$$\mathfrak{sl}_2^* = \begin{matrix} f^* \\ h^* \\ e^* \end{matrix}$$

$$\mathfrak{sl}_2^* \xrightarrow{\cong} \mathfrak{sl}_2, \qquad \begin{aligned} f^* &\longmapsto e \\ h^* &\longmapsto \tfrac{1}{2}h \\ e^* &\longmapsto f \end{aligned}$$

$$\Rightarrow \qquad \begin{aligned} Tr(e,f) &= 1 \\ Tr(h,h) &= 2 \end{aligned}$$

$$\begin{tikzcd} \mathfrak{sl}_2 \otimes \mathfrak{sl}_2 \arrow[dr, "\text{trace form}"] & \\ & \mathbb{C} \\ \mathfrak{sl}_2^* \otimes \mathfrak{sl}_2^* \arrow[uu, "\cong"] \arrow[ur, "\kappa"'] & \end{tikzcd}$$

$\kappa(e^*, f^*) = \kappa(f^*, e^*) = 1$

$\kappa(h^*, h^*) = \frac{1}{2}$ vanishes elsewhere

0 on all other basis pairs.

Have $ev : \mathfrak{sl}_2 \otimes \mathfrak{sl}_2 \xrightarrow{\cong} Hom_{\mathbb{C}}(\mathfrak{sl}_2^* \otimes \mathfrak{sl}_2^*, \mathbb{C})$

$$\Rightarrow \qquad \Omega \longleftarrow\!\shortmid \kappa$$

$$\Omega = \tfrac{1}{2}h^2 + ef + fe.$$

We now understand how $\mathfrak{sl}_2$-reps behave, essentially completely: i.e. all $\mathfrak{sl}_2$-reps V decomp. into eigenspaces for the action of $h \in \mathfrak{sl}_2(\mathbb{C})$

$$V = \bigoplus_{\mu \in \mathbb{Z}} V_\mu ,$$

are semisimple, and each simple $L(\lambda)$ is precisely structured according to some rules.

- The structure of $\mathfrak{sl}_n$ I

For $\mathfrak{sl}_2$ we have the decomp $\mathfrak{sl}_2 = \mathbb{C}f \oplus \mathbb{C}h \oplus \mathbb{C}e$ which we exploit repeatedly to assess the structure of $\mathfrak{sl}_2$-reps.

For $\mathfrak{sl}_n$ we have

$$\mathfrak{sl}_n = \underbrace{\mathrm{span}\{E_{ii} - E_{i+1,i+1} : 1 \le i < n\}}_{\text{diagonal stuff } =: \mathfrak{h}} \oplus \underbrace{\mathrm{span}\{E_{ij} : i > j\}}_{\text{strict lower } \Delta\text{'r} \; =: \mathfrak{n}^-} \oplus \underbrace{\mathrm{span}\{E_{ij} : i < j\}}_{\text{strict upper } \Delta\text{'r} \; =: \mathfrak{n}^+}$$

$$\mathfrak{sl}_n(\mathbb{C}) = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+ .$$

Lemma 1: The bracket on $\mathfrak{sl}_n \subseteq \mathfrak{gl}_n$ is given as

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{il} E_{kj} .$$

From this one sees directly that

Lemma 2: Each subspace $\mathfrak{n}^\pm, \mathfrak{h} \subseteq \mathfrak{sl}_n$ is a Lie subalg, and $\mathfrak{h}$ is abelian (has vanishing bracket).

Lemma 3: The Lie alg $\mathfrak{sl}_n$ is generated by the vectors $e_i = E_{i,i+1}$, $f_i = E_{i+1,i}$ [and $h_i = E_{ii} - E_{i+1,i+1}$]. Also the only subalg which cont. these elem is all of $\mathfrak{sl}_n(\mathbb{C})$ itself.

The subalg $\mathfrak{h}$ in $\mathfrak{sl}_n$ is called the (standard) Cartan subalg.
The subalg's $\mathfrak{n}^\pm$ are the pos. and neg. nilpotent sub lgs.

The subalgs $\mathfrak{b}^{\pm} := \mathfrak{h} \oplus \mathfrak{n}^{\pm}$ are called (2)
the pos. and neg. Borel subalgs.

- The structure of $\mathfrak{sl}_n$ II.

Def[n]: For any $\mathfrak{sl}_n$-rep V, a nonzero vector $v \in V$ is said to be an eigenvector for the action of the Cartan $\mathfrak{h} \subseteq \mathfrak{sl}_n$ if for each $x \in \mathfrak{h}$, $x \cdot v \in \mathbb{C} \cdot v$.

The corresp. eigenfunction $\lambda \in \mathfrak{h}^*$ is the unique linear fun which satisfies $x \cdot v = \lambda(x) \cdot v$ at all x in $\mathfrak{h}$. Given $\lambda \in \mathfrak{h}^*$ we let V_λ denote the corresponding weight space/eigenspace. $V_\lambda = \{w \in V : x \cdot w = \lambda(x) \cdot w$ at all $x \in \mathfrak{h}\}$

Lemma 4: Each vector $E_{ij} \in \mathfrak{sl}_n(\mathbb{C})$, $i \neq j$ is an eigenvector for the adj action of $\mathfrak{h}$ assoc. to nonzero eigenfunction $\gamma : \mathfrak{h} \to \mathbb{C}$. Further, if $v \in \mathfrak{sl}_n(\mathbb{C})$ is an eigenvector for the Cartan then $v \in \mathfrak{h}$ or $v \in \mathbb{C} \cdot E_{ij}$ for unique i, j.

