KERODON REMIX PART III: A SMALL STUDY OF THE DERIVED ∞-CATEGORY

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Abstract. This third installment is essentially a continuation of Part II. We employ the generally abstract shenanegans from Part II to provide a baseline analysis of the derived category, as a stable ∞-category. We also give a more responsible presentation of adjunction, and use adjunction to realize the derived ∞-category as the expected localization of the homotopy ∞-category. We provide a few other examples which demonstrate how one leverages, in practice, some of the mechanisms developed and studied in Part II. In particular, we show how the indification operation assembles into a functor, and describe how one manages the theory of monoidal ∞-categories via cocartesian fibrations over a fixed base.

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1. Limits and colimits in the homotopy ∞-category

1.1. Recollections of the homotopy ∞-category.

Definition 1.1. For an abelian category $\mathbb{A}$ we let $K\mathbf{Ch}(\mathbb{A})$ denote the simplicial category associated to the dg category of cochains over $\mathbb{A}$, and we take

$$\mathscr{K}(\mathbb{A})' := \mathrm{N}^{\mathrm{hc}}(K\mathbf{Ch}(\mathbb{A})).$$

We recall from Theorem II-10.4 that there is an equivalence of ∞-categories to the standard homotopy ∞-category $\mathfrak{Z} : \mathscr{K}(\mathbb{A})' \overset{\sim}{\to} \mathscr{K}(\mathbb{A})$ which is the identity on the underlying discrete categories

$$\begin{tikzcd} \mathscr{K}(\mathbb{A})' \arrow[rr, "\mathfrak{Z}"] & & \mathscr{K}(\mathbb{A}) \\ & \mathrm{Ch}(\mathbb{A}) \arrow[ul] \arrow[ur] & \end{tikzcd}$$

According to Proposition II-11.6 the functor

$$\underline{\mathrm{Hom}}_{\mathbb{A}}(V,-) := \underline{\mathrm{Hom}}_{K\mathbf{Ch}(\mathbb{A})}(V,-) : \mathscr{K}(\mathbb{A})' \to \mathscr{K}an$$

is corepresented by the given complex V. We recall that these morphism complexes in the simplicial category $\underline{\mathrm{Ch}}(\mathbb{Z})^{\leq 0}$ are explicitly given by the Eilenberg-MacLane spaces

$$\underline{\mathrm{Hom}}_{\mathbb{A}}(V,W) := K\,\mathrm{Hom}^*_{\mathbb{A}}(V,W).$$

In this particular setting Corollary II-16.17 appears as follows.

Proposition 1.2. *Given a diagram $p : K \to \mathscr{K}(\mathbb{A})'$, a given extension $\widetilde{p} : \{0\} \star K \to \mathscr{K}(\mathbb{A})'$ is a limit diagram in $\mathscr{K}(\mathbb{A})'$ if and only if, at each cochain complex V, the composite functor*

$$\underline{\mathrm{Hom}}_{\mathbb{A}}(V,-) \circ \widetilde{p} : \{0\} \star K \to \mathscr{K}an$$

is a limit diagram in $\mathscr{K}an$.

We are most interested in pullback diagrams in $\mathscr{K}(\mathbb{A})$ and/or $\mathscr{K}(\mathbb{A})'$. To get our wheels rolling in this direction, let us take a moment to recall the construction of 2-simplices in the ∞-categories $\mathscr{K}(\mathbb{A})'$ and $\mathscr{K}(\mathbb{A})$.

Directly, 2-simplices in $\underline{\mathrm{Hom}}_{\mathbb{A}}(V,W)$ are triples

$$\widetilde{h} = (h : V \to W,\ h_0 : V \to W,\ h_1 : V \to W)$$

with h of degree -1, the h_i cochain maps of degree 0, and

$$d(h) = d_W h + h d_V = h_0 - h_1.$$

The restrictions along the inclusions $\{i\} \to \Delta^2$ are as expected $\widetilde{h}|_{\{i\}} = h_i$. Now, according to the definition of the homotopy coherent nerve, a 2-simplex $\sigma : \Delta^2 \to \mathscr{K}(\mathbb{A})'$ is a not-necessarily-commuting diagram of cochain maps

$$\begin{tikzcd} & V_1 \arrow[dr, "f_{12}"] & \\ V_0 \arrow[ur, "f_{01}"] \arrow[rr, "f_{02}"'] & & V_2 \end{tikzcd}$$

and a 2-simplex $\widetilde{h}$ in $\underline{\mathrm{Hom}}_{\mathbb{A}}(V_0, V_2)$ with $\widetilde{h}|_0 = f_{12}f_{01}$ and $\widetilde{h}|_1 = f_{02}$, i.e. a choice of a degree -1 map h which establishes homotopy commutativity $d(h) = f_{12}f_{01} - f_{02}$. (See Lemma I-2.16).

To compare, a 2-simplex in the usual ∞-category $\mathscr{K}(\mathbb{A})$ is a choice of a not-necessarily-commuting diagram of cochain maps

$$\begin{tikzcd} & V_1 \arrow[dr, "f_{12}"] & \\ V_0 \arrow[ur, "f_{01}"] \arrow[rr, "f_{02}"'] & & V_2 \end{tikzcd}$$

and a degree -1 map $h : V_0 \to V_2$ satisfying $d(h) = f_{12}f_{01} - f_{02}$. These are clearly the same thing. The following is observed directly from the construction [5,] of the equivalence $\mathfrak{Z}$ from Theorem II-10.4.

1.2. Kan fibrations for simplicial abelian groups.

lem:5490 **Lemma 1.3.** (1) *If A is a discrete simplicial abelian group, then the inclusion $0 \to A$ is a Kan fibration.*
(2) *Any surjective map of simplicial abelian groups $f : X \to Y$ is a Kan fibration.*
(3) *If A is a discrete simplicial abelian group, and $f : X \to Y$ is a surjective, then the map $[0\ f]^t : X \to A \times Y$ is a Kan fibration.*

Proof. (1) Follows from the fact that, in this case, any simplex $\Delta^n \to A$ in which a single vertex maps to 0 is of constant value 0. (2) Consider a lifting diagram

$$\begin{tikzcd} \Lambda^n_i \arrow[r, "\tau"] \arrow[d] & X \arrow[d, "f"] \\ \Delta^n \arrow[r, "\bar{\sigma}"'] & Y. \end{tikzcd}$$

We can lift $\bar{\sigma}$ arbitrarily to an n-simplex $\sigma : \Delta^n \to X$, via surjectivity of f. We can now replace $\bar{\sigma}$ with the 0 simplex and τ with $\sigma|_{\Lambda^n_i} - \tau$ to reduce to the case $Y = 0$. In this case the desired solution exists since X is a Kan complex (Proposition I-10.1). (3) In this case $[0\ f]^t$ can be identified with a product of Kan fibrations $0 \times f : 0 \times X \to A \times Y$, and is thus a Kan fibration. $\square$

cor:K_kanfib **Corollary 1.4.** *For any map $f : V \to W$ of cochains of abelian groups in which $f^n : V^n \to W^n$ is surjective at all $n < 0$, the corresponding map $Kf : KV \to KW$ is a Kan fibration.*

Proof. We can factor f as the inclusion $V \to Z^0(W) \times V$ composed with the map $[i\ f] : Z^0(W) \times V = Z^0(W) \oplus V \to W$, where i here is the inclusion $i : Z^0(W) \to W$. Then Kf factors as the sequence

$$KV \overset{[0\ id]^t}{\longrightarrow} KZ^0(W) \times KV \overset{[i\ Kf]}{\longrightarrow} KW$$

in which the latter map is surjective, since the Eilenbergh-MacLane functor $K = K\tau_0$ is an equivalence on connective cochains and the map $\tau_0(Z^0(W) \times V) \to \tau_0(W)$ is surjective by construction. By Lemma 1.3 it follows that Kf is a Kan fibration. $\square$

1.3. **Claims: pullbacks in the homotopy ∞-category.**

Definition 1.5. For maps of cochains $f : V \to W$ and $f' : V' \to W$, we take

$$C(f, f') := \Sigma^{-1} \operatorname{cone}(\ [-f\ \ f'] : V \times V' \to W\ ).$$

For a single map $f : V \to W$ we take $C(f) = \Sigma^{-1} \operatorname{cone}(-f)$.

The complex $C(f, f')$ appears as

$$\left((V \times V') \oplus \Sigma^{-1} W,\ \begin{bmatrix} d_{V \times V'} & [f\ \ -f'] \\ 0 & -d_W \end{bmatrix} \right).$$

Note that we have the embedding of cochains

$$V \times_W V' = \ker([f\ \ -f']) \to C(f, f')$$

We have the degree -1 map of graded objects in $\mathbb{A}$

$$h_W = [0\ id_W] : C(f, f') \to W \qquad (1)$$ eq:hw

with

$$d_{\mathrm{Hom}}(h_W) = \big(C(f, f') \overset{\pi}{\to} V \times V' \overset{[f\ -f']}{\longrightarrow} W \big),$$

where π is the obvious projection. This homotopy defines a 2-simplex $h_W : \Delta^2 \to \mathscr{K}(\mathbb{A})$ which appears as

$$\begin{tikzcd} & V \arrow[dr, "f"] & \\ C(f, f') \arrow[ur, "p"] \arrow[rr, "f'p'"'] & & W. \end{tikzcd}$$

We append a strictly commuting diagram for V' to obtain a square $\Delta^1 \times \Delta^1 \to \mathscr{K}(\mathbb{A})$ which appears as

$$\begin{tikzcd} C(f, f') \arrow[r, "p"] \arrow[d, "p'"'] \arrow[dr, "f'p'" description] & V \arrow[d, "f"] \\ V' \arrow[r, "f'"'] & W. \end{tikzcd} \qquad (2)$$ eq:5570

def:std_pullback **Definition 1.6.** Given arbitrary morphisms $f : V \to W$ and $f' : V' \to W$ of $\mathbb{A}$-cochains, the corresponding standard pullback diagram is the diagram (2) in $\mathscr{K}(\mathbb{A})$ produced from the shifted mapping cone $C(f, f')$ and the homotopy h_Q of (1).

We prove in Proposition 1.12 below that any standard pullback diagram is in fact a limit diagram in $\mathscr{K}(\mathbb{A})$. We note that the construction of the standard pullback is natural in the sense that a strictly commuting diagram

$$\begin{tikzcd} V \arrow[r] \arrow[dr, "f"'] & X \arrow[dr, "g"] & V' \arrow[r] \arrow[dl, "f'"] & X' \arrow[dl, "g'"] \\ & W \arrow[r] & Y & \end{tikzcd}$$

extends to a diagram of the form

$$\begin{tikzcd} & C(f,f') \arrow[r] \arrow[dl] \arrow[dr] & C(g,g') \arrow[dl] \arrow[dr] & \\ V' \arrow[r, "="] \arrow[dr] & X' \arrow[dr] & V \arrow[r] \arrow[dl] & X \arrow[dl] \\ & W \arrow[r] & Y & \end{tikzcd}$$

in $\mathscr{K}(\mathbb{A})$.

1.4. **Pullbacks in the homotopy ∞-category.** Throughout the subsection we fix $\mathbb{A}$ an abelian category. We establish some background materials before returing to address the issue of pullbacks.

def:truncate

Definition 1.7. For any abelian category $\mathbb{A}$ we let

$$\tau_0 : \mathrm{Ch}(\mathbb{A}) \to \mathrm{Ch}(\mathbb{A})^{\leq 0}$$

denote the truncation functor, $\tau_0 V = \cdots \to V^{-2} \to V^{-1} \to Z^0(V) \to 0$.

If we let $\mathbf{Ch}'(\mathbb{A})$ denote the dg category of cochains with mapping complexes $\mathrm{Hom}_{\mathbb{Z}}^{\leq 0}(X, Y)$, then the functor τ_0 enhances to a dg functor

$$\tau_0 : \mathbf{Ch}'(\mathbb{A}) \to \mathbf{Ch}(\mathbb{A})^{\leq 0}.$$

In particular, τ_0 respects homotopy and homotopy equivalences. Also, since τ_0 is right adjoint to the inclusion $\mathrm{Ch}(\mathbb{A})^{\leq 0} \to \mathrm{Ch}(\mathbb{A})$ this functor commutes with limits.

Let us say that a map of $\mathbb{A}$-cochains $f : V \to W$ is termwise split surjective if, for each integer n, the map $f^n : V^n \to W^n$ is split surjective.

lem:5721

Lemma 1.8. *For a map of $\mathbb{A}$-cochains $f : V \to W$ the following are equivalent:*

(1) *f is termwise split surjective.*
(2) *f is split surjective as a map of graded objects in $\mathbb{A}$.*
(3) *For each cochain X, the induced map $f_* : \mathrm{Hom}^*_{\mathbb{A}}(X, V) \to \mathrm{Hom}^*_{\mathbb{A}}(X, W)$ is surjective.*

Proof. Omitted. □

The following might be seen as an algebraic analog of Corollary II-14.32.

prop:split_pullback

Proposition 1.9. *Consider maps of $\mathbb{A}$-cochains $f : V \to W$ and $f' : V' \to W$, and suppose one of f or f' is termwise split surjective. Then the strictly commuting* *pullback diagram*

$$\begin{tikzcd} V \times_W V' \arrow[r] \arrow[d] & V \arrow[d] \\ V' \arrow[r] & W \end{tikzcd} \tag{3}$$ eq:5645

is a limit diagram in $\mathscr{K}(\mathbb{A})'$.

Proof. Assume arbitrarily that f is graded split. Take

$$\mathrm{Hom}^\bullet_{\mathbb{A}}(X,Y) = \tau_0 \,\mathrm{Hom}^*_{\mathbb{A}}(X,Y)$$
$$= \cdots \to \mathrm{Hom}^{-2}_{\mathbb{A}}(X,Y) \to \mathrm{Hom}^{-1}_{\mathbb{A}}(X,Y) \to Z^0\,\mathrm{Hom}^*_{\mathbb{A}}(X,Y) \to 0.$$

For each cochain complex X we have

$$\mathrm{Hom}^\bullet_{\mathbb{A}}(X, V \times_W V') = \mathrm{Hom}^\bullet_{\mathbb{A}}(X,V) \times_{\mathrm{Hom}^\bullet_{\mathbb{A}}(X,W)} \mathrm{Hom}^\bullet_{\mathbb{A}}(X,V')$$

so that the induced diagram

$$\begin{tikzcd} \mathrm{Hom}^\bullet_{\mathbb{A}}(X, V \times_W V') \arrow[r] \arrow[d] & \mathrm{Hom}^\bullet_{\mathbb{A}}(X,V) \arrow[d, "f_*"] \\ \mathrm{Hom}^\bullet_{\mathbb{A}}(X,V') \arrow[r, "f'_*"'] & \mathrm{Hom}^\bullet_{\mathbb{A}}(X,W) \end{tikzcd} \tag{4}$$ eq:5660

is a pullback diagram. Furthermore, by our splitting assumption, the map f_* is split in each strictly negative degrees. In particular, f_* is surjective in all strictly negative degrees.

We now apply the Eilenbergh-MacLane functor K to obtain a pullback diagram

$$\begin{tikzcd} \underline{\mathrm{Hom}}_{\mathbb{A}}(X, V \times_W V') \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}_{\mathbb{A}}(X,V) \arrow[d, "Kf_*"] \\ \underline{\mathrm{Hom}}_{\mathbb{A}}(X,V') \arrow[r, "Kf'_*"'] & \underline{\mathrm{Hom}}_{\mathbb{A}}(X,W) \end{tikzcd}$$

in which the map Kf_* is a Kan fibration by Corollary 1.4. The above diagram is therefore a pullback diagram in $\mathscr{K}an$ by Corollary II-14.32. Since X was chosen arbitrarily we apply Corollary II-16.17 to observe that the diagram (9) is a pullback diagram in $\mathscr{K}(\mathbb{A})$. $\square$

Note that the above strict pullback diagram is in the image of the inclusion $\mathrm{Ch}(\mathbb{A}) \to \mathscr{K}(\mathbb{A})'$. Since the equivalence $\mathfrak{Z} : \mathscr{K}(\mathbb{A})' \to \mathscr{K}(\mathbb{A})$ restricts to the identity on $\mathrm{Ch}(\mathbb{A})$ we see that it preserves all strict pullback diagrams. Since equivalences preserve limits, by Proposition II-13.10, Proposition 1.9 implies the following.

cor:split_pullback **Corollary 1.10.** *Consider maps of connective cochains* $f : V \to W$ *and* $f' : V' \to W$, *and suppose one of* f *or* f' *splits as a graded morphism. Then the strictly commuting pullback diagram*

$$\begin{tikzcd} V \times_W V' \arrow[r] \arrow[d] & V \arrow[d] \\ V' \arrow[r] & W \end{tikzcd} \tag{5}$$ eq:5645

is a limit diagram in $\mathscr{K}(\mathbb{A})$.

In this split setting the shifted mapping cone $C(f, f')$ one sees furthermore that the fiber product are identified.

lem:pb_cone **Lemma 1.11.** *Consider maps of cochains $f : V \to W$ and $f' : V' \to W$, and suppose one of f or f' splits as a map of graded objects in $\mathbb{A}$. Then the inclusion*

$$V \times_W V' \to C(f, f')$$

is a homotopy equivalence.

Proof. By replacing V with $V \oplus V'$ and V' with 0, it suffices to prove that the inclusion $\ker(f) \to C(f)$ is a homotopy equivalence in the case that $f : V \to W$ is split surjective as a graded map.

