[All "Lie algebras" are finite-dimensional!] (1)

# Ch. 3: Solvable and Nilpotent Lie alg

**Def$^{\underline{n}}$**: A Lie alg $\mathfrak{g}$ is said to be <u>solvable</u> if the so-called derived series

$$\mathfrak{g} = \mathfrak{g}^{(0)} \supseteq \mathfrak{g}^{(1)} \supseteq \mathfrak{g}^{(2)} \supseteq \cdots, \quad \mathfrak{g}^{(i+1)} = [\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}]$$

satisfies $\mathfrak{g}^{(n)} = 0$ at sufficiently large $n$.

A Lie alg $\mathfrak{g}$ is called <u>nilpotent</u> if the so-called descending central series

$$\mathfrak{g} = \mathfrak{g}^{0} \supseteq \mathfrak{g}^{1} \supseteq \mathfrak{g}^{2} \supseteq \cdots, \quad \mathfrak{g}^{i+1} = [\mathfrak{g}, \mathfrak{g}^{i}]$$

satisfies $\mathfrak{g}^{n} = 0$ at sufficiently large $n$.

**Proposition 3.1**: (a) If $\mathfrak{g}$ is solvable, then any Lie subalg $\mathfrak{f}$ in $\mathfrak{g}$ is also solvable.

(b) If $I \leq \mathfrak{g}$ is an ideal in $\mathfrak{g}$, $I$ is solvable, and $\mathfrak{g}/I$ is solvable, then $\mathfrak{g}$ is solvable.

(c) If $\mathfrak{g}$ is solvable and $\pi : \mathfrak{g} \to \mathfrak{g}'$ is a surjective Lie alg map, then $\mathfrak{g}'$ is solvable.

(d) If $I$ and $J$ are nilpotent ideals in $\mathfrak{g}$, then the ideal $I+J$ is also solvable.

Proof: (a) We have $\mathfrak{f}^{(0)} \leq \mathfrak{g}^{(0)}$ and by induction see $\mathfrak{f}^{(i)} \leq \mathfrak{g}^{(i)}$ at all $i \geq 0$.
So $\mathfrak{g}^{(n)} = 0$ at large $n$ implies $\mathfrak{f}^{(n)} = 0$ at large $n$.

(c) In this case we have $(\mathfrak{g}')^{(0)} = \pi(\mathfrak{g}^{(0)})$ and by induction $(\mathfrak{g}')^{(n)} = \pi(\mathfrak{g}^{(n)})$, giving $(\mathfrak{g}')^{(n)} = 0$ whenever $\mathfrak{g}^{(n)} = 0$.

(b) $(\mathfrak{g}/I)^{(m)} = 0$ implies $\mathfrak{g}^{(m)} \leq I$, and for $m'$ with $I^{(m')} = 0$ we have

$$\mathfrak{g}^{(m+m')} = (\mathfrak{g}^{(m)})^{(m')} \leq I^{(m')} = 0.$$

(d) Take $\mathfrak{g}' = I+J$ to obtain (d) from (b). ∎

Example: Consider

$\mathfrak{g} = \mathfrak{b}^+ = \mathfrak{h} \oplus \mathfrak{n}^+$ in $\mathfrak{sl}_n(\mathbb{C})$. These are upper $\triangle$ matrices. Then

$\mathfrak{n}^+$ is seen to be solvable, and in fact nilpotent, with

$$(\mathfrak{n}^+)^k = \mathrm{Span}_{\mathbb{C}}\{E_{ij} : j-i > k\}$$

$$\Rightarrow (\mathfrak{n}^+)^{(n)} \subseteq (\mathfrak{n}^+)^n = 0.$$

We have $\mathfrak{n}^+$ an ideal in $\mathfrak{b}$ w/

$\mathfrak{b}/\mathfrak{n}^+ \cong \mathfrak{h}$ an abelian.

Thus $\mathfrak{b}$ is solvable by Prop 3.1 (b).

Prop 3.2: For any Lie alg $\mathfrak{g}$

(a) If $\mathfrak{g}$ is nilpotent then any Lie subalg $\mathfrak{f} \leq \mathfrak{g}$ is nilpotent, as is any quotient Lie alg $\pi : \mathfrak{g} \to \mathfrak{g}'$.

(b) If $\mathfrak{g}$ is nilpotent and nonzero then the center

$$Z(\mathfrak{g}) = \{x \in \mathfrak{g} : [x, y] = 0 \text{ at all } y \text{ in } \mathfrak{g}\}$$

is nonzero.

