

~~Lie alg's over \mathbb{C}~~

\mathbb{C} = abelian field of char 0

Def¹: A Lie alg over \mathbb{C} is a \mathbb{C} -vector

space of elements w/ a bilinear operator.

$\delta, J: \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$ are bracket operators
satisfying antisymmetry) $[x, y] = -[y, x]$

(Jacobi identity)

$$[x[y, z]] + [z[x, y]] + [y[z, x]] = 0$$

We can rewrite the Jacobi identity as

$$[x[y, z]] = [[x, y], z] + [y[x, z]] \quad (*)$$

- Example A^{Lie} .

Let Δ be any \mathbb{C} -algebra. Define A^{Lie}

to be the vector space Δ w/ commutator bracket

$$[a, b] := ab - ba.$$

A antisymmetric is obvious. For the Jacobi identity we have

Lemma 1: The comm bracket on Δ satisfies

$$[a, [b, c]] = [a, b]c + b[a, c].$$

Proof: See directly

$$RHS = [a, [b, c]] = bac - bac = LHS. \blacksquare$$

Lemma 2: The Jacobi identity holds.

$$\text{Proof: } [a[[b, c]] = [a, b]c + b[a, c]$$

$$= [a, b]c + b[a, c] - [a, c]b - c[a, b] \\ = [[a, b]c] + [b, [a, c]]. \blacksquare$$

Corollary 3: The pairings

$$A^{Lie} := (\Delta, [\cdot, \cdot]_{\text{comm}})$$

is a Lie alg.

Proof: By def¹:

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- Example [Abelian Lie alg]

For any vector space V we can endow V with

the trivial bracket $[,]_{\text{triv}}: V \otimes V \rightarrow V$

def'd by $[v, w] = 0$ at all $v, w \in V$. The Jacobi identity holds trivially ($0=0$) so that the pair $(V, [,]_{\text{triv}})$ forms a Lie algebra.

Def¹: A Lie algebra \mathfrak{h} is called abelian if

The bracket operator on \mathfrak{h} is identically 0, i.e. if

$$\mathcal{L} = (V, [,]_{\text{triv}})$$

for a vector space V .

Sub-example: The Lie alg \mathbb{A}^{Lie} corresponds to an abelian \mathbb{A} so abelian iff \mathbb{A} is commutative.

- Example [$\text{gl}(V)$].

For any vector space V we have the algebra of linear endomorphisms $\text{End}(V)$.

Def¹: The general linear Lie alg for V is

$$\text{gl}(V) := \text{End}(V)^{\text{Lie}}$$

$$= \left\{ \begin{array}{l} \text{linear endos } A: V \rightarrow V \text{ of comut.} \\ \text{bracket } [A, B] = AB - BA \end{array} \right\}.$$

In the particular case $V = \mathbb{C}^n$ we write

$$\text{gl}_n(\mathbb{C}) := \text{gl}(\mathbb{C}^n) = \text{End}(\mathbb{C}^n)^{\text{Lie}}.$$

- Lie subalgebra and ideals

Def¹: A Lie subalgebra is a Lie alg of

is a vector subspace $f \subseteq g$ for which

$(x, y) \in f$ whenever $x, y \in f$.

An ideal is \Rightarrow a subspace $I \subseteq g$ which satisfies $[x, z] \in I$ whenever one of x or z is in I .

Draft: A homomorphism of Lie alg's
 $\phi: \mathfrak{g} \rightarrow \mathfrak{g}'$
 is a linear map which satisfies:

$$\phi([x, y]) = [\phi(x), \phi(y)]' \text{ at all } x, y \in \mathfrak{g}. \quad [\text{also invariant}]$$

Lemma 4: a) Any Lie subalg $f \subseteq \mathfrak{g}$ is itself a Lie alg, w/ bracket inherited from that of \mathfrak{g} .

b) For any ideal $I \subseteq \mathfrak{g}$, \mathfrak{I} is a Lie subalg and the quotient $\mathfrak{g}/\mathfrak{I}$ inherits a unique Lie alg structure so that the quotient map $\pi: \mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{I}$ is a Lie alg homomorphism.

Prob: Exercise.

Lemma 5: The kernel $\ker \phi \subseteq \mathfrak{g}$ of any Lie alg homomorphism $\phi: \mathfrak{g} \rightarrow \mathfrak{g}'$ is an ideal in \mathfrak{g} .

Example [sl(n(C))]: Let $\mathbb{C} = \text{Indem}$ abelian Lie alg. Then the trace function:

$$\text{tr}: \text{gl}(n(\mathbb{C})) \rightarrow \mathbb{C}, \quad A \mapsto \text{tr}(A)$$

satisfies $\text{tr}([A, B]) = 0 = [\text{tr}A, \text{tr}B]$.

Hence the trace function is a Lie alg homomorphism, and the kernel

$$\text{sl}(n(\mathbb{C})) := \ker(\text{tr}) = \left\{ \begin{array}{l} \text{non traceless matrices} \\ \text{w/ commutator bracket} \end{array} \right\}.$$

We have

$$\dim \text{sl}(n(\mathbb{C})) = n^2$$

$$\dim \text{sl}_2(\mathbb{C}) = n^2 - 1.$$

In the particular case $n=2$, $\dim \text{sl}_2(\mathbb{C}) = 3$, and we have the spanning set

$$\text{sl}_2(\mathbb{C}) = \text{span}_{\mathbb{C}} \{ e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \}.$$

The Lie bracket is specified by the formulas:

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

$\Rightarrow \text{sl}_2(\mathbb{C})$ is a very special individual.

