# The Casimir and Semisimplicity.

**Lemma 5.2:** For semisimple $\mathfrak{g}$, $[\mathfrak{g},\mathfrak{g}]=\mathfrak{g}$.

**Proof:** In the simple case we have that $[\mathfrak{g},\mathfrak{g}]$ is a nonzero ideal in $\mathfrak{g}$, and hence all of $\mathfrak{g}$. Via the decomp into a product of simples, Thm 5.2, we have an eq $[\mathfrak{g},\mathfrak{g}]=\mathfrak{g}$ at all semisimple $\mathfrak{g}$ as well. ∎

**Corollary [6.3]:** For semisimple $\mathfrak{g}$, and any $\mathfrak{g}$-rep $V$, the action map $\rho_V:\mathfrak{g}\to\mathfrak{gl}(V)$ has image in $\mathfrak{sl}(V)$.

**Proof:** We have

$$\rho(\mathfrak{g})=\rho([\mathfrak{g},\mathfrak{g}])=[\rho(\mathfrak{g}),\rho(\mathfrak{g})]\subseteq\mathfrak{sl}(V).$$ ∎

**Corollary:** For semisimple $\mathfrak{g}$, the only 1-dimensional $\mathfrak{g}$-rep $V$ is the trivial rep

$$V\cong\mathbb{C}.$$

## – Casimir elements

Consider any semisimple $\mathfrak{g}$-rep $V$ we have corresp action

$$\rho:\mathfrak{g}\to\mathfrak{gl}(V)$$

and corresponding "Relative Killing form"

$$\kappa_V(x,y):=\mathrm{Tr}_V(\rho(x)\rho(y)).$$

**Lemma 6.2A:** If $V$ is a faithful $\mathfrak{g}$-rep then $\kappa_V$ is a nondegenerate $\mathfrak{g}$-inv. form on $\mathfrak{g}$.

**Proof:** As in the proof of Thm 5.1 we find $\mathrm{rad}(\kappa_V)\subseteq\mathrm{rad}(\mathfrak{g})=0$, via Cartan criterion. ∎

**Def$^n$:** For any faithful $\mathfrak{g}$-rep $V$ define the relative Casimir element $\Omega^V$ as follows:

Choose dual bases $\{x_i\}_{i=1}^n$ and $\{x^i\}$ for $\mathfrak{g}$, so that $\kappa_V(x_i, x^j) = \delta_{ij}$, and define

$$\Omega^V := \sum_i x_i \otimes x^i \in \mathfrak{g} \otimes \mathfrak{g}$$

Lemma 6.2 B: For semisimple $\mathfrak{g}$ and faithful $V$, the following hold:

(a) $\Omega^V$ is indep of the choice of basis $\{x_i, \ldots, x_n\}$ for $\mathfrak{g}$.

(b) $\Omega^V$ is $\mathfrak{g}$-invariant.

(c) $\mathrm{Tr}(\Omega^V : V \to V) = \dim(\mathfrak{g})$.

(d) If $V$ is simple then $\Omega^V$ acts as the scalar $\dim(\mathfrak{g}) / \dim V$

Proof: (a) + (b) Via non-degeneracy the form $\kappa_V$ defines an iso $f_V : \mathfrak{g} \to \mathfrak{g}^*$, $x \mapsto \kappa_V(x, -)$, and we obtain a form $\kappa^*_V$ on $\mathfrak{g}^*$ and bases

$$\{\xi_i := \kappa_V(x_i, -) : 1 \le i \le n\}$$

and $\{\xi^i = \kappa_V(x^i, -) : 1 \le i \le n\}$

so that $\kappa^*_V(\xi_i, \xi^j) = \delta_{ij}$. [redacted] Pictorially

$$\begin{tikzcd} \mathfrak{g} \otimes \mathfrak{g} \arrow[rr, "f_V \otimes f_V"] \arrow[dr, "\kappa"'] & & \mathfrak{g}^* \otimes \mathfrak{g}^* \arrow[dl, "\kappa^*"] \\ & \mathbb{C} & \end{tikzcd}$$

Via $\mathfrak{g}$ invariance of $\kappa_V$, $\kappa^*_V$ is $\mathfrak{g}$ inv as well, and we obtain an elem

$$\Omega \in (\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{g}} \underset{ev}{\overset{\cong}{\to}} \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}^* \otimes \mathfrak{g}^*, \mathbb{C})$$

w/ $ev(\Omega) = \kappa^*_V$. We now check

$$ev\left(\sum_i x_i \otimes x^i\right)$$
$$= \left(\sum_i c_{ij}\, \xi_i \otimes \xi^j \mapsto \sum_i c_{ii}\right)$$
$$= \kappa^*_V$$

To calculate $\Omega = \sum_i x_i \otimes x^i = \Omega^V$.

