– Abstract root systems

Def: A root system is a real inner-product space $\mathbb{E} = \{E, (-,-): E\otimes_{\mathbb{R}} E \to \mathbb{R}\}$ equipped w/ a finite subset $\Phi \subseteq \mathbb{E}$ which satisfies the following:

i) Φ spans $\mathbb{E}$.

ii) For each $\gamma \in \Phi$, $\mathbb{R}\cdot\Phi \cap \Phi = \{\gamma, -\gamma\}$.

iii) For each $\gamma \in \Phi$, the reflection

$$\sigma_\gamma : \mathbb{E} \to \mathbb{E}, \quad \sigma_\gamma(v) = v - \frac{2(v,\gamma)}{(\gamma,\gamma)}\cdot\gamma,$$

preserves Φ, $\sigma_\gamma(\Phi) = \Phi$.

iv) For each pair of elem $\gamma, \xi \in \Phi$,

$$2(\gamma,\xi)/(\gamma,\gamma) \in \mathbb{Z}.$$

For some notation, we call the elem $\gamma \in \Phi$ the roots in $\mathbb{E}$. The automorphism $\sigma_\gamma = ? - \frac{2(?,\gamma)}{(\gamma,\gamma)}\gamma$ is called the reflection across to γ.

Note that σ_γ preserves the orthogonal hyperplane

$$P_\gamma := \{v \in \mathbb{E} : (v,\gamma) = 0\} = (\mathbb{C}\gamma)^\perp$$

and negates vectors in the span of γ.

Def: Let $\Phi \subseteq \mathbb{E}$ be a root system, and $\gamma, \xi \in \Phi$ be roots. The Cartan integer for ξ paired w/ γ is the integer

$$\langle \xi, \gamma \rangle := 2(\xi,\gamma)/(\gamma,\gamma).$$

With this notation we have

$$\sigma_\gamma(\xi) = \xi - \langle \xi, \gamma\rangle \cdot \gamma.$$

Warning!: Some people take $\langle \gamma, \xi\rangle = \frac{2(\xi,\gamma)}{(\gamma,\gamma)}$. So "who comes first" and "who comes second" in this notation is not fixed in the literature.

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– Root systems from Lie algs.

By Thm 8.5 we understand that for any semisimple $\mathfrak{g}$ w/ choice of Cartan $\mathfrak{h} \subseteq \mathfrak{g}$ we have the roots $\Phi \subseteq \mathfrak{h}^*$, which are the nonzero wts appearing in the adj rep. We have the real space

$$\mathfrak{h}^*_{\mathbb{R}} := \mathbb{R} \cdot \Phi$$

w/ Killing form $(-,-) : \mathfrak{h}^*_{\mathbb{R}} \otimes_{\mathbb{R}} \mathfrak{h}^*_{\mathbb{R}} \to \mathbb{R}$.

Def^n: An isomorphism of root systems $\phi : (\Phi, E) \xrightarrow{\sim} (\Phi', E')$ is a linear iso $\phi : E \to E'$ which satisfies $\phi(\Phi) = \Phi'$ and

$$\langle \phi\gamma, \phi\xi^{\vee} \rangle = \langle \gamma, \xi^{\vee} \rangle \qquad (*)$$

at all pairs of simple roots.

Rem: The condition $(*)$ is satisfied whenever ϕ preserves the inner form "up to scaling". When Φ is "connected", then $(*)$ is precisely this statement.

Thm 8.5: For any semisimple $\mathfrak{g}$ w/ choice of Cartan $\mathfrak{h} \subseteq \mathfrak{g}$, the pairing $\Phi \subseteq \mathfrak{h}^*_{\mathbb{R}}$, along w/ the Killing form, is a root system.

Furthermore, for any other choice of Cartan $\mathfrak{t} \subseteq \mathfrak{g}$ the two root systems are isomorphic

$$(\Phi, \mathfrak{h}^*_{\mathbb{R}}) \cong (\Psi, \mathfrak{t}^*_{\mathbb{R}}).$$

Proof: Thm 8.5 says $(\Phi, \mathfrak{h}^*_{\mathbb{R}})$ is a root system, as is $(\Psi, \mathfrak{t}^*_{\mathbb{R}})$. Now, by uniqueness of the Cartan $\exists$ an auto $w : \mathfrak{g} \to \mathfrak{g}$ w/ $w(\mathfrak{h}) = \mathfrak{t}$, and hence $w(\mathfrak{g}_{\gamma}) = \mathfrak{g}_{\xi}$ for some ξ. We have at each $h \in \mathfrak{h}$, $x \in \mathfrak{g}_{\gamma}$

$$\xi(wh) \cdot w(x) = [w(h), w(x)] = w[h, x] = \gamma(h)\, w(x)$$

so that $w^*(\Xi) = \gamma$, and $w^*(\Phi) = \Psi$. (8)

Take now $\phi = (w^{-1})^*: \mathfrak{h}^* \to \mathfrak{t}^*$. We check finally the inner form. We have

$$w^{-1} \mathrm{ad}_{wx} w \;:\; \mathfrak{g} \to \mathfrak{g}$$
$$= y \longmapsto w^{-1}([wx, wy]) = [x,y]$$

$$\Rightarrow \quad w^{-1} \mathrm{ad}_{wx} w = \mathrm{ad}_x.$$

Thus $\kappa(wx, wy) = \mathrm{Tr}_{\mathfrak{g}}(w^{-1}\mathrm{ad}_x \mathrm{ad}_y w)$
$= \mathrm{Tr}_{\mathfrak{g}}(\mathrm{ad}_x, \mathrm{ad}_y) = \kappa(x,y)$..

