# Fin dim algebras and Jordan-Holder

$\sim$ I Aside: Group algebras!

Let's just take a moment to think about some interesting examples.

Let $G$ be a group and $k$ be a comm ring (generally a field)

**Def$^n$:** A $G$-representation is a vector space $V$ equipped with an action $\cdot : G \times V \to V$ which satisfies
$$g\cdot(h\cdot v) = (g\cdot h)\cdot v \quad \text{and} \quad g\cdot(cv + c'v') = c(g\cdot v) + c'(g\cdot v')$$
at all $g, h \in G$, $v, v' \in V$ and $c, c' \in k$.

Equivalently, we specify a group map
$$G \to \mathrm{Aut}_k(V).$$

For example we have $S_n$ and $D_n$ acting on $\mathbb{C}^n$ by permuting coordinates. $\sigma\left(\sum_{i=1}^n c_i e_i\right) = \sum_{i=1}^n c_i e_{\sigma(i)}$.

Also we have the 1-dimensional trivial representation
$$\mathbb{C}_{triv} = \mathbb{C} \text{ with } S_n\text{-action } \sigma\cdot 1 = 1$$
and the 1-dimensional sign representation
$$\mathbb{C}_{sign} = \mathbb{C} \text{ w/ } S_n\text{-action } \sigma\cdot 1 = \mathrm{sign}(\sigma)\, 1.$$

Def$^{n}$: A homomorphism of $G$-reps is a linear map $f: V \to W$ for which $f(g \cdot v) = g f(v)$ for all $v$ in $V$, $g \in G$.

Note that we have the inclusion of $S_n$-reps
$$\mathbb{C}_{triv} \to \mathbb{C}^n, \quad 1 \longmapsto \sum_{i=1}^{n} e_i$$
for example.

We can also define the group algebra $kG$ of arbitrary $G$ which is the free vect space with basis $G$ along w/ the expected multiplication
$$\left(\sum_{g \in G} c_g g\right) \cdot \left(\sum_{h \in G} c'_h h\right) \qquad (*)$$
$$= \sum_{g,h \in G} c_g \cdot c'_h (g \cdot h).$$
and unit $1 = 1_G$.

Def$^{n}$: For any ring $A$, a unit in $A$ is an element $a$ which admits $a^{-1}$ so that $a^{-1}a = aa^{-1} = 1$. We let $A^\times = \{a \in A : a \text{ is unit}\}$.

Note that $A^\times$ is a group under mult.

Ex: $M_n(k)^\times = GL_n(k)$, or in basis free notation $End_k(V)^\times = Aut_k(V)$ for any vect space $V$.

Ex: For each finite group $G$, we have a group embedding $G \to (kG)^\times$. This is not an isomorphism since, for example $-g$ is invertible at all $g$.

Obviously any ring map $A \to B$ induces a group map $A^\times \to B^\times$. In particular, any map of $k$-alg's $kG \to A$ restricts to a group map $G \to A^\times$.

Lemma 1: For any $k$-alg $A$ and finite group $G$, restriction provides a bijection
$$\{k\text{-alg maps } kG \to A\} \xrightarrow{\sim} \{\text{group maps } G \to A^\times\}.$$

Proof: This map is obviously injective, since $kG$ is spanned by $G$ as a $k$-module and any alg map is $k$-linear. Now, given a group map $\psi : G \to A^\times$ we understand, just via bilinearity of the product on $A$, that the elements $\sum_g c_g \psi(g)$ in $A$ multiply according to the formula $(*)$.

Hence the unique linear map
$$\phi : kG \to A$$

w/ $\phi|_G = \psi$ respects multiplication

$$\phi(x \cdot y) = \phi(x) \cdot \phi(y) \text{ for all } x, y \in G$$

and has

$$\phi(1_G) = \psi(1_G) = 1_{A^\times} = 1_A.$$

So $\phi$ is an algebra map w/ $\phi|_G = \psi$ and we see that restriction provides the claimed bijection. ∎

Theorem 2: A $G$-representation over $k$ is the same thing as a $kG$-module. More precisely, we have a (strictly invertible) equivalence of categories

$$kG\text{-mod} \xrightarrow{\sim} G\text{-rep}_k.$$

$$\left\{ \begin{matrix} V \text{ w/} \\ \phi: kG \to \mathrm{End}_k(V) \end{matrix} \right\} \longmapsto \left\{ \begin{matrix} V \text{ w/} \\ \phi|_G : G \to \mathrm{Aut}_k(V) \end{matrix} \right\}$$

$$\{ f: V \to W \} \longmapsto \{ f: V \to W \}.$$

Proof: ∎

Corollary 3: The category of $G$-reps has kernels and cokernels, quotients, subreps, etc. and they behave in the expected way.

