration. Then the induced map on functors*

$$\mathrm{Fun}(L, \mathcal{C}) \rightarrow \mathrm{Fun}(K, \mathcal{C}) \times_{\mathrm{Fun}(K, S)} \mathrm{Fun}(L, S) \quad (19)$$

eq:1597

*is an inner fibrations. Furthermore, if  $i : K \rightarrow L$  is inner anodyne, then the map (19) is a trivial Kan fibration.*

*Proof.* One replaces Lemma 4.7 with Lemma 5.6 and proceeds exactly as in the proof of Proposition 4.11.  $\square$

If one considers the inclusion  $\emptyset \rightarrow K$  and, when  $\mathcal{C}$  is an  $\infty$ -category, the inner fibration  $\mathcal{C} \rightarrow *$ , then one arrives at the following.

cor:Fun\_infty

**Corollary 5.8.** (1) *For any simplicial set  $K$ , and  $\infty$ -category  $\mathcal{C}$ , the simplicial set  $\mathrm{Fun}(K, \mathcal{C})$  is an  $\infty$ -category.*

- (2) *For any simplicial set  $K$ , and inner fibration  $f : \mathcal{C} \rightarrow S$ , the morphism  $f_* : \mathrm{Fun}(K, \mathcal{C}) \rightarrow \mathrm{Fun}(K, S)$  is an inner fibration.*
- (3) *For any monomorphism  $i : K \rightarrow L$ , and  $\infty$ -category  $\mathcal{C}$ , the map of  $\infty$ -categories  $i^* : \mathrm{Fun}(L, \mathcal{C}) \rightarrow \mathrm{Fun}(K, \mathcal{C})$  is an inner fibration.*
- (4) *For any inner anodyne morphism  $i : K \rightarrow L$ , and  $\infty$ -category  $\mathcal{C}$ , the map of  $\infty$ -categories  $i^* : \mathrm{Fun}(L, \mathcal{C}) \rightarrow \mathrm{Fun}(K, \mathcal{C})$  is a trivial Kan fibration.*

**5.3. Opposite categories.** For any linearly ordered set  $J$  define  $J^{\text{rev}}$  to be the set  $J$  with the reversed ordering,

$$j < j' \Leftrightarrow j >_{\text{rev}} j'.$$

Note that for any linearly ordered sets  $I$  and  $J$ , and set map  $r : I \rightarrow J$ , we have that  $r$  is weakly increasing as a function from  $I$  to  $J$  if and only if  $r$  is weakly increasing when considered as a function from  $I^{\text{rev}}$  to  $J^{\text{rev}}$ . Hence the reversing operation defines an autoequivalence

$$-^{\text{rev}} : \Delta \xrightarrow{\cong} \Delta.$$

Indeed,  $-^{\text{rev}}$  is an involution on  $\Delta$ .

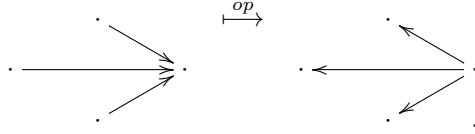
Clearly any linearly ordered set  $J$  admits a unique isomorphism  $J \cong J^{\text{rev}}$  in  $\Delta^{\text{op}}$ . This isomorphism is, however, *not* natural. One can interpret this failure of naturality as an assurance that the reversal operation has some non-trivial content.

We define the opposite  $K^{\text{op}}$  of a given simplicial set  $K$  to be the simplicial set

$$K^{\text{op}} := K \circ (-^{\text{rev}}).$$

So, the  $n$ -simplices  $K^{\text{op}}[n]$  in  $K^{\text{op}}$  are identified with the set  $K((\Delta^n)^{\text{rev}})$ .

**Example 5.9.** Let  $C$  be a plain category. Then  $N(C)^{\text{op}} = N(C^{\text{op}})$ . As a more explicit example, one takes opposites of diagrams



Clearly a natural transformation of simplicial sets  $f : K \rightarrow L$  induces a transformation  $f^{\text{op}} : K^{\text{op}} \rightarrow L^{\text{op}}$ , where  $f_J^{\text{op}} = f_{J^{\text{rev}}} : K^{\text{op}}(J) = K(J^{\text{rev}}) \rightarrow L(J^{\text{rev}}) = L^{\text{op}}(J)$ . So the opposite operation defines an involution on  $\text{sSet}$ .

**Lemma 5.10.** *The opposite  $\mathcal{X}^{\text{op}}$  of a Kan complex  $\mathcal{X}$  is another Kan complex. The opposite  $\mathcal{C}^{\text{op}}$  of an  $\infty$ -category is another  $\infty$ -category.*

*Proof.* This follows from the fact that the opposite of a horn inclusion  $j : \Lambda_i^n \rightarrow \Delta^n$  is identified with the horn inclusion  $\Lambda_{n-i}^n \rightarrow \Delta^n$ , so that a lifting problem

$$\begin{array}{ccc} \Lambda_i^n & \xrightarrow{\quad} & \mathcal{C}^{\text{op}} \\ \downarrow & \nearrow & \downarrow \\ \Delta^n & \xrightarrow{\quad} & * \end{array}$$

admits a solution if and only if the corresponding lifting problem on opposite categories

$$\begin{array}{ccc} \Lambda_{n-i}^n & \xrightarrow{\quad} & \mathcal{C} \\ \downarrow & \nearrow & \downarrow \\ \Delta^n & \xrightarrow{\quad} & * \end{array}$$

admits a solution. □

#### 5.4. Isomorphisms in $\infty$ -categories.

**Definition 5.11.** A morphism  $f : x \rightarrow y$  in an  $\infty$ -category  $\mathcal{C}$  is called an isomorphism if there exists another morphism  $g : y \rightarrow x$  for which one has (not necessarily unique) 2-simplices

$$\begin{array}{ccc} & y & \\ f \nearrow & & \searrow g \\ x & \xrightarrow{id_x} & x \end{array} \quad \text{and} \quad \begin{array}{ccc} & x & \\ g \nearrow & & \searrow f \\ y & \xrightarrow{id_y} & y \end{array}$$

Equivalently,  $f$  is an isomorphism in  $\mathcal{C}$  if its class  $\bar{f}$  in the homotopy category  $\mathbf{h}\mathcal{C}$  is an isomorphism.

**Remark 5.12.** In [8], such a map  $f$  is simply referred to as an equivalence, while in [10] the term isomorphism is explicitly used. In familiar settings, one might use the term isomorphism to refer to a map in some category  $\mathbf{C}$  which is literally invertible, while referring to some weaker relations as equivalences. However, one should note that in the generic  $\infty$ -categorical setting there is not stronger notion of equivalence than the one given above. This is because all 2-simplices are formally indistinguishable in  $\mathcal{C}$ . This is to say, there is no preferred method for “inverting” a given map  $f$ . Also, a map in the nerve  $N(\mathbf{C})$  of a plain category is an isomorphism if and only if the corresponding map in  $\mathbf{C}$  is an isomorphism.

**Example 5.13.** A morphism  $\alpha : x \rightarrow y$  in the homotopy  $\infty$ -category  $\mathcal{H}(A)$  of dg modules, over some dg algebra  $A$ , is an isomorphism if and only if  $\alpha$  is a homotopy equivalence.

def:assoc\_kan

**Definition 5.14.** Let  $\mathcal{C}$  be an  $\infty$ -category. The Kan complex associated to  $\mathcal{C}$  is the simplicial subset  $\mathcal{C}^{\text{Kan}}$  whose  $n$ -simplices  $\Delta^n \rightarrow \mathcal{C}^{\text{Kan}}$  consist of all  $n$ -simplices in  $\mathcal{C}$ ,  $s : \Delta^n \rightarrow \mathcal{C}$ , which restrict to an isomorphism  $s|_{\Delta^1} : \Delta^1 \rightarrow \mathcal{C}$  along each inclusion  $\Delta^1 \rightarrow \Delta^n$ .

Since there are no inclusions of  $\Delta^1$  into  $\Delta^0$ ,  $\mathcal{C}^{\text{Kan}}$  has the same objects as  $\mathcal{C}$ , and the morphisms in  $\mathcal{C}^{\text{Kan}}$  are precisely the isomorphisms in  $\mathcal{C}$ . Clearly  $\mathcal{C}^{\text{Kan}}$  is the largest simplicial subset in  $\mathcal{C}$  with this property.

**Lemma 5.15.** For any  $\infty$ -category  $\mathcal{C}$ , the associated Kan complex  $\mathcal{C}^{\text{Kan}}$  is an  $\infty$ -category, and the inclusion  $\mathcal{C}^{\text{Kan}} \rightarrow \mathcal{C}$  is an inner fibration.

*Proof.* It suffices to prove that the inclusion  $i : \mathcal{C}^{\text{Kan}} \rightarrow \mathcal{C}$  is an inner fibration. Consider a lifting problem

$$\begin{array}{ccc} \Lambda_i^n & \xrightarrow{\quad} & \mathcal{C}^{\text{Kan}} \\ \downarrow & \nearrow & \downarrow i \\ \Delta^n & \xrightarrow{s} & \mathcal{C} \end{array}$$

with  $0 < i < n$ . When  $n = 2$ , we simply note that isomorphisms in the homotopy category  $\mathbf{h}\mathcal{C}$  have the 2-of-3 property. So if  $s : \Delta^2 \rightarrow \mathcal{C}$  is a 2-simplex whose restriction to a horn  $\Lambda_i^2 \rightarrow \Delta^2 \rightarrow \mathcal{C}$  factors  $\Lambda_i^2 \rightarrow \mathcal{C}^{\text{Kan}} \rightarrow \mathcal{C}$ , then  $s$  itself factors through  $\mathcal{C}^{\text{Kan}}$ . For all higher simplices,  $n > 2$ , we note that all 1-simplices  $\Delta^1 \rightarrow \Delta^n$  all factor uniquely through the horn  $\Lambda_i^n \rightarrow \Delta^n$  so see that the simplex  $s$  is actually a simplex in  $\mathcal{C}^{\text{Kan}}$ . We therefore have a unique solution to the given lifting problem.  $\square$

One of the aims of this section is to prove that  $\mathcal{C}^{\text{Kan}}$  is not only an  $\infty$ -category, but a Kan complex. In order to prove this result we need to develop a number of notions which are quite important in their own right. We discuss overcategories, undercategories, and isofibrations then return to the topic of the associated Kan complex in Section 5.9.

**Remark 5.16.** Lurie refers to the subcategory  $\mathcal{C}^{\text{Kan}}$  as the *core* of  $\mathcal{C}$ , and denotes this subcategory  $\mathcal{C}^\simeq$ .

**5.5. Joins of simplicial sets.** For a linearly ordered set  $I$  let's take  $P_\pm(I)$  to be the collection of all partitions  $I = I_- \amalg I_+$  such that  $i < j$  for each  $i \in I_-$  and  $j \in I_+$ . Note that  $P_\pm(I)$  is identified with the collection of weakly increasing maps  $I \rightarrow [1]$ ,  $P_\pm(I) = \text{Hom}_\Delta(I, [2])$ .

**Definition 5.17.** Let  $A$  and  $B$  be simplicial sets. The join of  $A$  and  $B$ , denoted  $A \star B$ , is the simplicial set with  $I$ -simplices

$$(A \star B)(I) := \coprod_{(I_-, I_+) \in P_\pm(I)} A(I_-) \times B(I_+),$$

where one takes formally  $A(\emptyset) = B(\emptyset) = *$ . Given any weakly decreasing map  $r : I \rightarrow J$ , restriction

$$r^* : (A \star B)(J) \rightarrow (A \star B)(I)$$

sends a pair  $(s, t) \in A(J_-) \times B(J_+)$  to  $(r^*s, r^*t) \in A(r^{-1}J_-) \times B(r^{-1}J_+)$ .

One should note that we have canonical inclusions  $A \rightarrow A \star B$  and  $B \rightarrow A \star B$ . We note also that the join forms a bifunctor

$$\star : \text{sSet} \times \text{sSet} \rightarrow \text{sSet}.$$

Given two maps  $f : A \rightarrow A'$  and  $g : B \rightarrow B'$  the join  $f \star g : A \star B \rightarrow A' \star B'$  sends each pair of simplices  $(s, t)$  in  $A \star B$  to the corresponding pair  $(fs, gt)$  in  $A' \star B'$ . In terms of bifunctoriality, the two inclusions of the original simplicial sets into  $A \star B$  are deduced as the composites

$$A \cong A \star \emptyset \rightarrow A \star B \quad \text{and} \quad B \cong \emptyset \star B \rightarrow A \star B,$$

where the maps  $\emptyset \star B \rightarrow A \star B$  and  $A \star \emptyset \rightarrow A \star B$  are the joins of the unique morphisms from the emptyset with the respective identities.

**Example 5.18.** There are isomorphisms  $\Delta^0 \star \Delta^n \cong \Delta^{n+1}$  and  $\Delta^n \star \Delta^0 \cong \Delta^n$ . In the first case we take a pair of maps  $s_- : I_- \rightarrow \{0\}$  and  $s_+ : I_+ \rightarrow [n]$  to the unique map  $s : I \rightarrow [n+1]$  with  $s|_{I_-} = s_-$  and  $s(i) = s_+(i-1)$  for all  $i \in I_+$ . The second isomorphism is defined similarly. Indeed, we can argue as above to obtain an isomorphism

$$\Delta^m \star \Delta^n \xrightarrow{\cong} \Delta^{m+n+1}$$

We note that, since  $\Delta^{m+n+1}$  admits no nontrivial automorphism, the above isomorphism is *unique*.

If we take  $\Delta^{-1} = \emptyset$  we can describe  $n$ -simplices in the join  $s : \Delta^n \rightarrow A \star B$  as a choice of pair of integers  $m, m' \geq -1$  such that  $m + m' = n - 1$ , and a choice of simplices  $s_- : \Delta^m \rightarrow A$  and  $s_+ : \Delta^{m'} \rightarrow B$ . We reconstruct the original simplex as the composite of the unique isomorphism  $\Delta^n \cong \Delta^m \star \Delta^{m'}$  with the join  $s_- \star s_+ : \Delta^m \star \Delta^{m'} \rightarrow A \star B$ ,

$$s = \left( \Delta^n \cong \Delta^m \star \Delta^{m'} \xrightarrow{s_- \star s_+} A \star B \right).$$

One finds  $s_-$  and  $s_+$  as the maximal initial and terminal faces in  $\Delta^n$  which have images in the subcomplexes  $A \subset A \star B$  and  $B \subset A \star B$  respectively.

**prop:1713**

**Proposition 5.19.** *Suppose that  $f : \mathcal{C} \rightarrow S$  and  $g : \mathcal{D} \rightarrow T$  are inner fibrations. Then the join*

$$f \star g : \mathcal{C} \star \mathcal{D} \rightarrow S \star T$$

*is another inner fibration.*

*Proof.* The important point to keep in mind here is that an  $l$ -simplex  $t : \Delta^l \rightarrow \mathcal{C} \star \mathcal{D}$  has image in  $\mathcal{C}$  if and only if its composite  $\Delta^l \rightarrow \mathcal{C} \star \mathcal{D}$  along  $f \star g$  has image in  $S$ . Similarly,  $t$  has image in  $\mathcal{D}$  if and only if its composite along  $f \star g$  has image in  $T$ . So, one should keep this basic point in mind.

Consider a lifting problem

$$\begin{array}{ccc} \Lambda_i^n & \xrightarrow{\bar{s}} & \mathcal{C} \star \mathcal{D} \\ \downarrow & \nearrow & \downarrow f \star g \\ \Delta^n & \xrightarrow{s} & S \star T. \end{array}$$

where  $\Lambda_i^n \rightarrow \Delta^n$  is an inner horn. As was discussed above, we can decompose  $s$  uniquely into the join of simplices  $s_- : \Delta^m \rightarrow S$  and  $s_+ : \Delta^{m'} \rightarrow T$ , where  $m, m' \geq -1$  and  $m + m' = n - 1$ . If  $m$  or  $m'$  is less than 0, then  $s$  factors through  $S$  and  $\bar{s}$  factors through  $\mathcal{C}$ , or  $s$  factors through  $T$  and  $\bar{s}$  factors through  $\mathcal{D}$ . In either case, one can resolve this lifting problem. So let us suppose  $m, m' \geq 0$ .

We have the faces  $\Delta^m \rightarrow \Delta^m \star \Delta^{m'}$  and  $\Delta^{m'} \rightarrow \Delta^m \star \Delta^{m'}$  and, by the fact that our horn is inner, both of these faces factor through the horn

$$\Delta^m, \Delta^n \rightarrow \Lambda_i^n \rightarrow \Delta^m \star \Delta^{m'}.$$

We therefore have unique lifts  $s'_- : \Delta^m \rightarrow \mathcal{C} \star \mathcal{D}$  and  $s'_+ : \Delta^{m'} \rightarrow \mathcal{C} \star \mathcal{D}$ . Since the images of  $s'_-$  and  $s'_+$  in  $S \star T$  lie in the simplicial subsets  $S$  and  $T$  respectively, we see that  $s'_-$  and  $s'_+$  themselves factor uniquely through  $\mathcal{C}$  and  $\mathcal{D}$  respectively. So we consider the  $n$ -simplex

$$s' = s'_- \star s'_+ : \Delta^n \rightarrow \mathcal{C} \star \mathcal{D}.$$

We have  $(f \star g)s' = (fs'_-) \star (gs'_+) = s_- \star s_+ = s$ . We claim that  $s'|\Lambda_i^n = \bar{s}$ , and hence that  $s'$  solves our lifting problem.

To establish the equality  $s'|\Lambda_i^n = \bar{s}$  it suffices to establish equalities  $s'|\Delta^{n-1} = \bar{s}|\Delta^{n-1}$  for all of the  $(n-1)$ -faces  $\Delta^{n-1} \rightarrow \Lambda_i^n$ . But here we can factor each composite  $s'|\Delta^{n-1} : \Delta^{n-1} \rightarrow \mathcal{C} \star \mathcal{D}$  as

$$\Delta^{n-1} = \Delta^l \star \Delta^{l'} \xrightarrow{i \star i'} \Delta^m \star \Delta^{m'} \xrightarrow{s'_- \star s'_+} \mathcal{C} \star \mathcal{D},$$

where  $i : \Delta^l \rightarrow \Delta^m$  and  $i' : \Delta^{l'} \rightarrow \Delta^{m'}$  are face inclusions. By construction  $s'|\Delta^l = \bar{s}|\Delta^l$  and  $s'|\Delta^{l'} = \bar{s}|\Delta^{l'}$ . So  $\bar{s}$  maps  $\Delta^l$  into  $\mathcal{C}$ , and maps  $\Delta^{l'}$  into  $\mathcal{D}$ , and we therefore conclude

$$\bar{s}|\Delta^{n-1} = (\bar{s}|\Delta^l) \star (\bar{s}|\Delta^{l'}) = (s'_-|\Delta^l) \star (s'_+|\Delta^{l'}) = s'|\Delta^{n-1},$$

as required. So we see  $s'|\Lambda_i^n = \bar{s}$ , and hence that the original lifting problem admits a solution.  $\square$

**Corollary 5.20.** *If  $\mathcal{C}$  and  $\mathcal{D}$  are  $\infty$ -categories, then the join  $\mathcal{C} \star \mathcal{D}$  is an  $\infty$ -category.*



*Proof.* The map  $\mathcal{C} \star \mathcal{D} \rightarrow \Delta^0 \star \Delta^0 = \Delta^1$  is an inner fibration by Proposition 5.19. Since  $\Delta^1 = N(\{0 < 1\})$  has admits unique horn fillings, Lemma 5.5 now implies that  $\mathcal{C} \star \mathcal{D}$  is an  $\infty$ -category.  $\square$

**5.6. Overcategories and undercategories.** Let  $p : K \rightarrow \mathcal{C}$  be a map from a simplicial set  $K$  to another simplicial set  $\mathcal{C}$ . We will generally refer to  $p$  as a “diagram” in  $\mathcal{C}$ . (The case where  $\mathcal{C}$  is an  $\infty$ -category is most important for us, but it will be helpful to just consider simplicial sets for now.)

