

# Algebraic Groups and Representations

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Today: Algebraic group basics and examples

## - I. Complex Lie groups or algebraic groups

From an analytic perspective, an affine algebraic group is a complex analytic group (Lie group) which is equipped with a good class of algebraic functions.

$$\mathcal{O}_{\text{alg}} = \mathcal{O}_{\text{alg}}(G) = \left\{ \begin{array}{l} \text{Some specific subcollection of} \\ \text{analytic func } f: G \rightarrow \mathbb{C} \end{array} \right\}.$$

These functions must be stable under the group ops:

- For  $f: G \rightarrow \mathbb{C}$  alg., the composites

$$G \times G \xrightarrow{m} G \xrightarrow{f} \mathbb{C}$$

and  $G \xrightarrow{\text{inv.}} G \xrightarrow{f} \mathbb{C}$

must be alg. func on  $G \times G$ , and  $G$ , respectively.

(Here a func. on  $G \times G$  is alg if it's a product of alg func  $f_1, f_2(z_1, z_2) := f_1(z_1) \cdot f_2(z_2)$  or a sum of such things.)

Ex:  $\mathcal{O}_{\text{alg}}(\mathrm{GL}_n) = \left\{ \begin{array}{l} \text{Polynomial func in the} \\ \text{matrix entries } x_{ij} \end{array} \right\}$

$$= \mathbb{C}[x_{ij}: 1 \leq i, j \leq n] \text{ free } \mathbb{C}.$$

For each coord fun  $x_{ij}$  the composite

$$G_{\text{fun}} \times G_L \xrightarrow{\text{inj}} G_L \xrightarrow{x_{ij}} \mathbb{P}$$

is the alg fun  $\sum_{k=1}^n x_{ik} \cdot x_{kj}$ ,   
 first factor second factor

and

$$G_{\text{fun}} \xrightarrow{\text{inj}} G_{\text{fun}} \xrightarrow{x_{ij}} \mathbb{P}$$

$\Leftrightarrow (-1)^{i+j} \det^{-1} \cdot (\det [x_{kl}] \text{ minor } j\text{-th row and } i\text{-th col})$   
 $= (ij)\text{-th entry is } "x_{kl}"$ .

Since putting back along  $m$  and  $n$  preserve products and sums, it follows that

$f_m$  and  $f_n$  are alg whenever  $f$  is algebraic.

Other examples: -  $S/\text{fun}(\mathbb{C})$  w/ polys in the  $x_{ij}$ ,  
 $\cdot S_{\text{pol}}(\mathbb{C})$  w/ polys in the  $x_{ij}$ -  
 $\cdot \mathbb{C}^\times$  w/ polys in  $z$  and  $z^{-1}$ .

The class of algebraic functions must form a finitely generated  $\mathbb{C}$ -alg, i.e. must admit some finite generating collection, and must be "complete", in the following sense:

For each point  $p \in \mathbb{P}$  there must be some collection of func  $f_1, f_2, \dots$  in  $\mathbb{C}\text{-alg}$  with

$$\{q\} = \bigcup_{n \in \mathbb{N}} (f_1, \dots, f_n) \left( = \{q \in \mathbb{P} \mid \text{all } f_i(q) = 0\} \right).$$

Can talk about maps of alg groups or maps which  
preserve alg fun's, etc. 3

## - II. The pure algebraic perspective

**Def<sup>b</sup>:** An affine alg group  $G$ , over  $\mathbb{C}$ , is  
a group object in the category of affine  $\mathbb{C}$ -schemes.

So,  $G = \text{Spec } \mathcal{O}$  for some f.g.  $\mathbb{Q}$ -alg  
 $\mathcal{O}$ , and  $G$  comes equipped w/ an assoc. product  
map

$$G \times_{\text{Spec } \mathbb{C}} G \rightarrow G \quad (\dagger)$$

an inv. map  $\text{inv}: G \rightarrow G$  and unit

$$1: \text{Spec } \mathbb{C} \rightarrow G,$$

all of which are maps of schemes.

**Def<sup>a</sup>:** A map of affine alg groups  $\Sigma: A \rightarrow G$   
is a map of schemes which preserves the given group  
structures.

As affine  $\mathbb{C}$ -schemes are dual to comon (fr. gen)  
 $\mathbb{Q}$ -algebra,  
 $\text{Spec}: (\mathbb{C}\text{Alg})^{\text{op}} \xrightarrow{\sim} \text{AffSch}_{\mathbb{C}},$

The group structure ( $\star$ ) specifies, and is specified by, a certain  
structure on  $\mathcal{O}$ .

Def<sup>h</sup>:  $A$  (commutative) // first algebra is a comm.  
 (for us fin. gen'd)  $\mathbb{C}$ -alg  $\mathcal{O}$  w/ algebra map  
 (counit)  $\Delta: \mathcal{O} \xrightarrow{\sim} \mathcal{O} \otimes \mathcal{O}$   
 (counit)  $\epsilon: \mathcal{O} \rightarrow \mathbb{C}$   
 (antipode)  $S: \mathcal{O} \rightarrow \mathcal{O}$   
 which satisfy

$$\text{(coassoc.) } (\Delta \otimes \text{id}) \Delta = (\text{id} \otimes \Delta) \Delta$$

$$\text{(counit ax)} \quad (\epsilon \otimes \text{id}) \Delta = (\text{id} \otimes \epsilon) \Delta$$

$$\text{(antip. ax)} \quad \text{mult}(S \otimes \text{id}) \Delta = \text{mult}(\text{id} \otimes S) \Delta = \text{unit} \circ \epsilon.$$

$$\text{Ex.) } G_{\mathbb{A}^n} = \text{Spec}(\mathbb{C}[x_{ij}: 1 \leq i, j \leq n] / \det^{-1})$$

w/ structure specified on gen's by,

$$\Delta(x_{ij}) = \sum_k x_{ik} \otimes x_{kj}, \quad \epsilon(x_{ij}) = \delta_{ij},$$

$$S(x_{ij}) = \frac{(-1)^{i+j}}{\det} \cdot \det([x_{kl}] - \text{j-th row \& col}).$$

$$\cdot S_{\mathbb{A}^n} = \text{Spec}(\mathbb{C}[x_{ij}: 1 \leq i, j \leq n] / (\det - 1))$$

w same  $\Delta, \epsilon$  and  $S$ ,

$$S(x_{ij}) = (-1)^{i+j} \det([x_{kl}] - \text{j-th row \& col}).$$

We have the closed embedding

$$S_{\mathbb{A}^n} \rightarrow G_{\mathbb{A}^n}$$

closed to the first alg surjection  $\mathcal{O}(G_{\mathbb{A}^n}) \rightarrow \mathcal{O}(S_{\mathbb{A}^n})$ .

- $G_m = \text{Spec}(\mathbb{C}[x, x^{-1}])$ , 5  
 $\Delta(x) = x \otimes x, \quad \Delta(x^{-1}) = x^{-1} \otimes x^{-1}, \quad \epsilon(x^{\pm 1}) = 1,$   
 $S(x) = x^{-1}.$

- $G_a = \text{Spec}(\mathbb{C}[x])$  w/  $\Delta(x) = x \otimes 1 + 1 \otimes x$   
 $\epsilon(x) = 0, \quad S(x) = -x.$

### - III. Points in an algebraic group

**Proposition I:** For  $G$  an affine group scheme, and  $R$  any comm.  $\mathbb{R}$ -alg., the collection of  $R$ -points

$$G(R) := \{ \text{Scheme maps } \text{Spec}(R) \rightarrow G \}$$

$$= \{ \text{Alg maps } \mathcal{O}(G) \rightarrow R \}$$

naturally forms a discrete group, and for any group hom  $\xi: H \rightarrow G$ , evaluating at  $R$ -point provides a discrete group map

$$\xi(R): H(R) \rightarrow G(R).$$

Let's just explain that: Any two maps  
 $A, B: \text{Spec}(R) \rightarrow G$   
specify a single map to the product  $[A \ B]: \text{Spec}(R) \rightarrow G \times G$ ,  
and we can define  
 $A \cdot B = (\text{Spec}(R) \xrightarrow{[A \ B]} G \times G \xrightarrow{m} G).$

The unit is given by the composite

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$$1_{G(R)} = (\text{Spec}(R) \xrightarrow[\text{map}]{} \text{Spec}(\mathbb{C}) \xrightarrow{\mathbb{C}} G)$$

and inverse is

$$A^{-1} = (\text{Spec}(R) \xrightarrow{A} G \xrightarrow{\text{inv}} G).$$

Example: .  $G_{\text{GL}}(R)$  = invertible matrices over  $R$ ,

.  $G_m(R)$  = mult. group of units in  $R := R^\times$ ,

.  $G_a(R)$  = additive group underlying  $R = (R, +)$ .

