KERODON REMIX PART III: A SMALL STUDY OF THE DERIVED ∞-CATEGORY

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Abstract. We employ the materials from Part II to provide a baseline analysis of the derived ∞-category of an abelian category. We explicitly calculate pushouts and pullbacks in the homotopy (and derived) ∞-category, directly verify stability, and explicitly realize the derived ∞-category as a localization of the homotopy ∞-category relative to the class of quasi-isomorphisms. We also exhibit adjunctions between left and right derived functors. We conclude the text with a discussion of ind-completion and constructions of "renormalized" derived categories. In comparing with the previous installments, Parts I and II, we are somewhat more liberal in our treatment herein, as we occasionally employ results from higher topos theory or higher algebra without proof.

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1. Derived categories, Vamos!

This document is intended to provide an "intermediate" analysis of the derived ∞-category $\mathscr{D}(\mathbb{A})$, where $\mathbb{A}$ is a (generally Grothendieck) abelian category. We assume the reader has their own motivations for coming this topic, and so won't provide our own motivations.

This is our third and final contribution to the sequence "Kerodon remix" Parts I, II, and III. In Part I we provide an introduction some foundational aspects of ∞-categories, and Part II we provide an introduction to (co)cartesian fibrations, Hom functors, limits and colimits, and the Yoneda embedding.

1.1. **Derived recollections.**

As the title suggests, this is a text about the derived ∞-category. We recall some of the fundamentals.

Grothendieck abelian categories: In general we work with Grothendieck abelian categories. We recall that an abelian category $\mathbb{A}$ is called Grothendieck abelian if it admits small colimits and a generator, and if direct limits in $\mathbb{A}$ are exact. This restriction is not so important, other than the fact that it ensures the existence of enough K-injectives in the unbounded category of cochains. Such K-injectives can then be leveraged in a uniform way in the production and analysis of the (unbounded) derived ∞-category.

For those interested in finite categories, such as finite-dimensional representations over a finite group, one can move from the finite context to the Grothendieck abelian context by replacing the category of interest with its Ind-completion [11, Theorem 8.6.5]. In most hands-on situations, this move to the Ind-category is achieved simply by working within the category of infinite-dimensional rather than finite-dimensional representations.

∞-categories from dg categories: From any dg category $\mathbf{A}$ we can construct an associated, or maybe *the* associated, ∞-category by taking the dg nerve $\mathscr{A} := \mathrm{N}^{\mathrm{dg}}(K\mathbf{A})$. The dg nerve construction is described in [16, 00PK], or Section I-2.2. This construction is very straightforward, and somewhat intuitive from the

A_∞-perspective. An important point, however, is that one can equivalently construct the ∞-category $\mathscr{A}$–up to natural equivalence–by first factoring through the simplicial setting then applying the homotopy coherent nerve.

To elaborate, we can apply the Dold-Kan functor $K : \mathrm{Ch}(\mathbb{A}) \to \mathrm{Kan}_{\mathbb{Z}}$ to morphisms to produce from $\mathbf{A}$ a simplicial category $K\mathbf{A}$. This simplicial category has Hom spaces $K\,\mathrm{Hom}_{\mathbf{A}}(x, y)$ for each x and y. One then applies the homotopy coherent nerve [16, 00KM] to produce an ∞-category $\mathscr{A}' = \mathrm{N}^{\mathrm{hc}}(\mathbf{A})$ which admits an equivalence over the underlying plain category $Z^0(\mathbf{A})$,

$$\begin{tikzcd} \mathscr{A}' \arrow[rr, "\mathfrak{z}_{\mathbf{A}}", "\sim"'] & & \mathscr{A} \\ & Z^0(\mathbf{A}) \arrow[ul] \arrow[ur] & \end{tikzcd}$$

(Theorem II-10.4). This simplicial construction is much more convenient to work with when considering, say, Hom functors for the associated ∞-category. See for example the materials of Section II-11.3. So, from some practical perspectives, the approach via Dold-Kan is a better construction of the ∞-category associated to $\mathbf{A}$.

Definition 1.1. For an additive category $\mathbb{A}$, the homotopy ∞-category $\mathscr{K}(\mathbb{A})$ is the associated ∞-category for the dg category of cochains over $\mathbb{A}$,

$$\mathscr{K}(\mathbb{A}) := \mathrm{N}^{\mathrm{dg}}\big(\mathbf{Ch}(\mathbb{A})\big).$$

The derived ∞-category: For a Grothendieck abelian category $\mathbb{A}$ the discrete category of cochains $\mathrm{Ch}(\mathbb{A})$ admits enough K-injectives [22], i.e. complexes I for which the cochain-valued Hom functor $\mathrm{Hom}^*_{\mathbb{A}}(-, I) : \mathrm{Ch}(\mathbb{A}) \to \mathrm{Ch}(\mathbb{Z})$ preserves acyclicity. In Parts I and II we employed the following definition.

Proto-definition 1.2. For a Grothendieck abelian category $\mathbb{A}$, the derived ∞-category $\mathscr{D}(\mathbb{A})$ is the full ∞-subcategory of K-injective complexes in the homotopy ∞-category $\mathscr{K}(\mathbb{A})$.

We generally denoted this ∞-subcategory as $\mathscr{K}(\mathbb{A}) \supseteq \mathscr{D}_{\mathrm{Inj}} = \mathscr{D}(\mathbb{A})$.

Remark 1.3. It's shown in Section I-13 that one can equivalently construct the derived ∞-category via K-projective complexes, whenever such complexes exist in sufficient abundance.

From the perspective of this text however, one should not necessarily approach the derived ∞-category in such a precious manner, but instead employ a coordinate free realization of the derived ∞-category as a localization of the homotopy ∞-category $\mathscr{D}(\mathbb{A}) = \mathscr{K}(\mathbb{A})[\mathrm{Qiso}^{-1}]$ relative to the class of quasi-isomorphisms in $\mathscr{K}(\mathbb{A})$. See in particular Theorem 9.6 and Corollary 9.21 below.

1.2. **Contents.** The text proceeds as follows.

In Section 2 we provide a direct, hands on analysis of limits and colimits in the homotopy ∞-category $\mathscr{K}(\mathbb{A})$. Of special interest are pullbacks and pushouts, which are both realized explicitly via mapping cone constructions. We also explain how projective and injective resolutions are realized naturally as colimits and limits of their respective finite truncations. As the derived ∞-category $\mathscr{D}(\mathbb{A})$ can be constructed as a full subcategory in $\mathscr{K}(\mathbb{A})$, namely as the full subcategory of K-injectives or K-projectives (see Section I-13), our calculations for $\mathscr{K}(\mathbb{A})$ imply analogous calculations of limits and colimits in the derived ∞-category.

There are two general ambitions which we mean to realize in this section. The first is simply to calculate a number of important limits and colimits in homotopy and derived ∞-categories. The second is to demonstrate how the abstract shenanigans from Part II, regarding limits and colimits, can be applied to obtain explicitly describions of limits and colimits in a given explicit context.

In Section 3 we explain how the presence of a zero object in $\mathscr{D}(\mathbb{A})$ endows each Hom functor $H : \mathscr{D}(\mathbb{A})^{\mathrm{op}} \times \mathscr{D}(\mathbb{A}) \to \mathscr{K}an$ with a natural pointing. This pointing is, formally, a lift of H to a $\mathscr{K}an_*$-valued functor (where $\mathscr{K}an_*$ is the ∞-category of pointed spaces). This lift is furthermore uniquely determined by the requirement that the functor $\widetilde{H} : \mathscr{D}(\mathbb{A})^{\mathrm{op}} \times \mathscr{D}(\mathbb{A}) \to \mathscr{K}an_*$ preserves initial objects.

In Section 4 we study the ∞-category $\mathscr{K}an_*$ of pointed sets. We are particularly interested in completeness and cocompleness of this ∞-category, and in the ability of the forgetful functor $\mathscr{K}an_* \to \mathscr{K}an$ to preserve and detect (co)limits in the domain.

In Section 5 we introduce spectra and provide sufficient conditions for the pointed Hom functors from Section 3 to enhance to spectra-valued Hom functors. Some portion of this section is dedicated to a baseline analysis of spectra themselves and the production of spectrum objects in ∞-categories which admit a terminal object.

In Section 6 we show that the derived and homotopy ∞-categories are stable. This is an immediate consequence of the materials from Section 2. We then provide a basic overview of stable ∞-categories and recall some special phenomena which occur in the stable setting. In short, stability provides various shortcuts which allow one to make strong deductions about a stable ∞-category from a direct investigation of its homotopy category.

We find, for example, that the homotopy and derived ∞-categories are both complete and cocomplete. We also observe that the connective derived ∞-category $\mathscr{D}(\mathbb{A})^{\leq 0}$ of any Grothendieck abelian category is cocomplete.

In Section 8 we provide a generic analysis of adjoint functors. There are two main points here. The first is that any inclusion $\mathscr{C}_0 \to \mathscr{C}$ of a coreflective subcategory into an ∞-category $\mathscr{C}$ admits a left adjoint. The second is that the information of a pair of adjoint functors,

$$F : \mathscr{C} \to \mathscr{D} \ \text{ and } \ G : \mathscr{D} \to \mathscr{C},$$

can be codified into a single cartesian and cocartesian fibrations over the 1-simplex $\mathscr{E} \to \Delta^1$ whose fibers recover $\mathscr{C} = \mathscr{E}_0$ and $\mathscr{D} = \mathscr{E}_1$. This is a practical tool which allows one to both construct adjunctions, and check for the existence of adjunctions.

In Section 9 we prove that the derived ∞-category is identified as the localization $\mathscr{K}(\mathbb{A})[\mathrm{Qiso}^{-1}] = \mathscr{D}(\mathbb{A})$ relative to the class of quasi-isomorphisms in $\mathscr{K}(\mathbb{A})$. It then follows–from a particular result in [15]–that the derived ∞-category is furthermore identified with a localization of the discrete category of cochains $\mathrm{Ch}(\mathbb{A})[\mathrm{Qiso}^{-1}] = \mathscr{D}(\mathbb{A})$. This result is highly non-classical, and allows one to transfer structures directly from the abelian setting to the derived setting.

In Section 10 we discuss the process of deriving functors in the ∞-categorical context. We show that the left derived functor $\mathrm{L}\,F$ of a right exact functor $F : \mathbb{A} \to \mathbb{B}$ between Grothendieck abelian categories can be defined, as in the discrete derived setting, by taking F-acyclic resolutions on the domain. Right derived functors $\mathrm{R}\,G$ can be defined universally by taking K-injective resolutions. Given a pair of adjoint functors $F : \mathbb{A} \leftrightarrows \mathbb{B} : G$, we show that the associated left derived functor $\mathrm{L}\,F : \mathscr{D}(\mathbb{A}) \to \mathscr{D}(\mathbb{B})$ is left adjoint to the right derived functor $\mathrm{R}\,G : \mathscr{D}(\mathbb{B}) \to \mathscr{D}(\mathbb{A})$, as expected.

In Section 11 we introduce the notion of presentability for ∞-categories. Following [15] directly, we recall that the derived ∞-category $\mathscr{D}(\mathbb{A})$ is presentable, and following [14] directly we present the adjoint functor theorem, which characterizes those functors between presentable ∞-categories which are left or right adjoints. (This section is more of a rundown of results from [14] and [15] rather than an original presentation of the topic.)

In Section 12 we present the ind-completion functor and subsequent constructions of "renormalized" derived categories. These include derived categories of ind-coherent sheaves, and ind-finite representations for algebraic groups.

In Appendix A we discuss idempotent complete categories, the process of idempotent completion, and ($\aleph_0$-)accessibility.

At the center of our analysis is a description of the derived ∞-category as a *stable presentable* ∞-category. From this perspective Sections 2 through 5 appear as preliminaries to the main ambitions of the text, which occur later. These preliminaries constitute roughly one third of the text.

1.3. **Contextualization and originality.** Unlike Parts I and II, which essentially remixed and reorganized materials from Kerodon [16], the present text is a heterogenous collection of materials from Higher Topos Theory [14], Higher Algebra [15], and original content. For example the majority of the contents from Sections 3 and 10 were developed independently. All of our computations from Section 2 are also original.

Additionally, in this text we regularly employ "fundamental" results from [14] and [15] without outlining or speaking to their proofs in any way. Hence we do not pursue a self-contained treatment of the topic at hand, and instead attempt to provide a treatment which is most effective in a practical sense. This deviates from the point-by-point, completionist approach taken in Parts I and II.

1.4. **Omissions et al.** In this text one finds various small omissions, such as a discussion of t-structures, and the generic omission of, say, 98% of higher algebra [15]. This point will be evident to anyone who is mildly familiar with the topic. A point which might not be evident, however, is that the general framings of higher algebra [15] and, say, spectral algebraic geometry [17] are not the same.

From the perspective of higher algebra [15] one locates the the derived ∞-category $\mathscr{D}(\mathbb{A})$ as a point of interest within the class of stable, presentable ∞-categories. We have an effective algebra of stable presentable categories via the stable product of [15, Section 4.8.1]. From the perspective of [17], on the other hand, one emphasizes *Grothendieck* presentable rather that stable presentable ∞-categories. In the Grothendieck context we replace the full derived category $\mathscr{D}(\mathbb{A})$

with its connective subcategory $\mathscr{D}(\mathbb{A})^{\leq 0}$ [17, Section C.1.4]. (Note that the connective derived category is not stable.) In the Grothendieck setting we have an effective algebra of categories [17, Section C.4], and such categories seemingly provide a more functional location in which to do algebraic geometry.

In any case, we only alert the reader to the existence of this alternate universe of Grothendieck prestable ∞-categories, make no further mention of the topic, and adopt and develop the stable perspective throughout this text. So, let us begin.

2. Preliminaries I: Limits and colimits in the homotopy ∞-category

sect:lim_KA

We directly compute fundamental classes of limits and colimits in the homotopy ∞-category $\mathscr{K}(\mathbb{A})$, where $\mathbb{A}$ is an abelian category. Of special interest are pullbacks and pushouts. We approach these computations via the explicit descriptions of limits in $\mathscr{K}an$ provided in Section II-14.2, and the fact that limits and colimits can be detected by checking values under the application of Hom functors (Corollary II-16.16).

We use the results from this section to verify stability, and cocompleteness, of homotopy and derived ∞-categories in Section 6 below.

2.1. Orienting comments on the homotopy ∞-category.

Definition 2.1. For an abelian category $\mathbb{A}$ we let $K\mathbf{Ch}(\mathbb{A})$ denote the simplicial category associated to the dg category of cochains over $\mathbb{A}$, and we take

$$\mathscr{K}(\mathbb{A})' := \mathrm{N}^{\mathrm{hc}}(K\mathbf{Ch}(\mathbb{A})).$$

Let us begin with a simple comparison between the ∞-categories $\mathscr{K}(\mathbb{A})$ and $\mathscr{K}(\mathbb{A})'$, as the category $\mathscr{K}(\mathbb{A})'$ provides a more convenient locale in which to do many of our calculations.

We recall from Theorem II-10.4 that there is an equivalence $\mathfrak{Z} : \mathscr{K}(\mathbb{A})' \overset{\sim}{\to} \mathscr{K}(\mathbb{A}) = \mathrm{N}^{\mathrm{dg}}(\mathbf{Ch}(\mathbb{A}))$ which restricts to the identity on the underlying discrete category of cochains

$$\begin{tikzcd} \mathscr{K}(\mathbb{A})' \arrow[rr, "\mathfrak{Z}"] & & \mathscr{K}(\mathbb{A}) \\ & \mathrm{Ch}(\mathbb{A}) \arrow[ul] \arrow[ur] & \end{tikzcd} \quad . \tag{1}$$

eq:equiv_Z

Since the inclusions from $\mathrm{Ch}(\mathbb{A})$ are surjective on 0 and 1-simplices, we see that the ∞-categories $\mathscr{K}(\mathbb{A})$ and $\mathscr{K}(\mathbb{A})'$ are indistinguishable at the level of objects and morphisms.

Though we generally avoid considering 2-simplices in $\mathscr{K}(\mathbb{A})'$, it is the case that these two categories are identified in dimension 2 as well. Before observing this identification, let us recall that the Hom complexes in the simplicial category $K\,\mathrm{Ch}(\mathbb{A})$ are given by the Eilenbergh-MacLane spaces

$$\underline{\mathrm{Hom}}_{\mathbb{A}}(V, W) = K\,\mathrm{Hom}^*_{\mathbb{A}}(V, W)$$

(see Section I-11.4). Directly, a 1-simplices in $\underline{\mathrm{Hom}}_{\mathbb{A}}(V, W)$ is a triple

$$\widetilde{h} = (h : V \to W,\ h_0 : V \to W,\ h_1 : V \to W)$$

with h of degree -1, the h_i cochain maps of degree 0, and which solves the equation

$$d(h) = d_W h + h d_V = h_0 - h_1.$$

The restrictions along the two inclusions $\{i\} \to \Delta^1$ are as expected, with $\widetilde{h}|_{\{i\}} = h_i$.

Now, according to the definition of the homotopy coherent nerve, a 2-simplex $\sigma : \Delta^2 \to \mathscr{K}(\mathbb{A})'$ is the data of a not-necessarily-commuting diagram of cochain maps

$$\begin{tikzcd} & V_1 \arrow[dr, "f_{12}"] & \\ V_0 \arrow[ur, "f_{01}"] \arrow[rr, "f_{02}"'] & & V_2 \end{tikzcd}$$

and a 1-simplex $\widetilde{h}$ in $\underline{\mathrm{Hom}}_{\mathbb{A}}(V_0, V_2)$ with $\widetilde{h}|_0 = f_{12}f_{01}$ and $\widetilde{h}|_1 = f_{02}$, i.e. a choice of a degree -1 map h which establishes homotopy commutativity $d(h) = f_{12}f_{01} - f_{02}$. (See Lemma I-2.19).

To compare, a 2-simplex in the usual homotopy ∞-category $\mathscr{K}(\mathbb{A})$ is a choice of a not-necessarily-commuting diagram of cochain maps

$$\begin{tikzcd} & V_1 \arrow[dr, "f_{12}"] & \\ V_0 \arrow[ur, "f_{01}"] \arrow[rr, "f_{02}"'] & & V_2 \end{tikzcd}$$

and a degree -1 map $h : V_0 \to V_2$ which satisfies $d(h) = f_{12}f_{01} - f_{02}$. These are clearly the same thing, and the equivalence $\mathfrak{Z}$ appearing in (1) sends each 2-simplex in $\mathscr{K}(\mathbb{A})'$ to "the same" 2-simplex in $\mathscr{K}(\mathbb{A})$.

2.2. **Limits and colimits in $\mathscr{K}(\mathbb{A})$ via Hom functors.** According to Proposition II-11.6 the functor

$$\underline{\mathrm{Hom}}_{\mathbb{A}}(V, -) : \mathscr{K}(\mathbb{A})' \to \mathscr{K}an \quad \text{and} \quad \underline{\mathrm{Hom}}_{\mathbb{A}}(-, W) : (\mathscr{K}(\mathbb{A})')^{\mathrm{op}} \to \mathscr{K}an$$

are respectively corepresented and represented by the given objects V and W in $\mathscr{K}(\mathbb{A})$. Hence Corollary II-16.17 appears in our setting as follows.

Proposition 2.2. *For a diagram $p : K \to \mathscr{K}(\mathbb{A})'$, a given extension $p' : \{*\} \star K \to \mathscr{K}(\mathbb{A})'$ is a limit diagram for p if and only if, at each cochain complex V, the composite*

$$\underline{\mathrm{Hom}}_{\mathbb{A}}(V, -) \circ p' : \{*\} \star K \to \mathscr{K}an$$

is a limit diagram in $\mathscr{K}an$. Similarly, an extension $p'' : K \star \{*\} \to \mathscr{K}(\mathbb{A})'$ is a colimit diagram if and only if, at each complex W, the composite

$$\underline{\mathrm{Hom}}_{\mathbb{A}}(-, W) \circ (p'')^{\mathrm{op}} : \{*\} \star K^{\mathrm{op}} \to \mathscr{K}an$$

is a limit diagram in $\mathscr{K}an$.

We use this result ad nauseam in order to calculate limits and colimits in $\mathscr{K}(\mathbb{A})'$, and subsequently in $\mathscr{K}(\mathbb{A})$.

2.3. **Strictly commuting diagrams and "discrete (co)limits".** By a *strictly commuting diagram* in $\mathscr{K}(\mathbb{A})$ we mean a diagram $p : K \to \mathscr{K}(\mathbb{A})$ which factors through the discrete category of cochains,

$$\begin{tikzcd} & \mathrm{Ch}(\mathbb{A}) \arrow[dr, "\mathrm{incl}"] & \\ K \arrow[ur, dashed, "\exists\ \bar{p}"] \arrow[rr, "p"'] & & \mathscr{K}(\mathbb{A}). \end{tikzcd}$$

For such a diagram the completing functor $\bar{p}$ is uniquely determined, as the map $\mathrm{incl} : \mathrm{Ch}(\mathbb{A}) \to \mathscr{K}(\mathbb{A})$ is an inclusion of simplicial sets. We similarly define strictly commuting diagrams in the ∞-category $\mathscr{K}(\mathbb{A})'$.

When $\mathbb{A}$ is Grothendieck abelian the category $\mathrm{Ch}(\mathbb{A})$ admits arbitrary coproducts and coequalizers, and hence is cocomplete as an ∞-category [15, Proposition 4.4.3.2]. It is similarly seen to be complete as well. Furthermore, since the inclusion $\mathrm{Set} \to \mathscr{K}an$ is right adjoint to the connected components functor $\pi_0 : \mathscr{K}an \to \mathrm{Set}$ we see that Set is stable under taking limits in $\mathscr{K}an$. (The existence of such an adjunction follows from Theorem I-14.13.) Hence, by a consideration of Hom functors, and Corollary II-16.17, we observe that the limit (resp. colimit) of any map $p : K \to \mathrm{Ch}(\mathbb{A})$ from a discrete category K is just equal to the limit (resp. colimit) which we calculate in $\mathrm{Ch}(\mathbb{A})$ as an abelian category.

Definition 2.3. Let $\mathbb{A}$ be a Grothendieck abelian category and $p : K \to \mathscr{K}(\mathbb{A})$ be a strictly commuting diagram. Let $\bar{p} : K \to \mathrm{Ch}(\mathbb{A})$ be the corresponding diagram in $\mathrm{Ch}(\mathbb{A})$. The discrete limit (resp. colimit) of p in $\mathscr{K}(\mathbb{A})$ is the image

$$\mathrm{incl}\,\bar{p}' : \{0\} \star K \to \mathscr{K}(\mathbb{A}) \ \ (\text{ resp. } \mathrm{incl}\,\bar{p}'' : K \star \{\infty\} \to \mathscr{K}(\mathbb{A})\ )$$

of the corresponding limit diagram $\bar{p}'$ (resp. colimit diagram $\bar{p}''$) in $\mathrm{Ch}(\mathbb{A})$.

As was the case when considering limits and colimits in $\mathscr{C}at_\infty$, part of our game here is to identify a nice class of (strictly commuting) diagrams $p : K \to \mathscr{K}(\mathbb{A})$ for which the discrete limit, or colimit, already calculates the limit, or colimit, of p in $\mathscr{K}(\mathbb{A})$.

2.4. **Kan fibrations for simplicial abelian groups.** Below we call a simplicial abelian group A *discrete* if all its n-simplices, for $n > 1$, are degenerate.

lem:5490 **Lemma 2.4.** (1) *If A is a discrete simplicial abelian group, then the inclusion $0 \to A$ is a Kan fibration.*

(2) *Any surjective map of simplicial abelian groups $f : X \to Y$ is a Kan fibration.*
(3) *If A is a discrete simplicial abelian group, and $f : X \to Y$ is a surjective, then the map $[0\ f]^t : X \to A \times Y$ is a Kan fibration.*

Proof. (1) Follows from the fact that, in this case, any simplex $\Delta^n \to A$ in which a single vertex maps to 0 is of constant value 0. (2) Consider a lifting diagram

$$\begin{tikzcd} \Lambda^n_i \arrow[r, "\tau"] \arrow[d] & X \arrow[d, "f"] \\ \Delta^n \arrow[r, "\bar{\sigma}"'] & Y. \end{tikzcd}$$

We can lift $\bar{\sigma}$ arbitrarily to an n-simplex $\sigma : \Delta^n \to X$, via surjectivity of f. We can now replace $\bar{\sigma}$ with the 0 simplex and τ with $\sigma|_{\Lambda^n_i} - \tau$ to reduce to the case $Y = 0$.

In this case the desired solution exists since X is a Kan complex (Proposition I-11.1). (3) In this case $[0\ f]^t$ can be identified with a product of Kan fibrations $0 \times f : \{*\} \times X \to A \times Y$, and is thus a Kan fibration. $\square$

cor:K_kanfib **Corollary 2.5.** *If $f : V \to W$ is a map of cochains in which $f^n : V^n \to W^n$ is surjective at all $n < 0$, then the corresponding map $Kf : KV \to KW$ is a Kan fibration.*

Proof. We can factor f as the inclusion $V \to Z^0(W) \times V$ composed with the map $[i\ f] : Z^0(W) \times V = Z^0(W) \oplus V \to W$, where i here is the inclusion $i : Z^0(W) \to W$. Then Kf factors as the sequence

$$KV \xrightarrow{[0\ id]^t} KZ^0(W) \times KV \xrightarrow{[i\ Kf]} KW$$

in which the latter map is surjective, since the Eilenbergh-MacLane functor $K = K\tau_0$ is an equivalence on connective cochains and the map $\tau_0(Z^0(W) \times V) \to \tau_0(W)$ is surjective by construction. By Lemma 2.4 it follows that Kf is a Kan fibration. $\square$

2.5. **Fundamental claims: pullbacks in the homotopy ∞-category.**

Definition 2.6. For maps of cochains $f : V \to W$ and $f' : V' \to W$, we take

$$C(f, f') := \Sigma^{-1} \operatorname{cone}(\ [-f\ \ f'] : V \times V' \to W\ ).$$

For a single map $f : V \to W$ we take $C(f) = \Sigma^{-1} \operatorname{cone}(-f)$.