(c) $\mathfrak{g}$ is nilpotent if and only if $\mathfrak{g}/Z(\mathfrak{g})$ is nilpotent.

Proof sketch: (a) + (b) Clear. (c) For $\pi : \mathfrak{g} \to \bar{\mathfrak{g}} = \mathfrak{g}/Z(\mathfrak{g})$ we have $\bar{\mathfrak{g}}$ nilpotent by (a), and $\bar{\mathfrak{g}}^k = 0$ implies $\mathfrak{g}^k \leq Z(\mathfrak{g})$, as

$$\bar{\mathfrak{g}}^k = \mathrm{im}(\mathfrak{g}^k) \text{ via induction.}$$

Now for the center we have

$$\mathfrak{g}^{k+1} = [\mathfrak{g}, \mathfrak{g}^k] \leq [\mathfrak{g}, Z(\mathfrak{g})] = 0. \qquad \blacksquare$$

Q: What is the center? It is the kernel of the adjoint rep

$\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ ...

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– Nilpotence vs. ad-nilpotence

We want to compare Nilpotence of Lie alg (or a subalg in a Lie alg) to a kind of element-wise nilpotence.

**Def$^n$**: Call an element $x \in \mathfrak{g}$ ad-nilpotent if the operator $ad_x = [x, -]: \mathfrak{g} \to \mathfrak{g}$ is a nilpotent endomorphism of $\mathfrak{g}$, i.e. if $ad_x^n = 0$ for large $n$.

We'll prove the following

**Theorem (Engel)**: Let $\mathfrak{g}$ be a Lie algebra in which all $x \in \mathfrak{g}$ are ad-nilpotent. Then $\mathfrak{g}$ is a nilpotent Lie algebra.

– Proving Engel

**Def$^n$**: Given a Lie subalg $\mathfrak{f} \subseteq \mathfrak{g}$ the normalizer of $\mathfrak{f}$ in $\mathfrak{g}$ is defined as

$$N_{\mathfrak{g}}(\mathfrak{f}) := \{x \in \mathfrak{g} : [x, y] \in \mathfrak{f} \text{ for all } y \text{ in } \mathfrak{f}\}.$$

Applying Jacobi identity, we see that $N_{\mathfrak{g}}(\mathfrak{f})$ is a Lie subalg in $\mathfrak{g}$ at arbitrary $\mathfrak{f}$. < Def$^n$: Faithful rep.

**Lemma 3.2**: Let $V$ be a fin-dim vector space and $x \in \mathfrak{gl}(V) = End_{\mathbb{C}}(V)^{Lie}$ act on $V$ as a nilpotent endomorphism. Then $x$ is also ad-nilpotent.

*Proof*: Suppose $x_V^n = 0$. Then at all $y$ in $End_{\mathbb{C}}(V)^{Lie}$ we have

$$ad_x^{2n}(y) = \sum_{i=0}^{2n} \binom{2n}{i} x^i y x^{2n-i}.$$

In the above exp one of $i$ or $2n-i$ is $\geq n$, so that the entire expression vanishes. Hence

$$ad_x^{2n} = 0.$$

$\blacksquare$

Lemma 3.3: Let $V$ be a fin-dim'l nonzero $\mathfrak{g}$-represent. on which $\mathfrak{g}$ acts by nilpotent endomorphisms. Then there exists a non-zero vector $v \in V$ with $\mathfrak{g} \cdot v = 0$.

Proof: Fix a rep $V$ as in the statement. By replacing $\mathfrak{g}$ with $\mathfrak{g}/\mathrm{Ann}_{\mathfrak{g}}(V)$ we may assume that the rep map $\mathfrak{g} \to \mathfrak{gl}(V)$ is injective.

The cases $\dim(\mathfrak{g}) \leq 1$ are clear. Assume now that $\dim(\mathfrak{g}) = n$ and the result holds for all Lie algs $\mathfrak{g}'$ of dim $< n$.

Consider any maximal proper subalg $\mathfrak{g}' \subseteq \mathfrak{g}$.

We claim that $\mathfrak{g} = N_{\mathfrak{g}}(\mathfrak{g}')$ and that $\dim(\mathfrak{g}) = \dim(\mathfrak{g}') + 1$.

