Algebraic Groups and Representations

Today: Algebraic group basics and examples

I. Complex Lie groups vs. algebraic groups

From an analytic perspective, an affine algebraic group is a complex analytic group (Lie group) which is equipped with a good class of algebraic functions

$$\mathcal{O}_{alg} = \mathcal{O}_{alg}(G) = \left\{ \begin{array}{l} \text{Some specific subcollection of} \\ \text{analytic funs } f : G \to \mathbb{C} \end{array} \right\}.$$

These functions must be stable under the group ops:

- For $f : G \to \mathbb{C}$ algebraic, the composites

$$G \times G \xrightarrow{m} G \xrightarrow{f} \mathbb{C}$$

and $G \xrightarrow{inv.} G \xrightarrow{f} \mathbb{C}$

must be alg. funs on $G \times G$, and G, respectively.

(Here a fun. on $G \times G$ is alg if it's a product of alg funs $f_1 \cdot f_2 (z_1, z_2) := f_1(z_1) \cdot f_2(z_2)$ or a sum of such things.)

Ex: $\mathcal{O}_{alg}(GL_n) = \left\{ \begin{array}{l} \text{Polynomial funs in the} \\ \text{matrix entries } x_{ij} \end{array} \right\}$

$= \mathbb{C}[x_{ij} : 1 \le i,j \le n][\det^{-1}]$.

For each coord fn x_{ij} the composite

$$GL_n \times GL_n \xrightarrow{m} GL_n \xrightarrow{x_{ij}} \mathbb{C}$$

is the alg fn

$$\sum_{k=1}^{n} \underbrace{x_{ik}}_{\text{first factor}} \cdot \underbrace{x_{kj}}_{\text{second factor}},$$

and

$$GL_n \xrightarrow{\text{inv}} GL_n \xrightarrow{x_{ij}} \mathbb{C}$$

is $(-1)^{i+j} \det^{-1} \cdot (\det [x_{kl}]$ minus j-th row and i-th col$)$

$= (i,j)$-th entry in "$[x_{kl}]^{-1}$".

Since pulling back along m and inv preserve products and sums, it follows that

$f \circ m$ and $f \circ \text{inv}$

are alg whenever f is algebraic.

Other examples: - $SL_n(\mathbb{C})$ w/ polys in the x_{ij},
- $Sp_{2n}(\mathbb{C})$ w/ polys in the x_{ij}.
- $\mathbb{C}^\times$ w/ polys in z and z^{-1}.

The class of algebraic functions must form a finitely generated $\mathbb{C}$-alg, i.e. must admit some finite generating collection, and must be "complete", in the following sense:

For each point $p \in G$ there must be some collection of funs $f_1, f_2, \ldots, f_r$ in $\mathcal{O}_{alg}$ with

$$\{p\} = \mathrm{Van}(f_1, \ldots, f_r) \; (= \{q \text{ in } G \text{ w/ all } f_i(q) = 0\}).$$

Can talk about maps of alg groups as maps which preserve alg fun's, etc.

II. The pure algebraic perspective

Def$^{\underline{n}}$: An affine alg group G, over $\mathbb{C}$, is a group object in the category of affine $\mathbb{C}$-schemes.

So, $G = \mathrm{Spec}(\mathcal{O})$ for some f.g. $\mathbb{C}$-alg $\mathcal{O}$, and G comes equipped w/ an assoc. product map

$$G \times_{\mathrm{Spec}(\mathbb{C})} G \to G \qquad (*)$$

an inv. map $\mathrm{inv}: G \to G$ and unit $1: \mathrm{Spec}(\mathbb{C}) \to G$,

all of which are maps of schemes.

Def$^{\underline{n}}$: A map of affine alg groups $\xi: H \to G$ is a map of schemes which preserves the given group structures.

As affine $\mathbb{C}$-schemes are dual to comm (fin. gen) $\mathbb{C}$-algebras,

$$\mathrm{Spec}: (\mathrm{CAlg}_{\mathbb{C}})^{op} \xrightarrow{\sim} \mathrm{AffSch}_{\mathbb{C}},$$

the group structure $(*)$ specifies, and is specified by, a certain structure on $\mathcal{O}$:

Def$^{\underline{n}}$: A (commutative) Hopf algebra is a comm. (for us fin. gen'd) $\mathbb{C}$-alg $\mathcal{O}$ w/ algebra maps

comult) $\Delta : \mathcal{O} \to \mathcal{O} \otimes \mathcal{O}$

counit) $\epsilon : \mathcal{O} \to \mathbb{C}$

antipode) $S : \mathcal{O} \to \mathcal{O}$

which satisfy

coassoc.) $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$

counit ax) $(\epsilon \otimes 1)\Delta = (1 \otimes \epsilon)\Delta$

antip. ax) $mult(S \otimes id)\Delta = mult(id \otimes S)\Delta = unit \circ \epsilon$.

Ex) • $GL_n = Spec(\mathbb{C}[x_{ij} : 1 \le i,j \le n][\det^{-1}])$

w/ structure specified on gen's by

$$\Delta(x_{ij}) = \sum_k x_{ik} \otimes x_{kj}, \quad \epsilon(x_{ij}) = \delta_{ij},$$

$$S(x_{ij}) = \frac{(-1)^{i+j}}{\det} \cdot \det([x_{kl}] - j\text{-th row } i\text{th col}).$$

• $SL_n = Spec(\mathbb{C}[x_{ij} : 1 \le i,j \le n]/(\det - 1))$

w/ same Δ, ϵ and S,

$$S(x_{ij}) = (-1)^{i+j} \det([x_{kl}] - j\text{-th row } i\text{th col}).$$

We have the closed embedding

$$SL_n \to GL_n$$

dual to the Hopf alg surjection $\mathcal{O}(GL_n) \to \mathcal{O}(SL_n)$.

- $\mathbb{G}_m = \mathrm{Spec}(\mathbb{C}[x, x^{-1}])$, 5

$\Delta(x) = x \otimes x$, $\Delta(x^{-1}) = x^{-1} \otimes x^{-1}$, $\epsilon(x^{\pm 1}) = 1$, $S(x) = x^{-1}$.

- $\mathbb{G}_a = \mathrm{Spec}(\mathbb{C}[x])$ w/ $\Delta(x) = x \otimes 1 + 1 \otimes x$, $\epsilon(x) = 0$, $S(x) = -x$.

— III. Points in an algebraic group

Proposition 1: For G an affine group scheme and R any comm. $\mathbb{C}$-alg, the collection of R-points

$$G(R) := \{\text{Scheme maps } \mathrm{Spec}(R) \to G\}$$

$$= \{\text{Alg maps } \mathcal{O}(G) \to R\}$$

naturally forms a discrete group, and for any group hom $\xi : H \to G$, evaluating at R-points provides a discrete group map

$$\xi(R) : H(R) \to G(R).$$

Let's just explain this: Any two maps

$$A, B : \mathrm{Spec}(R) \to G$$

specify a single map to the product $[A\ B] : \mathrm{Spec}(R) \to G \times G$, and we can define

$$A \cdot B = \left(\mathrm{Spec}(R) \xrightarrow{[A\ B]} G \times G \xrightarrow{m} G \right).$$

The unit is given by the composite

$$1_{G(R)} = \left(\mathrm{Spec}(R) \xrightarrow[\text{map}]{\text{structure}} \mathrm{Spec}(\mathbb{C}) \xrightarrow{1} G \right)$$

and inverse is

$$A^{-1} = \left(\mathrm{Spec}(R) \xrightarrow{A} G \xrightarrow{\text{inv}} G \right).$$

Example: • $GL_n(R)$ = invertible matrices over R,

• $\mathbb{G}_m(R)$ = mult. group of units in R = $R^\times$,

• $\mathbb{G}_a(R)$ = additive group underlying $R = (R, +)$.

