Aug 26: Def + Examples (1)

Lie alg's over $\mathbb{C}$

$\mathbb{C}$ = alg closed field of char 0

Def^{n}: A Lie alg over $\mathbb{C}$ is a $\mathbb{C}$-vector space $\mathfrak{g}$ equipped w/ a bilinear operator

$[,]: \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ "Lie bracket operator"

satisfying

antisymm) $[x,y] = -[y,x]$

Jacobi identity)

$$[x[y,z]] + [z[x,y]] + [y[z,x]] = 0$$

We can rewrite the Jacobi identity as

$$[x[y,z]] = [[x,y],z] + [y[x,z]] \quad (*)$$

- Example A^{Lie}.

Let A be any $\mathbb{C}$-algebra. Define A^{Lie} to be the vect space A w/ commutator bracket

$$[a,b] := ab - ba.$$

Antisymmetry is obvious. For the Jacobi identity we have

Lemma 1: The comm bracket on A satisfies

$$[a, bc] = [a,b]c + b[a,c].$$

Proof: See directly

$RHS = [a,bc] - bac + bac = LHS$. ∎

Lemma 2: The Jacobi identity holds.

Proof: $[a[b,c]] = [a, bc] - [a, cb]$

$= [a,b]c + b[a,c] - [a,c]b - c[a,b]$
$= [[a,b]c] + [b,[a,c]]$. ∎

Corollary 3: The pairing

$A^{Lie} := (A, [,]_{comm})$

is a Lie alg.

Proof: By def^{n}. ∎

- Example [Abelian Lie algs]

For any vector space V we can endow V w/ the trivial bracket $[\,,\,]_{triv} : V \otimes V \to V$ def by $[v,w] = 0$ at all $v,w \in V$. The Jacobi identity holds trivially $(0=0)$ so that the pairing

$$(V, [\,,\,]_{triv}) \text{ forms a Lie algebra.}$$

Def^{n}: A Lie algebra $\mathfrak{h}$ is called abelian if the bracket operation on $\mathfrak{h}$ is identically 0, i.e. if

$$\mathfrak{h} = (V, [\,,\,]_{triv})$$

for a vector space V.

Sub-example: The Lie alg A^{Lie} assoc to an alg A is abelian iff A is commutative.

- Example [$\mathfrak{gl}(V)$].

For any vector space V we have the algebra of linear endomorphisms $End_{\mathbb{C}}(V)$.

Def^{n}: The general linear Lie alg for V is

$$\mathfrak{gl}(V) := End(V)^{Lie}$$

$$= \left\{ \begin{matrix} \text{linear endos } A: V \to V \text{ w/ commutator} \\ \text{bracket } [A,B] = AB - BA \end{matrix} \right\}.$$

In the particular case $V = \mathbb{C}^n$ we write

$$\mathfrak{gl}_n(\mathbb{C}) := \mathfrak{gl}(\mathbb{C}^n) = M_n(\mathbb{C})^{Lie}.$$

- Lie subalgebras and ideals

Def^{n}: A Lie subalgebra is a Lie alg of is a vector subspace $\mathfrak{f} \subseteq \mathfrak{g}$ for which

$$[x,y] \in \mathfrak{f} \text{ whenever } x,y \in \mathfrak{f}.$$

An ideal is $\mathfrak{g}$ is a subspace $I \subseteq \mathfrak{g}$ which satisfies $[x,z] \in I$ whenever one of x or z is in I.

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Def$^{\underline{n}}$: A homomorphism of Lie alg's $\phi: \mathfrak{g} \to \mathfrak{g}'$ is a linear map which satisfies

$$\phi([x,y]) = [\phi(x), \phi(y)]' \quad \text{at all } x, y \in \mathfrak{g}.$$

[also iso's]

Lemma 4: a) Any Lie subalg $\mathfrak{f} \leq \mathfrak{g}$ is itself a Lie alg, w/ bracket inherited from that of $\mathfrak{g}$.

b) For any ideal $I \leq \mathfrak{g}$, I is a Lie subalg and the quotient $\mathfrak{g}/I$ inherits a unique Lie alg structure so that the quotient map $\pi: \mathfrak{g} \to \mathfrak{g}/I$ is a Lie alg homomorphism.

Proof: Exercise. ∎

Lemma 5: The kernel $\ker\phi \leq \mathfrak{g}$ of any Lie alg homomorphism $\phi: \mathfrak{g} \to \mathfrak{g}'$ is an ideal in $\mathfrak{g}$.