Via the splitting we can write $V \cong \Sigma L \oplus K$ with $L = \Sigma^{-1}W$ and $K = \ker(f)$. Here K is a subcomplex in V and the map $V \to W$ is just the projection onto the first factor. We may assume for simplicity that, in fact, this isomorphism is an equality of graded objects $V = \Sigma L \oplus K$.

The composite

$$\Sigma L \xrightarrow{\text{incl}} V \overset{d_V}{\to} V \xrightarrow{\text{proj}} K$$

defines a degree 1 map from ΣL, which is then a degree 0 map $g : L \to K$. This map is seen to be a cochain morphism so that

$$V = \text{cone}(g) = \left(\Sigma L \oplus K, \begin{bmatrix} -d_L & g \\ 0 & d_K \end{bmatrix}\right).$$

We now have

$$C(f) = \left(\Sigma L \oplus K \oplus L, \begin{bmatrix} -d_L & g & id \\ 0 & d_K & 0 \\ 0 & 0 & d_L \end{bmatrix}\right)$$

and observe the projection

$$\pi = [0 \ id_K \ \ -g] : C(f) \to K.$$

We have directly $\pi \,\text{incl} = id_K : K \to K$ and the composite $\text{incl}\,\pi : C(f) \to C(f)$ is homotopic to the identity via the degree -1 map

$$h = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ id_L & 0 & 0 \end{bmatrix} : C(f) \to C(f)$$

□

Now, we have the inclusion $V \times_W V' \to C(f, f')$ and general f and f', and the homotopy h_W from (1) has trivial restriction $h_W|_{V \times_W V'}$. This inclusion therefore extends to a natural transformation of diagram

$$\begin{tikzcd}
 & & V \times_W V' \arrow[r] \arrow[dll] \arrow[drr] & & C(f,f) \arrow[dl] \arrow[drr] \arrow[dd, dashed] & & \\
V' \arrow[rr, "="] \arrow[dr] & & V' \arrow[drr] & & V \arrow[rr, "="] \arrow[dll] & & V \arrow[dll] \\
 & W \arrow[rrr, "="'] & & & W & &
\end{tikzcd} \tag{6}$$

eq:5696

By Lemma 1.11 this transformation is an isomorphism whenever f or f' is graded split.

`prop:K_pullback` **Proposition 1.12.** *For arbitrary maps $f : V \to W$ and $f' : V' \to W$ in $\mathscr{K}(\mathbb{A})$, the standard pullback diagram*

$$\begin{tikzcd} C(f,f') \arrow[r] \arrow[d] \arrow[dr] & V \arrow[d, "f"] \\ V' \arrow[r, "f'"'] & W \end{tikzcd} \tag{7}$$ `eq:5707`

(see Definition 1.6) is a limit diagram in $\mathscr{K}(\mathbb{A})$. In particular, the diagram (7) is isomorphic to a diagram of the form

$$\begin{tikzcd} V_0 \arrow[r] \arrow[d] & V_1 \arrow[d, "f_{12}"] \\ V_1' \arrow[r, "f_{12}'"'] & V_2 \end{tikzcd}$$

in which f_{12} is termwise split surjective and f_{12}' is injective

Proof. Take $V_2 = \mathrm{cone}(id_{V'}) \oplus W$, $V_1' = V'$, and $f_{12}' = [i\ f']^t : V' \to \mathrm{cone}(id_{V'}) \oplus W$ where $i : V' \to \mathrm{cone}(id_{V'})$ is the usual inclusion. Take now $V_1 = V \oplus C(id_{V_2})$ and $f_{12} = [\pi\ f] : V \oplus C(id_{V_2}) \to V_2$ where $\pi : C(id_{V_2}) \to V_2$ is the usual projection. The map f_{12}' is injective and g is split as a graded morphism via the identity map $V_2 to V_2 \oplus \Sigma^{-1} V_2 = C(id_{V_2})$.

Since the mapping cone of any identity morphism is contractible, the summands $C(id_{V'})$ and $C(id_{V_2})$ are contractible. The inclusion $V \to V_1$ and $V' \to V_2$ are therefore homotopy equivalence and induce an isomorphism of diagrams

$$\begin{tikzcd} & C(f,f') \arrow[r] \arrow[dl] \arrow[dr] & C(f_{12},f_{12}') \arrow[dl] \arrow[dr] & \\ V' \arrow[r, "="] \arrow[dr] & V_1' \arrow[dr] & V \arrow[r, "\mathrm{incl}"] \arrow[dl] & V_1 \arrow[dl] \\ & W \arrow[r, "="'] & V_2 & \end{tikzcd}$$

in $\mathscr{K}(\mathbb{A})$. As argued at (6) the diagram for $C(f_{12}, f_{12}')$ is furthermore isomorphic to the discrete pullback diagram

$$\begin{tikzcd} V_0 = V_1 \times_{V_2} V_1' \arrow[r] \arrow[d] & V_1 \arrow[d, "g"] \\ V_1' \arrow[r] & V_2, \end{tikzcd} \tag{8}$$ `eq:5733`

so that in total the diagram (7) is isomorphic to the diagram (8). Since the latter diagram is a pullback diagram in $\mathscr{K}(\mathbb{A})$ by Proposition 1.9, it follows by Proposition II-13.18 that the diagram (8) is a pullback diagram as well. $\square$

As a corollary we find that any diagram

$$\begin{tikzcd} & V \arrow[d] \\ V' \arrow[r] & W \end{tikzcd}$$

in $\mathscr{K}(\mathbb{A})$ admits a limit, or, in slightly more informal terms, that $\mathscr{K}(\mathbb{A})$ has pullbacks.

cor:K_pullback

Corollary 1.13. *Every diagram $\Lambda^2_2 \to \mathscr{K}(\mathbb{A})$ admits a limit. That is to say, for any abelian category $\mathbb{A}$, the homotopy ∞-category $\mathscr{K}(\mathbb{A})$ admits all pullbacks.*

1.5. **Pushouts diagrams in the homotopy ∞-category.** We recall that, for a partial diagram of cochains

$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d, "g'"'] & V \\ V' & \end{tikzcd}$$

the pushout is the quotient $V \amalg_{W'} V' = \mathrm{coker}(W \to V \oplus V')$.

Let us call a general morphism of $\mathbb{A}$-cochains $g : W \to V$ termwise split injective if, at each integer n, $g^n : W^n \to V^n$ is a split injective morphism in $\mathbb{A}$. We have the expected analog of Lemma 1.8.

lem:5915

Lemma 1.14. *For a map of $\mathbb{A}$-cochains $g : W \to V$ the following are equivalent:*

(1) *g is termwise split injective.*
(2) *g is split injective as a map of graded objects in $\mathbb{A}$.*
(3) *For each cochain Y, the induced map $g^* : \mathrm{Hom}^*_{\mathbb{A}}(V, Y) \to \mathrm{Hom}^*_{\mathbb{A}}(W, Y)$ is surjective.*

For any abelian category $\mathbb{A}$, we apply Corollary 1.10 to the opposite category $\mathbb{B} = \mathbb{A}^{\mathrm{op}}$ to obtain the corresponding result for pushout diagrams in $\mathscr{K}(\mathbb{A})$.

prop:split_pushout

Proposition 1.15. *Consider maps of $\mathbb{A}$-cochains $g : V \to W$ and $g' : V' \to W$, and suppose one of g or g' splits as a graded morphism. Then the strictly commuting pullback diagram*

$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d, "g'"'] & V \arrow[d] \\ V' \arrow[r] & V \amalg_W V' \end{tikzcd} \tag{9}$$

eq:5645

is a colimit diagram in $\mathscr{K}(\mathbb{A})$.

In the pushout context we return to the standard, rather than shifted, mapping cone.

Definition 1.16. Given maps $g : W \to V$ and $g' : W \to V'$ of $\mathbb{A}$-cochains we take

$$\mathrm{cone}(g, g') = \mathrm{cone}\left(\ [g\ \ -g']^t : W \to V' \oplus V\ \right).$$

Of course, in the case $V' = 0$ we have $\mathrm{cone}(g) = \mathrm{cone}(g, 0)$. We have the two inclusions from V and V' into $\mathrm{cone}(g, g')$ which provide a generally noncommuting diagram

$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d, "g'"'] & V \arrow[d] \\ V' \arrow[r] & \mathrm{cone}(g,g'). \end{tikzcd}$$

The degree -1 map $h'_W : W \to \mathrm{cone}(g,g')$ defined by the identity on W satisfies

$$d_{\mathrm{Hom}}(h'_W) = [g \ \ -g']^t : W \to V \oplus V' \subseteq \mathrm{cone}(g,g')$$

and hence produces a diagram

$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d, "g'"'] \arrow[dr] & V \arrow[d] \\ V' \arrow[r] & \mathrm{cone}(g,g'). \end{tikzcd} \tag{10}$$ eq:5958

in the homotopy ∞-category $\mathscr{K}(\mathbb{A})$ in which the bottom simplex is degenerate and the top simplex is exhibited via h'_W.

Definition 1.17. Given maps $g : W \to V$ and $g' : W \to V'$ in $\mathscr{K}(\mathbb{A})$, we refer to the correpsonding diagram (10) as the standard pushout diagram associated to g and g'.

Of course, we will see momentarily that standard pushout diagrams are colimit diagram in $\mathscr{K}(\mathbb{A})$. We obtain the following by applying Lemma 1.11 to the opposite category.

lem:5987 **Lemma 1.18.** *Consider maps of $\mathbb{A}$-cochains $g : V \to W$ and $g' : V' \to W$, and suppose one of g or g' splits as a graded morphism. Then the projection $\pi : \mathrm{cone}(g,g') \to V \amalg_W V'$ is a homotopy equivalence.*

Since the homotopy h'_W vanishes when composed with the projection $\pi : \mathrm{cone}(g,g') \to V \amalg_W V'$, this projection extends to a diagram in $\mathscr{K}(\mathbb{A})$ which appears as

$$\begin{tikzcd} & W \arrow[rr, "="] \arrow[dl] \arrow[d, dashed] \arrow[drr] & & W \arrow[dll] \arrow[dr] & \\ V' \arrow[r, "="] \arrow[dr] & V' \arrow[drr] & & V \arrow[r, "="] \arrow[dll] & V \arrow[dl] \\ & \mathrm{cone}(g,g') \arrow[rr] & & V \amalg_W V' & . \end{tikzcd}$$

According to Lemma 1.18, in the case that one of g or g' is split injective this diagram realizes an isomorphism between the two square faces, so that the standard square

$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d, "g'"'] \arrow[dr] & V \arrow[d] \\ V' \arrow[r] & \mathrm{cone}(g,g'). \end{tikzcd}$$

is observed to be a pushout square via Proposition 1.15. This realization of a colimit for such partial pushout diagrams now generalizes to arbitrary pairs of maps.

prop:K_pushout

Proposition 1.19. *For arbitrary maps $g : W \to V$ and $g' : W \to V'$ in $\mathscr{K}(\mathbb{A})$, the standard pushout diagram*

$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d, "g'"'] \arrow[dr] & V \arrow[d] \\ V' \arrow[r] & \mathrm{cone}(g, g') \end{tikzcd} \tag{11}$$

eq:6010

is a colimit diagram in $\mathscr{K}(\mathbb{A})$. In particular, the diagram (11) is isomorphic to a diagram of the form

$$\begin{tikzcd} V_0 \arrow[r, "g_{01}"] \arrow[d, "g'_{01}"'] & V_1 \arrow[d] \\ V'_1 \arrow[r] & V_2 \end{tikzcd}$$

in which g_{01} is termwise split injective and g'_{01} is injective

cor:K_pushout

Corollary 1.20. *Every diagram $\Lambda^2_0 \to \mathscr{K}(\mathbb{A})$ admits a colimit. That is to say, for any abelian category $\mathbb{A}$, the homotopy ∞-category $\mathscr{K}(\mathbb{A})$ admits all pushouts.*

1.6. **Products, coproducts, and the zero complex.**

prop:K_prods_coprods

Proposition 1.21. *For any abelian category $\mathbb{A}$, the functor $\mathrm{Ch}(\mathbb{A}) \to \mathscr{K}(\mathbb{A})$ preserves all small products and coproducts. In particular, the category $\mathscr{K}(\mathbb{A})$ admits all small products and coproducts.*

Proof. Since the inclusion $\mathrm{Ch}(\mathbb{A}) \to \mathscr{K}(\mathbb{A})$ factors through the equivalence $\mathfrak{Z} : \mathscr{K}(\mathbb{A})' \to \mathscr{K}(\mathbb{A})$ it suffices to show that products and coproducts in $\mathrm{Ch}(\mathbb{A})$ are products and coproducts in the simplicial construction of the homotopy ∞-category $\mathscr{K}(\mathbb{A})'$. For this we employ the Hom functor $\mathrm{K}\,\mathrm{Hom}^\bullet_{\mathbb{A}}$, where $\mathrm{Hom}^\bullet_{\mathbb{A}}(X, Y) = \tau_0 \mathrm{Hom}^*_{\mathbb{A}}(X, Y)$ and check that this functor turns discrete coproducts into products of spaces through the first coordinate, and discrete products into products of spaces through the secord coordinate. (Recall that products of spaces are as expected, by Example II-14.17 and Theorem II-14.26.)

Since τ_0 is a right adjoint it commutes with limits, and the Dold-Kan equivalence commutes with limits as well, it suffices to show that the functor $\mathrm{Hom}^*_{\mathbb{A}}$ sends discrete coproducts in the first coordinate to products of linear cochains, and discrete products in the second coordinate to products of linear cochains. However this follows immediately by, say, the fact that $\mathrm{Hom}^*_{\mathbb{A}}$ provide inner-Homs for the action of $\mathrm{Ch}(\mathbb{Z})$ on $\mathrm{Ch}(\mathbb{A})$. □

As a corollary to Lemma II-9.18 we also see that the zero complex provides a zero object for the homotopy ∞-category.

cor:K_zero

Corollary 1.22. *The zero complex provides a simultaneous initial and terminal object in the ∞-category $\mathscr{K}(\mathbb{A})$.*

1.7. **Resolutions as (co)limits.**

prop:lim_res

Proposition 1.23. *Given a strictly commuting diagram $I(-) : \mathbb{Z}_{\leq 0} \to \mathscr{K}(\mathbb{A})$ in which each map $I(n-1) \to I(n)$ is termwise split surjective and with discrete limit*

I in $\mathrm{Ch}(\mathbb{A})$, the discrete limit diagram

$$\begin{tikzcd}
\cdots \arrow[r] & I(m-1) \arrow[r] & I(m) \arrow[r] & I(m+1) \arrow[r] & \cdots \\
 & & I \arrow[ul] \arrow[u] \arrow[ur] & &
\end{tikzcd}$$

is a limit diagram in $\mathscr{K}(\mathbb{A})$. In particular, the discrete limit is an ∞-categorical limit in this case.

Proof. Since the diagram factors through $\mathrm{Ch}(\mathbb{A})$ we can work in the simplicial construction $\mathscr{K}(\mathbb{A})$. Applying the standard corepresentable functor $K\,\mathrm{Hom}^*_{\mathbb{A}}(X,-)$ to the given squence produces a sequence of Kan fibrations

$$\cdots \to K\,\mathrm{Hom}^*_{\mathbb{A}}(X, I(m-1)) \to K\,\mathrm{Hom}^*_{\mathbb{A}}(X, I(m)) \to \cdots \tag{12}$$ eq:6055

in $\mathrm{Kan} \subseteq \mathscr{K}an$, by Corollary 1.4. Since K has a left adjoint, given by the normalized cochains functor, it commutes with discrete limits, so that

$$\lim_n K\,\mathrm{Hom}^*_{\mathbb{A}}(X, I(n)) = K \lim_n \mathrm{Hom}^*_{\mathbb{A}}(X, I(n)) = \mathrm{Hom}^*_{\mathbb{A}}(X, L).$$

By Proposition II-14.39 and Theorem II-14.43, in this case the discrete limit diagram for the sequence (12) in Kan provides a limit diagram in $\mathscr{K}an$. By the above formula this discrete limit diagram is the image of the discrete limit diagram for the functor $I : \mathbb{Z}_{\leq 0} \to \mathrm{Ch}(\mathbb{A}) \subseteq \mathscr{K}(\mathbb{A})$. So we see that, at each complex X, the functor $K\,\mathrm{Hom}^*_{\mathbb{A}}(X,-)$ sends the discrete limit diagram for I to a limit diagram in spaces. By Corollary II-16.17 it follows that the discrete limit diagram for I is a limit diagram in $\mathscr{K}(\mathbb{A})'$, and hence in $\mathscr{K}(\mathbb{A})$. □

By consulting the opposite category, or simply by repeating the above arguments, one obtains a statement for colimits of sequences of split injections.

prop:colim_res **Proposition 1.24.** *Given a strictly commuting diagram $P(-) : \mathbb{Z}_{\geq 0} \to \mathscr{K}(\mathbb{A})$ in which each map $P(n) \to P(n+1)$ is termwise split injective and with discrete colimit P in $\mathrm{Ch}(\mathbb{A})$, the discrete colimit diagram*

$$\begin{tikzcd}
 & & P \arrow[dl] \arrow[d] \arrow[dr] & & \\
\cdots \arrow[r] & P(m-1) \arrow[r] & P(m) \arrow[r] & P(m+1) \arrow[r] & \cdots
\end{tikzcd}$$

is a colimit diagram in $\mathscr{K}(\mathbb{A})$. In particular, the discrete colimit is an ∞-categorical limit in this case.

