

# KERODON REMIX PART III: A SMALL STUDY OF THE DERIVED $\infty$ -CATEGORY

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ABSTRACT. We employ the materials from Part II to provide a baseline analysis of the derived  $\infty$ -category of an abelian category, as a stable  $\infty$ -category. We explicitly calculate pushouts and pullbacks in the homotopy (and derived)  $\infty$ -category, directly verify stability, and explicitly realize the derived  $\infty$ -category as a localization of the homotopy  $\infty$ -category along the class of quasi-isomorphisms. We also exhibit adjunctions between left and right derived functors. We conclude the text with a discussion of ind-completion and constructions of “renormalized” derived categories. In comparing with the previous installments, Parts I and II, we are somewhat more liberal in our treatment herein, as we occasionally employ results from higher topos theory or higher algebra when needed.

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## 1. DERIVED CATEGORIES, VAMOS!

This document is intended to provide an “intermediate” analysis of the derived  $\infty$ -category  $\mathcal{D}(\mathbb{A})$ , where  $\mathbb{A}$  is a (generally Grothendieck) abelian category. We assume the reader has their own motivations for coming this topic, and so won’t provide our own motivations.

This document provides the third, and final, portion in the sequence “Kerodon remix” Parts I, II, and III. Part I provides an introduction some foundational aspects of  $\infty$ -categories, and Part II provides an introduction to (co)cartesian fibrations, Hom functors, limits and colimits, and the Yoneda embedding.

**1.1. Derived recollections.** As the title suggests, this is a text about the derived category. We recall the necessary fundamentals here.

**Grothendieck abelian categories:** In general we work with Grothendieck abelian categories. We recall that an abelian category  $\mathbb{A}$  is called a Grothendieck abelian category if it admits small colimits and a generator, and if direct limits in  $\mathbb{A}$  are exact. This restriction is not so important, other than the fact that it ensures the existence of enough  $K$ -injectives in the unbounded category of cochains. Such  $K$ -injectives can then be leveraged in a uniform way in the production and analysis of the (unbounded) derived  $\infty$ -category.

For those interested in finite categories, such as finite-dimensional representations over a finite group, one can move from the finite context to the Grothendieck abelian context by replacing the category of interest with its Ind-category [10, Theorem 8.6.5]. In most hands-on situations, this move to the Ind-category is achieved simply by working within the category of infinite-dimensional rather than finite-dimensional representations.

**$\infty$ -categories from dg categories:** From any dg category  $\mathbf{A}$  we can construct an associated, or maybe *the* associated,  $\infty$ -category by applying the dg nerve  $\mathcal{A} := N^{\text{dg}}(K\mathbf{A})$ . The dg nerve construction is described in [15, 00PK], or Section I-2.2. This construction is very straightforward, and somewhat intuitive from the  $A_\infty$ -perspective. An important point, however, is that one can equivalently construct the  $\infty$ -category  $\mathcal{A}$ —up to equivalence—by first factoring through the simplicial setting then apply the homotopy coherent nerve.

Specifically, we can apply the Dold-Kan functor  $K : \text{Ch}(\mathbb{A}) \rightarrow \text{Kan}_{\mathbb{Z}}$  to morphisms to produce from  $\mathbf{A}$  a simplicial category  $K\mathbf{A}$ . This simplicial category has

Hom spaces  $K\mathrm{Hom}_{\mathbf{A}}(x, y)$  for each  $x$  and  $y$ . One then applies the homotopy coherent nerve [15, 00KM] to produce an  $\infty$ -category  $\mathcal{A}' = \mathrm{N}^{\mathrm{hc}}(\mathbf{A})$  which admits an equivalence over the underlying plain category  $Z^0(\mathbf{A})$ ,

$$\begin{array}{ccc} \mathcal{A}' & \xrightarrow[\sim]{3_{\mathbf{A}}} & \mathcal{A} \\ & \nwarrow \quad \nearrow & \\ & Z^0(\mathbf{A}) & \end{array} .$$

See Theorem II-10.4. This simplicial construction is much more convenient to work with when considering, say, Hom functors for the associated  $\infty$ -category. See for example the materials of Section II-11.3. So, from a practical perspective, the approach via Dold-Kan and the homotopy coherent nerve is a “better” construction of the  $\infty$ -category associated to  $\mathbf{A}$ .

We recall that, for an additive category  $\mathbb{A}$ , have its corresponding dg category  $\mathbf{Ch}(\mathbb{A})$  of unbounded cochains.

**Definition 1.1.** For an additive category  $\mathbb{A}$ , the homotopy  $\infty$ -category  $\mathcal{K}(\mathbb{A})$  is the associated  $\infty$ -category for the dg category of cochains over  $\mathbb{A}$ ,

$$\mathcal{K}(\mathbb{A}) := \mathrm{N}^{\mathrm{dg}}(\mathbf{Ch}(\mathbb{A})).$$

**The derived  $\infty$ -category:** For a Grothendieck abelian category  $\mathbb{A}$  the discrete category of cochains  $\mathbf{Ch}(\mathbb{A})$  admits enough  $K$ -injectives [20], i.e. complexes  $I$  for which the cochain-valued Hom functor  $\mathrm{Hom}_{\mathbb{A}}^*(-, I) : \mathbf{Ch}(\mathbb{A}) \rightarrow \mathbf{Ch}(\mathbb{Z})$  preserves acyclicity. In Parts I and II of this work we employed the following definition.

**Proto-definition 1.2.** For a Grothendieck abelian category  $\mathbb{A}$ , the derived  $\infty$ -category  $\mathcal{D}(\mathbb{A})$  is the full  $\infty$ -subcategory of  $K$ -injective complexes in the homotopy  $\infty$ -category  $\mathcal{K}(\mathbb{A})$ .

We generally denoted this  $\infty$ -subcategory as  $\mathcal{K}(\mathbb{A}) \supseteq \mathcal{D}_{\mathrm{Inj}} = \mathcal{D}(\mathbb{A})$ .

**Remark 1.3.** It’s shown in Section I-13 that one can equivalently construct the derived  $\infty$ -category via  $K$ -projective complexes, whenever such complexes exist in sufficient abundance.

However, from the perspective of this text, one should not necessarily approach the derived  $\infty$ -category in such a precious manner, but instead employ a coordinate free realization of the derived  $\infty$ -category as a localization of the homotopy  $\infty$ -category  $\mathcal{D}(\mathbb{A}) = \mathcal{K}(\mathbb{A})[\mathrm{Qiso}^{-1}]$  against the class of quasi-isomorphisms in  $\mathcal{K}(\mathbb{A})$ . See in particular Theorem 8.6 and Corollary 8.21 below.

**1.2. Contents.** The text proceeds as follows.

**In Section 2** we provide a direct, hands on analysis of limits and colimits in the homotopy  $\infty$ -category  $\mathcal{K}(\mathbb{A})$ . Of special interest are pullbacks and pushouts, which are both realized explicitly via mapping cone constructions. We also explain how projective and injective resolutions are realized naturally as colimits and limits of their respective finite truncations. As the derived  $\infty$ -category  $\mathcal{D}(\mathbb{A})$  can be constructed as a full subcategory in  $\mathcal{K}(\mathbb{A})$ , namely as the full subcategory of  $K$ -injectives or  $K$ -projectives (see Section I-13), our calculations for  $\mathcal{K}(\mathbb{A})$  imply analogous calculations of limits and colimits in the derived  $\infty$ -category.

There are two general ambitions which we mean to realize in this section. The first is simply to calculate a number of important limits and colimits in the homotopy and derived  $\infty$ -categories. The second is to demonstrate how the abstract shenanigans from Part II, regarding limits and colimits, can be applied to obtain explicitly descriptions of limits and colimits in practice.

**In Section 3** we explain how the presence of a zero object in  $\mathcal{D}(\mathbb{A})$  endows each Hom functor  $H : \mathcal{D}(\mathbb{A})^{\text{op}} \times \mathcal{D}(\mathbb{A}) \rightarrow \mathcal{K}an$  with a natural pointing. This pointing is, formally, a lift of  $H$  to a  $\mathcal{K}an_*$ -valued functor, where  $\mathcal{K}an_*$  is the  $\infty$ -category of pointed spaces. This lift is furthermore uniquely determined by the requirement that the functor  $\tilde{H} : \mathcal{D}(\mathbb{A})^{\text{op}} \times \mathcal{D}(\mathbb{A}) \rightarrow \mathcal{K}an_*$  preserves initial objects. We subsequently introduce spectra, and explain how our pointed Hom functors lift further to spectra-valued Hom functors in all settings of interest.

**In Section 5** we show that the derived and homotopy  $\infty$ -categories are stable. This is an immediate consequence of our calculations from Section 2. We then provide a basic overview of stable  $\infty$ -categories and some special phenomena which occur in the stable setting. In short, stability provides various shortcuts which allow one to, in particular, make strong deductions for stable  $\infty$ -categories from direct investigations of their homotopy categories. We find, for example, that the homotopy and derived  $\infty$ -categories are both complete and cocomplete. We also observe that the connective derived  $\infty$ -category  $\mathcal{D}(\mathbb{A})^{\leq 0}$  of any Grothendieck abelian category is cocomplete.

**In Section 7** we provide a generic analysis of adjoint functors. There are two main points here. The first is that any inclusion  $\mathcal{C}_0 \rightarrow \mathcal{C}$  of a coreflective subcategory into an  $\infty$ -category  $\mathcal{C}$  admits a left adjoint. The second is that the information of a pair of adjoint functors,

$$F : \mathcal{C} \rightarrow \mathcal{D} \quad \text{and} \quad G : \mathcal{D} \rightarrow \mathcal{C},$$

can be codified into a single cartesian and cocartesian fibrations over the 1-simplex  $\mathcal{C} \rightarrow \Delta^1$  whose fibers recover  $\mathcal{C} = \mathcal{C}_0$  and  $\mathcal{D} = \mathcal{C}_1$ . This is a practical tool which allows one to both construct adjunctions, and check for the existence of adjunctions.

**In Section 8** we prove that the derived category is identified, at the level of  $\infty$ -categories, as the localization  $\mathcal{K}(\mathbb{A})[\text{Qiso}^{-1}] = \mathcal{D}(\mathbb{A})$  relative to the class of quasi-isomorphisms in  $\mathcal{K}(\mathbb{A})$ . It then follows—from a particular result in [14]—that the derived  $\infty$ -category is furthermore identified with a localization of the discrete category of cochains  $\text{Ch}(\mathbb{A})[\text{Qiso}^{-1}] = \mathcal{D}(\mathbb{A})$ . This result is highly non-classical, and allows one to transfer structures directly from the abelian category of cochains to the derived setting.

**In Section 9** we discuss the process of deriving functors in the  $\infty$ -categorical context. We show that the left derived functor  $L F$  of a right exact functor  $F : \mathbb{A} \rightarrow \mathbb{B}$  between Grothendieck abelian categories can be defined, as in the discrete derived setting, by taking  $F$ -acyclic resolutions on the domain. Right derived functors  $R G$  can be defined universally by taking injective resolutions. Given a pair of adjoint functors  $F : \mathbb{A} \rightleftarrows \mathbb{B} : G$ , we show that the associated left derived functor  $L F : \mathcal{D}(\mathbb{A}) \rightarrow \mathcal{D}(\mathbb{B})$  is left adjoint to the right derived functor  $R G : \mathcal{D}(\mathbb{B}) \rightarrow \mathcal{D}(\mathbb{A})$ , as expected.

In **Section 10** we introduce the notion of presentability for  $\infty$ -categories. Following [14] directly, we recall that the derived  $\infty$ -category  $\mathcal{D}(\mathbb{A})$  is presentable, and following [13] directly we present the adjoint functor theorem which characterizes those functors between presentable  $\infty$ -categories which are left or right adjoints. (This section is a bit more of a rundown of results from [13] and [14] rather than an original presentation of the topic.)

In **Section 11** we present the ind-completion functor and subsequent constructions of “renormalized” derived categories. These include derived categories of ind-coherent sheaves, and ind-finite representations for algebraic groups.

In **Appendix A** we discuss idempotent complete categories, the process of idempotent completion, and  $\aleph_0$ -accessibility.

**1.3. Contextualization and originality.** Unlike Parts I and II, which essentially remixed and reorganized materials from Kerodon [15], the present text is a heterogeneous collection of materials from Higher Topos Theory [13], Higher Algebra [14], and original content. For example the majority of the contents from Sections 3 and 9 were developed independently. All of our computations from Section 2 are also original.

Additionally, in this text we regularly employ “fundamental” results from [13] and [14] without outlining or speaking to their proofs in any way. Hence we do not pursue a self-contained treatment of the topic at hand, and instead attempt to provide a treatment which is most effective in a practical sense. This deviates from the point-by-point, completionist approach taken in Parts I and II.

**1.4. Omissions et al.** In this text one finds various small omissions, such as a discussion of  $t$ -structures, and the generic omission of, say, 98% of higher algebra [14]. In any case, we expect that this point is clear to most readers. A point which might not be clear to most readers, however, is that there is some distinction between the framing of higher algebra [14] and the framing of spectral algebraic geometry [16].

From the perspective of higher algebra [14] one might locate the the derived  $\infty$ -category  $\mathcal{D}(\mathbb{A})$  as a special object within the class of presentable stable  $\infty$ -categories, possibly with a  $t$ -structure. The presentation of [16] might instead emphasize the connective derived category  $\mathcal{D}(\mathbb{A})^{\leq 0}$  as a special object within the class of Grothendieck prestable  $\infty$ -categories [16, Section C.1.4]. In the Grothendieck prestable setting we again have an effective algebra of categories [16, Section C.4], and such categories seemingly provide a more functional location in which to do algebraic geometry.

In any case, we simply alert the reader to the existence of this alternate universe of Grothendieck prestable  $\infty$ -categories, and note that the topics discussed herein are as relevant in the Grothendieck prestable context as they are in the stable presentable context. So, let us begin.

sect:lim\_KA

## 2. LIMITS AND COLIMITS IN THE HOMOTOPY $\infty$ -CATEGORY

We directly compute various limits and colimits in the homotopy  $\infty$ -category  $\mathcal{H}(\mathbb{A})$ , where  $\mathbb{A}$  is an abelian category. Here we use the explicit descriptions of limits in  $\mathcal{H}an$  provided in Section II-14.2 and the fact that limits and colimits can be detected by checking values under the application of Hom functors.

We use the calculations from this section to verify stability, and cocompleteness, of the homotopy and derived  $\infty$ -categories in Section 5 below. In considering the materials of Part II, one might also interpret the analysis herein as an exercise in which we apply the methods from Part II in a “real world” setting to compute limits and limits in a particular  $\infty$ -category of interest.

### 2.1. Orienting comments on the homotopy $\infty$ -category.

**Definition 2.1.** For an abelian category  $\mathbb{A}$  we let  $K\mathbf{Ch}(\mathbb{A})$  denote the simplicial category associated to the dg category of cochains over  $\mathbb{A}$ , and we take

$$\mathcal{K}(\mathbb{A})' := N^{\mathrm{hc}}(K\mathbf{Ch}(\mathbb{A})).$$

Let us begin with a simple comparison between the  $\infty$ -categories  $\mathcal{K}(\mathbb{A})$  and  $\mathcal{K}(\mathbb{A})'$ , as the  $\infty$ -category  $\mathcal{K}(\mathbb{A})'$  provides a more convenient locale in which to do many of our calculations.

We recall from Theorem II-10.4 that there is an equivalence of  $\infty$ -categories to the standard homotopy  $\infty$ -category  $\mathfrak{Z} : \mathcal{K}(\mathbb{A})' \xrightarrow{\sim} \mathcal{K}(\mathbb{A}) = N^{\mathrm{dg}}(\mathbf{Ch}(\mathbb{A}))$  which restricts to the identity on the underlying discrete category of cochains

$$\begin{array}{ccc} \mathcal{K}(\mathbb{A})' & \xrightarrow{\mathfrak{Z}} & \mathcal{K}(\mathbb{A}) \\ & \nwarrow \quad \nearrow & \\ & \mathbf{Ch}(\mathbb{A}) & \end{array} \quad (1) \quad \boxed{\text{eq:equiv\_Z}}$$

Since the inclusions from  $\mathbf{Ch}(\mathbb{A})$  are surjective on 0 and 1-simplices, we see that the  $\infty$ -categories  $\mathcal{K}(\mathbb{A})$  and  $\mathcal{K}(\mathbb{A})'$  are indistinguishable at the level of objects and morphisms.

Though we generally avoid considering 2-simplices in  $\mathcal{K}(\mathbb{A})'$ , it is the case that these two categories are identified in dimension 2 as well. Before observing this identification, let us recall that the Hom complexes in the simplicial category  $K\mathbf{Ch}(\mathbb{A})$  are given by the Eilenbergh-MacLane spaces

$$\underline{\mathrm{Hom}}_{\mathbb{A}}(V, W) = K\mathrm{Hom}_{\mathbb{A}}^*(V, W)$$

(see Section I-11.4). Directly, a 1-simplices in  $\underline{\mathrm{Hom}}_{\mathbb{A}}(V, W)$  is a triple

$$\tilde{h} = (h : V \rightarrow W, h_0 : V \rightarrow W, h_1 : V \rightarrow W)$$

with  $h$  of degree  $-1$ , the  $h_i$  cochain maps of degree 0, and

$$d(h) = d_W h + h d_V = h_0 - h_1.$$

The restrictions along the inclusions  $\{i\} \rightarrow \Delta^1$  are as expected  $\tilde{h}|_{\{i\}} = h_i$ .

Now, according to the definition of the homotopy coherent nerve, a 2-simplex  $\sigma : \Delta^2 \rightarrow \mathcal{K}(\mathbb{A})'$  is a not-necessarily-commuting diagram of cochain maps

$$\begin{array}{ccc} & V_1 & \\ f_{01} \nearrow & & \searrow f_{12} \\ V_0 & \xrightarrow{f_{02}} & V_2 \end{array}$$

and a 1-simplex  $\tilde{h}$  in  $\underline{\mathrm{Hom}}_{\mathbb{A}}(V_0, V_2)$  with  $\tilde{h}|_0 = f_{12}f_{01}$  and  $\tilde{h}|_1 = f_{02}$ , i.e. a choice of a degree  $-1$  map  $h$  which establishes homotopy commutativity  $d(h) = f_{12}f_{01} - f_{02}$ . (See Lemma I-2.19).

To compare, a 2-simplex in the usual  $\infty$ -category  $\mathcal{K}(\mathbb{A})$  is a choice of a not-necessarily-commuting diagram of cochain maps

$$\begin{array}{ccc} & V_1 & \\ f_{01} \nearrow & & \searrow f_{12} \\ V_0 & \xrightarrow{f_{02}} & V_2 \end{array}$$

and a degree  $-1$  map  $h : V_0 \rightarrow V_2$  satisfying  $d(h) = f_{12}f_{01} - f_{02}$ . These are clearly the same thing, and the equivalence **3** appearing in (1) sends each 2-simplex in  $\mathcal{K}(\mathbb{A})'$  to “the same” 2-simplex in  $\mathcal{K}(\mathbb{A})$ .

**2.2. Limits and colimits in  $\mathcal{K}(\mathbb{A})$  via Hom functors.** According to Proposition II-11.6 the functor

$$\underline{\mathrm{Hom}}_{\mathbb{A}}(V, -) : \mathcal{K}(\mathbb{A})' \rightarrow \mathcal{K}an \quad \text{and} \quad \underline{\mathrm{Hom}}_{\mathbb{A}}(-, W) : (\mathcal{K}(\mathbb{A})')^{\mathrm{op}} \rightarrow \mathcal{K}an$$

are respectively corepresented and represented by the given objects  $V$  and  $W$ . Hence Corollary II-16.17 appears in our setting as follows.

**Proposition 2.2.** *Given a diagram  $p : K \rightarrow \mathcal{K}(\mathbb{A})'$ , a given extension  $p' : \{0\} \star K \rightarrow \mathcal{K}(\mathbb{A})'$  is a limit diagram in  $\mathcal{K}(\mathbb{A})'$  if and only if, at each cochain complex  $V$ , the composite*

$$\underline{\mathrm{Hom}}_{\mathbb{A}}(V, -) \circ p' : \{0\} \star K \rightarrow \mathcal{K}an$$

*is a limit diagram in  $\mathcal{K}an$ . Similarly, an extension  $p'' : K \star \{\infty\} \rightarrow \mathcal{K}(\mathbb{A})'$  is a colimit diagram if and only if, at each complex  $W$ , the composite*

$$\underline{\mathrm{Hom}}_{\mathbb{A}}(-, W) \circ (p'')^{\mathrm{op}} : \{\infty\} \star K^{\mathrm{op}} \rightarrow \mathcal{K}an$$

*is a limit diagram in  $\mathcal{K}an$ .*

We use this result ad nauseam in order to calculate limits and colimits in  $\mathcal{K}(\mathbb{A})'$ , and subsequently in  $\mathcal{K}(\mathbb{A})$  as well.

**2.3. Strictly commuting diagrams and “discrete (co)limits”.** By a *strictly commuting diagram* in  $\mathcal{K}(\mathbb{A})$  we mean a diagram  $p : K \rightarrow \mathcal{K}(\mathbb{A})$  which factors through the discrete category of cochains,

$$\begin{array}{ccc} & \mathrm{Ch}(\mathbb{A}) & \\ \exists \bar{p} \nearrow & & \searrow \mathrm{incl} \\ K & \xrightarrow{r} & \mathcal{K}(\mathbb{A}). \end{array}$$

For such a diagram the completing functor  $\bar{p}$  is uniquely determined, as the map  $\mathrm{incl} : \mathrm{Ch}(\mathbb{A}) \rightarrow \mathcal{K}(\mathbb{A})$  is an inclusion of simplicial sets. We adopt a similar phraseology for the diagrams in the  $\infty$ -category  $\mathcal{K}(\mathbb{A})'$ .

Consider  $\mathbb{A}$  Grothendieck abelian. Since the category  $\mathrm{Ch}(\mathbb{A})$  admits arbitrary coproducts and coequalizers it is cocomplete as an  $\infty$ -category [14, Proposition 4.4.3.2]. It is similarly seen to be complete as well. Furthermore, since the inclusion  $\mathrm{Set} \rightarrow \mathcal{K}an$  is right adjoint to the connected components functor  $\pi_0 : \mathcal{K}an \rightarrow \mathrm{Set}$  we see that this inclusion preserves all limits. (The existence of such adjunction follows as Theorem I-14.13.) Hence, by a consideration of Hom functors and Corollary II-16.17, we observe that the limit (resp. colimit) of any map  $p : K \rightarrow \mathrm{Ch}(\mathbb{A})$  from



a discrete category  $K$  is just equal to the limit (resp. colimit) which we calculate in  $\text{Ch}(\mathbb{A})$  as an abelian category.

**Definition 2.3.** Let  $\mathbb{A}$  be a Grothendieck abelian category and  $p : K \rightarrow \mathcal{K}(\mathbb{A})$  be a strictly commuting diagram. Let  $\bar{p} : K \rightarrow \text{Ch}(\mathbb{A})$  be the corresponding diagram in  $\text{Ch}(\mathbb{A})$ . The discrete limit (resp. colimit) of  $p$  in  $\mathcal{K}(\mathbb{A})$  is the image

$$\text{incl } \bar{p}' : \{0\} \star K \rightarrow \mathcal{K}(\mathbb{A}) \quad (\text{resp. } \text{incl } \bar{p}'' : K \star \{\infty\} \rightarrow \mathcal{K}(\mathbb{A}))$$

of the limit diagram  $\bar{p}'$  (resp. colimit diagram  $\bar{p}''$ ) in  $\text{Ch}(\mathbb{A})$  under the standard inclusion.

As was the case when considering limits and colimits in  $\mathcal{Cat}_\infty$ , part of our game here is to identify a nice class of (strictly commuting) diagrams  $p : K \rightarrow \mathcal{K}(\mathbb{A})$  for which the discrete limit, or colimit, already calculates the limit, or colimit, of  $p$  in  $\mathcal{K}(\mathbb{A})$ .

#### 2.4. Kan fibrations for simplicial abelian groups.

lem:5490

- Lemma 2.4.** (1) *If  $A$  is a discrete simplicial abelian group, then the inclusion  $0 \rightarrow A$  is a Kan fibration.*  
 (2) *Any surjective map of simplicial abelian groups  $f : X \rightarrow Y$  is a Kan fibration.*  
 (3) *If  $A$  is a discrete simplicial abelian group, and  $f : X \rightarrow Y$  is a surjective, then the map  $[0 \ f]^t : X \rightarrow A \times Y$  is a Kan fibration.*

*Proof.* (1) Follows from the fact that, in this case, any simplex  $\Delta^n \rightarrow A$  in which a single vertex maps to 0 is of constant value 0. (2) Consider a lifting diagram

$$\begin{array}{ccc} \Lambda_i^n & \xrightarrow{\tau} & X \\ \downarrow & & \downarrow f \\ \Delta^n & \xrightarrow{\bar{\sigma}} & Y. \end{array}$$

We can lift  $\bar{\sigma}$  arbitrarily to an  $n$ -simplex  $\sigma : \Delta^n \rightarrow X$ , via surjectivity of  $f$ . We can now replace  $\bar{\sigma}$  with the 0 simplex and  $\tau$  with  $\sigma|_{\Lambda_i^n} - \tau$  to reduce to the case  $Y = 0$ . In this case the desired solution exists since  $X$  is a Kan complex (Proposition I-11.1). (3) In this case  $[0 \ f]^t$  can be identified with a product of Kan fibrations  $0 \times f : 0 \times X \rightarrow A \times Y$ , and is thus a Kan fibration.  $\square$

cor:K\_kanfib

**Corollary 2.5.** *For any map  $f : V \rightarrow W$  of cochains of abelian groups in which  $f^n : V^n \rightarrow W^n$  is surjective at all  $n < 0$ , the corresponding map  $Kf : KV \rightarrow KW$  is a Kan fibration.*

*Proof.* We can factor  $f$  as the inclusion  $V \rightarrow Z^0(W) \times V$  composed with the map  $[i \ f] : Z^0(W) \times V = Z^0(W) \oplus V \rightarrow W$ , where  $i$  here is the inclusion  $i : Z^0(W) \rightarrow W$ . Then  $Kf$  factors as the sequence

$$KV \xrightarrow{[0 \ id]^t} KZ^0(W) \times KV \xrightarrow{[i \ Kf]} KW$$

in which the latter map is surjective, since the Eilenbergh-MacLane functor  $K = K\tau_0$  is an equivalence on connective cochains and the map  $\tau_0(Z^0(W) \times V) \rightarrow \tau_0(W)$  is surjective by construction. By Lemma 2.4 it follows that  $Kf$  is a Kan fibration.  $\square$

### 2.5. Fundamental claims: pullbacks in the homotopy $\infty$ -category.

**Definition 2.6.** For maps of cochains  $f : V \rightarrow W$  and  $f' : V' \rightarrow W$ , we take

$$C(f, f') := \Sigma^{-1} \text{cone}([ -f \ f' ] : V \times V' \rightarrow W).$$

For a single map  $f : V \rightarrow W$  we take  $C(f) = \Sigma^{-1} \text{cone}(-f)$ .

The complex  $C(f, f')$  appears as

$$\left( (V \times V') \oplus \Sigma^{-1}W, \begin{bmatrix} d_{V \times V'} & [f \ -f'] \\ 0 & -d_W \end{bmatrix} \right).$$

Note that we have the embedding of cochains

$$V \times_W V' = \ker([f \ -f']) \rightarrow C(f, f')$$

We have the degree  $-1$  map of graded objects in  $\mathbb{A}$

$$h_W = [0 \ id_W] : C(f, f') \rightarrow W \tag{2} \quad \boxed{\text{eq:hw}}$$

with

$$d_{\text{Hom}}(h_W) = (C(f, f') \xrightarrow{[p \ p']^t} V \times V' \xrightarrow{[f \ -f']} W),$$

where  $p$  and  $p'$  are the obvious projection. This homotopy defines a 2-simplex  $h_W : \Delta^2 \rightarrow \mathcal{K}(\mathbb{A})$  which appears as

$$\begin{array}{ccc} & V & \\ p \nearrow & & \searrow f \\ C(f, f') & \xrightarrow{f'p'} & W. \end{array}$$

We append a strictly commuting diagram for  $V'$  to obtain a square  $\Delta^1 \times \Delta^1 \rightarrow \mathcal{K}(\mathbb{A})$  which appears as

$$\begin{array}{ccc} C(f, f') & \xrightarrow{p} & V \\ p' \downarrow & \searrow f'p' & \downarrow f \\ V' & \xrightarrow{f'} & W. \end{array} \tag{3} \quad \boxed{\text{eq:5570}}$$

**def:std\_pullback**

**Definition 2.7.** Given arbitrary morphisms  $f : V \rightarrow W$  and  $f' : V' \rightarrow W$  of  $\mathbb{A}$ -cochains, the corresponding standard pullback diagram is the diagram (3) in  $\mathcal{K}(\mathbb{A})$ .

We prove in Proposition 2.13 below that any standard pullback diagram is in fact a limit diagram in  $\mathcal{K}(\mathbb{A})$ . We note that the construction of the standard pullback is natural in the sense that a strictly commuting diagram

$$\begin{array}{ccccc} V & \longrightarrow & X & & V' & \longrightarrow & X' \\ & \searrow f & & \swarrow g & & \searrow f' & \\ & & W & \longrightarrow & Y & & \end{array}$$

extends to a diagram of the form

$$\begin{array}{ccccc}
 & C(f, f') & \longrightarrow & C(g, g') & \\
 & \swarrow & & \searrow & \\
 V' & \xrightarrow{=} & X' & & V & \longrightarrow & X \\
 & \searrow & & \swarrow & \\
 & W & \longrightarrow & Y &
 \end{array}$$

in  $\mathcal{K}(\mathbb{A})$ .

**2.6. Pullbacks in the homotopy  $\infty$ -category.** Throughout the subsection we fix  $\mathbb{A}$  an abelian category. We establish some background materials before returning to address the issue of pullbacks.

**def:truncate**

**Definition 2.8.** For any abelian category  $\mathbb{A}$  we let

$$\tau_0 : \text{Ch}(\mathbb{A}) \rightarrow \text{Ch}(\mathbb{A})^{\leq 0}$$

denote the truncation functor,  $\tau_0 V = \cdots \rightarrow V^{-2} \rightarrow V^{-1} \rightarrow Z^0(V) \rightarrow 0$ .

If we let  $\mathbf{Ch}'(\mathbb{A})$  denote the dg category of cochains with mapping complexes  $\text{Hom}_{\mathbb{Z}}^{\leq 0}(X, Y)$ , then the functor  $\tau_0$  enhances to a dg functor

$$\tau_0 : \mathbf{Ch}'(\mathbb{A}) \rightarrow \mathbf{Ch}(\mathbb{A})^{\leq 0}.$$

In particular,  $\tau_0$  respects homotopy and homotopy equivalences. Also, since  $\tau_0$  is right adjoint to the inclusion  $\text{Ch}(\mathbb{A})^{\leq 0} \rightarrow \text{Ch}(\mathbb{A})$  this functor commutes with limits.

Let us say that a map of  $\mathbb{A}$ -cochains  $f : V \rightarrow W$  is *termwise split surjective* if, for each integer  $n$ , the map  $f^n : V^n \rightarrow W^n$  is a split surjection.

**lem:5721**

**Lemma 2.9.** *For a map of  $\mathbb{A}$ -cochains  $f : V \rightarrow W$  the following are equivalent:*

- (1)  *$f$  is termwise split surjective.*
- (2)  *$f$  is split surjective as a map of graded objects in  $\mathbb{A}$ .*
- (3) *For each cochain  $X$ , the induced map  $f_* : \text{Hom}_{\mathbb{A}}^*(X, V) \rightarrow \text{Hom}_{\mathbb{A}}^*(X, W)$  is surjective.*

*Proof.* Omitted. □

The following can be seen as an algebraic analog of Corollary II-14.31.

**prop:split\_pullback**

**Proposition 2.10.** *Consider maps of  $\mathbb{A}$ -cochains  $f : V \rightarrow W$  and  $f' : V' \rightarrow W$ , and suppose one of  $f$  or  $f'$  is termwise split surjective. Then the strictly commuting pullback diagram*

$$\begin{array}{ccc}
 V \times_W V' & \longrightarrow & V \\
 \downarrow & & \downarrow \\
 V' & \longrightarrow & W
 \end{array} \tag{4}$$

**eq:5645**

*is a limit diagram in  $\mathcal{K}(\mathbb{A})'$ .*

*Proof.* Assume arbitrarily that  $f$  is graded split. Take

$$\begin{aligned}
 \text{Hom}_{\mathbb{A}}^{\bullet}(X, Y) &= \tau_0 \text{Hom}_{\mathbb{A}}^*(X, Y) \\
 &= \cdots \rightarrow \text{Hom}_{\mathbb{A}}^{-2}(X, Y) \rightarrow \text{Hom}_{\mathbb{A}}^{-1}(X, Y) \rightarrow Z^0 \text{Hom}_{\mathbb{A}}^*(X, Y) \rightarrow 0.
 \end{aligned}$$

For each cochain complex  $X$  we have

$$\mathrm{Hom}_{\mathbb{A}}^{\bullet}(X, V \times_W V') = \mathrm{Hom}_{\mathbb{A}}^{\bullet}(X, V) \times_{\mathrm{Hom}_{\mathbb{A}}^{\bullet}(X, W)} \mathrm{Hom}_{\mathbb{A}}^{\bullet}(X, V')$$

so that the induced diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{A}}^{\bullet}(X, V \times_W V') & \longrightarrow & \mathrm{Hom}_{\mathbb{A}}^{\bullet}(X, V) \\ \downarrow & & \downarrow f_* \\ \mathrm{Hom}_{\mathbb{A}}^{\bullet}(X, V') & \xrightarrow{f'_*} & \mathrm{Hom}_{\mathbb{A}}^{\bullet}(X, W) \end{array} \quad (5) \quad \boxed{\text{eq:5660}}$$

is a pullback diagram. Furthermore, by our splitting assumption, the map  $f_*$  is split in each strictly negative degree.

We now apply the Eilenbergh-MacLane functor  $K$  to obtain a pullback diagram

$$\begin{array}{ccc} \underline{\mathrm{Hom}}_{\mathbb{A}}(X, V \times_W V') & \longrightarrow & \underline{\mathrm{Hom}}_{\mathbb{A}}(X, V) \\ \downarrow & & \downarrow Kf_* \\ \underline{\mathrm{Hom}}_{\mathbb{A}}(X, V') & \xrightarrow{Kf'_*} & \underline{\mathrm{Hom}}_{\mathbb{A}}(X, W) \end{array}$$

in which the map  $Kf_*$  is a Kan fibration by Corollary 2.5. The above diagram is therefore a pullback diagram in  $\mathcal{Kan}$  by Corollary II-14.31. Since  $X$  was chosen arbitrarily we apply Corollary II-16.17 to observe that the diagram (10) is a pullback diagram in  $\mathcal{K}(\mathbb{A})'$ .  $\square$

Note that the above strict pullback diagram is in the image of the inclusion  $\mathrm{Ch}(\mathbb{A}) \rightarrow \mathcal{K}(\mathbb{A})'$ . Since the equivalence  $\mathfrak{J} : \mathcal{K}(\mathbb{A})' \rightarrow \mathcal{K}(\mathbb{A})$  restricts to the identity on  $\mathrm{Ch}(\mathbb{A})$  we see that it preserves all strict pullback diagrams. Since equivalences preserve limits, by Proposition II-13.10, Proposition 2.10 implies the following.

**cor:split\_pullback**

**Corollary 2.11.** *Consider maps of connective cochains  $f : V \rightarrow W$  and  $f' : V' \rightarrow W$ , and suppose one of  $f$  or  $f'$  is termwise split surjective. Then the strictly commuting pullback diagram*

$$\begin{array}{ccc} V \times_W V' & \longrightarrow & V \\ \downarrow & & \downarrow \\ V' & \longrightarrow & W \end{array} \quad (6) \quad \boxed{\text{eq:5645}}$$

is a limit diagram in  $\mathcal{K}(\mathbb{A})$ .

We now compare the discrete fiber product  $V \times_W V'$  to the shifted mapping cone which appears in the “standard pullback diagram” (3).

**lem:pb\_cone**

**Lemma 2.12.** *Consider maps of cochains  $f : V \rightarrow W$  and  $f' : V' \rightarrow W$ , and suppose one of  $f$  or  $f'$  is termwise split surjective. Then the inclusion*

$$V \times_W V' \rightarrow C(f, f')$$

is a homotopy equivalence.

*Proof.* By replacing  $V$  with  $V \oplus V'$  and  $V'$  with 0, it suffices to prove that the inclusion  $\ker(f) \rightarrow C(f)$  is a homotopy equivalence in the case that  $f : V \rightarrow W$  is termwise split surjective.

Via the splitting we can write  $V \cong \Sigma L \oplus K$  with  $L = \Sigma^{-1}W$  and  $K = \ker(f)$ . Here  $K$  is a subcomplex in  $V$  and the map  $V \rightarrow W$  is just the projection onto the first factor. We may assume for simplicity that the isomorphism  $V \cong \Sigma L \oplus K$  is an equality on the underlying graded objects  $V = \Sigma L \oplus K$ .

The composite

$$\Sigma L \xrightarrow{\text{incl}} V \xrightarrow{d_V} V \xrightarrow{\text{proj}} K$$

defines a degree 1 map from  $\Sigma L$ , which is then a degree 0 map  $g : L \rightarrow K$ . This map is seen to be a cochain morphism so that

$$V = \text{cone}(g) = \left( \Sigma L \oplus K, \begin{bmatrix} -d_L & g \\ 0 & d_K \end{bmatrix} \right).$$

We now have

$$C(f) = \left( \Sigma L \oplus K \oplus L, \begin{bmatrix} -d_L & g & id \\ 0 & d_K & 0 \\ 0 & 0 & d_L \end{bmatrix} \right)$$

and observe the projection

$$\pi = [0 \ id_K \ -g] : C(f) \rightarrow K.$$

We have directly  $\pi \text{incl} = id_K : K \rightarrow K$  and the composite  $\text{incl} \pi : C(f) \rightarrow C(f)$  is homotopic to the identity via the degree  $-1$  map

$$h = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ id_L & 0 & 0 \end{bmatrix} : C(f) \rightarrow C(f).$$

□

Now, we have the inclusion  $V \times_W V' \rightarrow C(f, f')$  at general  $f$  and  $f'$ , and the homotopy  $h_W$  from (2) has trivial restriction  $h_W|_{V \times_W V'} = 0$ . This inclusion therefore extends to a natural transformation of diagrams

$$\begin{array}{ccccc} & V \times_W V' & \longrightarrow & C(f, f') & \\ & \swarrow & & \searrow & \\ V' & \xrightarrow{=} & V' & & V \\ & \searrow & \swarrow & \downarrow & \swarrow \\ & W & \xrightarrow{=} & W & \end{array} \quad (7) \quad \boxed{\text{eq:5696}}$$

By Lemma 2.12 and Theorem I-7.6 this transformation is an isomorphism whenever  $f$  or  $f'$  is termwise split surjective.

prop:K\_pullback

**Proposition 2.13.** *For arbitrary maps  $f : V \rightarrow W$  and  $f' : V' \rightarrow W$  in  $\mathcal{K}(\mathbb{A})$ , the standard pullback diagram*

$$\begin{array}{ccc} C(f, f') & \longrightarrow & V \\ \downarrow & \searrow & \downarrow f \\ V' & \xrightarrow{f'} & W \end{array} \quad (8) \quad \boxed{\text{eq:5707}}$$

(see Definition 2.7) is a limit diagram in  $\mathcal{K}(\mathbb{A})$ . In particular, the diagram (8) is isomorphic to a diagram of the form

$$\begin{array}{ccc} V_0 & \longrightarrow & V_1 \\ \downarrow & & \downarrow f_{12} \\ V'_1 & \xrightarrow{f'_{12}} & V_2 \end{array}$$

in which  $f_{12}$  is termwise split surjective and  $f'_{12}$  is injective

*Proof.* Take  $V_2 = \text{cone}(id_{V'}) \oplus W$ ,  $V'_1 = V'$ , and  $f'_{12} = [i \ f']^t : V' \rightarrow \text{cone}(id_{V'}) \oplus W$  where  $i : V' \rightarrow \text{cone}(id_{V'})$  is the usual inclusion. Take now  $V_1 = V \oplus C(id_{V_2})$  and  $f_{12} = [\pi \ f] : V \oplus C(id_{V_2}) \rightarrow V_2$  where  $\pi : C(id_{V_2}) \rightarrow V_2$  is the usual projection. The map  $f'_{12}$  is injective and  $g$  is split as a graded morphism via the identity map  $V_2 \rightarrow V_2 \oplus \Sigma^{-1}V_2 = C(id_{V_2})$ .

Since the mapping cone of the identity morphism is contractible, the summands  $C(id_{V'})$  and  $C(id_{V_2})$  are contractible. The inclusion  $V \rightarrow V_1$  and  $V' \rightarrow V_2$  are therefore homotopy equivalence and induce an isomorphism of diagrams

$$\begin{array}{ccccc} & C(f, f') & \longrightarrow & C(f_{12}, f'_{12}) & \\ & \swarrow & & \searrow & \\ V' & \xrightarrow{=} & V'_1 & & V \\ & \searrow & \swarrow & \swarrow & \searrow \\ & W & \xrightarrow{=} & V_2 & \end{array}$$

in  $\mathcal{K}(\mathbb{A})$ . As argued at (7) the diagram for  $C(f_{12}, f'_{12})$  is naturally isomorphic to the discrete pullback diagram

$$\begin{array}{ccc} V_0 = V_1 \times_{V_2} V'_1 & \longrightarrow & V_1 \\ \downarrow & & \downarrow g \\ V'_1 & \longrightarrow & V_2, \end{array} \tag{9} \quad \boxed{\text{eq:5733}}$$

so that in total the diagram (8) is isomorphic to the diagram (9). Since the latter diagram is a pullback diagram in  $\mathcal{K}(\mathbb{A})$  by Proposition 2.10, it follows by Proposition II-13.18 that the diagram (9) is a pullback diagram as well.  $\square$

As a corollary we find that any diagram

$$\begin{array}{ccc} & V & \\ & \downarrow & \\ V' & \longrightarrow & W \end{array}$$

in  $\mathcal{K}(\mathbb{A})$  admits a limit.

cor:K\_pullback

**Corollary 2.14.** *Given any abelian category  $\mathbb{A}$ , every diagram  $\Lambda_2^2 \rightarrow \mathcal{K}(\mathbb{A})$  admits a limit. That is to say, the homotopy  $\infty$ -category  $\mathcal{K}(\mathbb{A})$  admits all pullbacks.*

**2.7. Pushouts diagrams in the homotopy  $\infty$ -category.** We recall that, for a partial diagram of cochains

$$\begin{array}{ccc} W & \xrightarrow{g} & V \\ g' \downarrow & & \\ V' & & \end{array}$$

the pushout in  $\text{Ch}(\mathbb{A})$  is the quotient  $V \amalg_{W'} V' = \text{coker}(W \rightarrow V \oplus V')$ .

Let us call a general morphism of  $\mathbb{A}$ -cochains  $g : W \rightarrow V$  termwise split injective if, at each integer  $n$ ,  $g^n : W^n \rightarrow V^n$  is a split injective morphism in  $\mathbb{A}$ . We have the expected analog of Lemma 2.9.

lem:5915

**Lemma 2.15.** *For a map of  $\mathbb{A}$ -cochains  $g : W \rightarrow V$  the following are equivalent:*

- (1)  *$g$  is termwise split injective.*
- (2)  *$g$  is split injective as a map of graded objects in  $\mathbb{A}$ .*
- (3) *For each cochain  $Y$ , the induced map  $g^* : \text{Hom}_{\mathbb{A}}^*(V, Y) \rightarrow \text{Hom}_{\mathbb{A}}^*(W, Y)$  is surjective.*

For any abelian category  $\mathbb{A}$ , we apply Corollary 2.11 to the opposite category  $\mathbb{B} = \mathbb{A}^{\text{op}}$  to obtain the corresponding result for pushout diagrams in  $\mathcal{K}(\mathbb{A}) \cong \mathcal{K}(\mathbb{B})^{\text{op}}$  (Lemma II-11.14).

prop:split\_pushout

**Proposition 2.16.** *Consider maps of  $\mathbb{A}$ -cochains  $g : V \rightarrow W$  and  $g' : V' \rightarrow W$ , and suppose one of  $g$  or  $g'$  is termwise split injective. Then the discrete pushout diagram*

$$\begin{array}{ccc} W & \xrightarrow{g} & V \\ g' \downarrow & & \downarrow \\ V' & \longrightarrow & V \amalg_W V' \end{array} \quad (10) \quad \text{eq:5645}$$

is a colimit diagram in  $\mathcal{K}(\mathbb{A})$ .

In the pushout context we return to the standard, rather than shifted, mapping cone.

**Definition 2.17.** Given maps  $g : W \rightarrow V$  and  $g' : W \rightarrow V'$  of  $\mathbb{A}$ -cochains we take

$$\text{cone}(g, g') = \text{cone}([g \ -g']^t : W \rightarrow V' \oplus V).$$

Of course, in the case  $V' = 0$  we have  $\text{cone}(g) = \text{cone}(g, 0)$ . We have the two inclusions from  $V$  and  $V'$  into  $\text{cone}(g, g')$  which provide a generally noncommuting diagram

$$\begin{array}{ccc} W & \xrightarrow{g} & V \\ g' \downarrow & & \downarrow \\ V' & \longrightarrow & \text{cone}(g, g'). \end{array}$$

The degree  $-1$  map  $h'_W : W \rightarrow \text{cone}(g, g')$  defined by the identity on  $W$  satisfies

$$d_{\text{Hom}}(h'_W) = [g \ -g']^t : W \rightarrow V \oplus V' \subseteq \text{cone}(g, g')$$

and hence produces a diagram

$$\begin{array}{ccc}
 W & \xrightarrow{g} & V \\
 g' \downarrow & \searrow & \downarrow \\
 V' & \longrightarrow & \text{cone}(g, g').
 \end{array} \tag{11} \quad \boxed{\text{eq:5958}}$$

in the homotopy  $\infty$ -category  $\mathcal{K}(\mathbb{A})$  in which the bottom simplex strictly commutes and the top simplex is exhibited via  $h'_W$ .

**Definition 2.18.** Given maps  $g : W \rightarrow V$  and  $g' : W \rightarrow V'$  in  $\mathcal{K}(\mathbb{A})$ , we refer to the diagram (11) as the standard pushout diagram associated to  $g$  and  $g'$ .

We obtain the following by applying Lemma 2.12 to the opposite category.

lem:5987

**Lemma 2.19.** Consider maps of  $\mathbb{A}$ -cochains  $g : V \rightarrow W$  and  $g' : V' \rightarrow W$ , and suppose one of  $g$  or  $g'$  is termwise split injective. Then the projection  $\pi : \text{cone}(g, g') \rightarrow V \amalg_W V'$  is a homotopy equivalence.

Since the homotopy  $h'_W$  vanishes when composed with the projection  $\pi : \text{cone}(g, g') \rightarrow V \amalg_W V'$ , this projection extends to a diagram in  $\mathcal{K}(\mathbb{A})$  which appears as

$$\begin{array}{ccccc}
 & W & \xrightarrow{=} & W & \\
 & \swarrow & & \searrow & \\
 V' & \xrightarrow{=} & V' & & V \xrightarrow{=} V \\
 & \searrow & & \swarrow & \\
 & \text{cone}(g, g') & \longrightarrow & V \amalg_W V' & .
 \end{array}$$

According to Lemma 2.19, in the case that one of  $g$  or  $g'$  is split injective this diagram realizes an isomorphism between the two square faces, so that the standard square

$$\begin{array}{ccc}
 W & \xrightarrow{g} & V \\
 g' \downarrow & \searrow & \downarrow \\
 V' & \longrightarrow & \text{cone}(g, g').
 \end{array}$$

is observed to be a pushout square via Proposition 2.16. This realization pushouts in  $\mathcal{K}(\mathbb{A})$  via the standard pushout diagrams now generalizes to arbitrary pairs of maps.

prop:K\_pushout

**Proposition 2.20.** For arbitrary maps  $g : W \rightarrow V$  and  $g' : W \rightarrow V'$  in  $\mathcal{K}(\mathbb{A})$ , the standard pushout diagram

$$\begin{array}{ccc}
 W & \xrightarrow{g} & V \\
 g' \downarrow & \searrow & \downarrow \\
 V' & \longrightarrow & \text{cone}(g, g')
 \end{array} \tag{12} \quad \boxed{\text{eq:6010}}$$



is a colimit diagram in  $\mathcal{K}(\mathbb{A})$ . In particular, the diagram (12) is isomorphic to a diagram of the form

$$\begin{array}{ccc} V_0 & \xrightarrow{g_{01}} & V_1 \\ g'_{01} \downarrow & & \downarrow \\ V'_1 & \longrightarrow & V_2 \end{array}$$

in which  $g_{01}$  is termwise split injective and  $g'_{01}$  is injective

The proof is similar to that of Proposition 2.13, and is omitted.

cor:K\_pushout

**Corollary 2.21.** *Given any abelian category  $\mathbb{A}$ , every diagram  $\Lambda_0^2 \rightarrow \mathcal{K}(\mathbb{A})$  admits a colimit. That is to say, the homotopy  $\infty$ -category  $\mathcal{K}(\mathbb{A})$  admits all pushouts.*

## 2.8. Products, coproducts, and the zero complex.

prop:K\_prods\_coprods

**Proposition 2.22.** *For any abelian category  $\mathbb{A}$ , the functor  $\text{Ch}(\mathbb{A}) \rightarrow \mathcal{K}(\mathbb{A})$  preserves all small products and coproducts. In particular, the category  $\mathcal{K}(\mathbb{A})$  admits all small products and coproducts.*

*Proof.* Since the inclusion  $\text{Ch}(\mathbb{A}) \rightarrow \mathcal{K}(\mathbb{A})$  factors through the equivalence  $\mathfrak{Z} : \mathcal{K}(\mathbb{A})' \rightarrow \mathcal{K}(\mathbb{A})$  it suffices to show that products and coproducts in  $\text{Ch}(\mathbb{A})$  are products and coproducts in the simplicial construction of the homotopy  $\infty$ -category  $\mathcal{K}(\mathbb{A})'$ . For this we employ the Hom functor  $\text{K Hom}_{\mathbb{A}}^{\bullet}$ , where  $\text{Hom}_{\mathbb{A}}^{\bullet}(X, Y) = \tau_0 \text{Hom}_{\mathbb{A}}^*(X, Y)$ , and check that this functor turns discrete coproducts (resp. products) into products of spaces through the first (resp. second) coordinate. (Recall that products of spaces are as expected, by Example II-14.16 and Theorem II-14.25.)

Since  $\tau_0$  is a right adjoint it commutes with limits, and the Dold-Kan equivalence commutes with limits as well, it suffices to show that the functor  $\text{Hom}_{\mathbb{A}}^*$  sends discrete coproducts in the first coordinate to products of additive cochains, and discrete products in the second coordinate to products of additive cochains. However this follows immediately by, say, the fact that  $\text{Hom}_{\mathbb{A}}^*$  provide inner-Homs for the action of  $\text{Ch}(\mathbb{Z})$  on  $\text{Ch}(\mathbb{A})$ .  $\square$

As a corollary to Lemma II-9.18 we also see that the zero complex provides a zero object for the homotopy  $\infty$ -category.

cor:K\_zero

**Corollary 2.23.** *The zero complex provides a simultaneous initial and terminal object in the  $\infty$ -category  $\mathcal{K}(\mathbb{A})$ .*

## 2.9. Resolutions as (co)limits.

prop:lim\_res

**Proposition 2.24.** *Given a strictly commuting diagram  $I(-) : \mathbb{Z}_{\leq 0} \rightarrow \mathcal{K}(\mathbb{A})$  in which each map  $I(n-1) \rightarrow I(n)$  is termwise split surjective and with discrete limit  $I$  in  $\text{Ch}(\mathbb{A})$ , the discrete limit diagram*

$$\begin{array}{ccccccc} \cdots & \longrightarrow & I(m-1) & \longrightarrow & I(m) & \longrightarrow & I(m+1) \longrightarrow \cdots \\ & & \nwarrow & & \uparrow & & \nearrow \\ & & & & I & & \end{array}$$

is a limit diagram in  $\mathcal{K}(\mathbb{A})$ .