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- Lie alg in low dim.

Dim 1: In dim 1, the only Lie alg $\mathfrak{h} = \mathbb{C}x$ is the abelian one. This follows from antisymmetry

$$[ax, bx] = a.b [x, x] = 0.$$

Dim 2: In dim 2, have $\mathfrak{h} = \mathbb{C}x \oplus \mathbb{C}y$

$$[x, x] = [y, y] = 0, \quad [x, y] = ax + by,$$

If $a \neq 0$ then replace x w/ $x + \frac{b}{a}y$ to get all expresssions $[x, y] = ax$. Then

$$\begin{aligned} [y, [x, y]] &= -a^2x \\ &= [[y, x]y] + ([x, [y, y]] = a^2x, \end{aligned}$$

giving $0 = 2a^2x$, a contradiction.

Conclusion: The only 2-dim Lie alg, up to isomorphism, is the abelian one.

Dim 3: The dim 3 are more than non-abelian.

Lie alg

$$\mathfrak{a}_3 := \left\{ \begin{bmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & a_2 \\ 0 & 0 & 0 \end{bmatrix} : a_{ij} \in \mathbb{C} \right\} \subseteq \text{gl}_3(\mathbb{C}).$$

Precisely $\mathfrak{a}_3 = \Delta^{Lie}$ for the commutative alg of strictly upper Δ matrices.

Exercise: Prove that any 3-dimensional Lie alg \mathfrak{g} is either abelian, or isomorphic to \mathfrak{a}_3 .

This statement is false. Instead, classify all 3-dimensional Lie algebras.

- Representations of Lie algebras

Def!: A representation of a Lie alg \mathfrak{g} is a vector space V equipped w/ a linear map

$$\cdot: \mathfrak{g} \otimes V \rightarrow V$$

satisfying $[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v)$.

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Lemma: For any $\text{fg-}\mathcal{V}$, the map

$$\phi_V: \mathfrak{g} \rightarrow \mathfrak{gl}(V), \quad x \mapsto (v \mapsto x \cdot v),$$

is a Lie algebra homomorphism, and any Lie algebra law \circ of \mathfrak{g} naturally defines a group structure on V by $x \cdot v := \phi(x) \cdot v$.

Proof: Exercise. ■

Example [Adjoint rep] For any Lie algebra \mathfrak{g} ,

the adjoint action $x \cdot y := [x, y]$ gives \mathfrak{g} the structure of a \mathfrak{g} -representation. Indeed, the Jacobi identity is equivalent to the requisite formula $(x \cdot y) \cdot z = x \cdot (y \cdot z) - y \cdot (x \cdot z)$.

This is the adjoint representation.

Example [The standard rep] For any vector space V , $\mathfrak{gl}(V)$ acts on V "functorially",

$$x \cdot v = x(v) \in v \text{ viewed as linear end.}$$

This gives V the structure of a $\mathfrak{gl}(V)$ -representation, and we call it the "standard representation".

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Recall, we have some examples of Lie algs
 $gl(V)$, $gl_n(\mathbb{C})$
 $sl(V)$, $sl_n(\mathbb{C})$, $sl_2(\mathbb{C})$: $\left\{ \begin{array}{l} \text{span } e, f, h \\ [he] = 2e \\ [hf] = -2f \\ [ef] = h \end{array} \right.$

A \mathfrak{g} -representation is a vector space V equipped w/ an "action" of \mathfrak{g} , $\cdot: \mathfrak{g} \otimes V \rightarrow V$, which satisfies
 $[x, y] \cdot v = xy \cdot v - y \cdot x \cdot v$.

This map specifies, and is specified by, its corresponding map to $gl(V)$, $\rho_V: \mathfrak{g} \rightarrow gl(V)$, $\rho_V(x) = x \cdot -$.
Ex 5 Adjoint rep: Any Lie alg \mathfrak{g} acts on itself via the adjoint action $\text{adj}: \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$.

$$x \cdot \text{adj } y = [x, y].$$

The requisite eq $[x, y]z = x[y, z] - [y, x]z$

is equal to the Jacobi identity

$[x, [y, z]] = [[x, y], z] + [y, [x, z]]$,
so that the adj rep $(\mathfrak{g}, \text{adj})$ is seen to be a \mathfrak{g} -rep.

Defn: \mathfrak{g} is simple if \mathfrak{g} has no proper non-zero ideals, and \mathfrak{g} is not the 1-dim abelian Lie alg.

Observation 1: If \mathfrak{g} is simple, then the adj rep map $\text{adj}: \mathfrak{g} \rightarrow gl(\mathfrak{g})$ is an inv Lie alg hom.

Proof: We already know it's a Lie alg hom. Suppose it's not an isomorphism, so $\ker \text{adj} \neq \mathfrak{g}$.

The latter case occurs iff \mathfrak{g} is abelian, which contradicts simplicity of \mathfrak{g} . Hence $\ker = 0$.

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Short term plan:

- Provide complete analysis of rep (slow). (\mathbb{R} -char)
- Do more sln.
- Begin w/ general theory of Humphreys.

