Radicals and semisimplicity

To begin we discuss another class of rings/algebras –

I : Division algebras

Def^{n}: A division ring D, is a ring for which each nonzero $a \in D$ admits $a^{-1} \in D$ with $a^{-1}a = aa^{-1} = 1$. A division algebra over a field k is an algebra which is also a division ring.

Def^{n}: A ring R is called a domain if for each nonzero a in R an equation $a \cdot b = 0$ or $b \cdot a = 0$, at b arbitrary in R, implies $b = 0$.

Observation 1: Any division ring is a domain.

Ex: $\mathbb{H}$ = quaternions are a division alg over $\mathbb{R}$.

Lemma 2: The center of any division ring is a field.

(Proof: For a in $Z(D)$ we have for a^{-1} and arbitrary b in D,

$$a^{-1}b = a^{-1}baa^{-1} = a^{-1}aba^{-1} = ba^{-1}.$$

So $a^{-1} \in Z(D)$ as well.

Lemma 3: If k is an algebraically closed field, then the only finite dimensional k-algebra is k itself.

Proof: Take D a div. alg over k and suppose there exists $x \in D \setminus k$. Then we have the alg. map $\phi : k[X] \to D$, $X \mapsto x$. If $\ker(\phi) \neq 0$ then $k[X]/\ker(\phi) \subseteq D$ is a domain, and hence $\ker(\phi) = (p(x))$ for an irred poly $p(x)$. Consequently $K = k[X]/\ker(\phi)$ is a finite field extension of k, $k \to K \subseteq D$, and by alg closure $K = k$. Thus $x \in k \subseteq D$, a contradiction. So $\ker(\phi) = 0$ necessarily, and $\dim D \geq \dim k[X] = \infty$.

As a consequence, if $\dim D < \infty$ for div k-alg D then we must have $D = k$. ∎

Corollary 4: The only finite division alg's over $\overline{\mathbb{C}}$, $\overline{\mathbb{Q}}$, $\overline{\mathbb{F}}_p$ etc. are $\overline{\mathbb{C}}$, $\overline{\mathbb{Q}}$, $\overline{\mathbb{F}}_p$ themselves, respectively.

Proposition 5: Any finitely generated module M over a div alg D is of the form D^l for some unique $l \in \mathbb{Z}_{\geq 0}$, and if $M' \subseteq M$ then $M' = D^{l'}$ for $l' \leq l$. Equality holds if and only if $l = l'$.

For first claim

Proof: Proceed by induction on the min. # of generators, and we claim

$\ell = \{$the cardinality of any minimal gen set $m_1, \ldots, m_\ell$ for $M\}$.

For such a minimal gen set we have the surj module map

$$f : \bigoplus_{j=1}^{\ell} D \longrightarrow M, \quad e_j \longmapsto m_j,$$

which we claim is injective. Supposing $f|_{\bigoplus_{j=1}^{\ell'} D}$ is injective for $0 \le \ell' < \ell$ then for any $a_{\ell+1} \in D_{\ell'+1} \setminus \{0\}$ we have $f(a_{\ell+1}) = a_{\ell+1} \cdot m_{\ell+1} = \sum_{j=1}^{\ell'} a_j \cdot m_j$ gives $m_{\ell+1} = f(1_{\ell+1}) = a_{\ell+1}^{-1} f(a_{\ell+1}) \in$ Submod gen'd by $m_1, \ldots, m_{\ell'}$, a contradiction. So $f|_{\bigoplus_{j=1}^{\ell'+1} D}$ is injective as well. By induction on ℓ' we see f is injective, and thus an isomorphism. So

$$D^\ell \xrightarrow{\cong} M.$$

Now, if we have an isom $f : D^{\ell_1} \xrightarrow{\cong} D^{\ell_2}$ for $\ell_1, \ell_2 > 0$ then, after precomposing w/ an aut of $\phi \circlearrowright D^\ell$ we can assume $f|_{\text{first copy of } D} = id$. Then f induces an isomorphism

$$D^{\ell_1 - 1} = D^{\ell_1}/D \xrightarrow{\cong} D^{\ell_2 - 1} = D^{\ell_2}/D.$$

Repeating we obtain $0 \xrightarrow{\cong} D^{\ell_2 - \ell_1}$ which gives $\ell_2 = \ell_1$.