Ex: We saw the inclusion $k_{\mathrm{triv}} \to k^n$ into the permutation representation over $S_n$. In the case

$n=3$, we take the quotient to get a 2-dim rep
$$L(2) = k^3 / k_{triv}.$$
This 2-dim rep is actually simple [HW].

In fact, we'll see later that
$$\{ k_{triv}, k_{sign}, L(2) \}$$
provides a complete list of simple $kS_3$-modules / $S_3$-reps in characteristic other than 2.

Though $\overline{\mathbb{F}_3} S_3$ and $\mathbb{C}S_3$ have "the same" simples, the module categories
$$\overline{\mathbb{F}_3} S_3\text{-mod} \quad \text{and} \quad \mathbb{C}S_3\text{-mod}$$
are wild different.

**Theorem** (Maschke's Theorem) Let $k = \bar{k}$ be a field. If $\operatorname{char}(k) \nmid |G|$ then $kG$-mod is very easy to understand theoretically, but combinatorially interesting. If $\operatorname{char}(k) \mid |G|$ the module category $kG$-mod can (generally speaking) never be understood in any concrete terms by anyone ever.

Proof: Future. ∎

Rem: $\overline{\mathbb{F}_3} S_3$ is actually not so bad, but like $\overline{\mathbb{F}_3} S_6$ is an absolute disaster...

## II. Artinian and Noetherian rings and modules

Def$^{n}$: Let $A$ be a ring.

An $A$-module $M$, call $M$ Artinian (resp. Noetherian) if any sequence of submodules

$$M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots \quad (\text{resp. } M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots)$$

stabilizes.

We call $A$ Artinian (resp. Noetherian) if every finitely generated $A$-module is Artinian (resp. Noetherian.

Fake-Def$^{n}$: Call $A$ ring theoretically Artinian (resp. Noetherian) if $A$ is Artinian (resp. Noetherian) as a module over itself.

Future Theorem 4: Any Artinian ring is also Noetherian.

We will focus on a concrete setting where both Artinianity and Noetherianity are apparent.

Example: Any finite dimensional algebra $A$, i.e. algebra over a field $k$ w/ $\dim_k A < \infty$, is both Artinian and Noetherian. Indeed, for any $M$ is for general is to say $M$ admits a surj $A^r \to M$, giving dim $M < \infty$. So $M$ satisfies ACC / DCC for simple dimension reasons.

< Explain extensions

Proposition 5: Given any extension

$$0 \to M' \to N \xrightarrow{\pi} M \to 0,$$

$N$ is Artinian (resp. Noetherian) if and only if $M'$ and $M$ are Artinian (resp. Noetherian).

Proof: Any descending chain $\cdots \subseteq M'_3 \subseteq M'_2 \subseteq M$, is a descending chain in $N$.
Hence stabilization for $N$ implies stabilization for $M$.
Similarly, any desc. chain $\cdots \subseteq M_2 \subseteq M_1 \subseteq M$ pulls back to a desc. chain $\cdots \subseteq \pi^{-1}$
Since $\pi\pi^{-1}(M_i) = M_i$, stabilization for $N$ implies stabilization for $M$. So $N$ Artinian $\Rightarrow$ $M$ Artinian.

Conversely, suppose $M$ and $M'$ Artinian, and take a chain $\cdots \subseteq N_2 \subseteq N_1$ in $N$. Define

$$M'_i = M' \cap N_i \quad \text{and} \quad M_i = \pi(N_i)$$

to obtain desc. chains $\cdots \subseteq M'_2 \subseteq M'_1$ and $\cdots \subseteq M_2 \subseteq M_1$. Take $k_0$ w/ $M_{k_0} = M_k$ and $M'_{k_0} = M'_k$ whenever $k \geq k_0$. Then we have exact sequences of each $\kappa \in K$:

$$\begin{tikzcd} 0 \arrow[r] & M'_\kappa \arrow[r] \arrow[d, "\mathrm{incl}'_\kappa"'] & N_\kappa \arrow[r] \arrow[d, "\widetilde{\mathrm{incl}}_\kappa"] & M_\kappa \arrow[r] \arrow[d, "\mathrm{incl}_\kappa"] & 0 \\ 0 \arrow[r] & M'_K \arrow[r] & N_K \arrow[r] & M_K \arrow[r] & 0 \end{tikzcd}$$

in which $\mathrm{incl}'_\kappa$ and $\mathrm{incl}_\kappa$ are isomorphisms. Hence $\widetilde{\mathrm{incl}}_\kappa$ is an isomorphism by short five lemma, and thus an equality. So we see that the sequence $\cdots \subseteq N_3 \subseteq N_2 \subseteq N_1$ stabilizes, and hence that $N$ is Artinian.

