Errata for "Aug 30": Here we explain the origins of the Casimir element via the trace form on sl_2, and the induced form \kappa on its dual

$$\mathfrak{sl}_2^* = \begin{array}{c} f^* \\ h^* \\ e^* \end{array}$$

$$\mathfrak{sl}_2^* \xrightarrow{\cong} \mathfrak{sl}_2, \qquad \begin{array}{l} f^* \longmapsto e \\ h^* \longmapsto \tfrac{1}{2} h \\ e^* \longmapsto f \end{array}$$

$$\Rightarrow \qquad \begin{array}{l} \text{Trace}(e,f) = 1 \\ \text{Tr}(h,h) = 2 \end{array}$$

$$\mathfrak{sl}_2 \otimes \mathfrak{sl}_2 \xrightarrow{\text{trace form}} \mathbb{C}$$
$$\mathfrak{s}_{\kappa} \uparrow$$
$$\mathfrak{sl}_2^* \otimes \mathfrak{sl}_2^* \xrightarrow{\quad \kappa \quad}$$

$$\kappa(e^*, f^*) = \kappa(f^*, e^*) = 1$$

$$\kappa(h^*, h^*) = \tfrac{1}{2} \qquad\qquad \text{vanishes elsewhere}$$

0 on all other basic pairings.

$$\text{have } \quad \text{eval}: \mathfrak{sl}_2 \otimes \mathfrak{sl}_2 \xrightarrow{\cong} \text{Hom}_{\mathbb{C}}(\mathfrak{sl}_2^* \otimes \mathfrak{sl}_2^*, \mathbb{C})$$

$$\Rightarrow \qquad \Omega \xleftarrow{\;\; \kappa \;\;}$$
$$\Omega = \tfrac{1}{2} h^2 + ef + fe.$$

We now understand how sl_2-reps behave, essentially
completely : i.e all sl_2-reps V decomp into eigenspaces
for the action of $h \in sl_2(\mathbb{C})$

$$V = \bigoplus_{\mu \in \mathbb{Z}} V_\mu \ ,$$

are semisimple, and each simple $\wedge^{(i)}$ is precisely
structured according to some rules.

- The structure of sl_n I

For sl_2 we have the decomp $sl_2 = \mathbb{C}f \oplus \mathbb{C}h \oplus \mathbb{C}e$
which we exploit repeatedly to access the structure of sl_2 reps.

For sl_n we have
$$sl_n = span\{ E_{ii} - E_{i+1,i+1} : 1 \leq i < n\} \overset{=h}{\text{ are diagonal stuff}}$$

$$\oplus \underset{\underset{n^-}{\underset{strict \ lower \ \Delta's}{\uparrow}}}{span \{ E_{ij} : i > j\}} \oplus \underset{\underset{n^+}{\underset{strict \ upper \ \Delta's}{\uparrow}}}{span \{ E_{ij} : i < j\}}$$

$$sl_n(\mathbb{C}) = n^- \oplus h \oplus n^+ .$$

Lemma 1: The bracket on $sl_n \subseteq gl_n$ is given as
$$[E_{ij} , E_{kl}] = \delta_{jk} E_{il} - \delta_{il} E_{kj} .$$
From this one sees directly that

Lemma 2: Each subspace $n^{\pm}, h \subseteq sl_n$ is a
Lie subalg, and h is abelian (has vanishing bracket).

Lemma 3: The Lie alg sl_n is generated by
the vectors $e_i = E_{i,i+1}$, $f_i = E_{i+1,i}$ (and
$h_i = E_{ii} - E_{i+1,i+1}$). Also the only subalg which cont.
these elem ideals of $sl_n(\mathbb{C})$ itself.

(standard)

The subalg h in sl_n is called the Cartan subalg.
The subalg's $n^{\pm}$ are the pos. and neg nilpotent sub lg.

The subalgs $b^{\pm} = h \oplus n^{\pm}$ are called ②
the pos. and neg. Borel subalgs.

– The structure of sl_n II.

Def^n: For any sl_n-rep V, a nonzero vector $v \in V$
is said to be an eigenvector for the action of the Cartan
$h \subseteq sl_n$ if for each $x \in h$ $x \cdot v \in \mathbb{C} \cdot v$.
The corresp. eigenfunction $\lambda \in h^*$ is the unique
linear fun which satisfies $x \cdot v = \lambda(x) \cdot v$ at all
x in h. Given $\lambda \in h^*$ we let V_λ denote the
corresponding weight space / eigenspace. $\{ w \in V : x \cdot w = \lambda(x) w \text{ at } \atop \text{all } x \in h \}$

Lemma 4: Each vector $E_{ij} \in sl_n(\mathbb{C})$, $i \neq j$
is an eigenvector for the adj action of h assoc.
to nonzero eigenfunction $\lambda : h \to \mathbb{C}$. Further, if
$v \in sl_n(\mathbb{C})$ is an eigenvector for the Cartan then
$v \in h$ or $v \in \mathbb{C} \cdot E_{ij}$ for unique i, j.