We define the overcategory  $\mathcal{C}/p$  to be the simplicial set with simplices

$$\mathcal{C}/p[n] = \text{Hom}_{\text{sSet}}(\Delta^n \star K, \mathcal{C})_p,$$

where the subscript  $p$  indicates that we consider maps of simplicial sets  $p' : \Delta^n \star K \rightarrow \mathcal{C}$  for which  $p'|_K = p$ . Restriction in  $\mathcal{C}/p$  along some weakly increasing function  $[m] \rightarrow [n]$  is defined by restricting along the corresponding map of simplicial sets  $\Delta^m \rightarrow \Delta^n$ . We similarly define the undercategory  $\mathcal{C}_{p/}$  as the simplicial set with simplices

$$\mathcal{C}_{p/} = \text{Hom}_{\text{sSet}}(K \star \Delta^n, \mathcal{C})_p.$$

By construction there are identifications

$$\text{Hom}_{\text{sSet}}(\Delta^n, \mathcal{C}/p) \xrightarrow{\cong} \text{Hom}_{\text{sSet}}(\Delta^n \star K, \mathcal{C})_p, \quad f \mapsto f(id_{[n]}) \quad (20) \quad \boxed{\text{eq:1784}}$$

and

$$\text{Hom}_{\text{sSet}}(\Delta^n, \mathcal{C}_{p/}) \xrightarrow{\cong} \text{Hom}_{\text{sSet}}(K \star \Delta^n, \mathcal{C})_p, \quad g \mapsto g(id_{[n]}). \quad (21) \quad \boxed{\text{eq:1788}}$$

Since the join functor is seen to commute with colimits the above isomorphism extend to adjunctions which are natural in all factors.

**Lemma 5.21** ([10, 017Z]). *For any diagram  $p : K \rightarrow \mathcal{C}$ , which can similarly be considered as a diagram to the opposite category  $p : K^{op} \rightarrow \mathcal{C}^{op}$ , there is an identification of simplicial sets  $(\mathcal{C}/p)^{op} \cong (\mathcal{C}^{op})_{p/}$ .*

*Proof.* Follows from the identification  $(\Delta^n \star K)^{op} = K^{op} \star (\Delta^n)^{op} \cong K^{op} \star \Delta^n$ .  $\square$

lem:1798

**Lemma 5.22** ([10, 0189]). *Consider an arbitrary diagram  $p : K \rightarrow \mathcal{C}$  in a simplicial set  $\mathcal{C}$ . For any simplicial set  $L$ , there are unique isomorphisms*

$$\text{Hom}_{\text{sSet}}(L, \mathcal{C}/p) \xrightarrow{\cong} \text{Hom}_{\text{sSet}}(L \star K, \mathcal{C})_p \quad (22) \quad \boxed{\text{eq:1795}}$$

and

$$\text{Hom}_{\text{sSet}}(L, \mathcal{C}_{p/}) \xrightarrow{\cong} \text{Hom}_{\text{sSet}}(K \star L, \mathcal{C})_p$$

which are natural in  $L$  and recover the isomorphisms (20, 21) when  $L = \Delta^n$ .

*Construction.* We construct the isomorphism for  $\mathcal{C}/p$ . The construction for  $\mathcal{C}_{p/}$  is similar, or can be recovered by replacing  $\mathcal{C}$  with its opposite category. First we consider the map

$$ev : \mathcal{C}/p \star K \rightarrow \mathcal{C}, \quad (s : \Delta^l \rightarrow \mathcal{C}/p, t : \Delta^m \rightarrow K) \mapsto s(id_{[l]}, t).$$

(Here we’ve abused notation to identify the function  $s : \Delta^l \rightarrow \mathcal{C}/p$  with its value at  $id_{[l]}$ .) Then the isomorphism (22) sends a function  $f : L \rightarrow \mathcal{C}/p$  to the composite  $ev \circ (f \star id_K)$ . The inverse takes a map  $\eta : L \star K \rightarrow \mathcal{C}$  with appropriate restriction to the map

$$\eta' : L \rightarrow \mathcal{C}/p, \quad (s : \Delta^n \rightarrow L) \mapsto \eta \circ (s \star id_K).$$

$\square$

We note that the overcategory/undercategory construction does enjoy some functoriality. For example, if we have a map of simplicial sets  $i : A \rightarrow K$  and an arbitrary diagram  $p : K \rightarrow \mathcal{C}$ , then restricting along  $i$  defines maps

$$i_{\text{over}}^* : \mathcal{C}_{/p} \rightarrow \mathcal{C}_{/pi} \quad \text{and} \quad i_{\text{under}}^* : \mathcal{C}_{p/} \rightarrow \mathcal{C}_{pi/}.$$

Similarly, if we have a map of simplicial sets  $f : \mathcal{C} \rightarrow \mathcal{D}$  then composing with  $f$  defines a maps

$$f_*^{\text{over}} : \mathcal{C}_{/p} \rightarrow \mathcal{D}_{/fp} \quad \text{and} \quad f_*^{\text{under}} : \mathcal{C}_{p/} \rightarrow \mathcal{D}_{fp/}.$$

If we consider such induced morphisms, the isomorphism of Lemma 5.22 will be natural in the  $K$  and  $\mathcal{C}$  variables as well.

### 5.7. Directional fibrations and under/overcategories of $\infty$ -categories.

`def:lr_anodyne`

**Definition 5.23.** The class of left (resp. right) anodyne maps  $i : A \rightarrow B$  is the saturated class of morphisms in  $\mathbf{sSet}$  generated by the collection of horn inclusions  $\Lambda_i^n \rightarrow \Delta^n$ , where  $n$  is arbitrary and  $0 \leq i < n$  (resp.  $0 < i \leq n$ ).

A map of simplicial sets  $f : \mathcal{C} \rightarrow S$  is called a left fibration (resp. right fibration) if any lifting problem

$$\begin{array}{ccc} A & \longrightarrow & \mathcal{C} \\ i \downarrow & & \downarrow f \\ B & \longrightarrow & S \end{array}$$

in which  $i$  is left anodyne (resp. right anodyne) admits a solution.

So, it is *easier* to be a right/left anodyne map, and hence *harder* for a map of simplicial sets to be a left/right fibration. We have the join analog of Lemma 4.7, which is slightly more robust.

`lem:1806`

**Lemma 5.24** ([10, 018J]). *For monomorphisms  $i : A \rightarrow B$  and  $j : K \rightarrow L$ , the corresponding map from the pushout*

$$(A \star L) \coprod_{(A \star K)} (B \star K) \rightarrow B \star L$$

*is inner anodyne whenever  $i$  is right anodyne or  $j$  is left anodyne.*

One employs Lemma 5.24 and follows the arguments of Proposition 4.11 to obtain the following. We repeat these arguments, since we haven't seen them in a while, and we miss them.

`prop:tech3`

**Proposition 5.25.** *Let  $f : \mathcal{C} \rightarrow S$  be an inner fibration, and  $p : K \rightarrow \mathcal{C}$  be any diagram in  $\mathcal{C}$ . Then for any monomorphism  $j : K_0 \rightarrow K$ , and corresponding diagram  $\pi = pj : K_0 \rightarrow \mathcal{C}$ , the induced map on diagram overcategories*

$$\mathcal{C}_{/p} \rightarrow \mathcal{C}_{/\pi} \times_{S_{/\pi}} S_{/p}$$

*is a right fibration, and the induced map on undercategories*

$$\mathcal{C}_{p/} \rightarrow \mathcal{C}_{\pi/} \times_{S_{\pi/}} S_{p/}$$

*is a left fibration.*

Clearly we have abused notation here and written  $S_{/p}$  instead of  $S_{/pf}$  and  $S_{/\pi}$  instead of  $S_{/\pi f}$ , for example.

*Proof.* The claim about undercategories follows from the claim about overcategories, after applying the opposite involution on  $\mathbf{sSet}$ . So, we deal with the overcategories.

Any lifting problem

$$\begin{array}{ccc} A & \longrightarrow & \mathcal{C}_{/p} \\ \downarrow i & & \downarrow \\ B & \longrightarrow & \mathcal{C}_{/\pi} \times_{S_{/\pi}} S_{/p} \end{array}$$

defines a corresponding lifting problem

$$\begin{array}{ccc} (A \star K) \amalg_{(A \star K_0)} (B \star K_0) & \longrightarrow & \mathcal{C} \\ \downarrow & \nearrow \text{dashed} & \downarrow f \\ B \star K & \longrightarrow & S \end{array}$$

via adjunction, where the left vertical map is defined by the maps  $i \star id_K$  and  $id_B \star j$ . If we suppose that  $i : A \rightarrow B$  is right anodyne, then Lemma 5.24, and the fact that  $f$  is an inner fibration, assures us that the second lifting problem admits a solution. Hence the first lifting admits a solution.  $\square$

The same argument can be used to establish the following.

**prop:tech3.5**

**Proposition 5.26.** *Suppose that we are in the setting of Proposition 5.25. If the inclusion  $j : K_0 \rightarrow K$  is left anodyne, then the induced map on overcategories*

$$\mathcal{C}_{/p} \rightarrow \mathcal{C}_{/\pi} \times_{S_{/\pi}} S_{/p}$$

*is a trivial Kan fibration. Similarly, if  $j : K_0 \rightarrow K$  is right anodyne, the induced map*

$$\mathcal{C}_{p/} \rightarrow \mathcal{C}_{\pi/} \times_{S_{\pi/}} S_{p/}$$

*is a trivial Kan fibration.*

If we consider the case  $S = *$  and  $K_0 = \emptyset$ , then we have identifications  $\mathcal{C}_{/\pi} = \mathcal{C}$  and  $S_{/p} = S_{/\pi} = *$ , essentially by the adjunction of Lemma 5.22. Similarly  $\mathcal{C}_{\pi/} = \mathcal{C}$  and  $S_{p/} = S_{\pi/} = *$  in this case. So we obtain the following.

**cor:1947**

**Corollary 5.27.** *Let  $\mathcal{C}$  be an  $\infty$ -category and  $p : K \rightarrow \mathcal{C}$  be an arbitrary diagram. Then restricting along the inclusions  $\Delta^n \rightarrow \Delta^n \star K$  defines a map of simplicial sets*

$$\mathcal{C}_{/p} \rightarrow \mathcal{C}.$$

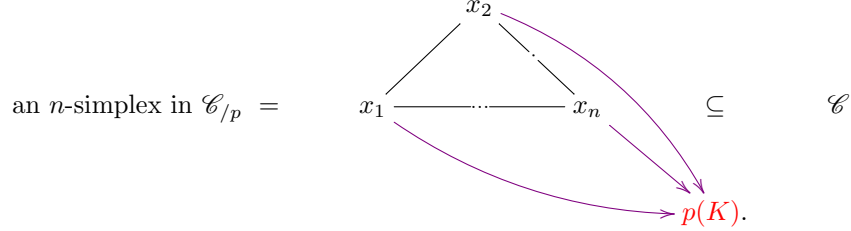
*This map is an right fibration. Similarly, restriction defines a map*

$$\mathcal{C}_{p/} \rightarrow \mathcal{C}$$

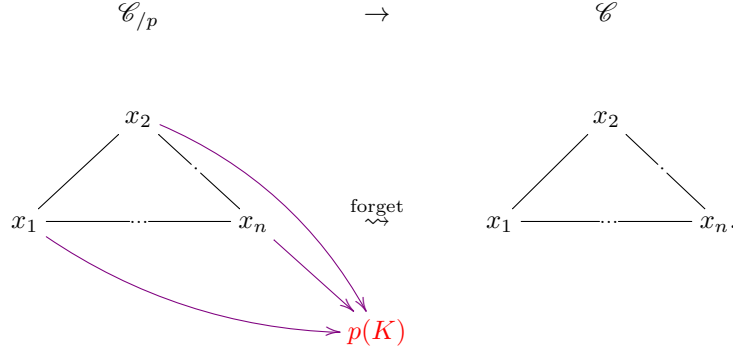
*which is a left fibration. In particular, both the overcategory  $\mathcal{C}_{/p}$  and the undercategory  $\mathcal{C}_{p/}$  are  $\infty$ -categories.*

Let's take a moment to explain clearly what this map  $\mathcal{C}_{/p} \rightarrow \mathcal{C}$  actually “looks like”. An  $n$ -simplex in  $\mathcal{C}$  can be thought of as a  $\Delta^n$ -shaped diagram in  $\mathcal{C}$ . Directly, an  $n$ -simplex in  $\mathcal{C}$  is identified with a map of simplicial sets  $s : \Delta^n \rightarrow \mathcal{C}$ . To compare, an  $n$ -simplex in  $\mathcal{C}_{/p}$  is a  $\Delta^n \star K$ -shaped diagram in  $\mathcal{C}$ , i.e. a map  $s' : \Delta^n \star K \rightarrow \mathcal{C}$ .

Now, the simplicial set  $\Delta^n \star K$  can be visualized as a copy of  $\Delta^n$  floating in space, a copy of  $K$  floating in space, and a bunch of connective simplices from  $\Delta^n$  to  $K$ . So a  $\Delta^n \star K$ -shaped diagram in  $\mathcal{C}$  appears as follows:



The map  $\mathcal{C}_{/p} \rightarrow \mathcal{C}$  simply keeps the uncolored portion of the above diagram and forgets about the colored portions,



Similarly, the map  $\mathcal{C}_{/p} \rightarrow \mathcal{C}_{/\pi}$  forgets *some* portion of the colored diagram, while remembering others. One can provide a similar describe the map  $\mathcal{C}_{p/} \rightarrow \mathcal{C}$ .

We record a final corollary to Proposition 5.26.

**cor:1998**

**Corollary 5.28.** *Let  $\mathcal{C}$  be an  $\infty$ -category and  $p : K \rightarrow \mathcal{C}$  be an arbitrary diagram. Adopt the notation from Proposition 5.25. If the inclusion  $j : K_0 \rightarrow K$  is left anodyne then the induced map*

$$\mathcal{C}_{/p} \rightarrow \mathcal{C}_{/\pi}$$

*is a trivial Kan fibration. If the inclusion  $j : K_0 \rightarrow K$  is right anodyne, then the induced map*

$$\mathcal{C}_{p/} \rightarrow \mathcal{C}_{\pi/}$$

*is a trivial Kan fibration.*

### 5.8. Isofibrations.

**def:isofib**

**Definition 5.29.** A map of  $\infty$ -categories  $F : \mathcal{C} \rightarrow \mathcal{D}$  is called an isofibrations if

- (a)  $F$  is an inner fibration.
- (b) For every object  $x$  in  $\mathcal{C}$ , and isomorphism  $\gamma' : y' \rightarrow F(x)$  in  $\mathcal{D}$ , there exists an object  $y$  in  $\mathcal{C}$  and an isomorphism  $\gamma : y \rightarrow x$  with  $F(\gamma) = \gamma'$ .
- (b') For every object  $y$  in  $\mathcal{C}$ , and isomorphism  $\theta' : F(y) \rightarrow x'$  in  $\mathcal{D}$ , there exists an object  $x$  in  $\mathcal{C}$  and an isomorphism  $\theta : y \rightarrow x$  with  $F(\theta) = \theta'$ .

Colloquially, an isofibration is an inner fibration along which one can lift isomorphisms.

lem:2034

**Lemma 5.30.** *Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be an inner fibration between  $\infty$ -categories. Then  $F$  satisfies property (b) from Definition 5.29 if and only if  $F$  satisfies property (b'). Rather, these two lifting properties are equivalent for any inner fibration between  $\infty$ -categories.*

*Proof.* Suppose  $F$  satisfies property (b), and consider a pairing of an object  $y$  in  $\mathcal{C}$  and an isomorphism  $\theta' : F(y) \rightarrow x'$  in  $\mathcal{D}$ . Take  $y' = F(y)$  and consider a homotopy inverse  $\zeta' : x' \rightarrow y'$  in  $\mathcal{D}$ . We can lift this inverse to an isomorphism  $\zeta : x \rightarrow y$  in  $\mathcal{C}$ , by hypothesis. In particular, we can lift  $x'$  to the object  $x$  in  $\mathcal{C}$ . Consider an inverse  $\vartheta : y \rightarrow x$  to  $\zeta$  in  $\mathcal{C}$ , write  $\theta'' = F(\vartheta) : y' \rightarrow x'$ . Note that  $\theta''$  provides an inverse to  $\zeta'$ . By uniqueness of inverses (up to 2-simplices) we have a 2-simplex  $s : \Delta^2 \rightarrow \mathcal{D}$  of the form

$$\begin{array}{ccc} & x' & \\ \theta'' \nearrow & & \searrow id \\ y' & \xrightarrow{\theta'} & x' \end{array}$$

of which the inner horn  $\Lambda_1^2 \rightarrow \Delta^2 \rightarrow \mathcal{D}$  lifts to an inner horn in  $\mathcal{C}$ ,  $\bar{s} : \Lambda_1^2 \rightarrow \mathcal{C}$ , which appears as

$$\begin{array}{ccc} & x & \\ \vartheta \nearrow & & \searrow id \\ y & & x. \end{array}$$

So we have a lifting problem

$$\begin{array}{ccc} \Lambda_1^2 & \xrightarrow{\bar{s}} & \mathcal{C} \\ \downarrow & \nearrow & \downarrow F \\ \Delta^2 & \xrightarrow{s} & \mathcal{D}. \end{array}$$

Since  $F$  is an inner fibration, by hypothesis, there exists a solution to this lifting problem  $s' : \Delta^2 \rightarrow \mathcal{C}$ ,

$$s' = \begin{array}{ccc} & x & \\ \vartheta \nearrow & & \searrow id \\ y & \xrightarrow{\theta} & x, \end{array}$$

and the map  $\theta := s'|_{\Delta^{\{1,3\}}} : y \rightarrow x$  is such that  $F(\theta) = \theta' : y' \rightarrow x'$ . By the two-of-three property for isomorphisms in  $\mathcal{C}$ , we see that  $\theta : y \rightarrow x$  is in fact an isomorphism, and hence provides the desired lift of  $\theta'$  to an isomorphism in  $\mathcal{C}$ . This shows that condition (b) of Definition 5.29 implies condition (b'). The opposite implication follows by applying the finding (b)  $\Rightarrow$  (b') to the opposite map  $F^{op} : \mathcal{C}^{op} \rightarrow \mathcal{D}^{op}$ .  $\square$

cor:2071

**Lemma 5.31.** *If  $\mathcal{C}$  and  $\mathcal{D}$  are  $\infty$ -categories, and  $F : \mathcal{C} \rightarrow \mathcal{D}$  is a left or right fibration, then  $F$  is an isofibration. Furthermore, if  $F$  is a left or right fibration then a given morphism  $\gamma$  in  $\mathcal{C}$  is an isomorphism if and only if its image in  $\mathcal{D}$  is an isomorphism.*

The proof just follows by fiddling with diagrams. We will give a detailed account in any account.

*Proof.* Suppose that  $F$  is a right fibration, for example. Then we can lift any map  $\gamma' : y' \rightarrow F(x)$  in  $\mathcal{D}$  to a map  $\gamma : y \rightarrow x$  in  $\mathcal{C}$ . This just follows by solving the corresponding lifting problem

$$\begin{array}{ccc} \Lambda_0^1 & \longrightarrow & \mathcal{C} \\ \downarrow & \nearrow & \downarrow F \\ \Delta^1 & \longrightarrow & \mathcal{D} \end{array}$$

One chooses an inverse  $\zeta'$  to  $\gamma'$  in  $\mathcal{D}$  and lifts the appropriate diagram to find a diagram

$$\begin{array}{ccc} & y & \\ \exists \zeta \nearrow & & \searrow \gamma \\ x & \xrightarrow{id} & x \end{array}$$

in  $\mathcal{C}$  to see that  $\gamma$  admits a right inverse  $\zeta$ . A similar lifting of a 2-simplex implies that  $\zeta$  admits a left inverse as well, and hence that  $\zeta$  is an isomorphism in  $\mathcal{C}$ . We see subsequently that  $\gamma$  is an isomorphism as well. So  $F$  is an inner fibration which satisfies condition (b) of Definition 5.29, and by Lemma 5.30 we see that  $F$  is an isofibration.