- Some category stuff $\text{finite-dimensional}$

Defⁿ: For any \mathfrak{g} -rep V we let $\text{rep}(\mathfrak{g})$ denote the category of \mathfrak{g} -representations. The objects are \mathfrak{g} -reps, and morphisms are homomorphisms of \mathfrak{g} -representations, i.e. linear maps $\varphi: V \rightarrow W$ which satisfy $\varphi(x \cdot v) = x \cdot \varphi(v)$ for all $x \in \mathfrak{g}$, $v \in V$.

A subrepresentation $V' \subseteq V$ is a linear subspace which is stable under the action of \mathfrak{g} .

Note that V' inherits a \mathfrak{g} -action, or \mathfrak{g} -rep structure, in this case. Call a \mathfrak{g} -rep simple if it has no proper, nonzero subrepresentations.

Example: The \mathfrak{g} -subrep in the adj rep are precisely the ideals $I \subseteq \mathfrak{g}$. If one of I simple it and only if \mathfrak{g} nonabelian w/ simple adjoint representation.

Lemma 2: If $\varphi: V \rightarrow W$ is a homomorph of \mathfrak{g} -repr then

- The kernel $(\ker \varphi) \subseteq V$ is a subrepresentation w/ V .
- The image $\varphi(V) \subseteq W$ is a subrep w/ W .
- The quotient $W/\varphi(V)$ inherits a unique \mathfrak{g} -rep structure so that the quotient map $\pi: W \rightarrow W/\varphi(V)$ is a map of \mathfrak{g} -reps.
- φ is an isomorphism iff $\ker(\varphi) = 0$ and $\varphi(V) = W$.

Proof: The proof just follows by standard arguments.

For example, (a) if $v \in \ker(\phi)$ then $\phi(x \cdot v) = x \cdot \phi(v) = x \cdot 0 = 0$. Hence the kernel is stable under the action of ϕ , and thus a ϕ -subrep. For (c) we have the right exact seq, $V \rightarrow W \rightarrow W' \rightarrow 0$ or $W' \cong W/\phi(v)$ (of vector spaces) and apply the right exact $\phi \otimes - \rightarrow -$ to get

$$\phi \otimes V \rightarrow \phi \otimes W \rightarrow \phi \otimes W' \rightarrow 0$$

and by our prop of colimit of functors a dir
 $\phi \otimes W' \rightarrow W'$ which completes the diag

$$\begin{array}{ccccccc} \phi \otimes V & \rightarrow & \phi \otimes W & \rightarrow & \phi \otimes W' & \rightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \exists! \\ V & \longrightarrow & W & \longrightarrow & W' & \longrightarrow & 0 \end{array}$$

This action is given in class by $x \cdot \bar{w} := \overline{x \cdot w}$,
and inherits the identity $(x, y) \cdot \bar{w} = x \cdot y \cdot \bar{w} - y \cdot x \cdot \bar{w}$
from the action id on W .

(d) For the linear inverse ϕ^{-1} we have

$$\phi^{-1}(x \cdot v) = \phi^{-1}(\phi(x \cdot \phi^{-1}(v)))$$

so that ϕ^{-1} seem to be a map of ϕ -reps. ✿

Also easy to check the following:

- An R -scaling $c \cdot \phi$ of a ϕ -rep has $\phi: V \rightarrow W$

as again a map of ϕ -reps, as it commutes $\phi \circ c \cdot \phi$ of
 ϕ -rep laws. Hence

$$\text{Hom}_\phi(V, W) := \text{Hom}_{\text{Rep}(\phi)}(V, W)$$

is a vector subspace of $\text{Hom}_R(V, W)$.

- The sum $V_1 \oplus V_2$ inherits a unique ϕ -rep
structure so that the two inclusions $V_i \rightarrow V_1 \oplus V_2$ are
maps of ϕ -reps. Furthermore, this sum is both a
product and coproduct in $\text{Rep}(\phi)$ (look it up).

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Taken together we conclude that

$\text{rep}(g)$ is a \mathbb{C} -linear abelian category.

can take linear cokernel
of morphisms
has kernels and
cokernels

Def^t: Call an abelian cat \mathbb{C} Artinian if every
seq of subobjects $V_0 \supseteq V_1 \supseteq V_2 \supseteq \dots$ stabilized.

Call \mathbb{C} semisimple if every exact sequence

$$0 \rightarrow V \xrightarrow{\phi} W \xrightarrow{\phi'} V' \rightarrow 0$$

spkt, i.e. if there exists $\psi: W \rightarrow V$ satisfying
 $\psi\phi = \text{id}_V$ or $\psi': V' \rightarrow W$ w/ $\phi'\psi = \text{id}_{V'}$.

Observe that $\mathbb{C} = \text{rep}(g)$ is Artinian.

Indeed, since each obj is fin-dim / \mathbb{C} and desc. seq
of subobj must stabilize for dim reasons. Goal:
 $\text{rep}(g)$ is
semisimple.

- Ando: Lengths in JH series.
Let \mathbb{C} be an Artinian cat, and V be an
object. A Toda-Gobler series for V is a seq
of proper submodules

$$0 = V_n \subsetneq V_{n-1} \subsetneq \dots \subsetneq V_0 = V \quad (*)$$

for which each quotient V_i/V_{i+1} is a nonzero
simple object in \mathbb{C} . (Here simple means cont.
no proper nonzero subobj.) The length of such a series is
to n .

