The Killing form and friends! ①

We begin to investigate the Killing form.

Defⁿ: For any Lie alg $\mathfrak{g}$ the Killing form on $\mathfrak{g}$ is the symmetric $\mathfrak{g}$-invariant form

$$\kappa: \mathfrak{g}\otimes\mathfrak{g} \to \mathbb{C}, \quad \kappa(x,y) := Tr(ad_x\, ad_y).$$

< But first...

- Aside: Solvable radicals and semisimplicity

Note from Ch 3 that sums of solvable ideals in a Lie alg $\mathfrak{g}$ are again solvable. So we observe

Lemma: Any Lie alg $\mathfrak{g}$ contains a maxl solvable ideal $I \subseteq \mathfrak{g}$.

Defⁿ: The radical $rad(\mathfrak{g})$ in $\mathfrak{g}$ is the maximal solvable ideal in $\mathfrak{g}$.

We call $\mathfrak{g}$ semisimple if its solvable radical vanishes, $rad(\mathfrak{g}) = 0$.

Ex: A) If $\mathfrak{g}$ is abelian eg. then $rad(\mathfrak{g}) = \mathfrak{g}$.

B) If $\mathfrak{g}$ is a simple Lie alg then it has no proper ideals, and $\mathfrak{g}$ is not solvable as $\mathfrak{g}$ is not abelian so that $\mathfrak{g}^{(1)} = [\mathfrak{g},\mathfrak{g}]$ must be all of $\mathfrak{g}$. Hence $rad(\mathfrak{g}) = 0$, and $\mathfrak{g}$ is semisimple.

c) If the adj rep for $\mathfrak{g}$ is semisimple, and $\mathfrak{g}$ has no abelian ideals, then $\mathfrak{g}$ is semisimple. Indeed in this case, we have unique decomp $\mathfrak{g} = \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_t$ for simple ideals $\mathfrak{g}_i$ so that $rad(\mathfrak{g}_i) = 0$ at all i, and hence the image under ad projection of $rad(\mathfrak{g})$

$$\pi_i : \mathfrak{g} \to \mathfrak{g}_i$$

must vanish. Hence $rad(\mathfrak{g}) = 0$.

Future Thm: The following are equivalent:

i) $rad(\mathfrak{g}) = 0$ ii) $\mathfrak{g}$ has semisimple adj rep, and no abelian ideals.

We embark on an adventure now to obtain the following two legendary items

Theorem A: A Lie alg $\mathfrak{g}$ is solvable $\Leftrightarrow$ $[\mathfrak{g},\mathfrak{g}] \subseteq \mathrm{rad}(\kappa_{\mathfrak{g}})$.

Theorem B: A Lie alg $\mathfrak{g}$ is semisimple $\Leftrightarrow$ κ is non-degenerate.

Our path will travel through various linear algebra cursor.

- Linear algebra I

Proposition 4.2 (JC decomposition): Let V be a finite dim vect space and $x \in \mathfrak{gl}(V)$.

a) There are unique $x_s, x_n \in \mathfrak{gl}(V)$ with x_s semisimple, x_n nilpotent, $[x_s, x_n] = 0$, and

$$x = x_s + x_n.$$

b) There are polynomials $p, q \in \mathbb{C}[t]$ with $p(x) = x_s$, $q(x) = x_n$, and $p(0) = q(0) = 0$.

c) If $x(V') \subseteq V''$ for subspaces $V', V'' \subseteq V$, then $x_s(V'), x_n(V') \subseteq V''$.

Proof: First, if (b) holds then (c) holds, ie if we can produce and for all y w/ $[y, x] = 0$ we have

$$[y, x_s] = [y, x_n] = 0$$ as well.

Now for any other commuting x_s' and x_n' with $x = x_s' + x_n'$ we have

$$[x_n', x] = [x_s', x] = 0 \Rightarrow [x_n', x_n] = [x_s', x_s] = 0$$

Hence $x_s - x_s' = x_n' - x_n$ w/ $x_s - x_s'$ semisimple and $x_n' - x_n$ nilpotent. Thus

$x_s - x_s' = 0 \Rightarrow x_s = x_s'$ and similarly $x_n = x_n'$.

Statement (c) is also clear from (b). So it suffices to prove that we can produce such x_s and x_n as in (b).

Let us decompose V into generalized eigenspaces $V = V_{\lambda_1}^{gen} \oplus \cdots \oplus V_{\lambda_m}^{gen}$ for the action of x and take n_i minimal so that $(x-\lambda_i)^{n_i}\big|_{V_{\lambda_i}^{gen}} = 0$ at each i.