Now suppose  $\gamma : x \rightarrow y$  is a map in  $\mathcal{C}$  with  $F(\gamma) : Fx \rightarrow Fy$  an isomorphism. By lifting a diagram for the right inverse to  $F(\gamma)$  we see that  $\gamma$  admits a right inverse  $\zeta$ , and we lift to find a diagram

$$\begin{array}{ccc} & x & \\ \exists \nearrow & & \searrow \zeta \\ y & \xrightarrow{id} & y \end{array}$$

which shows that  $\zeta$  itself admits a right inverse. So  $\zeta$  is invertible and  $\gamma$  is invertible as well.

The proof in the case that  $F$  is a left fibration follows by a similar lifting argument, or by considering opposite categories.  $\square$

sect:assoc\_kan

**5.9. The Kan complex associated to an  $\infty$ -category.** Let us recall the associated Kan complex functor, which associates to any  $\infty$ -category  $\mathcal{C}$  the  $\infty$ -subcategory  $\mathcal{C}^{\text{Kan}} \subset \mathcal{C}$  which is the maximal  $\infty$ -subcategory whose morphisms are the isomorphisms in  $\mathcal{C}$ . So, an  $n$ -simplex  $s : \Delta^n \rightarrow \mathcal{C}$  lies in  $\mathcal{C}^{\text{Kan}}$  if and only if each restriction  $s|_{\Delta^{\{i,j\}}} \in \mathcal{C}[1]$  is an isomorphism in  $\mathcal{C}$ . Since the image of an isomorphism under any map of  $\infty$ -categories remains an isomorphism, we simply restrict to observe functoriality of this assignment

$$F : \mathcal{C} \rightarrow \mathcal{D} \rightsquigarrow F^{\text{Kan}} : \mathcal{C}^{\text{Kan}} \rightarrow \mathcal{D}^{\text{Kan}}.$$

thm:assoc\_kan

**Theorem 5.32.** *For any  $\infty$ -category  $\mathcal{C}$ , the associated Kan complex  $\mathcal{C}^{\text{Kan}}$  is in fact a Kan complex.*

This theorem will essentially follow from the next proposition. We provide this proposition, a number of immediate corollaries, then give the proof of Theorem 5.32

prop:2112

**Proposition 5.33.** *Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be an inner fibration between  $\infty$ -categories. Suppose  $n \geq 2$  and consider a lifting problem of the form*

$$\text{Form A)} \quad \begin{array}{ccc} \Lambda_n^n & \xrightarrow{s} & \mathcal{C} \\ \downarrow & & \downarrow F \\ \Delta^n & \xrightarrow{\sigma} & \mathcal{D} \end{array} \quad \text{or} \quad \text{Form B)} \quad \begin{array}{ccc} \Lambda_0^n & \xrightarrow{s} & \mathcal{C} \\ \downarrow & & \downarrow F \\ \Delta^n & \xrightarrow{\sigma} & \mathcal{D}. \end{array} \quad (23) \quad \text{eq:2105}$$

A lifting problem of Form A admits a solution provided  $s|_{\Delta^{\{n-1,n\}}}$  is an isomorphism in  $\mathcal{C}$ . A lifting problem of Form B admits a solution provided  $s|_{\Delta^{\{0,1\}}}$  is an isomorphism in  $\mathcal{C}$ .

In the case of a diagram of Form A, the proof leverages of overcategories as a means of decomposing the  $\Lambda_n^n$ -shaped and  $\Delta^n$ -shaped diagrams in  $\mathcal{C}$  and  $\Delta^n \rightarrow \mathcal{D}$ . For a diagrams of Form B one employs undercategories in an analogous manner.

*Proof.* We assume  $s|_{\Delta^{\{n-1,n\}}}$  is an isomorphism in  $\mathcal{C}$ , and consider a lifting problem of Form A. The Form B case follows by considering opposite categories.

We consider the inclusion

$$\Delta^{n-1} = \Delta^{n-2} \star \emptyset \rightarrow \Delta^{n-2} \star \Delta^1 = \Delta^n$$

This inclusion factors through the horn  $\Lambda_n^n$  and, in terms of the decomposition  $\Delta^n = \Delta^{n-2} \star \Delta^1$ , this horn is the union

$$\Lambda_n^n = (\Delta^{n-2} \star \{1\}) \cup (\partial\Delta^{n-2} \star \Delta^1).$$

The external horn  $s : \Lambda_n^n \rightarrow \mathcal{C}$  restricts along the inclusion  $\Delta^{n-2} \rightarrow \Lambda_n^n$  to define a simplex  $f := s|_{\Delta^{n-2}} : \Delta^{n-2} \rightarrow \mathcal{C}$ , and we can restrict further to the boundary to get  $g = f|_{\partial\Delta^{n-2}} : \partial\Delta^{n-2} \rightarrow \mathcal{C}$ ,

$$\begin{array}{ccc} & \partial\Delta^{n-2} & \\ \text{incl} \swarrow & & \searrow g \\ \Delta^{n-2} & \xrightarrow{f} & \mathcal{C}. \end{array}$$

We have the various undercategories  $\mathcal{C}_{f/}$ ,  $\mathcal{C}_{g/}$ ,  $\mathcal{D}_{f/}$  and  $\mathcal{D}_{g/}$ , where we have abused notation and written simply  $f$  and  $g$  for the maps to  $\mathcal{D}$  given by composing with  $F$ . The functor  $F$  induces a map to the fiber product

$$\Theta_F : \mathcal{C}_{f/} \rightarrow \mathcal{C}_{g/} \times_{\mathcal{D}_{g/}} \mathcal{D}_{f/}. \quad (24) \quad \text{eq:2142}$$

We will construct the desired solution  $s' : \Delta^n \rightarrow \mathcal{C}$  to our lifting problem via an analysis of the map (24).

Let us take  $\mathcal{E} = \mathcal{C}_{g/} \times_{\mathcal{D}_{g/}} \mathcal{D}_{f/}$ , and for convenience identify  $\Delta^1$  in the formula  $\Delta^{n-2} \star \Delta^1 = \Delta^n$  with  $\Delta^{\{-1,1\}}$ . Note that  $\mathcal{E}$  is an  $\infty$ -category, by Corollary 5.27.

The restrictions

$$s|_{\text{subthing}} : \partial\Delta^{n-2} \star \{\pm 1\} \rightarrow \mathcal{C} \quad \text{and} \quad \sigma|_{\text{subthing}} : \Delta^{n-2} \star \{\pm 1\} \rightarrow \mathcal{D}$$

specify objects  $\bar{x}_-$  and  $\bar{x}_+$  in  $\mathcal{E}$ , and the corresponding restrictions

$$s|_{\text{subthing}} : \partial\Delta^{n-2} \star \Delta^1 \rightarrow \mathcal{C} \quad \text{and} \quad \sigma : \Delta^{n-2} \star \Delta^1 = \Delta^n \rightarrow \mathcal{D}$$

specify a morphism  $\bar{\gamma} : \bar{x}_- \rightarrow \bar{x}_+$  in  $\mathcal{E}$ . The restriction

$$s|_{\text{subthing}} : \Delta^{n-2} \star \{1\} \rightarrow \mathcal{C}$$

Specifies an object  $x_+$  in  $\mathcal{C}_{f/}$  with  $\Theta_F(x_+) = \bar{x}_+$ .

Now, a lifting of  $\bar{\gamma}$  to a(n object  $x_-$  with a specified) morphism  $\gamma : x_- \rightarrow x_+$  in  $\mathcal{C}_{f/}$  is the information of a simplex

$$s' : \Delta^{n-2} \star \Delta^1 = \Delta^n \rightarrow \mathcal{C}$$

with

$$s'|_{\Delta^{n-2} \star \{1\}} = s|_{\Delta^{n-2} \star \{1\}}, \text{ and } F(s') = \sigma, \text{ and } s'|\partial\Delta^{n-2} \star \Delta^1 = s|\partial\Delta^{n-2} \star \Delta^1.$$

The first and third equalities say precisely that  $s'$  restricts  $s$  along the inclusion  $\Lambda^n \rightarrow \Delta^n$ . So, there exists a solution to the lifting problem

$$\begin{array}{ccc} \Delta^0 = \Lambda_1^1 & \xrightarrow{x_+} & \mathcal{C}_{f/} \\ 1 \downarrow & \nearrow & \downarrow \Theta_F \\ \Delta^1 & \xrightarrow{\bar{\gamma}} & \mathcal{C} \end{array}$$

if and only if there exists a solution to the lifting problem (24). But now, we have already seen that  $\Theta_F$  is a left fibration, at Proposition 5.25, and hence an isofibration by Corollary 5.31. So such a lifting  $\gamma : \Delta^1 \rightarrow \mathcal{C}_{f/}$  exists, provided  $\bar{\gamma}$  is an isomorphism in  $\mathcal{C}$ .

To see that  $\bar{\gamma}$  is an isomorphism, we consider the functor  $w : \mathcal{E} \rightarrow \mathcal{C}$  which is obtained by composing the structural projection  $p_1 : \mathcal{E} \rightarrow \mathcal{C}_{g/}$  with the forgetful functor  $\mathcal{C}_{g/} \rightarrow \mathcal{C}$ . Both of the maps in this composite are left fibrations, so that  $w : \mathcal{E} \rightarrow \mathcal{C}$  is a left fibration. We have

$$w(\bar{\gamma}) = (s|\partial\Delta^{n-2} \star \Delta^1)|_{\Delta^1} = s|_{\Delta^{\{n-1, n\}}},$$

which is an isomorphism in  $\mathcal{C}$  by hypothesis. It follows that  $\bar{\gamma}$  is an isomorphism, by Corollary 5.31. Since  $\Theta_F$  is an isofibration we can then find our desired lift  $\gamma : x_- \rightarrow x_+$  in  $\mathcal{C}_{f/}$ , and hence find a solution to the lifting problem (23).  $\square$

cor:2209

**Corollary 5.34.** *If  $F : \mathcal{C} \rightarrow \mathcal{D}$  is an isofibration between  $\infty$ -categories, then the associated map  $F^{\text{Kan}} : \mathcal{C}^{\text{Kan}} \rightarrow \mathcal{D}^{\text{Kan}}$  is a Kan fibration.*

*Proof.* Consider a lifting problem

$$\begin{array}{ccc} \Delta_i^n & \longrightarrow & \mathcal{C}^{\text{Kan}} \\ \downarrow & \nearrow & \downarrow F^{\text{Kan}} \\ \Delta^1 & \longrightarrow & \mathcal{D}^{\text{Kan}} \end{array} \quad (25) \quad \text{eq:2215}$$

with  $0 \leq i \leq n$ . When  $n = 1$  there exists solutions since  $F$  is an isofibration. When  $n > 1$  we compose with the inclusions to  $\mathcal{C}$  and  $\mathcal{D}$  and find a solution  $s : \Delta^n \rightarrow \mathcal{C}$  to the corresponding lifting problem along for the original  $F : \mathcal{C} \rightarrow \mathcal{D}$ . Now, since  $n > 1$  all 2-simplices in  $\Delta^n$  lie in the horn  $\Lambda_i^n$ , and we conclude that all of the maps  $s|_{\Delta^{\{i, j\}}} \in \mathcal{C}[1]$  are isomorphisms. So  $s$  has image in  $\mathcal{C}^{\text{Kan}}$ , and thus provides a solution to the lifting problem (25). We are done.  $\square$

The proof of Theorem 5.32 is now transparent.

*Proof of Theorem 5.32.* Follows by Corollary 5.34, where we consider the case  $\mathcal{D} = *$ .  $\square$

We now can see this “Kan complex functor” in the appropriate light.



**Definition 5.35.** The (associated) Kan complex functor

$$-^{\text{Kan}} : \text{Cat}_\infty \rightarrow \text{Kan}$$

is defined by taking any  $\infty$ -category  $\mathcal{C}$  to its maximal subcategory of isomorphisms  $\mathcal{C}^{\text{Kan}}$ , and by restriction of functors to these maximal subcategories.

Evidently, any map  $\mathcal{X} \rightarrow \mathcal{C}$  from a Kan complex to an  $\infty$ -category factors through  $\mathcal{C}^{\text{Kan}}$ . Hence the Kan complex functor provides a right adjoint to the inclusion  $\text{incl}_{\text{Kan}} : \text{Kan} \rightarrow \text{Cat}_\infty$ . If we label everything explicitly, we have a bifunctorial identification

$$\text{Hom}_{\text{Cat}_\infty}(\text{incl}_{\text{Kan}} \mathcal{X}, \mathcal{C}) = \text{Hom}_{\text{Kan}}(\mathcal{X}, \mathcal{C}^{\text{Kan}}). \quad (26) \quad \text{eq:2258}$$

**5.10. Equivalences and the homotopy category of  $\infty$ -categories.** Given Theorem 5.32 and Corollary 5.8, we see that for any  $\infty$ -category  $\mathcal{C}$  and simplicial set  $K$  we can consider the mapping space

$$\text{Fun}(K, \mathcal{C})^{\text{Kan}} \subseteq \text{Fun}(K, \mathcal{C}).$$

This is the Kan complex parametrizing maps of simplicial set  $f : K \rightarrow \mathcal{C}$  with natural isomorphisms. Since  $\text{Fun}(K, \mathcal{C})^{\text{Kan}}$  is a Kan complex, and products of Kan complexes are Kan complexes, the adjunction (26) implies that composition for the simplicial category  $\text{Cat}_\infty$  restricts to provide composition maps

$$\circ : \text{Fun}(\mathcal{D}, \mathcal{E})^{\text{Kan}} \times \text{Fun}(\mathcal{C}, \mathcal{D})^{\text{Kan}} \rightarrow \text{Fun}(\mathcal{C}, \mathcal{E})^{\text{Kan}}.$$

We take connected components to obtain a well-defined, associative composition on the sets of connected components

$$\text{Hom}_{\text{hCat}_\infty}(-, -) := \pi_0(\text{Fun}(\mathcal{D}, \mathcal{E})^{\text{Kan}}), \quad (27) \quad \text{eq:2274}$$

and a well-defined category.

**Definition 5.36.** The homotopy category  $\text{hCat}_\infty$  of  $\infty$ -categories is the category whose objects are  $\infty$ -categories and whose morphisms are homotopy classes of maps, as defined at (27).

**Definition 5.37.** A functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  between  $\infty$ -categories is called an equivalence if there exists a functor  $G : \mathcal{D} \rightarrow \mathcal{C}$  such that  $FG$  and  $GF$  are isomorphic to  $\text{id}_{\mathcal{D}}$  and  $\text{id}_{\mathcal{C}}$  in  $\text{Fun}(\mathcal{D}, \mathcal{D})$  and  $\text{Fun}(\mathcal{C}, \mathcal{C})$  respectively.

The following lemma is transparent.

lem:equiv\_htop

**Lemma 5.38.** A map  $F : \mathcal{C} \rightarrow \mathcal{D}$  between  $\infty$ -categories is an equivalence if and only if its class  $\bar{F}$  in the homotopy category  $\text{hCat}_\infty$  is an isomorphism.

We end with some simple lemmas about equivalences of functors.

lem:2293

**Lemma 5.39.** Let  $v : \Delta^1 \times \mathcal{C} \rightarrow \mathcal{D}$  is an isomorphism between functors  $F_0$  and  $F_1$  in  $\text{Fun}(\mathcal{C}, \mathcal{D})$ . Then for any  $x : \Delta^0 \rightarrow \mathcal{C}$  the restriction  $v_x : \Delta^1 = \Delta^1 \times \Delta^0 \rightarrow \Delta^1 \times \mathcal{C} \rightarrow \mathcal{D}$  is an isomorphism in  $\mathcal{D}$ ,  $v_x : F_0(x) \rightarrow F_1(x)$ .

*Proof.* One chooses 2-simplices  $V, V' : \Delta^2 \times \mathcal{C} \rightarrow \mathcal{D}$  with

$$V|_{\Delta^{\{0,2\}}} = \text{id}_{F_0}, \quad V'|_{\Delta^{\{0,2\}}} = \text{id}_{F_1}, \quad V|_{\Delta^{\{0,1\}}} = v, \quad V'|_{\Delta^{\{1,2\}}} = v.$$

Evaluating these 2-simplex at  $x : \Delta^0 \rightarrow \mathcal{C}$  exhibit right and left inverses to  $v_x$ .  $\square$

lem:2305

**Lemma 5.40.** *If  $F_0, F_1 : \mathcal{C} \rightarrow \mathcal{D}$  are isomorphic functors between  $\infty$ -categories, then their restrictions  $F_0^{\text{Kan}}, F_1^{\text{Kan}} : \mathcal{C}^{\text{Kan}} \rightarrow \mathcal{D}^{\text{Kan}}$  are isomorphic as well. In particular, any isomorphism between the  $F_i$  restricts to an isomorphism between the  $F_i^{\text{Kan}}$ .*

*Proof.* It suffices to show that the restriction of  $v : \Delta^1 \times \mathcal{C}^{\text{Kan}} \rightarrow \mathcal{D}$  of an isomorphism  $\tilde{v} : F_0 \xrightarrow{\sim} F_1$  has image in  $\mathcal{D}^{\text{Kan}}$ . The 1-simplices in  $\Delta^1 \times \mathcal{C}^{\text{Kan}}$  are of the form  $\text{vert}_x = \Delta^1 \times \{x\}$ ,  $\alpha^0 : \Delta^{\{0\}} \times \alpha$ ,  $\alpha^1 : \Delta^{\{1\}} \times \alpha$ , and

$$\Delta^1 \xrightarrow{\delta} \Delta^1 \times \Delta^1 \xrightarrow{id \times \alpha} \Delta^1 \times \mathcal{C}, \quad (28)$$

eq:2311

where  $\delta$  is the diagonal map  $0 \mapsto (0, 0)$ ,  $1 \mapsto (1, 1)$ . By Lemma 5.39  $\text{vert}_x$  maps to an isomorphism  $v_x$  in  $\mathcal{D}$ , and the fact that the functors  $F_0 = v|_0$  and  $F_1 = v|_1$  preserve isomorphisms says that  $\alpha^0$  and  $\alpha^1$  map to isomorphisms in  $\mathcal{D}$ . We need to deal with the map (28).

Write  $\text{diag}_\alpha : (0, x) \rightarrow (1, y)$  for the map (28) in  $\Delta^1 \times \mathcal{C}$ . We have the 2-simplex  $\Delta^2 \rightarrow \Delta^1 \times \Delta^1$  defined by the function  $[2] \rightarrow [1] \times [1]$ ,  $0 \mapsto (0, 0)$ ,  $1 \mapsto (1, 0)$ ,  $2 \mapsto (1, 1)$ , and the composite

$$\Delta^2 \rightarrow \Delta^1 \times \Delta^1 \xrightarrow{id \times \alpha} \Delta^1 \times \mathcal{C}$$

defines a 2-simplex  $s$  in the product with faces  $s|_{\Delta^{\{0,1\}}} = \text{vert}_x$ ,  $s|_{\Delta^{\{1,2\}}} = \alpha^1$ , and  $s|_{\Delta^{\{0,2\}}} = \text{diag}_\alpha$ . The image of  $s$  in  $\mathcal{D}$  is a 2-simplex exhibiting  $v(\text{diag}_\alpha)$  as a composite of  $v_x$  with  $F_1(\alpha)$ . Since both of these maps are isomorphisms in  $\mathcal{D}$ ,  $v(\text{diag}_\alpha)$  is seen to be an isomorphism in  $\mathcal{D}$  as well. It follows that  $v$  has image in  $\mathcal{D}^{\text{Kan}}$ , as desired.  $\square$

**5.11. Equivalences via functor categories.** The point of this final subsection is to prove the following characterization of equivalences between  $\infty$ -categories via spaces of functors.

thm:Fun\_equiv\_infty

**Theorem 5.41.** *Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor between  $\infty$ -categories. The following are equivalent:*

- (a)  *$F$  is an equivalence.*
- (b) *For any simplicial set  $K$ , the map  $F_* : \text{Fun}(K, \mathcal{C}) \rightarrow \text{Fun}(K, \mathcal{D})$  is an equivalence.*
- (c) *For any  $\infty$ -category  $\mathcal{A}$ , the map  $F_* : \text{Fun}(\mathcal{A}, \mathcal{C}) \rightarrow \text{Fun}(\mathcal{A}, \mathcal{D})$  is an equivalence of  $\infty$ -categories.*
- (d) *For any  $\infty$ -category  $\mathcal{A}$ , the map  $F^* : \text{Fun}(\mathcal{D}, \mathcal{A}) \rightarrow \text{Fun}(\mathcal{C}, \mathcal{A})$  is an equivalence of  $\infty$ -categories.*
- (e) *For any  $\infty$ -category  $\mathcal{A}$ , the map  $F_* : \text{Fun}(\mathcal{A}, \mathcal{C})^{\text{Kan}} \rightarrow \text{Fun}(\mathcal{A}, \mathcal{D})^{\text{Kan}}$  is an equivalence of Kan complexes.*
- (f) *For any  $\infty$ -category  $\mathcal{A}$ , the map  $F^* : \text{Fun}(\mathcal{D}, \mathcal{A})^{\text{Kan}} \rightarrow \text{Fun}(\mathcal{C}, \mathcal{A})^{\text{Kan}}$  is an equivalence of Kan complexes.*

*Proof.* One employs homotopies directly, as in the proof of Lemma 4.19, to see that (a) implies (b), and the implication (b)  $\Rightarrow$  (c) follows since  $\infty$ -categories are simplicial sets. Lemma 5.40 shows that (c) implies (e). By applying  $\pi_0$  to (e) and considering Yoneda's Lemma we see that (e) implies  $\mathcal{C}$  and  $\mathcal{D}$  are isomorphic in the homotopy category of  $\infty$ -categories. By Lemma 5.38 it follows that  $F : \mathcal{C} \rightarrow \mathcal{D}$

is an equivalence. So (e)  $\Rightarrow$  (a). We thus find an equivalence between (a), (b), (c), and (e).