Proof: The eigenvector claim is clear since for
any diag matrix D, $D \cdot E_{ij} , E_{ij} \cdot D \subseteq \mathbb{C} \cdot E_{ij}$.
For the uniqueness claim consider E_{ij}, E_{kl} with $\atop \substack{\text{with} \\ i \neq j \text{ distinct}}$
$i \neq k$. Then $[E_{ii} - E_{jj}, E_{ij}] = 2 E_{ij}$
while $[E_{ii} - E_{jj}, E_{kl}] \in \cancel{\mathbb{Z}_{\leq 1}} \cdot E_{kl}$.
Hence the corresp. eigenfunctions for E_{ij} and
E_{kl} are distinct, since they take distinct values
on $E_{ii} - E_{jj} \in h$. ∎

Lemma 5: Let $\lambda \in h^*$ be the eigen fun for
E_{ij}. Then the eigenvector for E_{ji} is $-\lambda \in h^*$.
Proof: Follows from the bracket rule given in
Lemma 1. ∎
–

Def[n]: A functn $\gamma \in h^*$ is called a
root for $sl_n(\mathbb{C})$ if γ is nonzero and
the corresp. eigenspace $sl_n(\mathbb{C})_\gamma$ is nonzero.

We let $\Phi = \{$the coll. of all roots in $h^*\}$

$\Phi^+ = \{$All roots $\gamma \in \Phi$ for which $u_i^+ \gamma \neq 0\}$

$\Phi^- = \{$All roots $\gamma \in \Phi$ for which $u_i^- \gamma \neq 0\}$.

pos
roots   neg roots

Corollary 6: a) $\Phi^- = -\Phi^+$.

b) $\Phi = \Phi^+ \cup \Phi^-$.

c) Each weight space $sl_n(\mathbb{C})_\gamma$ is of dim 1.

d) $sl_n(\mathbb{C}) = \left(\bigoplus_{\mu \in \Phi^-} sl_n(\mathbb{C})_\mu\right) \oplus h \oplus \left(\bigoplus_{\gamma \in \Phi^+} sl_n(\mathbb{C})\right)$

Proof: (a) Follows by Lemma 5. (b) Follows by
Lemma 4. (c) Follows by Lemma 4. ∎

Rem: Note that Φ is not a subspace in
h^*, it is a subset.

We have $|\Phi^+| = \dim h^+ = \dfrac{n(n-1)}{2}$

and $|\Phi| = n(n-1)$.

e.g. $\Phi(sl_3) =$

E_{23} β $\alpha+\beta$ E_{33}
$-\alpha$ ─── ✗ ──o α E_{12}
$-\alpha-\beta$ $-\beta$
Φ^+
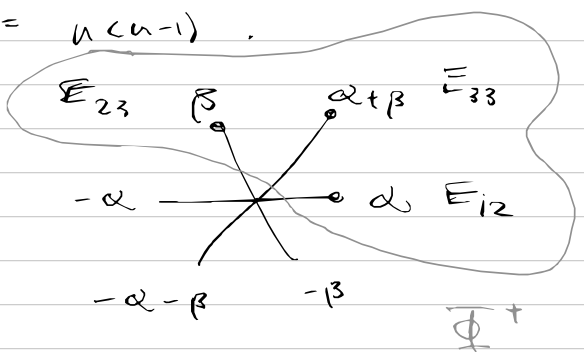
– Root subalgebras.

Proposition 7: For arbitrary $\gamma \in \Phi$, and
nonzero $e_\gamma \in sl_n(\mathbb{C})_\gamma$, there exists a unique
vector $f_\gamma \in sl_n(\mathbb{C})_{-\gamma}$ so that the unique linear
fn
$z_\gamma: sl_2(\mathbb{C}) \to sl_n(\mathbb{C}), \begin{cases} e \longmapsto e_\gamma \\ f \longmapsto f_\gamma \\ h \longmapsto [e_\gamma, f_\gamma] =: h_\gamma \end{cases}$

is an injective Lie alg homomorphism.

Furthermore, the vector h_γ is indep. of the choice (4)
of e_γ and the image $\mathfrak{sl}_2(i_\gamma) \le \mathfrak{sl}_n(\mathbb{C})$
is uniquely def. by γ (i.e. doesn't depend on e_γ).

Prof: First note that the triple (Take E_{ij} so that $\mathfrak{sl}_n(\mathbb{C})_\gamma = \mathbb{C} \cdot E_N$.)

$$E_\gamma = E_{ij} \quad, \quad F_\gamma = E_{ji} \quad, \quad h_\gamma = E_{ii} - E_{jj}$$

def. such a Lie alg. embedding

$$\mathfrak{sl}_2 \to \mathfrak{sl}_n, \quad e \mapsto E_\gamma, \quad f \mapsto F_\gamma, \quad h \mapsto h_\gamma.$$

Now for any choice of e_γ we have $e_\gamma = c \cdot E_\gamma$
for unique $c \in \mathbb{C}^\times$ and for any $d \in \mathbb{C}^\times$
we have $[e_\gamma, d F_\gamma] = c d \cdot h_\gamma$ so that

$$[[e_\gamma, d F_\gamma], e_\gamma] = (2 \cdot c d) \cdot e_\gamma.$$

Hence we have the unique relation $f_\gamma = c^{-1} \cdot F_\gamma$
so that the triple $\{e_\gamma, f_\gamma, h_\gamma\}$ specifies
such an embedding $i_\gamma: \mathfrak{sl}_2(\mathbb{C}) \hookrightarrow \mathfrak{sl}_n(\mathbb{C})$.