These propositions are relevant which expressing bounded complexes as filtered limits of bounded injectives, or filtered colimits of bounded projectives. In order to speak about this point precisely we should understand the derived ∞-category $\mathscr{D}(\mathbb{A})$ as a localization of the homotopy ∞-category $\mathscr{K}(\mathbb{A})$, i.e. as an ∞-category with some universal map $\mathscr{K}(\mathbb{A}) \to \mathscr{D}(\mathbb{A})$. This localization map will be realized, in the injective model, as a right adjoint to the inclusion $\mathscr{D}(\mathbb{A}) \to \mathscr{K}(\mathbb{A})$ and, in the projective model, as a left adjoint thereof. In particular, the localization map is seen to commute with limits in the first construction and colimits in the second construction. Via uniqueness it therefore commutes with both limits and colimits. So, we can simply provide the relevant statement at the homotopy level.

Example 1.25. Let $V \to I$ be an injective resolution of a bounded below complex V in $\mathrm{Ch}(\mathbb{A})$. Consider the complexes $I(n) = I/I^{\geq -n}$ and the correpsonding sequence of termwise split sujections of bounded complexes of injectives

$$\cdots \to I(-2) \to I(-1) \to I(0).$$

We have the corresponding strictly commuting diagram $I(-) : \mathbb{Z}_{\leq 0} \to \mathscr{K}(\mathbb{A})$ with discrete limit $\lim_n I(n) = I$. By Proposition 1.23 this discrete limit is limit diagram in $\mathscr{K}(\mathbb{A})$.

Example 1.26. Let $P \to W$ be a projective resolution of a bounded above complex in $\mathrm{Ch}(\mathbb{A})$. Consider the compelxes $P(n) = P^{\geq -n}$ and the corresponding sequence of termqise split inclusions

$$P(0) \to P(1) \to P(2) \to \cdots.$$

We have the associated strictly commuting diagram in $\mathscr{K}(\mathbb{A})$, which has colimit P by Proposition 1.24.

2. Stability and cocompleteness of homotopy and derived ∞-categories

2.1. **Pullbacks are puhsouts in the homotopy ∞-category.**

lem:6102 **Lemma 2.1.** *For a diagram*

$$\begin{tikzcd} V_0 \arrow[r, "g"] \arrow[d, "g'"'] & V_1 \arrow[d, "f"] \\ V_1' \arrow[r, "f'"'] & V_2 \end{tikzcd} \tag{13}$$

eq:5863

in $\mathrm{Ch}(\mathbb{A})$ *the following are equivalent:*

(a) *The map f is (termwise split) surjective, f' is injective, and* (13) *is a discrete pullback diagram.*
(b) *The map g is (termwise split) injective, g' is surjective, and* (13) *is a pushout diagram.*

Proof. (a) $\Rightarrow$ (b) If (13) is a pullback diagram with the prescribed properties then the map g is an inclusion which identifies V_0 as a kernel of the composite $V_1 \to V_2 \to V_2/V_1'$. This identification of V_0 with the kernel of f also tells us that g is termwise split whenever f is termwise split. Furthermore, in this case the map $g' : V_0 \to V_1'$ is simply the restriction of the projection $V_1 \to V_2$ to V_0, and hence g' is surjective as well. So we see that g is injective and g' is surjective in this case, and g is split when f is spit.

The implication (b) $\Rightarrow$ (a) is recovered by applying (a) $\Rightarrow$ (b) to the category $\mathrm{Ch}(\mathbb{A}^{\mathrm{op}}) = \mathrm{Ch}(\mathbb{A})^{\mathrm{op}}$. $\square$

prop:K_pullpush **Proposition 2.2.** *For any abelian category $\mathbb{A}$, a diagram*

$$\begin{tikzcd} V_0 \arrow[r, "g"] \arrow[d, "g'"'] \arrow[dr] & V_1 \arrow[d, "f"] \\ V_1' \arrow[r, "f'"'] & V_2 \end{tikzcd} \tag{14}$$

eq:6126

in $\mathscr{K}(\mathbb{A})$ is a limit (aka pullback) diagram if and only if it is a colimit (aka pushout) diagram.

Proof. Suppose that the diagram (14) is a limit diagram. Then, according to Proposition 1.12, we can assume (14) is a strictly commuting, discrete pullback diagram in which f' is termwise split surjective. In particular, any pullback diagram in $\mathscr{K}(\mathbb{A})$ is isomorphic to such a discrete pullback diagram. In this case (14) is also a discrete pushout diagram in which g is termwise split injective, by Lemma 2.1. By Proposition 1.15 such a discrete pushout diagram is a colimit diagram in $\mathscr{K}(\mathbb{A})$, so that the diagram (14) is a colimit diagram in the homotopy ∞-category.

The converse implication is proved similarly. Namely, if (14) is a colimit diagram then we can assume it is a discrete pushout diagram with g termwise split injective, by Proposition 1.15, at which point it is seen to be a limit diagram as well by Lemma 2.1 and Proposition 1.19. $\square$

2.2. **Stable ∞-categories.**

Definition 2.3. Let $\mathscr{C}$ be an ∞-category which admits an object 0 which is both initial and terminal. A diagram of the form

$$\begin{tikzcd} x \arrow[r, "\alpha"] \arrow[d] \arrow[dr] & y \arrow[d, "\beta"] \\ 0 \arrow[r] & z \end{tikzcd}$$

is called a fiber sequence in $\mathscr{C}$ if it is a limit diagram, and a cofiber sequence if it is a colimit diagram.

Let us recall that any ∞-category with a zero object has zero morphisms. For any pair $x_0, x_1 : * \to \mathscr{C}$, this zero morphism is the unique morphism $x_0 \to x_1$ which completes a 2-simplex

$$\begin{tikzcd} & 0 \arrow[dr] & \\ x_0 \arrow[ur] \arrow[rr] & & x_1. \end{tikzcd}$$

Here we recall that the space of maps to and from 0 are contractible, by definition, so that the space of such 2-simplices is contractible.

Now, if we consider a cofiber sequence in $\mathscr{C}$, for example, one can think of z as a cokernel for the morphism $\alpha : x \to y$. Here, for the diagram $p : \Lambda_0^2 \to \mathscr{C}$ obtained from the above square by deleting z, we have an equivalence between the mapping spaces

$$\mathrm{Hom}_{\mathscr{C}}(z, w) \overset{\sim}{\to} \mathrm{Hom}_{\mathrm{Fun}(\Lambda_0^2, \mathscr{C})}(p, \underline{w}) = \{p\} \times^{\mathrm{or}}_{\mathrm{Fun}(\Lambda_0^2, \mathscr{C})} \{w\},$$

simply by the definition of the colimit, and the latter space is identified with the fiber of the undercategory $\mathscr{C}_{p/}$ over w. So in total we obtain an equivalence

$$\mathrm{Hom}_{\mathscr{C}}(z, w) \overset{\sim}{\to} \mathscr{C}_{p/} \times_{\mathscr{C}} \{w\}$$

at arbitrary w in $\mathscr{C}$. The latter fiber can be viewed as the space of maps $y \to w$ whose restriction along $\alpha : x \to y$ is trivial. One can employ a similar understanding of fiber sequences.

Example 2.4. Let $\mathbb{A}$ be a discrete additive category. A diagram in $\mathbb{A}$ of the form

$$\begin{tikzcd} x \arrow[r, "\alpha"] \arrow[d] \arrow[dr] & y \arrow[d, "\beta"] \\ 0 \arrow[r] & z \end{tikzcd}$$

is a fiber sequence if and only if α is a kernel of β, and is a cofiber sequence if and only if β is a cokernel of α.

Definition 2.5. An ∞-category $\mathscr{C}$ is called stable if the following properties hold:

(a) $\mathscr{C}$ has an object 0 which is simultaneously initial and terminal.
(b) Every morphism $\alpha : x \to y$ in $\mathscr{C}$ extends to a cofiber sequence and every morphism $\beta : y \to z$ extends to a cofiber sequence.
(c) A diagram of the form

$$\begin{tikzcd} x \arrow[r, "\alpha"] \arrow[d] \arrow[dr] & y \arrow[d, "\beta"] \\ 0 \arrow[r] & z \end{tikzcd}$$

in $\mathscr{C}$ is a fiber sequence if and only if it is a cofiber sequence.

Definition 2.6. A functor between stable ∞-categories $F : \mathscr{C} \to \mathscr{D}$ is exact if F preserves initial/terminal objects and preserves fiber/cofiber sequences. A full ∞-subcategory $\mathscr{C}' \subseteq \mathscr{C}$ is called a stable subcategory if $\mathscr{C}'$ is stable and the inclusion $\mathscr{C}' \to \mathscr{C}$ is exact.

We note that $\mathscr{C}'$ is a stable ∞-subcategory in $\mathscr{C}$ if and only if, for each morphism $\alpha : x \to y$ in $\mathscr{C}'$, and pullback and pushout diagram

$$\begin{tikzcd} x \arrow[r, "\alpha"] \arrow[d] \arrow[dr] & y \arrow[d] \\ 0 \arrow[r] & \mathrm{cofib}(\alpha) \end{tikzcd} \qquad \begin{tikzcd} \mathrm{fib}(\alpha) \arrow[r] \arrow[d] \arrow[dr] & x \arrow[d] \\ 0 \arrow[r] & y \end{tikzcd}$$

in $\mathscr{C}$, there are objects z and w in $\mathscr{C}'$ which admit isomorphisms $z \cong \mathrm{cofib}(\alpha)$ and $w \cong \mathrm{fib}(\alpha)$.

We note that stability is a property, rather than a structure. For example, if $F : \mathscr{C} \to \mathscr{D}$ is an equivalence of ∞-categories, and one of $\mathscr{C}$ or $\mathscr{D}$ is stable, then both $\mathscr{C}$ and $\mathscr{D}$ are stable.

Lemma 2.7. *Suppose $F : \mathscr{C} \to \mathscr{D}$ is an equivalence of ∞-categories and that $\mathscr{C}$ is stable. Then $\mathscr{D}$ is stable.*

Proof. Left to the reader. □

The following is obvious.

Lemma 2.8. *If $\mathscr{C}$ is stable then the opposite category $\mathscr{C}^{\mathrm{op}}$ is also stable.*

2.3. **Stability of the homotopy and derived ∞-categories.**

thm:K_stable

Theorem 2.9. *For any abelian category $\mathbb{A}$, the homotopy ∞-category $\mathscr{K}(\mathbb{A})$ is stable.*

Proof. The zero complex is a zero object in $\mathscr{K}(\mathbb{A})$ by Corollary 1.22, and since $\mathscr{K}(\mathbb{A})$ admits all pullbacks and pushouts by Corollaries 1.13 and Corollary 1.20. In particular, one can complete any morphism in $\mathscr{K}(\mathbb{A})$ to both a fiber and cofiber sequence. Finally, fiber sequences and cofiber sequences in $\mathscr{K}(\mathbb{A})$ agree by Proposition 2.2. $\square$

By the description of pullbacks and pushouts in $\mathscr{K}(\mathbb{A})$ provided in Propositions 1.12 and 1.19 we see that any full subcategory $\mathscr{K} \subseteq \mathscr{K}(\mathbb{A})$ which is preserved under desuspension and the formation of mapping cones admits all pushouts an pullbacks. In particular, the inclusion $\mathscr{K} \to \mathscr{K}(\mathbb{A})$ preserves pushouts and pullbacks. Thus, under these conditions, and assuming additionally that $\mathscr{K}$ contains the zero complex, we see that $\mathscr{K}$ is a stable ∞-category as well.

cor:K_stable

Corollary 2.10. *Suppose a full ∞-subcategory $\mathscr{K} \subseteq \mathscr{K}(\mathbb{A})$ contains the zero complex, is closed under the formation of mapping cones, and is closed under the desuspension automorpihsm Σ^{-1}. Then $\mathscr{K}$ is a stable ∞-subcategory in $\mathscr{K}(\mathbb{A})$. In particular, $\mathscr{K}$ is stable.*

In the case of a Grothendieck abelian category, we have the injective construction of the (unbounded) derived ∞-category $\mathscr{D}(\mathbb{A})$, where $\mathscr{D}(\mathbb{A})$ is identified specifically as the full ∞-subcategory of K-injectives in $\mathscr{K}(\mathbb{A})$ (Definition I-2.9). As a particular instance of Corollary 2.10 we observe stability of the derived ∞-category.

cor:D_stable

Corollary 2.11. *If $\mathbb{A}$ is a Grothendieck abelian category then the derived ∞-category $\mathscr{D}(\mathbb{A})$ is a stable ∞-category.*

Proof. We construct the derived ∞-category $\mathscr{D}(\mathbb{A})$ specifically as the ∞-subcategory of K-injectives in $\mathscr{K}(\mathbb{A})$. For any map between K-injectives $f : I \to I'$, and arbitrary acyclic X, the Hom complex

$$\mathrm{Hom}^*_{\mathbb{A}}(X, \mathrm{cone}(f)) = \mathrm{cone}\left(f_* : \mathrm{Hom}^*_{\mathbb{A}}(X, I) \to \mathrm{Hom}^*_{\mathbb{A}}(X, I')\right).$$

Since both complexes in the latter cone are acyclic, the cone itself is acyclic, and we see that $\mathrm{cone}(f)$ is K-injective. By the above reasoning we now conclude that the derived ∞-category is stable. $\square$

Remark 2.12. As stated previously, stability of an ∞-category is a property not a structure. So, in the event that a Grothendieck abelian category $\mathbb{A}$ has enough projectives, one can prove stability by employing either the K-injective construction of the derived ∞-category $\mathscr{D}(\mathbb{A})$ or the K-projective construction. See Section I-12.

We similarly apply Corollary 2.10 to observe stability of the derived category under the standard bounding restrictions. We have the ∞-subcategories $\mathscr{D}^b(\mathbb{A})$, $\mathscr{D}^-(\mathbb{A})$, and $\mathscr{D}^+(\mathbb{A})$ of bounded, (cohomologically) bounded above, and (cohomologically) bounded below complexes.

Corollary 2.13. *If $\mathbb{A}$ is a Grothendieck abelian category then for $\star = b, +, -$, the appropriately bounded derived ∞-category $\mathscr{D}^\star(\mathbb{A}) \subseteq \mathscr{D}(\mathbb{A})$ is a stable ∞-subcategory. Furthermore, in the case where $\mathbb{A}$ is linear and locally finite, the full ∞-subcategory $\mathscr{D}(\mathbb{A})_{fin}$ of complexes with finite total length is a stable ∞-subcategory in $\mathscr{D}(\mathbb{A})$.*

To be clear, by a locally finite Grothendieck abelian category we mean a compactly generated abelian category all of whose compacts are of finite length, and whose morphisms between compacts $\mathrm{Hom}_{\mathbb{A}}(x, y)$ are finite-dimensional over the given base field. Such categories include representations $\mathbb{A} = \mathrm{Rep}(G)$ over an affine algebraic group G, for example.

Example 2.14 (An anti-example: The connective derived category)**.** An interesting anti-example, the connective derived ∞-category $\mathscr{D}^{\leq 0}(\mathbb{A})$ is not stable in $\mathscr{D}(\mathbb{A})$. Indeed, this ∞-category is not even stable.

We can consider explicitly a complex V which is concentrated in degree 0, and the pullback diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] \arrow[dr] & 0 \arrow[d] \\ 0 \arrow[r] & V. \end{tikzcd} \tag{15}$$

eq:859

To see that this diagram is a pullback diagram we note that the truncation functor $\tau_0 : \mathscr{D}(\mathbb{A}) \to \mathscr{D}^{\leq 0}(\mathbb{A})$ (Definition 1.7) is right adjoint to the inclusion $\mathscr{D}^{\leq 0}(\mathbb{A}) \to \mathscr{D}(\mathbb{A})$ (Theorem I-13.10). Truncation therefore preserves limits by Proposition II-13.23, so that we obtain the above limit diagram by truncating the standard limit diagram in $\mathscr{D}(\mathbb{A})$

$$\begin{tikzcd} \Sigma^{-1}V \arrow[r] \arrow[d] \arrow[dr] & 0 \arrow[d] \\ 0 \arrow[r] & V \end{tikzcd}$$

provided by Proposition 1.12. However, we see simultaneously that the diagram (15) is not a pushout diagram unless $V = 0$.