The complex $C(f, f')$ appears explicitly as

$$\left((V \times V') \oplus \Sigma^{-1} W,\ \begin{bmatrix} d_{V \times V'} & [f\ \ -f'] \\ 0 & -d_W \end{bmatrix} \right).$$

Note that we have the embedding of cochains

$$V \times_W V' = \ker([f\ \ -f']) \to C(f, f'),$$

and also the degree -1 map of graded objects in $\mathbb{A}$

$$h_W = [0\ id_W] : C(f, f') \to W \tag{2}$$ eq:hw

for which

$$d_{\mathrm{Hom}}(h_W) = \big(C(f, f') \xrightarrow{[p\ p']^t} V \times V' \xrightarrow{[f\ -f']} W\big),$$

where p and p' are the obvious projection. This homotopy defines a 2-simplex $h_W : \Delta^2 \to \mathscr{K}(\mathbb{A})$ which appears as

$$\begin{tikzcd} & V \arrow[dr, "f"] & \\ C(f,f') \arrow[ur, "p"] \arrow[rr, "f'p'"'] & & W. \end{tikzcd}$$

We append a strictly commuting diagram for V' to obtain a square $\Delta^1 \times \Delta^1 \to \mathscr{K}(\mathbb{A})$ which appears as

$$\begin{tikzcd} C(f,f') \arrow[r, "p"] \arrow[d, "p'"'] \arrow[dr, "f'p'"] & V \arrow[d, "f"] \\ V' \arrow[r, "f'"'] & W. \end{tikzcd} \tag{3}$$ eq:5570

def:std_pullback

Definition 2.7. Given arbitrary morphisms $f : V \to W$ and $f' : V' \to W$ of $\mathbb{A}$-cochains, the corresponding standard pullback diagram is the diagram (3) in $\mathscr{K}(\mathbb{A})$.

We prove in Proposition 2.13 below that any standard pullback diagram is in fact a limit diagram in $\mathscr{K}(\mathbb{A})$. We note that the construction of the standard pullback is natural, in the sense that a strictly commuting diagram

$$\begin{tikzcd}
V \arrow[r] \arrow[dr, "f"'] & X \arrow[dr, "g"] & V' \arrow[r] \arrow[dl, "f'"'] & X' \arrow[dl, "g'"] \\
& W \arrow[r] & Y &
\end{tikzcd}$$

extends to a diagram of the form

$$\begin{tikzcd}
& C(f,f') \arrow[r] \arrow[dl] \arrow[dr] & C(g,g') \arrow[dl] \arrow[dr] & \\
V' \arrow[r, "="] \arrow[dr] & X' \arrow[dr] & V \arrow[r] \arrow[dl] & X \arrow[dl] \\
& W \arrow[r] & Y &
\end{tikzcd}$$

in $\mathscr{K}(\mathbb{A})$.

2.6. **Pullbacks in the homotopy ∞-category.** Throughout the subsection we fix $\mathbb{A}$ an abelian category. We establish some background materials before returing to address the issue of pullbacks.

def:truncate

Definition 2.8. For any abelian category $\mathbb{A}$ we let

$$\tau_0 : \mathrm{Ch}(\mathbb{A}) \to \mathrm{Ch}(\mathbb{A})^{\leq 0}$$

denote the truncation functor, $\tau_0 V = \cdots \to V^{-2} \to V^{-1} \to Z^0(V) \to 0$.

One observes directly that the functor τ_0 respects homotopy and homotopy equivalences. Furthermore, since τ_0 is right adjoint to the inclusion $\mathrm{Ch}(\mathbb{A})^{\leq 0} \to \mathrm{Ch}(\mathbb{A})$ it commutes with all limits.

Definition 2.9. We call a map of $\mathbb{A}$-cochains $f : V \to W$ termwise split surjective if, for each integer n, the map $f^n : V^n \to W^n$ is a split surjection in $\mathbb{A}$.

lem:5721

Lemma 2.10. *For a map of $\mathbb{A}$-cochains $f : V \to W$ the following are equivalent:*

(1) *f is termwise split surjective.*
(2) *f is split surjective as a map of graded objects in $\mathbb{A}$.*
(3) *For each cochain X, the induced map $f_* : \mathrm{Hom}^*_{\mathbb{A}}(X, V) \to \mathrm{Hom}^*_{\mathbb{A}}(X, W)$ is surjective.*

Proof. Omitted. □

The following can be seen as an algebraic analog of Corollary II-14.31.

prop:split_pullback

Proposition 2.11. *Consider maps of $\mathbb{A}$-cochains $f : V \to W$ and $f' : V' \to W$ and suppose one of f or f' is termwise split surjective. Then the discrete pullback* *diagram*

$$\begin{tikzcd} V \times_W V' \arrow[r] \arrow[d] & V \arrow[d] \\ V' \arrow[r] & W \end{tikzcd} \tag{4}$$

eq:5645

is a limit diagram in $\mathscr{K}(\mathbb{A})$.

Proof. We first prove that the diagram (9) is a pullback diagram in the simplicial construction $\mathscr{K}(\mathbb{A})'$ of the homotopy ∞-category.

Assume arbitrarily that f is termwise split surjective. Take

$$\mathrm{Hom}^{\bullet}_{\mathbb{A}}(X,Y) = \tau_0 \mathrm{Hom}^{*}_{\mathbb{A}}(X,Y)$$
$$= \cdots \to \mathrm{Hom}^{-2}_{\mathbb{A}}(X,Y) \to \mathrm{Hom}^{-1}_{\mathbb{A}}(X,Y) \to Z^0 \mathrm{Hom}^{*}_{\mathbb{A}}(X,Y) \to 0.$$

For each cochain complex X we have

$$\mathrm{Hom}^{\bullet}_{\mathbb{A}}(X, V \times_W V') = \mathrm{Hom}^{\bullet}_{\mathbb{A}}(X,V) \times_{\mathrm{Hom}^{\bullet}_{\mathbb{A}}(X,W)} \mathrm{Hom}^{\bullet}_{\mathbb{A}}(X,V')$$

so that the induced diagram

$$\begin{tikzcd} \mathrm{Hom}^{\bullet}_{\mathbb{A}}(X, V \times_W V') \arrow[r] \arrow[d] & \mathrm{Hom}^{\bullet}_{\mathbb{A}}(X,V) \arrow[d, "f_*"] \\ \mathrm{Hom}^{\bullet}_{\mathbb{A}}(X,V') \arrow[r, "f'_*"'] & \mathrm{Hom}^{\bullet}_{\mathbb{A}}(X,W) \end{tikzcd} \tag{5}$$

eq:5660

is a pullback diagram. Furthermore, by our splitting assumption, the map f_* is split in each strictly negative degree.

We now apply the Eilenbergh-MacLane functor K to obtain a pullback diagram

$$\begin{tikzcd} \underline{\mathrm{Hom}}_{\mathbb{A}}(X, V \times_W V') \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}_{\mathbb{A}}(X,V) \arrow[d, "Kf_*"] \\ \underline{\mathrm{Hom}}_{\mathbb{A}}(X,V') \arrow[r, "Kf'_*"'] & \underline{\mathrm{Hom}}_{\mathbb{A}}(X,W) \end{tikzcd}$$

in which the map Kf_* is a Kan fibration, by Corollary 2.5. The above diagram is therefore a pullback diagram in $\mathscr{K}an$ by Corollary II-14.31. Since X was chosen arbitrarily we apply Corollary II-16.17 to observe that the diagram (9) is a pullback diagram in $\mathscr{K}(\mathbb{A})'$.

We now consider the corresponding diagram (9) in $\mathscr{K}(\mathbb{A})$. Since the equivalence $\mathfrak{Z} : \mathscr{K}(\mathbb{A})' \to \mathscr{K}(\mathbb{A})$ restricts to the identity on $\mathrm{Ch}(\mathbb{A})$ we see that it preserves all discrete pullback diagrams. Since equivalences preserve limits, by Proposition II-13.10, Proposition 2.11 implies that the diagram (9) is in fact a pullback diagram in $\mathscr{K}(\mathbb{A})$ as well. $\square$

We now compare the discrete fiber product $V \times_W V'$ to the shifted mapping cone which appears in the standard pullback diagram from (3).

lem:pb_cone **Lemma 2.12.** *Consider maps of cochains* $f : V \to W$ *and* $f' : V' \to W$, *and suppose one of* f *or* f' *is termwise split surjective. Then the inclusion*

$$V \times_W V' \to C(f, f')$$

is a homotopy equivalence.

Proof. By replacing V with $V \oplus V'$ and V' with 0, it suffices to prove that the inclusion $\ker(f) \to C(f)$ is a homotopy equivalence in the case that $f : V \to W$ is termwise split surjective. In this case the fiber product $V \times_W V'$ is simply the kernel of f.

Via the splitting we can write $V \cong \Sigma L \oplus K$ with $L = \Sigma^{-1} W$ and $K = \ker(f)$. Here K is a subcomplex in V and the map $V \to W$ is just the projection onto the first factor. We may assume for simplicity that the isomorphism $V \cong \Sigma L \oplus K$ is an equality on the underlying graded objects $V = \Sigma L \oplus K$.

The composite

$$\Sigma L \overset{\text{incl}}{\longrightarrow} V \overset{d_V}{\to} V \overset{\text{proj}}{\longrightarrow} K$$

defines a degree 1 map from ΣL, which is then a degree 0 map $g : L \to K$. This map is seen to be a cochain morphism so that

$$V = \operatorname{cone}(g) = \left(\Sigma L \oplus K, \begin{bmatrix} -d_L & g \\ 0 & d_K \end{bmatrix}\right).$$

We now have

$$C(f) = \left(\Sigma L \oplus K \oplus L, \begin{bmatrix} -d_L & g & id \\ 0 & d_K & 0 \\ 0 & 0 & d_L \end{bmatrix}\right)$$

and observe the projection

$$\pi = [0 \ id_K \ \ -g] : C(f) \to K.$$

We have directly $\pi \,\text{incl} = id_K : K \to K$ and the composite $\text{incl}\,\pi : C(f) \to C(f)$ is homotopic to the identity via the degree -1 map

$$h = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ id_L & 0 & 0 \end{bmatrix} : C(f) \to C(f).$$

□

Now we have the inclusion $V \times_W V' \to C(f, f')$ at general f and f', and the homotopy h_W from (2) has trivial restriction $h_W|_{V \times_W V'} = 0$. This inclusion therefore extends to a natural transformation of diagrams

$$\begin{tikzcd} & V \times_W V' \arrow[r] \arrow[dl] \arrow[dr] & C(f,f) \arrow[dl] \arrow[dr] \arrow[dd, dotted] & \\ V' \arrow[r, "="] \arrow[dr] & V' \arrow[dr] & V \arrow[r, "="] \arrow[dl] & V \arrow[dl] \\ & W \arrow[r, "="'] & W & \end{tikzcd} \tag{6}$$

eq:5696

By Lemma 2.12 and Theorem I-7.6 this transformation is an isomorphism whenever f or f' is termwise split surjective.

prop:K_pullback **Proposition 2.13.** *For arbitrary maps $f : V \to W$ and $f' : V' \to W$ in $\mathscr{K}(\mathbb{A})$, the standard pullback diagram*

$$\begin{tikzcd} C(f,f') \arrow[r] \arrow[d] \arrow[dr] & V \arrow[d, "f"] \\ V' \arrow[r, "f'"'] & W \end{tikzcd} \tag{7}$$ eq:5707

(see Definition 2.7) is a limit diagram in $\mathscr{K}(\mathbb{A})$. In particular, the diagram (7) is isomorphic to a diagram of the form

$$\begin{tikzcd} V_0 \arrow[r] \arrow[d] & V_1 \arrow[d, "f_{12}"] \\ V_1' \arrow[r, "f_{12}'"'] & V_2 \end{tikzcd}$$

in which f_{12} is termwise split surjective and f'_{12} is injective.

Proof. Take $V_2 = \mathrm{cone}(id_{V'}) \oplus W$, $V_1' = V'$, and $f'_{12} = [i\ f']^t : V' \to \mathrm{cone}(id_{V'}) \oplus W$ where $i : V' \to \mathrm{cone}(id_{V'})$ is the usual inclusion. Take now $V_1 = V \oplus C(id_{V_2})$ and $f_{12} = [\pi\ f] : V \oplus C(id_{V_2}) \to V_2$ where $\pi : C(id_{V_2}) \to V_2$ is the usual projection. The map f'_{12} is injective and g is split as a graded morphism via the identity map $V_2 \to V_2 \oplus \Sigma^{-1} V_2 = C(id_{V_2})$.

Since the mapping cone of the identity morphism is contractible, the summands $\mathrm{cone}(id_{V'})$ and $C(id_{V_2})$ are contractible. The inclusion $V \to V_1$ and $W \to V_2$ are therefore homotopy equivalence and induce an isomorphism of diagrams

$$\begin{tikzcd} & C(f,f') \arrow[r] \arrow[dl] \arrow[dr] & C(f_{12},f'_{12}) \arrow[dl] \arrow[dr] & \\ V' \arrow[r, "="] \arrow[dr] & V_1' \arrow[dr] & V \arrow[r, "\mathrm{incl}"] \arrow[dl] & V_1 \arrow[dl] \\ & W \arrow[r, "\mathrm{incl}"'] & V_2 & \end{tikzcd}$$

in $\mathscr{K}(\mathbb{A})$. As argued at (6) the diagram for $C(f_{12}, f'_{12})$ is naturally isomorphic to the discrete pullback diagram

$$\begin{tikzcd} V_0 = V_1 \times_{V_2} V_1' \arrow[r] \arrow[d] & V_1 \arrow[d, "g"] \\ V_1' \arrow[r] & V_2, \end{tikzcd} \tag{8}$$ eq:5733

so that in total the diagram (7) is isomorphic to the diagram (8). Since the latter diagram is a pullback diagram in $\mathscr{K}(\mathbb{A})$ by Proposition 2.11, it follows by Proposition II-13.18 that the diagram (7) is a pullback diagram as well. $\square$

As a corollary we find that any diagram of the form

$$\begin{tikzcd} & V \arrow[d] \\ V' \arrow[r] & W \end{tikzcd}$$

admits a limit in $\mathscr{K}(\mathbb{A})$.

cor:K_pullback

Corollary 2.14. *Every diagram of the form $\Lambda^2_2 \to \mathscr{K}(\mathbb{A})$ admits a limit in $\mathscr{K}(\mathbb{A})$. This is to say, the homotopy ∞-category $\mathscr{K}(\mathbb{A})$ of any abelian category $\mathbb{A}$ admits all pullbacks.*

2.7. **Pushouts diagrams in the homotopy ∞-category.** We recall that, for a partial diagram of cochains

$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d, "g'"'] & V \\ V' & {,} \end{tikzcd}$$

the pushout in $\mathrm{Ch}(\mathbb{A})$ is the quotient $V \amalg_{W'} V' = \mathrm{coker}(W \to V \oplus V')$.

Definition 2.15. We call a morphism of $\mathbb{A}$-cochains $g : W \to V$ termwise split injective if, at each integer n, $g^n : W^n \to V^n$ is a split injective morphism in $\mathbb{A}$.

We have the expected analog of Lemma 2.10.

lem:5915

Lemma 2.16. *For a map of $\mathbb{A}$-cochains $g : W \to V$ the following are equivalent:*

(1) *g is termwise split injective.*
(2) *g is split injective as a map of graded objects in $\mathbb{A}$.*
(3) *For each cochain Y, the induced map $g^* : \mathrm{Hom}^*_{\mathbb{A}}(V, Y) \to \mathrm{Hom}^*_{\mathbb{A}}(W, Y)$ is surjective.*

For any abelian category $\mathbb{A}$, we apply Proposition 2.11 to the opposite category $\mathbb{B} = \mathbb{A}^{\mathrm{op}}$ to obtain the corresponding result for pushout diagrams in $\mathscr{K}(\mathbb{A}) \cong \mathscr{K}(\mathbb{B})^{\mathrm{op}}$ (Lemma II-11.14).

prop:split_pushout

Proposition 2.17. *Consider maps of $\mathbb{A}$-cochains $g : V \to W$ and $g' : V' \to W$, and suppose one of g or g' is termwise split injective. Then the discrete pushout diagram*

$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d, "g'"'] & V \arrow[d] \\ V' \arrow[r] & V \amalg_W V' \end{tikzcd} \tag{9}$$

eq:5645

is a colimit diagram in $\mathscr{K}(\mathbb{A})$.

In the pushout context we return to the standard, rather than shifted, mapping cone.

Definition 2.18. Given maps $g : W \to V$ and $g' : W \to V'$ of $\mathbb{A}$-cochains we take

$$\mathrm{cone}(g, g') = \mathrm{cone}\left(\ [g \ \ -g']^t : W \to V' \oplus V \ \right).$$

In the case $V' = 0$ we recover the standard mapping cone $\mathrm{cone}(g) = \mathrm{cone}(g, 0)$. We have the two inclusions from V and V' into $\mathrm{cone}(g, g')$ which provide a generally noncommuting diagram

$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d, "g'"'] & V \arrow[d] \\ V' \arrow[r] & \mathrm{cone}(g, g'). \end{tikzcd}$$

The degree -1 map $h'_W : W \to \mathrm{cone}(g, g')$ defined by the identity on W satisfies

$$d_{\mathrm{Hom}}(h'_W) = [g\ \ -g']^t : W \to V \oplus V' \subseteq \mathrm{cone}(g, g')$$

and hence produces a diagram of the form

$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d, "g'"'] \arrow[dr] & V \arrow[d] \\ V' \arrow[r] & \mathrm{cone}(g, g'). \end{tikzcd} \tag{10}$$

eq:5958

in the homotopy ∞-category $\mathscr{K}(\mathbb{A})$ in which the bottom simplex strictly commutes and the top simplex is exhibited by h'_W.

Definition 2.19. Given maps $g : W \to V$ and $g' : W \to V'$ in $\mathscr{K}(\mathbb{A})$, we refer to the diagram (10) as the standard pushout diagram for g and g'.

We obtain the following by applying Lemma 2.12 to the opposite category.

lem:5987 **Lemma 2.20.** *Consider maps of $\mathbb{A}$-cochains $g : V \to W$ and $g' : V' \to W$, and suppose one of g or g' is termwise split injective. Then the projection $\pi : \mathrm{cone}(g, g') \to V \amalg_W V'$ is a homotopy equivalence.*

Since the homotopy h'_W vanishes when composed with the projection $\pi : \mathrm{cone}(g, g') \to V \amalg_W V'$, this projection extends to a diagram in $\mathscr{K}(\mathbb{A})$ which appears as

$$\begin{tikzcd} & W \arrow[rr, "="] \arrow[dl] \arrow[drr] & & W \arrow[dll] \arrow[dr] & \\ V' \arrow[r, "="] \arrow[dr] & V' \arrow[drr] & & V \arrow[r, "="] \arrow[dll] & V \arrow[dl] \\ & \mathrm{cone}(g, g') \arrow[rr] & & V \amalg_W V' & . \end{tikzcd}$$

According to Lemma 2.20, in the case that one of g or g' is split injective this diagram realizes an isomorphism between the two square faces, so that the standard square

$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d, "g'"'] \arrow[dr] & V \arrow[d] \\ V' \arrow[r] & \mathrm{cone}(g, g'). \end{tikzcd}$$

is observed to be a pushout square via Proposition 2.17. This realization of pushouts in $\mathscr{K}(\mathbb{A})$ via standard pushout diagrams follows.

prop:K_pushout

Proposition 2.21. *For arbitrary maps $g : W \to V$ and $g' : W \to V'$ in $\mathscr{K}(\mathbb{A})$, the standard pushout diagram*

$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d, "g'"'] \arrow[dr] & V \arrow[d] \\ V' \arrow[r] & \mathrm{cone}(g, g') \end{tikzcd} \tag{11}$$

eq:6010

is a colimit diagram in $\mathscr{K}(\mathbb{A})$. In particular, the diagram (11) is isomorphic to a diagram of the form

$$\begin{tikzcd} V_0 \arrow[r, "g_{01}"] \arrow[d, "g'_{01}"'] & V_1 \arrow[d] \\ V'_1 \arrow[r] & V_2 \end{tikzcd}$$

in which g_{01} is termwise split injective and g'_{01} is surjective.

The proof is similar to that of Proposition 2.13, and is omitted.

cor:K_pushout

Corollary 2.22. *Every diagram $\Lambda^2_0 \to \mathscr{K}(\mathbb{A})$ admits a colimit in $\mathscr{K}(\mathbb{A})$. That is to say, the homotopy ∞-category $\mathscr{K}(\mathbb{A})$ of any abelian category $\mathbb{A}$ admits all pushouts.*

2.8. **Products, coproducts, and the zero complex.**

prop:K_prods_coprods

Proposition 2.23. *For any abelian category $\mathbb{A}$, the functor $\mathrm{Ch}(\mathbb{A}) \to \mathscr{K}(\mathbb{A})$ preserves all small products and coproducts. In particular, the category $\mathscr{K}(\mathbb{A})$ admits all small products and coproducts whenever $\mathbb{A}$ is Grothendieck abelian.*

Proof. Since the inclusion $\mathrm{Ch}(\mathbb{A}) \to \mathscr{K}(\mathbb{A})$ factors through the equivalence $\mathfrak{Z} : \mathscr{K}(\mathbb{A})' \to \mathscr{K}(\mathbb{A})$ it suffices to show that products and coproducts in $\mathrm{Ch}(\mathbb{A})$ are products and coproducts in the simplicial construction of the homotopy ∞-category $\mathscr{K}(\mathbb{A})'$. For this we employ the Hom functor $\mathrm{K}\,\mathrm{Hom}^\bullet_{\mathbb{A}}$, where $\mathrm{Hom}^\bullet_{\mathbb{A}}(X, Y) = \tau_0 \mathrm{Hom}^*_{\mathbb{A}}(X, Y)$, and check that this functor turns discrete coproducts (resp. products) into products of spaces through the first (resp. second) coordinate. (Recall that products of spaces are as expected, by Example II-14.16 and Theorem II-14.25.)

Since τ_0 is a right adjoint it commutes with limits, and the Dold-Kan equivalence commutes with limits as well. So it suffices to show that the functor $\mathrm{Hom}^*_{\mathbb{A}}$ sends discrete coproducts in the first coordinate to products of additive cochains, and discrete products in the second coordinate to products of additive cochains as well. However this is immediate since $\mathrm{Hom}^*_{\mathbb{A}}$ provides inner-Homs for the action of $\mathrm{Ch}(\mathbb{Z})$ on $\mathrm{Ch}(\mathbb{A})$. □

Via vanishing of the Hom complexes $\mathrm{Hom}^*_{\mathbb{A}}(X, 0)$ and $\mathrm{Hom}^*_{\mathbb{A}}(0, X)$, and Lemma II-9.18, we also see that the zero complex is a zero object in the homotopy ∞-category.

cor:K_zero

Corollary 2.24. *The zero complex provides a simultaneous initial and terminal object in the ∞-category $\mathscr{K}(\mathbb{A})$.*

sect:add

2.9. **Additive categories.** Any additive category $\mathfrak{A}$ embeds fully faithfully into an abelian category $\mathbb{A}$, and hence $\mathscr{K}(\mathfrak{A})$ embeds fully faithfully into $\mathscr{K}(\mathbb{A})$. (One can take for example $\mathbb{A} = \mathrm{Fun}_{\mathrm{Add}}(\mathfrak{A}, \mathbb{Z}\text{-mod})$.) Since $\mathrm{Ch}(\mathfrak{A})$ is closed under the formation of mapping cones in $\mathrm{Ch}(\mathbb{A})$ it follows from Proposition 2.13 that $\mathscr{K}(\mathfrak{A})$ is closed under the formation of pullbacks in $\mathscr{K}(\mathbb{A})$. In particular, one can realize pullbacks in $\mathscr{K}(\mathfrak{A})$ via the standard pullback construction of (3). One similarly sees that $\mathscr{K}(\mathfrak{A})$ admits arbitrary pushouts.

prop:add_pullpush **Proposition 2.25.** *For any additive category $\mathfrak{A}$, the homotopy ∞-category $\mathscr{K}(\mathfrak{A})$ admits all pullbacks and pushouts. Furthermore, pullbacks and pushouts are realized explicitly via standard pullback and pushout constructions as in* (3) *and* (10)*.*

The same arguments as those provided in the abelian setting show that $\mathscr{K}(\mathfrak{A})$ admits finite products and coproducts as well, and that the zero complex is a zero object.