Proof: The eigenvector claim is clear since for any diag matrix D, $D \cdot E_{ij}, E_{ij} \cdot D \subseteq \mathbb{C} \cdot E_{ij}$.
For the uniqueness claim consider E_{ij}, E_{kl} with $i \neq k$ wlog. Then $[E_{ii} - E_{jj}, E_{ij}] = 2 E_{ij}$ while $[E_{ii} - E_{jj}, E_{kl}] \in \mathbb{Z}_{\leq 1} \cdot E_{kl}$.
Hence the corresp. eigenfunctions for E_{ij} and E_{kl} are distinct, since they take distinct values on $E_{ii} - E_{jj} \in \mathfrak{h}$. ∎

Lemma 5: Let $\gamma \in \mathfrak{h}^*$ be the eigenfun. for E_{ij}. Then the eigenvector for E_{ji} is $-\gamma \in \mathfrak{h}^*$.
Proof: Follows from the bracket rule given in Lemma 1. ∎

(3)

Def$^{\underline{n}}$: A function $\gamma \in \mathfrak{h}^*$ is called a root for $\mathfrak{sl}_n(\mathbb{C})$ if γ is nonzero and the corresp. eigenspace $\mathfrak{sl}_n(\mathbb{C})_\gamma$ is nonzero.

We let $\Phi = \{$the coll. of all roots in $\mathfrak{h}^*\}$

pos roots: $\Phi^+ = \{$All roots $\gamma \in \Phi$ for which $\mathfrak{n}^+_\gamma \neq 0\}$

neg roots: $\Phi^- = \{$All roots $\gamma \in \Phi$ for which $\mathfrak{n}^-_\gamma \neq 0\}$.

Corollary 6: a) $\Phi^- = -\Phi^+$.

b) $\Phi = \Phi^+ \cup \Phi^-$.

c) Each weight space $\mathfrak{sl}_n(\mathbb{C})_\gamma$ is of dim 1.

d) $\mathfrak{sl}_n(\mathbb{C}) = \left(\bigoplus_{\mu \in \Phi^-} \mathfrak{sl}_n(\mathbb{C})_\mu\right) \oplus \mathfrak{h} \oplus \left(\bigoplus_{\gamma \in \Phi^+} \mathfrak{sl}_n(\mathbb{C})_\gamma\right)$

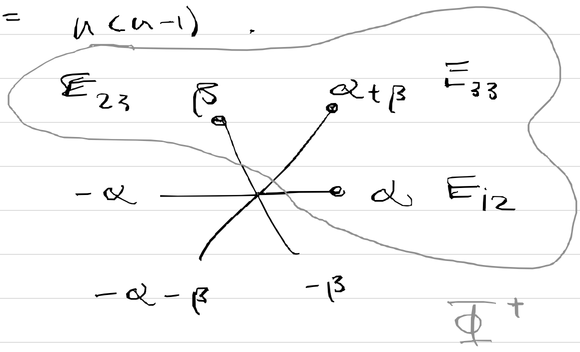
Proof: (a) Follows by Lemma 5. (b) Follows by Lemma 4. (c) Follows by Lemma 4. ∎

Rem: Note that Φ is not a subspace in $\mathfrak{h}^*$, it is a subset.

We have $|\Phi^+| = \dim \mathfrak{n}^+ = \frac{n(n-1)}{2}$.

and $|\Phi| = n(n-1)$.

e.g. $\Phi(\mathfrak{sl}_3) =$



– Root subalgebras.

Proposition 7: For arbitrary $\gamma \in \Phi$, and nonzero $e_\gamma \in \mathfrak{sl}_n(\mathbb{C})_\gamma$, there exists a unique vector $f_\gamma \in \mathfrak{sl}_n(\mathbb{C})_{-\gamma}$ so that the unique linear map

$$i_\gamma : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{sl}_n(\mathbb{C}), \quad \begin{cases} e \mapsto e_\gamma \\ f \mapsto f_\gamma \\ h \mapsto [e_\gamma, f_\gamma] =: h_\gamma \end{cases}$$

is an injective Lie alg homomorphism.

Furthermore, the vector h_γ is indep. of the choice (4)
of e_γ and the image $\mathrm{im}(i_\gamma) \subseteq sl_n(\mathbb{C})$
is uniquely det by γ (i.e. doesn't depend on e_γ).

Proof: First note that the triple Take E_{ij} so that $sl_n(\mathbb{C})_\gamma = \mathbb{C}\cdot E_{ij}$.

$$E_\gamma = E_{ij}\ ,\ F_\gamma = E_{ji}\ ,\ h_\gamma = E_{ii} - E_{jj}$$

det such a Lie alg embedding

$$sl_2 \to sl_n,\ e \mapsto E_\gamma,\ f \mapsto F_\gamma,\ h \mapsto h_\gamma.$$

Now for any choice of e_γ we have $e_\gamma = c\cdot E_\gamma$
for unique $c \in \mathbb{C}^\times$ and for any $d \in \mathbb{C}^\times$
we have $[e_\gamma, dF_\gamma] = c\cdot d\cdot h_\gamma$ so that
$$[[e_\gamma, dF_\gamma], e_\gamma] = (2\cdot c\cdot d)\cdot e_\gamma.$$
Hence we have the unique scaling $f_\gamma = c^{-1}\cdot F_\gamma$
so that the triple $\{e_\gamma, f_\gamma, h_\gamma\}$ specifies
such an embedding $i_\gamma : sl_2(\mathbb{C}) \hookrightarrow sl_n(\mathbb{C})$.