Rem: The  $\mathbb{R}$ -points  $G(\mathbb{C})$  have the natural structure of a complex Lie group, and you can just think of  $G$  as the pairing of the complex Lie group  $G(\mathbb{C})$  with the alg of fun's  $\mathcal{O}(G) \subseteq \{\text{Analytic fun's on } G(\mathbb{C})\}$ .

We'll elaborate more on this  $\mathbb{R}$ -point stuff next time

## - IV. Points continued

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Let's return to the claim:

For  $R$  comm  $\mathbb{R}$ -alg,  $G$  affine group scheme,

$$G(R) = \{A_g \text{ maps } \mathcal{O}(G) \rightarrow R\}$$

has group structure induced by group str. on  $G$  / (top str. on  $\mathcal{O}(G)$ ).

(\*)

Example:  $G_a(R) = (R, +)$ . Let's check.

$$\mathcal{O}(G_a) = (\mathbb{C}[x]) \text{ w/ } \Delta(x) = x \otimes 1 + 1 \otimes x.$$

$$\operatorname{Ham}_{A_g}(\mathbb{C}[x], R) \xrightarrow{\sim} R, \xi \mapsto \xi(x).$$

For two maps  $\xi_1, \xi_2$  w/  $\xi_i(x) = r_i$ , we have usually checked

$$[\xi_1, \xi_2]: (\mathbb{C}[x] \otimes \mathbb{C}[x]) = \mathbb{C}[x_1, x_2] \rightarrow R$$

$$x_i \mapsto r_i$$

and compose w/ cannot to get the group structure

$$\begin{aligned} \xi_1 \cdot \xi_2 &:= \left( \mathbb{C}[x] \xrightarrow{\Delta} \mathbb{C}[x] \otimes \mathbb{C}[x] \xrightarrow{[\xi_1, \xi_2]} R \right) \\ &x \mapsto x \otimes 1 + 1 \otimes x \mapsto r_1 + r_2. \end{aligned}$$

$$\xi_1 \cdot \xi_2(x) = r_1 + r_2. \quad \text{This gives (*)}.$$

Anyway, so we

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## - IV. Representations of algebraic groups

Def<sup>b</sup>: A  $G$ -representation, for a  $\text{alg}$  group  $G$ , is a vector space  $V$  equipped with a map of  $\text{alg}$  groups

$$\phi_V: G \rightarrow GL(V).$$

Here we take the usual free expression:

$$GL(V) = \text{Spec} \left\{ \text{Sym} \left( \underset{\mathbb{C}}{\text{End}(V)^*} \right) [\det^{-1}] \right\}$$

with

$$\Delta: \text{Sym}(\text{End}(V)^*) [\det^{-1}] \xrightarrow{\quad \otimes 2 \quad} \text{Sym}(\text{End}(V)^*) [\det^{-1}]$$

given a true generator by coevaluation

$$\Delta|_{\text{End}^* = \text{coev}_V}: \text{End}_{\mathbb{C}}(V)^* = V^* \otimes V \rightarrow V^* \otimes V \otimes V^* \otimes V$$
$$= \text{End}(V)^* \otimes \text{End}(V)^*$$

$$f \otimes v \mapsto \sum_i f(v_i \otimes v),$$

and  $\epsilon$  given on gen's by evaluation

doesn't depend on basis

$$\Delta|_{\text{End}^* = \text{ev}_V}: \text{End}_{\mathbb{C}}(V)^* = V^* \otimes V \rightarrow \mathbb{C}$$
$$f \otimes v \mapsto f(v).$$

Rem: The antipode  $S$  on a bialgebra  $\mathcal{O}$ , if it exists, is uniquely determined. Hence a Hopf algebra is, equiv, a bialgebra  $(\mathcal{O}, \Delta, \epsilon)$  for which an antipode exists. The antipode is also preserved under Hopf maps. So we generally ignore it.

We'll give a few, equivalent descriptions of the category of  $G$ -representations

## - VI. Corepresentations

Def<sup>b</sup>. Given a Hopf algebra (or coalg)  $\mathcal{O}$  a corepresentation over  $\mathcal{O}$  is a vector space equipped with a linear map (coaction)

$$\rho_V: V \rightarrow V \otimes \mathcal{O}$$

which is coassociative and counital

$$(1 \otimes \Delta) \rho_V = (\rho_{V \otimes 1}) \rho_V, \quad (1 \otimes \epsilon) \rho_V = \text{id}_V.$$

**Example** (Universal coaction): For

$\mathcal{O}_V = \mathcal{O}(\text{GL}(V))$ , I claim  $\mathcal{O}_V$  always coacts on  $V$  in a natural way, giving it the structure of a  $\mathcal{O}_V$ -corepresentation.

We define

$$\rho_V^{\text{uni}}: V \rightarrow V \otimes V^* \otimes V = V \otimes \text{End}_{\mathbb{C}}(V)^* \subset V \otimes \mathcal{O}_V$$

by

$$v \mapsto \text{coev}(1) \otimes v = \sum v_i \otimes v^i \otimes v.$$

From the formula for  $\Delta \mid_{\text{End}(V)^*}$  this 10  
 coaction is clearly coassociative, and for an expression  
 $v = \sum_i c_i \cdot v_i$   
 in a chosen basis  
 $(c \otimes \epsilon) \rho^{(2)}(v) = \sum_i v_i \cdot (v'_i c v)$   
 $= \sum_i c_i \cdot v_i = v,$   
 so comultiplication is well.

### - VII. Reps vs. Coreps

**Theorem 2:** Let  $G$  be an algebraic group over  $\mathbb{Q}$ .

- ① For a vector space  $V$ , the following data are equivalent:
  - a) The structure of a  $G$ -representation,  
 $\rho_V: G \rightarrow \text{GL}(V)$ .
  - b) The structure of an  $\mathcal{O}(G)$ -corepresentation,  
 $\rho_V: V \rightarrow V \otimes \mathcal{O}(G)$ .
  - c) The structure of an  $R$ -linear group action of  $G(\mathbb{Q})$  on the base change  $V_R = V \otimes_{\mathbb{Q}} R$ , at each common  $\mathbb{Q}$ -alg  $R$ , which varies naturally in  $R$ .

We'll sketch a proof.

Sketch proof: (a)  $\Rightarrow$  (b) Suppose we have a map

of algebraic groups  $\phi_v: G \rightarrow GL(V)$ . Then we obtain a Hopf algebra  $\chi_v: O(GL) \rightarrow O(G)$  (via res. of algebraic functions along  $\phi_v$ ). We compose the universal coaction with  $\chi_v$  to obtain a coaction of  $O(G)$  on  $V$ ,

$$\rho_V := (1 \otimes \chi_v) \rho_v^{\text{univ}}: V \rightarrow V \otimes O(GL) \rightarrow V \otimes O(G).$$

(b)  $\Rightarrow$  (c) Given a coaction  $\rho_V$  we need to produce a Hopf map  $\chi_v: O(GL_V) \rightarrow O(G)$ . For this we need to produce a map on the generators  $O(GL_V) = \text{End}(V)^*: V^* \otimes V \rightarrow O(G)$ .