Proposition 2.26. *For any additive category $\mathfrak{A}$, $\mathscr{K}(\mathfrak{A})$ admits all finite products and coproducts, and both are realized via discrete biproducts in* $\mathrm{Ch}(\mathfrak{A})$*. Also, the zero complex is both initial and terminal in $\mathscr{K}(\mathfrak{A})$.*

2.10. **Resolutions as (co)limits.**

prop:lim_res **Proposition 2.27.** *Given a strictly commuting diagram $I(-) : \mathbb{Z}_{\leq 0} \to \mathscr{K}(\mathbb{A})$ in which each map $I(n-1) \to I(n)$ is termwise split surjective, the discrete limit diagram*

$$\begin{tikzcd}
\cdots \arrow[r] & I(m-1) \arrow[r] & I(m) \arrow[r] & I(m+1) \arrow[r] & \cdots \\
 & & I \arrow[ul] \arrow[u] \arrow[ur] & &
\end{tikzcd}$$

is a limit diagram in $\mathscr{K}(\mathbb{A})$.

Of course in the above diagram I is the limit of $I(-)$, when considered as a functor valued in $\mathrm{Ch}(\mathbb{A})$.

Proof. Since the diagram $I(-)$ factors through $\mathrm{Ch}(\mathbb{A})$ it suffices to prove that its image provides a limit diagram in $\mathscr{K}(\mathbb{A})'$. Applying the corepresentable functor $K\,\mathrm{Hom}^*_{\mathbb{A}}(X,-)$ to the given squence $\cdots \to I(m) \to I(m+1) \to \cdots$ produces a sequence of Kan fibrations

$$\cdots \to K\,\mathrm{Hom}^*_{\mathbb{A}}(X, I(m-1)) \to K\,\mathrm{Hom}^*_{\mathbb{A}}(X, I(m)) \to \cdots, \qquad (12)$$ eq:6055

by Corollary 2.5. Since the functor $K : \mathrm{Ch}(\mathbb{A}) \to \mathrm{Kan}_{\mathbb{Z}}$ has a left adjoint, given by the normalized cochains functor, it commutes with limits, so that

$$\lim_n K\,\mathrm{Hom}^*_{\mathbb{A}}(X, I(n)) = K \lim_n \mathrm{Hom}^*_{\mathbb{A}}(X, I(n)) = \mathrm{Hom}^*_{\mathbb{A}}(X, I).$$

Also, the forgetful functor $\mathrm{Kan}_{\mathbb{Z}} \to \mathrm{Kan}$ preserves limits, as it has a left adjoint provided by the free module functor. Hence the above identification provides an identification of limits in Kan.

Given such a sequence of Kan fibrations, we know from Proposition II-14.38 and Theorem II-14.42 that the corresponding discrete limit diagram of Kan complexes is a limit diagram in $\mathscr{K}an$. So we see that, at each complex X, the functor $K\,\mathrm{Hom}^*_{\mathbb{A}}(X,-)$ sends the diagram $I(-)$ to a limit diagram in spaces. By Corollary

II-16.17 it follows that the diagram $I(-) : \mathbb{Z}_{\leq 0} \to \mathscr{K}(\mathbb{A})'$ is a limit diagram in $\mathscr{K}(\mathbb{A})'$, and hence its composite with the equivalence $\mathfrak{Z} : \mathscr{K}(\mathbb{A})' \to \mathscr{K}(\mathbb{A})$ is a limit diagram in $\mathscr{K}(\mathbb{A})$. □

By consulting the opposite category, or simply by repeating the above arguments, one obtains the analogous statement for colimits of sequences of split injections.

prop:colim_res

Proposition 2.28. *Given a strictly commuting diagram $P(-) : \mathbb{Z}_{\geq 0} \to \mathscr{K}(\mathbb{A})$ in which each map $P(n) \to P(n+1)$ is termwise split injective, the discrete colimit diagram*

$$\begin{tikzcd} & & & P \arrow[dl] \arrow[d] \arrow[dr] & & \\ \cdots \arrow[r] & P(m-1) \arrow[r] & P(m) \arrow[r] & P(m+1) \arrow[r] & \cdots \end{tikzcd}$$

is a colimit diagram in $\mathscr{K}(\mathbb{A})$.

Example 2.29. Let I be a bounded below complex in $\mathscr{K}(\mathbb{A})$. Consider the quotients $I(n) = I/I^{\geq -n}$ and the subsequent sequence of termwise split sujections

$$\cdots \to I(-2) \to I(-1) \to I(0).$$

We have the corresponding strictly commuting diagram $I(-) : \mathbb{Z}_{\leq 0} \to \mathscr{K}(\mathbb{A})$ with discrete limit $\lim_n I(n) = I$. By Proposition 2.27 this discrete limit is a limit for the diagram $I(-)$ in $\mathscr{K}(\mathbb{A})$.

From the example we see that the bounded below homotopy ∞-categories $\mathscr{K}(\mathbb{A})^+$ is generated by the full subcategory of bounded complexes under (filtered) limits. We similarly see that the bounded above homotopy ∞-category $\mathscr{K}(\mathbb{A})^-$ is generated by the full subcategory of bounded complexes under (filtered) colimits.

Example 2.30. Let P be a bounded above complex in $\mathscr{K}(\mathbb{A})$. Consider the subcomplexes $P(n) = P^{\geq -n}$ and the corresponding sequence of termqise split inclusions

$$P(0) \to P(1) \to P(2) \to \cdots.$$

We have the associated strictly commuting diagram in $\mathscr{K}(\mathbb{A})$, which has colimit P by Proposition 2.28.

sect:zero_hom

3. Preliminaries II: Zero objects and pointed Hom functors

Given an ∞-category $\mathscr{C}$, we obtain its unique Hom functor

$$\text{“}\mathrm{Hom}_{\mathscr{C}}\text{”} = H : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{K}an$$

via transport along the twisted arrows fibration (see Section II-12.1). In the event that $\mathscr{C}$ has a zero object, we expect each space $H(x,y)$ to have a distinguished zero morphism which is preserved under composition. In this way the Hom functor should inherit a natural pointing, i.e. a *lifting* to the ∞-category $\mathscr{K}an_*$ of pointed spaces.

We prove that such a canonical pointing for the Hom functor exist. In Section 5 we show, further, that these pointed Hom functors enhance to spectra-valued Hom functors whenever $\mathscr{C}$ is sufficiently symmetric around zero.

3.1. **Pointed Hom functors.** We begin with an expected defintion.

Definition 3.1. A zero object in an ∞-category is an object $0 : * \to \mathscr{C}$ which is both initial and terminal in $\mathscr{C}$.

Consider the left fibration

$$q : \mathscr{K}an_*(= \mathscr{K}an_{*/}) \to \mathscr{K}an$$

and any ∞-category $\mathscr{C}$ with a zero object. Recall that the identity morphisms $* \to *$ is simultaneously initial and terminal in the category of pointed spaces $\mathscr{K}an_*$, by Proposition II-9.15. Furthermore, any choice of a zero object $0 : * \to \mathscr{C}$ provides an initial object in the enveloping category

$$\vec{0} = (0, 0) : * \to \mathscr{C}^{\mathrm{op}} \times \mathscr{C}.$$

The following is a direct application of Corollary II-9.25.

prop:693 **Proposition 3.2.** *Let $\mathscr{C}$ be an ∞-category with a zero object 0, and take $\mathscr{C}^e = \mathscr{C}^{\mathrm{op}} \times \mathscr{C}$. The functors*

$$\mathrm{Fun}(\mathscr{C}^e, \mathscr{K}an_*) \to \mathscr{K}an_* \times_{\mathscr{K}an} \mathrm{Fun}(\mathscr{C}^e, \mathscr{K}an), \quad F \mapsto (F|_{\vec{0}}, qF), \tag{13}$$ eq:695

and

$$\mathrm{Fun}(\Delta^1 \times \mathscr{C}^e, \mathscr{K}an_*) \to \mathscr{K}an_* \times_{\mathscr{K}an} \mathrm{Fun}(\Delta^1 \times \mathscr{C}^e, \mathscr{K}an), \quad \zeta \mapsto (\zeta|_{(0,\vec{0})}, q\zeta),$$

are trivial Kan fibrations.

Given any Hom functor $H : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{K}an$ (Definition II-12.4), we take the fiber of the trivial Kan fibration (13) at the pairing of H with a choice of element $* \to H(0,0)$ in the contractible space $H(0,0) \cong \mathrm{Hom}_{\mathscr{C}}(0,0)$ to obtain a contractible space of pointings for the functor H, as outlined precisely below.

Corollary 3.3. *For an ∞-category $\mathscr{C}$ equipped with a choice of zero object $0 : * \to \mathscr{C}$, Hom functor $H : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{K}an$, and point $1 : * \to H(0,0)$, there is a unique functor $\widetilde{H} : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{K}an_*$ which fits into a strictly commuting diagram*

$$\begin{tikzcd} & \mathscr{K}an_* \arrow[d] \\ \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \arrow[ur, "\widetilde{H}"] \arrow[r, "H"'] & \mathscr{K}an, \end{tikzcd}$$

and for which $\widetilde{H}(0,0) = (1 : * \to H(0,0))$.

One can show furthermore that there is a unique pointing for any given Hom functor.

thm:pointed_hom **Theorem 3.4.** *Let $\mathscr{C}$ be an ∞-category which admits a zero object. Then for any choice of Hom functor $H : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{K}an$, the space*

$$\mathrm{Fun}(\mathscr{C}^{\mathrm{op}} \times \mathscr{C}, \mathscr{K}an_*) \times_{\mathrm{Fun}(\mathscr{C}^{\mathrm{op}} \times \mathscr{C}, \mathscr{K}an)} \{H\}$$

of lifts, i.e. the space of functors $\widetilde{H} : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{K}an_*$ which fit into a strictly commuting diagram

$$\begin{tikzcd} & \mathscr{K}an_* \arrow[d] \\ \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \arrow[ur, "\widetilde{H}"] \arrow[r, "H"'] & \mathscr{K}an, \end{tikzcd}$$

is a contractible Kan complex.

Proof. Take again $\mathscr{C}^e = \mathscr{C}^{\mathrm{op}} \times \mathscr{C}$ and fix a zero object $0 : * \to \mathscr{C}$. The sequence

$$* \xrightarrow{H} \mathrm{Fun}(\mathscr{C}^e, \mathscr{K}an) \xrightarrow{\vec{0}^*} \mathscr{K}an$$

picks out the space $H(0,0)$ in $\mathscr{K}an$, so that the fiber

$$(\mathscr{K}an_* \times_{\mathscr{K}an} \mathrm{Fun}(\mathscr{C}^e, \mathscr{K}an)) \times_{\mathrm{Fun}(\mathscr{C}^e, \mathscr{K}an)} \{H\} = \mathscr{K}an_* \times_{\mathscr{K}an} \{H(0,0)\}$$

is the left pinched mapping space $\mathrm{Hom}^{\mathrm{L}}_{\mathscr{K}an}(*, H(0,0))$. Since $H(0,0)$ is contractible, it is terminal in $\mathscr{K}an$ by Lemma II-9.3. Hence this mapping space $\mathrm{Hom}^{\mathrm{L}}_{\mathscr{K}an}(*, H(0,0))$ is contractible.

Now, by the above information the fiber

$$\mathrm{Fun}(\mathscr{C}^e, \mathscr{K}an_*) \times_{\mathrm{Fun}(\mathscr{C}^e, \mathscr{K}an)} \{H\}$$

$$= \mathrm{Fun}(\mathscr{C}^e, \mathscr{K}an_*) \times_{(\mathscr{K}an_* \times_{\mathscr{K}an} \mathrm{Fun}(\mathscr{C}^e, \mathscr{K}an))} (\mathscr{K}an_* \times_{\mathscr{K}an} \mathrm{Fun}(\mathscr{C}^e, \mathscr{K}an)) \times_{\mathrm{Fun}(\mathscr{C}^e, \mathscr{K}an)} \{H\}$$

fits into a pullback square

$$\begin{tikzcd}
\mathrm{Fun}(\mathscr{C}^e, \mathscr{K}an_*) \times_{\mathrm{Fun}(\mathscr{C}^e, \mathscr{K}an)} \{H\} \arrow[r] \arrow[d] & \mathrm{Fun}(\mathscr{C}^e, \mathscr{K}an_*) \arrow[d] \\
\mathrm{Hom}^{\mathrm{L}}_{\mathscr{C}}(0, H(0,0)) \arrow[r] & \mathscr{K}an_* \times_{\mathscr{K}an} \mathrm{Fun}(\mathscr{C}^e, \mathscr{K}an),
\end{tikzcd}$$

where the right vertical map is a trivial Kan fibration by Proposition 3.2. Hence we have a trivial Kan fibration over a contractible space

$$\mathrm{Fun}(\mathscr{C}^e, \mathscr{K}an_*) \times_{\mathrm{Fun}(\mathscr{C}^e, \mathscr{K}an)} \{H\} \to \mathrm{Hom}^{\mathrm{L}}_{\mathscr{C}}(0, H(0,0)),$$

from which we conclude that the fiber under consideration is a contractible Kan complex. □

`def:pointed_hom`

Definition 3.5. Let $\mathscr{C}$ be an ∞-category which admits a zero object. A pointed Hom functor for $\mathscr{C}$ is a functor $\widetilde{H} : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{K}an_*$ whose composite

$$\mathscr{C}^{\mathrm{op}} \times \mathscr{C} \xrightarrow{\widetilde{H}} \mathscr{K}an_* \xrightarrow{forget} \mathscr{K}an$$

is a Hom functor for $\mathscr{C}$.

Let us consider now the twisted arrows fibration $\lambda : \mathscr{T}w(\mathscr{C}) \to \mathscr{C}^{\mathrm{op}} \times \mathscr{C}$ from Section II-12.1, and the space of associated transport functors with a witnessing data $\mathscr{T}(\lambda)$ (Definition II-6.13). By Theorem II-6.14 we understand that the space $\mathscr{T}(\lambda)$ is contractible. One can rasonably define the space of pointed Hom functors now as the fiber product

$$\mathrm{Fun}(\mathscr{C}^{\mathrm{op}} \times \mathscr{C}, \mathscr{K}an_*) \times_{\mathrm{Fun}(\mathscr{C}^{\mathrm{op}} \times \mathscr{C}, \mathscr{K}an)} \mathscr{T}(\lambda).$$

We observe uniqueness of pointed Hom functors in the absolute sense, i.e. without the specification of the underlying unpointed Hom functor.

`prop:pointed_hom_uniq`

Proposition 3.6. *For any ∞-category $\mathscr{C}$ which admits a zero object, the space*

$$\mathrm{Fun}(\mathscr{C}^{\mathrm{op}} \times \mathscr{C}, \mathscr{K}an_*) \times_{\mathrm{Fun}(\mathscr{C}^{\mathrm{op}} \times \mathscr{C}, \mathscr{K}an)} \mathscr{T}(\lambda)$$

of pointed Hom functors is a contractible Kan complex.

Proof. Take $\mathscr{C}^e = \mathscr{C}^{\mathrm{op}} \times \mathscr{C}$. Since the map $\mathscr{K}an_* \to \mathscr{K}an$ is a left fibration, and in particular an isofibration, the induced map $\mathrm{Fun}(\mathscr{C}^e, \mathscr{K}an_*) \to \mathrm{Fun}(\mathscr{C}^e, \mathscr{K}an)$ is an isofibration, by Corollary I-6.14. For any choice of Hom functor H with witnessing data the corresponding map $H : * \to \mathscr{T}(\lambda)$ is a homotopy equivalence, and so the induced map on fiber products

$$\mathrm{Fun}(\mathscr{C}^e, \mathscr{K}an_*) \times_{\mathrm{Fun}(\mathscr{C}^e, \mathscr{K}an)} \{H\} \to \mathrm{Fun}(\mathscr{C}^e, \mathscr{K}an_*) \times_{\mathrm{Fun}(\mathscr{C}^e, \mathscr{K}an)} \mathscr{T}(\lambda)$$

is an equivalence by Corollay I-6.24. Since the domain space for this functor is a contractible Kan complex, by Theorem 3.4, if follows that the space of pointed Hom functors is a contractible Kan complex as well. □

3.2. **Naturality for pointed Hom functors.**

thm:pointed_natural

Theorem 3.7. *Let $F : \mathscr{C}_0 \to \mathscr{C}_1$ be a functor between ∞-categories with zero objects, and suppose that F preserves zero objects. Take $\mathscr{C}_i^e = \mathscr{C}_i^{\mathrm{op}} \times \mathscr{C}_i$ and*

$$\mathscr{L} = \mathrm{Fun}(\partial\Delta^1 \times \mathscr{C}_0^e, \mathscr{K}an_*) \times_{\mathrm{Fun}(\partial\Delta^1 \times \mathscr{C}_0^e, \mathscr{K}an)} \mathrm{Fun}(\Delta^1 \times \mathscr{C}_0^e, \mathscr{K}an).$$

Let $\widetilde{H}_i : \mathscr{C}_i^e \to \mathscr{K}an_*$ be pointed Hom functors, H_i be the underlying unpointed Hom functors, and $H_F : H_0 \to H_1 F$ be the transformation induced by F (Definition II-8.5). The space

$$\mathrm{Fun}(\Delta^1 \times \mathscr{C}_0^e, \mathscr{K}an_*) \times_{\mathscr{L}} \{(\widetilde{H}_0, \widetilde{H}_1 F, H_F)\}$$

of transformations lifting H_F, i.e. transformations $\widetilde{H}_F : \Delta^1 \times \mathscr{C}_0^e \to \mathscr{K}an_*$ which fit into a strictly commuting diagram

$$\begin{tikzcd} & \mathscr{K}an_* \arrow[d] \\ \Delta^1 \times \mathscr{C}_0 \arrow[ur, "\widetilde{H}_F"] \arrow[r, "H_F"'] & \mathscr{K}an \end{tikzcd} \tag{14}$$

eq:810

and satisfy $\widetilde{H}_F|_{\{0\}} = \widetilde{H}_0$ and $\widetilde{H}_F|_{\{1\}} = \widetilde{H}_1 F$, is a contractible Kan complex.

Before giving the proof we record a useful lemma.

lem:tech_fiber

Lemma 3.8. *Suppose $\mathscr{E}$ is an ∞-category with an initial object $e : * \to \mathscr{E}$, and let $G : \mathscr{E} \to \mathscr{K}an$ be a functor for which the value $G(e)$ is contractible. Then the fiber product $\mathrm{Fun}(\mathscr{E}, \mathscr{K}an_*) \times_{\mathrm{Fun}(\mathscr{E}, \mathscr{K}an)} \{G\}$ is a contractible Kan complex.*

Proof. Same as the proof of Theorem 3.4. □

We now return to the matter at hand

Proof of Theorem 3.7. Given a zero object w in $\mathscr{C}$, the ∞-category $\Delta^1 \times \mathscr{C}_0^e$ has the initial object $(0, \vec{w})$. Furthermore, since the space $H_F(0, \vec{w}) = H_0(w, w)$ is contractible, Lemma 3.8 tells us that the fiber product

$$\mathrm{Fun}(\Delta^1 \times \mathscr{C}_0^e, \mathscr{K}an_*) \times_{\mathrm{Fun}(\Delta^1 \times \mathscr{C}_0^e, \mathscr{K}an)} \{H_F\}$$

is a contractible Kan complex. We can rewrite this fiber product as

$$\mathrm{Fun}(\Delta^1 \times \mathscr{C}_0^e, \mathscr{K}an_*) \times_{\mathscr{L}} (\mathscr{L} \times_{\mathrm{Fun}(\Delta^1 \times \mathscr{C}_0^e, \mathscr{K}an)} \{H_F\})$$

and note that the fiber product

$$\mathscr{L} \times_{\mathrm{Fun}(\Delta^1 \times \mathscr{C}_0^e, \mathscr{K}an)} \{H_F\} = \mathrm{Fun}(\partial^1\Delta^1 \times \mathscr{C}_0^e, \mathscr{K}an_*) \times_{\mathrm{Fun}(\partial\Delta^1 \times \mathscr{C}_0^e, \mathscr{K}an)} \{(H_0, H_1 F)\}$$

is again contractible by Lemma 3.8.

We consider the point

$$(\widetilde{H}_0, \widetilde{H}_1 F, H_F) : * \to \mathscr{L} \times_{\mathrm{Fun}(\Delta^1 \times \mathscr{C}_0^e, \mathscr{K}an)} \{H_F\},$$

which is now a homotopy equivalence of Kan complexes. Since the map $\mathrm{Fun}(\Delta^1 \times \mathscr{C}_0^{\mathrm{e}}, \mathscr{K}an_*) \to \mathscr{L}$ is an isofibration, by Proposition I-6.13, pulling back yields an equivalence

$$\begin{aligned}\mathrm{Fun}(\Delta^1 \times \mathscr{C}_0^{\mathrm{e}}, \mathscr{K}an_*) \times_{\mathscr{L}} \{(\widetilde{H}_0, \widetilde{H}_1 F, H_F)\} \\ \xrightarrow{\sim} \mathrm{Fun}(\Delta^1 \times \mathscr{C}_0^e, \mathscr{K}an_*) \times_{\mathrm{Fun}(\Delta^1 \times \mathscr{C}_0^e, \mathscr{K}an)} \{H_F\},\end{aligned}$$

by Corollary I-6.24. As we argued above, the target space here is contractible, so that the fiber $\mathrm{Fun}(\Delta^1 \times \mathscr{C}_0^e, \mathscr{K}an_*) \times_{\mathscr{L}} \{(\widetilde{H}_0, \widetilde{H}_1 F, H_F)\}$ is seen to be a contractible Kan complex as well. □

def:pointed_transf

Definition 3.9. Let $F : \mathscr{C}_0 \to \mathscr{C}_1$ be a functor between ∞-categories with zero objects, and suppose furthermore that F preserves zero objects. Then for pointed Hom functors $\widetilde{H}_i : \mathscr{C}_i^{\mathrm{op}} \times \mathscr{C} \to \mathscr{K}an_*$, the tranformation $\widetilde{H}_F : \widetilde{H}_0 \to \widetilde{H}_1 F$ induced by F is any transformation whose composite

$$\Delta^1 \times \mathscr{C}_0^{\mathrm{op}} \times \mathscr{C}_0 \xrightarrow{\widetilde{H}_F} \mathscr{K}an_* \xrightarrow{forget} \mathscr{K}an$$

recovers the transformation $H_F : H_0 \to H_1 F$ induced by F on the underlying unpointed Hom functors (in the sense of Definition II-8.5).

Remark 3.10. We have abused language in speaking of "the" induced transformation rather than "an" induced transformation. One notes, however, that the space parametrizing such choices is again contractible.

3.3. **Pointed spaces vs. pointed spaces.** Below we consider pointed Hom functors for homotopy coherent nerves of simplicial categories. When working with simplicial categories it is convenient to have a simplicial construction for the ∞-category $\mathscr{K}an_*$ of pointed spaces.

We consider the simplicial category $\underline{\mathrm{Kan}}_*$ whose objects are Kan complexes $\mathscr{X}$ with a fixed point $x : * \to \mathscr{X}$, and whose morphism complexes are the fibers

$$\mathrm{Fun}_{*/}((\mathscr{X}, x), (\mathscr{Y}, y)) := \mathrm{Fun}(\mathscr{X}, \mathscr{Y}) \times_{\mathrm{Fun}(*, \mathscr{Y})} \{y\}. \tag{15}$$ eq:890

In particular, we have $\mathrm{Fun}_{*/}(*, (\mathscr{Y}, y)) = *$, and one sees that the point $*$ is both initial and terminal in the simplicial category $\underline{\mathrm{Kan}}_*$. Note also that the the functor spaces $\mathrm{Fun}(\mathscr{X}, \mathscr{Y})$ are Kan complexes, and that restriction along the point $x^* : \mathrm{Fun}(\mathscr{X}, \mathscr{Y}) \to \mathrm{Fun}(*, \mathscr{Y}) = \mathscr{Y}$ is a Kan fibration. Hence the fiber (15) is always a Kan complex.

Due to triviality of the mapping spaces from the one point space, one sees that any functor $\mathrm{Path}\,\Delta^n \to \underline{\mathrm{Kan}}_*$ extends uniquely to a functor $\mathrm{Path}(\{-1\} \star \Delta^n) \to \underline{\mathrm{Kan}}_*$ whose value at -1 is $*$. So we see that the left fibration

$$\mathrm{N}^{\mathrm{hc}}(\underline{\mathrm{Kan}}_*)_{*/} \to \mathrm{N}^{\mathrm{hc}}(\underline{\mathrm{Kan}}_*) \tag{16}$$ eq:1233

is an isomorphism of simplicial sets. We also have the map

$$\mathrm{N}^{\mathrm{hc}}(\underline{\mathrm{Kan}}_*) \to \mathrm{N}^{\mathrm{hc}}(\underline{\mathrm{Kan}}) = \mathscr{K}an$$

induced by the forgetful functor $\underline{\mathrm{Kan}}_* \to \underline{\mathrm{Kan}}$.