For the uniqueness claim, we always have

$$\mathfrak{sl}_2(i_\gamma) = \mathbb{C} \cdot E_\gamma \oplus \mathbb{C} \cdot h_\gamma \oplus \mathbb{C} \cdot F_\gamma$$
$$= \mathfrak{sl}_n(\mathbb{C})_\gamma \oplus \mathbb{C} \cdot h_\gamma \oplus \mathfrak{sl}_n(\mathbb{C})_{-\gamma}. \quad \blacksquare$$

Let's just collect what we've seen here:

For each positive root $\gamma \in \Phi^+$ we get a copy
of $\mathfrak{sl}_2(\mathbb{C})$ in $\mathfrak{sl}_n(\mathbb{C})$,

$$i_\gamma: \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{sl}_n(\mathbb{C}).$$

This is our root subalg. assoc. to γ. The map i_γ
itself is not def. by γ, but its image, i.e. the assoc.
subalg. in $\mathfrak{sl}_n(\mathbb{C})$ is.

Further, for each root γ we have a uniquely
assoc. vector $h_\gamma \in \mathfrak{h}$. There are $\frac{n(n-1)}{2}$
such vectors, and they span $\mathfrak{h}$.

Proposition 8: There is a unique subset of positive roots $\Delta \subseteq \Phi^+$ satisfying the following

a) Δ is lin. indep. in $\mathfrak{h}^*$, and in fact pow. classes.

b) $\Phi^+ \supseteq \mathbb{Z}_{\geq 0} \cdot \Delta$.

Proof: Consider

$\Delta = \{$ the weights for the superdiag'l elem's $E_{i,i+1} \}$

$= \{\alpha_1, \dots, \alpha_{n-1} : \alpha_i = \text{wt. for } E_{i,i+1}\}$.

Since for all $i < j$

$$E_{ij} = [[E_{i,i+1}, E_{i+1,i+2}] \cdots] E_{j-1,j}]$$

and $[e_\mu, e_\nu] \in (\mathfrak{sl}_n)_{\mu+\nu}$ via Jacobi, we see that (b) holds.

Since $|\Delta| = n-1 = \dim \mathfrak{h}^*$ it suffices to show now that Δ spans $\mathfrak{h}^*$. For this it suffices to show that for each $x \in \mathfrak{h}$ we have $x = 0$ iff $\alpha(x) = 0$ at all $\alpha \in \Delta$. Write

$$x = \sum_{i=0}^{i=n-1} c_i h_{\alpha_i}, \quad h_{\alpha_i} = E_{ii} - E_{i+1,i+1},$$

and observe

$$\alpha_i(x) E_{i,i+1} = [x, E_{i,i+1}] = (-c_{i-1} + 2c_i - c_{i+1}) \cdot E_{i,i+1}$$

so that $\alpha_i(x) = 0$ at all $i \Leftrightarrow$

$$\text{Cart}_n \cdot [c_1 \cdots c_{n-1}]^t = \vec{0} \qquad (*)$$

where $\text{Cart}_n = \begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & 0 \\ & -1 & \ddots & \\ 0 & & 2 & -1 \\ & & -1 & 2 \end{bmatrix}$.

We calculate by induction $\det(\text{Cart}_n) = n \neq 0$ so that the eq. $(*)$ forces $x = 0$, as desired.

We leave uniqueness as an exercise. ∎

Def^n: The subset $\Delta \subseteq \Phi^+$ as in

Proposition 8 is called the (or a, depending) base

for Φ. The elements of Δ are called the

simple roots for $\mathfrak{sl}_n$.

Recall explicitly

$$\Delta = \{\alpha_1, \ldots, \alpha_{n-1} : \mathfrak{sl}(\mathbb{C})_{\alpha_i} = \mathbb{C} E_{i,i+1}\}$$

We have also $h_{\alpha_i} = E_{ii} - E_{i+1,i+1}$.

Observation / Corollary 9: For the simple

roots $\alpha \in \Delta$, the corresponding vectors

$$\{h_\alpha : \alpha \in \Delta\}$$

provide a basis for the Cartan subalgebra $\mathfrak{h}$, and for

all $\gamma \in \Phi^+$ h_γ is the non-neg span

$$h_\gamma \in \mathbb{Z}_{\geq 0} \{h_\alpha : \alpha \in \Delta\}.$$

- **Weights and dominant weights.**

Def^n: A weight for $\mathfrak{sl}_n(\mathbb{C})$ is a

function $\lambda \in \mathfrak{h}^*$ which takes integer values

$$\lambda(h_\gamma) \in \mathbb{Z} \quad \text{at all } \gamma \in \Phi^+.$$

A weight λ is called dominant if it takes

nonnegative integer values

$$\lambda(h_\gamma) \in \mathbb{Z}_{\geq 0} \text{ at all } \gamma \in \Phi^+.$$

We take

$$P := \{\text{all weights in } \mathfrak{h}^*\}$$

$$P^+ := \{\text{all dominant weights in } \mathfrak{h}^*\}.$$

Lemma 10: $\lambda \in \mathfrak{h}^*$ is a wt iff $\lambda(h_\alpha) \in \mathbb{Z}$ for

all simple α, and a wt $\lambda \in P$ is dominant iff

$$\lambda(h_\alpha) \geq 0 \text{ at all simple } \alpha.$$

Proof: Immediate from Corollary 9.

Lemma 11: Every root $\gamma \in \Phi$ is also a weight.

Proof: Since $\Phi = \Phi^+ \cup \Phi^-$ and $\Phi^- = -\Phi^+$, and $\Phi^+ \subseteq \mathbb{Z}_{\geq 0} \Delta$, it suffices to show that each simple root α is a weight. But we have

$$\alpha_j(\langle h_{\alpha_i}\rangle) = \text{the coeff of } E_{\alpha_j} \text{ in } [h_{\alpha_i}, E_{\alpha_j}]$$

$$= \begin{cases} 2 & \text{if } i=j \\ -1 & \text{if } |i-j|=1 \\ 0 & \text{else.} \end{cases}$$

We're done.