2.4. **Overview: Suspension and desuspension in stable ∞-categories.** Fix $\mathscr{C}$ a stable ∞-category. We first sketch a construction the shift automorphism [4, Section 1.1.2]: Let

$$\mathscr{M} = \mathscr{M}_{\mathscr{C}} \subseteq \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathscr{C})$$

denote the full ∞-subcategory spanned by fiber/cofiber sequences of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & Y. \end{tikzcd}$$

Then evaluation at both the initial and terminal vertices induce trivial Kan fibrations

$$ev_0, ev_1 : \mathscr{M} \to \mathscr{C}$$

by [3, Proposition 4.3.2.15]. For any sections $s_i : \mathscr{C} \to \mathscr{M}$ of ev_i we obtain the suspension and desuspension functors

$$\Sigma := ev_2 s_0 : \mathscr{C} \to \mathscr{C} \quad \text{and} \quad \Omega := ev_0 s_2 : \mathscr{C} \to \mathscr{C}.$$

Proposition 2.15. *If $\mathscr{C}$ is stable, then the endofunctors $\Sigma, \Omega : \mathscr{C} \to \mathscr{C}$ are autoequivalences which are mutually inverse.*

Proof. Since the evaluation morphisms are equivalences so are their sections s_i. Furthermore each s_i is an inverse to ev_i. So we have

$$\Sigma\Omega = \mathrm{ev}_2\, s_0\, \mathrm{ev}_0\, s_2 \cong id_{\mathscr{C}}, \quad \Omega\Sigma = \mathrm{ev}_0\, s_2\, \mathrm{ev}_2\, s_0 \cong id_{\mathscr{C}}.$$

□

In the algebriac setting the dg automorphism $\Sigma : \mathbf{Ch}(\mathbb{A}) \to \mathbf{Ch}(\mathbb{A})$ induces an equivalence of ∞-categories $\Sigma' : \mathscr{K}(\mathbb{A}) \to \mathscr{K}(\mathbb{A})$, and one can construct an explicit section s_0 for the homotopy ∞-category $\mathrm{ev}_0 : \mathscr{K}(\mathbb{A}) \to \mathscr{M}$ by sending each object V to the explicit diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma X \end{tikzcd}$$

and each n-simplex σ to the n-simplex $s_0\sigma : \Delta^n \times (\Delta^1 \times \Delta^1) \to \mathscr{C}$ which restricts to σ and $\Sigma'\sigma$ at the vertices $(0,0)$ and $(1,1)$ respectively, 0 at the vertices $(1,0)$ and $(0,1)$, and vanishes on all other non-degenerate vertices. (Here one should interpret the word vanishes correctly, based on the explicit construction of the dg nerve.) In this way we recover $\Sigma' = \mathrm{ev}_2\, s_0 = \Sigma$, and one similarly recovers $(\Sigma')^{-1} = \mathrm{ev}_0\, s_2 = \Omega$.

In general, we simply write Σ and Σ^{-1} for the shifts, or suspension and desuspension, on $\mathscr{K}(\mathbb{A})$ and $\mathscr{D}(\mathbb{A})$.

Lemma 2.16. *Let $F : \mathscr{C} \to \mathscr{D}$ be a functor between stable categories. If F respects fiber sequences then there is a uniquely associated isomorphism $F \circ \Sigma \cong \Sigma \circ F$.*

Proof. We have the induced map on the restricted functor spaces $\mathscr{M}_F : \mathscr{M}_{\mathscr{C}} \to \mathscr{M}_{\mathscr{D}}$ which fits into diagrams

$$\begin{tikzcd} \mathscr{M}_{\mathscr{C}} \arrow[r, "\mathscr{M}_F"] \arrow[d, "\mathrm{ev}_0"'] & \mathscr{M}_{\mathscr{D}} \arrow[d, "\mathrm{ev}_0"] \\ \mathscr{C} \arrow[r, "F"'] & \mathscr{D}. \end{tikzcd}$$

Since the evaluation functor is a trivial Kan fibration there is, furthermore, a unique lift $\widetilde{F}_0 : \mathscr{C} \to \mathscr{M}_{\mathscr{D}}$ of the functor $F : \mathscr{C} \to \mathscr{D}$ along ev_0. For any sections s_0 of ev_0 we then have the two lifts $\mathscr{M}_F s_0$ and $s_0 \mathscr{M}_F$, which are therefore identified, which gives

$$F\Sigma = F\,\mathrm{ev}_2\, s_0 = \mathrm{ev}_2\, \mathscr{M}_F s_0 \cong \mathrm{ev}_2\, s_0 F = \Sigma F.$$

□

prop:891 **Proposition 2.17.** *For any stable ∞-category $\mathscr{C}$, and all pairs of objects $x, y : * \to \mathscr{C}$, we have canonical isomorphisms*

$$\Omega \operatorname{Hom}_{\mathscr{C}}(x,y) \overset{\sim}{\to} \operatorname{Hom}_{\mathscr{C}}(\Sigma x, y) \quad \textit{and} \quad \Omega \operatorname{Hom}_{\mathscr{C}}(x,y) \overset{\sim}{\to} \operatorname{Hom}_{\mathscr{C}}(x, \Omega y)$$

in the homotopy category $\mathrm{h}\,\mathscr{K}an$.

Here Ω is the space of pointed loops at the 0 map in $\operatorname{Hom}_{\mathscr{C}}(x,y)$ (Section I-6.4).

Proof. Given a Hom functor $H : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{K}an$ the functors $H(-,y)$ and $H(x,-)$ commute with limits. So we have the pullback diagrams

$$\begin{tikzcd} H(\Sigma x, y) \arrow[r] \arrow[d] \arrow[dr] & H(0,y) \arrow[d] \\ H(0,y) \arrow[r] & H(x,y) \end{tikzcd} \qquad \begin{tikzcd} H(x, \Omega y) \arrow[r] \arrow[d] \arrow[dr] & H(x,0) \arrow[d] \\ H(x,0) \arrow[r] & H(x,y), \end{tikzcd}$$

in which the mapping spaces $H(0,y)$ and $H(x,0)$ are contractible. By definition, the pointed loops are the pullback of the diagram

$$\begin{tikzcd} & * \arrow[d] \\ * \arrow[r] & H(x,y) \end{tikzcd}$$

in $\mathscr{K}an$. We replace this diagram with the isomorphic diagrams

$$\begin{tikzcd} & H(0,y) \arrow[d] & & H(x,0) \arrow[d] \\ H(0,y) \arrow[r] & H(x,y) & H(x,0) \arrow[r] & H(x,y), \end{tikzcd}$$

and reference Proposition II-13.17, to obtain the claimed identifications. □

Stable categories have various additivity properties, which are of course expressed a kind of topological guise. As a hint of this structure, we have the following.

Corollary 2.18. *If $\mathscr{C}$ is stable, then at any pair of objects the fundamental group $\pi_1 \operatorname{Hom}_{\mathscr{C}}(x,y)$ (based at 0) is abelian, and the connected components $\pi_0 \operatorname{Hom}_{\mathscr{C}}(x,y)$ admit an abelian group structure.*

Proof. We have, in the homotopy category,

$$\operatorname{Hom}_{\mathscr{C}}(x,y) \cong \Omega^2 \operatorname{Hom}_{\mathscr{C}}(x,\Sigma^2 y)$$

by Proposition 2.17, which provides an isomorphism of groups

$$\pi_i \operatorname{Hom}_{\mathscr{C}}(x,y) \cong \pi_{i+2} \operatorname{Hom}_{\mathscr{C}}(x,\Sigma^2 y)$$

at all $i > 0$, and a bijection of sets at $i = 0$, by Proposition I-6.10. □

One can show that this additive structure on $\pi_0 \operatorname{Hom}_{\mathscr{C}}(x,y)$ is natural, in the sense that any exact functor between stable categories respects this additive structure. This is part of a general claim that the homotopy category of a stable category is natrually traingulated.

2.5. **Overview: Limits and colimits in stable ∞-categories.** Our main aims of this section are two-fold: First, we want to prove homotopy and derived ∞-categories are stable. Second, we want to explain how one observes cocompleteness of the derived ∞-category. We hope this provides the reader with the proper entrance point need to begin to apply some of the essential findings from [4]. For the moment, we simply record some of the basic properties of stable categories, with appropriate referenceing to [4].

Proposition 2.19 ([4, Lemma 1.1.2.9])**.** *Any stable ∞-category $\mathscr{C}$ admits finite products and coproducts, and for any pair of object $x, y : * \to \mathscr{C}$ the canonical map*

$$\begin{bmatrix} id_x & 0 \\ 0 & id_y \end{bmatrix} : x \times y \to x \amalg y$$

is an isomorphism in $\mathscr{C}$.

Proposition 2.20 ([4, Propositions 1.1.3.4 & 1.1.4.1])**.** *Any stable ∞-category $\mathscr{C}$ admits all finite limits and colimits. Furthermore, for a functor between ∞-categories $F : \mathscr{C} \to \mathscr{D}$ the following are equivalent:*

(a) *F is exact, i.e. preserves fiber sequences.*
(b) *F preserves all finite limits.*
(c) *F preserves all finite colimits.*

Proposition 2.21 ([4, Proposition 1.1.3.4])**.** *A diagram*

$$\begin{tikzcd} x \arrow[r] \arrow[d] \arrow[dr] & y \arrow[d] \\ y' \arrow[r] & z \end{tikzcd}$$

in a stable ∞-category $\mathscr{C}$ is a pullback diagram if and only if it is a pushout diagram.

We are especially interested in cocompleteness and cocontinuity for stable ∞-categories. As in the abelian setting, existence of all small coproducts in the stable setting ensures the existence of all small colimits.

prop:stable_cocomp **Proposition 2.22** ([4, Proposition 1.4.4.1])**.** *Let $\mathscr{C}$ be a stable ∞-category. The following are equivalent:*

(1) *$\mathscr{C}$ admits all small coproducts.*
(2) *$\mathscr{C}$ is cocomplete, i.e. admits all small colimits.*

Furthermore, if $F : \mathscr{C} \to \mathscr{D}$ is functor between stable ∞-categories, and $\mathscr{C}$ admits all small coproducts, then F is cocontinuous if and only if F is exact and preserves small coproducts.

By considering opposite categories we similarly find that a stable category $\mathscr{C}$ is complete if and only if $\mathscr{C}$ admits all small products.

prop:1042 **Proposition 2.23.** *Consider a full ∞-subcategory $\mathscr{C}'$ in a cocomplete stable ∞-category $\mathscr{C}$. If $\mathscr{C}'$ contains the zero object, and is closed under small coproducts and the formation of cofiber sequences in $\mathscr{C}$, then $\mathscr{C}'$ is cocomplete and the inclusion $\mathscr{C}' \to \mathscr{C}$ is cocontinuous.*

Sketch proof. By [3, Proposition 4.4.3.2] and ∞-category is cocomplete if and only if it admits small coproducts and coequalizaers. As $\mathscr{C}'$ is closed under the formation of small coproducts in $\mathscr{C}$, it suffices to prove that $\mathscr{C}'$ is similarly closed under the formation of coequalizers.

As argued in the proof of [4, Proposition 1.1.3.1], the coequalizer c of a pair of maps $\alpha, \alpha' : x \to y$ is the cofiber of the difference $c = \mathrm{cofib}(\alpha - \alpha')$. Indeed, we have the standard pushout diagram

$$\begin{tikzcd} x \amalg x \arrow[r, "{[\alpha\ \alpha']}"] \arrow[d, "\mathrm{diag}"'] & y \arrow[d] \\ x \arrow[r] & c \end{tikzcd}$$

which then extends to a concatenation of pushout diagrams

$$\begin{tikzcd} x \arrow[r, "{[1\ -1]^t}"] \arrow[d] & x \times x \arrow[r] \arrow[d] & y \arrow[d] \\ 0 \arrow[r] & x \arrow[r] & c \end{tikzcd}$$

and the outer square recovers the claimed fiber sequence

$$\begin{tikzcd} x \arrow[r, "(\alpha-\alpha')"] \arrow[d] & y \arrow[d] \\ 0 \arrow[r] & c. \end{tikzcd}$$

So we see that stability under the formation of cofibers implies the existence of coequalizers in any full ∞-subcategory, and we conclude that $\mathscr{C}'$ is is fact cocomplete.

As for the claim that the inclusion $\mathscr{C}' \to \mathscr{C}$ is cocontinuous, for object $x, y, y' : * \to \mathscr{C}'$ the pushout z of a diagram

$$\begin{tikzcd} x \arrow[r, "\alpha"] \arrow[d, "\alpha'"'] & y \\ y' & \end{tikzcd}$$

in $\mathscr{C}$ is the coequalizer of the maps $\alpha, \alpha' : x \to y \amalg y'$. This follows by [3, Proposition 4.4.3.1]. Since $\mathscr{C}'$ is stable under the formation of coequalizers, we now find that $\mathscr{C}'$ is stable under the formation of pushouts in $\mathscr{C}$ as well. Cocontinuity of the inclusion follows by [3, Proposition 4.4.2.7]. $\square$

Remark 2.24. Clearly the argument for Proposition 2.23 is simpler if we just assume that $\mathscr{C}'$ is stable under the formation of coproducts and pushouts in $\mathscr{C}$. In the only example we are interested in, that of connective cochains, such pushout stability is clear.

As in the usual discrete setting, one has the notion of compact objects in an ∞-category.

Definition 2.25. Let $\mathscr{C}$ be a cocomplete ∞-category. An object x in an $\mathscr{C}$ is called compact if there exists a functor $h^x : \mathscr{C} \to \mathscr{K}an$ which is represented by x and which preserves small filtered colimits [5, 02PB].

Of course, since all functors which are represented by x are isomorphic, there exists such a functor h^x which preserves filtered colimits if and only if all functors which are represented by x preserve small filtered colimits.

prop:compact

Proposition 2.26 ([4, Proposition 1.4.4.1])**.** *Let $\mathscr{C}$ be a cocomplete, stable ∞-category. An object $x : * \to \mathscr{C}$ is compact if and only if, for each small coproduct $\amalg_{\lambda \in \Lambda} y_\lambda$ and map $\alpha : x \to \coprod_\lambda y_\lambda$, there exists a finite subset $\{\lambda_0, \ldots, \lambda_m\} \subseteq \Lambda$ for which α factors as a composite*

$$x \to (y_{\lambda_0} \amalg \cdots \amalg y_{\lambda_m}) \to \coprod_{\lambda \in \Lambda} y_\lambda.$$

2.6. **Overview: The homotopy category under stability.** We record the following for the sake of completeness.

Theorem 2.27 ([4, Theorem 1.1.2.14])**.** *For any stable ∞-category $\mathscr{C}$, the homotopy category $\mathrm{h}\mathscr{C}$ inherits a natural triangulated structure in which the exact triangles*

$$x \to y \to z$$

are exactly the images of fiber sequences in $\mathscr{C}$. The connecting morphism $\delta : z \to \Sigma x$ for such a triangle is provided by the universal property of the pushout applied to the diagram

$$\begin{tikzcd} x \arrow[r] \arrow[d] \arrow[dr] & y \arrow[d, "0"] \\ 0 \arrow[r] & \Sigma x. \end{tikzcd}$$

One can check directly that this "natural" triangulated structure on $\mathrm{h}\,\mathscr{C}$ recovers the well-known, and oft-employed, triangulated structure on the discrete homotopy category $K(\mathbb{A}) = \mathrm{h}\,\mathscr{K}(\mathbb{A})$ and discrete derived category $D(\mathbb{A}) = \mathrm{h}\,\mathscr{D}(\mathbb{A})$. (See for example [7, Sections 10.2 & 10.4].) We leave this as an exercise for the interested reader.

Corollary 2.28. *Any exact functor between stable ∞-categories $F : \mathscr{C} \to \mathscr{D}$ induces an exact functor of triangulated categories $\mathrm{h}\,F : \mathrm{h}\,\mathscr{C} \to \mathrm{h}\,\mathscr{D}$.*

Of course, it is not the case, generally speaking, that one can detect equivalences between ∞-categories at the level of the homotopy category. Consider, for example, the inclusion $0 : * \to \mathrm{Sing}(S^2)$ of a point into the circle. We have $\mathrm{h}\,\mathrm{Sing}(S^2) = *$, so that the map 0 induces an equivalence on homotopy categories. However, this map is not an equivalence since, using PropositionI-6.10, we have

$$\pi_1 \mathrm{Hom}_{\mathrm{Sing}(S^2)}(0, 0) \cong \pi_2 S^2 = \mathbb{Z}.$$

In the stable setting such phenomena never occurs, as all of the higher homotopy groups in the mapping spaces $\mathrm{Hom}_{\mathscr{C}}(x, y)$ are realized as the 0-th homotopy group of a shifted space $\mathrm{Hom}_{\mathscr{C}}(\Sigma^n x, y)$. We have the following fundamental fact.

Proposition 2.29 ([1, Proposition 5.10])**.** *An exact functor between stable ∞-categories $F : \mathscr{C} \to \mathscr{D}$ is fully faithful (resp. an equivalence) if and only if the induced $\mathrm{h}\,F : \mathrm{h}\,\mathscr{C} \to \mathrm{h}\,\mathscr{D}$ is fully faithful (resp. an equivalence).*

Additionally, Proposition 2.26 tells us that compactness of objects in a stable ∞-category can be checked at the level of the homotopy category. We recall that an object x in a triangulated category $\mathbb{C}$ which admits small sums is called compact if the functor $\mathrm{Hom}_{\mathbb{C}}(x, -)$ commutes with small sums. The following is now simply a repackaging of Proposition 2.26.

`cor:compact` **Corollary 2.30.** *An object x in a stable ∞-category $\mathscr{C}$ is compact if and only if its image is compact in the homotopy category $\mathrm{h}\,\mathscr{C}$.*

2.7. **Cocompleteness of the derived ∞-category.** We consider again a Grothendieck abelian category $\mathbb{A}$. As explained in the proof of Corollary 2.11, the class of K-injectives in $\mathrm{Ch}(\mathbb{A})$ is stable under the formation of mapping cones. It is also clearly stable under suspension and desuspension. So, as was already argued implicitly, the full ∞-subcategory of K-injectives $\mathscr{D}(\mathbb{A})$ in $\mathscr{K}(\mathbb{A})$ is stable under the formation of both pullbacks and pushouts. This follows by the explicit constructions provided in Propositions 1.12 and 1.19 above.