Then $\prod_{i=1}^{m} (x-\lambda_i)^{n_i} = 0$ in $\mathrm{End}_{\mathbb{C}}(V)$. Here the λ_i are distinct so that the polys $(t-\lambda_i)^{n_i}$ are rel. prime in $\mathbb{C}[t]$, and by CRT we can find a poly $p(t)$ so that

$$p(t) \equiv \lambda_i \mod (t-\lambda_i)^{n_i} \quad \text{at all } i,$$

and if no $\lambda_i = 0$ also $p(t) = 0 \mod t$. (Note that if some $\lambda_i = 0$ then the last constraint is superfluous.)

Then $p(x)\big|_{V_{\lambda_i}^{gen}} = \lambda_i$ at all i so that $p(x) \in \mathrm{End}_{\mathbb{C}}(V)$ is semisimple. Furthermore

$$x - p(x)\big|_{V_{\lambda_i}^{gen}} = x - \lambda_i\big|_{V_{\lambda_i}^{gen}}$$

is nilpotent at all i. Hence for $q = t - p(t)$ we can take $x_s = p(x)$, $x_n = q(x)$ to obtain the desired decomp $x = x_s + x_n$. ∎

Lemma 4.2: If $x \in \mathfrak{g} = \mathfrak{gl}(V)$ has JC decomp $x = x_s + x_n$, then ad_{x_s} and ad_{x_n} are semisimple and nilpotent resp., and

Proof: One sees ad_{x_s} and ad_{x_n} are semisi/nilp resp, as in HW 2. Further $[\mathrm{ad}_{x_s}, \mathrm{ad}_{x_n}] = \mathrm{ad}_{[x_s, x_n]} = 0$,

So that
$$\mathrm{ad}_x = \mathrm{ad}_{(x_s+x_n)} = \mathrm{ad}_{x_s} + \mathrm{ad}_{x_n}$$
is the JC decomp by uniqueness. ▣

– Lie algebra II

Lemma 4.3 (Technical Lemma): Let
$$\mathfrak{g}_0 \subseteq \mathfrak{g}_1 \subseteq \mathfrak{g} = \mathfrak{gl}(V),$$
and take $N_{01} = \{x \in \mathfrak{g} \mid [x, \mathfrak{g}_1] \subseteq \mathfrak{g}_0\}$.
If an element $x \in N_{01}$ satisfies $tr(xy)$ at all $y \in N_{01}$, then x is nilpotent.

Proof: Let $x = x_0 + x_1$ be the Jordan decomp. We want to show that x_0, the semisimple part, vanishes. Let $\{v_1, \ldots, v_n\}$ be a basis of V under which $x_0 \cdot v_i = \lambda_i \cdot v_i$ for some λ_i's, and let e_{ij} be the corresp basis of $gl(V)$ so that
$$\mathrm{ad}_{x_0}(e_{ij}) = (\lambda_i - \lambda_j) \cdot e_{ij}.$$
Let $E = \mathbb{Z} \cdot \{\lambda_1, \ldots \lambda_n\} \subseteq \mathbb{C}$.

This is a finitely gend torsion free $\mathbb{Z}$-mod, and hence a free $\mathbb{Z}$-mod of rank $\leq n$. We claim $E = 0$, so that all $\lambda_i = 0$, and hence $x_0 = 0$. To see this, let us consider an arbitrary $\mathbb{Z}$-linear function
$$f : E \to \mathbb{Z}.$$
We claim, equivalently, that this f must be 0.

For this we consider the semisimple y with $y \cdot v_i = f(\lambda_i) \cdot v_i$ at each i, so that
$$\mathrm{ad}_y(e_{ij}) = (f(\lambda_i) - f(\lambda_j)) \cdot e_{ij}.$$
To see that $y \in N_{01}$ consider the points
$$\{\lambda_i - \lambda_j : 1 \leq i, j \leq n\} \subseteq \mathbb{C}$$

—

and let $p \in \mathbb{C}[t]$ be any polynomial with

$p(t) = f(\lambda_i - \lambda_j) = f(\lambda_i) - f(\lambda_j) \mod (t - (\lambda_i - \lambda_j))$,

i.e., with $p(\lambda_i - \lambda_j) = f(\lambda_i) - f(\lambda_j)$ at all i,j,

which exists by CRT. Then

$$p(ad_{x_0}) = ad_y$$

implying $ad_y(\mathfrak{g}_0) \subseteq \mathfrak{g}_1$, as desired.

