

# KERODON REMIX PART II: CARTESIAN AND COCARTESIAN FIBRATIONS, LIMITS AND COLIMITS [UNDER CONSTRUCTION]

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ABSTRACT. These are notes on  $\infty$ -categories which are (mostly) adapted from Lurie’s digital text Kerodon [3]. The main distinctions are the length of the document, the order of presentation, and selective omissions. We also deviate from [3] in that we focus on derived categories and dg categories as our primary examples of interest. A distinction from the related text [2] would be the complete avoidance of model structures, though this approach is already adopted in [3].

Following Part I, which presented the basic foundations for studies of  $\infty$ -categories, we discuss herein cartesian and cocartesian fibrations, transport functors (i.e. Grothendieck straightening and unstraightening), and limits and colimits. Specific topics include Hom functions and Yoneda embedding, calculations of limits and colimits in categories of spaces and cochains, and stable  $\infty$ -categories. While we follow the analytic, rather than synthetic, approach, we’ve attempted to communicate these topics in a manner which is consumable to the working mathematician.

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## 1. CARTESIAN AND COCARTESIAN FIBRATIONS

### 1.1. Definitions.

**Definition 1.1.** Consider a map of simplicial sets  $q : X \rightarrow S$ . We call a 1-simplex  $\alpha : x \rightarrow y$  in  $X$  a  $q$ -cartesian morphism if any lifting problem

$$\begin{array}{ccc}
 \Lambda^n & \xrightarrow{\bar{\sigma}} & X \\
 \downarrow & \nearrow & \downarrow q \\
 \Delta^n & \longrightarrow & S
 \end{array}
 \tag{1} \quad \boxed{\text{eq:121}}$$

with  $n \geq 2$  and  $\bar{\sigma}|\Delta^{\{n-1,n\}} = \alpha$  admits a solution. We say  $\alpha : x \rightarrow y$  is  $q$ -cocartesian if any lifting problem

$$\begin{array}{ccc} \Lambda_0^n & \xrightarrow{\bar{\tau}} & X \\ \downarrow & \nearrow & \downarrow q \\ \Delta^n & \longrightarrow & S \end{array} \quad (2) \quad \boxed{\text{eq:128}}$$

with  $n \geq 2$  and  $\bar{\tau}|\Delta^{\{0,1\}} = \alpha$  admits a solution.

Though at some specific moments we will consider a case where  $X$  and  $S$  are not  $\infty$ -categories, we are primarily invested in the  $\infty$ -categorical setting.

**Definition 1.2.** We call a map of  $\infty$ -categories  $q : \mathcal{C} \rightarrow \mathcal{D}$  a cartesian fibration if  $q$  is an inner fibration and, for any map  $\bar{\alpha} : \bar{x} \rightarrow \bar{y}$  in  $\mathcal{D}$  and  $y$  in  $\mathcal{C}$  with  $q(y) = \bar{y}$ , there is a  $q$ -cartesian map  $\alpha : x \rightarrow y$  in  $\mathcal{C}$  with  $q(\alpha) = \bar{\alpha}$ .

Similarly, we call  $q$  a cocartesian fibration if it is an inner fibration and, for any map  $\bar{\beta} : \bar{x} \rightarrow \bar{y}$  in  $\mathcal{D}$  and  $x$  with  $q(x) = \bar{x}$ , there is a  $q$ -cocartesian fibration  $\beta : x \rightarrow y$  with  $q(\beta) = \bar{\beta}$ .

The following is obvious. Recall our definitions of right and left fibrations from Definition I-4.23.

**Proposition 1.3.** *If  $q : \mathcal{C} \rightarrow \mathcal{D}$  is a right fibration (resp. left fibration) then  $q$  is a cartesian (resp. cocartesian).*

*Proof.* In this case any lifting problem of the form (1), or (2) respectively, admits a solution simply by the definition.  $\square$

Obviously when  $q : \mathcal{C} \rightarrow \mathcal{D}$  is a Kan fibration it is both cartesian and cocartesian.

**Example 1.4.** Consider a diagram  $p : K \rightarrow \mathcal{C}$ , with  $K$  some simplicial set. Then we have the overcategory  $\mathcal{C}_{/p}$  and the undercategory  $\mathcal{C}_{p/}$ . The two forgetful functors

$$\mathcal{C}_{/p} \rightarrow \mathcal{C} \quad \text{and} \quad \mathcal{C}_{p/} \rightarrow \mathcal{C}$$

are, respectively, a right and left fibration Proposition I-4.25. Hence these maps are respectively a cartesian and cocartesian fibration.

In the case where  $K$  is a point  $x : * \rightarrow \mathcal{C}$  we recall that the fibers of the fibration  $\mathcal{C}_{/x} \rightarrow \mathcal{C}$  and  $\mathcal{C}_{x/} \rightarrow \mathcal{C}$  are the right and left pinched mapping spaces  $\text{Hom}_{\mathcal{C}}^R(w, x)$  and  $\text{Hom}_{\mathcal{C}}^L(x, y)$ .

**Example 1.5** ([3, 01T8]). Consider an  $\infty$ -category  $q : \mathcal{C} \rightarrow *$ . A morphism  $\alpha : x \rightarrow y$  is  $q$ -cartesian if and only if  $\alpha$  is an isomorphism. To see this consider fillings of the horns

$$\begin{array}{ccc} & x & \\ \exists \beta \nearrow & & \searrow \alpha \\ y & \xrightarrow{id} & y \end{array} \quad \text{and} \quad \begin{array}{ccccc} & & y & & \\ & \nearrow \alpha & & \searrow \beta & \\ x & & id & & x \\ & \searrow \alpha & & \nearrow \alpha & \\ & & y & & \end{array}$$

One similarly finds that a morphism is  $q$ -cocartesian if and only if it is an isomorphism.

Via the existence of identity morphisms the structure map  $q : \mathcal{C} \rightarrow *$  is always a cartesian and cocartesian fibration. Note that this map is not a left or right fibration unless  $\mathcal{C}$  is a Kan fibration.

**1.2. Imaginings: Cartesian fibrations as lax moduli.** Give a cartesian fibration  $q : \mathcal{C} \rightarrow \mathcal{D}$  one might think of  $\mathcal{C}$  as a lax moduli of “stuff” varying over the objects in  $\mathcal{D}$ . The cartesian lifts of morphisms in  $\mathcal{D}$  provide transition functions between these fibers, i.e. the stuff we are parametrizing, over  $\mathcal{D}$ . In the case of the cartesian fibration  $\mathcal{C}/_x \rightarrow \mathcal{C}$  the category  $\mathcal{C}/_x$  is, in an obvious sense, the “moduli of maps to  $x$ ”. Let us leave the latter point about lifting maps for now, and try to make some comment on the moduli point.

Let us just consider how one classically constructs a moduli space. Here we consider the base space  $\mathcal{D} = \text{Sch}_k$  of schemes over  $k$ , which we can endow with some Grothendieck topology if we like, though we don’t care at the moment. Then a pre-stack is a choice of a functor of plain categories  $q : \mathbb{M} \rightarrow \text{Sch}_k$  which makes  $\mathbb{M}$  into a category fibered in groupoids over  $\text{Sch}_k$  [5, 003S]. One simply compares definitions to see that

$$\left\{ \begin{array}{c} \mathbb{M} \text{ is fibered in} \\ \text{groupoids over } \text{Sch}_k \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{c} q \text{ is a cartesian fibration} \\ \text{in which all maps in } \mathbb{M} \\ \text{are } q\text{-cartesian} \end{array} \right\}.$$

In this familiar setting one can now “invert” this functor  $q$  to produce an associated 2-functor

$$q^\vee : (\text{Sch}_k)^{\text{op}} \rightarrow \text{Groupoids} \subseteq \text{Cat}, \quad Y \mapsto \mathbb{M}_Y.$$

One establishes this functor via an abuse of the axiom of choice.

To elaborate a bit more, for any map of schemes  $\alpha : X \rightarrow Y$  we take a lift  $\alpha^*y \rightarrow y$  in  $\mathbb{M}$ . This lift is unique up to unique isomorphism, and via unique filling defines a functor between the fibers

$$\alpha^* : \mathbb{M}_Y \rightarrow \mathbb{M}_X, \quad y \mapsto \alpha^*y.$$

On morphisms  $\xi : y_1 \rightarrow y_2$  in the fiber  $\mathbb{M}_Y$ , we note that the cartesian property for maps in  $\mathbb{M}$  asserts the existence of a unique map  $\alpha^*\xi : \alpha^*y_1 \rightarrow \alpha^*y_2$  completing the diagram

$$\begin{array}{ccc} \alpha^*y_1 & \longrightarrow & y_1 \\ \downarrow \exists! & & \downarrow \xi \\ \alpha^*y_2 & \longrightarrow & y_2, \end{array}$$

where we note that uniqueness comes from filling the appropriate 3-simplex in  $\mathbb{M}$ . Hence  $\alpha^*$  is well-defined on morphisms via the assignment  $\xi \mapsto \alpha^*\xi$ .

We note that this inversion of  $q : \mathbb{M} \rightarrow \text{Sch}_k$  into a functor  $q^\vee : (\text{Sch}_k)^{\text{op}} \rightarrow \text{Cat}$  does not require all maps in  $\mathbb{M}$  to be cartesian. This is simply a consequence of  $q$  being a cartesian fibration between plain categories.

In the general  $\infty$ -context, we again have this inversion property for (co)cartesian fibrations. Here a cartesian fibration  $q : \mathcal{C} \rightarrow \mathcal{D}$  will specify, and be specified by, a functor

$$q^\vee : \mathcal{D}^{\text{op}} \rightarrow \text{Cat}_\infty$$

whose values over objects  $y$  in  $\mathcal{D}$  are the fibers  $\mathcal{C}_y$ . The functors between fibers  $\alpha^* : \mathcal{C}_y \rightarrow \mathcal{C}_x$  are what we’ve referred to as *transport* along  $\alpha$  (following Kerodon [3]).

While such fibrations play an essentially non-existent role in plain category theory, from the perspective of the working mathematician, they play an extraordinarily important role in the development of  $\infty$ -category theory. The main point is that they tame choices in the  $\infty$ -categorical setting. While in the plain category setting we can simply make a choice, and if that choice is not unique we can simply say it's unique up to a unique isomorphism, and then if I make two of the same types of choices then any ambiguities will vanish due to sufficient uniqueness, etc. etc., such a laissez faire attitude will lead to immediate intractable problems in the  $\infty$ -context. So one generally bundles all choices of a certain “type” into a cartesian or cocartesian fibrations, and manipulates these bundles in order to make global movements between choices of different types.

### 1.3. Cartesian morphisms via overcategories.

prop:232

**Proposition 1.6** ([3, 01TF]). *Let  $q : \mathcal{C} \rightarrow \mathcal{D}$  be a map between  $\infty$ -categories. A morphism  $\alpha : x \rightarrow y$  in  $\mathcal{C}$  is  $q$ -cartesian if and only if the natural map*

$$\mathcal{C}_{/\alpha} \rightarrow \mathcal{C}_{/y} \times_{\mathcal{D}_{/q(y)}} \mathcal{D}_{/q(\alpha)}$$

*is a trivial Kan fibration. Similarly,  $\alpha$  is  $q$ -cocartesian if and only if the map*

$$\mathcal{C}_{/\alpha} \rightarrow \mathcal{C}_{x/} \times_{\mathcal{D}_{q(x)/}} \mathcal{D}_{q(\alpha)/}$$

*is a trivial Kan fibration.*

For the proof we employ a specific deconstruction of the relevant horn inclusions.

lem:joyal3.3

**Lemma 1.7** ([1, Lemma 3.3]). *For non-negative integers  $p$  and  $q$ , and  $n = p+q+1$ , the inclusions*

$$(\Delta_0^p \star \Delta^q) \coprod_{\Lambda_q^p \star \partial \Delta^q} (\Delta^p \star \partial \Delta^q) \rightarrow \Delta^p \star \Delta^q \cong \Delta^n$$

*and*

$$(\partial \Delta^p \star \Delta^q) \coprod_{\partial \Delta^p \star \Lambda_q^q} (\partial \Delta^p \star \Lambda_q^q) \rightarrow \Delta^p \star \Delta^q \cong \Delta^n$$

*are identified with the inclusions of the extremal horns  $\Lambda_0^n \rightarrow \Delta^n$  and  $\Lambda_n^n \rightarrow \Delta^n$  respectively.*

One can see the text [1], or [3, 018N] for the details. We now proceed with the proof of Proposition 1.6.

*Proof of Proposition 1.6.* We address the cartesian situation, the cocartesian one being similar. Let  $F : \mathcal{C}_{/\alpha} \rightarrow \mathcal{C}_{/y} \times_{\mathcal{D}_{/q(y)}} \mathcal{D}_{/q(\alpha)}$  denote the map under consideration. A solution to a lifting problems of the form

$$\begin{array}{ccc} \partial \Delta^m & \xrightarrow{\quad} & \mathcal{C}_{/\alpha} \\ \downarrow & \nearrow & \downarrow F \\ \Delta^m & \xrightarrow{\quad} & \mathcal{C}_{/y} \times_{\mathcal{D}_{/q(y)}} \mathcal{D}_{/q(\alpha)}, \end{array} \quad (3) \quad \text{eq:262}$$

with  $m \geq 0$ , admit a solution if and only if the equivalent lifting problem

$$\begin{array}{ccc} (\partial \Delta^m \star \Delta^1) \cup (\Delta^m \star \Lambda_1^1) & \xrightarrow{\quad} & \mathcal{C} \\ \downarrow & \nearrow & \downarrow q \\ \Delta^m \star \Delta^1 & \xrightarrow{\quad} & \mathcal{D} \end{array}$$

obtained by way of adjunction Lemma I-4.22 admits a solution. Via direct inspection the final edge  $\Delta^1 \cong \emptyset \star \Delta^1 \rightarrow \mathcal{C}$  in the latter diagram is  $\alpha$ , so that this diagram is identified, via Lemma 1.7, with a diagram of the form

$$\begin{array}{ccc} \Lambda_n^n & \longrightarrow & \mathcal{C} \\ \downarrow & \nearrow & \downarrow q \\ \Delta^n & \longrightarrow & \mathcal{D} \end{array} \quad (4) \quad \text{eq:276}$$

in which  $n \geq 2$  the edge  $\Delta^{\{n-1, n\}} \rightarrow \mathcal{C}$  is  $\alpha$ . It follows that all lifting problems of the form (3) admit a solution if and only if all lifting problems of the form (4) admit a solution, i.e. that the map  $F$  is a trivial Kan fibration if and only if the map  $\alpha$  is  $q$ -cartesian.  $\square$

#### 1.4. Cartesian morphisms via mapping spaces.

prop:cocart\_maps

**Proposition 1.8** ([3, 01TL]). *Consider an inner fibration  $q : \mathcal{C} \rightarrow \mathcal{D}$ , and a morphism  $\alpha : x_1 \rightarrow x_2$  in  $\mathcal{C}$  with image  $\bar{\alpha} : \bar{x}_1 \rightarrow \bar{x}_2$  in  $\mathcal{D}$ . The morphism  $\alpha$  is  $q$ -cartesian if and only if for each third object  $x_0$  in  $\mathcal{C}$ , with corresponding triples  $x : \{0, 1, 2\} \rightarrow \mathcal{C}$  and  $\bar{x} : \{0, 1, 2\} \rightarrow \mathcal{D}$ , the diagram*

$$\begin{array}{ccc} \text{Fun}(\Delta^2, \mathcal{C})_x \times_{\text{Hom}_{\mathcal{C}}(x_1, x_2)} \{\alpha\} & \longrightarrow & \text{Hom}_{\mathcal{C}}(x_1, x_2) \\ \downarrow q & & \downarrow q \\ \text{Fun}(\Delta^2, \mathcal{D})_{\bar{x}} \times_{\text{Hom}_{\mathcal{D}}(\bar{x}_1, \bar{x}_2)} \{\bar{\alpha}\} & \longrightarrow & \text{Hom}_{\mathcal{D}}(\bar{x}_1, \bar{x}_2) \end{array}$$

is a homotopy pullback diagram of Kan complexes.