For the second claim, given $M' \subseteq M$ we extend a minimal gen set $m_1, m_2, \dots, m_{l'}$ for M' to a min gen set $m_1, \dots, m_l$ for M to get

$$M' \cong D^{l'} \text{ and } M \cong D^{l} \text{ with}$$

$$l' \leq l \text{ and equality holding iff } M' = M.$$ $\blacksquare$

Def!: The rank of a D-module, for given M, for D a div ring, is the unique integer l at which we have an iso of D-modules $M \cong D^l$.

~ II. Matrices over division rings

On the HW we saw k^n is simple over $M_n(k)$, for k a field. Similarly, for ~~any~~ division ring D, D^n is simple module over $M_n(D)$, and $M_n(D)$ admits no ideals other than 0 and $M_n(D)$ itself.

Furthermore we have

$$M_n(D) = \bigoplus_{i=1}^{n} D^n.$$

as a $M_n(D)$-module.

Here, for any ring R, $M_n(R)$ is the ring with $1 = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$ and mult $[a_{ij}][b_{ij}] = [c_{ij}]$ w/ $c_{ij} = \sum_k a_{ik} b_{kl}$.

III. Semisimple modules

Def^{n}: A module M over an algebra A is called semisimple if
$$M = \bigoplus_{\lambda \in \Lambda} L_\lambda$$
for simple submodules $L_\lambda \subseteq M$.

Equiv: M is semisimple if and only if M admits an A-module isomorphism
$$\bigoplus_{\lambda \in \Lambda} L_\lambda \xrightarrow{\sim} M$$
for simple M_λ.

Observation 6: For a semisimple module M the following are equivalent

a) M is finitely generated

b) M is finite length (admits a composition series)

c) For any isomorphism $\bigoplus_{\lambda \in \Lambda} L_\lambda \xrightarrow{\sim} M$ w/ the L_λ simple, the indexing set Λ is finite.

Proof: Exercise. $\blacksquare$

Lemma (Schur) If L and L' are simple A-modules, then any A-module map $f: L \to L'$ is either

0, or an isomorphism.

Proof: By simplicity either $\ker(f) = 0$ or L. In the first case f is injective with

$$0 \subsetneq \operatorname{im}(f) \subseteq L$$

and by simplicity again $\operatorname{im}(f) = L$. Hence f is bijective as a map of sets, and thus invertible as a map of A-modules. In the second case, $\ker(f) = L$, we have $f = 0$. ∎

Corollary 7: For any simple A-module L, $\operatorname{End}_A(L)$ is a division ring.

Corollary 8: If A is a finite dimensional algebra over an algebraically closed field $k = \bar{k}$, and L is simple over A, then

$$\operatorname{End}_A(L) = k \cdot \mathrm{id}_L \cong k.$$

Proof: Since L is cyclic we have a surjection $A \to L$ and hence $\dim_k(L) \leq \dim_k(A) < \infty$. Thus

$$\dim_k \operatorname{End}_A(L) \leq \dim_k \operatorname{End}_k(L) < \infty.$$

Thus $\operatorname{End}_A(L)$ is a finite-dimensional division alg over k. Since $k = \bar{k}$, we now have

$$\mathrm{End}_A(L) = k$$

by Lemma 3. ∎

Example: For $L_{\mathbb{C}}(2)$ the 2-dim simple mod over $\mathbb{C}S_3$, we have

$$\mathrm{End}_{\mathbb{C}S_3}(L_{\mathbb{C}}(2)) = \mathbb{C}.$$

Also,

$$\mathrm{End}_{\bar{\mathbb{F}}_3 S_3}(L_{\bar{\mathbb{F}}_3}(2)) = \bar{\mathbb{F}}_3.$$

More generally, for any finite group G, any simple G-rep L over an alg closed field k has

$$\mathrm{End}_{kG}(L) = k.$$

Proposition 9: Let M be a finitely generated semisimple A-module.

i) Every submodule $M' \le M$ is semisimple.

ii) Every quotient module M/M' is semisimple.