NOW! we have $f \in E$

$$Tr(xy) = \sum' \lambda_i f(\lambda_i) = 0$$

by hypothesis, but applying f gives

$$0 = \sum_i f(\lambda_i)^2 \Rightarrow f(\lambda_i) = 0 \text{ at all } i.$$

Thus f vanishes. ∎

– Cartan's Criterion

Theorem 4.3: Let V be a $\mathfrak{g}$-representation and suppose that $Tr_V(xy) = 0$ for each $x \in [\mathfrak{g},\mathfrak{g}]$, $y \in \mathfrak{g}$. Then $im(\mathfrak{g}) \subseteq \mathfrak{gl}(V)$ is a solvable subalg.

Proof: Take $\mathfrak{g}_0 = im(\mathfrak{g})$ and $\mathfrak{g}_1 = [\mathfrak{g}_0, \mathfrak{g}_0]$.

For an z in the relative normalizer

$$N_{\mathfrak{g}_1} = \{z \in \mathfrak{gl} : [z, y] \in \mathfrak{g}_1 \text{ for each } y \in \mathfrak{g}_0\}$$

and spanning elem $[y, y'] \in \mathfrak{g}_1$ we have

$$Tr(z[y,y']) = Tr([z,y] \cdot y') = 0,$$

by hypothesis. Hence the Technical Lemma implies all $x \in \mathfrak{g}_1$ are nilpotent, that $\mathfrak{g}_1$ is nilpotent by Engel, and hence that $\mathfrak{g}$ is solvable by Cor 4.1 c. ∎

Applying to the case V = adjoint rep gives

— Thm (Cartan's Criterion) $\mathfrak{g}$ is solvable iff

$\kappa(x,y)$ at all $x \in [\mathfrak{g},\mathfrak{g}]$, $y \in \mathfrak{g}$

(6)

Proof: If all such $\kappa(x,y)=0$ then $\mathfrak{g}/Z(\mathfrak{g})$ solvable by Thm 4.3 $\Rightarrow$ $\mathfrak{g}$ solvable. If on the other hand, $\mathfrak{g}$ is solvable then under a basis on $V=\mathfrak{g}$, $\mathfrak{g}$ acts on itself via upper Δ matrices via the adj action. (This follows by Lie's Theorem.) Hence all $x\in[\mathfrak{g},\mathfrak{g}]$ act by strictly upper Δ matrices, and thus all such

$$\kappa(x,y)=\mathrm{Tr}(\mathrm{ad}_x\,\mathrm{ad}_y)=0. \quad \blacksquare$$

– Killing form and semisimplicity

Lemma 5.1A: If $I\subseteq\mathfrak{g}$ is any ideal in $\mathfrak{g}$, then

$$\kappa_{\mathfrak{g}}\big|_{I\times I}=\kappa_I,$$

where $\kappa_{\mathfrak{g}}$ and κ_I are the respective Killing forms.

Proof: Take a linear splitting $V\oplus I=\mathfrak{g}$ and note that under this splitting, for each $x\in I$,

$$\mathrm{ad}_x=\left[\begin{array}{c|c} 0 & * \\ \hline 0 & \mathrm{ad}_x|_I \end{array}\right].$$

Hence for $x,y\in I$

$$\kappa_{\mathfrak{g}}(x,y)=\mathrm{Tr}(\mathrm{ad}_x\,\mathrm{ad}_y)=\mathrm{Tr}(\mathrm{ad}_x|_I\cdot\mathrm{ad}_y|_I)$$
$$=\kappa_I(x,y). \quad \blacksquare$$

Lemma 5.1B: For each nilpotent ideal $I\subseteq\mathfrak{g}$, $I\subseteq\mathrm{rad}(\kappa)$.

Proof: For $x\in I$ and $y\in\mathfrak{g}$ we have

$$(\mathrm{ad}_x\,\mathrm{ad}_y)^n(\mathfrak{g})\subseteq I^{n-1}$$

by induction and the fact that each $I^m=[I,I^{m-1}]$ is an ideal in $\mathfrak{g}$ (Jacobi id). By nilpotence of I, each $\mathrm{ad}_x\,\mathrm{ad}_y$ is nilpotent, and hence $\kappa(x,-)=0$ for each $x\in I$, giving $I\subseteq\mathrm{rad}(\kappa)$. $\blacksquare$

Now! We begin to cook!