Since the construction of $\Omega$ is basis indep, (5)

$\Omega_V$ is basis indep, and since $\Omega$ is $\mathfrak{g}$-invariant

$\Omega_V$ is as well.

(c) $Tr(\Omega_V) = \sum_i Tr(x_i x^i)$

$= \sum_i \kappa_V(x_i, x^i)$

$= \sum_i 1 = \dim \mathfrak{g}$.

(d) Since $\Omega_V$ is $\mathfrak{g}$ invariant & acts on

$V$ via a $\mathfrak{g}$-invariant endo, i.e. via a map

of $\mathfrak{g}$-reps. $V$ is irreducible then

$$\Omega_V = c \cdot \mathrm{id}_V : V \to V,$$

and the trace constraint for (c) gives $c = \frac{\dim(\mathfrak{g})}{\dim(V)}$.

∎

- Aside: $\mathfrak{g}$-invariants and inner Homs

Def$^n$: For each $\mathfrak{g}$-rep $V$ define

$V^{\mathfrak{g}} := \{ v \in V : \mathfrak{g} \cdot v = 0 \}$

$[\cong Hom_{\mathfrak{g}}(\mathbb{C}, V)]$.

Def$^n$ (Inner Hom): For $\mathfrak{g}$-reps $V$ and

$W$, let $Hom_{\mathbb{C}}(V, W)$ = linear maps w/ specified

$\mathfrak{g}$-action
$x \cdot f := (v \mapsto x \cdot f(v) - f(x \cdot v))$

It's easy to check directly that $Hom_{\mathbb{C}}(V, W)$ is

in fact a $\mathfrak{g}$-rep.

Proposition A: a) The linear iso

$W \otimes V^* \to Hom_{\mathbb{C}}(V, W)$, $w \otimes \xi \mapsto (v \mapsto w \cdot \xi(v))$,

is an $\cong$ of $\mathfrak{g}$-reps.

(b) The inclusion $Hom_{\mathfrak{g}}(V, W) \hookrightarrow Hom_{\mathbb{C}}(V, W)$

provides an equality $Hom_{\mathfrak{g}}(V, W) = Hom_{\mathbb{C}}(V, W)^{\mathfrak{g}}$.

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Proof: (a) ~~Easy~~ direct check. (b) We have $f \in \operatorname{Hom}_{\mathbb{C}}(V,W)$ iff at all $v$ and $x$

$$x \cdot f(v) - f(x \cdot v) = 0$$

iff $x \cdot f(v) = f(x \cdot v)$ iff $f \in \operatorname{Hom}_{\mathfrak{g}}(V,W)$. ∎

– Semisimplicity vs semisimplicity

Lemma 6.3A: If $V$ and $V'$ are trivial $\mathfrak{g}$-reps, and $\mathfrak{g}$ is semisimple, then any exact seq

$$0 \to V \to W \to V' \to 0$$

splits. // Trivial $\mathfrak{g}$-rep means $\mathfrak{g}$ acts as $0$.

Proof: In this case $W$ admits a flag

$$W_0 \subseteq \cdots \subseteq W_m = V \subseteq W_{m+1} \subseteq \cdots \subseteq W_n = W$$

on which $\mathfrak{g} \cdot W_l \subseteq W_{l-1}$ at all $l$. Hence under the action map $\rho : \mathfrak{g} \to \mathfrak{gl}(W)$

$\rho(\mathfrak{g}) \subseteq \{$strictly upper $\Delta$ matrices$\}$ is nilpotent, and thus solvable. Since any quotient of a semisimp Lie alg is semisimple (or zero) this forces $\rho(\mathfrak{g}) = 0$, i.e. forces triviality of $W$. Hence the sequence splits. ∎

Lemma 6.3B: If $\mathfrak{g} = \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_t$ for Lie algebras $\mathfrak{g}_i$, and one of the $\mathfrak{g}_i$ is abelian, then $\operatorname{rep}(\mathfrak{g})$ is <u>not</u> semisimple.