Theorem 3 (JH series) For any two JH
series $0 = V_m \subsetneq V_{m-1} \subsetneq \dots \subsetneq V_0 = V$

$$0 = V_n \subsetneq V_{n-1} \subsetneq \dots \subsetneq V_0 = V$$

we have $n = m$, and for some permutation $\sigma \in S_m$

there are items $V_{\sigma(i)} / V_{\sigma(i)+1} \cong V_{\sigma(\tau(i))} / V_{\sigma(\tau(i))+1}$
in \mathbb{C} .

Proof: Exercise.



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Def^t: For any object V in an Abelian cat \mathcal{C} ,
the length of V is the length in of any TTF exact sequence

$$0 = V_n \subseteq V_{n-1} \subseteq \dots \subseteq V_0 = V.$$

The composition factors are, up to isomorphism, the simple objects V_i/V_{i+1} in the collection $\{V_j/V_{j+1} : 0 \leq i < n\}$.

Proposition 4: For an Abelian category \mathcal{C} the following are equivalent.

a) \mathcal{C} is semisimple.

c) Every obj V decomposes into a sum of simples $V = \bigoplus_{i=1}^m L_i$.

b) Any extension $0 \rightarrow V \rightarrow W \rightarrow V' \rightarrow 0$ is

such that V and V' are simple objects.

Sketch Proof: (a) \Rightarrow (b) is trivial. Assume now that

(b) holds.

$$0 \rightarrow V \rightarrow W \rightarrow V' \rightarrow 0$$

(*)

in which

$$\text{length}(W) = \text{length}(V) + \text{length}(V') \leq 2$$

is split. Suppose now that a seq (*) is s.t.

$$\text{length}(W) = n+1 \text{ and that all seq w/ middle term of}$$

length $\leq n$ split. We can assume $n > 2$, so

that one of $\text{length}(V)$ or $\text{length}(V') > 1$. Assume

first that $\text{length}(V') > 1$, and consider

an exact sequence

$$0 \rightarrow V' \rightarrow V' \rightarrow V'_0 \rightarrow 0$$

with V'_0 simple.

By taking fiber products we obtain an exact

seq

$$0 \rightarrow V \rightarrow W = W \times_{V'} V' \rightarrow V' \rightarrow 0,$$

which is split since $\text{length}(W) = \text{length}(V) + \text{length}(V') = n$.

So we have a splitting

$$W \cong V \oplus V'.$$

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Take now $W_0 = W / \text{im } V'$, under splitting map $V' \rightarrow W_0 \hookrightarrow W$, and note the exact seq $0 \rightarrow V \rightarrow W_0 \rightarrow V'_0 \rightarrow 0$ and a diagram

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & \sqsupseteq & \downarrow \\ V & \longrightarrow & V \\ \downarrow & & \downarrow \\ W & \longrightarrow & W_0 \\ \downarrow & & \downarrow \\ V' & \longrightarrow & V'_0 \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

and the induced map to the fiber product

$$W \rightarrow V' \times_{V'_0} W_0.$$

is an isomorphism. So we see that the projection $W \rightarrow V'$ is split if the prech $W_0 \rightarrow V'_0$ is split.

However, the latter splitting occurs by our induction hypothesis - so that the seq $0 \rightarrow V \rightarrow W \rightarrow V' \rightarrow 0$ is in fact split.

The argument in the case $\text{length}(V) > 1$ is similar.

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Eratta to Aug 28:

In order for all obj. in an Artinian cat C to have comp. series C must be both Artinian and Noetherian. So, in the statement of Prop 4, Aug 28⁷⁵, which clear semisimplicity via ext. of simpler, we should replace "Let C be Artinian" with "Let C be Artinian and Noetherian". It is in the Art + Noeth setting that all objects have specified length and composition factors. We note that all familiar Art. cats are already Noeth. as well:

Ex 1: The cat Vect of finite-dim vect spaces is both Art. and Noeth.

Ex 2: The cat $A\text{-mod}_f$ of fin-dim modules over any $\mathbb{C}\text{-alg}$ A is both Art and Noeth.

Ex 3: The cat $R\text{-mod}_f$ of fin. gen'dl mod's over an Artinian ring R is both Art. and Noeth.

Ex 4: The cat $\text{rep}(G)$ of fin-dim G -repr. for any Lie alg G is both Art. and Noeth.

Anti-Ex 5: The opposite cat $(C\text{-mod}_f)^{op}$ is Artinian but not Noeth.

These fin. constraints hold for Examples 1, 2, 4 for simple dimension reasons.

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①

- sl₂-repr of \mathbb{I} : weight

$$\begin{aligned} \text{ch. } e\mathbb{J} &= 2e \\ \text{ch. } f\mathbb{J} &= -2f \\ \text{ch. } g\mathbb{J} &= h. \end{aligned}$$

Let V be a fin. dim sl₂-rep. V

decomposes into generalized wt. spaces for the action of h

$$V = \bigoplus_{i=1}^n V_{\lambda_i}^{\text{gen}}$$

where each λ_i is a complex scalar and

$$V_{\lambda_i} = \ker((ch - \lambda_i \cdot id_V)^{\gg 0}: V \rightarrow V).$$

This is clear from a consideration of the Jordan form.