Hence we have

$$\kappa(wx, -) : \mathfrak{t} \to \mathbb{C}$$
$$= \kappa(x, w^{-1}-) : \mathfrak{t} \to \mathbb{C}$$

giving a diagram

$$\begin{tikzcd} \mathfrak{h} \arrow[r, "w"] \arrow[d, "\mathrm{form}_{\mathfrak{h}}"'] & \mathfrak{t} \arrow[d, "\mathrm{form}_{\mathfrak{t}}"] \\ \mathfrak{h}^* \arrow[r, "\phi"'] & \mathfrak{t}^* \end{tikzcd}$$

and from the diag

$$\begin{tikzcd} \mathfrak{h} \otimes \mathfrak{h} \arrow[rr, "w^{\otimes 2}"] \arrow[dr, "\kappa"'] & & \mathfrak{t} \otimes \mathfrak{t} \arrow[dl, "\kappa"] \\ & \mathbb{C} & \end{tikzcd}$$

we conclude a diag

$$\begin{tikzcd} \mathfrak{h}^* \otimes \mathfrak{h}^* \arrow[rr, "\phi^{\otimes 2}"] \arrow[dr, "(,)"'] & & \mathfrak{t}^* \otimes \mathfrak{t}^* \arrow[dl, "(-,-)"] \\ & \mathbb{C} & \end{tikzcd}.$$

So we discover that ϕ provides an $\cong$ of root systems

$$\phi : (\Phi, \mathfrak{h}^*_{\mathbb{R}}) \xrightarrow{\cong} (\Psi, \mathfrak{t}^*_{\mathbb{R}}).$$ ∎

Def^{n}: For any semisimple Lie alg $\mathfrak{g}$, let $E(\mathfrak{g}) = (\Phi, E(\mathfrak{g}))$ denote the assoc. root system, which we accept is only defined up to $\cong$.

Theorem 14.2: For semisimple Lie alg $\mathfrak{g}$ and $\mathfrak{g}'$, there is an $\cong$ of root systems $\phi : E(\mathfrak{g}) \xrightarrow{\cong} E(\mathfrak{g}')$ if and only if there is a Lie alg iso $\tilde{\phi} : \mathfrak{g} \xrightarrow{\cong} \mathfrak{g}'$ which lifts ϕ.

By considering root preserving autos of $\mathfrak{g}$, one sees that this lift $\tilde{\theta} : \mathfrak{g} \to \mathfrak{g}$ is completely determined up to some $(\mathbb{C}^*)^{\times n}$-action.

Corollary: The associated root system $\mathfrak{g} \mapsto \Phi(\mathfrak{g})$ is a complete invariant for a semisimple Lie alg $\mathfrak{g}$.

This is to say to classify semisimple Lie alg over $\mathbb{C}$ it suffices to classify (indecomposable) root systems.

Ex: Consider $\mathfrak{sl}_n(\mathbb{C})$, γ the root for E_{ij} & ξ the root for E_{kl}. Then

$$\langle \xi, \gamma \rangle = \xi(h_\gamma) = \text{coeff of } h_\gamma = E_{ii} - E_{jj} \text{ acting on } E_{kl}.$$

$$h_\gamma \cdot E_{kl} = (\delta_{ik} - \delta_{jk}) - (\delta_{il} - \delta_{jl})$$

$$= \begin{cases} \pm 1 & \text{if 1 shared index} \\ \pm 2 & \text{if 2 shared indices.} \end{cases}$$

Restriction to our preferred basis $\Delta \subseteq \Phi$,

$$\Delta = \{\alpha_i \,;\, \alpha_i = \text{root for } E_{i,i+1}\}$$

we get

$$\langle \alpha_i, \alpha_j \rangle = \begin{cases} 2 & \text{if } i = j \\ -1 & \text{if } j = i+1 \text{ or } i = j+1. \end{cases} \quad (*)$$

Note that $\langle \alpha_i, \alpha_j \rangle = \langle \alpha_j, \alpha_i \rangle$ so that

$$\langle -, - \rangle = \begin{cases} \text{a rescaling of } (-,-) \\ \text{so that all } \langle \gamma, \gamma \rangle = 2. \end{cases}$$

Can encode $(*)$ in a graph (Dynkin diagram)

α_1 — α_2 — $\cdots$ — α_{n-2} — α_{n-1}

where # of edges between α_i and α_j = $\langle \alpha_i, \alpha_j \rangle$.

Will see that these numbers / this diagram completely determine the root system.

Now, back to the general setting.

$\rightsquigarrow$ The Weyl group

Def^{n}: Given a root system (Φ, E), the associated Weyl group W is the subgroup in $GL(E)$ generated by the reflections σ_γ, for $\gamma \in \Phi$.