Now the coaction gives an element

$$\rho_V \in \text{Hom}_\mathbb{C}(V, V \otimes O(G))$$

and via adjunction this determines a correspond. linear function  $\chi_v|_{\text{End}(V)^*}: \text{End}(V)^*: V^* \otimes V \rightarrow O(G)$ .

This extends to an algebra map

$$\bar{\chi}_v: \text{Sym}(\text{End}(V)^*) \rightarrow O(G)$$

which I promise sends the determinant to a unit and thus gives  $\chi_v: O(GL_V) \rightarrow O(G)$ . This map is a Hopf algebra map, again I promise.

(b)  $\Rightarrow$  (c) Given  $\xi \in G(\mathbb{Q})$ ,  $\xi: O(G) \rightarrow \mathbb{R}$ ,

We have the corresponding  $\mathbb{C}$ -linear functor

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$$\overline{\text{act}}_g: V \xrightarrow{\rho} V \otimes \mathcal{O}(G) \xrightarrow{\text{act}_g} V \otimes \mathbb{Q}.$$

This functor determines a unique  $\mathbb{Q}$ -linear map

$$\text{act}_g: V_{\mathbb{Q}} \rightarrow V_{\mathbb{Q}} \quad \text{w/ } \text{act}_g|_V = \overline{\text{act}}_g.$$

This map is invertible with inverse  $\text{act}_{g^{-1}}$ , and the assignment  $\xi \mapsto \text{act}_{\xi}$  determines a natural action

$$G(\mathbb{Q}) \rightarrow \text{Aut}_{\mathbb{Q}}(V_{\mathbb{Q}}).$$

(c)  $\Rightarrow$  (a)  $GL(V)$  has functor of pts

$$GL(V)(\mathbb{Q}) = \text{Aut}_{\mathbb{Q}}(V_{\mathbb{Q}}),$$

and (c) determines a transformation

$$G(-) \rightarrow GL(V)(-)$$

of functors  $\text{CAlg}_{\mathbb{C}}^{\text{op}} \rightarrow \text{Group}$ . By Yoneda this gives a group map  $\gamma_v: G \rightarrow GL(V)$ .

Rem: Note that morphisms of groups, in terms of maps  $f: G \rightarrow GL(V)$ , are somewhat opaque. However, morphisms of comodules, or in terms of (c), are clear.

## — VIII. The category of $G$ -representations

Defn: We define, for a group  $R$ ,  $\text{Rep}(G) = \text{Comod}(\mathcal{O}(G))$ .

Next Time: I'll explain clearly what a map  
of cospans is, and elaborate more on various things.

⇒  $\text{Lie}(G)$  in oly terms

⇒ Examples

⇒ The functor  $\text{Rep}(R) \hookrightarrow \text{Rep}(\text{Lie}(G))$ .

## - IX. More fundamentals

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For an abelian group  $G$  recall that the  $\mathbb{Q}$ -point

$$G(\mathbb{Q}) = \lim_{\text{pro-}} \text{Sch}(\text{Spec } \mathbb{Q}, G)$$

and a  $V_{\mathbb{Q}}$ , for  $\mathcal{O}(G)$ -comod  $V$ , as

$$\times_{\mathbb{Q}} : V_{\mathbb{Q}} \rightarrow V_{\mathbb{Q}},$$

$$\times_{\mathbb{Q}}(v \otimes r) = \sum_i v_i \otimes \times(v_i) \cdot r$$

where  $\Delta(v) = \sum_i v_i \otimes v_i \in V \otimes \mathcal{O}(G)$ . In particular,  
the complex points  $G(\mathbb{C})$  act as

$$\times \cdot v = \sum_i \times(v_i) v_i. \quad (*)$$

Lemma 3: Let  $\rho_v^{\text{num}} : V \rightarrow V \otimes \mathcal{O}(\mathcal{GL}(V))$  be  
the numerical coaction and  $A \in \mathcal{GL}(V)(\mathbb{C}) = \text{Aut}_{\mathbb{C}}(V)$   
be a complex point. Then

$$\underbrace{A \cdot v}_{\text{from } (*)} = \underbrace{\Delta(v)}_{\text{expected action}}$$

Proof: In a basis  $\{v_1, \dots, v_n\}$  we have  $A = [a_{ij}]$  and  
then

$$\begin{aligned} A \cdot v_j &= \sum_{i=1}^n v_i \otimes \Delta(v_i \otimes v_j) = \sum_{i \in \text{End}(V)^*} v_i (\Delta(v_j)) \cdot v_i \\ &= \sum_i a_{ij} v_i \\ &= A(v_j). \end{aligned}$$

By linearity the  $A \cdot v = A(v)$  at all  $v \in V$ .



Establishing the equivalence between (a) and (b)  
in Theorem 2.

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**Lemma 4:** Let  $\rho: G \rightarrow G \otimes_{\mathbb{C}} V$  be a Group  
with corresponding coaction  $\rho = (\text{id}_G \otimes \rho^*) \rho_V^{\text{univ}}: V \rightarrow V \otimes \mathcal{O}(G)$ .  
For each  $x \in G(\mathbb{C})$  and  $v \in V$  we have  
 $x \cdot v = \rho(x)(v)$ .

**Proof:** We have  $\rho(x) = (\mathcal{O}(G \otimes_{\mathbb{C}} V))^{\rho^*} \xrightarrow{\rho^*} \mathcal{O}(G) \xrightarrow{x} \mathbb{C}$   
and  $x \cdot -: V \rightarrow V$  is given by the composite  

$$V \xrightarrow{\rho_V^{\text{univ}}} V \otimes \mathcal{O}(G \otimes_{\mathbb{C}} V) \xrightarrow{1 \otimes \rho^*} V \otimes \mathcal{O}(G) \xrightarrow{1 \otimes x} V$$
  
 $= \rho(x) \cdot -: V \rightarrow V.$

Thus, by Lemma 3,  $x \cdot v = \rho(x) \cdot v = \rho(x)(v)$ . ■

## - I. Morphisms of $G$ -repr via closed points

By a map of  $\mathcal{O}$ -repr,  $f: V \rightarrow W$ , we mean the ex-  
pected thing: A linear map which fits into a diagram

$$\begin{array}{ccc} V & \xrightarrow{\rho_V} & V \otimes \mathcal{O} \\ f \downarrow & & \downarrow f \otimes 1 \\ W & \xrightarrow{\rho_W} & W \otimes \mathcal{O} \end{array}$$

**Proposition 5:** Let  $\rho_V: G \rightarrow GL(V)$  and  $\rho_W: G \rightarrow GL(W)$  be  $G$ -representations, with corresponding comodules  $(V, \rho_V)$  and  $(W, \rho_W)$ . A linear map  $f: V \rightarrow W$  is a map of  $\mathcal{O}(G)$ -comodules if and only if, at each complex point  $x \in G(\mathbb{C})$ , we have  $f(x \cdot v) = x \cdot f(v)$ .

Before the proof we need an algebra fact.

**Theorem (Reducedness):** Any affine algebraic group  $G$  over  $\mathbb{C}$  is reduced, i.e. has no nilpotent elements in its algebra of functions.

OK.

**Proof of Proposition 5:** If  $f$  is a map of  $\mathcal{O}(G)$ -comodules then

$$(*) \quad f(x \cdot v) = f\left(\sum_i x(v_i) v_i\right) = \sum_i x(v_i) f(v_i)$$

and by comodules

$$\sum_i f(v_i) \otimes v_i = \rho_W(f(v)),$$

so that  $(*) = x \cdot f(v)$  as well.

Suppose conversely  $f(x \cdot v) = x \cdot f(v)$  at all complex points.

In a basis  $\{w_1, \dots, w_m\}$  for  $V$ , we have at each  $v \in V$

$$(f \otimes 1) \rho_V(v) = \sum_{j=1}^m w_j \otimes \xi_j$$

$$\rho_W(f(v)) = \sum_{j=1}^m w_j \otimes \xi'_j$$

for some function  $\xi_j, \xi'_j \in \mathcal{O}(G)$ . Now

the difference is given by

$$(\cancel{f_{\alpha}})(\rho_v(v)) - \rho_w(\cancel{f_{\alpha}}) = \sum_j w_j \otimes (\xi_j - \xi'_j) \quad (*)$$

and we know that at each  $\mathbb{P}$ -point  $x: \text{Spec}(\mathbb{C}) \rightarrow G$

$$f(x \cdot v) - x \cdot f(v) = (1 \otimes x)(*) = \sum_j x(\xi_j - \xi'_j) w_j = 0,$$

by assumption.