For the uniqueness claim, we always have
$$\mathrm{im}(i_\gamma) = \mathbb{C}\cdot E_\gamma \oplus \mathbb{C}\, h_\gamma \oplus \mathbb{C}\cdot F_\gamma$$
$$= sl_n(\mathbb{C})_\gamma \oplus \mathbb{C}\cdot h_\gamma \oplus sl_n(\mathbb{C})_{-\gamma}.$$

Let's just collect what we've seen here:
For each positive root $\gamma \in \Phi^+$ we get a copy
of $sl_2(\mathbb{C})$ in $sl_n(\mathbb{C})$,
$$i_\gamma : sl_2(\mathbb{C}) \to sl_n(\mathbb{C}).$$
This is our root subalg assoc. to γ. The map i_γ
itself is not det. by γ, but its image, i.e. the assoc
subalg in $sl_n(\mathbb{C})$ is.

Further, for each root γ we have a uniquely
assoc. vector $h_\gamma \in h$. There are $\frac{n(n-1)}{2}$
such vectors, and they span h. They are not lin indep
when $n > 2$.

- Simple root vectors

Proposition 8: There is a unique subset of positive roots $\Delta \subseteq \Phi^+$ satisfying the following

a) Δ is lin. indep in $\mathfrak{h}^*$, and in fact provide a basis.

b) $\Phi^+ \subseteq \mathbb{Z}_{\geq 0} \cdot \Delta$.

Proof: Consider

$\Delta = \{$ the weights for the superdiag elem's $E_{i,i+1}\}$

$= \{\alpha_1, \ldots, \alpha_{n-1} : \alpha_i = \text{wt. for } E_{i,i+1}\}$.

Since for all $i<j$

$$E_{ij} = [[[E_{i,i+1}, E_{i+1,i+2}] \cdots] E_{j-1,j}]$$

and $[e_\mu, e_\nu] \in (\mathfrak{sl}_n)_{\mu+\nu}$ via Jacobi, we see that (b) holds.

Since $|\Delta| = n-1 = \dim \mathfrak{h}^*$ it suffices to show now that Δ spans $\mathfrak{h}^*$. For this it suffices to show that for each $x \in \mathfrak{h}$ we have $x=0$ iff $\alpha(x)=0$ at all $\alpha \in \Delta$. Write

$$x = \sum_{i=1}^{n-1} c_i h_{\alpha_i}, \quad h_{\alpha_i} = E_{ii} - E_{i+1,i+1},$$

and observe

$$\alpha_i(x) E_{i,i+1} = [x, E_{i,i+1}] = (-c_{i-1} + 2c_i - c_{i+1}) \cdot E_{i,i+1}$$

so that $\alpha_i(x) = 0$ at all $i \Leftrightarrow$

$$\mathrm{Cart}_n \cdot [c_1 \cdots c_{n-1}]^t = \vec{0} \qquad (*)$$

where $\mathrm{Cart}_n = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & 0 \\ & -1 & \ddots & -1 & \\ 0 & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}$.

We calculate by induction $\det(\mathrm{Cart}_n) = n \neq 0$ so that the eq. (*) forces $x=0$, as desired.

We leave uniqueness as an exercise. ∎

—

Def^n: The subset $\Delta \subseteq \Phi^+$ as in Proposition 8 is called the (or a, depending) base for Φ. The elem of Δ are called the simple roots for $\mathfrak{sl}_n$.

Recall explicitly

$$\Delta = \{\alpha_1, \ldots, \alpha_{n-1} : \mathfrak{sl}_n(\mathbb{C})_{\alpha_i} = \mathbb{C}E_{i,i+1}\}.$$

We have also $h_{\alpha_i} = E_{ii} - E_{i+1,i+1}$.

Observation / Corollary 9: For the simple roots $\alpha \in \Delta$, the corresponding vectors

$$\{h_\alpha : \alpha \in \Delta\}$$

provide a basis for the Cartan subalg $\mathfrak{h}$, and for all $\gamma \in \Phi^+$ h_γ is in the non-neg span

$$h_\gamma \in \mathbb{Z}_{\geq 0}\{h_\alpha : \alpha \in \Delta\}.$$

- Weights and dominant weights.

Def^n: A weight for $\mathfrak{sl}_n(\mathbb{C})$ is a function $\lambda \in \mathfrak{h}^*$ which takes integer values

$$\lambda(h_\gamma) \in \mathbb{Z} \quad \text{at all } \gamma \in \Phi^+.$$

A weight λ is called dominant if it takes nonnegative integer values

$$\lambda(h_\gamma) \in \mathbb{Z}_{\geq 0} \quad \text{at all } \gamma \in \Phi^+.$$

We take

$$P := \{\text{all weights in } \mathfrak{h}^*\}$$

$$P^+ := \{\text{all dominant weights in } \mathfrak{h}^*\}.$$

Lemma 10: $\lambda \in \mathfrak{h}^*$ is a wt. iff $\lambda(h_\alpha) \in \mathbb{Z}$ for all simple α, and a wt $\lambda \in P$ is dominant iff

$$\lambda(h_\alpha) \geq 0 \quad \text{at all simple } \alpha.$$

Proof: Immediate from Corollary 9. ∎

Lemma 11: Every root $\gamma \in \Phi$ is also a weight.