Note that P is a lattice in h^*, i.e. a $\mathbb{Z}$-submodule w/ $\mathbb{C} \otimes_{\mathbb{Z}} P \xrightarrow{\cong} h^*$ via the natural map. We have

$$Q := \mathbb{Z} \cdot \Phi \subseteq P, \text{ another lattice}$$

We call Q the root lattice and P the weight lattice.

- A partial ordering on the weights

Def: For weights $\mu, \lambda \in P$ we write
$$\mu \leq \lambda \quad \text{if} \quad \lambda - \mu \in \mathbb{Z}_{\geq} \cdot \Delta,$$
i.e. if λ is obtained by a positive shift via simple roots
$$\lambda = \mu + \sum_i d_i \cdot \alpha_i, \quad d_i \in \mathbb{Z}_{\geq 0}.$$

Example: Take $\mathfrak{g}$ so that $\mathfrak{sl}_n(\mathbb{C}) \subseteq$ is $\mathbb{C} \cdot E_{nn}$. Then all roots γ satisfy $\mathfrak{g} \geq \gamma$. It is the longest root for $\mathfrak{sl}_n(\mathbb{C})$.

— Weights for $\mathfrak{sl}_n$-representations

Proposition 12: Let V be any finite dimensional $\mathfrak{sl}_n$-representation.

a) The Cartan subalgebra $\mathfrak{h}$ acts semisimply on V, i.e. V decomposes into eigenspaces for the $\mathfrak{h}$-action.

b) If the eigenspace V_λ is nonzero, for $\lambda \in \mathfrak{h}^*$, then λ lies in the weight lattice.

Proof: We have
$$\mathfrak{h} = \mathbb{C} \cdot \{ h_\alpha : \alpha \text{ simple} \}$$
and these h_α provide n commuting endomorphisms on V. Hence V decomposes into generalized eigenspaces for the action of these h_α, and hence for the action of $\mathfrak{h}$,
$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda^{gen}.$$

For each simple α we have the root subalgebra $\mathfrak{sl}_\alpha \to \mathfrak{sl}_n$, and restricting along α realizes an $\mathfrak{sl}_2$-action on V for which $h \in \mathfrak{sl}_2$ acts by h_α. Since h acts semisimply on any $\mathfrak{sl}_2$-rep, we conclude that each h_α acts semisimply on V. Hence V decomposes into its eigenspaces $V = \bigoplus_\lambda V_\lambda$.

Now, if $V_\lambda \neq 0$, then we again consider $\mathfrak{sl}_2$ acting on V_λ via α to see that $\lambda(\alpha) \in \mathbb{Z}$ at each simple α, via our classification of (simple) $\mathfrak{sl}_2$-representations. (Corollary 5, "Aug 25"). So we see that λ is in the weight lattice. ∎

Def^n: A weight vector of V is a nonzero vector which lies in some weight space V_λ, $\lambda \in P$. We call a weight vector $v \in V_\lambda$ a highest wt. vector if $e_\alpha \cdot v = 0$ for each simple root α.

Lemma 13: Any (fin. dim.) $\mathfrak{sl}_n$-rep V admits a highest wt. vector v.

Proof: Take any wt μ with V_μ non-vanishing. Since the base Δ provides a basis for $\mathfrak{h}^*$, we have for any tuple of non-neg integers $c: \Delta \to \mathbb{Z}_{\geq 0}$ and $c': \Delta \to \mathbb{Z}_{\geq 0}$, $\mu + \sum'_\alpha c_\alpha \alpha = \mu + \sum'_\alpha c'_\alpha \alpha$ iff $c_\alpha = c'_\alpha$ for all α. Hence the space of wts. $\{\lambda : \lambda \geq \mu$ and $V_\lambda \neq 0\}$ is finite and thus contains a max elem. λ under the ordering $\geq_0$. Any nonzero vector $v \in V_\lambda$ provides a highest wt. vect. in V. ∎

Theorem 14: If $v \in V$ is a highest wt. vector, with assoc. wt. $\lambda \in P$, then λ is dominant. Furthermore the subspace

$$\mathcal{L}(\lambda) = \mathbb{C} \cdot \{ f_{\beta_1} \cdots f_{\beta_t} \cdot v : t \geq 0, \ \forall_{\{1,\ldots,t\}} \to \Phi^+ \}$$

forms a $\mathfrak{sl}_n(\mathbb{C})$-subrep in V.

Proof: By restricting along any root subalg. $\gamma_\alpha : \mathfrak{sl}_2 \to \mathfrak{sl}_n$ we realize V as a $\mathfrak{sl}_2$-rep w/ h acting by the root vector h_α and e acting by e_α. Hence v is a highest wt vector for that $\mathfrak{sl}_2$-action, and we conclude [Cor 5, Aug 28] that the value $\lambda(h_\alpha)$ is a nonneg integer. Since α was chosen arbitrarily we see that λ is dominant.