From the above information we obtain a unique functor $\rho : \mathrm{N}^{\mathrm{hc}}(\underline{\mathrm{Kan}}_*) \to \mathscr{K}an_*$ which fits into a strictly commuting diagram

$$\begin{tikzcd} \mathrm{N}^{\mathrm{hc}}(\underline{\mathrm{Kan}}_*)_{*/} \arrow[r, "forget_{*/}"] \arrow[d] & \mathscr{K}an_* \arrow[d] \\ \mathrm{N}^{\mathrm{hc}}(\underline{\mathrm{Kan}}_*) \arrow[ur, "\rho"] \arrow[r, "forget"'] & \mathscr{K}an. \end{tikzcd} \tag{17}$$ eq:1055

As the forgetful functor from $\mathrm{N}^{\mathrm{hc}}(\underline{\mathrm{Kan}}_*)$ to $\mathscr{K}an$ is an injective map of simplicial sets, the completing map ρ is injective as well.

prop:ptd_id **Proposition 3.11** ([16, 0200]). *The inclusion $\rho : \mathrm{N}^{\mathrm{hc}}(\underline{\mathrm{Kan}}_*) \to \mathscr{K}an_*$ is an equivalence of ∞-categories.*

We refer the reader to the text [16] for the details.

Remark 3.12. The image of the functor ρ consists of those n-simplices in $\mathscr{K}an_*$, i.e. those maps $\{-1\} \star \Delta^n \to \mathscr{K}an$, whose restriction $\{-1\} \star \mathrm{Sk}_1(\Delta^n) \to \mathscr{K}an$ strictly commutes.

sect:simp_ptd_hom 3.4. **Pointed Hom functors via the homotopy coherent nerve.** Let $\underline{A}$ be a Kan-enriched category with a strict zero object 0, in the sense that the Hom complexes to and from 0 are just points

$$* = \underline{\mathrm{Hom}}_{\underline{A}}(x, 0) = \underline{\mathrm{Hom}}_{\underline{A}}(0, x).$$

Then the simplicial Hom functor $\underline{\mathrm{Hom}}_{\underline{A}} : \underline{A}^{\mathrm{op}} \times \underline{A} \to \underline{\mathrm{Kan}}$ admits a natural pointing via the 0 morphisms $0 : * \to \underline{\mathrm{Hom}}_{\underline{A}}(x, y)$,

$$\underline{\mathrm{Hom}}_{\underline{A}} : \underline{A}^{\mathrm{op}} \times \underline{A} \to \underline{\mathrm{Kan}}_*.$$

For $\mathscr{A} = \mathrm{N}^{\mathrm{hc}}(\underline{A})$ and $\underline{\mathrm{Hom}}_{\mathscr{A}} = \mathrm{N}^{\mathrm{hc}}\, \underline{\mathrm{Hom}}_{\underline{A}}$ we therefore obtain a functor

$$\underline{\mathrm{Hom}}_{\mathscr{A}} : \mathscr{A}^{\mathrm{op}} \times \mathscr{A} \to \mathrm{N}^{\mathrm{hc}}(\underline{\mathrm{Kan}}_*).$$

We compose with the equivalence $\rho : \mathrm{N}^{\mathrm{hc}}(\underline{\mathrm{Kan}}_*) \to \mathscr{K}an_*$ to obtain a pointed Hom functor for the homotopy coherent nerve $\mathscr{A}$, via commutativity of the diagram (17). We refer to this functor as the canonical pointed Hom functor for $\mathscr{A} = \mathrm{N}^{\mathrm{hc}}(\underline{A})$.

Proposition 3.13. *If $\underline{A}$ is a simplicial category with a strict zero object, and $\mathscr{A} = \mathrm{N}^{\mathrm{hc}}(\mathscr{A})$, then the canonical pointed Hom functor*

$$\underline{\mathrm{Hom}}_{\mathscr{A}} : \mathscr{A}^{\mathrm{op}} \times \mathscr{A} \to \mathscr{K}an_*$$

constructed above is a pointed Hom functor for $\mathscr{A} = \mathrm{N}^{\mathrm{hc}}(\underline{A})$.

Proof. By commutativity of the diagram (17). composing $\underline{\mathrm{Hom}}_{\mathscr{A}}$ with the forgetful functor to $\mathscr{K}an$ recovers the standard unpointed Hom functor from Section II-12.4. So, simply by defintion, $\underline{\mathrm{Hom}}_{\mathscr{A}}$ is a pointed Hom functor for $\mathscr{A}$. □

We are of course most interested in the case of the homotopy ∞-category $\mathscr{K}(\mathbb{A})$. Here we consider the simplicial category $K\mathbf{Ch}(\mathbb{A})$ which is strictly pointed via the zero complex. We then have the homotopy coherent nerve $\mathscr{K}(\mathbb{A})' = \mathrm{N}^{\mathrm{hc}}(K\mathbf{Ch}(\mathbb{A}))$ which is now equipped with its pointed Hom functor

$$\underline{\mathrm{Hom}}_{\mathscr{K}'} : \mathscr{K}(\mathbb{A})'^{\mathrm{op}} \times \mathscr{K}(\mathbb{A}) \to \mathscr{K}an_*.$$

We transfer this pointed Hom functor to $\mathscr{K}(\mathbb{A})$ via the equivalence $\mathfrak{Z} : \mathscr{K}(\mathbb{A})' \to \mathscr{K}(\mathbb{A})$ of Theorem II-10.4, and restrict to obtain a pointed Hom functor for the derived ∞-category $\mathscr{D}(\mathbb{A}) \subseteq \mathscr{K}(\mathbb{A})$ as well.

3.5. **Homotopy groups via $\mathscr{K}an_*$.** Of course, analyses of mapping spaces via their homotopy groups play an essential role in assessing fully faithfulness of functors. To conclude the section we show that homotopy groups of pointed spaces can be calculated via processes which are internal to the ∞-category $\mathscr{K}an_*$.

Let us recall now the geometric realization functor $|-| : \mathrm{sSet} \to \mathrm{Top}$, which is left adjoint to the singular complex functor $\mathrm{Sing} : \mathrm{Top} \to \mathrm{sSet}$. Being a left adjoint, the functor $|-|$ commutes with colimits, and so one calculates

$$|K| = \mathrm{colim}_{n,\ \Delta^n \to K} |\Delta^n|.$$

We have the following fundamental result of Milnor.

thm:milnor

Theorem 3.14 ([18])**.** *For any simplicial set K, the unit map $K \to \mathrm{Sing}\,|K|$ is a weak homotopy equivalence.*

Consequently, for any simplicial set K, we observe an equivalence of Kan complexes $\mathrm{Fun}(\mathrm{Sing}\,|K|, \mathscr{Y}) \to \mathrm{Fun}(K, \mathscr{Y})$ at arbitrary $\mathscr{Y}$ in $\mathscr{K}an$. In the pointed setting, restriction along the marked point $k : * \to K$ provides a Kan fibration

$$k^* : \mathrm{Fun}(K, \mathscr{Y}) \to \mathrm{Fun}(*, \mathscr{Y}) = \mathscr{Y}$$

by Proposition I-3.11 so that taking the fiber provides an equivalence

$$\mathrm{Fun}_{*/}(\mathrm{Sing}\,|K|, \mathscr{Y}) \to \mathrm{Fun}_{*/}(K, \mathscr{Y}).$$

As a corollary to Proposition 3.11 we now observe an identification of homotopy groups via mapping spaces in the ∞-category $\mathscr{K}an_*$.

cor:pi_ptd

Corollary 3.15. *Take $\mathbb{S}^n = \mathrm{Sing}\,|\Delta^n/\partial\Delta^n|$. For any pointed space $x : * \to \mathscr{X}$, there is a natural bijection*

$$\pi_n(\mathscr{X}, x) \xrightarrow{\cong} \pi_0 \operatorname{Hom}_{\mathscr{K}an_*}(\mathbb{S}^n, \mathscr{X}).$$

Proof. Via Theorem 3.14 restricting along the unit map $\Delta^n/\partial\Delta^n \to \mathbb{S}^n$ provides an equivalence $\mathrm{Fun}_{*/}(\mathbb{S}^n, \mathscr{X}) \xrightarrow{\sim} \mathrm{Fun}_{*/}(\Delta^n/\partial\Delta^n, \mathscr{X})$. We have also the natural equivalence

$$\mathrm{Fun}_{*/}(\mathbb{S}^n, \mathscr{X}) \xrightarrow{\sim} \operatorname{Hom}_{\mathrm{N}^{\mathrm{hc}}(\underline{\mathrm{Kan}}_*)}(\mathbb{S}^n, \mathscr{X}) \xrightarrow{\rho} \operatorname{Hom}_{\mathscr{K}an_*}(\mathbb{S}^n, \mathscr{X})$$

which provides a roof of equivalences

$$\mathrm{Fun}_{*/}(\Delta^n/\partial\Delta^n, \mathscr{X}) \leftarrow \mathrm{Fun}_{*/}(\mathbb{S}^n, \mathscr{X}) \to \operatorname{Hom}_{\mathscr{K}an_*}(\mathbb{S}^n, \mathscr{X}).$$

Taking connected components therefore provides an isomorphism

$$\begin{tikzcd}
\pi_0 \mathrm{Fun}_{*/}(\mathbb{S}^n, \mathscr{X}) \arrow[r, "\cong"] \arrow[d, "\cong"'] & \pi_0 \operatorname{Hom}_{\mathscr{K}an_*}(\mathbb{S}^n, \mathscr{X}) \\
\pi_n(\mathscr{X}) = \pi_0 \mathrm{Fun}_{*/}(\Delta^n/\partial\Delta^n, \mathscr{X}) \arrow[ur, dashed, "\cong", "\exists !"'] &
\end{tikzcd}$$

which is natural in the $\mathscr{X}$ coordinate. $\square$

sect:pointed_space

4. Preliminaries III: Pointed spaces

Before continuing with our analysis we take a moment to study the category of pointed spaces itself. Of particular interest are limits and colimits in $\mathscr{K}an_*$, and the behaviors of limits and colimits under forgetting $\mathscr{K}an_* \to \mathscr{K}an$ to the ∞-category of unpointed spaces.

4.1. **Limits and colimits of pointed spaces.** Our first order of business is to show that the category $\mathscr{K}an_*$ is complete. We begin with a supporting lemma.

lem:875

Lemma 4.1. *Let $\mathscr{C}$ be an ∞-category and $\mathscr{C}_{\mathrm{Term}}$ be the full ∞-subcategory of terminal objects in $\mathscr{C}$. The category $\mathscr{C}_{\mathrm{Term}}$ is complete and the inclusion $\mathscr{C}_{\mathrm{Term}} \to \mathscr{C}$ preserves limits.*

Proof. We have $\mathscr{C}_{\mathrm{Term}} \cong *$, by Lemma II-9.2, so that it is both complete and cocomplete. To see that the inclusion $\mathscr{C}_{\mathrm{Term}} \to \mathscr{C}$ is continuous, it suffices to show that the constant diagram $\underline{t} : K \to \mathscr{C}$ at a given terminal object t is terminal in the full subcategory of constant diagrams in $\mathrm{Fun}(K, \mathscr{C})$. This just follows by the definition of K shaped limits in $\mathscr{C}$ via the functor category (Definition II-13.1).

We have that the space

$$\mathrm{Fun}(\Delta^1, \mathscr{C}) \times_{\mathrm{Fun}(\partial\Delta^1, \mathscr{C})} \{(x,t)\} = \mathrm{Hom}_{\mathscr{C}}(x,t)$$

is contractible at each x in $\mathscr{C}$, by definition, so that the functor space

$$\begin{aligned}\mathrm{Fun}(K, \mathrm{Hom}_{\mathscr{C}}(x,t)) &= \mathrm{Fun}(\Delta^1 \times K, \mathscr{C}) \times_{\mathrm{Fun}(\partial\Delta^1 \times K, \mathscr{C})} \{(\underline{x}, \underline{t})\} \\ &= \mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{C})}(\underline{x}, \underline{t})\end{aligned}$$

is contractible as well. Hence $\underline{t}$ is terminal in $\mathrm{Fun}(K, \mathscr{C})$, and we see that the map

$$* \cong \mathrm{Hom}_{\mathscr{C}}(x,t) \to \mathrm{Hom}_{\mathrm{Fun}}(\underline{x}, \underline{t}) \overset{(id_{\underline{t}})_*}{\to} \mathrm{Hom}_{\mathrm{Fun}}(\underline{x}, \underline{t}) \cong *$$

is an equivalence at each x in $\mathscr{C}$. Therefore t is a limit for its own constant diagram. $\square$

prop:pointed_ccpt

Proposition 4.2. (a) *The category $\mathscr{K}an_*$ is complete.*

(b) *The forgetful functor $\mathscr{K}an_* \to \mathscr{K}an$ is continuous.*

(c) *A diagram $\{*\} \star K \to \mathscr{K}an_*$ is a limit diagram if and only if its image in $\mathscr{K}an$ is a limit diagram.*

Proof. Via the coslice equivalence

$$\mathscr{K}an_* \overset{\sim}{\to} \{*\} \times^{\mathrm{or}}_{\mathscr{K}an} \mathscr{K}an$$

of Theorem I-10.15 it suffices to show that the oriented fiber product $\{*\} \times^{\mathrm{or}}_{\mathscr{K}an} \mathscr{K}an$ is complete and that the projection to $\mathscr{K}an$ both preserves and detects limits.

By Proposition II-13.29 the category $\mathrm{Fun}(\Delta^1, \mathscr{K}an)$ is complete, and a diagram $p : \{*\} \star K \to \mathrm{Fun}(\Delta^1, \mathscr{K}an)$ is a limit diagram if and only if it evaluates to a limit diagram in $\mathscr{K}an$ at both 0 and 1. Supposing that 0^*p takes constant value $*$, the evaluation at 0 is already a limit diagram by Lemma 4.1. This shows that the fiber

$$\{*\} \times_{\mathrm{Fun}(\{0\}, \mathscr{K}an)} \mathrm{Fun}(\Delta^1, \mathscr{K}an) = \{*\} \times^{\mathrm{or}}_{\mathscr{K}an} \mathscr{K}an$$

is a cocomplete subcategory in $\mathrm{Fun}(\Delta^1, \mathscr{K}an)$, and that the projection to $\mathscr{K}an$, i.e. the evaluation at 1 in Δ^1, both preserves and detects limits in the oriented fiber product. $\square$

Recall that a simplicial set K is said to be weakly contractible if the terminal map $K \to *$ induces an equivalence

$$\mathscr{X} = \mathrm{Fun}(*, \mathscr{X}) \to \mathrm{Fun}(K, \mathscr{X})$$

at each Kan complex $\mathscr{X}$. We record the following result without proof.

prop:03PK **Proposition 4.3** ([16, 03PK])**.** *The class of weakly contractible simplicial sets is closed under the formation of filtered colimits in* sSet*.*

We also understand from Proposition II-9.24 that any ∞-category with an initial or terminal object is weakly contractible. We therefore observe the following.

prop:932 **Proposition 4.4.** *The class of weakly contractible simplicial sets contains Δ^{op} as well as all filtered ∞-categories (Definition 11.2).*

Proof. The category Δ^{op} has the initial object $[0]$ and is thus weakly contractible by Proposition II-9.24. For a filtered ∞-category $\mathscr{K}$, the inclusion of any finite subset $K \to \mathscr{K}$ extends to a map of simplicial sets $K \star \{1\} \to \mathscr{K}$. Furthermore each inclusion $\{1\} \to K \star \{1\}$ is right anodyne, by Lemma II-9.23, so that $K \star \{1\}$ is weakly contractible. Hence $\mathscr{K}$ can be written as a filtered colimit of weakly contractible simplicial sets, and is thus weakly contractible by Proposition 4.3. $\square$

Proposition 4.5 ([16, 02KR])**.** *Let $q : \mathscr{E} \to \mathscr{C}$ be any left fibration of ∞-categories, K be a weakly contractible simplicial set, and suppose that $\mathscr{C}$ admits all K-indexed colimits.*

1. *$\mathscr{E}$ admits all K-indexed colimits.*
2. *A diagram $p : K \star \{1\} \to \mathscr{E}$ is a colimit diagram if and only if the composite $qp : K \star \{1\} \to \mathscr{C}$ is a colimit diagram.*

We only outline the proof.

Outline of proof. Let $\bar{p} : K \to \mathscr{E}$ be a diagram. Take $\bar{p}_0 = q\bar{p} : K \to \mathscr{C}$. By a general result [16, 0179] the inclusion $K \to K \star \{1\}$ is left anodyne, so that any colimit diagram $p_0 : K \star \{1\} \to \mathscr{C}$ lifts uniquely to a diagram $p : K \star \{1\} \to \mathscr{E}$ whose restriction to K recovers $\bar{p}$. One applies [16, 02KR] (see also [16, 02KN]) to find that p is a colimit in $\mathscr{E}$.

We now have that $\mathscr{E}$ admits the proposed colimits and that q preserves colimits of the given type. Uniqueness of solutions to the the lifting problems

$$\begin{tikzcd} K \arrow[r] \arrow[d] & \mathscr{E} \arrow[d, "q"] \\ K \star \{1\} \arrow[r] \arrow[ur, dotted] & \mathscr{C} \end{tikzcd}$$

however, which we observe from [16, 0179] and Proposition I-3.11, implies that q detects such colimits in $\mathscr{E}$ as well. $\square$

We apply this result to our favorite left fibration $\mathscr{K}an_* \to \mathscr{K}an$.

Corollary 4.6. *For any weakly contractible simplicial set K, the category $\mathscr{K}an_*$ admits all K-indexed colimits, and a diagram $K \star \{1\} \to \mathscr{K}an_*$ is a colimit diagram if and only if the composite $K \star \{1\} \to \mathscr{K}an_* \to \mathscr{K}an$ is a colimit diagram in $\mathscr{K}an$.*

By Proposition 4.4 we specifically observe the following.

cor:ptd_geom

Corollary 4.7. *The category $\mathscr{K}an_*$ admits geometric realizations, filtered colimits, and pushouts. Furthermore, the forgetful functor $\mathscr{K}an_* \to \mathscr{K}an$ both preserves and detects such colimits in $\mathscr{K}an_*$.*

Proof. All is clear save for pushouts. However, this follows from the fact that the diagram $K = \Lambda^2_0$ admits an initial object, and is therefore weakly contractible (Proposition II-9.24). □

Remark 4.8. We see below, in Theorem 4.13, that the category $\mathscr{K}an_*$ in fact admits all small colimits, i.e. is cocomplete. It is not the case, however, that the forgetful functor $\mathscr{K}an_* \to \mathscr{K}an$ preserves general colimits. Indeed, the coproduct $* \amalg *$ in the category $\mathscr{K}an_*$ is just a point, by Lemma 4.12 below, while in $\mathscr{K}an$ it is the disjoint union of two points.

From the above information we see that one can detect limits and colimits in a pointed ∞-category via pointed Hom functors, in addition to the unpointed Hom functors.

prop:lim_ptd

Proposition 4.9. *Let $\mathscr{C}$ be an ∞-category with a zero object, and $\widetilde{H} : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{K}an_*$ be a pointed* Hom *functor for $\mathscr{C}$. For a small diagram $p : \{*\} \star K \to \mathscr{C}$ the following are equivalent:*

1. *p is a limit diagram in $\mathscr{C}$.*
2. *At each object x in $\mathscr{C}$, the diagram $\widetilde{H}(x, -)p : \{*\} \star K \to \mathscr{K}an_*$ is a limit diagram in $\mathscr{K}an_*$.*

Similarly, for a diagram $q : K \star \{*\} \to \mathscr{C}$ the following are equivalent:

(1′) *q is a colimit diagram in $\mathscr{C}$.*

(2′) *At each object y in $\mathscr{C}$, the diagram $\widetilde{H}(-, y)q : \{*\} \star K^{\mathrm{op}} \to \mathscr{K}an_*$ is a limit diagram in $\mathscr{K}an_*$.*

Proof. Follows immediately from Proposition 4.2 and Corollary II-16.16. □

We similarly observe the following.

prop:1210

Proposition 4.10. *Let $\mathscr{C}$ be an ∞-category with a zero object, $\widetilde{H} : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{K}an_*$ be a pointed Hom functor for $\mathscr{C}$, and H be the underlying unpointed Hom functor. Suppose K is filtered, or that $K = \Delta^{\mathrm{op}}$. At a given object $x : * \to \mathscr{C}$, the functor $\widetilde{H}(x, -) : \mathscr{C} \to \mathscr{K}an_*$ preserves K-indexed colimits if and only if the functor $H(x, -) : \mathscr{C} \to \mathscr{K}an$ preserves K-indexed colimits.*

Remark 4.11. Proposition 4.10 is relevant when assessing compactness and/or projectivity of objects in an ∞-category, in the sense of [14, Definition 5.3.4.5, 5.5.8.18].

4.2. **Completeness and cocompleteness of $\mathscr{K}an_*$.** We can also use Proposition 3.11 to see that the category of pointed spaces is cocomplete. We first observe the existence of arbitrary coproducts. In the statement of the following lemma we write, for any collection of pointed spaces $\{(\mathscr{X}_\lambda, x_\lambda) : \lambda \in \Lambda\}$,

$$\mathscr{X}_\Lambda = \coprod_{\lambda \in \Lambda} \mathscr{X}_\lambda \quad \text{and} \quad \mathscr{X}_\Lambda / * = \mathscr{X}_\Lambda / (\coprod_{\lambda \in \Lambda} x_\lambda).$$

lem:ptd_coprod

Lemma 4.12. *The category $\mathscr{K}an_*$ admits all small coproducts. Specificially, for Λ a small discrete set and $\mathscr{X}_? : \Lambda \to \mathscr{K}an_*$ any functor, the maps*

$$i_\lambda = \big(\ \mathscr{X}_\lambda \to \mathscr{X}_\Lambda \to \mathrm{Sing}\,|\mathscr{X}_\Lambda / *\,|\ \big)$$

realizes tha space $\mathrm{Sing}\,|\mathscr{X}_\Lambda/*|$ *as a colimit for the functor* $\mathscr{X}_?$.

Proof. Take $\mathscr{X}_{\Lambda,*} = \mathrm{Sing}\,|(\coprod_\lambda \mathscr{X}_\lambda)/*|$. Via the equivalence $\mathrm{N}^{\mathrm{hc}}(\underline{\mathrm{Kan}_*}) \cong \mathscr{K}an_*$ of Proposition 3.11 it suffices to show that the given morphism is a coproduct in the nerve $\mathrm{N}^{\mathrm{hc}}(\underline{\mathrm{Kan}_*})$. From Proposition II-11.6 and Corollary II-16.17 it then suffices to prove that the map

$$[i_\lambda^*;\lambda\in\Lambda]^t : \mathrm{Fun}_{*/}(\mathscr{X}_{\Lambda,*},\mathscr{Y}) \to \prod_\lambda \mathrm{Fun}_{*/}(\mathscr{X}_\lambda,\mathscr{Y})$$

is a homotopy equivalence at all pointed spaces $\mathscr{Y}$. As the inclusion $\mathscr{X}_\Lambda/* \to \mathscr{X}_{\Lambda,*}$ is a weak homotopy equivalence [16, 0142], and the structure maps for $\mathscr{X}_{\Lambda,*}$ factor through $\mathscr{X}_\Lambda/*$, it then suffices to prove that the map

$$[\mathrm{incl}_\lambda^*;\lambda\in\Lambda]^t : \mathrm{Fun}_{*/}(\mathscr{X}_\Lambda/*,\mathscr{Y}) \to \prod_\lambda \mathrm{Fun}_{*/}(\mathscr{X}_\lambda,\mathscr{Y}) \tag{18}$$

eq:1072

is an equivalence.

The space $\mathrm{Fun}_{*/}(\mathscr{X}_\Lambda/*,\mathscr{Y})$ completes a pullback square

$$\begin{tikzcd}
\mathrm{Fun}_{*/}(\mathscr{X}_\Lambda/*,\mathscr{Y}) \arrow[r] \arrow[d] & \mathrm{Fun}(\mathscr{X}_\Lambda/*,\mathscr{Y}) \arrow[d, "x_\Lambda^*"] \\
* \arrow[r, "y_*"'] & \mathscr{Y}
\end{tikzcd}$$

and we have the additional pullback square

$$\begin{tikzcd}
\mathrm{Fun}(\mathscr{X}_\Lambda/*,\mathscr{Y}) \arrow[r] \arrow[d] & \mathrm{Fun}(\mathscr{X}_\Lambda,\mathscr{Y}) \arrow[d, "(\amalg_\lambda x_\lambda)^*"] \\
\mathscr{Y} \arrow[r, "\mathrm{diag}"'] & \prod_\lambda \mathscr{Y}.
\end{tikzcd}$$

We therefore observe a pullback square

$$\begin{tikzcd}
\mathrm{Fun}_{*/}(\mathscr{X}_\Lambda/*,\mathscr{Y}) \arrow[r] \arrow[d] & \mathrm{Fun}(\mathscr{X}_\Lambda,\mathscr{Y}) \arrow[d, "(\amalg_\lambda x_\lambda)^*"] \\
* \arrow[r, "\prod_\lambda y_*"'] & \prod_\lambda \mathscr{Y}.
\end{tikzcd}$$

We similarly have a pullback square

$$\begin{tikzcd}
\prod_\lambda \mathrm{Fun}_{*/}(\mathscr{X}_\lambda,\mathscr{Y}) \arrow[r] \arrow[d] & \prod_\lambda \mathrm{Fun}(\mathscr{X}_\lambda,\mathscr{Y}) \arrow[d, "\prod_\lambda x_\lambda^*"] \\
* \arrow[r, "\prod_\lambda y_*"'] & \prod_\lambda \mathscr{Y}
\end{tikzcd}$$

so that the unpointed isomorphism

$$[\mathrm{incl}_\lambda^*;\lambda\in\Lambda]^t : \mathrm{Fun}(\mathscr{X}_\Lambda/*,\mathscr{Y}) \overset{\cong}{\to} \prod_\lambda \mathrm{Fun}(\mathscr{X}_\lambda,\mathscr{Y})$$

induces an isomorphism $\mathrm{Fun}_{*/}(\mathscr{X}_\Lambda/*,\mathscr{Y}) \overset{\cong}{\to} \prod_\lambda \mathrm{Fun}_{*/}(\mathscr{X}_\lambda,\mathscr{Y})$ which is explicitly given by the map (18). $\square$

thm:ptd_co_comp

Theorem 4.13. *The ∞-category $\mathscr{K}an_*$ is both complete and cocomplete.*

Proof. Completeness was covered in Proposition 4.2 above. For cocompleteness, we know from Corollary 4.7 and Lemma 4.12 that $\mathscr{K}an_*$ also admits pushouts and small coproducts. It follows from [14, Proposition 4.4.2.6] that $\mathscr{K}an_*$ admits all small colimit. $\square$

sect:spectra_hom

5. Preliminaries IV: Spectral Hom functors

We show that if a pointed ∞-category $\mathscr{C}$ is suffiently symmetric about the zero objects–in particular stable, see Definiiton 6.5–then any pointed Hom functor for $\mathscr{C}$ lifts to a functor valued in spectra. In fact, a pointed Hom functor for $\mathscr{C}$ is more-or-less the same thing as a spectra valued Hom functor. Of course, our first order of business here is to define these strange figures, "spectra".