Proof: Since $\Phi = \Phi^+ \cup \Phi^-$ w/ $\Phi^- = -\Phi^+$, and $\Phi^+ \subseteq \mathbb{Z}_{\geq 0} \cdot \Delta$, it suffices to show that each simple root α is a weight. But we have

$$\alpha_j(h_{\alpha_i}) = \text{the coeff of } E_{\alpha_j} \text{ in } [h_{\alpha_i}, E_{\alpha_j}]$$

$$= \begin{cases} 2 & \text{if } i=j \\ -1 & \text{if } |i-j|=1 \\ 0 & \text{else.} \end{cases}$$

We're done. ∎

Note that P is a lattice in $\mathfrak{h}^*$, i.e. a $\mathbb{Z}$-submodule w/ $\mathbb{C} \otimes_{\mathbb{Z}} P \xrightarrow{\cong} \mathfrak{h}^*$ via the natural map. We have

$Q := \mathbb{Z} \cdot \Phi \subseteq P$, another lattice.

We call Q the <u>root lattice</u> and P the <u>weight lattice</u>.

- A partial ordering on the weights

Def: For weights $\mu, \lambda \in P$ we write

$$\mu \leq \lambda \quad \text{if} \quad \lambda - \mu \in \mathbb{Z}_{\geq} \cdot \Delta,$$

i.e. if λ is obtained by a positive shift via simple roots $\lambda = \mu + \sum_i d_i \cdot \alpha_i$, $d_i \in \mathbb{Z}_{\geq 0}$.

Example: Take ϑ so that $\mathfrak{sl}_n(\mathbb{C})_\vartheta$ is $\mathbb{C} \cdot E_{1n}$. Then all roots γ satisfy

$$\vartheta \geq \gamma$$

It is the longest root for $\mathfrak{sl}_n(\mathbb{C})$.

– Weights for $\mathfrak{sl}_n$-representations

Proposition 12: Let V be any finite-dimensional $\mathfrak{sl}_n$-representation.

a) The Cartan subalg. $\mathfrak{h}$ acts semisimply on V, i.e. V decomposes into eigenspaces for the $\mathfrak{h}$-action.

b) If the eigenspace V_λ is nonzero, for $\lambda \in \mathfrak{h}^*$, then λ lies in the weight lattice.

Proof: We have

$$\mathfrak{h} = \mathbb{C}\cdot\{h_\alpha : \alpha \text{ simple}\}$$

and the h_α provide n commuting endomorphisms on V. Hence V decomposes into generalized eigensp's for the action of these h_α, and hence for the action of $\mathfrak{h}$, $V = \bigoplus_{\lambda\in\mathfrak{h}^*} V_\lambda^{gen}$.

For each simple α we have the root subalg $i_\alpha: \mathfrak{sl}_2 \to \mathfrak{sl}_n$, and restricting along α realizes an $\mathfrak{sl}_2$-action on V for which $h \in \mathfrak{sl}_2$ acts by h_α. Since h acts semisimply on any $\mathfrak{sl}_2$-rep, we conclude that each h_α acts semisimply on V. Hence V decomposes into its eigenspaces $V = \bigoplus_\lambda V_\lambda$.

Now, if $V_\lambda \neq 0$, then we again consider $\mathfrak{sl}_2$ acting on V_λ via i_α to see that $\lambda(\alpha) \in \mathbb{Z}$ at each simple α, via our classification of (simple) $\mathfrak{sl}_2$-representations. [Corollary 5, "Aug 28th"].

So we see that λ is in the weight lattice. ∎

Def^{n}: A weight vector of V is a nonzero vector which lies in some weight space V_λ, $\lambda\in P$.

We call a weight vector $v \in V_\lambda$ a highest wt. vector if $e_\alpha \cdot v = 0$ for each simple root α.

(9)

Lemma 13: Any nonzero (fin-dim) $\mathfrak{sl}_n$-rep V admits a highest wt. vector v.

Proof: Take any wt μ with V_μ nonvanishing. Since the base Δ provides a basis for $\mathfrak{h}^*$, we have for any tuple of non-neg integers $c: \Delta \to \mathbb{Z}_{\geq 0}$ and $c': \Delta \to \mathbb{Z}_{\geq 0}$, $\mu + \sum' c_\alpha \alpha = \mu + \sum'_\alpha c'_\alpha \alpha$ iff $c_\alpha = c'_\alpha$ for all α. Hence the space of wts. $\{\lambda : \lambda \geq \mu \text{ and } V_\lambda \neq 0\}$ is finite and thus contains a max elem. λ under the ordering $\geq$. Any nonzero vector $v \in V_\lambda$ provides a highest wt. vect in V. ∎

Theorem 14: If $v \in V$ is a highest wt vector, with assoc. wt. $\lambda \in P$, then λ is dominant. Furthermore the subspace

$$L(\lambda) = \mathbb{C} \cdot \{ f_{\gamma_1} \cdots f_{\gamma_t} \cdot v : t \geq 0, \gamma : \{1, \ldots, t\} \to \Phi^+ \}$$

forms a $\mathfrak{sl}_n(\mathbb{C})$-subrep in V.

Proof: By restricting along any root subalg, $\iota_\alpha : \mathfrak{sl}_2 \to \mathfrak{sl}_n$ we realize V as a $\mathfrak{sl}_2$-rep w/ h acting by the root vector h_α and e acting by e_α. Hence v is a highest wt vector for that $\mathfrak{sl}_2$-action, and we conclude [Cor 5, Prop 2?] that the value $\lambda(h_\alpha)$ is a nonneg integer. Since α was chosen arbitrarily we see that λ is dominant.