*Proof.* Since the diagram  $I(-)$  factors through  $\text{Ch}(\mathbb{A})$  it suffices to prove that its image provides a limit diagram in  $\mathcal{K}(\mathbb{A})'$ . Applying the corepresentable functor  $K \text{Hom}_{\mathbb{A}}^*(X, -)$  to the given sequence  $\cdots \rightarrow I(m) \rightarrow I(m+1) \rightarrow \cdots$  produces a sequence of Kan fibrations

$$\cdots \rightarrow K \text{Hom}_{\mathbb{A}}^*(X, I(m-1)) \rightarrow K \text{Hom}_{\mathbb{A}}^*(X, I(m)) \rightarrow \cdots \quad (13)$$

eq:6055

in  $\text{Kan} \subseteq \mathcal{K}an$ , by Corollary 2.5. Since the functor  $K : \text{Ch}(\mathbb{A}) \rightarrow \text{Kan}_{\mathbb{Z}}$  has a left adjoint, given by the normalized cochains functor, it commutes with limits, so that

$$\lim_n K \text{Hom}_{\mathbb{A}}^*(X, I(n)) = K \lim_n \text{Hom}_{\mathbb{A}}^*(X, I(n)) = \text{Hom}_{\mathbb{A}}^*(X, I).$$

In this case the limit diagram for the sequence (13) in  $\text{Kan}$  provides a limit diagram in  $\mathcal{K}an$ . This follows by Proposition II-14.38 and Theorem II-14.42. So we see that, at each complex  $X$ , the functor  $K \text{Hom}_{\mathbb{A}}^*(X, -)$  sends the diagram  $I(-)$  to a limit diagram in spaces. By Corollary II-16.17 it follows that the diagram  $I(-) : \mathbb{Z}_{\leq 0} \rightarrow \mathcal{K}(\mathbb{A})'$  is a limit diagram in  $\mathcal{K}(\mathbb{A})'$ , and hence its composite with the equivalence  $\mathfrak{J} : \mathcal{K}(\mathbb{A})' \rightarrow \mathcal{K}(\mathbb{A})$  is a limit diagram in  $\mathcal{K}(\mathbb{A})$ .  $\square$

By consulting the opposite category, or simply by repeating the above arguments, one obtains a statement for colimits of sequences of split injections.

prop:colim\_res

**Proposition 2.25.** *Given a strictly commuting diagram  $P(-) : \mathbb{Z}_{\geq 0} \rightarrow \mathcal{K}(\mathbb{A})$  in which each map  $P(n) \rightarrow P(n+1)$  is termwise split injective, the discrete colimit diagram*

$$\begin{array}{c} P \\ \swarrow \downarrow \searrow \\ \cdots \longrightarrow P(m-1) \longrightarrow P(m) \longrightarrow P(m+1) \longrightarrow \cdots \end{array}$$

*is a colimit diagram in  $\mathcal{K}(\mathbb{A})$ .*

**Example 2.26.** Let  $I$  be a bounded below complex in  $\mathcal{K}(\mathbb{A})$ . (From a practical perspective, we might consider  $I$  a bounded below complex of injectives.) Consider the complexes  $I(n) = I/I^{\geq -n}$  and the corresponding sequence of termwise split injections of bounded complexes of injectives

$$\cdots \rightarrow I(-2) \rightarrow I(-1) \rightarrow I(0).$$

We have the corresponding strictly commuting diagram  $I(-) : \mathbb{Z}_{\leq 0} \rightarrow \mathcal{K}(\mathbb{A})$  with discrete limit  $\lim_n I(n) = I$ . By Proposition 2.24 this discrete limit is limit diagram in  $\mathcal{K}(\mathbb{A})$ . In particular  $I$  is the limit  $I(-)$  in  $\mathcal{K}(\mathbb{A})$ .

From the example we see that the bounded below homotopy  $\infty$ -categories  $\mathcal{K}(\mathbb{A})^+$  is generated under limits, and in fact filtered limits, by bounded complexes. We similarly see that the bounded above  $\infty$ -category  $\mathcal{K}(\mathbb{A})^-$  is generated under colimits by bounded complexes.

**Example 2.27.** Let  $P$  be a bounded above complex in  $\mathcal{K}(\mathbb{A})$ . Consider the complexes  $P(n) = P^{\geq -n}$  and the corresponding sequence of termwise split inclusions

$$P(0) \rightarrow P(1) \rightarrow P(2) \rightarrow \cdots$$

We have the associated strictly commuting diagram in  $\mathcal{K}(\mathbb{A})$ , which has colimit  $P$  by Proposition 2.25.

sect:zero\_hom

## 3. ZERO OBJECTS AND POINTED HOM FUNCTORS

Given an  $\infty$ -category  $\mathcal{C}$ , we obtain the unique Hom functor

$$\text{“Hom}_{\mathcal{C}}”} = H : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{K}an$$

as the transport functor along the twisted arrows fibration (see Section II-12.1). In the event that  $\mathcal{C}$  has a zero object, we expect each space  $H(x, y)$  to have a distinguished zero morphism which is preserved under composition. In this way the Hom functor should inherit a natural pointing, i.e. a *lifting* of  $H$  to a functor valued in the  $\infty$ -category  $\mathcal{K}an_*$ .

We prove that such a canonical pointing for the Hom functor exist. In Section 4 we show, further, that these pointed Hom functors enhance to spectra-valued Hom functors whenever  $\mathcal{C}$  is sufficiently symmetric around zero.

## 3.1. Pointed Hom functors.

**Definition 3.1.** A zero object in an  $\infty$ -category is an object  $0 : * \rightarrow \mathcal{C}$  which is both initial and terminal in  $\mathcal{C}$ .

Consider the left fibration

$$q : \mathcal{K}an_*(= \mathcal{K}an_{*/}) \rightarrow \mathcal{K}an$$

and any  $\infty$ -category  $\mathcal{C}$  with a zero object. Recall that the identity morphisms  $* \rightarrow *$  is simultaneously initial and terminal in the category of pointed spaces  $\mathcal{K}an_*$ , by Proposition II-9.15. Furthermore, any choice of a zero object  $0 : * \rightarrow \mathcal{C}$  then provides an initial object in the enveloping category

$$\vec{0} = (0, 0) : * \rightarrow \mathcal{C}^{\text{op}} \times \mathcal{C}.$$

The following is a direct application of Corollary II-9.25.

prop:693

**Proposition 3.2.** Let  $\mathcal{C}$  be an  $\infty$ -category with a zero object  $0$ , and take  $\mathcal{C}^e = \mathcal{C}^{\text{op}} \times \mathcal{C}$ . The functors

$$\text{Fun}(\mathcal{C}^e, \mathcal{K}an_*) \rightarrow \mathcal{K}an_* \times_{\mathcal{K}an} \text{Fun}(\mathcal{C}^e, \mathcal{K}an), \quad F \mapsto (F|_{\vec{0}}, qF), \quad (14)$$

eq:695

and

$$\text{Fun}(\Delta^1 \times \mathcal{C}^e, \mathcal{K}an_*) \rightarrow \mathcal{K}an_* \times_{\mathcal{K}an} \text{Fun}(\Delta^1 \times \mathcal{C}^e, \mathcal{K}an), \quad \zeta \mapsto (\zeta|_{(0, \vec{0})}, q\zeta),$$

are trivial Kan fibrations.

Given any Hom functor  $H : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{K}an$  (Definition II-12.4), we take the fiber of the trivial Kan fibration (14) at the pairing of  $H$  with a choice of element  $* \rightarrow H(0, 0)$  in the contractible space  $H(0, 0) \cong \text{Hom}_{\mathcal{C}}(0, 0)$  to obtain a contractible space of pointings for the functor  $H$ , as outlined precisely below.

**Corollary 3.3.** For an  $\infty$ -category  $\mathcal{C}$  equipped with a choice of zero object  $0 : * \rightarrow \mathcal{C}$ , Hom functor  $H : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{K}an$ , and point  $1 : * \rightarrow H(0, 0)$ , there is a unique functor  $\tilde{H} : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{K}an_*$  which fits into a strictly commuting diagram

$$\begin{array}{ccc} & & \mathcal{K}an_* \\ & \nearrow \tilde{H} & \downarrow \\ \mathcal{C}^{\text{op}} \times \mathcal{C} & \xrightarrow{H} & \mathcal{K}an \end{array}$$

and for which  $\tilde{H}(0, 0) = (1 : * \rightarrow H(0, 0))$ .

One can show furthermore that there is a unique pointing for any given Hom functor.

thm:pointed\_hom

**Theorem 3.4.** *Let  $\mathcal{C}$  be an  $\infty$ -category which admits a zero object. Then for any choice of Hom functor  $H : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{K}an$ , the space*

$$\text{Fun}(\mathcal{C}^{\text{op}} \times \mathcal{C}, \mathcal{K}an_*) \times_{\text{Fun}(\mathcal{C}^{\text{op}} \times \mathcal{C}, \mathcal{K}an)} \{H\}$$

*of lifts, i.e. the space of functors  $\tilde{H} : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{K}an_*$  which fit into a strictly commuting diagram*

$$\begin{array}{ccc} & & \mathcal{K}an_* \\ & \nearrow \tilde{H} & \downarrow \\ \mathcal{C}^{\text{op}} \times \mathcal{C} & \xrightarrow{H} & \mathcal{K}an, \end{array}$$

*is a contractible Kan complex.*

*Proof.* Take again  $\mathcal{C}^e = \mathcal{C}^{\text{op}} \times \mathcal{C}$  and fix any zero object  $0 : * \rightarrow \mathcal{C}$ . The sequence

$$* \xrightarrow{H} \text{Fun}(\mathcal{C}^e, \mathcal{K}an) \xrightarrow{\tilde{0}^*} \mathcal{K}an$$

picks out the space  $H(0, 0)$  in  $\mathcal{K}an$ , so that the fiber

$$(\mathcal{K}an_* \times_{\mathcal{K}an} \text{Fun}(\mathcal{C}^e, \mathcal{K}an)) \times_{\text{Fun}(\mathcal{C}^e, \mathcal{K}an)} \{H\} = \mathcal{K}an_* \times_{\mathcal{K}an} \{H(0, 0)\}$$

is the left pinched mapping space  $\text{Hom}_{\mathcal{K}an}^L(*, H(0, 0))$ . Since  $H(0, 0)$  is contractible, it is terminal in  $\mathcal{K}an$  by Lemma II-9.3. Hence this mapping space  $\text{Hom}_{\mathcal{K}an}^L(*, H(0, 0))$  is contractible.

Now, by the above information the fiber

$$\text{Fun}(\mathcal{C}^e, \mathcal{K}an_*) \times_{\text{Fun}(\mathcal{C}^e, \mathcal{K}an)} \{H\}$$

$= \text{Fun}(\mathcal{C}^e, \mathcal{K}an_*) \times_{(\mathcal{K}an_* \times_{\mathcal{K}an} \text{Fun}(\mathcal{C}^e, \mathcal{K}an))} (\mathcal{K}an_* \times_{\mathcal{K}an} \text{Fun}(\mathcal{C}^e, \mathcal{K}an)) \times_{\text{Fun}(\mathcal{C}^e, \mathcal{K}an)} \{H\}$   
fits into a pullback square

$$\begin{array}{ccc} \text{Fun}(\mathcal{C}^e, \mathcal{K}an_*) \times_{\text{Fun}(\mathcal{C}^e, \mathcal{K}an)} \{H\} & \longrightarrow & \text{Fun}(\mathcal{C}^e, \mathcal{K}an_*) \\ \downarrow & & \downarrow \\ \text{Hom}_{\mathcal{C}}^L(0, H(0, 0)) & \longrightarrow & \mathcal{K}an_* \times_{\mathcal{K}an} \text{Fun}(\mathcal{C}^e, \mathcal{K}an), \end{array}$$

where the right vertical map is a trivial Kan fibration by Proposition 3.2. Hence we have a trivial Kan fibration over a contractible space

$$\text{Fun}(\mathcal{C}^e, \mathcal{K}an_*) \times_{\text{Fun}(\mathcal{C}^e, \mathcal{K}an)} \{H\} \rightarrow \text{Hom}_{\mathcal{C}}^L(0, H(0, 0)),$$

from which we conclude that the fiber under consideration is a contractible Kan complex.  $\square$

def:pointed\_hom

**Definition 3.5.** Given an  $\infty$ -category  $\mathcal{C}$  which admits a zero object, a pointed Hom functor for  $\mathcal{C}$  is a functor  $\tilde{H} : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{K}an_*$  whose composite

$$\mathcal{C}^{\text{op}} \times \mathcal{C} \xrightarrow{\tilde{H}} \mathcal{K}an_* \xrightarrow{\text{forget}} \mathcal{K}an$$

is a Hom functor for  $\mathcal{C}$ .

Let us consider now the twisted arrows fibration  $\lambda : \mathcal{T}w(\mathcal{C}) \rightarrow \mathcal{C}^{\text{op}} \times \mathcal{C}$ , and the space of associated transport functors with a witnessing diagram  $\mathcal{T}(\lambda)$  (Definition II-6.13). By Theorem II-6.14 we understand that the space  $\mathcal{T}(\lambda)$  is contractible. One can reasonably define the space of pointed Hom functors as the fiber product

$$\text{Fun}(\mathcal{C}^{\text{op}} \times \mathcal{C}, \mathcal{K}an_*) \times_{\text{Fun}(\mathcal{C}^{\text{op}} \times \mathcal{C}, \mathcal{K}an)} \mathcal{T}(\lambda).$$

We now observe uniqueness of pointed Hom functors in the absolute sense.

prop:pointed\_hom\_uniq

**Proposition 3.6.** *For any  $\infty$ -category  $\mathcal{C}$  which admits a zero object, the space*

$$\text{Fun}(\mathcal{C}^{\text{op}} \times \mathcal{C}, \mathcal{K}an_*) \times_{\text{Fun}(\mathcal{C}^{\text{op}} \times \mathcal{C}, \mathcal{K}an)} \mathcal{T}(\lambda)$$

*of pointed Hom functors is a contractible Kan complex.*

*Proof.* Take  $\mathcal{C}^e = \mathcal{C}^{\text{op}} \times \mathcal{C}$ . Since the map  $\mathcal{K}an_* \rightarrow \mathcal{K}an$  is a left fibration, and in particular an isofibration, the induced map  $\text{Fun}(\mathcal{C}^e, \mathcal{K}an_*) \rightarrow \text{Fun}(\mathcal{C}^e, \mathcal{K}an)$  is an isofibration, by Corollary I-6.14. For any choice of Hom functor  $H$  with witnessing data the corresponding map  $H : * \rightarrow \mathcal{T}(\lambda)$  is a homotopy equivalence, and so the induced map on fiber products

$$\text{Fun}(\mathcal{C}^e, \mathcal{K}an_*) \times_{\text{Fun}(\mathcal{C}^e, \mathcal{K}an)} \{H\} \rightarrow \text{Fun}(\mathcal{C}^e, \mathcal{K}an_*) \times_{\text{Fun}(\mathcal{C}^e, \mathcal{K}an)} \mathcal{T}(\lambda)$$

is an equivalence by Corollary I-6.24. Since the domain space for this functor is a contractible Kan complex, by Theorem 3.4, it follows that the space of pointed Hom functors is a contractible Kan complex as well.  $\square$

### 3.2. Naturality for pointed Hom functors.

thm:pointed\_natural

**Theorem 3.7.** *Let  $F : \mathcal{C}_0 \rightarrow \mathcal{C}_1$  be a functor between  $\infty$ -categories with zero objects, and suppose that  $F$  preserves zero objects. Take  $\mathcal{C}_i^e = \mathcal{C}_i^{\text{op}} \times \mathcal{C}_i$  and*

$$\mathcal{L} = \text{Fun}(\partial\Delta^1 \times \mathcal{C}_0^e, \mathcal{K}an_*) \times_{\text{Fun}(\partial\Delta^1 \times \mathcal{C}_0^e, \mathcal{K}an)} \text{Fun}(\Delta^1 \times \mathcal{C}_0^e, \mathcal{K}an).$$

*Let  $\tilde{H}_i : \mathcal{C}_i^e \rightarrow \mathcal{K}an_*$  be pointed Hom functors,  $H_i$  be the underlying unpointed Hom functors, and  $H_F : H_0 \rightarrow H_1 F$  be the transformation induced by  $F$  (Definition II-8.5). The space*

$$\text{Fun}(\Delta^1 \times \mathcal{C}_0^e, \mathcal{K}an_*) \times_{\mathcal{L}} \{(\tilde{H}_0, \tilde{H}_1 F, H_F)\}$$

*of transformations lifting  $H_F$ , i.e. transformations  $\tilde{H}_F : \Delta^1 \times \mathcal{C}_0^e \rightarrow \mathcal{K}an_*$  which fit into a strictly commuting diagram*

$$\begin{array}{ccc} & & \mathcal{K}an_* \\ & \nearrow \tilde{H}_F & \downarrow \\ \Delta^1 \times \mathcal{C}_0 & \xrightarrow{H_F} & \mathcal{K}an \end{array} \quad (15) \quad \text{eq:810}$$

*and satisfy  $\tilde{H}_F|_{\{0\}} = \tilde{H}_0$  and  $\tilde{H}_F|_{\{1\}} = \tilde{H}_1 F$ , is a contractible Kan complex.*

Before giving the proof we record a useful lemma.

lem:tech\_fiber

**Lemma 3.8.** *Suppose  $\mathcal{E}$  is an  $\infty$ -category with an initial object  $e : * \rightarrow \mathcal{E}$ , and let  $G : \mathcal{E} \rightarrow \mathcal{K}an$  be a functor for which the value  $G(e)$  is contractible. Then the fiber product  $\text{Fun}(\mathcal{E}, \mathcal{K}an_*) \times_{\text{Fun}(\mathcal{E}, \mathcal{K}an)} \{G\}$  is a contractible Kan complex.*

*Proof.* Same as the proof of Theorem 3.4.  $\square$

We now return to the matter at hand

*Proof of Theorem 3.7.* Given a zero object  $w$  in  $\mathcal{C}$ , the  $\infty$ -category  $\Delta^1 \times \mathcal{C}_0^e$  has the initial object  $(0, \bar{w})$ . Furthermore, since the space  $H_F(0, \bar{w}) = H_0(w, w)$  is contractible, Lemma 3.8 tells us that the fiber product

$$\mathrm{Fun}(\Delta^1 \times \mathcal{C}_0^e, \mathcal{K}an_*) \times_{\mathrm{Fun}(\Delta^1 \times \mathcal{C}_0^e, \mathcal{K}an)} \{H_F\}$$

is a contractible Kan complex. We can rewrite this fiber product as

$$\mathrm{Fun}(\Delta^1 \times \mathcal{C}_0^e, \mathcal{K}an_*) \times_{\mathcal{L}} (\mathcal{L} \times_{\mathrm{Fun}(\Delta^1 \times \mathcal{C}_0^e, \mathcal{K}an)} \{H_F\})$$

and note that the fiber product

$$\mathcal{L} \times_{\mathrm{Fun}(\Delta^1 \times \mathcal{C}_0^e, \mathcal{K}an)} \{H_F\} = \mathrm{Fun}(\partial^1 \Delta^1 \times \mathcal{C}_0^e, \mathcal{K}an_*) \times_{\mathrm{Fun}(\partial \Delta^1 \times \mathcal{C}_0^e, \mathcal{K}an)} \{(H_0, H_1 F)\}$$

is again contractible by Lemma 3.8.

We consider the point

$$(\tilde{H}_0, \tilde{H}_1 F, H_F) : * \rightarrow \mathcal{L} \times_{\mathrm{Fun}(\Delta^1 \times \mathcal{C}_0^e, \mathcal{K}an)} \{H_F\},$$

which is now a homotopy equivalence of Kan complexes. Since the map  $\mathrm{Fun}(\Delta^1 \times \mathcal{C}_0^e, \mathcal{K}an_*) \rightarrow \mathcal{L}$  is an isofibration, by Proposition I-6.13, pulling back yields an equivalence

$$\begin{aligned} & \mathrm{Fun}(\Delta^1 \times \mathcal{C}_0^e, \mathcal{K}an_*) \times_{\mathcal{L}} \{(\tilde{H}_0, \tilde{H}_1 F, H_F)\} \\ & \xrightarrow{\sim} \mathrm{Fun}(\Delta^1 \times \mathcal{C}_0^e, \mathcal{K}an_*) \times_{\mathrm{Fun}(\Delta^1 \times \mathcal{C}_0^e, \mathcal{K}an)} \{H_F\}, \end{aligned}$$

by Corollary I-6.24. As we argued above, the target space here is contractible, so that the fiber  $\mathrm{Fun}(\Delta^1 \times \mathcal{C}_0^e, \mathcal{K}an_*) \times_{\mathcal{L}} \{(\tilde{H}_0, \tilde{H}_1 F, H_F)\}$  is seen to be a contractible Kan complex as well.  $\square$

`def:pointed_transf`

**Definition 3.9.** Let  $F : \mathcal{C}_0 \rightarrow \mathcal{C}_1$  be a functor between  $\infty$ -categories with zero objects, and suppose furthermore that  $F$  preserves zero objects. Then for pointed Hom functors  $\tilde{H}_i : \mathcal{C}_i^{\mathrm{op}} \times \mathcal{C} \rightarrow \mathcal{K}an_*$ , the transformation  $\tilde{H}_F : \tilde{H}_0 \rightarrow \tilde{H}_1 F$  induced by  $F$  is any transformation whose composite

$$\Delta^1 \times \mathcal{C}_0^{\mathrm{op}} \times \mathcal{C}_0 \xrightarrow{\tilde{H}_F} \mathcal{K}an_* \xrightarrow{\mathrm{forget}} \mathcal{K}an$$

recovers the transformation  $H_F : H_0 \rightarrow H_1 F$  induced by  $F$  on the underlying unpointed Hom functors (in the sense of Definition II-8.5).

**Remark 3.10.** We have abused language in speaking of “the” induced transformation rather than “an” induced transformation. One notes, however, that the space parametrizing such choices is again contractible.

**3.3. Pointed spaces vs. pointed spaces.** When working with simplicial categories it is convenient to have a simplicial construction of the  $\infty$ -category  $\mathcal{K}an_*$  of pointed spaces.

We consider the simplicial category  $\underline{\mathcal{K}an}_*$  whose objects are Kan complexes  $\mathcal{X}$  with a fixed point  $x : * \rightarrow \mathcal{X}$ , and whose morphism complexes are the fibers

$$\mathrm{Fun}_*/((\mathcal{X}, x), (\mathcal{Y}, y)) := \mathrm{Fun}(\mathcal{X}, \mathcal{Y}) \times_{\mathrm{Fun}(*, \mathcal{Y})} \{y\}. \quad (16)$$

`eq:890`

In particular, we have  $\mathrm{Fun}_*/(*, (\mathcal{Y}, y)) = *$ , and one sees that the point  $*$  is both initial and terminal in the simplicial category  $\underline{\mathcal{K}an}_*$ . Note also that the the functor spaces  $\mathrm{Fun}(\mathcal{X}, \mathcal{Y})$  are Kan complexes, and that restriction along the point  $x^* : \mathrm{Fun}(\mathcal{X}, \mathcal{Y}) \rightarrow \mathrm{Fun}(*, \mathcal{Y}) = \mathcal{Y}$  is a Kan fibration. Hence the fiber (16) is always a Kan complex.

Due to triviality of the mapping spaces from the one point space, one sees that any functor  $\text{Path } \Delta^n \rightarrow \underline{\text{Kan}}_*$  extends uniquely to a functor  $\text{Path}(\{-1\} \star \Delta^n) \rightarrow \underline{\text{Kan}}_*$  whose value at  $-1$  is  $*$ . So we see that the left fibration

$$\text{N}^{\text{hc}}(\underline{\text{Kan}}_*)_{*/} \rightarrow \text{N}^{\text{hc}}(\underline{\text{Kan}}_*) \quad (17) \quad \text{eq:1233}$$

is an isomorphism of simplicial sets. We also have the map

$$\text{N}^{\text{hc}}(\underline{\text{Kan}}_*) \rightarrow \text{N}^{\text{hc}}(\underline{\text{Kan}}) = \mathcal{K}an$$

induced by the forgetful functor  $\underline{\text{Kan}}_* \rightarrow \underline{\text{Kan}}$ .

From the above information we obtain a unique functor  $\rho : \text{N}^{\text{hc}}(\underline{\text{Kan}}_*) \rightarrow \mathcal{K}an_*$  which fits into a strictly commuting diagram

$$\begin{array}{ccc} \text{N}^{\text{hc}}(\underline{\text{Kan}}_*)_{*/} & \xrightarrow{\text{forget}_{*/}} & \mathcal{K}an_* \\ \downarrow & \nearrow \rho & \downarrow \\ \text{N}^{\text{hc}}(\underline{\text{Kan}}_*) & \xrightarrow{\text{forget}} & \mathcal{K}an. \end{array} \quad (18) \quad \text{eq:1055}$$

As the forgetful functor to  $\mathcal{K}an$  is an injective map of simplicial sets, the completing map  $\rho$  is injective as well.

prop:ptd\_id

**Proposition 3.11** ([15, 0200]). *The inclusion  $\rho : \text{N}^{\text{hc}}(\underline{\text{Kan}}_*) \rightarrow \mathcal{K}an_*$  is an equivalence of  $\infty$ -categories.*

We refer the reader to the text [15] for the details.

**Remark 3.12.** The image of the functor  $\rho$  consists of those  $n$ -simplices in  $\mathcal{K}an_*$ , i.e. those maps  $\{-1\} \star \Delta^n \rightarrow \mathcal{K}an$ , whose restriction  $\{-1\} \star \text{Sk}_1(\Delta^n) \rightarrow \mathcal{K}an$  strictly commutes.

sect:simp\_ptd\_hom

**3.4. Pointed Hom functors via the homotopy coherent nerve.** Let  $\underline{A}$  be a Kan-enriched category with a strict zero object 0, in the sense that the Hom complexes to and from 0 are just points

$$* = \underline{\text{Hom}}_{\underline{A}}(x, 0) = \underline{\text{Hom}}_{\underline{A}}(0, x).$$

Then the simplicial Hom functor  $\underline{\text{Hom}}_{\underline{A}} : \underline{A}^{\text{op}} \times \underline{A} \rightarrow \underline{\text{Kan}}$  admits a natural pointing via the 0 morphisms  $0 : * \rightarrow \underline{\text{Hom}}_{\underline{A}}(x, y)$ ,

$$\underline{\text{Hom}}_{\underline{A}} : \underline{A}^{\text{op}} \times \underline{A} \rightarrow \underline{\text{Kan}}_*.$$

For  $\mathcal{A} = \text{N}^{\text{hc}}(\underline{A})$  and  $\underline{\text{Hom}}_{\mathcal{A}} = \text{N}^{\text{hc}} \underline{\text{Hom}}_{\underline{A}}$  we therefore obtain a functor

$$\underline{\text{Hom}}_{\mathcal{A}} : \mathcal{A}^{\text{op}} \times \mathcal{A} \rightarrow \text{N}^{\text{hc}}(\underline{\text{Kan}}_*).$$

We compose with the equivalence  $\rho : \text{N}^{\text{hc}}(\underline{\text{Kan}}_*) \rightarrow \mathcal{K}an_*$  to obtain a pointed Hom functor for the homotopy coherent nerve  $\mathcal{A}$ , via commutativity of the diagram (18). We refer to this functor as the *canonical pointed Hom functor* for  $\mathcal{A} = \text{N}^{\text{hc}}(\underline{A})$ .

**Proposition 3.13.** *If  $\underline{A}$  is a simplicial category with a strict zero object, then the canonical pointed Hom functor*

$$\underline{\text{Hom}}_{\mathcal{A}} : \mathcal{A}^{\text{op}} \times \mathcal{A} \rightarrow \mathcal{K}an_*$$

*constructed above is a pointed Hom functor for  $\mathcal{A} = \text{N}^{\text{hc}}(\underline{A})$ .*

We are of course most interested in the case of the homotopy  $\infty$ -category  $\mathcal{K}(\mathbb{A})$ . Here we consider the simplicial category  $K\mathbf{Ch}(\mathbb{A})$  which is strictly pointed via the zero complex. We then have the homotopy coherent nerve  $\mathcal{K}(\mathbb{A})' = N^{\mathrm{hc}}(K\mathbf{Ch}(\mathbb{A}))$  which is now equipped with its pointed Hom functor

$$\underline{\mathrm{Hom}}_{\mathcal{K}'} : \mathcal{K}(\mathbb{A})'^{\mathrm{op}} \times \mathcal{K}(\mathbb{A}) \rightarrow \mathcal{K}an_*.$$

We transfer this pointed Hom functor to  $\mathcal{K}(\mathbb{A})$  via the equivalence  $\mathfrak{Z} : \mathcal{K}(\mathbb{A})' \rightarrow \mathcal{K}(\mathbb{A})$  of Theorem II-10.4, and restrict to obtain a pointed Hom functor for the derived  $\infty$ -category  $\mathcal{D}(\mathbb{A}) \subseteq \mathcal{K}(\mathbb{A})$  as well.

**3.5. Homotopy groups via  $\mathcal{K}an_*$ .** Of course, analyses of mapping spaces via their homotopy groups play an essential role in assessing fully faithfulness of functors. We show below that homotopy groups can be calculated via processes which are internal to the  $\infty$ -category  $\mathcal{K}an_*$ .

Let us recall now the geometric realization functor  $|-| : \mathbf{sSet} \rightarrow \mathbf{Top}$ , which is left adjoint to the singular complex functor  $\mathrm{Sing} : \mathbf{Top} \rightarrow \mathbf{sSet}$ . Being a left adjoint, the functor  $|-|$  commutes with colimits, and so one calculates

$$|K| = \mathrm{colim}_{n, \Delta^n \rightarrow K} |\Delta^n|.$$

We have the following fundamental result of Milnor.

thm:milnor

**Theorem 3.14** ([17]). *For any simplicial set  $K$ , the unit map  $K \rightarrow \mathrm{Sing} |K|$  is a weak homotopy equivalence.*

Consequently, for any simplicial set  $K$ , we observe an equivalence of Kan complexes  $\mathrm{Fun}(\mathrm{Sing} |K|, \mathcal{Y}) \rightarrow \mathrm{Fun}(K, \mathcal{Y})$  at arbitrary  $\mathcal{Y}$  in  $\mathcal{K}an$ . In the pointed setting, restriction along the marked point  $k : * \rightarrow K$  provides a Kan fibration

$$k^* : \mathrm{Fun}(K, \mathcal{Y}) \rightarrow \mathrm{Fun}(*, \mathcal{Y}) = \mathcal{Y}$$

by Proposition I-3.11 so that taking the fiber provides an equivalence

$$\mathrm{Fun}_*/(\mathrm{Sing} |K|, \mathcal{Y}) \rightarrow \mathrm{Fun}_*(K, \mathcal{Y}).$$

As a corollary to Proposition 3.11 we now observe an identification of homotopy groups via mapping spaces in the  $\infty$ -category  $\mathcal{K}an_*$ .

cor:pi\_ptd

**Corollary 3.15.** *Take  $\mathbb{S}^n = \mathrm{Sing} |\Delta^n / \partial \Delta^n|$ . For any pointed space  $x : * \rightarrow \mathcal{X}$ , there is a natural bijection*

$$\pi_n(\mathcal{X}, x) \xrightarrow{\cong} \pi_0 \mathrm{Hom}_{\mathcal{K}an_*}(\mathbb{S}^n, \mathcal{X}).$$

*Proof.* Via Theorem 3.14 restricting along the unit map  $\Delta^n / \partial \Delta^n \rightarrow \mathbb{S}^n$  provides an equivalence  $\mathrm{Fun}_*/(\mathbb{S}^n, \mathcal{X}) \xrightarrow{\sim} \mathrm{Fun}_*/(\Delta^n / \partial \Delta^n, \mathcal{X})$ . We have also the natural equivalence

$$\mathrm{Fun}_*/(\mathbb{S}^n, \mathcal{X}) \xrightarrow{\sim} \mathrm{Hom}_{N^{\mathrm{hc}}(\underline{\mathbf{Kan}}_*)}(\mathbb{S}^n, \mathcal{X}) \xrightarrow{\rho} \mathrm{Hom}_{\mathcal{K}an_*}(\mathbb{S}^n, \mathcal{X})$$

which provides a roof of equivalences

$$\mathrm{Fun}_*/(\Delta^n / \partial \Delta^n, \mathcal{X}) \leftarrow \mathrm{Fun}_*/(\mathbb{S}^n, \mathcal{X}) \rightarrow \mathrm{Hom}_{\mathcal{K}an_*}(\mathbb{S}^n, \mathcal{X}).$$



Taking connected components therefore provides an isomorphism

$$\begin{array}{ccc} \pi_0 \operatorname{Fun}_{*/}(\mathbb{S}^n, \mathcal{X}) & \xrightarrow{\cong} & \pi_0 \operatorname{Hom}_{\mathcal{K}an*}(\mathbb{S}^n, \mathcal{X}) \\ \cong \downarrow & \dashrightarrow \exists! & \uparrow \cong \\ \pi_n(\mathcal{X}) = \pi_0 \operatorname{Fun}_{*/}(\Delta^n / \partial \Delta^n, \mathcal{X}) & & \end{array}$$

which is natural in the  $\mathcal{X}$  coordinate.  $\square$

**3.6. Limits and colimits of pointed spaces.** We conclude the section with a study of limits and colimits in the category of pointed sets.

lem:875

**Lemma 3.16.** *Let  $\mathcal{C}$  be an  $\infty$ -category and  $\mathcal{C}_{\text{Term}}$  be the full  $\infty$ -subcategory of terminal objects in  $\mathcal{C}$ . The category  $\mathcal{C}_{\text{Term}}$  is complete and the inclusion  $\mathcal{C}_{\text{Term}} \rightarrow \mathcal{C}$  preserves limits.*

*Proof.* We have  $\mathcal{C}_{\text{Term}} \cong *$  so that it is both complete and cocomplete. To see that the inclusion  $\mathcal{C}_{\text{Term}} \rightarrow \mathcal{C}$  is continuous, it suffices to show that the constant diagram  $\underline{t} : K \rightarrow \mathcal{C}$  at a given terminal object  $t$  is terminal in the full subcategory of constant diagrams in  $\operatorname{Fun}(K, \mathcal{C})$ . We have that the space

$$\operatorname{Fun}(\Delta^1, \mathcal{C}) \times_{\operatorname{Fun}(\partial \Delta^1, \mathcal{C})} \{(x, t)\} = \operatorname{Hom}_{\mathcal{C}}(x, t)$$

is contractible at each  $x$  in  $\mathcal{C}$ , by definition, so that the functor space

$$\begin{aligned} \operatorname{Fun}(K, \operatorname{Hom}_{\mathcal{C}}(x, t)) &= \operatorname{Fun}(\Delta^1 \times K, \mathcal{C}) \times_{\operatorname{Fun}(\partial \Delta^1 \times K, \mathcal{C})} \{(\underline{x}, \underline{t})\} \\ &= \operatorname{Hom}_{\operatorname{Fun}(K, \mathcal{C})}(\underline{x}, \underline{t}) \end{aligned}$$

is contractible as well. Hence  $\underline{t}$  is terminal in  $\operatorname{Fun}(K, \mathcal{C})$ , and we see that the map

$$* \cong \operatorname{Hom}_{\mathcal{C}}(x, t) \rightarrow \operatorname{Hom}_{\operatorname{Fun}}(\underline{x}, \underline{t}) \xrightarrow{(id_{\underline{t}})^*} \operatorname{Hom}_{\operatorname{Fun}}(\underline{x}, \underline{t}) \cong *$$

is an equivalence at each  $x$  in  $\mathcal{C}$ . Therefore  $t$  is a limit for its own constant diagram.  $\square$

prop:pointed\_ccpt

**Proposition 3.17.** (a) *The category  $\mathcal{K}an_*$  is complete.*

(b) *The forgetful functor  $\mathcal{K}an_* \rightarrow \mathcal{K}an$  is continuous.*

(c) *A diagram  $\{0\} \star K \rightarrow \mathcal{K}an_*$  is a limit diagram if and only if its image in  $\mathcal{K}an$  is a limit diagram.*

*Proof.* Via the coslice equivalence

$$\mathcal{K}an_* \xrightarrow{\sim} \{*\} \times_{\mathcal{K}an}^{\text{or}} \mathcal{K}an$$

of Theorem I-10.15 it suffices to show that the oriented fiber product  $\{*\} \times_{\mathcal{K}an}^{\text{or}} \mathcal{K}an$  is complete and that the projection to  $\mathcal{K}an$  both preserves and detects limits.

By Proposition II-13.29 the category  $\operatorname{Fun}(\Delta^1, \mathcal{K}an)$  is complete, and a diagram  $p : \{0\} \star K \rightarrow \operatorname{Fun}(\Delta^1, \mathcal{K}an)$  is a limit diagram if and only if it evaluates to a limit diagram in  $\mathcal{K}an$  at both  $0$  and  $1 : * \rightarrow \Delta^1$ . Supposing that  $0^*p$  takes constant value  $*$ , the evaluation at  $*$  is already a limit diagram by Lemma 3.16. This shows that the fiber

$$\{*\} \times_{\operatorname{Fun}(\{0\}, \mathcal{K}an)} \operatorname{Fun}(\Delta^1, \mathcal{K}an) = \{*\} \times_{\mathcal{K}an}^{\text{or}} \mathcal{K}an$$

is a cocomplete subcategory in  $\operatorname{Fun}(\Delta^1, \mathcal{K}an)$ , and that the projection to  $\mathcal{K}an$ , i.e. the evaluation at  $1$  in  $\Delta^1$ , both preserves and detects limits in the oriented fiber product.  $\square$

Recall that a simplicial set  $K$  is said to be weakly contractible if the terminal map  $K \rightarrow *$  induces an equivalence

$$\mathcal{X} = \mathrm{Fun}(*, \mathcal{X}) \rightarrow \mathrm{Fun}(K, \mathcal{X})$$

at each Kan complex  $\mathcal{X}$ . We record the following result without proof.

**prop:03PK**

**Proposition 3.18** ([15, 03PK]). *The class of weakly contractible simplicial sets is closed under the formation of filtered colimits in  $\mathbf{sSet}$ .*

We also understand from Proposition II-9.24 that any simplicial set with an initial or terminal object is weakly contractible. We therefore observe the following.

**prop:932**

**Proposition 3.19.** *The class of weakly contractible simplicial sets contains the simplicial set  $\Delta^{\mathrm{op}}$ , as well as all filtered simplicial sets (Definition 10.1).*

*Proof.* The category  $\Delta^{\mathrm{op}}$  has the initial object  $[0]$  and is thus weakly contractible by Proposition II-9.24. For a filtered simplicial sets  $K$ , the inclusion of any subset  $K' \rightarrow K$  extends to a map of simplicial sets  $K' \star \{1\} \rightarrow K$ . Hence  $K$  can be written as a filtered colimit of simplicial sets with a terminal object. As any simplicial set with a terminal object is contractible, again by Proposition II-9.24, we see that  $K$  is weakly contractible by Proposition 3.18.  $\square$

**Proposition 3.20** ([15, 02KR]). *Let  $q : \mathcal{E} \rightarrow \mathcal{C}$  be any left fibration of  $\infty$ -categories,  $K$  be a weakly contractible simplicial set, and suppose that  $\mathcal{C}$  admits all  $K$ -indexed colimits.*

- (1)  $\mathcal{E}$  admits all  $K$ -indexed colimits.
- (2) A diagram  $p : K \star \{1\} \rightarrow \mathcal{E}$  is a colimit diagram if and only if the composite  $qp : K \star \{1\} \rightarrow \mathcal{C}$  is a colimit diagram.

*Outline of proof.* Let  $\bar{p} : K \rightarrow \mathcal{E}$  be a diagram. Take  $\bar{p}_0 = q\bar{p} : K \rightarrow \mathcal{C}$ . By a general result [15, 0179] the inclusion  $K \rightarrow K \star \{1\}$  is left anodyne, so that any  $K$ -colimit diagram  $p_0 : K \star \{1\} \rightarrow \mathcal{C}$  lifts uniquely to a diagram  $p : K \star \{1\} \rightarrow \mathcal{E}$  whose restriction to  $K$  recovers  $\bar{p}$ . One applies [15, 02KR] (see also [15, 02KN]) to find that  $p$  is a colimit in  $\mathcal{E}$ .

We now have that  $\mathcal{E}$  admits the proposed colimits and that  $q$  preserves colimits of the given type. Uniqueness of solutions to the the lifting problems

$$\begin{array}{ccc} K & \xrightarrow{\quad} & \mathcal{E} \\ \downarrow & \nearrow & \downarrow q \\ K \star \{1\} & \xrightarrow{\quad} & \mathcal{C} \end{array}$$

however, which we observe from [15, 0179] and Proposition I-3.11, implies that  $q$  detects such colimits in  $\mathcal{E}$  as well.  $\square$

We apply this result to our favorite left fibration  $\mathcal{K}an_* \rightarrow \mathcal{K}an$ .

**Corollary 3.21.** *For any weakly contractible simplicial set  $K$ , the category  $\mathcal{K}an_*$  admits all  $K$ -indexed colimits, and a diagram  $K \star \{1\} \rightarrow \mathcal{K}an_*$  is a colimit diagram if and only if the composite  $K \star \{1\} \rightarrow \mathcal{K}an_* \rightarrow \mathcal{K}an$  is a colimit diagram in  $\mathcal{K}an$ .*

By Proposition 3.19 we specifically observe the following.

cor:ptd\_geom

**Corollary 3.22.** *The category  $\mathcal{K}an_*$  admits geometric realizations, filtered colimits, and pushouts. Furthermore, the forgetful functor  $\mathcal{K}an_* \rightarrow \mathcal{K}an$  both preserves and detects such colimits in  $\mathcal{K}an_*$ .*

*Proof.* All is clear save for pushouts. However, this follows from the fact that the diagram  $K = \Lambda_0^2$  admits an initial object, and is therefore weakly contractible (Proposition II-9.24).  $\square$

**Remark 3.23.** We see below, in Theorem 3.28 that the category  $\mathcal{K}an_*$  in fact admits all small colimits, i.e. is cocomplete. It is not the case, however, that the forgetful functor  $\mathcal{K}an_* \rightarrow \mathcal{K}an$  preserves general colimits. Indeed, the coproduct  $* \amalg *$  in the category  $\mathcal{K}an_*$  is just a point, by Lemma 3.27 below, while in  $\mathcal{K}an$  it is the disjoint union of two points.

From the above information we see that one can detect limits and colimits in a pointed  $\infty$ -category via pointed Hom functors, in addition to the unpointed Hom functors.

**Proposition 3.24.** *Let  $\mathcal{C}$  be an  $\infty$ -category with a zero object, and  $\tilde{H} : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{K}an_*$  be a pointed Hom functor for  $\mathcal{C}$ . For a small diagram  $p : \{0\} \star K \rightarrow \mathcal{C}$  the following are equivalent:*

- (a)  *$p$  is a limit diagram in  $\mathcal{C}$ .*
- (b) *At each object  $x$  in  $\mathcal{C}$ , the diagram  $\tilde{H}(x, -)p : \{0\} \star K \rightarrow \mathcal{K}an_*$  is a limit diagram in  $\mathcal{K}an_*$ .*

*Similarly, for a diagram  $q : K \star \{1\} \rightarrow \mathcal{C}$  the following are equivalent:*

- (a')  *$q$  is a colimit diagram in  $\mathcal{C}$ .*
- (b') *At each object  $y$  in  $\mathcal{C}$ , the diagram  $\tilde{H}(-, y)q : \{1\} \star K^{\text{op}} \rightarrow \mathcal{K}an_*$  is a limit diagram in  $\mathcal{K}an_*$ .*

*Proof.* Follows immediately from Proposition 3.17 and Corollary II-16.16.  $\square$

We similarly observe the following.

prop:1210

**Proposition 3.25.** *Let  $\mathcal{C}$  be an  $\infty$ -category with a zero object,  $\tilde{H} : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{K}an_*$  be a pointed Hom functor for  $\mathcal{C}$ , and  $H$  be the underlying unpointed Hom functor. Suppose  $K$  is filtered, or that  $K = \Delta^{\text{op}}$ . At a given object  $x : * \rightarrow \mathcal{C}$ , the functor  $\tilde{H}(x, -) : \mathcal{C} \rightarrow \mathcal{K}an_*$  preserves  $K$ -indexed colimits if and only if the functor  $H(x, -) : \mathcal{C} \rightarrow \mathcal{K}an$  preserves  $K$ -indexed colimits.*

**Remark 3.26.** Proposition 3.25 is relevant when assessing compactness and/or “projectivity” of objects in an  $\infty$ -category.

**3.7. Completeness and cocompleteness of  $\mathcal{K}an_*$ .** We can also use Proposition 3.11 to see that the category of pointed spaces is cocomplete. We first observe the existence of arbitrary coproducts. In the statement of the following lemma we write, for any collection of pointed spaces  $\{(\mathcal{X}_\lambda, x_\lambda) : \lambda \in \Lambda\}$ ,

$$\mathcal{X}_\Lambda = \coprod_{\lambda \in \Lambda} \mathcal{X}_\lambda \quad \text{and} \quad \mathcal{X}_\Lambda / * = \mathcal{X}_\Lambda / (\coprod_{\lambda \in \Lambda} x_\lambda).$$

lem:ptd\_coprod

**Lemma 3.27.** *The category  $\mathcal{K}an_*$  admits all small coproducts. Specifically, for  $\Lambda$  a small discrete set and  $\mathcal{X}_\gamma : \Lambda \rightarrow \mathcal{K}an_*$  any functor, the maps*

$$i_\lambda = ( \mathcal{X}_\lambda \rightarrow \mathcal{X}_\Lambda \rightarrow \text{Sing} | \mathcal{X}_\Lambda / * | )$$

*realizes the space  $\text{Sing} | \mathcal{X}_\Lambda / * |$  as a colimit for the functor  $\mathcal{X}_\gamma$ .*

*Proof.* Take  $\mathcal{X}_{\Lambda,*} = \text{Sing} |(\coprod_{\lambda} \mathcal{X}_{\lambda}) / *|$ . Via the equivalence  $\text{N}^{\text{hc}}(\mathbf{Kan}_*) \cong \mathcal{K}an_*$  of Proposition 3.11 it suffices to show that the given morphism is a coproduct in the given nerve. From Proposition II-11.6 and Corollary II-16.17 it then suffices to prove that the map

$$[i_{\lambda}^*; \lambda \in \Lambda]^t : \text{Fun}_{*/}(\mathcal{X}_{\Lambda,*}, \mathcal{Y}) \rightarrow \prod_{\lambda} \text{Fun}_{*/}(\mathcal{X}_{\lambda}, \mathcal{Y})$$

is a homotopy equivalence at all pointed spaces  $\mathcal{Y}$ . As the inclusion  $\mathcal{X}_{\Lambda}/* \rightarrow \mathcal{X}_{\Lambda,*}$  is a weak homotopy equivalence [15, 0142], and the structure maps for  $\mathcal{X}_{\Lambda,*}$  factor through  $\mathcal{X}_{\Lambda}/*$ , it then suffices to prove that the map

$$[\text{incl}_{\lambda}^*; \lambda \in \Lambda]^t : \text{Fun}_{*/}(\mathcal{X}_{\Lambda}/*, \mathcal{Y}) \rightarrow \prod_{\lambda} \text{Fun}_{*/}(\mathcal{X}_{\lambda}, \mathcal{Y}) \quad (19) \quad \boxed{\text{eq:1072}}$$

is an equivalence.

The space  $\text{Fun}_{*/}(\mathcal{X}_{\Lambda}/*, \mathcal{Y})$  completes a pullback square

$$\begin{array}{ccc} \text{Fun}_{*/}(\mathcal{X}_{\Lambda}/*, \mathcal{Y}) & \longrightarrow & \text{Fun}(\mathcal{X}_{\Lambda}/*, \mathcal{Y}) \\ \downarrow & & \downarrow x_{\Lambda}^* \\ * & \xrightarrow{y_*} & \mathcal{Y} \end{array}$$

and we have the additional pullback square

$$\begin{array}{ccc} \text{Fun}(\mathcal{X}_{\Lambda}/*, \mathcal{Y}) & \longrightarrow & \text{Fun}(\mathcal{X}_{\Lambda}, \mathcal{Y}) \\ \downarrow & & \downarrow (\Pi_{\lambda} x_{\lambda})^* \\ \mathcal{Y} & \xrightarrow{\text{diag}} & \prod_{\lambda} \mathcal{Y}. \end{array}$$

We therefore observe a pullback square

$$\begin{array}{ccc} \text{Fun}_{*/}(\mathcal{X}_{\Lambda}/*, \mathcal{Y}) & \longrightarrow & \text{Fun}(\mathcal{X}_{\Lambda}, \mathcal{Y}) \\ \downarrow & & \downarrow (\Pi_{\lambda} x_{\lambda})^* \\ * & \xrightarrow{\Pi_{\lambda} y_*} & \prod_{\lambda} \mathcal{Y}. \end{array}$$

We similarly have a pullback square

$$\begin{array}{ccc} \prod_{\lambda} \text{Fun}_{*/}(\mathcal{X}_{\lambda}, \mathcal{Y}) & \longrightarrow & \prod_{\lambda} \text{Fun}(\mathcal{X}_{\lambda}, \mathcal{Y}) \\ \downarrow & & \downarrow \Pi_{\lambda} x_{\lambda}^* \\ * & \xrightarrow{\Pi_{\lambda} y_*} & \prod_{\lambda} \mathcal{Y} \end{array}$$

so that the unpointed isomorphism

$$[\text{incl}_{\lambda}^*; \lambda \in \Lambda]^t : \text{Fun}(\mathcal{X}_{\Lambda}, \mathcal{Y}) \xrightarrow{\cong} \prod_{\lambda} \text{Fun}(\mathcal{X}_{\lambda}, \mathcal{Y})$$

induces an isomorphism  $\text{Fun}_{*/}(\mathcal{X}_{\Lambda}/*, \mathcal{Y}) \xrightarrow{\cong} \prod_{\lambda} \text{Fun}_{*/}(\mathcal{X}_{\lambda}, \mathcal{Y})$  which is explicitly given by the map (19).  $\square$

thm:ptd\_co\_comp

**Theorem 3.28.** *The  $\infty$ -category  $\mathcal{K}an_*$  is both complete and cocomplete.*

*Proof.* Completeness was covered in Proposition 3.17 above. For cocompleteness, we know from Corollary 3.22 and Lemma 3.27 that  $\mathcal{K}an_*$  also admits pushouts and small coproducts. It follows from [13, Proposition 4.4.2.6] that  $\mathcal{K}an_*$  admits all small colimit.  $\square$

sect:spectra\_hom

#### 4. SPECTRAL HOM FUNCTORS

We show that if a pointed  $\infty$ -category  $\mathcal{C}$  is sufficiently symmetric about the zero objects—in particular stable, see Definition 5.5—then any pointed Hom functor for  $\mathcal{C}$  lifts to a functor valued in spectra. In fact, a pointed Hom functor for  $\mathcal{C}$  is more-or-less the same thing as a spectra valued Hom functor. Of course, our first order of business here is to define these strange figures, “spectra”.