Proof: In this case pulling back along the proj $\pi : \mathfrak{g} \to \mathfrak{g}_i$ provides a fully faithful emb.

$$\operatorname{res}_\pi : \operatorname{rep}(\mathfrak{g}_i) \to \operatorname{rep}(\mathfrak{g}).$$

Hence non-semisimplicity of $\mathfrak{g}_i$ implies non-semisimp of $\mathfrak{g}$. So we reduce to the case of abelian $\mathfrak{g}$.

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Here we have
$$\mathrm{rep}(\mathfrak{g}) = \left( \begin{matrix} \{\text{Vector spaces w/ } n \text{ commuting} \\ \text{endos } x_1^V, \ldots, x_n^V : V \to V \} \end{matrix} \right),$$
and we observe the non-split extension of trivial reps
$$W = \left( \begin{matrix} \{\mathbb{C}^2 \text{ w/ } x_i \text{ acting as the} \\ \text{endo } x_i^V = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \} \end{matrix} \right),$$
$$0 \to \mathbb{C} \to W \to \mathbb{C} \to 0$$
$$c \mapsto c \cdot e_1$$
$$c_1 e_1 + c_2 e_2 \mapsto c_2$$
Lem 6.3C $>$ ∎

Theorem 6.3: For a Lie alg $\mathfrak{g}$, the following are equivalent:

(a) $\mathrm{rep}(\mathfrak{g})$ is a semisimple abelian category.

(b) $\mathfrak{g}$ is a semisimple Lie algebra.

Proof: (a) $\Rightarrow$ (b) We decompose the adj rep to obtain a decomp of $\mathfrak{g}$ into a product of simple ideals
$$\mathfrak{g} = \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_t .$$
Each $\mathfrak{g}_i$ is a Lie alg which admits no proper nonzero ideals itself, and hence is simple or abelian. Abelianness cannot occur, by Lemma 6.3 B, so that $\mathfrak{g}$ is a product of simple Lie algs, and thus semisimple.

(b) $\Rightarrow$ (a) We first show that all extensions from the trivial rep split. Consider such an extension
$$0 \to V \to W \to \mathbb{C} \to 0. \quad (*)$$
Supposing that $V$ is simple, if $\mathfrak{g}$ acts trivially on $V$ then $(*)$ splits by Lemma 6.3 A. Supposing $V$ nontrivial, we can replace $\mathfrak{g}$ by $\mathfrak{g}/\mathrm{Ann}_{\mathfrak{g}}(V)$ to assume $V$ is a faithful $\mathfrak{g}$-rep.

(6)

Then we have the relative Casimir $\Omega^W$ w/ $Tr_W(\Omega^W) = \dim(\mathfrak{g})$ and $\Omega^V$ acting as $0$ on $W/V \cong \mathbb{C}$.

Hence $Tr_W(\Omega^W) = Tr_V(\Omega^W)$ and we conclude via simplicity that $\Omega^W$ acts on $V$ (subrep) as the scalar $\dim(\mathfrak{g})/\dim(V)$. Hence

$$V^\perp := \ker(\Omega^W : V \to V)$$

provides a $\mathfrak{g}$-subrep in $W$ w/ a decomp.

$$V \oplus V^\perp \subseteq W.$$

For dim reasons $V \oplus V^\perp = W$ and the seq

$$V^\perp \to W \to \mathbb{C} \text{ is an } \cong \text{ of reps,}$$

from which we obtain the desired splitting

$$\mathbb{C} \to V^\perp \subseteq W.$$

Suppose now that all seq

$$0 \to V' \to W' \to \mathbb{C} \to 0$$

split when $\text{length}(V') < n$ and that $\text{length}(V) = n$. Then for any simple subrep $L \subseteq V$ the seq

$$0 \to V/L \to W/L \to \mathbb{C} \to 0$$

splits via some map $f : \mathbb{C} \to W/L$. Taking

$$W' = \text{preimage of } f(\mathbb{C}) \text{ along } W \to W/L$$

we now have an exact seq

$$0 \to L \to W' \to \mathbb{C} \to 0$$

which we split again to obtain the desired splitting of (*), $\mathbb{C} \to W' \subseteq W$.

The general case now follows by Lemma 6.3C. ∎

Lemma 6.3C: Suppose that $\mathfrak{g}$ satisfies $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$ and that all extensions of the form $0 \to V \to W \to \mathbb{C} \to 0$

split. The $\mathrm{rep}(\mathfrak{g})$ is semisimple.