The subspace $L(\lambda)$ is clearly stable under the action of each f_α and h_α, for simple α, and for each e_α we have

$$e_\alpha \cdot f_{\gamma_1} \cdots f_{\gamma_t} \cdot v = [e_\alpha, f_{\gamma_1} \cdots f_{\gamma_t}] \cdot v$$

$$= \sum_i f_{\gamma_1} \cdots f_{\gamma_{i-1}} [e_\alpha, f_{\gamma_i}] \cdots f_{\gamma_t} \cdot v.$$

Each commutator $[e_\alpha, f_{\gamma_i}] \in (\mathfrak{sl}_n)_{-\gamma_i + \alpha}$ with one of three things occurring, by Proposition 8, case I) $-\gamma_i+\alpha$ is not a root, and $[e_\alpha, f_{\gamma_i}]=0$.

case II) $-\gamma_i+\alpha$ is a negative root, and

$$[e_\alpha, f_{\gamma_i}] = c_\alpha f_\gamma$$

for some scalar c_α and γ.

case III) $-\gamma_i+\alpha=0$, i.e. $\gamma_i=\alpha$, and

$$[e_\alpha, f_{\gamma_i}] = h_\alpha.$$

In each case the term $f_{\gamma_1}\cdots[e_\alpha, f_{\gamma_i}]\cdots f_{\gamma_t}\cdot v$ lies in $L(\lambda)$, so that

$$e_\alpha\cdot f_{\gamma_1}\cdots f_{\gamma_t}\cdot v \in L(\lambda).$$

Since the e_α, f_α generate $sl_n(\mathbb{C})$ as a Lie alg, we see that $L(\lambda)\leq V$ is a sl-subrep. ∎

Corollary 15: Each simple $sl_n(\mathbb{C})$-rep L admits a unique highest wt. vector $v\in L$, up to scaling.

Proof: L has some highest wt vector v. For any other highest wt vector v', Prop 14 tells us that L contains a subrep $L'\leq L$ with highest wt vector w and $(L')_\mu\neq 0$ implying $\mu\leq\lambda'=wt(v')$. Since $L'=L$ necessarily we have $\lambda\leq\lambda'$ and similarly $\lambda'\leq\lambda$, giving $\lambda'=\lambda$. Prop 14 also gives $\dim[(L')_\lambda]=1$ so that in fact $v'=c\cdot v$ for nonzero scalar c. ∎

- Characterizing rep$(\mathfrak{sl}_n(\mathbb{C}))$

Main Theorem: a) For each dominant weight $\lambda \in P^+$ there is a unique simple $\mathfrak{sl}_n(\mathbb{C})$-rep $L(\lambda)$ of highest weight λ.

b) The map

$$P^+ \to \mathrm{Irrep}(\mathfrak{sl}_n(\mathbb{C}))/\cong, \quad \lambda \mapsto L(\lambda),$$

is a bijection.

c) The category rep$(\mathfrak{sl}_n(\mathbb{C}))$ is semisimple.

We'll deal w/ some details of the proof later in the class. Let me sketch some details here however

First we deal w/ existence of largest wt simples / P^+.

As a starting point we again consider the standard rep.

- Fundamental wts and the standard rep < In foll. disc. we take many of simpler for granted.

Def^n: Take $\Delta = \{\alpha_1, \ldots, \alpha_{n-1}\}$ the standard basis for $\mathfrak{sl}_n$, and let $h_{\alpha_i} \in \mathfrak{h}$ be the corresp vectors in the Cartan ($h_{\alpha_i} = E_{ii} - E_{i+1,i+1}$).
The i^{th} fundamental weight $\varpi_i \in P^+ \subseteq \mathfrak{h}^*$ is the unique weight satisfying

$$\varpi_i(h_{\alpha_j}) = \delta_{ij}.$$

Remark: Clearly $P = \mathbb{Z}\cdot\{\varpi_i : 1 \le i \le n-1\}$.

By tensoring, one can obtain simples of arbitrary highest wts if we can construct simples $L(\varpi_i)$ of highest wt ϖ_i at each fundamental wt.

Example: For $\mathfrak{sl}_n(\mathbb{C})$ we have the standard rep $\mathbb{V} = \mathbb{C}^n$ w/ natural action.

For $v_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ ← ith row, the standard basis vector, we have

$$e_{\alpha_i} \cdot v_j = \begin{cases} v_{i-1} & \text{if } j = i+1 \\ 0 & \text{else} \end{cases}$$

$$f_{\alpha_i} \cdot v_j = \begin{cases} v_{i+1} & \text{if } j = i \\ 0 & \text{else} \end{cases}$$

$$h_{\alpha_i} \cdot v_j = \begin{cases} 1 & \text{if } j = i \\ -1 & \text{if } j = i+1 \\ 0 & \text{else} \end{cases} \quad (*)$$

$\Rightarrow$ $\mathbb{V}$ has unique highest wt. vector v_1.

From $(*)$ we see $h_{\alpha_i} \cdot v_1 = \begin{cases} 1 & \text{if } i = 1 \\ 0 & \text{else.} \end{cases}$

so that v_1 has weight w_1.

Proposition 16: The standard rep $\mathbb{V}$ over $\mathfrak{sl}_n(\mathbb{C})$ is the first fundamental simple

$$\mathbb{V} = L(w_1),$$

i.e. it is a simple rep of highest wt w_1.

(Proof: We only note that $\mathbb{V}$ has a unique highest wt vector, up to scaling, and is hence simple by Theorem 14. We calculated the highest wt as w_1 above. ∎

- Aside: Symmetric and exterior powers of representations.

One discovers directly that for any Lie alg $\mathfrak{g}$, and $\mathfrak{g}$ reps V and W, the vect. space swap

$$\tau_{V,W} : V \otimes W \to W \otimes V, \quad \tau_{V,W}(v, w) := w \otimes v,$$

is an isomorphism of $\mathfrak{g}$-representations.