For the first claim, we have

$$N_{\mathfrak{g}}(\mathfrak{g}') = \left\{ \begin{array}{l} \text{preimage of the collection} \\ \{\bar{x} \in \mathfrak{g}/\mathfrak{g}' : y \cdot \bar{x} = 0 \ \forall y \in \mathfrak{g}'\} \\ \text{along the projection } \mathfrak{g} \to \mathfrak{g}/\mathfrak{g}'. \end{array} \right.$$

Here we consider the adj. action of $\mathfrak{g}'$ on $\mathfrak{g}$ and the induced $\mathfrak{g}'$-action on the quotient $\mathfrak{g}/\mathfrak{g}'$.

By Lemma 3.2 and faithfulness of the $\mathfrak{g}$-action on $V$, we conclude that $\mathfrak{g}'$ acts on $\mathfrak{g}$ and hence $\mathfrak{g}/\mathfrak{g}'$ by nilpotent endos. Hence, by induction, there exists nonzero $\bar{x} \in \mathfrak{g}/\mathfrak{g}'$ which is annihilated by all of $\mathfrak{g}'$. Thus there exists some vector $x \in \mathfrak{g} - \mathfrak{g}'$ with $x \in N_{\mathfrak{g}}(\mathfrak{g}')$. By assumption $x$ stabilizes $\mathfrak{g}'$ under the bracket, and obviously $[x, x] = 0$, so that we have a sequence of inclusions of Lie alg's

$$\mathfrak{g}' \subsetneq \mathfrak{g}' + \mathbb{C} \cdot x \subseteq N_{\mathfrak{g}}(\mathfrak{g}') \subseteq \mathfrak{g}.$$

Our maximality hypothesis now forces
$$\mathfrak{g} = N_{\mathfrak{g}}(\mathfrak{g}') = \mathfrak{g}' + \mathbb{C}\cdot x.$$

Now for any vector $v \in V$ in our original rep $V$ with $\mathfrak{g}'\cdot v = 0$. Then for all $y \in \mathfrak{g}'$
$$y \cdot x \cdot v = [y,x]\cdot v \subseteq \mathfrak{g}'\cdot v = 0,$$
so that the subspace $V' \subseteq V$ on which $\mathfrak{g}'$ acts trivially is a $\mathfrak{g}$-subrep. Since $\mathfrak{g}$ acts nilpotently on $V$ $x$ acts as a nilpotent endo on $V'$, and has only $0$-eigenvalues as a linear endo on $V'$.

Hence any eigenvector $v' \in V'$ for the action of $x$ gives a nonzero $v' \in V$ with $\mathfrak{g}\cdot v' = 0$. ∎

**Corollary 3.3:** If $\mathfrak{g}$ acts on $V$ by nilpotent endomorphisms, then there exists a flag in $V$, $0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = V$ which satisfies $\mathfrak{g}\cdot V_i \subseteq V_{i-1}$ at all $i$.

Here by a *flag* we mean an ascending sequence of linear subspaces $V_0 \subsetneq \cdots \subsetneq V_n = V$ which satisfy $\dim V_i = i$ at all $0 \leq i \leq n$.

*Proof:* $V_0$ is for free, and supposing we have such $V_0 \subseteq \cdots \subseteq V_i$ $[\subseteq V]$ we have that $V_i$ is a $\mathfrak{g}$-subrep in $V$, $\mathfrak{g}$ acts by nilpotent endos on the quotient $V/V_i$ and we can take

$$V_{i+1} = \left\{\begin{array}{l}\text{The preimage of any span } \mathbb{C}\cdot\bar{v} \text{ of a nonzero vector } \bar{v} \in V/V_i \\ \text{with } \mathfrak{g}\cdot\bar{v} = 0.\end{array}\right.$$

$$= \left\{\begin{array}{l}V_i + \mathbb{C}\cdot v \text{ where } v \text{ descends to} \\ \text{a } 0\text{-eigenvector in } V/V_i\end{array}\right.$$

∎

Proof of Engel [All elem ad-nilpotent ⇒ $\mathfrak{g}$ is a nilpotent Lie alg]:
By the previous corollary we obtain a flag
$$0 = \mathfrak{g}_0 \subseteq \dots \subseteq \mathfrak{g}_n = \mathfrak{g}$$
which has $[\mathfrak{g}, \mathfrak{g}_i] \subseteq \mathfrak{g}_{i-1}$. Then by induction we see $\mathfrak{g}^j \subseteq \mathfrak{g}_{n-j}$ at all $0 \leq j \leq n$, and hence observe the requisite vanishing $\mathfrak{g}^n \subseteq \mathfrak{g}_0 = 0$. ∎