Now, to be clear, one can provide a coherent treatment of the derived  $\infty$ -category without ever speaking of spectra. Furthermore, in this text we avoid directly employing spectra in our arguments. However, avoiding spectra while engaging with the literature on this topic is impossible. So, we provide an abridged discussions here and leave it to the interested reader to connect the dots between our treatment and some of the more spectra-forward treatments in the literature. (Somewhat surprisingly, it’s not at all difficult to connect the required dots when needed. . .)

**Remark 4.1.** Our presentation in this section is fairly coarse, as we rely on technical results from both higher topos theory [13] and higher algebra [14] in order to “digest” many of the details.

**4.1. Introduction to spectra.** In short, the  $\infty$ -category of spectra  $\mathcal{S}p$  is the localization of the category of pointed spaces  $\mathcal{S}p = \mathcal{K}an_*[\Omega^{-1}]$  relative to the the looping functor. (See Section I-7.4 and the discussions preceding [14, Remark 1.1.2.6].) In order to meet the demands of actually doing mathematics however, we take a moment to describe this  $\infty$ -category in more detail.

We first consider the full subcategory  $\mathcal{K}an_*^{\text{fin}}$  of  $\mathcal{K}an_*$  which is generated by the one point space  $*$  under finite colimits. We note that we have pushout diagrams

$$\begin{array}{ccc} \mathbb{S}^n & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathbb{S}^{n+1} \end{array} \quad (20) \quad \text{eq:spush}$$

at all  $n$ , so that all spheres appear in  $\mathcal{K}an_*^{\text{fin}}$ . (For details see Example 5.20.) We define the category of spectra as the full subcategory of so-called reduced excisive functors from  $\mathcal{K}an_*^{\text{fin}}$  to  $\mathcal{K}an_*$ .

**Definition 4.2.** Let  $F : \mathcal{K} \rightarrow \mathcal{C}$  be a functor between  $\infty$ -categories, and suppose that  $\mathcal{K}$  admits all finite colimits and a terminal object  $*$ . We call  $F$  excisive if it sends pushout diagrams in  $\mathcal{K}$  to pullback diagrams in  $\mathcal{C}$ . We call  $F$  reduced if  $F(*)$  is terminal in  $\mathcal{C}$ .

For  $\mathcal{K}$  and  $\mathcal{C}$  as above, we let  $\text{Exc}_*(\mathcal{K}, \mathcal{C})$  denote the full  $\infty$ -subcategory in  $\text{Fun}(\mathcal{K}, \mathcal{C})$  spanned by those functors which are both excisive and reduced. We are particularly interested in the case  $\mathcal{K} = \mathcal{K}an_*^{\text{fin}}$ .

**Definition 4.3.** For any  $\infty$ -category  $\mathcal{C}$ , the  $\infty$ -category of spectrum objects in  $\mathcal{C}$  is defined as the  $\infty$ -category of reduced excisive functors from finite pointed spaces

$$\mathcal{S}p(\mathcal{C}) = \text{Exc}_*(\mathcal{K}an_*^{\text{fin}}, \mathcal{C}).$$

In the particular case  $\mathcal{C} = \mathcal{K}an_*$  we take

$$\mathcal{S}p := \mathcal{S}p(\mathcal{K}an_*) = \text{Exc}_*(\mathcal{K}an_*^{\text{fin}}, \mathcal{K}an_*).$$

For any object  $\mathcal{X}_* : \mathcal{K}an_*^{\text{fin}} \rightarrow \mathcal{K}an_*$  in the category of spectra, we consider the values of this functor on the  $n$ -spheres  $\mathcal{X}_n := \mathcal{X}_*(\mathbb{S}^n)$ . From the pushout diagram (20) and excisiveness of  $\mathcal{X}_*$ , we have identifications with the loop spaces  $\mathcal{X}_n = \Omega \mathcal{X}_{n+1}$  at each nonnegative integer  $n$ .

**Definition 4.4.** For each nonnegative integer  $n$  we take  $\Omega^{\infty-n} : \mathcal{S}p \rightarrow \mathcal{K}an_*$  the evaluation functor at the  $n$ -sphere  $\mathbb{S}^n$ , and take in particular  $\Omega^\infty := \Omega^{\infty-0}$ .

**Proposition 4.5.** *The category  $\mathcal{S}p$  is complete and cocomplete, and the forgetful functor  $\Omega^\infty : \mathcal{S}p \rightarrow \mathcal{K}an_*$  is continuous.*

*Sketch proof.* The category  $\mathcal{K}an$  is generated by the subcategory of finite discrete sets under geometric realization. In particular, any space  $\mathcal{X}$  is the colimit of its own simplicial functor  $\mathcal{X} = \text{colim}_{\Delta^{\text{op}}} \mathcal{X}[n]$  and, under this identification, the structure map  $\mathcal{X}[0] \rightarrow \mathcal{X}$  is the natural inclusion. Since the forgetful functor  $\mathcal{K}an_* \rightarrow \mathcal{K}an$  preserves geometric realization (Corollary 3.22) we see similarly that  $\mathcal{K}an_*$  is generated by the subcategory of finite pointed sets under geometric realization. In particular, the category  $\mathcal{K}an_*$  is seen to be presentable. It follows that the category  $\mathcal{S}p$  is presentable as well [14, Proposition 1.4.4.4], and thus complete and cocomplete, and also that the functor  $\Omega^\infty : \mathcal{S}p \rightarrow \mathcal{K}an_*$  admits a left adjoint [14, Proposition 1.4.4.4]. The functor  $\Omega^\infty$  is therefore continuous by Proposition II-13.24.  $\square$

For  $\infty$ -categories  $\mathcal{C}$  and  $\mathcal{D}$  which admit finite limits, we consider the  $\infty$ -category  $\text{Fun}^{\text{lex}}(\mathcal{C}, \mathcal{D})$  of left exact functors. By definition, this is the full subcategory in  $\text{Fun}(\mathcal{C}, \mathcal{D})$  whose objects are functors which preserve finite limits.

`prop:spectra_lift`

**Proposition 4.6** ([14, Corollary 1.4.2.23]). *Suppose  $\mathcal{C}$  is a pointed  $\infty$ -category which admits finite limits, and that pushout diagrams agree with pullback diagrams in  $\mathcal{C}$ . Then the functor*

$$\Omega_*^\infty : \text{Fun}^{\text{lex}}(\mathcal{C}, \mathcal{S}p) \rightarrow \text{Fun}^{\text{lex}}(\mathcal{C}, \mathcal{K}an_*)$$

*is an equivalence of  $\infty$ -categories.*

Now, one can show further that any spectrum is determined by its values on the  $n$ -spheres. To elaborate on this point, let us take  $\mathcal{L}\mathcal{K}an_*$  the limit of the diagram  $\omega : \mathbb{Z}_{\leq 0} \rightarrow \mathcal{C}at_\infty^{\text{big}}$ ,

$$\mathcal{L}\mathcal{K}an_* = \lim(\cdots \rightarrow \mathcal{K}an_* \xrightarrow{\Omega} \mathcal{K}an_* \xrightarrow{\Omega} \mathcal{K}an_*).$$

Here one can take this looping functor as specifically realized by the construction of Section I-7.4 at the level of the simplicial category  $\underline{\mathcal{K}an}_*$ , which then induces a functor on  $\infty$ -categories via an application of the homotopy coherent nerve and Proposition 3.11.

In terms of the explicit formulae given in Section II-14.2, this limit category is the  $\infty$ -category of sections of the weighted nerve

$$\mathcal{L}\mathcal{K}an_* = \text{Fun}_{\mathbb{Z}_{\leq 0}}^{\text{CCart}}(\mathbb{Z}_{\leq 0}, \mathcal{N}^\omega(\mathbb{Z}_{\leq 0})).$$

At each nonpositive integer  $i$ , evaluation provides a structure map

$$p_i = (ev_i)^* : \text{Fun}_{\mathbb{Z}_{\leq 0}}^{\text{CCart}}(\mathbb{Z}_{\leq 0}, \mathcal{N}^\omega(\mathbb{Z}_{\leq 0})) \rightarrow \mathcal{N}^\omega(\mathbb{Z}_{\leq 0})_i = \mathcal{K}an_*$$

For each object  $\mathcal{X}_*$  in the limit, we take  $\mathcal{X}_i = p_{-i}\mathcal{X}_*$  to see that  $\mathcal{X}_*$  is defined by a sequence of pointed spaces  $\{\mathcal{X}_i : i \geq 0\}$  which are equipped with homotopy equivalences  $\Omega\mathcal{X}_{i+1} \xrightarrow{\sim} \mathcal{X}_i$ . A morphism is a sequence of maps  $f_i : \mathcal{X}_i \rightarrow \mathcal{Y}_i$  which fit into commuting diagrams

$$\begin{array}{ccc} \Omega\mathcal{X}_{i+1} & \xrightarrow{\Omega f_{i+1}} & \Omega\mathcal{Y}_{i+1} \\ \sim \downarrow & & \downarrow \sim \\ \mathcal{X}_i & \xrightarrow{f_i} & \mathcal{Y}_i \end{array} \quad (21) \quad \boxed{\text{eq:1318}}$$

in  $\mathcal{K}an_*$ .

`prop:sp_looper`

**Proposition 4.7.** *There is a unique equivalence of  $\infty$ -categories  $t : \mathcal{L}\mathcal{K}an_* \xrightarrow{\sim} \mathcal{S}p$  which fits into a 2-simplex*

$$\begin{array}{ccc} & \mathcal{S}p & \\ t \nearrow & & \searrow \Omega^{\infty-n} \\ \mathcal{L}\mathcal{K}an_* & \xrightarrow{p_{-n}} & \mathcal{K}an_* \end{array}$$

at each nonnegative integer  $n$ .

*Proof outline.* We have the shift inclusion  $-1 : \mathbb{Z}_{\leq 0} \rightarrow \mathbb{Z}_{\leq 0}$  and restricting along  $-1$  provides a functor  $\Sigma : \mathcal{L}\mathcal{K}an_* \rightarrow \mathcal{L}\mathcal{K}an_*$ . This functor sends an object  $\mathcal{X}_*$  to the object  $\Sigma\mathcal{X}_* = \mathcal{X}_{*+1}$ . We also have the looping functor  $\Omega : \mathcal{L}\mathcal{K}an_* \rightarrow \mathcal{L}\mathcal{K}an_*$  which is induced by the transformation

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \mathcal{K}an_* & \xrightarrow{\Omega} & \mathcal{K}an_* & \xrightarrow{\Omega} & \mathcal{K}an_* \\ & & \uparrow \Omega & & \uparrow \Omega & & \uparrow \Omega \\ \cdots & \longrightarrow & \mathcal{K}an_* & \xrightarrow{\Omega} & \mathcal{K}an_* & \xrightarrow{\Omega} & \mathcal{K}an_* \end{array}$$

and one checks that these functors are mutually inverse. It's furthermore argued in [14, Proof of Proposition 1.4.2.24] that the functor  $\Omega$  is the looping functor for  $\mathcal{L}\mathcal{K}an_*$  in the sense of [14, Remark 1.1.2.8]. We have now  $p_{-n} = p_0 \Sigma^n$ .

It is argued in [14, Proof of Proposition 1.4.2.24] that we have a unique equivalence  $t$  which fits into a diagram

$$\begin{array}{ccc} & \mathcal{S}p & \\ t \nearrow & & \searrow \Omega^\infty \\ \mathcal{L}\mathcal{K}an_* & \xrightarrow{p_0} & \mathcal{K}an_* \end{array}$$

For the delooping functor  $S : \mathcal{K}an_*^{\text{fin}} \rightarrow \mathcal{K}an_*^{\text{fin}}$  [14, Remark 1.1.2.6] the corresponding functor  $\Sigma = \text{Fun}(S, \mathcal{K}an_*)$  is the delooping functor on  $\mathcal{S}p$ , so that we have a commutative diagram

$$\begin{array}{ccc} \mathcal{L}\mathcal{K}an_* & \xrightarrow{t} & \mathcal{S}p \\ \Sigma^n \downarrow & \searrow & \downarrow \Sigma^n \\ \mathcal{L}\mathcal{K}an_* & \xrightarrow{t} & \mathcal{S}p \end{array}$$

in  $\mathcal{Cat}_\infty^{\text{big}}$  at each integer  $n$ . This gives an isomorphism of functors  $p_{-n} \cong \Omega^\infty \Sigma^n$ . The latter functor evaluates a spectrum  $\mathcal{X}_*$  at  $S^n(\mathbb{S}^0) \cong \mathbb{S}^n$  (see Example 5.20). We therefore have an isomorphism  $\Omega^\infty \Sigma^n \cong \Omega^{\infty-n}$ , as desired.  $\square$

**Remark 4.8.** Our structural equivalences  $\Omega \mathcal{X}_{n+1} \xrightarrow{\sim} \mathcal{X}_n$  are going in the “wrong direction” relative to standard practice, cf. [1, Section 2]. The direction we’ve employed herein is simply the direction which is implied by the description of the limit given in Section II-14.2.

#### 4.2. Spectral Hom functors.

**Definition 4.9.** An  $\infty$ -category  $\mathcal{C}$  is called finitely complete (resp. cocomplete) if every diagram  $p : K \rightarrow \mathcal{C}$  from a finite simplicial set admits a limit (resp. colimit) in  $\mathcal{C}$ .

We begin with a helpful little lemma.

lem:funlex\_equiv

**Lemma 4.10.** *If a functor  $F : \mathcal{D} \rightarrow \mathcal{D}'$  is an equivalence of finitely complete  $\infty$ -categories, and  $\mathcal{C}$  is finitely complete as well, then the induced functor*

$$F_* : \text{Fun}^{\text{lex}}(\mathcal{C}, \mathcal{D}) \rightarrow \text{Fun}^{\text{lex}}(\mathcal{C}, \mathcal{D}') \quad (22)$$

eq:1246

*is an equivalence.*

*Proof.* The functor  $\text{Fun}(\mathcal{C}, \mathcal{D}) \rightarrow \text{Fun}(\mathcal{C}, \mathcal{D}')$  is an equivalence, and equivalences preserve and detect limit diagrams. It follows that taking the fiber along the inclusion  $\text{incl} : \text{Fun}^{\text{lex}}(\mathcal{C}, \mathcal{D}') \rightarrow \text{Fun}(\mathcal{C}, \mathcal{D}')$  returns the map (22). Since the class of left exact functors is stable under isomorphism, by Proposition II-13.19, we see that the map  $\text{incl}$  is an isofibration. It follows that the map (22) is an equivalence, by Corollary I-6.24 for example.  $\square$

We now observe the following.

prop:spectra\_hom

**Proposition 4.11.** *Suppose  $\mathcal{C}$  is a pointed  $\infty$ -category which admits finite limits, and that pushout diagrams agree with pullback diagrams in  $\mathcal{C}$ . Then for any pointed Hom functor  $\tilde{H} : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{K}an_*$  there is a functor  $\tilde{H}_* : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{S}p$  which fits into a 2-simplex*

$$\begin{array}{ccc} & & \mathcal{S}p \\ & \nearrow \tilde{H}_* & \downarrow \\ \mathcal{C}^{\text{op}} \times \mathcal{C} & \xrightarrow{\tilde{H}} & \mathcal{K}an_* \end{array}$$

*in  $\mathcal{Cat}_\infty$ . Furthermore, this functor  $\tilde{H}_*$  is uniquely determined up to a contractible space of choices.*

*Proof.* Take  $\mathcal{C}^e = \mathcal{C}^{\text{op}} \times \mathcal{C}$  and, for any  $\infty$ -category  $\mathcal{D}$  which admits finite limits, let  $\text{Fun}'(\mathcal{C}^e, \mathcal{D})$  denote the full subcategory of functors which are left exact in each factor independently. The adjunction  $\text{Fun}(\mathcal{C}, \text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{D})) \cong \text{Fun}(\mathcal{C}^e, \mathcal{D})$  then restricts to an adjunction

$$\text{Fun}'(\mathcal{C}^e, \mathcal{D}) \cong \text{Fun}^{\text{lex}}(\mathcal{C}, \text{Fun}^{\text{lex}}(\mathcal{C}^{\text{op}}, \mathcal{D})),$$

where here we note that  $\text{Fun}^{\text{lex}}(\mathcal{C}^{\text{op}}, \mathcal{D})$  is closed under the formation of finite limits in the ambient category  $\text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{D})$  [14, Remark 1.4.2.3]. By Proposition 4.6 and Lemma 4.10 we now see that the functor  $\Omega^\infty$  induces an equivalence

$$\Omega_*^\infty : \text{Fun}'(\mathcal{C}^e, \mathcal{S}p) \xrightarrow{\sim} \text{Fun}'(\mathcal{C}^e, \mathcal{K}an_*). \quad (23)$$

eq:1277



Now, the space of lift  $\tilde{H}_*$  which fit into a 2-simplex as proposed can be parametrized by the homotopy fiber

$$\mathrm{Fun}'(\mathcal{C}^e, \mathcal{S}\mathcal{P}) \times_{\mathrm{Fun}'(\mathcal{C}^e, \mathcal{K}an_*)}^{\mathrm{htop}} \{\tilde{H}\}.$$

Since the map (23) is an equivalence it follows, by Proposition I-6.23, that the projection

$$\mathrm{Fun}'(\mathcal{C}^e, \mathcal{S}\mathcal{P}) \times_{\mathrm{Fun}'(\mathcal{C}^e, \mathcal{K}an_*)}^{\mathrm{htop}} \{\tilde{H}\} \rightarrow *$$

is an equivalence as well. In particular, the fiber in question is a contractible Kan complex.  $\square$

**Remark 4.12.** Categories satisfying the hypotheses of Proposition 4.11 are called stable  $\infty$ -categories. We see in Section 5 that the homotopy and derived  $\infty$ -categories of an abelian category are both stable. Hence the pointed Hom functors from Section 3.4 uniquely enhance to spectra valued Hom functors.

#### 4.3. Whitehead's theorem in the spectral setting.

**Proposition 4.13** ([14, Proposition 1.4.4.4]). *The forgetful functor  $\Omega^\infty : \mathcal{S}\mathcal{P} \rightarrow \mathcal{K}an_*$  admits a left adjoint  $\Sigma^\infty : \mathcal{K}an_* \rightarrow \mathcal{S}\mathcal{P}$ .*

**Definition 4.14.** The  $n$ -th sphere spectrum  $\underline{\mathbb{S}}^n$  is the object  $\Sigma^\infty \mathbb{S}^n$  in  $\mathcal{S}\mathcal{P}$ .

We can define the  $n$ -th homotopy group  $\pi_n(X_*)$  of a spectrum  $X_*$  as the set

$$\pi_n(X_*) = \mathrm{Hom}_{\mathrm{h}\mathcal{S}\mathcal{P}}(\underline{\mathbb{S}}^n, X_*).$$

Via the  $(\Sigma^\infty, \Omega^\infty)$ -adjunction we have the natural identification

$$\pi_n(X_*) = \pi_n(X_0, x_0) \cong \pi_{n+i}(X_i, x_i),$$

where  $x_k : * \rightarrow X_k$  is the implicit pointing on the  $k$ -th space. From the second perspective it is clear that we can define the homotopy groups at negative  $n$ .

**Definition 4.15.** For any spectrum  $\mathcal{X}_*$  we define the  $n$ -th homotopy group, for  $n \in \mathbb{Z}$ , as the colimit

$$\pi_n(X_*) = \mathrm{colim}_{n+i \geq 0} \pi_{n+i}(X_i, x_i).$$

The following is a consequence of Whitehead's theorem for Kan complexes.

thm:sp\_whitehead

**Theorem 4.16** (Spectral Whitehead's theorem). *For a map of spectra  $f : \mathcal{X}_* \rightarrow \mathcal{Y}_*$  the following are equivalent:*

- (a)  *$f$  is an isomorphism in  $\mathcal{S}\mathcal{P}$ .*
- (b) *At each integer  $i \geq 0$ , the induced map  $f_i : \mathcal{X}_i \rightarrow \mathcal{Y}_i$  is an equivalence of Kan complexes.*
- (c) *At each integer  $n \in \mathbb{Z}$ , the map  $\pi_n f : \pi_n(\mathcal{X}_*) \rightarrow \pi_n(\mathcal{Y}_*)$  is an isomorphism.*

*Proof.* We identify the category of spectra with the limit category  $\mathcal{L}\mathcal{K}an_*$  as in Proposition 4.7. From this perspective we have a transformation  $f : \mathcal{X}_* \rightarrow \mathcal{Y}_*$  between functors  $\mathcal{X}_*, \mathcal{Y}_* : \mathbb{Z}_{\leq 0} \rightarrow \mathcal{N}^\omega(\mathbb{Z}_{\leq 0})$ , where  $\omega$  is the diagram

$$\cdots \rightarrow \mathcal{K}an_* \xrightarrow{\Omega} \mathcal{K}an_* \xrightarrow{\Omega} \mathcal{K}an_*$$

in  $\mathcal{C}at_\infty^{\mathrm{big}}$ . Each map  $f_i : \mathcal{X}_i \rightarrow \mathcal{Y}_i$  is obtained by evaluating the transformation  $f$  at  $-i \in \mathbb{Z}_{\leq 0}$ . Hence the equivalence between (a) and (b) follows from the fact that a transformation between functors is an equivalence if and only if it evaluates to an

equivalence at each vertex in the domain, by Proposition I-7.6. The fact that (b) implies (c) is also immediate.

For the implication from (c) to (b), consider a map of Kan complexes  $g : \mathcal{S} \rightarrow \mathcal{T}$  and a point  $s : * \rightarrow \mathcal{S}$  with image  $t = g(s) : * \rightarrow \mathcal{T}$ . Let  $\mathcal{S}_0$  and  $\mathcal{T}_0$  be the components of  $\mathcal{S}$  and  $\mathcal{T}$  containing  $s$  and  $t$  respectively. By Whitehead's theorem (Theorem I-4.15) and base point independence of the homotopy groups [15, 04GF], the restriction  $f|_{\mathcal{S}_0}$  is a homotopy equivalence if and only if each map

$$\pi_m f : \pi_m(\mathcal{S}, s) \rightarrow \pi_m(\mathcal{T}, t),$$

for  $m \geq 0$ , is an equivalence. This occurs if and only if the induced map on loop spaces  $\Omega f : \Omega(\mathcal{S}, s) \rightarrow \Omega(\mathcal{T}, t)$  is a homotopy equivalence.

Now, at a fixed integer  $i$ , the condition that each map  $\pi_n f : \pi_n(\mathcal{X}_*) \rightarrow \pi_n(\mathcal{Y}_*)$  is an isomorphism implies that  $\pi_m(f_{i+1}) : \pi_m(\mathcal{X}_{i+1}) \rightarrow \pi_m(\mathcal{Y}_{i+1})$  is an isomorphism at all nonnegative integers  $m$ . Hence the induced map on loop spaces  $\Omega(f_{i+1}) : \Omega(\mathcal{X}_{i+1}) \rightarrow \Omega(\mathcal{Y}_{i+1})$  is an equivalence. From the diagram (21) we conclude that each map  $f_i : \mathcal{X}_i \rightarrow \mathcal{Y}_i$  is an equivalence.  $\square$

sect:stable

## 5. STABILITY OF HOMOTOPY AND DERIVED $\infty$ -CATEGORIES

Given a Grothendieck abelian category  $\mathbb{A}$ , we prove that the homotopy and derived  $\infty$ -categories,  $\mathcal{K}(\mathbb{A})$  and  $\mathcal{D}(\mathbb{A})$ , are both stable and cocomplete. In a very vague sense, one might view stability as an analog of abelianness in the derived/homotopical setting. From a practical perspective, stability allows one, quite often, to reduce analyses at the level of  $\infty$ -categories to corresponding analyses at the discrete level, via an application of the homotopy category functor.

### 5.1. Pullbacks are pushouts in the homotopy $\infty$ -category.

lem:6102

**Lemma 5.1.** *For a diagram*

$$\begin{array}{ccc} V_0 & \xrightarrow{g} & V_1 \\ g' \downarrow & & \downarrow f \\ V'_1 & \xrightarrow{f'} & V_2 \end{array} \quad (24) \quad \text{eq:5863}$$

in  $\text{Ch}(\mathbb{A})$  the following are equivalent:

- (a) *The map  $f$  is (termwise split) surjective,  $f'$  is injective, and (24) is a discrete pullback diagram.*
- (b) *The map  $g$  is (termwise split) injective,  $g'$  is surjective, and (24) is a pushout diagram.*

*Proof.* (a)  $\Rightarrow$  (b) If (24) is a pullback diagram with the prescribed properties then the map  $g$  is an inclusion which identifies  $V_0$  as a kernel of the composite  $V_1 \rightarrow V_2 \rightarrow V_2/V'_1$ . This identification of  $V_0$  with the kernel of  $f$  also tells us that  $g$  is termwise split whenever  $f$  is termwise split. Furthermore, in this case the map  $g' : V_0 \rightarrow V'_1$  is simply the restriction of the projection  $V_1 \rightarrow V_2$  to  $V_0$ , and hence  $g'$  is surjective as well. So we see that  $g$  is injective and  $g'$  is surjective in this case, and  $g$  is split when  $f$  is split.

The implication (b)  $\Rightarrow$  (a) is recovered by applying (a)  $\Rightarrow$  (b) to the category  $\text{Ch}(\mathbb{A}^{\text{op}}) = \text{Ch}(\mathbb{A})^{\text{op}}$ .  $\square$

prop:K\_pullpush

**Proposition 5.2.** *For any abelian category  $\mathbb{A}$ , a diagram*

$$\begin{array}{ccc} V_0 & \xrightarrow{g} & V_1 \\ g' \downarrow & \searrow & \downarrow f \\ V'_1 & \xrightarrow{f'} & V_2 \end{array} \quad (25) \quad \text{eq:6126}$$

in  $\mathcal{K}(\mathbb{A})$  is a limit (aka pullback) diagram if and only if it is a colimit (aka pushout) diagram.

*Proof.* Suppose that the diagram (25) is a limit diagram. Then, according to Proposition 2.13, we can assume (25) is a strictly commuting, discrete pullback diagram in which  $f'$  is termwise split surjective. In particular, any pullback diagram in  $\mathcal{K}(\mathbb{A})$  is isomorphic to such a discrete pullback diagram. In this case (25) is also a discrete pushout diagram in which  $g$  is termwise split injective, by Lemma 5.1. By Proposition 2.16 such a discrete pushout diagram is a colimit diagram in  $\mathcal{K}(\mathbb{A})$ , so that the diagram (25) is a colimit diagram in the homotopy  $\infty$ -category.

The converse implication is proved similarly. Namely, if (25) is a colimit diagram then we can assume it is a discrete pushout diagram with  $g$  termwise split injective, by Proposition 2.16, at which point it is seen to be a limit diagram as well by Lemma 5.1 and Proposition 2.20.  $\square$

## 5.2. Fibers and cofibers.

**Definition 5.3.** Let  $\mathcal{C}$  be an  $\infty$ -category which admits a zero object  $0$ . A diagram of the form

$$\begin{array}{ccc} x & \xrightarrow{\alpha} & y \\ \downarrow & \searrow & \downarrow \beta \\ 0 & \longrightarrow & z \end{array} \quad (26) \quad \text{eq:1670}$$

is called a fiber sequence in  $\mathcal{C}$  if it is a limit diagram, and a cofiber sequence if it is a colimit diagram. Given a fiber (resp. cofiber) sequence of the form (26) we refer to the object  $x$  as the fiber of the morphism  $\beta$  (resp.  $z$  as the cofiber of the morphism  $\alpha$ ).

Under any reasonable interpretation of the terms, the fiber of a morphism in an  $\infty$ -category  $\mathcal{C}$  is a “kernel” for the given morphism, and the cofiber of a morphism is a “cokernel”. Let us examine this perspective in the case of the cofiber.

Consider the diagram  $p : \Lambda_0^2 \rightarrow \mathcal{C}$  obtained from (26) by deleting  $z$ , and let  $w$  be an arbitrary object in  $\mathcal{C}$ . We have an equivalence of mapping spaces

$$\text{Hom}_{\mathcal{C}}(z, w) \xrightarrow{\sim} \text{Hom}_{\text{Fun}(\Lambda_0^2, \mathcal{C})}(p, \underline{w}) = \{p\} \times_{\text{Fun}(\Lambda_0^2, \mathcal{C})}^{\text{or}} \{w\},$$

simply by the definition of the colimit, and the latter space is identified with the fiber of the undercategory  $\mathcal{C}_{p/}$  over  $w$  by Theorem I-10.15. We therefore obtain an equivalence

$$\text{Hom}_{\mathcal{C}}(z, w) \xrightarrow{\sim} \mathcal{C}_{p/} \times_{\mathcal{C}} \{w\}.$$

Now, the space  $\mathcal{C}_{p/} \times_{\mathcal{C}} \{w\}$  parametrizes maps  $\zeta : y \rightarrow w$  which are equipped with a trivialization of the composite along  $\alpha : x \rightarrow y$ . In this way the cofiber has the universal property one expects of a “cokernel” for the morphism  $\alpha$ .

In the discrete setting, our vague associations with kernels and cokernels become precise.

**Example 5.4.** Let  $\mathbb{A}$  be a discrete additive category. A diagram in  $\mathbb{A}$  of the form

$$\begin{array}{ccc} x & \xrightarrow{\alpha} & y \\ \downarrow & \searrow & \downarrow \beta \\ 0 & \longrightarrow & z \end{array}$$

is a fiber sequence if and only if  $\alpha$  is a kernel of  $\beta$ , and is a cofiber sequence if and only if  $\beta$  is a cokernel of  $\alpha$ .

### 5.3. Stable $\infty$ -categories and exact functors.

**def:stable**

**Definition 5.5.** An  $\infty$ -category  $\mathcal{C}$  is called stable if the following properties hold:

- (a)  $\mathcal{C}$  admits a zero object.
- (b) Every morphism in  $\mathcal{C}$  extends to both a fiber sequence and a cofiber sequence.
- (c) A diagram of the form

$$\begin{array}{ccc} x & \xrightarrow{\alpha} & y \\ \downarrow & \searrow & \downarrow \beta \\ 0 & \longrightarrow & z \end{array}$$

in  $\mathcal{C}$  is a fiber sequence if and only if it is a cofiber sequence.

**Definition 5.6.** A functor between stable  $\infty$ -categories  $F : \mathcal{C} \rightarrow \mathcal{D}$  is exact if  $F$  preserves zero objects and also preserves fiber/cofiber sequences. A full  $\infty$ -subcategory  $\mathcal{C}' \subseteq \mathcal{C}$  is called a stable  $\infty$ -subcategory if  $\mathcal{C}'$  is stable and the inclusion  $\mathcal{C}' \rightarrow \mathcal{C}$  is exact.

Speaking liberally, a full subcategory  $\mathcal{C}'$  in a stable  $\infty$ -category  $\mathcal{C}$  is a stable subcategory if it is closed under the formation of fibers and cofibers in  $\mathcal{C}$ . We note that stability is a property, rather than a structure. For example, if  $F : \mathcal{C} \rightarrow \mathcal{D}$  is an equivalence of  $\infty$ -categories, and one of  $\mathcal{C}$  or  $\mathcal{D}$  is stable, then both  $\mathcal{C}$  and  $\mathcal{D}$  are stable and  $F$  is exact.

**Lemma 5.7.** *Suppose  $F : \mathcal{C} \rightarrow \mathcal{D}$  is an equivalence of  $\infty$ -categories and that  $\mathcal{C}$  is stable. Then  $\mathcal{D}$  is stable.*

*Proof.* Omitted. □

The following is obvious.

**Lemma 5.8.** *If  $\mathcal{C}$  is stable then the opposite category  $\mathcal{C}^{\text{op}}$  is also stable.*

The quintessential stable  $\infty$ -category is the category of  $\mathcal{S}_{\mathcal{P}}$  of spectra.

**Theorem 5.9.** *The category  $\mathcal{S}_{\mathcal{P}}$  of spectra is stable and has zero object  $0 = \Sigma^{\infty}*$ .*

*Proof.* By definition, the category of spectra is the stabilization  $\mathcal{S}_{\mathcal{P}} = \mathcal{S}_{\mathcal{P}}(\mathcal{H}an_*)$  of the  $\infty$ -category of pointed spaces. Since the category of pointed spaces is complete, by Proposition 3.17, it follows that  $\mathcal{S}_{\mathcal{P}}$  is stable by [14, Corollary 1.4.2.17]. As for the calculation of the zero object, the left adjoint  $\Sigma^{\infty} : \mathcal{H}an_* \rightarrow \mathcal{S}_{\mathcal{P}}$  commutes with colimits, so that it preserves the colimit over the empty diagram, i.e. the initial object. By Proposition II-9.15 the object  $*$  is initial in  $\mathcal{H}an_*$ , so that  $\Sigma^{\infty}*$  is an initial in  $\mathcal{S}_{\mathcal{P}}$ . □

#### 5.4. Stability of homotopy and derived $\infty$ -categories.

thm:K\_stable

**Theorem 5.10.** *For any abelian category  $\mathbb{A}$ , the homotopy  $\infty$ -category  $\mathcal{K}(\mathbb{A})$  is stable.*

*Proof.* The zero complex is a zero object in  $\mathcal{K}(\mathbb{A})$  by Corollary 2.23, and  $\mathcal{K}(\mathbb{A})$  admits all pullbacks and pushouts by Corollaries 2.14 and Corollary 2.21. In particular, one can complete any morphism in  $\mathcal{K}(\mathbb{A})$  to both a fiber and cofiber sequence. Finally, fiber sequences and cofiber sequences in  $\mathcal{K}(\mathbb{A})$  agree by Proposition 5.2.  $\square$

By the description of pullbacks and pushouts in  $\mathcal{K}(\mathbb{A})$  provided in Propositions 2.13 and 2.20 we see that any full subcategory  $\mathcal{K} \subseteq \mathcal{K}(\mathbb{A})$  which is preserved under desuspension and the formation of mapping cones admits all pushouts and pullbacks. In particular, the inclusion  $\mathcal{K} \rightarrow \mathcal{K}(\mathbb{A})$  preserves pushouts and pullbacks. Thus, under these conditions, and assuming additionally that  $\mathcal{K}$  contains the zero complex, we see that  $\mathcal{K}$  is a stable  $\infty$ -category as well.

cor:K\_stable

**Corollary 5.11.** *Suppose a full  $\infty$ -subcategory  $\mathcal{K} \subseteq \mathcal{K}(\mathbb{A})$  contains the zero complex, is closed under the formation of mapping cones, and is closed under the desuspension automorphism  $\Sigma^{-1}$ . Then  $\mathcal{K}$  is a stable  $\infty$ -subcategory in  $\mathcal{K}(\mathbb{A})$ . In particular,  $\mathcal{K}$  is stable.*

In the case of a Grothendieck abelian category, we have the injective construction of the (unbounded) derived  $\infty$ -category  $\mathcal{D}(\mathbb{A})$ , where  $\mathcal{D}(\mathbb{A})$  is identified specifically as the full  $\infty$ -subcategory of  $K$ -injectives in  $\mathcal{K}(\mathbb{A})$  (Definition I-2.10). As a particular instance of Corollary 5.11 we observe stability of  $\mathcal{D}(\mathbb{A})$ .

cor:D\_stable

**Corollary 5.12.** *If  $\mathbb{A}$  is a Grothendieck abelian category then the derived  $\infty$ -category  $\mathcal{D}(\mathbb{A})$  is stable.*

*Proof.* We construct the derived  $\infty$ -category  $\mathcal{D}(\mathbb{A})$  specifically as the  $\infty$ -subcategory of  $K$ -injectives in  $\mathcal{K}(\mathbb{A})$ . For any map between  $K$ -injectives  $f : I \rightarrow I'$ , and arbitrary acyclic  $X$ , we have

$$\mathrm{Hom}_{\mathbb{A}}^*(X, \mathrm{cone}(f)) = \mathrm{cone}(f_* : \mathrm{Hom}_{\mathbb{A}}^*(X, I) \rightarrow \mathrm{Hom}_{\mathbb{A}}^*(X, I')).$$

Since both complexes in the latter mapping cone are acyclic. So we see that  $\mathrm{cone}(f)$  is  $K$ -injective. By the above reasoning we conclude that the derived  $\infty$ -category is stable.  $\square$

**Remark 5.13.** As stated previously, stability of an  $\infty$ -category is a property not a structure. So, in the event that a Grothendieck abelian category  $\mathbb{A}$  has enough projectives, stability holds whether one constructs the derived  $\infty$ -category via  $K$ -injectives or  $K$ -projectives. See Section I-13.

We similarly apply Corollary 5.11 to observe stability of the derived  $\infty$ -category under all standard bounding restrictions. We have the  $\infty$ -subcategories  $\mathcal{D}^b(\mathbb{A})$ ,  $\mathcal{D}^-(\mathbb{A})$ , and  $\mathcal{D}^+(\mathbb{A})$  of bounded, bounded above, and bounded below complexes.

**Corollary 5.14.** *If  $\mathbb{A}$  is a Grothendieck abelian category then for  $\star = b, +, -$ , the appropriately bounded derived  $\infty$ -category  $\mathcal{D}^\star(\mathbb{A}) \subseteq \mathcal{D}(\mathbb{A})$  is a stable  $\infty$ -subcategory. Furthermore, in the case where  $\mathbb{A}$  is locally finite, the full  $\infty$ -subcategory  $\mathcal{D}(\mathbb{A})_{\mathrm{fin}}$  of complexes with finite total length is a stable  $\infty$ -subcategory in  $\mathcal{D}(\mathbb{A})$ .*

To be clear, by a locally finite Grothendieck abelian category we mean a compactly generated abelian category in which the compact objects form an abelian subcategory, and in which all compact objects are of finite length. Such categories include representations over an affine algebraic group, and locally finite modules over a  $C_2$ -cofinite vertex operator algebra, for example.

**Example 5.15** (An anti-example). We consider the connective derived  $\infty$ -category  $\mathcal{D}^{\leq 0}(\mathbb{A})$ , i.e. the full subcategory in  $\mathcal{D}(\mathbb{A})$  spanned by those complexes whose cohomology vanishes in all positive degrees. We claim that the  $\infty$ -category  $\mathcal{D}^{\leq 0}(\mathbb{A})$  is not not stable. Explicitly, for a nonzero complex  $V$  which is concentrated in degree 0, we claim that the discrete pullback diagram

$$\begin{array}{ccc} 0 & \longrightarrow & 0 \\ \downarrow & \searrow & \downarrow \\ 0 & \longrightarrow & V \end{array} \quad (27) \quad \boxed{\text{eq:859}}$$

is a pullback diagram in  $\mathcal{D}^{\leq 0}(\mathbb{A})$  which is not a pushout.

To see this, note that the truncation functor  $\tau_0 : \mathcal{D}(\mathbb{A}) \rightarrow \mathcal{D}^{\leq 0}(\mathbb{A})$  (Definition 2.8) is right adjoint to the inclusion  $\mathcal{D}^{\leq 0}(\mathbb{A}) \rightarrow \mathcal{D}(\mathbb{A})$  (Theorem I-14.9). Truncation therefore preserves limits by Proposition II-13.24, so that we obtain the above limit diagram by truncating the standard limit diagram

$$\begin{array}{ccc} \Sigma^{-1}V & \longrightarrow & 0 \\ \downarrow & \searrow & \downarrow \\ 0 & \longrightarrow & V \end{array}$$

in  $\mathcal{D}(\mathbb{A})$ . However, we see simultaneously that the diagram (27) is not a pushout diagram when  $V$  is nonzero, since the inclusion  $\mathcal{D}^{\leq 0}(\mathbb{A}) \rightarrow \mathcal{D}(\mathbb{A})$  preserves colimits and hence the image of any pushout diagram in  $\mathcal{D}^{\leq 0}(\mathbb{A})$  must be a pushout diagram in  $\mathcal{D}(\mathbb{A})$ .

**5.5. Suspension and desuspension.** Fix  $\mathcal{C}$  an  $\infty$ -category which admits a zero object. We first sketch a construction the suspension and desuspension functors, following [14, Section 1.1.2]: Let  $\mathcal{M}^{\Sigma} = \mathcal{M}_{\mathcal{C}}^{\Sigma}$  and  $\mathcal{M}^{\Omega} = \mathcal{M}_{\mathcal{C}}^{\Omega}$  denote the full subcategories

$$\mathcal{M}^{\Sigma}, \mathcal{M}^{\Omega} \subseteq \text{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$$

whose objects are diagrams

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0' & \longrightarrow & Y \end{array}$$

which are pushout diagrams, in the case of  $\mathcal{M}^{\Sigma}$ , and pullback diagrams, in the case of  $\mathcal{M}^{\Omega}$ . Here 0 and 0' are arbitrary zero objects in  $\mathcal{C}$ .

We have the evaluation functors at the initial and terminal vertices

$$ev_0 : \mathcal{M}^{\Sigma} \rightarrow \mathcal{C} \quad \text{and} \quad ev_1 : \mathcal{M}^{\Omega} \rightarrow \mathcal{C}, \quad (28) \quad \boxed{\text{eq:1529}}$$

which are both isofibrations. The following is an application of [13, Proposition 4.3.2.15]. See the discussion preceding [14, Remark 1.1.2.6].

**Proposition 5.16.** *For  $\mathcal{C}$  a pointed  $\infty$ -category which admits arbitrary pushouts and pullbacks, the evaluation functors of (28) are both trivial Kan fibrations.*

We can now define the suspension and looping operations as endofunctors on  $\mathcal{C}$ .

**Definition 5.17.** Let  $\mathcal{C}$  be a pointed  $\infty$ -category which admits all pullbacks and pushouts. A suspension functor  $\Sigma : \mathcal{C} \rightarrow \mathcal{C}$  is any composite

$$\mathcal{C} \xrightarrow{s_0} \mathcal{M}_{\mathcal{C}}^{\Sigma} \xrightarrow{ev_1} \mathcal{C},$$

where  $s_0$  is a section of the trivial fibration  $ev_0$ . Similarly, a looping functor  $\Omega : \mathcal{C} \rightarrow \mathcal{C}$  is any composite

$$\mathcal{C} \xrightarrow{s_1} \mathcal{M}_{\mathcal{C}}^{\Omega} \xrightarrow{ev_0} \mathcal{C},$$

where  $s_1$  is a section of  $ev_1$ .

We note that suspension and desuspension are uniquely determined, up to a contractible space of choices.

**Example 5.18.** We saw in Theorem 3.28 that the category of pointed spaces  $\mathcal{K}an_*$  is both complete and cocomplete. It furthermore has the zero object  $*$  provided by the one point space, by Proposition II-9.15. We therefore have the suspension and looping functors  $\Sigma, \Omega : \mathcal{K}an_* \rightarrow \mathcal{K}an_*$ . While the suspension functor is slightly mysterious, the looping functor can be realized explicitly on objects as the assignment

$$\Omega : (\mathcal{X}, x) \mapsto \text{Hom}_{\mathcal{X}}(x, x),$$

according to the materials of Section I-7.4. If we replace  $\mathcal{K}an_*$  with its more rigid model  $\text{N}^{\text{hc}}(\mathbf{Kan}_*)$ , then  $\Omega$  can also be clearly defined on morphisms,

$$\Omega : (F : \mathcal{X} \rightarrow \mathcal{Y}) \mapsto (F_* : \text{Hom}_{\mathcal{X}}(x, x) \rightarrow \text{Hom}_{\mathcal{Y}}(y, y)).$$

In general, the suspensions and loops functors are adjoint. Specifically, the suspension functor is left adjoint to the loops functor [14, Remark 1.1.2.8]. However, we are most interested in the stable setting. In this case fiber and cofiber diagrams are identified, so that the looping and delooping spaces are identified

$$\mathcal{M}^{\Sigma} = \mathcal{M}^{\Omega} =: \mathcal{M},$$

and we observe the following.

**Proposition 5.19.** *If  $\mathcal{C}$  is stable, then the endofunctors  $\Sigma, \Omega : \mathcal{C} \rightarrow \mathcal{C}$  are mutually inverse equivalences.*

*Proof.* In this case the two evaluation maps are defined on the same space,  $ev_i : \mathcal{M} \rightarrow \mathcal{C}$ . Since the evaluation morphisms are equivalences so are their sections  $s_i$ . Furthermore each  $s_i$  is an inverse to  $ev_i$ . So we have directly

$$\Sigma\Omega = ev_1 s_0 ev_0 s_1 \cong id_{\mathcal{C}}, \quad \Omega\Sigma = ev_0 s_1 ev_1 s_0 \cong id_{\mathcal{C}}.$$

□

We cover another example.

ex:susphere

**Example 5.20** (Suspensions of spheres). Consider the  $n$ -sphere  $\mathbb{S}^n = \Sigma^\infty \mathbb{S}^n$  in  $\mathcal{S}\mathcal{P}$ . Since the functor  $\Sigma^\infty$  is a left adjoint, and hence commutes with colimits, it suffices to compute the pushout

$$\begin{array}{ccc} \mathbb{S}^n & \longrightarrow & * \\ \downarrow & \searrow & \downarrow \\ * & \longrightarrow & \mathcal{X}_{\mathbb{S}^n} \end{array}$$

in the category  $\mathcal{K}an_*$ . By Corollary 3.22 it suffices further to compute such pushouts in the unpointed category  $\mathcal{K}an$ . By the explicit formula given in Section II-14.5, we have  $\mathcal{X}_{\mathbb{S}^n} = Q(N^p(\Lambda_2^0)) = Q(\mathbb{S}^n \star \partial\Delta^1)$  where  $Q(K)$  denotes any weak homotopy replacement for the given simplicial set [15, 00UV]. This weak homotopy replacement can be given as the singular complex of the geometric realization, at which point one calculates

$$\mathcal{X}_{\mathbb{S}^n} = \text{Sing} |\mathbb{S}^n \star \partial\Delta^1| \cong \mathbb{S}^{n+1}.$$

Since the pushout is only defined up to isomorphism, we can take simply  $\mathcal{X}_{\mathbb{S}^n} = \mathbb{S}^{n+1}$ .

If we adopt the explicit expression given by the singular complex  $\mathbb{S}^k = \text{Sing } S^k$ , then the inclusion  $S^n \rightarrow S^{n+1}$  along with the two contractions onto the north and south poles provide an explicit pushout diagram

$$\begin{array}{ccc} \text{Sing } S^n & \longrightarrow & * \\ \downarrow & \searrow & \downarrow \scriptstyle \infty \\ * & \xrightarrow{-\infty} & \text{Sing } S^{n+1}. \end{array}$$

This pushout diagram determines an isomorphism  $\Sigma \mathbb{S}^n \rightarrow \mathbb{S}^{n+1}$  which is unique up to a contractible space of choices.

One notes that the above analysis generalizes to provide a calculation of the suspension  $\Sigma \mathcal{X}_*$  at any spectrum  $\mathcal{X}_*$  which is in the image of the functor  $\Sigma^\infty$ . As a corollary to our example we find the following.

**Corollary 5.21.** *At any non-negative integer  $n$ , there are isomorphisms*

$$\Sigma(\mathbb{S}^n) \cong \mathbb{S}^{n+1}, \quad \Omega(\mathbb{S}^{n+1}) \cong \mathbb{S}^n, \quad \text{and} \quad \Sigma^n(\mathbb{S}^0) \cong \mathbb{S}^n.$$

**5.6. Suspension and desuspension for cochains.** Let  $\mathbb{A}$  be an additive category, and  $\Sigma : \mathbf{Ch}(\mathbb{A}) \rightarrow \mathbf{Ch}(\mathbb{A})$  be the shift automorphism on the dg category of cochains. On objects  $\Sigma V$  is the expected shifted complex, and on mapping complexes we take specifically

$$\Sigma : \text{Hom}_{\mathbb{A}}^*(V, W) \rightarrow \text{Hom}_{\mathbb{A}}^*(\Sigma V, \Sigma W), \quad f \mapsto (-1)^{\deg(f)} f.$$

def:shift

**Definition 5.22.** For an additive discrete category  $\mathbb{A}$ , we let  $\Sigma : \mathcal{K}(\mathbb{A}) \rightarrow \mathcal{K}(\mathbb{A})$  denote the autoequivalence on the homotopy  $\infty$ -category provided by taking the dg nerve of the shift automorphism on the dg category of  $\mathbb{A}$ -cochains. Similarly, we let  $\Sigma^{-1} : \mathcal{K}(\mathbb{A}) \rightarrow \mathcal{K}(\mathbb{A})$  denote the dg nerve of the inverse to the shift automorphism.

Of course, we claim that  $\Sigma$  and  $\Sigma^{-1}$  realize suspension and desuspension for  $\mathcal{K}(\mathbb{A})$ .



prop:shift\_chains

**Proposition 5.23.** *Let  $\mathbb{A}$  be an additive discrete category  $\mathbb{A}$ . The functors  $\Sigma$  and  $\Sigma^{-1}$  from Definition 5.22 provide, respectively, suspension and desuspension functors for the homotopy  $\infty$ -category  $\mathcal{K}(\mathbb{A})$ .*

*Proof.* We prove that  $\Sigma$  is a suspension functor. The proof for  $\Sigma^{-1}$  is similar.