The subspace $\mathcal{L}(\lambda)$ is clearly stable under the action of each f_α and h_α, for simple α, and for each e_α we have

$$e_\alpha \cdot f_{\beta_1} \cdots f_{\beta_t} \cdot v = [e_\alpha, f_{\beta_1} \cdots f_{\beta_t}] \cdot v$$

$$= \sum_i f_{\beta_1} \cdots f_{\beta_{i-1}} [e_\alpha, f_{\beta_i}] \cdots f_{\beta_t} \cdot v.$$

Each commutator $[e_\alpha, f_{\beta_i}] \in (\mathfrak{sl}_n)_{-\beta_i + \alpha}$ with one of three things occurring, by Proposition 8.

(10)

case I) $-\gamma_i + \alpha$ is not a root, and $[e_\alpha, f_{\gamma_i}] = 0$.

case II) $-\gamma_i + \alpha$ is a negative root, and
$$[e_\alpha, f_{\gamma_i}] = c_\alpha f_\beta$$
for some scalar c_α and β.

case III) $-\gamma_i + \alpha = 0$, i.e. $\gamma_i = \alpha$, and
$$[e_\alpha, f_{\gamma_i}] = h_\alpha.$$

In each case the term $f_{\gamma_i} \cdots [e_\alpha, f_{\gamma_i}] \cdots f_{\gamma_t} v$
lies in $L(\lambda)$, so that
$$e_\alpha \cdot f_{\gamma_i} \cdots f_{\gamma_t} \cdot v \in L(\lambda).$$
Since the e_α, f_α generate $\mathfrak{sl}_n(\mathbb{C})$ as a Lie algebra, we see that $L(\lambda) \leq V$ is a $\mathfrak{sl}_n$-subrep. ∎

Corollary 15: Each simple $\mathfrak{sl}_n(\mathbb{C})$-rep L admits a unique highest wt. vector $v \in L$, up to scaling.

Proof: L has some highest wt vector v. For any other highest wt vector v', Prop 14 tells us that L contains a subrep $L' \leq L$ with highest wt vector w and $(L')_\mu \neq 0$ implying, $\mu \leq \lambda' = wt(v')$. Since $L' = L$ necessarily we have $\lambda \leq \lambda'$ and similarly $\lambda' \leq \lambda$, giving $\lambda' = \lambda$. Prop 14 also gives $\dim[(L')_\lambda] = 1$ so that in fact $v' = c \cdot v$ for nonzero scalar c. ∎

— Characterizing rep $(\mathfrak{sl}_n(\mathbb{C}))$

Main Theorem: a) For each dominant weight $\lambda \in P^+$ there is a unique simple $\mathfrak{sl}_n(\mathbb{C})$-rep $L(\lambda)$ of highest weight λ.

b) The map
$$P^+ \longrightarrow Irrep\,(\mathfrak{sl}_n(\mathbb{C}))/\cong, \quad \lambda \longmapsto L(\lambda),$$
is a bijection.

c) The category $rep(\mathfrak{sl}_n(\mathbb{C}))$ is semisimple.

We'll deal w/ some details of the proof later in the class. Let me sketch some details here however

First an ideal of existence of largest wt samples $/P^+$. As a starting point we again consider the standard rep.

— Fundamental wts and the standard rep $\langle$ In full
discussion we
take many & simpler

for granted!

Def^n: Take $\Delta = \{\alpha_1 \to \alpha_{n-1}\}$ the standard base for $\mathfrak{sl}_n$, and let $h_{\alpha_i} \in \mathfrak{h}$ be the corresp vectors in the Cartan $\langle h_{\alpha_i} = E_{ii} - E_{i+1,i+1}\rangle$. The ith fundamental weight $w_i \in P^+ \subseteq \mathfrak{h}^*$ is the unique weight satisfying
$$w_i(h_{\alpha_j}) = \delta_{ij}.$$

Remark: Clearly $P = \mathbb{Z} \cdot \{w_i : 1 \le i \le n\}$.

By tensoring, we can observe samples of arbitrary highest wts if we can construct samples $L(w_i)$ of highest wt w_i at each fundamental wt.

Example: For $\mathfrak{sl}_n(\mathbb{C})$ we have the standard rep $V = \mathbb{C}^n$ w/ natural action.

For $v_i = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ ←i-th row the standard

basis vector, we have

$$e_{\alpha_i} \cdot v_j = \begin{cases} v_{i-1} & \text{if } j = i+1 \\ 0 & \text{else} \end{cases}$$

$$f_{\alpha_i} \cdot v_j = \begin{cases} v_{i+1} & \text{if } j = i \\ 0 & \text{else} \end{cases}$$

$$h_{\alpha_i} \cdot v_j = \begin{cases} 1 & \text{if } j = i \\ -1 & \text{if } j = i+1 \\ 0 & \text{else} \end{cases} \qquad (*)$$

$\Rightarrow \mathbb{V}$ has unique highest wt. vector v_1.

From $(*)$ we see $h_{\alpha_i} \cdot v_1 = \begin{cases} 1 & \text{if } i = 1 \\ 0 & \text{else.} \end{cases}$

so that v_1 has weight ω_1.

Proposition 16: The standard rep $\mathbb{V}$ over $\mathfrak{sl}_n(\mathbb{C})$

is the first fundamental simple

$$\mathbb{V} = \mathcal{L}(\omega_1),$$

i.e. it is a simple rep of highest wt ω_1.

Proof: We only note that $\mathbb{V}$ has a

unique highest wt vector, up to scaling, and is hence

simple by Theorem 14. We calculated the

highest wt as ω_1 above. ▨

– Aside: Symmetric and exterior powers of

representations.