Lemma: a) W is a finite group

b) For each $\gamma \in \Phi$, $v, w \in E$, $(\sigma_\gamma v, \sigma_\gamma w) = (v, w)$.

Proof: a) Each elem is determined by its value on the finite set Φ, so that by restricting the action on E we obtain a group embedding $W \hookrightarrow S_\Phi$.

b) Given directly

$$(\sigma_\gamma v, \sigma_\gamma w) = (v,w) + \frac{4(v,\gamma)(w,\gamma)}{(\gamma,\gamma)^2}(\gamma,\gamma) - \frac{2(v,\gamma)}{(\gamma,\gamma)}(\gamma, w) - \frac{2(w,\gamma)}{(\gamma,\gamma)}(\gamma, v)$$

$$= (v, w). \quad \blacksquare$$

The following will help us to observe some relations in W.

Lemma 9.1: Let $\sigma \in GL(E)$ preserve Φ, and fix some hyperplane P, and have $\sigma(\gamma) = -\gamma$ for some $\gamma \in \Phi$, then $\sigma = \sigma_\gamma$.

Proof: It suffices to show $\sigma = \sigma_\gamma$ on the complexification $E_\mathbb{C}$. Here we exploit the fact that any finite order linear auto on a complex vect space is semi-simple. ($\phi = \phi_s + \phi_n$, $1 = \phi^N = \phi_s^N + \underbrace{(N-1)\phi_n \phi^{N-1}}_{\text{nilpotent}}$ so that $\phi_s = 1$ or $\phi_n \phi^{N-1} = 0 \Rightarrow \phi_n = 0$. So in either case $\phi = \phi_s$.)

Consider $\sigma\sigma_\gamma$. On $\mathbb{C}\cdot\gamma$ we have $\sigma\sigma_\gamma|_{\mathbb{C}\gamma} = id$ and for the induced map on $E_\mathbb{C}/\mathbb{C}\cdot\gamma$ we have $\sigma\sigma_\gamma$ acting as id.

So $$\sigma\sigma_\gamma = \left(\begin{array}{c|c} 1 & * \\ \hline & Id \end{array}\right)$$ in the

$\rightarrow (\sigma\sigma_\gamma - 1)^2 = 0$ on $E_\mathbb{C}$.

But since $\sigma\sigma_\gamma$ is determined by its values on Φ, and Φ is finite, $\sigma\sigma_\gamma$ is of finite order. Thus $\sigma\sigma_\gamma = id \Rightarrow \sigma = \sigma_\gamma^{-1} = \sigma_\gamma$. ∎

Lemma 9.2: Let (Φ, E) be a root system, $\gamma \in \Phi$, and $\sigma : E \to E$ be a linear auto with $\sigma(\Phi) = \Phi$. Then $\sigma\sigma_\gamma\sigma^{-1} = \sigma_{\sigma(\gamma)}$, and $\langle \xi, \gamma \rangle = \langle \sigma(\xi), \sigma(\gamma) \rangle$ at each pair of roots.

Proof: $\sigma\sigma_\gamma\sigma^{-1}$ fixes the hyperplane σP_γ and has $\sigma(\gamma) \mapsto -\sigma(\gamma)$, so that $\sigma\sigma_\gamma\sigma^{-1} = \sigma_{\sigma(\gamma)}$ by Lemma 9.1. For the Cartan integers

$$\langle \sigma\xi, \sigma\gamma \rangle \sigma\gamma = \sigma(\xi) - \sigma_{\sigma(\gamma)}(\sigma\xi)$$
$$= \sigma(\xi - \sigma_\gamma\sigma^{-1}(\sigma\xi))$$
$$= \sigma(\xi - \sigma_\gamma(\xi))$$
$$= \sigma(\langle \xi, \gamma \rangle \gamma) = \langle \xi, \gamma \rangle \sigma\gamma.$$

Hence $\langle \sigma\xi, \sigma\gamma \rangle = \langle \xi, \gamma \rangle$. ∎

– Coroots and dual root systems

Def^n: For each $\gamma \in \Phi$ define
$$\gamma^\vee = 2\gamma / (\gamma, \gamma).$$
$\subseteq E$

Define $\Phi^\vee = \{ \gamma^\vee : \gamma \in \Phi \}$. This is the (Langlands) dual root system to Φ.

Direct calculation verifies $\Phi^\vee$ is another root system

w/
$$(\gamma^\vee, \gamma^\vee) = 4(\gamma, \gamma) / (\gamma, \gamma)^2$$
$$= 4 / (\gamma, \gamma).$$
$$\langle \xi^\vee, \gamma^\vee \rangle = 2 (\xi^\vee, \gamma^\vee) / (\gamma^\vee, \gamma^\vee)$$
$$= 2 (\xi, \gamma)(\xi, \xi)^{-1} = \langle \gamma, \xi \rangle.$$

Lemma: If $|\gamma|^2 \le |\xi|^2$ in Φ, then $|\gamma^\vee|^2 \ge |\xi^\vee|^2$ in $\Phi^\vee$. Furthermore
$$\langle \xi^\vee, \gamma^\vee \rangle = \langle \gamma, \xi \rangle.$$

Lemma: $\langle \gamma, \xi^\vee \rangle = \gamma(h_\xi)$.