We similarly see that (a)  $\Rightarrow$  (d)  $\Rightarrow$  (f)  $\Rightarrow$  (a), and hence that (a), (d), and (f) are equivalent. We are done.  $\square$

## 6. MAPPING SPACES AND EQUIVALENCES

### 6.1. Mapping spaces and fully faithful functors.

**Definition 6.1.** Let  $x$  and  $y$  be objects in an  $\infty$ -category  $\mathcal{C}$ . The mapping space  $\mathrm{Hom}_{\mathcal{C}}(x, y)$  is the simplicial subset in  $\mathrm{Fun}(\Delta^1, \mathcal{C})$  whose  $n$ -simplices are all maps  $f : \Delta^n \times \Delta^1 \rightarrow \mathcal{C}$  with

$$f|_{\Delta^n \times \{0\}} = id_x \quad \text{and} \quad f|_{\Delta^n \times \{1\}} = id_y.$$

We've abused notation to write, for any object  $z$  in  $\mathcal{C}$ ,  $id_z : \Delta^n \rightarrow \mathcal{C}$  for the composite of the terminal map  $\Delta^n \rightarrow *$  with the map  $z : * \rightarrow \mathcal{C}$ . We can reproduce  $\mathrm{Hom}_{\mathcal{C}}(x, y)$  as a pullback

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(x, y) & \longrightarrow & \mathrm{Fun}(\Delta^1, \mathcal{C}) \\ \downarrow & & \downarrow \text{res} \\ * & \xrightarrow{x \amalg y} & \mathrm{Fun}(\partial\Delta^1, \mathcal{C}). \end{array} \quad (29) \quad \boxed{\text{eq:2459}}$$

**Lemma 6.2.** *The mapping space  $\mathrm{Hom}_{\mathcal{C}}(x, y)$  is an  $\infty$ -category.*

*Proof.* By Proposition 5.25 the restriction map from  $\mathrm{Fun}(\Delta^1, \mathcal{C})$  is an inner fibration. So the terminal map  $\mathrm{Hom}_{\mathcal{C}}(x, y) \rightarrow *$  is an inner fibration, and hence  $\mathrm{Hom}_{\mathcal{C}}(x, y)$  is an  $\infty$ -category.  $\square$

We will see momentarily that these morphism spaces  $\mathrm{Hom}_{\mathcal{C}}(x, y)$  are Kan complexes.

Now, for any functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  the induced map  $F_* : \mathrm{Fun}(\Delta^1, \mathcal{C}) \rightarrow \mathrm{Fun}(\Delta^1, \mathcal{D})$  restricts to provide a map

$$F_* : \mathrm{Hom}_{\mathcal{C}}(x, y) \rightarrow \mathrm{Hom}_{\mathcal{D}}(Fx, Fy).$$

**Remark 6.3.** These Hom spaces do not admit a natural composition operation, at least before taking some homotopy truncation. So we are *not* providing an object and morphism description of our  $\infty$ -category  $\mathcal{C}$ .

**Definition 6.4.** A functor between  $\infty$ -categories  $F : \mathcal{C} \rightarrow \mathcal{D}$  is called fully faithful if, for each pair of objects  $x$  and  $y$  in  $\mathcal{C}$ , the induced map on Hom spaces  $F_* : \mathrm{Hom}_{\mathcal{C}}(x, y) \rightarrow \mathrm{Hom}_{\mathcal{D}}(Fx, Fy)$  is an equivalence.

### 6.2. Aside on natural isomorphisms.

**thm:natty\_isom**

**Theorem 6.5.** *Let  $f, f' : K \rightarrow \mathcal{C}$  be two functors maps from a simplicial set to an  $\infty$ -category  $\mathcal{C}$ , and consider a natural transformation  $u : f \rightarrow f'$ , i.e. a map from  $f$  to  $f'$  in  $\mathrm{Fun}(K, \mathcal{C})$ . Then  $t$  is an isomorphism if and only if, for all vertices  $x$  in  $K$ , the map  $u_x : f(x) \rightarrow f'(x)$  is an isomorphism in  $\mathcal{C}$ .*

The proof relies on certain technical results about simplicial sets, which we recall here.

lem:2504

**Lemma 6.6** ([10, 01DN]). *For any integers  $m \geq 0$  and  $n \geq 2$ , there is a sequence of simplicial sets*

$$(\Delta^m \times \Lambda_0^n \cup \partial\Delta^m \times \Delta^n) = X(0) \subset X(1) \subset \cdots \subset X(l) = \Delta^m \times \Delta^n$$

*such that, for each positive integer  $k \leq l$ , there are integers  $2 \leq q$  and  $p < q$  which admit a pushout diagram*

$$\begin{array}{ccc} \Lambda_p^q & \longrightarrow & X(k-1) \\ \downarrow & & \downarrow \text{incl} \\ \Delta^q & \xrightarrow{s} & X(k). \end{array}$$

*Furthermore, if  $p = 0$  then the map  $s$  can be chosen so that  $s(0) = (0, 0)$  and  $s(1) = (0, 1)$ .*

In the expressions for  $s(0)$  and  $s(1)$ , we identify the map  $s : \Delta^p \rightarrow X(k) \subset \Delta^m \times \Delta^n$  with a choice of function  $[p] \rightarrow [m] \times [n]$ . We refer the reader directly to [10] for the proof.

lem:2521

**Lemma 6.7.** *Let  $Y \rightarrow S$  be an inner fibration and  $\bar{F} : B \rightarrow S$  be any map of simplicial sets. Consider any simplicial subset  $A \subset B$  and integer  $n \geq 2$ , and let  $B \times \Delta^n \rightarrow S$  be the composite of the projection  $B \times \Delta^n \rightarrow B$  composed with  $\bar{F}$ . Suppose that we have a lifting problem*

$$\begin{array}{ccc} (A \times \Delta^n) \amalg_{A \times \Lambda_0^n} (B \times \Lambda_0^n) & \xrightarrow{F} & Y \\ \downarrow & \nearrow \text{dashed} & \downarrow \\ B \times \Delta^n & \xrightarrow{\quad} & S \end{array} \quad (30) \quad \text{eq:2523}$$

*for which, at every choice of vertex  $b$  in  $B$ , the corresponding edge*

$$\Delta^1 \cong \{b\} \times \Delta^{\{0,1\}} \rightarrow \{b\} \times \Lambda_0^n \rightarrow \{Fb\} \times_S Y = Y_b$$

*is an isomorphism in the  $\infty$ -category  $Y_b$ . Then the problem (30) admits a solution.*

*Proof.* We can replace  $B$  by any intermediate complex  $A \subseteq K \subseteq B$ , and consider the corresponding lifting problem  $\mathbb{L}_K$  obtained by restricting  $F$  and  $\bar{F}$  to  $(A \times \Delta^n) \amalg_{A \times \Lambda_0^n} (K \times \Lambda_0^n)$  and  $K$  respectively. We consider the collection  $\mathcal{P}$  of pairs  $(K, F_K)$  consisting of a choice of an intermediate complexes  $K$  and a map  $F_K : K \times \Delta^n \rightarrow Y$  which solves the lifting problem  $\mathbb{L}_K$ . This collection  $\mathcal{P}$  admits a natural partially ordered, where we take  $(K, F_K) \leq (L, F_L)$  if and only if  $K \subset L$  and  $F_K = F_L|_{(K \times \Delta^n)}$ . By taking unions we see that any chain in  $\mathcal{P}$  admits an upper bound, so we apply Zorns lemma to see that  $\mathcal{P}$  admits a maximal element  $(K_{\max}, F_{\max})$ , and we claim that  $K_{\max} = B$ , so that the original lifting problem (30) admits a solution.

Suppose, by way of contradiction, that  $K_{\max}$  is not  $B$ , and choose a non-degenerate simplex  $z : \Delta^m \rightarrow B$  of minimal dimension which does not factor through  $K_{\max}$ . The minimality condition tells us that the restriction  $z|_{\partial\Delta^m}$  factors through  $K_{\max}$ , and we take  $L = (K_{\max} \cup z) \subset B$ . We therefore have the pushout diagram

$$\begin{array}{ccc} \partial\Delta^m & \longrightarrow & K_{\max} \\ \downarrow & & \downarrow \\ \Delta^m & \longrightarrow & L \end{array}$$

and consider the lifting problem

$$\begin{array}{ccc}
 (K_{\max} \times \Delta^n) \amalg_{K_{\max} \times \Lambda_0^n} (L \times \Lambda_0^n) & \xrightarrow{\quad} & Y \\
 \downarrow & \nearrow \text{dashed} & \downarrow \\
 L \times \Delta^n & \xrightarrow{\quad} & S
 \end{array} \quad (31) \quad \boxed{\text{eq:2548}}$$

where top map is obtained from  $F_{\max} : K_{\max} \times \Delta^n \rightarrow Y$  and  $F|(L \times \Lambda_0^n)$ , and the bottom map is  $\bar{F}|_L \circ \text{proj}_1$ . Via the above pushout diagram we can solve the lifting problem (31) by solving a corresponding pushout diagram

$$\begin{array}{ccc}
 (\partial \Delta^m \times \Delta^n) \amalg_{\partial \Delta^m \times \Lambda_0^n} (\Delta^m \times \Lambda_0^n) & \xrightarrow{f} & \Delta^m \times_S Y \\
 \downarrow & \nearrow \text{dashed} & \downarrow \\
 \Delta^m \times \Delta^n & \xrightarrow{\bar{f}} & \Delta^m
 \end{array} \quad (32) \quad \boxed{\text{eq:2555}}$$

We note that at this point the base category  $\Delta^m$  is an  $\infty$ -category, as is the fiber product  $\Delta^m \times_S Y$ . This is our main goal.

We break the inclusion

$$(\partial \Delta^m \times \Delta^n) \amalg_{\partial \Delta^m \times \Lambda_0^n} (\Delta^m \times \Lambda_0^n) = (\partial \Delta^m \times \Delta^n) \cup (\Delta^m \times \Lambda_0^n) \rightarrow \Delta^m \times \Delta^n$$

into a sequence of inclusion

$$(\partial \Delta^m \times \Delta^n) \cup (\Delta^m \times \Lambda_0^n) = X(0) \subset X(1) \subset \cdots \subset X(l) = \Delta^m \times \Delta^n$$

which satisfy the conclusions of Lemma 6.6. We have a map  $f_0 = f : X(0) \rightarrow \Delta^m \times_S Y$  and we claim that for each  $0 \leq k \leq l$  there is a map  $f_k : X(k) \rightarrow \Delta^m \times_S Y$  which solve the lifting problem

$$\begin{array}{ccc}
 X(0) & \xrightarrow{f} & \Delta^m \times_S Y \\
 \downarrow & \nearrow \text{dashed } f_k & \downarrow \\
 X(k) & \xrightarrow{\bar{f}|_{X(k)}} & \Delta^m
 \end{array}$$

We prove this claim by induction.

Suppose that we have the desired map  $f_{k-1} : X(k-1) \rightarrow \Delta^m \times_S Y$ , for some  $k > 0$ . By hypotheses, we have a pushout diagram

$$\begin{array}{ccc}
 \Lambda_p^q & \longrightarrow & X(k-1) \\
 \downarrow & & \downarrow \\
 \Delta^q & \xrightarrow{s} & X(k),
 \end{array}$$

where  $p < q$ . Supposing  $0 < p$ , the fact that the map  $\Delta^m \times_S Y \rightarrow \Delta^m$  is an inner fibration tells us that we can lift  $\bar{f} : X(k) \rightarrow \Delta^m$  to a map  $f_k : X(k) \rightarrow \Delta^m \times_S Y$  with  $f_k|_{X(k-1)} = f_{k-1}$ . Supposing  $p = 0$ , then we can find a lift  $f_k : X(k) \rightarrow \Delta^m \times_S Y$  by our hypotheses that  $s(0) = (0, 0)$ ,  $s(1) = (0, 1)$ , and the hypotheses that  $f_{k-1} : \Delta^{\{0,1\}} \rightarrow \{f0\} \times_S Y$  is an isomorphism, and a subsequent application of Proposition 5.33.

In any case, we can always extend the map  $f_{k-1}$  to  $f_k$ , as desired, and the map  $f_l : X(l) = \Delta^m \times \Delta^n \rightarrow \Delta^m \times_S Y$  solves the lifting problem (32). We can therefore

solve the original lifting problem (31), which contradicts the maximality of the pair  $(K_{\max}, F_{\max})$ . So we have necessarily  $K_{\max} = B$ , and  $F_{\max} : B \times \Delta^n \rightarrow Y$  solves our lifting problem (30).  $\square$

Theorem 6.5 is now obtained as a special case of the following.

prop:2597

**Proposition 6.8.** *Consider an inner fibration  $\mathcal{C} \rightarrow S$  and an arbitrary map of simplicial sets  $a : K \rightarrow S$ , and let  $u : F \rightarrow F'$  be a map in the  $\infty$ -category  $\text{Fun}_{/S}(K, \mathcal{C})$ . Then  $u$  is an isomorphism if and only if, at every vertex  $z$  in  $K$ , the map  $u_z : F(z) \rightarrow F'(z)$  is an isomorphism in  $\mathcal{C}_z = \{a(z)\} \times_S \mathcal{C}$ .*

To be clear, the space  $\text{Fun}_{/S}(K, \mathcal{C})$  is the fiber product

$$\text{Fun}_{/S}(K, \mathcal{C}) = \text{Fun}(K, \mathcal{C}) \times_{\text{Fun}(K, S)} \{a\}$$

Since  $\mathcal{C} \rightarrow S$  is an inner fibration, the map  $\text{Fun}(K, \mathcal{C}) \rightarrow \text{Fun}(K, S)$  is an inner fibration by Corollary 5.8. So the fiber  $\text{Fun}_{/S}(K, \mathcal{C})$  is in fact an  $\infty$ -category.

*Proof.* If  $u$  is an isomorphism, then one can evaluate any choice of inverse map  $v : F' \rightarrow F$  at the vertices in  $K$  to produce inverses to the  $u_z$ . For the converse claim, let's suppose that  $u_z$  is an isomorphism at all  $z \in B[0]$ . We consider the diagrams

$$\begin{array}{ccc} K \times \Lambda_0^2 & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow \\ K \times \Delta^2 & \longrightarrow & S \end{array} \quad (33) \quad \text{eq:2609}$$

where the bottom map is the projection  $K \times \Delta^2 \rightarrow K$  composed with  $a$ , and the top map has restrictions  $F$ ,  $F'$ , and  $F$  to  $K \times \Delta^{\{0\}}$ ,  $K \times \Delta^{\{1\}}$ , and  $K \times \Delta^{\{2\}}$  respectively. The restrictions to  $K \times \Delta^{\{0,1\}}$  and  $K \times \Delta^{\{0,2\}}$  are  $u$  and  $id_F$  respectively. By the hypothesis that  $u_z$  is an isomorphism at all  $z$ , we may apply Lemma 6.7 to find a map  $\eta : K \times \Delta^2 \rightarrow \mathcal{C}$  with solves the lifting problem corresponding to (33). For  $v = \eta|_{K \times \Delta^{\{1,2\}}} : F' \rightarrow F$  we therefore have  $v \circ u \sim id_F$ . So  $u$  admits a left inverse in  $\text{Fun}_{/S}(K, \mathcal{C})$ . We consider opposite categories to similarly find that the opposite map  $u^{op} : (F')^{op} \rightarrow F^{op}$  admits a left inverse  $w^{op}$ , and hence that  $u$  admits a right inverse  $w$ . Hence  $u$  is an isomorphism.  $\square$

*Proof of Theorem 6.5.* Apply Proposition 6.8 in the case  $S = *$ .  $\square$

### 6.3. The mapping spaces $\text{Hom}_{\mathcal{C}}(x, y)$ are spaces.

**Theorem 6.9.** *For any objects  $x, y$  in an  $\infty$ -category  $\mathcal{C}$ , the  $\infty$ -category of maps  $\text{Hom}_{\mathcal{C}}(x, y)$  is a Kan complex.*

*Proof.* It suffices to show that all morphisms in  $\text{Hom}_{\mathcal{C}}(x, y)$  are isomorphisms. Objects in  $\text{Hom}_{\mathcal{C}}(x, y)$  are maps  $f : \Delta^1 \rightarrow \mathcal{C}$  with  $f(0) = x$  and  $f(1) = y$ , and a morphism  $f \rightarrow f'$  in  $\text{Hom}_{\mathcal{C}}(x, y)$  is a natural transformation  $u : f \rightarrow f'$  with  $u_0 = id_x : f(0) \rightarrow f(0)$  and  $u_1 = id_y : f(1) \rightarrow f(1)$ . So we just apply Theorem 6.5 directly to see that any map in  $\text{Hom}_{\mathcal{C}}(x, y)$  is an isomorphism.  $\square$

#### 6.4. Fully faithful functors and equivalences.

**Definition 6.10.** We say a functor between  $\infty$ -categories  $F : \mathcal{C} \rightarrow \mathcal{D}$  is fully-faithful if the induced maps

$$F_* : \mathrm{Hom}_{\mathcal{C}}(x, y) \rightarrow \mathrm{Hom}_{\mathcal{D}}(Fx, Fy)$$

are equivalences of Kan complexes, at all  $x, y$  in  $\mathcal{C}$ . We say  $F$  is essentially surjective if  $F$  induces a surjection

$$\pi_0(F^{\mathrm{Kan}}) : \pi_0(\mathcal{C}^{\mathrm{Kan}}) \rightarrow \pi_0(\mathcal{D}^{\mathrm{Kan}})$$

on isoclasses of objects.

The main result of the section is to prove the following essential result.

thm:ffes\_equiv

**Theorem 6.11.** *A functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  between  $\infty$ -categories is an equivalence if and only if  $F$  is fully faithful and essentially surjective.*

While any equivalence  $F$  is easily seen to be essentially surjective, the fact that the operation  $\mathrm{Hom}_{\mathcal{C}}(x, -) : \mathcal{C}[1] \rightarrow \mathrm{Kan}$  is not a priori defined on morphisms  $\mathcal{C}[2]$  makes fully faithfulness slightly unclear. So, we are interested in both of the claims in Theorem 6.11. We first deal with the “easier” implication

$F$  is an equivalence  $\Rightarrow F$  is fully faithful and essentially surjective,

which still requires quite a significant analysis.