`prop:D_pullpush` **Proposition 2.31.** *Let $\mathbb{A}$ be a Grothendieck abelian category. The derived ∞-category $\mathscr{D}(\mathbb{A})$ admits all pullbacks and pushouts, and the inclusion $\mathscr{D}(\mathbb{A}) \to \mathscr{K}(\mathbb{A})$ preserves all pullback and pushout diagrams.*

Since products of acyclic complexes are acyclic, it is clear that products of K-injectives are K-injective. It follows tha te derived ∞-category $\mathscr{D}(\mathbb{A})$ is stable under products in $\mathscr{K}(\mathbb{A})$, and in particular admits all products. One can show that this ∞-category admits all coproducts as well.

prop:D_sumprod **Proposition 2.32.** *For any Grothendieck abelian category $\mathbb{A}$, the derived ∞-category $\mathscr{D}(\mathbb{A})$ admits all small products and coproducts.*

In the case that $\mathbb{A}$ has enough projectives, one can simply employ the construction of the derived ∞-category via K-projectives, and note that arbitrary sums of K-projectives remain K-projective. In a case like $\mathbb{A} = \operatorname{Rep} G$ for a smooth affine algebraic group of positive dimension, for example, we have no such projectives. As we demonstrate in the proof, in this case we form sums of K-injectives by taking the ordinary sum, then taking a K-projective replacement.

Proof of Proposition 2.32. The situation with products is as explained above. Consider now any collection of K-injectives $I_- : \Lambda \to \mathscr{D}(\mathbb{A})$ indexed over a small discrete set Λ. Take any injective resolution of the resulting coproduct $k : \oplus_{\lambda\in\Lambda} I_\lambda \to I$ and for each index λ let $k_\lambda : I_\lambda \to I$ be the composition of the structural map $I_\lambda \to \oplus_{\lambda\in\Lambda} I_\lambda$ with k.

Take $\mathscr{D}(\mathbb{A})'$ the simplicial construction of $\mathscr{D}(\mathbb{A})$, with natural equivalence $\mathfrak{Z} : \mathscr{D}(\mathbb{A})' \to \mathscr{D}(\mathbb{A})$ as in Theorem II-10.4. It suffices to show that the extension diagram $\{0\} \star \Lambda \to \mathscr{D}(\mathbb{A})'$, with cone point I and maps given by the k_λ, is a limit diagram.

We apply the representable functor

$$\underline{\operatorname{Hom}}_{\mathbb{A}}(-, J) = K\operatorname{Hom}^*_{\mathbb{A}}(-, J) : (\mathscr{D}(\mathbb{A})')^{\mathrm{op}} \subseteq (\mathscr{K}(\mathbb{A})')^{\mathrm{op}} \to \mathscr{K}an$$

to obtain an equivalence

$$\underline{\operatorname{Hom}}_{\mathbb{A}}(I, J) \xrightarrow{\sim} \underline{\operatorname{Hom}}_{\mathbb{A}}(\oplus_\lambda I_\lambda, J) = \prod_{\lambda\in\Lambda} \underline{\operatorname{Hom}}_{\mathbb{A}}(I_\lambda, J).$$

By considering a corresponding diagram in $\operatorname{Ch}(\mathbb{A})$, this equivalence is seen to extend to a strictly commuting diagram $\Delta^1 \times (\{0\} \star \Lambda) \to \mathscr{K}an$ which realizes an isomorphism between the standard product diagram

$$\begin{tikzcd} & & \underline{\operatorname{Hom}}_{\mathbb{A}}(I, J) \arrow[dl, "(k_{\mu'})^*"'] \arrow[d, "(k_\mu)^*"] \arrow[dr, "(i_{\mu''})^*"] & & \\ \dots & \underline{\operatorname{Hom}}_{\mathbb{A}}(I_{\mu'}, J) & \underline{\operatorname{Hom}}_{\mathbb{A}}(I_\mu, J) & \underline{\operatorname{Hom}}_{\mathbb{A}}(I_{\mu''}, J) & \dots \end{tikzcd} \tag{16}$$

eq:6280

and the standard product diagram

$$\begin{tikzcd} & & \prod_\lambda \underline{\operatorname{Hom}}_{\mathbb{A}}(I_\lambda, J) \arrow[dl] \arrow[d] \arrow[dr] & & \\ \dots & \underline{\operatorname{Hom}}_{\mathbb{A}}(I_{\mu'}, J) & \underline{\operatorname{Hom}}_{\mathbb{A}}(I_\mu, J) & \underline{\operatorname{Hom}}_{\mathbb{A}}(I_{\mu''}, J) & \dots. \end{tikzcd} \tag{17}$$

eq:6287

Since products in $\mathscr{K}an$ are realized via the discrete product in Kan, the diagram (17) is a limit diagram at all J, and it follows from Proposition II-13.18 that the diagram (16) is a limit diagram at all J. By Corollaries II-16.17 and II-11.13, it follows that I provides a coproduct for the diagram $I_- : \Lambda \to \mathscr{D}(\mathbb{A})'$, and hence for I is a coproduct for the original diagram in $\mathscr{D}(\mathbb{A})$. $\square$

Corollary 2.33. *For any Grothendieck abelian category $\mathbb{A}$, the homotopy ∞-category $\mathscr{K}(\mathbb{A})$ is both complete and cocomplete, as is the derived ∞-category $\mathscr{D}(\mathbb{A})$.*

Corollary 2.34. *For any Grothendieck abelian category, the connective derived category $\mathscr{D}^{\leq 0}(\mathbb{A})$ is cocomplete and the inclusion $\mathscr{D}^{\leq 0}(\mathbb{A}) \to \mathscr{D}(\mathbb{A})$ is cocontinuous.*

3. Adjoints again

3.1. Reflexive subcategories.

Definition 3.1 ([5, 02F6])**.** Let $\mathscr{C}' \subseteq \mathscr{C}$ be a full ∞-subcategory. Given an object x in $\mathscr{C}$, a morphism $f : x \to y$ with y in $\mathscr{C}'$ is said to exhibit y as a $\mathscr{C}'$-reflection of x if, for each third object z in $\mathscr{C}'$, the precomposition function

$$f^* : \mathrm{Hom}_{\mathscr{C}}(y, z) \to \mathrm{Hom}_{\mathscr{C}}(x, z)$$

is an isomorphism in $\mathrm{h}\,\mathscr{K}an$. Similarly, a morphism $g : y \to x$ with y in $\mathscr{C}'$, is said to exhibit y as a $\mathscr{C}'$-coreflection of x if, for each third object z in $\mathscr{C}'$, the composition function

$$g_* : \mathrm{Hom}_{\mathscr{C}}(z, y) \to \mathrm{Hom}_{\mathscr{C}}(z, x)$$

is an isomorphism in $\mathrm{h}\,\mathscr{K}an$.

We say $\mathscr{C}'$ itself is a reflexive (resp. coreflexive) subcategory in $\mathscr{C}$ if every object x in $\mathscr{C}$ admits a morphism $x \to y$ (resp. $y \to x$) which exhibits y as a $\mathscr{C}'$-reflection (resp. $\mathscr{C}'$-coreflection) of x.

One sees directly that taking opposites $\mathscr{C} \mapsto \mathscr{C}^{\mathrm{op}}$ exchanges reflections and coreflections, and exchanges reflexive and coreflexive subcategories as well. So, throughout the section, we may prove a result only for reflections with the understanding that the analogous result for coreflections is obtained by applying opposites.

ex:inj_reflex

Example 3.2 (K-injectives)**.** Let $\mathbb{A}$ be a Grothendiek abelian category, and let $\mathscr{D}_{\mathrm{Inj}} \subseteq \mathscr{K}(\mathbb{A})$ denote the full subcategory of K-injective complexes. Let $\mathscr{D}'_{\mathrm{Inj}} \subseteq \mathscr{K}(\mathbb{A})'$ be the corresponding full subcategory in the simplicial construction of the homotopy ∞-category.

Every complex V in $\mathrm{Ch}(\mathbb{A})$ admits a K-injective resolution $f : V \to I_V$. This map induces a quasi-isomorphism

$$f^* : \mathrm{Hom}^*_{\mathbb{A}}(I_V, J) \to \mathrm{Hom}^*_{\mathbb{A}}(V, J),$$

which then induces a homotopy equivalence

$$f^* : K\,\mathrm{Hom}^*_{\mathbb{A}}(I_V, J) \to K\,\mathrm{Hom}^*_{\mathbb{A}}(V, J)$$

so that f is a $\mathscr{D}'_{\mathrm{Inj}}$-reflection in $\mathscr{K}(\mathbb{A})'$, by Proposition II-**??**. It follows via the natural equivalence $\mathfrak{Z}_?$ of Theorem II-10.4 that $f : V \to I_V$ is also a $\mathscr{D}_{\mathrm{Inj}}$-reflection in $\mathscr{K}(\mathbb{A})$. So we conclude that $\mathscr{D}_{\mathrm{Inj}}$ is a reflexive subcategory in $\mathscr{K}(\mathbb{A})$.

ex:proj_coreflex

Example 3.3 (K-projectives)**.** Suppose that an abelian category $\mathbb{A}$ has enough projectives, and let $\mathscr{D}_{\mathrm{Proj}}$ be the full ∞-subcategory of K-projectives in $\mathscr{K}(\mathbb{A})$. Each complex V admits a K-projective resolution $g : P_V \to V$. One argues as in Example 3.2 to see that g is a $\mathscr{D}_{\mathrm{Proj}}$-coflection, and hence to see that $\mathscr{D}_{\mathrm{Proj}}$ is a coreflexive subcategory in $\mathscr{K}(\mathbb{A})$.

Lemma 3.4. *Given any morphism $f : x \to y$, the forgetful functor $\mathscr{C}_{f/} \to \mathscr{C}_{y/}$ is a trivial Kan fibration.*

Proof. By [2, Lemma 3.3] the apparent map

$$(\Delta^1 \star \partial\Delta^n) \coprod_{(\{1\}\star\partial\Delta^n)} (\{1\} \star \Delta^n) \to \Delta^1 \star \Delta^n \cong \Delta^{n+2}$$

is an isomorphism onto the inner horn $\Lambda^{n+2}_1 \subseteq \Delta^{n+2}$. Hence solving a lifting problem of the form

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & \mathscr{C}_{f/} \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & \mathscr{C}_{y/} \end{tikzcd}$$

is equivalent to solving a lifting problem of the form

$$\begin{tikzcd} \Lambda^{n+2}_1 \arrow[r] \arrow[d] & \mathscr{C} \arrow[d] \\ \Delta^{n+2} \arrow[r] \arrow[ur, dotted] & *. \end{tikzcd}$$

Since $\mathscr{C}$ is an ∞-category, the latter problem always admits a solution. □

lem:1253 **Lemma 3.5** ([5, 02LL])**.** *For any morphism $f : x \to y$ and object $z : * \to \mathscr{C}$, we have a commuting diagram*

$$\begin{tikzcd} \mathscr{C}_{y/} \times_{\mathscr{C}} \{z\} \arrow[d, "\cong"'] & \mathscr{C}_{f/} \times_{\mathscr{C}} \{z\} \arrow[l, "\cong"'] \arrow[r] & \mathscr{C}_{x/} \times_{\mathscr{C}} \{z\} \arrow[d, "\cong"] \\ \mathrm{Hom}_{\mathscr{C}}(y, z) \arrow[rr, "f^*"'] & & \mathrm{Hom}_{\mathscr{C}}(x, z). \end{tikzcd}$$

in $\mathrm{h}\,\mathscr{K}an$*.*

Here the vertical maps specifically those induced by the coslice diagonal equivalences, where we observe that the fibers of this equivalence remain equivalences by Corollary I-5.22 and Proposition I-5.23.

Idea of proof. One produces a morphism

$$i : \mathscr{C}_{f/} \times_{\mathscr{C}} \{z\} \to \{f\} \times_{\mathrm{Hom}_{\mathscr{C}}(x,y)} \mathrm{Fun}(\Delta^2, \mathscr{C})_{\vec{x}}$$

which bisects the diagram

$$\begin{tikzcd} \mathscr{C}_{y/} \times_{\mathscr{C}} \{z\} \arrow[d, "\cong"'] & \mathscr{C}_{f/} \times_{\mathscr{C}} \{z\} \arrow[l, "\cong"'] \arrow[r] \arrow[d] & \mathscr{C}_{x/} \times_{\mathscr{C}} \{z\} \arrow[d, "\cong"] \\ \mathrm{Hom}_{\mathscr{C}}(y, z) & \{f\} \times_{\mathrm{Hom}_{\mathscr{C}}(x,y)} \mathrm{Fun}(\Delta^2, \mathscr{C})_{\vec{x}} \arrow[l] \arrow[r] & \mathrm{Hom}_{\mathscr{C}}(x, z). \end{tikzcd}$$

at the level of the discrete category Kan. □

prop:reflex_char **Proposition 3.6.** *Let $\mathscr{C}' \subseteq \mathscr{C}$ be a full ∞-subcategory. For a fixed morphism $f : x \to y$, with y in $\mathscr{C}'$, the following are equivalent:*

(a) *f exhibits y as a $\mathscr{C}'$-reflection of x.*
(b) *At each object z in $\mathscr{C}'$ the forgetful functor $\mathscr{C}_{f/} \times_{\mathscr{C}} \{z\} \to \mathscr{C}_{x/} \times_{\mathscr{C}} \{z\}$ is an equivalence.*
(c) *The map $\mathscr{C}_{f/} \times_{\mathscr{C}} \mathscr{C}' \to \mathscr{C}_{f/} \times_{\mathscr{C}} \mathscr{C}'$ is a trivial Kan fibration.*

(d) *Each lifting problem*

$$\begin{tikzcd} \Lambda^n_0 \arrow[r, "\tau"] \arrow[d] & \mathscr{C} \arrow[d] \\ \Delta^n \arrow[r] & * \end{tikzcd}$$

with $n \geq 2$ and $\tau|_{\Delta^{\{1,\dots,n\}}}$ having imagine in $\mathscr{C}'$, admits a solution.

Proof. The equivalence (a) $\Leftrightarrow$ (b) is a consequence of Lemma 3.5. For (b) $\Leftrightarrow$ (c), we recall that the map

$$\mathscr{C}_{f/} = (\mathscr{C}_{x/})_{y/} \to \mathscr{C}_{x/}$$

is a left fibration by Corollary I-4.27, and hence its base change $\mathscr{C}_{f/} \times_{\mathscr{C}} \mathscr{C}' \to \mathscr{C}_{f/} \times_{\mathscr{C}} \mathscr{C}'$ is a left fibration as well. This left fibration furthermore fits into a diagram of left fibrations

$$\begin{tikzcd} \mathscr{C}_{f/} \times_{\mathscr{C}} \mathscr{C}' \arrow[rr] \arrow[dr] & & \mathscr{C}_{f/} \times_{\mathscr{C}} \mathscr{C}' \arrow[dl] \\ & \mathscr{C}' & \end{tikzcd}.$$

Theorem II-3.3 and Corollary II-9.8 together now imply that the forgetful functor $\mathscr{C}_{f/} \times_{\mathscr{C}} \mathscr{C}' \to \mathscr{C}_{x/} \times_{\mathscr{C}} \mathscr{C}'$ is a trivial Kan fibration if and only if at each object $z : * \to \mathscr{C}'$ the fiber

$$\mathscr{C}_{f/} \times_{\mathscr{C}} \{z\} \to \mathscr{C}_{x/} \times_{\mathscr{C}} \{z\}$$

is an equivalence. Statement (d) is identified with (c) via the identification of lifting problems

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & \mathscr{C}_{f/} \times_{\mathscr{C}} \mathscr{C}' \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & \mathscr{C}_{x/} \times_{\mathscr{C}} \mathscr{C}' \end{tikzcd} \quad = \quad \begin{tikzcd} \Lambda^{n+2}_0 \arrow[r] \arrow[d] & \mathscr{C} \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & * \end{tikzcd}$$

which is implied by Lemma II-9.10. $\square$

3.2. **Reflexivity and adjoints.**

prop:1327 **Proposition 3.7.** *Let $\mathscr{C}' \subseteq \mathscr{C}$ be a full ∞-subcategory and $i : \mathscr{C}' \to \mathscr{C}$ be the inclusion. The subcategory $\mathscr{C}'$ is reflexive if and only if there is a functor $L : \mathscr{C} \to \mathscr{C}'$ and a transformation $u : id_{\mathscr{C}} \to iL$ for which, at each x in $\mathscr{C}$, the map $u_x : x \to L(x)$ exhibits $L(x)$ as a $\mathscr{C}'$-reflection of x.*

Proof. Let $\mathscr{E} \subseteq \mathscr{C} \times \Delta^1$ be the full ∞-subcategory whose objects $\mathscr{E}[0]$ are the union $(\mathscr{C}[0] \times \{0\}) \cup (\mathscr{C}'[0] \times \{1\})$. By Proposition 3.6 the projection

$$q : \mathscr{E} \to \Delta^1$$

is a cocartesian fibration, and a map $f : (x, 0) \to (y, 1)$ in $\mathscr{E}$ is q-cocartesian if and only if the underlying map $f : x \to y$ in $\mathscr{C}$ exhibits y as a $\mathscr{C}$-reflection of x.