Span. of the endo

$$h|_V =$$

$$\begin{bmatrix} \overset{\lambda_1}{\ddots} & & & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & & & \ddots & \lambda_n \\ & & & & \ddots & \lambda_n \end{bmatrix} \quad \sqrt{\lambda_n}$$

Def^h: A wt vector in V is an eigenvector

$v \in V$ for the action of h , and the assoc. wt.

$\lambda = wt(v)$ is the unique scalar so that $h \cdot v = \lambda \cdot v$.

We say a wt vector v is a highest wt

vector if $e \cdot v = 0$. Define lowest wt vector ... $f \cdot v = 0$.

Given a scalar $\lambda \in \mathbb{C}$, the assoc. wt.

space in V is the subspace $V_\lambda \subseteq V$ of all λ -eigenvectors in V , for the action of h .

We say V is weight graded if $V = \bigoplus_{i=1}^n V_{\lambda_i}$

for scalars λ_i .

Ex. For the adjoint rep $V = \text{sl}_2$

we have

$$V = V_{-2} \oplus V_0 \oplus V_2$$

w/ each V_i of dim 1, and $v \in V$ is the unique highest wt vector, up to scaling.

Ex: For $\text{sl}_2(\mathbb{C}) = \text{sl}_2(\mathbb{C}^2)$ we have the

standard representation

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$\nabla = \bigoplus_{i=1}^r V_i$ w.r.t. \mathfrak{h} acts nat. on V_i .
 $V_i = \{v_1, v_2, \dots, v_r\}$ resp. Then

$\nabla = V_{\lambda_1} \oplus V_{\lambda_2} \oplus \dots \oplus V_{\lambda_n}$ w.r.t. highest wt. vector $v_i \in V_{\lambda_i}$.

[Rem: For $\text{sl}_2(\mathbb{C})$ we always have the standard rep. $\nabla = \bigoplus_{i=1}^r V_i$ w.r.t. action of $\text{sl}_2(\mathbb{C}) \subset \text{End}(V_i)$]

- Existence of highest wt. vector $\xleftarrow{\text{Trivial rep}}$

Lemma 1: If ∇ is a sl₂-rep and v is a wt. vector of wt. λ , then the following hold:

a) $e \cdot v \in V_{\lambda+2}$.

b) $h \cdot v \in V_{\lambda}$.

c) $f \cdot v \in V_{\lambda-2}$.

Proof: (a) is clear by

rel. Jacobi
a) $h \cdot (e \cdot v) = [h, e] v + e \cdot (h \cdot v)$
 $= 2e \cdot v + \lambda e \cdot v = (\lambda+2)e \cdot v$

b) $h \cdot v = \lambda \cdot v \in V_{\lambda}$

c) $h \cdot (f \cdot v) = [h, f] v + f \cdot (h \cdot v)$
 $= -2f \cdot v + \lambda f \cdot v = (\lambda-2)f \cdot v$

Weir alone.

Proposition 2: Any nonzero sl₂-rep ∇ contains a highest wt. vector $v \in \nabla$, and the subspace

$$\bigoplus_{i \geq 1} V_{\lambda_i} \subseteq \nabla$$

spanned by wt. vectors in ∇ is a nontrivial sl₂-subrep.

Proof: Let $\nabla' = \bigoplus_{i \geq 1} V_{\lambda_i}$ be the span of the wt. vectors in ∇ . By considering the Tard. norm given for \mathfrak{h} it is clear that ∇' has one unique wt. vector $v \in \nabla'$, since $\dim \nabla' > 0$ by hypothesis. Take λ w/r $v \in V_{\lambda}$.

By Lemma 1 $e^{\lambda} \cdot v \in \sqrt{\lambda+2n}$, and (3)
 by fin. dim. $\sqrt{\lambda+2n} = 0$ for $\lambda > n$.

So there exists some minimal no. n_0 w/ $e^{n_0} v \neq 0$
 and $e^{n_0+1} \cdot v = 0$. $e^{n_0} v$ is therefore a
 highest wt. vector in $\sqrt{ }$.

The latter claim, that $\sqrt{ } \subseteq V$ as a maximal
 subp. follows by Lemma 1. □

Corollary 3: Every simple sl₂-representation V
 is weight graded,

$$V = \bigoplus_{\lambda \in \mathbb{Z}} V_{\lambda}.$$

- Constraining weights

Structure Theorem for sl₂-reps: Let V be a
 simple sl₂-rep. Then

- a) V has a unique highest wt. vector v , up to scaling.
- b) The highest wt. vector v has wt $\lambda \in \mathbb{Z}_{\geq 0}$.
- c) The nonzero subspaces i.e. V are precisely
 $V_{\lambda-2m}$ for $0 \leq m \leq \lambda$.
- d) For each $0 \leq m \leq \lambda$, $V_{\lambda-2m}$ is 1-dim.
 and spanned by $f^m \cdot v$.

We decompose this proof into a seq. of Lemmas
 and their consequences.