—

Lemma 17: For any maps of $\mathfrak{g}$-reps $\phi : V_0 \to V_1$, $\psi : W_0 \to W_1$, the map $\phi \otimes \psi$ is a map of $\mathfrak{g}$-reps and the diagram

$$\begin{array}{ccc} V_0 \otimes W_0 & \xrightarrow{\tau} & W_0 \otimes V_0 \\ {\scriptstyle \phi\otimes\psi}\downarrow & & \downarrow{\scriptstyle \psi\otimes\phi} \\ V_1 \otimes W_1 & \xrightarrow{\tau} & W_1 \otimes V_1 \end{array}$$

commutes.

Proof: Clear by inspection. $\blacksquare$

For any $\mathfrak{g}$-rep V and $n \geq 1$, the automorphisms

$$\tau_i := id^{\otimes i-1} \otimes \tau_{V,V} \otimes id^{\otimes n-i-1} : V^{\otimes n} \circlearrowleft$$

satisfy the relations

$$(*) \quad \begin{cases} \tau_i^2 = id \\ \tau_i \tau_j = \tau_j \tau_i & \text{when } |i-j| > 1 \\ \tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1} \end{cases}$$

and hence define an action of

$$S_n = \langle \tau_1, \dots, \tau_{n-1} \mid \text{rels } (*) \rangle$$

via $\mathfrak{g}$-automorphisms,

$$S_n \longrightarrow \mathrm{Aut}_{\mathfrak{g}}(V^{\otimes n}).$$

Def^{n}: Given any $\mathfrak{g}$-representation V, the n-th symmetric power is the $\mathfrak{g}$-representation

$$S^n(V) := (V^{\otimes n})_{S^n} \quad \text{an } S^n\text{-coinvariants}$$

and the n-th exterior power

$$\Lambda^n(V) := \left[(V^{\otimes n}) \otimes \mathrm{sgn}\right]_{S^n}$$

Explicitly, $S^n(V) = (V^{\otimes n}) \otimes_{\mathbb{C}S^n} \mathbb{C}$

$$= V^{\otimes n} / (w - \tau_i w : w \in V^{\otimes n} \text{ and } i = 1, \dots, n-1)$$

$$= \{\text{span of ordered monomials } [v_{j_1} \cdots v_{j_n}] \text{ in some ordered basis}\}$$

$$\Lambda^n(V) = (V^{\otimes n}) \otimes_{\mathbb{C}S^n} \mathrm{sgn}$$

$$= V^{\otimes n} / (w + \tau_i w : w \in V^{\otimes n} \text{ and } i = 1, \dots, n-1)$$

$$= \{\text{Span of wedges } v_{j_1} \wedge \cdots \wedge v_{j_n} \text{ w/ } j_i < j_{i+1} \text{ strictly, in some ordered basis}\}$$

Exercise: Verify directly that $S^n(V)$ is a $\mathfrak{g}$-subrep in $V^{\otimes n}$ and that $\Lambda^n(V)$ is a quotient $\mathfrak{g}$-rep of $V^{\otimes n}$.

Example: For $\mathbb{V} = L(1)$ the standard rep for $\mathfrak{sl}_2(\mathbb{C})$, $S^n(L(1))$ has highest wt. vector $v_1^{\otimes n} = \begin{bmatrix}1\\0\end{bmatrix}^{\otimes n}$ which is of wt $n \cdot 1 = n$. We have

$$\dim S^n(L(1))$$
$$= \dim \mathbb{C}\cdot\{[v_1^{\otimes n-m} \otimes v_{-1}^{\otimes m}] : 0 \le m \le n\}$$
$$= n+1.$$

For wt. reasons now $L(n) \subseteq S^n(L(1))$ and for dim reasons this inclusion is an isomorphism;

$$S^n(L(1)) \cong L(n) \text{ at all } n.$$

For the exterior powers,

$$\Lambda^1(L(1)) = L(1), \quad \Lambda^2(L(2)) = \mathbb{C}\cdot\{v_1 \wedge v_{-1}\} \cong L(0),$$
$$\Lambda^{\ge 3}(L(2)) = 0.$$

- Exterior power of the standard rep at higher n

Consider the n-dimensional standard rep

$$\mathbb{V} = \mathbb{C}^n, \text{ w/ basis vect } v_i = \begin{bmatrix}0\\ \vdots\\1\\0\\ \vdots\\0\end{bmatrix} \leftarrow i\text{th pos},$$

we have the highest wt. vector

$$\eta_k := v_1 \wedge v_2 \wedge \cdots \wedge v_k \in \Lambda^k(\mathbb{V}),$$

whenever $k \le n$. $\left\{\begin{array}{l}\text{we exploit the basis}\\ \eta(\vec{\imath}) = v_{i_1} \wedge \cdots \wedge v_{i_k} \text{ indexed by increas. seq}\end{array}\right.$

Lemma 18: η_k is of weight ϖ_k, and is the unique highest wt. vector in $\Lambda^k(\mathbb{V})$.

Proof: We have for $i < k$ directly

$$h_{\alpha_i} \cdot \eta_k = v_1 \wedge \cdots \wedge h_{\alpha_i} v_i \wedge \cdots \wedge v_k$$
$$+ v_1 \wedge \cdots \wedge h_{\alpha_i} v_{i+1} \wedge \cdots \wedge v_k$$
$$= \eta_k - \eta_k = 0$$

and for $k < i$ $\quad h_{\alpha_i} \cdot \eta_k = 0$ as well. At $i = k$

$$h_{\alpha_k} \cdot \eta_k = v_1 \wedge \cdots \wedge h_{\alpha_k} v_k = \eta_k.$$

Then $h \cdot \eta_k = w_k(h) \cdot \eta_k$ at all $h \in \mathfrak{h}$, giving η_k of weight w_k.