Proof: Consider any extension

$$0 \to V \xrightarrow{i} W \xrightarrow{\pi} V' \to 0$$

and the corresp exact seq

$$0 \to \mathrm{Hom}_{\mathbb{C}}(V', V) \xrightarrow{i_*} \mathrm{Hom}_{\mathbb{C}}(V', W) \xrightarrow{\pi_*} \mathrm{Hom}_{\mathbb{C}}(V', V') \to 0.$$

Take $\mathcal{W} = \pi_*^{-1}(\mathbb{C} \cdot id_{V'})$

$\mathcal{V} = i_*^{-1}(\mathcal{W})$

to obtain an exact seq

$$0 \to \mathcal{V} \to \mathcal{W} \to \mathbb{C} \to 0.$$

A $\mathfrak{g}$-linear splitting $\bar{f}: \mathbb{C} \to \mathcal{W} \subseteq \mathrm{Hom}_{\mathbb{C}}(V', W)$

then choose a $\mathfrak{g}$-invariant element

$$f = \bar{f}(1) \in \mathrm{Hom}_{\mathfrak{g}}(V', W)$$

with $\pi f = \pi_*(f) = id_{V'}$. This $f: V' \to W$

provides the requisite splitting. ∎

Corollary: $\mathfrak{sl}_n(\mathbb{C})$ has semisimple representation theory

– Aside: Representation Theory under products

Theorem: For any product of semisimple Lie algebras

$\mathfrak{g} = \mathfrak{g}_1 \times \mathfrak{g}_2$, and simple reps $L_1$ and $L_2$ over

$\mathfrak{g}$ and $\mathfrak{g}$ respectively, the tensor product

$$L_1 \otimes L_2 = \begin{cases} L_1 \otimes_{\mathbb{C}} L_2 \text{ w/ } \mathfrak{g}\text{-action} \\ (x + x)\cdot(v \otimes w) = (x \cdot v) \otimes v \\ \qquad + v \otimes x \cdot v \end{cases}$$

is simple over $\mathfrak{g}$.

Furthermore, any simple $L$ over $\mathfrak{g}$ decompose

into such a product $L \cong L_1 \otimes L_2$ for unique

simple $\mathfrak{g}_i$-repr $L_i$.

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Proof: Exercise. ∎

Example: For $\mathfrak{g} = \overbrace{\mathfrak{sl}_n \times \cdots \times \mathfrak{sl}_n}^{m\text{-fold}}$,

$\mathfrak{h} = \mathfrak{h}_1 \times \cdots \times \mathfrak{h}_m$, $\mathfrak{h}_i =$ diag's in $i$th factor of $\mathfrak{sl}_n$

$\mathfrak{n} = \mathfrak{n}_1 \times \cdots \times \mathfrak{n}_m$ each simple $\mathfrak{g}$-rep decomp into weights $V = \bigoplus_{\lambda \in P} V_\lambda$,

$P = P_1 \oplus \cdots \oplus P_m \subseteq \mathfrak{h}_1^* \oplus \cdots \oplus \mathfrak{h}_m^* = \mathfrak{h}^*$.

We then have dominant weights

$P^+ = P_1^+ + \cdots + P_m^+$.

Thm: For $\mathfrak{g} = \mathfrak{sl}_n \times \cdots \times \mathfrak{sl}_n$, simple reps are described by their highest wts,

$$P^+ \to \operatorname{Irrep}(\mathfrak{g}), \quad \lambda \mapsto L(\lambda).$$

Further, for each $\lambda \in P^+ = P_1^+ + \cdots + P_m^+$,

$\lambda = \lambda_1 + \cdots + \lambda_m$, we

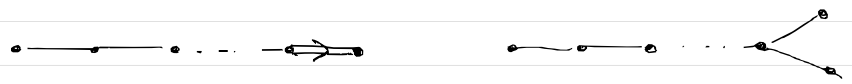
$$L(\lambda) = L(\lambda_1) \otimes \cdots \otimes L(\lambda_m).$$

– What are the simple Lie alg's

Classics: • Type A $\mathfrak{sl}_n(\mathbb{C}) = \operatorname{Lie} SL_n(\mathbb{C})$



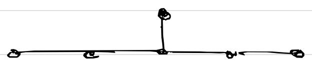
• Type B + D: $\mathfrak{so}_n(\mathbb{C}) = \operatorname{Lie} SO_n(\mathbb{C})$



• Type C: $\mathfrak{sp}_{2n}(\mathbb{C}) = \operatorname{Lie} Sp_{2n}(\mathbb{C})$.

DLC:

$F_4$  $G_2$ 

$E_6$ 

$E_7$ 

$E_8$

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