Example: • $[\Phi(\mathfrak{sp}_{2n}(\mathbb{C}))]^\vee = \Phi(\mathfrak{so}_{2n+1}(\mathbb{C}))$.

• $[\Phi(\mathfrak{sl}_n(\mathbb{C}))]^\vee = \Phi(\mathfrak{sl}_n(\mathbb{C}))$.

• $[\Phi(\mathfrak{so}_{2n}(\mathbb{C}))]^\vee = \Phi(\mathfrak{so}_{2n}(\mathbb{C}))$.

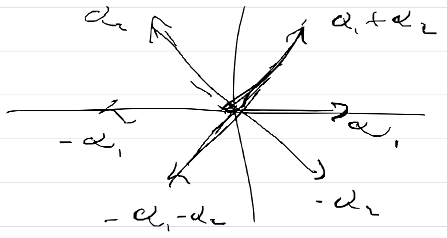
Proposition: $\mathcal{W}(E, \Phi) = \mathcal{W}(E, \Phi^\vee)$.

Proof: We have $\sigma_\gamma = \sigma_{\gamma^\vee}$ since they both fix $P_\gamma = P_{\gamma^\vee}$ and negate γ (and $\gamma^\vee$). ∎

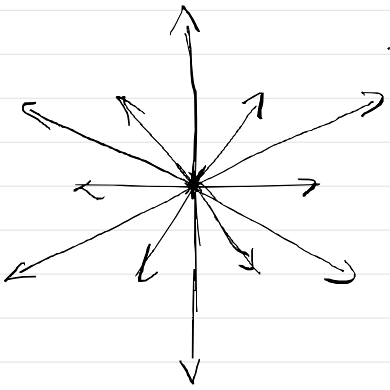
~ Constraining angles and magnitudes in a root system.

Via the scalarity constraint $\mathbb{R}\gamma \cap \Phi = \{\pm\gamma\}$ and stability under reflection, we see that $\Phi \subseteq E$ is some highly symmetric star like configuration.

E.g. A_2 / $\mathfrak{sl}_3$



G_2



Since we're in a Euclidean/inner prod space we can speak of angles between roots, and magnitudes.

The angle is only det mod π, clearly, so that for roots γ, ξ angle between γ, ξ in E det by value $\cos(\theta)$.

We have $|\gamma| = \sqrt{\langle \gamma, \gamma \rangle}$ and

$$\cos\theta = (\xi, \gamma) / (|\xi| \cdot |\gamma|)$$

$$= \frac{|\gamma|}{2|\xi|} \langle \xi, \gamma \rangle.$$

$$\Rightarrow \quad 2|\xi||\gamma|^{-1} \cdot \cos\theta = \langle \xi, \gamma \rangle.$$

Observation 9.4A: For any pair of roots ξ, γ,
$$\langle \xi, \gamma \rangle \langle \gamma, \xi \rangle = 4 \cos^2 \theta \in \mathbb{Z}.$$

But now, $0 \le \cos^2\theta \le 1$ with 1 occurring iff $\xi = \pm \gamma$.

Observation 9.4B: If $\xi \neq \pm \gamma$ and $\langle \gamma, \xi \rangle \neq 0$, then

- $\langle \xi, \gamma \rangle \in \{0, 1, 2, 3\}$.
- $\cos \theta \in \frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{2}$
- If $\langle \xi, \gamma \rangle = \langle \gamma, \xi \rangle$ then $\langle \xi, \gamma \rangle = \langle \xi, \gamma \rangle = \pm 1$

and $|\xi| = |\gamma|$.

- If $\langle \xi, \gamma \rangle \neq \langle \gamma, \xi \rangle$ then, supposing $|\gamma| > |\xi|$,
$\langle \xi, \gamma \rangle = \pm 1$ and $\langle \gamma, \xi \rangle = \pm 2, \pm 3$.

Working things out we observe the following table

Summing $|\gamma| \ge |\xi|$

$\langle \xi, \gamma \rangle$	$\langle \gamma, \xi \rangle$	θ	$\frac{\|\gamma\|^2}{\|\xi\|^2}$
0	0	$\pi/2$	undet
1	1	$\pi/3$	1
-1	-1	$2\pi/3$	1
1	2	$\pi/4$	2
-1	-2	$3\pi/4$	2
1	3	$\pi/6$	3
-1	-3	$5\pi/6$	3

[Stuff about root strings which I'll skip]

Ex. In type A, all roots are same length, so that only angles $\pi/3$ and $2\pi/3$ are realized.
Both angles realized already in A_2.

- In type C and B, rel root lengths of 2 occur, and hence more acute angles occur.
- In type G_2, rel root length of 3 occurs.

—

Lemma 9.4: Suppose $\xi \neq \pm\gamma$. Then

- $(\gamma, \xi) > 0$ then $\gamma - \xi$ is a root.
- $(\gamma, \xi) < 0$ then $\gamma + \xi$ is a root.