Hence  $x(\xi_j - \xi'_j) = 0$  at each  $j$  and all  $x$ .

Thus the vanishing locus of  $\xi_j - \xi'_j$  is all of  $G(\mathbb{C})$ ,

$$\text{Van}(\xi_j - \xi'_j) = G(\mathbb{C}),$$

which gives by Nullsatz (sp?)  $\xi_j - \xi'_j \in \sqrt{0} \subseteq \mathcal{O}$ ,

at each  $j$ . But finally by reducedness of  $G$  this implies

$$\xi_j - \xi'_j = 0 \text{ at all } j, \text{ and thus}$$

$$\cancel{(\cancel{f_{\alpha}})(\rho_v(v))} = \rho_w(\cancel{f_{\alpha}}).$$

Since this holds at all  $v \in V$ , we conclude that  $f_{\alpha}$   
is a map of  $\mathcal{O}(G)$ -comrepresentations. ■

**Theorem 6:** Let  $G$  be an affine algebraic group, and  
 $\text{Rep}(G)$  be the category whose objects are fin-dim vect  
 spaces  $V$  equipped with a map of alg groups  $\sigma: G \rightarrow \text{GL}(V)$ ,  
 and whose morphisms are linear maps  $f: V \rightarrow W$  which  
 satisfy  $f(x \cdot v) = x \cdot f(v)$  at all  $v \in V$  and  
 $x \in G(\mathbb{C})$ .

There is a canonical equivalence of categories

$$t: \text{Rep}(G) \xrightarrow{\sim} \mathcal{O}(G)\text{-crops}$$

$$(V, \sigma) \mapsto (V, (\iota \otimes \sigma^*)_{\rho}^{\text{univ}})$$

$$f \mapsto f.$$

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Proof: By Theorem 2,  $t$  is bijective on objects and by Proposition 5,  $t$  is bijective on morphisms.

The fact that  $t$  respects composition is automatic by construction  $t(f \circ g) = f \circ g \circ t(g) \circ t(f)$ . ■

**Exercise:** Prove that for an affine algebraic group  $G$ , the forgetful functor

$$\text{Rep}(G) \rightarrow \text{Rep}(G(\mathbb{C}))$$

is fully faithful.

Considered as a  
discrete group!

[This is just me asking if you payed attention in Aravind's AG class.]

## - XII. The tangent at the identity (pre-Lie)

Let  $G$  be an analytic group. We would think of the tangent space here as equivalence classes of functions.

$$g: \mathbb{C} \ni 0 \rightarrow G$$

for a small disk  $D$  around 0 with  $g(0) = 1$ . Here

$$[g] \sim [h] \text{ if } g'(0) = h'(0).$$

If we let  $\varepsilon$  be the parameter on  $D$ , we have 19

an expansion in coords around 1

$$g(\varepsilon) = 1 + g_1 \varepsilon + O(\varepsilon^2)$$

and  $[g] = [h] \in T_1 G$  iff  $g_1 = h_1$ . So, only the linear term matters.

In algebraic perspective we replace the infinitesimally small slice with the "dual number"  $\mathbb{C}[\varepsilon]/\varepsilon^2$ . We note that  $\mathbb{C}[\varepsilon]/\varepsilon^2$  has only one map to  $\mathbb{C}$ , namely  $\varepsilon \mapsto 0$ , and hence a canonical map for the functor of points

$$G(\mathbb{C}[\varepsilon]/\varepsilon^2) \rightarrow G(\mathbb{C})$$

at algebraic  $G$ . The tangent space at 1 is then

$$T_1 G := \{1_G\} \times_{G(\mathbb{C})} G(\mathbb{C}[\varepsilon]/\varepsilon^2)$$

=  $\left\{ \begin{array}{l} \text{Algebra maps } \mathcal{O}(G) \rightarrow \mathbb{C}[\varepsilon]/\varepsilon^2 \\ \text{whose composite along } \mathbb{C}[\varepsilon]/\varepsilon^2 \rightarrow \mathbb{C} \\ \text{recovers } L_G = e \end{array} \right\}$

where  $m_G \subset \mathcal{O}$  = the kernel of counit  $e$ .

Example: For  $G_{\mathrm{ln}}$

$G_{\mathrm{ln}}(\mathbb{C}[\varepsilon]/\varepsilon^2) = \text{Inv. un-mat over } \mathbb{C}[\varepsilon]/\varepsilon^2$   
and

$$T_1 \text{GL}_n = \left\{ \begin{array}{l} \text{Inv. mat. } [a_{ij} + a_{ij}\varepsilon] \text{ over } \\ \mathbb{C}[\varepsilon]/\varepsilon^2 \text{ w/ } [a_{ij}] = I_n \end{array} \right\} \quad 20$$

$$= \left\{ \begin{array}{l} I_n + A\varepsilon \text{ where } A \text{ is arbitrary} \\ \text{in } \text{GL}_n(\mathbb{C}) \end{array} \right\}.$$

Then we have

$$\text{GL}_n(\mathbb{C}) \xrightarrow{\cong} T_1 \text{GL}_n, A \mapsto I_n + A\varepsilon.$$

Remark:

Note that, if we consider  $\varepsilon$  a complex number of sufficiently small magnitude,  $I_n + A\varepsilon$  is actually invertible. So we literally have corresponding complex paths

$$I_n + A\varepsilon : \mathbb{D} \rightarrow \text{GL}_n(\mathbb{C})$$

which realize an identification between algebraic  $T_1 \text{GL}_n$  and analytic  $T_1 \text{GL}_n(\mathbb{C})$ .

Example : We have  $\text{SL}_n(\mathbb{C}[\varepsilon]/\varepsilon^2) \subseteq \text{GL}_n(\mathbb{C}[\varepsilon]/\varepsilon^2)$  as matrices of determinant 1. Then

$$T_1 \text{SL}_n = \left\{ I_n + A\varepsilon \in T_1 \text{GL}_n \text{ with } \det(I_n + A\varepsilon) = 1 \right\}.$$

We check now

$$\det(I_n + A\varepsilon) = \det \begin{bmatrix} 1 + a_{11}\varepsilon & & & \\ & \ddots & & a_{1j}\varepsilon \\ & & \ddots & \\ a_{j1}\varepsilon & & & 1 + a_{nn}\varepsilon \end{bmatrix} = 1 + \text{tr}(A)\varepsilon.$$

Rees

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$$T_1 S_{2n} = \{ I + A\varepsilon : f_r(A) = 0 \} \subseteq \mathfrak{gl}_n(\mathbb{C})$$
$$= \mathfrak{so}_n(\mathbb{C})$$

Example :  $\text{Sp}_{2n}(\mathbb{C}[\varepsilon]/\varepsilon^2) \subseteq GL_{2n}(\mathbb{C}[\varepsilon]/\varepsilon^2)$

is given by matrices  $\tilde{A}$  over  $\mathbb{C}[\varepsilon]/\varepsilon^2$  with

$$\langle \tilde{A} \cdot v, w \rangle = \langle v, \tilde{A}^{-1} \cdot w \rangle \quad (*)$$

where

$$\langle , \rangle : \mathbb{C}^{2n} \otimes \mathbb{C}^{2n} \rightarrow \mathbb{C}$$

is the anti-symmetric form  $\langle v_i, v_j \rangle = \delta_{j,i+n} - \delta_{i,j+n}$ .