#### 6.5. The map $\mathrm{Fun}(\Delta^1, \mathcal{C}) \rightarrow \mathrm{Fun}(\partial\Delta^1, \mathcal{C})$ . We claim the following.

prop:2663

**Proposition 6.12.** *Consider the inclusion  $i : \partial\Delta^1 \rightarrow \Delta^1$ . For any  $\infty$ -category  $\mathcal{C}$ , the restriction map*

$$i^* : \mathrm{Fun}(\Delta^1, \mathcal{C}) \rightarrow \mathrm{Fun}(\partial\Delta^1, \mathcal{C})$$

*is an isofibration.*

Let us recall that a functor between  $\infty$ -categories is called an isofibration if it is an inner fibration of simplicial sets along which one can lift isomorphisms (Definition 5.29). We already understand that  $i^*$  is an inner fibration, by Corollary 5.8, so we need only establish the desired lifting property.

Proposition 6.12 is in fact a specific case of a more general result, which proposes that restricting along any inclusion  $A \rightarrow B$  of simplicial sets produces an isofibration between  $\infty$ -categories [10, 01F3]. However, the situation of Proposition 6.12 is simple enough that we can just deal with it directly.

*Proof.* Take  $A = \partial\Delta^1$  and let  $i : A \rightarrow \Delta^1$  denote the inclusion. Consider an isomorphism  $w : a \rightarrow b$  in  $\mathrm{Fun}(A, \mathcal{C})$ . This is a choice of two pairs of objects  $a = (a_1, a_2)$ ,  $b = (b_1, b_2)$ , and two isomorphisms  $w_1 : a_1 \rightarrow b_1$  and  $w_2 : a_2 \rightarrow b_2$  in  $\mathcal{C}$ . Suppose that we have an object  $\beta$  in  $\mathrm{Fun}(\Delta^1, \mathcal{C})$  with  $i^*(\beta) = b$ . According to Lemma 5.30, and Proposition 6.8, it suffices to show that there exists an object  $\alpha$  and a morphism  $\omega : \alpha \rightarrow \beta$  in  $\mathrm{Fun}(\Delta^1, \mathcal{C})$  such that  $i^*(\alpha) = a$  and  $i^*(\omega) = w$ .

Now,  $\beta$  is precisely the information of a map  $f : b_1 \rightarrow b_2$  in  $\mathcal{C}$ , and we want to find a morphism  $g : a_1 \rightarrow a_2$  which fits into a diagram of the form

$$\begin{array}{ccc} a_1 & \xrightarrow{g?} & a_2 \\ w_1 \downarrow & \searrow ? & \downarrow w_2 \\ b_1 & \xrightarrow{f} & b_2 \end{array} \quad (34) \quad \text{eq:2681}$$

in  $\mathcal{C}$ . The map  $g : a_1 \rightarrow a_2$  is our lift  $\alpha$  for  $a$ , and the above diagram is our map  $\omega : \Delta^1 \times \Delta^1 \rightarrow \mathcal{C}$  which realizes the desired morphism in  $\text{Fun}(\Delta^1, \mathcal{C})$  over  $w : a \rightarrow b$ . Now, the map  $w : a \rightarrow b$  provides us with a partial diagram

$$\begin{array}{ccc} a_1 & & a_2 \\ w_1 \downarrow & & \downarrow w_2 \\ b_1 & \xrightarrow{f} & b_2 \end{array}$$

and we choose a composite  $h = f \circ w_1$  to produce a partial diagram

$$\begin{array}{ccc} a_1 & & a_2 \\ w_1 \downarrow & \searrow h & \downarrow w_2 \\ b_1 & \xrightarrow{f} & b_2 \end{array}$$

(35)

eq:2695

To complete the partial diagram (35) to the desired morphism (34) in  $\text{Fun}(\Delta^1, \mathcal{C})$  it suffices to solve the lifting problem

$$\begin{array}{ccc} \Lambda_2^2 & \xrightarrow{r} & \text{Fun}(\Delta^1, \mathcal{C}) \\ \downarrow & \nearrow & \downarrow \\ \Delta^2 & \xrightarrow{\quad} & * \end{array}$$

where  $r$  is specified by  $r|_{\Delta^{\{0,3\}}} = h$  and  $r|_{\Delta^{\{2,3\}}} = w_2$ . Since  $w_2$  is an isomorphism, Proposition 5.33 tells us that the above lifting problem has a solution, and hence that the desired lift  $\omega$  in  $\text{Fun}(\Delta^1, \mathcal{C})$  exists.  $\square$

We apply Corollary 5.34 to obtain the following.

**Corollary 6.13.** *For any  $\infty$ -category  $\mathcal{C}$ , restricting along the inclusion  $i : \partial\Delta^1 \rightarrow \Delta^1$  produces a Kan fibration*

$$\text{Fun}(\Delta^1, \mathcal{C})^{\text{Kan}} \rightarrow \text{Fun}(\partial\Delta^1, \mathcal{C})^{\text{Kan}}$$

between the associated Kan complexes.

We now find ourselves in a particularly advantageous situation. Since  $\text{Hom}_{\mathcal{C}}(x, y)$  is itself a Kan complex we see that the inclusion  $\text{Hom}_{\mathcal{C}}(x, y) \rightarrow \text{Fun}(\Delta^1, \mathcal{C})$  factors through the associated Kan complex, and we therefore realize  $\text{Hom}_{\mathcal{C}}(x, y)$  as the pullback of a point along a Kan fibration

$$\begin{array}{ccc} \text{Hom}_{\mathcal{C}}(x, y) & \longrightarrow & \text{Fun}(\Delta^1, \mathcal{C})^{\text{Kan}} \\ \downarrow & \text{pullback sq} & \downarrow \text{Kan fib} \\ * & \longrightarrow & \text{Fun}(\partial\Delta^1, \mathcal{C})^{\text{Kan}} \end{array}$$

At this point it is believable, at least, that any equivalence between  $\infty$ -categories is fully faithful. Indeed, any equivalence  $F : \mathcal{C} \rightarrow \mathcal{D}$  induces equivalences of Kan complexes  $F_* : \text{Fun}(K, \mathcal{C})^{\text{Kan}} \rightarrow \text{Fun}(K, \mathcal{D})^{\text{Kan}}$ , for  $K = \Delta^1$  and  $\partial\Delta^1$ , so that we expect the induced map on the fibers

$$F_* : \text{Hom}_{\mathcal{C}}(x, y) \rightarrow \text{Hom}_{\mathcal{D}}(Fx, Fy)$$

to also be equivalences.



In order to realize our intuition we travel back to our discussion of Kan complexes in order to cover a topic which we omitted from our initial investigations from Section 4.

sect:les

**6.6. A deviation on long exact sequences and homotopy groups.** Consider a Kan fibration  $f : \mathcal{X} \rightarrow \mathcal{S}$  between Kan complexes. Furthermore, let's choose a point  $x : * \rightarrow \mathcal{X}$ , and  $s = f \circ x : * \rightarrow \mathcal{S}$ , in order to get our homotopy group machine started. We then have a pullback square

$$\begin{array}{ccc} \mathcal{X}_s & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow f \\ * & \xrightarrow{s} & \mathcal{S}, \end{array} \quad (36) \quad \text{eq:1587}$$

and a sequence of maps of homotopy groups

$$\pi_n(\mathcal{X}_s, x) \rightarrow \pi_n(\mathcal{X}, x) \xrightarrow{\pi_n f} \pi_n(\mathcal{S}, s) \quad (37) \quad \text{eq:1594}$$

with  $\text{im}(\pi_n(\mathcal{X}_s)) \subset \ker(f_*)$ . In the case  $n = 0$ , we take specifically  $\ker(\pi_0 f) := (\pi_0 f)^{-1}([s])$ .

Very coarsely, we might view a pullback diagram (36) as analogous to an exact sequence of cochain complexes. In continued analogy with the abelian setting, a diagram as in (36) in fact produces a *long exact sequence* of homotopy groups

$$\begin{aligned} \cdots \rightarrow \pi_{n+1}(\mathcal{S}, s) \xrightarrow{\partial_{n+1}} \pi_n(\mathcal{X}_s, x) \rightarrow \pi_n(\mathcal{X}, x) \rightarrow \pi_n(\mathcal{S}, s) \xrightarrow{\partial_n} \pi_{n-1}(\mathcal{X}_s, x) \rightarrow \\ \cdots \xrightarrow{\partial_1} \pi_0(\mathcal{X}_s, x) \rightarrow \pi_0(\mathcal{X}, x) \rightarrow \pi_0(\mathcal{S}, s). \end{aligned} \quad (38) \quad \text{eq:les}$$

which extends the sequence (37).

thm:les\_pin

**Theorem 6.14** ([10, 00WM]). *Given a Kan fibration  $f : (\mathcal{X}, x) \rightarrow (\mathcal{S}, s)$  and corresponding pullback diagram as in (36), there is a connecting morphism*

$$\partial_n : \pi_n(\mathcal{S}, s) \rightarrow \pi_{n-1}(\mathcal{X}_s, x)$$

*which extends the sequence (37) to a long exact sequence of homotopy groups. This connecting morphism is a morphism of groups whenever  $n > 1$ .*

In the case where  $f : \mathcal{X} \rightarrow \mathcal{S}$  is a trivial Kan fibration, so that the fiber  $\mathcal{X}_s$  is contractible, the proposed long exact sequence of Theorem 6.14 recovers Proposition 4.27. Note that this theorem implicitly claims that the short sequence (37) is in fact exact [10, 00WN]. In this subsection we won't argue the exactness of the above sequence—for those details the reader should see [10]—but instead focus on the construction of the connecting morphism  $\partial_n$  and its “naturality”.

For the remainder of the section we fix pointed spaces  $(\mathcal{X}, x)$  and  $(\mathcal{S}, s)$ , a Kan fibration  $f : \mathcal{X} \rightarrow \mathcal{S}$  between these pointed spaces, and the corresponding pullback diagram (36). The connecting map is specified precisely by the following information.

prop:1622

**Proposition 6.15** ([10, 00WG]). *There is a unique map of sets  $\partial_n : \pi_n(\mathcal{S}, s) \rightarrow \pi_{n-1}(\mathcal{X}_s, x)$  for which  $\pi_n([\gamma]) = [\gamma']$  precisely when the representing morphisms  $\gamma : \Delta^n \rightarrow \mathcal{S}$  and  $\gamma' : \Delta^{n-1} \rightarrow \mathcal{X}$  precisely when there exists a third map  $\vartheta : \Delta^n \rightarrow \mathcal{X}$  with the following properties:*

$$(a) \quad f\vartheta = \gamma : \Delta^n \rightarrow \mathcal{S};$$

$$(b) \vartheta|\Delta^{\{1,\dots,n\}} = \gamma';$$

$$(c) \vartheta|\Lambda_0^n = x.$$

Furthermore,  $\partial_n$  is a group map whenever  $n > 1$ .

As a sanity check, so to speak, let's think about conditions (a) and (b). Condition (a) requires  $\vartheta(\partial\Delta^n) \in f^{-1}(s) = \mathcal{X}_s$ , so that the  $(n-1)$ -simplex  $\vartheta|\Delta^{\{1,\dots,n\}}$  necessarily has image in  $\mathcal{X}_s$ . We require that this  $(n-1)$ -simplex agrees with  $\gamma'$ . Now, (c) says we simply crush the remainder of the boundary at  $x$ . So, topologically, Proposition 6.15 says  $\partial([\gamma]) = [\gamma']$  if there is a map  $|\vartheta| : D^n \rightarrow |\mathcal{X}|$  which lifts  $|\gamma| : S^n \rightarrow |\mathcal{S}|$  and has prescribed boundary  $|\vartheta|_{|\mathbb{S}^{n-1}} = |\gamma'| : \mathbb{S}^{n-1} \rightarrow |\mathcal{X}_s|$ . From the topological perspective the definition of the connecting morphism then becomes quite intuitive.

The explicit description of the connecting morphism provided by Proposition 6.15 allows one to witness certain naturality properties for the long exact sequence of homotopy groups (38).

prop:2796

**Proposition 6.16.** *Suppose we have a diagram of pointed Kan complexes*

$$\begin{array}{ccc} (\mathcal{X}, x) & \xrightarrow{\mu} & (\mathcal{X}', x') \\ f \downarrow & & \downarrow f' \\ (\mathcal{S}, s) & \xrightarrow{\nu} & (\mathcal{S}', s') \end{array}$$

in which both  $f$  and  $f'$  are Kan fibrations. Then the induced maps on homotopy groups

$$\pi_n(\mu_s) : \pi_n(\mathcal{X}_s, x) \rightarrow \pi_n(\mathcal{X}'_s, x'), \quad \pi_n(\mu) : \pi_n(\mathcal{X}, x) \rightarrow \pi_n(\mathcal{X}', x'),$$

$$\pi_n(\nu) : \pi_n(\mathcal{S}, s) \rightarrow \pi_n(\mathcal{S}', s')$$

fit into a map between the corresponding long exact sequences

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \pi_{n+1}(\mathcal{S}, s) & \xrightarrow{\partial} & \pi_n(\mathcal{X}_s, x) & \longrightarrow & \pi_n(\mathcal{X}, x) \longrightarrow \pi_n(\mathcal{S}, s) \xrightarrow{\partial} \cdots \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & \pi_{n+1}(\mathcal{S}', s') & \xrightarrow{\partial} & \pi_n(\mathcal{X}'_s, x') & \longrightarrow & \pi_n(\mathcal{X}', x') \longrightarrow \pi_n(\mathcal{S}', s') \xrightarrow{\partial} \cdots \end{array}$$

sect:loop

**6.7. Mapping spaces as loop spaces.** We consider an example of such a fiber diagram (36) which relates mapping spaces and for a Kan complex  $\mathcal{X}$  to the underlying space  $\mathcal{X}$  itself, and in particular allows us to access the (higher) homotopy group for  $\mathcal{X}$  via the homotopy groups of its mapping spaces.

Consider any Kan complex  $\mathcal{X}$ , and the inclusion  $i : \Delta^{\{0\}} \rightarrow \Delta^1$ . Since  $i$  is anodyne the corresponding restriction map  $i^* : \text{Fun}(\Delta^1, \mathcal{X}) \rightarrow \mathcal{X}$  is a trivial Kan fibration, by Corollary 4.12. Taking the fiber along any point  $x : * \rightarrow \mathcal{X}$  provides another trivial Kan fibration

$$\{x\} \times_{\text{Fun}(\Delta^{\{0\}}, \mathcal{X})} \text{Fun}(\Delta^1, \mathcal{X}) \rightarrow *,$$

so that the domain is seen to be contractible.

Now, restriction along the inclusion  $j : \partial\Delta^1 \rightarrow \Delta^1$  provides a Kan fibration  $j^* : \text{Fun}(\Delta^1, \mathcal{X}) \rightarrow \text{Fun}(\partial\Delta^1, \mathcal{X})$  and we have the pullback diagram

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\quad} & * \\ \downarrow & & \downarrow x \\ \text{Fun}(\partial\Delta^1, \mathcal{X}) & \longrightarrow & \text{Fun}(\Delta^{\{0\}}, \mathcal{X}) \cong \mathcal{X}, \end{array}$$

identifying  $\mathcal{X}$  with the subcomplex of maps  $\partial\Delta^1 \rightarrow \mathcal{X}$  which take constant value  $x$  on  $\Delta^{\{0\}}$ . Pulling back  $j^*$  along the inclusion  $\mathcal{X} \rightarrow \text{Fun}(\partial\Delta^1, \mathcal{X})$  then yields the map

$$\{x\} \times_{\text{Fun}(\Delta^{\{0\}}, \mathcal{X})} \text{Fun}(\Delta^1, \mathcal{X}) \rightarrow \mathcal{X}, \quad (s, t) \mapsto t|_{\Delta^{\{1\}}},$$

which we now conclude is a Kan fibration as well. Let us call this map  $f$ .

We now, finally, have a pullback diagram

$$\begin{array}{ccc} \text{Hom}_{\mathcal{X}}(x, x) & \longrightarrow & \{x\} \times_{\text{Fun}(\Delta^{\{0\}}, \mathcal{X})} \text{Fun}(\Delta^1, \mathcal{X}) \\ \downarrow & & \downarrow f \\ * & \xrightarrow{x} & \mathcal{X} \end{array} \quad (39) \quad \boxed{\text{eq:maps\_loop}}$$

with  $f$  a Kan fibration from a contractible domain. The long exact sequence on homotopy groups now provides the following.

**prop:based**

**Proposition 6.17.** *For any Kan complex  $\mathcal{X}$ , point  $x : * \rightarrow \mathcal{X}$ , and integer  $n \geq 0$ , there is a natural isomorphisms of homotopy groups*

$$\partial_{n+1} : \pi_{n+1}(\mathcal{X}, x) \xrightarrow{\sim} \pi_n(\text{Hom}_{\mathcal{X}}(x, x), id_x).$$

We only note that naturality (with respect to maps of Kan complexes  $\mathcal{X} \rightarrow \mathcal{Y}$ ) comes from naturality of the above constructions in conjunction with Proposition 6.16.

**Remark 6.18** ([10, 01JE]). The diagram (39) identifies the mapping space  $\text{Hom}_{\mathcal{X}}(x, x)$  as the based loop space for  $\mathcal{X}$ , in the homotopy category of spaces.

Via the naturality claim from Proposition 6.17 and Whitehead's Theorem we observe that any map of Kan complexes which is essentially surjective and fully faithful is also an equivalence.

**cor:ffes\_kan**

**Corollary 6.19.** *If  $f : \mathcal{X} \rightarrow \mathcal{Y}$  is a map of Kan complexes which is fully faithful and essentially surjective, then  $f$  is an equivalence.*

**6.8. Equivalences are fully faithful.** We first provide a general result, which we deduce from naturality of the long exact sequence for homotopy groups.

**prop:weee**

**Proposition 6.20.** *Suppose we have a diagram of Kan complexes*

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \\ \downarrow & & \downarrow \\ \mathcal{S} & \xrightarrow{q} & \mathcal{T} \end{array}$$

in which the vertical maps are Kan fibrations, and  $q : \mathcal{S} \rightarrow \mathcal{T}$  is an equivalence. Then the following are equivalent:

- (a) The map  $f : \mathcal{X} \rightarrow \mathcal{Y}$  is an equivalence of Kan complexes.

- (b) For any choice of point  $s : * \rightarrow \mathcal{S}$ , and  $t = qs : * \rightarrow \mathcal{T}$ , the induced map on the fibers  $f_s : \mathcal{X}_s \rightarrow \mathcal{Y}_t$  is an equivalence.

*Proof.* (a)  $\Rightarrow$  (b) By Whitehead's theorem it suffices to show that, for some choice of point  $x : * \rightarrow \mathcal{X}_s$  and  $y = fx : * \rightarrow \mathcal{Y}_t$ , all of the maps on connected components and homotopy groups  $\pi_n f_s : \pi_n(\mathcal{X}_s, x) \rightarrow \pi_n(\mathcal{Y}_t, y)$  are all isomorphisms.

To begin, we know that the induced maps  $\pi_n f : \pi_n(\mathcal{X}, x) \rightarrow \pi_n(\mathcal{Y}, y)$  and  $\pi_n q : \pi_n(\mathcal{S}, s) \rightarrow \pi_n(\mathcal{T}, t)$  are all isomorphisms. So the five lemma and naturality of the long exact sequence on homotopy groups can be employed to see that  $\pi_n f_s$  is an isomorphism whenever  $n > 1$ . When  $n = 1$  a version of the five lemma for not-necessarily-abelian groups still holds, so that  $\pi_1 f_s$  is also seen to be an isomorphism. So we need only show that  $\pi_0 f_s$  is a bijection.

Note that the map  $\pi_0 f_s$  does not depend on the choice of base point  $x : * \rightarrow \mathcal{X}_s$ , but the long exact sequence on homotopy groups does. Suppose we have a point  $[z] \in \pi_0(\mathcal{Y}_t)$  and note that the image of  $[z]$  under the composite

$$\pi_0(\mathcal{Y}_t) \rightarrow \pi_0(\mathcal{Y}) \rightarrow \pi_0(\mathcal{T}, t)$$

is trivial, i.e. is equal to  $[t] \in \pi_0(\mathcal{T}, t)$ .