Lemma 4: If $v \in V$ a highest weight
 vector of weight λ then, for each $m \geq 0$,
 $e \cdot (f^m \cdot v) = m(\lambda - m + 1) f^{m-1} \cdot v$
 and

$$e^m \cdot (f^m \cdot v) = [\prod_{k=1}^m (\lambda - k + 1)] \cdot v.$$

Prof: We have

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$$\begin{aligned}
 e.f^m \cdot v &= [e, f]^{m-1} \cdot v \\
 &= \sum_{i=0}^{m-1} f^i [e, f] f^{m-i-1} \cdot v \\
 &= \sum_{i=0}^{m-1} f^i (\lambda - 2i) \cdot f^{m-i-1} \cdot v \\
 &= (\lambda m - 2 \left(\sum_{i=0}^{m-1} i \right)) \cdot f^{m-1} \cdot v \\
 &= m(\lambda - m + 1) \cdot f^{m-1} \cdot v.
 \end{aligned}$$

The second eq is an immediate consequence of the first.



Corollary 5: If \sqrt{V} is a nonzero, fin-dim sl₂-rep, then for any highest wt. vector $v \in V$, $\text{wt}(v) \in \mathbb{Z}_{\geq 0}$.

Proof: Take $\lambda = \text{wt}(v)$. Since \sqrt{V} is fin-dim $V_{\lambda-2m}$ vanishes for $m > 0$.

Hence $f^m \cdot v = 0$ for $m > 0$, and by Lemma 4 we see

$$k \cdot (\lambda - k + 1) = 0 \quad \text{for some } k > 0.$$

$$\Rightarrow \lambda = k - 1 \quad \text{for some } k > 0$$

$$\Rightarrow \lambda \text{ is nonnegative integral.}$$

Proposition 6: If \sqrt{V} is a fin-dim sl₂-rep.

w/ highest wt vector v s.t. $\text{wt } v \geq 0$. Then

$f^m \cdot v = 0$ if and only if $m > \lambda$,

and the vectors $\{v, f \cdot v, \dots, f^\lambda \cdot v\}$ span

a simple sl₂ subrep $L(\lambda) \subseteq V$ which has

unique highest wt. vector v , up to scaling, and

is of dim $\dim_L L(\lambda) = \lambda + 1$.

Proof: By the formula from Lemma 4,

$$e^\lambda f^{\lambda \cdot v} = w \cdot v \text{ for a nonzero scalar } w,$$

so that $f^m \cdot v \neq 0$ whenever $m \leq \lambda$.

At $\lambda+1$ we have

$$e \cdot f^{\lambda+1} \cdot v = (\lambda+1)(\lambda-\lambda) \cdot f^\lambda \cdot v \\ = 0$$

so that either $f^{\lambda+1} \cdot v = 0$ or $f^{\lambda+1} \cdot v$

is a highest wt vector of v .

$$\lambda - 2(\lambda+1) = -\lambda - 2 < 0.$$

By Corollary 5 \nexists highest wt vector of negative wt

in V , so that $f^{\lambda+1} \cdot v = 0$ and all $f^m \cdot v = 0$

when $m > \lambda$.

The fact that $L^{(\lambda)}$ is a subrep, i.e.

is closed under the actions of e , f , and h , is

immediate from Lemma 4. For simplicity, any

nonzero subrep $L \subseteq L^{(\lambda)}$ has a highest

wt vector $w \in L$, which it therefore has a

highest wt vector in $L^{(\lambda)}$. But the only highest

wt vector in $L^{(\lambda)}$ is v , up to scaling, so that

$v = c \cdot w$ for some scalar c , $w \in L$, and

hence $L^{(\lambda)} = \text{Span}\{f^m \cdot v\} \subseteq L$, so

that $L = L^{(\lambda)}$. □

Corollary 7: Any simple sl_n-rep L

has a unique highest wt vector v , up to scaling,

$\lambda = \text{wt}(v)$ is a non-negative integer,

$$L = \text{Span}_{\mathbb{C}}\{f^m \cdot v : 0 \leq m \leq \lambda\},$$

and

$$\dim L = \lambda + 1.$$

Defⁿ: For any simple sl_n-rep L , w/
highest wt vector of wt $\lambda \geq 0$, we say

L is a simple of highest wt λ .

(6)

- Uniqueness of highest wt simple.

Proposition 8: For simple L and L' with highest wt vectors v and v' of $\text{wt}(v) = \lambda = \text{wt}(v')$,

there exists a unique isomorphism of sl_n-repr

$$\phi: L \rightarrow L'$$

with $\phi(v) = v'$.

On the other hand, if L and L' have distinct highest wt then L and L' are not isom. of sl_n-reprs.

Suppose $\lambda = \text{wt}(v) < \text{wt}(v')$.

Proof: We have $L = \text{Span}\{\mathbf{e}_2, f_2, \dots, f_n\}$ and $L' = \text{Span}\{\mathbf{e}_{v'}, f_{v'}, \dots, f_{v'}\}$ so that, by the action formulas of Lemma 4, the unique linear map

$$\phi: L \rightarrow L' \text{ w/ } \phi(f^u \cdot v) = f^{u'} \cdot v'$$

provides the desired isomorphism. ■

Example (adj rep): The adjoint rep

$$V_{\text{adj}} = V_{-2} \oplus V_0 \oplus V_2$$

is the unique simple of highest wt 2, $V_{\text{adj}} = L(2)$

Example (standard rep): The standard rep

$$V = V_{-1} \oplus V_1$$

is the unique simple of highest wt 1, $V = L(1)$.