Let  $c : \mathbf{Ch}(\mathbb{A}) \rightarrow \mathbf{Ch}(\mathbb{A})$  be the dg functor which sends each complex  $X$  to  $c(V) = \text{cone}(id_V)$  and each homogeneous map  $f : X \rightarrow Y$  to the map

$$\begin{bmatrix} \Sigma f & 0 \\ 0 & f \end{bmatrix} : \text{cone}(id_V) \rightarrow \text{cone}(id_V).$$

The inclusion  $\iota_V : V \rightarrow c(V)$  and projection  $\pi_V : c(V) \rightarrow \Sigma V$  provide natural transformations of dg functors

$$id_{\mathbf{Ch}(\mathbb{A})} \xrightarrow{\iota} c \xrightarrow{\pi} \Sigma,$$

and the composite is the zero transformation  $0 : id_{\mathbf{Ch}(\mathbb{A})} \rightarrow \Sigma$ . We therefore have induced transformations between the associated functors on the dg nerve

$$I, \Pi, 0 : \Delta^1 \times \mathcal{K}(\mathbb{A}) \rightarrow \mathcal{K}(\mathbb{A}),$$

as in Proposition I-14.6. By Lemma I-14.8 these transformations furthermore provide the faces for a diagram  $\Theta_{01} : \Delta^2 \times \mathcal{K}(\mathbb{A}) \rightarrow \mathcal{K}(\mathbb{A})$  which evaluates at each complex  $V$  to the strictly commuting diagram

$$\begin{array}{ccc} & \text{cone}(V) & \\ \iota_V \nearrow & & \searrow \pi_V \\ V & \xrightarrow{0} & \Sigma V. \end{array}$$

We now consider the zero functor  $z : \mathbf{Ch}(\mathbb{A}) \rightarrow \mathbf{Ch}(\mathbb{A})$ ,  $z(V) = 0$ , and induced functor  $Z : \mathcal{K}(\mathbb{A}) \rightarrow \mathcal{K}(\mathbb{A})$ . We have the unique dg transformations  $id_{\mathbf{Ch}(\mathbb{A})} \rightarrow z$  and  $z \rightarrow \Sigma$  whose composite is the zero transformation  $0 : id_{\mathbf{Ch}(\mathbb{A})} \rightarrow \Sigma$ . We again have the induced transformations and subsequent 2-simplex  $\Theta_{10} : \Delta^2 \times \mathcal{K}(\mathbb{A}) \rightarrow \mathcal{K}(\mathbb{A})$  which evaluates at each complex to the strictly commuting diagram

$$\begin{array}{ccc} V & \xrightarrow{0} & \Sigma V \\ & \searrow & \nearrow \\ & 0 & \end{array}.$$

By the specific claim of Lemma I-14.8 these two simplices glue along the diagonal to provide a diagram

$$\Theta : \Delta^1 \times \Delta^1 \times \mathcal{K}(\mathbb{A}) \rightarrow \mathcal{K}(\mathbb{A}) \tag{29}$$

eq:1978

which evaluates at each complex  $V$  to the strictly commuting diagram

$$\begin{array}{ccc} V & \xrightarrow{\iota_V} & \text{cone}(id_V) \\ \downarrow & \searrow 0 & \downarrow \pi_V \\ 0 & \xrightarrow{\quad} & \Sigma V. \end{array}$$

By Proposition 2.16 the above diagram is a pushout diagram in  $\mathcal{K}(\mathbb{A})$ , and by contractibility of  $\text{cone}(id_V)$  it is furthermore an object in the full subcategory

$$\mathcal{M} = \mathcal{M}^\Sigma \subseteq \text{Fun}(\Delta^1 \times \Delta^1, \mathcal{K}(\mathbb{A})).$$

Hence the map (29) defines a functor  $\Theta : \mathcal{K}(\mathbb{A}) \rightarrow \mathcal{M}^\Sigma$  which evaluates at the  $(0, 0)$  vertex to recover the identity,

$$ev_0 \Theta = id_{\mathcal{K}(\mathbb{A})} : \mathcal{K}(\mathbb{A}) \rightarrow \mathcal{K}(\mathbb{A}).$$

Hence the composite  $ev_1 \Theta : \mathcal{K}(\mathbb{A}) \rightarrow \mathcal{K}(\mathbb{A})$  is a suspension functor for  $\mathcal{K}(\mathbb{A})$ . By construction this composite recovers the shift autoequivalence  $\Sigma : \mathcal{K}(\mathbb{A}) \rightarrow \mathcal{K}(\mathbb{A})$ .  $\square$

Given a stable subcategory  $\mathcal{C}' \subseteq \mathcal{C}$ , and suspension functor  $\Sigma$  for  $\mathcal{C}$ ,  $\Sigma$  restricts to provide a suspension functor for  $\mathcal{C}'$  whenever  $\Sigma$  preserves objects in  $\mathcal{C}'$ . Hence Proposition 5.23 tells us that the shift operations at the level of the dg category provide suspension and desuspension functors for the derived  $\infty$ -category as well.

**Corollary 5.24.** *Let  $\mathbb{A}$  be a Grothendieck abelian category. The functors  $\Sigma$  and  $\Sigma^{-1}$  from Definition 5.22 restrict to provide, respectively, suspension and desuspension functors for the derived  $\infty$ -category  $\mathcal{D}(\mathbb{A})$ .*

### 5.7. Suspension and exact functors.

prop:2007

**Proposition 5.25.** *Let  $\mathcal{C}$  and  $\mathcal{D}$  be  $\infty$ -categories which admit zero objects, and which admit all pullbacks and pushouts. Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor which preserves zero objects.*

- (1) *If  $F$  respects fiber sequences (resp. cofiber sequences) then there is a uniquely associated isomorphism*

$$F \circ \Omega \xrightarrow{\sim} \Omega \circ F \quad (\text{resp. } F \circ \Sigma \xrightarrow{\sim} \Sigma \circ F).$$

- (2) *For any simplicial set  $K$ , map  $F : K \times \mathcal{C} \rightarrow \mathcal{D}$  for which  $F_x = F|_{\{x\} \times \mathcal{C}}$  respects fiber sequences (resp. cofiber sequences) at each vertex  $x : * \rightarrow K$ , and collection of isomorphisms  $\zeta_x$  as in (1), there is a uniquely associated isomorphism*

$$F \circ (\Delta^n \times \Omega) \xrightarrow{\sim} \Omega \circ F \quad (\text{resp. } F \circ (\Delta^n \times \Sigma) \xrightarrow{\sim} \Sigma \circ F)$$

*which evaluates to  $\zeta_x$  at each vertex  $x$  in  $K$ .*

The above commutativity isomorphisms are constructed in the proof.

**Construction 5.25.** Let us consider the case where  $F$  preserves fiber sequences. Given an  $\infty$ -category  $\mathcal{A}$ , let  $\mathcal{M}_{\mathcal{A}} = \mathcal{M}_{\mathcal{A}}^\Omega$  denote the full subcategory in  $\text{Fun}(\Delta^1 \times \Delta^1, \mathcal{A})$  spanned by fiber diagrams  $D$  in which both  $D|_{(0,1)}$  and  $D|_{(1,0)}$  are zero objects. We recall that the evaluation map  $ev_1 = ev_{(1,1)} : \mathcal{M}_{\mathcal{A}} \rightarrow \mathcal{A}$  is a trivial Kan fibration whenever  $\mathcal{A}$  admits a zero object and all pullbacks. For  $\mathcal{A} = \mathcal{C}$  and  $\mathcal{D}$  we suppose we've chosen sections  $s_1 : \mathcal{A} \rightarrow \mathcal{M}_{\mathcal{A}}$  from which we define the looping functors  $\Omega = ev_0 s_1$ .

- (1) In our setting the map  $F$  induces a functor  $\mathcal{M}_F : \mathcal{M}_{\mathcal{C}} \rightarrow \mathcal{M}_{\mathcal{D}}$  which fits into diagrams

$$\begin{array}{ccc} \mathcal{M}_{\mathcal{C}} & \xrightarrow{\mathcal{M}_F} & \mathcal{M}_{\mathcal{D}} \\ ev_i \downarrow & & \downarrow ev_i \\ \mathcal{C} & \xrightarrow{F} & \mathcal{D}. \end{array}$$

Now, we have the two functors  $\mathcal{M}_F s_1, s_1 F : \mathcal{C} \rightarrow \mathcal{M}_{\mathcal{D}}$  which both map to  $F$  under the trivial Kan fibration

$$(ev_1)_* : \text{Fun}(\mathcal{C}, \mathcal{M}_{\mathcal{D}}) \rightarrow \text{Fun}(\mathcal{C}, \mathcal{D}).$$

Hence the identity on  $F$  lifts to a natural isomorphism  $t : \mathcal{M}_F s_1 \xrightarrow{\sim} s_1 F$ . From  $t$  we now obtain the desired isomorphism

$$ev_0(t) : F\Omega = F ev_0 s_1 = ev_0 \mathcal{M}_F s_1 \xrightarrow{\sim} ev_0 s_1 F = \Omega F. \quad (30)$$

eq:2034

(2) We have the exponentiation map  $-_* : \text{Fun}(\mathcal{C}, \mathcal{D}) \rightarrow \text{Fun}(\mathcal{C}^L, \mathcal{D}^L)$  of Proposition II-13.21 which, at  $L = \Delta^1 \times \Delta^1$ , sends the  $K$ -shaped diagram  $F : K \times \mathcal{C} \rightarrow \mathcal{D}$  to a map

$$F_* : K \times \text{Fun}(\Delta^1 \times \Delta^1, \mathcal{C}) \rightarrow \text{Fun}(\Delta^1 \times \Delta^1, \mathcal{D}).$$

By construction, the exponential map satisfies  $F_*|_{\{x\} \times \text{Fun}} = (F_x)_*$  at each index vertex  $x$  in  $K$ . It now follows from our hypotheses on the  $F_x$ , and the naturality claim of Proposition II-13.21, that  $F_*$  restricts to a functor  $\mathcal{M}_F : K \times \mathcal{M}_{\mathcal{C}} \rightarrow \mathcal{M}_{\mathcal{D}}$  which fits into diagrams

$$\begin{array}{ccc} K \times \mathcal{M}_{\mathcal{C}} & \xrightarrow{\mathcal{M}_F} & \mathcal{M}_{\mathcal{D}} \\ K \times ev_i \downarrow & & \downarrow ev_i \\ K \times \mathcal{C} & \xrightarrow{F} & \mathcal{D} \end{array}$$

at  $i = 1, 2$ . Since  $ev_1$  is a trivial Kan fibration the map  $K \times ev_1$  is a trivial Kan fibration as well.

Fix now isomorphisms  $t_x : \mathcal{M}_{F_x} s_1 \xrightarrow{\sim} s_1 F_x$  at each vertex  $x$  so that  $\zeta_x$  is obtained from  $t_x$  via the composite (30). Let  $\bar{t}$  denote the induced transformation on the zero skeleton

$$\bar{t} : \Delta^1 \times \text{sk}_0(K) \times \mathcal{C} \rightarrow \mathcal{M}_{\mathcal{D}}.$$

By construction the composite  $ev_0(\bar{t})$  is the identity transformation on  $F|_{\text{sk}_0(K)}$ . For  $A = \text{sk}_0(K)$  we now have the trivial Kan fibration

$$\text{Fun}(K \times \mathcal{C}, \mathcal{M}_{\mathcal{D}}) \rightarrow \text{Fun}(A \times \mathcal{C}, \mathcal{M}_{\mathcal{D}}) \times_{\text{Fun}(A \times \mathcal{C}, \mathcal{D})} \text{Fun}(K \times \mathcal{C}, \mathcal{D})$$

provided by restriction and  $ev_1$  (Proposition I-3.11), and lift the natural isomorphism  $(\bar{t}, id_F)$  to obtain an isomorphism  $t : \mathcal{M}_F s_1 \xrightarrow{\sim} s_1 F$  which restricts to  $t_x$  at each vertex  $x$  in  $K$ . From  $t$  we obtain the desired natural isomorphism

$$ev_0(t) : F(K \times \Omega) = F(K \times ev_0)(K \times s_1) = ev_0 \mathcal{M}_F(K \times s_1) \xrightarrow{\sim} ev_0 s_1 F = \Omega F.$$

□

Applying Proposition 5.25 to the stable setting yields the following.

**Corollary 5.26.** *For any exact functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  between stable  $\infty$ -categories there are natural isomorphisms  $F\Omega \xrightarrow{\sim} \Omega F$  and  $F\Sigma \xrightarrow{\sim} \Sigma F$ .*

We also apply Proposition 5.25 to the case where  $F$  is a pointed Hom functor.

prop:891

**Proposition 5.27.** *For any stable  $\infty$ -category  $\mathcal{C}$ , and any pairs of objects  $x, y : * \rightarrow \mathcal{C}$ , we have canonical isomorphisms*

$$\text{can}_{\Sigma}^n : \Omega^n \text{Hom}_{\mathcal{C}}(x, y) \xrightarrow{\sim} \text{Hom}_{\mathcal{C}}(\Sigma^n x, y) \text{ and } \text{can}_{\Omega}^n : \Omega^n \text{Hom}_{\mathcal{C}}(x, y) \xrightarrow{\sim} \text{Hom}_{\mathcal{C}}(x, \Omega^n y)$$

in the homotopy category  $\mathbf{h}\mathcal{K}an_*$ . Furthermore, for any exact functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  these isomorphisms fit into diagrams

$$\begin{array}{ccc} \Omega^n \mathrm{Hom}_{\mathcal{D}}(Fx, Fy) & \xrightarrow{\mathrm{can}^n} & \mathrm{Hom}_{\mathcal{D}}(\Sigma^n Fx, Fy) \\ \uparrow F & & \uparrow F \\ \Omega^n \mathrm{Hom}_{\mathcal{C}}(x, y) & \xrightarrow{\mathrm{can}^n} & \mathrm{Hom}_{\mathcal{C}}(\Sigma^n x, y) \end{array}$$

and

$$\begin{array}{ccc} \Omega^n \mathrm{Hom}_{\mathcal{D}}(Fx, Fy) & \xrightarrow{\mathrm{can}^n} & \mathrm{Hom}_{\mathcal{D}}(Fx, \Omega^n Fy) \\ \uparrow F & & \uparrow F \\ \Omega^n \mathrm{Hom}_{\mathcal{C}}(x, y) & \xrightarrow{\mathrm{can}^n} & \mathrm{Hom}_{\mathcal{C}}(x, \Omega^n y), \end{array}$$

which commute in  $\mathbf{h}\mathcal{K}an_*$ .

Here, to be clear,  $\Omega \mathrm{Hom}_{\mathcal{D}}(a, b)$  is the space of pointed loops at the zero map (Section I-7.4).

*Proof.* One obtains the claim about suspension  $\Sigma$  from that of  $\Omega$  by taking opposites. So it suffices to prove the claim about  $\Omega$ . Furthermore, the claim at  $n > 1$  is obtained from the claim at  $n = 1$  via iteration. So we consider the case of a single desuspension.

Consider a pointed Hom functor  $H_{\mathcal{C}}$  for  $\mathcal{C}$ . As  $H_{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \rightarrow \mathcal{K}an_*$  is left exact in both coordinates, we have a natural isomorphism

$$H_{\mathcal{C}}(x, \Omega -) \xrightarrow{\sim} \Omega H_{\mathcal{C}}(x, -)$$

at each object  $x$  in  $\mathcal{C}$ , by Proposition 5.25 (1). All naturality claims now follow by Lemma 5.25 (2).  $\square$

**Remark 5.28.** Let us be clear that Proposition 5.27 is completely redundant for those who wish to employ spectra in their analysis of stable categories. It provides, however, a means of *circumventing* spectra for those who may be less familiar with the subject, and provides simultaneously an indication of the kinds of information which spectra encode.

## 6. MORE ON STABLE $\infty$ -CATEGORIES

We continue our study of stable  $\infty$ -categories. We discuss completeness and cocompleteness, triangulation of the homotopy category, and exact equivalences. We prove that the unbounded homotopy and derived  $\infty$ -categories are complete and cocomplete.

**6.1. Limits and colimits in stable  $\infty$ -categories.** We provide an overview of limits and colimits in the stable setting, following directly [13, 14]. To begin, we record the following beautiful result.

`prop:stable_finfo`

**Proposition 6.1** ([14, Propositions 1.1.3.4 & 1.1.4.1]). *Any stable  $\infty$ -category admits all finite limits and colimits. Furthermore, for a functor between stable  $\infty$ -categories  $F : \mathcal{C} \rightarrow \mathcal{D}$  the following are equivalent:*

- (a)  *$F$  is exact, i.e. preserves zero objects and fiber sequences.*
- (b)  *$F$  preserves all finite limits.*

(c)  $F$  preserves all finite colimits.

Recall that, in our study of the homotopy  $\infty$ -category, we found that pullbacks and pushouts agree in  $\mathcal{K}(\mathbb{A})$  (Proposition 5.2). It turns out that this type of symmetry holds in any stable  $\infty$ -category.

**Proposition 6.2** ([14, Proposition 1.1.3.4]). *A diagram*

$$\begin{array}{ccc} x & \longrightarrow & y \\ \downarrow & \searrow & \downarrow \\ y' & \longrightarrow & z \end{array}$$

*in a stable  $\infty$ -category  $\mathcal{C}$  is a pullback diagram if and only if it is a pushout diagram.*

It is also the case that products and coproducts agree in the stable setting. This identification of products and coproducts reflects a natural additive structure—or even, triangulated structure—which exists at the level of the homotopy category. See Theorem 6.11 below.

**Proposition 6.3** ([14, Lemma 1.1.2.9]). *Any stable  $\infty$ -category  $\mathcal{C}$  admits finite products and coproducts, and for any pair of object  $x, y : * \rightarrow \mathcal{C}$  the map*

$$\begin{bmatrix} id_x & 0 \\ 0 & id_y \end{bmatrix} : x \amalg y \rightarrow x \times y$$

*is an isomorphism in  $\mathcal{C}$ .*

Now, we are especially interested in cocompleteness and cocontinuity in the stable setting. In considering such issues the following result from [13] proves invaluable.

thm:push\_cocomp

**Theorem 6.4.** *For an  $\infty$ -category  $\mathcal{C}$  the following are equivalent:*

- (a)  $\mathcal{C}$  admits pushouts and small coproducts.
- (b)  $\mathcal{C}$  is cocomplete.

*Furthermore, for any functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  between cocomplete  $\infty$ -categories, the following are equivalent.*

- (a')  $F$  commutes with pushouts and small coproducts.
- (b')  $F$  is cocontinuous.

*Proof.* The equivalences between (a) and (b), and (a') and (b'), are [13, Proposition 4.4.3.2] and [13, Proposition 4.4.2.7] respectively.  $\square$

We apply Proposition 6.1 and Theorem 6.4 to observe the following.

cor:stable\_cocomp

**Corollary 6.5.** *A stable  $\infty$ -category  $\mathcal{C}$  is cocomplete if and only if  $\mathcal{C}$  admits all small coproducts. Furthermore, an exact functor between stable, cocomplete  $\infty$ -categories is cocontinuous if and only if it commutes with small coproducts.*

**Remark 6.6.** Applying Theorem 6.4 and Corollary 6.5 to the opposite category  $\mathcal{C}^{\text{op}}$  recovers analogous characterizations for cocompleteness and continuity. We take this point for granted below.

ex:K\_cocomp

**Example 6.7** (Completeness and cocompleteness for  $\mathcal{K}(\mathbb{A})$ ). For Grothendieck abelian  $\mathbb{A}$ , we saw in Proposition 2.22 that  $\mathcal{K}(\mathbb{A})$  admits all small products and coproducts. Hence  $\mathcal{K}(\mathbb{A})$  is both complete and cocomplete by Corollary 6.5.

## 6.2. Compact objects.

**Definition 6.8.** An  $\infty$ -category  $\mathcal{K}$  is called filtered if any map  $A \rightarrow \mathcal{K}$  from a finite simplicial set  $A$  admits an extension to the cone  $A \star \{1\} \rightarrow \mathcal{K}$ .

As in the usual discrete setting, one has the expected notion of compact objects in an  $\infty$ -category.

**Definition 6.9.** Let  $\mathcal{C}$  be a cocomplete  $\infty$ -category. An object  $x$  in  $\mathcal{C}$  is called compact if there is a functor  $h^x : \mathcal{C} \rightarrow \mathcal{K}an$  which is represented by  $x$  and which preserves small filtered colimits.

Of course, since all functors which are represented by  $x$  are isomorphic, there exists such a functor  $h^x$  which preserves filtered colimits if and only if all functors which are represented by  $x$  preserve small filtered colimits.

`prop:compact`

**Proposition 6.10** ([14, Proposition 1.4.4.1]). *Let  $\mathcal{C}$  be a cocomplete, stable  $\infty$ -category. An object  $x : * \rightarrow \mathcal{C}$  is compact if and only if, for each small coproduct  $\coprod_{\lambda \in \Lambda} y_\lambda$  and map  $\alpha : x \rightarrow \coprod_{\lambda \in \Lambda} y_\lambda$ , there exists a finite subset  $\{\lambda_0, \dots, \lambda_m\} \subseteq \Lambda$  for which  $\alpha$  factors as a composite*

$$x \rightarrow (y_{\lambda_0} \amalg \dots \amalg y_{\lambda_m}) \rightarrow \coprod_{\lambda \in \Lambda} y_\lambda.$$

**6.3. The homotopy category under stability.** We record the following for the sake of completeness.

`thm:triangles`

**Theorem 6.11** ([14, Theorem 1.1.2.14]). *For any stable  $\infty$ -category  $\mathcal{C}$ , the homotopy category  $h\mathcal{C}$  inherits a natural triangulated structure in which the exact triangles*

$$x \rightarrow y \rightarrow z$$

*are exactly the images of fiber sequences in  $\mathcal{C}$ . The connecting morphism  $\delta : z \rightarrow \Sigma x$  for such a triangle is provided by the universal property of the pushout applied to the diagram*

$$\begin{array}{ccc} x & \xrightarrow{\quad} & y \\ \downarrow & \searrow & \downarrow 0 \\ 0 & \xrightarrow{\quad} & \Sigma x. \end{array}$$

One can check directly that this “natural” triangulated structure on  $h\mathcal{C}$  recovers the standard triangulated structure on the discrete homotopy category  $K(\mathbb{A}) = h\mathcal{K}(\mathbb{A})$  and discrete derived category  $D(\mathbb{A}) = h\mathcal{D}(\mathbb{A})$ . (See for example [23, Sections 10.2 & 10.4].) We leave this as an exercise for the interested reader.

**Corollary 6.12.** *Any exact functor between stable  $\infty$ -categories  $F : \mathcal{C} \rightarrow \mathcal{D}$  induces an exact functor of triangulated categories  $hF : h\mathcal{C} \rightarrow h\mathcal{D}$ .*

Additionally, Proposition 6.10 tells us that compactness of objects in a stable  $\infty$ -category can be checked at the level of the homotopy category. We recall that an object  $x$  in a triangulated category  $\mathbb{A}$  which admits small coproducts is called compact if the functor  $\mathrm{Hom}_{\mathbb{A}}(x, -)$  commutes with small coproducts. The following is simply a repackaging of Proposition 6.10.

`cor:compact`

**Corollary 6.13.** *An object  $x$  in a stable  $\infty$ -category  $\mathcal{C}$  is compact if and only if its image in the homotopy category  $h\mathcal{C}$  is compact.*

**6.4. Fully faithfulness via the homotopy category.** Of course, it is not the case, generally speaking, that one can detect equivalences between  $\infty$ -categories at the level of the homotopy category. Consider, for example, the inclusion  $0 : * \rightarrow \text{Sing}(S^2)$  of a point into the circle. We have  $\text{hSing}(S^2) = *$ , so that the map  $0$  induces an equivalence on homotopy categories. However, this map is not an equivalence since, using Proposition I-7.11, we have

$$\pi_1 \text{Hom}_{\text{Sing}(S^2)}(0, 0) \cong \pi_2 S^2 = \mathbb{Z}.$$

In the stable setting such phenomena never occurs, as all of the higher homotopy groups in the mapping spaces  $\text{Hom}_{\mathcal{C}}(x, y)$  are realized as the 0-th homotopy group of some shifted space  $\text{Hom}_{\mathcal{C}}(\Sigma^n x, y)$ . We have the following fundamental fact.

`prop:ff_hff`

**Proposition 6.14** ([2, Proposition 5.10]). *For an exact functor between stable  $\infty$ -categories  $F : \mathcal{C} \rightarrow \mathcal{D}$ , the following are equivalent:*

- (a)  *$F$  is fully faithful (resp. an equivalence).*
- (b)  *$\text{h}F : \text{h}\mathcal{C} \rightarrow \text{h}\mathcal{D}$  is fully faithful (resp. an equivalence).*

*Proof.* It is clear that if  $F$  is fully faithful, or an equivalence, then the map on homotopy categories  $\text{h}F$  is also fully faithful, or an equivalence. So we have the implication (a)  $\Rightarrow$  (b). For the converse claim (b)  $\Rightarrow$  (a), essential surjectivity can be checked at the level of the homotopy category. So we need only deal with fully faithfulness.

Suppose that  $\text{h}F$  is fully faithful. Let  $H_{\mathcal{C}}$  and  $H_{\mathcal{D}}$  be pointed Hom functors for  $\mathcal{C}$  and  $\mathcal{D}$  respectively, and  $H_F : H_{\mathcal{C}} \rightarrow H_{\mathcal{D}}$  be the transformation induced by  $F$ . We have that the maps

$$\pi_0(H_F) : \pi_0(H_{\mathcal{C}}(x, y)) \rightarrow \pi_0(H_{\mathcal{D}}(Fx, Fy))$$

are isomorphisms at all  $x$  and  $y$  in  $\mathcal{C}$ . By applying shifts, and considering Proposition 5.27, it follows that the maps

$$\pi_n(H_F) : \pi_n(H_{\mathcal{C}}(x, y)) \rightarrow \pi_n(H_{\mathcal{D}}(Fx, Fy))$$

are isomorphisms at all  $n \geq 0$  as well, where we again base at 0. Hence the induced map on loop spaces  $\Omega H_F : \Omega H_{\mathcal{C}}(x, y) \rightarrow \Omega H_{\mathcal{D}}(Fx, Fy)$  is a homotopy equivalence, by Whitehead's theorem. By replacing  $y$  with  $\Sigma y$ , and considering the natural isomorphisms of Proposition 5.25, we see now that  $H_F$  itself is an equivalence at all  $x$  and  $y$  in  $\mathcal{C}$ . Hence  $F$  is fully faithful.  $\square$

**Remark 6.15.** If one employs spectral, rather than pointed, Hom functors, then the proof of Proposition 6.14 follows immediately from spectral Whitehead.

**6.5. Cocompleteness of the derived  $\infty$ -category.** Fix a Grothendieck abelian category  $\mathbb{A}$ . As explained in the proof of Corollary 5.12, the class of  $K$ -injectives in  $\text{Ch}(\mathbb{A})$  is stable under the formation of mapping cones. It is also clearly stable under suspension and desuspension. So, as was already argued implicitly, the full  $\infty$ -subcategory of  $K$ -injectives  $\mathcal{D}(\mathbb{A})$  in  $\mathcal{K}(\mathbb{A})$  is stable under the formation of both pullbacks and pushouts. This follows by the explicit constructions provided in Propositions 2.13 and 2.20 above.

`prop:D_pullpush`

**Proposition 6.16.** *Let  $\mathbb{A}$  be a Grothendieck abelian category. The derived  $\infty$ -category  $\mathcal{D}(\mathbb{A})$  admits all pullbacks and pushouts, and the inclusion  $\mathcal{D}(\mathbb{A}) \rightarrow \mathcal{K}(\mathbb{A})$  preserves all pullback and pushout diagrams.*

Recall from example 6.7 that  $\mathcal{K}(\mathbb{A})$  is both complete and cocomplete. Since products of acyclic complexes are acyclic, it is clear that products of  $K$ -injectives are  $K$ -injective. It follows that the derived  $\infty$ -category  $\mathcal{D}(\mathbb{A})$  is stable under the formation of small products in  $\mathcal{K}(\mathbb{A})$ , and in particular admits all small products. By Corollary 6.5 it follows that  $\mathcal{D}(\mathbb{A})$  is complete and that the inclusion  $\mathcal{D}(\mathbb{A}) \rightarrow \mathcal{K}(\mathbb{A})$  is continuous. We now consider coproducts and cocompleteness.

`prop:D_co_comp`

**Proposition 6.17.** *For any Grothendieck abelian category  $\mathbb{A}$ , the derived  $\infty$ -category  $\mathcal{D}(\mathbb{A})$  is both complete and cocomplete, and the inclusion  $\mathcal{D}(\mathbb{A}) = \mathcal{D}_{\text{Inj}} \rightarrow \mathcal{K}(\mathbb{A})$  is continuous.*

*Proof.* Completeness was discussed above, as was continuity of the inclusion. For cocompleteness, it suffices to show that  $\mathcal{D}(\mathbb{A})$  admits small coproducts, by Corollary 6.5.

Consider any collection of  $K$ -injectives  $I_- : \Lambda \rightarrow \mathcal{D}(\mathbb{A})$  indexed over a small discrete set  $\Lambda$ . Take a  $K$ -injective resolution of the resulting coproduct  $\kappa : \bigoplus_{\lambda \in \Lambda} I_{\lambda} \rightarrow I$  and for each index  $\lambda$  let  $\kappa_{\lambda} : I_{\lambda} \rightarrow I$  be the composition of the structural map  $I_{\lambda} \rightarrow \bigoplus_{\lambda \in \Lambda} I_{\lambda}$  with  $\kappa$ .

Take  $\mathcal{D}(\mathbb{A})'$  the simplicial construction of  $\mathcal{D}(\mathbb{A})$ , with specified equivalence  $\mathfrak{z} : \mathcal{D}(\mathbb{A})' \rightarrow \mathcal{D}(\mathbb{A})$  as in Theorem II-10.4. It suffices to show that the extension  $i : \Lambda \star \{*\} \rightarrow \mathcal{D}(\mathbb{A})'$  of  $I_-$ , with cone point  $I$  and connecting maps provided by the  $\kappa_{\lambda}$ , is a limit diagram. We check this by composing with the corepresentable functor  $\underline{\text{Hom}}_{\mathbb{A}}(-, J) = \text{K Hom}_{\mathbb{A}}^*(-, J)$  at any  $K$ -injective complex  $J$ .

Applying such  $\underline{\text{Hom}}_{\mathbb{A}}(-, J)$  to the strictly commuting diagram

$$\begin{array}{ccccc}
 \cdots & & I_{\mu'} & & I_{\mu} & & I_{\mu''} & & \cdots \\
 & & \searrow & & \downarrow & & \swarrow & & \\
 & & & & \bigoplus_{\lambda} I_{\lambda} & & & & \\
 & & \searrow & & \downarrow \kappa & & \swarrow & & \\
 & & & & I & & & & 
 \end{array}$$

in  $\mathcal{K}(\mathbb{A})'$  produces a strictly commuting diagram

$$\begin{array}{ccccc}
 & & \underline{\text{Hom}}(I, J) & & \\
 & \swarrow & \downarrow \cong & \searrow & \\
 & \underline{\text{Hom}}(\bigoplus_{\lambda} I_{\lambda}, J) = \prod_{\lambda} \underline{\text{Hom}}(I_{\lambda}, J) & & & \\
 \swarrow & & \downarrow & & \searrow \\
 \cdots \underline{\text{Hom}}_{\mathbb{A}}(I_{\mu'}, J) & & \underline{\text{Hom}}_{\mathbb{A}}(I_{\mu}, J) & & \underline{\text{Hom}}_{\mathbb{A}}(I_{\mu''}, J) \cdots
 \end{array} \tag{31}$$

`eq:6280`

in  $\mathcal{K}an$ . Since the bottom diagram in (31) is a limit diagram (Example II-14.16 & Theorem II-14.25), it follows that the top diagram is a limit diagram as well. In particular, the diagram (31) establishes an isomorphism between  $\prod_{\lambda} \underline{\text{Hom}}_{\mathbb{A}}(I_{\lambda}, J)$  and  $\underline{\text{Hom}}_{\mathbb{A}}(I, J)$  in the overcategory  $\mathcal{K}an_{/\underline{\text{Hom}}(\Lambda, J)}$  so that  $\underline{\text{Hom}}_{\mathbb{A}}(I, J)$  is seen to be terminal in the overcategory, since the product  $\prod_{\lambda} \underline{\text{Hom}}_{\mathbb{A}}(I_{\lambda}, J)$  is terminal.

Since  $J$  was chosen arbitrarily, it follows by Corollary II-16.16 that the original diagram  $i : \Lambda \star \{*\} \rightarrow \mathcal{D}(\mathbb{A})'$  is a colimit diagram, and hence that  $I$  is a coproduct



of the  $\{I_\lambda : \lambda \in \Lambda\}$  in  $\mathcal{D}(\mathbb{A})'$ . Applying the equivalence  $\mathfrak{Z} : \mathcal{D}(\mathbb{A})' \rightarrow \mathcal{D}(\mathbb{A})$ , we find that the diagram

$$\begin{array}{ccccc} \cdots & I_{\mu'} & I_\mu & I_{\mu''} & \cdots \\ & \searrow \kappa_{\mu'} & \downarrow \kappa_\mu & \swarrow \kappa_{\mu''} & \\ & & I & & \end{array}$$

realizes  $I$  as a coproduct of the  $\{I_\lambda : \lambda \in \Lambda\}$  in  $\mathcal{D}(\mathbb{A})$  as well.  $\square$

We also observe completeness and cocompleteness of the connective derived category.

**Corollary 6.18.** *Consider the connective derived category  $\mathcal{D}^{\leq 0}(\mathbb{A})$  for  $\mathbb{A}$  Grothendieck abelian. The following hold:*

- (1) *The category  $\mathcal{D}^{\leq 0}(\mathbb{A})$  is both complete and cocomplete.*
- (2) *The inclusion  $\mathcal{D}^{\leq 0}(\mathbb{A}) \rightarrow \mathcal{D}(\mathbb{A})$  is cocontinuous.*

*Proof.* For completeness, the inclusion  $i : \mathcal{D}^{\leq 0}(\mathbb{A}) \rightarrow \mathcal{D}(\mathbb{A})$  has a right adjoint which is given by the truncation  $\tau_0 : \mathcal{D}(\mathbb{A}) \rightarrow \mathcal{D}^{\leq 0}(\mathbb{A})$ . This follows as an application of Theorem I-14.9 for example. By Proposition II-13.24 the functor  $\tau_0$  is continuous, and hence any diagram  $p : K \rightarrow \mathcal{D}^{\leq 0}(\mathbb{A})$  has a limit which can be computed as

$$\lim(p) = \tau_0(\lim(ip)).$$

So we see that the connective derived category of complete.

For cocompleteness, and cocontinuity of the inclusion  $i$ , we observe directly from the formula for pushouts in  $\mathcal{D}(\mathbb{A})$  given in Proposition 2.20 that  $\mathcal{D}^{\leq 0}(\mathbb{A})$  is stable under the formation of pushouts in  $\mathcal{D}(\mathbb{A})$ . Similarly, from the formula for the coproduct given in the proof of Proposition 6.17, we also see that  $\mathcal{D}^{\leq 0}(\mathbb{A})$  is stable under the formation of small coproducts in  $\mathcal{D}(\mathbb{A})$ . By Theorem 6.4 it follows that  $\mathcal{D}^{\leq 0}(\mathbb{A})$  is cocomplete and that the inclusion  $\mathcal{D}^{\leq 0}(\mathbb{A}) \rightarrow \mathcal{D}(\mathbb{A})$ .  $\square$

**6.6. Cocompleteness via the homotopy category.** If one accepts that the discrete derived category  $D(\mathbb{A})$  admits both small coproducts and products, then we can alternatively approach the proofs of Proposition 6.17 via the homotopy category.

prop:cocomp\_via\_h

**Proposition 6.19.** *A stable  $\infty$ -category  $\mathcal{C}$  is cocomplete (resp. complete) if and only if its homotopy category  $h\mathcal{C}$  admits all small coproducts (resp. products).*

*Proof.* We address cocompleteness. Completeness follows by considering the opposite category.

Let  $\{x_\lambda : \lambda \in \Lambda\}$  be a small collection of objects in  $\mathcal{C}$ , and let  $x_\Lambda$  be a coproduct for this collection in  $h\mathcal{C}$  along with the structure maps  $\bar{i}_\lambda : x_\lambda \rightarrow x_\Lambda$ . Let  $i_\lambda : x_\lambda \rightarrow x_\Lambda$  be a lift of each  $\bar{i}_\lambda$  to a map in  $\mathcal{C}$ . Fix a pointed Hom functor  $H_{\mathcal{C}}$  for  $\mathcal{C}$  as well.

At each  $y$  in  $\mathcal{C}$  the maps  $i_\lambda$  induce isomorphisms

$$\pi_0[i_\lambda^* : \lambda \in \Lambda]^t : \pi_0(H_{\mathcal{C}}(x_\Lambda, y)) \rightarrow \prod_{\lambda \in \Lambda} \pi_0(H_{\mathcal{C}}(x_\lambda, y)) = \pi_0\left(\prod_{\lambda \in \Lambda} H_{\mathcal{C}}(x_\lambda, y)\right),$$

by hypothesis. Via the identifications of Proposition 5.27 it follows that the induced maps on all higher homotopy groups

$$\pi_n[i_\lambda^* : \lambda \in \Lambda]^t : \pi_n(H_{\mathcal{C}}(x_\Lambda, y)) \rightarrow \prod_\lambda \pi_n(H_{\mathcal{C}}(x_\lambda, y)) = \pi_n\left(\prod_\lambda H_{\mathcal{C}}(x_\lambda, y)\right),$$

all based at 0, are isomorphisms as well. Hence the map on loop spaces

$$\Omega(H_{\mathcal{C}}(x_\Lambda, y)) \rightarrow \Omega\left(\prod_\lambda H_{\mathcal{C}}(x_\lambda, y)\right)$$

is a homotopy equivalence, by Whitehead's theorem. By replacing  $y$  with  $\Sigma y$ , and consulting the natural isomorphism of Proposition 5.27, we see now that the original map  $H_{\mathcal{C}}(x_\Lambda, y) \rightarrow \prod_\lambda H_{\mathcal{C}}(x_\lambda, y)$  is an equivalence. It follows by Corollary II-16.16 that  $x_\Lambda$  provides a coproduct of the  $\{x_\lambda : \lambda \in \Lambda\}$  in  $\mathcal{C}$ .  $\square$

sect:adjointer

## 7. ADJOINTS, AGAIN

Our next goal is to realize the derived  $\infty$ -category  $\mathcal{D}(\mathbb{A})$  as a localization of the homotopy  $\infty$ -category  $\mathcal{K}(\mathbb{A})$  relative to the class of quasi-isomorphisms. The localization functor in this case is obtained as a left adjoint  $\text{loc} : \mathcal{K}(\mathbb{A}) \rightarrow \mathcal{D}(\mathbb{A})$  to the inclusion  $\mathcal{D}(\mathbb{A}) = \mathcal{D}_{\text{Inj}} \rightarrow \mathcal{K}(\mathbb{A})$ .

In order to facilitate such an analysis we return to the topic of adjunctions, which we saw previously in Section I-14. We prove that adjoint pairs of functors can be identified with simultaneous cartesian and cocartesian fibrations over the 1-simplex, and determine when the inclusion  $\mathcal{C}' \rightarrow \mathcal{C}$  of a full  $\infty$ -subcategory admits a left (or right) adjoint. Our findings are not only applied to address the localization problem discussed above, but also in our discussions of derived functors, indization of small  $\infty$ -categories, and idempotent completions.

### 7.1. Reflexive subcategories.

**Definition 7.1** ([15, 02F6]). Let  $\mathcal{C}' \subseteq \mathcal{C}$  be a full  $\infty$ -subcategory. Given an object  $x$  in  $\mathcal{C}$ , a morphism  $f : x \rightarrow y$  with  $y$  in  $\mathcal{C}'$  is said to exhibit  $y$  as a  $\mathcal{C}'$ -reflection of  $x$  if, for each third object  $z$  in  $\mathcal{C}'$ , the precomposition function

$$f^* : \text{Hom}_{\mathcal{C}}(y, z) \rightarrow \text{Hom}_{\mathcal{C}}(x, z)$$

is an isomorphism in  $\text{h}\mathcal{K}an$ . Similarly, a morphism  $g : y \rightarrow x$  with  $y$  in  $\mathcal{C}'$ , is said to exhibit  $y$  as a  $\mathcal{C}'$ -coreflection of  $x$  if, for each third object  $z$  in  $\mathcal{C}'$ , the composition function

$$g_* : \text{Hom}_{\mathcal{C}}(z, y) \rightarrow \text{Hom}_{\mathcal{C}}(z, x)$$

is an isomorphism in  $\text{h}\mathcal{K}an$ .

We say  $\mathcal{C}'$  itself is a reflective (resp. coreflective) subcategory in  $\mathcal{C}$  if every object  $x$  in  $\mathcal{C}$  admits a morphism  $x \rightarrow y$  (resp.  $y \rightarrow x$ ) which exhibits  $y$  as a  $\mathcal{C}'$ -reflection (resp.  $\mathcal{C}'$ -coreflection) of  $x$ .

One sees directly that taking opposites  $\mathcal{C} \mapsto \mathcal{C}^{\text{op}}$  exchanges reflections and coreflections, and exchanges reflective and coreflective subcategories as well. So, throughout the section, we can safely prove a result only for reflections with the understanding that the analogous result for coreflections is obtained by applying opposites.

**ex:inj\_reflex**

**Example 7.2** ( $K$ -injectives). Let  $\mathbb{A}$  be a Grothendieck abelian category, and let  $\mathcal{D}_{\text{Inj}} \subseteq \mathcal{K}(\mathbb{A})$  denote the full subcategory of  $K$ -injective complexes. Let  $\mathcal{D}'_{\text{Inj}} \subseteq \mathcal{K}(\mathbb{A})'$  be the corresponding full subcategory in the simplicial construction of the homotopy  $\infty$ -category.

Every complex  $V$  in  $\text{Ch}(\mathbb{A})$  admits a  $K$ -injective resolution  $f : V \rightarrow I_V$ . This map induces a quasi-isomorphism

$$f^* : \text{Hom}_{\mathbb{A}}^*(I_V, J) \rightarrow \text{Hom}_{\mathbb{A}}^*(V, J),$$

at each  $K$ -injective complex  $J$ , which then induces a homotopy equivalence

$$f^* : K \text{Hom}_{\mathbb{A}}^*(I_V, J) \rightarrow K \text{Hom}_{\mathbb{A}}^*(V, J)$$

so that  $f$  is a  $\mathcal{D}'_{\text{Inj}}$ -reflection in  $\mathcal{K}(\mathbb{A})'$ . It follows via the natural equivalence  $\mathfrak{Z}_?$  of Theorem II-10.4 that  $f : V \rightarrow I_V$  is also a  $\mathcal{D}_{\text{Inj}}$ -reflection in  $\mathcal{K}(\mathbb{A})$ . We conclude that  $\mathcal{D}_{\text{Inj}}$  is a reflective subcategory in  $\mathcal{K}(\mathbb{A})$ .

**ex:proj\_coreflex**

**Example 7.3** ( $K$ -projectives). Suppose that an abelian category  $\mathbb{A}$  has enough projectives, and let  $\mathcal{D}_{\text{Proj}}$  be the full  $\infty$ -subcategory of  $K$ -projectives in  $\mathcal{K}(\mathbb{A})$ . Each complex  $V$  admits a  $K$ -projective resolution  $g : P_V \rightarrow V$ . One argues as in Example 7.2 to see that  $g$  is a  $\mathcal{D}_{\text{Proj}}$ -coreflection, and therefore to see that  $\mathcal{D}_{\text{Proj}}$  is a coreflective subcategory in  $\mathcal{K}(\mathbb{A})$ .

**Lemma 7.4.** *Given any morphism  $f : x \rightarrow y$ , the forgetful functor  $\mathcal{C}_{f/} \rightarrow \mathcal{C}_{y/}$  is a trivial Kan fibration.*

*Proof.* By [9, Lemma 3.3] the apparent map

$$(\Delta^1 \star \partial \Delta^n) \coprod_{(\{1\} \star \partial \Delta^n)} (\{1\} \star \Delta^n) \rightarrow \Delta^1 \star \Delta^n \cong \Delta^{n+2}$$

is an isomorphism onto the inner horn  $\Lambda_1^{n+2} \subseteq \Delta^{n+2}$ . Hence solving a lifting problem of the form

$$\begin{array}{ccc} \partial \Delta^n & \longrightarrow & \mathcal{C}_{f/} \\ \downarrow & \nearrow & \downarrow \\ \Delta^n & \longrightarrow & \mathcal{C}_{y/} \end{array}$$

is equivalent to solving a lifting problem of the form

$$\begin{array}{ccc} \Lambda_1^{n+2} & \longrightarrow & \mathcal{C} \\ \downarrow & \nearrow & \downarrow \\ \Delta^{n+2} & \longrightarrow & * \end{array}$$

Since  $\mathcal{C}$  is an  $\infty$ -category, the latter problem always admits a solution.  $\square$

**lem:1253**

**Lemma 7.5** ([15, 02LL]). *For any morphism  $f : x \rightarrow y$  and object  $z : * \rightarrow \mathcal{C}$ , we have a commuting diagram*

$$\begin{array}{ccc} \mathcal{C}_{y/} \times_{\mathcal{C}} \{z\} & \xleftarrow{\cong} \mathcal{C}_{f/} \times_{\mathcal{C}} \{z\} & \longrightarrow \mathcal{C}_{x/} \times_{\mathcal{C}} \{z\} \\ \cong \downarrow & & \downarrow \cong \\ \text{Hom}_{\mathcal{C}}(y, z) & \xrightarrow{f^*} & \text{Hom}_{\mathcal{C}}(x, z). \end{array}$$

in  $\mathbf{h}\mathcal{K}an$ .

Here the vertical maps are specifically those induced by the coslice diagonal equivalences. We only recall the main idea of the proof here.

*Idea of proof.* One produces a morphism

$$i : \mathcal{C}_{f/} \times_{\mathcal{C}} \{z\} \rightarrow \{f\} \times_{\mathrm{Hom}_{\mathcal{C}}(x,y)} \mathrm{Fun}(\Delta^2, \mathcal{C})_{\vec{x}}$$

which bisects the diagram

$$\begin{array}{ccccc} \mathcal{C}_{y/} \times_{\mathcal{C}} \{z\} & \xleftarrow{\cong} & \mathcal{C}_{f/} \times_{\mathcal{C}} \{z\} & \xrightarrow{\quad} & \mathcal{C}_{x/} \times_{\mathcal{C}} \{z\} \\ \downarrow \cong & & \downarrow & & \downarrow \cong \\ \mathrm{Hom}_{\mathcal{C}}(y, z) & \xleftarrow{\quad} & \{f\} \times_{\mathrm{Hom}_{\mathcal{C}}(x,y)} \mathrm{Fun}(\Delta^2, \mathcal{C})_{\vec{x}} & \xrightarrow{\quad} & \mathrm{Hom}_{\mathcal{C}}(x, z). \end{array}$$

at the level of the discrete category  $\mathbf{Kan}$ .  $\square$

`prop:reflex_char`

**Proposition 7.6.** *Let  $\mathcal{C}' \subseteq \mathcal{C}$  be a full  $\infty$ -subcategory. For a fixed morphism  $f : x \rightarrow y$ , with  $y$  in  $\mathcal{C}'$ , the following are equivalent:*

- (a)  *$f$  exhibits  $y$  as a  $\mathcal{C}'$ -reflection of  $x$ .*
- (b) *At each object  $z$  in  $\mathcal{C}'$  the forgetful functor  $\mathcal{C}_{f/} \times_{\mathcal{C}} \{z\} \rightarrow \mathcal{C}_{x/} \times_{\mathcal{C}} \{z\}$  is an equivalence.*
- (c) *The map  $\mathcal{C}_{f/} \times_{\mathcal{C}} \mathcal{C}' \rightarrow \mathcal{C}_{f/} \times_{\mathcal{C}} \mathcal{C}'$  is a trivial Kan fibration.*
- (d) *Each lifting problem*

$$\begin{array}{ccc} \Lambda_0^n & \xrightarrow{\tau} & \mathcal{C} \\ \downarrow & & \downarrow \\ \Delta^n & \longrightarrow & * \end{array}$$

*with  $n \geq 2$  and  $\tau|_{\Delta_{\{1, \dots, n\}}}$  having image in  $\mathcal{C}'$  admits a solution.*

*Proof.* The equivalence (a)  $\Leftrightarrow$  (b) is a consequence of Lemma 7.5. For (b)  $\Leftrightarrow$  (c), we recall that the map

$$\mathcal{C}_{f/} = (\mathcal{C}_{x/})_{y/} \rightarrow \mathcal{C}_{x/}$$

is a left fibration by Corollary I-5.27, and hence its base change  $\mathcal{C}_{f/} \times_{\mathcal{C}} \mathcal{C}' \rightarrow \mathcal{C}_{x/} \times_{\mathcal{C}} \mathcal{C}'$  is a left fibration as well. This left fibration furthermore fits into a diagram

$$\begin{array}{ccc} \mathcal{C}_{f/} \times_{\mathcal{C}} \mathcal{C}' & \xrightarrow{\quad} & \mathcal{C}_{f/} \times_{\mathcal{C}} \mathcal{C}' \\ & \searrow & \swarrow \\ & \mathcal{C}' & \end{array}.$$

Theorem II-3.8 and Corollary II-9.8 together now imply that the forgetful functor  $\mathcal{C}_{f/} \times_{\mathcal{C}} \mathcal{C}' \rightarrow \mathcal{C}_{x/} \times_{\mathcal{C}} \mathcal{C}'$  is a trivial Kan fibration if and only if at each object  $z : * \rightarrow \mathcal{C}'$  the fiber

$$\mathcal{C}_{f/} \times_{\mathcal{C}} \{z\} \rightarrow \mathcal{C}_{x/} \times_{\mathcal{C}} \{z\}$$

is an equivalence. Statement (d) is identified with (c) via the identification of lifting problems

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & \mathcal{C}_f / \times_{\mathcal{C}} \mathcal{C}' \\ \downarrow & \nearrow & \downarrow \\ \Delta^n & \longrightarrow & \mathcal{C}_x / \times_{\mathcal{C}} \mathcal{C}' \end{array} = \begin{array}{ccc} \Lambda_0^{n+2} & \longrightarrow & \mathcal{C} \\ \downarrow & \nearrow & \downarrow \\ \Delta^n & \longrightarrow & * \end{array}$$

which is implied by Lemma II-9.10.  $\square$

## 7.2. Reflexivity and adjoints.

prop:1327

**Proposition 7.7.** *Let  $\mathcal{C}' \subseteq \mathcal{C}$  be a full  $\infty$ -subcategory and  $i : \mathcal{C}' \rightarrow \mathcal{C}$  be the inclusion. The subcategory  $\mathcal{C}'$  is reflective if and only if there is a functor  $L : \mathcal{C} \rightarrow \mathcal{C}'$  and a transformation  $u : id_{\mathcal{C}} \rightarrow iL$  for which, at each  $x$  in  $\mathcal{C}$ , the map  $u_x : x \rightarrow L(x)$  exhibits  $L(x)$  as a  $\mathcal{C}'$ -reflection of  $x$ .*

*Proof.* Let  $\mathcal{E} \subseteq \mathcal{C} \times \Delta^1$  be the full  $\infty$ -subcategory whose objects  $\mathcal{E}[0]$  are the union  $(\mathcal{C}[0] \times \{0\}) \cup (\mathcal{C}'[0] \times \{1\})$ . By Proposition 7.6 the projection

$$q : \mathcal{E} \rightarrow \Delta^1$$

is a cocartesian fibration, and a map  $f : (x, 0) \rightarrow (y, 1)$  in  $\mathcal{E}$  is  $q$ -cocartesian if and only if the underlying map  $f : x \rightarrow y$  in  $\mathcal{C}$  exhibits  $y$  as a  $\mathcal{C}'$ -reflection of  $x$ .

Now, by Theorem II-2.7 there exists a unique functor  $U : \Delta^1 \times \mathcal{C} \rightarrow \mathcal{E}$  which splits the diagram

$$\begin{array}{ccc} \{0\} \times \mathcal{C} & \xrightarrow{\text{incl}} & \mathcal{E} \\ \downarrow & \nearrow U & \downarrow q \\ \Delta^1 \times \mathcal{C} & \xrightarrow{p_1} & \Delta^1 \end{array}$$

and sends each map  $\Delta^1 \times \{x\}$  to a  $q$ -cocartesian morphism in  $\mathcal{E}$ . For  $L : \mathcal{C} \rightarrow \mathcal{C}' \subseteq \mathcal{C}$  defined as the composite

$$\mathcal{C} \cong \{1\} \times \mathcal{C} \rightarrow \Delta^1 \times \mathcal{C} \rightarrow \mathcal{E} \xrightarrow{p_1} \mathcal{C},$$

the transformation  $u = p_1 U : \Delta^1 \times \mathcal{C} \rightarrow \mathcal{C}$  has the prescribed property.  $\square$

lem:1357

**Lemma 7.8** ([15, 02DK]). *Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  and  $G : \mathcal{D} \rightarrow \mathcal{C}$  be functors between  $\infty$ -categories, and  $u : id_{\mathcal{C}} \rightarrow GF$  be a transformation. Suppose that the induced transformations*

$$Fu : F \rightarrow F(GF) \quad \text{and} \quad uG : G \rightarrow (GF)G$$

*are isomorphisms in  $\text{Fun}(\mathcal{C}, \mathcal{D})$  and  $\text{Fun}(\mathcal{D}, \mathcal{C})$  respectively, and that  $G$  is fully faithful. Then  $u$  is the unit of an adjunction, and the counit  $\epsilon : FG \rightarrow id_{\mathcal{D}}$  is a natural isomorphism.*

*Sketch proof.* Since  $G$  is fully faithful the induced map  $G_* : \text{Fun}(\mathcal{C}, \mathcal{D}) \rightarrow \text{Fun}(\mathcal{C}, \mathcal{C})$  is fully faithful. Hence there is a unique transformation  $\epsilon : FG \rightarrow id_{\mathcal{D}}$  which lifts the isomorphism  $(uG)^{-1} : GFG \rightarrow G$ . Fully faithfulness implies that  $\epsilon$  is an isomorphism as well.