For a general nonzero vector $\eta \in \Lambda^k(V)$ take $\eta(i) = v_{i_1} \wedge \cdots \wedge v_{i_k}$ for i an increasing function $i : \{1, \dots, k\} \to \{1, \dots, n\}$, and order such fns via the dictionary ordering. Then

$$\eta = c_i \cdot \eta(i) + \sum'_{j<i} c_j \cdot \eta(j)$$

with c_i nonzero. Supposing $\eta \neq \eta_k$ we have a final index $i_\ell \in \{i_1, \dots, i_k\}$ with $i_\ell - i_{\ell-1} > 1$, where we take formally $i_0 = 0$, and for

$$i' = \{i_1, \dots, i_\ell - 1, i_{\ell+1}, \dots, i_k\}$$

we have

$$e_{\alpha_{i_\ell - 1}} \cdot \eta = c_i \, \eta(i') + \sum'_{j'<i} d_{j'} \cdot \eta(j').$$

In particular $e_{\alpha_{i_\ell-1}} \cdot \eta \neq 0$, and η is not a highest wt. vector. ∎

Theorem 19: For each integer $k = 1, \dots, n-1$ the exterior power $\Lambda^k(V)$ is a simple $\mathfrak{sl}_n(\mathbb{C})$-rep. of highest wt. w_k,

$$\Lambda^k(V) = L(w_k).$$

Proof: By Theorem 14 any nonzero $\mathfrak{sl}_n(\mathbb{C})$-subrep $L \subseteq \Lambda^k(V)$ contains a highest wt vector, and hence contains η_k by Lemma 18.

We claim now that

$$\Lambda^k(V) = \mathbb{C}\{ f_{\alpha_{m_1}} \cdots f_{\alpha_{m_t}} \cdot \eta_k : t \geq 0, \alpha_m : \{1, \dots, t\} \to \Delta \}, \quad (*)$$

so that any subrep containing η_k must be all of $\Lambda^k(V)$. For this consider again the basis vectors

$$\{ \eta(i) = v_{i_1} \wedge \cdots \wedge v_{i_k} : \text{increasing fns } i : \{1, \dots, k\} \to \{1, \dots, n\} \}$$

$\to$.

We claim that each basic vector $\xi(i)$ is in the span $(*)$, so that any subrep which contains $\xi_k = \xi(i_{min}) = \xi(1,2,\dots,k)$ is necessarily $= \Lambda^k(\mathbb{V})$. We proceed by ind. under the dictionary order on the set of increasing fns $i : \{1,\dots,k\} \to \{1,\dots,n\}$.

For the min index $i_{min} = (1,\dots,k)$ we have $\xi_k = \xi(i_{min}) \in \mathrm{span}\,(*)$, and suppose now $i > i_{min}$ with $\xi(j) \in \mathrm{span}\,(*)$ for all $j < i$. Since $i > i_{min}$ there is a first index

$$i_\ell \in \{i_1, \dots, i_k\}$$

at which $i_\ell - i_{\ell-1} > 1$, where we take $i_0 = 0$. Then for the index $i' < i$ def by

$$i' = (i_1, \dots, i_\ell - 1, \dots, i_k)$$

we have $\xi(i') \in \mathrm{span}\,(*)$ and

$$\xi(i) = f_{\alpha_{i_\ell - 1}} \cdot \xi(i')$$

giving $\xi(i) \in \mathrm{span}\,(*)$ as well. Hence all basic vectors $\xi(i) \in \mathrm{span}\,(*)$ by induction.

Consequently, any subrep $W \subseteq \Lambda^k(\mathbb{V})$ which contains the highest wt. vector ξ_k is equal to $\Lambda^k(\mathbb{V})$ and we conclude our arbitrary subrep L is in fact all of $\Lambda^k(\mathbb{V})$. This establishes simplicity. $\blacksquare$

Conclusion: For each fundamental wt ϖ_k, $k = 1, \dots, n-1$,

$$\varpi_k(h_{\alpha_i}) = \delta_{ik},$$

the k-th exterior power of the standard rep $\mathbb{V}$ realizes a simple $\mathfrak{sl}_n(\mathbb{C})$-rep of highest wt ϖ_k.

In particular, such highest wt. simples exist.

- Existence of highest wt. simples over P^+

Proposition 20: For each dominant weight $\lambda \in P^+$, there is a simple representation $L(\lambda)$ with a simple highest wt. vector of wt. λ.

Proof: Since λ is dominant we have unique nonnegative integers $m_1, \ldots, m_{n-1}$ so that

$$\lambda = m_1 \varpi_1 + \cdots + m_{n-1} \varpi_{n-1}.$$

By Thm 19 there exist simple sl_n-reps $L(\varpi_k)$ of highest wt ϖ_k at each fundamental wt. ϖ_k.

Let's take

$$V(\lambda) := L(\varpi_1)^{\otimes m_1} \otimes \cdots \otimes L(\varpi_k)^{\otimes m_k}.$$

Note that $\dim V(\lambda)_\lambda = 1$ and by Thm 14 applied to the $L(\varpi_k)$ we see

$$V(\lambda)_\mu \neq 0 \Rightarrow \mu \leq \lambda. \qquad (*)$$

Thus, for any composition series

$$0 = V_t \subseteq V_{t-1} \subseteq \cdots \subseteq V_0 = V(\lambda)$$

there exists a unique index i at which the (simple) composition factor

$$L = V_i / V_{i+1}$$

has $\dim_{\mathbb{C}} L_\lambda = 1$ and $L_\mu \neq 0 \Rightarrow \mu \leq \lambda$.