Rem: The $\mathbb{C}$-points $G(\mathbb{C})$ have the natural structure of a complex Lie group, and you can just think of G as the pairing of this complex Lie group $G(\mathbb{C})$ with the alg of fun's

$$\mathcal{O}(G) \subseteq \{\text{Analytic funs on } G(\mathbb{C})\}.$$

We'll elaborate more on this R-point stuff next time

IV. Points continued

Lets return to the claim:

For R comm $\mathbb{C}$-alg, G affine group scheme,

$$G(R) = \{\text{Alg maps } \mathcal{O}(G) \to R\}$$

has group structure induced by group str. on G / Hopf str. on $\mathcal{O}(G)$.

Example: $\mathbb{G}_a(R) = (R, +)$. (*) Lets check.

$\mathcal{O}(\mathbb{G}_a) = \mathbb{C}[x]$ w/ $\Delta(x) = x \otimes 1 + 1 \otimes x$.

$$\mathrm{Hom}_{\mathrm{Alg}}(\mathbb{C}[x], R) \xrightarrow{\sim} R, \quad \xi \mapsto \xi(x).$$

For two maps ξ_1, ξ_2 w/ $\xi_i(x) = r_i$, we have uniquely def

$$[\xi_1, \xi_2] : \mathbb{C}[x] \otimes \mathbb{C}[x] = \mathbb{C}[x_1, x_2] \to R$$
$$x_i \longmapsto r_i$$

and compose w/ comult to get the group structure

$$\xi_1 \cdot \xi_2 := \left(\mathbb{C}[x] \xrightarrow{\Delta} \mathbb{C}[x] \otimes \mathbb{C}[x] \xrightarrow{[\xi_1, \xi_2]} R \right)$$
$$x \longmapsto x \otimes 1 + 1 \otimes x \longmapsto r_1 + r_2.$$

$\xi_1 \cdot \xi_2(x) = r_1 + r_2$. This gives (*).

Anyway, to repr

— V. Representations of algebraic groups

Def$^{\underline{n}}$: A G-representation, for alg group G, is a vector space V equipped with a map of alg groups

$$\rho_V : G \to GL(V).$$

Then we take the coord free expression

$$GL(V) = \mathrm{Spec}\left\{ \mathrm{Sym}\left(\mathrm{End}_{\mathbb{C}}(V)^*\right)[\det^{-1}] \right\}$$

with

$$\Delta : \mathrm{Sym}(\mathrm{End}(V)^*)[\det^{-1}] \to \mathrm{Sym}(\mathrm{End}(V)^*)[\det^{-1}] \otimes 2$$

given on the generators by coevaluation

$$\Delta\big|_{\mathrm{End}^*} = \mathrm{coev}_V : \mathrm{End}_{\mathbb{C}}(V)^* = V^* \otimes V \longrightarrow \underbrace{V^* \otimes V \otimes V^* \otimes V}_{= \mathrm{End}(V)^* \otimes \mathrm{End}(V)^*}$$

$$f \otimes v \longmapsto \sum_i f \otimes \underbrace{v_i \otimes v^i}_{\text{doesn't depend on basis}} \otimes v.$$

and ϵ given on gen's by evaluation

$$\Delta\big|_{\mathrm{End}^*} = \mathrm{ev}_V : \mathrm{End}_{\mathbb{C}}(V)^* = V^* \otimes V \longrightarrow \mathbb{C}$$

$$f \otimes v \longmapsto f(v).$$

Rem: The antipode S on a bialgebra $\mathcal{O}$, if it exists, is uniquely determined. Hence a Hopf alg is, equiv, a bialgebra $(\mathcal{O}, \Delta, \epsilon)$ for which an antipode exists. The antipode is also preserved under Hopf maps. So we generally ignore it.

We'll give a few, equivalent descriptions of the category of G-representations

– VI. Corepresentations

Def$^{\underline{n}}$. Given a Hopf algebra (or coalg) $\mathcal{O}$ a corepresentation over $\mathcal{O}$ is a vector space V equipped with a linear map (coaction)

$$\rho_V : V \longrightarrow V \otimes \mathcal{O}$$

which is coassociative and counital

$$(1 \otimes \Delta)\rho_V = (\rho_V \otimes 1)\rho_V, \quad (1 \otimes \epsilon)\rho_V = id_V.$$

Example (Universal coaction): For $\mathcal{O}_V = \mathcal{O}(GL(V))$, I claim $\mathcal{O}_V$ always coacts on V in a natural way, giving it the structure of a $\mathcal{O}_V$-corepresentation.

We define

$$\rho_V^{univ} : V \longrightarrow V \otimes V^* \otimes V = V \otimes \mathrm{End}_{\mathbb{C}}(V)^* \subseteq V \otimes \mathcal{O}_V$$

by

$$v \longmapsto \mathrm{coev}(1) \otimes v = \sum_i v_i \otimes v^i \otimes v.$$

From the formula for $\Delta|_{\mathrm{End}(V)^*}$ this coaction is clearly coassociative, and for an expression

$$v = \sum_i c_i \cdot v_i$$

in a chosen basis

$$(1 \otimes \epsilon)\rho^{\mathrm{univ}}(v) = \sum_i v_i (v^i(v))$$

$$= \sum_i c_i \cdot v_i = v,$$

so counital as well.

VII. Reps vs. Coreps

Theorem 21: Let G be an algebraic group over $\mathbb{C}$. For a vector space V, the following data are equivalent:

a) The structure of a G-representation,

$$\rho_V : G \to GL(V).$$

b) The structure of an $\mathcal{O}(G)$-corepresentation,

$$\rho_V : V \to V \otimes \mathcal{O}(G).$$

c) The structure of an R-linear group action of $G(R)$ on the base change $V_R = V \otimes_{\mathbb{C}} R$, at each comm $\mathbb{C}$-alg R, which varies naturally in R.

We'll sketch a proof.

Sketch proof: (a) $\Rightarrow$ (b) Suppose we have a map

of algebraic groups $\phi_V : G \to GL(V)$. Then we obtain a Hopf algebra $\psi_V : \mathcal{O}(GL) \to \mathcal{O}(G)$ (via res. of algebraic functions along ϕ_V). We compose the universal coaction with ψ_V to obtain a coaction of $\mathcal{O}(G)$ on V,

$$\rho_V := (1 \otimes \psi_V)\rho_V^{univ} : V \to V \otimes \mathcal{O}(GL) \to V \otimes \mathcal{O}(G).$$

(b) $\Rightarrow$ (a) Given a coaction ρ_V we need to produce a Hopf map $\psi_V : \mathcal{O}(GL_V) \to \mathcal{O}(G)$. For this we need to produce a map on the generators

$$\mathcal{O}(GL_V) \supseteq \mathrm{End}(V)^* = V^* \otimes V \to \mathcal{O}(G).$$

Now the coaction gives an element

$$\rho_V \in \mathrm{Hom}_{\mathbb{C}}(V, V \otimes \mathcal{O}(G))$$

and via adjunction this determines a corresp. linear function

$$\psi_V|_{\mathrm{End}(V)^*} : \mathrm{End}(V)^* = V^* \otimes V \to \mathcal{O}(G).$$

This extends to an algebra map

$$\bar{\psi}_V : \mathrm{Sym}(\mathrm{End}(V)^*) \to \mathcal{O}(G)$$

which I promise sends the determinant to a unit and thus gives $\psi_V : \mathcal{O}(GL_V) \to \mathcal{O}(G)$. This map is a Hopf algebra map, again I promise.