We cover the proof of Proposition ?? in Section 1.5 below. Let us record now a number of examples.

sect:cocart\_maps\_proof

#### 1.5. Proof of Proposition 1.8.

### 2. DEVIATION ON $(\infty, 2)$ -CATEGORIES

#### 2.1. $(\infty, 2)$ -categories.

**Definition 2.1.** Let  $X$  be a simplicial set. A 2-simplex  $\tau : \Delta^2 \rightarrow X$  is called thin if any horn for any  $n > 2$ , index  $0 < i < n$ , and inner horn

$$\bar{\sigma} : \Delta_i^n \rightarrow X \quad \text{with} \quad \bar{\sigma}|_{\Delta^{\{i-1, i, i+1\}}} = \tau,$$

the lifting problem

$$\begin{array}{ccc} \Lambda_i^n & \xrightarrow{\bar{\sigma}} & X \\ \downarrow & \nearrow & \downarrow \\ \Delta^n & \longrightarrow & * \end{array}$$

admits a solution.

One sees immediately that every 2-simplex in an  $\infty$ -category is thin, for example.

Recall our notation  $s_i : [n] \rightarrow [n-1]$  for the weakly increasing surjection with  $s_i(i) = s_i(i+1) = i$ , for  $0 \leq i \leq n-1$ , and the corresponding degeneracies  $s_i^* : \Delta^n \rightarrow \Delta^{n-1}$ . We call an  $n$ -simplex  $\sigma : \Delta^n \rightarrow X$  in a simplicial set *left degenerate* if  $\sigma$  factors through the degeneracy  $s_0^* : \Delta^n \rightarrow \Delta^{n-1}$ , and *right degenerate* if  $\sigma$  factors through the degeneracy  $s_{n-1}^* : \Delta^n \rightarrow \Delta^{n-1}$ .

def:infty2

**Definition 2.2** ([3, 01W9, 01Y5]). A simplicial set  $X$  is called an  $(\infty, 2)$ -category if the following hold:

- (a) Any horn  $\Lambda_1^2 \rightarrow X$  admits an extension to a thin 2-simplex.
- (b) Every degenerate 2-simplex in  $X$  is thin.
- (c.l) For  $n > 2$ , any horn  $\bar{\sigma} : \Lambda_0^n \rightarrow X$  in which the 2-simplex  $\bar{\sigma}|_{\Delta^{\{0,1,n\}}}$  is left degenerate admits an extension to an  $n$ -simplex in  $X$ .
- (c.r) For  $n > 2$ , any horn  $\bar{\sigma}' : \Lambda_n^n \rightarrow X$  in which the 2-simplex  $\bar{\sigma}'|_{\Delta^{\{0,n-1,n\}}}$  is right degenerate admits an extension to an  $n$ -simplex in  $X$ .

A functor, or map, between  $(\infty, 2)$ -categories is a map of simplicial sets  $F : X \rightarrow Y$  which preserves thin 2-simplices.

**Remark 2.3.** Having introduced this notion, let us recall that the term  $\infty$ -category is used interchangeably with the term  $(\infty, 1)$ -category.

If we consider an  $\infty$ -category  $\mathcal{C}$ , then in any horn  $\Lambda_0^n \rightarrow \mathcal{C}$  as in (c.l) the initial edge  $\Delta^{\{0,1\}} \rightarrow \mathcal{C}$  is degenerate, and hence an isomorphism in  $\mathcal{C}$ . Hence we have the proposed completion to an  $n$ -simplex  $\Delta^n \rightarrow \mathcal{C}$ , by Proposition I-4.33. Similarly any horn  $\Lambda_n^n \rightarrow \mathcal{C}$  as in (c.r) completes to an  $n$ -simplex as well. So we observe the following.

**Lemma 2.4.** *Any  $\infty$ -category is an  $(\infty, 2)$ -category. Furthermore, an  $(\infty, 2)$ -category  $X$  is an  $\infty$ -category if and only if every 2-simplex in  $X$  is thin.*

Recall that each simplex  $\Delta^n$  is an  $\infty$ -category, and hence an  $(\infty, 2)$ -category.

**Example 2.5.** Since any degenerate 2-simplex in an  $(\infty, 2)$ -category is thin, any map of simplicial sets  $* = \Delta^0 \rightarrow X$  is a map of  $(\infty, 2)$ -categories. Similarly, any map of simplicial sets  $\Delta^1 \rightarrow X$  is a map of  $(\infty, 2)$ -categories.

One has the following practical check for maps between  $(\infty, 2)$ -categories.

prop:infty2\_check

**Proposition 2.6** ([3, 01YC]). *Let  $X$  and  $Y$  be  $(\infty, 2)$ -categories, and  $F : X \rightarrow Y$  be a map of simplicial sets. Then  $F$  is a functor, i.e. preserves thin 2-simplices, if and only if any horn  $\Lambda_1^2 \rightarrow X$  can be completed to a thin simplex with thin image in  $Y$ .*

*Idea of proof.* The result is a consequence of stability of thin simplices under various conditions. Namely one establishes an inner-exchange property for thin simplices, which we recall below, and a 4-of-5 property which one can find at [3, 01XX].  $\square$

## 2.2. The pith of an $(\infty, 2)$ -category.

**Definition 2.7.** Given an  $(\infty, 2)$ -category  $X$ , the pith in  $X$  is the simplicial subset  $X^{\text{Pith}} \subseteq X$  whose simplices  $\Delta^n \rightarrow X^{\text{Pith}}$  consist of all simplices  $\sigma : \Delta^n \rightarrow X$  in which each restriction along a 2-simplex

$$\Delta^2 \rightarrow \Delta^n \xrightarrow{\sigma} X$$

is thin.

Since functors between  $(\infty, 2)$ -categories preserve thin simplices, by definition, we see that any map  $F : \mathcal{C} \rightarrow X$  from an  $\infty$ -category to an  $(\infty, 2)$ -category factors through the pith.

lem:360

**Lemma 2.8** ([3, 01XL], Inner-exchange property). *Consider a 3-simplex  $\sigma : \Delta^3 \rightarrow X$  in an  $(\infty, 2)$ -category, and suppose that the associated 2-simplices  $\sigma|_{\Delta^{\{1,2,3\}}}$  and  $\sigma|_{\Delta^{\{0,1,2\}}}$  are thin. Then the 2-simplex  $\sigma|_{\Delta^{\{0,2,3\}}}$  is thin if and only if the 2-simplex  $\sigma|_{\Delta^{\{0,1,3\}}}$  is thin.*

The proof employs certain facts about interior fibrations (see below), and is omitted. From Lemma 2.8 the proof of the following is immediate.

**Proposition 2.9.** *For any  $(\infty, 2)$ -category  $X$ , the subcomplex  $X^{\text{Pith}}$  is an  $\infty$ -category.*

*Proof.* For any completion  $\Delta^3 \rightarrow X$  of an inner horn  $\Lambda_i^3 \rightarrow X$  in which all of the associated face  $\Delta^2 \rightarrow \Lambda_i^3 \rightarrow X$  are thin, the final face  $\Delta^{[3] \setminus \{i\}} \rightarrow X$  is also thin, by Lemma 2.8. This shows that the pith is stable under the completion of inner horns  $\Lambda_i^3 \rightarrow X^{\text{Pith}}$ . Stability under completion of all inner horns  $\Lambda_i^n \rightarrow X^{\text{Pith}}$  with  $n > 3$  is immediate, since the horn  $\Lambda_i^n$  already contains all 2-faces in  $\Delta^n$  in this case. Taken together with condition (a) of Definition 2.2, we see that any lifting problem

$$\begin{array}{ccc} \Lambda_i^n & \longrightarrow & X^{\text{Pith}} \\ \downarrow & & \downarrow \\ \Delta^n & \longrightarrow & * \end{array}$$

with  $0 < i < n$  admits a solution, as required.  $\square$

**2.3.  $(\infty, 2)$ -category via simplicial categories.** Recall that one can associate to any simplicial category  $\underline{S}$  its associated homotopy coherent nerve  $N^{\text{hc}}(\underline{S})$  Section I-2.7. The  $n$ -simplices in  $N^{\text{hc}}(\underline{S})$  are simplicial functors from the path category  $\text{Path}[n]$ . For  $S = N^{\text{hc}}(\underline{S})$  we have in low dimension

$$S[0] = \{ \text{objects in } \underline{S} \}$$

$$S[1] = \{ \text{pairs of object } (x_0, x_1) \text{ along with a map } f \in \underline{\text{Hom}}_{\underline{S}}(x_0, x_1)[0] \}$$

$$S[2] = \left\{ \begin{array}{l} \text{triples of objects } (x_0, x_1, x_2), \text{ maps } f_{ij} : x_i \rightarrow x_j \text{ for each } i < j, \text{ and} \\ \text{a 1-simplex } h : \Delta^1 \rightarrow \underline{\text{Hom}}_{\underline{S}}(x_0, x_2) \text{ with } h|_0 = f_{02}, h|_1 = f_{12}f_{01} \end{array} \right\}$$

Lemma I-2.16. We take the following theorem for granted.

thm:hc\_infty2

**Theorem 2.10** ([3, 01YM]). *Let  $\underline{S}$  be a simplicial category in which, at each pair of objects  $x$  and  $y$  in  $\underline{S}$ , the mapping complex  $\underline{\text{Hom}}_{\underline{S}}(x, y)$  is an  $\infty$ -category. Then the homotopy coherent nerve  $N^{\text{hc}}(\underline{S})$  is an  $(\infty, 2)$ -category.*

What we are most interested in here is the  $(\infty, 2)$ -category of  $\infty$ -categories. Recall that for any  $\infty$ -categories  $\mathcal{C}$  and  $\mathcal{D}$  the simplicial set of functors  $\text{Fun}(\mathcal{C}, \mathcal{D})$ , whose simplicial are as expected

$$\text{Fun}(\mathcal{C}, \mathcal{D})[n] = \text{Hom}_{\text{sSet}}(\Delta^n \times \mathcal{C}, \mathcal{D}),$$

form another  $\infty$ -category Corollar I-4.8. With these morphisms we obtain the simplicial category  $\underline{\text{Cat}}_{\infty}$  of  $\infty$ -categories and their functor categories. We note that  $\underline{\text{Cat}}_{\infty}$  is a full simplicial subcategory in the ambient category  $\underline{\text{sSet}}$  of simplicial sets.



#### 2.4. The $(\infty, 2)$ -category of $\infty$ -categories.

**Theorem 2.11.** *The homotopy coherent nerve*

$$\mathrm{Cat}_\infty := \mathrm{N}^{\mathrm{hc}}(\underline{\mathrm{Cat}}_\infty)$$

is an  $(\infty, 2)$ -category.

According to the above analysis the 0-simplices in  $\mathrm{Cat}_\infty$  are  $\infty$ -categories, the 1-simplices are functors between  $\infty$ -categories, and 2-simplices are triples of functors and a natural transformation

$$\begin{array}{ccc} & \mathcal{C}_2 & \\ F_{12} \nearrow & & \searrow F_{02} \\ \mathcal{C}_1 & & \mathcal{C}_0 \\ & F_{01} \nwarrow & \end{array} \xrightarrow{\zeta} \begin{array}{c} \mathcal{C}_2 \\ \uparrow F_{02} \\ \mathcal{C}_0 \end{array}$$

**Definition 2.12.** The  $(\infty, 2)$ -category  $\mathrm{Cat}_\infty$  is called the  $(\infty, 2)$ -category of  $\infty$ -categories.

**Remark 2.13.** The  $(\infty, 2)$ -category  $\mathrm{Cat}_\infty$  is in our universe of “large” sets, which is strictly larger than our universe of “normal sized” set in which all other  $\infty$ -categories are assumed to live.

We recall our  $\infty$ -category  $\mathcal{Cat}_\infty$  of  $\infty$ -categories, which we obtain by restricting the morphisms  $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$  to the associated Kan can complex  $\mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\mathrm{Kan}}$  then applying the simplicial nerve. The inclusions of  $\infty$ -categories

$$\mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\mathrm{Kan}} \rightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

imply an inclusion of  $(\infty, 2)$ -categories  $\mathcal{Cat}_\infty \rightarrow \mathrm{Cat}_\infty$ , and hence an inclusion into the pith

$$\mathcal{Cat}_\infty \rightarrow (\mathrm{Cat}_\infty)^{\mathrm{Pith}}. \quad (5) \quad \boxed{\text{eq:444}}$$

By a general result one finds that this inclusion is an equality.