Proof: For (i) we claim that M' and M admit decomps into simples $L_1 \oplus \cdots \oplus L_s = M'$ and $M = L_1 \oplus \cdots \oplus L_s \oplus \cdots \oplus L_t$ under which the inclusion

$M' \to M$ is just the matrix $\left[\frac{I_s}{0}\right] : \oplus_{i=1}^{s} L_i \to \oplus_{j=1}^{t} L_j$.

We proceed by induction on length(M). When length$(M) = 0$ there's nothing to do. Take now length$(M) = t$ and assume the result holds for semisimple N with length$(N) < t$. Since M' is of finite length [Cor 6, Fuchs] we have some simple $L \subseteq M'$ and each composite

$$L \to M \to L_i$$

is either 0 or an isomorphism, by Schur's Lemma. Since the inclusion $L \to M$ is nonzero we can find minimal index i_0 at which $L \to M \to L_{i_0}$ is nonzero. Then the map

$$(\oplus_{i < i_0} L_i) \oplus L \oplus (\oplus_{j > i_0} L_j) \to M$$

induced by the inclusions is an isomorphism (?) and after replacing L_{i_0} with L and reindexing we can assume $i_0 = 1$ and observe

$$\begin{tikzcd} & L_1 \arrow[dl] \arrow[dr] & \\ M' \arrow[rr] & & M \end{tikzcd}.$$

Take $L_1^{\perp} \subseteq M'$ the kernel of the sequence

$$M' \to M \to M / \oplus_{i>1} L_i.$$

Then we have

$$M' = L_1 \oplus L_1^{\perp}$$

and $L_1^{\perp} \subseteq \oplus_{i>1} L_i \subseteq M$. Since

Length $(\bigoplus_{i\geq 1} L_i) = t-1$ we obtain consistent decomposition

$$L_1^{\perp} = L_2' \oplus \cdots \oplus L_s'$$

$$\bigoplus_{i>1} L_i = L_2' \oplus \cdots \oplus L_t'$$

and thus consistent decomp.

$$M = L_1 \oplus L_2' \oplus \cdots \oplus L_s' \qquad (*)$$

$$M = L_1 \oplus L_2' \oplus \cdots \oplus L_t'.$$

(ii) Taking consistent decomp.

$$M = \bigoplus_{i\leq s} L_i \ , \ M' = \bigoplus_{i\leq t} L_i$$

we obtain

$$M/M' = \bigoplus_{i>s} L_i. \qquad \blacksquare$$

The following was already demonstrated in the proof.

Theorem 10: If M is semisimple, any exact sequence $0 \to M' \to M \to M'' \to 0$ splits, i.e....

Remark: The analog of Prop 9 holds for infinite length semisimple modules as well. One can argue the point from the finite case and some Zorn's lemma argument.

Proposition 9½: For any semisimple A-module M, the following hold.

i) Any submodule $M' \subseteq M$ is semisimple.

ii) Any quotient module

Proof: Omitted. ∎

~ III ½ An example: Matrix rings

Example: Consider any division ring D and $M_n(D)$. We have the simple module

$$L_{standard} = D^n$$

provided by the columnar module and the decomp. of the regular module

$$M_n(D) = \bigoplus_{i=1}^{n} L_{standard}.$$

Hence $M_n(D)$ itself is semisimple.

Now, since any fin. gen'd $M_n(D)$-module M admits a surjection

$$\bigoplus_{i=1}^{m} M_n(D) = \bigoplus_{i=1}^{n \cdot m} L_{standard} \to M$$

Prop 9 (ii) tells us that M is also semisimple, and [Prop 9, Findim] forces (as expected)

$$M = \bigoplus_{i=1}^{t} L_{standard}.$$

Corollary 11: For any division ring D, any finitely generated $M_n(D)$-module is semisimple, and

The standard column module is the only simple $M_n(D)$ module, up to isomorphism.