It follows that $L = L(\lambda)$ is of highest wt λ. ∎

- Sketch proof of Uniqueness

Thm 21: Given $sl_n(\mathbb{C})$-simples L and L' of highest wt. λ, with highest wt. vectors v and v', there exists a unique isom $\phi: L \to L'$ over $sl_n(\mathbb{C})$ with $\phi(v) = v'$.

-

We'll be more careful about the proof when we deal w/ the general case. Let me sketch the details however.

Sketch Proof: We have the univ. env. alg. $\mathcal{U}(\mathfrak{sl}_2)$ and the subalg. env. alg for the Borel $\mathcal{U}(\mathfrak{b}^+)$, $\mathfrak{b}^+ = \mathfrak{h} \oplus \mathfrak{n}^+$. We have the 1-dim simple $\mathbb{C}_\lambda$ of wt. λ over $\mathfrak{b}^+$ and define the Verma module $\mathcal{M}(\lambda) := \mathcal{U}(\mathfrak{sl}_2) \otimes_{\mathcal{U}(\mathfrak{b}^+)} \mathbb{C}_\lambda$.

This is a highest weight, ∞-dim, weight graded, $\mathfrak{sl}_2$-rep and restriction along the inclusion $\mathbb{C}_\lambda \to \mathcal{M}(\lambda)$ provides a linear $\cong$

$$\mathrm{Hom}_{\mathfrak{sl}_2}(\mathcal{M}(\lambda), V) \xrightarrow{\sim} \mathbb{C}\cdot\left\{\begin{array}{c}\text{highest wt. vectors}\\ v \in V \text{ of wt. } \lambda\end{array}\right\}.$$
$$f \longmapsto f(1_\lambda).$$

For gradation reasons, there is a unique simp quotient $\pi : \mathcal{M}(\lambda) \to \overline{\mathcal{M}}(\lambda)$ $(= L(\lambda))$, and hence the unique $\mathfrak{sl}_2$-maps

$$f : \mathcal{M}(\lambda) \to L, \quad f' : \mathcal{M}(\lambda) \to L'$$
$$f(1_\lambda) = v_\lambda \qquad f'(1_\lambda) = v'_\lambda$$

induce isoms

$$\begin{tikzcd} & \overline{\mathcal{M}}(\lambda) \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & \\ L \arrow[rr, dashed, "\exists!", "\phi"'] & & L' \end{tikzcd}.$$

The induced $\cong$ $\phi : L \to L'$ completing the above diagram does the desired job. ∎

Corollary 22: The assignment
$$P^+ \longrightarrow \mathrm{Irrep}(\mathfrak{sl}_2(\mathbb{C})), \quad \lambda \longmapsto L(\lambda)$$
is a bijection, i.e. classifies all irreducible $\mathfrak{sl}_2(\mathbb{C})$-reps up to isom.

— Semisimplicity for $\mathrm{rep}(\mathfrak{sl}_n(\mathbb{C}))$.

Prop 22: For simple $\mathfrak{sl}_n(\mathbb{C})$-reps $L(\lambda)$ and $L(\mu)$, any extension

$$0 \to L(\mu) \to V \to L(\lambda) \to 0$$

is split.

Again, will cover the details more cleanly in the general setting. We again sketch the details.

Proof: The adjoint rep $\mathfrak{sl}_n$ is again self dual, so that the trace form on $\mathfrak{sl}_n$ induces a symm non-deg $\mathfrak{sl}_n$-invariant form

$$\kappa : \mathfrak{sl}_n^* \otimes \mathfrak{sl}_n^* \to \mathbb{C},$$

and under the natural $\mathfrak{sl}_n$ iso

$$\mathfrak{sl}_n \otimes \mathfrak{sl}_n \cong (\mathfrak{sl}_n^* \otimes \mathfrak{sl}_n^*)^*$$

κ determines an elem $\Omega \in \mathfrak{sl}_n \otimes \mathfrak{sl}_n$,

$$\Omega = \Big(\sum_{\gamma \in \Phi^+} c_\gamma (f_\gamma \cdot e_\gamma + e_\gamma f_\gamma)\Big) + \Big(\sum_i \text{Cartan terms}\Big),$$

w/ $x \cdot \Omega = 0$ at all $x \in \mathfrak{sl}_n(\mathbb{C})$, aka Ω is an invariant elem in $\mathfrak{sl}^{\otimes 2}$.

The elem Ω therefore acts by $\mathfrak{sl}_n$-linear endos on all reps $\Omega_V = \Omega \cdot - : V \to V$, and we can use Ω to split extensions of simples, just as in the case of $\mathfrak{sl}_2$. [Exercise] ∎

Using [Prop 6, Aug 28] we now see that the category $\mathrm{rep}(\mathfrak{sl}_n(\mathbb{C}))$ is semisimple.

Theorem 23: The cat $\mathrm{rep}(\mathfrak{sl}_n(\mathbb{C}))$ is semisimple, and the simple $\mathfrak{sl}_n(\mathbb{C})$-reps are in bij correspondence with dominant wts, $\lambda \mapsto L(\lambda)$.

— We've now recovered our "Main Theorem" for $\mathfrak{sl}_n(\mathbb{C})$.

End.