**Proposition 2.14** ([3, 01YT]). *The inclusion (5) is an equality,  $\mathcal{Cat}_\infty = (\mathrm{Cat}_\infty)^{\mathrm{Pith}}$ .*

#### 2.5. Interior fibrations.

`def:interior`

**Definition 2.15.** A map of simplicial sets  $q : X \rightarrow S$  is called an interior fibration if the following hold:

- (a) At each 0-simplex  $x$  in  $X$ , the identity  $id_x : x \rightarrow x$  is both  $q$ -cartesian and cocartesian.
- (b) For any lifting problem

$$\begin{array}{ccc} \Lambda_i^n & \xrightarrow{\quad} & X \\ \downarrow & \nearrow \text{dotted} & \downarrow \\ \Delta^n & \xrightarrow[\sigma]{} & S \end{array} \quad (6) \quad \boxed{\text{eq:400}}$$

in which  $0 < i < n$  and  $\sigma|_{\Delta^{\{i-1, i, i\}}}$  is thin in  $S$ , (6) admits a solution.

It is clear that if  $f : S' \rightarrow S$  is a map of simplicial sets which preserves thin 2-simplices, and the diagram

$$\begin{array}{ccc} X' & \longrightarrow & X \\ q' \downarrow & & \downarrow q \\ S' & \xrightarrow{f} & S \end{array}$$

is a pullback diagram of simplicial sets in which  $q$  is an interior fibration, then the map  $q' : X' \rightarrow S'$  is an interior fibration as well.

One also observes the following.

1em:495

**Lemma 2.16.** *If  $S$  is an  $(\infty, 2)$ -category, and  $q : X \rightarrow S$  is an interior fibration, then  $X$  is also an  $(\infty, 2)$ -category and  $q$  is a functor between  $(\infty, 2)$ -categories.*

*Proof.* One sees via the lifting property for  $q$  that any 2-simplex  $\Delta^2 \rightarrow X$  which has thin image in  $S$  is thin in  $X$ . From this we see that any horn  $\Lambda_1^2 \rightarrow X$  can be completed to a thin 2-simplex in  $X$ . One obtains this thin completion by lifting a thin completion  $\Delta^2 \rightarrow S$ . We are left to prove that any appropriate degenerate horn  $\Lambda_0^n \rightarrow X$  or  $\Lambda_n^n \rightarrow X$ , at  $n > 2$ , completes to an  $n$ -simplex. However this follows from the fact that identity maps in  $X$  are both  $q$ -cartesian and cocartesian, and the fact that the corresponding horns in  $S$  admit completions. We now see that  $X$  is an  $(\infty, 2)$ -category. One sees that  $q$  is a functor, i.e. preserves thin 2-simplices, by applying Proposition 2.6.  $\square$

As we see in the above proof, given an interior fibration  $q : X \rightarrow S$  over an  $(\infty, 2)$ -category, one can detect thin simplices in  $X$  by considering their images in  $S$  along  $q$ .

1em:505

**Lemma 2.17.** *If  $q : X \rightarrow S$  is an interior fibration then a 2-simplex in  $X$  is thin if and only if its image in  $S$  is thin.*

cor:interior\_pullback

**Corollary 2.18.** *Consider a pullback diagram*

$$\begin{array}{ccc} Z & \xrightarrow{p_2} & X \\ p_1 \downarrow & & \downarrow q \\ Y & \xrightarrow{F} & S \end{array}$$

*in which  $q$  is an interior fibration and  $F$  is a map between  $(\infty, 2)$ -categories. Then  $Z$  is an  $(\infty, 2)$ -category,  $p_1$  is an interior fibration, and  $p_2$  is a map of  $(\infty, 2)$ -categories.*

*Proof.* The fact that  $Z$  is an  $(\infty, 2)$ -category and  $p_1$  is a map of  $(\infty, 2)$ -categories follows by Lemma 2.16. As for  $p_1$ , we consider a thin 2-simplex  $\Delta^2 \rightarrow Z$ , and note that its image in  $Y$  is thin. Hence its image in  $S$  is thin, and so its image in  $X$  is thin by Lemma 2.17. It follows that  $p_2$  is a map of  $(\infty, 2)$ -categories, by definition.  $\square$

We are especially interested in the fiberings of interior fibrations over  $\infty$ -categories.

1em:502

**Lemma 2.19.** *Let  $\mathcal{C}$  be an  $\infty$ -category. A map of simplicial sets  $q : X \rightarrow \mathcal{C}$  is an interior fibration if and only if it is an inner fibration.*

*Proof.* If  $q$  is an interior fibration then it is an inner fibration since all 2-simplices in  $\mathcal{C}$  are thin. Conversely, if  $q$  is an inner fibration then  $X$  is an  $\infty$ -category and  $q$  is therefore an inner fibration between  $\infty$ -categories. Condition (a) of Definition 2.15 now follows from the fact that the identity in an  $\infty$ -category is an isomorphism, and an application of Proposition I-4.33.  $\square$

One combines Lemma 2.19 with the above discussion of fiber products to obtain the following corollary.

**Corollary 2.20.** *Consider an interior fibration  $q : X \rightarrow S$  over an  $(\infty, 2)$ -category  $S$ .*

- (a) *For any  $\infty$ -category  $\mathcal{C}$ , and any functor of  $(\infty, 2)$ -categories  $\mathcal{C} \rightarrow S$ , the fiber product  $X \times_S \mathcal{C}$  is an  $\infty$ -category. Furthermore, the projection  $X \times_S \mathcal{C} \rightarrow \mathcal{C}$  is an inner fibration.*
- (b) *At each point  $s : * \rightarrow S$  the fiber  $X_s$  is an  $\infty$ -category.*

cor:interior\_pith

**Corollary 2.21.** *If  $q : X \rightarrow S$  is an interior fibration over an  $(\infty, 2)$ -category  $S$  then the diagram*

$$\begin{array}{ccc} X^{\text{Pith}} & \longrightarrow & X \\ \downarrow & & \downarrow \\ S^{\text{Pith}} & \longrightarrow & S \end{array}$$

*is a pullback diagram, and the map  $X^{\text{Pith}} \rightarrow S^{\text{Pith}}$  is an interior fibration.*

*Proof.* In this case the pullback  $X \times_S S^{\text{Pith}}$  is an  $\infty$ -category and the projection to  $X$  is a map of  $(\infty, 2)$ -categories. So the identification

$$X^{\text{Pith}} = X \times_S S^{\text{Pith}}$$

follows via an application of the universal property for the pullback and the universal property for the pith.  $\square$

**2.6. Undercategories and overcategories and pointed  $\infty$ -categories.** In the  $(\infty, 2)$ -setting we can define overcategories and undercategories exactly as in the  $\infty$ -setting. Namely, for a map of simplicial sets  $p : K \rightarrow X$  the overcategory  $X_{p/}$  is the simplicial set with  $n$ -simplices provided by the join

$$X_{p/}[n] := \text{Hom}_{\text{sSet}}(K \star \Delta^n, X)_p,$$

and similarly for the undercategory

$$X_{/p}[n] := \text{Hom}_{\text{sSet}}(\Delta^n \star K, X)_p$$

Section I-4.6. In the case in which  $X$  is an  $\infty$ -category, we saw that the forgetful functors

$$\begin{array}{ccc} X_{p/} & & X_{/p} \\ & \searrow & \swarrow \\ & X & \end{array}$$

obtained by restricting along the inclusions  $\Delta^n \rightarrow \Delta^n \star K$  and  $\Delta^n \rightarrow K \star \Delta^n$  are directional fibrations, and in particular isofibrations. We have a similar result in the 2-categorical context.

prop:504

**Proposition 2.22** ([3, 01WU]). *Let  $X$  be an  $(\infty, 2)$ -category and  $p : K \rightarrow X$  be a map of simplicial sets. The the forgetful maps*

$$X_{p/} \rightarrow X \quad \text{and} \quad X_{/p} \rightarrow X$$

*are both interior fibrations.*

At this point we'll begin to leave many of the details unaccounted for. In particular, we direct the reader to the original text [3] for the details on Proposition 2.22. In any case, we record some corollaries.

**Corollary 2.23.** *For an  $(\infty, 2)$ -category  $X$  and a diagram  $p : K \rightarrow X$ , the simplicial sets  $X_{p/}$  and  $X_{/p}$  are  $(\infty, 2)$ -category and the forgetful maps are both functors between  $(\infty, 2)$ -categories.*

cor:601

**Corollary 2.24.** *Let  $p : K \rightarrow X$  be a map from a simplicial set into an  $(\infty, 2)$ -category. At any point  $x : * \rightarrow X$  the fibers  $(X_{p/})_x$  and  $(X_{/p})_x$  are both  $\infty$ -categories.*

We apply this corollary in the case where the diagram  $p$  is a point  $x : * \rightarrow X$  to obtain mapping categories for any  $(\infty, 2)$ -category  $X$ .

**Definition 2.25.** For any  $(\infty, 2)$ -category  $X$ , and objects  $x, y : * \rightarrow X$ , the left pinched mapping  $\infty$ -category is the fiber

$$\mathrm{Hom}_X^L(x, y) := (X_{x/}) \times_X \{y\}.$$

Similarly, the right pinched mapping  $\infty$ -category is the fiber

$$\mathrm{Hom}_X^R(x, y) = \{x\} \times_X (X_{/y}).$$

As with any interior fibration, we can restrict the forgetful functor to the piths to obtain inner fibrations of  $\infty$ -categories

$$(X_{p/})^{\mathrm{Pith}} \rightarrow X^{\mathrm{Pith}} \quad \text{and} \quad (X_{/p})^{\mathrm{Pith}} \rightarrow X^{\mathrm{Pith}}.$$

In this particular instance one can observe a stronger characterization of these functors.

prop:over\_cartesian

**Proposition 2.26** ([3, 01YE]). *For  $X$  and  $p : K \rightarrow X$  as above, the restrictions of the forgetful functors*

$$(X_{p/})^{\mathrm{Pith}} \rightarrow X^{\mathrm{Pith}} \quad \text{and} \quad (X_{/p})^{\mathrm{Pith}} \rightarrow X^{\mathrm{Pith}}.$$

*are, respectively, a cocartesian fibration and a cartesian fibration.*

One might view this result in analogy with the  $\infty$ -setting, where the forgetful functors were observed to be right and left fibrations Corollary I-4.27.

**2.7. Mapping categories in the homotopy coherent nerve.** Let  $\underline{S}$  be a simplicial category whose morphism complexes are weak Kan complexes, and let  $S$  be the homotopy coherent nerve,  $S = \mathrm{N}^{\mathrm{hc}}(\underline{S})$ . We recall that  $S$  is an  $(\infty, 2)$ -category in this case. By an abuse of notation take

$$\underline{\mathrm{Hom}}_S(x, y) = \underline{\mathrm{Hom}}_{\underline{S}}(x, y)$$

for any given pair of objects in  $S$ . We construct a map of simplicial sets

$$\theta : \underline{\mathrm{Hom}}_S(x, y) \rightarrow \mathrm{Hom}_S^L(x, y)$$

[3, 01LD] which is subsequently found to be an equivalence of  $\infty$ -categories.

To begin, for any simplicial set  $K$  we consider the simplicial category  $E(K)$  with objects  $x_-$  and  $x_+$  and morphisms

$$\mathrm{Hom}_{E(K)}(x_-, x_-) = \mathrm{Hom}_{E(K)}(x_+, x_+) = * \text{ and } \mathrm{Hom}_{E(K)}(x_-, x_+) = K.$$

We consider the  $(n+1)$ -simplex  $\{-1\} \star \Delta^n \cong \Delta^{n+1}$  and the simplicial path category  $\mathrm{Path}(\{-1\} \star \Delta^n)$  whose morphisms are given by the nerves

$$\mathrm{Hom}_{\mathrm{Path}(\{-1\} \star \Delta^n)}(l, m) = \mathrm{N}(\mathrm{Subsets}_{l,m}^{\mathrm{op}})$$

where  $\mathrm{Subsets}_{l,m}$  is the partially ordered set of subsets  $S \subseteq [n]$  with  $\min S = l$  and  $\max S = m$ , ordered by inclusion.

At each integer  $n$  we have a simplicial functor

$$\theta_n^* : \mathrm{Path}(\{-1\} \star \Delta^n) \rightarrow E\Delta^n$$

which is define on objects by taking  $\theta_n^*(-1) = x_-$ , and  $\theta_n^*(i) = x_+$  for all  $i \geq 0$ , and defined on morphisms by the simplicial map

$$\theta_n^* : \mathrm{Hom}_{\mathrm{Path}(\{-1\} \star \Delta^n)}(l, m) = \mathrm{N}(\mathrm{Subsets}_{l,m}^{\mathrm{op}}) \rightarrow \mathrm{Hom}_E(x_-, x_+) = \Delta^n = \mathrm{N}([n])$$

associated to the functor  $\mathrm{Subsets}_{l,m}^{\mathrm{op}} \rightarrow [n]$  which sends each subset  $S = \{l < s_1 < \dots < s_r < m\}$  to  $s_1$  and each inclusion  $S' \supseteq S$  to the inequality  $s'_1 \leq s_1$ .