We can replace the functor categories  $\text{Fun}(\mathcal{A}, \mathcal{B})$  with their homotopy categories  $\text{hFun}(\mathcal{A}, \mathcal{B})$  and work with the corresponding 2-category  $\text{Cat}_\infty^2$  obtained from the simplicial category  $\underline{\text{Cat}}_\infty$ . At this level we consider the composites

$$F \xrightarrow{Fu} FGF \xrightarrow{\epsilon F} F \quad \text{and} \quad G \xrightarrow{uG} GFG \xrightarrow{G\epsilon} G.$$

The latter composite is the identity by the definition of  $\epsilon$ . We consider now the composite  $\beta = (\epsilon F)(Fu)$ . By our assumptions,  $Fu$  is an isomorphism, so that  $\beta$  is an isomorphism as well. One now argues, using [15, 02CX], that  $\beta$  also satisfies  $\beta^2 = \beta$  and hence  $\beta = id_F$  necessarily.  $\square$

rem:1376

**Remark 7.9.** If a transformation  $u : id_{\mathcal{C}} \rightarrow GF$  admits some transformation  $FG \rightarrow id_{\mathcal{D}}$  which exhibits  $F$  as left adjoint to  $G$ , then this transformation is fixed up to the action of  $\text{Aut}_{\text{Fun}(\mathcal{D}, \mathcal{C})}(G)$  [15, 02D7]. In particular, any transformation  $\epsilon' : FG \rightarrow id_{\mathcal{D}}$  which pairs with  $u$  to realize  $F$  as left adjoint to  $G$ , in the situation of Lemma 7.8, must be a natural isomorphism.

prop:1380

**Proposition 7.10.** *Let  $i : \mathcal{C}' \rightarrow \mathcal{C}$  be the inclusion of a full  $\infty$ -subcategory into an  $\infty$ -category  $\mathcal{C}$ . Consider any functor  $L : \mathcal{C} \rightarrow \mathcal{C}'$  and transformation  $u : id_{\mathcal{C}} \rightarrow iL$ . The following are equivalent:*

- (a) *The transformation  $u$  is part of an adjunction which exhibits  $L$  as left adjoint to the inclusion  $i : \mathcal{C}' \rightarrow \mathcal{C}$ .*
- (b) *At each object  $x$  in  $\mathcal{C}$ , the map  $u_x : x \rightarrow L(x)$  exhibits  $L(x)$  as a  $\mathcal{C}'$ -reflection of  $x$ .*
- (c) *At each  $x$  in  $\mathcal{C}$  the map  $L(u_x) : L(x) \rightarrow LL(x)$  is an isomorphism and, at each  $y$  in  $\mathcal{C}'$ , the map  $u_y : y \rightarrow L(y)$  is an isomorphism in  $\mathcal{C}'$ .*

Furthermore, in this case, any transformation  $\epsilon : Li \rightarrow id_{\mathcal{C}'}$  which pairs with  $u$  to realize  $L$  as left adjoint to  $i$  is a natural isomorphism.

*Proof.* Supposing (a) and (c) hold, the claim about  $\epsilon$  follows by Lemma 7.8 and Remark 7.9. Now, suppose (a) holds. Then by Corollary I-14.3 the transformation  $u$  realizes the  $\text{hKan}$ -enriched functor  $\pi L : \pi\mathcal{C} \rightarrow \pi\mathcal{C}'$  as left adjoint to the enriched embedding  $\pi i : \pi\mathcal{C}' \rightarrow \pi\mathcal{C}$ . Hence, at each  $z$  in  $\mathcal{C}'$ ,  $u$  induces isomorphisms

$$u^* : \text{Hom}_{\mathcal{C}}(L(x), z) \rightarrow \text{Hom}_{\mathcal{C}}(x, z)$$

in  $\text{hKan}$ . Thus (b) holds.

Supposing (b) holds. When  $y$  is in  $\mathcal{C}'$  applying  $u$  yields an isomorphism of sets

$$u^* : \pi_0 \text{Hom}_{\mathcal{C}'}(L(y), z) \rightarrow \pi_0 \text{Hom}_{\mathcal{C}'}(y, z)$$

which shows, via Yoneda, that  $u_y : y \rightarrow L(y)$  is an isomorphism in  $\text{h}\mathcal{C}'$ . By definition this implies that  $u_y$  is an isomorphism in  $\mathcal{C}'$ .

As for the transformation  $L(u_x) : L(x) \rightarrow LL(x)$  at general  $x$ , we have the diagram

$$\begin{array}{ccc} x & \xrightarrow{u_x} & L(x) \\ u_x \downarrow & & \downarrow u_{L(x)} \\ L(x) & \xrightarrow{L(u_x)} & LL(x) \end{array}$$

in  $\mathbf{h}\mathcal{C}$  and apply  $\mathrm{Hom}_{\mathbf{h}\mathcal{C}}(-, z)$  at arbitrary  $z$  in  $\mathcal{C}'$  to obtain a diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{h}\mathcal{C}}(L(x), z) & \xrightarrow{\cong} & \mathrm{Hom}_{\mathbf{h}\mathcal{C}'}(x, z) \\ \cong \downarrow & & \downarrow \cong \\ \mathrm{Hom}_{\mathbf{h}\mathcal{C}'}(LL(x), z) & \xrightarrow{L(u)^*} & \mathrm{Hom}_{\mathbf{h}\mathcal{C}'}(L(x), z). \end{array}$$

From this we conclude that  $L(u_x)^*$  is an isomorphism at all  $z$ , and hence that  $L(u_x)$  is an isomorphism in  $\mathbf{h}\mathcal{C}$ . It follows that  $L(u_x)$  is an isomorphism in  $\mathcal{C}$ .

Finally, Lemma 7.8 tells us directly that (c) implies (a). This completes the proof.  $\square$

One combines Propositions 7.7 and 7.10 to obtain the following.

thm:reflex\_adj

**Theorem 7.11.** *Consider a full  $\infty$ -subcategory  $\mathcal{C}' \subseteq \mathcal{C}$  in an arbitrary  $\infty$ -category  $\mathcal{C}$ . Then  $\mathcal{C}'$  is reflective in  $\mathcal{C}$  if and only if the inclusion  $i : \mathcal{C}' \rightarrow \mathcal{C}$  admits a left adjoint  $L : \mathcal{C} \rightarrow \mathcal{C}'$  whose unit and counit transformations have the properties outlined in Proposition 7.10 above. Similarly,  $\mathcal{C}'$  is coreflective if and only if the inclusion admits a right adjoint  $R : \mathcal{C} \rightarrow \mathcal{C}'$ .*

*Proof.* The claim about reflective subcategories is clear, on the claim about coreflective subcategories is obtain by applying opposites.  $\square$

ex:1454

**Example 7.12.** As an application of Theorem 7.11 to Example 7.2 we observe, for Grothendieck abelian  $\mathbb{A}$ , the existence of a left adjoint  $L : \mathcal{K}(\mathbb{A}) \rightarrow \mathcal{D}_{\mathrm{Inj}} = \mathcal{D}(\mathbb{A})$  to the inclusion  $i : \mathcal{D}(\mathbb{A}) \rightarrow \mathcal{K}(\mathbb{A})$ . This left adjoint exhibits  $\mathcal{D}(\mathbb{A})$  as a localization  $\mathcal{K}(\mathbb{A})[\mathrm{Qiso}^{-1}]$  of the homotopy  $\infty$ -category at the class of quasi-isomorphisms. We return to this topic in Section 8 below.

### 7.3. Adjoints via simultaneous fibrations.

lem:cocart\_ref

**Lemma 7.13.** *Let  $q : \mathcal{E} \rightarrow \Delta^1$  be an inner fibration of  $\infty$ -categories, with fibers  $\mathcal{E}_i = \mathcal{E} \times_{\Delta^1} \{i\}$ . The following hold:*

- (1) *The subcategory  $\mathcal{E}_1$  is a reflective subcategory in  $\mathcal{E}$  if and only if  $q$  is a cocartesian fibration, and in this case a map  $f : x \rightarrow y$  over  $0 < 1$  is  $q$ -cocartesian if and only if it exhibits  $y$  as a  $\mathcal{E}_1$ -reflection of  $x$ .*
- (2) *The subcategory  $\mathcal{E}_0$  is a coreflective subcategory in  $\mathcal{E}$  if and only if  $q$  is a cartesian fibration, and in this case a map  $g : y \rightarrow x$  over  $0 < 1$  is  $q$ -cartesian if and only if it exhibits  $y$  as a  $\mathcal{E}_0$ -coreflection of  $x$ .*

*Proof.* Follows by Proposition 7.6 (c).  $\square$

prop:adj\_fibration

**Proposition 7.14.** *Let  $q : \mathcal{E} \rightarrow \Delta^1$  be a cocartesian fibration, and  $F : \mathcal{E}_0 \rightarrow \mathcal{E}_1$  be the functor given by covariant transport along  $q$  (Definition II-7.1). The functor  $F$  admits a right adjoint  $G : \mathcal{E}_1 \rightarrow \mathcal{E}_0$  if and only if  $q$  is a cartesian fibration as well, and in this case  $G$  is given by contravariant transport along  $q$ .*

Our claim that  $G$  is “given by contravariant transport” should be interpreted in a strict sense. Namely, we claim that when  $F$  admits a right adjoint  $G$ , there is a cartesian transformation  $\Delta^1 \times \mathcal{E}_1 \rightarrow \mathcal{E}$ , i.e. a cartesian solution to the lifting

problem

$$\begin{array}{ccc} \{1\} \times \mathcal{E}_1 & \longrightarrow & \mathcal{E} \\ \downarrow & \nearrow & \downarrow q \\ \Delta^1 \times \mathcal{E}_1 & \longrightarrow & \Delta^1 \end{array}$$

(see Proposition II-2.6) whose restriction to  $\{0\} \times \mathcal{E}_1$  recovers  $G$ .

*Proof.* First suppose that  $F$  admits such a right adjoint  $G : \mathcal{E}_1 \rightarrow \mathcal{E}_0$ , and consider the unit and counit transformations

$$u : id_{\mathcal{E}_0} \rightarrow GF \quad \text{and} \quad \epsilon : FG \rightarrow id_{\mathcal{E}_1}$$

respectively. By Proposition 7.6 (c) we understand that  $q$  is cartesian if and only if the subcategory  $\mathcal{E}_0$  is coreflective in  $\mathcal{E}$ . So we seek to demonstrate  $\mathcal{E}_0$ -coreflections  $g_x : y \rightarrow x$  at each  $x$  in  $\mathcal{E}$ . When  $x$  is in  $\mathcal{E}_0$  we can just take  $g_x = id_x$ , so that we can assume here that  $x$  is in the fiber  $\mathcal{E}_1$ .

First, let us consider the extended fibration  $q^l : \mathcal{E}^l \rightarrow \Delta^1$  where  $\mathcal{E}^l \subseteq \mathcal{E} \times \Delta^1$  is the full  $\infty$ -subcategory with objects  $(\mathcal{E}[0] \times \{0\}) \cup (\mathcal{E}_1[0] \times \{1\})$ . Here  $q^l$  is specifically the projection onto the second factor. As in the proof of Proposition 7.7, we see that  $q^l$  is a cocartesian fibration and one observes directly the inclusion of cocartesian fibrations  $\mathcal{E} \rightarrow \mathcal{E}^l$  provided by restricting the image of the product map  $[id_{\mathcal{E}} q]^t : \mathcal{E} \rightarrow \mathcal{E} \times \Delta^1$ .

The transport functor  $L : \mathcal{E} \rightarrow \mathcal{E}_1$  along  $q^l$  comes equipped with a transformation  $\eta : id_{\mathcal{E}} \rightarrow L$  which evaluates to a  $q^l$ -cocartesian morphism at each  $x$  in  $\mathcal{E}$ , by definition. By uniqueness of transport functors and the fact that the restriction functor

$$\text{Fun}(\Delta^1 \times \mathcal{E}, \mathcal{E}) \rightarrow \text{Fun}(\Delta^1 \times \mathcal{E}_0, \mathcal{E})$$

is an isofibration (Corollary I-6.14), we can choose  $L$  so that  $F = L|_{\mathcal{E}_0}$ . Note also that  $\eta$  exhibits  $L$  as left adjoint to the inclusion  $i_1 : \mathcal{E}_1 \rightarrow \mathcal{E}$ , by Lemma 7.8.

At each  $x$  in  $\mathcal{E}_1$  define the map  $g_x : G(x) \rightarrow x$  as a composite

$$G(x) \xrightarrow{\eta} LG(x) = FG(x) \xrightarrow{\epsilon} x.$$

We then have at each  $z$  in  $\mathcal{E}_0$ , in the homotopy category of spaces, the sequence of maps

$$\text{Hom}_{\mathcal{E}_0}(z, G(x)) \xrightarrow{F} \text{Hom}_{\mathcal{E}_1}(F(z), FG(x)) \xrightarrow{\epsilon_*} \text{Hom}_{\mathcal{E}_1}(F(z), x) \xrightarrow{\eta^*} \text{Hom}_{\mathcal{E}}(z, x).$$

As the first two maps compose to an isomorphism, and the third map is also an isomorphism, this composite is an isomorphism. By commutativity of the operations  $\epsilon_* = \epsilon \circ -$  and  $\eta^* = - \circ \eta$ , i.e. by associativity of composition in  $\mathbf{h}\mathcal{K}an$ , and naturality of  $\eta$  (Lemma I-14.2), the above composite is equal to the map

$$(g_x)_* : \text{Hom}_{\mathcal{E}_0}(z, G(x)) \rightarrow \text{Hom}_{\mathcal{E}}(z, x),$$

which we conclude is an isomorphism. So each  $g_x : G(x) \rightarrow x$  exhibits  $G(x)$  as a  $\mathcal{E}_0$ -coreflection,  $\mathcal{E}_0$  is seen to be coreflective in  $\mathcal{E}$ , and  $q$  is therefore a cartesian fibration.

Suppose now that  $q : \mathcal{E} \rightarrow \Delta^1$  is cartesian. Then the subcategory  $\mathcal{E}_0$  is coreflective in  $\mathcal{E}$  by Proposition 7.6 and we have the associated cartesian fibration  $q^r : \mathcal{E}^r \rightarrow \Delta^1$  obtained by considering the full  $\infty$ -subcategory  $\mathcal{E}^r \subseteq \mathcal{E} \times \Delta^1$  whose objects are provided by the union  $(\mathcal{E}_0[0] \times \{0\}) \cup (\mathcal{E}[0] \times \{1\})$ . We have



the contravariant transport functor  $R : \mathcal{E} \rightarrow \mathcal{E}_0$  for  $q^r$  along with the transformation  $\pi : i_0 R \rightarrow id_{\mathcal{E}}$  which evaluates to a  $q^r$ -cartesian morphism at each  $x$  in  $\mathcal{E}$ . By Lemma 7.8 the transformation  $\pi$  exhibits  $R$  as right adjoint to the inclusion  $\mathcal{E}_0 \rightarrow \mathcal{E}$ .

Define  $G : \mathcal{E}_1 \rightarrow \mathcal{E}_0$  as the composite  $R i_1 : \mathcal{E}_1 \rightarrow \mathcal{E} \rightarrow \mathcal{E}_0$ . The inclusion  $\mathcal{E}_1 \rightarrow \mathcal{E}$  has left adjoint  $L : \mathcal{E} \rightarrow \mathcal{E}_1$  which restricts to  $F$  on  $\mathcal{E}_0$ , and we see that  $G$  is a composite of right adjoints. Hence  $G$  itself is right adjoint to the functor  $L i_0 = F$ , as desired (see [15, 02DT]). We see also that  $G$  is a contravariant transport functor for  $q : \mathcal{E} \rightarrow \Delta^1$  since restricting along the inclusion of cartesian fibrations  $\mathcal{E} \rightarrow \mathcal{E}^r$  recovers transport functors for  $q$  from transport functors for  $q^r$ .

As for the claim that any right adjoint to  $F$  is given by contravariant transport, consider  $G : \mathcal{E}_1 \rightarrow \mathcal{E}_0$  a right adjoint to  $F$  with counit transformation  $\epsilon : FG \rightarrow id_{\mathcal{E}_1}$ . Consider the transformation  $\tilde{\epsilon} : \Delta^1 \times \mathcal{E}_1 \rightarrow \mathcal{E}$  which sends an  $n$ -simplex  $(\alpha, \sigma) : \Delta^n \rightarrow \Delta^1 \times \mathcal{E}_1$  to the triple

$$(\alpha, G\sigma|_{\Delta^{\alpha^{-1}(0)}}, \sigma) : \Delta^n \rightarrow \mathcal{E} = N^F(\Delta^1).$$

At each  $y : * \rightarrow \mathcal{E}_1$ ,  $\tilde{\epsilon}$  sends the edge  $\Delta^1 \times \{y\}$  to the edge  $(G(y), \epsilon_y : FG(y) \rightarrow y)$  in  $\mathcal{E}$ . Let us denote this edge  $\tilde{\epsilon}_y : \Delta^1 \rightarrow \mathcal{E}$ .

By Corollary I-14.4,  $\epsilon$  realizes the  $\mathbf{h}\mathcal{K}an$ -enriched functor  $\pi F$  and left adjoint to  $\pi G$ , so that the composite

$$\mathrm{Hom}_{\mathcal{E}_0}(z, G(y)) \xrightarrow{F} \mathrm{Hom}_{\mathcal{E}_1}(F(z), FG(y)) \xrightarrow{\epsilon} \mathrm{Hom}_{\mathcal{E}}(F(z), y)$$

is an isomorphism at arbitrary  $z$  in  $\mathcal{E}$  and  $y$  in  $\mathcal{D}$ . One checks that this sequence recovers composition  $(\tilde{\epsilon}_y)_* : \mathrm{Hom}_{\mathcal{E}}(z, G(y)) \rightarrow \mathrm{Hom}_{\mathcal{E}}(z, y)$  in  $\pi \mathcal{E}$  to see that  $\tilde{\epsilon}_y$  is an  $\mathcal{E}_0$ -coreflection in  $\mathcal{E}$ . By Lemma 7.13 we conclude that each  $\tilde{\epsilon}_y$  is  $q$ -cartesian in  $\mathcal{E}$ , and hence that  $\tilde{\epsilon}$  is a cartesian solution to the lifting problem

$$\begin{array}{ccc} \{1\} \times \mathcal{E}_1 & \xrightarrow{\quad} & \mathcal{E} \\ \downarrow & \nearrow & \downarrow q \\ \Delta^1 \times \mathcal{E}_1 & \xrightarrow{\quad} & \Delta^1. \end{array}$$

By construction  $\tilde{\epsilon}|_{\{0\} \times \mathcal{E}_1} = G$ . □

We now obtain a characterization of adjunctions via fibrations over the 1-simplex.

thm:adj\_fibration

**Theorem 7.15.** *Given a pair of functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  and  $G : \mathcal{D} \rightarrow \mathcal{C}$ , the following are equivalent:*

- (a) *The functors  $F$  and  $G$  admit transformations which exhibit  $F$  as left adjoint to  $G$ .*
- (b) *There is a simultaneous cartesian and cocartesian fibration  $q : \mathcal{E} \rightarrow \Delta^1$  with fixed isomorphisms at the fibers  $\mathcal{C} \cong \mathcal{E}_0$  and  $\mathcal{D} \cong \mathcal{E}_1$ , and for which  $F$  and  $G$  are recovered respectively as covariant and contravariant transport along  $q$ .*

*Proof.* Note that  $F$  defines a functor  $F : \Delta^1 \rightarrow \mathcal{Cat}_{\infty}$  and consider the weighted nerve  $q : \mathcal{E} = N^F(\Delta^1) \rightarrow \Delta^1$ . The fact that  $F$  is recovered by covariant transport along  $q$  is implicit in the claim that there is an isomorphism of fibrations  $\mathcal{E} \cong \int_{\Delta^1} F$  (Theorem II-6.28). However, we can just observe this fact directly.

For an  $n$ -simplex  $\sigma = (\sigma', \sigma'') : \Delta^n \rightarrow \Delta^1 \times \mathcal{C} = \mathcal{C}_0$  take  $\Delta^{n_0} = (\sigma')^{-1}(0)$ . Define now

$$\tilde{\sigma} : \Delta^n \rightarrow \mathcal{E}$$

as the pairing of the  $n$ -simplex  $\sigma' : \Delta^n \rightarrow \Delta^1$  with the pair of  $n$ -simplices  $\sigma''|_{\Delta^{n_0}} : \Delta^{n_0} \rightarrow \mathcal{C}$  and  $F\sigma'' : \Delta^n \rightarrow \mathcal{D}$ . The assignment  $\sigma \mapsto \tilde{\sigma}$  defines a functor cocartesian lift of the inclusion  $\{0\} \times \mathcal{C} \rightarrow \mathcal{E}$ , and so recovers  $F : \mathcal{C} \rightarrow \mathcal{E}$  as covariant transport along  $q$ .

In any case, we recover the claimed equivalence of (a) and (b) by applying Proposition 7.14 to the weighted nerve for  $F$ .  $\square$

#### 7.4. Local criterion for adjunction.

thm:local\_adj

**Theorem 7.16.** *A functor between  $\infty$ -categories  $F : \mathcal{C} \rightarrow \mathcal{D}$  admits a right adjoint if and only if, for each  $y$  in  $\mathcal{D}$ , there exists a morphism  $g_y : F(x) \rightarrow y$  from some object  $x$  in  $\mathcal{C}$  such that, at any other  $z$  in  $\mathcal{C}$ , the sequence*

$$\mathrm{Hom}_{\mathcal{E}}(z, x) \xrightarrow{F} \mathrm{Hom}_{\mathcal{D}}(F(z), F(x)) \xrightarrow{(g_y)_*} \mathrm{Hom}_{\mathcal{D}}(F(z), y) \quad (32)$$

eq:1550

is an isomorphism in  $\mathbf{h}\mathcal{K}\mathbf{an}$ .

*Proof.* If there exists an adjoint  $G$  then we can take  $x = G(y)$  and  $g$  the counit morphism. Conversely, suppose we can always find such a  $g_y$  at each  $y$  in  $\mathcal{D}$ . Then for the weighted nerve  $q : \mathcal{E} = N^F(\Delta^1) \rightarrow \Delta^1$  covariant transport along  $q$  recovers the functor  $F$ , as was argued in the proof of Theorem 7.15.

In the weighted nerve

$$\mathrm{Hom}_{\mathcal{E}}(z, x) = \mathrm{Hom}_{\mathcal{C}}(z, x) \quad \text{and} \quad \mathrm{Hom}_{\mathcal{E}}(z, y) = \mathrm{Hom}_{\mathcal{E}}(F(z), y),$$

and one can check directly that the composition function

$$\mathrm{Hom}_{\mathcal{E}}(x, y) \times \mathrm{Hom}_{\mathcal{E}}(z, x) \rightarrow \mathrm{Hom}_{\mathcal{E}}(z, y)$$

is obtained by applying  $F : \mathrm{Hom}_{\mathcal{C}}(z, x) \rightarrow \mathrm{Hom}_{\mathcal{D}}(F(z), F(x))$  then composing in  $\mathcal{D}$ . Hence if we view  $g_y$  as a morphism in  $\mathcal{E}$ ,  $g_y : x \rightarrow y$ , then the operation

$$(g_y)_* : \mathrm{Hom}_{\mathcal{E}}(z, x) \rightarrow \mathrm{Hom}_{\mathcal{E}}(z, y)$$

is identified with the sequence (32). So, we conclude that the sequence (32) is an isomorphism at some  $g_y$ , for each  $y$ , if and only if the fiber  $\mathcal{D} = \mathcal{E}_1$  is coreflective in  $\mathcal{E}$ , which then occurs if and only if  $q : \mathcal{E} \rightarrow \Delta^1$  is a cartesian fibration by Proposition 7.6 (c). We apply Proposition 7.14 to see that  $F$  has a right adjoint in this case.  $\square$

Taking opposites, we observe the analagous local criterion for the existence of left adjoints.

**Theorem 7.17.** *A functor between  $\infty$ -categories  $G : \mathcal{D} \rightarrow \mathcal{C}$  admits a left adjoint if and only if, for each  $x$  in  $\mathcal{C}$ , there exists a morphism  $f_x : x \rightarrow G(y)$  from some object  $y$  in  $\mathcal{D}$  such that, at any other  $z$  in  $\mathcal{D}$ , the sequence*

$$\mathrm{Hom}_{\mathcal{D}}(y, z) \xrightarrow{G} \mathrm{Hom}_{\mathcal{C}}(G(y), G(z)) \xrightarrow{f_x^*} \mathrm{Hom}_{\mathcal{C}}(x, G(z))$$

is an isomorphism in  $\mathbf{h}\mathcal{K}\mathbf{an}$ .

### 7.5. Going halvesies on adjoints.

`prop:adj_half`

**Proposition 7.18.** *For functors  $F : \mathcal{C} \rightarrow \mathcal{D}$  and  $G : \mathcal{D} \rightarrow \mathcal{C}$  between  $\infty$ -categories the following are equivalent:*

- (a) *The transformation  $F$  is left adjoint to  $G$ .*
- (b) *There is a transformation  $\epsilon : FG \rightarrow id_{\mathcal{D}}$  for which, at each  $z$  in  $\mathcal{C}$  and  $y$  in  $\mathcal{D}$ , the composite*

$$\mathrm{Hom}_{\mathcal{C}}(z, G(y)) \xrightarrow{F} \mathrm{Hom}_{\mathcal{D}}(F(z), FG(y)) \xrightarrow{(\epsilon_y)^*} \mathrm{Hom}_{\mathcal{D}}(F(z), y)$$

*is an isomorphism in  $\mathbf{h}\mathcal{K}\mathbf{an}$ .*

- (c) *There is a transformation  $\eta : id_{\mathcal{C}} \rightarrow GF$  for which, at each  $x$  in  $\mathcal{C}$  and  $z$  in  $\mathcal{D}$ , the composite*

$$\mathrm{Hom}_{\mathcal{D}}(F(x), z) \xrightarrow{G} \mathrm{Hom}_{\mathcal{C}}(GF(x), G(z)) \xrightarrow{\eta_x^*} \mathrm{Hom}_{\mathcal{C}}(x, G(z))$$

*is an isomorphism in  $\mathbf{h}\mathcal{K}\mathbf{an}$ .*

Furthermore, in the cases of (b), the transformation  $\epsilon$  is the counit of an adjunction between  $F$  and  $G$ , and in the case of (c), the transformation  $\eta$  is the unit of such an adjunction.

*Proof.* The implications (a)  $\Rightarrow$  (b) and (a)  $\Rightarrow$  (c) are immediate. For the implication (b)  $\Rightarrow$  (a), suppose we have a transformation  $\epsilon : FG \rightarrow id_{\mathcal{D}}$  as in (b). Consider the weighted nerve  $q : \mathcal{E} = N^F(\Delta^1) \rightarrow \Delta^1$ . As was argued in the proof of Theorem 7.16,  $\mathcal{D} = \mathcal{E}_1$  is coreflective in  $\mathcal{C}$  and by Lemma 7.13 the fibration  $q$  is both cartesian and cocartesian. Furthermore Lemma 7.13 tells us that, at each  $y$  in  $\mathcal{D}$ , the morphism  $(G(y), \epsilon_y : FG(y) \rightarrow y)$  is  $q$ -cartesian in  $\mathcal{E}$ .

Define the functor  $\tilde{\epsilon} : \Delta^1 \times \mathcal{D} = \Delta^1 \times \mathcal{E}_1 \rightarrow \mathcal{E}$  which takes a simplex  $\tilde{\sigma} = (\alpha, \sigma) : \Delta^n \rightarrow \Delta^1 \times \mathcal{D}$  to the triple

$$\{\alpha : \Delta^n \rightarrow \Delta^1, G\sigma|_{\Delta^{\alpha^{-1}(0)}} : \Delta^{\alpha^{-1}(0)} \rightarrow \mathcal{C}, \epsilon\tilde{\sigma} : \Delta^n \rightarrow \mathcal{D}\}.$$

By direct inspection  $\tilde{\epsilon}$  fits into a diagram

$$\begin{array}{ccc} \{1\} \times \mathcal{E}_1 & \xrightarrow{\mathrm{incl}} & \mathcal{E} \\ \downarrow & \searrow \tilde{\epsilon} & \downarrow q \\ \Delta^1 \times \mathcal{D} = \Delta^1 \times \mathcal{E}_1 & \xrightarrow{\mathrm{proj}} & \Delta^1 \end{array} \quad (33) \quad \boxed{\text{eq:2513}}$$

and at each  $y$  in  $\mathcal{D}$  the edge  $\tilde{\epsilon}|_{\Delta^1 \times \{y\}} = (G(y), \epsilon_y : FG(y) \rightarrow y) : \Delta^1 \rightarrow \mathcal{E}$  is  $q$ -cartesian, as explained above. So  $\tilde{\epsilon}$  is the unique cartesian transformation which splits the above square (see Corollary II-2.8), and by Theorem 7.15 the functor  $G = \tilde{\epsilon}|_{\{0\} \times \mathcal{E}_1}$  is seen to be right adjoint to  $F$ .

We claim now that  $\epsilon : FG \rightarrow id_{\mathcal{D}}$  is the counit transformation in a pair of transformation which  $(\epsilon, \epsilon)$  which realize  $G$  as right adjoint to  $F$ . However, this follows by uniqueness of the cartesian transformation which splits the diagram (33). Specifically, the counit  $\epsilon' : FG \rightarrow id_{\mathcal{D}}$  can be used to define a cocartesian transformation  $\tilde{\epsilon}' : \Delta^1 \times \mathcal{D} \rightarrow \mathcal{E}$  exactly as above, so that we obtain an isomorphism  $\tilde{\epsilon} \cong \tilde{\epsilon}'$  in  $\mathrm{Fun}(\Delta^1 \times \mathcal{D}, \mathcal{E})$ .

We have the transformation  $k : F \rightarrow id_{\mathcal{D}}$  in  $\text{Fun}(\Delta^1, \mathcal{Cat}_{\infty})$  which one simply observes by the existence of the strictly commuting diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ F \downarrow & & \downarrow id_{\mathcal{D}} \\ \mathcal{D} & \xrightarrow{id_{\mathcal{D}}} & \mathcal{D}. \end{array}$$

This tranformation defines a functor  $N^k : \mathcal{E} = N^F(\Delta^1) \rightarrow N^{id_{\mathcal{D}}}(\Delta^1) = \Delta^1 \times \mathcal{D}$ , and we compose with the projection  $\Delta^1 \times \mathcal{D} \rightarrow \mathcal{D}$  to obtain a functor

$$\pi : \mathcal{E} \rightarrow \mathcal{D}.$$

The compositions recover our original functors  $\pi\tilde{\epsilon} = \epsilon$  and  $\pi\tilde{\epsilon}' = \epsilon'$  so that we obtain an isomorphism  $\epsilon \cong \epsilon'$  in  $\text{Fun}(\Delta^1 \times \mathcal{D}, \mathcal{D})$ . Since compositions of morphisms are stable under isomorphisms of morphisms, in any  $\infty$ -category, the fact that  $\epsilon'$  can be paired with a transformation  $\eta : id_{\mathcal{C}} \rightarrow GF$  with witnesses  $G$  as right adjoint to  $F$  implies that the pair  $(\epsilon, \eta)$  also witnesses  $G$  as right adjoint to  $F$ .

We now obtain the converse implication (b)  $\Rightarrow$  (a), and hence the equivalence (a)  $\Leftrightarrow$  (b). The equivalence (a)  $\Rightarrow$  (c) follows by taking opposites.  $\square$

## 7.6. Universal properties of adjoints.

prop:univprop\_right

**Proposition 7.19.** *Suppose a functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  admits a right adjoint  $G : \mathcal{D} \rightarrow \mathcal{C}$ , and let  $\epsilon : FG \rightarrow id_{\mathcal{D}}$  be the counit transformation for this adjunction. Suppose that we have another functor  $G' : \mathcal{D} \rightarrow \mathcal{C}$  and a transformation  $\epsilon' : FG' \rightarrow id_{\mathcal{D}}$ . Then the following hold:*

- (1) *There exists a transformation  $\zeta : G' \rightarrow G$  and an identification of  $\epsilon'$  as a composite  $\epsilon' = \epsilon(F\zeta)$  in  $\text{Fun}(\Delta^1 \times \mathcal{D}, \mathcal{D})$ .*
- (2) *The transformation  $\zeta$  from (1) is an isomorphism if and only if  $\epsilon'$  realizes  $G'$  as a(nother) right adjoint  $F$ .*

*Proof.* (1) We have the cartesian (and cocartesian) fibration  $q : \mathcal{E} = N^F(\Delta^1) \rightarrow \Delta^1$  and the cocartesian lift

$$\begin{array}{ccc} \{1\} & \xrightarrow{\text{incl}} & \text{Fun}(\mathcal{D}, \mathcal{E}) \\ \downarrow & \nearrow \tilde{\epsilon} & \downarrow q_* \\ \Delta^1 & \xrightarrow{\text{proj}} & \text{Fun}(\mathcal{D}, \Delta^1) \end{array}$$

as in the proof of Proposition 7.18. (Here we recall that  $q_*$  is a cartesian fibration, by Proposition II-2.6.) The map  $\tilde{\epsilon} : \Delta^1 \times \mathcal{D} \rightarrow \mathcal{E}$  has restrictions

$$\tilde{\epsilon}|_0 = (\mathcal{D} \xrightarrow{G} \mathcal{C} = \mathcal{E}_0 \rightarrow \mathcal{E}) \quad \text{and} \quad \tilde{\epsilon}|_1 = (\mathcal{D} = \mathcal{E}_1 \rightarrow \mathcal{E}).$$

Let us denote these maps  $i_0G$  and  $i_1$  respectively.

Pulling back, we obtain a cocartesian (and cocartesian) fibration  $\mathcal{F} \rightarrow \Delta^1$  and a cartesian lift  $\tilde{\epsilon}$  of the morphism  $0 < 1$ . From  $\epsilon'$  we obtain a note-necessarily-cartesian lift  $\tilde{\epsilon}' : \Delta^1 \rightarrow \mathcal{F}$  which is defined in exactly the same manner.

By Lemma 7.13 the map  $\tilde{\epsilon}$  provides an equivalence

$$\tilde{\epsilon}_* : \text{Hom}_{\text{Fun}(\mathcal{D}, \mathcal{C})}(G', G) \rightarrow \text{Hom}_{\text{Fun}(\mathcal{D}, \mathcal{E})}(i_0G', i_1)$$

in  $\mathcal{H}an$ , so that the transformation  $\tilde{\epsilon}' : i_0 G' \rightarrow i_1$  lifts uniquely to a map  $\zeta : G' \rightarrow G$  with  $\tilde{\epsilon}\zeta = \tilde{\epsilon}'$  in  $\text{Fun}(\Delta^1, \text{Fun}(\mathcal{D}, \mathcal{E})) = \text{Fun}(\Delta^1 \times \mathcal{D}, \mathcal{E})$ .

Again we consider the transformation  $k : \Delta^1 \rightarrow \text{Fun}(\Delta^1, \mathcal{C}at_\infty)$  between  $F$  and  $id_{\mathcal{D}}$  provided by the strictly commuting diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ F \downarrow & & \downarrow id \\ \mathcal{D} & \xrightarrow{id} & \mathcal{D} \end{array}$$

to obtain a projection  $\pi : \mathcal{E} \rightarrow \mathcal{D}$  via the composite

$$\mathcal{E} = N^F(\Delta^1) \xrightarrow{N^k} N^{id_{\mathcal{D}}}(\Delta^1) = \Delta^1 \times \mathcal{D} \xrightarrow{p_2} \mathcal{D}.$$

Composing with  $\pi$  provides the claimed identification  $\epsilon(F\zeta) = \pi\tilde{\epsilon}\zeta = \pi\tilde{\epsilon}' = \epsilon'$ . The proof of (2) is left to the interested reader.  $\square$

Of course we have the analogous universal property for the left adjoint, which one obtains by taking opposites.

prop:univprop\_left

**Proposition 7.20.** *Suppose a functor  $G : \mathcal{D} \rightarrow \mathcal{C}$  admits a left adjoint  $F : \mathcal{C} \rightarrow \mathcal{D}$ , and let  $\eta : id_{\mathcal{C}} \rightarrow GF$  be the unit transformation for this adjunction. Suppose that we have another functor  $F' : \mathcal{D} \rightarrow \mathcal{C}$  and a transformation  $\eta' : id_{\mathcal{C}} \rightarrow GF'$ . Then the following hold:*

- (1) *There exists a transformation  $\zeta : F \rightarrow F'$  and an identification of  $\eta'$  as a composite  $\eta' = (G\zeta)\eta$  in  $\text{Fun}(\Delta^1 \times \mathcal{C}, \mathcal{C})$ .*
- (2) *The transformation  $\zeta$  from (1) is an isomorphism if and only if  $\eta'$  realizes  $F'$  as a(nother) left adjoint  $F$ .*

sect:D\_loc

## 8. THE DERIVED $\infty$ -CATEGORY VIA LOCALIZATION

We show that the derived  $\infty$ -category  $\mathcal{D}(\mathbb{A})$  is obtainable as a localization  $\mathcal{D}(\mathbb{A}) = \mathcal{K}(\mathbb{A})[\text{Qiso}^{-1}]$ . After applying some fundamental results from higher algebra [14], we observe a further identification of the derived  $\infty$ -category as a localization of the *discrete* category of cochains

$$\mathcal{D}(\mathbb{A}) = \text{Ch}(\mathbb{A})[\text{Qiso}^{-1}]. \quad (34)$$

eq:ch\_loc

This latter characterization is, at least from the perspective of the author, mind blowing, as it is not premeditated by anything observed in classical representation theory or geometry.

Though it is outside of the scope of the present document, the characterization from (34) can be used to transfer various structures, e.g. symmetric or braided monoidal structures, from the abelian setting to the derived setting.

**8.1. The setup.** Throughout this section  $\mathbb{A}$  is a Grothendieck abelian category. We recall that  $\mathbb{A}$  admits enough injectives in this case, and that every complex  $V$  admits a quasi-isomorphism  $V \rightarrow I$  to a  $K$ -injectives [20, Theorem 3.13]. From this we conclude that the full subcategory

$$\mathcal{D}_{\text{Inj}} := \mathcal{K}(\mathbb{A})_{\text{Inj}} = \left\{ \begin{array}{l} \text{The full } \infty\text{-subcategory of } K\text{-injective} \\ \text{complexes in } \mathcal{K}(\mathbb{A}). \end{array} \right.$$

is reflective in  $\mathcal{K}(\mathbb{A})$  (Example 7.2). Of course, the  $\infty$ -category  $\mathcal{D}_{\text{Inj}}$  is one of our standard models for the derived  $\infty$ -category of  $\mathbb{A}$  (Section I-2.5). However, the point of this section is to argue that the derived category should be characterized, or defined, via a universal property. So we emphasize throughout the section the particular nature of  $\mathcal{D}_{\text{Inj}}$  as the  $\infty$ -subcategory of  $K$ -injectives in  $\mathcal{K}(\mathbb{A})$ .

We are also interested in the cases where  $\mathbb{A}$  admits enough projectives. For examples, one might consider:

- $\mathbb{A} = A\text{-Mod}$  for a ring  $A$ .
- $\mathbb{A} = \text{QCoh}(\mathfrak{X})$  where  $\mathfrak{X} = [X/G]$  is the quotient stack of an affine scheme by the action of a reductive algebraic group  $G$ , in characteristic 0. In this case the projective objects are identified with equivariant vector bundles on  $X$ , under the pullback equivalence between  $\text{QCoh}(\mathfrak{X})$  and equivariant vector bundles on  $X$ .
- $\mathbb{A} = \text{Rep}_q(G)$  the category of quantum group representations for a semisimple algebraic group  $G$  at an arbitrary complex parameter  $q \in \mathbb{C}$ .

In this case the full subcategory of  $K$ -projectives

$$\mathcal{D}_{\text{Proj}} := \mathcal{K}(\mathbb{A})_{\text{Proj}} = \left\{ \begin{array}{l} \text{The full } \infty\text{-subcategory of } K\text{-injective} \\ \text{complexes in } \mathcal{K}(\mathbb{A}) \end{array} \right.$$

forms a coreflective subcategory in  $\mathcal{K}(\mathbb{A})$  (Example 7.3), and we have a canonical equivalence

$$\mathcal{D}_{\text{Proj}} \cong \mathcal{D}_{\text{Inj}}$$

by Theorem I-13.5. So in this case we have reflective and coreflective subcategories

$$\mathcal{D}_{\text{Proj}}, \mathcal{D}_{\text{Inj}} \subseteq \mathcal{K}(\mathbb{A})$$

which provide two distinct, but categorically indistinguishable, “models” for the derived  $\infty$ -category.

## 8.2. Reflections and coreflections as resolutions.

lem:injres\_refl

**Lemma 8.1.** *For a morphism  $f : V \rightarrow X$  in  $\mathcal{K}(\mathbb{A})$ , the following are equivalent:*

- The object  $X$  is  $K$ -injective and  $f$  is a quasi-isomorphism.*
- The morphism  $f$  is a  $\mathcal{D}_{\text{Inj}}$ -reflection.*

*Proof.* The implication (a)  $\Rightarrow$  (b) follows from the fact that maps into any  $K$ -injective  $\text{Hom}_{\mathbb{A}}^*(-, Z)$  preserve quasi-isomorphisms. Hence the functor

$$\text{Hom}_{\mathcal{K}(\mathbb{A})}(-, Z) \cong K \text{Hom}_{\mathbb{A}}^*(-, Z) : \mathcal{K}(\mathbb{A}) \rightarrow \mathbf{h}\mathcal{K}an$$

sends quasi-isomorphisms to isomorphisms.

For (b)  $\Rightarrow$  (a), suppose  $f$  is a  $\mathcal{D}_{\text{Inj}}$ -reflection. Then  $X$  is  $K$ -injective, by definition. Suppose, by way of contradiction, that  $f$  is not a quasi-isomorphism. Then the mapping cone  $\text{cone}(f)$  is not acyclic, and there is some integer  $i$  so that

$$H^i(\text{cone}(f)) \neq 0.$$

Let  $\alpha'' : H^i(\text{cone}(f)) \rightarrow I^0$  be an inclusion into an injective object (which exists since  $\mathbb{A}$  has enough injectives),  $\alpha' : Z^0(\text{cone}(f)) \rightarrow I^0$  be the restriction along the projection from the cocycles, and  $\alpha : (\text{cone}(f))^i \rightarrow I^0$  be an arbitrary lift to degree  $i$  cochains. We note that such a lift exists via injectivity of  $I^0$ .

Take now  $I = \Sigma^{-i}I^0$ , considered as a complex. The map  $\alpha$  now defines a map of cochains

$$\alpha : \text{cone}(f) \rightarrow I$$

which recovers  $\alpha''$  on cohomology. In particular,  $\alpha$  is not homotopically trivial, and hence realizes a nonzero class in cohomology

$$\bar{\alpha} \in H^0 \text{Hom}_{\mathbb{A}}^*(\text{cone}(f), I) \cong H^0(\Sigma^{-1} \text{cone}(f^*)).$$

It follows that the induced map  $f^* : \text{Hom}_{\mathbb{A}}^*(X, I) \rightarrow \text{Hom}_{\mathbb{A}}^*(V, I)$  is not a quasi-isomorphism.

We have in particular

$$\begin{aligned} \text{gr } H^0(\Sigma^{-1} \text{cone}(f^*)) &= \ker(H^0 \text{Hom}_{\mathbb{A}}^*(X, I) \rightarrow H^0 \text{Hom}_{\mathbb{A}}^*(V, I)) \\ &\quad \oplus \\ &\text{coker}(H^{-1} \text{Hom}_{\mathbb{A}}^*(X, I) \rightarrow H^{-1} \text{Hom}_{\mathbb{A}}^*(V, I)) \end{aligned}$$

under the apparent filtration on the mapping cone so that the above arguments show that at least one of the maps

$$H^\varepsilon(f^*) : H^\varepsilon \text{Hom}_{\mathbb{A}}^*(X, I) \rightarrow H^\varepsilon \text{Hom}_{\mathbb{A}}^*(V, I)$$

at  $\varepsilon = 0, -1$ , is not an isomorphism. It follows that the induced map of simplicial abelian groups

$$f^* : \text{Hom}_{\mathcal{K}(\mathbb{A})}(X, I) \cong K \text{Hom}_{\mathbb{A}}^*(X, I) \rightarrow K \text{Hom}_{\mathbb{A}}^*(V, I) \cong \text{Hom}_{\mathcal{K}(\mathbb{A})}(V, I)$$

is not a isomorphism in  $\mathbf{h}\mathcal{K}\mathbf{an}$  by Theorem I-11.13. Since the complex  $I$  is  $K$ -injective, this contradicts the assumption that  $f$  is a  $\mathcal{D}_{\text{Inj}}$ -reflection, and we conclude that reflection-ness of  $f$  forces  $f$  to be a quasi-isomorphism.  $\square$

Completely similar arguments apply in the projective situation.

lem:projres\_corefl

**Lemma 8.2.** *Suppose that  $\mathbb{A}$  has enough projectives. Then for a morphism  $g : X \rightarrow V$  in  $\mathcal{K}(\mathbb{A})$  the following are equivalent:*

- (a) *The object  $X$  is  $K$ -projective and  $g$  is a quasi-isomorphism.*
- (b) *The morphism  $g$  is a  $\mathcal{D}_{\text{Proj}}$ -coreflection.*

### 8.3. Precomposition and natural isomorphisms in functor categories.

lem:1696

**Lemma 8.3.** *Let  $\zeta : F_0 \rightarrow F_1$  be a natural transformation between functors  $F_i : \mathcal{K} \rightarrow \mathcal{K}'$ .*

- (1) *For each  $\infty$ -category  $\mathcal{C}$  the functors  $\zeta$  induces a natural transformation  $\zeta^* : F_0^* \rightarrow F_1^*$  between the corresponding functors  $F_i^* : \text{Fun}(\mathcal{K}', \mathcal{C}) \rightarrow \text{Fun}(\mathcal{K}, \mathcal{C})$ .*
- (2) *If  $\zeta$  is an isomorphism, then  $\zeta^*$  is an isomorphism as well.*

**Construction 8.3.** The transformation  $\zeta$  is a 2-simplex  $\zeta : \Delta^1 \rightarrow \text{Fun}(\mathcal{K}, \mathcal{K}')$  which restricts to  $F_i$  at  $\{i\}$ , for  $i = 0, 1$ . So composition in the simplicial category  $\underline{\text{Cat}}_\infty$  provides us with a map

$$\zeta^* : \text{Fun}(\mathcal{K}', \mathcal{C}) \times \Delta^1 \rightarrow \text{Fun}(\mathcal{K}', \mathcal{C}) \times \text{Fun}(\mathcal{K}, \mathcal{K}') \xrightarrow{\circ} \text{Fun}(\mathcal{K}, \mathcal{C})$$

whose restrictions to  $\{i\} \subseteq \Delta^1$  recover the maps  $F_i^*$ . So  $\zeta^*$  is a transformation  $\zeta^* : F_0^* \rightarrow F_1^*$ .

Similarly, if we have an  $n$ -simplex  $\sigma : \Delta^n \rightarrow \text{Fun}(\mathcal{K}, \mathcal{K}')$  with vertices  $G_i : \mathcal{K} \rightarrow \mathcal{K}'$  we get an  $n$ -simplex  $\sigma^* : \text{Fun}(\mathcal{K}', \mathcal{C}) \times \Delta^n \rightarrow \text{Fun}(\mathcal{K}, \mathcal{C})$  with vertices

$G_i^*$ , and one sees that  $\sigma^*$  is degenerate whenever  $\sigma$  is degenerate. Hence diagrams of the form

$$\begin{array}{ccc} & F_1 & \\ \zeta \nearrow & & \searrow \eta \\ F_0 & \xrightarrow{id_{F_0}} & F_0 \end{array} \quad \begin{array}{ccc} & F_0 & \\ \eta \nearrow & & \searrow \zeta \\ F_1 & \xrightarrow{id_{F_1}} & F_1 \end{array}$$

in  $\text{Fun}(\mathcal{K}, \mathcal{K}')$  imply diagrams of the form

$$\begin{array}{ccc} & F_1^* & \\ \zeta^* \nearrow & & \searrow \eta^* \\ F_0^* & \xrightarrow{id_{F_0^*}} & F_0^* \end{array} \quad \begin{array}{ccc} & F_0^* & \\ \eta^* \nearrow & & \searrow \zeta^* \\ F_1^* & \xrightarrow{id_{F_1^*}} & F_1^* \end{array}$$

in  $\text{Fun}(\text{Fun}(\mathcal{K}', \mathcal{C}), \text{Fun}(\mathcal{K}, \mathcal{C}))$ . So we see directly that  $\zeta^*$  is isomorphism whenever  $\zeta$  is an isomorphism.  $\square$

sect:loclocloc

#### 8.4. Localizing the homotopy $\infty$ -category against quasi-isomorphisms.

prop:pre\_loc

**Proposition 8.4.** *Let  $L : \mathcal{K}(\mathbb{A}) \rightarrow \mathcal{D}_{\text{Inj}}$  be the left adjoint to the inclusion  $i : \mathcal{D}_{\text{Inj}} \rightarrow \mathcal{K}(\mathbb{A})$ , along with the unit and counit transformations*

$$u : id_{\mathcal{K}(\mathbb{A})} \rightarrow iL \quad \text{and} \quad \epsilon : Li \rightarrow id_{\mathcal{D}_{\text{Inj}}}$$

as in Proposition 7.10.

- (1) *At each  $x$  in  $\mathcal{K}(\mathbb{A})$  the unit transformation  $u_x : x \rightarrow L(x)$  is a quasi-isomorphism.*
- (2) *A map  $\alpha : x \rightarrow y$  in  $\mathcal{K}(\mathbb{A})$  is a quasi-isomorphism if and only if  $L(\alpha) : L(x) \rightarrow L(y)$  is an isomorphism in  $\mathcal{D}_{\text{Inj}}$ .*
- (3) *The counit transformation is a natural isomorphism from the composite*

$$\mathcal{D}_{\text{Inj}} \xrightarrow{i} \mathcal{K}(\mathbb{A}) \xrightarrow{L} \mathcal{D}_{\text{Inj}}$$

*to the identity  $id_{\mathcal{D}_{\text{Inj}}}$ , and so defines a 2-simplex in the mapping complex  $\epsilon : \Delta^2 \rightarrow \text{Fun}(\mathcal{D}_{\text{Inj}}, \mathcal{D}_{\text{Inj}})^{\text{Kan}}$ .*

*Proof.* Statement (1) follows from the characterization of  $\mathcal{D}_{\text{Inj}}$ -reflections provided in Lemma 8.1 and Theorem 7.11. For (2), naturality of  $u$  implies, for each morphism  $\alpha : x \rightarrow y$ , the existence of a diagram

$$\begin{array}{ccc} L(x) & \xrightarrow{L(\alpha)} & L(y) \\ u_x \uparrow & & \uparrow u_y \\ x & \xrightarrow{\alpha} & y \end{array}$$

in the discrete homotopy category  $K(\mathbb{A}) = \text{h}\mathcal{K}(\mathbb{A})$ . The vertical maps in this diagram are quasi-isomorphisms by (1), so that  $\alpha$  is a quasi-isomorphism if and only if  $L(\alpha)$  is a quasi-isomorphism. However, a map between  $K$ -injectives is a quasi-isomorphism if and only if it is a homotopy equivalence, i.e. an isomorphism in  $\mathcal{D}_{\text{Inj}}$ . So we conclude that  $\alpha$  is a quasi-isomorphism if and only if  $L(\alpha)$  is an isomorphism in  $\mathcal{D}_{\text{Inj}}$ . Statement (3) is implied directly by the generic description of  $\epsilon$  given in Proposition 7.10  $\square$



Let us recall that the localization  $\mathcal{C}[W^{-1}]$  of an  $\infty$ -category at a class of morphisms  $W \subseteq \mathcal{C}[1]$ , with all degenerate 1-simplices in  $W$ , is any  $\infty$ -category  $\mathcal{D}$  equipped with a functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  which induces, at all  $\infty$ -categories  $\mathcal{E}$ , a fully faithful functor

$$F^* : \text{Fun}(\mathcal{D}, \mathcal{E}) \rightarrow \text{Fun}(\mathcal{C}, \mathcal{E})$$

whose image is the full  $\infty$ -subcategory spanned by all functors  $\mathcal{C} \rightarrow \mathcal{E}$  which sends all maps in  $W$  to isomorphisms in  $\mathcal{E}$  (Definition II-14.17). In this case we write, somewhat ambiguously,  $\mathcal{D} = \mathcal{C}[W^{-1}]$ .