(b) $\Rightarrow$ (c) Given $\xi \in G(R)$, $\xi : \mathcal{O}(G) \to R$,

We have the corresponding $\mathbb{C}$-linear function

$$\overline{act}_\xi : V \xrightarrow{\rho_V} V \otimes \mathcal{O}(G) \xrightarrow{1 \otimes \xi} V \otimes R .$$

This function determines a unique R-linear map

$$act_\xi : V_R \longrightarrow V_R \quad w/ \quad act_\xi |_V = \overline{act}_\xi .$$

This map is invertible with inverse $act_{\xi \circ S}$, and the assignment $\xi \mapsto act_\xi$ determines a natural action

$$G(R) \longrightarrow Aut_R(V_R) .$$

(c) $\Rightarrow$ (a) $GL(V)$ has functor of pts

$$GL(V)(R) = Aut_R(V_R),$$

and (c) determines a transformation

$$G(-) \longrightarrow GL(V)(-)$$

of functors $CAlg_{\mathbb{C}}^{op} \to Group$. By Yoneda this gives a group map $\chi_V : G \to GL(V)$. ∎

Rem: Note that morphisms of Groups, in terms of maps $\chi_V : G \to GL(V)$, are somewhat opaque. However, morphisms of corep, or in terms of (c), are clear.

— VIII. The category of G-representations

Def$^{\underline{n}}$: We define, for alg group G, $Rep(G) = Corep(\mathcal{O}(G))$.

Next Time: I'll explain clearly what a map of corep's is, and elaborate more on various things.

$\Rightarrow$ $\mathrm{Lie}(G)$ in alg terms

$\Rightarrow$ Examples

$\Rightarrow$ The functor $\mathrm{Rep}(G) \hookrightarrow \mathrm{Rep}(\mathrm{Lie}\, G)$.

IX. More fundamentals

For an alg group G recall that the R-point
$$G(R) = \mathrm{Hom}_{Sch}(\mathrm{Spec}\, R, G)$$
act on V_R, for $\mathcal{O}(G)$-comod V, as
$$x \cdot - : V_R \to V_R,$$

$$x(v \otimes r) = \sum_i v_{i_0} \otimes x(v_{i_1}) \cdot r$$
where $\Delta(v) = \sum_i v_{i_0} \otimes v_{i_1} \in V \otimes \mathcal{O}(G)$. In particular, the complex points $G(\mathbb{C})$ act as
$$x \cdot v = \sum_i x(v_{i_1})\, v_{i_0}. \qquad (*)$$

Lemma 3: Let $\rho_V^{univ} : V \to V \otimes \mathcal{O}(GL(V))$ be the universal coaction and $A \in GL(V)(\mathbb{C}) = \mathrm{Aut}_{\mathbb{C}}(V)$ be a complex point. Then
$$\underbrace{A \cdot v}_{\text{from } (*)} = \underbrace{A(v)}_{\text{expected action}}$$

Proof: In a basis $\{v_1, \dots, v_n\}$ we have $A = [a_{ij}]$ and then
$$A \cdot v_j = \sum_{i=1}^{n} v_i \otimes A(\underbrace{v^i \otimes v_j}_{\text{in } \mathrm{End}(V)^*}) = \sum_i v^i(A(v_j)) \cdot v_i$$
$$= \sum_i a_{ij} v_i$$
$$= A(v_j).$$
By linearity then $A \cdot v = A(v)$ at all $v \in V$. ∎

Elaborating on the equivalence between (a) and (b) in Theorem 2.

Lemma 4: Let $\phi : G \to GL(V)$ be a G-rep. with corresponding coaction $\rho = (1 \otimes \phi^*)\rho_V^{univ} : V \to V \otimes \mathcal{O}(G)$. For each $x \in G(\mathbb{C})$ and $v \in V$ we have

$$x \cdot v = \phi(x)(v).$$

Proof: We have $\phi(x) = (\mathcal{O}(GL(V)) \xrightarrow{\phi^*} \mathcal{O}(G) \xrightarrow{x} \mathbb{C})$ and $x \cdot - : V \to V$ is given by the composite

$$V \xrightarrow{\rho_V^{univ}} V \otimes \mathcal{O}(GL(V)) \xrightarrow{1 \otimes \phi^*} V \otimes \mathcal{O}(G) \xrightarrow{1 \otimes x} V \otimes \mathbb{C} \cong V$$

$$= \phi(x) \cdot - : V \to V.$$

Thus, by Lemma 3, $x \cdot v = \phi(x) \cdot v = \phi(x)(v)$. ∎

– V. Morphisms of G-repr via closed points

By a map of $\mathcal{O}$-coreps, $f : V \to W$, we mean the expected thing: A linear map which fits into a diagram

$$\begin{tikzcd} V \arrow[r, "\rho_V"] \arrow[d, "f"'] & V \otimes \mathcal{O} \arrow[d, "f \otimes 1"] \\ W \arrow[r, "\rho_W"'] & W \otimes \mathcal{O} \end{tikzcd}$$

Proposition 5: Let $\rho_V : G \to GL(V)$ and $\rho_W : G \to GL(W)$ be G-representations, with corresponding comodules (V, ρ_V) and (W, ρ_W). A linear map $f : V \to W$ is a map of $\mathcal{O}(G)$-comods if and only if, at each complex point $x \in G(\mathbb{C})$, we have

$$f(x \cdot v) = x \cdot f(v).$$

Before the proof we need an algebra fact.

Theorem (Reducedness): Any affine algebraic group G over $\mathbb{C}$ is reduced, i.e. has no nilpotent elements in its algebra of functions.

GK.

Proof of Proposition 5: If f is a map of $\mathcal{O}(G)$-comods then

$$(*) \quad f(x \cdot v) = f\left(\sum_i x(v_{i_1}) v_{i_0}\right) = \sum_i x(v_{i_1}) f(v_{i_0})$$

and by comodness

$$\sum_i f(v_{i_0}) \otimes v_{i_1} = \rho_W(f(v)),$$

so that $(*) = x \cdot f(v)$ as well.

Suppose conversely $f(x \cdot v) = x \cdot f(v)$ at all complex points. In a basis $\{w_1, \dots, w_m\}$ for W, we have at each $v \in V$

$$(f \otimes 1)\rho_V(v) = \sum_{j=1}^m w_j \otimes \xi_j$$

$$\rho_W(f(v)) = \sum_{j=1}^m w_j \otimes \xi_j'$$

for some functions $\xi_j, \xi_j' \in \mathcal{O}(G)$. Now the difference is given by

$$(f\otimes 1)(\rho_V(v)) - \rho_W(f(v)) = \sum_j w_j \otimes (\xi_j - \xi_j') \qquad (*)$$

and we know that at each $\mathbb{C}$-point $x: \mathrm{Spec}(\mathbb{C}) \to G$

$$f(x\cdot v) - x\cdot f(v) = (1\otimes x)\cdot(*) = \sum_j x(\xi_j - \xi_j')\, w_j = 0,$$

by assumption.

Hence $x(\xi_j - \xi_j') = 0$ at each j and all x. Thus the vanishing locus of $\xi_j - \xi_j'$ is all of $G(\mathbb{C})$,

$$\mathrm{Van}(\xi_j - \xi_j') = G(\mathbb{C}),$$

which gives by Nullsensatz (sp?) $\xi_j - \xi_j' \in \sqrt{0} \subseteq \mathcal{O}$, at each j. But finally by reducedness of G this implies $\xi_j - \xi_j' = 0$ at all j, and thus

$$(f\otimes 1)(\rho_V(v)) = \rho_W(f(v)).$$

Since this holds at all $v \in V$, we conclude that f is a map of $\mathcal{O}(G)$-corepresentations. ∎

Theorem 6: Let G be an affine algebraic group, and $\mathrm{Rep}(G)$ be the category whose objects are fin-dim vect spaces V equipped with a map of alg groups $\rho: G \to GL(V)$, and whose morphisms are linear maps $f: V \to W$ which satisfy $f(x\cdot v) = x\cdot f(v)$ at all $v \in V$ and $x \in G(\mathbb{C})$.