For objects  $x$  and  $y$  in  $S$ ,  $n$ -simplices in  $\underline{\mathrm{Hom}}_S(x, y)$  are identified with simplicial functors  $\mathrm{Fun}_{\mathrm{sCat}}(E\Delta^n, \underline{S})$  in the fiber over  $(x, y)$  in  $\mathrm{Fun}(E\emptyset, \underline{S})$ . Each such functor now defined an  $(n+1)$ -simplex in  $S$  via a consideration of the identification

$$S[n+1] = \mathrm{Fun}_{\mathrm{sCat}}(\mathrm{Path}(\{-1\} \star \Delta^n), \underline{S})$$

and restricting along  $\theta_n^*$ . One sees, by the definiton of  $\theta_n^*$  that this associated  $(n+1)$ -simplex has initial vertex  $x$  and all other vertices  $y$ , and restricts trivially to  $\Delta^n \subseteq \{-1\} \star \Delta^n$ . So we obtain a map of sets

$$\theta_n : \underline{\mathrm{Hom}}_S(x, y)[n] \rightarrow (S_{x/}) \times_S \{y\} = \mathrm{Hom}_S^L(x, y)[n],$$

$$(f : E\Delta^n \rightarrow \underline{S}) \mapsto (f\theta_n^* : \mathrm{Path}\{-1\} \star \Delta^n \rightarrow \underline{S}).$$

One observes directly that any increasing function  $t : [n] \rightarrow [n']$  produces a commutative diagram

$$\begin{array}{ccc} \mathrm{Path}(\{-1\} \star \Delta^n) & \xrightarrow{\theta_n} & E\Delta^n \\ t_* \downarrow & & \downarrow t_* \\ \mathrm{Path}(\{-1\} \star \Delta^{n'}) & \xrightarrow{\theta_{n'}} & E\Delta^{n'}, \end{array}$$

from which we see that the  $\theta_n$  assemble into a map of simplicial sets, or a map of  $\infty$ -categories,

$$\theta : \underline{\mathrm{Hom}}_S(x, y) \rightarrow \mathrm{Hom}_S^L(x, y).$$

thm:pinched\_simplicial

**Theorem 2.27** ([3, 01LG]). *Let  $\underline{S}$  be a simplicial category whose morphism complexes are  $\infty$ -categories. Take  $S = \mathrm{N}^{\mathrm{hc}}(\underline{S})$ . For any objects  $x, y : * \rightarrow S$  there is a natural equivalence of  $\infty$ -categories*

$$\theta : \underline{\mathrm{Hom}}_S(x, y) \rightarrow \mathrm{Hom}_S^L(x, y).$$

We do not cover the details, and refer the reader to the text [3].

cor:simplicial\_pullback

**Corollary 2.28.** *Take  $\underline{S}$  and  $S$  as above. For any pair of points  $x, y : * \rightarrow S$  there is a categorical pullback diagram*

$$\begin{array}{ccc} \underline{\mathrm{Hom}}_S(x, y) & \xrightarrow{\theta} & (S_{x/})^{\mathrm{Pith}} \\ \downarrow & & \downarrow \\ * & \xrightarrow{y} & S^{\mathrm{Pith}}. \end{array} \quad (7)$$

eq:704

*Proof.* By Corollary 2.18 and Proposition 2.22 the projection map  $\mathrm{Hom}_S^L(x, y) \rightarrow S_{x/}$  has image in the  $\mathrm{Pith} (S_{x/})^{\mathrm{Pith}}$ . Applying this fact in conjunction with Corollary 2.21, we observe a pullback diagram of  $\infty$ -categories

$$\begin{array}{ccc} \mathrm{Hom}_S^L(x, y) & \xrightarrow{\theta} & (S_{x/})^{\mathrm{Pith}} \\ \downarrow & & \downarrow \\ * & \xrightarrow{y} & S^{\mathrm{Pith}}. \end{array}$$

in which the right-hand map is an inner fibration. This diagram is additionally a categorical pullback square by Corollary I-5.22 and Proposition 2.26. Since  $\theta : \underline{\mathrm{Hom}}_S(x, y) \rightarrow \mathrm{Hom}_S^L(x, y)$  is an equivalence of  $\infty$ -categories it follows that the corresponding diagram (7) is a categorical pullback square as well (see Proposition I-5.23).  $\square$

sect:transport

### 3. TRANSPORT FUNCTORS

**3.1. Preliminary discussion.** In analogy with the plain categorical setting, we claim that cocartesian fibrations  $q : \mathcal{E} \rightarrow \mathcal{C}$  over a given  $\infty$ -category are “the same thing” as functors into the  $\infty$ -category of  $\infty$ -categories  $F : \mathcal{C} \rightarrow \mathcal{Cat}_\infty$ . In our imaginations, the functor  $F$  should evaluate as the fibers  $F(x) \cong \mathcal{E}_x$  and the image of a given map  $\alpha : x \rightarrow y$  should be some kind of pushforward functor  $\alpha_* : \mathcal{E}_x \rightarrow \mathcal{E}_y$  which “moves along” cartesian lifts  $\tilde{\alpha} : \tilde{x} \rightarrow \tilde{y}$ , so that  $\alpha_*(\tilde{x}) \cong \tilde{y}$ .

Of course one can not simply construct the desired functor  $F : \mathcal{C} \rightarrow \mathcal{Cat}_\infty$  by hand. We (or rather, Lurie) instead proceed(s) by establishing a universal cocartesian fibration over  $\infty$ -categories

$$U : \mathcal{Z} \rightarrow \mathcal{Cat}_\infty.$$

It is then shown that each cocartesian fibration is realized as a (categorical) pullback along  $U$ ,

$$\begin{array}{ccc} \mathcal{E} & \longrightarrow & \mathcal{Z} \\ q \downarrow & & \downarrow \\ \mathcal{C} & \xrightarrow{F} & \mathcal{Cat}_\infty, \end{array}$$

and furthermore that the space of such pullback diagrams assembles into a contractible space. We refer to this uniquely determined functor  $F$  as the covariant transport functor along  $q$ , or as the functor which classifies  $q$ . One obtains a completely similar analysis of cartesian fibrations and classification via an applications of the opposite involution.

In this section we outline the above construction. Unlike at other points in this text we are not especially concerned with (all of) the technical details, and seek

only to provide a coherent narrative which explains clearly what's going on and how this stuff works.

We begin with a detour into  $(\infty, 2)$ -categories. We then construct the universal cocartesian fibration  $\mathcal{Z}$  via a certain “category of objects”, and explain how each fiber  $\mathcal{Z}_{\mathcal{E}}$  over a given  $\infty$ -category  $\mathcal{E} : * \rightarrow \mathcal{Cat}_{\infty}$  reproduces  $\mathcal{E}$  itself, up to equivalence. We define the space  $\mathcal{T}(q)$  of classifying diagrams and recall the contractibility of this space from [3]. The section concludes with a description of the pushforward functors  $\alpha_*$  appearing the transport  $F$ .

**3.2. Categories with objects and the universal cocartesian fibration.** From the  $(\infty, 2)$ -category  $\mathcal{Cat}_{\infty}$  we can produce the simplicial of pointed  $\infty$ -categories

$$(\mathcal{Cat}_{\infty})_* := (\mathcal{Cat}_{\infty})_{\Delta^0/}.$$

By Proposition 2.22 the simplicial set  $(\mathcal{Cat}_{\infty})_*$  is an  $(\infty, 2)$ -category and the forgetful functor  $(\mathcal{Cat}_{\infty})_* \rightarrow \mathcal{Cat}_{\infty}$  is a interior fibration.

**Definition 3.1.** The  $\infty$ -category of  $\infty$ -categories with a distinguished object is the pith of the  $(\infty, 2)$ -category of pointed  $\infty$ -categories,

$$\mathcal{P}\mathcal{Cat}_{\infty} := ((\mathcal{Cat}_{\infty})_*)^{\text{Pith}}.$$

**Remark 3.2.** The  $\mathcal{P}$  suffix stands for “pointed”, though we heed the warning from [3, 020W] and do not label this  $\infty$ -category as such.

**Remark 3.3.** There is a comparison functor  $\mathcal{P}\mathcal{Cat}_{\infty} \rightarrow (\mathcal{Cat}_{\infty})_{*/}$  which is, apparently, bijective on objects. However this map is not bijective on 1-morphisms so that it is not an isomorphism [3, 020Z].

Via an application of Corollary 2.21 we see that the forgetful functor restricts to provide a pullback diagram

$$\begin{array}{ccc} \mathcal{P}\mathcal{Cat}_{\infty} & \longrightarrow & (\mathcal{Cat}_{\infty})_* \\ \downarrow & & \downarrow \\ \mathcal{Cat}_{\infty} & \longrightarrow & \mathcal{Cat}_{\infty}. \end{array}$$

The forgetful functor  $\mathcal{P}\mathcal{Cat}_{\infty} \rightarrow \mathcal{Cat}_{\infty}$  is furthermore seen to be a cocartesian fibration in this case, via an application of Proposition 2.26. We record this result.

**Proposition 3.4** ([3, 0213]). *The forgetful functor  $\mathcal{P}\mathcal{Cat}_{\infty} \rightarrow \mathcal{Cat}_{\infty}$  is a cocartesian fibration.*

We call the above forgetful functor the *universal cocartesian fibration*, for reasons which will be apparent shortly.

**Definition 3.5.** We let  $\text{univ} : \mathcal{P}\mathcal{Cat}_{\infty} \rightarrow \mathcal{Cat}_{\infty}$  denote the cocartesian fibration induced by the forgetful functor  $(\mathcal{Cat}_{\infty})_* \rightarrow \mathcal{Cat}_{\infty}$ , as considered above.

**3.3. A remark on notation.** Our  $(\infty, 2)$ -category  $\mathcal{Cat}_{\infty}$  is the  $(\infty, 2)$ -category denoted by a bold  $\mathcal{QC}$  in [3, 020K]. Our  $(\mathcal{Cat}_{\infty})_*$  is the  $(\infty, 2)$ -category denoted by a bold  $\mathcal{QC}_{\text{Obj}}$  in [3, 0210]. The associated piths, which we've denoted  $\mathcal{Cat}_{\infty}$  and  $\mathcal{P}\mathcal{Cat}_{\infty}$  respectively, are the non-bolded  $\infty$ -categories  $\mathcal{QC}$  and  $\mathcal{QC}_{\text{Obj}}$  in [3].

**3.4. Fibers of the universal map  $\mathcal{P}\mathcal{C}at_\infty \rightarrow \mathcal{C}at_\infty$ .** In considering the universal cocartesian fibration  $\mathcal{P}\mathcal{C}at_\infty \rightarrow \mathcal{C}at_\infty$ , any point  $e : \Delta^0 \rightarrow \mathcal{C}at$  corresponds to an  $\infty$ -category  $\mathcal{E} = e(0)$  and we have the pullback

$$\mathcal{P}\mathcal{C}at_\infty \times_{\mathcal{C}at_\infty} \{e\}$$

which is some other  $\infty$ -category. Now, objects in this fiber are simply maps of  $\infty$ -categories  $* \rightarrow \mathcal{E}$ , and hence are identified with objects in  $\mathcal{E}$ . Similarly, 1-simplices in the fiber are identified 2-simplices in the  $\infty$ -category of  $\infty$ -categories

$$\begin{array}{ccc} & * & \\ x \swarrow & & \searrow y \\ \mathcal{E} & \xrightarrow{=} & \mathcal{E}. \end{array}$$

These are, by definition, natural transformations  $\alpha \in \text{Hom}_{\text{sSet}}(\Delta^1, \mathcal{E})$  with  $\alpha|_0 = x$  and  $\alpha|_1 = y$ , i.e. 1-simplices  $\alpha : x \rightarrow y$ . So we observe an identification of 1-skeleta

$$\mathcal{E}[\leq 1] = \mathcal{P}\mathcal{C}at_\infty \times_{\mathcal{C}at_\infty} \{e\}[\leq 1].$$

As an application of Theorem 2.27 and Corollary 2.28, we see that this direct identification of simplices in low-dimension expands to an equivalence of  $\infty$ -categories which calculates the fiber.

`prop:univ_fibs`

**Proposition 3.6.** *For any  $\infty$ -category  $\mathcal{E}$ , which we can understand as a point  $\mathcal{E} : * \rightarrow \mathcal{C}at_\infty$ , we have a categorical pullback square*

$$\begin{array}{ccc} \mathcal{E} & \longrightarrow & \mathcal{P}\mathcal{C}at_\infty \\ \downarrow & & \downarrow \\ * & \xrightarrow[\mathcal{E}]{} & \mathcal{C}at_\infty, \end{array}$$

and a corresponding equivalence between  $\infty$ -categories  $\theta : \mathcal{E} \xrightarrow{\sim} \mathcal{P}\mathcal{C}at_\infty \times_{\mathcal{C}at_\infty} \{\mathcal{E}\}$ .

### 3.5. Covariant transport: classifying cocartesian fibrations.

**Definition 3.7.** Let  $q : \mathcal{E} \rightarrow \mathcal{C}$  be a cocartesian fibration of  $\infty$ -categories. We say a functor  $F : \mathcal{C} \rightarrow \mathcal{C}at_\infty$  classifies the cocartesian fibration  $q$  if the functors  $q$  and  $F$  fit into a categorical pullback diagram

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\tilde{F}} & \mathcal{P}\mathcal{C}at_\infty \\ q \downarrow & & \downarrow \text{univ} \\ \mathcal{C} & \xrightarrow{F} & \mathcal{C}at_\infty. \end{array} \tag{8} \quad \boxed{\text{eq:672}}$$

In this case we say the above diagram witnesses  $F$  as a (covariant) transport functor along  $q$ .

**Remark 3.8.** The term “classifies” is used with some frequency in the works [2, ?]. However, [3] seems to prefer the term “transport representation” for a functor  $F$  as above. We will usually just refer to  $F$  as a, or the, transport functor for  $q$ .

We have an alternate characterization of transport functors via the fiber product  $\mathcal{C} \times_{\mathcal{C}at_\infty} \mathcal{P}\mathcal{C}at_\infty$ .



**Lemma 3.9.** *Let  $q : \mathcal{E} \rightarrow \mathcal{C}$  be a cocartesian fibration. A diagram (11) witnesses  $F : \mathcal{C} \rightarrow \mathcal{Cat}_\infty$  as transport along  $q$  if and only if the induced map to the fiber product*

$$\mathcal{E} \rightarrow \mathcal{C} \times_{\mathcal{Cat}_\infty} \mathcal{P}\mathcal{Cat}_\infty$$

*is an equivalence of cocartesian fibrations over  $\mathcal{C}$ .*

*Proof.* The fact that the map to the fiber is an equivalence follows by Proposition I-5.23. It follows by Proposition ?? and the diagram

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\sim} & \mathcal{C} \times_{\mathcal{Cat}_\infty} \mathcal{P}\mathcal{Cat}_\infty \\ & \searrow q & \swarrow \\ & \mathcal{C} & \end{array}$$

that the equivalence in question is an equivalence of cocartesian fibrations. □

The fiber product considered above is often denoted

$$\int_{\mathcal{C}} F := \mathcal{C} \times_{\mathcal{Cat}_\infty} \mathcal{P}\mathcal{Cat}_\infty,$$

so that any functors  $F : \mathcal{C} \rightarrow \mathcal{Cat}_\infty$  determines an associated cocartesian fibration  $\int_{\mathcal{C}} F \rightarrow \mathcal{C}$ . We find that  $F$  is a transport functor for  $q : \mathcal{E} \rightarrow \mathcal{C}$  if there is an equivalence

$$\mathcal{E} \xrightarrow{\sim} \int_{\mathcal{C}} F \text{ in the overcategory } \mathbf{sSet}_{/\mathcal{C}}.$$

We also notes that transport functors are stable under restriction.