Now, to be clear, one can provide a coherent treatment of the derived ∞-category without ever speaking of spectra. Furthermore, in this text we avoid directly employing spectra in our arguments. However, avoiding spectra while engaging with the literature on this topic is not practical. So, we provide an abridged discussions here and leave it to the interested reader to connect the dots between our treatment and some of the more spectra-forward treatments in the literature. (Somewhat surprisingly, it's not at all difficult to connect the required dots when needed.)

Remark 5.1. Our presentation in this section is fairly coarse, as we rely on technical results from both higher topos theory [14] and higher algebra [15] in order to "digest" many of the details. The reader won't be harmed in just skimming the contents.

5.1. **Introduction to spectra.** In short, the ∞-category of spectra $\mathscr{S}p$ is the localization of the category of pointed spaces $\mathscr{S}p = \mathscr{K}an_*[\Omega^{-1}]$ relative to the the looping functor. (See Section I-7.4 and the discussions preceding [15, Remark 1.1.2.6].) In order to meet the demands of actually doing mathematics however, we take a moment to describe this ∞-category in more detail.

We first consider the full subcategory $\mathscr{K}an_*^{\mathrm{fin}}$ of $\mathscr{K}an_*$ which is generated by the one point space $*$ under finite colimits. We note that we have pushout diagrams

$$\begin{tikzcd} \mathbb{S}^n \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \mathbb{S}^{n+1} \end{tikzcd} \tag{19}$$

eq:spush

at all n, so that all spheres appear in $\mathscr{K}an_*^{\mathrm{fin}}$. (For details see Example 6.19.) We define the category of spectra as the full subcategory of so-called reduced excisive functors from $\mathscr{K}an_*^{\mathrm{fin}}$ to $\mathscr{K}an_*$.

Definition 5.2. Let $F : \mathscr{K} \to \mathscr{C}$ be a functor between ∞-categories, and suppose that $\mathscr{K}$ admits all finite colimits and a terminal object k. We call F excisive if it sends pushout diagrams in $\mathscr{K}$ to pullback diagrams in $\mathscr{C}$. We call F reduced if $F(k)$ is terminal in $\mathscr{C}$.

For $\mathscr{K}$ and $\mathscr{C}$ as above, we let $\mathrm{Exc}_*(\mathscr{K}, \mathscr{C})$ denote the full ∞-subcategory in $\mathrm{Fun}(\mathscr{K}, \mathscr{C})$ spanned by those functors which are both excisive and reduced. We are particularly interested in the case $\mathscr{K} = \mathscr{K}an_*^{\mathrm{fin}}$.

Definition 5.3. For any ∞-category $\mathscr{C}$, the ∞-category of spectrum objects in $\mathscr{C}$ is defined as the ∞-category of reduced excisive functors from finite pointed spaces

$$\mathscr{S}\!p(\mathscr{C}) = \mathrm{Exc}_*(\mathscr{K}an_*^{\mathrm{fin}}, \mathscr{C}).$$

In the particular case $\mathscr{C} = \mathscr{K}an_*$ we take

$$\mathscr{S}\!p := \mathscr{S}\!p(\mathscr{K}an_*) = \mathrm{Exc}_*(\mathscr{K}an_*^{\mathrm{fin}}, \mathscr{K}an_*).$$

For any object $\mathscr{X}_* : \mathscr{K}an_*^{\mathrm{fin}} \to \mathscr{K}an_*$ in the category of spectra, we consider the values of this functor on the n-spheres $\mathscr{X}_n := \mathscr{X}_*(\mathbb{S}^n)$. From the pushout diagram (19) and excisiveness of $\mathscr{X}_*$, we have identifications with the loop spaces $\mathscr{X}_n = \Omega\mathscr{X}_{n+1}$ at each nonnegative integer n.

Definition 5.4. For each nonnegative integer n we take $\Omega^{\infty-n} : \mathscr{S}\!p \to \mathscr{K}an_*$ the evaluation functor at the n-sphere $\mathbb{S}^n$, and take in particular $\Omega^\infty := \Omega^{\infty-0}$.

Proposition 5.5. *The category $\mathscr{S}\!p$ is complete and cocomplete, and the forgetful functor $\Omega^\infty : \mathscr{S}\!p \to \mathscr{K}an_*$ is continuous.*

Sketch proof. The category $\mathscr{K}an$ is generated by the subcategory of finite discrete sets under geometric realization. In particular, any space $\mathscr{X}$ is the colimit of its own simplicial functor $\mathscr{X} = \mathrm{colim}_{\Delta^{\mathrm{op}}} \mathscr{X}[n]$ and, under this identification, the structure map $\mathscr{X}[0] \to \mathscr{X}$ is the natural inclusion. Since the forgetful functor $\mathscr{K}an_* \to \mathscr{K}an$ preserves geometric realization (Corollary 4.7) we see similarly that $\mathscr{K}an_*$ is generated by the subcategory of finite pointed sets under geometric realization. In particular, the category $\mathscr{K}an_*$ is seen to be presentable [14, Section 5.0]. It follows that the category $\mathscr{S}\!p$ is presentable as well [15, Proposition 1.4.4.4], and thus complete and cocomplete, and also that the functor $\Omega^\infty : \mathscr{S}\!p \to \mathscr{K}an_*$ admits a left adjoint [15, Proposition 1.4.4.4]. The functor Ω^∞ is therefore continuous by Proposition II-13.24. $\square$

For ∞-categories $\mathscr{C}$ and $\mathscr{D}$ which admit finite limits, we consider the ∞-category $\mathrm{Fun}^{\mathrm{lex}}(\mathscr{C}, \mathscr{D})$ of left exact functors. By definition, this is the full subcategory in $\mathrm{Fun}(\mathscr{C}, \mathscr{D})$ whose objects are functors which preserve finite limits.

prop:spectra_lift

Proposition 5.6 ([15, Corollary 1.4.2.23])**.** *Suppose $\mathscr{C}$ is a pointed ∞-category which admits finite limits, and that pushout diagrams agree with pullback diagrams in $\mathscr{C}$. Then the functor*

$$\Omega^\infty_* : \mathrm{Fun}^{\mathrm{lex}}(\mathscr{C}, \mathscr{S}\!p) \to \mathrm{Fun}^{\mathrm{lex}}(\mathscr{C}, \mathscr{K}an_*)$$

is an equivalence of ∞-categories.

Now, one can show further that any spectrum is determined by its values on the n-spheres. To elaborate on this point, let us take $\mathscr{L}\mathscr{K}an_*$ the limit of the diagram $\omega : \mathbb{Z}_{\le 0} \to \mathscr{C}at_\infty^{\mathrm{big}}$,

$$\mathscr{L}\mathscr{K}an_* = \lim(\cdots \to \mathscr{K}an_* \xrightarrow{\Omega} \mathscr{K}an_* \xrightarrow{\Omega} \mathscr{K}an_*).$$

Here one can take this looping functor as specifically realized by the construction of Section I-7.4 at the level of the simplicial category $\underline{\mathrm{Kan}}_*$, which then induces a functor on ∞-categories via an application of the homotopy coherent nerve and Proposition 3.11.

In terms of the explicit formulae given in Section II-14.2, this limit category is the ∞-category of sections of the weighted nerve

$$\mathscr{L}\mathscr{K}an_* = \mathrm{Fun}^{\mathrm{CCart}}_{\mathbb{Z}_{\leq 0}}(\mathbb{Z}_{\leq 0}, \mathrm{N}^{\omega}(\mathbb{Z}_{\leq 0})).$$

At each nonpositive integer i, evaluation provides a structure map

$$p_i = (ev_i)^* : \mathrm{Fun}^{\mathrm{CCart}}_{\mathbb{Z}_{\leq 0}}(\mathbb{Z}_{\leq 0}, \mathrm{N}^{\omega}(\mathbb{Z}_{\leq 0})) \to \mathrm{N}^{\omega}(\mathbb{Z}_{\leq 0})_i = \mathscr{K}an_*$$

For each object $\mathscr{X}_*$ in the limit, we take $\mathscr{X}_i = p_{-i}\mathscr{X}_*$ to see that $\mathscr{X}_*$ is defined by a sequence of pointed spaces $\{\mathscr{X}_i : i \geq 0\}$ which are equipped with homotopy equivalences $\Omega\mathscr{X}_{i+1} \overset{\sim}{\to} \mathscr{X}_i$. A morphism is a sequence of maps $f_i : \mathscr{X}_i \to \mathscr{Y}_i$ which fit into commuting diagrams

$$\begin{tikzcd} \Omega\mathscr{X}_{i+1} \arrow[r, "\Omega f_{i+1}"] \arrow[d, "\sim"'] & \Omega\mathscr{Y}_{i+1} \arrow[d, "\sim"] \\ \mathscr{X}_i \arrow[r, "f_i"'] & \mathscr{Y}_i \end{tikzcd} \tag{20}$$

eq:1318

in $\mathscr{K}an_*$.

prop:sp_looper

Proposition 5.7. *There is a unique equivalence of ∞-categories $t : \mathscr{L}\mathscr{K}an_* \overset{\sim}{\to} \mathscr{S}\!p$ which fits into a 2-simplex*

$$\begin{tikzcd} & \mathscr{S}\!p \arrow[dr, "\Omega^{\infty-n}"] & \\ \mathscr{L}\mathscr{K}an_* \arrow[ur, "t"] \arrow[rr, "p_{-n}"'] & & \mathscr{K}an_*, \end{tikzcd}$$

at each nonegative integer n.

Sketch proof. We have the shifted inclusion $-1 : \mathbb{Z}_{\leq 0} \to \mathbb{Z}_{\leq 0}$ and restricting along -1 provides a functor $\Sigma : \mathscr{L}\mathscr{K}an_* \to \mathscr{L}\mathscr{K}an_*$. This functor sends an object $\mathscr{X}_*$ to the object $\Sigma\mathscr{X}_* = \mathscr{X}_{*+1}$. We also have the looping functor $\Omega : \mathscr{L}\mathscr{K}an_* \to \mathscr{L}\mathscr{K}an_*$ which is induced by the transformation

$$\begin{tikzcd} \cdots \arrow[r] & \mathscr{K}an_* \arrow[r, "\Omega"] & \mathscr{K}an_* \arrow[r, "\Omega"] & \mathscr{K}an_* \\ \cdots \arrow[r] & \mathscr{K}an_* \arrow[r, "\Omega"'] \arrow[u, "\Omega"'] & \mathscr{K}an_* \arrow[r, "\Omega"'] \arrow[u, "\Omega"'] & \mathscr{K}an_*, \arrow[u, "\Omega"'] \end{tikzcd}$$

and one checks that these functors are mutually inverse. It's furthermore argued in [15, Proof of Proposition 1.4.2.24] that the functor Ω is the looping functor for $\mathscr{L}\mathscr{K}an_*$ in the sense of [15, Remark 1.1.2.8]. We have now $p_{-n} = p_0\Sigma^n$.

It is argued in [15, Proof of Proposition 1.4.2.24] that we have a unique equivalence t which fits into a diagram

$$\begin{tikzcd} & \mathscr{S}\!p \arrow[dr, "\Omega^{\infty}"] & \\ \mathscr{L}\mathscr{K}an_* \arrow[ur, "t"] \arrow[rr, "p_0"'] & & \mathscr{K}an_*. \end{tikzcd}$$

For the delooping functor $S : \mathscr{K}an_*^{\mathrm{fin}} \to \mathscr{K}an_*^{\mathrm{fin}}$ [15, Remark 1.1.2.6] the corresponding functor $\Sigma = \mathrm{Fun}(S, \mathscr{K}an_*)$ is the delooping functor on $\mathscr{S}\!p$, so that we have a commutative diagram

$$\begin{tikzcd} \mathscr{L}\mathscr{K}an_* \arrow[r, "t"] \arrow[d, "\Sigma^n"'] \arrow[dr] & \mathscr{S}\!p \arrow[d, "\Sigma^n"] \\ \mathscr{L}\mathscr{K}an_* \arrow[r, "t"'] & \mathscr{S}\!p \end{tikzcd}$$

in $\mathscr{C}at_\infty^{\mathrm{big}}$ at each integer n. This gives an isomorphism of functors $p_{-n} \cong \Omega^\infty\Sigma^n$. The latter functor evaluates a spectrum $\mathscr{X}_*$ at $S^n(\mathbb{S}^0) \cong \mathbb{S}^n$ (see Example 6.19). We therefore have an isomorphism $\Omega^\infty\Sigma^n \cong \Omega^{\infty-n}$, as desired. □

Remark 5.8. Our structural equivalences $\Omega\mathscr{X}_{n+1} \xrightarrow{\sim} \mathscr{X}_n$ are going in the "wrong direction" relative to standard practice, cf. [1, Section 2]. The direction we've employed here is simply the direction which is implied by the description of the limit given in Section II-14.2.

5.2. **Spectral Hom functors.**

Definition 5.9. An ∞-category $\mathscr{C}$ is called finitely complete (resp. cocomplete) if every diagram $p : K \to \mathscr{C}$ from a finite simplicial set admits a limit (resp. colimit) in $\mathscr{C}$.

We begin with a helpful little lemma.

`lem:funlex_equiv` **Lemma 5.10.** *If a functor $F : \mathscr{D} \to \mathscr{D}'$ is an equivalence of finitely complete ∞-categories, and $\mathscr{C}$ is finitely complete as well, then the induced functor*

$$F_* : \mathrm{Fun}^{\mathrm{lex}}(\mathscr{C}, \mathscr{D}) \to \mathrm{Fun}^{\mathrm{lex}}(\mathscr{C}, \mathscr{D}') \tag{21}$$ `eq:1246`

is an equivalence.

Proof. The functor $\mathrm{Fun}(\mathscr{C}, \mathscr{D}) \to \mathrm{Fun}(\mathscr{C}, \mathscr{D}')$ is an equivalence, and equivalences preserve and detect limit diagrams. It follows that taking the fiber along the inclusion $\mathrm{incl} : \mathrm{Fun}^{\mathrm{lex}}(\mathscr{C}, \mathscr{D}') \to \mathrm{Fun}(\mathscr{C}, \mathscr{D}')$ returns the map (21). Since the class of left exact functors is stable under isomorphism, by Proposition II-13.19, we see that the map incl is an isofibration. It follows that the map (21) is an equivalence, by Corollary I-6.24 for example. □

We now observe the following.

`prop:spectra_hom` **Proposition 5.11.** *Suppose $\mathscr{C}$ is a pointed ∞-category which admits finite limits, and that pushout diagrams agree with pullback diagrams in $\mathscr{C}$. Then for any pointed Hom functor $\widetilde{H} : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{K}an_*$ there is a functor $\widetilde{H}_* : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{S}\!p$ which fits into a 2-simplex*

$$\begin{tikzcd} & \mathscr{S}\!p \arrow[d] \\ \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \arrow[ur, "\widetilde{H}_*"] \arrow[r, "\widetilde{H}"'] & \mathscr{K}an_* \end{tikzcd}$$

in $\mathscr{C}at_\infty$. Furthermore, this functor $\widetilde{H}_*$ is uniquely determined up to a contractible space of choices.

Proof. Take $\mathscr{C}^e = \mathscr{C}^{\mathrm{op}} \times \mathscr{C}$ and, for any ∞-category $\mathscr{D}$ which admits finite limits, let $\mathrm{Fun}'(\mathscr{C}^e, \mathscr{D})$ denote the full subcategory of functors which are left exact in each factor independently. The adjunction $\mathrm{Fun}(\mathscr{C}, \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathscr{D})) \cong \mathrm{Fun}(\mathscr{C}^e, \mathscr{D})$ then restricts to an adjunction

$$\mathrm{Fun}'(\mathscr{C}^e, \mathscr{D}) \cong \mathrm{Fun}^{\mathrm{lex}}(\mathscr{C}, \mathrm{Fun}^{\mathrm{lex}}(\mathscr{C}^{\mathrm{op}}, \mathscr{D})),$$

where here we note that $\mathrm{Fun}^{\mathrm{lex}}(\mathscr{C}^{\mathrm{op}}, \mathscr{D})$ is closed under the formation of finite limits in the ambient category $\mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathscr{D})$ (see [15, Remark 1.4.2.3]). By Proposition 5.6 and Lemma 5.10 we now see that the functor Ω^∞ induces an equivalence

$$\Omega^\infty_* : \mathrm{Fun}'(\mathscr{C}^e, \mathscr{S}\!p) \overset{\sim}{\to} \mathrm{Fun}'(\mathscr{C}^e, \mathscr{K}an_*). \tag{22}$$ eq:1277

Now, the space of lift $\widetilde{H}_*$ which fit into a 2-simplex as proposed can be parametrized by the homotopy fiber

$$\mathrm{Fun}'(\mathscr{C}^e, \mathscr{S}\!p) \times^{\mathrm{htop}}_{\mathrm{Fun}'(\mathscr{C}^e, \mathscr{K}an_*)} \{\widetilde{H}\}.$$

Since the map (22) is an equivalence it follows, by Proposition I-6.23, that the projection

$$\mathrm{Fun}'(\mathscr{C}^e, \mathscr{S}\!p) \times^{\mathrm{htop}}_{\mathrm{Fun}'(\mathscr{C}^e, \mathscr{K}an_*)} \{\widetilde{H}\} \to *$$

is an equivalence as well. In particular, the fiber in question is a contractible Kan complex. □

Remark 5.12. Categories satisfying the hypotheses of Proposition 5.11 are called *stable* ∞-categories. We observe in Section 6 that the homotopy and derived ∞-categories of an abelian category are both stable. Hence the pointed Hom functors from Section 3.4 uniquely enhance to spectra valued Hom functors in this setting.

5.3. **Whitehead's theorem in the spectral setting.**

Proposition 5.13 ([15, Proposition 1.4.4.4])**.** *The forgetful functor $\Omega^\infty : \mathscr{S}\!p \to \mathscr{K}an_*$ admits a lift adjoint $\Sigma^\infty : \mathscr{K}an_* \to \mathscr{S}\!p$.*

Proof. Omitted. □

Definition 5.14. The n-th sphere spectrum $\underline{\mathbb{S}}^n$ is the object $\Sigma^\infty \mathbb{S}^n$ in $\mathscr{S}\!p$.

We can define the n-th homotopy group $\pi_n(X_*)$ of a spectrum X_* as the set

$$\pi_n(X_*) = \mathrm{Hom}_{\mathrm{h}\,\mathscr{S}\!p}(\underline{\mathbb{S}}^n, X_*).$$

Via the $(\Sigma^\infty, \Omega^\infty)$-adjunction we have the natural identification

$$\pi_n(X_*) = \pi_n(X_0, x_0) \cong \pi_{n+i}(X_i, x_i),$$

where $x_k : * \to X_k$ is the implicit pointing on the k-th space. From the second perspective it is clear that we can define homotopy groups for spectra at negative n.

Definition 5.15. For any spectrum $\mathscr{X}_*$ we define the n-th homotopy group, for $n \in \mathbb{Z}$, as the colimit

$$\pi_n(X_*) = \mathrm{colim}_{n+i\geq 0}\, \pi_{n+i}(X_i, x_i).$$

The following is a consequence of Whitehead's theorem for Kan complexes.

thm:sp_whitehead **Theorem 5.16** (Spectral Whitehead's theorem)**.** *For a map of spectra $f : \mathscr{X}_* \to \mathscr{Y}_*$ the following are equivalent:*

(a) *f is an isomorphism in $\mathscr{Sp}$.*

(b) *At each integer $i \geq 0$, the induced map $f_i : \mathscr{X}_i \to \mathscr{Y}_i$ is an equivalence of Kan complexes.*

(c) *At each integer $n \in \mathbb{Z}$, the map $\pi_n f : \pi_n(\mathscr{X}_*) \to \pi_n(\mathscr{Y}_*)$ is an isomorphism.*

Proof. We identify the category of spectra with the limit category $\mathscr{L}\mathscr{K}an_*$ as in Proposition 5.7. From this perspective we have a transformation $f : \mathscr{X}_* \to \mathscr{Y}_*$ between functors $\mathscr{X}_*, \mathscr{Y}_* : \mathbb{Z}_{\leq 0} \to \mathrm{N}^\omega(\mathbb{Z}_{\leq 0})$, where ω is the diagram

$$\cdots \to \mathscr{K}an_* \overset{\Omega}{\to} \mathscr{K}an_* \overset{\Omega}{\to} \mathscr{K}an_*$$

in $\mathscr{C}at_\infty^{\mathrm{big}}$. Each map $f_i : \mathscr{X}_i \to \mathscr{Y}_i$ is obtained by evaluating the transformation f at $-i \in \mathbb{Z}_{\leq 0}$. Hence the equivalence between (a) and (b) follows from the fact that a transformation between functors is an equivalence if and only if it evaluates to an equivalence at each vertex in the domain, by Proposition I-7.6. The fact that (b) implies (c) is also immediate.

For the implication from (c) to (b), consider a map of Kan complexes $g : \mathscr{S} \to \mathscr{T}$ and a point $s : * \to \mathscr{S}$ with image $t = g(s) : * \to \mathscr{T}$. Let $\mathscr{S}_0$ and $\mathscr{T}_0$ be the components of $\mathscr{S}$ and $\mathscr{T}$ containing s and t respectively. By Whitehead's theorem (Theorem I-4.15) and base point independence of the homotopy groups [16, 04GF], the restriction $f|_{\mathscr{S}_0}$ is a homotopy equivalence if and only if each map

$$\pi_m f : \pi_m(\mathscr{S}, s) \to \pi_m(\mathscr{T}, t),$$

for $m \geq 0$, is an equivalence. This occurs if and only if the induced map on loop spaces $\Omega f : \Omega(\mathscr{S}, s) \to \Omega(\mathscr{T}, t)$ is a homotopy equivalence.

Now, at a fixed integer i, the condition that each map $\pi_n f : \pi_n(\mathscr{X}_*) \to \pi_n(\mathscr{Y}_*)$ is an isomorphism implies that $\pi_m(f_{i+1}) : \pi_m(\mathscr{X}_{i+1}) \to \pi_m(\mathscr{Y}_{i+1})$ is an isomorphism at all nonnegative integers m. Hence the induced map on loop spaces $\Omega(f_{i+1})$: $\Omega(\mathscr{X}_{i+1}) \to \Omega(\mathscr{Y}_{i+1})$ is an equivalence. From the diagram (20) we conclude that each map $f_i : \mathscr{X}_i \to \mathscr{Y}_i$ is an equivalence. □

sect:stable

6. Stability of homotopy and derived ∞-categories

We now begin our study of the derived ∞-category in earnest. Given a Grothendieck abelian category $\mathbb{A}$, we prove that the homotopy and derived ∞-categories $\mathscr{K}(\mathbb{A})$ and $\mathscr{D}(\mathbb{A})$ are both stable. We then study some basic structures on stable categories. In a very vague sense, one might view stability as a homotopical analog of abelianness in the discrete setting.

In Section 7 we explain how stability can be employed to reduce various analyses at the ∞-categorical level to corresponding analyses at the discrete level, via an application of the homotopy category functor.

6.1. **Pullbacks are puhsouts in the homotopy ∞-category.**

lem:6102

Lemma 6.1. *For a diagram*

$$\begin{tikzcd} V_0 \arrow[r, "g"] \arrow[d, "g'"'] & V_1 \arrow[d, "f"] \\ V_1' \arrow[r, "f'"'] & V_2 \end{tikzcd} \tag{23}$$

eq:5863

in $\mathrm{Ch}(\mathbb{A})$ *the following are equivalent:*

(a) *The map f is surjective, f' is injective, and (23) is a discrete pullback diagram.*
(b) *The map g is injective, g' is surjective, and (23) is a pushout diagram.*

The same implications hold when we replace, simultaneously, injective and surjective with termwise split injective and termwise split surjective.

Proof. (a) $\Rightarrow$ (b) If (23) is a pullback diagram with the prescribed properties then the map g is an inclusion which identifies V_0 as a kernel of the composite $V_1 \to V_2 \to V_2/V_1'$. Furthermore, in this case the map $g' : V_0 \to V_1'$ is simply the restriction of the projection $V_1 \to V_2$ to V_0, and hence g' is surjective as well. So we see that g is injective and g' is surjective in this case, and g is split when f is spit. The identification of V_0 with the aforementioned kernel also tells us that g and g' is termwise split whenever f and f' are termwise split.