We consider the image of  $[z]$  in  $\pi_0(\mathcal{Y})$ , which we also denote  $[z]$  by abuse of notation, and take the unique lift of  $[z]$  to an element  $[x] \in \pi_0(\mathcal{X})$  along the isomorphism  $\pi_0(\mathcal{X}) \xrightarrow{\cong} \pi_0(\mathcal{Y})$ . We via the diagram

$$\begin{array}{ccccc} \pi_0(\mathcal{X}_s) & \longrightarrow & \pi_0(\mathcal{X}) & \longrightarrow & \pi_0(\mathcal{S}, s) \\ \downarrow & & \downarrow \cong & & \downarrow \cong \\ \pi_0(\mathcal{Y}_t) & \longrightarrow & \pi_0(\mathcal{Y}) & \longrightarrow & \pi_0(\mathcal{T}, t) \end{array} \quad (40)$$

we see that  $[x]$  maps to  $[s] \in \pi_0(\mathcal{S}, s)$  so that, by exactness of the rows in the above diagram we see that  $[x]$  lifts to an element in  $\pi_0(\mathcal{X}_s)$ . Rather, we see that  $[x] \in \pi_0(\mathcal{X})$  is represented by a point in  $\mathcal{X}_s$ , so that we may assume that  $x$  itself is a point in  $\mathcal{X}_s$  and write simply  $[x] \in \pi_0(\mathcal{X}_s)$  by an abuse of notation. Now, we claim that  $\pi_0 f_s([x]) = [z] \in \pi_0(\mathcal{Y}_t)$ . To see this we fix  $x : * \rightarrow \mathcal{X}_s$  as our base point,  $y = f_s x : * \rightarrow \mathcal{Y}_t$ , and consider the map between long exact sequences

$$\begin{array}{ccccccc} \pi_1(\mathcal{X}, x) & \longrightarrow & \pi_1(\mathcal{S}, s) & \xrightarrow{\partial} & \pi_0(\mathcal{X}_s, x) & \longrightarrow & \pi_0(\mathcal{X}, x) \longrightarrow \pi_0(\mathcal{S}, s) \\ \cong \downarrow & & \cong \downarrow & & \downarrow & & \downarrow \cong \\ \pi_1(\mathcal{Y}, y) & \longrightarrow & \pi_1(\mathcal{T}, t) & \xrightarrow{\partial} & \pi_0(\mathcal{Y}_t, y) & \longrightarrow & \pi_0(\mathcal{Y}, y) \longrightarrow \pi_0(\mathcal{T}, t) \end{array} \quad (41)$$

eq:2856

Via the above diagram, and the fact that  $[x] \mapsto [z]$  under the map  $\pi_0(\mathcal{X}) \rightarrow \pi_0(\mathcal{Y})$ , we see that  $[z] \mapsto [y]$  under the map  $\pi_0(\mathcal{Y}_t, y) \rightarrow \pi_0(\mathcal{Y}, y)$ . This is to say,  $[z]$  is in the kernel of  $\pi_0(\mathcal{Y}_t, y) \rightarrow \pi_0(\mathcal{Y}, y)$ . Via exactness  $[z] \in \pi_0(\mathcal{Y}_t, y)$  lifts to an element  $[\zeta] \in \pi_1(\mathcal{T}, t)$ , which then lifts uniquely to some element  $[\zeta'] \in \pi_1(\mathcal{S}, s)$ . The above diagram then says  $\partial[\zeta'] \in \pi_0(\mathcal{X}_s, x)$  provides provides a lift of  $[z] \in \pi_0(\mathcal{Y}_t, y)$  along  $\pi_0 f_s$ . So we see that  $\pi_0 f_s : \pi_0(\mathcal{X}_s) \rightarrow \pi_0(\mathcal{Y}_t)$  is in fact surjective.

As for injectivity, we suppose  $\pi_0 f_s : \pi_0(\mathcal{X}_s) \rightarrow \pi_0(\mathcal{Y}_t)$  sends two points  $[x], [x'] \in \pi_0(\mathcal{X}_s)$  to the same element  $[y] \in \pi_0(\mathcal{Y}_t)$ , then fix  $x : * \rightarrow \mathcal{X}_s$  as our base point. Via the diagram (41) we see that  $[x']$  is in the kernel of the map  $\pi_0(\mathcal{X}_s, x) \rightarrow \pi_0(\mathcal{X}, s)$  and hence has a preimage in  $[\gamma] \in \pi_1(\mathcal{S}, s)$ . The image  $[\gamma'] = \pi_1 q[\gamma] \in \pi_1(\mathcal{T}, t)$  is in the kernel of the connecting morphism, and so lifts to an element  $[\tilde{\gamma}] \in \pi_1(\mathcal{Y}, y)$ .

This element has a unique preimage  $[\tilde{\gamma}] \in \pi_1(\mathcal{X}, x)$  which necessarily maps to  $[\gamma] \in \pi_1(\mathcal{S}, s)$ , via the above diagram. So

$$[\gamma] \in \ker(\pi_1(\mathcal{S}, s) \rightarrow \pi_0(\mathcal{X}_s, x)),$$

and hence has image  $[x] \in \pi_0(\mathcal{X}_s, x)$ . But we chose  $[\gamma]$  as a lift of  $[x'] \in \pi_0(\mathcal{X}_s, x)$ ! We therefore have  $[x] = [x']$ , and conclude that  $\pi_0 f_s$  is injective. This establishes bijectivity of  $\pi_0 f_s$ ,

$$\pi_0 f_s : \pi_0(\mathcal{X}_s) \xrightarrow{\cong} \pi_0(\mathcal{Y}_t),$$

completing our proof.

(b)  $\Rightarrow$  (a) This follows by a similar analysis of the long exact sequence(s) for homotopy groups, as in (41), and an application of Whitehead's theorem.  $\square$

We now deduce fully faithfulness for equivalences of  $\infty$ -categories.

cor:equiv\_ff

**Corollary 6.21.** *Any equivalence of  $\infty$ -categories  $F : \mathcal{C} \rightarrow \mathcal{D}$  is fully faithful.*

*Proof.* At any pair of objects  $x, y : * \rightarrow \mathcal{C}$  we have the diagram

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathcal{C}}(x, y) & \xrightarrow{\quad\quad\quad} & \mathrm{Hom}_{\mathcal{D}}(Fx, Fy) & & (42) \\ \downarrow & \searrow & \downarrow & \searrow & \\ * & & \mathrm{Fun}(\Delta^1, \mathcal{C})^{\mathrm{Kan}} & \xrightarrow{\quad\sim\quad} & * \\ & \searrow & \downarrow & & \searrow \\ & & \mathrm{Fun}(\partial\Delta^1, \mathcal{C})^{\mathrm{Kan}} & \xrightarrow{\quad\sim\quad} & \mathrm{Fun}(\partial\Delta^1, \mathcal{D})^{\mathrm{Kan}}. \end{array}$$

eq:2914

Here the horizontal (red) maps are induced by  $F$  and those maps labeled  $\sim$  are equivalences of  $\infty$ -categories, by Theorem 5.41, and the (white) squares involving the point  $*$  are pullback diagrams. Hence the map

$$F_* : \mathrm{Hom}_{\mathcal{C}}(x, y) \rightarrow \mathrm{Hom}_{\mathcal{D}}(Fx, Fy)$$

is an equivalence, by Proposition 6.20.  $\square$

**6.9. The Kan complex functor and equivalences.** The aim of this subsection is to prove the following.

prop:2881

**Proposition 6.22.** *Suppose  $F : \mathcal{C} \rightarrow \mathcal{D}$  is an equivalence between  $\infty$ -categories. Then the induced map  $F^{\mathrm{Kan}} : \mathcal{C}^{\mathrm{Kan}} \rightarrow \mathcal{D}^{\mathrm{Kan}}$  is also an equivalence.*

Proposition 6.22 is used in our proof of the fact that equivalences of  $\infty$ -categories are essentially surjective. The proof is relatively straightforward, but requires some technical details. We provide the necessary details then return to provide the proof of Proposition 6.22.

lem:2822

**Lemma 6.23.** *Consider  $\infty$ -categories  $\mathcal{A}$  and  $\mathcal{B}$ , and morphisms  $\alpha : a_1 \rightarrow a_2$  and  $\beta : b_1 \rightarrow b_2$  in  $\mathcal{A}$  and  $\mathcal{B}$  respectively. Then the product map  $[\alpha, \beta] : \Delta^1 \rightarrow \mathcal{A} \times \mathcal{B}$  is a composition  $[\alpha, \beta] = [a_1, \beta] \circ [\alpha, b_1]$  in  $\mathcal{A} \times \mathcal{B}$*

*Proof.* We need to construct a 2-simplex  $c : \Delta^2 \rightarrow \mathcal{A} \times \mathcal{B}$  with the three maps above appearing as the boundary morphisms for  $c$ . By definition, any such  $c$  is specified by two 2-simplices  $c_1 : \Delta^2 \rightarrow \mathcal{A}$  and  $c_2 : \Delta^2 \rightarrow \mathcal{B}$ . We take

$$c_1 = \Delta^2 \xrightarrow{d_1} \Delta^1 \xrightarrow{\alpha} \mathcal{A}, \quad c_2 = \Delta^2 \xrightarrow{d_0} \Delta^1 \xrightarrow{\beta} \mathcal{B},$$

to produce our 2-simplex  $c = [c_1, c_2]$  with

$$c|\Delta^{\{0,1\}} = [\alpha, id_{b_1}], \quad c|\Delta^{\{1,2\}} = [id_{a_2}, \beta], \quad \text{and} \quad c|\Delta^{\{0,3\}} = [\alpha, \beta],$$

as desired.  $\square$

Recall that for any Kan complexes  $\mathcal{X}$  and  $\mathcal{Y}$  the functor space  $\text{Fun}(\mathcal{X}, \mathcal{Y})$  is a Kan complex. Hence for any Kan complex  $\mathcal{X}$  and  $\infty$ -category  $\mathcal{E}$  the inclusion  $\mathcal{E}^{\text{Kan}} \rightarrow \mathcal{E}$  induces a natural map of Kan complexes

$$\text{Fun}(\mathcal{X}, \mathcal{E}^{\text{Kan}}) \rightarrow \text{Fun}(\mathcal{X}, \mathcal{E})^{\text{Kan}}. \quad (43)$$

eq:2841

lem:2845

**Lemma 6.24.** *For any Kan complex  $\mathcal{X}$ , and  $\infty$ -category  $\mathcal{E}$ , the inclusion (43) is a isomorphism of Kan complexes.*

*Proof.* Take any  $n$ -simplex  $w : \Delta^n \times \mathcal{X} \rightarrow \mathcal{E}$  in  $\text{Fun}(\mathcal{X}, \mathcal{E})^{\text{Kan}}$ . We want to show that  $w$  has image in  $\mathcal{E}^{\text{Kan}}$ , so that  $w$  factors uniquely through an  $n$ -simplex  $\tilde{w} : \Delta^n \times \mathcal{X} \rightarrow \mathcal{E}^{\text{Kan}}$ . We just need to show that the image  $w(\xi)$  of every morphism  $\xi$  in the  $\infty$ -category  $\Delta^n \times \mathcal{X}$  is an isomorphism in  $\mathcal{E}$ .

Now, morphisms  $\xi : \Delta^1 \rightarrow \Delta^n \times \mathcal{X}$  are of three flavors. In the first case  $\xi = [\xi_1, x]$  for some point  $x : \Delta^0 \rightarrow \mathcal{X}$ , and

$$w(\xi) : \xi_1|_{\Delta^{\{0\}}}(x) \rightarrow \xi_1|_{\Delta^{\{1\}}}(x)$$

is the evaluation of a natural transformation  $\xi|\Delta^1 \times \mathcal{X}$  between the functors  $\xi_1|_{\Delta^{\{0\}}}$  and  $\xi_1|_{\Delta^{\{1\}}}$  at  $x$ . Since  $\xi$  is a 2-simplex in  $\text{Fun}(\mathcal{X}, \mathcal{E})^{\text{Kan}}$ , this natural transformation is in particular a natural isomorphism. By Proposition 6.8 it follows that  $w(\xi)$  is an isomorphism. In the second case  $\xi = [i, \xi_2]$  for a morphism  $\xi_2 : \Delta^2 \rightarrow \mathcal{X}$ , so that  $w(\xi)$  is the image of a morphism in  $\mathcal{X}$  under the composite functor

$$\mathcal{X} \xrightarrow{\cong} \Delta^{\{i\}} \times \mathcal{X} \xrightarrow{w_i} \mathcal{E}.$$

Since functors between  $\infty$ -categories preserve isomorphisms, we see that  $w(\xi)$  is an isomorphism. Lastly, when  $\xi = [\xi_1, \xi_2]$  for morphisms  $\xi_1$  in  $\Delta^1$  and  $\xi_2$  in  $\mathcal{X}$ , we realize  $w(\xi)$  as a composite

$$w(\xi) = w([i, \xi_2]) \circ w([\xi_1, x])$$

by Lemma 6.23. We have just argued that  $w([\xi_1, x])$  and  $w([i, \xi_2])$  are isomorphisms in  $\mathcal{E}$ , so that  $w(\xi)$  is again an isomorphism in  $\mathcal{E}$ .

We now see any  $n$ -simplex  $w : \Delta^n \times \mathcal{X} \rightarrow \mathcal{E}$  has image in  $\mathcal{E}$ . This show that all  $n$ -simplices in  $\text{Fun}(\mathcal{X}, \mathcal{E})^{\text{Kan}}$  lie in the subcomplex  $\text{Fun}(\mathcal{X}, \mathcal{E}^{\text{Kan}})$ , and hence that these two complexes are identified.  $\square$

The above Proposition tells us that the associated Kan complex functor  $-^{\text{Kan}}$  lifts to a functor between simplicial categories, and subsequently between  $\infty$ -categories

$$-^{\text{Kan}} : \mathcal{Cat}_{\infty} \rightarrow \mathcal{Kan} = \mathcal{Spaces}.$$

We then get an induced map on homotopy categories

$$\text{h}(-^{\text{Kan}}) : \text{h}\mathcal{Cat}_{\infty} \rightarrow \text{h}\mathcal{Spaces}.$$

This functor necessarily sends isomorphisms to isomorphisms, which implies Proposition 6.22. We record a more direct proof in any case.

*Proof of Proposition 6.22.* Consider an inverse  $F' : \mathcal{D} \rightarrow \mathcal{C}$  and natural isomorphisms

$$\xi : \Delta^1 \rightarrow \text{Fun}(\mathcal{C}, \mathcal{C})^{\text{Kan}}, \quad \eta : \Delta^1 \rightarrow \text{Fun}(\mathcal{D}, \mathcal{D})^{\text{Kan}}$$

between  $F'F$  and the identity on  $\mathcal{C}$ , and  $FF'$  and the identity on  $\mathcal{D}$ , respectively. We restrict the domains to produce natural isomorphisms

$$\bar{\xi} : \Delta^1 \rightarrow \text{Fun}(\mathcal{C}^{\text{Kan}}, \mathcal{C})^{\text{Kan}} = \text{Fun}(\mathcal{C}^{\text{Kan}}, \mathcal{C}^{\text{Kan}})$$

and

$$\bar{\eta} : \Delta^1 \rightarrow \text{Fun}(\mathcal{D}^{\text{Kan}}, \mathcal{D})^{\text{Kan}} = \text{Fun}(\mathcal{D}^{\text{Kan}}, \mathcal{D}^{\text{Kan}})$$

which realize  $(F')^{\text{Kan}}$  as an inverse to  $F^{\text{Kan}}$ .  $\square$

#### 6.10. Equivalences are essentially surjective.

`prop:equiv_ffes`

**Proposition 6.25.** *An equivalence between  $\infty$ -categories  $F : \mathcal{C} \rightarrow \mathcal{D}$  is essentially surjective.*

*Proof.* The induced map on Kan complexes  $F^{\text{Kan}} : \mathcal{C}^{\text{Kan}} \rightarrow \mathcal{D}^{\text{Kan}}$  is also an equivalence, by Proposition 6.22. By Whitehead's theorem, the induced map on connected components  $\pi_0 F^{\text{Kan}} : \pi_0(\mathcal{C}^{\text{Kan}}) \rightarrow \pi_0(\mathcal{D}^{\text{Kan}})$  is an isomorphism, so that  $F$  is essentially surjective.  $\square$

We've now established the “easier” direction for Theorem 6.11,

$$\text{equivalence} \Rightarrow \text{fully faithful} + \text{essentially surjective}.$$

#### 6.11. Restricting fully faithful functors.

We now want to establish the implication

$$\text{fully faithful} + \text{essentially surjective} \Rightarrow \text{equivalence}.$$

In the most basic terms, this implication follows via a reduction argument from the  $\infty$ -category setting to the Kan complex setting. We first prove that any fully faithful functor between  $\infty$ -categories restricts to a fully faithful functor on the associated Kan complexes.

`lem:3002`

**Lemma 6.26.** *Let  $F : \mathcal{X} \rightarrow \mathcal{Y}$  be an equivalence between Kan complexes and let  $\mathcal{X}' \subseteq \mathcal{X}$  and  $\mathcal{Y}' \subseteq \mathcal{Y}$  be full subcategories. Suppose that  $F$  restricts to a map  $F' : \mathcal{X}' \rightarrow \mathcal{Y}'$  and that the induced map on connected components  $\pi_0(\mathcal{X}') \rightarrow \pi_0(\mathcal{Y}')$  is a bijection. Then the restriction  $F'$  is an equivalence of Kan complexes.*

*Proof.* For any  $x \in \mathcal{X}'$  and  $y \in \mathcal{Y}'$  the respective inclusions induce equalities

$$\pi_n(\mathcal{X}', x) = \pi_n(\mathcal{X}, x) \quad \text{and} \quad \pi_n(\mathcal{Y}', y) = \pi_n(\mathcal{Y}, y)$$

for all positive integers  $n$ . Via Whitehead's theorem we understand that  $F$  induces an isomorphism on all homotopy groups for  $\mathcal{X}$  and  $\mathcal{Y}$ , and therefore that  $F'$  induces isomorphisms on all homotopy groups for  $\mathcal{X}'$  and  $\mathcal{Y}'$ . Apply Whitehead again to observe that  $F'$  is an equivalence.  $\square$

`lem:3082`

**Lemma 6.27.** *Suppose that a functor between  $\infty$ -categories  $F : \mathcal{C} \rightarrow \mathcal{D}$  is fully faithful. Then the induced map  $F^{\text{Kan}} : \mathcal{C}^{\text{Kan}} \rightarrow \mathcal{D}^{\text{Kan}}$  is fully faithful as well.*

*Proof.* For objects  $x$  and  $x'$  in  $\mathcal{C}$ , with images  $y$  and  $y'$  in  $\mathcal{D}$ , we consider the map of Kan complexes

$$\text{Hom}_{\mathcal{C}^{\text{Kan}}}(x, x') \rightarrow \text{Hom}_{\mathcal{D}^{\text{Kan}}}(y, y').$$

These Kan complexes sit as fully subcategories in the ambient categories  $\text{Hom}_{\mathcal{C}}(x, x')$  and  $\text{Hom}_{\mathcal{D}}(y, y')$ , and we have a diagram

$$\begin{array}{ccc} \text{Hom}_{\mathcal{C}^{\text{Kan}}}(x, x') & \xrightarrow{F^{\text{Kan}}} & \text{Hom}_{\mathcal{D}^{\text{Kan}}}(y, y') \\ \uparrow & & \uparrow \\ \text{Hom}_{\mathcal{C}}(x, x') & \xrightarrow{F} & \text{Hom}_{\mathcal{D}}(y, y'). \end{array}$$

So it suffices to show that  $F^{\text{Kan}}$  induces a bijection on connected components, by Lemma 6.26. For this it suffices to prove that every equivalence  $\beta : y \rightarrow y'$  lifts to an equivalence  $\alpha : x \rightarrow x'$  in  $\mathcal{C}$ . Let us choose an arbitrary map  $\alpha : x \rightarrow x'$  with  $F\alpha \simeq \beta$ . We claim that  $\alpha$  is an equivalence.