Now, by Theorem II-2.7 there exists a unique functor

$$U : \Delta^1 \times \mathscr{C} \to \mathscr{E}$$

which splits the diagram

$$\begin{tikzcd} \{0\} \times \mathscr{C} \arrow[r, "\mathrm{incl}"] \arrow[d] & \mathscr{E} \arrow[d, "q"] \\ \Delta^1 \times \mathscr{C} \arrow[r, "p_1"'] \arrow[ur, "U"] & \Delta^1 \end{tikzcd}$$

and sends each map $\Delta^1 \times \{x\}$ to a q-cocartesian morphism in $\mathscr{E}$. For $L : \mathscr{C} \to \mathscr{C}' \subseteq \mathscr{C}$ defined as the composite

$$\mathscr{C} \cong \{1\} \times \mathscr{C} \to \Delta^1 \times \mathscr{C} \to \mathscr{E} \overset{p_1}{\to} \mathscr{C},$$

the transformation $u = p_1 U : \Delta^1 \times \mathscr{C} \to \mathscr{C}$ has the prescribed property. □

lem:1357

Lemma 3.8 ([5, 02DK]). *Let $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ be functors between ∞-categories, and $u : id_{\mathscr{C}} \to GF$ be a transformation. Suppose that the induced transformations*

$$Fu : F \to F(GF) \quad \textit{and} \quad uG : G \to (GF)G$$

are isomorphisms in $\mathrm{Fun}(\mathscr{C}, \mathscr{D})$ and $\mathrm{Fun}(\mathscr{D}, \mathscr{C})$ respectively, and that G is fully faithful. Then u is the unit of an adjunction, and the counit $\epsilon : FG \to id_{\mathscr{D}}$ is a natural isomorphism.

Sketch proof. Since G is fully faithful the induced map $G_* : \mathrm{Fun}(\mathscr{C}, \mathscr{D}) \to \mathrm{Fun}(\mathscr{C}, \mathscr{C})$ is fully faithful. Hence there is a unique transformation $\epsilon : FG \to id_{\mathscr{D}}$ which lifts the isomorphism $(uG)^{-1} : GFG \to G$. Fully faithfulness implies that ϵ is an isomorphism as well.

We can replace the functor categories $\mathrm{Fun}(\mathscr{A}, \mathscr{B})$ with their homotopy categories $\mathrm{h}\,\mathrm{Fun}(\mathscr{A}, \mathscr{B})$ and work with the corresponding 2-category Cat^2_∞ obtained from the simplicial category $\underline{\mathrm{Cat}}_\infty$. At this level we consider the composites

$$F \overset{Fu}{\to} FGF \overset{\epsilon F}{\to} F \quad \text{and} \quad G \overset{uG}{\to} GFG \overset{G\epsilon}{\to} G.$$

The latter composite is the identity by the definition of ϵ. We consider now the composite $\beta = (\epsilon F)(Fu)$. By our assumptions, Fu is an isomorphism, so that β is an isomorphism as well. One now argues, using [5, 02CX], that β also satisfies $\beta^2 = \beta$ and hence $\beta = id_F$ necessarily. □

rem:1376

Remark 3.9. If a transformation $u : id_{\mathscr{C}} \to GF$ admits some transformation $FG \to id_{\mathscr{D}}$ which exhibits F as left adjoint to G, then this transformation is fixed up to the action of $\mathrm{Aut}_{\mathrm{Fun}(\mathscr{D},\mathscr{C})}(G)$ [5, 02D7]. In particular, any transformation $\epsilon' : FG \to id_{\mathscr{D}}$ which pairs with u to realize F as left adjoint to G, in the situation of Lemma 3.8, must be a natural isomorphism.

prop:1380

Proposition 3.10. *Let $i : \mathscr{C}' \to \mathscr{C}$ be the inclusion of a full ∞-subcategory into an ∞-category $\mathscr{C}$. Consider any functor $L : \mathscr{C} \to \mathscr{C}'$ and transformation $u : id_{\mathscr{C}} \to iL$. The following are equivalent:*

(a) *The transformation u is part of an adjunction which exhibits L as left adjoint to the inclusion $i : \mathscr{C}' \to \mathscr{C}$.*
(b) *At each object x in $\mathscr{C}$, the map $u_x : x \to L(x)$ exhibits $L(x)$ as a $\mathscr{C}'$-reflection of x.*
(c) *At each x in $\mathscr{C}$ the map $L(u_x) : L(x) \to LL(x)$ is an isomorphism and, at each y in $\mathscr{C}'$, the map $u_y : y \to L(y)$ is an isomorphism in $\mathscr{C}'$.*

Furthermore, in this case, any transformation $\epsilon : Li \to id_{\mathscr{C}'}$ which pairs with u to realize L as left adjoint to i is a natural isomorphism.

Proof. Supposing (a) and (c) hold, the claim about ϵ follows by Lemma 3.8 and Remark 3.9. Now, suppose (a) holds. Then by Corollary I-13.4 the transformation u realizes the $\mathrm{h}\mathscr{K}an$-enriched functor $\pi L : \pi\mathscr{C} \to \pi\mathscr{C}'$ as left adjoint to the enriched embedding $\pi i : \pi\mathscr{C}' \to \pi\mathscr{C}$. Hence, at each z in $\mathscr{C}'$, u induces isomorphisms

$$u^* : \mathrm{Hom}_{\mathscr{C}}(L(x), z) \to \mathrm{Hom}_{\mathscr{C}}(x, z)$$

in $\mathrm{h}\mathscr{K}an$. Thus (b) holds.

Supposing (b) holds. When y is in $\mathscr{C}'$ applying u yields an isomorphism of sets

$$u^* : \pi_0 \mathrm{Hom}_{\mathscr{C}'}(L(y), z) \to \pi_0 \mathrm{Hom}_{\mathscr{C}'}(y, z)$$

which shows, via Yoneda, that $u_y : y \to L(y)$ is an isomorphism in $\mathrm{h}\mathscr{C}'$. By definition this implies that u_y is an isomorphism in $\mathscr{C}'$.

As for the transformation $L(u_x) : L(x) \to LL(x)$ at general x, we have the diagram

$$\begin{tikzcd} x \arrow[r, "u_x"] \arrow[d, "u_x"'] & L(x) \arrow[d, "u_{L(x)}"] \\ L(x) \arrow[r, "L(u_x)"'] & LL(x) \end{tikzcd}$$

in $\mathrm{h}\mathscr{C}$ and apply $\mathrm{Hom}_{\mathrm{h}\mathscr{C}}(-, z)$ at arbitrary z in $\mathscr{C}'$ to obtain a diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathrm{h}\mathscr{C}}(L(x), z) \arrow[r, "\cong"] \arrow[d, "\cong"'] & \mathrm{Hom}_{\mathrm{h}\mathscr{C}'}(x, z) \arrow[d, "\cong"] \\ \mathrm{Hom}_{\mathrm{h}\mathscr{C}'}(LL(x), z) \arrow[r, "L(u)^*"'] & \mathrm{Hom}_{\mathrm{h}\mathscr{C}'}(L(x), z). \end{tikzcd}$$

From this we conclude that $L(u_x)^*$ is an isomorphism at all z, and hence that $L(u_x)$ is an isomorphism in $\mathrm{h}\mathscr{C}$. It follows that $L(u_x)$ is an isomorphism in $\mathscr{C}$.

Finally, Lemma 3.8 tells us directly that (c) implies (a). This completes the proof. □

One combines Propositions 3.7 and 3.10 to obtain the following.

thm:reflex_adj

Theorem 3.11. *Consider a full ∞-subcategory $\mathscr{C}' \subseteq \mathscr{C}$ in an arbitrary ∞-category $\mathscr{C}$. Then $\mathscr{C}'$ is reflexive in $\mathscr{C}$ if and only if the inclusion $i : \mathscr{C}' \to \mathscr{C}$ admits a left adjoint $L : \mathscr{C} \to \mathscr{C}'$ whose unit and counit transformations have the properties outlined in Proposition 3.10 above. Similarly, $\mathscr{C}'$ is coreflexive if and only if the inclusion admits a right adjoint $R : \mathscr{C} \to \mathscr{C}'$.*

Proof. The claim about reflexive subcategories is clear, on the claim about coreflexive subcategories is obtain by applying opposites. □

ex:1454

Example 3.12. As an application of Theorem 3.11 to Example 3.2 we observe, for Grothendieck abelian $\mathbb{A}$, the existence of a left adjoint $L : \mathscr{K}(\mathbb{A}) \to \mathscr{D}_{\mathrm{Inj}} = \mathscr{D}(\mathbb{A})$ to the inclusion $i : \mathscr{D}(\mathbb{A}) \to \mathscr{K}(\mathbb{A})$. This left adjoint exhibits $\mathscr{D}(\mathbb{A})$ as a localization $\mathscr{K}(\mathbb{A})[\mathrm{Qiso}^{-1}]$ of the homotopy ∞-category at the class of quasi-isomorphisms. We return to this topic in Section 4 below.

3.3. **Adjoints via simultaneous fibrations.**

Lemma 3.13. *Let $q : \mathscr{E} \to \Delta^1$ be an inner fibration of ∞-categories, with fibers $\mathscr{E}_i = \mathscr{E} \times_{\Delta^1} \{i\}$. The following hold:*

(1) *The subcategory $\mathscr{E}_1$ is a reflexive subcategory in $\mathscr{E}$ if and only if q is a cocartesian fibration, and in this case a map $f : x \to y$ over $0 < 1$ is q-cocartesian if and only if it exhibits y as a $\mathscr{E}_1$-reflection of x.*

(2) *The subcategory $\mathscr{E}_0$ is a coreflexive subcategory in $\mathscr{E}$ if and only if q is a cartesian fibration, and in this case a map $g : y \to x$ over $0 < 1$ is q-cartesian if and only if it exhibits y as a $\mathscr{E}_0$-coreflection of x.*

Proof. Follows by Proposition 3.6 (c). □

prop:adj_fibration **Proposition 3.14.** *Let $q : \mathscr{E} \to \Delta^1$ be a cocartesian fibration, and $F : \mathscr{E}_0 \to \mathscr{E}_1$ be the functor given by covariant transport along q (Definiiton II-7.1). The functor F admits a right adjoint $G : \mathscr{E}_1 \to \mathscr{E}_0$ if and only if q is a cartesian fibration as well, and in this case G is given by contravariant transport along q.*

Our claim that G is "given by contravariant transport" should be interpreted in a strict sense. Namely, we claim that when F admits a right adjoint G, there is a global contravariant transport functor $R : \mathscr{E} \to \mathscr{E}_0$ whose restriction to the fiber $\mathscr{E}_1$ recovers G exactly $R|_{\mathscr{E}_1} = G$.

Proof. First suppose that F admits such a right adjoint $G : \mathscr{E}_1 \to \mathscr{E}_0$, and consider the unit and counit transformations

$$u : id_{\mathscr{E}_0} \to GF \quad \text{and} \quad \epsilon : FG \to id_{\mathscr{E}_1}$$

respectively. By Proposition 3.6 (c) we understand that q is cartesian if and only if the subcategory $\mathscr{E}_0$ is coreflexive in $\mathscr{E}$. So we seek to demonstrate $\mathscr{E}_0$-coreflections $g_x : y \to x$ at each x in $\mathscr{E}$. When x is in $\mathscr{E}_0$ we can just take $g_x = id_x$, so that we can assume here that x is in the fiber $\mathscr{E}_1$.

First, let us consider the extended fibration $q^l : \mathscr{E}^l \to \Delta^1$ where $\mathscr{E}^l \subseteq \mathscr{E} \times \Delta^1$ is the full ∞-subcategory with objects $(\mathscr{E}[0] \times \{0\}) \cup (\mathscr{E}_1[0] \times \{1\})$. Here q^l is specifically the projection onto the second factor. As in the proof of Proposition 3.7, we see that q^l is a cocartesian fibration and one observes directly the inclusion of cocartesian fibrations $\mathscr{E} \to \mathscr{E}^l$ provided by restricting the image of the product map $[id_{\mathscr{E}}\ q]^t : \mathscr{E} \to \mathscr{E} \times \Delta^1$.

The transport functor $L : \mathscr{E} \to \mathscr{E}_1$ along q^l comes equipped with a transformation $\eta : id_{\mathscr{E}} \to L$ which evaluates to a q^l-cocartesian morphism at each x in $\mathscr{E}$, by definition. By uniqueness of transport functors and the fact that the restriction functor

$$\mathrm{Fun}(\Delta^1 \times \mathscr{E}, \mathscr{E}) \to \mathrm{Fun}(\Delta^1 \times \mathscr{E}_0, \mathscr{E})$$

is an isofibration (Corollary I-5.14), we can choose L so that $F = L|_{\mathscr{E}_0}$. Note also that η exhibits L as left adjoint to the inclusion $i_1 : \mathscr{E}_1 \to \mathscr{E}$, by Lemma 3.8.

At each x in $\mathscr{E}_1$ define the map $g_x : G(x) \to x$ as a composite

$$G(x) \xrightarrow{\eta} LG(x) = FG(x) \xrightarrow{\epsilon} x.$$

We then have at each z in $\mathscr{E}_0$, in the homotopy category of spaces, the sequence of maps

$$\mathrm{Hom}_{\mathscr{E}_0}(z, G(x)) \xrightarrow{F} \mathrm{Hom}_{\mathscr{E}_1}(F(z), FG(x)) \xrightarrow{\epsilon_*} \mathrm{Hom}_{\mathscr{E}_1}(F(z), x) \xrightarrow{\eta^*} \mathrm{Hom}_{\mathscr{E}}(z, x).$$

As the first two maps compose to an isomorpihsm, and the third map is also an isomorphism, this composite is an isomorphism. By commutativity of the operations $\epsilon_* = \epsilon \circ -$ and $\eta^* = - \circ \eta$, i.e. by associativity of composition in $\mathrm{h}\,\mathscr{K}an$, and naturality of η (Lemma I-13.3), the above composite is equal to the map

$$(g_x)_* : \mathrm{Hom}_{\mathscr{E}_0}(z, G(x)) \to \mathrm{Hom}_{\mathscr{E}}(z, x),$$

which we conclude is an isomorphism. So each $g_x : G(x) \to x$ exhibits $G(x)$ as a $\mathscr{E}_0$-coreflecton, $\mathscr{E}_0$ is seen to be coreflexive in $\mathscr{E}$, and q is therefore a cartesian fibration.

Suppose now that $q : \mathscr{E} \to \Delta^1$ is cartesian. Then the subcategory $\mathscr{E}_0$ is coreflexive in $\mathscr{E}$ by Proposition 3.6 and we have the associated cartesian fibration $q^r : \mathscr{E}^r \to \Delta^1$ obtained by considering the full ∞-subcategory $\mathscr{E}^r \subseteq \mathscr{E} \times \Delta^1$ whose objects are provided by the union $(\mathscr{E}_0[0] \times \{0\}) \cup (\mathscr{E}[0] \times \{1\})$. We have the contravariant transport functor $R : \mathscr{E} \to \mathscr{E}_0$ for q^r along with the transformation $\pi : i_0 R \to id_{\mathscr{E}}$ which evaluates to a q^r-cartesian morphism at each x in $\mathscr{E}$. By Lemma 3.8 the transformation π exhibits R as right adjoint to the inclusion $\mathscr{E}_0 \to \mathscr{E}$.

Define $G : \mathscr{E}_1 \to \mathscr{E}_0$ as the composite $Ri_1 : \mathscr{E}_1 \to \mathscr{E} \to \mathscr{E}_0$. The inclusion $\mathscr{E}_1 \to \mathscr{E}$ has left adjoint $L : \mathscr{E} \to \mathscr{E}_1$ which restricts to F on $\mathscr{E}_0$, and we see that G is a composite of right adjoints. Hence G itself is right adjoint to the functor $Li_0 = F$, as desired (see [5, 02DT]). We see also that G is a contravariant trasport functor for $q : \mathscr{E} \to \Delta^1$ since restricting along the inclusion of cartesian fibrations $\mathscr{E} \to \mathscr{E}^r$ recovers transport functors for q from transport functors for q^r.

As for the claim that any right adjoint to F is given by contravariant transport, this follows by uniqueness of adjoints. Specifically, consider $G : \mathscr{E}_1 \to \mathscr{E}_0$ obtained by contravariant transport, i.e. obtained by restricting our particular cartesian lift $\pi|_{\mathscr{E}_1} : \Delta^1 \times \mathscr{E}_1 \to \mathscr{E}$ of the inclusion $\{1\} \times \mathscr{E}_1 \to \mathscr{E}$, and with counit ϵ constructed as above. For any other right adjoint G' with counit transformation $\epsilon' : FG \to id_{\mathscr{E}_1}$ there is a natural isomoprhism $\xi : G' \to G$ for which ϵ' is recovered as a composite $\epsilon(F\xi) = \epsilon'$ [5, 02D7]. Since the restriction functor

$$\mathrm{Fun}(\Delta^1 \times \mathscr{E}_1, \mathscr{E}) \to \mathrm{Fun}(\{1\} \times \mathscr{E}_1 \amalg \{0\} \times \mathscr{E}_1, \mathscr{E})$$

is an isofibration, by Corollary I-5.14, we can simultaneously lift the pair (i_1, G') to a transformation $\pi' : \Delta^1 \times \mathscr{E}_1 \to \mathscr{E}$ and the isomorphism ξ to an isomorphism $\Xi : \pi' \to \pi|_{\mathscr{E}_1}$. The map π' therefore provides a cartesian lift of the inclusion $\{1\} \times \mathscr{E}_1 \to \mathscr{E}$ as well, and we realize G' as explicitly obtained by contravariant trasport for q. □

We now obtain a characterization of adjunctions via fibrations over the 1-simplex.

thm:adj_fibration

Theorem 3.15. *Given a pair of functor $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$, the following are equivalent:*

(a) *The functors F and G admit transformations which exhibit F as left adjoint to G.*

(b) *There is a simultaneous cartesian and cocartesian fibration $q : \mathscr{E} \to \Delta^1$ with fixed isomorphisms at the fibers $\mathscr{C} \cong \mathscr{E}_0$ and $\mathscr{D} \cong \mathscr{E}_1$, and for which F and G are recovered respectivley as covariant and contravariant transport along q.*

Proof. Note that F defines a functor $F : \Delta^1 \to \mathscr{C}at_\infty$ and consider the weighted nerve $q : \mathscr{E} = \mathrm{N}^F(\Delta^1) \to \Delta^1$. The fact that F is recovered by covariant transport along q is implicit in the claim that there is an isomorphism of fibrations $\mathscr{E} \cong \int_{\Delta^1} F$ (Theorem II-6.28). However, we can just observe this fact directly.