1em:687

**Lemma 3.10.** *Suppose we have a pullback square*

$$\begin{array}{ccc} \mathcal{E}' & \xrightarrow{g} & \mathcal{E} \\ q' \downarrow & & \downarrow q \\ \mathcal{C}' & \xrightarrow{f} & \mathcal{C} \end{array}$$

(9) eq:689

*in which  $q$  and  $q'$  are cocartesian fibrations between  $\infty$ -categories, and consider a diagram of the form (11) which witnesses a functor  $F : \mathcal{C} \rightarrow \mathcal{Cat}_\infty$  as transport along  $q$ . Then for  $\tilde{F}' = \tilde{F}g$  and  $F' = Ff$ , the diagram*

$$\begin{array}{ccc} \mathcal{E}' & \xrightarrow{\tilde{F}'} & \mathcal{P}\mathcal{Cat}_\infty \\ q' \downarrow & & \downarrow \\ \mathcal{C}' & \xrightarrow{F'} & \mathcal{Cat}_\infty. \end{array}$$

*witnesses  $F'$  as a transport functor along  $q'$ .*

*Proof.* Since  $q$  is a cocartesian fibration, and in particular an isofibration, the diagram (9) is a categorical pullback square Corollary I-5.22. The result now follows from the fact that categorical pullback squares are stable under composition [3, 033J]. □

Given a cocartesian fibration we can now consider the simplicial subset in the functor category

$$\mathrm{Fun}(\mathcal{C}, \mathcal{Cat}_\infty) \times_{\mathrm{Fun}(\mathcal{E}, \mathcal{Cat}_\infty)} \mathrm{Fun}(\mathcal{E}, \mathcal{P}\mathcal{Cat}_\infty) \quad (10)$$

eq:710

which consists of diagrams witnessing transport for a given cocartesian fibration  $q : \mathcal{E} \rightarrow \mathcal{C}$ .

**Definition 3.11.** For a given cocartesian fibration  $q : \mathcal{E} \rightarrow \mathcal{C}$ , we let  $\mathcal{T}(q)$  denote the simplicial subset in the fiber product (10) whose simplices correspond to diagrams

$$\begin{array}{ccc} \Delta^n \times \mathcal{E} & \xrightarrow{\tilde{F}} & \mathcal{P}\mathcal{Cat}_\infty \\ \Delta^n \times q \downarrow & & \downarrow \\ \Delta^n \times \mathcal{C} & \xrightarrow{F} & \mathcal{Cat}_\infty. \end{array}$$

which witness  $F$  as a covariant transport functor along  $\Delta^n \times q$ .

Stability of such diagrams under restriction (Lemma 3.10) assures us that  $\mathcal{T}(q)$  is in fact a simplicial subset in the given fiber product.

thm:transport

**Theorem 3.12** (Universality theorem [3, 02SC]). *For any cocartesian fibration  $q : \mathcal{E} \rightarrow \mathcal{C}$ , the terminal map  $\mathcal{T}(q) \rightarrow *$  is a trivial Kan fibration.*

This result says that any cocartesian fibration  $q$  admits a covariant transport functor  $F : \mathcal{C} \rightarrow \mathcal{Cat}_\infty$ , and that this functor is uniquely determined up to a contractible space of choices.

**3.6. Transport for cartesian fibrations.** Given any cartesian fibration  $p : \mathcal{E} \rightarrow \mathcal{C}$ , we have the associated cocartesian fibration  $p^{\mathrm{op}} : \mathcal{E}^{\mathrm{op}} \rightarrow \mathcal{C}^{\mathrm{op}}$ . So our analysis of classifying functors for cocartesian fibrations dualizes in the obvious ways to provide an analysis of classifying functors for cartesian fibrations.

**Definition 3.13.** Let  $p : \mathcal{E} \rightarrow \mathcal{C}$  be a cartesian fibration of  $\infty$ -categories. We say a functor  $F : \mathcal{C}^{\mathrm{op}} \rightarrow \mathcal{Cat}_\infty$  classifies the cartesian fibration  $p$  if the functors  $p^{\mathrm{op}}$  and  $F$  fit into a categorical pullback diagram

$$\begin{array}{ccc} \mathcal{E}^{\mathrm{op}} & \xrightarrow{\tilde{F}} & \mathcal{P}\mathcal{Cat}_\infty \\ q \downarrow & & \downarrow \mathrm{univ} \\ \mathcal{C}^{\mathrm{op}} & \xrightarrow{F} & \mathcal{Cat}_\infty. \end{array} \quad (11)$$

eq:672

In this case we say the above diagram witnesses  $F$  as a (contravariant) transport functor along  $p$ .

Given a cartesian fibration  $p : \mathcal{E} \rightarrow \mathcal{C}$ , we define the space of transport functors with witness in the obvious way  $\mathcal{T}(p) := \mathcal{T}(p^{\mathrm{op}})$ . Theorem 3.12 implies contractibility of this space immediately.

thm:uniq\_transp

**Theorem 3.14** (Contravariant universality). *For any cartesian fibration  $p : \mathcal{E} \rightarrow \mathcal{C}$ , the terminal map  $\mathcal{T}(p) \rightarrow *$  from the space of transport functors along  $p$  is a trivial Kan fibration.*

Again this establishes both the existence and uniqueness of contravariant transport.

**3.7. Transport along edges.** Let  $q : \mathcal{E} \rightarrow \Delta^1$  be a cocartesian fibration, and  $f : \Delta^1 \rightarrow \mathcal{Cat}_\infty$  be a transport functor for  $q$ . The functor  $f$  specifies a pair of  $\infty$ -categories and a functor  $F' : \mathcal{E}'_0 \rightarrow \mathcal{E}'_1$ . We then have partial diagrams

$$\begin{array}{ccc} \mathcal{E}_i & & \mathcal{E}'_i \\ & \searrow & \swarrow \theta \\ & \{i\} \times_{\Delta^1} (\int_{\Delta^1} f), & \end{array}$$

where as before  $\int_{\Delta^1} f$  is the fiber product  $\Delta^1 \times_{\mathcal{Cat}_\infty} \mathcal{P}\mathcal{Cat}_\infty$ ,  $\theta$  is the equivalence of Section ??, and  $\mathcal{E}_i \rightarrow \int_{\Delta^1} f$  is the equivalence witnessing  $f$  as a transport functor. We choose equivalences  $\xi_i : \mathcal{E}_i \rightarrow \mathcal{E}'_i$  which complete a 2-simplex

$$\begin{array}{ccc} \mathcal{E}_i & \xrightarrow{\xi_i} & \mathcal{E}'_i \\ & \searrow & \swarrow \theta \\ & \{i\} \times_{\Delta^1} (\int_{\Delta^1} f), & \end{array}$$

in  $\mathcal{Cat}_\infty$ . By pulling back along the  $\xi_i$ , the defining functor  $F' : \mathcal{E}_0 \rightarrow \mathcal{E}_1$  is identified with a functor between the  $\mathcal{E}_i$ . The following determines this functor via implicit criterion.

**prop:1\_transport**

**Proposition 3.15.** *Fix  $q : \mathcal{E} \rightarrow \Delta^1$  and  $f : \Delta^1 \rightarrow \mathcal{Cat}_\infty$  as above. There is a functor  $F : \mathcal{E}_1 \rightarrow \mathcal{E}_2$  and a natural transformation*

$$\tilde{F} : \Delta^1 \times \mathcal{E}_0 \rightarrow \mathcal{E}$$

*from the inclusion  $\mathcal{E}_1 \rightarrow \mathcal{E}$  to the composite of  $F$  with the inclusion  $\mathcal{E}_1 \rightarrow \mathcal{E}$  for which, at each object  $s : * \rightarrow \mathcal{E}_0$ , the corresponding map*

$$\tilde{F}_s : \Delta^1 = \Delta^1 \times \{s\} \rightarrow \mathcal{E}$$

*is  $q$ -cocartesian. This functor is unique up to equivalence, and fits into a diagram*

$$\begin{array}{ccc} \mathcal{E}_0 & \xrightarrow{F} & \mathcal{E}_1 \\ \xi_0 \downarrow & \searrow & \downarrow \xi_1 \\ \mathcal{E}'_0 & \xrightarrow{F'} & \mathcal{E}'_1, \end{array}$$

*in the  $\infty$ -category  $\mathcal{Cat}_\infty$ , where the  $\xi_i$  are as above.*

*Proof.* Combine [3, 01VS] and [3, 027K]. □

Now consider the global situation, where we have some cocartesian fibration  $q : \mathcal{E} \rightarrow \mathcal{C}$  and classifying functor  $\mathcal{C} \rightarrow \mathcal{Cat}_\infty$ . At any morphism  $\alpha : x \rightarrow y$  in  $\mathcal{C}$ , transport restricts to a functor  $F' : \mathcal{E}'_x \rightarrow \mathcal{E}'_y$ . According to Proposition 3.15, this functor is uniquely determined by a functorial choice of  $q$ -cocartesian maps between the fiber categories  $\mathcal{E}_x$  and  $\mathcal{E}_y$ . There is, in particular, a unique functor

$F : \mathcal{E}_x \rightarrow \mathcal{E}_y$  and a transformation which fits into a diagram

$$\begin{array}{ccc} \Delta^1 \times \mathcal{E}_x & \longrightarrow & \mathcal{E} \\ \downarrow & & \downarrow \\ \Delta^1 & \xrightarrow{\alpha} & \mathcal{C} \end{array}$$

and sends each edge  $\Delta^1 \times \{s\}$  in  $\Delta^1 \times \mathcal{E}_x$  to a  $q$ -cartesian morphism in  $\mathcal{E}$  which lies over  $\alpha$ . This functor fits into a diagram

$$\begin{array}{ccc} \mathcal{E}_x & \xrightarrow{F} & \mathcal{E}_y \\ \downarrow \sim & \searrow & \downarrow \sim \\ \mathcal{E}'_x & \xrightarrow{F'} & \mathcal{E}'_y \end{array}$$

**3.8. Classification of left and right fibrations.** We have the simplicial subcategory  $\underline{\mathbf{Kan}} \rightarrow \mathbf{Cat}_\infty$  and subsequent simplicial subset  $\mathcal{K}an \subseteq \mathbf{Cat}_\infty$ . This simplicial subset is the full  $(\infty, 2)$ -subcategory whose objects are precisely those  $\infty$ -categories which are Kan complexes, and so the inclusion preserves thin 2-simplices. We now have the full  $(\infty, 2)$ -subcategory  $\mathcal{K}an_* \rightarrow (\mathbf{Cat}_\infty)_*$  of pointed Kan complexes and the pullback diagram

$$\begin{array}{ccc} \mathcal{K}an_* & \longrightarrow & (\mathbf{Cat}_\infty)_* \\ \downarrow & & \downarrow \\ \mathcal{K}an & \longrightarrow & \mathbf{Cat}_\infty \end{array}$$

which restricts to a pullback diagram into the piths

$$\begin{array}{ccc} \mathcal{K}an_* & \longrightarrow & (\mathbf{Cat}_\infty)_* \\ \downarrow & & \downarrow \\ \mathcal{K}an & \longrightarrow & \mathbf{Cat}_\infty. \end{array}$$

We recall that the map  $\mathcal{K}an_* \rightarrow \mathcal{K}an$  is a left fibration, by Corollary I-??.

**Proposition 3.16.** *A cocartesian fibration  $q : \mathcal{E} \rightarrow \mathcal{C}$  is a left fibration if and only if the corresponding transport functor  $F : \mathcal{C} \rightarrow \mathbf{Cat}_\infty$  has image in  $\mathcal{K}an$ . Similarly, a cartesian fibration  $p : \mathcal{E} \rightarrow \mathcal{C}$  is a right fibration if and only if the corresponding transport functor  $G : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Cat}_\infty$  has image in  $\mathcal{K}an$ .*

*Proof.* By Proposition ??, a cocartesian (resp. cartesian) fibration  $\mathcal{E} \rightarrow \mathcal{C}$  is a left (resp. right) fibration if and only if its fibers over object  $\text{sin } \mathcal{C}$  are Kan complexes. So the result follows by the calculation of the fibers of the pullback fibration  $\int_{\mathcal{C}} F \rightarrow \mathcal{C}$  provided in Proposition 3.6.  $\square$

This proposition tells us that any left fibration  $q : \mathcal{E} \rightarrow \mathcal{C}$  fits into a categorical pullback square

$$\begin{array}{ccc} \mathcal{E} & \longrightarrow & \mathcal{K}an_* \\ q \downarrow & & \downarrow \\ \mathcal{C} & \xrightarrow{F} & \mathcal{K}an \end{array}$$

for some functor  $F$ , and that  $F$ , considered as a functor into  $\mathcal{Cat}_\infty$ , is the transport functor for  $q$ . In this way left fibrations are classified by maps into the  $\infty$ -category of Kan complexes. One obtains similar statements for right fibrations by applying the opposite functor.

#### 4. NATURALITY OF TRANSPORT: STRAIGHTENING AND UNSTRAIGHTENING

In this section we explain how the assignment which sends a functor  $F : \mathcal{C} \rightarrow \mathcal{Cat}_\infty$  to the corresponding cocartesian fibration

$$\int_{\mathcal{C}} F = \mathcal{C} \times_{\mathcal{Cat}_\infty} \mathcal{P}\mathcal{Cat}_\infty \rightarrow \mathcal{C}$$

extends to an equivalence of  $\infty$ -categories

$$\mathrm{Un} : \mathrm{Fun}(\mathcal{C}, \mathcal{Cat}_\infty) \xrightarrow{\sim} \mathrm{Cocart}(\mathcal{C}),$$

where  $\mathrm{Cocart}(\mathcal{C})$  is an  $\infty$ -category of cocartesian fibrations over  $\mathcal{C}$ . Similarly, we have an equivalence for cartesian fibrations

$$\mathrm{Un} : \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{Cat}_\infty) \xrightarrow{\sim} \mathrm{Cart}(\mathcal{C}),$$

which one obtains simply by applying the opposite involution. These functors are referred to as the unstraightening functors [2].

Our investigation centers not the unstraightening functor per-se, but its discrete inverse, the aptly named straightening functor. In our discussion we generalize our base to a simplicial set  $S$  rather than an  $\infty$ -category.

**4.1. Marked simplicial sets.** A marked simplicial set is a pair  $(K, W)$  consisting of a simplicial set  $K$  and a choice of 1-simplices  $W \subseteq K[1]$  which contains all degenerate 1-simplices. A map between marked simplicial sets  $f : (K, W) \rightarrow (K', W')$  is a map of simplicial sets which sends  $W$  into  $W'$ . In this way we obtain the category  $\mathrm{sSet}^+$  of marked simplicial sets.