Proof: Supposing $(\gamma, \xi) > 0$ then one of $\langle \gamma, \xi \rangle$ or $\langle \xi, \gamma \rangle = 1$. Supposing arbitrarily that $\langle \xi, \gamma \rangle = 1$, we get

$$\sigma_\gamma(\xi) = \xi - \langle \xi, \gamma \rangle \gamma = \xi - \gamma \text{ a root,}$$

and thus $\gamma - \xi$ a root as well by stability under negation. If $(\gamma, \xi) < 0$ then one of $\langle \xi, \gamma \rangle$ or $\langle \gamma, \xi \rangle = -1$. Supposing arbitrarily $\langle \xi, \gamma \rangle = -1$ we get $\sigma_\gamma(\xi) = \xi + \gamma \in \Phi$. $\blacksquare$

~ Bases for root system.

Fix a root system (E, Φ). We seek now that what has been promised to us by the ancients:

(1) A good basis of elem $\Delta \subseteq \Phi$ for E.

(2) Δ splitting $\Phi = \Phi^+ \sqcup \Phi^-$ w/ $\Phi^- = -\Phi^+$ and $\Delta \subseteq \Phi^+$.

Def^n: A base for a root system (E, Φ) is a choice of subset $\Delta \subseteq \Phi$ which satisfies the following:

a) Δ is a basis for E.

b) For each root γ, $\gamma = \sum_{\alpha \in \Delta} c_\alpha \alpha$ with all $c_\alpha \in \mathbb{Z}_{\geq 0}$, or all $c_\alpha \in \mathbb{Z}_{\leq 0}$.

Ex: α_i = root for $E_{i,i+1}$ forms here $\{\alpha_1, \dots, \alpha_{n-1}\}$ for $\Phi(sl_n)$.

Def^n: For a root system (E, Φ) w/ chosen base Δ, we define

$$\Phi^+ = \left\{ \gamma \in \Phi : \gamma = \sum_{\alpha \in \Delta} c_\alpha \alpha \text{ w/ all } c_\alpha \geq 0 \right\}$$

$$\Phi^- = \{ \gamma \in \Phi : \gamma = \text{" } \quad \text{" } c_\alpha \leq 0 \}.$$

Clearly $\Phi^- = -\Phi^+$, and by def of a base

$$\Phi = \Phi^+ \sqcup \Phi^-.$$

Ex: For $\Phi(\mathfrak{sl}_n)$ w/ chosen base $\Delta = \{\alpha_i : 1 \leq i \leq n-1\}$, $\Phi^+ = \{$roots for upper Δ elem $E_{ij}\}$.

Lemma 10.1: If $\alpha, \beta \in \Delta$ are distinct, then $(\alpha, \beta) \leq 0$.

Proof: $\alpha - \beta$ is not a root, by def of base. Hence the constraint $(\alpha, \beta) \leq 0$ occurs by Lemma 9.4. ∎

~ Existence of a base.

Consider (E, Φ) and the collection of hyperplanes $\{P_\gamma : \gamma \in \Phi\}$, $P_\gamma = \{v \in E : (v, \gamma) = 0\}$.

The union $\bigcup_{\gamma \in \Phi} P_\gamma \subseteq E$ is not an equality, by linear algebra, so that $\bigcup_{\gamma \in \Phi} P_\gamma$ is a non-trivial closed alg subspace in E, and the complement

$$E - \bigcup_\gamma P_\gamma$$

is an open alg subspace, and hence dense in E. Anyway, one can find elem v in the complement.

Def^n: Call $v \in E$ regular if $v \in E - \bigcup_\gamma P_\gamma$, and singular otherwise. For regular v we write

$$\Phi(v)^+ = \{\gamma \in \Phi : (\gamma, v) > 0\}$$

$$\Phi(v)^- = \{\gamma \in \Phi : (\gamma, v) < 0\}.$$

Clearly each $\gamma \in \Phi$ is either in $\Phi(v)^+$ or $\Phi(v)^-$, but not both, and $\Phi(v)^- = -\Phi(v)^+$.

Def$^{\underline{n}}$: Call $\alpha \in \Phi(v)^+$ indecomposable if $\alpha \neq \alpha_1 + \alpha_2$ for any $\alpha_i \in \Phi(v)^+$.

Note that such indecomp roots exist, by choosing $\alpha \in \Phi(v)^+$ w/ minimal value $(\alpha, v) > 0$, for example.

Theorem 10.1: For any regular $v \in E$, the set
$$\Delta(v) = \{\alpha \in \Phi(v)^+ : \alpha \text{ indecomposable}\}$$
provides a basis for (E, Φ). Furthermore, all bases occur in this way.

Proof: Step 1: Note that the collection
$$\{(v, \gamma) : \gamma \in \Phi(v)^+\} = \{m_1 < \cdots < m_r\} \quad (*)$$
is a finite and naturally linearly ordered. If α is such that (α, v) is minimal in $(*)$ then α indecomp.