There is a canonical isomorphism of categories

$$t : \mathrm{Rep}(G) \xrightarrow{\sim} \mathcal{O}(G)\text{-comod}$$
$$(V, \rho) \longmapsto (V, (1 \otimes \rho^*)\rho_V^{univ})$$
$$f \longmapsto f.$$

Proof: By Theorem 2 t is bijective on objects and by Proposition 5 t is bijective on morphisms.

The fact that t respects composition is automatic by construction $t(f \circ g) = f \circ g = t(f) \circ t(g)$. ∎

Exercise: Prove that for an affine algebraic group G, the forgetful functor

$$\mathrm{Rep}(G) \longrightarrow \mathrm{Rep}(\underbrace{G(\mathbb{C})}_{\text{Considered as a discrete group!}})$$

is fully faithful.

[This is just me checking if you payed attention in Arunil's AG class.]

— XII. The tangent at the identity (pre-Lie)

Let G be an analytic group. We would think of the tangent space here as equivalence classes of functions

$$g : \mathbb{C} \supseteq D \to G$$

from a small disk D around 0 with $g(0) = 1$. Here

$$[g] \sim [h] \quad \text{if} \quad g'(0) = h'(0),$$

If we let ε be the parameter on D, we have an expansion in coords around 1

$$g(\varepsilon) = 1 + g_1 \varepsilon + O(\varepsilon^2)$$

and $[g] = [h]$ in $T_1 G$ iff $g_1 = h_1$. So, only the linear term matters.

In algebraic perspective we replace the infinitesimally small disk with the "dual number" $\mathbb{C}[\varepsilon]/\varepsilon^2$. We note that $\mathbb{C}[\varepsilon]/\varepsilon^2$ has only one map to $\mathbb{C}$, namely $\varepsilon \to 0$, and hence a canonical map for the functor of points

$$G(\mathbb{C}[\varepsilon]/\varepsilon^2) \to G(\mathbb{C})$$

at algebraic G. The tangent space at 1 is then

$$T_1 G := \{1_G\} \times_{G(\mathbb{C})} G(\mathbb{C}[\varepsilon]/\varepsilon^2)$$

$$= \left\{ \begin{array}{l} \text{Algebra maps } \mathcal{O}(G) \to \mathbb{C}[\varepsilon]/\varepsilon^2 \\ \text{whose composite along } \mathbb{C}[\varepsilon]/\varepsilon^2 \to \mathbb{C} \\ \text{recovers } 1_G = \epsilon \end{array} \right\}$$

where $m_G \subseteq \mathcal{O}$ = the kernel of counit ϵ.

Example: For GL_n

$$GL_n(\mathbb{C}[\varepsilon]/\varepsilon^2) = \text{Inv. } n\times n\text{-mat over } \mathbb{C}[\varepsilon]/\varepsilon^2$$

and

$$T_1 GL_n = \left\{ \begin{array}{l} \text{Inv. mat. } [a'_{ij} + a_{ij}\varepsilon] \text{ over} \\ \mathbb{C}[\varepsilon]/\varepsilon^2 \text{ w/ } [a'_{ij}] = I_n \end{array} \right\}$$

$$= \left\{ \begin{array}{l} I_n + A\varepsilon \text{ where } A \text{ is arbitrary} \\ \text{in } M_{n\times n}(\mathbb{C}) \end{array} \right\}.$$

Thus we have

$$M_{n\times n}(\mathbb{C}) \xrightarrow{\cong} T_1 GL_n, \quad A \longmapsto I_n + A\varepsilon.$$

Remark:

Note that, if we consider ε a complex number of sufficiently small magnitude, $I_n + A\varepsilon$ is actually invertible. So we literally have corresponding complex paths

$$I_n + A\varepsilon : \mathbb{D} \longrightarrow GL_n(\mathbb{C})$$

which realize an identification between algebraic $T_1 GL_n$ and analytic $T_1 GL_n(\mathbb{C})$.

Example: We have $SL_n(\mathbb{C}[\varepsilon]/\varepsilon^2) \subseteq GL_n(\mathbb{C}[\varepsilon]/\varepsilon^2)$ as matrices of determinant 1. Thus

$$T_1 SL_n = \left\{ \begin{array}{l} I_n + A\varepsilon \in T_1 GL_n \text{ with} \\ \det(I_n + A\varepsilon) = 1 \end{array} \right\}.$$

We check now

$$\det(I_n + A\varepsilon) = \det \begin{pmatrix} 1 + a_{11}\varepsilon & & a_{ij}\varepsilon \\ & \ddots & \\ a_{ji}\varepsilon & & 1 + a_{nn}\varepsilon \end{pmatrix} = 1 + \mathrm{tr}(A)\varepsilon.$$

Thus

$$T_1 SL_n = \{ 1 + A\varepsilon : \operatorname{tr}(A) = 0 \} \subseteq gl_n(\mathbb{C})$$
$$= sl_n(\mathbb{C})$$

Example: $Sp_{2n}(\mathbb{C}[\varepsilon]/\varepsilon^2) \subseteq GL_{2n}(\mathbb{C}[\varepsilon]/\varepsilon^2)$ is given by matrices $\tilde{A}$ over $\mathbb{C}[\varepsilon]/\varepsilon^2$ with

$$\langle \tilde{A} \cdot v, w \rangle = \langle v, \tilde{A}^{-1} \cdot w \rangle \qquad (*)$$

where

$$\langle , \rangle : \mathbb{C}^{2n} \otimes \mathbb{C}^{2n} \to \mathbb{C}$$

is the anti-symmetric form $\langle v_i, v_j \rangle = \delta_{j\,i+n} - \delta_{i\,j+n}$.

Now for an element of the tangent space

$$\tilde{A} = I_n + A\varepsilon \in T_1 GL_n$$

we have

$$\tilde{A}^{-1} = I_n - A\varepsilon$$

so that $(*)$ becomes

$$\langle v, w \rangle + \langle A \cdot v, w \rangle = \langle v, w \rangle - \langle v, A \cdot w \rangle$$
$$\Longleftrightarrow \quad \langle A \cdot v, w \rangle = - \langle v, A \cdot w \rangle.$$

Hence we calculate

$$T_1 Sp_{2n} = sp_{2n}(\mathbb{C}) \subseteq gl_{2n}(\mathbb{C}).$$

Similarly $T_I SO_n = so_n(\mathbb{C}) \subseteq gl_n(\mathbb{C})$

– ~~XIII~~ The Lie structure on the tangent space

Def^n: For functions $\alpha, \beta : \mathcal{O}(G) \to \mathbb{C}$ (here I just mean linear functions) the convolution product $\alpha * \beta : \mathcal{O}(G) \to \mathbb{C}$ is the new function defined by

$$\alpha * \beta(\xi) = \sum_i \alpha(\xi_{i_1})\beta(\xi_{i_2}),$$

where

$$\Delta(\xi) = \sum_i \xi_{i_1} \otimes \xi_{i_2}.$$

Note now that we have the inclusion

$$\mathrm{incl} : T_1 G \hookrightarrow \mathrm{Hom}_{\mathbb{C}}(\mathcal{O}(G), \mathbb{C})$$

$$\left(\tilde{\alpha} = 1_G + \alpha\varepsilon : \mathcal{O}(G) \to \mathbb{C}[\varepsilon]/\varepsilon^2\right) \longmapsto \left(\alpha : \mathcal{O}(G) \to \mathbb{C}\right).$$

This inclusion identifies $T_1 G$ with functions $\alpha : \mathcal{O}(G) \to \mathbb{C}$ which satisfy

T1) $\alpha|_{m_G^2} = 0$

T2) $\alpha|_{\mathbb{C}1_G} = 0$.