The forgetful functor  $\mathrm{sSet}^+ \rightarrow \mathrm{sSet}$  has a right adjoint  $-^\# : \mathrm{sSet} \rightarrow \mathrm{sSet}^+$ , which sends a simplicial set  $K$  to the simplicial set  $K$  with all 1-simplices marked. The category  $\mathrm{sSet}^+$  also admits products, with  $(K, W) \times (K', W') = (K \times K', W \times W')$ , and the right adjoint  $-^\#$  is seen to be symmetric monoidal. Via this monoidal functor we obtain an action of  $\mathrm{sSet}$  on  $\mathrm{sSet}^+$ .

As a notational point, from this point on we often omit markings from our notation, and simply say  $K$  is a marked simplicial set to indicate that  $K$  is a simplicial set equipped with some specified marking  $(K, W)$ .

**Lemma 4.1.** *The  $\mathrm{sSet}$ -module category  $\mathrm{sSet}^+$  admits inner-Homs  $\underline{\mathrm{Hom}}_{\mathrm{sSet}^+}(K, K')$ . The  $n$ -simplices in the underlying simplicial set maps*

$$\underline{\mathrm{Hom}}_{\mathrm{sSet}^+}(K, K')[n] := \mathrm{Hom}_{\mathrm{sSet}^+}((\Delta^n)^\# \times K, K').$$

Note that this simplicial set is a simplicial subset in the usual inner-Homs for simplicial sets.

lem:1074

**Lemma 4.2.** *Let  $(K, W)$  and  $(K', W')$  be marked simplicial sets and suppose that the marked vertices in  $K'$  are stable under compositions, i.e. that for any simplex*

$$s : \Delta^2 \rightarrow K'$$

*in which edges  $s|_{\Delta^{\{i, i+1\}}}$  are marked, the edge  $s|_{\Delta^{\{0, 2\}}}$  is marked as well. A map of unmarked simplicial sets  $F : \Delta^n \times K \rightarrow K'$  is an  $n$ -simplex in  $\underline{\mathrm{Hom}}_{\mathrm{sSet}^+}(K, K')$  if and only if the following hold:*

- (a) The restrict to each vertex  $F|_{\Delta^{\{i\}}} : K \rightarrow K'$  is a map of marked simplicial sets.
- (b) At each vertex  $x : * \rightarrow K$ , and each  $0 \leq i < j \leq n$ , the edge  $F|_{\Delta^{\{i,j\}}} \times \{x\} \rightarrow K$  is marked in  $K'$ .

*Proof.* The marked edges in  $(\Delta^n)^\# \times K$  are all pairs  $(\alpha_{ij}, w)$  where  $\alpha_{ij} : [1] \rightarrow [n]$  is the unique increasing map with image  $\{i, j\}$  and  $w : x \rightarrow y$  is any marked edge in  $K$ . Since all degenerate 1-simplices are marked, it is clear that any marked map  $F$  must satisfy (a) and (b). So let us suppose now that  $F$  satisfies (a) and (b), and consider a marked edge  $(\alpha_{ij}, w)$  with  $i \leq j$ . Such an edge appears as the  $\{0, 3\}$  edge in a 2-simplex

$$t : \Delta^2 \rightarrow \Delta^n \times K \text{ with } t|_{\Delta^{\{0,1\}}} = (\alpha_{i,j}, x) \text{ and } t|_{\Delta^{\{1,2\}}} = (\alpha_{i,j}, w).$$

and  $Ft : \Delta^2 \rightarrow K'$  is now a 2-simplex in  $K'$  with both edges  $Ft|_{\Delta^{\{i,i+1\}}}$  marked. (In the case  $i = j$  the simplex  $Ft|_{\Delta^{\{i,j\}}}$  is marked simply because it is degenerate, otherwise this follows by (a).) It follows by stability under composition that  $Ft|_{\Delta^{\{0,2\}}} = F(\alpha_{ij}, w)$  is marked as well. So  $F$  preserves markings.  $\square$

The following is an alternate phrasing of this lemma

lem:1098

**Lemma 4.3.** *The mapping complex  $\underline{\text{Hom}}_{\text{sSet}^+}(K, K')$  is a subcomplex in the complex of simplicial maps  $\text{Fun}(K, K')$ . An  $n$ -simplex  $F : \Delta^n \rightarrow \text{Fun}(K, K')$  lies in  $\underline{\text{Hom}}_{\text{sSet}^+}(K, K')$  if and only if each functor  $F_i : \Delta^{\{i\}} \rightarrow \text{Fun}(K, K')$  preserved markings and, at each  $x : * \rightarrow K'$  and  $i < j$ , the image*

$$\Delta^1 \cong \Delta^{\{i,j\}} \xrightarrow{F_{ij}} \text{Fun}(K, K') \xrightarrow{x^*} \text{Fun}(*, K') = K'$$

*is a marked map in  $K'$ .*

We are also interested in the nature of 2-simplices in  $\underline{\text{Hom}}_{\text{sSet}^+}(K, K')$ . The following is obtained from Lemma 4.3, essentially immediately.

cor:1108

**Corollary 4.4.** *Suppose that  $K$  and  $K'$  are marked simplicial sets, and that the markings in  $K'$  satisfy the 2-of-3 property. A 2-simplex  $\Delta^2 \rightarrow \text{Fun}(K, K')$  lies in  $\underline{\text{Hom}}_{\text{sSet}^+}(K, K')$  if and only if each constituent map  $\Delta^{\{i\}} \rightarrow \text{Fun}(K, K')$  is marked, and 2 of the 3 edges  $\Delta^{\{i < j\}} \rightarrow \text{Fun}(K, K')$  is marked.*

Given a simplicial set  $S$  we let  $\text{sSet}_S^+$  denote the simplicial category whose objects are morphisms of marked simplicial sets  $p : K \rightarrow S^\#$ , whose morphism complexes  $\underline{\text{Hom}}_S(K, K')$  are the fiber products

$$\begin{array}{ccc} \underline{\text{Hom}}_S(K, K') & \longrightarrow & \underline{\text{Hom}}_{\text{sSet}^+}(K, K') \\ \downarrow & & \downarrow (p')_* \\ * & \xrightarrow{p} & \underline{\text{Hom}}_{\text{sSet}^+}(K, S^\#) = \text{Fun}(K, S) \end{array}$$

prop:cocart\_kan

**Proposition 4.5.** *Let  $q : X \rightarrow S$  be a cocartesian fibration, and consider the associated map of marked simplicial sets  $X^q \rightarrow S^\#$  where  $X^q$  is  $X$  paired with the collection of all  $q$ -cocartesian maps. Then for any marked simplicial set  $K$  over  $S^\#$ , the mapping complex  $\underline{\text{Hom}}_S(K, X^q)$  is a Kan complex.*

*Proof.* By 4.8 the map  $\text{Fun}(K, X) \rightarrow \text{Fun}(K, S)$  is an inner fibration, so that the fiber  $\text{Fun}_S(K, X)$  over the diagram  $p$  is an  $\infty$ -category. We have now that

$\underline{\text{Hom}}_S(K, X^q)$  is a subcomplex in the  $\infty$ -category  $\text{Fun}_S(K, X)$  whose  $n$ -simplices are precisely those  $n$ -simplices in  $\text{Fun}_S(K, X)$  which satisfy the conditions specified in Lemma 4.2 (a) and (b), since  $q$ -cocartesian maps in  $X$  are stable under composition. This stability under composition in  $X$  also implies that any completion  $\Delta^2 \times K \rightarrow X$  of an inner horn  $\Lambda_1^2 \times K \rightarrow X$  which lies in  $\underline{\text{Hom}}_S(K, X^q)$  also lies in  $\underline{\text{Hom}}_S(K, X^q)$ . Since the subcomplex  $\underline{\text{Hom}}_S(K, X^q)$  in  $\text{Fun}(K, X)$  is characterized by a restriction on the 1-simplices, it follows that all higher dimensional horns in  $\underline{\text{Hom}}_S(K, X^q)$  complete to simplices in  $\underline{\text{Hom}}_S(K, X^q)$ . So  $\underline{\text{Hom}}_S(K, X^q)$  is an  $\infty$ -subcategory in  $\text{Fun}_S(K, X)$ .

Now, for any 1-simplex  $\zeta : \Delta^1 \rightarrow \underline{\text{Hom}}_S(K, X^q)$  and vertex  $x : * \rightarrow K$  with image  $s$  in  $S$ , the composite

$$\Delta^1 \rightarrow \underline{\text{Hom}}_S(K, X^q) \xrightarrow{x^*} X$$

has  $q$ -cocartesian image in the  $\infty$ -category  $X_s$ , and in particular is an isomorphism in  $X_s$ . This implies that  $\zeta$  is an isomorphism in the ambient category  $\text{Fun}_S(K, X)$ , by Proposition I-6.8. From the 2 of 3 property for  $q$ -cocartesian morphisms, and Corollary 4.4, it follows that any inverse  $\zeta^{-1} : \Delta^1 \rightarrow \text{Fun}_S(K, X)$  to  $\zeta$  is also in  $\underline{\text{Hom}}_S(K, X^q)$ . So we see that every morphism in  $\underline{\text{Hom}}_S(K, X^q)$  is an isomorphism, and so this complex is a Kan complex.  $\square$

**4.2. The  $\infty$ -category of cocartesian fibrations over a base.** We consider each cocartesian fibration  $q : X \rightarrow S$  as a morphism in  $\text{sSet}^+$  by applying the  $q$ -cocartesian marking  $X^q$  on  $X$  and the maximal marking  $S^\#$  on  $S$ , and we have a the full simplicial subcategory  $\underline{\text{Cocart}}(S) \subseteq \text{sSet}_{/S}^+$  of cocartesian fibrations over  $S$ . By Proposition 4.5 this simplicial subcategory is enriched in Kan complexes, so that the homotopy coherent nerve is an  $\infty$ -category.

**Definition 4.6.** Given a simplicial set  $S$ , the  $\infty$ -category of cocartesian fibrations over  $S$  is the homotopy coherent nerve

$$\text{Cocart}(S) := \text{N}^{\text{hc}}(\underline{\text{Cocart}}(S))$$

**Example 4.7.** When  $S = *$  a cocartesian fibration over  $*$  is an  $\infty$ -category  $\mathcal{C}$ . The associated cocartesian marking marks equivalences in  $\mathcal{C}$ . Since all functors between  $\infty$ -categories preserve equivalences we have

$$\underline{\text{Hom}}_*(\mathcal{C}, \mathcal{D}) = \text{Fun}(\mathcal{C}, \mathcal{D}) \quad \text{and} \quad \underline{\text{Hom}}_{\text{Cocart}(*)}(\mathcal{C}, \mathcal{D}) = \text{Fun}(\mathcal{C}, \mathcal{D})^{\text{Kan}}.$$

Hence  $\text{Cocart}(*) = \mathcal{Cat}_\infty$ .

**Remark 4.8.** Of course, we have an  $(\infty, 2)$ -category of cocartesian fibrations, which we obtain by applying the homotopy coherent nerve to the simplicial category  $\underline{\text{Cocart}}(S)'$ . We have no intentions of using this  $(\infty, 2)$ -category in this work, and so disregard it.

To state things clearly, objects in  $\text{Cocart}(S)$  are cocartesian fibrations  $X \rightarrow S$  and morphisms are functors  $F : X \rightarrow Y$  which preserve cocartesian maps and fit into a diagram over the base

$$\begin{array}{ccc} X & \xrightarrow{F} & Y \\ & \searrow & \swarrow \\ & S & \end{array} .$$

A 2-simplex  $\Delta^2 \rightarrow \text{Cocart}(S)$  is a pair of partial diagrams of cocartesian fibrations

$$\begin{array}{ccc} X_0 & \xrightarrow{F_{01}} & X_1 & \xrightarrow{F_{12}} & X_2 \\ & \searrow q_0 & \downarrow q_1 & \swarrow q_2 & \\ & & S & & \end{array} \quad \begin{array}{ccc} X_0 & \xrightarrow{F_{02}} & X_2 \\ & \searrow q_0 & \swarrow q_2 & \\ & & S & \end{array}$$

and a map

$$\begin{array}{ccc} \Delta^1 \times X_0 & \xrightarrow{F} & X_2 \\ & \searrow q_0 p_2 & \swarrow q_2 & \\ & & S & \end{array}$$

which satisfies  $F|_0 = F_{12}F_{01}$  and  $F|_1 = F_{02}$ , and for which  $F|_{\Delta^1 \times \{x\}}$  is an isomorphism in the  $\infty$ -category  $X_2 \times_S \{s\}$ .

**4.3. Simplicial enrichment for simplicial functors.** For any simplicial set  $K$ , and any simplicial functor  $F : \underline{A} \rightarrow \underline{\text{sSet}}^+$  we define  $K \times F : \underline{A} \rightarrow \underline{\text{sSet}}^+$  to be the composite

$$\underline{A} \xrightarrow{F} \underline{\text{sSet}}^+ \xrightarrow{K \times -} \underline{\text{sSet}}^+.$$

One sees that the action of  $K$  is in fact a simplicial functor via the symmetry  $K \times \Delta^n \times - \cong \Delta^n \times K \times -$ .

For two functors  $F, F' : \underline{A} \rightarrow \underline{\text{sSet}}^+$  we let  $\text{Nat}(F, F')$  denote the simplicial set with  $n$ -simplices

$$\text{Nat}(F, F')[n] := \{\text{Natural transformations } \Delta^n \times F \rightarrow F'\}.$$

The composition operation

$$\text{Nat}(F', F'') \times \text{Nat}(F, F') \rightarrow \text{Nat}(F, F'')$$

takes transformations  $\Delta^n \times F' \rightarrow F''$  and  $\Delta^n \times F \rightarrow F'$  to the transformation

$$\Delta^n \times F \xrightarrow{\delta \times 1} (\Delta \times \Delta) \times F = \Delta^n \times (\Delta^n \times F) \rightarrow \Delta^n \times F' \rightarrow F''.$$

**Definition 4.9.** For any simplicial category  $\underline{A}$ , we let  $\text{Fun}(\underline{A}, \underline{\text{sSet}}^+)$  denote the simplicial category of simplicial functors, with morphism complexes  $\text{Nat}(F, F')$ .