IV Radicals for modules

We call a proper submodule $M_0 \subsetneq M$ maximal if any larger module $M_0 \subseteq M_1 \subseteq M$ has either $M_1 = M_0$ or $M_1 = M$.

Lemma 11: A proper submodule $M_0 \subseteq M$ is maximal if and only if M/M_0 is simple.

Proof: Consider the surjection $\pi : M \to N$, with $N = M/M_0$. We have the bijection

$$\left\{\begin{matrix}\text{Submodules } M_1 \\ \text{with } M_0 \subseteq M_1 \subseteq M\end{matrix}\right\} \rightleftarrows \left\{\begin{matrix}\text{Submodules} \\ N_1 \subseteq N\end{matrix}\right\}$$

$$M_1 \longmapsto \pi(M_1)$$

$$\pi^{-1}(N) \longleftarrow N_1 .$$

Indeed, for $m \in M$ with $\pi(m) = \pi(m')$ for m' in M_1, we have $m - m' \in \ker(\pi) = M_0$ so that $m = m' + (m - m') \in M_1$. This gives

$$\pi^{-1}(\pi(M_1)) = M_1 .$$

The equality $\pi(\pi^{-1}(N_1)) = N_1$, at any submodule $N_1 \subseteq N$ follows by surjectivity of π. Hence

The only submodules M_1 between M_0 and M are M_0 and M itself if and only if the only submodules in N are 0 and N itself. Rather, M_0 is maximal if and only if $N = M/M_0$ is simple. ∎

Def^t: Given a ring A and a nonzero A-mod M, the radical in M is the intersection

$$\mathrm{Rad}(M) = \bigcap_{M_0 \subseteq_{\max} M} M_0 .$$

Example: For an infinite field k (e.g. $\mathbb{Q}$, $\mathbb{C}$, or $\overline{\mathbb{F}}_p$) and any $\alpha \in k$ we have the max ideal / submodule $(x-\alpha)$ in the regular module $k[x]$, and the quotient

$$k[x]/(x-\alpha) \xrightarrow{\sim} k\cdot(\alpha)$$

is the 1-dimensional $k[x]$-module k on which $x \cdot 1_\alpha = \alpha 1_\alpha$. Now, for any poly

$$p(x) \in \mathrm{Rad}(k[x])$$

we have $p(x)\cdot 1_\alpha = p(\alpha)\cdot 1_\alpha = 0$ at all $\alpha \in k$. Hence $p(x)$ has as many zeros, and we conclude $p(x) = 0$. So

$$\mathrm{Rad}(k[x]) = 0 .$$

Example: At each $i = 1, \ldots, n$, and k a field, we have the surjection
$$\pi_i : M_n(k) \longrightarrow L_{\text{standard}} = k^n$$
with kernel
$$\ker(\pi_i) = \left\{ \begin{array}{c} \left[\text{stuff} \,\middle|\, 0 \,\middle|\, \text{stuff} \right] \\ \uparrow \\ i\text{th column} \end{array} \right\}$$

Since L_{standard} is simple this gives
$$\mathrm{Rad}(M_n(k)) \subseteq \bigcap_i \ker(\pi_i) = 0.$$
Similarly, for any division ring D, $\mathrm{Rad}(M_n(D)) = 0$.

Example: For $M = k[x]/(x^{10})$ as a module over $k[x]$, any simple quotient
$$\pi : M \to L$$
we have $x \cdot - : L \to L$ an endomorphism. By Schur either $x \cdot -$ is 0 or an isomorphism. Since
$$(x \cdot -)^{10} = x^{10} \cdot - = 0$$
we conclude $x \cdot - = 0$. Rather, x annihilates L. We see now $x \in \ker(\pi)$ at all such π, giving
$$\bar{x} \in \mathrm{Rad}(M) \Rightarrow k[x] \cdot \bar{x} = x \cdot M \subseteq \mathrm{Rad}(M).$$
Since the quotient $M / x \cdot M \cong k$ is simple, we get $\mathrm{Rad}(M) = x \cdot M$.