Example: For $G = GL_n$, the inclusion incl sends A in $\mathfrak{gl}_n(\mathbb{C}) \cong T_1 GL_n$ to the unique linear function

"A": $\mathcal{O}(GL_n) = \mathbb{C}[x_{ij} : 1 \le i,j \le n][\det^{-1}] \to \mathbb{C}$

which satisfies T1 and T2 and has

"A"$(x_{ij}) = a_{ij}$.

Theorem 7: Considering $\mathrm{Hom}_{\mathbb{C}}(\mathcal{O}(G), \mathbb{C})$ as an assoc. alg under convolution, the inclusion

$$\mathrm{incl}: T_1 G \hookrightarrow \mathrm{Hom}_{\mathbb{C}}(\mathcal{O}(G), \mathbb{C})$$

identifies $T_1 G$ with a Lie subalg in

$$\underbrace{\mathrm{Hom}_{\mathbb{C}}(\mathcal{O}(G), \mathbb{C})}_{\text{Lie alg under convolution commutator } \alpha * \beta - \beta * \alpha.}{}^{\mathrm{Lie}}.$$

Proof: Take any two such functions α, β which satisfy (T1) and (T2). As $\Delta(1) = 1 \otimes 1$

$$\alpha * \beta - \beta * \alpha \big|_{\mathbb{C} \cdot 1_{\mathcal{O}}} = 0,$$

and for $x \in \mathfrak{m}$ we have

$$\Delta(x) = 1 \otimes x + x \otimes 1 \mod (\mathfrak{m} \otimes \mathfrak{m})$$

since $(\epsilon \otimes 1)\Delta(x) = (1 \otimes \epsilon)\Delta(x) = x$. Thus for $x \cdot y$ with $x, y \in \mathfrak{m}$

$$\begin{aligned}\Delta(xy) &= \Delta(x) \cdot \Delta(y) \\ &= xy \otimes 1 + 1 \otimes xy + x \otimes y + y \otimes x \mod (\mathfrak{m}^2 \otimes \mathfrak{m}^2) \\ &= x \otimes y + y \otimes x \mod (\mathfrak{m}^2 \otimes \mathfrak{m}^2)\end{aligned}$$

so that

$$\begin{aligned}\alpha * \beta(x \cdot y) &= \alpha(x)\beta(y) + \alpha(y)\beta(x) \\ &= \beta * \alpha(xy).\end{aligned}$$

Hence $[\alpha, \beta](xy) = 0$. Since $\mathfrak{m}^2 = \mathrm{Span}_{\mathbb{C}}(xy : x, y \in \mathfrak{m})$ we have $[\alpha, \beta]\big|_{\mathfrak{m}^2} = 0$. ∎

Def$^{\underline{n}}$: For G an affine algebraic group

$$\mathrm{Lie}\, G := \{ T_1 G \text{ w/ convolution bracket} \}$$

– XIV. Functoriality of Lie(–)

For $\phi : H \to G$ a map of algebraic groups the corresp map on functions

$$\phi^* : \mathcal{O}(G) \to \mathcal{O}(H)$$

is a Hopf map. Thus restricting along ϕ^* gives a map of (Lie) algebras

$$\phi_* : \mathrm{Hom}_{\mathbb{C}}(\mathcal{O}(H), \mathbb{C})^{\mathrm{Lie}} \to \mathrm{Hom}_{\mathbb{C}}(\mathcal{O}(G), \mathbb{C})^{\mathrm{Lie}}$$

which furthermore fits into a diagram

$$\begin{tikzcd} T_1 H \arrow[r, "T_1\phi"] \arrow[d, hook] & T_1 G \arrow[d, hook] \\ \mathrm{Hom}(\mathcal{O}(H), \mathbb{C})^{\mathrm{Lie}} \arrow[r, "\phi_*"'] & \mathrm{Hom}(\mathcal{O}(G), \mathbb{C})^{\mathrm{Lie}} \end{tikzcd}$$

(Just follow by $(\phi^*)^{-1}(\mathfrak{m}_H) = \mathfrak{m}_G$.) Thus

$T_1 \phi : T_1 H \to T_1 G$ is a map of Lie algebras,

$\mathrm{Lie}\, \phi : \mathrm{Lie}\, H \to \mathrm{Lie}\, G$.

XV. Examples

Theorem 8: The linear identification

$$\mathfrak{gl}_n(\mathbb{C}) \xrightarrow{\sim} \mathrm{Lie}\, GL_n, \quad A \longmapsto I_n + A\varepsilon,$$

is an isomorphism of Lie algebras.

Proof: The above isomorphism identifies, further, a matrix A with the function "A" $: \mathcal{O} \to \mathbb{C}$ w/ "A"$(x_{ij}) = a_{ij}$ as before. Then for matrices A, B we have

$$[\text{"}A\text{"}, \text{"}B\text{"}](x_{ij}) = \sum_k A(x_{ik})B(x_{kj}) - \sum_k B(x_{ik})A(x_{kj})$$

$$= \sum_k a_{ik} \cdot b_{kj} - b_{ik} \cdot a_{kj}$$

$$= \text{"}[A, B]\text{"}(x_{ij}).$$

We're done. ∎

Corollary 9: The identifications

- $\mathfrak{sl}_n(\mathbb{C}) \xrightarrow{\sim} \mathrm{Lie}\, SL_n$
- $\mathfrak{sp}_{2n}(\mathbb{C}) \xrightarrow{\sim} \mathrm{Lie}\, Sp_{2n}$
- $\mathfrak{so}_n(\mathbb{C}) \xrightarrow{\sim} \mathrm{Lie}\, SO_n$

are all isomorphisms of Lie algebras.

Proof: For $G = SL_n, Sp_{2n}, SO_n$ and $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C}), \mathfrak{sp}_{2n}(\mathbb{C}), \mathfrak{so}_n(\mathbb{C})$ we've established

diagrams of linear inclusions

$$\begin{tikzcd} \mathfrak{g} \arrow[r] \arrow[d, "\wr"] & \mathfrak{gl}_m(\mathbb{C}) \arrow[d, "\wr"] \\ \mathrm{Lie}\, G \arrow[r] & \mathrm{Lie}\, GL_m \end{tikzcd}$$

where each map but $\mathfrak{g} \xrightarrow{\sim} \mathrm{Lie}\, G$ is now understood to be a map of Lie algebras. But now the diagram forces this final map to be an isomorphism of Lie algebras. ∎

– XVI. Infinitesimal actions for G-reps

Let fix G algebraic and $\mathfrak{g} = \mathrm{Lie}(G)$. Fix also $\mathcal{O} = \mathcal{O}(G)$. Of our many options, we consider now $\mathrm{Rep}\, G$ via $\mathrm{Corep}(\mathcal{O})$ and consider

$$\mathfrak{g} \subseteq \{\text{linear funct } \mathcal{O} \to \mathbb{C}\}$$

consisting of all maps which

T1) Vanish on m_G^2

T2) Vanish on $1 \in \mathcal{O}$.

algebra under convolution

Proposition 10: Take $H_G = \mathrm{Hom}_{\mathbb{C}}(\mathcal{O}, \mathbb{C})$.

a) For any G-rep V, considered as $\mathcal{O}$-corep, the formula

$$\alpha \cdot v = (1 \otimes \alpha)\rho_V(v) = \sum_i \alpha(v_{i_1}) \cdot v_{i_0} \in V$$

defines an assoc. action of H_G on V.

b) Restricting to elements A in $\mathfrak{g}$, the action $A \cdot v =$ as in (a) defines an action of $\mathfrak{g}$ on V, i.e. gives it the structure of a $\mathfrak{g}$-representation.