Example (trivial rep): The trivial rep \mathbb{C}

or The unique simple of highest wt 0, $\mathbb{C} = L(0)$.

- A side: Tensor products of \mathfrak{g} -reprs.

Theorem 9: Let \mathfrak{g} be an arbitrary Lie algebra. For any two \mathfrak{g} -reps V and W the tensor product $V \otimes W = V \otimes_{\mathbb{C}} W$ admits a unique \mathfrak{g} -rep structure under the action

$$x \cdot (v \otimes w) := (x \cdot v) \otimes w + v \otimes (x \cdot w).$$

Proof: For each $x \in \mathfrak{g}$ we have the endos $x_V : V \rightarrow V$ and $x_W : W \rightarrow W$

so that we have the assoc. linear endos

$$x_W \otimes id_W + id_V \otimes x_W : V \otimes W \rightarrow V \otimes W,$$

via naturality of the tensor product. We claim

that the assoc. linear map

$$\phi_{V \otimes W} : \mathfrak{g} \rightarrow \mathfrak{gl}(V \otimes W), \quad \phi_{V \otimes W} = \phi_V \otimes id_{V \otimes W},$$

defines a \mathfrak{g} -rep structure on the tensor product.

We check relations Tadd directly on generators

$$\text{in } V \otimes_{\mathbb{C}} W,$$

$$\begin{aligned} \tau_{x,y} \cdot (v \otimes w) &= [x,y] \cdot v \otimes w + v \otimes [x,y] \cdot w \\ &= xyv \otimes w + v \otimes xyw - yxv \otimes w \\ &\quad - v \otimes yxw \\ &= xyv \otimes w + x \otimes yw + y \cdot v \otimes x \cdot w + v \otimes yw \\ &\quad - yxv \otimes w - yv \otimes xw - xv \otimes yw - v \otimes yxw \\ &= x \cdot y \cdot (v \otimes w) - y \cdot x \cdot (v \otimes w). \end{aligned}$$

Example: Let L and L' be simple sl₂-reps of highest wt λ and λ' resp. Let $v \in L$ and $v' \in L'$ be highest wt vector.

Then $v \otimes v'$ is a highest wt vector in

$L \otimes L'$ and

$$\begin{aligned} h \cdot (v \otimes v') &= hv \otimes v' + v \otimes h \cdot v' \\ &= (\lambda + \lambda') (v \otimes v'). \end{aligned}$$

So $L \otimes L'$ contains a highest wt vector of wt $\lambda + \lambda'$.

- Existence and uniqueness for simple sl₂-reps

Theorem 10: For each $\lambda \geq 0$, there exists a unique simple sl₂-representation $L(\lambda)$ of highest wt λ . Furthermore, for any highest wt v , we have $f^m \cdot v = 0$ for all $m \leq \lambda$ and

$$L(\lambda) = \text{span}_{\mathbb{C}} \{ v, f \cdot v, \dots, f^{\lambda} \cdot v \}. \quad (*)$$

Proof: Uniqueness was covered in Proposition 8,

and the structure (*) follows by Corollary 7.

So we need only establish existence.

(8)

At low wt we have

$L(0)$: trivial rep, $L(1)$: standard rep
 $L(2)$: adjoint rep.

Now for all $\lambda \geq 1$ we note that

$L(1)^{\otimes \lambda}$ has a highest wt vector $v = [1] \otimes \dots \otimes [1]$ of wt. λ , so $L(1)^{\otimes \lambda}$ contains v , and hence true, simple sl₂-rep $L(\lambda) \subseteq L(1)^{\otimes \lambda}$ of highest wt. λ by Proposition 6. \blacksquare

We've now classified simple sl₂-representations:

$$\begin{matrix} \mathbb{Z}_{\geq 0} & \xrightarrow{\cong} & \{ \text{simple sl}_2\text{-reps} \} / \cong \\ \lambda & \longmapsto & L(\lambda). \end{matrix}$$

$$L(\lambda) = \begin{matrix} & & \curvearrowleft h \\ & e(v_\lambda) f & \curvearrowright h \\ & v_{\lambda-2} & \curvearrowleft h \\ \vdots & & \vdots \\ & e(v_{-\lambda+2}) f & \curvearrowright h \\ & v_{-\lambda} & \curvearrowleft h \end{matrix}$$

- Next: semisimplification of rep(sl_n(C)).

We know, from (Prop 4, Aug 28),

that rep(C) is semisimple if each extension of simpler

$$0 \rightarrow L(\lambda) \rightarrow V \rightarrow L(\lambda) \rightarrow 0$$

split. Since we know so much about simpler, one can observe such splittings directly. However, let us take an approach which mimics the higher-rank setting.