Theorem 5.1: A Lie algebra $\mathfrak{g}$ is semisimple if and only if the Killing form κ on $\mathfrak{g}$ is nondegenerate.

Proof: First note that, by $\mathfrak{g}$-invariance of κ, the radical

$$\mathrm{rad}(\kappa) = \{x \in \mathfrak{g} : \kappa(x, -) = 0\}$$ is an ideal in $\mathfrak{g}$.

Hence by Lemma 5.2 and the Cartan Criterion we see also that $\mathrm{rad}(\kappa)$ is solvable. Hence

$$\mathrm{rad}(\kappa) \subseteq \mathrm{rad}(\mathfrak{g})$$

and vanishing of $\mathrm{rad}(\mathfrak{g})$ implies $\mathrm{rad}(\kappa) = 0$, i.e. non-degeneracy of κ.

Conversely if $\mathrm{rad}(\kappa) = 0$, then

$$[\mathrm{rad}(\mathfrak{g}), \mathrm{rad}(\mathfrak{g})] \subseteq \mathrm{rad}(\kappa)$$

by Lemma 5.1 B and Corollary 4.1 C. Hence $[\mathrm{rad}(\mathfrak{g}), \mathrm{rad}(\mathfrak{g})] = 0$ and $\mathrm{rad}(\mathfrak{g})$ is therefore abelian. Subsequently, we apply 5.1 B again to get

$$\mathrm{rad}(\mathfrak{g}) \subseteq \mathrm{rad}(\kappa) \Rightarrow \mathrm{rad}(\mathfrak{g}) = 0.$$

So $\mathfrak{g}$ is semisimple. ∎

- Semisimplicity implies semisimplicity.

Theorem 5.2: For a Lie alg $\mathfrak{g}$ the following are equivalent:

a) $\mathrm{rad}(\mathfrak{g}) = 0$.

b) $\mathfrak{g}$ has semisimple adj rep and no abelian ideals.

c) $\mathfrak{g} = \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_t$ where the $\{\mathfrak{g}_1, \ldots, \mathfrak{g}_t\}$ are the collection of minimal ideals in $\mathfrak{g}$, and each such $\mathfrak{g}_i$ is a simple Lie alg.

Proof: (b) $\Leftrightarrow$ (c) Clear [Exercise]. (c) $\Rightarrow$ (a)

We covered last time. (a) $\Rightarrow$ (c) We proceed by nil. on dim($\mathfrak{g}$)), the dim 1 case being vacuously true. Now suppose dim $\mathfrak{g} = n$ and that the result holds for all $\mathfrak{g}'$ of dim $< n$. Take now $\mathfrak{g}_1$ minimal ideal in $\mathfrak{g}$. Then for

$$\mathfrak{g}' = \mathfrak{g}_1^{\perp} = \{y \in \mathfrak{g} \mid \kappa(x,y) = 0 \text{ for all } x \in \mathfrak{g}_1\}$$

we have that $\mathfrak{g}'$ is an ideal in $\mathfrak{g}$ by invariance of κ. We have for $x \in \mathfrak{g}_1$, $y \in \mathfrak{g}'$, $z \in \mathfrak{g}$

$$\kappa(z, [x,y]) = -\kappa([x,z], y) \in \kappa(\mathfrak{g}_1, \mathfrak{g}') = 0.$$

Thus $[x,y] \in \mathrm{rad}(\kappa) = 0$, and we see that the linear isomorphism

$$\mathfrak{g}_1 \times \mathfrak{g}' \to \mathfrak{g}, \quad (x,y) \mapsto x+y,$$

is an isomorphism of Lie algebras. Hence any ideal in $\mathfrak{g}_1$ is an ideal in $\mathfrak{g}$ from which we conclude that $\mathfrak{g}_1$ has no proper nonzero ideals and is not abelian by Lemma 5.1B. Thus $\mathfrak{g}_1$ is simple.

Now since $\mathfrak{g}'$ is orthogonal to $\mathfrak{g}_1$ we have

$$\mathrm{rad}(\kappa_{\mathfrak{g}'}) \subseteq \mathrm{rad}(\kappa) = 0$$

implying nondegen of $\kappa_{\mathfrak{g}'}$. By induction we therefore obtain a decomp

$$\mathfrak{g} = \underbrace{\mathfrak{g}_1 \times \mathfrak{g}_2}_{= \mathfrak{g}_1 \times \mathfrak{g}'} \times \cdots \times \mathfrak{g}_t \quad \text{with each } \mathfrak{g}_i \text{ simple.} \; \blacksquare$$