Proof: a) Recall the product on $\mathcal{H}_G$,

$$\alpha * \beta(\xi) := (\alpha * \beta)\Delta(\xi)$$

at $\xi \in \mathcal{O}$. Then for $v \in V$,

$$(\alpha * \beta)(v) = (1 \otimes \alpha \otimes \beta)(1 \otimes \Delta)\rho_V(v)$$

$$= (1 \otimes \alpha \otimes \beta)(\rho_V \otimes 1)\rho_V(v) \quad \text{(Coassoc.)}$$

$$= (1 \otimes \alpha \otimes 1)(\rho_V \otimes \beta)\rho_V(v)$$

$$= (1 \otimes \alpha \otimes 1)(\rho_V \otimes 1)(1 \otimes \beta)\rho_V(v)$$

$$\begin{tikzcd}
V \arrow[r, "\rho_V"] \arrow[drrr, "\beta\cdot -"'] & V \otimes \mathcal{O} \arrow[r, "1\otimes\beta"] & V \otimes \mathbb{C} \arrow[r, "\rho_V \otimes 1"] & V \otimes \mathcal{O} \otimes \mathbb{C} \arrow[r, "1\otimes\alpha\otimes 1"] & V \otimes \mathbb{C} \otimes \mathbb{C} \\
 & & V \arrow[u, "\wr"] \arrow[r] & V \otimes \mathcal{O} \arrow[u, "\wr"] \arrow[r] & V \arrow[u, "\wr"] \\
 & & & \alpha\cdot - \arrow[ur, brace] &
\end{tikzcd}$$

$$= \alpha \cdot (\beta \cdot v).$$

We thus obtain associativity.

b) Any $\mathcal{H}_G$-module becomes a $\mathcal{H}_G^{Lie}$-rep, under the same action. Hence restricting to the subalgebra

$$\mathfrak{g} \hookrightarrow \mathcal{H}_G^{Lie}$$

produces a $\mathfrak{g}$-rep structure on any $\mathcal{O}$-corepresentation V. ∎

Def$^{\underline{n}}$: For a G-representation V we call the corresponding action of $\mathfrak{g} = \mathrm{Lie}\, G$ on V, as in Proposition 10, the *infinitesimal action* of $\mathfrak{g}$ on V.

Clearly the infinitesimal action is natural, in the sense that a map $f: V \to W$ of G-representations is also a map of $\mathfrak{g}$-reps, under the infinitesimal action. So we get a <u>functor</u> (!)

$$\inf = \inf_G : \mathrm{Rep}(G) \to \mathrm{Rep}(\mathfrak{g}).$$

Proposition 11: Let $\phi: H \to G$ be a map of algebraic groups and $\mathrm{Lie}\,\phi : \mathrm{Lie}\, H \to \mathrm{Lie}\, G$ be the induced map on Lie algebras. Then for any G-rep V with restriction $V|_H$ to H

$$\inf_H(V|_H) = \inf_G(V)|_{\mathrm{Lie}\, H} \qquad (*)$$

Proof: $\phi: H \to G \Rightarrow \phi^*: \mathcal{O}(G) \to \mathcal{O}(H)$
$\Rightarrow \phi_*: \mathcal{H}_H \to \mathcal{H}_G$
which fits into a diagram

$$\begin{tikzcd} & \mathcal{H}_H \arrow[r, "\phi_*"] & \mathcal{H}_G \\ \mathfrak{h} \arrow[ur] \arrow[rr, two heads, "\mathrm{Lie}\,\phi"'] & & \mathfrak{g} \arrow[u] \end{tikzcd}.$$

This is sufficient for $(*)$. ∎

Example (The universal example):
We have the standard representation V over $GL(V)$ with corresponding universal coaction
$$\rho_V^{univ}(v) = \sum_{i=1}^{n} v_i \otimes v^i \otimes v \in V \otimes \mathrm{End}(V)^* \subseteq V \otimes \mathcal{O}.$$

In coordinates, A in $gl(V) = \mathrm{Lie}\, GL(V)$ is given by
$$A = [a_{ij}] = \sum_{i,j} a_{ij}\, v_i \otimes v^j .$$
Then for the expression
$$v = \sum_j c_j v_j$$
we get for the infinitesimal action of $gl(V)$ on V,
$$A \cdot v = \sum_i A(v^i \otimes v) \cdot v_i$$
$$= \sum c_j a_{ij} \cdot v_i = A(v).$$

This is to say, the standard rep for $GL(V)$ becomes the standard rep for $gl(V) = \mathrm{Lie}\, GL(V)$ under the infinitesimal action.

As a corollary to this Example and Proposition 11 we find an alternate definition of the infinitesimal action.

Theorem 12: For a G-representation $\rho_V : G \to GL(V)$ the infinitesimal action of $\mathfrak{g} = \mathrm{Lie}\, G$ on V is determined by the map $\mathrm{Lie}\,\rho_V : \mathfrak{g} \to gl(V)$, and the standard

action of $\mathfrak{gl}(V)$ on V.

Remark: The description of the infinitesimal action as in Proposition 10 is good because it makes it clear that we have a functor

$$\inf : \mathrm{Rep}(G) \to \mathrm{Rep}(\mathfrak{g}).$$

On the other hand Theorem 12 makes it clear that the infinitesimal action of $\mathfrak{g}$ on V is defined in the expected way: An infinitesimal path through the identity

$$\mathbb{C} \supseteq D \xrightarrow{t} G$$

composes to give an infinitesimal path in $GL(V)$

$$\begin{tikzcd} & G \arrow[dr, "\phi"] & \\ D \arrow[ur, "t"] \arrow[rr] & & GL(V) \end{tikzcd}$$

$$\varepsilon \longmapsto I_n + A\varepsilon + O(\varepsilon^2),$$

and the element $[t]$ in $\mathfrak{g}$ acts "infinitesimally" on V via the matrix

$$\partial \phi \circ t / \partial z \,\big|_{z=0} = A \in \mathfrak{gl}(V).$$

— XVII. Symmetric monoidality of $\inf_G$

For G-representations V and W we have the apparent $G(\mathbb{C})$-action on $V \otimes W$ given by the expected formula $g.(v \otimes w) = (g.v) \otimes (g.w)$.

This "diagonal" action of $G(\mathbb{C})$ extends uniquely

to a G-representation structure on the tensor product. In terms of comodules,

$$\rho_{V\otimes W} = (1\otimes 1\otimes \mathrm{mult}_{\mathcal{O}})(1\otimes \mathrm{swap}\otimes 1)(\rho_V\otimes\rho_W):$$

$$V\otimes W \xrightarrow{\rho's} (V\otimes W)\otimes(\mathcal{O}\otimes\mathcal{O}) \xrightarrow{1\otimes \mathrm{mult}} (V\otimes W)\otimes\mathcal{O}.$$

The standard vector space swap

$$\tau_{VW}: V\otimes W \to W\otimes V, \quad v\otimes w \mapsto w\otimes v,$$

is an isomorphism of G-representations. Thus we have a symmetric tensor structure on $\mathrm{Rep}\, G$.

Theorem 13: The functor

$$\inf: \mathrm{Rep}(G) \to \mathrm{Rep}(\mathfrak{g})$$

is symmetric monoidal.

Proof: Exercise. ∎

— XVIII. Full faithfulness of $\inf_G$

Theorem 14: For any algebraic group G with Lie algebra $\mathfrak{g} = \mathrm{Lie}\, G$, the functor

$$\inf_G: \mathrm{Rep}(G) \to \mathrm{Rep}(\mathfrak{g})$$

is <u>fully faithful</u> provided G is connected.