We have the evaluation functor

$$ev : \text{Fun}(\underline{A}, \underline{\text{sSet}}^+) \times \underline{A} \rightarrow \underline{\text{sSet}}^+.$$

This functor sends a pair  $(F, a)$  of a functor and an object in  $\underline{A}$  to  $F(a)$ , and on morphisms the map of simplicial sets

$$\text{Nat}(F, F') \times \underline{\text{Hom}}_{\underline{A}}(a, a') \rightarrow \underline{\text{Hom}}_{\underline{\text{sSet}}}^+(Fa, F'a')$$

sends a pair

$$(\Delta^n \times F \rightarrow F', \sigma : \Delta^n \rightarrow \underline{\text{Hom}}_{\underline{A}}(a, a'))$$

to the composite

$$\Delta^n \times Fa \xrightarrow{\delta \times 1} \Delta^n \times \Delta^n \times Fa \xrightarrow{1 \times F \sigma} \Delta^n \times Fa' \rightarrow F'a'.$$

**Lemma 4.10.** *There is a natural isomorphism*

$$\begin{aligned} \text{Hom}_{\text{sCat}}(\underline{A}', \text{Fun}(\underline{A}, \underline{\text{sSet}}^+)) &\xrightarrow{\sim} \text{Hom}_{\text{sCat}}(\underline{A}' \times \underline{A}, \underline{\text{sSet}}^+) \\ G &\mapsto ev(G \times id_{\underline{A}}). \end{aligned}$$



*Proof.* The inverse sends a functor  $\Theta : A' \times A \rightarrow \underline{\mathbf{sSet}}^+$  to the functor

$$\begin{aligned} \theta : \underline{A}' &\rightarrow \text{Fun}(\underline{A}, \underline{\mathbf{sSet}}^+), \quad \theta(a) = \Theta(a, -) \\ \theta_{ab} : \underline{\text{Hom}}_{\underline{A}'}(a, b) &\rightarrow \text{Nat}(\Theta(a, -), \Theta(b, -)), \end{aligned}$$

where  $\theta_{ab}$  sends an  $n$ -simplex  $\sigma$  to the transformation which evaluates at each  $x$  in  $\underline{A}$  as

$$\Theta(\sigma, x) : \Delta^n \times \Theta(a, x) \rightarrow \Theta(b, x).$$

□

**4.4. Simplicial functors as functor categories.** As the category of simplicial categories is cocomplete [3, 00K3], one sees that the path category construction admits a unique extension from the class of simplices to the entire category of simplicial sets.

**Lemma 4.11** ([3, 00L4]). *The association  $\Delta^n \mapsto \text{Path } \Delta^n$  extends to a functor  $\text{Path} : \mathbf{sSet} \rightarrow \mathbf{sCat}$  which provides a left adjoint to the homotopy coherent nerve,*

$$\text{Hom}_{\mathbf{sSet}}(-, \mathbf{N}^{\text{hc}}-) \cong \text{Hom}_{\mathbf{sCat}}(\text{Path } -, -).$$

The product of the unit map  $S \rightarrow \mathbf{N}^{\text{hc}} \text{Path } S$ , and commutativity of the homotopy coherent nerve with products, provide natural maps

$$\text{Path}(S' \times S) \rightarrow \text{Path}(S') \times \text{Path}(S)$$

from which the path category functor becomes op-lax monoidal. Via this op-lax structure we obtain a map

$$\begin{aligned} \text{Hom}_{\mathbf{sCat}}(\text{Path } \Delta^n, \text{Fun}(\text{Path } S, \underline{\mathbf{sSet}}^+)) &\xrightarrow{\sim} \text{Hom}_{\mathbf{sCat}}(\text{Path } \Delta^n \times \text{Path } S, \underline{\mathbf{sSet}}^+) \\ &\rightarrow \text{Hom}_{\mathbf{sCat}}(\text{Path}(\Delta^n \times S), \underline{\mathbf{sSet}}^+) \cong \text{Hom}_{\mathbf{sSet}}(\Delta^n \times S, \mathbf{N}^{\text{hc}} \underline{\mathbf{sSet}}^+). \end{aligned}$$

Taking these maps collectively across various  $n$  provides a map of simplicial sets

$$\mathbf{N}^{\text{hc}} \text{Fun}(\text{Path } S, \underline{\mathbf{sSet}}^+) \rightarrow \text{Fun}(S, \mathbf{N}^{\text{hc}} \underline{\mathbf{sSet}}^+).$$

We restrict to consider those functors which land in the non-full simplicial subcategory of  $\infty$ -categories, with Kanified morphism complexes, and consider the resultant map

$$\text{comp} : \mathbf{N}^{\text{hc}} \text{Fun}(\text{Path } S, \underline{\mathbf{Cat}}_\infty^+) \rightarrow \text{Fun}(S, \underline{\mathcal{C}at}_\infty). \quad (12)$$

eq:comp\_NN

prop:nerv\_to\_nerv

**Proposition 4.12.** *The simplicial category  $\text{Fun}(\text{Path } S, \underline{\mathbf{Cat}}_\infty^+)$  is enriched in Kan complexes, and the comparison functor (12) is an equivalence of  $\infty$ -categories.*

We outline how this result occurs, according to the logic of [2]. So the proof is not so much a proof as an “authentication ticket” which the reader might verify for themselves.

*Proof.* The categories  $\underline{\mathbf{sSet}}^+$  and  $\text{Fun}(\text{Path } S, \underline{\mathbf{sSet}}^+)$  admit combinatorial simplicial model structures under which the subcategories of fibrant-cofibrant objects are precisely the subcategories

$$\underline{\mathbf{Cat}}_\infty^+ \subseteq \underline{\mathbf{sSet}}^+ \quad \text{and} \quad \text{Fun}(\text{Path } S, \underline{\mathbf{Cat}}_\infty^+) \subseteq \text{Fun}(\text{Path } S, \underline{\mathbf{sSet}}^+)$$

[2, Proposition 3.1.3.7, Corollary 3.1.4.4, Proposition A.3.3.2, Remark A.3.3.4]. It follows that the simplicial category  $\text{Fun}(\text{Path } S, \underline{\mathbf{Cat}}_\infty^+)$  is enriched in Kan complexes [2, Remark A.3.1.8], and also that  $\underline{\mathbf{sSet}}^+$  provides a  $\text{Path } S$ -chunk [2, Definition A.3.4.9] of itself [2, Example A.3.4.4]. We now see from [2, Proposition 4.2.4.4] that the comparison functor.

□

**Remark 4.13.** The specific claim of Proposition 4.12 seems not to appear explicitly in [2], though it may be implicit. We found this particular claim in [?, Remark 3.7].

**4.5. Non-enriched Straightening and unstraightening.** Let  $p : K \rightarrow S$  be a cocartesian fibration from a marked simplicial set  $(K, W)$ , and consider the pushout  $S_p$  of the following diagram

$$\begin{array}{ccc} K & \xrightarrow{i} & \{*\} \star K \\ p \downarrow & & \downarrow \\ S & \xrightarrow{i_p} & S_p. \end{array}$$

We now have the functor of plain categories

$$\mathrm{St}[0] : \underline{\mathrm{sSet}}_{/S}^+[0] \rightarrow \mathrm{Fun}(\mathrm{Path} S, \underline{\mathrm{sSet}}_{\infty}^+[0])$$

which sends each object  $q : K \rightarrow S$  to the representable functor  $\underline{\mathrm{Hom}}_{\mathrm{Path} S_p}(*, -)$ , where each value  $\underline{\mathrm{Hom}}_{\mathrm{Path} S_p}(*, s)$  is equipped with a natural marking  $W_s$ . (Here we are writing  $s$  for  $i_p(s)$  by an abuse of notation.) The marked edges are all those which appear as follows:

- Take a marked edge  $\alpha : x' \rightarrow y'$  in  $K$ , and consider the extension to a 2-simplex

$$\mathrm{Ext}(\alpha) := \{*\} \star \alpha : \Delta^2 \rightarrow \{*\} \star K$$

which then has image  $\mathrm{Ext}'(\alpha) : \Delta^2 \rightarrow S_p$ . This 2-simplex appears as

$$\mathrm{Ext}'(\alpha) : \begin{array}{ccc} & px & \\ * \nearrow & & \searrow p\alpha \\ & py & \end{array}$$

which then determines a 1-simplex  $E(\alpha) : \Delta^1 \rightarrow \underline{\mathrm{Hom}}_{\mathrm{Path} S_p}(*, py)$ .

- Take any 1-simplex  $B : \Delta^1 \rightarrow \underline{\mathrm{Hom}}_{\mathrm{Path} S}(py, s)$ .
- Consider the composite  $BE(\alpha) : \Delta^1 \rightarrow \underline{\mathrm{Hom}}_{\mathrm{Path} S_p}(*, s)$ .

The marked edges in  $W_s$  in  $\underline{\mathrm{Hom}}_{\mathrm{Path} S_p}(*, s)$  are precisely those edges which appear as above.

**Definition 4.14.** The functor  $\mathrm{St}[0] : \underline{\mathrm{sSet}}_{/S}^+[0] \rightarrow \mathrm{Fun}(\mathrm{Path} S, \underline{\mathrm{sSet}}_{\infty}^+[0])$  constructed above is called the non-enriched straightening functor.

It can be shown that the straightening functor is cocontinuous [2, Proposition 3.2.1.4] and hence admits a right adjoint.

**Proposition 4.15** ([2, Corollary 3.2.1.5]). *The functor  $\mathrm{St}[0]$  admits a right adjoint*

$$\mathrm{Un}[0] : \mathrm{Fun}(\mathrm{Path} S, \underline{\mathrm{sSet}}_{\infty}^+[0]) \rightarrow \underline{\mathrm{sSet}}_{/S}^+[0].$$

**4.6. Enriched unstraightening.** At each simplicial set  $L$  and map  $p : K \rightarrow S$  from a marked simplicial set, we have the new map

$$Lp : L^{\#} \times K \rightarrow K \rightarrow S$$

and hence a new object in  $\underline{\mathrm{sSet}}_{/S}^+$ . This product construction endows  $\underline{\mathrm{sSet}}_{/S}^+$  with a module category structure over  $\underline{\mathrm{sSet}}_{/S}^+$  whose inner-Homs recover the simplicial mapping complexes for the enhancement  $\underline{\mathrm{sSet}}_{/S}^+$ .

We have the natural map of simplicial sets

$$\mathrm{St}[0](L^\# \times K) \rightarrow L^\# \times \mathrm{St}[0](K)$$

[2, Corollary 3.2.1.15] which endows the straightening functor with a op-lax module category structure. The op-lax structure on  $\mathrm{St}[0]$  endow the adjoint  $\mathrm{Un}[0]$  with a lax module category structure, and this lax structure provides a canonical enrichment on the unstraightening functor to the simplicial setting. Specifically, we have the natural maps

$$\begin{aligned} \mathrm{Hom}_{\mathrm{sSet}}(L, \underline{\mathrm{Hom}}(F, G)) &= \mathrm{Hom}_{\mathrm{Fun}}(L^\# \times F, G) \rightarrow \mathrm{Hom}_{\mathrm{sSet}_{/S}^+}(\mathrm{Un}[0](L^\# \times F) \\ &\rightarrow \mathrm{Hom}_{\mathrm{sSet}_{/S}^+}(L^\# \times \mathrm{Un}[0]F, \mathrm{Un}[0]G) = \mathrm{Hom}_{\mathrm{sSet}}(L, \underline{\mathrm{Hom}}(\mathrm{Un}[0]F, \mathrm{Un}[0]G)) \end{aligned}$$

across all simplicial sets  $L$ . Via Yoneda this lifts the application of the unstraightening functor on morphism sets to the simplicial level,

$$\mathrm{Un} : \underline{\mathrm{Hom}}_{\mathrm{Fun}(\mathrm{Path } S, \mathrm{sSet}^+)}(F, G) \rightarrow \underline{\mathrm{Hom}}_{\mathrm{sSet}_{/S}^+}(\mathrm{Un}[0]F, \mathrm{Un}[0]G),$$

and so enriches the non-simplicial unstraightening functor.

**Definition 4.16.** The unstraightening functor

$$\underline{\mathrm{Un}} : \mathrm{Fun}(\mathrm{Path } S, \mathrm{sSet}^+) \rightarrow \mathrm{sSet}_{/S}^+$$

is the simplicial enrichment of the non-enriched unstraightening functor, as constructed above.

#### 4.7. The straightening and unstraightening equivalences.

thm:1382

**Theorem 4.17.** *The unstraightening functor restricts to an equivalence of simplicial categories*

$$\underline{\mathrm{Un}} : \mathrm{Fun}(\mathrm{Path } S, \underline{\mathrm{Cat}}_\infty^+) \rightarrow \underline{\mathrm{Cocart}}(S).$$

*Proof.* Follows by [2, Lemma 3.2.4.1] and [2, Theorem 3.2.0.1].  $\square$

In the statement of Theorem 4.17 by an equivalence of simplicial categories we mean an equivalence specifically in the sense of [2, Definition A.3.2.1]. Equivalently, we are saying that the induced functor on homotopy categories is an equivalence and that the maps on morphism complexes are equivalences of Kan complexes. We refer to this latter property as fully faithfulness.