Now for an element of the tangent space

$$\tilde{A} = I_n + A\varepsilon \in T_1 GL_n$$

we have

$$\tilde{A}^{-1} = I_n - A\varepsilon$$

so that (\*) becomes

$$\langle v, w \rangle + \langle A \cdot v, w \rangle = \langle v, w \rangle - \langle v, A \cdot w \rangle$$

$$\Leftrightarrow \langle A \cdot v, w \rangle = - \langle v, A \cdot w \rangle.$$

Hence we calculate

$$T_1 \text{Sp}_{2n} = \text{sp}_{2n}(\mathbb{C}) \subseteq \mathfrak{gl}_n(\mathbb{C}).$$

Similarly  $T_1 SO_n = \mathfrak{so}_n(\mathbb{C}) \subseteq \mathfrak{gl}_n(\mathbb{C})$

- ~~XIII~~ The Lie structure on the tangent space

Def<sup>n</sup>: For functions  $\alpha, \beta: \mathcal{O}(G) \rightarrow \mathbb{C}$

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(here I just mean linear functions) the convolution product  $\alpha * \beta: \mathcal{O}(G) \rightarrow \mathbb{C}$  is the new function defined by

$$\alpha * \beta(\xi) = \sum; \alpha(\xi_{i_1}) \beta(\xi_{i_2}),$$

where

$$\Delta(\xi) = \sum; \xi_{i_1} \otimes \xi_{i_2}.$$

Note now that we have the inclusion

$$\text{incl}: T_1 G \hookrightarrow \text{Hom}_{\mathbb{C}}(\mathcal{O}(G), \mathbb{C})$$

$$(\tilde{\alpha} = 1_G + d\epsilon: \mathcal{O}(G) \rightarrow \mathbb{C}[\epsilon]/\epsilon^2) \mapsto (\alpha: \mathcal{O}(G) \rightarrow \mathbb{C}).$$

This inclusion identifies  $T_1 G$  with function  $\alpha: \mathcal{O}(G) \rightarrow \mathbb{C}$  which satisfy

$$T1) \quad \alpha|_{m_G^2} = 0$$

$$T2) \quad \alpha|_{\mathbb{C}1_0} = 0.$$

**Example**: For  $G = GL_n$ , the inclusion incl

sends  $A$  in  $gl_n(\mathbb{C}) \cong T_1 GL_n$  to the unique linear function

$$"A": \mathcal{O}(GL_n) = \mathbb{C}\{x_{ij} : 1 \leq i, j \leq n\} \{\det^{-1}\} \rightarrow \mathbb{C}$$

which satisfies T1 and T2 and has

$$"A"(x_{ij}) = a_{ij}.$$

Theorem 7 : Considering  $\text{Fun}_C(O(G), \mathbb{C})$

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as an assoc. alg under convolution, the inclusion

$$\text{incl. } T_1 G \hookrightarrow \text{Fun}_C(O(G), \mathbb{C})$$

identifies  $T_1 G$  with a Lie subalg in  
 $\text{Fun}_C(O(G), \mathbb{C})^{\text{Lie}}$ .

Lie alg under convolution commutator  $\alpha * \beta - \beta * \alpha$ .

Proof: Take any two such functions  $\alpha, \beta$  which satisfy  $(T_1)$  and  $(T_2)$ . As  $\Delta(1) = 1 \otimes 1$

$$\alpha * \beta - \beta * \alpha \mid_{C[1]} = 0,$$

and for  $x \in m$  we have

$$\Delta(x) = 1 \otimes x + x \otimes 1 \pmod{m \otimes m}$$

since  $(\epsilon \otimes 1) \Delta(x) = (1 \otimes \epsilon) \Delta(x) = x$ . Thus for

$x \cdot y$  with  $x, y \in m$

$$\Delta(xy) = \Delta(x) \cdot \Delta(y)$$

$$= xy \otimes 1 + 1 \otimes xy + x \otimes y + y \otimes x \pmod{m^2 \otimes m^2}$$

$$= x \otimes y + y \otimes x \pmod{m^2 \otimes m^2}$$

so that

$$\begin{aligned} \alpha * \beta(xy) &= \alpha(x)\beta(y) + \alpha(y)\beta(x) \\ &= \beta * \alpha(xy). \end{aligned}$$

Hence  $[\alpha, \beta](xy) = 0$ . Since

$$m^2 = \text{Span}_C(xy : x, y \in m)$$

we have  $[\alpha, \beta] \mid_{m^2} = 0$ .



Def<sup>L</sup>: For  $G$  an affine algebraic group

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$$\text{Lie } G := \{T_1 G \text{ w/ convolution bracket}\}$$

### - XIV. Functoriality of $\text{Lie}(-)$

For  $\phi: H \rightarrow G$  a map of algebraic groups the  
corresp map on functors

$$\phi^*: \mathcal{O}(G) \rightarrow \mathcal{O}(H)$$

is a Hopf map. Thus restricting along  $\phi^*$  gives a  
map of (Lie) algebras

$$\phi_*: \text{Ham}_{\mathbb{C}}(\mathcal{O}(H), \mathbb{C}) \xrightarrow{\text{Lie}} \text{Ham}_{\mathbb{C}}(\mathcal{O}(G), \mathbb{C})$$

which furthermore fits into a diagram

$$\begin{array}{ccc} T_1 H & \xrightarrow{\quad \phi_* \quad} & T_1 G \\ \downarrow & & \downarrow \\ \text{Ham}(\mathcal{O}(H), \mathbb{C})^{\text{Lie}} & \xrightarrow{\phi_*} & \text{Ham}(\mathcal{O}(G), \mathbb{C})^{\text{Lie}} \end{array}$$

(Just follows by  $(\phi^*)^{-1}(m_H) = m_G$ .) Thus

$T_1 \phi: T_1 H \rightarrow T_1 G$  is a map of Lie algebras.

$$\text{Lie } \phi: \text{Lie } H \rightarrow \text{Lie } G.$$

## - XV. Examples

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Theorem 8: The linear identification

$$\text{gl}_n(\mathbb{C}) \xrightarrow{\sim} \text{Lie } \text{GL}_n, A \mapsto I_n + A\epsilon,$$

is an isomorphism of Lie algebras.

Proof: The above isomorphism identifies, further, a matrix  $A$  with the function " $A$ " :  $\mathbb{C} \rightarrow \mathbb{C}$  w/ " $A$ "( $x_{ij}$ ) =  $a_{ij}$  as before. Then for matrices  $A, B$  we have

$$\begin{aligned} ["A", "B"] (x_{ij}) &= \sum_{ik} A(x_{ik}) B(x_{kj}) - \sum_{ik} B(x_{ik}) A(x_{kj}) \\ &= \sum_{ik} a_{ik} \cdot b_{kj} - b_{ik} \cdot a_{kj} \\ &= "[A, B]" (x_{ij}). \end{aligned}$$

We're done. ■

Corollary 9: The identifications

- $\text{sl}_n(\mathbb{C}) \xrightarrow{\sim} \text{Lie } \text{SL}_n$
- $\text{sp}_{2n}(\mathbb{C}) \xrightarrow{\sim} \text{Lie } \text{Sp}_{2n}$
- $\text{so}_n(\mathbb{C}) \xrightarrow{\sim} \text{Lie } \text{SO}_n$

are all isomorphisms of Lie algebras.

Proof: For  $G = \text{SL}_n, \text{Sp}_{2n}, \text{SO}_n$  and  $\mathfrak{g} = \text{sl}_n(\mathbb{C}), \text{sp}_{2n}(\mathbb{C}), \text{so}_n(\mathbb{C})$  we've established

$$\begin{array}{ccc} \mathfrak{g} & \longrightarrow & \mathfrak{gl}_m(\mathbb{C}) \\ \downarrow & & \downarrow s \\ \mathfrak{Lie} G & \longrightarrow & \mathfrak{Lie} GL_m \end{array}$$

where each map but  $\mathfrak{g} \xrightarrow{\sim} \mathfrak{Lie} G$  is now understood to be a map of Lie algebras. But now the diagram forces the final map to be an isomorphism of Lie algebras.