Remark: Corollary 3.2 says that if $\mathfrak{g}$ acts on $V$ by nilpotent endomorphisms, then under some basis on $V$ $\mathfrak{g}$ acts by strictly upper triangular matrices. Rather under that basis $V \cong \mathbb{C}^n$ the corresponding representation
$$\rho_V : \mathfrak{g} \to \mathfrak{gl}_n(\mathbb{C})$$
has image in the standard nilpotent subalg $\mathfrak{n}^+ = \begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix}$ of strictly upper △'r matrices

– Solvability and Lie's Theorem [Ch 4]

We want to prove an extension of Corollary 3.3 to the solvable setting.

Theorem (Lie's Theorem):
Let $\mathfrak{g}$ be a solvable Lie alg, and $V$ be an arbitrary (fin-dim'l) $\mathfrak{g}$-representation. Then $\mathfrak{g}$ stabilizes a flag
$$0 = V_0 \subseteq V_1 \subseteq \dots \subseteq V_n = V$$
in $V$.

Proof: It suffices to show that any fin-dim $\mathfrak{g}$-rep $V$ contains a common eigenvector $v \in V \setminus \{0\}$ for the action of $\mathfrak{g}$. The cases $\dim(\mathfrak{g}) \leq 1$ are trivial. Suppose now $\dim(\mathfrak{g}) = n > 1$ and that the result holds for all solvable Lie algs of dim $< n$. We proceed in steps.

Step 1 [Find an ideal $\mathfrak{g}'$ of codim 1 in $\mathfrak{g}$] (7)

In this case $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}] \subsetneq \mathfrak{g}$ by solvability (and $\mathfrak{g} \neq 0$). Now $\mathfrak{g}^{(1)}$ is an ideal in $\mathfrak{g}$ via Jacobi, with $\mathfrak{g}/\mathfrak{g}^{(1)}$ abelian. Hence any codim 1 subspace $K_0 \subseteq \mathfrak{g}/\mathfrak{g}^{(1)}$ is a Lie subalg of codim 1, and the preimage $\mathfrak{g}' = \pi^{-1}(K_0)$ along the projection

$$\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{g}^{(1)}.$$

provides a codim 1 ideal in $\mathfrak{g}$.

Step 2 [$\mathfrak{g}$ stabilizes an eigenspace $V_\lambda$ for the $\mathfrak{g}'$-action]

Take $\lambda : \mathfrak{g}' \to \mathbb{C}$ a function for which the eigenspace $V_\lambda$ is nonvanishing. (Such a function exists by our induction hypothesis.) For $z \in \mathfrak{g} - \mathfrak{g}'$ and $v \in V_\lambda$ and any $x \in \mathfrak{g}'$ we have

$$\begin{aligned} x \cdot z \cdot v &= \lambda(x) \cdot z \cdot v + [x, z] \cdot v \qquad (*) \\ &= \lambda(x)(z \cdot v) + \lambda([x, z]) \cdot v. \end{aligned}$$

We claim that the value $\lambda([x,z]) = 0$. For this take

$$W_l = \mathrm{Span}\{v, z\cdot v, z^2 \cdot v, \dots, z^l \cdot v\}$$

and $W = W_m$ at the maximal index $m$ at which $W_{m+1} = W_m$, i.e. the maximal index at which the vectors $\{v, z\cdot v, \dots, z^m \cdot v\}$ are lin. indep in $V$. The $W$ is a $\mathfrak{g} = \mathbb{C}\cdot z + \mathfrak{g}'$ - subrep in $V$, and for all $y \in \mathfrak{g}'$, $y$ acts on $W = W_m$ by an upper $\Delta$ matrix

$$y = \begin{bmatrix} \lambda(y) & & * \\ & \ddots & \\ 0 & & \lambda(y) \end{bmatrix}$$

in the given basis $\{v, z\cdot v, \dots, z^m \cdot v\}$ (observe by ind on $l$). In particular, this holds for $y = [x, z]$, giving $Tr_W([x,z]) = \dim(W)\, \lambda([x,z])$. But by cyclic invariance of the trace

$$\dim(W)\, \lambda([x,z]) = Tr_W([x,y]) = Tr_W(xy) - Tr(yx) = 0.$$

This forces $\lambda([x,z]) = 0$ at all $x \in \mathfrak{g}'$, and by (*) and induction on $l$, all $z^l v$ lie in $V_\lambda$. Rather $V_\lambda \subseteq V$ is a $\mathfrak{g}$-subrep. Taking an eigenvector for the action of $z$ on $V_\lambda$, we obtain an eigenvector $v \in V$ for the action of $\mathfrak{g}$.