One discovers directly that for any

Lie alg $\mathfrak{g}$, and $\mathfrak{g}$ reps V and W, the vect.

space swap

$$\tau_{V,W} : V \otimes W \longrightarrow W \otimes V, \quad \tau_{V,W}(v,w) := w \otimes v,$$

is an isomorphism of $\mathfrak{g}$-representations.

—

Lemma 17: For any maps of $\mathfrak{g}$-reps

$$\phi : V_0 \to V_1 \, , \quad \psi : W_1 \to W_0 \, , \quad \text{the map } \psi \otimes \phi \text{ is}$$

a map of $\mathfrak{g}$-reps and the diagram

$$
\begin{array}{ccc}
V_0 \otimes W_0 & \xrightarrow{\;\tau\;} & W_0 \otimes V_0 \\
\phi \otimes \psi \downarrow & & \downarrow \psi \otimes \phi \\
V_1 \otimes W_1 & \xrightarrow{\;\tau\;} & W_1 \otimes V_1
\end{array}
$$

commutes.

Proof: Clear by inspection. ▨

For any $\mathfrak{g}$-rep V and $n \geq 1$, the automorphisms

$$\tau_i = id^{\otimes i-1} \otimes \tau_{V,V} \otimes id^{\otimes n-i-1} : V^{\otimes n} \circlearrowleft$$

satisfy the relations

$$(*) \quad \begin{cases} \tau_i^2 = id \\ \tau_i \tau_j = \tau_j \tau_i & \text{when } |i-j| > 1 \\ \tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1} \end{cases}$$

and hence define an action of

$$S_n = \langle \tau_1, \ldots, \tau_{n-1} \mid \text{rels } (*) \rangle$$

via $\mathfrak{g}$-automorphisms,

$$S_n \longrightarrow \text{Aut}_{\mathfrak{g}}(V^{\otimes n}).$$

Def: Given any $\mathfrak{g}$-representation V, the n-th

symmetric power is the $\mathfrak{g}$-representation

$$S^n(V) := (V^{\otimes n})^{S_n} \quad \text{as } S_n\text{-invariants}$$

and the n-th exterior power

$$\Lambda^n(V) := \left[(V^{\otimes n}) \otimes \text{sgn} \right]_{S_n} \quad \text{as } S_n\text{-covariants}.$$

Explicitly, $\quad S^n(V) = \left\{ w \in V^{\otimes n} : w = \tau_i w \text{ at all} \atop i = 1, \ldots, n-1 \right\}$

$$= \text{Hom}_{S_n}(\mathbb{C}, V^{\otimes n})$$

$$\overset{\text{or}}{=} \text{equalizer} \left(V^{\otimes n} \underset{\tau_i}{\overset{id}{\rightrightarrows}} V^{\otimes n} : i = 1, \ldots, n-1 \right)$$

$$\Lambda^n(V) = V^{\otimes n} / (w - \varepsilon_i w : w \in V^{\otimes n} \text{ and } i=1, \ldots, n-1)$$

$$\overset{\text{or}}{=} (V^{\otimes n}) \otimes_{\mathbb{C} S_n} \text{sgn}$$

$$\overset{\text{or}}{=} \text{coequalizer} \left(V^{\otimes n} \underset{-\tau_i}{\overset{id}{\rightrightarrows}} V^{\otimes n} : i=1, \ldots, n-1 \right)$$

Exercise: Verify directly that $S^n(V)$ is a subrep in $V^{\otimes n}$ and that $\Lambda^n(V)$ is a quotient of $V^{\otimes n}$.

Example: For V the standard rep for $sl_2(\mathbb{C})$, $= L(1)$
$S^n(L(1))$ has highest wt. vector $v_1^{\otimes n} = \begin{bmatrix}1\\0\end{bmatrix}^{\otimes n}$ which is of wt $n \cdot 1 = n$. We have

$$\dim S^n(L(1))$$
$$= \dim \left(\mathbb{C} \cdot \left\{ \tfrac{1}{n!} (\Sigma_{\sigma \in S_n} \sigma) \cdot v_1^{\otimes n-m} \otimes v_{-1}^{\otimes m} \mid 0 \le m \le n \right\} \right)$$
$$= n+1.$$

For wt. reasons now $L(n) \le S^n(L(1))$ and for dim reasons this inclusion is an isomorphism;
$$S^n(L(\mathbb{C})) \cong L(n) \text{ at all } n.$$

For the exterior powers,
$$\Lambda^1(L(1)) = L(1), \quad \Lambda^2(L(2)) = \mathbb{C}\{v_1 \wedge v_{-1}\} \cong L(0),$$
$$\Lambda^{\ge 3}(L(2)) = 0.$$

- Exterior power of the standard rep
  Consider the n-dimensional standard rep
  $$V = \mathbb{C}^n,$$
we have the highest wt. vector
$$z_k := v_1 \wedge v_2 \wedge \cdots \wedge v_k \in \Lambda^k(V),$$
whenever $k \le n$. we exploit basis $\langle z_{(i)} = v_{i_1} \wedge \cdots \wedge v_{i_k}$ with

Lemma 18: z_k is of weight ϖ_k, and is the unique highest wt. vector in $\Lambda^k(V)$.