Now suppose $\gamma \in \Phi^+(v)$ is such that each $\xi \in \Phi^+$ w/ $(v, \xi) < (v, \gamma)$ is expressible as non-neg linear combo of elem in $\Delta(v)$. If $\gamma \in \Delta(v)$ then $\gamma \in \mathbb{Z}_{\geq 0} \cdot \Delta(v)$ and otherwise $\gamma = \gamma_1 + \gamma_2$ w/ $\gamma_i \in \Phi^+$ and nec. $(v, \gamma_i) < (v, \gamma)$ so that both $\gamma_i \in \mathbb{Z}_{\geq 0} \Delta(v)$, giving $\gamma = \gamma_1 + \gamma_2 \in \mathbb{Z}_{\geq 0} \Delta(v)$. So we see $\Phi^+(v) \subseteq \mathbb{Z}_{\geq 0} \cdot \Delta(v)$. By negation $\Phi^-(v) \subseteq \mathbb{Z}_{\leq 0} \cdot \Delta(v)$ as well.

Step 2: For distinct $\alpha, \beta \in \Delta(v)$ we have $\alpha - \beta$ not a root. Otherwise, up to swapping α and β, $\alpha - \beta \in \Phi^+(v)$ and we decompose α

or $\alpha = \beta + (\alpha - \beta)$, which is crap.

So $\alpha - \beta$ is not a root, and we conclude

$(\alpha, \beta) \leq 0$ whenever $\alpha, \beta \in \Delta(v)$ are distinct

by Lemma 4.4.

Step 3: By Step 1

$$\mathbb{R}_{\geq} \cdot \Delta(v) \supseteq \mathbb{R}_{\geq} \cdot \Phi^{+}(v) + \mathbb{R}_{\geq} \Phi^{-}(v)$$

$$= \mathbb{R}_{\geq} \cdot \Phi = E,$$

so that $\Delta(v)$ spans E. By step 1 again

we see now $\Delta(v)$ is a base iff its linearly indep.

For this consider an expression

$$0 = \sum_{\alpha}' r_\alpha \alpha \quad \text{w/ all } r_\alpha \text{ real.}$$

Then for $\Delta' = \{\alpha \text{ w/ } r_\alpha > 0\}$ we have

$$\tau := \sum_{\alpha \in \Delta'} r_\alpha \cdot \alpha = \sum_{\beta \notin \Delta'}' s_\alpha \beta$$

w/ all $r_\alpha, s_\beta \geq 0$. Then

$$(\tau, \tau) = \sum_{\alpha, \beta}' r_\alpha \cdot s_\beta (\alpha, \beta) \leq 0$$

$\Rightarrow \tau = 0$. So we reduce to the case of a sum

$0 = \sum' r_\alpha \alpha$ with all $r_\alpha \geq 0$. In this case

$0 = (v, \sum' r_\alpha \alpha) = \sum' r_\alpha (v, \alpha) \Rightarrow$ all $r_\alpha = 0$.

So $\Delta(v)$ is in fact a lin. indep set. ∎

Rest = Exercise

~ Weyl Chambers

Def! The Weyl Chambers $C \subseteq E$ are the components of the complement $E - \bigcup_\gamma P_\gamma$.

Since each C is path connected (in fact convex) we see that C is determined by any choice of regular $v \in C$, and for any $v' \in C$ we have

$$\Delta(v) = \Delta(v').$$

Given base $\Delta = \Delta(v)$ we write $C(\Delta) := \left\{ \begin{array}{l} \text{the unique} \\ \text{chamber} \\ \text{containing } v. \end{array} \right.$

We now show a bijection

$$\{\text{Bases for } (E,\Phi)\} \longleftrightarrow \left\{\begin{array}{c}\text{Components of}\\ E - \bigcup_\gamma P_\gamma\end{array}\right\}$$
$$= \text{Weyl chambers.}$$

~ Some neat-o bits and bobs

Fix throughout (Φ, E) a root system with specified base $\Delta \subseteq \Phi$.

Def$^{\underline{n}}$: We call a reflection $\sigma_\alpha \in W$ a simple reflection if $\alpha \in \Delta$.

We'll see in a moment that W is generated by simple reflections.

Lemma 10.2 A: For positive non-simple $\gamma \in \Phi^+$, there exists $\alpha \in \Delta$ with $\gamma - \alpha \in \Phi^+$.

Proof: Write $\gamma = \sum'_\alpha c_\alpha \cdot \alpha$ for non-neg c_α. Then $0 < (\gamma, \gamma) = \sum'_\beta c_\beta (\gamma, \beta)$ so that $(\gamma, \alpha) > 0$ at some α. In that case $\gamma - \alpha \in \Phi$ by Lemma 9.4. Since $\gamma \neq \alpha$ by hypothesis, and $\gamma - \alpha \in \Phi^+$ or Φ^- necessarily, the expression $\gamma - \alpha = (c_\alpha - 1)\alpha + \sum'_{\beta \neq \alpha} c_\beta \beta$ forces $\gamma - \alpha \in \Phi^+$. ∎

Corollary 10.2 A: Each $\overset{\text{pos}}{\gamma}$ admits an expression $\gamma = \sum_{i=1}^{k} \alpha_i$ with all α_i simple and each sub-sum $\sum_{i=1}^{k-m} \alpha_i \in \Phi^+$ as well.