The Noetherian arguments are completely similar. ∎

Corollary 6: i) For $M$ Artinian, any quotient module or submodule of $M$ is Artinian

ii) Any finite sum $\bigoplus_{i=1}^n M_i$ of Artinian modules is Artinian.

Furthermore, the same result holds when Artinian is replaced w/ Noetherian.

Theorem 7: A ring $A$ is Artinian (resp. Noetherian) if and only if $A$ is ring theoretically Artinian (resp. Noeth).

Proof: A module is finitely generated iff $M$ admits a surjection $\bigoplus_{i=1}^n A \to M$. So done by Corollary 3. ∎

Corollary 8: Any principle ideal domain is Noetherian.

Proof: H/W. ∎

Observation 9: If $A \to B$ is a ring map, $B$ is finite as a module over $A$, and $A$ is Artinian (resp. Noetherian) then $B$ is also Artinian (resp. Noetherian).

Example: $\mathbb{Z}$ is Noetherian, but not Artinian. For example, we have the infinite descending chain of ideals $(p) \supsetneq (p^2) \supsetneq (p^3) \supsetneq \cdots$ at any prime $p$. Similarly, for $k$ a field, $k[x]$ is Noetherian but not Artinian.

Example: If $R$ is a commutative Noeth. ring then, for any finitely generated $R$-module $M$, the $R$-algebra $\mathrm{End}_R(M)$ is Noetherian (?)

Example: For any group $G$ and commutative ring $R$ we have the group ring

$$RG = \bigoplus_{g \in G} R \cdot g$$

with mult $\left(\sum_{g \in G} a_g g\right) \cdot \left(\sum_{h \in G} b_h h\right)$
$= \sum_{g,h} a_g \cdot b_h (gh)$.

When $G$ is finite $\mathbb{Q}G$ is finite over $\mathbb{Q}$. Hence $\mathbb{Z}G$ is Noetherian and, for any field $k$, $kG$ is Artinian and Noetherian.

Example: For $X$ a finite CW complex, the rational cohomology $H^*(X, \mathbb{Q})$ is Artinian and Noetherian, under the cup product.

## — VI. Composition series and Jordan-Hölder

Def: A composition series for a module $M$ is a sequence of submodules

composition factors

$$0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_\ell = M \qquad (*)$$

in which each subquotient $M_{i+1}/M_i$ is simple. The number $\ell$ is called the length of the series $(*)$.

Ex: $M$ has length 0 comp. series $\Leftrightarrow$ $M = 0$

$M$ has length 1 comp. series $\Leftrightarrow$ $M$ is simple.

We do not claim that all modules admit composition series.

Lemma 10: An $A$-module $M$ admits a composition series $(*)$ if and only if $M$ is both Artinian and Noetherian.

Proof: Suppose $M$ has a composition series of length $l$, and that that any module w/ a comp series of length $< l$ is both Artinian and Noetherian. From the supposed series

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{l-1} \subseteq M_l = M$$

we obtain $M$ as an extension

$$0 \to M_{l-1} \to M \to M/M_{l-1} \to 0$$

with $M/M_{l-1}$ Art and Noeth since it's simple, and $M_{l-1}$ Art. and Noeth. by our assumption. Then $M$ is both Art and Noeth by Proposition 2. Since every length 0 module is Art and Noeth, naturally, we see that all modules which admit a composition series are both Artinian and Noetherian.

Conversely, suppose $M$ is both Artinian and Noetherian. If $M = 0$ then it clearly has a comp series $0 = M$, so we assume $M \neq 0$. By Artinianness, $M$ admits a simple submodule $M_1 \leq M$. Taking the quotient and noting that $M/M_1$ remains Artinian, by Corollary 3, we find a simple module $\bar{M}_2 \leq M/M_1$, OR take $\bar{M}_2 = 0$ if $M/M_1 = 0$ aka $M_1 = M$. Pulling back along the projection $\pi_1 : M \to M/M_1$ we obtain a submodule $M_2 = \pi^{-1}(\bar{M}_2) \leq M$ with

$\mathcal{M}_1 \subseteq \mathcal{M}_2$ and $\mathcal{M}_2 / \mathcal{M}_1 = \bar{\mathcal{M}}_2$ simple. Proceeding in this way we obtain an ascending sequence

$$0 \subseteq \mathcal{M}_1 \subseteq \mathcal{M}_2 \subseteq \mathcal{M}_3 \subseteq \cdots \subseteq \mathcal{M}$$