Proof: We have for $i < k$ directly
$$h_{\alpha_i} z_k = v_1 \wedge \cdots \wedge h_{\alpha_i} v_i \wedge \cdots \wedge v_k$$
$$+ v_1 \wedge \cdots \wedge h_{\alpha_i} v_{i+1} \wedge \cdots v_k$$
$$= z_k - z_k = 0$$
and for $k < i$ $h_{\alpha_i} z_k = 0$ as well. At $i=k$
$$h_{\alpha_k} z_k = v_1 \wedge \cdots \wedge h_{\alpha_k} v_k = z_k.$$

Then $h_i \cdot z_k = w_k(h_i) z_k$ at all $h_i \in h$, giving z_k of weight w_k.

For a general nonzero vector $z \in \bigwedge^k(\overline{V}/)$ take $z(i) = v_{i_1} \wedge \cdots \wedge v_{i_k}$ for i an increasing function $i: \{1, \ldots, k\} \to \{1, \ldots, n\}$, and order such functions via the dictionary ordering. Then

$$z = c_i \cdot z(i^\circ) + \sum_{j < i}' c_j \cdot z(j)$$

with c_i nonzero. Supposing $z \neq z_k$ we have a final index $i_\ell \in \{i_1, \ldots, i_k\}$ with $i_\ell - i_{\ell-1} > 1$, where we take formally $i_0 = 0$, and for

$$i' = \{i_1, \ldots, i_\ell - 1, i_{\ell+1}, \ldots, i_k\}$$

we have

$$e \alpha_{i_\ell - 1} \cdot z = c_i \cdot z(i') + \sum_{j' < i}' d_{j'} \cdot z(j')$$

In particular $e \alpha_{i_\ell - 1} z \neq 0$, and z is not a highest wt. vector. ▨

Theorem 19: For each integer $k = 1, \ldots, n-1$ the exterior power $\bigwedge^k(\overline{V}/)$ is a simple $sl_n(\mathbb{C})$-rep. of highest wt. w_k,

$$\bigwedge^k(\overline{V}/) = L_1(w_k).$$

Proof: By Theorem 14 any nonzero $sl_n(\mathbb{C})$-subrep $L \subseteq \bigwedge^k(\overline{V}/)$ contains a highest wt vector, and hence contains z_k by Lemma 18.

We claim now that $\qquad\qquad (*)$
$$\bigwedge^k(\overline{V}/) = \mathbb{C}\{f_{\alpha_{m_1}} \cdots f_{\alpha_{m_t}} \cdot z_k : t \geq 0, \alpha_{m_i} \in \{1, \ldots, t\} \to k\},$$
so that any subrep containing z_k must be all of $\bigwedge^k(\overline{V}/)$. For this consider again the basis vectors
$$\{z(i) = v_{i_1} \wedge \cdots \wedge v_{i_k} : \text{increasing fun } i: \{1, \ldots, k\} \to \{1, \ldots, n\}\}$$

We claim that each basic vector $z(i)$ is in the span (*), so that any subrep which contains $z_k = z(i_{min}) = z(1,2,\to k)$ is necessarily = to $\Lambda^k(\overline{V})$. We proceed by ind. under the dictionary order on the set of increasing funcs $i: \{1,\to k\} \to \{1,\to n\}$.

For the min index $i_{min} = (1,\to k)$ we have $z_k = z(i_{min}) \in \text{span}(*)$, and suppose now $i > i_{min}$ with $z(j) \in \text{span}(*)$ for all $j < i$. Since $i > i_{min}$ there is a first index
$$i_\ell \in \{i_1, \ldots, i_k\}$$
at which $i_\ell - i_{\ell-1} > 1$, where we take $i_0 = 0$. Then for the index $i' < i$ def by
$$i' = (i_1, \ldots, i_\ell - 1, \ldots, i_k)$$
we have $z(i') \in \text{span}(*)$ and
$$z(i) = f_{a_{i_\ell - 1}} \cdot z(i')$$
giving $z(i) \in \text{span}(*)$ as well. Hence all basis vectors $z(i) \in \text{span}(*)$ by induction. Consequently, any subrep $W \leq \Lambda^k(\overline{V})$ which contains the highest wt. vector z_k is equal to $\Lambda^k(\overline{V})$ and we conclude our arbitrary subrep L is in fact all of $\Lambda^k(\overline{V})$. This establishes simplicity.

Conclusion: For each fundamental wt w_k, $k = 1, \ldots, n-1$,
$$w_k(h_{a_i}) = \delta_{ik},$$
The k-th exterior power of the standard rep $\overline{V}$ realizes a simple $\mathfrak{sl}_n(\mathbb{C})$-rep of highest wt w_k. In particular, such highest wt. simples exist.

— Existence of highest wt. simples over P^+

Proposition 20: For each dominant weight $\lambda \in P^+$, there is a simple representation $L(\lambda)$ with a unique highest wt. vector of wt. λ.

Proof: Since λ is dominant we have unique nonnegative integers $m_1, \ldots, m_{n-1}$ so that

$$\lambda = m_1 \varpi_1 + \cdots + m_{n-1} \varpi_{n-1}.$$

By Thm 19 there exist simple sub-reps $L(\varpi_k)$ of highest wt. ϖ_k at each fundamental wt. ϖ_k. Let's take

$$V(\lambda) := L(\varpi_1)^{\otimes m_1} \otimes \cdots \otimes L(\varpi_k)^{\otimes m_k}.$$

Note that $\dim V(\lambda)_\lambda = 1$ and by Thm 14 applied to the $L(\varpi_k)$ we see

$$V(\lambda)_\mu \neq 0 \implies \mu < \lambda. \qquad (*)$$

Thus, for any composition series

$$0 \subset V_t \subseteq V_{t-1} \subseteq \cdots \subseteq V_0 = V(\lambda)$$

There exists a unique index i at which the (simple) composition factor

$$L = V_i / V_{i+1}$$

has $\dim L_\lambda = 1$ and $L_\mu \neq 0 \implies \mu < \lambda$. It follows that $L = L(\lambda)$ is of highest wt. λ. ∎

— Sketch proof of Uniqueness

Thm 21: Given simples L and L' of $\text{sl}_n(\mathbb{C})$- highest wt. λ, with highest wt. vectors v and v', there exists a unique iso $\phi: L \to L'$ over $\text{sl}_n(\mathbb{C})$ with $\phi(v) = v'$.