- The Casimir element

For each sl_n(C)-rep V define

- $\Omega_V : V \rightarrow V$ as:

$$\Omega_V := \frac{1}{2} h^2 + cf + fe \in \mathrm{End}(V)$$

Lemma 11: a) For each map $\xi: V \rightarrow W$ (9)

of sl₂-reps, the diagram

$$\begin{array}{ccc} V & \xrightarrow{\xi} & W \\ \Omega_V \downarrow & & \downarrow \Omega_W \\ V & \xrightarrow{\xi} & W \end{array}$$

commutes.

b) Each linear endo Ω_V is in fact an sl₂-linear endo of V .

c) For each simple rep $L(\lambda)$, $\lambda \in \mathbb{Z}_{\geq 0}$,

$$\Omega_{L(\lambda)} = \frac{1}{2} \lambda(\lambda+2) \cdot \text{id}_{L(\lambda)}.$$

Proof: a) It is clear as at each $v \in V$ we have

$$\begin{aligned} & \left(\left(\frac{1}{2} h \cdot h + e \cdot f + f \cdot e \right) \cdot v \right) \\ &= \left(\frac{1}{2} h^2 + e \cdot f + f \cdot e \right) \cdot \xi(v), \end{aligned}$$

via sl₂-linearity of ξ . b) We want

to show $x \cdot \Omega_V = \Omega_V x$ for each $x \in \text{sl}_2$,

i.e. $[x, \Omega_V] = 0$, i.e. $\Omega_V(x) = \text{End}(V)^{\text{sl}_2}$.

Hence this follows by the calculations

$$[h, \frac{1}{2} h^2 + ef + fe] = 2ef + (-2)ef + (-2)fe = 2fe$$

$$[e, \frac{1}{2} h^2 + ef + fe] = -eh - he + eh = 0$$

$$[f, \frac{1}{2} h^2 + ef + fe] = fh + hf - hf - fh = 0.$$

c) By Schur's Lemma $\text{End}_{\text{sl}_2}(L(\lambda)) = \mathbb{C}$,

so that $\Omega_{L(\lambda)} = c \cdot \text{id}$ for some scalar c .

We can find the scalar c by evaluating on the highest wt. vector $v \in L(\lambda)$. We have

$$\begin{aligned} \left(\frac{1}{2} h^2 + ef + fe \right) \cdot v &= \frac{1}{2} \lambda^2 v + ef \cdot v \\ &= \frac{1}{2} \lambda^2 v + [e, f] \cdot v \\ &= \frac{1}{2} \lambda^2 v + \lambda v \end{aligned}$$

$$= \frac{1}{2} \lambda(\lambda+2) \cdot v. \quad \blacksquare$$

(10)

Remark: Ω_{ν} is the action of the element $\Omega = \frac{1}{2} h^2 + ef + fe$ in $\mathcal{L}(ch)$ on the given $sl_2(\mathbb{C})$ -rep V . This element

$$\Omega \in \mathcal{L}(ch)$$

is central, by (d), so it is called the Casimir element.

- Splitting extension:

Proposition 12: Any extension of simple subrepr

$$0 \rightarrow L(\lambda) \rightarrow V \rightarrow L(\lambda) \rightarrow 0 \quad (*)$$

is split.

Proof: If $\lambda = \mu$ then $V(\lambda) = (w \oplus \mathbb{C}w)$ where w is the image of the highest wt. vector $v \in L(\lambda)$ under the given inclusion and w' maps to v under the projection $V \rightarrow L(\lambda)$. By Proposition 6 we have two simple subrepr

$$L, L' \subseteq V, L, L' \cong L(\lambda),$$

with highest wt. vectors w and w' respectively.

The map $L(\lambda) \rightarrow V$ is therefore an \cong onto L and the map $V \rightarrow L(\lambda)$ restricted to an isomorphism: $L \rightarrow V \rightarrow L(\lambda)$. The inverse morphism $L(\lambda) \rightarrow L \hookrightarrow V$ provides the desired splitting.

If $\mu \neq \lambda$ then $\frac{1}{2}\mu(\mu+1) \neq \frac{1}{2}\lambda(\lambda+1)$.

By Lemma 11 the operator $\Omega_{\nu}: V \rightarrow V$ has eigenvalues $\frac{1}{2}\mu(\mu+1)$ and $\frac{1}{2}\lambda(\lambda+1)$ and the ^{resp} generalised eigenspaces $V(\mu)$ and $V(\lambda)$ are nonvanishing subps in V and

$$V(\mu) \oplus V(\lambda) = V.$$

(11)

Since $\text{Length}(V) = 2$ we have

$$V(\lambda) = \text{im } L(\lambda)$$

and the composite $V(\lambda) \rightarrow V \rightarrow L(\lambda)$ is
an isomorphism of SL_2 -repr. Then we're

$$L(\lambda) \xrightarrow{\cong} V(\lambda) \hookrightarrow V$$

Then provides the required splitting. 

Theorem (semisimplicity of $\text{rep}(\text{sl}_2)$):

- a) The category $\text{rep}(\text{sl}_2(\mathbb{C}))$ is semisimple.
- b) The simple in $\text{rep}(\text{sl}_2(\mathbb{C}))$ are classified by their highest wt.

$$\mathbb{Z}_{\geq 0} \xrightarrow{\cong} \left\{ \text{simple } \text{sl}_2(\mathbb{C})\text{-rep} \right\} / \sim$$

- c) Every fin-dim $\text{sl}_2(\mathbb{C})$ -rep V decomposes uniquely into a sum

$$V = \bigoplus_{i=1}^n m(\lambda_i) \cdot L(\lambda_i)$$

with $m(\lambda_i) = \dim \text{Hom}_{\text{sl}_2(\mathbb{C})}(L(\lambda_i), V)$.

Proof: Immediate from Prop 12 and [Prop 4,