## - XVI. Infinitesimal actions for $G$ -reps

Let  $\mathfrak{g}$  fix  $G$  algebraic and  $\mathfrak{g} = \mathfrak{Lie}(G)$ . Fix also  $\mathcal{O} = \mathcal{O}(G)$ . Of our many options, we consider now  $\text{Rep } G$  via  $\text{Corep}(\mathcal{O})$  and consider

$$\mathfrak{g} \subseteq \{ \text{Linear functions } \mathcal{O} \rightarrow \mathbb{C} \}$$

consisting of all maps which

- T1) Vanish on  $m_{\mathcal{O}}^2$
- T2) Vanish on  $\mathfrak{g} \otimes \mathcal{O}$ .

*algebra under convolution*

**Proposition 10:** Take  $H_G = \text{Fun}_{\mathcal{O}}(\mathcal{O}, \mathbb{C})$ .

- a) For any  $G$ -rep  $V$ , considered as  $\mathcal{O}$ -corep, the formula

$$\alpha \cdot v = (\mathbb{1} \otimes \alpha) \rho_V(v) = \sum_i \alpha(v_i) \cdot v_i \in V$$

defines an assoc. action of  $H_G$  on  $V$ .

b) Restricting to elements  $A \in \mathcal{O}$ , the action  $A \cdot v = a_s \cdot v$  defines an action of  $\mathcal{O}$  on  $V$ , i.e. gives it the structure of a  $\mathcal{O}$ -representation.

Proof: a) Recall the product on  $\mathcal{H}_G$ ,

$$\alpha * \beta \in \mathcal{E} := (\alpha * \beta) \Delta(\mathcal{E})$$

at  $\mathcal{E} \in \mathcal{O}$ . Then for  $v \in V$ ,

$$(\alpha * \beta)(v) = ((\mathbb{1} \otimes \alpha \otimes \beta)(\mathbb{1} \otimes \Delta)) \rho_V(v)$$

$$= ((\mathbb{1} \otimes \alpha \otimes \beta)(\rho_V \otimes \mathbb{1})) \rho_V(v) \quad (\text{Coassoc.})$$

$$= ((\mathbb{1} \otimes \alpha \otimes \mathbb{1})(\rho_V \otimes \beta)) \rho_V(v)$$

$$= ((\mathbb{1} \otimes \alpha \otimes \mathbb{1})(\rho_V \otimes \mathbb{1}))((\mathbb{1} \otimes \beta)) \rho_V(v)$$

$$V \xrightarrow{\rho_V} V \otimes \mathcal{O} \xrightarrow{1 \otimes \beta} V \otimes \mathcal{C} \xrightarrow{\rho_V \otimes 1} V \otimes \mathcal{O} \otimes \mathcal{C} \xrightarrow{1 \otimes \alpha \otimes 1} V \otimes \mathcal{C} \otimes \mathcal{C}.$$

$$\begin{array}{ccccc} & & & & \\ & \searrow \beta \cdot - & & & \\ & V & \longrightarrow & V \otimes \mathcal{O} & \longrightarrow V \\ & & & \curvearrowright \alpha \cdot - & \\ & & & & \end{array}$$

$$= \alpha \cdot (\beta \cdot v).$$

We thus obtain associativity.

b) Any  $\mathcal{H}_G$ -module becomes a  $\mathcal{H}_G^{\text{Lie}}$ -rep, under the same action. Hence restricting to the subalgebra  $\mathfrak{g} \hookrightarrow \mathcal{H}_G^{\text{Lie}}$

produces a  $\mathfrak{g}$ -rep structure on any  $\mathcal{O}$ -corepresentation  $V$ .

Def<sup>n</sup>: For a  $G$ -representation  $V$  we call 28

the corresponding action of  $\mathfrak{g} = \text{Lie } G$  on  $V$ , as in  
Proposition 10, the infinitesimal action of  $\mathfrak{g}$  on  $V$ .

Clearly the infinitesimal action is natural, in the sense that a map  $f: V \rightarrow W$  of  $G$ -representations is also a map of  $\mathfrak{g}$ -reps, under the infinitesimal action.  
So we get a functor (!)

$$\inf = \inf_G: \text{Rep}(G) \rightarrow \text{Rep}(\mathfrak{g}).$$

Proposition 11: Let  $\varphi: H \rightarrow G$  be a map of algebraic groups and  $\text{Lie } \varphi: \text{Lie } H \rightarrow \text{Lie } G$  be the induced map on Lie algebras. Then for any  $G$ -rep  $V$  with restriction  $V|_H$  to  $H$

$$\inf_H(V|_H) = \inf_G(V)|_{\text{Lie } H} \quad (*)$$

Proof:  $\varphi: H \rightarrow G \Rightarrow \varphi^*: \mathcal{O}(G) \rightarrow \mathcal{O}(H)$   
 $\Rightarrow \varphi_*: \mathcal{Z}_H \rightarrow \mathcal{Z}_G$

which fits into a diagram

$$\begin{array}{ccc} \mathcal{Z}_H & \xrightarrow{\varphi^*} & \mathcal{Z}_G \\ \uparrow & & \uparrow \\ H & \xrightarrow{\text{Lie } \varphi} & G \end{array}$$

This is sufficient for (\*). ■

**Example (The universal example):**

We have the standard representation  $\check{\rho}_V$  over  $GL(V)$  with corresponding universal coaction

$$\rho_V^{\text{univ}}(v) = \sum_{i=1}^n v_i \otimes v^i \otimes v \in V \otimes \text{End}(V)^* = V \otimes \mathcal{O}.$$

In coordinates,  $A \in gl(V) = \text{Lie } GL(V)$  is given by

$$A = [a_{ij}] = \sum_{i,j} a_{ij} v_i \otimes v^j.$$

Then for the expression

$$v = \sum_j c_j v^j$$

we get for the infinitesimal action of  $gl(V)$  on  $V$ ,

$$A \cdot v = \sum_i A(v^i \otimes v) \cdot v_i$$

$$= \sum c_j a_{ij} \cdot v_i = A(v).$$

This is to say, the standard rep for  $GL(V)$  becomes the standard rep for  $gl(V) = \text{Lie } GL(V)$  under the infinitesimal action.

As a corollary to this Example and Proposition 11 we find an alternate definition of the infinitesimal action.

**Theorem 12:** For a  $G$ -representation  $\phi: G \rightarrow GL(V)$  the infinitesimal action of  $\mathfrak{g} = \text{Lie } G$  on  $V$  is determined by the map  $\text{Lie } \phi: \mathfrak{g} \rightarrow gl(V)$ , and the standard

action of  $gl(V)$  on  $V$ .

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Remark: The description of the infinitesimal action as in Proposition 10 is good because it makes it clear that we have a functor

$$inf: \text{Rep}(G) \rightarrow \text{Rep}(\mathfrak{g}).$$

On the other hand Theorem 12 makes it clear that the infinitesimal action of  $\mathfrak{g}$  on  $V$  is defined in the expected way: An infinitesimal path through the identity

$$\mathbb{C} \ni t \xrightarrow{\quad} G$$

composes to give an infinitesimal path in  $GL(V)$

$$\begin{array}{ccc} & t \nearrow G & \searrow \phi \\ D & \longrightarrow & GL(V) \end{array}$$

$$\varepsilon \mapsto I_n + A\varepsilon + O(\varepsilon^2),$$

and the element  $(t)$  in  $\mathfrak{g}$  acts "infinitesimally" on  $V$  via the matrix

$$\left. \frac{\partial \phi \circ t}{\partial z} \right|_{z=0} = A \in gl(V).$$

## - XVII. Symmetric monoidality of $inf_G$

For  $G$ -representations  $V$  and  $W$  we have the apparent  $G(\mathbb{C})$ -action on  $V \otimes W$  given by the expected formula  $g \cdot (v \otimes w) = (g \cdot v) \otimes (g \cdot w)$ .