~ IV ½ Radicals and semisimplicity

Theorem 12: For any finite length module M, $M/\mathrm{Rad}(M)$ is semisimple. Furthermore, for any surjective A-module map $\pi: M \to N$, N is semisimple if and only if $\mathrm{Rad}(M) \subseteq \ker(\pi)$.

Before we prove the result, we note the following

Lemma 13: For any surjective module map $\pi: M \to N$ to nonzero N, $\mathrm{Rad}(M) \subseteq \pi^{-1}(\mathrm{Rad}(N))$.

Proof: For maximal $N_0 \subseteq N$, $\pi^{-1}(N_0)$ is max in M. Hence

$$\pi^{-1}\mathrm{Rad}(N) = \pi^{-1}\Big(\bigcap_{N_0 \max} N_0\Big)$$

$$= \bigcap_{N_0 \max} \pi^{-1}(N_0) \supseteq \bigcap_{M_0 \max} M_0 = \mathrm{Rad}(M). \blacksquare$$

Lemma 14: If N is finite length and semisimple, then $\mathrm{Rad}(N) = 0$.

Proof: Given an expression $N = \bigoplus_{i=1}^{r} L_i$ with the L_i simple, the kernel $R_i \subseteq N$ of each projection $p_i : N \to L_i$ satisfying

$\bigcap_{i=1}^r K_i = \ker\left([p_1 \cdots p_r]^t : N \to \bigoplus_{i=1}^r L_i \right)$.

But! $[p_1 \cdots p_r]^t$ just recover the identity on N, and hence $\mathrm{Rad}(N) \subseteq \bigcap_{i=1}^r K_i = 0 \Rightarrow \mathrm{Rad}(N) = 0$. ∎

We now prove our main theorem.

Proof of Theorem 12: We have, via Artinianness of the quotient $M/\mathrm{Rad}(M)$ the existence of finitely many maximal submodules $K_1, \dots, K_r \subseteq M$ for which $\mathrm{Rad}(M) = K_1 \cap \cdots \cap K_r$. Hence, $\mathrm{Rad}(M)$ is the kernel of the map

$$M \to \bigoplus_{i=1}^r L_i, \quad L_i := M/K_i,$$

induced by the individual quotients $M \to L_i$, giving

$$M/\mathrm{Rad}(M) \subseteq \bigoplus_{i=1}^r L_i \text{ semisimple.}$$

By Proposition 9 we conclude $M/\mathrm{Rad}(M)$ is semisimple.

As for the second claim, for semisimple N we have $\mathrm{Rad}(N) = 0$ by Lemma 14 and hence

$\mathrm{Rad}(M) \subseteq \ker(\pi) = \pi^{-1}\mathrm{Rad}(N)$ by Lemma 13.

Conversely, if $\pi : M \to N$ is surj with $\mathrm{Rad}(M) \subseteq \ker$ then N is the quotient of the semisimple module $M/\mathrm{Rad}(M)$.

By Proposition 9 we conclude that N is semisimple. ∎

~ V Socles for modules

Theorem 15: For a Noetherian module M (e.g. a fin gen'd module over fin dim A), the sum

$$\mathrm{soc}(M) = \sum_{\lambda \in \Lambda} L_\lambda \subseteq M \qquad (*)$$

over the collection Λ of all simple submodules $L_\lambda \subseteq M$ is a finite length semisimple submodule in M.

Furthermore, for any semisimple module N, any module map $f: N \to M$ has image in $\mathrm{soc}(M)$.