For an n-simplex $\sigma = (\sigma', \sigma'') : \Delta^n \to \Delta^1 \times \mathscr{C} = \mathscr{E}_0$ take $\Delta^{n_0} = (\sigma')^{-1}(0)$. Define now

$$\widetilde{\sigma} : \Delta^n \to \mathscr{E}$$

as the pairing of the n-simplex $\sigma' : \Delta^n \to \Delta^1$ with the pair of n-simplices $\sigma''|_{\Delta^{n_0}} : \Delta^{n_0} \to \mathscr{C}$ and $F\sigma'' : \Delta^n \to \mathscr{D}$. The assignment $\sigma \mapsto \widetilde{\sigma}$ defines a functor cocartesian lift of the inclusion $\{0\} \times \mathscr{C} \to \mathscr{E}$, and so recovers $F : \mathscr{C} \to \mathscr{E}$ as covariant transport along q.

In any case, we recover the claimed equivalence of (a) and (b) by applying Proposition 3.14 to the weighted nerve for F. $\square$

3.4. **Local criterion for adjunction.**

Theorem 3.16. *A functor between ∞-categories $F : \mathscr{C} \to \mathscr{D}$ admits a right adjoint if and only if, for each y in $\mathscr{D}$, there exists a morphism $g_y : F(x) \to y$ from some object x in $\mathscr{C}$ such that, at any other z in $\mathscr{C}$, the sequence*

$$\mathrm{Hom}_{\mathscr{C}}(z, x) \xrightarrow{F} \mathrm{Hom}_{\mathscr{D}}(F(z), F(x)) \xrightarrow{(g_y)_*} \mathrm{Hom}_{\mathscr{D}}(F(z), y) \tag{18}$$

eq:1550

is an isomorphism in $\mathrm{h}\mathscr{K}an$.

Proof. If there exists an adjoint G then we can take $x = G(y)$ and g the counit morphism. Conversely, suppose we can always find such a g_y at each y in $\mathscr{D}$. Then for the weighted nerve $q : \mathscr{E} = \mathrm{N}^F(\Delta^1) \to \Delta^1$ covariant trasport along q recovers the functor F, as was argued in the proof of Theorem 3.15.

In the weighted nerve

$$\mathrm{Hom}_{\mathscr{E}}(z, x) = \mathrm{Hom}_{\mathscr{C}}(z, x) \quad \text{and} \quad \mathrm{Hom}_{\mathscr{E}}(z, y) = \mathrm{Hom}_{\mathscr{E}}(F(z), y),$$

and one can check directly that the composition function

$$\mathrm{Hom}_{\mathscr{E}}(x, y) \times \mathrm{Hom}_{\mathscr{E}}(z, x) \to \mathrm{Hom}_{\mathscr{E}}(z, y)$$

is obtained by applying $F : \mathrm{Hom}_{\mathscr{C}}(z, x) \to \mathrm{Hom}_{\mathscr{D}}(F(z), F(x))$ then composing in $\mathscr{D}$. Hence if we view g_y as a morphism in $\mathscr{E}$, $g_y : x \to y$, then the operation

$$(g_y)_* : \mathrm{Hom}_{\mathscr{E}}(z, x) \to \mathrm{Hom}_{\mathscr{E}}(z, y)$$

is identified with the sequence (18). So, we conclude that the sequence (18) is an isomorphism at some g_y, for each y, if and only if the fiber $\mathscr{D} = \mathscr{E}_1$ is coreflexive in $\mathscr{E}$, which then occurs if and only if $q : \mathscr{E} \to \Delta^1$ is a cartesian fibration by Proposition 3.6 (c). We apply Proposition 3.14 to see that F has a right adjoint in this case. $\square$

Taking opposites, we observe the analagous local criterion for the existence of left adjoints.

Theorem 3.17. *A functor between ∞-categories $G : \mathscr{D} \to \mathscr{C}$ admits a left adjoint if and only if, for each x in $\mathscr{C}$, there exists a morphism $f_x : x \to G(y)$ from some object y in $\mathscr{D}$ such that, at any other z in $\mathscr{D}$, the sequence*

$$\mathrm{Hom}_{\mathscr{D}}(y, z) \xrightarrow{G} \mathrm{Hom}_{\mathscr{C}}(G(y), G(z)) \xrightarrow{f_x^*} \mathrm{Hom}_{\mathscr{C}}(x, G(z))$$

is an isomoprhism in $\mathrm{h}\mathscr{K}an$.

sect:D_loc

4. The derived ∞-category as a localization

4.1. **The setup.** Throughout this section $\mathbb{A}$ is a Grothendieck abelian category. We recall that $\mathbb{A}$ admits enough injectives in this case, and that every complex V admits a quasi-isomorphism $V \to I$ to a K-injectives [6, Theorem 3.13]. From this we conclude that the full subcategory

$$\mathscr{D}_{\mathrm{Inj}} := \mathscr{K}(\mathbb{A})_{\mathrm{Inj}} = \left\{ \begin{array}{c} \text{The full } \infty\text{-subcategory of } K\text{-injective} \\ \text{complexes in } \mathscr{K}(\mathbb{A}). \end{array} \right.$$

is reflexive in $\mathscr{K}(\mathbb{A})$ (Example 3.2). Of course, the ∞-category $\mathscr{D}_{\mathrm{Inj}}$ is one of our standard models for the derived ∞-category of $\mathbb{A}$ (Section I-2.5). However, the point of this section is to argue that the derived category should be characterized, or defined, via a universal property. So we emphasize throughout the section the particular nature of $\mathscr{D}_{\mathrm{Inj}}$ as the ∞-subcategory of K-injectives in $\mathscr{K}(\mathbb{A})$.

We are also interested in the cases where $\mathbb{A}$ admits enough projectives. For examples, one might consider:

- $\mathbb{A} = A$-Mod for a ring A.
- $\mathbb{A} = \mathrm{QCoh}(\mathfrak{X})$ where $\mathfrak{X} = [X/G]$ is the quotient stack of an affine scheme by the action of a reductive algebraic group G, in characteristic 0. In this case the projective objects are identified with equivariant vector bundles on X, under the pullback equivalence between $\mathrm{QCoh}(\mathfrak{X})$ and equivariant vector bundles on X.
- $\mathbb{A} = \mathrm{Rep}_q(G)$ the category of quantum group representations for a semisimple algebraic group G at an arbitrary complex parameter $q \in \mathbb{C}$.

In this case the full subcategory of K-projectives

$$\mathscr{D}_{\mathrm{Proj}} := \mathscr{K}(\mathbb{A})_{\mathrm{Proj}} = \left\{ \begin{array}{c} \text{The full } \infty\text{-subcategory of } K\text{-injective} \\ \text{complexes in } \mathscr{K}(\mathbb{A}) \end{array} \right.$$

forms a coreflexive subcategory in $\mathscr{K}(\mathbb{A})$ (Example 3.3), and we have a canonical equivalence

$$\mathscr{D}_{\mathrm{Proj}} \cong \mathscr{D}_{\mathrm{Inj}}$$

by Theorem I-12.5. So in this case we have reflexive a coreflexive subcategories

$$\mathscr{D}_{\mathrm{Proj}},\ \mathscr{D}_{\mathrm{Inj}}\ \subseteq\ \mathscr{K}(\mathbb{A})$$

which provide two distinct, but categorically indestinguishable, "models" for the derived ∞-category.

Remark 4.1. The proposed use of projectives is, based on our experience with triangulated categories, clearly too restrictive. For example, if we have a functor $F : \mathscr{K}(\mathbb{A}) \to \mathscr{K}(A)$ to the homotopy ∞-category of dg moduels over a dg algebra A, and F induces an equivalence on derived categories $\mathscr{D}(\mathbb{A}) \to \mathscr{D}(\mathbb{A})$ (however we "derive" this functor), then we might expect to transfer K-projectives in $\mathscr{K}(A)$ to a class of "projectives" in $\mathscr{K}(\mathbb{A})$. We will return to this topic later in the section.

4.2. **Reflections and coreflections as reoslutions.**

lem:injres_refl

Lemma 4.2. *For a morphism $f : V \to X$ in $\mathscr{K}(\mathbb{A})$, the following are equivalent:*

(a) *The object X is K-injective and f is a quasi-isomorpihsm.*
(b) *The morphism f is a $\mathscr{D}_{\mathrm{Inj}}$-reflection.*

Proof. The implication (a) $\Rightarrow$ (b) follows from the fact that maps into any K-injective $\operatorname{Hom}^*_{\mathbb{A}}(-,Z)$ preserve quasi-isomoprphisms. Hence the functor

$$\operatorname{Hom}_{\mathscr{K}(\mathbb{A})}(-,Z) \cong K\operatorname{Hom}^*_{\mathbb{A}}(-,Z) : \mathscr{K}(\mathbb{A}) \to \mathrm{h}\mathscr{K}an$$

sends quasi-isomorphisms to isomorphisms.

For (b) $\Rightarrow$ (a), suppose f is a $\mathscr{D}_{\mathrm{Inj}}$-reflection. Then X is K-injective, by definition. Suppose, by way of contradiction, that f is not a quasi-isomorphism. Then the mapping cone $\operatorname{cone}(f)$ is not acyclic, and there is some integer i so that

$$H^i(\operatorname{cone}(f)) \neq 0.$$

Let $\alpha'' : H^i(\operatorname{cone}(f)) \to I^0$ be an inclusion into an injective object (which exists since $\mathbb{A}$ has enough injectives), $\alpha' : Z^0(\operatorname{cone}(f)) \to I^0$ be the restriction along the projection from the cocycles, and $\alpha : (\operatorname{cone}(f))^i \to I^0$ be an arbitrary lift to degree i cochains. We note that such a lift exists via injectivity of I^0.

Take now $I = \Sigma^{-i} I^0$, considered as a complex. The map α now defines a map of cochains

$$\alpha : \operatorname{cone}(f) \to I$$

which recovers α'' on cohomology. In particular, α is not homotopically trivial, and hence realizes a nonzero class in cohomology

$$\bar{\alpha} \in H^0 \operatorname{Hom}^*_{\mathbb{A}}(\operatorname{cone}(f), I) \cong H^0\left(\Sigma^{-1}\operatorname{cone}(f^*)\right).$$

It follows that the induced map $f^* : \operatorname{Hom}^*_{\mathbb{A}}(X,I) \to \operatorname{Hom}^*_{\mathbb{A}}(V,I)$ is not a quasi-isomorphism.

We have in particular

$$\operatorname{gr} H^0\left(\Sigma^{-1}\operatorname{cone}(f^*)\right) = \begin{array}{c} \ker\left(H^0\operatorname{Hom}^*_{\mathbb{A}}(X,I) \to H^0\operatorname{Hom}^*_{\mathbb{A}}(V,I)\right) \\ \oplus \\ \operatorname{coker}\left(H^{-1}\operatorname{Hom}^*_{\mathbb{A}}(X,I) \to H^{-1}\operatorname{Hom}^*_{\mathbb{A}}(V,I)\right) \end{array}$$

under the apparent filtation on the mapping cone so that the above arguments show that at least one of the maps

$$H^\varepsilon(f^*) : H^\varepsilon\operatorname{Hom}^*_{\mathbb{A}}(X,I) \to H^\varepsilon\operatorname{Hom}^*_{\mathbb{A}}(V,I)$$

at $\varepsilon = 0, -1$, is not an isomorphism. It follows that the induced map of simplicial abelian groups

$$f^* : \operatorname{Hom}_{\mathscr{K}(\mathbb{A})}(X,I) \cong K\operatorname{Hom}^*_{\mathbb{A}}(X,I) \to K\operatorname{Hom}^*_{\mathbb{A}}(V,I) \cong \operatorname{Hom}_{\mathscr{K}(\mathbb{A})}(V,I)$$

is not a isomorphism in $\mathrm{h}\mathscr{K}an$ by Theorem I-10.13. Since the complex I is K-injective, this contradicts the assumption that f is a $\mathscr{D}_{\mathrm{Inj}}$-reflection, and we conclude that reflection-ness of f forces f to be a quasi-isomorphism. $\square$

Completely similar arguments apply in the projective situation.

lem:projres_corefl

Lemma 4.3. *Suppose that $\mathbb{A}$ has enough projectives. Then for a morphism $g : X \to V$ in $\mathscr{K}(\mathbb{A})$ the following are equivalent:*

(a) *The object X is K-projective and g is a quasi-isomorphism.*
(b) *The morphism g is a $\mathscr{D}_{\mathrm{Proj}}$-coreflection.*

4.3. **Precomposition and natural isomorphisms in functor categories.**

lem:1696

Lemma 4.4. *Let $\zeta : F_0 \to F_1$ be a natural transformation between functors $F_i : \mathscr{K} \to \mathscr{K}'$.*

(1) *For each ∞-category $\mathscr{C}$ the functors ζ induces a natural transformation $\zeta^* : F_0^* \to F_1^*$ between the corresponding functors $F_i^* : \mathrm{Fun}(\mathscr{K}', \mathscr{C}) \to \mathrm{Fun}(\mathscr{K}, \mathscr{C})$.*

(2) *If ζ is an isomorphism, then ζ^* is an isomorphism as well.*

Construction 4.4. The transformation ζ is a 2-simplex $\zeta : \Delta^1 \to \mathrm{Fun}(\mathscr{K}, \mathscr{K}')$ which restricts to F_i at $\{i\}$, for $i = 0, 1$. So composition in the simplicial category $\underline{\mathrm{Cat}}_\infty$ provides us with a map

$$\zeta^* : \mathrm{Fun}(\mathscr{K}', \mathscr{C}) \times \Delta^1 \to \mathrm{Fun}(\mathscr{K}', \mathscr{C}) \times \mathrm{Fun}(\mathscr{K}, \mathscr{K}') \xrightarrow{\circ} \mathrm{Fun}(\mathscr{K}, \mathscr{C})$$

whose restrictions to $\{i\} \subseteq \Delta^1$ recover the maps F_i^*. So ζ^* is a transformation $\zeta^* : F_0^* \to F_1^*$.

Similarly, if we have an n-simplex $\sigma : \Delta^n \to \mathrm{Fun}(\mathscr{K}, \mathscr{K}')$ with vertices $G_i : \mathscr{K} \to \mathscr{K}'$ we get an n-simplex $\sigma^* : \mathrm{Fun}(\mathscr{K}', \mathscr{C}) \times \Delta^n \to \mathrm{Fun}(\mathscr{K}, \mathscr{C})$ with vertices G_i^*, and one sees that σ^* is degenerate whenever σ is degenerate. Hence diagrams of the form

$$\begin{tikzcd} & F_1 \arrow[dr, "\eta"] & \\ F_0 \arrow[ur, "\zeta"] \arrow[rr, "id_{F_0}"'] & & F_0 \end{tikzcd} \qquad \begin{tikzcd} & F_0 \arrow[dr, "\zeta"] & \\ F_1 \arrow[ur, "\eta"] \arrow[rr, "id_{F_1}"'] & & F_1 \end{tikzcd}$$

in $\mathrm{Fun}(\mathscr{K}, \mathscr{K}')$ imply diagrams of the form

$$\begin{tikzcd} & F_1^* \arrow[dr, "\eta^*"] & \\ F_0^* \arrow[ur, "\zeta^*"] \arrow[rr, "id_{F_0^*}"'] & & F_0^* \end{tikzcd} \qquad \begin{tikzcd} & F_0^* \arrow[dr, "\zeta^*"] & \\ F_1^* \arrow[ur, "\eta^*"] \arrow[rr, "id_{F_1^*}"'] & & F_1^* \end{tikzcd}$$

in $\mathrm{Fun}(\mathrm{Fun}(\mathscr{K}', \mathscr{C}), \mathrm{Fun}(\mathscr{K}, \mathscr{C}))$. So we see directly that ζ^* is isomorphism whenever ζ is an isomorphism. $\square$

4.4. **Localizating the homotopy ∞-category against quasi-isomorphisms.**

prop:pre_loc

Proposition 4.5. *Let $L : \mathscr{K}(\mathbb{A}) \to \mathscr{D}_{\mathrm{Inj}}$ be the left adjoint to the inclusion $i : \mathscr{D}_{\mathrm{Inj}} \to \mathscr{K}(\mathbb{A})$, along with the unit and counit transformations*

$$u : id_{\mathscr{K}(\mathbb{A})} \to iL \quad \text{and} \quad \epsilon : Li \to id_{\mathscr{D}_{\mathrm{Inj}}}$$

as in Proposition 3.10.