We'll be more careful about the proof when we deal w/ the general case. Let me sketch the details however.

Sketch Proof: We have the univ. env. alg. $\mathcal{U}(\mathfrak{sl}_2)$ and the subalg. env. alg. for the Borel $\mathcal{U}(\mathfrak{b}^+)$, $\mathfrak{b}^+ = h \oplus \mathfrak{n}^+$. We have the 1-dim simple $\mathbb{C}_\lambda$ of wt. λ over $\mathfrak{b}^+$ and define the Verma module $M(\lambda) = \mathcal{U}(\mathfrak{sl}_2) \otimes_{\mathcal{U}(\mathfrak{b}^+)} \mathbb{C}_\lambda$.

This is a highest weight, ∞-dim, weight graded, $\mathfrak{sl}_2$-rep and restriction along the inclusion $\mathbb{C}_\lambda \to M(\lambda)$ provides a linear $\cong$

$$\operatorname{Hom}_{\mathfrak{sl}_2}(M(\lambda), V) \xrightarrow{\sim} \mathbb{C} \cdot \left\{ \begin{array}{l} \text{highest wt. vectors} \\ v \in V \text{ of wt. } \lambda \end{array} \right\}.$$

$$f \longmapsto f(\mathfrak{t}_\lambda).$$

For generic reasons, there is a unique simple quotient $\pi: M(\lambda) \to \overline{M}(\lambda) \; (= L(\lambda))$, and hence the nonzero $\mathfrak{sl}_2$-maps

$$f: M(\lambda) \to L, \qquad f': M(\lambda) \to L'$$

$$f(\mathfrak{t}_\lambda) = v_\lambda \qquad \qquad f'(\mathfrak{t}_\lambda) = v'_\lambda$$

induce isoms

$$\begin{array}{ccc} & M(\lambda) & \\ \swarrow^{\exists !} & \vert & \searrow^{\exists !} \\ L & - - - \overset{\exists !}{\underset{\phi}{-} \to} & L' \end{array}$$

The induced $\cong$ $\phi: L \to L'$ completing the above diagram does the desired job. □

Corollary 22: The assignment

$$P^+ \longrightarrow \operatorname{Irrep}(\mathfrak{sl}_2(\mathbb{C})), \qquad \lambda \longmapsto L(\lambda)$$

is a bijection, i.e. classifies all irreducible $\mathfrak{sl}_2(\mathbb{C})$-reps up to isom.

— Semisimplicity for rep $(\mathfrak{sl}_n(\mathbb{C}))$.

Prop 22: For simple $\mathfrak{sl}_n(\mathbb{C})$-reps $L(\lambda)$ and $L(\mu)$, and extension

$$0 \to L(\mu) \to V \to L(\lambda) \to 0$$

is split.

Again, we'll cover the details more clearly in the general setting. We again sketch the details.

Proof: The adjoint rep $\mathfrak{sl}_n$ is again self-dual, so that the trace form on $\mathfrak{sl}_n$ induces a symm non-deg $\mathfrak{sl}_n$-invariant form

$$\kappa: \mathfrak{sl}_n^* \otimes \mathfrak{sl}_n^* \to \mathbb{C},$$

and under the natural $\mathfrak{sl}_n$ iso

$$\mathfrak{sl}_n \otimes \mathfrak{sl}_n \cong (\mathfrak{sl}_n^* \otimes \mathfrak{sl}_n^*)^*$$

κ determines an elem $\Omega \in \mathfrak{sl}_n \otimes \mathfrak{sl}_n$,

$$\Omega = \left(\sum_{\gamma \in \Phi^+} \langle f_\gamma \cdot e_\gamma + e_\gamma f_\gamma \rangle \right)$$
$$+ \left(\sum_i \text{Cartan terms} \right),$$

w/ $x \cdot \Omega = 0$ at all $x \in \mathfrak{sl}_n(\mathbb{C})$, aka Ω is an invariant elem in $\mathfrak{sl}_n^{\otimes 2}$.

The elem Ω therefore acts by $\mathfrak{sl}_n$-lin endos on all reps $\Omega_V = \Omega \cdots: V \to V$, and we can use Ω to split extensions of simples, just as in the case of $\mathfrak{sl}_2$. [Exercise]

Using [Prop 6, Aug 28] we now see that the category $\text{rep}(\mathfrak{sl}_n(\mathbb{C}))$ is semisimple.

Theorem 23: The cat $\text{rep}(\mathfrak{sl}_n(\mathbb{C}))$ is semisimple, and the simple $\mathfrak{sl}_n(\mathbb{C})$-reps are in bij correspondence with dominant wts, $\lambda \mapsto L(\lambda)$.

— We've now recovered our "Main Theorem" for $\mathfrak{sl}_n(\mathbb{C})$.

End.