Proof: Since M is Noetherian, the sum in $(*)$ is finite, $\sum_\lambda L_\lambda = L_1 + \cdots + L_r$ for some simple $L_i \subseteq M$. This sum is the image of the map

$$\oplus_{i=1}^r L_i \to M$$

induced by the inclusions, giving $\mathrm{soc}(M)$ as a quotient of a finite length semisimple module. Hence $\mathrm{soc}(M)$ is finite length semisimple, by Proposition 9. For the second claim write $N = \oplus_\mu V_\mu$ for simple V_μ to get $\mathrm{im}(N) = \sum_\mu \mathrm{im}(V_\mu)$ with each $\mathrm{im}(V_\mu)$

either simple or zero, by simplicity. Hence $\operatorname{im}(N) \subseteq \operatorname{soc}(M)$.

Def$^{\underline{n}}$: The sum of all simple submodules in a Noeth. A-module M is called the socle in M.

Ex: For $k[x]/(x^n)$, considered as a module over $k[x]$, any simple submod $M \subseteq k[x]/(x^n)$ is annihilated by x^n, and thus simplicity forces $x \cdot M = 0$. This gives

$$\operatorname{soc}(k[x]/(x^n)) = k \cdot \bar{x}^{n-1} \cong k_0.$$

the 1-dim simple on which x acts as 0.

HW

1. For the regular A-module A, prove that there is a canonical ring isomorphism

$$A^{op} \xrightarrow{\sim} \mathrm{End}_A(A).$$

3. Let G be a group and suppose G acts on a k-algebra A by algebra automorphisms. Suppose k is a field, and take $A \rtimes G := A \otimes_k kG$ with the unique bilinear map $\cdot : A \rtimes G \times A \rtimes G \longrightarrow A \rtimes G$ satisfying

$$(a \otimes g) \cdot (b \otimes h) = a(g * b) \otimes gh$$

on monomials. Prove that $A \rtimes G$ is a k-algebra.

4. Let $\mathbb{Z}/n\mathbb{Z}$ act on $\mathbb{C}[x]$ via the automorphisms $\overline{m} \cdot p(x) = p(q \cdot x)$ for $q \in \mathbb{C}^{\times}$ an n-th root of unity. Take $A_q = \mathbb{C}[x] \rtimes \mathbb{Z}/n\mathbb{Z}$.

a) Prove that $M_r = A_q / A_q \cdot x^r$ is a free module over $\mathbb{C}\mathbb{Z}/n\mathbb{Z}$ of rank r. Give a basis for M_r over $\mathbb{C}\mathbb{Z}/n\mathbb{Z}$.

b) Calculate $\mathrm{Rad}(M_r)$, $M_r/\mathrm{Rad}(M_r)$, and $\mathrm{soc}(M_r)$.

4. a) Fix $\mathbb{Z}/n\mathbb{Z}$. Prove that $\mathbb{Z}/n\mathbb{Z}$ has precisely n non-isomorphic (simple) 1-dimensional representations L_i over $\mathbb{C}$.

b) Prove that each simple 1-dimensional representation L_i over $\mathbb{Z}/n\mathbb{Z}$ admits an injective $\mathbb{C}\mathbb{Z}/n\mathbb{Z}$-module map $L_i \to \mathbb{C}\mathbb{Z}/n\mathbb{Z}$, and that that map is unique up to rescaling.

c) Provide an isomorphism of $\mathbb{C}\mathbb{Z}/n\mathbb{Z}$-modules

$$\bigoplus_{i=1}^{n} L_i \xrightarrow{\sim} \mathbb{C}\mathbb{Z}/n\mathbb{Z}.$$

d) Prove that every finite-dimensional $\mathbb{Z}/n\mathbb{Z}$-rep M over $\mathbb{C}$ decomposes as

$$M \cong \bigoplus_{i\in I}^{\circ} m_i \cdot L_i$$

where each $m_i = [L_i : M]$.

5. For any finite length semisimple module M over a ring A, prove that there are division rings $D_1, \ldots, D_t$ and integers $n_1, \ldots, n_t$ for which

$$\mathrm{End}_A(M) \cong \prod_i M_{n_i}(D_i)$$

as rings.