By Noetherianity there must be an index $l$ at which $\mathcal{M}_l = \mathcal{M}_n$ at all $n \geq l$, and hence at which $\mathcal{M}_l = \mathcal{M}$. We thus obtain a composition series for $\mathcal{M}$,

$$0 = \mathcal{M}_0 \subseteq \mathcal{M}_1 \subseteq \cdots \subseteq \mathcal{M}_l = \mathcal{M}.$$

Example: We know each simple module over $\mathbb{Q}[x]$ is finite dimensional (though there is no bound on the dimension). Hence a $\mathbb{Q}[x]$ module $\mathcal{M}$ is both Artinian and Noetherian, equivalently, admits a composition series if and only if $\mathcal{M}$ is finite dimensional.

For a specific example, given distinct irred polys $p(x)$ and $q(x)$ with roots $\alpha$ and $\beta$, the mod $\mathcal{M} = \mathbb{Q}[x]/(p^2 q^2)$ has composition series

$$0 = (p^2q^2)\cdot\mathcal{M} \subseteq (pq^2)\mathcal{M} \subseteq (pq)\mathcal{M} \subseteq q\cdot\mathcal{M} \subseteq \mathcal{M}$$

$$0 = (p^2q^2)\cdot\mathcal{M} \subseteq (p^2q)\mathcal{M} \subseteq p^2\cdot\mathcal{M} \subseteq p\mathcal{M} \subseteq \mathcal{M}$$

for example, w/ resp. subquotients

$\mathbb{Q}(\alpha)$, $\mathbb{Q}(\beta)$, $\mathbb{Q}(\alpha)$, $\mathbb{Q}(\beta)$ and

$\mathbb{Q}(\beta)$, $\mathbb{R}(\beta)$, $\mathbb{Q}(\alpha)$, $\mathbb{R}(\alpha)$.

So we see, composition series are not unique. Though in this example we find that

(a) The length of the two series are the same

(b) The simple modules which appear as subq. in these series agree.

Ex: A module $M$ over a finite dim alg $A$ admits a comp. series if and only if $M$ is finite dimensional.

Theorem (Jordan-Hölder): Let $M$ be an $A$ module w/ composition series

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_\ell = M$$

and

$$0 = M_0' \subseteq M_1' \subseteq \cdots \subseteq M_r' = M.$$

Then $\ell = r$ and, for some permutation $\sigma \in S_\ell$, we have

$$M_{i+1}/M_i \cong M'_{\sigma(i)+1}/M'_{\sigma(i)}$$

at each $0 \le i < \ell$.

Proof: For a module $N$ which admits a composition series, define the length of $N$ to be the minimal length of a composition series for $N$. Note that a module is of length 1 if and only if it is simple.

The result holds for any length 0 or length 1 module trivially. Suppose now that the result holds for all modules of length $< l$ and take $M$ of length $l$. Consider comp. series as in the statement.

If $M_{l-1} = M'_{r-1}$ then $r = l$ as $M_{l-1}$ is of length $l-1$. Otherwise we have proper inclusions

$$M_{l-1} \supseteq (\overset{M''}{\overset{\shortparallel}{M_{l-1} \cap M'_{r-1}}}) \to M_{r-1}$$

and hence nonzero injections

$$M_{l-1}/M'' \to M/M'_{r-1} \qquad (*)$$

and

$$M'_{r-1}/M'' \to M/M_{l-1}.$$

By simplicity of the target modules these injections are both isomorphisms, so that both quotients by $M''$ are simple.

From any comp series for $M''$

$$0 = M''_0 \subseteq \cdots \subseteq M''_t = M''$$

we obtain comp series

$$0 = M''_0 \subseteq \cdots \subseteq M''_t \overset{\subseteq M_{l-1}}{\underset{\subseteq M'_{r-1}}{}} .$$

This gives
$$r-1 = \text{length}(M'_{r-1}) = \text{length}(M_{\ell-1}) = \ell-1$$
$\Rightarrow$ $r=\ell$.