Take any lift  $\alpha' : x' \rightarrow x$  of an inverse  $\beta^{-1} : y' \rightarrow y$ , and consider a composites  $\alpha'\alpha$  and  $\alpha\alpha'$ . We have  $F(\alpha'\alpha) \simeq id_y$  and since  $F$  induces an equivalence on morphism spaces we have  $\alpha'\alpha \simeq id_x$ , and similarly find  $\alpha\alpha' \simeq id_y$ . So  $\alpha$  is in fact an equivalence, i.e. a map in  $\mathcal{C}^{\text{Kan}}$ , as desired.  $\square$

As we saw in the proof, for a given fully faithful functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  we find that a given map  $\alpha$  in  $\mathcal{C}$  is an equivalence if and only if  $F\alpha$  is an equivalence.

**Definition 6.28.** A functor between  $\infty$ -categories  $F : \mathcal{C} \rightarrow \mathcal{D}$  is called conservative if a map  $\alpha : x \rightarrow x'$  in  $\mathcal{C}$  is an equivalence if and only if  $F\alpha : Fx \rightarrow Fx'$  is an equivalence.

We have the following.

**Lemma 6.29** ([10, 01JN]). *Any fully faithful functor between  $\infty$ -categories is conservative.*

**6.12. Proof of Theorem 6.11.** We begin with a refinement of Proposition 6.22.

lem:3123

**Lemma 6.30.** *Suppose  $F : \mathcal{C} \rightarrow \mathcal{D}$  is a trivial Kan fibration between  $\infty$ -categories. Then the map  $F^{\text{Kan}} : \mathcal{C}^{\text{Kan}} \rightarrow \mathcal{D}^{\text{Kan}}$  is a trivial Kan fibration as well.*

*Proof.* The lifting property applies to the inclusions  $\Lambda_0^2 \rightarrow \Delta^2$  and  $\Lambda_2^2$  imply that  $F$  is conservative. Hence the lifting property for  $F$  immediately restricts to a lifting property for  $F^{\text{Kan}}$ .  $\square$

The following result allows us to descend from the Kan setting to the  $\infty$ -setting.

thm:OK

**Theorem 6.31** ([10, 01HG]). *A map between  $\infty$ -categories  $F : \mathcal{C} \rightarrow \mathcal{D}$  is an equivalence if and only if the associated map of Kan complexes*

$$\text{Fun}(\Delta^1, F)^{\text{Kan}} : \text{Fun}(\Delta^1, \mathcal{C})^{\text{Kan}} \rightarrow \text{Fun}(\Delta^1, \mathcal{D})^{\text{Kan}}$$

*is an equivalence.*

We sketch the proof, and note that it involves the notion of a categorical pushout square for simplicial sets. This notion is covered in Section 7.

*Sketch proof.* Take

$$F^K = \text{Fun}(K, F)^{\text{Kan}} : \text{Fun}(K, \mathcal{C})^{\text{Kan}} \rightarrow \text{Fun}(K, \mathcal{D})^{\text{Kan}}$$

at a given simplicial set  $K$ . By Theorem 5.41,  $F$  is an equivalence if and only if  $F^K$  is an equivalence at all  $K$ . In particular, if  $F$  is an equivalence then  $F^{\Delta^1}$  is an equivalence.



For the converse claim, suppose that  $F^{\Delta^1}$  is an equivalence and let  $\Upsilon$  denote the class of all simplicial sets  $K$  at which  $F^K$  is an equivalence. One can show the following: (O1) If we have a directed sequence  $K_0 \rightarrow K_1 \rightarrow \cdots$  is maps of simplicial sets, and all  $F^{K_i}$  are equivalences, then the map  $F^K$  is an equivalence for  $K = \varprojlim_n K_n$ . (O2) If we have a retract  $L \rightarrow K \rightarrow L$ , and  $F^K$  is an equivalence, then  $F^L$  is an equivalence. (O3) If  $\{K_\lambda\}_\lambda$  is a set of simplicial sets for which all  $F^{K_\lambda}$  are equivalences, then the map  $F^{\prod_\lambda K_\lambda}$  is an equivalence. (O4) If  $L \rightarrow K$  is inner anodyne, then  $F^L$  is an equivalence if and only if  $F^K$  is an equivalence (Corollary 5.8 and Lemma 6.30). (O5) If

$$\begin{array}{ccc} L_0 & \xrightarrow{\mu_1} & L_1 \\ \mu_2 \downarrow & & \downarrow \\ L_2 & \longrightarrow & K \end{array}$$

is a categorical pushout square and all of the  $F^{L_i}$  are equivalences, then  $F^K$  is an equivalence (Proposition 7.10). This applies, in particular, to the case of a standard pushout square in which one of the  $\mu_i$  is injective (Corollary 7.18).

We now see that  $\Upsilon$  is a class of simplicial sets which contains  $\Delta^1$  and which is stable under the operations (O1)–(O5). Since the 0-simplex is a retract of  $\Delta^1$ , we see that  $\Upsilon$  contains  $\Delta^0$ . Now, supposing all  $\Delta^m$  are in  $\Upsilon$  for  $m < n$ , and  $n \geq 2$ , one sees that all horns  $\Lambda_i^{n+1}$  are in  $\Upsilon$  by considering the appropriate pushout diagram, and subsequently sees that  $\Delta^n$  is in  $\Upsilon$  by considering the inner anodyne morphism  $\Lambda_1^n \rightarrow \Delta^n$  and applying (O4). It follows by induction that all standard simplices are in  $\Upsilon$ , and subsequently that  $\Upsilon$  contains all simplicial sets via applications of (O5) and (O1).  $\square$

We also have the following.

lem:3103

**Lemma 6.32.** *For any  $\infty$ -category  $\mathcal{C}$ , the restriction functor  $\text{Fun}(\Delta^1, \mathcal{C}) \rightarrow \text{Fun}(\partial\Delta^1, \mathcal{C})$  is an isofibration (Definition 5.29).*

*Proof.* Suppose we a functor  $f : \Delta^1 \rightarrow \mathcal{C}$ , i.e. a map in  $\mathcal{C}$ , and object  $y : \partial\Delta^1 \rightarrow \mathcal{C}$ , i.e. a pair of objects  $(y_1, y_2)$ , and an isomorphism  $\alpha : f|_{\partial\Delta^1} \xrightarrow{\sim} y$ . Such an  $\alpha$  is the information of a partial diagram

$$\begin{array}{ccc} x_1 & \xrightarrow{f} & x_2 \\ \alpha_1 \downarrow & & \downarrow \alpha_2 \\ y_1 & & y_2 \end{array}$$

in which the  $\alpha_i$  are isomorphisms. By Proposition 5.33 we can complete this partial diagram to a diagram

$$\begin{array}{ccc} x_1 & \xrightarrow{f} & x_2 \\ \alpha_1 \downarrow & \searrow h & \downarrow \alpha_2 \\ y_1 & \xrightarrow{g} & y_2 \end{array}$$

in  $\mathcal{C}$ . Such a diagram defines, or really is defined by, a map  $z : \Delta^1 \times \Delta^1 \rightarrow \mathcal{C}$  with the appropriate restrictions. This is to say, we can find a map  $g : \Delta^1 \rightarrow \mathcal{C}$  and a natural transformation  $z : f \rightarrow g$  with  $z|_{\partial\Delta^1} = \alpha$ . Since  $\alpha$  is an isomorphism  $z$  is

an isomorphism, by Theorem 6.5. By Lemma 5.30 we conclude that the restriction functor  $\text{Fun}(\Delta^1, \mathcal{C}) \rightarrow \text{Fun}(\partial\Delta^1, \mathcal{C})$  is an isofibration.  $\square$

Supposing Theorem 6.31, we can now provide a proof of Theorem 6.11.

*Proof.* Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor between  $\infty$ -categories. If  $F$  is an equivalence, the  $F$  is fully faithful and essentially surjective by Corollary 6.21 and Proposition 6.25. Suppose conversely that  $F$  is fully faithful and essentially surjective.

Let us consider the diagram

$$\begin{array}{ccc} \text{Fun}(\Delta^1, \mathcal{C})^{\text{Kan}} & \xrightarrow{F_*} & \text{Fun}(\Delta^1, \mathcal{D})^{\text{Kan}} \\ \downarrow & & \downarrow \\ \text{Fun}(\partial\Delta^1, \mathcal{C})^{\text{Kan}} & \xrightarrow{F_*} & \text{Fun}(\partial\Delta^1, \mathcal{D})^{\text{Kan}}. \end{array}$$

We know that the bottom map is an fully faithful and essentially surjective, by Lemma 6.27, and hence an equivalence by Corollary 6.19. Also, by Corollary 5.34 and Lemma 6.32, the vertical maps are both Kan fibrations.

Since the fibers of the vertical maps over points in  $\text{Fun}(\partial\Delta^1, \mathcal{C})$  and  $\text{Fun}(\partial\Delta^1, \mathcal{D})$  are the respective mapping spaces, and  $F_*$  restricts the induced morphism

$$F_{x,y} : \text{Hom}_{\mathcal{C}}(x, y) \rightarrow \text{Hom}_{\mathcal{D}}(Fx, Fy)$$

on these fibers. So, via fully faithfulness of  $F$ , we apply Proposition 6.20 to see that  $F_* : \text{Fun}(\Delta^1, \mathcal{C})^{\text{Kan}} \rightarrow \text{Fun}(\Delta^1, \mathcal{D})^{\text{Kan}}$  is an equivalence. By Theorem 6.31 it follows that  $F$  is an equivalence. We are done.  $\square$

sect:htop\_pullback

## 7. HOMOTOPY PULLBACK AND CATEGORICAL PUSHOUT

**7.1. Fibrant replacement.** While we are not especially intersted in introducing the general notion of a model structure, let us provide some basic structuring of the category of Kan complexes.

**Lemma 7.1.** *If  $f : \mathcal{X} \rightarrow \mathcal{X}'$  is anodyne morphism between Kan complexes then  $f$  is a homotopy equivalence.*

*Proof.* Follows by Corollary 4.12, Proposition 4.29, and Proposition 4.18.  $\square$

**Proposition 7.2.** *Any map of Kan complexes  $f : \mathcal{X} \rightarrow \mathcal{Y}$  admits a factorization*

$$\mathcal{X} \xrightarrow{t} \mathcal{X}' \xrightarrow{f'} \mathcal{Y}$$

*with  $t$  an anodyne equivalence and  $f'$  a Kan fibration.*

We do not cover the proof, and refer directly to the text [10, 00UU] for the details. We only note that, most immediately, [10] provides a factorization as above where  $\mathcal{X}'$  is only assumed to be a simplicial set. However, since  $\mathcal{X}' \rightarrow \mathcal{Y}$  is a Kan fibration and  $\mathcal{Y}$  is itself a Kan complex, it follows that  $\mathcal{X}'$  is a Kan complex as well.

There are points at which the fact that  $t$  is an anodyne map will be important. However, we will often only use the fact that any map factors  $\mathcal{X} \rightarrow \mathcal{X}' \rightarrow \mathcal{Y}$  where the second map is a Kan fibration and the first map is simply an equivalence. In such a situation we refer to  $\mathcal{X}'$  as a “fibrant replacement” for  $\mathcal{X}$  in the category of Kan complexes over  $\mathcal{Y}$ .

**7.2. Introductory discussion for (homotopy) pullback.** In general, for a pullback diagram

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathcal{Y} \\ q' \downarrow & & \downarrow q \\ \mathcal{X}' & \longrightarrow & \mathcal{S} \end{array}$$

the fact that  $q$  is a homotopy equivalence does not imply that  $q'$  is a homotopy equivalence. Consider for example the case where  $q : \mathcal{Y} \rightarrow \mathcal{S}$  is a map between two contractible spaces which is not surjective on points. In this case we can consider a point  $x : * \rightarrow \mathcal{S}$  which does not lie in the image of  $\mathcal{Y}$  to obtain a pullback diagram

$$\begin{array}{ccc} \emptyset & \longrightarrow & \mathcal{Y} \\ q' \downarrow & & \downarrow p \\ * & \longrightarrow & \mathcal{S} \end{array} \quad (44) \quad \boxed{\text{eq:3224}}$$

in which the map  $q'$  is, obviously, not a equivalence.

In comparison with the dg setting, for dg schemes (or affine dg schemes if one likes), the preservation of equivalences under base change might be understood as a kind of flatness condition on morphisms. The Kan condition provides such a condition.

`prop:kan_basechange`

**Proposition 7.3** ([10, 0109]). *Suppose we have a diagram*

$$\begin{array}{ccccc} \mathcal{X} & & & & \mathcal{Y} \\ & \searrow f & & \swarrow u & \\ & & \mathcal{S} & & \\ & \swarrow f' & & \searrow & \\ \mathcal{X}' & & & & \mathcal{Y}' \\ & \searrow & & \swarrow & \\ & & \mathcal{S}' & & \end{array}$$

in which the maps  $f$  and  $f'$  are Kan fibrations, and all vertical maps are homotopy equivalences. Then the induced map on fiber products  $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y} \rightarrow \mathcal{X}' \times_{\mathcal{S}'} \mathcal{Y}'$  is an equivalence.

*Proof.* We consider the diagram

$$\begin{array}{ccc} \mathcal{X} \times_{\mathcal{S}} \mathcal{Y} & \longrightarrow & \mathcal{X}' \times_{\mathcal{S}'} \mathcal{Y}' \\ \downarrow & & \downarrow \\ \mathcal{Y} & \xrightarrow{u} & \mathcal{Y}' \end{array}$$

in which the vertical maps are Kan fibrations, and the bottom map is a homotopy equivalence. Hence, by Proposition 6.20, the map on fiber products is an equivalence if and only if we have an equivalence

$$\mathcal{X}_s = (\mathcal{X} \times_{\mathcal{S}} \mathcal{Y})_y \rightarrow (\mathcal{X}' \times_{\mathcal{S}'} \mathcal{Y}')_y = \mathcal{X}'_s$$

at all points  $y : * \rightarrow \mathcal{Y}$  with corresponding point  $s : * \rightarrow \mathcal{S}$ . However, the above map is the fiber of the equivalence  $t$  over  $s$ , which we know to be an equivalence by

Proposition 6.20 applied to the diagram

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{t} & \mathcal{X}' \\ f \downarrow & & \downarrow f' \\ \mathcal{S} & \xrightarrow{v} & \mathcal{S}'. \end{array}$$

□

As a particular application of Proposition 7.3 we see that the base change  $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y} \rightarrow \mathcal{X} \times_{\mathcal{S}} \mathcal{Y}'$  of an equivalence  $\mathcal{Y} \rightarrow \mathcal{Y}'$ , for spaces over  $\mathcal{S}$ , remains an equivalence provided  $\mathcal{X}$  is a Kan fibration over  $\mathcal{S}$ .

In vague analogy to the dg setting, one might define the homotopy (aka “derived”) pullback of spaces by first taking a fibrant replacement

$$\mathcal{X} \rightarrow \mathcal{S} \rightsquigarrow \mathcal{X} \overset{\sim}{\dashrightarrow} \mathcal{X}' \rightarrow \mathcal{S}$$

and replacing the usual fiber product  $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$  with the product  $\mathcal{X}' \times_{\mathcal{S}} \mathcal{Y}$ , at least up to some first order. As one might understand from experience, this construction has the drawback of being ambiguous and non-functorial.

In some sense, one of the points of working in the  $\infty$ , rather than dg, setting is to provide explicit control on such ambiguities. However, let us leave this issue for now and simply proceed with our presentation.

### 7.3. A functorial construction.

**Definition 7.4.** Given maps of Kan complexes  $\mathcal{X} \rightarrow \mathcal{S}$  and  $\mathcal{Y} \rightarrow \mathcal{S}$  we define the homotopy pullback as

$$\mathcal{X} \times_{\mathcal{S}}^{\text{htop}} \mathcal{Y} := \mathcal{X} \times_{\text{Fun}(\{0\}, \mathcal{S})} \text{Fun}(\Delta^1, \mathcal{S}) \times_{\text{Fun}(\{1\}, \mathcal{S})} \mathcal{Y}.$$

Note that we can write, alternatively,

$$\mathcal{X} \times_{\mathcal{S}}^{\text{htop}} \mathcal{Y} = \text{Fun}(\Delta^1, \mathcal{S}) \times_{\text{Fun}(\partial\Delta^1, \mathcal{S})} (\mathcal{X} \times \mathcal{Y}).$$

Let us be clear that the homotopy pullback  $\mathcal{X} \times_{\mathcal{S}}^{\text{htop}} \mathcal{Y}$  does not fit into a diagram over  $\mathcal{X}$  and  $\mathcal{Y}$  in general. Instead we have a diagram

$$\begin{array}{ccc} \mathcal{X} \times_{\mathcal{S}}^{\text{htop}} \mathcal{Y} & \longrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{S} \end{array}$$

in the  $\infty$ -category of spaces, where the necessary homotopy between the two maps to  $\mathcal{S}$  is given by evaluation

$$\Delta^1 \times (\text{Fun}(\Delta^1, \mathcal{S}) \times_{\text{Fun}(\partial\Delta^1, \mathcal{S})} (\mathcal{X} \times \mathcal{Y})) \xrightarrow{\text{project}} \Delta^1 \times \text{Fun}(\Delta^1, \mathcal{S}) \xrightarrow{\text{eval}} \mathcal{S}$$

**Example 7.5.** Consider two distinct points in a contractible space  $x, y : * \rightarrow \mathcal{S}$ . The homotopy fiber product in this case is the mapping space

$$\{x\} \times_{\mathcal{S}}^{\text{htop}} \{y\} = \text{Hom}_{\mathcal{S}}(x, y).$$

Since the terminal map  $\mathcal{S} \rightarrow *$  is an equivalence, it is in particular fully faithful. So the induced map  $\text{Hom}_{\mathcal{S}}(x, y) \rightarrow \text{Hom}_*(*, *) = *$  is an equivalence. In particular, the homotopy fiber product is a non-empty contractible space.

More generally, when  $\mathcal{S}$  is non-contractible the homotopy fiber product  $x \times_{\mathcal{S}}^{\text{htop}} y$  is empty if and only if the two points live in different components in  $\mathcal{S}$ . In the event that  $x$  and  $y$  are in the same component we claim, without proof, that the homotopy fiber product is equivalent to the loop space for  $\mathcal{S}$  at either  $x$  or  $y$  (cf. Section 6.7).