**Remark 8.5.** One can think of localization fairly clearly from the perspective of marked simplicial sets. A functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is a localization of  $\mathcal{C}$  relative to a class of maps  $W$  if and only if it induces an equivalence between the category of marked functors

$$\text{Fun}(\mathcal{C}, \mathcal{D}) = \text{Fun}^b((\mathcal{C}, \text{Isom}), (\mathcal{D}, \text{Isom})),$$

at general  $\mathcal{E}$ , and the category of marked functors  $\text{Fun}^b((\mathcal{C}, W), (\mathcal{E}, \text{Isom}))$  [13, Section 3.1.3].

thm:DInj\_as\_loc

**Theorem 8.6.** *Let  $\mathbb{A}$  be a Grothendieck abelian category.*

- (1) *Given any  $\infty$ -category  $\mathcal{E}$ , restriction along the left adjoint  $L : \mathcal{K}(\mathbb{A}) \rightarrow \mathcal{D}_{\text{Inj}}$  to the inclusion provides a fully faithful functor*

$$L^* : \text{Fun}(\mathcal{D}_{\text{Inj}}, \mathcal{E}) \rightarrow \text{Fun}(\mathcal{K}(\mathbb{A}), \mathcal{E})$$

*which is an equivalence onto the full subcategory spanned by functors which sends all quasi-isomorphisms in  $\mathcal{K}(\mathbb{A})$  to isomorphisms in  $\mathcal{E}$ .*

- (2) *For  $\text{Fun}(\mathcal{K}(\mathbb{A}), \mathcal{E})^{\text{Qiso}}$  the full subcategory spanned by all functors which send quasi-isomorphisms to isomorphisms in  $\mathcal{E}$ , the inverse to the equivalence  $L^*$  is provided by restriction  $i^* : \text{Fun}(\mathcal{K}(\mathbb{A}), \mathcal{E})^{\text{Qiso}} \rightarrow \text{Fun}(\mathcal{D}_{\text{Inj}}, \mathcal{E})$  along the inclusion  $i : \mathcal{D}_{\text{Inj}} \rightarrow \mathcal{K}(\mathbb{A})$ .*

- (3) *the functor  $L : \mathcal{K}(\mathbb{A}) \rightarrow \mathcal{D}_{\text{Inj}}$  exhibits  $\mathcal{D}_{\text{Inj}}$  as a localization  $\mathcal{D}_{\text{Inj}} = \mathcal{K}(\mathbb{A})[\text{Qiso}^{-1}]$ .*

*Proof.* (3) Follows from (1), simply by the definition of a localization. We prove (1) and (2). Take  $\mathcal{K} = \mathcal{K}(\mathbb{A})$  and  $\mathcal{D} = \mathcal{D}_{\text{Inj}}$ , and let  $\mathcal{E}$  be arbitrary. Take  $\text{Fun}(\mathcal{K}, \mathcal{E})^{\text{Qiso}}$  the full  $\infty$ -subcategory of functors  $T : \mathcal{K} \rightarrow \mathcal{E}$  which send quasi-isomorphisms in  $\mathcal{K}$  to isomorphisms in  $\mathcal{E}$ . Then, by Proposition 8.4 (2), the functor  $L^*$  has image in  $\text{Fun}(\mathcal{K}, \mathcal{E})^{\text{Qiso}}$  and so restricts to a map

$$L^* : \text{Fun}(\mathcal{D}, \mathcal{E}) \rightarrow \text{Fun}(\mathcal{K}, \mathcal{E})^{\text{Qiso}}.$$

We have the functor  $i^* : \text{Fun}(\mathcal{K}, \mathcal{E})^{\text{Qiso}} \rightarrow \text{Fun}(\mathcal{D}, \mathcal{E})$  provided by restricting along the inclusion  $i : \mathcal{D} \rightarrow \mathcal{K}$ . We claim that these functors are mutually inverse, and so realize the claimed equivalence. More precisely, we claim that the counit and unit transformations  $\epsilon$  and  $u$  induce isomorphisms

$$\epsilon^* : i^* L^* \rightarrow \text{id}_{\text{Fun}(\mathcal{D}, \mathcal{E})} \quad \text{and} \quad u^* : \text{id}_{\text{Fun}(\mathcal{K}, \mathcal{E})^{\text{Qiso}}} \rightarrow L^* i^*.$$

The fact that  $\epsilon^*$  is an isomorphism just follows from the fact that  $\epsilon$  itself is a isomorphism. See Proposition 8.4 and Lemma 8.3. So we need only address the transformation  $u^*$ .

First note that

$$u^* : \text{Fun}(\mathcal{K}, \mathcal{E}) \times \Delta^1 \rightarrow \text{Fun}(\mathcal{K}, \mathcal{E})$$

sends each object in the subcategory  $\text{Fun}(\mathcal{K}, \mathcal{E})^{\text{Qiso}} \times \Delta^1$  to an object in  $\text{Fun}(\mathcal{K}, \mathcal{E})^{\text{Qiso}}$ , since  $u^*$  is a transformation between  $\text{id}_{\text{Fun}(\mathcal{K}, \mathcal{E})}$  and  $L^*i^*$  and these endofunctors preserve the subcategory  $\text{Fun}(\mathcal{K}, \mathcal{E})^{\text{Qiso}}$ . Since  $\text{Fun}(\mathcal{K}, \mathcal{E})^{\text{Qiso}}$  is full in  $\text{Fun}(\mathcal{K}, \mathcal{E})$  it follows that  $u^*$  does in fact restrict to a transformation

$$u^* : \text{Fun}(\mathcal{K}, \mathcal{E})^{\text{Qiso}} \times \Delta^1 \rightarrow \text{Fun}(\mathcal{K}, \mathcal{E})^{\text{Qiso}}$$

between the identity and  $L^*i^*$ .

We need to show that  $u^*$  is a natural isomorphism. By Proposition I-7.9 it suffices to show that  $u^*$  evaluates to an isomorphism in  $\text{Fun}(\mathcal{K}, \mathcal{E})^{\text{Qiso}}$  at each functor  $T : \mathcal{K} \rightarrow \mathcal{E}$  in  $\text{Fun}(\mathcal{K}, \mathcal{E})^{\text{Qiso}}$ . By the definition of  $u^*$  from Construction 8.3 we have

$$u_T^* : \Delta^1 \xrightarrow{u} \text{Fun}(\mathcal{K}, \mathcal{K}) \xrightarrow{T} \text{Fun}(\mathcal{K}, \mathcal{E})^{\text{Qiso}} \subseteq \text{Fun}(\mathcal{K}, \mathcal{E}),$$

and to see that  $u_T^*$  is an isomorphism it again suffices to show that  $u_T^*$  evaluates to an isomorphism at each  $x$  in  $\mathcal{K}$ . At any such  $x$  we have

$$(u_T^*)_x = T(u_x) : T(x) \rightarrow TL(x).$$

By Proposition 8.4 (1) each map  $u_x$  is a quasi-isomorphism, and since  $T$  sends quasi-isomorphisms to isomorphism we have that  $(u_T^*)_x$  is an isomorphism in  $\mathcal{E}$ , as desired. So we see that  $u^*$  itself is a natural isomorphism.

We now have natural isomorphisms  $\epsilon^* : i^*L^* \rightarrow \text{id}_{\text{Fun}(\mathcal{D}, \mathcal{E})}$  and  $(u^*)^{-1} : L^*i^* \rightarrow \text{id}_{\text{Fun}(\mathcal{K}, \mathcal{E})^{\text{Qiso}}}$ . By the definition of  $\mathcal{Cat}_\infty$  as the homotopy coherent nerve of the simplicial category  $\underline{\text{Cat}}_\infty^+$ , these natural isomorphisms provide 2-simplices

$$\begin{array}{ccc} & \text{Fun}(\mathcal{K}, \mathcal{E})^{\text{Qiso}} & \\ L^* \nearrow & & \searrow i^* \\ \text{Fun}(\mathcal{D}, \mathcal{E}) & \xrightarrow{\text{id}_{\text{Fun}(\mathcal{K}, \mathcal{D})}} & \text{Fun}(\mathcal{D}, \mathcal{E}) \end{array}$$

and

$$\begin{array}{ccc} & \text{Fun}(\mathcal{D}, \mathcal{E}) & \\ i^* \nearrow & & \searrow L^* \\ \text{Fun}(\mathcal{K}, \mathcal{E})^{\text{Qiso}} & \xrightarrow{\text{id}_{\text{Fun}(\mathcal{K}, \mathcal{E})^{\text{Qiso}}}} & \text{Fun}(\mathcal{K}, \mathcal{E})^{\text{Qiso}} \end{array}$$

in  $\mathcal{Cat}_\infty$  which realize  $L^*$  and  $i^*$  as mutually inverse.  $\square$

In the event that  $\mathbb{A}$  has enough projectives, we can consider the right adjoint  $R : \mathcal{K}(\mathbb{A}) \rightarrow \mathcal{D}_{\text{Proj}}$  along with its unit and counit transformations  $u : \text{id}_{\mathcal{D}_{\text{Proj}}} \rightarrow Ri$  and  $\epsilon : iR \rightarrow \text{id}_{\mathcal{K}(\mathbb{A})}$ . We have that  $u$  is a natural isomorphism, that  $\epsilon$  evaluates to a quasi-isomorphism  $\epsilon_x : R(x) \rightarrow x$  at each  $x$  in  $\mathcal{K}(\mathbb{A})$ , and that a map  $\alpha : x \rightarrow y$  in  $\mathcal{K}(\mathbb{A})$  is a quasi-isomorphism if and only if  $R(\alpha) : R(x) \rightarrow R(y)$  is an isomorphism. To observe these properties one argues exactly as in the proof of Proposition 8.4. We can therefore argue as in the proof of Theorem 8.6 to realize the projective construction of the derived  $\infty$ -category as a localization.

**thm:DProj\_as\_loc**

**Theorem 8.7.** *Let  $\mathbb{A}$  be a Grothendieck abelian category, and suppose that  $\mathbb{A}$  has enough projectives.*

- (1) For any  $\infty$ -category  $\mathcal{E}$ , restriction along the right adjoint  $R : \mathcal{K}(\mathbb{A}) \rightarrow \mathcal{D}_{\text{Proj}}$  to the inclusion provides a fully faithful functor

$$R^* : \text{Fun}(\mathcal{D}_{\text{Proj}}, \mathcal{E}) \rightarrow \text{Fun}(\mathcal{K}(\mathbb{A}), \mathcal{E})$$

which is an equivalence onto the full  $\infty$ -subcategory spanned by those functors  $T : \mathcal{K}(\mathbb{A}) \rightarrow \mathcal{E}$  which send quasi-isomorphisms in  $\mathcal{K}(\mathbb{A})$  to isomorphisms in  $\mathcal{E}$ .

- (2) For  $\text{Fun}(\mathcal{K}(\mathbb{A}), \mathcal{E})^{\text{Qiso}}$  the full subcategory spanned by all functors which send quasi-isomorphisms to isomorphisms in  $\mathcal{E}$ , the inverse to  $R^*$  is given by restriction  $i^* : \text{Fun}(\mathcal{K}(\mathbb{A}), \mathcal{E})^{\text{Qiso}} \rightarrow \text{Fun}(\mathcal{D}_{\text{Proj}}, \mathcal{E})$  along the inclusion  $i : \mathcal{D}_{\text{Proj}} \rightarrow \mathcal{K}(\mathbb{A})$ .
- (3) The functor  $R : \mathcal{K}(\mathbb{A}) \rightarrow \mathcal{D}_{\text{Proj}}$  realizes  $\mathcal{D}_{\text{Proj}}$  as a localization  $\mathcal{D}_{\text{Proj}} = \mathcal{K}(\mathbb{A})[\text{Qiso}^{-1}]$ .

As in the case of  $\mathcal{D}_{\text{Inj}}$ , one sees that the inverse to  $R^*$  is given by restricting along the inclusion  $\mathcal{D}_{\text{Proj}} \rightarrow \mathcal{K}(\mathbb{A})$ .

cor:1844

**Corollary 8.8.** For any Grothendieck abelian category  $\mathbb{A}$  which has enough projectives, there is a unique equivalence  $\psi : \mathcal{D}_{\text{Inj}} \xrightarrow{\sim} \mathcal{D}_{\text{Proj}}$  which fits into a diagram

$$\begin{array}{ccc} & \mathcal{K}(\mathbb{A}) & \\ L^* \swarrow & & \searrow R^* \\ \mathcal{D}_{\text{Inj}} & \xrightarrow[\psi]{\sim} & \mathcal{D}_{\text{Proj}} \end{array}$$

in  $\mathcal{Cat}_\infty$ .

We leave the following exercise to the interested reader.

**Exercise 8.9.** Prove that the equivalence  $\psi : \mathcal{D}_{\text{Inj}} \rightarrow \mathcal{D}_{\text{Proj}}$  is precisely the equivalence realized previously in Section I-13. (Here of course we accept that the functor  $\psi$  is only defined up to equivalence.)

### 8.5. Re-defining the derived $\infty$ -category.

**Definition 8.10.** Given a Grothendieck abelian category  $\mathbb{A}$ , the derived  $\infty$ -category  $\mathcal{D}(\mathbb{A})$  is the localization of the homotopy  $\infty$ -category relative to the class of quasi-isomorphisms

$$\mathcal{D}(\mathbb{A}) := \mathcal{K}(\mathbb{A})[\text{Qiso}^{-1}].$$

Theorems 8.6 and 8.7 say that we can construct the derived  $\infty$ -category via  $K$ -injectives in  $\text{Ch}(\mathbb{A})$ , or via  $K$ -projectives when they exist. Up to equivalence, and relative specifically to the universal property of localization, it's all the same.

**8.6. Stable localization.** In the triangulated setting, for a triangulated subcategory  $T$  in a triangulated category  $C$  we have the Verdier localization

$$C/T := C[W_T^{-1}],$$

where  $W_T$  is the collection of all morphisms whose mapping cone lies in  $T$ . We have the canonical functor  $C \rightarrow C/T$  whose kernel is  $T$  and which is universal amongst all exact functors which annihilate  $T$ .

**Definition 8.11.** Given a stable subcategory  $\mathcal{T}$  in a stable  $\infty$ -category  $\mathcal{C}$ , let  $W_{\mathcal{T}}$  be the collection of all morphisms  $\alpha : x \rightarrow y$  in  $\mathcal{C}$  whose cofiber  $\text{cofib}(\alpha)$  is isomorphic to an object in  $\mathcal{T}$ . (Note that identity morphisms are all  $\mathcal{T}$ -acyclic since  $\mathcal{T}$  contains the zero object in  $\mathcal{C}$ .) The Verdier quotient of  $\mathcal{C}$  by  $\mathcal{T}$  is the localization

$$\mathcal{C}/\mathcal{T} := \mathcal{C}[W_{\mathcal{T}}^{-1}].$$

Given a discrete category  $\mathbb{A}$  and a class of morphisms  $S$  in  $\mathbb{A}$ , which we assume contains all identity maps, we can consider the discrete localization  $\mathbb{A}[S^{-1}]_{\text{disc}}$ . This is any discrete category equipped with a functor  $\mathbb{A} \rightarrow \mathbb{A}[S^{-1}]_{\text{disc}}$  which is universal amongst all functors to a discrete target which turn morphisms in  $S$  into isomorphisms. One can construct such a discrete localization by taking the homotopy category of the  $\infty$ -categorical (Dwyer-Kan) localization, for example.

**Lemma 8.12.** Consider an  $\infty$ -category  $\mathcal{C}$  and class of morphisms  $W \subseteq \mathcal{C}[1]$  which contains all degenerate edges. Let  $F : \mathcal{C} \rightarrow \mathcal{C}[W^{-1}]$  be the localization functor. Then the unique map

$$\bar{F} : (\text{h}\mathcal{C})[W^{-1}]_{\text{disc}} \rightarrow \text{h}(\mathcal{C}[W^{-1}])$$

induced by  $\text{h}F$  is an equivalence of categories.

*Proof.* Take  $\mathbb{C} = \text{h}\mathcal{C}$  and consider the discrete localization  $\mathbb{C}[W^{-1}]_{\text{disc}}$ . For any discrete category  $\mathbb{E}$  let  $\text{Fun}(\mathbb{C}, \mathbb{E})^W$  and  $\text{Fun}(\mathcal{C}, \mathbb{E})^W$  denote the full subcategories of functors which send maps in  $W$  to isomorphisms in  $\mathbb{E}$ . It follows that, for any discrete category  $\mathbb{E}$ , restriction along the localization map  $\mathbb{C} \rightarrow \mathbb{C}[W^{-1}]$ , which exists and is produced via a calculus of localization [23, Theorem 10.3.7], provides an equivalence

$$\text{Fun}(\mathbb{C}[W^{-1}]_{\text{disc}}, \mathbb{E}) \xrightarrow{\sim} \text{Fun}(\mathbb{C}, \mathbb{E})^W \xrightarrow{\sim} \text{Fun}(\mathcal{C}, \mathbb{E})^W.$$

The latter equivalence is ensured since the homotopy category functor is left adjoint to the inclusion  $\text{Cat} \rightarrow \text{Cat}_{\infty}$ .

Similarly, we have an equivalence

$$\text{Fun}(\text{h}(\mathcal{C}[W^{-1}]), \mathbb{E}) \xrightarrow{\sim} \text{Fun}(\mathcal{C}[W^{-1}], \mathbb{E}) \xrightarrow{\sim} \text{Fun}(\mathcal{C}, \mathbb{E})^W.$$

These two equivalences fit into a diagram

$$\begin{array}{ccc} & \text{Fun}(\mathcal{C}, \mathbb{E})^W & \\ \uparrow \sim & \nwarrow \sim & \\ \text{Fun}(\mathbb{C}[W^{-1}]_{\text{disc}}, \mathbb{E}) & \xleftarrow{\bar{F}^*} & \text{Fun}(\mathcal{C}[W^{-1}], \mathbb{E}), \end{array}$$

at arbitrary discrete  $\mathbb{E}$ , from which we conclude that  $\bar{F}^*$  is an equivalence at all  $\mathbb{E}$ . This implies that  $\bar{F}$  itself is an equivalence.  $\square$

As a particular example, we find that the homotopy category of the Verdier localization  $\mathcal{C}/\mathcal{T}$ , where  $\mathcal{C}$  is stable and  $\mathcal{T}$  is thick in  $\mathcal{C}$ , recovers the discrete Verdier localization of the homotopy category,

$$\text{h}\mathcal{C}/\text{h}\mathcal{T} \xrightarrow{\sim} \text{h}(\mathcal{C}/\mathcal{T}).$$

See for example [12, Lemma 4.6.1, Proposition 4.6.2].

From this observation one sees that the homotopy category of the localization carries a unique triangulated structure so that the localization map  $F : \mathcal{C} \rightarrow \mathcal{C}/\mathcal{T}$

induces an exact functor  $hF : h\mathcal{C} \rightarrow h\mathcal{C}/\mathcal{T}$ . One can show that this discrete triangulated structure lifts, in the most advantageous fashion, to the  $\infty$ -categorical level.

thm:stable\_loc

**Theorem 8.13** ([19, Theorem I.3.3]). *Given a thick subcategory  $\mathcal{T}$  in a stable  $\infty$ -category  $\mathcal{C}$ , the following hold:*

- (1) *The Verdier quotient  $\mathcal{C}/\mathcal{T}$  is stable.*
- (2) *The localization functor  $l : \mathcal{C} \rightarrow \mathcal{C}/\mathcal{T}$  is exact.*
- (3) *The induced map on the homotopy category  $\bar{l} : h\mathcal{C}/h\mathcal{T} \rightarrow h(\mathcal{C}/\mathcal{T})$  is an equivalence of triangulated categories.*
- (4) *For any stable  $\infty$ -category  $\mathcal{D}$ , and exact functor  $\mathcal{C} \rightarrow \mathcal{D}$  which sends  $\mathcal{T}$  to the subcategory  $\mathcal{D}_{\text{Zero}}$  of zero objects, the induced map  $F' : \mathcal{C}/\mathcal{T} \rightarrow \mathcal{D}$  is also exact.*

*Proof.* All is covered in [19] save for (4). For (4), since  $F$  and  $l$  preserve zero objects,  $F'$  preserves zero objects as well. So we need only show that  $F'$  preserves cofiber sequences. Any morphism  $\alpha' : x \rightarrow y'$  in  $\mathcal{C}/\mathcal{T}$  fits into a diagram

$$\begin{array}{ccc} x & \xrightarrow{\alpha} & y \\ \downarrow = & \searrow & \downarrow \sim \\ x & \xrightarrow{\alpha'} & y' \end{array}$$

where  $\alpha$  is in the image of the localization map. This just follows from the fact that morphisms in the homotopy category  $h\mathcal{C}/h\mathcal{T}$  are determined by a calculus of fractions.

Taking cofibers we obtain an isomorphism of pushout diagrams

$$\begin{array}{ccc} \begin{array}{ccc} & y & \\ \alpha \nearrow & & \searrow \\ x & \xrightarrow{\quad} & \text{cofib}(\alpha) \\ & 0 \searrow & \nearrow \end{array} & \xrightarrow{\sim} & \begin{array}{ccc} & y' & \\ \alpha' \nearrow & & \searrow \\ x & \xrightarrow{\quad} & \text{cofib}(\alpha') \\ & 0 \searrow & \nearrow \end{array} \end{array}$$

by Proposition II-13.20. Applying  $F$  we obtain another isomorphism of pushout diagrams

$$\begin{array}{ccc} \begin{array}{ccc} & Fy & \\ \alpha \nearrow & & \searrow \\ Fx & \xrightarrow{\quad} & F\text{cofib}(\alpha) \\ & F0 \searrow & \nearrow \end{array} & \xrightarrow{\sim} & \begin{array}{ccc} & F'y' & \\ \alpha' \nearrow & & \searrow \\ F'x & \xrightarrow{\quad} & F'\text{cofib}(\alpha') \\ & F'0 \searrow & \nearrow \end{array} \end{array}$$

in  $\mathcal{D}$ . Since  $F$  is exact the left hand diagram is a cofiber sequence and we conclude that the right hand diagram is a cofiber diagram as well, by Proposition II-13.18.  $\square$

sect:Dsmall\_loc

### 8.7. Small derived categories via localization.

prop:Dsmall\_loc

**Proposition 8.14.** *Let  $\mathbb{A}$  be a Grothendieck abelian category,  $\mathcal{K} \subseteq \mathcal{K}(\mathbb{A})$  be a full stable subcategory in the homotopy  $\infty$ -category, and take  $K = h\mathcal{K}$ . Let  $\text{Acyc}$  and  $\text{Acyc}_K$  be the triangulated subcategories of acyclic complexes in  $K(\mathbb{A})$  and  $K$ , respectively, and suppose that the induced map on discrete Verdier quotients*

$$K / \text{Acyc}_K \rightarrow K(\mathbb{A}) / \text{Acyc} = D(\mathbb{A})$$

is fully faithful. Then for  $\mathcal{D} \subseteq \mathcal{D}(\mathbb{A})$  the full subcategory spanned by the image of  $\mathcal{K}$  under the localization map, the functor

$$\mathcal{K}[\text{Qiso}^{-1}] \rightarrow \mathcal{D}$$

induced by the sequence  $\mathcal{K} \rightarrow \mathcal{K}(\mathbb{A}) \rightarrow \mathcal{D}(\mathbb{A})$  is an equivalence.

In each of the following examples one employs standard arguments to prove that the relevant map  $K / \text{Acyc}_K \rightarrow D(\mathbb{A})$  is fully faithful. Since I cannot find a reference for such arguments, we record all of the details for (only) the first example.

ex:D\_bounded

**Example 8.15** (The bounded derived category). Let  $\mathbb{A}$  be any Grothendieck abelian category and  $\mathcal{K}^b(\mathbb{A})$  be the homotopy  $\infty$ -category of bounded complexes in  $\mathbb{A}$  and  $\mathcal{D}^b(\mathbb{A})$  be the full subcategory in  $\mathcal{D}(\mathbb{A})$  spanned by complexes with bounded cohomology. We also consider the  $\infty$ -categories  $\mathcal{K}^-(\mathbb{A})$  of bounded above complexes. We claim that the functor  $\mathcal{K}^b(\mathbb{A})[\text{Qiso}^{-1}] \rightarrow \mathcal{D}^b(\mathbb{A})$  is an equivalence. For this it suffices to show, via Proposition 8.14, that the functor  $K(\mathbb{A}) / \text{Acyc}^b \rightarrow D(\mathbb{A})$  is fully faithful. We express within this example morphisms in  $D(\mathbb{A})$  by either left or right fractions, following [12, Section 3].

Since any cochain complex  $X$  which has bounded above cohomology admits a quasi-isomorphism  $X' \rightarrow X$  from a bounded above complex, we see that every morphism  $V \leftarrow X \rightarrow W$  between bounded above complexes in  $D(\mathbb{A})$  is equivalent to a morphism  $V \leftarrow X' \rightarrow W$  which only involves bounded above complexes. This shows, from the perspective of the calculus of left fractions, that the functor  $K^-(\mathbb{A}) / \text{Acyc}^- \rightarrow D(\mathbb{A})$  is full. This same fact, and a direct consideration of the equivalence relation on morphisms  $V \leftarrow X \rightarrow W$  in the calculus of left fractions, also tells us that the given functor is full. We conclude that the functor  $K^-(\mathbb{A}) / \text{Acyc}^- \rightarrow D(\mathbb{A})$  is fully faithful.

It now suffices to show that the functor  $K^b(\mathbb{A}) / \text{Acyc}^b \rightarrow K^-(\mathbb{A}) / \text{Acyc}^-$  is fully faithful. For this one notes that any bounded above complex  $X$  which has bounded cohomology admits a quasi-isomorphism  $X \rightarrow X''$  onto a bounded complex. From this fact, and a direct consideration of the calculus of right fractions which describes morphisms in the latter category, we see that the given functor is in fact fully faithful. In total, this recovers the well-known fact that the map on discrete categories

$$K^b(\mathbb{A}) / \text{Acyc}^b \rightarrow D(\mathbb{A})$$

is fully faithful, and hence that the restriction  $\mathcal{K}^b(\mathbb{A}) \rightarrow \mathcal{D}^b(\mathbb{A})$  of the localization functor  $L : \mathcal{K}(\mathbb{A}) \rightarrow \mathcal{D}(\mathbb{A})$  induces an equivalence of  $\infty$ -categories

$$\mathcal{K}^b(\mathbb{A})[\text{Qiso}^{-1}] \xrightarrow{\sim} \mathcal{D}^b(\mathbb{A}).$$

ex:D\_finite

**Example 8.16** (The finite-dimensional derived category). Let  $G$  be an algebraic group in finite characteristic, say, and consider the representation category  $\text{Rep}(G)$ . Take  $\mathcal{K}(\mathbb{A})_{fin}$  the homotopy  $\infty$ -category of bounded complexes of finite-dimensional representations. Take  $\mathcal{D}(G)_{fin}$  the full subcategory in  $\mathcal{D}(G) = \mathcal{D}(\text{Rep}(G))$  consisting of all complexes with finite-dimensional cohomology. By Proposition 8.14, the functor  $L : \mathcal{K}(G)_{fin} \rightarrow \mathcal{D}(G)_{fin}$  given by restricting the localization functor on  $\mathcal{K}(G)$  induces an equivalence of  $\infty$ -categories

$$\mathcal{K}(G)_{fin}[\text{Qiso}^{-1}] \rightarrow \mathcal{D}(G)_{fin}.$$

ex:D\_coh

**Example 8.17** (The coherent derived category). Let  $X$  be a Noetherian scheme, for example a quasi-projective scheme over a Noetherian ring  $k$ , and let  $\mathcal{K}(X)_{coh}$

denote the homotopy  $\infty$ -category of bounded complexes of coherent sheaves on  $X$ . Let  $\mathcal{D}(X)_{coh}$  be the full subcategory of complexes in  $\mathcal{D}(X) = \mathcal{D}(\mathrm{QCoh}(X))$  with coherent total cohomology. Then the restriction of the localization functor  $L : \mathcal{K}(X)_{coh} \rightarrow \mathcal{D}(X)_{coh}$  induces an equivalence

$$\mathcal{K}(X)_{coh}[\mathrm{Qiso}^{-1}] \xrightarrow{\sim} \mathcal{D}(X)_{coh}$$

**ex:D\_flat**

**Example 8.18** (The derived category via flat sheaves). Let  $X$  be a quasi-compact quasi-separated scheme and  $\mathcal{K}(X)_{flat}$  be the homotopy  $\infty$ -category of  $K$ -flat quasi-coherent sheaves on  $X$ . Then the restriction of the localization functor  $L : \mathcal{K}(X)_{flat} \rightarrow \mathcal{D}(X)$  induces an equivalence

$$\mathcal{K}(X)_{flat}[\mathrm{Qiso}^{-1}] \xrightarrow{\sim} \mathcal{D}(X)$$

[21, Lemma 3.3].

**ex:D\_perf**

**Example 8.19** (The perfect derived category). Let  $X$  be any Noetherian algebraic stack with the resolution property, i.e. for which every coherent sheaf  $M$  admits a surjection  $V \rightarrow M$  from a finite rank vector bundle. Let  $mscD(X)_{perf}$  be the full  $\infty$ -subcategory of perfect sheaves in  $\mathcal{D}(X) = \mathcal{D}(\mathrm{QCoh}(X))$ , i.e. sheaves whose image in  $D(X)$  is dualizable for the product  $\otimes_{\mathcal{O}_X}^L$ , and  $\mathcal{K}(X)_{vec}$  be the homotopy  $\infty$ -category of bounded complexes of finite rank vector bundles on  $X$ . Then the restriction of the localization functor  $L : \mathcal{K}(X)_{vec} \rightarrow \mathcal{D}(X)_{perf}$  induces an equivalence

$$\mathcal{K}(X)_{vec}[\mathrm{Qiso}^{-1}] \xrightarrow{\sim} \mathcal{D}(X)_{perf}.$$

### 8.8. Localizing directly from cochains.

**thm:K\_loc\_chain**

**Theorem 8.20** ([14, Proposition 1.3.4.5]). *Consider an additive category  $\mathbb{A}$  and any full subcategory  $\mathrm{Ch}^*(\mathbb{A}) \subseteq \mathrm{Ch}(\mathbb{A})$  which is closed under finite sums and the formation of mapping cones. Then the inclusion  $\mathrm{Ch}^*(\mathbb{A}) \rightarrow \mathcal{K}(\mathbb{A})$  induces an fully faithful functor from the localization of  $\mathrm{Ch}^*(\mathbb{A})$  against the class of homotopy equivalences*

$$\mathrm{Ch}^*(\mathbb{A})[\mathrm{Htop}^{-1}] \xrightarrow{\sim} \mathcal{K}(\mathbb{A}).$$

*which is an equivalence onto the full subcategory in  $\mathcal{K}(\mathbb{A})$  spanned by the complexes in  $\mathrm{Ch}^*(\mathbb{A})$ .*

**cor:D\_loc\_chain**

**Corollary 8.21.** *Let  $\mathbb{A}$  be a Grothendieck abelian category and  $\mathrm{Ch}^*(\mathbb{A}) \subseteq \mathrm{Ch}(\mathbb{A})$  be a full subcategory which is stable under suspension, desuspension, and the formation of mapping cones. Suppose also that the, for the corresponding homotopy  $\infty$ -category  $\mathcal{K}^*(\mathbb{A})$ , the induced map  $\mathrm{h}\mathcal{K}^*(\mathbb{A})/\mathrm{Acyc} \rightarrow \mathrm{D}(\mathbb{A})$  is fully faithful. Then the inclusion of simplicial sets  $\mathrm{Ch}^*(\mathbb{A}) \rightarrow \mathcal{D}(\mathbb{A})$  induces a fully faithful functor*

$$\mathrm{Ch}^*(\mathbb{A})[\mathrm{Qiso}^{-1}] \rightarrow \mathcal{D}(\mathbb{A})$$

*which is an equivalence onto the full subcategory in  $\mathcal{D}(\mathbb{A})$  spanned by the (images of the) complexes in  $\mathrm{Ch}^*(\mathbb{A})$ .*

Of course, in the absolute setting we obtain an equivalence

$$\mathrm{Ch}(\mathbb{A})[\mathrm{Qiso}^{-1}] \xrightarrow{\sim} \mathcal{D}(\mathbb{A}).$$

Corollary 8.21 applies to all of the examples discussed in Section 8.7 above. We employ the notations from Examples 8.15–8.19 and recall few of these instances here.

**Example 8.22** (The bounded derived category). For  $\mathbb{A}$  Grothendieck abelian, the canonical functor

$$\mathrm{Ch}^b(\mathbb{A})[\mathrm{Qiso}^{-1}] \rightarrow \mathcal{D}^b(\mathbb{A})$$

is an equivalence.

**Example 8.23** (The finite-dimensional derived category). For any algebraic group  $G$ , the canonical functor

$$\mathrm{Ch}(G)_{fin}[\mathrm{Qiso}^{-1}] \rightarrow \mathcal{D}(G)_{fin}$$

is an equivalence.

**Example 8.24** (The perfect derived category). For any Noetherian algebraic stack  $X$  with the resolution property, the canonical functor

$$\mathrm{Ch}(X)_{vec}[\mathrm{Qiso}^{-1}] \rightarrow \mathcal{D}(X)_{perf}$$

is an equivalence.

sect:deriving\_fun

## 9. LEFT AND RIGHT DERIVED FUNCTORS

We define the left and right derived functors

$$\mathrm{L}F : \mathcal{D}(\mathbb{A}) \rightarrow \mathcal{D}(\mathbb{B}) \quad \text{and} \quad \mathrm{R}G : \mathcal{D}(\mathbb{B}) \rightarrow \mathcal{D}(\mathbb{A})$$

for pairs of adjoint functors which exist at the level of the homotopy  $\infty$ -category. We show, in particular, that the left derived functor can be computed by taking  $F$ -acyclic resolutions in the domain. This approach mirrors directly the approach taken in the discrete derived setting.

**9.1. Derived functors in ideal situations.** Consider Grothendieck abelian  $\mathbb{A}$  with enough projectives, and  $\mathbb{B}$  Grothendieck abelian. Let

$$\bar{F} : \mathbf{Ch}(\mathbb{A}) \rightarrow \mathbf{Ch}(\mathbb{B}) \quad \text{and} \quad \bar{G} : \mathbf{Ch}(\mathbb{B}) \rightarrow \mathbf{Ch}(\mathbb{A})$$

be dg functors with  $\bar{F}$  left adjoint to  $\bar{G}$ . For example, we can consider the case where  $\bar{F}$  and  $\bar{G}$  are induced by adjoint functors between  $\mathbb{A}$  and  $\mathbb{G}$ . In this situation, by Theorem I-14.9, the induced functors on homotopy  $\infty$ -categories

$$F : \mathcal{K}(\mathbb{A}) \rightarrow \mathcal{K}(\mathbb{B}) \quad \text{and} \quad G : \mathcal{K}(\mathbb{B}) \rightarrow \mathcal{K}(\mathbb{A})$$

are such that  $F$  is left adjoint to  $G$ .

In this situation, we define the left and right derived functors in the expected ways.

def:der\_fun1

**Definition 9.1.** Let  $\mathbb{A}$  and  $\mathbb{B}$  be Grothendieck abelian categories. Take  $\mathrm{R} : \mathcal{D}(\mathbb{B}) \rightarrow \mathcal{K}(\mathbb{B})$  the right adjoint to the localization functor for  $\mathbb{B}$ .

For any continuous functor  $G : \mathcal{K}(\mathbb{B}) \rightarrow \mathcal{K}(\mathbb{A})$  we define the right derived functor  $\mathrm{R}G : \mathcal{D}(\mathbb{B}) \rightarrow \mathcal{D}(\mathbb{A})$  as the composite

$$\mathrm{R}G : \mathcal{D}(\mathbb{B}) \xrightarrow{\mathrm{R}} \mathcal{K}(\mathbb{B}) \xrightarrow{G} \mathcal{K}(\mathbb{A}) \xrightarrow{\mathrm{loc}} \mathcal{D}(\mathbb{A}).$$

We note that  $G$  is continuous whenever it admits a left adjoint  $F : \mathcal{D}(\mathbb{A}) \rightarrow \mathcal{D}(\mathbb{B})$ .

def:der\_fun2

**Definition 9.2.** Let  $\mathbb{A}$  and  $\mathbb{B}$  be Grothendieck abelian categories, and suppose the localization functor  $\mathcal{K}(\mathbb{A}) \rightarrow \mathcal{D}(\mathbb{A})$  has a left adjoint  $\mathrm{L} : \mathcal{D}(\mathbb{A}) \rightarrow \mathcal{K}(\mathbb{A})$ .

For any cocontinuous functor  $F : \mathcal{K}(\mathbb{A}) \rightarrow \mathcal{K}(\mathbb{B})$  We define the left derived functor  $\mathrm{L}F : \mathcal{D}(\mathbb{A}) \rightarrow \mathcal{D}(\mathbb{B})$  as the composite

$$\mathrm{L}F : \mathcal{D}(\mathbb{A}) \xrightarrow{\mathrm{L}} \mathcal{K}(\mathbb{A}) \xrightarrow{F} \mathcal{K}(\mathbb{B}) \xrightarrow{\mathrm{loc}} \mathcal{D}(\mathbb{B}).$$



**Proposition 9.3.** *Take  $\mathbb{A}$  and  $\mathbb{B}$  Grothendieck abelian and let  $F : \mathcal{K}(\mathbb{A}) \rightarrow \mathcal{K}(\mathbb{B})$  be left adjoint to a functor  $G : \mathcal{K}(\mathbb{B}) \rightarrow \mathcal{K}(\mathbb{A})$ . Suppose that the localization functor  $\mathcal{K}(\mathbb{A}) \rightarrow \mathcal{D}(\mathbb{A})$  admits a left adjoint. Then the functor  $L F : \mathcal{K}(\mathbb{A}) \rightarrow \mathcal{K}(\mathbb{B})$  is left adjoint to  $R G : \mathcal{D}(\mathbb{B}) \rightarrow \mathcal{D}(\mathbb{A})$ .*

*Proof.* This just follows from the fact that a composite of left adjoints is left adjoint to the respective composite of right adjoints (Proposition ??).  $\square$

**9.2. The candidate left derived functor.** In the general setting we do not have enough projective in  $\mathbb{A}$ , so that we cannot define the left derived functor as in the ideal setting discussed above. (One might consider for example the case of sheaves  $\mathbb{A} = \text{QCoh}(X)$  on a projective variety  $X$ .) However, we can also approach the situation via relatively acyclic complexes.

def:enough\_F

**Definition 9.4.** Let  $\mathbb{A}$  and  $\mathbb{B}$  be Grothendieck abelian categories, and  $F : \mathcal{K}(\mathbb{A}) \rightarrow \mathcal{K}(\mathbb{B})$  be any cocontinuous functor. We say  $\mathcal{K}(\mathbb{A})$  has enough  $F$ -acyclic objects if there is a stable subcategory  $\mathcal{K}(\mathbb{A})_{F\text{-ac}}$  in  $\mathcal{K}(\mathbb{A})$  which satisfies the following:

- (a)  $\mathcal{K}(\mathbb{A})_{F\text{-ac}}$  admits small colimits and the inclusion  $\mathcal{K}(\mathbb{A})_{F\text{-ac}} \rightarrow \mathcal{K}(\mathbb{A})$  is cocontinuous.
- (b) Every object  $V$  in  $\mathcal{K}(\mathbb{A})$  admits a quasi-isomorphism  $W \rightarrow V$  from an object  $W$  in  $\mathcal{K}(\mathbb{A})_{F\text{-ac}}$ .
- (c) If an object  $W$  in  $\mathcal{K}(\mathbb{A})_{F\text{-ac}}$  is acyclic then the image  $F(W)$  in  $\mathcal{K}(\mathbb{B})$  is acyclic as well.

The following is more-or-less apparent.

lem:LF\_candidate

**Lemma 9.5.** *Let  $\mathbb{A}$  and  $\mathbb{B}$  be Grothendieck abelian,  $F : \mathcal{K}(\mathbb{A}) \rightarrow \mathcal{K}(\mathbb{B})$  be a cocontinuous functor, and suppose that  $\mathcal{K}(\mathbb{A})$  has enough  $F$ -acyclic objects. Let  $\mathcal{K}(\mathbb{A})_{F\text{-ac}} \subseteq \mathcal{K}(\mathbb{A})$  be a full subcategory of  $F$ -acyclics which satisfies the constraints of Definition 9.4.*

*There is a unique functor  $L F : \mathcal{D}(\mathbb{A}) \rightarrow \mathcal{D}(\mathbb{B})$  which fits into a 2-simplex*

$$\begin{array}{ccc} \mathcal{K}(\mathbb{A})_{F\text{-ac}} & \xrightarrow{F} & \mathcal{K}(\mathbb{B}) \\ \text{loc} \downarrow & \searrow & \downarrow \text{loc} \\ \mathcal{D}(\mathbb{A}) & \xrightarrow{L F} & \mathcal{D}(\mathbb{B}) \end{array}$$

in  $\mathcal{Cat}_\infty$ .

*Proof.* As in all of the examples from Section 8.7, one observes that the sequence

$$\mathcal{K}(\mathbb{A})_{F\text{-ac}} \xrightarrow{\text{incl}} \mathcal{K}(\mathbb{A}) \xrightarrow{\text{loc}} \mathcal{D}(\mathbb{A})$$

identifies the derived  $\infty$ -category with the localization  $\mathcal{K}(\mathbb{A})_{F\text{-ac}}[\text{Qiso}^{-1}] = \mathcal{D}(\mathbb{A})$ . Furthermore, by our assumption that  $F$  preserves acyclics in  $\mathcal{K}(\mathbb{A})_{F\text{-ac}}$  exactness implies that  $F$  preserves quasi-isomorphisms. Hence the restriction functor

$$\text{Fun}(\mathcal{D}(\mathbb{A}), \mathcal{D}(\mathbb{B})) \rightarrow \text{Fun}(\mathcal{K}(\mathbb{A})_{F\text{-ac}}, \mathcal{D}(\mathbb{B}))$$

is an equivalence. Existence and uniqueness of the functor  $L F$  now follows by contractibility of the homotopy fiber

$$\text{Fun}(\mathcal{D}(\mathbb{A}), \mathcal{D}(\mathbb{B})) \times_{\text{Fun}(\mathcal{K}(\mathbb{A})_{F\text{-ac}}, \mathcal{D}(\mathbb{B}))}^{\text{htop}} \{\text{loc } F|_{F\text{-ac}}\}.$$

$\square$

### 9.3. Checking adjoints at the homotopy level.

prop:adj\_htop

**Proposition 9.6** ([15, 02EY]). *Suppose  $G : \mathcal{D} \rightarrow \mathcal{C}$  is a functor between  $\infty$ -categories which admits a left adjoint. Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be any functor which we pair with a transformation  $\eta : id_{\mathcal{C}} \rightarrow GF$ . Then the following are equivalent:*

- (1) *The transformation  $\eta$  exhibits  $F$  as left adjoint to  $G$ .*
- (2) *The induced transformation  $h\eta : id_{h\mathcal{C}} \rightarrow hG hF$  exhibits  $hF$  as left adjoint to  $hG$ .*

*Proof.* Let  $F' : \mathcal{C} \rightarrow \mathcal{D}$  be left adjoint to  $G$  with unit transformation  $\eta' : id \rightarrow GF$ . Then by Proposition 7.20 there is a transformation  $\zeta : F' \rightarrow F$  with  $\eta = (G\zeta)\eta'$ , and  $\eta$  is a unit transformation which realizes  $F$  as left adjoint to  $G$  if and only if  $\zeta$  is a natural isomorphism. Taking the homotopy categories, we see that  $h\zeta : hF' \rightarrow hF$  is an isomorphism since  $hF$  is left adjoint to  $hG$ .

We have that  $\zeta$  is a natural isomorphism if and only if, at each  $x$  in  $\mathcal{C}$ , the map  $\zeta_x : F'(x) \rightarrow F(x)$  is an isomorphism in  $\mathcal{C}$ . See Theorem I-7.6. This property can be checked at the level of the homotopy category. So we see that  $\zeta$  is in fact a natural isomorphism since  $h\zeta$  is a natural isomorphism.  $\square$

The obvious analog of Proposition 9.6 holds for right adjoints, simply by taking opposites. In the stable setting we can forgo the presupposition that a left adjoint to  $G$  exists.

prop:adj\_htop\_stab

**Proposition 9.7.** *Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  and  $G : \mathcal{D} \rightarrow \mathcal{C}$  be exact functors between stable  $\infty$ -categories, and consider a transformation  $\eta : id_{\mathcal{C}} \rightarrow GF$ . Then the following are equivalent:*

- (a) *The transformation  $\eta$  exhibits  $F$  as left adjoint to  $G$ .*
- (b) *The induced transformation  $h\eta$  exhibits  $hF$  as left adjoint to  $hG$ .*

*Proof.* We consider the composite

$$\mathrm{Hom}_{\mathcal{D}}(Fx, y') \xrightarrow{G} \mathrm{Hom}_{\mathcal{C}}(GFx, Gy') \xrightarrow{\eta^*} \mathrm{Hom}_{\mathcal{C}}(x, Gy')$$

in  $h\mathcal{K}an_*$ . By exactness of  $G$  and Proposition 5.27, the fact that the composite

$$\pi_0 \mathrm{Hom}_{\mathcal{D}}(Fx, y') \xrightarrow{G} \pi_0 \mathrm{Hom}_{\mathcal{C}}(GFx, Gy') \xrightarrow{\eta^*} \pi_0 \mathrm{Hom}_{\mathcal{C}}(x, Gy')$$

is an isomorphism at arbitrary  $y'$  implies that at all higher homotopy groups, based at 0, the composite

$$\pi_n \mathrm{Hom}_{\mathcal{D}}(Fx, y') \xrightarrow{G} \pi_n \mathrm{Hom}_{\mathcal{C}}(GFx, Gy') \xrightarrow{\eta^*} \pi_n \mathrm{Hom}_{\mathcal{C}}(x, Gy')$$

are isomorphisms. Hence the induced map on loop spaces  $\Omega \mathrm{Hom}_{\mathcal{D}}(Fx, y') \rightarrow \Omega \mathrm{Hom}_{\mathcal{C}}(x, Gy')$  is an equivalence. We considering the case  $y' = \Sigma y$ , and apply the isomorphism Proposition 5.27, so see that the composite map  $\mathrm{Hom}_{\mathcal{D}}(Fx, y) \rightarrow \mathrm{Hom}_{\mathcal{C}}(x, Gy)$  is an equivalence at all objects  $x, y : * \rightarrow \mathcal{D}$ . By Proposition 7.18 it follows that  $\eta$  realizes  $F$  as left adjoint to  $G$ .  $\square$

### 9.4. Left adjoints to right derived functors.

prop:left\_derived

**Proposition 9.8.** *Let  $\mathbb{A}$  and  $\mathbb{B}$  be Grothendieck abelian categories and  $F : \mathcal{K}(\mathbb{A}) \rightarrow \mathcal{K}(\mathbb{B})$  be left adjoint to a functor  $G : \mathcal{K}(\mathbb{B}) \rightarrow \mathcal{K}(\mathbb{A})$ . Suppose that  $\mathcal{K}(\mathbb{A})$  has enough  $F$ -acyclic objects, and let  $LF : \mathcal{D}(\mathbb{A}) \rightarrow \mathcal{D}(\mathbb{B})$  be as in the statement of Lemma 9.5. Then  $LF$  is left adjoint to the right derived functor  $RG : \mathcal{D}(\mathbb{B}) \rightarrow \mathcal{D}(\mathbb{A})$ .*

Before giving the proof let us consider the situation at the level of homotopy categories. We have the left adjoint  $i : D(\mathbb{B}) \rightarrow K(\mathbb{B})$  to the localization functor, and recall that the composite  $i \text{ loc} : K(\mathbb{B}) \rightarrow K(\mathbb{B})$ , along with the transformation  $u_V : V \rightarrow i \text{ loc}(V)$ , takes  $K$ -injective resolutions of complexes. We now have the commuting diagram

$$\begin{array}{ccccc} K(\mathbb{A})_{F\text{-ac}} & \xrightarrow{F} & K(\mathbb{B}) & \xrightarrow{Gi \text{ loc}} & K(\mathbb{B}) \\ \text{loc} \downarrow & & \text{loc} \downarrow & & \text{loc} \downarrow \\ D(\mathbb{A}) & \xrightarrow{L F} & D(\mathbb{B}) & \xrightarrow{R G} & D(\mathbb{B}) \end{array}$$

and the unit transformation  $id_{K(\mathbb{A})} \rightarrow GF$  induces a unique transformation

$$\text{loc} |_{F\text{-ac}} \rightarrow \text{loc} GF|_{F\text{-ac}} \xrightarrow{u} \text{loc} Gi \text{ loc} F|_{F\text{-ac}} \cong R G \text{ loc} F|_{F\text{-ac}} \cong R G L F \text{ loc} |_{F\text{-ac}}.$$

(Here we have omitted the  $h$  from the notations, and taken for example  $F = h F$ , to ease notation.) The above transformation induces a unique transformation for the discrete derived category

$$\eta : id_{D(\mathbb{A})} \rightarrow R G L F. \quad (35)$$

eq:3454

lem:3628

**Lemma 9.9.** *The transformation  $\eta : id_{D(\mathbb{A})} \rightarrow R G L F$  realizes  $L F$  as left adjoint to  $R G$  at the level of the discrete derived category.*

*Proof.* At the discrete level the derived category is realizable via a calculus of fractions, so that the localization functor  $\text{loc} : K(\mathbb{A}) \rightarrow D(\mathbb{A})$  essentially does nothing. Let  $u : id_{K(\mathbb{B})} \rightarrow i (= i \text{ loc})$  denote the unit transformation for the  $(\text{loc}, i)$ -adjunction and  $\bar{\eta} : id_{K(\mathbb{B})} \rightarrow GF$  denote the unit of the  $(F, G)$ -adjunction.