One obtains the desired flag
$$0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = V$$
precisely as in the nilpotent case [proof of Corollary 3.3]. ∎

— Corollaries to Lie's Theorem.

Corollary A.I: If $\mathfrak{g}$ is solvable, then there are ideals $0 = \mathfrak{g}_0 \subseteq \mathfrak{g}_1 \subseteq \cdots \subseteq \mathfrak{g}_n = \mathfrak{g}$ with $\dim(\mathfrak{g}_i) = i$.

Corollary A.II: If $\mathfrak{g}$ is solvable, then under some basis on $\mathfrak{g}$, $\mathfrak{g} \cong \mathbb{C}^n$, the adjoint representation
$$\mathrm{ad}: \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) \cong \mathfrak{gl}_n(\mathbb{C})$$
has image in the standard Borel of upper $\Delta$'r matrices
$$\mathrm{ad}(\mathfrak{g}) \subseteq \left\{ \begin{bmatrix} a_{11} & & a_{1n} \\ & \ddots & \\ 0 & & a_{nn} \end{bmatrix} : a_{ij} \in \mathbb{C} \right\}.$$

Corollary B: If $\mathfrak{g}$ is solvable and $V$ is any rep, $\exists$ a basis on $V$ under which the the action map
$$\rho_V : \mathfrak{g} \to \mathfrak{gl}(V)$$
has image in the subalg of upper $\Delta$'r matrices.

Corollary C: $\mathfrak{g}$ is solvable if and only if the ideal $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}]$ is nilpotent.

Proof: If $\mathfrak{g}^{(1)}$ is nilpotent, then it is solvable.

and we place $\mathfrak{g}$ into an exact sequence

$$0 \to \mathfrak{g}^{(1)} \to \mathfrak{g} \to \mathfrak{h} \to 0$$

of Lie algs in which $\mathfrak{h}$ is abelian (and thus solvable as well). By Prop 3.1 we see that $\mathfrak{g}$ is solvable.

Conversely, if $\mathfrak{g}$ is solvable then under the adj rep $\mathrm{ad}: \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ $\mathfrak{g}$ acts on itself by upper $\Delta$'r matrices, and hence $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}]$ acts by strictly upper $\Delta$'r matrices. Consequently $\mathfrak{g}^{(1)}$ acts by nilpotent endos, and

$$\mathrm{im}(\mathfrak{g}^{(1)}) \subseteq \mathfrak{gl}(\mathfrak{g})$$

is a nilpotent Lie subalg by Engel's Theorem. We therefore observe an exact seq of Lie algs

$$0 \to Z(\mathfrak{g}^{(1)}) \to \mathfrak{g}^{(1)} \to \mathrm{im}(\mathfrak{g}^{(1)}) \to 0$$

and obtain Nilpotence of $\mathfrak{g}^{(1)}$ by [Prop 3.2, 4]. ∎

– Lie's Theorem and $\mathfrak{sl}_n(\mathbb{C})$

Theorem: If $\mathfrak{b}'$ is a maximal solvable subalg in $\mathfrak{sl}_n(\mathbb{C})$, then there exists $A \in SL_n(\mathbb{C})$ with $A\mathfrak{b}'A^{-1} = \mathfrak{b}$ = standard Borel of upper $\Delta$'r matrices.

Proof: $\mathfrak{b}'$ acts by upper $\Delta$'r matrices on $V$ = standard rep, under some basis. Take $A$ to be the corresp. change of basis matrix, normalized so that $\det(A) = 1$, to get

$$A\mathfrak{b}'A^{-1} \subseteq \mathfrak{b}. \quad (*)$$

Since $\mathfrak{b}' \subseteq A^{-1}\mathfrak{b}A$ is solvable, the inclusion $(*)$ is seen to be an equality, via maximality. ∎

Conclusion: $\exists$ a unique maximal solvable subalg in $\mathfrak{sl}_n(\mathbb{C})$, up to the natural action of $SL_n(\mathbb{C})$.
This maximal subalg is the Borel of upper $\Delta$'r matrices.

— Jordan normal form and Cartan's Criterion [4.3 and 4.4].