One sees that restriction along the map  $\Delta^1 \rightarrow *$  provides a binatural embedding  $\mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{X} \times_{\mathcal{S}}^{\text{htop}} \mathcal{Y}$ . We refer to this embedding informally as the “comparison map”.

prop:3319

**Proposition 7.6** ([10, 0329]). *For any partial diagram*

$$\begin{array}{ccc} \mathcal{X} & & \mathcal{Y} \\ & \searrow f & \swarrow g \\ & \mathcal{S} & \end{array}$$

*in which either  $f$  or  $g$  is a Kan fibration the comparison map  $\mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{X} \times_{\mathcal{S}}^{\text{htop}} \mathcal{Y}$  is an equivalence.*

*Proof.* Suppose that  $\mathcal{X} \rightarrow \mathcal{S}$  is a Kan fibration. In this case the base changed map  $\mathcal{X} \times_{\mathcal{S}} \text{Fun}(\Delta^1, \mathcal{S}) \rightarrow \text{Fun}(\Delta^1, \mathcal{S})$  is also a Kan fibration. Furthermore, since the inclusion  $1 : \Delta^0 \rightarrow \Delta^1$  is anodyne the induced map  $\text{Fun}(\Delta^1, \mathcal{S}) \rightarrow \mathcal{S}$  is a trivial Kan fibration, by Corollary 4.12, and in particular an equivalence by Proposition 4.29. We consider also the map

$$h = \left( \mathcal{Y} \xrightarrow{g} \mathcal{S} \xrightarrow{\text{const}} \text{Fun}(\Delta^1, \mathcal{S}) \right).$$

We now have a diagram

$$\begin{array}{ccccc} \mathcal{X} \times_{\mathcal{S}} \text{Fun}(\Delta^1, \mathcal{S}) & \longrightarrow & \text{Fun}(\Delta^1, \mathcal{S}) & \xleftarrow{h} & \mathcal{Y} \\ \downarrow = & & \downarrow 1^* & & \downarrow = \\ \mathcal{X} \times_{\mathcal{S}} \text{Fun}(\Delta^1, \mathcal{S}) & \longrightarrow & \mathcal{S} & \xleftarrow{g} & \mathcal{Y} \end{array}$$

in which the vertical maps are equivalences (in particular trivial Kan fibrations) and the maps from  $\mathcal{X} \times_{\mathcal{S}} \text{Fun}(\Delta^1, \mathcal{S})$  are Kan fibrations. So, by Proposition 7.3, the induced map

$$\begin{aligned} \mathcal{X} \times_{\mathcal{S}} \mathcal{Y} &\cong (\mathcal{X} \times_{\mathcal{S}} \text{Fun}(\Delta^1, \mathcal{S})) \times_{\text{Fun}(\Delta^1, \mathcal{S})} \mathcal{Y} \\ &\rightarrow \mathcal{X} \times_{\mathcal{S}} \text{Fun}(\Delta^1, \mathcal{S}) \times_{\mathcal{S}} \mathcal{Y} = \mathcal{X} \times_{\mathcal{S}}^{\text{htop}} \mathcal{Y} \end{aligned}$$

is a homotopy equivalence. The case where the  $g$  is a Kan fibration is dealt with similarly.  $\square$

As one imagines at this point, we have a homotopy analog of Proposition 7.3 in which the Kan condition is now obviated.

prop:htop\_basechange

**Proposition 7.7** ([10, 032B]). *Suppose we have a diagram*

$$\begin{array}{ccccc}
 \mathcal{X} & & & & \mathcal{Y} \\
 \downarrow t & \searrow & & \swarrow & \downarrow u \\
 & & \mathcal{S} & & \\
 & \swarrow & \downarrow v & \searrow & \\
 \mathcal{X}' & & \mathcal{S}' & & \mathcal{Y}'
 \end{array}$$

in which all vertical maps are homotopy equivalences. Then the induced map on homotopy fiber products

$$\mathcal{X} \times_{\mathcal{S}}^{\text{htop}} \mathcal{Y} \rightarrow \mathcal{X}' \times_{\mathcal{S}'}^{\text{htop}} \mathcal{Y}'$$

is a homotopy equivalence.

*Proof.* In this case the map on functor spaces  $v^* : \text{Fun}(K, \mathcal{S}) \rightarrow \text{Fun}(K, \mathcal{S}')$  is an equivalence at all simplicial sets  $K$ , by Proposition 4.18. Furthermore restricting along any inclusion  $L \rightarrow K$  produces a Kan fibration  $\text{Fun}(K, \mathcal{S}) \rightarrow \text{Fun}(L, \mathcal{S})$  by Corollary 4.12. So the result follows by applying Proposition 7.3 to the diagram

$$\begin{array}{ccccc}
 \text{Fun}(\Delta^1, \mathcal{S}) & \longrightarrow & \text{Fun}(\partial\Delta^1, \mathcal{S}) & \longleftarrow & \mathcal{X} \times \mathcal{Y} \\
 \downarrow & & \downarrow & & \downarrow \\
 \text{Fun}(\Delta^1, \mathcal{S}') & \longrightarrow & \text{Fun}(\partial\Delta^1, \mathcal{S}') & \longleftarrow & \mathcal{X}' \times \mathcal{Y}'
 \end{array}$$

□

#### 7.4. Homotopy pullback squares.

**Definition 7.8.** A commutative diagram of Kan complexes

$$\begin{array}{ccc}
 \mathcal{Z} & \longrightarrow & \mathcal{Y} \\
 \downarrow & & \downarrow \\
 \mathcal{X} & \longrightarrow & \mathcal{S}
 \end{array}$$

is called a homotopy pullback square if the induced map

$$\mathcal{Z} \rightarrow \mathcal{X} \times_{\mathcal{S}} \mathcal{Y} \rightarrow \mathcal{X} \times_{\mathcal{S}}^{\text{htop}} \mathcal{Y}$$

is a homotopy equivalence.

We make no claim that any partial diagram  $\mathcal{X} \rightarrow \mathcal{S} \leftarrow \mathcal{Y}$  can be completed to a homotopy pullback square. Proposition 7.6 says that the usual pullback provides a homotopy pullback square whenever either of the maps  $\mathcal{X} \rightarrow \mathcal{S}$  or  $\mathcal{Y} \rightarrow \mathcal{S}$  is a Kan fibration.

We have the following interpretation of homotopy pullback squares via fibrant replacements.

**Proposition 7.9.** *Consider a diagram of Kan complexes*

$$\begin{array}{ccc}
 \mathcal{Z} & \longrightarrow & \mathcal{Y} \\
 \downarrow & & \downarrow g \\
 \mathcal{X} & \xrightarrow{f} & \mathcal{S}
 \end{array}$$

The following are equivalent:

- (a) The above diagram is a homotopy pullback diagram.
- (b) For each factorization  $\mathcal{X} \xrightarrow{t} \mathcal{X}' \xrightarrow{f'} \mathcal{S}$  in which  $t$  is a homotopy equivalence and  $f'$  is a Kan fibration, the induced map  $\mathcal{Z} \rightarrow \mathcal{X}' \times_{\mathcal{S}} \mathcal{Y}$  is an equivalence.
- (c) For each factorization  $\mathcal{Y} \xrightarrow{u} \mathcal{Y}' \xrightarrow{g'} \mathcal{S}$  in which  $u$  is a homotopy equivalence and  $g'$  is a Kan fibration, the induced map  $\mathcal{Z} \rightarrow \mathcal{X} \times_{\mathcal{S}} \mathcal{Y}'$  is an equivalence.
- (d) for any two factorizations as in (b) and (c) the induced map  $\mathcal{Z} \rightarrow \mathcal{X}' \times_{\mathcal{S}} \mathcal{Y}'$  is an equivalence.

*Proof.* The result follows by a consideration of the diagram of equivalences

$$\begin{array}{ccccc}
 \mathcal{X}' \times_{\mathcal{S}} \mathcal{Y} & \longrightarrow & \mathcal{X}' \times_{\mathcal{S}} \mathcal{Y}' & \longleftarrow & \mathcal{X} \times_{\mathcal{S}} \mathcal{Y}' \\
 \downarrow & & \downarrow & & \downarrow \\
 \mathcal{X}' \times_{\mathcal{S}}^{\text{htop}} \mathcal{Y} & \longrightarrow & \mathcal{X}' \times_{\mathcal{S}}^{\text{htop}} \mathcal{Y}' & \longleftarrow & \mathcal{X} \times_{\mathcal{S}}^{\text{htop}} \mathcal{Y}' \\
 & \nwarrow & \uparrow & \nearrow & \\
 & & \mathcal{X} \times_{\mathcal{S}}^{\text{htop}} \mathcal{Y} & & .
 \end{array}$$

We leave it to the interested reader to fill in the details.  $\square$

As with the functorial construction of the homotopy pullback, homotopy pullback squares enjoy invariance under homotopy equivalence.

**prop:htopy\_invariance**

**Proposition 7.10** (Homotopy invariance, [10, 0111]). *Consider a diagram of Kan complexes*

$$\begin{array}{ccccc}
 \mathcal{Z} & \xrightarrow{\quad} & \mathcal{Y} & & \\
 \downarrow & \searrow t & \downarrow & \searrow t_2 & \\
 & \mathcal{Z}' & \xrightarrow{\quad} & \mathcal{Y}' & \\
 \downarrow & \downarrow & \downarrow & \downarrow & \\
 \mathcal{X} & \xrightarrow{\quad} & \mathcal{S} & & \\
 \downarrow & \searrow t_1 & \downarrow & \searrow t_0 & \\
 & \mathcal{X}' & \xrightarrow{\quad} & \mathcal{S}'' & 
 \end{array} \tag{45}$$

**eq:3432**

in which all of the  $t_i$  is a equivalence. Then the following are equivalent:

- (a) Both the back and front faces in (45) are homotopy pullback squares.
- (b) Either the back face or the front face in (45) is a homotopy pullback square, and the map  $t$  is an equivalence.

*Proof.* Follows by homotopy invariance of the homotopy fiber product, by Proposition 7.7, and the diagram

$$\begin{array}{ccc}
 \mathcal{X} \times_{\mathcal{S}}^{\text{htop}} \mathcal{Y} & \xrightarrow{\sim} & \mathcal{X}' \times_{\mathcal{S}'}^{\text{htop}} \mathcal{Y}' \\
 \uparrow & & \uparrow \\
 \mathcal{Z} & \xrightarrow{t} & \mathcal{Z}' .
 \end{array}$$

□

**7.5. Simplicial injections, Kan fibrations, and homotopy pullback.** We claim that homotopy pullback squares appear naturally when one applies the functor  $\text{Fun}(-, \mathcal{C})^{\text{Kan}}$  to from certain pushout squares for simplicial sets. We begin our analysis with a technical lifting result.

lem:3465

**Lemma 7.11** ([10, 01NX]). *Consider a lifting problem*

$$\begin{array}{ccc} (\Delta^1 \times A) \amalg_{(\{1\} \times A)} (\{1\} \times B) & \xrightarrow{\quad} & \mathcal{C} \\ \downarrow & \nearrow \text{dashed} & \downarrow \\ \Delta^1 \times B & \xrightarrow{\quad \nu \quad} & \mathcal{D} \end{array}$$

associated to an inner fibration of  $\infty$ -categories  $\mathcal{C} \rightarrow \mathcal{D}$  and injective map of simplicial sets  $i : A \rightarrow B$ . Suppose that for every simplex  $s : \Delta^n \rightarrow B$  which is not contained in the image of  $i$  the associated map in  $\mathcal{C}$

$$\nu(id_{\Delta^1} \times s(n))\Delta^1 = \Delta^1 \times * \rightarrow \mathcal{C}$$

is an isomorphism, and that  $i(A)$  contains all vertices in  $B$ . (Here  $n : * \rightarrow \Delta^n$  is the terminal point in  $\Delta^n$  and  $s(n) : * \rightarrow B$  is the composite map to  $B$ .) Then the above lifting problem has a solution.

We only relay the basic idea of the proof, and refer the reader to the source [10] for any details.

*Idea of proof.* One reduces to the case where  $A = \partial\Delta^n$ ,  $B = \Delta^n$ , and  $i : A \rightarrow B$  is the inclusion. One then factors this inclusion into a sequence

$$(\Delta^1 \times \partial\Delta^n) \cup (\{0\} \times \Delta^n) \subseteq X_1 \subseteq \cdots \subseteq X_n = \Delta^1 \times \Delta^n$$

in which each  $X_{i+1}$  is obtained from  $X_i$  via a pushout diagram of the form

$$\begin{array}{ccc} \Lambda_i^{n+1} & \longrightarrow & X_i \\ \downarrow & & \downarrow \\ \Delta_i^{n+1} & \xrightarrow{s_i} & X_{i+1}. \end{array}$$

One can furthermore assume that the final simplex we adjoin  $s_{n+1} : \Delta^{n+1} \rightarrow X_{n+1} = \Delta^1 \times \Delta^n$  we adjoin has  $s_{n+1}(\Delta^{\{n, n+1\}}) = \Delta^1 \times \{n\}$  [10, Proof of 00TH]. We therefore reduce the lifting problem for to the inclusion  $(\Delta^1 \times \partial\Delta^n) \cup (\{0\} \times \Delta^n) \rightarrow \Delta^1 \times \Delta^n$  to a sequence of lifting problems of the form

$$\begin{array}{ccc} \Lambda_i^{n+1} & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow \\ \Delta_i^{n+1} & \xrightarrow{\tau_i} & \mathcal{D}, \end{array}$$

with  $0 < i \leq n+1$  and  $\tau_n$  sending the final edge in  $\Delta^{n+1}$  to an isomorphism in  $\mathcal{D}$ . We can solve all such lifting problems via the weak Kan condition and Proposition 5.33. □



Now, for any inclusion of simplicial sets  $i : A \rightarrow B$  we can expand  $i$  to an inclusion  $i' : A' \rightarrow B$  for which all vertices in  $B$  lie in the image of  $i'$  by simply adjoining the 0 skeleton of  $B$  to  $A$ . By considering such minimal extensions one obtains the following adjacent lifting property from that of Lemma 7.11.

cor:3506

**Corollary 7.12** ([10, 01NY]). *Consider a lifting problem*

$$\begin{array}{ccc} (\Delta^1 \times A) \amalg_{(\{1\} \times A)} (\{1\} \times B) & \xrightarrow{\quad} & \mathcal{C} \\ \downarrow & \nearrow \text{dashed} & \downarrow \\ \Delta^1 \times B & \xrightarrow{\quad \nu \quad} & \mathcal{D} \end{array}$$

associated to an isofibration of  $\infty$ -categories  $\mathcal{C} \rightarrow \mathcal{D}$  and an arbitrary injective map of simplicial sets  $i : A \rightarrow B$ . The above lifting problem admits a solution provided the following conditions hold:

- (a) For each vertex  $a : * \rightarrow A$  the corresponding edge  $\Delta^1 \times \{a\} \rightarrow \mathcal{C}$  is an isomorphism in  $\mathcal{C}$ .
- (b) For each vertex  $b : * \rightarrow B$  the corresponding edge  $\Delta^1 \times \{b\} \rightarrow \mathcal{D}$  is an isomorphism in  $\mathcal{D}$ .

Furthermore, when the above hypotheses are satisfied, there exists a solution  $\eta : \Delta \times B \rightarrow \mathcal{C}$  for which each edge  $\Delta^1 \times \{b\} \rightarrow \mathcal{C}$  is an isomorphism in  $\mathcal{C}$ .

prop:inj\_isofib

**Proposition 7.13.** *Let  $L \rightarrow K$  be an injective map of simplicial sets and  $\mathcal{C} \rightarrow \mathcal{D}$  be an isofibration of  $\infty$ -categories. Then the induced map*

$$\theta : \text{Fun}(K, \mathcal{C}) \rightarrow \text{Fun}(L, \mathcal{C}) \times_{\text{Fun}(L, \mathcal{D})} \text{Fun}(K, \mathcal{D})$$

is an isofibration of  $\infty$ -categories.

*Proof.* First note that the restriction map  $\text{Fun}(K, \mathcal{D}) \rightarrow \text{Fun}(L, \mathcal{D})$  is an inner fibration by Corollary 5.8, so that the projection  $\text{Fun}(L, \mathcal{C}) \times_{\text{Fun}(L, \mathcal{D})} \text{Fun}(K, \mathcal{D}) \rightarrow \text{Fun}(L, \mathcal{C})$  is an inner fibration as well. It follows that the fiber product is in fact an  $\infty$ -category. We also know that  $\theta$  is an inner fibration by Proposition 5.7. So we need only know that for any object  $\xi$  in  $\text{Fun}(K, \mathcal{C})$  and isomorphism  $a : \eta \rightarrow \theta(\xi)$  in the fiber product, we can lift  $a$  to an isomorphism  $a' : \eta' \rightarrow \xi$  in  $\text{Fun}(K, \mathcal{C})$ .

The map  $a$  in the fiber product is precisely the information of a diagram

$$\begin{array}{ccc} (\Delta^1 \times L) \amalg_{(\{1\} \times L)} (\{1\} \times K) & \xrightarrow{\quad} & \mathcal{C} \\ \downarrow & \nearrow \text{dashed} & \downarrow \\ \Delta^1 \times K & \xrightarrow{\quad \nu \quad} & \mathcal{D} \end{array}$$

and the desired lift  $\tilde{a}$  is precisely the information of a solution to the corresponding lifting problem. Furthermore, since  $a$  is a natural isomorphism the maps  $\Delta^1 \times L \rightarrow \mathcal{C}$  and  $\Delta^1 \times K \rightarrow \mathcal{D}$  both restrict to isomorphisms on all edges of the form  $\Delta^1 \times \{x\}$ . So we see that the desired solution  $\tilde{a}$  exists by Corollary 7.12.  $\square$

We note that an analog of Proposition 7.13 in the case where the map  $\mathcal{C} \rightarrow \mathcal{D}$  is simply an inner fibration and  $L \rightarrow K$  is an injection which contains all vertices in  $K$  in its image. The following corollary is immediate.

`cor:inj_isofib`

**Corollary 7.14.** (1) For any injection  $i : L \rightarrow K$ , and any  $\infty$ -category  $\mathcal{C}$ , the map

$$i^* : \mathrm{Fun}(K, \mathcal{C}) \rightarrow \mathrm{Fun}(L, \mathcal{C})$$

is an isofibration.

(2) For any simplicial set  $K$ , and any isofibration  $F : \mathcal{C} \rightarrow \mathcal{D}$ , the map

$$F_* : \mathrm{Fun}(K, \mathcal{C}) \rightarrow \mathrm{Fun}(K, \mathcal{D})$$

is an isofibration.

We recall that any isofibration produces a Kan fibration on associated Kan complexes, by Corollary 5.34. So we have the following.

`cor:inj_kan`

**Corollary 7.15.** (1) For any injection  $i : L \rightarrow K$ , and any  $\infty$ -category  $\mathcal{C}$ , the map

$$i^* : \mathrm{Fun}(K, \mathcal{C})^{\mathrm{Kan}} \rightarrow \mathrm{Fun}(L, \mathcal{C})^{\mathrm{Kan}}$$

is a Kan fibration.

(2) For any simplicial set  $K$ , and any isofibration  $F : \mathcal{C} \rightarrow \mathcal{D}$ , the map

$$F_* : \mathrm{Fun}(K, \mathcal{C})^{\mathrm{Kan}} \rightarrow \mathrm{Fun}(K, \mathcal{D})^{\mathrm{Kan}}$$

is a Kan fibration.

`cor:3681`

**Corollary 7.16.** Suppose we have a pushout diagram of simplicial sets

$$\begin{array}{ccc} T & \longrightarrow & A \\ \mu \downarrow & & \downarrow \\ B & \longrightarrow & K \end{array}$$

in which the map  $\mu$  is injective. Then the corresponding diagram

$$\begin{array}{ccc} \mathrm{Fun}(K, \mathcal{C})^{\mathrm{Kan}} & \longrightarrow & \mathrm{Fun}(A, \mathcal{C})^{\mathrm{Kan}} \\ \downarrow & & \downarrow \\ \mathrm{Fun}(B, \mathcal{C})^{\mathrm{Kan}} & \xrightarrow{\mu^*} & \mathrm{Fun}(T, \mathcal{C})^{\mathrm{Kan}} \end{array}$$

is a homotopy pullback diagram.

*Proof.* Follows from the fact that  $\mu^*$  is a Kan fibration, by Corollary 7.15, and Proposition 7.6.  $\square$

## 7.6. Categorical pushout for simplicial sets.

**Definition 7.17** ([10, 01F7]). We call a diagram of simplicial sets

$$\begin{array}{ccc} T & \longrightarrow & A \\ \downarrow & & \downarrow \\ B & \longrightarrow & K \end{array}$$

a categorical pushout square if, for any  $\infty$ -category  $\mathcal{C}$ , the corresponding diagram of Kan complexes

$$\begin{array}{ccc} \mathrm{Fun}(K, \mathcal{C})^{\mathrm{Kan}} & \longrightarrow & \mathrm{Fun}(A, \mathcal{C})^{\mathrm{Kan}} \\ \downarrow & & \downarrow \\ \mathrm{Fun}(B, \mathcal{C})^{\mathrm{Kan}} & \longrightarrow & \mathrm{Fun}(T, \mathcal{C})^{\mathrm{Kan}} \end{array}$$

is a homotopy pullback square

Corollary 7.16 can now be rephrased as follows.

cor:3659

**Corollary 7.18.** *A pushout diagram of simplicial sets*

$$\begin{array}{ccc} T & \longrightarrow & A \\ \mu \downarrow & & \downarrow \\ B & \longrightarrow & K \end{array}$$

in which  $\mu$  is injective is a categorical pushout diagram.

## 8. COMPOSITION FUNCTIONS FOR MAPPING SPACES

We conclude with an indication, so to speak, of a composition function for mapping spaces in an  $\infty$ -category.

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