For  $F$ -acyclic  $M$  and  $K$ -injective  $N$  we have the diagram

$$\begin{array}{ccccc} \text{Hom}_{K(\mathbb{B})}(FM, N) & \xrightarrow{i} & \text{Hom}_{K(\mathbb{B})}(iFM, iN) & \xrightarrow{G} & \text{Hom}_{K(\mathbb{A})}(GiFM, GiN) \\ & & \downarrow u^* & & \downarrow Gu^* \\ & & \text{Hom}_{K(\mathbb{B})}(FM, iN) & \xrightarrow{G} & \text{Hom}_{K(\mathbb{A})}(GFM, GiN) \\ & & & & \downarrow \bar{\eta}^* \\ & & & & \text{Hom}_{K(\mathbb{A})}(M, GiN) \end{array}$$

in which the maps  $i$  and  $u^*$  are isomorphisms, since  $N$  is  $K$ -injective. Hence the top sequence is an isomorphism if and only if the composite

$$\text{Hom}_{K(\mathbb{B})}(FM, iN) \xrightarrow{G} \text{Hom}_{K(\mathbb{A})}(GFM, GiN) \xrightarrow{\bar{\eta}^*} \text{Hom}_{K(\mathbb{A})}(M, GiN)$$

is an isomorphism. However, this holds since  $\bar{\eta}$  is the unit of the relevant adjunction.

We apply the localization functor to obtain a diagram

$$\begin{array}{ccccc} \text{Hom}_{K(\mathbb{B})}(FM, N) & \longrightarrow & \text{Hom}_{K(\mathbb{A})}(GiFM, GiN) & \xrightarrow{\bar{\eta}^*} & \text{Hom}_{K(\mathbb{A})}(M, GiN) \\ \text{loc} \downarrow \cong & & \text{loc} \downarrow & & \text{loc} \downarrow \\ \text{Hom}_{D(\mathbb{B})}(L F M, N) & \longrightarrow & \text{Hom}_{D(\mathbb{A})}(R G L F M, R G N) & \xrightarrow{\eta^*} & \text{Hom}_{D(\mathbb{A})}(M, R G N). \end{array}$$

To see that the bottom sequence is an isomorphism it now suffices to prove that the localization functor induces an isomorphism

$$\text{loc} : \text{Hom}_{K(\mathbb{A})}(M, GN') \rightarrow \text{Hom}_{D(\mathbb{A})}(M, GN') \quad (36) \quad \text{eq:3489}$$

whenever  $N'$  is  $K$ -injective and  $M$  is  $F$ -acyclic.

Since  $K(\mathbb{A})$  has enough  $F$ -acyclics we can write maps in the derived category as the colimit

$$\text{colim}_\alpha \text{Hom}_{K(\mathbb{A})}(M_\alpha, GN') = \text{Hom}_{D(\mathbb{A})}(M, GN')$$

over quasi-isomorphisms  $\alpha : M_\alpha \rightarrow M$  between  $F$ -acyclics, and under this identification the localization map

$$\text{loc} : \text{Hom}_{K(\mathbb{A})}(M, GN') \rightarrow \text{colim}_\alpha \text{Hom}_{K(\mathbb{A})}(M_\alpha, GN')$$

is just the structure map at  $\alpha = id_M$ . But now, via  $(F, G)$ -adjunction and  $F$ -acyclicity each transition function

$$\alpha^* : \text{Hom}_{K(\mathbb{A})}(M, GN') \rightarrow \text{Hom}_{K(\mathbb{A})}(M_\alpha, GN')$$

is an isomorphism, so that this colimit diagram is essentially constant. Hence the map (36) is seen to be an isomorphism, and that the composite

$$\text{Hom}_{D(\mathbb{B})}(FM, N) \xrightarrow{R^G} \text{Hom}_{D(\mathbb{A})}(RGLFM, RGN) \xrightarrow{\eta^*} \text{Hom}_{D(\mathbb{A})}(M, RGN) \quad (37) \quad \text{eq:3508}$$

is an isomorphism at all  $F$ -acyclic  $M$  and  $K$ -injective  $N$ . Since we have enough  $F$ -acyclics and enough  $K$ -injectives, it follows that the sequence (37) is an isomorphism at all  $M$  and  $N$ , and hence that  $\eta$  realizes the claimed adjunction.  $\square$

We now provide the proof of Proposition 9.8.

*Proof of Proposition 9.8.* Let  $\eta_{\mathcal{K}} : id_{\mathcal{K}(\mathbb{A})} \rightarrow GF$  denote the unit of the adjunction. We obtain a transformation  $\eta' : \Delta^1 \times \mathcal{K}(\mathbb{A}) \rightarrow \mathcal{D}(\mathbb{A})$  as the composite

$$\text{loc}|_{F\text{-ac}} \xrightarrow{\eta_{\mathcal{K}}} \text{loc} GF|_{F\text{-ac}} \xrightarrow{u} \text{loc} Gi \text{loc} F|_{F\text{-ac}} \cong RGLF|_{F\text{-ac}} \cong RGLF \text{loc}|_{F\text{-ac}}, \quad (38) \quad \text{eq:3316}$$

where  $i : \mathcal{D}(\mathbb{B}) \rightarrow \mathcal{K}(\mathbb{B})$  is the right adjoint to localization. (Recall that the composite  $i \text{loc}$  takes functorial injective resolutions of objects.) We claim that this transformation induces a unique transformation at the level of derived categories  $\eta : id_{\mathcal{D}(\mathbb{A})} \rightarrow RGLF$  which fits into a diagram

$$\begin{array}{ccc} \Delta^1 \times \mathcal{K}(\mathbb{A}) & & \\ \downarrow \text{loc} & \searrow \eta' & \\ \Delta^1 \times \mathcal{D}(\mathbb{A}) & \xrightarrow{\eta} & \mathcal{D}(\mathbb{A}) \end{array} \quad (39) \quad \text{eq:3320}$$

in  $\mathcal{Cat}_\infty$ .

Take  $\mathcal{D} = \mathcal{D}(\mathbb{A})$  and  $\mathcal{K} = \mathcal{K}(\mathbb{A})$ . First note that the restriction functor

$$\text{Fun}(\Delta^1 \times \mathcal{D}, \mathcal{D}) \rightarrow \text{Fun}(\Delta^1 \times \mathcal{K}, \mathcal{D})^{\text{Qiso}}$$

is an equivalence, so that we can find some functor  $\eta'' : \Delta^1 \times \mathcal{D} \rightarrow \mathcal{D}$  which fits into a 2-simplex as in (39). On the boundary  $\partial\Delta^1 \times \mathcal{D}$  we obtain unique functors

(40) eq:3334

$$\begin{array}{ccc} \mathcal{K} & & \mathcal{K} \\ \swarrow \text{loc} & \downarrow \text{loc} & \swarrow \text{R } f_* \mathbb{L} f^* \text{loc} \\ \mathcal{D} & \xleftarrow{F_0} \mathcal{D} & \mathcal{D} \xrightarrow{F_1} \mathcal{D} \end{array}$$

Since the restriction functor

$$\mathrm{Fun}(\Delta^1 \times \mathcal{D}, \mathcal{D}) \rightarrow \mathrm{Fun}(\partial\Delta^1 \times \mathcal{D}, \mathcal{D})$$

$$\mathrm{Fun}(\Delta^1 \times \Delta^1 \times \mathcal{K}, \mathcal{D}) \rightarrow \mathrm{Fun}(\Delta^1 \times \partial\Delta^1 \times \mathcal{K}, \mathcal{D})$$

Finally, Proposition 9.7 tells us that the transformation  $\eta : id_{\mathcal{D}} \rightarrow RGLF$  is the unit of an adjunction between  $LF$  and  $RG$  if and only if the induced transformation on the homotopy category exhibits  $hLF$  as left adjoint to  $hRG$ . However, this was already argued in Lemma 9.9, so that the transformation  $\eta$  is seen to realize  $LF : \mathcal{D}(\mathbb{A}) \rightarrow \mathcal{D}(\mathbb{B})$  the left adjoint to  $RG$ .  $\square$

### 9.5. Left derived functors via acyclic resolutions.

In this setting, the left derived functor  $LF: \mathcal{D}(\mathbb{A}) \rightarrow \mathcal{D}(\mathbb{B})$  for  $F$  is defined as the left adjoint to the right derived functor  $RG: \mathcal{D}(\mathbb{B}) \rightarrow \mathcal{D}(\mathbb{A})$ .

We note that, by Proposition 9.8, such a left adjoint exists. Furthermore, by uniqueness of adjoints, the functor  $L F$  can be computed via any sufficiently large collection of  $F$ -acyclic complexes  $\mathcal{K}(\mathbb{A})_{F\text{-ac}} \subseteq \mathcal{K}(\mathbb{A})$ , as in Lemma 9.5.

**Example 9.11** (The push-pull adjunction). Let  $f : X \rightarrow Y$  be a map of quasi-compact and quasi-separated schemes. Then we have the push-pull adjunction on quasi-coherent sheaves, which implies an adjunction between functors

$$f^* : \mathcal{K}(\mathrm{QCoh}(Y)) \rightarrow \mathcal{K}(\mathrm{QCoh}(X)) \quad \text{and} \quad f_* : \mathcal{K}(\mathrm{QCoh}(X)) \rightarrow \mathcal{K}(\mathrm{QCoh}(Y))$$

at the level of homotopy  $\infty$ -categories. This follows by Theorem I-14.9, for example.

The category  $\mathcal{K}(\mathrm{QCoh}(Y))$  admits enough  $K$ -flat complexes, so that we can define the derived pullback functor

$$Lf^* : \mathcal{D}(\mathrm{QCoh}(Y)) \rightarrow \mathcal{D}(\mathrm{QCoh}(X)),$$

loosely speaking, by taking  $K$ -flat resolutions over  $Y$  and applying the underived pullback functor. This is precisely what one expects from standard algebraic practices, as outlined in [23] for example.

This example generalizes to the setting where  $X$  and  $Y$  are geometric stacks. Here we note that the categories of quasi-coherent sheaves remain Grothendieck abelian [22, Corollary 5.10] and that the categories of complexes of quasi-coherent sheaves admit enough  $K$ -flat objects by [7, Theorem 3.5.5].

sect:pr

## 10. PRESENTABILITY OF THE DERIVED $\infty$ -CATEGORY

We provide a very concise discussion of presentability and the adjoint functor theorem. The main deliverable here is that the derived  $\infty$ -category of a Grothendieck abelian category  $\mathcal{D}(\mathbb{A})$  is presentable.

Presentability is a kind of lax compact generation condition which places an  $\infty$ -category in a functional algebraic context, where the term algebraic here indicates a kind of “algebra of categories”. In this context we can also begin to speak of linear categories [16, Section D.1.2], sheaves of categories [16, Section 10.2], etc. One can see for example results from higher algebra [14, section 4.8], which are pre-mediated by earlier results of Kelley in the discrete setting [11, Section 6.5] [4].

Our main ambitions of the section are simply to define presentability and introduce the adjoint functor theorem. The explicit and implicit points appearing in this section are then relevant for our analysis of the indization functor, and renormalization of derived categories as presented in Section 11 below.

**10.1. Filtering and compactness at a regular cardinal.** By a regular cardinal we mean, formally, a cardinal  $\kappa$  for which satisfies the following: If  $\Lambda$  is an indexing set of cardinality  $|\Lambda| < \kappa$ , and  $\{S_\lambda : \lambda \in \Lambda\}$  is a collection of sets with  $|S_\lambda| < \kappa$  at all  $\lambda$ , then the union  $\cup_{\lambda \in \Lambda} S_\lambda$  also has cardinality less than  $\kappa$ . For some examples,  $\aleph_0$ , which characterizes countably infinite sets, is regular. Under the continuum hypothesis,  $2^{\aleph_0}$  is also regular, and in general any successor cardinal is regular. Informally, regular cardinals just provide a mechanism through which, in some instances, to impose functional size constraints.

We say a simplicial set  $K$  is  $\kappa$ -small, for a regular cardinal  $\kappa$ , if the collection of non-degenerate simplices in  $\coprod_{n \geq 0} K[n]$  has cardinality  $< \kappa$ .

def:filtered

**Definition 10.1.** An  $\infty$ -category  $\mathcal{K}$  is called  $\kappa$ -filtered if, for any  $\kappa$ -small simplicial set  $K$  and map  $i : A \rightarrow \mathcal{K}$ , there a map  $i^+ : A \star \{*\} \rightarrow \mathcal{K}$  with  $i^+|_A = i$ . We call  $\mathcal{K}$  filtered if it is  $\aleph_0$ -filtered.

We note that, via functoriality of the join, all maps  $i : A \rightarrow \mathcal{K}$  from  $\kappa$ -small  $K$  admit an extension as proposed if and only if all injective maps from  $\kappa$ -small  $K$  admit such an extension. Hence  $\mathcal{K}$  is filtered if and only if each finite simplicial subset in  $\mathcal{K}$  admits a cone point in  $\mathcal{K}$ . In particular, we recover the familiar notion of a filtered category in the discrete setting.

**Remark 10.2.** Though we won’t use the notion here, in [13, Remark 5.3.1.11] Lurie defines a simplicial set  $S$  to be  $\kappa$ -filtered if it admits a categorical equivalence  $S \rightarrow \mathcal{C}$  to a  $\kappa$ -filtered  $\infty$ -category  $\mathcal{C}$ .

We say an  $\infty$ -category  $\mathcal{C}$  admits  $\kappa$ -filtered colimits if each diagram  $p : \mathcal{K} \rightarrow \mathcal{C}$  from a  $\kappa$ -filtered  $\infty$ -category admits a colimit in  $\mathcal{C}$ .

def:k\_compact

**Definition 10.3.** Let  $\mathcal{C}$  be an  $\infty$ -category which admits  $\kappa$ -filtered colimits. We call a functor  $F : \mathcal{C} \rightarrow \mathcal{D}$   $\kappa$ -cocontinuous if  $F$  preserves  $\kappa$ -filtered limits. We call an object  $x$  in  $\mathcal{C}$   $\kappa$ -compact if each functor  $h^x : \mathcal{C} \rightarrow \mathcal{K}an$  which is corepresented by  $x$  is  $\kappa$ -cocontinuous. We call  $x$  compact if it is  $\aleph_0$ -compact.

**Proposition 10.4.** *For an  $\infty$ -category  $\mathcal{C}$ , the following are equivalent:*

- (a)  $\mathcal{C}$  is  $\kappa$ -cocomplete.
- (b) For each  $\kappa$ -filtered partially ordered set  $A$ , and diagram  $p : A \rightarrow \mathcal{C}$ ,  $p$  admits a colimit in  $\mathcal{C}$ .

*Supposing  $\mathcal{C}$  is  $\kappa$ -cocomplete, and  $F : \mathcal{C} \rightarrow \mathcal{D}$  is any functor, the following are equivalent:*

- (a')  $F$  is  $\kappa$ -cocontinuous.
- (b') For each  $\kappa$ -filtered partially ordered set  $A$ ,  $F$  preserves  $A$ -filtered colimits.

*Outline.* [15, 02QA, 02NU]. □

## 10.2. Compact generation and presentability of derived categories.

**Definition 10.5.** A cocomplete  $\infty$ -category  $\mathcal{C}$  is called compactly generated if  $\mathcal{C}$  is generated under small colimits by an essentially small subcategory of compact objects.

lem:compgen\_viah

**Lemma 10.6.** *A locally small stable  $\infty$ -category  $\mathcal{C}$  is compactly generated if and only if its homotopy category  $\mathrm{h}\mathcal{C}$  is compactly generated. More precisely, if  $\mathcal{C}_0 \subseteq \mathrm{h}\mathcal{C}$  is a small subcategory which generates  $\mathrm{h}\mathcal{C}$ , and  $\mathcal{C}_0$  is the full  $\infty$ -subcategory in  $\mathcal{C}$  spanned by the objects of  $\mathcal{C}_0$ , then*

- (1)  $\mathcal{C}_0$  consists of compact objects in  $\mathrm{h}\mathcal{C}$  if and only if  $\mathcal{C}_0$  consists of compact objects in  $\mathcal{C}$ , and
- (2)  $\mathcal{C}_0$  generates  $\mathrm{h}\mathcal{C}$  under small coproducts and the formation of mapping cones if and only if  $\mathcal{C}_0$  generates  $\mathcal{C}$  under small colimits.

*Proof.* Let  $\mathrm{h}\mathcal{C}_0 \subseteq \mathrm{h}\mathcal{C}$  be any full subcategory, and  $\mathcal{C}_0$  be the full subcategory in  $\mathcal{C}$  spanned by the objects in  $\mathrm{h}\mathcal{C}_0$ . Since  $\mathcal{C}$  is locally small, we understand that  $\mathcal{C}_0$  is essentially small if and only if  $\mathrm{h}\mathcal{C}_0$  is essentially small. Also, by Corollary 6.13,  $\mathcal{C}_0$  consists entirely of compact objects in  $\mathcal{C}$  if and only if  $\mathrm{h}\mathcal{C}_0$  consists entirely of compact objects in  $\mathrm{h}\mathcal{C}$ . Let us suppose that  $\mathcal{C}_0$  is in fact essentially small and consists of compact objects.

Take now  $\mathrm{h}\mathrm{Loc}(\mathcal{C}_0)$  the smallest subcategory in  $\mathrm{h}\mathcal{C}$  which is stable under the formation of triangles and small sums, as well as isomorphisms, and  $\mathrm{Loc}(\mathcal{C}_0)$  the corresponding lift to a full  $\infty$ -subcategory in  $\mathcal{C}$ . Then  $\mathrm{Loc}(\mathcal{C}_0)$  is the smallest full subcategory in  $\mathcal{C}$  which is stable under sums, the formation of cofibers, and isomorphisms in  $\mathcal{C}$ . It follows by Corollary 6.5 that  $\mathrm{Loc}(\mathcal{C}_0)$  is the smallest  $\infty$ -subcategory in  $\mathcal{C}$  which is stable under the formation of small colimits, contains  $\mathcal{C}_0$ , and is closed under isomorphism. Clearly we have

$$\mathrm{h}\mathrm{Loc}(\mathcal{C}_0) = \mathrm{h}\mathcal{C} \Leftrightarrow \mathrm{Loc}(\mathcal{C}_0) = \mathcal{C}.$$

By definition,  $\mathrm{h}\mathcal{C}$  is compactly generated if we can find such an essentially small subcategory of compact objects  $\mathcal{C}_0$  with  $\mathrm{h}\mathrm{Loc}(\mathcal{C}_0) = \mathrm{h}\mathcal{C}$ . By the information above we also see that  $\mathcal{C}$  is compactly generated if and only if we can find such  $\mathcal{C}_0$  with  $\mathrm{Loc}(\mathcal{C}_0) = \mathcal{C}$ . So we observe that  $\mathrm{h}\mathcal{C}$  is compactly generated if and only if  $\mathcal{C}$  is compactly generated. □

We apply Lemma 10.6 to observe a number of familiar examples.

**Example 10.7.** Let  $R$  be any ring and take  $\mathcal{D}(R) = \mathcal{D}(R\text{-Mod})$ . All bounded complexes of projectives are compact in the discrete derived category  $D(R)$ , and  $D(R)$  is generated by this subcategory under small sums and the formation of cones. Hence  $D(R)$  is compactly generated, and we conclude that  $\mathcal{D}(R)$  is compactly generated.

**Example 10.8.** If  $X$  is a quasi-projective scheme, or more generally quasi-compact and separated, then the discrete derived category  $D(X) = D(\text{QCoh}(X))$  is compactly generated [3, Theorems 3.1.1, 3.1.3]. It is, in particular, generated by the full subcategory  $D(X)_{\text{perf}}$  of perfect sheaves. It follows that the derived  $\infty$ -category  $\mathcal{D}(X)$  is generated by the essentially small subcategory of compacts  $\mathcal{D}(X)_{\text{perf}}$ , and in particular is compactly generated.

We consider the following generalization of compact generation via  $\kappa$ -filtered simplicial sets and  $\kappa$ -compact objects.

**Definition 10.9** ([13, Theorem 5.5.1.1]). An  $\infty$ -category  $\mathcal{C}$  is called *presentable* if it satisfies the following:

- (a)  $\mathcal{C}$  is cocomplete.
- (b) For some fixed regular cardinal  $\kappa$ ,  $\mathcal{C}$  is generated under colimits by a small, finitely cocomplete subcategory  $\mathcal{C}_0$  of  $\kappa$ -small objects.

In “tame” situations, the derived category  $\mathcal{D}(\mathbb{A})$  is simply compactly generated. However, there are very reasonable settings where this is not the case. For example, if we take  $\mathbb{A} = \text{Rep } G$  for a reductive algebraic group  $G$  in finite characteristic, then the discrete derived category  $D(\text{Rep } G)$  is not compactly generated [8, Theorem 1.1] and so the derived  $\infty$ -category  $\mathcal{D}(\text{Rep } G)$  is not compactly generated either. We do, however, have the following consolation prize.

**Theorem 10.10** ([14, Proposition 1.3.5.21]). *For any Grothendieck abelian category  $\mathbb{A}$ , the derived category  $\mathcal{D}(\mathbb{A})$  is presentable.*

**10.3. Presentability and the adjoint functor theorem.** As an example application of presentability, we record below the adjoint functor theorem.

`def:acc_F`

**Definition 10.11.** Let  $\mathcal{C}$  be a cocomplete  $\infty$ -category. A functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is called *accessible* if there exists a regular cardinal for which  $F$  preserves all  $\kappa$ -filtered colimits.

**Remark 10.12.** One generally defines accessibility of a functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  in the situation where  $\mathcal{C}$  is only accessible [13, Definition 5.4.2.5], not necessarily cocomplete. As all of our categories in this text are cocomplete, and we don’t want to introduce this additional notion, we stick with the more restrictive setting of Definition 10.11.

The easiest way for a functor to be accessible is for it to be cocomplete. We have the following fundamental characterization of left and right adjoints.

`thm:AFT`

**Theorem 10.13** (Adjoint functor theorem, [13, Corollary 5.5.2.9]). *Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor between presentable categories. The following hold:*

- (1)  *$F$  admits a right adjoint if and only if it is cocontinuous.*
- (2)  *$F$  admits a left adjoint if and only if it is accessible and continuous.*



**10.4. Derived functors and the adjoint functor theorem.** As an immediate consequence of Theorem 10.13 and Proposition 9.8 we obtain the following.

**Corollary 10.14.** *In the situation of Lemma 9.5, the left derived functor  $L F : \mathcal{D}(\mathbb{A}) \rightarrow \mathcal{D}(\mathbb{B})$  is cocontinuous and the right derived functor  $R G : \mathcal{D}(\mathbb{B}) \rightarrow \mathcal{D}(\mathbb{A})$  is accessible.*

It is relatively easy to check that the functor  $L F$  is cocontinuous by hand. Accessibility of the right derived functor  $R G$  is not clear to us in general, however.

**Exercise 10.15.** Prove accessibility of the right derived functor  $R G$  without reference to Proposition 9.8.

sect:ind\_deriver

## 11. IND-COMPLETION AND RENORMALIZED DERIVED CATEGORIES

In concluding, we discuss the ind-completion functor  $\text{Ind} : \mathcal{Cat}_{\infty}^{\text{sm}} \rightarrow \mathcal{Cat}_{\infty}$  and its relatively recent appearances in studies of geometry and representation theory. The main point here is fairly benign; Namely, ind-completion provides an alternate means of “compactifying” small derived categories of interest. This compactifying process produces, from any essentially small  $\infty$ -category  $\mathcal{C}$  with all finite colimits, a tautological cocompletion  $\text{Ind}(\mathcal{C})$  of  $\mathcal{C}$  which is presentable and recovers  $\mathcal{C}$  as its subcategory of compact objects.

Such alternate compactifications, or renormalizations of derived categories, have actually appeared constantly in the discrete derived setting, though we’ve not recognized them as such. For example, Koszul duality provides an equivalence between the discrete derived category  $D(\Lambda)_{\text{fin}}$  of finite-dimensional representations over the exterior algebra and the derived category  $D(S)_{\text{coh}}$  of coherent dg modules over the polynomial ring. This equivalence, however, does not extend to the unbounded setting as there is a disagreement between the compact objects in  $D(\Lambda)$  and in  $D(S)$ . Instead, one sees clearly from the indization perspective that Koszul duality calculates the ind-completion of  $\mathcal{D}(\Lambda)_{\text{fin}}$  as the unbounded category of dg modules over  $S$ ,

$$\text{Ind } \mathcal{D}(\Lambda)_{\text{fin}} \cong \mathcal{D}(S).$$

This example is discussed, along with a few others, in the final subsection below.

### 11.1. Ind-completion of small $\infty$ -categories.

**Notation 11.1.** For  $\infty$ -categories  $\mathcal{A}$  and  $\mathcal{B}$  which admit small  $(\aleph_0)$ -filtered colimits, we let  $\text{Fun}^{\aleph_0}(\mathcal{A}, \mathcal{B})$  denote the full  $\infty$ -subcategory of functors in  $\text{Fun}(\mathcal{A}, \mathcal{B})$  which preserve small filtered colimits.

For  $\mathcal{A}$  admitting small filtered colimits, we recall that an object  $a : * \rightarrow \mathcal{A}$  is compact if any functor  $h^a : \mathcal{A} \rightarrow \mathcal{K}an$  which is corepresented by  $a$  commutes with all small filtered colimits.

thm:ind\_C

**Theorem 11.2** ([13, Corollary 5.3.5.4 & Proposition 5.3.5.11]). *For any essentially small  $\infty$ -category  $\mathcal{C}$ , there is an  $\infty$ -category  $\mathcal{C}'$  and a functor  $i : \mathcal{C} \rightarrow \mathcal{C}'$  which has the following properties:*

- (1)  $\mathcal{C}'$  admits all small filtered colimits.
- (2)  $i$  is fully faithful, and for each  $x$  in  $\mathcal{C}$  the image  $F(x)$  is compact in  $\mathcal{C}'$ .
- (3) Every object  $z$  in  $\mathcal{C}'$  admits a filtered diagram  $p : \mathcal{K} \rightarrow \mathcal{C}$  for which  $z$  is a colimit of the corresponding diagram  $ip : \mathcal{K} \rightarrow \mathcal{C}'$ .

Furthermore, the category  $\mathcal{C}'$  is uniquely determined, as an object in the undercategory  $(\mathcal{Cat}_\infty)_{\mathcal{C}/}$ , up to equivalence.

Let us outline how this theorem works, according to the fundamentals laid out in [13].

*Outline proof.* Define the full subcategory  $\mathcal{C}' = \text{Ind}(\mathcal{C}) \subseteq \text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{H}an)$  spanned by those functors  $\xi : \mathcal{C}^{\text{op}} \rightarrow \mathcal{H}an$  which classify a cartesian fibration  $\mathcal{E} \rightarrow \mathcal{C}$  for which  $\mathcal{E}$  is filtered as an  $\infty$ -category. We note that all representable functors lie in  $\text{Ind}(\mathcal{C})$  since each fibration  $\mathcal{C}_{/x} \rightarrow \mathcal{C}$ , for  $x$  in  $\mathcal{C}$ , has terminal object  $id_x : x \rightarrow x$  (Proposition II-9.11). Hence the Yoneda embedding  $i : \mathcal{C} \rightarrow \text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{H}an)$  has image in  $\text{Ind}(\mathcal{C})$ . Furthermore,  $i$  is fully faithful since the Yoneda embedding is fully faithful, by Theorem II-16.1.

The fact that  $\text{Ind}(\mathcal{C})$  is stable under taking filtered limits in  $\text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{H}an)$  is covered in [13, Proposition 5.3.5.3], and the fact that  $\text{Ind}(\mathcal{C})$  is generated by the image of  $\mathcal{C}$  under small filtered limits is covered in [13, Corollary 5.3.5.4]. Compactness of the image  $i(x)$  for each  $x$  in  $\mathcal{C}$  is covered in [13, Proposition 5.3.5.5].

As for the uniqueness claim, for any  $\infty$ -category  $\mathcal{A}$  which admits small filtered limits, and any functor  $f : \mathcal{C} \rightarrow \mathcal{A}$  with compact image,  $f$  admits a unique extension to a  $\aleph_0$ -cocontinuous functor  $F : \text{Ind}(\mathcal{C}) \rightarrow \mathcal{A}$  which is obtained via left Kan extension [13, Lemma 5.3.5.8]. This functor is an equivalence if and only if  $f$  is fully faithful and  $\mathcal{A}$  is generated by the image of  $\mathcal{C}$  under filtered limits [13, Proposition 5.3.5.11].  $\square$

**Remark 11.3.** We note that essential smallness of  $\mathcal{C}$  is required to ensure the existence of the Kan extension  $F : \text{Ind}(\mathcal{C}) \rightarrow \mathcal{A}$  employed in the proof of Theorem 11.2.

def:ind\_completion

**Definition 11.4.** Given an essentially small  $\infty$ -category  $\mathcal{C}$ , any  $\aleph_0$ -cocomplete  $\infty$ -category  $\mathcal{C}'$  equipped with a fully faithful functor  $i : \mathcal{C} \rightarrow \mathcal{C}'$  as in Theorem 11.2 is called an ind-completion of  $\mathcal{C}$ .

We have the following universal property for the ind-completion.

thm:ind\_univ

**Theorem 11.5** ([13, Proposition 5.3.5.10]). *Let  $\mathcal{C}$  be a small  $\infty$ -category and  $i : \mathcal{C} \rightarrow \mathcal{C}'$  be any ind-completion of  $\mathcal{C}$ . Then for any  $\infty$ -category  $\mathcal{A}$  which admits small filtered colimits, restriction along  $i$  provides an equivalence*

$$i^* : \text{Fun}^{\aleph_0}(\mathcal{C}', \mathcal{A}) \xrightarrow{\sim} \text{Fun}(\mathcal{C}, \mathcal{A}).$$

We can be more precise about our uniqueness claim from Theorem 11.2.

thm:ind\_equiv

**Theorem 11.6** ([13, Proposition 5.3.5.11]). *Let  $\mathcal{A}$  be an  $\infty$ -category which admits all small filtered colimits, and  $f : \mathcal{C} \rightarrow \mathcal{A}$  be a functor from an essentially small  $\infty$ -category. Consider any ind-completion  $i : \mathcal{C} \rightarrow \mathcal{C}'$  and the unique extension  $F : \mathcal{C}' \rightarrow \mathcal{A}$  of  $f$  to an  $\aleph_0$ -cocontinuous map from  $\mathcal{C}'$ . Then*

- (1)  $F$  is fully faithful provided  $f$  is fully faithful and has compact image.
- (2)  $F$  is an equivalence if and only if  $f$  is fully faithful with compact image, and  $\mathcal{A}$  is generated by the image of  $\mathcal{C}$  under small filtered colimits.

We consider some examples. Below we take for granted that, for any ring  $R$ , the compact objects in the discrete derived category  $D(R) = \mathcal{D}(R\text{-mod})$  are precisely those objects which are quasi-isomorphic to a bounded complex of projectives,

and for any quasi-compact quasi-separated scheme  $X$  the compacts in the discrete derived category  $D(X) = \mathcal{D}(\mathrm{QCoh}(X))$  are the perfect complexes, i.e. those complexes which are isomorphic over any affine open to a bounded complex of locally free sheaves.

**Example 11.7** (Ind-finite representations I). Let  $R$  be a finite-dimensional algebra and  $\mathcal{D}(R)$  be the derived  $\infty$ -category. Let  $\mathcal{D}(R)_{fin}$  be the full subcategory of complexes with finite-dimensional cohomology. As  $\mathcal{D}(R)$  is cocomplete it admits all small filtered colimits, and by Corollary 6.13 the compact objects in  $\mathcal{D}(R)$  are precisely those complexes of finite projective dimension, i.e. which are quasi-isomorphic to bounded complexes of projectives. Hence, if  $R$  is of finite global dimension, then all objects in  $\mathcal{D}(R)_{fin}$  are compact in  $\mathcal{D}(R)$  and the inclusion  $\mathcal{D}(R)_{fin} \rightarrow \mathcal{D}(R)$  identifies  $\mathcal{D}(R)$  as an ind-completion of  $\mathcal{D}(R)_{fin}$ .

**Example 11.8** (Ind-coherent sheaves I). Let  $X$  be a complex variety with a singular point  $x : \mathrm{Spec}(\mathbb{C}) \rightarrow X$ . Then the residue field  $k(x)$  is a coherent sheaf on  $X$  which is non-compact in the derived  $\infty$ -category  $\mathcal{D}(X) = \mathcal{D}(\mathrm{QCoh}(X))$ . Hence, for  $\mathcal{D}(X)_{coh}$  the full subcategory of complexes with coherent cohomology, the inclusion  $\mathcal{D}(X)_{coh} \rightarrow \mathcal{D}(X)$  does *not* identify the derived  $\infty$ -category as an ind-completion of  $\mathcal{D}(X)_{coh}$ .

We note, however, that in this case we can still consider the ind-completion  $\mathrm{Ind} \mathcal{D}(X)_{coh}$  and we have the unique extension  $F : \mathrm{Ind} \mathcal{D}(X)_{coh} \rightarrow \mathcal{D}(X)$  of the inclusion. To say that  $\mathcal{D}(X)$  is not an ind-completion of  $\mathcal{D}(X)_{coh}$  is, more precisely, to say that the functor  $F$  is not an equivalence.

**Example 11.9** (Ind-coherent sheaves II). For  $X$  a smooth Noetherian scheme,  $\mathcal{D}(X)_{perf} = \mathcal{D}(X)_{coh}$ , and  $\mathcal{D}(X)$  is an ind-completion of  $\mathcal{D}(X)_{coh}$ .

**Example 11.10** (Ind-finite representations II). Let  $G$  be a finite group with  $p$  dividing the order of  $G$ , for some prime  $p$ . Let  $k$  be a field of characteristic  $p$ . Then the trivial representation  $k$  is not compact in  $\mathcal{D}(G) = \mathcal{D}(\mathrm{Rep}_k(G))$ , and the unique extension

$$F : \mathrm{Ind} \mathcal{D}(G)_{fin} \rightarrow \mathcal{D}(G)$$

of the inclusion  $\mathcal{D}(G)_{fin} \rightarrow \mathcal{D}(G)$  is not an equivalence.

sect:ind\_fun

**11.2. Ind-completion as a functor.** Let  $\mathcal{Cat}_{\infty}^{\aleph_0}$  denote the non-full  $\infty$ -subcategory whose objects are  $\infty$ -categories which admit all small filtered colimits, and whose maps are those functors  $F : \mathcal{A} \rightarrow \mathcal{B}$  which commute with small filtered colimits. Consider the non-full subcategory

$$\mathcal{M}_{\mathrm{Ind}} \subseteq \mathcal{Cat}_{\infty} \times \Delta^1$$

whose objects are the union

$$\mathcal{M}_{\mathrm{Ind}}[0] = (\mathcal{Cat}_{\infty}^{\mathrm{sm}}[0] \times \{0\}) \cup (\mathcal{Cat}_{\infty}^{\aleph_0}[0] \times \{1\}).$$

The morphisms (edges) over  $\{0\}$  and over  $0 < 1$ , under the projection  $\mathcal{Cat}_{\infty} \times \Delta^1 \rightarrow \Delta^1$ , are arbitrary maps between essentially small  $\infty$ -categories and from essentially small  $\infty$ -categories to  $\aleph_0$ -cocomplete categories. The morphisms over 1 are  $\aleph_0$ -cocontinuous functors. The  $n$ -simplices  $\sigma : \Delta^n \rightarrow \mathcal{M}_{\mathrm{Ind}}$  are precisely those  $n$ -simplices in  $\mathcal{Cat}_{\infty} \times \Delta^1$  whose vertices  $\sigma|_{\Delta^{\{i\}}}$  and edges  $\sigma|_{\Delta^{\{i,i+1\}}}$  satisfy the above restrictions.

One sees that  $\mathcal{M}_{\text{Ind}}$  is an  $\infty$ -category since composites of  $\kappa$ -cocomplete functors are  $\kappa$ -cocomplete. The fibers over  $\Delta^1$  under the projection  $q : \mathcal{M}_{\text{Ind}} \rightarrow \Delta^1$  are precisely

$$(\mathcal{M}_{\text{Ind}})_0 = \mathcal{Cat}_{\infty}^{\text{sm}} \quad \text{and} \quad (\mathcal{M}_{\text{Ind}})_1 = \mathcal{Cat}_{\infty}^{\text{N}_0}.$$

One sees that this projection is an inner fibration, as it is a composite of the inner fibrations provided by the inclusion  $\mathcal{M}_{\text{Ind}} \rightarrow \mathcal{Cat}_{\infty} \times \Delta^1$  and the projection  $\mathcal{Cat}_{\infty} \times \Delta^1 \rightarrow \Delta^1$ .

**lem:3693**

**Lemma 11.11.** *The projection  $q : \mathcal{M}_{\text{Ind}} \rightarrow \Delta^1$  is a cocartesian fibration, and for any  $\infty$ -category  $\mathcal{C}$  over 0, i.e. any essentially small  $\infty$ -category, an edge  $i : \mathcal{C} \rightarrow \mathcal{C}'$  over  $0 < 1$  is  $q$ -cocartesian if and only if  $i$  realizes  $\mathcal{C}'$  as an ind-completion of  $\mathcal{C}$ .*

*Proof.* The category  $\mathcal{Cat}_{\infty} \times \Delta^1$  is the homotopy coherent nerve of the simplicial category  $\underline{\text{Cat}}_{\infty}^+ \times \{0 < 1\}$ , and  $\mathcal{M}_{\text{Ind}}$  is the homotopy coherent nerve of the simplicial uncategor  $\underline{\mathbf{M}}$  with prescribed objects and morphisms given by

$$\underline{\text{Hom}}_{\underline{\mathbf{M}}}(\mathcal{C}, \mathcal{A}) = \begin{cases} \text{Fun}(\mathcal{C}, \mathcal{A}) & \text{if } \mathcal{C} \text{ lies over } 0 \\ \text{Fun}^{\text{N}_0}(\mathcal{C}, \mathcal{A}) & \text{if } \mathcal{C} \text{ and } \mathcal{A} \text{ lie over } 1 \\ \emptyset & \text{otherwise.} \end{cases}$$

We therefore identify the  $\text{h}\mathcal{K}\text{an}$ -enriched category  $\pi\mathcal{M}_{\text{Ind}}$  with the category  $\pi\underline{\mathbf{M}}$  obtained by applying the symmetric monoidal functor  $\pi : \mathcal{K}\text{an} \rightarrow \text{h}\mathcal{K}\text{an}$  to the morphisms, by Proposition II-7.6. By Theorem 11.5 we therefore conclude that the subcategory

$$\mathcal{Cat}_{\infty}^{\text{N}_0} = (\mathcal{M}_{\text{Ind}})_1 \subseteq \mathcal{M}_{\text{Ind}}$$

is reflective in  $\mathcal{M}_{\text{Ind}}$ . By Lemma 7.13 it follows that the projection  $q : \mathcal{M}_{\text{Ind}} \rightarrow \Delta^1$  is a cocartesian fibration, and a map  $i : \mathcal{C} \rightarrow \mathcal{C}'$  is a  $q$ -cocartesian edge in  $\mathcal{M}_{\text{Ind}}$ , over  $0 < 1$ , if and only if  $i$  realizes  $\mathcal{C}'$  as an ind-completion of  $\mathcal{C}$ .  $\square$

We now have the unique cocartesian transformation  $I : \Delta^1 \times \mathcal{Cat}_{\infty}^{\text{sm}} \rightarrow \mathcal{M}_{\text{Ind}}$  which solves the lifting problem

$$\begin{array}{ccc} \{0\} \times \mathcal{Cat}_{\infty}^{\text{sm}} & \xrightarrow{\quad} & \mathcal{M}_{\text{Ind}} \\ \downarrow & \nearrow I & \downarrow q \\ \Delta^1 \times \mathcal{Cat}_{\infty}^{\text{sm}} & \xrightarrow{\quad} & \Delta^1, \end{array} \quad (41) \quad \boxed{\text{eq:indize}}$$

i.e. the unique functor which solves the lifting problem and evaluates to a cocartesian functor in  $\mathcal{M}_{\text{Ind}}$  at each edge  $\Delta^1 \times \{\mathcal{C}\}$ . By Lemma 11.11 this cocartesian edge is an ind-completion  $i : \mathcal{C} \rightarrow \mathcal{C}'$  of  $\mathcal{C}$ .

We now consider the other projection

$$p : \mathcal{M}_{\text{Ind}} \subseteq \mathcal{Cat}_{\infty} \times \Delta^1 \rightarrow \mathcal{Cat}_{\infty}$$

and the composition  $\widetilde{\text{Ind}} := pI : \Delta^1 \times \mathcal{Cat}_{\infty}^{\text{sm}} \rightarrow \mathcal{Cat}_{\infty}$ .

**Definition 11.12.** The indization functor is the pairing  $(\text{Ind}, \widetilde{\text{Ind}})$  of the restriction

$$\text{Ind} := \widetilde{\text{Ind}}|_{\{1\} \times \mathcal{Cat}_{\infty}^{\text{sm}}} : \mathcal{Cat}_{\infty}^{\text{sm}} \rightarrow \mathcal{Cat}_{\infty}^{\text{N}_0}$$

along with the transformation  $\widetilde{\text{Ind}} : \Delta^1 \times \mathcal{Cat}_{\infty}^{\text{sm}} \rightarrow \mathcal{Cat}_{\infty}$  constructed above.

We note that the cocartesian solution from (41) is unique up to a contractible space of choices, by Theorem II-2.7. Hence the indization functor is determined up to a contractible space of choices.

**Remark 11.13.** By an abuse of notation we often refer to the functor  $\text{Ind} : \mathcal{Cat}_\infty^{\text{sm}} \rightarrow \mathcal{Cat}_\infty$  itself as the indization functor. We also generally let  $i$  denote the structural transformation  $i = \widetilde{\text{Ind}}$  in order to ease notation.

**Remark 11.14.** Despite the apparent differences in the constructions, our indization functor is the same as the one introduced in [13, Proposition 5.4.2.19]. This follows by Proposition A.21 below.

To be very clear, the functor  $\widetilde{\text{Ind}}$  picks, at each object  $(1, \mathcal{C})$  in  $\Delta^1 \times \mathcal{Cat}_\infty^{\text{sm}}$ , an ind-completion  $\text{Ind}(\mathcal{C})$  and  $\widetilde{\text{Ind}}$  itself is a transformation between the inclusion  $\mathcal{Cat}_\infty^{\text{sm}} \rightarrow \mathcal{Cat}_\infty$  and the functor  $\text{Ind} : \mathcal{Cat}_\infty^{\text{sm}} \rightarrow \mathcal{Cat}_\infty^{\text{N}_0} \subseteq \mathcal{Cat}_\infty$  which evaluates at each essentially small  $\infty$ -category  $\mathcal{C}$  to a functor  $i : \mathcal{C} \rightarrow \text{Ind}(\mathcal{C})$  which specifically witnesses  $\text{Ind}(\mathcal{C})$  as an ind-completion of  $\mathcal{C}$ .

### 11.3. Ind-completion and products.

**Proposition 11.15.** *The indization functor  $\text{Ind} : \mathcal{Cat}_\infty^{\text{sm}} \rightarrow \mathcal{Cat}_\infty$  commutes with arbitrary products.*

In the proof we employ the notations and constructions introduced in Appendix A.

*Proof.* By [13, Proposition 5.4.7.3] the category  $\text{Acc}_{\text{N}_0}$  admits all small limits and the inclusion  $\text{Acc}_{\text{N}_0} \rightarrow \mathcal{Cat}_\infty$  is continuous. Since the restriction to idempotent complete categories

$$\text{Ind} : (\mathcal{Cat}_\infty^{\text{sm}})_{\text{split}} \xrightarrow{\sim} \text{Acc}_{\text{N}_0}$$

is an equivalence by Corollary A.20, we conclude that the category of idempotent split  $\infty$ -categories is complete as well and that indization from  $(\mathcal{Cat}_\infty^{\text{sm}})_{\text{split}}$ . From the factorization of Proposition A.21 it suffices to provide that the idempotent completion functor  $(-)^{\vee} : \mathcal{Cat}_\infty^{\text{sm}} \rightarrow (\mathcal{Cat}_\infty^{\text{sm}})_{\text{split}}$  preserves products.

For any simplicial set  $K$  and small collection of categories  $\{\mathcal{C}_\lambda : \lambda \in \Lambda\}$  we have the natural isomorphism

$$\text{Fun}(K, \prod_{\lambda} \mathcal{C}_\lambda) \xrightarrow{\sim} \prod_{\lambda} \text{Fun}(K, \mathcal{C}_\lambda)$$

so that, by the characterization of idempotent complete categories provided in Proposition A.5,  $(\mathcal{Cat}_\infty^{\text{sm}})_{\text{split}}$  is stable under small products in  $\mathcal{Cat}_\infty$ , and hence products in the complete category  $(\mathcal{Cat}_\infty^{\text{sm}})_{\text{split}}$  are the usual cartesian product.

We consider a small collection  $\{\mathcal{C}_\lambda : \lambda \in \Lambda\}$  of arbitrary essentially small  $\infty$ -categories and consider the idempotent completions  $i_\lambda : \mathcal{C}_\lambda \rightarrow \mathcal{C}_\lambda^{\vee}$ . Consider also the idempotent completion of the product  $i : \prod_{\lambda} \mathcal{C}_\lambda \rightarrow (\prod_{\lambda} \mathcal{C}_\lambda)^{\vee}$ . Via naturality of the transformation  $i : \text{id}_{\mathcal{Cat}_\infty} \rightarrow (-)^{\vee}$ , each projection  $p_\mu^{\vee} : (\prod_{\lambda} \mathcal{C}_\lambda)^{\vee} \rightarrow \mathcal{C}_\mu^{\vee}$  fits into a diagram

$$\begin{array}{ccc} (\prod_{\lambda} \mathcal{C}_\lambda)^{\vee} & \xrightarrow{p_\mu^{\vee}} & \mathcal{C}_\mu^{\vee} \\ \uparrow i & & \uparrow i_\mu \\ \prod_{\lambda} \mathcal{C}_\lambda & \xrightarrow{p_\mu} & \mathcal{C}_\mu^{\vee} \end{array}$$

Hence the induced map to the product fits into a diagram

$$\begin{array}{ccc} (\prod_{\lambda} \mathcal{C}_{\lambda})^{\vee} & \xrightarrow{\text{comp}} & \prod_{\lambda} \mathcal{C}_{\lambda}^{\vee} \\ & \swarrow i \quad \searrow \prod i_{\lambda} & \\ & \prod_{\lambda} \mathcal{C}_{\lambda} & \end{array}$$

To show that this comparison map is an equivalence it suffices to show that the product  $\prod_{\lambda} \mathcal{C}_{\lambda}^{\vee}$  is an idempotent completion of  $\prod_{\lambda} \mathcal{C}_{\lambda}$ .

For this final point, since each functor  $i_{\lambda} : \mathcal{C} \rightarrow \mathcal{C}_{\lambda}$  is fully faithful, the product map  $\prod_{\lambda} i_{\lambda}$  is also fully faithful. Furthermore, for a tuple of objects  $y = (y_{\lambda} : \lambda \in \Lambda)$  in  $\prod_{\lambda} \mathcal{C}_{\lambda}^{\vee}$  any collection of retract diagrams  $r_{\lambda} : \text{Ret} \rightarrow \mathcal{C}_{\lambda}^{\vee}$  which realize each  $y_{\lambda}$  as a retract of some  $x_{\lambda}$  in  $\mathcal{C}_{\lambda}$ , the uniquely associated diagram

$$r = [r_{\lambda} : \lambda \in \Lambda]^t : \text{Ret} \rightarrow \prod_{\lambda} \mathcal{C}_{\lambda}^{\vee}$$

expresses  $y$  as a retract of the object  $(x_{\lambda} : \lambda \in \Lambda)$  in  $\prod_{\lambda} \mathcal{C}_{\lambda}$ . Therefore, by definition, the product map

$$\prod_{\lambda} i_{\lambda} : \prod_{\lambda} \mathcal{C}_{\lambda} \rightarrow \prod_{\lambda} \mathcal{C}_{\lambda}^{\vee}$$

exhibits the target category as an idempotent completion of the product  $\prod_{\lambda} \mathcal{C}_{\lambda}$ . as desired.  $\square$

**Remark 11.16.** This preservation of products is important when considering the transfer of monoidal structures, or actions, through the ind-completion functor. See for example [14, Corollary 2.4.1.8].

#### 11.4. Ind-completion in the $\aleph_0$ -cocomplete and stable settings.

thm:ind\_present

**Theorem 11.17.** *Consider an essentially small  $\infty$ -category  $\mathcal{C}$ . If  $\mathcal{C}$  admits all finite colimits then the following hold:*

- (1) *The ind-completion  $\text{Ind}(\mathcal{C})$  is presentable and the structure map  $\mathcal{C} \rightarrow \text{Ind}(\mathcal{C})$  preserves finite colimits.*
- (2) *For any cocomplete  $\infty$ -category, and functor  $\mathcal{C} \rightarrow \mathcal{A}$  which commutes with finite colimits, the induced functor  $\text{Ind}(\mathcal{C}) \rightarrow \mathcal{A}$  commutes with all small colimits.*

We first offer a supporting lemma regarding idempotent completion. (See Appendix A.)

lem:3814

**Lemma 11.18.** *If  $i : \mathcal{C} \rightarrow \mathcal{D}$  is an idempotent completion in which  $\mathcal{C}$  and  $\mathcal{D}$  both admit  $\kappa$ -small limits (resp. colimits), then  $i$  commutes with  $\kappa$ -small limits (resp. colimits).*

*Proof.* We argue the point for limits. The case of colimits follows by taking opposites.

By identifying  $\mathcal{C}$  with its essential image in  $\mathcal{C}^{\vee}$  we may assume that  $i$  is an inclusion of simplicial sets. For any diagram  $p : K \rightarrow \mathcal{C}$  the corresponding inclusion  $\mathcal{C}_{/p} \rightarrow \mathcal{D}_{/p}$  is an idempotent completion of  $\mathcal{C}_{/p}$  by [13, Lemma 5.1.4.4]. So it suffices to prove that idempotent completion preserves terminal objects. So, let us suppose

that an object  $z$  is terminal in  $\mathcal{C}$ . For any  $y$  in  $\mathcal{D}$  express  $y$  as a retract  $y \rightarrow x \rightarrow y$  of an object  $x$  in  $\mathcal{C}$  to obtain a retract diagram

$$\begin{array}{ccc} & \text{Hom}_{\mathcal{C}}(x, z) & \\ \nearrow & & \searrow \\ \text{Hom}_{\mathcal{D}}(y, z) & \xrightarrow{\quad id \quad} & \text{Hom}_{\mathcal{D}}(y, z) \end{array}$$

in  $\mathbf{h}\mathcal{K}an$ . Contractibility of  $\text{Hom}_{\mathcal{C}}(x, z)$  now implies contractibility of  $\text{Hom}_{\mathcal{D}}(y, z)$ . Since  $y$  was chosen arbitrarily we see that, by definition,  $z$  is terminal in  $\mathcal{D}$ .  $\square$

As we explain in Remark 11.19 below, it is in fact the case that the idempotent completion  $\mathcal{C}^{\vee}$  admits  $\kappa$ -small limits (resp. colimits) whenever  $\mathcal{C}$  admits such limits. So the proper definitive statement is that any idempotent completion admits  $\kappa$ -small limits (colimits) whenever  $\mathcal{C}$  does, and that the structure map  $i : \mathcal{C} \rightarrow \mathcal{D}$  always respects such limits (colimits). In any case, we can now record the proof of Theorem 11.17.