Proof: Immediate ∎

Lemma 10.2 B: For α simple,
$$\sigma_\alpha(\Phi^+ - \{\alpha\}) = \Phi^+ - \{\alpha\}.$$

Proof: For $\gamma \in \Phi^+ - \{\alpha\}$ write $\gamma = \sum_\beta c_\beta \beta$ to get $\sigma_\alpha(\gamma) = \sum'_\beta c_\beta \beta - \left(\sum'_\beta c_\beta \langle \beta, \alpha \rangle\right)\alpha$.

Since $\gamma \neq \alpha$ we have $c_\beta \neq 0$ at some $\beta \neq \alpha$

giving $\sigma_\alpha(\gamma) \notin \Phi^-$. Hence $\sigma_\alpha(\gamma) \in \Phi^+$. ∎

Corollary 10.2B: For the half-sum of the positive roots $\rho = \frac{1}{2}\sum'_{\gamma\in\Phi^+}\gamma$ we have

a) $\langle \rho, \alpha \rangle = 1$ at all simple α.

b) $\sigma_\alpha(\rho) = \rho - \alpha$.

Proof: (a) Follows from (b). For (b) we have

$$\rho = \left(\frac{1}{2}\sum'_{\gamma\neq\alpha}\gamma\right) + \frac{1}{2}\alpha$$

$$\Rightarrow \sigma_\alpha(\rho) = \left(\frac{1}{2}\sum'_{\gamma\neq\alpha}\sigma_\alpha(\gamma)\right) - \frac{1}{2}\alpha$$

$$= \left(\frac{1}{2}\sum'_{\gamma\neq\alpha}\gamma\right) - \frac{1}{2}\alpha$$

$$= \rho - \alpha.$$ ∎

Lemma 10.2C: Let $\alpha_1, \ldots, \alpha_t \in \Delta$ be simple, and $\sigma_j = \sigma_{\alpha_j}$ at all j. If

$$\sigma_1 \cdots \sigma_{t-1}(\alpha_t) \in \Phi^-$$

then there exists an index $1 \leq s < t$ for which

$$\sigma_1 \cdots \sigma_t = \sigma_1 \cdots \sigma_{s-1}\sigma_{s+1} \cdots \sigma_{t-1}.$$

Proof: Let s be maximal so that

$$\sigma_s\sigma_{s+1} \cdots \sigma_{t-1}(\alpha_t) \text{ is negative.}$$

Then $\beta = \sigma_{s+1} \cdots \sigma_{t-1}(\alpha_t)$ is positive, and further $\alpha_s = \beta$. For $\sigma = \sigma_{s+1} \cdots \sigma_{t-1}$ one now have

$$\sigma_s = \sigma_{\sigma(\alpha)} = \sigma\sigma_\alpha\sigma^{-1}$$

so that
$$\sigma_1 \cdots \sigma_s \cdots \sigma_{t-1}\sigma_t = \sigma_1 \cdots \sigma\sigma_t\sigma^{-1}\sigma\sigma_t$$
$$= \sigma_1 \cdots \sigma_{s-1}\sigma_{s+1} \cdots \sigma_{t-1}$$

as claimed. ∎

Corollary 10.2C: For $\sigma = \sigma_1 \cdots \sigma_t \in W$ with the σ_i simple reflection $\sigma_i = \sigma_{\alpha_i}$, and t minimal so that σ admits such an expression, we have $\sigma(\alpha_t)$ a negative root.

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Proof: Otherwise $\sigma\sigma_t(\alpha_t)$ is negative implying $\sigma = \sigma_1 \cdots \sigma_{s-1}\sigma_{s+1} \cdots \sigma_{t-1}$. ∎

Lemma 10.2 D: If $\sigma = \sigma_1 \cdots \sigma_t$ is a minimal such exp in terms of simple reflections, then $t = |\{\gamma \in \Phi^+ : \sigma(\gamma) \in \Phi^-\}|$.

Proof: Proceed by ind. on t, the $t=1$ case following by Lemma 10.2.B. Define now for

$$\sigma \in \langle \sigma_\alpha : \alpha \in \Delta \rangle \subseteq \mathcal{W}$$

$\ell(\sigma) = \min\{t : \sigma = \sigma_1 \cdots \sigma_t$ for some simple $\sigma_i\}$, and suppose the result holds for all τ w/ $\ell(\tau) < t$, and take σ of $\ell(\sigma) = t$.

Then $\tau = \sigma_1 \cdots \sigma_{t-1}$ is necessarily a min length exp, so that $|\{\gamma : \tau(\gamma)$ negative$\}| = t-1$.

If $\tau(\alpha_t)$ negative then $\sigma = \tau\sigma_t$ has a length $t-2$ exp by Lemma 10.2.C, a contradiction.

So $\tau(\alpha_t)$ is positive, and $\sigma(\alpha_t)$ is negative and $\{\gamma : \sigma(\gamma)$ negative$\}$

$= \{\alpha_t\} \sqcup \{\sigma_t(\gamma) : \gamma \in \Phi^+$ and $\tau(\gamma) < 0\}$

$\Rightarrow |\{\gamma : \sigma(\gamma)$ negative$\}| = t$, as desired. ∎

~ Generating the Weyl group.