However one approaches this, there are two basic principles we need to deal with. We sketch some details

Principle I (Actions are determined in an infinitesimal neighborhood of the identity)

By restricting along the embedding $\hat{G}_1 \to G$ from the formal neighborhood around the identity we obtain a functor

$$\mathrm{res}: \mathrm{Rep}(G) \to \mathrm{Rep}(\hat{G}_1).$$

Now, rep $\hat{G}_1 = \mathrm{Corep}(\hat{\mathcal{O}})$ where

$$\hat{\mathcal{O}} = \varprojlim_n \hat{\mathcal{O}}/\mathfrak{m}_G^n, \quad \mathfrak{m}_G = \ker(1_G).$$

complete local ring at $1_G \in G(\mathbb{C})$.

The Hopf structure on $\mathcal{O}$ becomes a topological Hopf structure on $\hat{\mathcal{O}}$, $\hat{\Delta}: \hat{\mathcal{O}} \to \hat{\mathcal{O}} \hat{\otimes} \hat{\mathcal{O}} = \varprojlim_n \mathcal{O}/\mathfrak{m}^n \otimes \mathcal{O}/\mathfrak{m}^n$

so we can consider

$$\mathrm{Rep}(\hat{G}_1) := \mathrm{Corep}(\hat{\mathcal{O}})$$

and the restriction functor is just given by "corestricting"

$$\mathrm{res}: \mathrm{Corep}(\mathcal{O}) \to \mathrm{Corep}(\hat{\mathcal{O}}).$$

Since G is smooth [a fact] and connected the canonical map

$$\mathcal{O} \to \hat{\mathcal{O}}$$

is injective, hence res is fully faithful. This is to say, any representation $\phi: G \to GL(V)$ is determined by its restriction to the "infinitesimal nbd" $\phi|\hat{G}_1$; and

and any linear map $f: V \to W$ is a map of G-repr if and only if f is a map of $\hat{G}_1$ repr.

Rem: Any nbd (Zariski or analytic) $1 \subseteq \mathcal{U} \subseteq G$ contains $\hat{G}_1 \subseteq \mathcal{U} \subseteq G$. So any algebraic rep V has G-action det in any nbd $\mathcal{U} \subseteq G$, and G-linearity of any map holds iff

$$f(x \cdot v) = x \cdot f(v) \text{ for all } x \in \mathcal{U}(\mathbb{C}).$$

Principle II (The Lie algebra $\mathfrak{g}$ knows everything about the formal nbd $\hat{G}_1$)

Consider the topological dual

$$\mathcal{U}_G := \varinjlim_n (\mathcal{O}/m_G^n)^* \subseteq \mathcal{H}_G.$$

This is a subalgebra with

$$\mathfrak{g} \hookrightarrow \mathcal{U}_G \subseteq \mathcal{H}_G,$$

(Lie, Lie)

since all $\mathrm{ins}(x)$ for x in $\mathfrak{g}$ vanish on m^2. Now, we have

$$\mathcal{U}_G^* = \left(\varinjlim_n (\mathcal{O}/m_G^n)^*\right)^* = \varprojlim (\mathcal{O}/m^n)^{**}$$

$$= \varprojlim \mathcal{O}/m^n = \hat{\mathcal{O}},$$

so finally we have an identification of categories

$$\mathrm{Rep}(\hat{G}_1) = \mathcal{U}_G\text{-mod}_{\text{fin dim}},$$

(Cf.

$$\mathrm{Hom}_{\mathbb{C}}(V \otimes \mathcal{U}_G, V) = \mathrm{Hom}_{\mathbb{C}}(V, \mathrm{Hom}_{\mathbb{C}}(\mathcal{U}_G, V))$$

$$= \mathrm{Hom}_{\mathbb{C}}(V, V \otimes \mathcal{U}_G^*)$$
$$= \mathrm{Hom}_{\mathbb{C}}(V, V \otimes \hat{\mathcal{O}}).\ )$$

Now, the Lie embedding

$$\mathfrak{g} \hookrightarrow \mathcal{U}_G^{Lie}$$

determines an algebra map

$$\mathrm{can}: \mathcal{U}(\mathfrak{g}) \to \mathcal{U}_G^{Lie}.$$

Using smoothness of G, one can prove that in fact this map can is an <u>isomorphism</u>, giving

$$\begin{tikzcd}
\mathrm{Rep}(G) \arrow[r, "\inf"] \arrow[d, "\text{full}", "\text{faithful}"'] & \mathrm{Rep}(\mathfrak{g}) \\
\mathrm{Rep}(\hat{G}_1) = \mathrm{Corep}(\hat{\mathcal{O}}) \arrow[r, "\cong"] & \mathcal{U}(\mathfrak{g})\text{-mod}_{\text{findim}} \arrow[u, "\wr"']
\end{tikzcd}.$$

Thus $\inf$ is fully faithful. ∎

– XIX. Integrable representations

Def$^{\underline{n}}$: For $\mathfrak{g} = \mathrm{Lie}\, G$, we say a $\mathfrak{g}$-representation V is integrable (for G) if there exists V' in $\mathrm{Rep}(G)$ with $\inf(V') = V$.

In this case note that $V' = V$, and that the G-action on V is actually fixed, due to full faithfulness of $\inf: \mathrm{Rep}(G) \to \mathrm{Rep}(\mathfrak{g})$.

So we are simply asking if we can "integrate" the $\mathfrak{g}$-action on V to an action of the group G.

Another way of saying Thm 14 is as follows.

Theorem 15: The functor

$$\inf_G : \mathrm{Rep}(G) \longrightarrow \mathrm{Rep}(\mathfrak{g})$$

is a symmetric monoidal equivalence (in fact isomorphism) onto the full subcategory of integrable $\mathfrak{g}$-reps in $\mathrm{Rep}(\mathfrak{g})$,

$$\inf_G : \mathrm{Rep}(G) \xrightarrow{\sim} \mathrm{Rep}(\mathfrak{g})_{\mathrm{int}}.$$

The following is also clear from the proof of Thm 14.

Corollary (to proof) 16: The subcategory of integrable reps in $\mathrm{Rep}(\mathfrak{g})$ is closed under taking subobjects and quotients.

~~XX~~. Examples

Example 1: For discrete (finite) G we have $\mathrm{Lie}(G) = 0$ so that $\mathrm{Rep}(\mathfrak{g}) = \mathrm{Vect}$, and the infinitesimal action functor

is just the forgetful functor

$$\mathrm{inf}_G : \mathrm{Rep}(G) \to \mathrm{Vect}.$$

This is faithful, but not full, and clearly not an equivalence when G is nontrivial.

Example 2: Consider $B \subseteq GL_n$ consisting of non-strictly upper Δ inv. matrices. Then an element

$$\tilde{A}(\varepsilon) = I_n + A\varepsilon \in \mathrm{Lie}\, GL_n$$

lies in $\mathrm{Lie}\, B \subseteq \mathrm{Lie}\, GL_n$ if and only if $\tilde{A}(\varepsilon)$ is upper Δ. This occurs if and only if A is upper Δ's (with arbitrary entries in a_{ij}, $i<j$). Hence

$$\mathrm{Lie}\, B = \mathfrak{b} \subseteq \mathfrak{gl}_n(\mathbb{C}).$$

Take $T \subseteq GL_n$ the inv. diagonal matrices. We have $\mathrm{Lie}\, T = \mathfrak{h} = \mathrm{diags} \subseteq \mathfrak{gl}_n(\mathbb{C})$, and there is a diagram

$$\begin{tikzcd}
\mathrm{Rep}(B) \arrow[r, "\mathrm{inf}_B"] \arrow[d, "\mathrm{res}"'] & \mathrm{Rep}(\mathfrak{b}) \arrow[d, "\mathrm{res}"] \\
\mathrm{Rep}(T) \arrow[r, "\mathrm{inf}_T"'] & \mathrm{Rep}(\mathfrak{h})
\end{tikzcd}$$

by Prop 11. Since $T = \mathbb{G}_m^{\times n}$ we see an $\mathfrak{h}$-rep V is integrable if and only if $\mathfrak{h}$ acts semisimply on V and all the $h_i := e_{ii}$ act as integers on V.