By our ind. hyp. the comp factors for the resp. series are
$$M''_{i+1}/M''_i, \quad M_{\ell-1}/M'', \quad M/M_{\ell-1}$$
or
$$M''_{i+1}/M''_i, \quad M'_{\ell-1}/M'', \quad M/M'_{\ell-1}.$$
We already calculated isomorphism
$$M_{\ell-1}/M'' \cong M/M'_{\ell-1}$$
and
$$M'_{\ell-1}/M'' \cong M/M_{\ell-1},$$
so that all of the factors are identified (after a permutation). ∎

**Def$^{\underline{n}}$:** Given finite length $M$ over a ring $A$, the length of $M$ is the length of any comp. series for $M$. For any simple $A$-module $L$ the multiplicity of $L$ in a comp. series for $M$ is the integer
$$[L:M] := \left\{ \begin{array}{l} \text{the number of distinct indices} \\ i \text{ at which } L \cong M_{i+1}/M_i \\ \text{in a given comp. series } M_0 \leq M_1 \leq \cdots \leq M. \end{array} \right.$$
Note that this integer is indep. of the choice of comp series

for $M$, by Jordan-Holder.

Example: For distinct irreducible $p_1, \dots, p_t$ in $\mathbb{Q}[x]$, and $M = \mathbb{Q}[x]/(p_1^{m_1} \cdots p_t^{m_t})$ has length
$\operatorname{length}(M) = \sum_{i=1}^{t} m_i$ and

$$[\mathbb{Q}(\alpha) : M] = \begin{cases} m_i & \text{if } \alpha \text{ is a root for } p_i \\ 0 & \text{if all } p_i(\alpha) \neq 0. \end{cases}$$

Proposition 4: Given an extension of finite length modules

$$0 \to M' \to N \xrightarrow{\pi} M \to 0$$

we have $\operatorname{length}(N) = \operatorname{length}(M) + \operatorname{length}(M')$ and for any simple module $L$ we have

$$[L : N] = [L : M] + [L : M'].$$

Proof: From comp series $M_0 \subseteq \cdots \subseteq M_l = M$ and $M'_0 \subseteq \cdots \subseteq M'_t = M'$ we obtain a comp series

$$N_0 \subseteq \cdots \subseteq N_t = M' \subseteq N_{t+1} \subseteq \cdots \subseteq N_{t+l} = N$$

w/ $N_i = M'_i$ for $i \leq t$ and $N_{t+j} = \pi^{-1}(M_j)$ and subsequently,

$$N_{i+1}/N_i = M'_{i+1}/M'_i \text{ for } i < t \text{ and}$$

$$N_{t+j+1}/N_{t+j} \cong M_{j+1}/M_j.$$

This gives the proposed result. ∎

# HW

1. Let $k$ be a field of characteristic $\neq 2, 3$. Prove that the quotient mod $L(2) = k^3 / k_{triv}$ of the permutation module over $kS_3$ along the inclusion $k_{triv} \to k^3$, $1 \mapsto e_1 + e_2 + e_3$, is a simple mod over $kS_3$.

2. Prove that the action map $kS_3 \to \mathrm{End}_k(L(2))$ is surjective, in particular, observe that the matrix ring

3. a) Prove that any PID is Noetherian.
b) Prove that $\mathbb{Z}$ and $k[X]$ are Noetherian but not Artinian, for any field $k$.

4. For distinct irreducible polynomials $p_1, \dots, p_r$ in $\mathbb{Q}[X]$, and integers $m_i \geq 0$, take
$$M = \mathbb{Q}[X] / (p_1^{m_1} \cdots p_r^{m_r}).$$
For $\alpha \in \overline{\mathbb{Q}}$, prove that $[\mathbb{Q}(\alpha) : M] > 0$ if and only if $p_i(\alpha) = 0$ at some $i$, and in this case $[\mathbb{Q}(\alpha) : M] = m_i$.

5. a) For any finite dimensional $\mathbb{C}[x]$-module $M$, prove that $\mathrm{length}(M) = \dim_{\mathbb{C}}(M)$.

b) Prove that there are finitely generated $\mathbb{F}_3 S_3$-modules $M$ for which $\mathrm{length}(M) < \dim(M)$.

6. For any collection of modules $\{M_\lambda : \lambda \in \Lambda\}$, the module inclusions $i_\mu : M_\mu \to \bigoplus_{\lambda\in\Lambda} M_\lambda$ and projections $p_\mu : \prod_{\lambda\in\Lambda} M_\lambda \to M_\mu$ induce isomorphisms of abelian groups

$$\{i_\lambda^* : \lambda\in\Lambda\} : \mathrm{Hom}_A\Big(\bigoplus_{\lambda\in\Lambda} M_\lambda, N\Big) \to \prod_{\lambda\in\Lambda} \mathrm{Hom}_A(M_\lambda, N)$$

and

$$\{p_{\lambda *} : \lambda\in\Lambda\} : \mathrm{Hom}_A\Big(N, \prod_\lambda M_\lambda\Big) \longrightarrow \prod_\lambda \mathrm{Hom}_A(N, M_\lambda)$$

at arbitrary $N$.