**Lemma 4.18.** *If  $\Theta : \underline{A} \rightarrow \underline{B}$  is an equivalence of Kan-enriched simplicial categories, then the induced functor on homotopy coherent nerves*

$$\mathrm{N}^{\mathrm{hc}} \Theta : \mathrm{N}^{\mathrm{hc}} \underline{A} \rightarrow \mathrm{N}^{\mathrm{hc}} \underline{B}$$

*is an equivalence of  $\infty$ -categories.*

*Proof.* Since the induced functor on homotopy categories is an equivalence,  $\mathrm{N}^{\mathrm{hc}} \Theta$  is essentially surjective. Fully faithfulness follows from fully faithfulness of  $\Theta$  and the calculation of the (left pinched) mapping spaces in the homotopy coherent nerve via the mapping complexes in the original categories (see Theorem 2.27).  $\square$

We now conclude that unstraightening induces an equivalence on the associated  $\infty$ -categories

$$\mathrm{N}^{\mathrm{hc}} \underline{\mathrm{Un}} : \mathrm{Fun}(\mathrm{Path } S, \underline{\mathrm{Cat}}_\infty^+) \rightarrow \mathrm{N}^{\mathrm{hc}} \underline{\mathrm{Cocart}}(S) = \mathrm{Cocart}(S)$$

is an equivalence of  $\infty$ -categories. Finally, we compose with the inverse comparison equivalence to obtain an equivalence of  $\infty$ -categories

$$\mathrm{Fun}(S, \mathcal{C}at_\infty) \xrightarrow{\sim} \mathrm{Cocart}(S).$$

We record this finding.

thm:unstrt\_equiv

**Theorem 4.19.** *The functor*

$$\mathrm{Un} := \mathrm{comp}^{-1} \mathrm{N}^{\mathrm{hc}} \mathrm{Un} : \mathrm{Fun}(S, \mathcal{C}at_\infty) \xrightarrow{\sim} \mathrm{Cocart}(S).$$

*is an equivalence of  $\infty$ -categories.*

**Definition 4.20.** The unstraightening equivalence is the equivalence of Theorem 4.19. The straightening equivalence is the inverse functor

$$\mathrm{St} : \mathrm{Cocart}(S) \xrightarrow{\sim} \mathrm{Fun}(S, \mathcal{C}at_\infty).$$

#### 4.8. A remark on uniqueness.

**4.9. Recovering transport via straightening.** For any map of simplicial sets  $f : S \rightarrow S'$ , we have the enriched pullback functor

$$f^* : \mathrm{Cocart}(S') \rightarrow \mathrm{Cocart}(S), \quad (K \rightarrow S') \mapsto (K \times_{S'} S \rightarrow S).$$

lem:1434

**Lemma 4.21** ([?, Observation 2.13]). *For any map of simplicial sets  $f : S \rightarrow S'$ , there is a commutative diagram at the level of homotopy categories*

$$\begin{array}{ccc} \mathrm{hCocart}(S) & \xleftarrow{\mathrm{Un}} & \mathrm{hFun}(S, \mathcal{C}at_\infty) \\ f^* \uparrow & & \uparrow f^* \\ \mathrm{hCocart}(S') & \xleftarrow{\mathrm{Un}} & \mathrm{hFun}(S', \mathcal{C}at_\infty) \end{array}$$

*Proof.* The functor  $f^* : \mathrm{Fun}(\mathrm{Path} S', \mathrm{sSet}^+) \rightarrow \mathrm{Fun}(\mathrm{Path} S, \mathrm{sSet}^+)$  has a left adjoint  $f_!$  and we have a natural isomorphism  $\mathrm{St}_f[0] \cong f_! \mathrm{St}[0]$  [2, Proposition 3.2.1.4], where the functor  $\mathrm{St}_f$  is as in [2]. This implies an identification of non-enriched functors  $\mathrm{Un}_f[0] \cong \mathrm{Un}[0]f^*$ . But, by construction,  $\mathrm{St}_f[0] = \mathrm{St}[0]f$ , where

$$f : \mathrm{sSet}_{/S}^+ \rightarrow \mathrm{sSet}_{/S'}^+$$

just composes maps over  $S$  with  $f$ . Now one can see directly that the pullback functor on marked simplicial sets over  $S'$  is right adjoint to this composition functor  $f$ , so that we have  $\mathrm{Un}_f[0] = f^* \mathrm{Un}[0]$ . So we have finally  $f^* \mathrm{Un}[0] \cong \mathrm{Un}[0]f^*$ , and this identification of non-enriched functors implies a corresponding identification of the  $\infty$ -functors at the level of homotopy categories.  $\square$

**Remark 4.22.** We expect that all of the identifications employed in the proof are compatible with (op)-lax module category structures, so that the identification  $f^* \mathrm{Un}[0] \cong \mathrm{Un}[0]f^*$  enriches to an identification at the simplicial level, and hence at the  $\infty$ -level  $f^* \mathrm{Un} \cong \mathrm{Un} f^*$ . This homotopy-level identification suffices for our purposes however.

We can now consider the universal cocartesian fibration, i.e. the cocartesian fibration over  $\mathcal{C}at_\infty$  which is associated to the identity functor

$$\mathcal{Z} := \mathrm{Un}(\mathrm{id}_{\mathcal{C}at_\infty}).$$

By the materials of Section 3 we understand that there is an equivalence of fibrations

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{\sim} & \mathcal{P}\mathcal{C}at_\infty \\ & \searrow & \swarrow \\ & \mathcal{C}at_\infty & \end{array} .$$

Now the diagram of Lemma 4.21 implies the existence of an equivalence of cocartesian fibrations

$$\mathrm{Un}(F) \cong S \times_{\mathcal{C}at_\infty} \mathcal{Z} =: \int_S F$$

at any functor  $F$  in  $\mathrm{Fun}(S, \mathcal{C}at_\infty)$ .

`prop:str_t_transport`

**Proposition 4.23** ([2, §3.3.2]). *For any cocartesian fibration  $q : X \rightarrow S$  there is an equivalence of cocartesian fibrations*

$$\begin{array}{ccc} X & \xrightarrow{\sim} & \int_S \mathrm{St}(q) \\ & \searrow q & \swarrow \\ & S & \end{array} .$$

*In particular, when the base  $S$  is an  $\infty$ -category the straightening functor  $\mathrm{St}(q) : S \rightarrow \mathcal{C}at_\infty$  is a transport functor for  $q : X \rightarrow S$ .*

*Proof.* We have  $q \cong \mathrm{Un}\mathrm{St}(q) \cong \int_S \mathrm{St}(q)$ . □

We can consider now the  $\infty$ -category of left fibrations.

**Definition 4.24.** Let  $\mathrm{LFib}(S)$  denote the full  $\infty$ -subcategory in  $\mathrm{Cocart}(S)$  whose objects are precisely the left fibrations  $q : X \rightarrow S$ .

**4.10. Straightening and unstraightening for left fibrations.** Since the straightening functor  $\mathrm{St}(q)$  produces a transport functor for any cocartesian fibration  $q : X \rightarrow S$ , and the unstraightening functor pulls back along the universal fibrations, Proposition ?? and the fibration calculation of Proposition 3.6 now imply that unstraightening and straightening restrict to equivalences between left fibrations over  $S$  and functors into the  $\infty$ -subcategory of spaces  $\mathcal{K}an \subseteq \mathcal{C}at_\infty$ .

`cor:str_t_kan`

**Corollary 4.25.** *The straightening and unstraightening equivalences restrict to inverse equivalences*

$$\mathrm{St} : \mathrm{LFib}(S) \xrightarrow{\sim} \mathrm{Fun}(S, \mathcal{K}an) \quad \text{and} \quad \mathrm{Un} : \mathrm{Fun}(S, \mathcal{K}an) \xrightarrow{\sim} \mathrm{LFib}(S).$$

**4.11. Some naturality over the base.**

`prop:small_nat`

**Proposition 4.26.** *Any diagram of cocartesian fibrations*

$$\begin{array}{ccc} Y & \longrightarrow & X \\ p \downarrow & & \downarrow q \\ T & \xrightarrow{\xi} & S \end{array}$$

*determines a uniquely associated transformation  $\mathrm{St}(p) \rightarrow \mathrm{St}(q)\xi$  in the  $\infty$ -category  $\mathrm{Fun}(T, \mathcal{C}at_\infty)$ .*

In the case where the bases are  $\infty$ -categories, this transformation is visualized as a diagram

$$\begin{array}{ccc} T & \xrightarrow{\quad} & S \\ & \searrow \text{St}(p) & \swarrow \text{St}(q) \\ & \mathcal{Cat}_\infty & \end{array}$$

in the  $(\infty, 2)$ -category of (big)  $\infty$ -categories.

*Construction.* Take  $F = \text{St}(q)$ ,  $G = \text{St}(p)$ . The composite  $F\xi$  is a transport functor for the fibration  $q' : X \times_S T \rightarrow T$ , which then determines an isomorphism  $F\xi \cong \text{St}(q')$  which is unique up to a contractible space of choices, by Theorem 3.14 (or rather its generalized form [3, 02SC]). Additionally the given diagram specifies, and is specified by, a morphism

$$\begin{array}{ccc} Y & \xrightarrow{\quad l \quad} & X \times_S T \\ & \searrow p & \swarrow q' \\ & T & \end{array}$$

in  $\text{Cocart}(T)$ . Via straightening this morphism determines a map

$$\text{St}(l) : G \rightarrow \text{St}(q') \cong F\xi.$$

□

**4.12. Straightening for cartesian fibrations.** As with the cocartesian case, one can show that the full simplicial subcategory  $\underline{\text{Cart}}(S) \subseteq \underline{\text{sSet}}/S$  consisting of cartesian fibrations over  $S$  is enriched in Kan complexes. We can therefore consider the  $\infty$ -category

$$\text{Cart}(S) = \text{N}^{\text{hc}} \underline{\text{Cart}}(S).$$

An application of the opposite functor provides an isomorphism of  $\infty$ -categories  $\text{Cart}(S) \cong \text{Cocart}(S^{\text{op}})$ . We therefore obtain the following results for cartesian fibrations:

There are mutually inverse equivalences  $\text{St} : \text{Cart}(S) \xrightarrow{\sim} \text{Fun}(S^{\text{op}}, \mathcal{Cat}_\infty)$  and  $\text{Un} : \text{Fun}(S, \mathcal{Cat}_\infty) \xrightarrow{\sim} \text{Cart}(S)$  (Theorem 4.19). These equivalences restrict to equivalences

$$\text{St} : \text{RFib}(S) \xrightarrow{\sim} \text{Fun}(S^{\text{op}}, \mathcal{Kan}) \quad \text{and} \quad \text{Un} : \text{Fun}(S^{\text{op}}, \mathcal{Kan}) \xrightarrow{\sim} \text{RFib}(S)$$

(Corollary 4.25). The value  $\text{St}(q) : S^{\text{op}} \rightarrow \mathcal{Cat}_\infty$  at any cartesian fibration  $q : X \rightarrow S$  is a transport functor for  $q$  (Proposition 4.23). Finally, any diagram of cocartesian fibrations

$$\begin{array}{ccc} Y & \xrightarrow{\quad} & X \\ p \downarrow & & \downarrow q \\ T & \xrightarrow[\xi]{} & S \end{array}$$

determines a uniquely associated transformation between the transport functors  $\text{St}(p) \rightarrow \text{St}(q)\xi$  (Proposition 4.26).

## 5. INITIAL AND TERMINAL OBJECTS

Before beginning with our study in earnest, with the introduction of Hom functors and the Yoneda embedding for  $\infty$ -categories, we discuss the notions of initial and terminal objects in an  $\infty$ -category.

## 5.1. Initial and terminal objects.

**Definition 5.1.** Let  $\mathcal{C}$  be an  $\infty$ -category. An object  $x$  in  $\mathcal{C}$  is called initial if, for each object  $z$  in  $\mathcal{C}$ , the left pinched mapping space  $\mathrm{Hom}_{\mathcal{C}}^{\mathrm{L}}(x, z)$  is contractible. An object  $z$  in  $\mathcal{C}$  is called terminal if, for each object  $x$  in  $\mathcal{C}$ , the right pinched mapping space  $\mathrm{Hom}_{\mathcal{C}}^{\mathrm{R}}(x, z)$  is contractible.

One sees that an object  $x$  is initial (resp. terminal) in  $\mathcal{C}$  if and only if  $x$  is terminal (resp. initial) in the opposite category  $\mathcal{C}^{\mathrm{op}}$ . So we can freely translate between results for initial versus terminal objects.

**Lemma 5.2.** *If  $x$  and  $x'$  are initial in  $\mathcal{C}$ , then every morphism*

$$\alpha : * \rightarrow \mathrm{Hom}_{\mathcal{C}}^{\mathrm{L}}(x, x') = \mathcal{C}_{x/} \times_{\mathcal{C}} \{x'\}$$

*is an isomorphism and the space  $\mathrm{Hom}_{\mathcal{C}}^{\mathrm{L}}(x, x')$  itself is contractible. The obvious dual statement holds for terminal objects.*

This is to say, the initial (or terminal) object in an  $\infty$ -category  $\mathcal{C}$  is unique provided any such object exists.

*Proof.* Contractibility of the morphism space just follows by definition. So we need only show that any map between initial objects is an isomorphism. But this follows from the fact that initial objects in  $\mathcal{C}$  are also initial in the homotopy category  $\mathrm{h}\mathcal{C}$ , and such objects are unique in  $\mathrm{h}\mathcal{C}$ .  $\square$

## 5.2. The initial and terminal spaces.

5.3. Zero object in the derived  $\infty$ -category.

## 6. THE YONEDA EMBEDDING

**Definition 6.1.** Given an  $\infty$ -category we define the twisted arrow category  $\mathcal{Tw}(\mathcal{C})$  as the simplicial set whose  $n$ -simplices are

$$\mathcal{Tw}(\mathcal{C})[n] := \mathrm{Hom}_{\mathrm{sSet}}((\Delta^n)^{\mathrm{op}} \star \Delta^n, \mathcal{C}).$$

Restricting along the inclusions

$$(\Delta^n)^{\mathrm{op}} \rightarrow (\Delta^n)^{\mathrm{op}} \star \Delta^n \quad \text{and} \quad \Delta^n \rightarrow (\Delta^n)^{\mathrm{op}} \star \Delta^n$$

provides a natural map to the product  $\mathcal{Tw}(\mathcal{C}) \rightarrow \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ . In this way the twisted arrow category is understood as an object in the overcategory  $(\mathcal{Cat}_{\infty})_{/\mathcal{C}^{\mathrm{op}} \times \mathcal{C}}$ .

## 6.1. Twisted arrows as a cocartesian fibration.

**Theorem 6.2.** *The structure map  $\lambda : \mathcal{Tw}(\mathcal{C}) \rightarrow \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$  is a left fibration.*

**Corollary 6.3.** *The simplicial set  $\mathcal{Tw}(\mathcal{C})$  is an  $\infty$ -category.*

**Corollary 6.4.** *The map  $\lambda : \mathcal{Tw}(\mathcal{C}) \rightarrow \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$  is a cocartesian fibration.*

**Definition 6.5.** A Hom-functor  $h : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \rightarrow \mathcal{H}an$  for an  $\infty$ -category  $\mathcal{C}$  is any functor which classifies, i.e. is a transport functor for, the cocartesian fibration  $\lambda : \mathcal{Tw}(\mathcal{C}) \rightarrow \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ .

7. LIMITS AND COLIMITS
8. COPRODUCTS AND COFIBERS
9. HOMOTOPY LIMITS AND COLIMITS KAN COMPLEXES
10. LIMITS AND COLIMITS IN  $\mathcal{K}an$
11. STABILITY AND COCOMPLETENESS
12. LIMITS VIA MAPPING SPACES
13. COCOMPLETION AND PRESENTABILITY
14. EXACTNESS AND ADJOINTS

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