The implication (b) $\Rightarrow$ (a) is recovered by applying (a) $\Rightarrow$ (b) to the category $\mathrm{Ch}(\mathbb{A}^{\mathrm{op}}) = \mathrm{Ch}(\mathbb{A})^{\mathrm{op}}$. $\square$

prop:K_pullpush **Proposition 6.2.** *For any abelian category $\mathbb{A}$, a diagram*

$$\begin{tikzcd} V_0 \arrow[r, "g"] \arrow[d, "g'"'] \arrow[dr] & V_1 \arrow[d, "f"] \\ V_1' \arrow[r, "f'"'] & V_2 \end{tikzcd} \tag{24}$$ eq:6126

in $\mathscr{K}(\mathbb{A})$ is a limit (aka pullback) diagram if and only if it is a colimit (aka pushout) diagram.

Proof. Suppose that the diagram (24) is a limit diagram. Then, according to Proposition 2.13 and Proposition II-13.20, we can assume (24) is a strictly commuting, discrete pullback diagram in which f is termwise split surjective. We can also assume that f' is termwise split injective as well, by appending a copy of the mapping cone for the identity on V_1' to both V_1 and V_2 if necessary. In this case (24) is also a discrete pushout diagram in which g is termwise split injective, by Lemma 6.1. By Proposition 2.17 such a discrete pushout diagram is a colimit diagram in $\mathscr{K}(\mathbb{A})$, so that the diagram (24) is a colimit diagram in the homotopy ∞-category.

The converse implication is proved similarly. Namely, if (24) is a colimit diagram then we can assume it is a discrete pushout diagram with g termwise split injective, by Proposition 2.17, at which point it is seen to be a limit diagram as well by Lemma 6.1 and Proposition 2.21. $\square$

6.2. **Fibers and cofibers.**

Definition 6.3. Let $\mathscr{C}$ be an ∞-category which admits a zero object 0. A diagram of the form

$$\begin{tikzcd} x \arrow[r, "\alpha"] \arrow[d] \arrow[dr] & y \arrow[d, "\beta"] \\ 0 \arrow[r] & z \end{tikzcd} \tag{25}$$ eq:1670

is called a fiber sequence in $\mathscr{C}$ if it is a limit diagram, and a cofiber sequence if it is a colimit diagram. Given a fiber (resp. cofiber) sequence of the form (25) we refer to the object x as the fiber of the morphism β (resp. z as the cofiber of the morphism α).

Under any reasonable interpretation of the terms, the fiber of a morphism in an ∞-category $\mathscr{C}$ is a "kernel" for the given morphism, and the cofiber of a morphism is a "cokernel". Let us examine this perspective in the case of the cofiber.

Consider the diagram $p : \Lambda^2_0 \to \mathscr{C}$ obtained from (25) by deleting z, and let w be an arbitrary object in $\mathscr{C}$. We have an equivalence of mapping spaces

$$\mathrm{Hom}_{\mathscr{C}}(z, w) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Fun}(\Lambda^2_0,\mathscr{C})}(p, \underline{w}) = \{p\} \times^{\mathrm{or}}_{\mathrm{Fun}(\Lambda^2_0,\mathscr{C})} \{w\},$$

simply by the definition of the colimit, and the latter space is identified with the fiber of the undercategory $\mathscr{C}_{p/}$ over w by Theorem I-10.15. We therefore obtain an equivalence

$$\mathrm{Hom}_{\mathscr{C}}(z, w) \xrightarrow{\sim} \mathscr{C}_{p/} \times_{\mathscr{C}} \{w\}.$$

Now, the space $\mathscr{C}_{p/} \times_{\mathscr{C}} \{w\}$ parametrizes maps $\zeta : y \to w$ which are equipped with a trivialization of the composite along $\alpha : x \to y$. In this way the cofiber has the universal property one expects of a "cokernel" for the morphism α.

In the discrete setting, our vague associations with kernels and cokernels become precise.

Example 6.4. Let $\mathbb{A}$ be a discrete, additive category. A diagram in $\mathbb{A}$ of the form

$$\begin{tikzcd} x \arrow[r, "\alpha"] \arrow[d] \arrow[dr] & y \arrow[d, "\beta"] \\ 0 \arrow[r] & z \end{tikzcd}$$

is a fiber sequence if and only if α is a kernel of β, and is a cofiber sequence if and only if β is a cokernel of α.

6.3. Stable ∞-categories and exact functors.

def:stable **Definition 6.5.** An ∞-category $\mathscr{C}$ is called stable if the following properties hold:

(a) $\mathscr{C}$ admits a zero object.
(b) Every morphism in $\mathscr{C}$ extends to both a fiber sequence and a cofiber sequence.
(c) A diagram of the form

$$\begin{tikzcd} x \arrow[r, "\alpha"] \arrow[d] \arrow[dr] & y \arrow[d, "\beta"] \\ 0 \arrow[r] & z \end{tikzcd}$$

in $\mathscr{C}$ is a fiber sequence if and only if it is a cofiber sequence.

Definition 6.6. A functor between stable ∞-categories $F : \mathscr{C} \to \mathscr{D}$ is called exact if F preserves zero objects and also preserves fiber/cofiber sequences. A full ∞-subcategory $\mathscr{C}' \subseteq \mathscr{C}$ is called a stable ∞-subcategory if $\mathscr{C}'$ is stable and the inclusion $\mathscr{C}' \to \mathscr{C}$ is exact.

We note that stability is a *property*, not a structure. The reader might compare, in their own mind, stability to abelianness and triangulated structures in this regard. Also, speaking liberally, a full subcategory $\mathscr{C}'$ in a stable ∞-category $\mathscr{C}$ is a stable subcategory if it is closed under the formation of fibers and cofibers in $\mathscr{C}$. We leave it to the interested reader to check the following basic properties.

Lemma 6.7. (1) *Suppose $F : \mathscr{C} \to \mathscr{D}$ is an equivalence of ∞-categories and that $\mathscr{C}$ is stable. Then $\mathscr{D}$ is stable.*
(2) *If $\mathscr{C}$ is stable then the opposite category $\mathscr{C}^{\mathrm{op}}$ is also stable.*

Proof. Omitted. $\square$

The quintessential example of a stable ∞-category is the ∞-category of spectra.

Theorem 6.8. *The category $\mathscr{S}\!p$ of spectra is stable and has zero object $0 = \Sigma^\infty *$.*

Sketch proof. By definition, the category of spectra is the stabilization $\mathscr{S}\!p = \mathscr{S}\!p(\mathscr{K}an_*)$ of the ∞-category of pointed spaces. Since the category of pointed spaces is complete, by Proposition 4.2, it follows that $\mathscr{S}\!p$ is stable by [15, Corollary 1.4.2.17]. As for the calculation of the zero object, the left adjoint $\Sigma^\infty : \mathscr{K}an_* \to \mathscr{S}\!p$ commutes with colimits, so that it preserves the colimit over the empty diagram, i.e. the initial object. By Proposition II-9.15 the object $*$ is initial in $\mathscr{K}an_*$, so that $\Sigma^\infty *$ is an initial in $\mathscr{S}\!p$, and hence a terminal object in $\mathscr{S}\!p$ as well. $\square$

As stated perviously, we do not employ spectra directly in this text. So we center our analysis of stability on the algebraic examples discussed below.

6.4. **Stability of homotopy and derived ∞-categories.**

thm:K_stable **Theorem 6.9.** *For any abelian category $\mathbb{A}$, the homotopy ∞-category $\mathscr{K}(\mathbb{A})$ is stable.*

Proof. The zero complex is a zero object in $\mathscr{K}(\mathbb{A})$ by Corollary 2.24, and $\mathscr{K}(\mathbb{A})$ admits all pullbacks and pushouts by Corollaries 2.14 and Corollary 2.22. In particular, one can complete any morphism in $\mathscr{K}(\mathbb{A})$ to both a fiber and cofiber sequence. Finally, fiber sequences and cofiber sequences in $\mathscr{K}(\mathbb{A})$ agree by Proposition 6.2. $\square$

By the description of pullbacks and pushouts in $\mathscr{K}(\mathbb{A})$ provided in Propositions 2.13 and 2.21 we see that any full subcategory $\mathscr{K} \subseteq \mathscr{K}(\mathbb{A})$ which is preserved under desuspension and the formation of mapping cones admits all pushouts and pullbacks. In particular, the inclusion $\mathscr{K} \to \mathscr{K}(\mathbb{A})$ preserves pushouts and pullbacks. Thus, under these conditions, and assuming additionally that $\mathscr{K}$ contains the zero complex, we see that $\mathscr{K}$ is a stable ∞-category as well.

cor:K_stable **Corollary 6.10.** *Suppose a full ∞-subcategory $\mathscr{K} \subseteq \mathscr{K}(\mathbb{A})$ contains the zero complex, is closed under the formation of mapping cones, and the desuspension automorpihsm Σ^{-1}. Then $\mathscr{K}$ is a stable ∞-subcategory in $\mathscr{K}(\mathbb{A})$. In particular, $\mathscr{K}$ is stable.*

In the case of a Grothendieck abelian category, we have the injective construction of the (unbounded) derived ∞-category $\mathscr{D}(\mathbb{A})$ via the subcategory of K-injectives in $\mathscr{K}(\mathbb{A})$ (Definition I-2.10). As a particular instance of Corollary 6.10 we observe stability of $\mathscr{D}(\mathbb{A})$.

cor:D_stable **Corollary 6.11.** *If $\mathbb{A}$ is a Grothendieck abelian category, then the derived ∞-category $\mathscr{D}(\mathbb{A})$ is stable.*

Proof. Up to equivalence, the derived ∞-category $\mathscr{D}(\mathbb{A})$ is identified with the subcategory of K-injectives in $\mathscr{K}(\mathbb{A})$. Clearly this subcategory is closed under suspension and desuspension. So, according to Corollary 6.10, the derived ∞-category is stable provided mapping cones of K-injectives remain K-injective in $\mathscr{K}(\mathbb{A})$.

For any map between K-injectives $f : I \to I'$, and arbitrary acyclic X, we have

$$\mathrm{Hom}^*_{\mathbb{A}}(X, \mathrm{cone}(f)) = \mathrm{cone}\left(f_* : \mathrm{Hom}^*_{\mathbb{A}}(X, I) \to \mathrm{Hom}^*_{\mathbb{A}}(X, I')\right).$$

Since both complexes in the latter mapping cone are acyclic. So we see that $\mathrm{cone}(f)$ is K-injective, as desired. □

Remark 6.12. As stated previously, stability of an ∞-category is a property not a structure. So, in the event that a Grothendieck abelian category $\mathbb{A}$ has enough projectives, stability holds whether one constructs the derived ∞-category via K-injectives or K-projectives. See Section I-13.

We similarly apply Corollary 6.10 to observe stability of the derived ∞-category under all standard bounding restrictions. We consider specifically the ∞-subcategories $\mathscr{D}^b(\mathbb{A})$, $\mathscr{D}^-(\mathbb{A})$, and $\mathscr{D}^+(\mathbb{A})$ of bounded, bounded above, and bounded below complexes.

Corollary 6.13. *If $\mathbb{A}$ is a Grothendieck abelian category, then for $\star = b, +,$ and $-$ the appropriately bounded derived ∞-category $\mathscr{D}^\star(\mathbb{A}) \subseteq \mathscr{D}(\mathbb{A})$ is a stable ∞-subcategory. Furthermore, in the case where $\mathbb{A}$ is locally finite, the full ∞-subcategory $\mathscr{D}(\mathbb{A})_{fin}$ of complexes with finite total length is a stable ∞-subcategory in $\mathscr{D}(\mathbb{A})$.*

To be clear, by a locally finite Grothendieck abelian category we mean a compactly generated abelian category in which the compact objects form an abelian subcategory, and in which all compacts objects are of finite length. Such categories include representations over an affine algebraic group, and locally finite modules over a C_2-cofinite vertex operator algebra, for example.

We can also work in the additive setting.

Corollary 6.14. *For any additive category $\mathfrak{A}$, the homotopy ∞-category $\mathscr{K}(\mathfrak{A})$ is stable.*

Proof. Choose a fully faithful embedding $\mathfrak{A} \to \mathbb{A}$ into an abelian category, as in Section 2.9, to realize $\mathscr{K}(\mathfrak{A})$ as a stable subcategory in $\mathscr{K}(\mathbb{A})$. □

6.5. **Connective cochains, and anti-example.** We consider the connective derived ∞-category $\mathscr{D}^{\leq 0}(\mathbb{A})$, i.e. the full subcategory in $\mathscr{D}(\mathbb{A})$ spanned by those complexes whose cohomology vanishes in all positive degrees. We claim that the ∞-category $\mathscr{D}^{\leq 0}(\mathbb{A})$ is not not stable, though it is certainly an interesting object to study (see e.g. [17, Appendix C]). Explicitly, for a nonzero complex V which is concentrated in degree 0, we claim that the discrete pullback diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] \arrow[dr] & 0 \arrow[d] \\ 0 \arrow[r] & V \end{tikzcd} \tag{26}$$

eq:859

is a pullback diagram in $\mathscr{D}^{\leq 0}(\mathbb{A})$ which is not a pushout.

To see this, note that the truncation functor $\tau_0 : \mathscr{D}(\mathbb{A}) \to \mathscr{D}^{\leq 0}(\mathbb{A})$ (Definition 2.8) is right adjoint to the inclusion $\mathscr{D}^{\leq 0}(\mathbb{A}) \to \mathscr{D}(\mathbb{A})$ (Theorem I-14.9). Truncation therefore preserves limits by Proposition II-13.24, so that we obtain the above limit diagram by truncating the standard limit diagram

$$\begin{tikzcd} \Sigma^{-1}V \arrow[r] \arrow[d] \arrow[dr] & 0 \arrow[d] \\ 0 \arrow[r] & V \end{tikzcd}$$

in $\mathscr{D}(\mathbb{A})$. However, we see simultaneously that the diagram (26) is not a pushout diagram when V is nonzero, since the inclusion $\mathscr{D}^{\leq 0}(\mathbb{A}) \to \mathscr{D}(\mathbb{A})$ preserves colimits and hence the image of any pushout diagram in $\mathscr{D}^{\leq 0}(\mathbb{A})$ must be a pushout diagram in $\mathscr{D}(\mathbb{A})$.

sect:sus_loop

6.6. **Suspension and desuspension.** Fix $\mathscr{C}$ an ∞-category which admits a zero object. We sketch a construction of the suspension and desuspension functors, following [15, Section 1.1.2]: Let $\mathscr{M}^{\Sigma} = \mathscr{M}^{\Sigma}_{\mathscr{C}}$ and $\mathscr{M}^{\Omega} = \mathscr{M}^{\Omega}_{\mathscr{C}}$ denote the full subcategories of the functor category

$$\mathscr{M}^{\Sigma},\ \mathscr{M}^{\Omega} \ \subseteq\ \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathscr{C})$$

whose objects are diagrams of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0' \arrow[r] & Y \end{tikzcd}$$

which are pushout diagrams, in the case of $\mathscr{M}^{\Sigma}$, and pullback diagrams, in the case of $\mathscr{M}^{\Omega}$. Here 0 and $0'$ are arbitrary zero objects in $\mathscr{C}$.

We have the evaluation functors at the initial and terminal vertices in the above diagram

$$ev_0 : \mathscr{M}^{\Sigma} \to \mathscr{C} \ \text{ and } \ ev_1 : \mathscr{M}^{\Omega} \to \mathscr{C}, \tag{27}$$ eq:1529

which are both isofibrations. The following is an application of [14, Proposition 4.3.2.15]. (See the discussion preceding [15, Remark 1.1.2.6].)

Proposition 6.15. *For $\mathscr{C}$ a pointed ∞-category which admits arbitrary pushouts and pullbacks, the evaluation functors of (27) are both trivial Kan fibrations.*

We can now define the suspension and looping operations as endofunctors on $\mathscr{C}$.

Definition 6.16. Let $\mathscr{C}$ be a pointed ∞-category which admits all pullbacks and pushouts. A suspension functor $\Sigma : \mathscr{C} \to \mathscr{C}$ is any composite

$$\mathscr{C} \xrightarrow{s_0} \mathscr{M}^{\Sigma}_{\mathscr{C}} \xrightarrow{ev_1} \mathscr{C},$$

where s_0 is a section of the trivial fibration ev_0. Similarly, a looping functor $\Omega : \mathscr{C} \to \mathscr{C}$ is any composite

$$\mathscr{C} \xrightarrow{s_1} \mathscr{M}^{\Omega}_{\mathscr{C}} \xrightarrow{ev_0} \mathscr{C},$$

where s_1 is a section of ev_1.

We note that suspension and desuspension are uniquely determined up to a contractible space of choices.

Example 6.17. We saw in Theorem 4.13 that the category of pointed spaces $\mathscr{K}an_*$ is both complete and cocomplete. It furthermore has the zero object $*$ provided by the one point space, by Proposition II-9.15. We therefore have the suspension and looping functors $\Sigma, \Omega : \mathscr{K}an_* \to \mathscr{K}an_*$. While the suspension functor is slightly mysterious, the looping functor can be realized explicitly on objects as the assignment

$$\Omega : (\mathscr{X}, x) \mapsto \mathrm{Hom}_{\mathscr{X}}(x, x),$$

according to the materials of Section I-7.4. If we replace $\mathscr{K}an_*$ with its more rigid model $\mathrm{N}^{\mathrm{hc}}(\underline{\mathrm{Kan}}_*)$, then Ω can also be clearly defined on morphisms,

$$\Omega : (F : \mathscr{X} \to \mathscr{Y}) \mapsto (F_* : \mathrm{Hom}_{\mathscr{X}}(x, x) \to \mathrm{Hom}_{\mathscr{Y}}(y, y)).$$

In general, the suspensions functor is left adjoint to the looping functors [15, Remark 1.1.2.8]. However, we are most interested in the stable setting. In this case fiber and cofiber diagrams are identified, so that the looping and delooping spaces are identified

$$\mathscr{M}^{\Sigma} = \mathscr{M}^{\Omega} =: \mathscr{M}.$$

We observe the following.

Proposition 6.18. *If $\mathscr{C}$ is stable, then the endofunctors $\Sigma, \Omega : \mathscr{C} \to \mathscr{C}$ are mutually inverse equivalences.*

Proof. In this case the two evaluation maps are defined on the same domain $ev_i : \mathscr{M} \to \mathscr{C}$. Since the evaluation morphisms are equivalences so are their sections s_i, and each s_i is an inverse to ev_i. So we have directly

$$\Sigma\Omega = \mathrm{ev}_1\, s_0\, \mathrm{ev}_0\, s_1 \cong id_{\mathscr{C}}, \quad \Omega\Sigma = \mathrm{ev}_0\, s_1\, \mathrm{ev}_1\, s_0 \cong id_{\mathscr{C}}.$$

□

We cover another example.

ex:susphere

Example 6.19 (Suspensions of spheres)**.** Consider the n-sphere $\underline{\mathbb{S}}^n = \Sigma^\infty \mathbb{S}^n$ in $\mathscr{S}\!p$. Since the functor Σ^∞ is a left adjoint, and hence commutes with colimits, it suffices to compute the pushout

$$\begin{tikzcd} \mathbb{S}^n \arrow[r] \arrow[d] \arrow[dr] & * \arrow[d] \\ * \arrow[r] & \mathrm{Sus}(\mathbb{S}^n) \end{tikzcd}$$

in the category $\mathscr{K}an_*$. By Corollary 4.7 it suffices further to compute such pushouts in the unpointed category $\mathscr{K}an$. By the explicit formula given in Section II-14.5, we have $\mathrm{Sus}(\mathbb{S}^n) = Q(\mathrm{N}^p(\Lambda^0_2)) = Q(\mathbb{S}^n \star \partial\Delta^1)$ where $Q(K)$ denotes any weak homotopy replacement for the given simplicial set [16, 00UV]. This weak homotopy replacement can be given as the singular complex of the geometric realization, at which point one calculates

$$\mathrm{Sus}(\mathbb{S}^n) = \mathrm{Sing}\, |\mathbb{S}^n \star \partial\Delta^1| \cong \mathbb{S}^{n+1}.$$

Since the pushout is only defined up to isomorphism, we can take simply $\mathrm{Sus}(\mathbb{S}^n) = \mathbb{S}^{n+1}$.

If we adopt the explicit expression given by the singular complex $\mathbb{S}^k = \mathrm{Sing}\, S^k$, then the inclusion $S^n \to S^{n+1}$, along with the two contractions onto the north and south poles, provide an explicit pushout diagram

$$\begin{tikzcd} \mathrm{Sing}\, S^n \arrow[r] \arrow[d] \arrow[dr] & * \arrow[d, "\infty"] \\ * \arrow[r, "-\infty"'] & \mathrm{Sing}\, S^{n+1}. \end{tikzcd}$$

This pushout diagram determines an isomorphism $\Sigma\mathbb{S}^n \to \mathbb{S}^{n+1}$ which is unique up to a contractible space of choices.

One notes that the above analysis generalizes to provide a calculation of the suspension $\Sigma\mathscr{X}_*$ at any spectrum $\mathscr{X}_*$ which is in the image of the functor Σ^∞. As a corollary to our example we find the following.

Corollary 6.20. *At any non-negative integer n, there are isomorphisms of spectra*

$$\Sigma(\underline{\mathbb{S}}^n) \cong \mathbb{S}^{n+1},\ \Omega(\underline{\mathbb{S}}^{n+1}) \cong \mathbb{S}^n,\ \textit{and}\ \Sigma^n(\underline{\mathbb{S}}^0) \cong \underline{\mathbb{S}}^n.$$

6.7. **Suspension and desuspension for cochains.** Let $\mathbb{A}$ be an additive category and $\Sigma : \mathbf{Ch}(\mathbb{A}) \to \mathbf{Ch}(\mathbb{A})$ be the shift automorphism on the dg category of cochains. On objects ΣV is the expected shifted complex, and on mapping complexes we take specifically

$$\Sigma : \mathrm{Hom}^*_{\mathbb{A}}(V, W) \to \mathrm{Hom}^*_{\mathbb{A}}(\Sigma V, \Sigma W), \quad f \mapsto (-1)^{\deg(f)} f.$$

def:shift

Definition 6.21. For an discrete, additive category $\mathbb{A}$, we let $\Sigma : \mathscr{K}(\mathbb{A}) \to \mathscr{K}(\mathbb{A})$ denote the autoequivalence on the homotopy ∞-category realized by taking the dg nerve of the shift automorphism on the dg category of $\mathbb{A}$-cochains. Similarly, we let $\Sigma^{-1} : \mathscr{K}(\mathbb{A}) \to \mathscr{K}(\mathbb{A})$ denote the dg nerve of the inverse to the shift automorphism.

Of course, we claim that Σ and Σ^{-1} realize suspension and desuspension for $\mathscr{K}(\mathbb{A})$, in the sense of Section 6.6.

prop:shift_chains

Proposition 6.22. *Let $\mathbb{A}$ be an discrete, additive category $\mathbb{A}$. The functors Σ and Σ^{-1} from Definition 6.21 provide, respectively, suspension and desuspension functors on the homotopy ∞-category $\mathscr{K}(\mathbb{A})$.*

Proof. We prove that Σ is a suspension functor. The proof that Σ^{-1} is a desuspension functor is similar.