This "diagonal" action of  $G(\mathbb{C})$  extends uniquely

for a  $G$ -representation structure on the tensor product. 81  
In terms of composites,

$$\rho_{V \otimes W} = (1 \otimes 1 \otimes \text{mult}_G^H) (\text{swap} \otimes 1) (\rho_V \otimes \rho_W) :$$

$$V \otimes W \xrightarrow{\rho_V \otimes \rho_W} (V \otimes W) \otimes (O \otimes O) \xrightarrow{\text{swap} \otimes \text{mult}} (V \otimes W) \otimes O.$$

The standard vector space swap

$$\tau_{VW} : V \otimes W \rightarrow W \otimes V, v \otimes w \mapsto w \otimes v,$$

is an isomorphism of  $G$ -representations. Thus we have a symmetric tensor structure on  $\text{Rep } G$ .

**Theorem 13:** The functor

$$\text{inf} : \text{Rep}(G) \rightarrow \text{Rep}(\mathfrak{g})$$

is symmetric monoidal.

Proof: Exercise. 

- XVIII. Full faithfulness of  $\text{inf}_G$

**Theorem 14:** For any algebraic group  $G$  with Lie algebra  $\mathfrak{g} = \text{Lie}(G)$ , the functor

$$\text{inf}_G : \text{Rep}(G) \rightarrow \text{Rep}(\mathfrak{g})$$

is fully faithful provided  $G$  is connected.

However one approaches this, there are two basic principles we need to deal with. (1) - sketch some details

**Principle I (Actions are determined in an infinitesimal neighborhood of the identity)**

By restricting along the embedding  $\hat{G}_1 \rightarrow G$  from the formal neighborhood around the identity we obtain a functor

$$\text{res: } \text{Rep}(G) \rightarrow \text{Rep}(\hat{G}_1).$$

Now,  $\text{rep } \hat{G}_1 = \text{Corep}(\hat{\mathcal{O}})$  where

$$\hat{\mathcal{O}} = \varprojlim \hat{\mathcal{O}} / m_G^n, \quad m_G = \text{Ker}(1_G).$$

complete local ring at  $t_G \in G(\mathbb{C})$ .

The first structure on  $\mathcal{O}$  becomes a topological first structure on  $\hat{\mathcal{O}}$ ,  $\hat{\wedge}: \hat{\mathcal{O}} \rightarrow \hat{\mathcal{O}} \hat{\otimes} \hat{\mathcal{O}} = \varprojlim \mathcal{O}/m \otimes \mathcal{O}/m$  so we can consider

$$\text{Rep}(\hat{\mathcal{O}}_1) := \text{Corep}(\hat{\mathcal{O}})$$

and the restriction functor is just given by "corestricting"

$$\text{res: } \text{Corep}(\mathcal{O}) \rightarrow \text{Corep}(\hat{\mathcal{O}}).$$

Since  $G$  is smooth [a fact] and connected the canonical map

$$\mathcal{O} \rightarrow \hat{\mathcal{O}}$$

is injective, hence res is fully faithful. This is to say, any representation  $\phi: G \rightarrow GL(V)$  is determined by its restriction to the "infinitesimal neighborhood"  $\phi|_{\hat{G}_1}$ ; and

and any linear map  $f: V \rightarrow W$  is a map of  $G$ -repr if and only if  $f$  is a map of  $\hat{G}_1$ -repr. 33

**Rew:** Any nbd (Zariski or analytic)  $1 \subseteq U \subseteq G$  contains  $\hat{G}_1 \subseteq U \subseteq G$ . So any algebraic rep  $V$  has  $G$ -action def in any nbd  $U \subseteq G$ , and  $G$ -invariance of any map holds iff  $f(x \cdot v) = x \cdot f(v)$  for all  $x \in U(\mathbb{C})$ .

**Principle II** (The Lie algebra  $\mathfrak{g}$  knows everything about the formal nbd  $\hat{G}_1$ )

Consider the topological dual

$$U_G := \varprojlim_n (\mathcal{O}/m_G^n)^* \subseteq H_G.$$

This is a subalgebra with  $\overset{\text{Lie}}{\mathfrak{g}} \hookrightarrow U_G \subseteq H_G$ ,

since all  $\tau_{\alpha}(x)$  for  $x \in \mathfrak{g}$  vanish on  $m^2$ . Now,

we have

$$U_G^* = \left( \varprojlim_n (\mathcal{O}/m_G^n)^* \right)^* = \varprojlim (\mathcal{O}/m^n)^*$$

$$= \varprojlim \mathcal{O}/m^n = \hat{\mathcal{O}},$$

so that we have an identification of categories

$\text{Rep}(\hat{G}_1) = U_G\text{-mod finitely}$ ,  
(Cf.

$$\text{Hom}(V \otimes U_G, V) = \text{Hom}_{\mathbb{C}}(V, \text{Hom}_{\mathbb{C}}(U_G, V))$$

$$= \text{Hom}_{\mathbb{C}}(V, V \otimes U_G^*)$$

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$$= \text{Hom}_{\mathbb{C}}(V, V \otimes \hat{\mathcal{O}}). )$$

Now, the Lie embedding

$$\mathfrak{g} \hookrightarrow \mathfrak{U}_G^{\text{Lie}}$$

determines an algebra map

$$\text{can}: \mathcal{U}(g) \rightarrow \mathfrak{U}_G^{\text{Lie}}$$

Using smoothness of  $G$ , one can prove that in fact this map can be an isomorphism, giving

$$\text{Rep}(G) \xrightarrow{\text{inf}} \text{Rep}(g)$$

full /faith

↑ "

$$\text{Rep}(\hat{G}) = \text{Cores}(\hat{\mathcal{O}}) \xrightarrow{\cong} \mathcal{U}(g)\text{-mod fidim}.$$

Thus  $\text{inf}$  is fully faithful. 

## - ~~XIX~~. Integrable representations

Def<sup>+</sup>: For  $g = \text{Lie } G$ , we say a  $g$ -representation  $V$  is integrable (for  $G$ ) if there exists  $V'$  in  $\text{Rep}(G)$  with  $\text{inf}(V') = V$ .

In this case note that  $V' = V$ , and that the  $G$ -action on  $V$  is actually fixed, due to full faithfulness of  $\text{inf}: \text{Rep}(G) \rightarrow \text{Rep}(g)$ .

So we are simply asking if we can "integrate" 35  
the  $\mathfrak{g}$ -action on  $V$  to an action of the group  
 $G$ .

Another way of saying Thm 14 is as follows.

**Theorem 15:** The functor

$$\inf_G : \text{Rep}(G) \rightarrow \text{Rep}(\mathfrak{g})$$

is a symmetric monoidal equivalence (in fact isomorphism)  
onto the full subcategory of integrable  $\mathfrak{g}$ -reps  
in  $\text{Rep}(\mathfrak{g})$ ,

$$\inf_G : \text{Rep}(G) \xrightarrow{\sim} (\text{Rep}(\mathfrak{g}))_{\text{Int}}.$$

The following is also clear from the proof of Thm 14.

**Corollary (to proof) 16:** The subcategory of integrable  
reps in  $\text{Rep}(\mathfrak{g})$  is closed under taking subobjects  
and quotients.

## ~~- XX.~~ Examples

**Example 1:** For discrete (finite)  $G$  we have  
 $L^*(G) = 0$  so that  $\text{Rep}(G) = \text{Vect}$ , and  
the infinitesimal action functor

is just the forgetful functor  
 $\inf_G : \text{Rep}(G) \rightarrow \text{Vect}$ .

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This is faithful, but not full, and clearly not an equivalence when  $G$  is nontrivial.

**Example 2:** Consider  $B \subseteq \text{GL}_n$  consisting of non-strictly upper  $\Delta^{\text{inv.}}$  matrices. Then an element

$$\tilde{A}(\varepsilon) = I_n + A\varepsilon \in \text{Lie } \text{GL}_n$$

$b \in \text{Lie } B \subseteq \text{Lie } \text{GL}_n$  if and only if  $\tilde{A}(\varepsilon)$  is upper  $\Delta$ . This occurs if and only if  $A$  is upper  $\Delta$ 'r (with arbitrary entries  $a_{ij}, i < j$ ). Hence

$$\text{Lie } B = b \subseteq \text{gl}_n(\mathbb{C}).$$

Take  $T \subseteq \text{GL}_n$  the irr. diagonal matrices. We have  $\text{Lie } T = h = \text{diag} \subseteq \text{gl}_n(\mathbb{C})$ , and there

is a diagram

$$\begin{array}{ccc} \text{Rep}(B) & \xrightarrow{\inf_B} & \text{Rep}(b) \\ \text{res} \swarrow & & \searrow \text{res} \\ \text{Rep}(T) & \xrightarrow{\inf_T} & \text{Rep}(h) \end{array}$$

by Prop 11. Since  $T = \mathbb{G}_m^{xn}$  we see an  $h$ -rep  $V$  is integrable if and only if  $h$  acts semisimply on  $V$  and all the  $h_i = e_{ii}$  act as integers on  $V$ .