Theorem 10.3: Let Δ be a base for (Φ, E).

i) If $\lambda \in E$ is regular then there exist $\sigma \in \mathcal{W}$ so that $(\sigma(\lambda), \alpha) > 0$ at all $\alpha \in \Delta$.

ii) For any other base Δ' there exists $\sigma \in \mathcal{W}$ with $\sigma(\Delta') = \Delta$.

iii) For arbitrary $\gamma \in \Phi$ there exists $\sigma \in \mathcal{W}$ so that $\sigma(\gamma) \in \Delta$.

iv) $\mathcal{W}$ is generated by the σ_α w/ $\alpha \in \Delta$.

v) $\sigma(\Delta) = \Delta$ iff $\sigma = \mathrm{id}$.

Proof: We prove (i)–(iii) for $\mathcal{W}' = \langle \sigma_\alpha : \alpha \in \Delta \rangle$ which will imply (iv).

(i) For $\rho = \frac{1}{2} \sum_{\gamma \in \Phi^+} \gamma$ take $\sigma \in \mathcal{W}'$ so that $(\sigma(\lambda), \rho)$ is maximal. Then for each simple root σ_α we have

$(\sigma_\alpha \sigma(\lambda), \rho) = (\sigma(\lambda), \sigma_\alpha(\rho)) = (\sigma(\lambda), \rho) - (\sigma(\lambda), \alpha)$.

w/ $(\sigma_\alpha \sigma(\lambda), \rho) \leq (\sigma(\lambda), \rho)$. Hence $(\sigma(\lambda), \alpha) > 0$ by regularity of λ, and thus reg of $\sigma(\lambda)$.

(ii) Write $\Delta' = \Delta(\lambda)$ and choose $\sigma \in \mathcal{W}'$ so that $(\sigma(\lambda), \alpha) > 0$ at all simple $\alpha \Rightarrow$

$\sigma(\Delta') = \Delta(\sigma\lambda) = \Delta$.

(iii) Take first $\lambda' \in P_\gamma$ with

$\lambda' \notin \bigcup_{\xi \neq \gamma} P_\xi$.

Such λ' exists since the P_ξ intersect P_γ at a lower dim space and $(\bigcup_\xi P_\xi) \cap P_\gamma = \bigcap_\xi (P_\xi \cap P_\gamma)$.

Now we can choose $\lambda = \lambda' + \varepsilon\gamma$ with ε such small so that $(\gamma, \lambda) = \varepsilon(\gamma, \gamma) > 0$ sat

$(\gamma, \lambda) < |(\xi, \lambda)|$ at all $\xi \in \Phi - \{\gamma\}$.

Then $\gamma \in \Delta(\lambda)$ by minimality of $(*)$, and for σ as in (ii) we get $\sigma(\gamma) \in \Delta$.

(iv) For any $\gamma \in \Phi$ choose $\sigma \in \mathcal{W}'$ so that $\sigma(\gamma) \in \Delta$. Then $\sigma\sigma_\gamma\sigma^{-1} = \sigma_{\sigma(\gamma)} \in \mathcal{W}'$ and thus $\sigma_\gamma = \sigma^{-1}\sigma_{\sigma(\gamma)}\sigma \in \mathcal{W}'$. It follows that $\mathcal{W} = \langle \sigma_\gamma : \gamma \in \Phi \rangle = \mathcal{W}'$.

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cf.) If $\sigma(\Delta) = \Delta$ then $\sigma(\Phi^+) = \Phi^+$. (17)

By Corollary 10.2 C we have $\sigma = id$ necessarily. ∎

— Lengths of elem in the Weyl group.

Def^n: For $\sigma \in W = \langle \sigma_\alpha : \alpha \text{ simple} \rangle$, the length of σ is

$$l(\sigma) = |\{\gamma \in \Phi^+ : \sigma(\gamma) \in \Phi^-\}|$$

Lemma 10.2 D
$= \min\{t : \sigma \text{ admits an expression } \sigma = \sigma_1 \cdots \sigma_t \text{ with the } \sigma_i \text{ simple refl}\}$.

We call an expression $\sigma = \sigma_1 \cdots \sigma_t$ reduced if $t = l(\sigma)$

Lemma (the longest word): There is a unique element $w \in W$ with $l(w) = |\Phi^+|$.

Furthermore, for all other $\sigma \in W$ $l(\sigma) < l(w)$.

Proof: For Δ the given base we have $-\Delta$ another base. Hence, by Thm 10.3 there is an elem $w \in W$ with $w(\Delta) = -\Delta$ and thus $w(\Phi^+) = \Phi^-$.

So $l(w) = |\Phi^+|$ is maximal. For any other w' w/ $l(w') = |\Phi^+|$ we have $w'(\Phi^-) = \Phi^+$ so that $w'w^{-1}(\Delta) = w'(-\Delta) = \Delta$. Then by Thm 10.3 (e) $w'w^{-1} = 1 \Rightarrow w' = w$.

For any other $\sigma \neq w$ we have $l(\sigma) < l(w)$ by uniqueness of w and the def. of length. ∎