Consequently, any integrable $\mathfrak{b}$-rep V must have $\mathfrak{h} \subseteq \mathfrak{g}$ acting semisimply with integral eigenvalues. One can show that in fact this integral Cartan condition characterizes integrable $\mathfrak{b}$-reps,

$$\inf_B : \mathrm{Rep}(B) \xrightarrow{\sim} \left\{ \begin{array}{c} \mathfrak{b}\text{-reps } V \text{ on} \\ \text{which } \mathfrak{h} \subseteq \mathfrak{b} \\ \text{acts semisimply w/} \\ \text{the } h_i = e_{ii} \text{ acting} \\ \text{as integers} \end{array} \right\} \subseteq \mathrm{Rep}(\mathfrak{b}).$$

As a general consequence of Thm 14 we have the following.

Corollary 17: Suppose G is connected w/ Lie alg $\mathfrak{g}$. If $\mathrm{Rep}(\mathfrak{g})$ is semisimple then $\mathrm{Rep}(G)$ is also semisimple.

So we see, for example that $\mathrm{Rep}(SL_n)$ is semisimple. Also for any finite quotient

$$\pi : SL_n \longrightarrow SL_n/\mu = G$$

$\mathrm{Rep}\, G$ is semisimple, as $\mathrm{Lie}\, G = \mathrm{Lie}\, SL_n = \mathfrak{sl}_n(\mathbb{C})$ in this case.

Corollary 18: $\mathrm{Rep}(PGL_n)$ is semisimple.

Example 3: For $\mathrm{Rep}(SL_n)$, we've already seen that the standard representation $\mathbb{V}$ for $\mathfrak{sl}_n(\mathbb{C})$ integrates to the standard rep for SL_n, using Theorem 12.

Since $\inf: \mathrm{Rep}(SL_n) \to \mathrm{Rep}(\mathfrak{sl}_n(\mathbb{C}))$ is symmetric monoidal and has image stable under subquotients, we see

$$\{\text{Symm } \otimes\text{-subcat gen'd by } \mathbb{V} \text{ in } \mathrm{Rep}(\mathfrak{sl}_n)\}$$

$$\subseteq \text{image of } \mathrm{Rep}(SL_n).$$

But, this $\otimes$-subcat is all of $\mathrm{Rep}(\mathfrak{sl}_n)$, so that $\inf_{SL_n}$ is <u>essentially surjective</u>, and thus an equivalence

$$\inf_{SL_n}: \mathrm{Rep}(SL_n) \xrightarrow{\sim} \mathrm{Rep}(\mathfrak{sl}_n(\mathbb{C})).$$

Example 4: We have

$$PGL_n = SL_n/\mu_n$$

where

$$\mu_n = \left\{ \begin{bmatrix} \zeta & & 0 \\ & \ddots & \\ 0 & & \zeta \end{bmatrix} : \exists \text{ an } n\text{-th root of } 1 \right\}.$$

Thus $\mathrm{Rep}\, PGL_n \subseteq \mathrm{Rep}\, SL_n$ identifies w/ the full subcat of reps on which $\mu_n \curvearrowright V$ trivially.

Now, $H(\zeta) = \mathrm{diag}\{\zeta, \dots \zeta\} = H_1(\zeta) H_2(\zeta^2) \cdots H_{n-1}(\zeta^{n-1})$

for the root subgroups

$$H_i : \mathbb{C}^\times \to SL_n(\mathbb{C}),\ z \mapsto \begin{array}{c} \text{i-th pos} \\ \begin{bmatrix} 1 & & & & & & 0 \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & z & & & \\ & & & & z^{-1} & & \\ & & & & & 1 & \\ & & & & & & \ddots \\ 0 & & & & & & & 1 \end{bmatrix} \end{array}.$$

Thus for a weight $\lambda = c_1\omega_1 + \cdots + c_{n-1}\omega_{n-1} \in P$

$H_i(z) \cdot v = z^{(\sum_k k c_k)} \cdot v$ for $v \in V_\lambda$,

and we find V is in $\mathrm{Rep}(PGL_n) \subseteq \mathrm{Rep}(SL_n)$

if and only if V is graded by the sublattice

$$X_{PGL_n} := \{ \sum_k c_k \cdot \omega_k : \sum_k k \cdot c_k \text{ is } n \cdot \mathbb{Z} \} \subseteq P.$$

Now for the weight $\alpha_i = -\omega_{i-1} + 2\omega_i - \omega_{i+1}$ we have $-(i-1) + 2i - (i+1) = 0$. Hence we have a seq. of inclusions

$$\mathbb{Z} \cdot \Phi = \text{Root lattice} \subseteq X_{PGL_n} \subseteq P$$

Since P has a $\mathbb{Z}$-basis $\{\omega_n, \alpha_2, \ldots, \alpha_{n-1}\}$

with

$$\omega_n = \frac{1}{n}(\alpha_1 + 2\alpha_2 + \cdots + (n-1)\alpha_{n-1})$$

and Root lattice has basis

$$\{\alpha_1 + 2\alpha_2 + \cdots + (n-1)\alpha_{n-1}, \alpha_2, \ldots, \alpha_{n-1}\}$$

we have $P / \text{Root Lattice} \cong \mathbb{Z}/n\mathbb{Z}$ and

by construction X_{PGL_n} is the kernel of the map

$[1\ 2\ \cdots\ (n-1)] : P \to \mathbb{Z}/n\mathbb{Z}$

so that

$$P / X_{PGL_n} \cong \mathbb{Z}/n\mathbb{Z} \text{ as well.}$$

Since $|P/X_{PGL_n}| = |P/\text{Root Lattice}|$ we find the inclusion

$\text{Root Lattice} \subseteq X_{PGL_n}$

is an equality.

Proposition 19: For $pgl_n(\mathbb{C}) = sl_n(\mathbb{C}) =$ Lie PGL_n, the infinitesimal action functor

$$inf_{PGL_n} : Rep(PGL_n) \to Rep(pgl_n(\mathbb{C}))$$

restricts to an equivalence

$$inf_{PGL} : Rep(PGL_n) \xrightarrow{\sim} \left\{ \begin{array}{l} pgl_n(\mathbb{C}) = sl_n(\mathbb{C})\text{-reps} \\ \text{whose } P\text{-grading is} \\ \text{supported on the sublattice} \\ \text{spanned by the roots} \end{array} \right\}.$$

This is the $\otimes$-subcat generd by the <u>adjoint</u> representation.

Other classical examples

$$\mathrm{inf}_G : \mathrm{Rep}(G) \xrightarrow{\sim} \mathrm{Rep}(\mathfrak{g})$$

for $G = Sp_{2n}$, SO_{2n} and

$$\mathrm{inf}_{GL_n} : \mathrm{Rep}(GL_n) \xrightarrow{\sim} \left\{ \begin{array}{l} \mathfrak{gl}_n(\mathbb{C})\text{-reps on which} \\ \text{the identity matrix } I_n \text{ acts} \\ \text{semisimply w/ integral eigen-} \\ \text{values} \end{array} \right\}.$$

Since $\mathfrak{gl}_n(\mathbb{C}) = \mathfrak{sl}_n(\mathbb{C}) \oplus \mathbb{C} \cdot I_n$ we observe now that $\mathrm{Rep}(GL_n)$ is semisimple.