Consequently, any integrable  $b$ -rep  $\mathcal{V}$  must have  $h \in \mathfrak{t}_g^*$  acting semisimply with integral eigenvalues. One can show that in fact this integral Cartan condition characterizes integrable  $b$ -reps,

$$\inf_B : \text{Rep}(B) \xrightarrow{\sim} \left\{ \begin{array}{l} \text{b-reps } \mathcal{V} \text{ on} \\ \text{which } h \in \mathfrak{t}_g^* \\ \text{act semisimply w/} \\ \text{the } h_i = e_i \text{ acting} \\ \text{as integers} \end{array} \right\} \subset \text{Rep}(B).$$

As a general consequence of Theorem 14 we have the following.

**Corollary 17:** Suppose  $G$  is connected w/ Lie alg  $\mathfrak{g}$ . If  $\text{Rep}(\mathfrak{g})$  is semisimple then  $\text{Rep}(G)$  is also semisimple.

So we see, for example that  $\text{Rep}(SL_n)$  is semisimple. Also for any finite quotient

$$\pi : SL_n \longrightarrow SL_n/\mu = G$$

$\text{Rep } G$  is semisimple, as  $Lie(G) = Lie(SL_n) = sl_n(\mathbb{C})$  in this case.

**Corollary 18:**  $\text{Rep}(PGL_n)$  is semisimple.

**Example 3:** For  $\text{Rep}(\text{SL}_n)$ , we've already seen that the standard representation  $\pi/\!\!/$  for  $\text{sl}_n(\mathbb{C})$  integrates to the standard rep for  $\text{SL}_n$ , using Theorem 12.

Since  $\inf: \text{Rep}(\text{SL}_n) \rightarrow \text{Rep}(\text{sl}_n(\mathbb{C}))$  is symmetric monoidal and has image stable under subquotients, we see

$$\begin{aligned} & \{ \text{Symm } \otimes\text{-subcat} \text{ gen'd by } \pi/\!\!/ \text{ in } \text{Rep}(\text{sl}_n) \} \\ & \subseteq \text{image of } \text{Rep}(\text{SL}_n). \end{aligned}$$

But, this  $\otimes$ -subcat is all of  $\text{Rep}(\text{sl}_n)$ , so that  $\inf_{\text{SL}_n}$  is essentially surjective, and thus an equivalence

$$\inf_{\text{SL}_n}: \text{Rep}(\text{SL}_n) \xrightarrow{\sim} \text{Rep}(\text{sl}_n(\mathbb{P})).$$

**Example 4:** We have

$$\text{PGl}_n = \text{SL}_n / \mu_n$$

where

$$\mu_n = \left\{ \begin{bmatrix} \zeta^j & 0 \\ 0 & \zeta^{-j} \end{bmatrix} : j \text{ an } n\text{-th root of 1} \right\}.$$

Thus  $\text{Rep} \text{PGl}_n \subseteq \text{Rep} \text{SL}_n$  identified w/ the full subcat of repr on which  $\mu_n \rightarrow \text{GL}$  trivially.

$$\text{Now, } \theta(s) = \det \{ s, \dots, s^n \} = f_1(s) f_2(s^2) \cdots f_{n-1}(s^{n-1})$$

for the root subgroups

sitzen 39

$$f_i: \mathbb{C}^\times \rightarrow \mathrm{SL}_n(\mathbb{C}), z \mapsto \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & z & z^{-1} \\ & & 0 & \ddots \\ & & & & 1 \end{bmatrix}.$$

Now for a weight  $\lambda = c_1 w_1 + \dots + c_{n-1} w_{n-1} \in P$

$$f_i(\zeta) \cdot v = (\sum_{k=1}^n c_k e_k) \cdot v \quad \text{for } v \in V_i,$$

and we find  $V_{\lambda} = \mathrm{Ran}(\mathrm{PGL}_n) \subseteq \mathrm{Ran}(c_\lambda)$   
if and only if  $V_\lambda$  is generated by the sublattice

$$X_{\mathrm{PGL}_n} := \left\{ \sum_k c_k \cdot w_k : \sum_k n \cdot c_k \in \mathbb{Z} \right\} \subseteq P.$$

Now for the weight  $\alpha_i = -w_{i-1} + 2w_i - w_{i+1}$  we have  
 $-(i-1) + 2i - (i+1) = 0$ . Hence we have a seq. of inclusions

$$\mathbb{Z} \cdot \overline{\alpha}_i = \text{Root lattice} \subseteq X_{\mathrm{PGL}_n} \subseteq P$$

Since  $P$  has a  $\mathbb{Z}$ -basis  $\{w_n, \alpha_1, \dots, \alpha_{n-1}\}$  with

$$w_n = \frac{1}{n}(\alpha_1 + 2\alpha_2 + \dots + \alpha_{n-1})\alpha_{n-1}$$

and Root lattice has basis

$$\{\alpha_1 + 2\alpha_2 + \dots + \alpha_{n-1}, \alpha_n, \alpha_2, \dots, \alpha_{n-1}\}$$

we have  $P/\text{Root lattice} \cong \mathbb{Z}/n\mathbb{Z}$  and

by construction  $\mathcal{X}_{\text{PGL}_n}$  is the kernel of the 40 map

$$[1 \ 2 \ \dots \ n]: P \rightarrow \mathbb{Z}/n\mathbb{Z}$$

so that

$$P/\mathcal{X}_{\text{PGL}_n} \cong \mathbb{Z}/n\mathbb{Z} \text{ as well.}$$

Since  $|P/\mathcal{X}_{\text{PGL}_n}| = |\text{Root Lattice}|$

we find the inclusion

$$\text{Root Lattice} \subseteq \mathcal{X}_{\text{PGL}_n}$$

it an equality.

**Proposition 19:** For  $\text{pgln}(C) = \text{sl}_n(C) =$   
Lie  $\text{PGL}_n$ , the infinitesimal action functor

$$\text{inf}_{\text{PGL}_n}: \text{Rep}(\text{PGL}_n) \rightarrow \text{Rep}(\text{pgln}(C))$$

restricts to an equivalence

$$\text{inf}_{\text{PGL}}: \text{Rep}(\text{PGL}_n) \xrightarrow{\sim} \left\{ \begin{array}{l} \text{pgln}(C) = \text{sl}_n(C) - \text{repr} \\ \text{whose } P\text{-grading is} \\ \text{supported on the sublattice} \\ \text{spanned by the roots} \end{array} \right\}$$

This is the  $\otimes$ -subcategory  
generated by the adjoint representation.

## Other classical examples

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$$\inf_G : \text{Rep}(G) \xrightarrow{\sim} \text{Rep}(\mathfrak{g})$$

for  $G = \text{Sp}_{2n}, \text{SO}_{2n}$  and

$$\inf_{GL_n} : \text{Rep}(GL_n) \xrightarrow{\sim} \left\{ \begin{array}{l} \text{gl}_n(\mathbb{C})\text{-rep on which} \\ \text{the identity matrix } I_n \text{ acts} \\ \text{semisimply w/ integral eigen-} \\ \text{values} \end{array} \right\}$$

Since  $\text{gl}_n(\mathbb{C}) = \text{sl}_n(\mathbb{C}) \oplus \mathbb{C} \cdot I_n$  we observe  
now that  $\text{Rep}(GL_n)$  is semisimple.