*Proof of Theorem 11.17.* (1) By [13, Theorem 5.5.1.1] the category  $\text{Ind}(\mathcal{C})$  is presentable and by [13, Corollary 5.3.4.15] the subcategory of compacts  $\text{Ind}(\mathcal{C})^c$  is stable under finite colimits in  $\text{Ind}(\mathcal{C})$ . By Lemma A.16 the fully faithful functor  $\mathcal{C} \rightarrow \text{Idem}(\mathcal{C})^c$  exhibits  $\text{Idem}(\mathcal{C})^c$  as an idempotent completion of  $\mathcal{C}$ . By Lemma 11.18 the map  $\mathcal{C} \rightarrow \text{Idem}(\mathcal{C})$  preserves finite colimits, so that the composite

$$\mathcal{C} \rightarrow \text{Ind}(\mathcal{C})^c \subseteq \text{Ind}(\mathcal{C})$$

preserves finite colimits as well. Claim (2) follows by [13, Proposition 5.5.1.9].  $\square$

rem:idem\_colim

**Remark 11.19.** Since  $\text{Ind}(\mathcal{C})^c$  is an idempotent completion of  $\mathcal{C}$ , one can employ [13, Corollary 5.3.4.15] and Lemma 11.18 to see that the idempotent completion  $\mathcal{C}^{\vee}$  of any finitely cocomplete  $\infty$ -category  $\mathcal{C}$  admits all finite colimits, and the structure map  $i : \mathcal{C} \rightarrow \mathcal{C}^{\vee}$  commutes with finite colimits. One can similarly argue, by replacing  $\text{Ind} = \text{Ind}_{\aleph_0}$  with  $\text{Ind}_{\kappa}$  at a generic regular cardinal, that the idempotent completion  $\mathcal{C}^{\vee}$  admits  $\kappa$ -small colimits whenever  $\mathcal{C}$  admits  $\kappa$ -small colimits and that the structure map  $i : \mathcal{C} \rightarrow \mathcal{C}^{\vee}$  respects such colimits. By taking opposite categories the analogous results are seen to hold for  $(\kappa-)$ small limits as well.

**Remark 11.20.** As in the accessible situation, represented in Corollary A.20, one observes that the indization functor  $\text{Ind}$  provides an equivalence between the category of idempotent complete, essentially small, and finitely cocomplete  $\infty$ -categories and a certain category  $\text{Pr}_{\aleph_0}$  of compactly generated presentable  $\infty$ -categories. See [14, Lemma 5.3.2.9].

As one expects, stability is preserved under ind-completion as well.

prop:ind\_stable

**Proposition 11.21** ([14, Proposition 1.1.3.6]). *If  $\mathcal{C}$  is essentially small and stable, then the ind-completion  $\text{Ind}(\mathcal{C})$  is presentable and stable. Furthermore, the structure map  $i : \mathcal{C} \rightarrow \text{Ind}(\mathcal{C})$  is exact by Theorem 11.17 (2).*

*Proof.* Stability is covered in [14], and exactness of  $i$  follows by  $\square$

**11.5. Renormalized derived categories.** One can employ ind-completion to produce presentable and stable alternatives to the unbounded derived category. The fundamental point here is the following.

**Proposition 11.22.** *If  $\mathcal{C}$  is an idempotent complete essentially small  $\infty$ -category, then the structure map  $i : \mathcal{C} \rightarrow \mathrm{Ind}(\mathcal{C})$  restricts to an equivalence*

$$\mathcal{C} \xrightarrow{\sim} \mathrm{Ind}(\mathcal{C})^c.$$

*Proof.* This is Lemma A.17 (2).  $\square$

Hence, for  $\mathcal{D}(\mathbb{A})_{fav}$  some essentially small stable subcategory of objects in  $\mathcal{D}(\mathbb{A})$  with a “favored” property, the ind-completion  $\mathrm{Ind} \mathcal{D}(\mathbb{A})_{fav}$  is a stable completion of  $\mathcal{D}(\mathbb{A})_{fav}$  which is freely generated by  $\mathcal{D}(\mathbb{A})_{fav}$  and from which one recovers  $\mathcal{D}(\mathbb{A})_{fav}$  as the subcategory of compacts. Taking the homotopy category, we obtain a triangulated category  $D = \mathrm{hInd} \mathcal{D}(\mathbb{A})_{fav}$  which contains the discrete derived category  $D(\mathbb{A})_{fav}$ , and with

$$\mathrm{Loc} D(\mathbb{A})_{fav} = D.$$

We note furthermore that we have, via the universal property of ind-completion, a unique cocontinuous and exact functor  $\mathrm{Ind} \mathcal{D}(\mathbb{A})_{fin} \rightarrow \mathcal{D}(\mathbb{A})$  which fits into a diagram

$$\begin{array}{ccc} \mathrm{Ind} \mathcal{D}(\mathbb{A})_{fin} & \xrightarrow{\quad} & \mathcal{D}(\mathbb{A}) \\ & \nwarrow \quad \nearrow & \\ & \mathcal{D}(\mathbb{A})_{fav} & \end{array}.$$

One can view the construction  $\mathrm{Ind} \mathcal{D}(\mathbb{A})_{fin}$  as a kind of “renormalization” of the derived  $\infty$ -category.

We do not attempt to convince the reader that this kind of construction is of any particular relevance to them, but cover two examples which already appear in the literature.

**Example 11.23** (Ind-coherent sheaves). Let  $X$  be a reasonable scheme or stack, and  $\mathcal{D}(X)_{coh}$  be the derived category of dg quasi-coherent sheaves with (bounded) coherent cohomology. Take

$$\mathbf{IndCoh}(X) := \mathrm{Ind} \mathcal{D}(X)_{coh},$$

where the derived-ness is implicit in the left-hand notation. This is the category of so-called ind-coherent sheaves on  $X$ .

Ind-coherent sheaves play an essential role in the geometric Langlands program [5, 6].

**Example 11.24** (Ind-finite representations). Let  $\mathbb{G}$  be an affine algebraic group in finite characteristic. Take  $\mathcal{D}(\mathbb{G})_{fin}$  the derived category of dg  $\mathbb{G}$ -representations with (bounded) finite dimensional cohomology, and take also

$$\mathbf{Rep}(\mathbb{G}) = \mathrm{Ind} \mathcal{D}(\mathbb{G})_{fin}.$$

This is a compactly generated stable  $\infty$ -category with compacts  $\mathbf{Rep}(\mathbb{G})_{fin} := \mathcal{D}(\mathbb{G})_{fin}$ .

In comparing with the derived  $\infty$ -category, we have that  $\mathcal{D}(\mathbb{G})$  is not even compactly generated except under very specific circumstances [8]. Furthermore,



if we consider  $\mathcal{D}(\mathbb{G})$  as a symmetric monoidal  $\infty$ -category, it has dualizable objects  $\mathcal{D}(\mathbb{G})_{fin}$ , so that its dualizable and compact objects disagree horribly. In comparing with ind-coherent sheaves, we have in this case  $\mathbf{Rep}(\mathbb{G}) = \mathbf{IndCoh}(*/\mathbb{G})$  where here the quotient is specifically the quotient stack.

The following example demonstrates that ind-constructions appear very naturally, and should be expected to appear generically, when considering equivalences of derived categories.

ex:kosz\_dual

**Example 11.25** (Koszul duality). Let  $V$  be a finite dimensional vector space. Take  $A = \wedge^*(V)$  and  $S = k[\Sigma^{-1}V^*]$ , where we consider  $S$  as a dg algebra generated in degree 1. Koszul duality provides an equivalence

$$\mathrm{Kos} = \mathrm{RHom}_A(k, -) : \mathcal{D}(A)_{fin} \rightarrow \mathcal{D}(S)_{coh}.$$

To be clear,  $\mathcal{D}(S)$  is obtained as the dg nerve of the dg category  $S\text{-dgm}$  of arbitrary dg  $S$ -modules, and  $\mathcal{D}(S)_{coh}$  is the full subcategory of complexes with finitely generated cohomology.

The equivalence  $\mathrm{Kos}$  sends the trivial representation  $k$  to  $\mathrm{Kos}(k) = S$ , and we note that the trivial representation is non-compact in  $\mathcal{D}(A)$ , since  $A$  is Frobenius. Specifically, an  $A$ -module  $M$  is compact in  $\mathcal{D}(A)$  if and only if  $M$  is projective. So we see that  $\mathrm{Kos}$  does *not* extend to an equivalence on unbounded derived categories, from  $\mathcal{D}(A)$  to  $\mathcal{D}(S)$ . However, trivially,  $\mathrm{Kos}$  does extend to an equivalence

$$\mathrm{Ind} \mathrm{Kos} : \mathrm{Ind} \mathcal{D}(A)_{fin} \rightarrow \mathrm{Ind} \mathcal{D}(S)_{coh} \cong \mathcal{D}(S).$$

So this Ind-category  $\mathrm{Ind} \mathcal{D}(A)_{fin}$  is seen to be the natural cocompletion of the finite derived category which is implied by Koszul duality.

sect:idem

## APPENDIX A. IDEMPOTENTS AND IDEMPOTENT COMPLETION

**A.1. Idempotents and retracts.** By a *linear equivalence relation* on an ordered set  $I$  we mean an equivalence relation  $\sim$  which satisfies the following:

$$\text{if } a \sim c \text{ and } a \leq b \leq c, \text{ then } a \sim b \sim c.$$

Take  $\mathrm{Idem}$  the simplicial set with  $n$ -simplices

$$\mathrm{Idem}[n] = \{\text{the set of linear equivalence relations } \sim \text{ on } [n]\}$$

The structure maps are the apparent ones. Note that  $\mathrm{Idem}$  contains a single non-degenerate simplex in each dimension, which is given specifically by the equivalence relation with  $a \sim b$  if and only if  $a = b$ . Furthermore, each face map  $d_i^* : \mathrm{Idem}[n] \rightarrow \mathrm{Idem}[n-1]$  sends the unique nondegenerate simplex in dimension  $n$  to the nondegenerate simplex in dimension  $n-1$ .

A diagram in an  $\infty$ -category  $F : \mathrm{Idem} \rightarrow \mathcal{C}$  can be seen as an “infinitely coherent idempotent” in  $\mathcal{C}$ , and consists of a choice of object  $x$ , endomorphism  $e : x \rightarrow x$ , 2-simplex

$$\begin{array}{ccc} & x & \\ e \nearrow & & \searrow e \\ x & \xrightarrow{e} & x \end{array}$$

and furthermore an  $n$ -simplex  $F_n : \Delta^n \rightarrow \mathcal{C}$  at all  $n \geq 0$  all of whose faces are equal in  $\mathcal{C}$ .

**Definition A.1.** An idempotent in an  $\infty$ -category  $\mathcal{C}$  is a diagram  $e : \mathrm{Idem} \rightarrow \mathcal{C}$ .

We consider also the simplicial set  $\text{Ret} = \Delta^2 / \Delta^{\{0,2\}}$  whose diagrams  $\text{Ret} \rightarrow \mathcal{C}$  classify retracts in  $\mathcal{C}$ . Explicitly, a diagram  $\text{Ret} \rightarrow \mathcal{C}$  is a choice of a 2-simplex of the form

$$\begin{array}{ccc} & x & \\ l \nearrow & & \searrow \pi \\ y & \xrightarrow{id_y} & y. \end{array}$$

We claim that, as in the discrete setting, any retract diagram  $r : \text{Ret} \rightarrow \mathcal{C}$  determines a uniquely associated idempotent  $e : \text{Idem} \rightarrow \mathcal{C}$ . In establishing this relation we factor through an intermediate construction  $\text{Idem}^+$ .

We define the complex  $\text{Idem}^+$  whose  $n$ -simplices are

$$\text{Idem}^+[n] = \{\text{subsets } I \subseteq [n] \text{ equipped with a linear equivalence relation } \sim\}.$$

Given a map  $\alpha : [m] \rightarrow [n]$  the function  $\alpha^* : \text{Idem}^+[n] \rightarrow \text{Idem}^+[m]$  sends the pair  $(I, \sim)$  to  $\alpha^{-1}I \subseteq [m]$  paired with the relation  $\sim_\alpha$  where  $a \sim_\alpha b$  in  $\alpha^{-1}I$  if and only if  $\alpha(a) \sim \alpha(b)$ .

We have the inclusion  $\text{Idem} \rightarrow \text{Idem}^+$  and in low dimensions we have non-degenerate vertices

$$\text{Idem}^+[0] = \{[0], \emptyset\}, \quad \text{Idem}^+[1]_{\text{nd}} = \{[1], \{0\}, \{1\}\},$$

$$\text{Idem}^+[2]_{\text{nd}} = \{[2], \{0, 1\}, \{0, 2\}, \{1, 2\}, \{1\}\}$$

where in each case above we give the set  $I$  the equivalence relation  $a \sim b$  if and only if  $a = b$ . We have additionally the inclusion  $j : \text{Ret} = \Delta^2 / \Delta^{\{0,2\}} \rightarrow \text{Idem}^+$  which has

$$j(0) = j(2) = \emptyset, \quad j(1) = [1], \quad j(0 < 1) = \{1\}, \quad j(1 < 2) = \{0\}, \quad j(0 < 2) = \emptyset$$

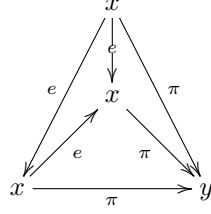
and  $j(0 < 1 < 2) = \{1\}$ . The union  $\text{Idem} \cup \text{Ret}$  covers all 2-simplices save for the  $[1]$  and  $[2]$ .

We see now that a diagram  $e^+ : \text{Idem}^+ \rightarrow \mathcal{C}$  specifies an idempotent  $e : \text{Idem} \rightarrow \mathcal{C}$  and a retract  $r : \text{Ret} \rightarrow \mathcal{C}$ , which one might display as above, along with additional simplices which validate the equation “ $e = \pi l : x \rightarrow y \rightarrow x$ ” up to all higher levels of compatibility. For example  $e^+$  specifies diagrams of the form

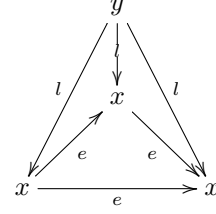
$$\begin{aligned} e^+(\{0, 1\} \subseteq [2]) &= \begin{array}{ccc} & x & \\ e \nearrow & & \searrow \pi \\ x & \xrightarrow{\pi} & y \end{array} & e^+(\{0, 2\} \subseteq [2]) &= \begin{array}{ccc} & y & \\ \pi \nearrow & & \searrow l \\ x & \xrightarrow{e} & x \end{array} \\ \\ e^+(\{1, 2\} \subseteq [2]) &= \begin{array}{ccc} & x & \\ l \nearrow & & \searrow e \\ y & \xrightarrow{l} & x \end{array} \end{aligned}$$

in  $\mathcal{C}$ , in dimension 2, and also diagrams of the form

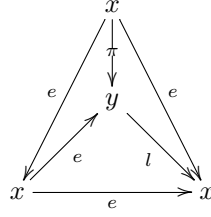
$$e^+(\{0, 1, 2\} \subseteq [3]) =$$



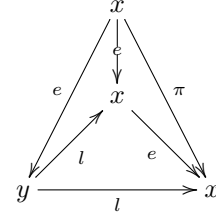
$$e^+(\{1, 2, 3\} \subseteq [3]) =$$



$$e^+(\{0, 1, 3\} \subseteq [3]) =$$



$$e^+(\{0, 2, 3\} \subseteq [3]) =$$



in dimension 3.

**Proposition A.2** ([13, Proposition 4.4.5.6]). *The inclusion  $\text{Ret} \rightarrow \text{Idem}^+$  is inner anodyne.*

We can now reasonably speak of split idempotents.

cor:split

**Corollary A.3.** *At any  $\infty$ -category  $\mathcal{C}$  there is a functor  $\text{split} : \text{Fun}(\text{Ret}, \mathcal{C}) \rightarrow \text{Fun}(\text{Idem}, \mathcal{C})$  which into a 2-simplex*

$$\begin{array}{ccc} & \text{Fun}(\text{Idem}^+, \mathcal{C}) & \\ \text{restrict} \swarrow & & \searrow \text{restrict} \\ \text{Fun}(\text{Ret}, \mathcal{C}) & \xrightarrow{\text{split}} & \text{Fun}(\text{Idem}, \mathcal{C}). \end{array}$$

in  $\mathcal{Cat}_\infty$ . Furthermore, this functor  $\text{split}$  is uniquely determined up to a contractible space of choices.

*Proof.* Since  $\text{Ret} \rightarrow \text{Idem}^+$  is inner anodyne, the map  $\text{Fun}(\text{Idem}^+, \mathcal{C}) \rightarrow \text{Fun}(\text{Ret}, \mathcal{C})$  is a trivial Kan fibration, by Proposition I-5.7. Hence the functor

$$\text{Fun}(\text{Fun}(\text{Ret}, \mathcal{C}), \text{Fun}(\text{Idem}, \mathcal{C})) \rightarrow \text{Fun}(\text{Fun}(\text{Idem}^+, \mathcal{C}), \text{Fun}(\text{Idem}, \mathcal{C}))$$

is an equivalence, and the space

$$\{\text{restrict}\} \times_{\text{Fun}(\text{Fun}(\text{Idem}^+, \mathcal{C}), \text{Fun}(\text{Idem}, \mathcal{C}))}^{\text{htop}} \text{Fun}(\text{Fun}(\text{Ret}, \mathcal{C}), \text{Fun}(\text{Idem}, \mathcal{C}))$$

of functors completing the given diagram is contractible.  $\square$

**Definition A.4.** An idempotent  $e : \text{Idem} \rightarrow \mathcal{C}$  is called split if  $e$  is in the essential image of the functor  $\text{split} : \text{Fun}(\text{Ret}, \mathcal{C}) \rightarrow \text{Fun}(\text{Idem}, \mathcal{C})$ .

prop:idem\_split

**Proposition A.5** ([13, Corollary 4.4.5.14]). *For a given  $\infty$ -category  $\mathcal{C}$ , the following are equivalent:*

- (a) *Every idempotent in  $\mathcal{C}$  is split.*
- (b) *The restriction functor  $\text{Fun}(\text{Idem}^+, \mathcal{C}) \rightarrow \text{Fun}(\text{Idem}, \mathcal{C})$  is a trivial Kan fibration.*

(c) *The functor  $\text{split} : \text{Fun}(\text{Ret}, \mathcal{C}) \rightarrow \text{Fun}(\text{Idem}, \mathcal{C})$  is an equivalence.*

*Proof.* The equivalence between (a) and (b) is covered in [13, Corollary 4.4.5.14]. For the equivalence between (b) and (c), the diagram from Corollary A.3 and the fact that the restriction  $\text{Fun}(\text{Idem}^+, \mathcal{C}) \rightarrow \text{Fun}(\text{Ret}, \mathcal{C})$  is an equivalence tells us that the map  $\text{split}$  is an equivalence if and only if the restriction functor  $\text{Fun}(\text{Idem}^+, \mathcal{C}) \rightarrow \text{Fun}(\text{Idem}, \mathcal{C})$  is an equivalence.

Since the map  $\text{Idem} \rightarrow \text{Idem}^+$  is injective, the restriction functor

$$\text{Fun}(\text{Idem}^+, \mathcal{C}) \rightarrow \text{Fun}(\text{Idem}, \mathcal{C})$$

is an isofibration (Corollary I-6.14). To close we recall that, by Proposition II-16.12, an isofibration between  $\infty$ -categories is a trivial Kan fibration if and only if it is an equivalence.  $\square$

**Definition A.6.** We call an  $\infty$ -category  $\mathcal{C}$  idempotent complete if any of the equivalent conditions from Proposition A.5 hold.

The following tells us that any cocomplete  $\infty$ -category is idempotent complete.

`prop:cocomp_split`

**Proposition A.7** ([13, Corollary 4.4.5.16]). *Let  $\kappa$  be a regular cardinal. If  $\mathcal{A}$  is a  $\kappa$ -cocomplete  $\infty$ -category then  $\mathcal{A}$  is idempotent complete.*

## A.2. Idempotent completions.

**Definition A.8.** An idempotent completion of an  $\infty$ -category  $\mathcal{C}$  is a functor  $i : \mathcal{C} \rightarrow \mathcal{C}^\vee$  to an  $\infty$ -category  $\mathcal{C}^\vee$  for which:

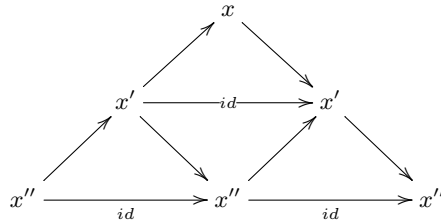
- (a) The functor  $i$  is fully faithful.
- (b)  $\mathcal{C}^\vee$  is idempotent complete.
- (c) Every object in  $\mathcal{C}^\vee$  is a retract of an object in the image of  $\mathcal{C}$ .

By a retract of an object  $x$  in  $\mathcal{C}^\vee$  we mean, of course, an object  $y$  which admits a retract diagram  $r : \text{Ret} \rightarrow \mathcal{C}^\vee$  with  $r(0) = y$  and  $r(1) = x$ . The following tells us that the collection of retracts for a class of objects  $O \subset \mathcal{C}^\vee[0]$  is stable under all expected operations.

`lem:3745`

**Lemma A.9.** *Let  $\mathcal{C}$  be any  $\infty$ -category, and  $x, x', x'' : * \rightarrow \mathcal{C}$  be arbitrary objects. If  $x'$  is a retract of  $x$ , and  $x''$  is a retract of  $x'$ , then  $x''$  is a retract of  $x$ .*

*Proof.* Follows by a consideration of the diagram



in  $\mathcal{C}$ .  $\square$

We note that idempotent completions exist in complete generality.

`lem:idem_exist`

**Lemma A.10.** (1) *For any  $\infty$ -category  $\mathcal{C}$ , an idempotent completion  $i : \mathcal{C} \rightarrow \mathcal{C}^\vee$  exists.*

- (2) If  $\mathcal{C}$  is essentially small, then there is an idempotent completion  $i : \mathcal{C} \rightarrow \mathcal{C}^\vee$  for which  $\mathcal{C}^\vee$  is also essentially small.

*Proof of Lemma A.10.* We will prove that any essentially small  $\infty$ -category  $\mathcal{C}$  admits an essentially small idempotent completion. For (1) one just repeats the same arguments without regard for size constraints.

Suppose that  $\mathcal{C}$  is essentially small. Since the category  $\text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{K}an^{\text{sm}})$  is cocomplete it is idempotent split. Let  $\mathcal{C}' \subseteq \text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{K}an^{\text{sm}})$  denote the essential image of the Yoneda embedding  $\mathcal{C} \rightarrow \text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{K}an^{\text{sm}})$ . Now, the collection of idempotents  $\text{Fun}(\text{Idem}, \mathcal{C}')$  is also essentially small, as the simplicial set  $\text{Idem}$  is idempotent complete.

Let us take

$$\mathcal{Z} = \text{Fun}(\text{Idem}^+ \times \mathcal{C}^{\text{op}}, \mathcal{K}an^{\text{sm}}) \times_{\text{Fun}(\text{Idem} \times \mathcal{C}^{\text{op}}, \mathcal{K}an^{\text{sm}})} \text{Fun}(\text{Idem}, \mathcal{C}').$$

As the restriction functor  $\text{Fun}(\text{Idem}^+ \times \mathcal{C}^{\text{op}}, \mathcal{K}an^{\text{sm}}) \rightarrow \text{Fun}(\text{Idem} \times \mathcal{C}^{\text{op}}, \mathcal{K}an^{\text{sm}})$  is a trivial Kan fibration [13, Corollary 4.4.5.16.], the projection  $p : \mathcal{Z} \rightarrow \text{Fun}(\text{Idem}, \mathcal{C}')$  is a trivial Kan fibration as well.

We note that  $\mathcal{Z}$  is a fully subcategory in  $\text{Fun}(\text{Idem}^+ \times \mathcal{C}^{\text{op}}, \mathcal{K}an^{\text{sm}})$ . The restriction functor  $\mathcal{Z} \rightarrow \text{Fun}(\text{Ret} \times \mathcal{C}^{\text{op}}, \mathcal{K}an^{\text{sm}})$  is a trivial Kan fibration onto its image as well [13, Proposition 4.4.5.6]. (Here we mean strict image, not essential image.) This image, call it  $\mathcal{Y}$ , is precisely the collection of diagrams  $F : \text{Ret} \rightarrow \text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{K}an^{\text{sm}})$  with  $F(1)$  in  $\mathcal{C}'$ , and we deduce an equivalence  $\mathcal{Y} \xrightarrow{\sim} \text{Fun}(\text{Idem}, \mathcal{C}')$ . In particular  $\mathcal{Y}$  is essentially small.

Evaluating at  $0 = 2$  we obtain a functor

$$ev_0 : \mathcal{Y} \rightarrow \text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{K}an^{\text{sm}})$$

and let  $\mathcal{C}^\vee \subseteq \text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{K}an^{\text{sm}})$  denote the full  $\infty$ -subcategory spanned by the image of  $\mathcal{Y}$ . Since  $\mathcal{Y}$  is essentially small and  $\text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{K}an^{\text{sm}})$  is locally small, we conclude that  $\mathcal{C}^\vee$  is essentially small. From a consideration of degenerate diagram

$$\text{Ret} \rightarrow * \xrightarrow{x} \text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{K}an^{\text{sm}})$$

at any  $x$  in  $\mathcal{C}'$  it is clear that  $\mathcal{C}^\vee$  contains  $\mathcal{C}'$ , and we see directly that  $\mathcal{C}^\vee$  is spanned by all objects which are retracts of objects in  $\mathcal{C}'$ . Hence the Yoneda embedding provides a fully faithful functor  $i : \mathcal{C} \rightarrow \mathcal{C}^\vee$  for at which all objects in  $\mathcal{C}^\vee$  are obtained by taking retracts of objects in the image of  $i$ . Furthermore, by Proposition A.7 the functor category  $\text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{K}an^{\text{sm}})$  is idempotent split, and hence  $\mathcal{C}^\vee$  is idempotent complete by Lemma A.9. So the Yoneda embedding realizes  $\mathcal{C}^\vee$  as an idempotent completion of  $\mathcal{C}$ .  $\square$

We have the following universal property for idempotent completion.

prop: idemcomp\_univ

**Proposition A.11** ([13, Proposition 5.1.4.9]). *If  $i : \mathcal{C} \rightarrow \mathcal{C}^\vee$  is an idempotent completion of an  $\infty$ -category  $\mathcal{C}$ , then for any idempotent complete  $\infty$ -category  $\mathcal{A}$  restricting along  $i$  provides an equivalence of  $\infty$ -categories*

$$i^* : \text{Fun}(\mathcal{C}^\vee, \mathcal{A}) \rightarrow \text{Fun}(\mathcal{C}, \mathcal{A}).$$

**Corollary A.12.** *For any  $\infty$ -category  $\mathcal{C}$ , the idempotent completion  $i : \mathcal{C} \rightarrow \mathcal{C}^\vee$  is uniquely determined up to isomorphism in the undercategory  $(\mathcal{Cat}_\infty)_{\mathcal{C}/}$ .*

We combine uniqueness with the smallness assertion from Lemma A.10 to observe that idempotent completion preserves essentially small  $\infty$ -categories.

cor:split\_small

**Corollary A.13.** *If  $\mathcal{C}$  is essentially small, then any idempotent completion  $\mathcal{C}^\vee$  of  $\mathcal{C}$  is also essentially small.*

Using Proposition A.11 one can also establish the existence of a unique idempotent completion functor

$$(-)^\vee : \mathcal{Cat}_\infty \rightarrow \mathcal{Cat}_\infty$$

which comes equipped with a transformation  $i : id_{\mathcal{Cat}_\infty} \rightarrow (-)^\vee$  which evaluates to an idempotent completion  $i : \mathcal{C} \rightarrow \mathcal{C}^\vee$  at each  $\infty$ -category  $\mathcal{C}$ . To see this one constructs a cocartesian fibration  $q : \mathcal{M}_{\text{Idem}} \rightarrow \Delta^1$  for idempotent completions and proceeds exactly as in Section 11.2, where we construct the indization functor.

Below we let  $(\mathcal{Cat}_\infty)_{\text{split}}$  denote the full subcategory of idempotent split  $\infty$ -categories in  $\mathcal{Cat}_\infty$ . We leave the proof of the following as an exercise for the interested reader.

prop:idem\_functr

**Proposition A.14.** *There is a functor  $(-)^\vee : \mathcal{Cat}_\infty \rightarrow (\mathcal{Cat}_\infty)_{\text{split}}$  is right adjoint to the inclusion  $(\mathcal{Cat}_\infty)_{\text{split}} \rightarrow \mathcal{Cat}_\infty$ , and the unit transformation  $i : id_{\mathcal{Cat}_\infty} \rightarrow (-)^\vee$  evaluates to an idempotent completion  $i : \mathcal{C} \rightarrow \mathcal{C}^\vee$  at each  $\infty$ -category  $\mathcal{C}$ .*

### A.3. $\aleph_0$ -accessibility and idempotent completion.

**Definition A.15.** An  $\infty$ -category  $\mathcal{A}$  is called  $\aleph_0$ -accessible if  $\mathcal{A}$  admits a functor  $i : \mathcal{C} \rightarrow \mathcal{A}$  from an essentially small  $\infty$ -category  $\mathcal{C}$  which exhibits  $\mathcal{A}$  as an ind-completion of  $\mathcal{C}$  (Definition 11.4).

The category  $\text{Acc}_{\aleph_0}$  of  $\aleph_0$ -accessible categories is the non-full subcategory in  $\mathcal{Cat}_\infty$  whose  $n$ -simplices  $\sigma : \Delta^n \rightarrow \text{Acc}_{\aleph_0}$  are those simplices  $\sigma : \Delta^n \rightarrow \mathcal{Cat}_\infty$  for which each vertex  $\mathcal{A}_i = \sigma(\{i\})$  is  $\aleph_0$ -accessible, and for which each edge  $F_{ij} = \sigma|_{\Delta^{[i,j]}} : \mathcal{A}_i \rightarrow \mathcal{A}_j$  is  $\aleph_0$ -cocontinuous and preserves  $(\aleph_0-)$ compact objects.

We can construct the category  $\text{Acc}_{\aleph_0}$  as the homotopy coherent nerve of the simplicial subcategory  $\underline{\text{Acc}}_{\aleph_0} \subseteq \underline{\text{Cat}}_\infty^+$  whose objects are  $\aleph_0$ -accessible categories and whose mapping complexes are the full subcategories

$$\text{Fun}_{\aleph_0}(\mathcal{A}, \mathcal{B}) \subseteq \text{Fun}(\mathcal{A}, \mathcal{B})^{\text{Kan}}$$

spanned by accessible functors.

Taking compacts provides a simplicial functor  $\underline{\text{Acc}}_{\aleph_0} \rightarrow \underline{\text{Cat}}_\infty^+$  and we apply the homotopy coherent nerve to obtain a functor  $(-)^c : \text{Acc}_{\aleph_0} \rightarrow \mathcal{Cat}_\infty$ . The following gives essential information about the image of the compacts functor.

lem:indc\_idem

**Lemma A.16** ([13, Lemma 5.4.2.4]). *If  $\mathcal{C}$  is essentially small, then the subcategory  $\text{Ind}(\mathcal{C})^c$  of compact objects in  $\text{Ind}(\mathcal{C})$  is an idempotent completion of  $\mathcal{C}$ .*

Applying Corollary A.13, we see that the full subcategory of compacts  $\mathcal{A}^c$  in any accessible  $\infty$ -category form an essentially small  $\infty$ -category.

lem:aleph\_prequiv

**Lemma A.17.** (1) *If  $\mathcal{A}$  is an  $\aleph_0$ -accessible  $\infty$ -category then the full subcategory of compacts  $\mathcal{A}^c$  is essentially small, and the inclusion  $\mathcal{A}^c \rightarrow \mathcal{A}$  induces an equivalence  $\text{Ind}(\mathcal{A}^c) \rightarrow \mathcal{A}$ .*

(2) *If  $\mathcal{C}$  is essentially small and idempotent complete, then the inclusion  $i : \mathcal{C} \rightarrow \text{Ind}(\mathcal{C})^c$  is an equivalence.*

*Proof.* Statement (2) is clear, by uniqueness of idempotent completion. For (1) we have an equivalence  $\text{Ind}(\mathcal{C}) \xrightarrow{\sim} \mathcal{A}$  for some essentially small  $\mathcal{C}$ , so that the inclusion  $\mathcal{C} \rightarrow \text{Ind}(\mathcal{C}) \rightarrow \mathcal{A}^c$  exhibits  $\mathcal{A}^c$  as an idempotent completion of  $\mathcal{C}$ . By

Corollary A.13 it follows that  $\mathcal{A}^c$  is essentially small. Furthermore, since  $\mathcal{A}$  is generated by the image of  $\mathcal{C}$  under filtered colimits, we see that  $\mathcal{A}$  is generated by  $\mathcal{A}^c$  under filtered colimits. Hence the induced map  $\text{Ind}(\mathcal{A}^c) \rightarrow \mathcal{A}$  is an equivalence by Theorem 11.6.  $\square$

We note that, via the structural transformation  $i : \text{incl} \rightarrow \text{Ind}$  we have a diagram

$$\begin{array}{ccc} \text{Ind}(\mathcal{C}) & \xrightarrow{\text{Ind}(f)} & \text{Ind}(\mathcal{C}') \\ \uparrow i & \nearrow & \uparrow i \\ \mathcal{C} & \xrightarrow{f} & \mathcal{C}' \end{array}$$

at any map  $f : \mathcal{C} \rightarrow \mathcal{C}'$  between essentially small  $\infty$ -categories. It follows that the functor  $\text{Ind}(f)$  preserves compact objects, and is therefore  $\aleph_0$ -accessible. We can therefore restrict the codomain to obtain a functor  $\text{Ind} : \mathcal{Cat}_{\infty}^{\text{sm}} \rightarrow \text{Acc}_{\aleph_0}$ .

Let us take

$(\mathcal{Cat}_{\infty}^{\text{sm}})_{\text{split}} :=$  the full subcategory of idempotent complete  $\infty$ -categories in  $\mathcal{Cat}_{\infty}^{\text{sm}}$ .

We have functors

$$(-)^c : \text{Acc}_{\aleph_0} \rightarrow (\mathcal{Cat}_{\infty}^{\text{sm}})_{\text{split}} \quad \text{and} \quad \text{Ind} : (\mathcal{Cat}_{\infty}^{\text{sm}})_{\text{split}} \rightarrow \text{Acc}_{\aleph_0},$$

and by Lemma A.17 the functor  $(-)^c$ , taken now to have image in idempotent complete categories, is essentially surjective. Furthermore, for the simplicial category  $\underline{\text{Acc}}_{\aleph_0}$ , Lemma A.17 and Theorem 11.5 tell us that restricting to the compacts provides a fully faithful functor

$$\text{Fun}_{\aleph_0}(\mathcal{A}, \mathcal{B}) \rightarrow \text{Fun}(\mathcal{A}^c, \mathcal{B}^c)$$

whose essential image is precisely the subcategory  $\text{Fun}(\mathcal{A}^c, \mathcal{B}^c)$ . So, at the level of simplicial categories, the functor given by restricting to the compacts

$$\underline{\text{Acc}}_{\aleph_0} \rightarrow (\underline{\text{Cat}}_{\infty}^{\text{sm}})_{\text{split}}$$

is fully faithful. By Proposition II-7.6 it follows that the functor  $(-)^c : \text{Acc}_{\aleph_0} \rightarrow (\mathcal{Cat}_{\infty}^{\text{sm}})_{\text{split}}$  is fully faithful as well.

thm:compacts\_equiv

**Theorem A.18.** *The functor  $(-)^c : \text{Acc}_{\aleph_0} \rightarrow (\mathcal{Cat}_{\infty}^{\text{sm}})_{\text{split}}$  is an equivalence.*

*Proof.* By the information above, this functor is fully faithful and essentially surjective, and hence an equivalence by Theorem I-8.2.  $\square$

We claim finally that the inverse to  $(-)^c$  is provided by the indization functor. To prove this it suffices to prove that indization is left adjoint to the compact objects functor.

prop:ind\_compacts\_adj

**Proposition A.19.** *The functor  $(-)^c : \text{Acc}_{\aleph_0} \rightarrow \mathcal{Cat}_{\infty}^{\text{sm}}$  is right adjoint to the indization functor  $\text{Ind} : \mathcal{Cat}_{\infty}^{\text{sm}} \rightarrow \text{Acc}_{\aleph_0}$ .*

*Proof.* Let  $\mathcal{M}_{\aleph_0} \subseteq \Delta^1 \times \mathcal{Cat}_{\infty}$  denote the non-full subcategory whose objects are

$$\mathcal{M}_{\aleph_0}[0] = (\{0\} \times \mathcal{Cat}_{\infty}^{\text{sm}}[0]) \amalg (\{1\} \times \text{Acc}_{\aleph_0}[0])$$

and whose edges  $(i, \mathcal{C}) \rightarrow (j, \mathcal{A})$  are arbitrary maps of essentially small  $\infty$ -categories when  $i = j = 0$ , maps in  $\text{Acc}_{\aleph_0}$  when  $i = j = 1$ , and maps  $\mathcal{C} \rightarrow \mathcal{A}$  which have image in  $\mathcal{A}^c$  when  $i < j$ . We place no other restrictions on  $n$ -simplices in  $\mathcal{M}_{\aleph_0}$ . One checks directly that the inclusion  $\mathcal{M}_{\aleph_0} \rightarrow \Delta^1 \times \mathcal{Cat}_{\infty}$  is an inner fibration, so that

the composite  $q : \mathcal{M}_{\aleph_0} \rightarrow \Delta^1$  of the inclusion with the projection  $\Delta^1 \times \mathcal{Cat}_\infty \rightarrow \Delta^1$  is also an inner fibration. One argues exactly as in the proof of Lemma ?? to see that the projection  $q : \mathcal{M}_{\aleph_0} \rightarrow \Delta^1$  is a cocartesian fibration with  $q$ -cocartesian edges over  $0 < 1$  provided by those functors  $\mathcal{C} \rightarrow \mathcal{A}$  which realize  $\mathcal{A}$  as an ind-completion of  $\mathcal{C}$ .

(To recall, we can identify  $\mathcal{M}_{\aleph_0}$  with the homotopy coherent nerve of the apparent simplicial subcategory  $\mathcal{M}_{\aleph_0} \subseteq \{0 < 1\} \times \underline{\mathcal{Cat}}_\infty^{\text{sm}}$ , and use this simplicial construction along with Proposition II-7.6 to check reflexivity of  $\text{Acc}_{\aleph_0}$  in  $\mathcal{M}_{\aleph_0}$  directly via the functor categories and the isomorphism

$$i^* : \text{Fun}_{\aleph_0}(\mathcal{C}', \mathcal{A}) \rightarrow \text{Fun}_c(\mathcal{C}, \mathcal{A})$$

given by restricting along any ind-completion  $i : \mathcal{C} \rightarrow \mathcal{C}'$  at essentially small  $\mathcal{C}$ , where  $\text{Fun}_c(\mathcal{C}, \mathcal{A})$  here denotes the full subcategory of functors with image in  $\mathcal{A}^c$ .)

From this characterization of  $q$ -cocartesian edges we see that the indization functor, as defined in Section 11.2, provides the unique cocartesian solution to the lifting problem

$$\begin{array}{ccc} \{0\} \times \mathcal{Cat}_\infty^{\text{sm}} & \xrightarrow{\quad} & \mathcal{M}_{\aleph_0} \\ \downarrow & \nearrow I & \downarrow q \\ \Delta^1 \times \mathcal{Cat}_\infty^{\text{sm}} & \xrightarrow{\quad} & \Delta^1, \end{array}$$

and hence that the functor  $\text{Ind} : \mathcal{Cat}_\infty^{\text{sm}} \rightarrow \text{Acc}_{\aleph_0}$  is realized as homotopy transport for along  $q$ .

We claim now that  $q : \mathcal{M}_{\aleph_0} \rightarrow \Delta^1$  is also cartesian. For this it suffices to show that  $\mathcal{Cat}_\infty^{\text{sm}} \rightarrow \mathcal{M}_{\aleph_0}$  is coreflexive. However, this is clear since for any  $\aleph_0$ -accessible  $\infty$ -category  $\mathcal{A}$  the inclusion  $k : \mathcal{A}^c \rightarrow \mathcal{A}$  provides an equivalence  $k_* : \text{Hom}_{\mathcal{M}}(\mathcal{C}, \mathcal{A}^c) \rightarrow \text{Hom}_{\mathcal{M}}(\mathcal{C}, \mathcal{A})$ . One observes this fact precisely by considering the diagram

$$\begin{array}{ccc} \text{Hom}_{\mathcal{M}}(\mathcal{C}, \mathcal{A}^c) & \xrightarrow{k_*} & \text{Hom}_{\mathcal{M}}(\mathcal{C}, \mathcal{A}) \\ \uparrow \sim & & \uparrow \sim \\ \text{Fun}(\mathcal{C}, \mathcal{A}^c) & \xrightarrow[k_*]{\sim} & \text{Fun}_c(\mathcal{C}, \mathcal{A}) \end{array}$$

in  $\mathbf{h}\mathcal{Kan}$ , which exists by Proposition II-7.6.

Using Lemma 7.13 we furthermore characterize  $q$ -cartesian edges over  $0 < 1$  as those map  $k : \mathcal{A}' \rightarrow \mathcal{A}$  which are an equivalence onto the compacts in  $\mathcal{A}$ . At the level of simplicial categories, the inclusions  $\mathcal{A}^c \rightarrow \mathcal{A}$  provide a transformation  $(-)^c \rightarrow \text{incl}$  between the simplicial functor  $(-)^c : \underline{\text{Acc}}_{\aleph_0} \rightarrow \underline{\mathcal{Cat}}_\infty$  and the inclusion  $\text{incl} : \underline{\text{Acc}}_{\aleph_0} \rightarrow \underline{\mathcal{Cat}}_\infty^+$ , and we obtain an induced transformation  $k : (-)^c \rightarrow \text{incl}$  between objects in  $\text{Fun}(\text{Acc}_\lambda, \mathcal{Cat}_\infty)$ . This transformation provides the unique cartesian solution to the lifting problem

$$\begin{array}{ccc} \{1\} \times \text{Acc}_{\aleph_0} & \xrightarrow{\quad} & \mathcal{M}_{\aleph_0} \\ \downarrow & \nearrow K & \downarrow q \\ \Delta^1 \times \text{Acc}_{\aleph_0} & \xrightarrow{\quad} & \Delta^1, \end{array}$$

which is given explicitly by the sequence

$$\Delta^1 \times \text{Acc}_{\aleph_0} \xrightarrow{\delta \times 1} \Delta^1 \times \Delta^1 \times \text{Acc}_{\aleph_0} \xrightarrow{1 \times k} \mathcal{M}_{\aleph_0} \subseteq \Delta^1 \times \mathcal{Cat}_\infty.$$



Evaluating at 0 recovers the functor which takes compacts

$$K|_{\{0\} \times \text{Acc}_{\mathbb{N}_0}} = (-)^c : \text{Acc}_{\mathbb{N}_0} \rightarrow \mathcal{Cat}_{\infty}^{\text{sm}}$$

so that this functor is realized as homotopy contravariant transport along  $q$ . It follows by Theorem 7.15 that  $(-)^c$  is right adjoint to the indization functor.  $\square$

Since taking the compacts in  $\text{Acc}_{\mathbb{N}_0}$  has image in  $(\mathcal{Cat}_{\infty}^{\text{sm}})_{\text{split}}$ , we see that the restricted functor

$$\text{Ind} : (\mathcal{Cat}_{\infty}^{\text{sm}})_{\text{split}} \rightarrow \text{Acc}_{\mathbb{N}_0}$$

remains left adjoint to the functor  $(-)^c$ . By uniqueness of adjoints, we conclude that  $\text{Ind}$  is in fact the inverse functor to the equivalence  $(-)^c$ .

cor:ind\_inverse

**Corollary A.20.** *The functors*

$$\text{Ind} : (\mathcal{Cat}_{\infty}^{\text{sm}})_{\text{split}} \rightarrow \text{Acc}_{\mathbb{N}_0} \quad \text{and} \quad (-)^c : \text{Acc}_{\mathbb{N}_0} \rightarrow (\mathcal{Cat}_{\infty}^{\text{sm}})_{\text{split}}$$

*are mutually inverse equivalences.*

The following identifies our indization functor with the corresponding functor appearing in [13, Proposition 5.4.2.19].

prop:ind\_composite

**Proposition A.21.** *In  $\text{Fun}(\mathcal{Cat}_{\infty}^{\text{sm}}, \mathcal{Cat}_{\infty})$ , there is a natural isomorphism of functors*

$$\text{Ind} \cong (\text{Ind}|_{(\mathcal{Cat}_{\infty}^{\text{sm}})_{\text{split}}} \circ (-)^{\vee}.$$

*Proof.* If  $F_i$  is left adjoint to  $G_i$  then the composite  $F_1 F_0$  is left adjoint to  $G_0 G_1$ . Hence, by Proposition A.14 and Corollary A.20. Hence the composite  $(\text{Ind}|_{(\mathcal{Cat}_{\infty}^{\text{sm}})_{\text{split}}} \circ (-)^{\vee}$  is left adjoint to the composite

$$(-)^c = \text{incl} \circ (-)^c : \text{Acc}_{\mathbb{N}_0} \rightarrow (\mathcal{Cat}_{\infty}^{\text{sm}})_{\text{split}} \rightarrow \mathcal{Cat}_{\infty}^{\text{sm}}.$$

By uniqueness of left adjoints and Proposition A.19.  $\square$

#### A.4. Idempotent splitting in the stable setting.

**Proposition A.22.** *A stable  $\infty$ -category  $\mathcal{C}$  is idempotent complete if and only if its homotopy category  $\text{h}\mathcal{C}$  is idempotent complete.*

*Proof.* By enlarging our universe if necessary, we may assume  $\mathcal{C}$  is essentially small. Taking  $\mathcal{A} = \text{Ind}(\mathcal{C})$ , we have the exact embedding  $\mathcal{C} \rightarrow \mathcal{A}$  into a presentable stable idempotent complete  $\infty$ -category, by Proposition 11.21. As  $\mathcal{A}^c$  is an idempotent completion of  $\mathcal{C}$ , we understand that  $\mathcal{C}$  is idempotent complete if and only if the map  $\mathcal{C} \rightarrow \mathcal{A}^c$  is an equivalence.

Note that  $\text{h}\mathcal{A}$  is idempotent complete, as it is triangulated with all small coproducts [18, Proposition 1.6.8]. So we have a bijection between isoclasses of retract diagrams in  $\text{h}\mathcal{A}$  and idempotents in  $\text{h}\mathcal{A}$ . Since every retract diagram in  $\text{h}\mathcal{A}$  lifts to a retract diagram in  $\mathcal{A}$ , we obtain a bijection between isoclasses of retract diagrams in  $\mathcal{A}$  and idempotents in  $\text{h}\mathcal{A}$ . This bijection simply sends a retract  $r = (y \rightarrow x \rightarrow y)$  to the idempotent  $e_r : x \rightarrow y \rightarrow x$  in  $\text{h}\mathcal{A}$ .

Since  $\text{h}\mathcal{A}$  is idempotent complete, we have that the subcategory of compacts  $(\text{h}\mathcal{A})^c$  is idempotent complete. Now, at the level of the homotopy category we have  $\text{h}(\mathcal{A}^c) = (\text{h}\mathcal{A})^c$ , by Corollary 6.13. So if  $\mathcal{C}$  is idempotent complete we have that the homotopy category  $\text{h}\mathcal{C} \cong (\text{h}\mathcal{A})^c$  is idempotent complete as well.

Suppose conversely that  $\text{h}\mathcal{C}$  is idempotent complete. By the definition of idempotent completion we have that every object in  $\text{h}(\mathcal{A}^c) = (\text{h}\mathcal{A})^c$  is a retract of an

object in the image of  $h\mathcal{C}$ . By idempotent completeness of  $h(\mathcal{A}^c)$  it follows that every object is obtained by splitting an idempotent in the image of  $h\mathcal{C}$ . In the case that  $h\mathcal{C}$  is idempotent split, it follows that every object in  $h\mathcal{A}^c$  is in the essential image of  $h\mathcal{C}$ , and hence that the map  $h\mathcal{C} \rightarrow h(\mathcal{A}^c)$  is an equivalence. So we see that  $\mathcal{C}$  is idempotent complete.  $\square$

## REFERENCES

- [1] J. F. Adams. *Stable homotopy and generalised homology*. University of Chicago press, 1974.
- [2] A. J. Blumberg, D. Gepner, and G. Tabuada. A universal characterization of higher algebraic k-theory. *Geom. Topol.*, 17(2):733–838, 2013.
- [3] A. Bondal and M. V. den Bergh. Generators and representability of functors in commutative and noncommutative geometry. *Mosc. Math. J.*, 3(1):1–36, 2003.
- [4] A. Chirvasitu and T. Johnson-Freyd. The fundamental pro-groupoid of an affine 2-scheme. *Appl. Categor. Struct.*, 21(5):469–522, 2013.
- [5] D. Gaitsgory. Ind-coherent sheaves. *preprint arXiv:1105.4857*.
- [6] D. Gaitsgory and N. Rozenblyum. *A study in derived algebraic geometry: Volume I: correspondences and duality*, volume 221. American Mathematical Society, 2019.
- [7] P. Gross. *Vector bundles as generators on schemes and stacks*. PhD thesis, Düsseldorf, Univ., Diss., 2010, 2010.
- [8] J. Hall, A. Neeman, and D. Rydh. One positive and two negative results for derived categories of algebraic stacks. *J. Inst. Math. Jussieu*, 18(5):1087–1111, 2019.
- [9] A. Joyal. Quasi-categories and kan complexes. *J. Pure Appl. Algebra*, 175(1-3):207–222, 2002.
- [10] M. Kashiwara and P. Schapira. *Categories and Sheaves*. Springer Berlin Heidelberg, 2006.
- [11] G. M. Kelly. *Basic concepts of enriched category theory*, volume 64. CUP Archive, 1982.
- [12] H. Krause. *Localization theory for triangulated categories*, pages 161–235. London Mathematical Society Lecture Note Series. Cambridge University Press, 2010.
- [13] J. Lurie. *Higher topos theory*. Number 170. Princeton University Press, Princeton, New Jersey 08540, 2009.
- [14] J. Lurie. Higher algebra. [www.math.ias.edu/~lurie/papers/HA.pdf](http://www.math.ias.edu/~lurie/papers/HA.pdf), 2017.
- [15] J. Lurie. Kerodon. <https://kerodon.net>, 2024.
- [16] J. Lurie. Spectral algebraic geometry. <https://www.math.ias.edu/~lurie/papers/SAG-rootfile.pdf>, updated 2018.
- [17] J. Milnor. The geometric realization of a semi-simplicial complex. *Ann. of Math.*, 65(2):357–362, 1957.
- [18] A. Neeman. *Triangulated categories*. Number 148. Princeton University Press, 2001.
- [19] T. Nikolaus and P. Scholze. On topological cyclic homology. *Acta Math.*, 221(2):203 – 409, 2018.
- [20] C. Serpé. Resolution of unbounded complexes in grothendieck categories. *J. Pure Appl. Algebra*, 177(1):103–112, 2003.
- [21] L. A. Tarrío, A. J. López, M. P. Rodríguez, and M. J. V. Gonsalves. The derived category of quasi-coherent sheaves and axiomatic stable homotopy. *Adv. Math.*, 218(4):1224–1252, 2008.
- [22] L. A. Tarrío, A. J. López, M. P. Rodríguez, and M. J. V. Gonsalves. A functorial formalism for quasi-coherent sheaves on a geometric stack. *Expo. Math.*, 33(4):452–501, 2015.
- [23] C. A. Weibel. *An introduction to homological algebra*, volume 38. Cambridge university press, 1994.

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