Let $c : \mathbf{Ch}(\mathbb{A}) \to \mathbf{Ch}(\mathbb{A})$ be the dg functor which sends each complex X to $c(V) = \mathrm{cone}(id_V)$ and each homogeneous map $f : X \to Y$ to the map

$$\begin{bmatrix} \Sigma f & 0 \\ 0 & f \end{bmatrix} : \mathrm{cone}(id_V) \to \mathrm{cone}(id_V).$$

The inclusion $\iota_V : V \to c(V)$ and projection $\pi_V : c(V) \to \Sigma V$ provide natural transformations of dg functors

$$id_{\mathbf{Ch}(\mathbb{A})} \overset{\iota}{\to} c \overset{\pi}{\to} \Sigma,$$

and the composite is the zero transformation $0 : id_{\mathbf{Ch}(\mathbb{A})} \to \Sigma$. We therefore have induced transformations between the associated functors on the dg nerve

$$I, \Pi, 0 : \Delta^1 \times \mathscr{K}(\mathbb{A}) \to \mathscr{K}(\mathbb{A}),$$

as in Proposition I-14.6. By Lemma I-14.8 these transformations furthermore provide the faces for a diagram $\Theta_{01} : \Delta^2 \times \mathscr{K}(\mathbb{A}) \to \mathscr{K}(\mathbb{A})$ which evaluates at each complex V to the strictly commuting diagram

$$\begin{tikzcd} & \mathrm{cone}(id_V) \arrow[dr, "\pi_V"] & \\ V \arrow[ur, "\iota_V"] \arrow[rr, "0"'] & & \Sigma V. \end{tikzcd}$$

We now consider the zero functor $z : \mathbf{Ch}(\mathbb{A}) \to \mathbf{Ch}(\mathbb{A})$, $z(V) = 0$, and induced functor $\mathrm{Z} : \mathscr{K}(\mathbb{A}) \to \mathscr{K}(\mathbb{A})$. We have the unique dg transformations $id_{\mathbf{Ch}(\mathbb{A})} \to z$ and $z \to \Sigma$ whose composite is the zero transformation $0 : id_{\mathbf{Ch}(\mathbb{A})} \to \Sigma$. We again have the induced transformations and subsequent 2-simplex $\Theta_{10} : \Delta^2 \times \mathscr{K}(\mathbb{A}) \to \mathscr{K}(\mathbb{A})$ which evaluates at each complex to the strictly commuting diagram

$$\begin{tikzcd} V \arrow[rr, "0"] \arrow[dr] & & \Sigma V \\ & 0 \arrow[ur] & \end{tikzcd} .$$

By the specific claim of Lemma I-14.8 these two simplices glue along the diagonal to provide a diagram

$$\Theta : \Delta^1 \times \Delta^1 \times \mathscr{K}(\mathbb{A}) \to \mathscr{K}(\mathbb{A}) \tag{28}$$ eq:1978

which evaluates at each complex V to the strictly commuting diagram

$$\begin{tikzcd} V \arrow[r, "\iota_V"] \arrow[d] \arrow[dr, "0"] & \mathrm{cone}(id_V) \arrow[d, "\pi_V"] \\ 0 \arrow[r] & \Sigma V. \end{tikzcd}$$

By Proposition 2.17 the above diagram is a pushout diagram in $\mathscr{K}(\mathbb{A})$, and by contractibility of $\mathrm{cone}(id_V)$ it is furthermore an object in the full subcategory

$$\mathscr{M} = \mathscr{M}^\Sigma \subseteq \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathscr{K}(\mathbb{A})).$$

Hence the map (28) defines a functor $\Theta : \mathscr{K}(\mathbb{A}) \to \mathscr{M}^\Sigma$ which evaluates at the $(0,0)$ vertex to recover the identity,

$$ev_0\Theta = id_{\mathscr{K}(\mathbb{A})} : \mathscr{K}(\mathbb{A}) \to \mathscr{K}(\mathbb{A}).$$

Hence the composite $ev_1\Theta : \mathscr{K}(\mathbb{A}) \to \mathscr{K}(\mathbb{A})$ is a suspension functor for $\mathscr{K}(\mathbb{A})$. By construction this composite recovers the shift autoequivalence $\Sigma : \mathscr{K}(\mathbb{A}) \to \mathscr{K}(\mathbb{A})$, and hence we realize Σ as a suspension functor. □

Given a stable subcategory $\mathscr{C}' \subseteq \mathscr{C}$, and suspension functor Σ for $\mathscr{C}$, Σ restricts to a suspension functor for $\mathscr{C}'$ whenever Σ preserves objects in $\mathscr{C}'$. Hence Proposition 6.22 tells us that the shift operations at the level of the dg category provide suspension and desuspension functors for the derived ∞-category as well.

Corollary 6.23. *Let $\mathbb{A}$ be a Grothendieck abelian category. The functors Σ and Σ^{-1} from Definition 6.21 restrict to provide, respectively, suspension and desuspension functors for the derived ∞-category $\mathscr{D}(\mathbb{A})$.*

6.8. **Suspension and exact functors.**

prop:2007

Proposition 6.24. *Let $\mathscr{C}$ and $\mathscr{D}$ be ∞-categories which admit zero objects, and which admit all pullbacks and pushouts. Let $F : \mathscr{C} \to \mathscr{D}$ be a functor which preserves zero objects.*

(1) *If F respects fiber sequences (resp. cofiber sequences) then there is a canonically defined isomorphism*
$$\zeta : F \circ \Omega \overset{\sim}{\to} \Omega \circ F \quad (\textit{resp. } F \circ \Sigma \overset{\sim}{\to} \Sigma \circ F).$$

(2) *For any simplicial set K, map $F : K \times \mathscr{C} \to \mathscr{D}$ for which $F_x = F|_{\{x\}\times\mathscr{C}}$ respects fiber sequences (resp. cofiber sequences) at each vertex $x : * \to K$, and collection of isomorphisms ζ_x as in (1), there is a uniquely associated isomorphism*
$$\zeta_K : F \circ (\Delta^n \times \Omega) \overset{\sim}{\to} \Omega \circ F \quad (\textit{resp. } F \circ (\Delta^n \times \Sigma) \overset{\sim}{\to} \Sigma \circ F)$$
which evaluates to ζ_x at each vertex x in K.

The above commutativity isomorphisms are constructed in the proof.

Construction 6.24. Let us consider the case where F preserves fiber sequences. Given an ∞-category $\mathscr{A}$, let $\mathscr{M}_{\mathscr{A}} = \mathscr{M}^{\Omega}_{\mathscr{A}}$ denote the full subcategory in $\mathrm{Fun}(\Delta^1 \times \Delta^1, \mathscr{A})$ spanned by fiber diagrams D in which both $D|_{(0,1)}$ and $D|_{(1,0)}$ are zero objects. We recall that the evaluation map $ev_1 = ev_{(1,1)} : \mathscr{M}_{\mathscr{A}} \to \mathscr{A}$ is a trivial Kan fibration whenever $\mathscr{A}$ admits a zero object and all pullbacks. For $\mathscr{A} = \mathscr{C}$ and $\mathscr{D}$ we suppose we've chosen sections $s_1 : \mathscr{A} \to \mathscr{M}_{\mathscr{A}}$ from which we define the looping functors $\Omega = ev_0 s_1$.

(1) In our setting the map F induces a functor $\mathscr{M}_F : \mathscr{M}_{\mathscr{C}} \to \mathscr{M}_{\mathscr{D}}$ which fits into diagrams
$$\begin{tikzcd} \mathscr{M}_{\mathscr{C}} \arrow[r, "\mathscr{M}_F"] \arrow[d, "ev_i"'] & \mathscr{M}_{\mathscr{D}} \arrow[d, "ev_i"] \\ \mathscr{C} \arrow[r, "F"'] & \mathscr{D}. \end{tikzcd}$$
Now, we have the two functors $\mathscr{M}_F s_1, s_1 F : \mathscr{C} \to \mathscr{M}_{\mathscr{D}}$ which both map to F under the trivial Kan fibration
$$(ev_1)_* : \mathrm{Fun}(\mathscr{C}, \mathscr{M}_{\mathscr{D}}) \to \mathrm{Fun}(\mathscr{C}, \mathscr{D}).$$
Hence the identity on F lifts to a natural isomorphism $t : \mathscr{M}_F s_1 \overset{\sim}{\to} s_1 F$. From t we now obtain the desired isomorphism
$$ev_0(t) : F\Omega = F ev_0 s_1 = ev_0 \mathscr{M}_F s_1 \overset{\sim}{\to} ev_0 s_1 F = \Omega F. \tag{29}$$
eq:2034

(2) We have the exponentiation map $-_* : \mathrm{Fun}(\mathscr{C}, \mathscr{D}) \to \mathrm{Fun}(\mathscr{C}^L, \mathscr{D}^L)$ of Proposition II-13.21 which, at $L = \Delta^1 \times \Delta^1$, sends the K-shaped diagram $F : K \times \mathscr{C} \to \mathscr{D}$ to a map
$$F_* : K \times \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathscr{C}) \to \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathscr{D}).$$
By construction, the exponential map satisfies $F_*|_{\{x\}\times\mathrm{Fun}} = (F_x)_*$ at each vertex x in K. It now follows from our hypotheses on the F_x, and the naturality claim of

Proposition II-13.21, that F_* restricts to a functor $\mathscr{M}_F : K \times \mathscr{M}_{\mathscr{C}} \to \mathscr{M}_{\mathscr{D}}$ which fits into diagrams

$$\begin{tikzcd} K \times \mathscr{M}_{\mathscr{C}} \arrow[r, "\mathscr{M}_F"] \arrow[d, "K \times ev_i"'] & \mathscr{M}_{\mathscr{D}} \arrow[d, "ev_i"] \\ K \times \mathscr{C} \arrow[r, "F"'] & \mathscr{D} \end{tikzcd}$$

at $i = 1, 2$. Since ev_1 is a trivial Kan fibration the map $K \times ev_1$ is a trivial Kan fibration as well.

Fix now isomorphisms $t_x : \mathscr{M}_{F_x} s_1 \overset{\sim}{\to} s_1 F_x$ at each vertex x so that ζ_x is obtained from t_x via the composite (29). Let $\bar{t}$ denote the induced transformation on the zero skeleton

$$\bar{t} : \Delta^1 \times \mathrm{sk}_0(K) \times \mathscr{C} \to \mathscr{M}_{\mathscr{D}}.$$

By construction the composite $ev_0(\bar{t})$ is the identity transformation on $F|_{\mathrm{sk}_0(K)}$. For $A = \mathrm{sk}_0(K)$ we now have the trivial Kan fibration

$$\mathrm{Fun}(K \times \mathscr{C}, \mathscr{M}_{\mathscr{D}}) \to \mathrm{Fun}(A \times \mathscr{C}, \mathscr{M}_{\mathscr{D}}) \times_{\mathrm{Fun}(A \times \mathscr{C}, \mathscr{D})} \mathrm{Fun}(K \times \mathscr{C}, \mathscr{D})$$

provided by restriction and ev_1 (Proposition I-3.11), and lift the natural isomorphism $(\bar{t}, id_F)$ to obtain an isomorphism $t : \mathscr{M}_F s_1 \overset{\sim}{\to} s_1 F$ which restricts to t_x at each vertex x in K. From t we obtain the desired natural isomorphism

$$ev_0(t) : F(K \times \Omega) = F(K \times ev_0)(K \times s_1) = ev_0 \mathscr{M}_F(K \times s_1) \overset{\sim}{\to} ev_0 s_1 F = \Omega F.$$

□

Applying Proposition 6.24 to the stable setting yields the following.

Corollary 6.25. *For any exact functor $F : \mathscr{C} \to \mathscr{D}$ between stable ∞-categories there are natural isomorphisms $F\Omega \overset{\sim}{\to} \Omega F$ and $F\Sigma \overset{\sim}{\to} \Sigma F$.*

We also apply Proposition 6.24 to the case where F is a pointed Hom functor.

`prop:891` **Proposition 6.26.** *For any stable ∞-category $\mathscr{C}$, and any pairs of objects $x, y : * \to \mathscr{C}$, we have canonical isomorphisms*

$$\mathrm{can}^n_\Sigma : \Omega^n \mathrm{Hom}_{\mathscr{C}}(x, y) \overset{\sim}{\to} \mathrm{Hom}_{\mathscr{C}}(\Sigma^n x, y) \ \ \textit{and} \ \ \mathrm{can}^n_\Omega : \Omega^n \mathrm{Hom}_{\mathscr{C}}(x, y) \overset{\sim}{\to} \mathrm{Hom}_{\mathscr{C}}(x, \Omega^n y)$$

in the homotopy category $\mathrm{h}\mathscr{K}an_*$. Furthermore, for any exact functor $F : \mathscr{C} \to \mathscr{D}$ these isomorphisms fit into diagrams

$$\begin{tikzcd} \Omega^n \mathrm{Hom}_{\mathscr{D}}(Fx, Fy) \arrow[r, "\mathrm{can}^n"] & \mathrm{Hom}_{\mathscr{D}}(\Sigma^n Fx, Fy) \\ \Omega^n \mathrm{Hom}_{\mathscr{C}}(x, y) \arrow[r, "\mathrm{can}^n"'] \arrow[u, "F"] & \mathrm{Hom}_{\mathscr{C}}(\Sigma^n x, y) \arrow[u, "F"'] \end{tikzcd}$$

and

$$\begin{tikzcd} \Omega^n \mathrm{Hom}_{\mathscr{D}}(Fx, Fy) \arrow[r, "\mathrm{can}^n"] & \mathrm{Hom}_{\mathscr{D}}(Fx, \Omega^n Fy) \\ \Omega^n \mathrm{Hom}_{\mathscr{C}}(x, y) \arrow[r, "\mathrm{can}^n"'] \arrow[u, "F"] & \mathrm{Hom}_{\mathscr{C}}(x, \Omega^n y), \arrow[u, "F"'] \end{tikzcd}$$

which commute in $\mathrm{h}\mathscr{K}an_*$.

Here, to be clear, $\Omega \operatorname{Hom}_{\mathscr{A}}(a,b)$ is the space of pointed loops at the zero map (Section I-7.4).

Proof. One obtains the claim about suspension Σ from that of Ω by taking opposites. So it suffices to prove the claim about Ω. Furthermore, the claim at $n > 1$ is obtained from the claim at $n = 1$ via iteration. So we consider the case of a single looping.

Consider a pointed Hom functor $H_{\mathscr{C}}$ for $\mathscr{C}$. As $H_{\mathscr{C}} : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{K}an_*$ is left exact in both coordinates, by Proposition 4.9, we have a natural isomorphism

$$H_{\mathscr{C}}(x, \Omega -) \overset{\sim}{\to} \Omega H_{\mathscr{C}}(x, -)$$

at each object x in $\mathscr{C}$, by Proposition 6.24 (1). All naturality claims now follow by Lemma 6.24 (2). $\square$

Remark 6.27. Let us be clear that Proposition 6.26 is completely redundant for those who wish to employ spectra in their analysis of stable categories. It provides, however, a means of circumventing spectra for those who may be less familiar with the subject, and provides simultaneously an indication of the kinds of information which spectra encode.

sect:more_stable

7. More on stable ∞-categories

We continue our study of stable ∞-categories. We discuss completeness and cocompleteness, triangulation of the homotopy category, and exact equivalences. We also prove that the unbounded homotopy and derived ∞-categories are complete and cocomplete.

7.1. Limits and colimits in stable ∞-categories.

We provide an overview of limits and colimits in the stable setting, following [14, 15] directly. To begin, we record the following beautiful result.

prop:stable_finco

Proposition 7.1 ([15, Propositions 1.1.3.4 & 1.1.4.1])**.** *Any stable ∞-category admits all finite limits and colimits. Furthermore, for a functor between stable ∞-categories $F : \mathscr{C} \to \mathscr{D}$ the following are equivalent:*

(a) *F is exact, i.e. preserves zero objects and fiber sequences.*
(b) *F preserves all finite limits.*
(c) *F preserves all finite colimits.*

Recall that, in our study of the homotopy ∞-category, we found that all pullbacks and pushouts agree in $\mathscr{K}(\mathbb{A})$ (Proposition 6.2). It turns out that this type of symmetry holds in any stable ∞-category.

Proposition 7.2 ([15, Proposition 1.1.3.4])**.** *A diagram*

$$\begin{tikzcd} x \arrow[r] \arrow[d] \arrow[dr] & y \arrow[d] \\ y' \arrow[r] & z \end{tikzcd}$$

in a stable ∞-category $\mathscr{C}$ is a pullback diagram if and only if it is a pushout diagram.

It is also the case that finite products and coproducts agree in the stable setting. This identification of products and coproducts reflects a natural additive structure–or even, triangulated structure–which exists at the level of the homotopy category. See Theorem 7.12 below.

Proposition 7.3 ([15, Lemma 1.1.2.9])**.** *Any stable ∞-category $\mathscr{C}$ admits finite products and coproducts, and for any pair of object $x, y : * \to \mathscr{C}$ the map*

$$\begin{bmatrix} id_x & 0 \\ 0 & id_y \end{bmatrix} : x \amalg y \to x \times y$$

is an isomorphism in $\mathscr{C}$.

We are especially interested in cocompleteness and cocontinuity in the stable setting. In considering such issues the following result from [14] proves invaluable.

thm:push_cocomp **Theorem 7.4.** *For an ∞-category $\mathscr{C}$ the following are equivalent:*

(a) *$\mathscr{C}$ admits pushouts and small coproducts.*
(b) *$\mathscr{C}$ is cocomplete.*

Furthermore, for any functor $F : \mathscr{C} \to \mathscr{D}$ between cocomplete ∞-categories, the following are equivalent.

(a′) *F commutes with pushouts and small coproducts.*
(b′) *F is cocontinuous.*

Proof. The equivalences between (a) and (b), and (a′) and (b′), are [14, Proposition 4.4.3.2] and [14, Proposition 4.4.2.7] respectively. □

We apply Proposition 7.1 and Theorem 7.4 to observe the following.

cor:stable_cocomp **Corollary 7.5.** *A stable ∞-category $\mathscr{C}$ is cocomplete if and only if $\mathscr{C}$ admits all small coproducts. Furthermore, an exact functor between stable, cocomplete ∞-categories is cocontinuous if and only if it commutes with small coproducts.*

Remark 7.6. Applying Theorem 7.4 and Corollary 7.5 to the opposite category $\mathscr{C}^{\mathrm{op}}$ recovers analogous characterizations for completeness and continuity.

ex:K_cocomp **Example 7.7** (Completeness and cocompleteness for $\mathscr{K}(\mathbb{A})$)**.** For Grothendieck abelian $\mathbb{A}$, we saw in Proposition 2.23 that $\mathscr{K}(\mathbb{A})$ admits all small products and coproducts. Hence $\mathscr{K}(\mathbb{A})$ is both complete and cocomplete, by Corollary 7.5.

7.2. **Compact objects.**

def:compact **Definition 7.8.** An ∞-category $\mathscr{K}$ is called filtered if any map $A \to \mathscr{K}$ from a finite simplicial set A (Definition II-13.25) admits an extension to the cone $A \star \Delta^0 \to \mathscr{K}$.

As in the usual discrete setting, one has the expected notion of compact objects in an ∞-category.

Definition 7.9. Let $\mathscr{C}$ be a cocomplete ∞-category. An object x in $\mathscr{C}$ is called compact if there is a functor $h^x : \mathscr{C} \to \mathscr{K}an$ which is represented by x, and which preserves small filtered colimits.

Remark 7.10. Since all functors which are represented by x are isomorphic, there exists such a functor h^x which preserves filtered colimits if and only if all functors which are represented by x preserve small filtered colimits.

prop:compact **Proposition 7.11** ([15, Proposition 1.4.4.1]). *Let $\mathscr{C}$ be a cocomplete, stable ∞-category. An object $x : * \to \mathscr{C}$ is compact if and only if, for each small coproduct $\amalg_{\lambda\in\Lambda} y_\lambda$ and map $\alpha : x \to \coprod_\lambda y_\lambda$, there exists a finite subset $\{\lambda_0, \ldots, \lambda_m\} \subseteq \Lambda$ for which α factors as a composite*

$$x \to (y_{\lambda_0} \amalg \cdots \amalg y_{\lambda_m}) \to \coprod_{\lambda\in\Lambda} y_\lambda.$$

7.3. **The homotopy category under stability.** We record the following for the sake of completeness.

thm:triangles **Theorem 7.12** ([15, Theorem 1.1.2.14]). *For any stable ∞-category $\mathscr{C}$, the homotopy category $\mathrm{h}\mathscr{C}$ inherits a natural triangulated structure in which the exact triangles*

$$x \to y \to z$$

are exactly the images of fiber sequences in $\mathscr{C}$. The connecting morphism $\delta : z \to \Sigma x$ for such a triangle is provided by the universal property of the pushout applied to the diagram

$$\begin{tikzcd} x \arrow[r] \arrow[d] \arrow[dr] & y \arrow[d, "0"] \\ 0 \arrow[r] & \Sigma x. \end{tikzcd}$$

One can check directly that this "natural" triangulated structure on $\mathrm{h}\mathscr{C}$ recovers the standard triangulated structure on the discrete homotopy category $K(\mathbb{A}) = \mathrm{h}\mathscr{K}(\mathbb{A})$ and discrete derived category $D(\mathbb{A}) = \mathrm{h}\mathscr{D}(\mathbb{A})$. (See for example [25, Sections 10.2 & 10.4].) We leave this as an exercise for the interested reader.

Corollary 7.13. *Any exact functor between stable ∞-categories $F : \mathscr{C} \to \mathscr{D}$ induces an exact functor of triangulated categories $\mathrm{h}F : \mathrm{h}\mathscr{C} \to \mathrm{h}\mathscr{D}$.*

Additionally, Proposition 7.11 tells us that compactness of objects in a stable ∞-category can be checked at the level of the homotopy category. We recall that an object x in a triangulated category $\mathbb{A}$ which admits small coproducts is called compact if the functor $\mathrm{Hom}_{\mathbb{A}}(x, -)$ commutes with small coproducts. The following is simply a repackaging of Proposition 7.11.

cor:compact **Corollary 7.14.** *An object x in a stable ∞-category $\mathscr{C}$ is compact if and only if its image in in the homotopy category $\mathrm{h}\mathscr{C}$ is compact.*

7.4. **Fully faithfulness via the homotopy category.** It is not the case, generally speaking, that one can detect equivalences between ∞-categories at the level of the homotopy category. Consider, for example, the inclusion $0 : * \to \mathrm{Sing}(S^2)$ of a point into the circle. We have $\mathrm{h}\,\mathrm{Sing}(S^2) = *$, so that the map 0 induces an equivalence on homotopy categories. However, this map is not an equivalence since, using PropositionI-7.11, we have

$$\pi_1 \mathrm{Hom}_{\mathrm{Sing}(S^2)}(0, 0) \cong \pi_2 S^2 = \mathbb{Z}.$$

In the stable setting such phenomena never occurs, as all of the higher homotopy groups in the mapping spaces $\mathrm{Hom}_{\mathscr{C}}(x, y)$ are realized as the 0-th homotopy group of some shifted space $\mathrm{Hom}_{\mathscr{C}}(\Sigma^n x, y)$.

prop:ff_hff **Proposition 7.15** ([2, Proposition 5.10]). *For an exact functor between stable ∞-categories $F : \mathscr{C} \to \mathscr{D}$, the following are equivalent:*

(a) *F is fully faithful (resp. an equivalence).*

(b) $\mathrm{h}\,F : \mathrm{h}\,\mathscr{C} \to \mathrm{h}\,\mathscr{D}$ *is fully faithful (resp. an equivalence).*

Proof. It is clear that if F is fully faithful, or an equivalence, then the map on homotopy categories $\mathrm{h}\,F$ is also fully faithful, or an equivalence. So we have the implication (a) $\Rightarrow$ (b). For the converse claim (b) $\Rightarrow$ (a), essential surjectivity can be checked at the level of the homotopy category. So we need only deal with fully faithfulness.

Suppose that $\mathrm{h}\,F$ is fully faithful. Let $H_{\mathscr{C}}$ and $H_{\mathscr{D}}$ be pointed Hom functors for $\mathscr{C}$ and $\mathscr{D}$ respectively, and $H_F : H_{\mathscr{C}} \to H_{\mathscr{D}}$ be the transformation induced by F. We have that the maps

$$\pi_0(H_F) : \pi_0\left(H_{\mathscr{C}}(x,y)\right) \to \pi_0\left(H_{\mathscr{D}}(Fx,Fy)\right)$$

are isomorphisms at all x and y in $\mathscr{C}$. By applying shifts, and considering Proposition 6.26, it follows that the maps

$$\pi_n(H_F) : \pi_n\left(H_{\mathscr{C}}(x,y)\right) \to \pi_n\left(H_{\mathscr{D}}(Fx,Fy)\right)$$

are isomorphisms at all $n \geq 0$ as well, where again we base at 0. Hence the induced map on loop spaces $\Omega H_F : \Omega H_{\mathscr{C}}(x,y) \to \Omega H_{\mathscr{D}}(Fx,Fy)$ is a homotopy equivalence, by Whitehead's theorem. By replacing y with Σy, and considering the natural isomorphisms of Proposition 6.24, we see now that H_F itself is an equivalence at all x and y in $\mathscr{C}$. Hence F is fully faithful. $\square$

Remark 7.16. If one employs spectral, rather than pointed Hom functors, then the proof of Proposition 7.15 follows immediately from spectral Whitehead.

7.5. **Cocompleteness of the derived ∞-category.** Fix a Grothendieck abelian category $\mathbb{A}$. As explained in the proof of Corollary 6.11, the class of K-injectives in $\mathrm{Ch}(\mathbb{A})$ is stable under the formation of mapping cones. It is also clearly stable under suspension and desuspension. So, as was already argued implicitly, the full ∞-subcategory of K-injectives $\mathscr{D}(\mathbb{A})$ in $\mathscr{K}(\mathbb{A})$ is stable under the formation of both pullbacks and pushouts. This follows by the explicit constructions provided in Propositions 2.13 and 2.21.

prop:D_pullpush **Proposition 7.17.** *Let $\mathbb{A}$ be a Grothendieck abelian category. The derived ∞-category $\mathscr{D}(\mathbb{A})$ admits all pullbacks and pushouts, and the inclusion $\mathscr{D}(\mathbb{A}) \to \mathscr{K}(\mathbb{A})$ preserves all pullback and pushout diagrams.*

Recall from Example 7.7 that $\mathscr{K}(\mathbb{A})$ is both complete and cocomplete. Since products of acyclic complexes are acyclic, it is clear that products of K-injectives are K-injective. It follows tha the derived ∞-category $\mathscr{D}(\mathbb{A})$ is stable under the formation of small products in $\mathscr{K}(\mathbb{A})$, and in particular admits all small products. By Corollary 7.5 it follows that $\mathscr{D}(\mathbb{A})$ is complete and that the inclusion $\mathscr{D}(\mathbb{A}) \to \mathscr{K}(\mathbb{A})$ is continuous. We now consider coproducts and cocompleteness.

prop:D_co_comp **Proposition 7.18.** *For any Grothendieck abelian category $\mathbb{A}$, the derived ∞-category $\mathscr{D}(\mathbb{A})$ is both complete and cocomplete, and the inclusion $\mathscr{D}(\mathbb{A}) = \mathscr{D}_{\mathrm{Inj}} \to \mathscr{K}(\mathbb{A})$ is continuous.*

Proof. Completeness was argued above, as was continuity of the inclusion. For cocompleteness, it suffices to show that $\mathscr{D}(\mathbb{A})$ admits small coproducts, by Corollary 7.5.