(1) *At each x in $\mathscr{K}(\mathbb{A})$ the unit transformation $u_x : x \to L(x)$ is a quasi-isomorphism.*

(2) *A map $\alpha : x \to y$ in $\mathscr{K}(\mathbb{A})$ is a quasi-isomorphism if and only if $L(\alpha) : L(x) \to L(y)$ is an isomoprhism in $\mathscr{D}_{\mathrm{Inj}}$.*

(3) *The counit transformation is a natural isomoprhism from the composite*

$$\mathscr{D}_{\mathrm{Inj}} \xrightarrow{i} \mathscr{K}(\mathbb{A}) \xrightarrow{L} \mathscr{D}_{\mathrm{Inj}}$$

to the identity $id_{\mathscr{D}_{\text{Inj}}}$*, and so defines a* 2*-simplex in the mapping complex* $\epsilon : \Delta^2 \to \mathrm{Fun}(\mathscr{D}_{\text{Inj}}, \mathscr{D}_{\text{Inj}})^{\text{Kan}}$.

Proof. Statement (1) follows from the characterization of $\mathscr{D}_{\text{Inj}}$-reflections provided in Lemma 4.2 and Theorem 3.11. For (2), naturality of u implies, for each morphism $\alpha : x \to y$, the existence of a diagram

$$\begin{tikzcd} L(x) \arrow[r, "L(\alpha)"] & L(y) \\ x \arrow[u, "u_x"] \arrow[r, "\alpha"'] & y \arrow[u, "u_y"'] \end{tikzcd}$$

in the discrete homotopy category $K(\mathbb{A}) = \mathrm{h}\mathscr{K}(\mathbb{A})$. The vertical maps in this diagram are quasi-isomorpihsms by (1), so that α is a quasi-isomorphism if and only if $L(\alpha)$ is a quasi-isomorphism. However, a map between K-injectives is a quasi-isomorphism if and only if it is a homotopy equivalence, i.e. an isomoprhism in $\mathscr{D}_{\text{Inj}}$. So we conclude that α is a quasi-isomorphism if and only if $L(\alpha)$ is an isomorphism in $\mathscr{D}_{\text{Inj}}$. Statement (3) is implied directly by the generic description of ϵ given in Proposition 3.10 $\square$

Let us recall that the localization $\mathscr{C}[W^{-1}]$ of an ∞-category at a class of morphisms $W \subseteq \mathscr{C}[1]$, with all degenerate 1-simplices in W, is any ∞-category $\mathscr{D}$ equipped with a functor $F : \mathscr{C} \to \mathscr{D}$ which induces, at all ∞-categories $\mathscr{E}$, a fully faithful functor

$$F^* : \mathrm{Fun}(\mathscr{D}, \mathscr{E}) \to \mathrm{Fun}(\mathscr{C}, \mathscr{E})$$

whose image is the full ∞-subcategory spanned by all functors $\mathscr{C} \to \mathscr{E}$ which sends all maps in W to isomorphisms in $\mathscr{E}$ (Definition II-14.18). In this case we write, somewhat ambiguously, $\mathscr{D} = \mathscr{C}[W^{-1}]$.

Remark 4.6. One can think of localization fairly clearly from the perspective of marked simplicial sets. A functor $F : \mathscr{C} \to \mathscr{D}$ is a localization of $\mathscr{C}$ relative to a class of maps W if and only if it induces an equivalence between the category of marked functors

$$\mathrm{Fun}(\mathscr{C}, \mathscr{D}) = \mathrm{Fun}^\flat((\mathscr{C}, \mathrm{Isom}), (\mathscr{D}, \mathrm{Isom})),$$

at general $\mathscr{E}$, and the category of marked functors $\mathrm{Fun}^\flat((\mathscr{C}, W), (\mathscr{E}, \mathrm{Isom}))$ [3, Section 3.1.3].

thm:DInj_as_loc

Theorem 4.7. *Let* $\mathbb{A}$ *be a Grothendieck abelian category.*

(1) *Given any* ∞*-category* $\mathscr{E}$*, restriction along the left adjoint* $L : \mathscr{K}(\mathbb{A}) \to \mathscr{D}_{\text{Inj}}$ *to the inclusion provides a fully faithful functor*

$$L^* : \mathrm{Fun}(\mathscr{D}_{\text{Inj}}, \mathscr{E}) \to \mathrm{Fun}(\mathscr{K}(\mathbb{A}), \mathscr{E})$$

which is an equivalence onto the full subcategory spanned by fnuctors which sends all quasi-isomoprihsms in $\mathscr{K}(\mathbb{A})$ *to isomoprhisms in* $\mathscr{E}$.

(2) *For* $\mathrm{Fun}(\mathscr{K}(\mathbb{A}), \mathscr{E})^{\text{Qiso}}$ *the full subcategory spanned by all functors which send quasi-isomorphisms to isomorphisms in* $\mathscr{E}$*, the inverse to the equivalence* L^* *is provided by restiction* $i^* : \mathrm{Fun}(\mathscr{K}(\mathbb{A}), \mathscr{E})^{\text{Qiso}} \to \mathrm{Fun}(\mathscr{D}_{\text{inj}}, \mathscr{E})$ *along the inclusion* $i : \mathscr{D}_{\text{Inj}} \to \mathscr{K}(\mathbb{A})$.

(3) *the functor* $L : \mathscr{K}(\mathbb{A}) \to \mathscr{D}_{\text{Inj}}$ *exhibits* $\mathscr{D}_{\text{Inj}}$ *as a localization* $\mathscr{D}_{\text{Inj}} = \mathscr{K}(\mathbb{A})[\mathrm{Qiso}^{-1}]$.

Proof. (3) Follows from (1), simply by the definition of a localization. We prove (1) and (2). Take $\mathscr{K} = \mathscr{K}(\mathbb{A})$ and $\mathscr{D} = \mathscr{D}_{\mathrm{Inj}}$, and let $\mathscr{E}$ be arbitrary. Take $\mathrm{Fun}(\mathscr{K},\mathscr{E})^{\mathrm{Qiso}}$ the full ∞-subcategory of functors $T : \mathscr{K} \to \mathscr{E}$ which send quasi-isomoprhisms in $\mathscr{K}$ to isomorpsms in $\mathscr{E}$. Then, by Proposition 4.5 (2), the functor L^* has image in $\mathrm{Fun}(\mathscr{K},\mathscr{E})^{\mathrm{Qiso}}$ and so restricts to a map

$$L^* : \mathrm{Fun}(\mathscr{D},\mathscr{E}) \to \mathrm{Fun}(\mathscr{K},\mathscr{E})^{\mathrm{Qiso}}.$$

We have the functor $i^* : \mathrm{Fun}(\mathscr{K},\mathscr{E})^{\mathrm{Qiso}} \to \mathrm{Fun}(\mathscr{D},\mathscr{E})$ provided by restricting along the inclusion $i^* : \mathscr{D} \to \mathscr{K}$. We claim that these functors are mutually inverse, and so realize the claimed equivalence. More prescisely, we claim that the counit and unit transformations ϵ and u induce isomorphisms

$$\epsilon^* : i^*L^* \to id_{\mathrm{Fun}(\mathscr{D},\mathscr{E})} \quad \text{and} \quad u^* : id_{\mathrm{Fun}(\mathscr{K},\mathscr{E})^{\mathrm{Qiso}}} \to L^*i^*.$$

The fact that ϵ^* is an isomorphism just follows from the fact that ϵ itself is a isomorphism. See Proposition 4.5 and Lemma 4.4. So we need only address the transformation u^*.

First note that

$$u^* : \mathrm{Fun}(\mathscr{K},\mathscr{E}) \times \Delta^1 \to \mathrm{Fun}(\mathscr{K},\mathscr{E})$$

sends each object in the subcategory $\mathrm{Fun}(\mathscr{K},\mathscr{E})^{\mathrm{Qiso}} \times \Delta^1$ to an object in $\mathrm{Fun}(\mathscr{K},\mathscr{E})^{\mathrm{Qiso}}$, since u^* is a transformation between $id_{\mathrm{Fun}(\mathscr{K},\mathscr{E})}$ and L^*i^* and these endofunctors preserve the subcategory $\mathrm{Fun}(\mathscr{K},\mathscr{E})^{\mathrm{Qiso}}$. Since $\mathrm{Fun}(\mathscr{K},\mathscr{E})^{\mathrm{Qiso}}$ is full in $\mathrm{Fun}(\mathscr{K},\mathscr{E})$ it follows that u^* does in fact restrict to a transformation

$$u^* : \mathrm{Fun}(\mathscr{K},\mathscr{E})^{\mathrm{Qiso}} \times \Delta^1 \to \mathrm{Fun}(\mathscr{K},\mathscr{E})^{\mathrm{Qiso}}$$

between the identity and L^*i^*.

We need to show that u^* is a natural isomorphism. By Proposition I-6.8 it suffices to show that u^* evaluates to an isomorphism in $\mathrm{Fun}(\mathscr{K},\mathscr{E})^{\mathrm{Qiso}}$ at each functor $T : \mathscr{K} \to \mathscr{E}$ in $\mathrm{Fun}(\mathscr{K},\mathscr{E})^{\mathrm{Qiso}}$. By the definition of u^* from Construction 4.4 we have

$$u^*_T : \Delta^1 \overset{u}{\to} \mathrm{Fun}(\mathscr{K},\mathscr{K}) \overset{T_*}{\to} \mathrm{Fun}(\mathscr{K},\mathscr{E})^{\mathrm{Qiso}} \subseteq \mathrm{Fun}(\mathscr{K},\mathscr{E}),$$

and to see that u^*_T is an isomoprhism it again suffices to show that u^*_T evaluates to an isomorphism at each x in $\mathscr{K}$. At any such x we have

$$(u^*_T)_x = T(u_x) : T(x) \to TL(x).$$

By Proposition 4.5 (1) each map u_x is a quasi-isomorphism, and since T sends quasi-isomorphisms to isomorphism we have that $(u^*_T)_x$ is an isomoprhism in $\mathscr{E}$, as desired. So we see that u^* itself is a natural isomorpihsm.

We now have natural isomorphisms $\epsilon^* : i^*L^* \to id_{\mathrm{Fun}(\mathscr{D},\mathscr{E})}$ and $(u^*)^{-1} : L^*i^* \to id_{\mathrm{Fun}(\mathscr{K},\mathscr{E})^{\mathrm{Qiso}}}$. By the definition of $\mathscr{C}at_\infty$ as the homotopy coherent nerve of the simplicial category $\underline{\mathrm{Cat}}^+_\infty$, these natural isomorphsms provide 2-simplices

$$\begin{tikzcd}
& \mathrm{Fun}(\mathscr{K},\mathscr{E})^{\mathrm{Qiso}} \arrow[dr, "i^*"] & \\
\mathrm{Fun}(\mathscr{D},\mathscr{E}) \arrow[ur, "L^*"] \arrow[rr, "id_{\mathrm{Fun}(\mathscr{K},\mathscr{D})}"'] & & \mathrm{Fun}(\mathscr{D},\mathscr{E})
\end{tikzcd}$$

and

$$\begin{tikzcd} & \mathrm{Fun}(\mathscr{D},\mathscr{E}) \arrow[dr, "L^*"] & \\ \mathrm{Fun}(\mathscr{K},\mathscr{E})^{\mathrm{Qiso}} \arrow[ur, "i^*"] \arrow[rr, "id_{\mathrm{Fun}(\mathscr{K},\mathscr{E})^{\mathrm{Qiso}}}"'] & & \mathrm{Fun}(\mathscr{K},\mathscr{E})^{\mathrm{Qiso}} \end{tikzcd}$$

in $\mathscr{C}at_\infty$ which realize L^* and i^* as mutually inverse. □

In the event that $\mathbb{A}$ has enough projectives, we can consider the right adjoint $R : \mathscr{K}(\mathbb{A}) \to \mathscr{D}_{\mathrm{Proj}}$ along with its unit and counit transformations $u : id_{\mathscr{D}_{\mathrm{Proj}}} \to Ri$ and $\epsilon : iR \to id_{\mathscr{K}(\mathbb{A})}$. We have that u is a natural isomorphism, that ϵ evaluates to a quasi-isomorphism $\epsilon_x : R(x) \to x$ at each x in $\mathscr{K}(\mathbb{A})$, and that a map $\alpha : x \to y$ in $\mathscr{K}(\mathbb{A})$ is a quasi-isomorphism if and only if $R(\alpha) : R(x) \to R(y)$ is an isomorphism. To observe these properties one argues exactly as in the proof of Proposition 4.5. We can therefore argue as in the proof of Theorem 4.7 to realize the projective construction of the derived ∞-category as a localization.

thm:DProj_as_loc **Theorem 4.8.** *Let $\mathbb{A}$ be a Grothendieck abelian category, and suppose that $\mathbb{A}$ has enough projectives.*

(1) *For any ∞-category $\mathscr{E}$, restriction along the right adjoint $R : \mathscr{K}(\mathbb{A}) \to \mathscr{D}_{\mathrm{Proj}}$ to the inclusion provides a fully faithful functor*
$$R^* : \mathrm{Fun}(\mathscr{D}_{\mathrm{Proj}},\mathscr{E}) \to \mathrm{Fun}(\mathscr{K}(\mathbb{A}),\mathscr{E})$$
which is an equivalence onto the full ∞-subcategory spanned by those functors $T : \mathscr{K}(\mathbb{A}) \to \mathscr{E}$ which send quasi-isomorphisms in $\mathscr{K}(\mathbb{A})$ to isomorphisms in $\mathscr{E}$.

(2) *For $\mathrm{Fun}(\mathscr{K}(\mathbb{A}),\mathscr{E})^{\mathrm{Qiso}}$ the full subcategory spanned by all functors which send quasi-isomorphisms to isomorphisms in $\mathscr{E}$, the inverse to R^* is given by restriction $i^* : \mathrm{Fun}(\mathscr{K}(\mathbb{A}),\mathscr{E})^{\mathrm{Qiso}} \to \mathrm{Fun}(\mathscr{D}_{\mathrm{Proj}},\mathscr{E})$ along the inclusion $i : \mathscr{D}_{\mathrm{Proj}} \to \mathscr{K}(\mathbb{A})$.*

(3) *The functor $R : \mathscr{K}(\mathbb{A}) \to \mathscr{D}_{\mathrm{Proj}}$ realizes $\mathscr{D}_{\mathrm{Proj}}$ as a localization $\mathscr{D}_{\mathrm{Proj}} = \mathscr{K}(\mathbb{A})[\mathrm{Qiso}^{-1}]$.*

As in the case of $\mathscr{D}_{\mathrm{Inj}}$, one sees that the inverse to R^* is given by restricting along the inclusion $\mathscr{D}_{\mathrm{Proj}} \to \mathscr{K}(\mathbb{A})$.

cor:1844 **Corollary 4.9.** *For any Grothendieck abelian category $\mathbb{A}$ which has enough projectives, there is a unique equivalence $\psi : \mathscr{D}_{\mathrm{Inj}} \xrightarrow{\sim} \mathscr{D}_{\mathrm{Proj}}$ which fits into a diagram*

$$\begin{tikzcd} & \mathscr{K}(\mathbb{A}) \arrow[dl, "L^*"'] \arrow[dr, "R^*"] & \\ \mathscr{D}_{\mathrm{Inj}} \arrow[rr, "\sim", "\psi"'] & & \mathscr{D}_{\mathrm{Proj}} \end{tikzcd}$$

in $\mathscr{C}at_\infty$.

We leave the following exercise to the interested reader.

Exercise 4.10. Provie that the equivalence $\psi : \mathscr{D}_{\mathrm{Inj}} \to \mathscr{D}_{\mathrm{Proj}}$ is precisely the equivalence realized previously in Section I-12. (Here of course we accept that the functor ψ is only defined up to equivalence.)

4.5. **Redefinining the derived ∞-category.**

Definition 4.11. Given a Grothendieck abelian category $\mathbb{A}$, the derived ∞-category $\mathscr{D}(\mathbb{A})$ is the localization of the homotopy ∞-category relaztive to the class of quasi-isomorphisms

$$\mathscr{D}(\mathbb{A}) := \mathscr{K}(\mathbb{A})[\mathrm{Qiso}^{-1}].$$

Theorems 4.7 and 4.8 say that we can construct the derived ∞-category via K-injectives in $\mathrm{Ch}(\mathbb{A})$, or via K-projectives when they exist. Up to equivalence, and relative specifically to the universal property of localization, it's all the same.

4.6. **Localization for small derived categories.**

Theorem 4.12. *Let $\mathscr{K}' \subseteq \mathscr{K}(\mathbb{A})$ be a full stable subcategory, $\mathrm{K}' = \mathrm{h}\,\mathscr{K}'$, and suppose that the induced map on Veridier localizations*

$$\mathrm{K}' / \mathrm{Acyc} \to \mathrm{K}(\mathbb{A}) / \mathrm{Acyc} = \mathrm{D}(\mathbb{A})$$

is fully faithful. Then for $\mathscr{D}' \subseteq \mathscr{D}(\mathbb{A})$ the full subcategory spanned by the image of $\mathscr{K}'$ under the localization map, the induced functor

$$\mathscr{K}'[\mathrm{Qiso}^{-1}] \to \mathscr{D}'$$

is an equivalence.

5. Derived functors

5.1. **Indification and renormalized derived categories.**

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