Example [$\mathfrak{sl}_n(\mathbb{C})$]: Let $\mathbb{C}$ = 1-dim abelian Lie alg. Then the trace function

$$tr: \mathfrak{gl}_n(\mathbb{C}) \to \mathbb{C}, \quad A \mapsto tr(A)$$

satisfies $tr([A,B]) = 0 = [trA, trB]$.

Hence the trace function is a Lie alg homomorphism, and the kernel

$$\mathfrak{sl}_n(\mathbb{C}) := \ker(tr) = \left\{ \begin{array}{c} n\times n \text{ traceless matrices} \\ \text{w/ commutator} \\ \text{bracket} \end{array} \right\}.$$

We have

$$\dim \mathfrak{gl}_n(\mathbb{C}) = n^2$$

$$\dim \mathfrak{sl}_n(\mathbb{C}) = n^2 - 1.$$

In the particular case $n=2$, $\dim \mathfrak{sl}_2(\mathbb{C}) = 3$, and we have the spanning set

$$\mathfrak{sl}_2(\mathbb{C}) = \mathrm{span}_{\mathbb{C}}\left\{ e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}.$$

The Lie bracket is specified by the formulas:

$$[h,e] = 2e, \quad [h,f] = -2f, \quad [e,f] = h.$$

$\mathfrak{sl}_2(\mathbb{C})$ is a very special individual.

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– Lie alg in low dim.

Dim 1: In dim 1, the only Lie alg $\mathfrak{h} = \mathbb{C}\cdot x$ is the abelian one. This follows from antisymmetry

$$[a\cdot x, b\cdot x] = a\cdot b\,[x,x] = 0.$$

Dim 2: In dim 2, have $\mathfrak{h} = \mathbb{C}x \oplus \mathbb{C}y$

$$[x,x] = [y,y] = 0, \qquad [x,y] = ax + by.$$

If $a \neq 0$ then replace x w/ $x + \frac{b}{a}y$ to get alt expression $[x,y] = a\cdot x$. Then

$$[y,[x,y]] = -a^2 x$$

$$= [[y,x],y] + [x,[y,y]] = a^2 x,$$

giving $0 = 2a^2 x$, a contradiction.

Conclusion: The only 2-dim Lie alg, up to isomorphism, is the abelian one.

Dim 3: In dim 3 we have the non-abelian Lie alg

$$\mathfrak{n}_3 := \left\{ \begin{bmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{bmatrix} : a_{ij} \in \mathbb{C} \right\} \leq \mathfrak{gl}_3(\mathbb{C}).$$

Precisely $\mathfrak{n}_3 = A^{Lie}$ for the (non-unital) alg of strictly upper Δ matrices.

Exercise: Prove that any 3-dimensional Lie alg $\mathfrak{g}$ is either abelian, or isomorphic to $\mathfrak{n}_3$.

– Representations of Lie algebras

Def[!]: A representation of a Lie alg $\mathfrak{g}$ is a vector space V equipped w/ a a linear map

$$\cdot : \mathfrak{g} \otimes V \to V$$

satisfying $[x,y]\cdot v = x\cdot(y\cdot v) - y\cdot(x\cdot v)$.

Lemma: For any $\mathfrak{g}$-rep V, the map

$$\rho_V : \mathfrak{g} \to \mathfrak{gl}(V), \quad x \longmapsto (v \longmapsto x \cdot v),$$

is a Lie alg homomorphism, and any Lie alg hom $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ defines a $\mathfrak{g}$-rep structure on V by $x \cdot v := \rho(x) \cdot v$.

Proof: Exercise. ∎

Example [Adjoint rep] For any Lie alg $\mathfrak{g}$, the adjoint action $x \cdot y := [x, y]$ gives $\mathfrak{g}$ the structure of a $\mathfrak{g}$-representation. Indeed, the Jacobi identity is equiv to the requisite formula $[x, y] \cdot z = x \cdot (y \cdot z) - y \cdot (x \cdot z)$. This is the adjoint representation.

Example [The standard rep] For any vector space V, $\mathfrak{gl}(V)$ acts on V "tautologically",

$$x \cdot v = x(v) \leftarrow \text{viewed as linear endo.}$$

This gives V the structure of a $\mathfrak{gl}(V)$-representation, and we call it the "standard representation".