Consider any collection of K-injectives $I_- : \Lambda \to \mathscr{D}(\mathbb{A})$ indexed over a small discrete set Λ. Take a K-injective resolution of the resulting coproduct $\kappa : \oplus_{\lambda\in\Lambda} I_\lambda \to I$ and for each index λ let $\kappa_\lambda : I_\lambda \to I$ be the composition of the structural map $I_\lambda \to \oplus_{\lambda\in\Lambda} I_\lambda$ with κ.

Take $\mathscr{D}(\mathbb{A})'$ the simplicial construction of $\mathscr{D}(\mathbb{A})$, with specified equivalence $\mathfrak{Z} : \mathscr{D}(\mathbb{A})' \to \mathscr{D}(\mathbb{A})$ as in Theorem II-10.4. It suffices to show that the extension $i : \Lambda \star \{*\} \to \mathscr{D}(\mathbb{A})'$ of I_-, with cone point I and connecting maps provided by the κ_λ, is a limit diagram. We check this by composing with the corepresentable functor $\underline{\mathrm{Hom}}_{\mathbb{A}}(-, J) = \mathrm{K}\,\mathrm{Hom}^*_{\mathbb{A}}(-, J)$ at any K-injective complex J.

Applying such $\underline{\mathrm{Hom}}_{\mathbb{A}}(-, J)$ to the strictly commuting diagram

$$\begin{tikzcd}
\cdots & I_{\mu'} \arrow[dr] \arrow[ddr] & I_\mu \arrow[d] \arrow[dd, bend left] & I_{\mu''} \arrow[dl] \arrow[ddl] & \cdots \\
& & \oplus_\lambda I_\lambda \arrow[d, "\kappa"'] & & \\
& & I & &
\end{tikzcd}$$

in $\mathscr{K}(\mathbb{A})'$ produces a strictly commuting diagram

$$\begin{tikzcd}
& \underline{\mathrm{Hom}}(I,J) \arrow[d, "\cong"] \arrow[ddl] \arrow[dd, bend right] \arrow[ddr] & \\
& \underline{\mathrm{Hom}}(\oplus_\lambda I_\lambda, J) = \prod_\lambda \underline{\mathrm{Hom}}(I_\lambda, J) \arrow[dl] \arrow[d] \arrow[dr] & \\
\cdots\ \underline{\mathrm{Hom}}_{\mathbb{A}}(I_{\mu'}, J) & \underline{\mathrm{Hom}}_{\mathbb{A}}(I_\mu, J) & \underline{\mathrm{Hom}}_{\mathbb{A}}(I_{\mu''}, J)\ \cdots
\end{tikzcd} \tag{30}$$

eq:6280

in $\mathscr{K}an$. Since the bottom diagram in (30) is a limit diagram (Example II-14.16 & Theorem II-14.25), it follows that the top diagram is a limit diagram as well. In particular, the diagram (30) establishes an isomorphism between $\prod_\lambda \underline{\mathrm{Hom}}_{\mathbb{A}}(I_\lambda, J)$ and $\underline{\mathrm{Hom}}_{\mathbb{A}}(I, J)$ in the overcategory $\mathscr{K}an_{/\underline{\mathrm{Hom}}(\Lambda,J)}$ so that $\underline{\mathrm{Hom}}_{\mathbb{A}}(I, J)$ is seen to be terminal in the overcategory, since the product $\prod_\lambda \underline{\mathrm{Hom}}_{\mathbb{A}}(I_\lambda, J)$ is terminal.

Since J was chosen arbitrarily, it follows by Corollary II-16.16 that the original diagram $i : \Lambda \star \{*\} \to \mathscr{D}(\mathbb{A})'$ is a colimit diagram, and hence that I is a coproduct of the $\{I_\lambda : \lambda \in \Lambda\}$ in $\mathscr{D}(\mathbb{A})'$. Applying the equivalence $\mathfrak{Z} : \mathscr{D}(\mathbb{A})' \to \mathscr{D}(\mathbb{A})$, we find that the diagram

$$\begin{tikzcd}
\cdots & I_{\mu'} \arrow[dr, "\kappa_{\mu'}"'] & I_\mu \arrow[d, "\kappa_\mu"] & I_{\mu''} \arrow[dl, "\kappa_{\mu''}"] & \cdots \\
& & I & &
\end{tikzcd}$$

realizes I as a coproduct of the $\{I_\lambda : \lambda \in \Lambda\}$ in $\mathscr{D}(\mathbb{A})$ as well. □

We also obtain completeness and cocompleteness of the connective derived category.

Corollary 7.19. *Consider the connective derived category $\mathscr{D}^{\leq 0}(\mathbb{A})$ for $\mathbb{A}$ Grothendieck abelian. The following hold:*

(1) *The category* $\mathscr{D}^{\leq 0}(\mathbb{A})$ *is both complete and cocomplete.*
(2) *The inclusion* $\mathscr{D}^{\leq 0}(\mathbb{A}) \to \mathscr{D}(\mathbb{A})$ *is cocontinuous.*

Proof. For completeness, the inclusion $i : \mathscr{D}^{\leq 0}(\mathbb{A}) \to \mathscr{D}(\mathbb{A})$ has a right adjoint which is given by the truncation $\tau_0 : \mathscr{D}(\mathbb{A}) \to \mathscr{D}^{\leq 0}(\mathbb{A})$. This follows by an application of Theorem I-14.9 for example. By Proposition II-13.24 the functor τ_0 is continuous, and hence any diagram $p : K \to \mathscr{D}^{\leq 0}(\mathbb{A})$ has a limit which can be computed as

$$\lim(p) = \tau_0(\lim(ip)).$$

So we see that the connective derived category of complete.

For cocompleteness, and cocontinuity of the inclusion i, we observe directly from the formula for pushouts in $\mathscr{D}(\mathbb{A})$ given in Proposition 2.21 that $\mathscr{D}^{\leq 0}(\mathbb{A})$ is stable under the formation of pushouts in $\mathscr{D}(\mathbb{A})$. Similarly, from the formula for the coproduct given in the proof of Proposition 7.18, we also see that $\mathscr{D}^{\leq 0}(\mathbb{A})$ is stable under the formation of small coproducts in $\mathscr{D}(\mathbb{A})$. By Theorem 7.4 it follows that $\mathscr{D}^{\leq 0}(\mathbb{A})$ is cocomplete and that the inclusion $\mathscr{D}^{\leq 0}(\mathbb{A}) \to \mathscr{D}(\mathbb{A})$ is cocontinuous. □

7.6. **Cocompleteness via the homotopy category.** If one accepts that the discrete derived category $D(\mathbb{A})$ admits both small coproducts and products, then we can approach the proofs of Proposition 7.18, alternatively, via the homotopy category.

prop:cocomp_via_h

Proposition 7.20. *A stable ∞-category $\mathscr{C}$ is cocomplete (resp. complete) if and only if its homotopy category $\mathrm{h}\mathscr{C}$ admits all small coproducts (resp. products).*

Proof. We address cocompleteness. Completeness follows by considering the opposite category.

Let $\{x_\lambda : \lambda \in \Lambda\}$ be a small collection of objects in $\mathscr{C}$, and let x_Λ be a coproduct for this collection in $\mathrm{h}\mathscr{C}$ along with the structure maps $\bar{i}_\lambda : x_\lambda \to x_\Lambda$. Let $i_\lambda : x_\lambda \to x_\Lambda$ be a lift of each $\bar{i}_\lambda$ to a map in $\mathscr{C}$. Fix a pointed Hom functor $H_\mathscr{C}$ for $\mathscr{C}$ as well.

At each y in $\mathscr{C}$ the maps i_λ induce isomorphisms

$$\pi_0[i_\lambda^* : \lambda \in \Lambda]^t : \pi_0\left(H_\mathscr{C}(x_\Lambda, y)\right) \to \prod_{\lambda\in\Lambda} \pi_0\left(H_\mathscr{C}(x_\lambda, y)\right) = \pi_0\left(\prod_{\lambda\in\Lambda} H_\mathscr{C}(x_\lambda, y)\right),$$

by hypothesis. Via the identifications of Proposition 6.26 it follows that the induced maps on all higher homotopy groups

$$\pi_n[i_\lambda^* : \lambda \in \Lambda]^t : \pi_n\left(H_\mathscr{C}(x_\Lambda, y)\right) \to \prod_{\lambda} \pi_n\left(H_\mathscr{C}(x_\lambda, y)\right) = \pi_n\left(\prod_{\lambda} H_\mathscr{C}(x_\lambda, y)\right),$$

all based at 0, are isomorphisms as well. Hence the map on loop spaces

$$\Omega\left(H_\mathscr{C}(x_\Lambda, y)\right) \to \Omega\left(\prod_{\lambda} H_\mathscr{C}(x_\lambda, y)\right)$$

is a homotopy equivalence, by Whitehead's theorem. By replacing y with Σy, and consulting the natural isomorphism of Proposition 6.26, we found that the original map $H_\mathscr{C}(x_\Lambda, y) \to \prod_\lambda H_\mathscr{C}(x_\lambda, y)$ is an equivalence. It follows by Corollary II-16.16 that x_Λ is a coproduct of the $\{x_\lambda : \lambda \in \Lambda\}$ in $\mathscr{C}$. □

sect:adjointer

8. Adjoints, again

Our next goal is to realize the derived ∞-category as a localization of the homotopy ∞-category relative to the class of quasi-isomorphisms. The localization functor in this case is obtained as a left adjoint $\mathrm{loc} : \mathscr{K}(\mathbb{A}) \to \mathscr{D}(\mathbb{A})$ to the inclusion $\mathscr{D}(\mathbb{A}) = \mathscr{D}_{\mathrm{Inj}} \to \mathscr{K}(\mathbb{A})$.

In order to facilitate such an analysis we return to the topic of adjunctions, which we saw previously in Section I-14. We prove that adjoint pairs of functors can be identified with simultaneous cartesian and cocartesian fibrations over the 1-simplex, and determine when the inclusion $\mathscr{C}' \to \mathscr{C}$ of a full ∞-subcategory admits a left (or right) adjoint. Our findings are not only applied to address the localization problem discussed above, but also in our analysis of derived functors, indization of small ∞-categories, and idempotent completions.

8.1. **Reflective subcategories.**

The approach to adjoints which we explore in this section in based on a consideration of so-called reflective and coreflective subcategories.

Definition 8.1 ([16, 02F6])**.** Let $\mathscr{C}' \subseteq \mathscr{C}$ be a full ∞-subcategory. Given an object x in $\mathscr{C}$, a morphism $f : x \to y$ with y in $\mathscr{C}'$ is said to exhibit y as a $\mathscr{C}'$-reflection of x if, for each third object z in $\mathscr{C}'$, the precomposition function

$$f^* : \mathrm{Hom}_{\mathscr{C}}(y, z) \to \mathrm{Hom}_{\mathscr{C}}(x, z)$$

is an isomorphism in $\mathrm{h}\,\mathscr{K}an$. Similarly, a morphism $g : y \to x$ with y in $\mathscr{C}'$, is said to exhibit y as a $\mathscr{C}'$-coreflection of x if, for each third object z in $\mathscr{C}'$, the composition function

$$g_* : \mathrm{Hom}_{\mathscr{C}}(z, y) \to \mathrm{Hom}_{\mathscr{C}}(z, x)$$

is an isomorphism in $\mathrm{h}\,\mathscr{K}an$.

We say $\mathscr{C}'$ itself is a reflective (resp. coreflective) subcategory in $\mathscr{C}$ if every object x in $\mathscr{C}$ admits a morphism $x \to y$ (resp. $y \to x$) which exhibits y as a $\mathscr{C}'$-reflection (resp. $\mathscr{C}'$-coreflection) of x.

One sees directly that taking opposites $\mathscr{C} \mapsto \mathscr{C}^{\mathrm{op}}$ exchanges reflections and coreflections, and exchanges reflective and coreflective subcategories as well. So, throughout the section, we can address reflections with the understanding that analogous result for coreflections are obtained by applying opposites.

As we see in the following examples, $\mathscr{C}'$-reflections and coreflections in $\mathscr{C}$ might be thought of, vaguely, as resolutions by objects in $\mathscr{C}'$.

ex:inj_reflex

Example 8.2 (K-injectives)**.** Let $\mathbb{A}$ be a Grothendiek abelian category, and let $\mathscr{D}_{\mathrm{Inj}} \subseteq \mathscr{K}(\mathbb{A})$ denote the full subcategory of K-injective complexes. Let $\mathscr{D}'_{\mathrm{Inj}} \subseteq \mathscr{K}(\mathbb{A})'$ be the corresponding full subcategory in the simplicial construction of the homotopy ∞-category.

Every complex V in $\mathrm{Ch}(\mathbb{A})$ admits a K-injective resolution $f : V \to I_V$. This map induces a quasi-isomorphism

$$f^* : \mathrm{Hom}^*_{\mathbb{A}}(I_V, J) \to \mathrm{Hom}^*_{\mathbb{A}}(V, J),$$

at each K-injective complex J, which then induces a homotopy equivalence

$$f^* : K\,\mathrm{Hom}^*_{\mathbb{A}}(I_V, J) \to K\,\mathrm{Hom}^*_{\mathbb{A}}(V, J)$$

so that f is a $\mathscr{D}'_{\mathrm{Inj}}$-reflection in $\mathscr{K}(\mathbb{A})'$. It follows via the equivalence $\mathfrak{Z}$ of Theorem II-10.4 that $f : V \to I_V$ is also a $\mathscr{D}_{\mathrm{Inj}}$-reflection in $\mathscr{K}(\mathbb{A})$. We conclude that $\mathscr{D}_{\mathrm{Inj}}$ is a reflective subcategory in $\mathscr{K}(\mathbb{A})$.

ex:proj_coreflex

Example 8.3 (K-projectives)**.** Suppose that an abelian category $\mathbb{A}$ has enough projectives, and let $\mathscr{D}_{\mathrm{Proj}}$ be the full ∞-subcategory of K-projectives in $\mathscr{K}(\mathbb{A})$. Each complex V admits a K-projective resolution $g : P_V \to V$. One argues as in Example 8.2 to see that g is a $\mathscr{D}_{\mathrm{Proj}}$-coflection, and hence to see that $\mathscr{D}_{\mathrm{Proj}}$ is a coreflective subcategory in $\mathscr{K}(\mathbb{A})$.

8.2. **Some technical stuff.** We record a few technical results which will prove helpful in a moment.

Lemma 8.4. *Given any morphism $f : x \to y$, the forgetful functor $\mathscr{C}_{f/} \to \mathscr{C}_{y/}$ is a trivial Kan fibration.*

Proof. By [10, Lemma 3.3] the apparent map

$$(\Delta^1 \star \partial\Delta^n) \coprod_{(\{1\}\star\partial\Delta^n)} (\{1\} \star \Delta^n) \to \Delta^1 \star \Delta^n \cong \Delta^{n+2}$$

is an isomorphism onto the inner horn $\Lambda_1^{n+2} \subseteq \Delta^{n+2}$. Hence solving a lifting problem of the form

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & \mathscr{C}_{f/} \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & \mathscr{C}_{y/} \end{tikzcd}$$

is equivalent to solving a lifting problem of the form

$$\begin{tikzcd} \Lambda_1^{n+2} \arrow[r] \arrow[d] & \mathscr{C} \arrow[d] \\ \Delta^{n+2} \arrow[r] \arrow[ur, dotted] & *. \end{tikzcd}$$

Since $\mathscr{C}$ is an ∞-category, the latter problem always admits a solution. □

lem:1253

Lemma 8.5 ([16, 02LL])**.** *For any morphism $f : x \to y$ and object $z : * \to \mathscr{C}$, we have a commuting diagram*

$$\begin{tikzcd} \mathscr{C}_{y/} \times_{\mathscr{C}} \{z\} \arrow[d, "\cong"'] & \mathscr{C}_{f/} \times_{\mathscr{C}} \{z\} \arrow[l, "\cong"'] \arrow[r] & \mathscr{C}_{x/} \times_{\mathscr{C}} \{z\} \arrow[d, "\cong"] \\ \mathrm{Hom}_{\mathscr{C}}(y, z) \arrow[rr, "f^*"'] & & \mathrm{Hom}_{\mathscr{C}}(x, z). \end{tikzcd}$$

in $\mathrm{h}\mathscr{K}an$.

Here the vertical maps are specifically those induced by the coslice diagonal equivalences. We only recall the main idea of the proof here.

Idea of proof. One produces a morphism

$$i : \mathscr{C}_{f/} \times_{\mathscr{C}} \{z\} \to \{f\} \times_{\mathrm{Hom}_{\mathscr{C}}(x,y)} \mathrm{Fun}(\Delta^2, \mathscr{C})_{\vec{x}}$$

which bisects the diagram

$$\begin{tikzcd}
\mathscr{C}_{y/} \times_{\mathscr{C}} \{z\} \arrow[d, "\cong"'] & \mathscr{C}_{f/} \times_{\mathscr{C}} \{z\} \arrow[l, "\cong"'] \arrow[r] \arrow[d] & \mathscr{C}_{x/} \times_{\mathscr{C}} \{z\} \arrow[d, "\cong"] \\
\mathrm{Hom}_{\mathscr{C}}(y,z) & \{f\} \times_{\mathrm{Hom}_{\mathscr{C}}(x,y)} \mathrm{Fun}(\Delta^2, \mathscr{C})_{\vec{x}} \arrow[l] \arrow[r] & \mathrm{Hom}_{\mathscr{C}}(x,z).
\end{tikzcd}$$

at the level of the discrete category of Kan complexes. □

prop:reflex_char

Proposition 8.6. *Let $\mathscr{C}' \subseteq \mathscr{C}$ be a full ∞-subcategory. For a fixed morphism $f : x \to y$, with y in $\mathscr{C}'$, the following are equivalent:*

(a) *f exhibits y as a $\mathscr{C}'$-reflection of x.*
(b) *At each object z in $\mathscr{C}'$ the forgetful functor $\mathscr{C}_{f/} \times_{\mathscr{C}} \{z\} \to \mathscr{C}_{x/} \times_{\mathscr{C}} \{z\}$ is an equivalence.*
(c) *The map $\mathscr{C}_{f/} \times_{\mathscr{C}} \mathscr{C}' \to \mathscr{C}_{x/} \times_{\mathscr{C}} \mathscr{C}'$ is a trivial Kan fibration.*
(d) *Each lifting problem*

$$\begin{tikzcd}
\Lambda^n_0 \arrow[r, "\tau"] \arrow[d] & \mathscr{C} \arrow[d] \\
\Delta^n \arrow[r] \arrow[ur, dotted] & *
\end{tikzcd}$$

with $n \geq 2$, $\tau|_{\Delta^{\{0,1\}}} = f$, and $\tau|_{\Delta^{\{1,\dots,n\}}}$ having image in $\mathscr{C}'$ admits a solution.

Proof. The equivalence (a) ⇔ (b) is a consequence of Lemma 8.5. For (b) ⇔ (c), we recall that the map

$$\mathscr{C}_{f/} = (\mathscr{C}_{x/})_{y/} \to \mathscr{C}_{x/}$$

is a left fibration by Corollary I-5.27, and hence its base change $\mathscr{C}_{f/} \times_{\mathscr{C}} \mathscr{C}' \to \mathscr{C}_{f/} \times_{\mathscr{C}} \mathscr{C}'$ is a left fibration as well. This left fibration furthermore fits into a diagram

$$\begin{tikzcd}
\mathscr{C}_{f/} \times_{\mathscr{C}} \mathscr{C}' \arrow[rr] \arrow[dr] & & \mathscr{C}_{f/} \times_{\mathscr{C}} \mathscr{C}' \arrow[dl] \\
& \mathscr{C}' &
\end{tikzcd}.$$

Theorem II-3.8 and Corollary II-9.8 together now imply that the forgetful functor $\mathscr{C}_{f/} \times_{\mathscr{C}} \mathscr{C}' \to \mathscr{C}_{x/} \times_{\mathscr{C}} \mathscr{C}'$ is a trivial Kan fibration if and only if at each object $z : * \to \mathscr{C}'$ the fiber

$$\mathscr{C}_{f/} \times_{\mathscr{C}} \{z\} \to \mathscr{C}_{x/} \times_{\mathscr{C}} \{z\}$$

is an equivalence. Statement (d) is identified with (c) via the identification of lifting problems

$$\begin{tikzcd}
\partial\Delta^n \arrow[r] \arrow[d] & \mathscr{C}_{f/} \times_{\mathscr{C}} \mathscr{C}' \arrow[d] \\
\Delta^n \arrow[r] \arrow[ur, dotted] & \mathscr{C}_{x/} \times_{\mathscr{C}} \mathscr{C}'
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
\Lambda^{n+2}_0 \arrow[r] \arrow[d] & \mathscr{C} \arrow[d] \\
\Delta^n \arrow[r] \arrow[ur, dotted] & *
\end{tikzcd}$$

which is implied by Lemma II-9.10. □

8.3. **Reflectivity and adjoints.** We now pursue a characterization of reflective and coreflective subcategories via adjunctions.

prop:1327

Proposition 8.7. *Let $\mathscr{C}' \subseteq \mathscr{C}$ be a full ∞-subcategory and $i : \mathscr{C}' \to \mathscr{C}$ be the inclusion. The subcategory $\mathscr{C}'$ is reflective if and only if there is a functor $L : \mathscr{C} \to \mathscr{C}'$ and a transformation $u : id_{\mathscr{C}} \to iL$ for which, at each x in $\mathscr{C}$, the map $u_x : x \to L(x)$ exhibits $L(x)$ as a $\mathscr{C}'$-reflection of x.*

Proof. It is clear that $\mathscr{C}'$ is reflective whenever such a functor L with transformation u exists. Conversely, if $\mathscr{C}'$ is reflective in $\mathscr{C}$, let $\mathscr{E} \subseteq \mathscr{C} \times \Delta^1$ be the full subcategory whose objects $\mathscr{E}[0]$ are the union $(\mathscr{C}[0] \times \{0\}) \cup (\mathscr{C}'[0] \times \{1\})$. By Proposition 8.6 the projection

$$q : \mathscr{E} \to \Delta^1$$

is a cocartesian fibration, and a map $f : (x, 0) \to (y, 1)$ in $\mathscr{E}$ is q-cocartesian if and only if the underlying map $f : x \to y$ in $\mathscr{C}$ exhibits y as a $\mathscr{C}$-reflection of x.

Now, by Theorem II-2.7 there exists a unique functor $U : \Delta^1 \times \mathscr{C} \to \mathscr{E}$ which splits the diagram

$$\begin{tikzcd} \{0\} \times \mathscr{C} \arrow[r, "\mathrm{incl}"] \arrow[d] & \mathscr{E} \arrow[d, "q"] \\ \Delta^1 \times \mathscr{C} \arrow[r, "p_1"'] \arrow[ur, "U"] & \Delta^1 \end{tikzcd}$$

and sends each map $\Delta^1 \times \{x\}$ to a q-cocartesian morphism in $\mathscr{E}$. For $L : \mathscr{C} \to \mathscr{C}' \subseteq \mathscr{C}$ defined as the composite

$$\mathscr{C} \cong \{1\} \times \mathscr{C} \to \Delta^1 \times \mathscr{C} \to \mathscr{E} \xrightarrow{p_1} \mathscr{C},$$

the transformation $u = p_1 U : \Delta^1 \times \mathscr{C} \to \mathscr{C}$ has the prescribed property. □

For $\mathscr{C}'$ reflective in $\mathscr{C}$, we claim that a tranformation $u : id_{\mathscr{C}} \to iL$ as in Proposition 8.7 exhibits L as a left adjoint to the inclusion $i : \mathscr{C}' \to \mathscr{C}$. In order to establish this claim, it is helpful to have some sufficient condition which allows one to check that a given transformation induces an adjunction between functors.

lem:1357

Lemma 8.8 ([16, 02DK])**.** *Let $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ be functors between ∞-categories, and $u : id_{\mathscr{C}} \to GF$ be a transformation. Suppose that the induced transformations*

$$Fu : F \to F(GF) \quad \textit{and} \quad uG : G \to (GF)G$$

are isomorphisms in $\mathrm{Fun}(\mathscr{C}, \mathscr{D})$ *and* $\mathrm{Fun}(\mathscr{D}, \mathscr{C})$ *respectively, and that G is fully faithful. Then u is the unit of an adjunction in which the counit $\epsilon : FG \to id_{\mathscr{D}}$ is a natural isomorphism.*

Let us recall here that a tranformation between functors in a given functor category $\mathrm{Fun}(K, \mathscr{E})$ is a natural isomorphism if and only if it evaluates to an isomorphism at each object in K (Theorem I-7.6).

Sketch proof. Since G is fully faithful the induced map $G_* : \mathrm{Fun}(\mathscr{C}, \mathscr{D}) \to \mathrm{Fun}(\mathscr{C}, \mathscr{C})$ is fully faithful. Hence there is a unique transformation $\epsilon : FG \to id_{\mathscr{D}}$ which lifts the isomorphism $(uG)^{-1} : GFG \to G$. Fully faithfulness implies that ϵ is an isomorphism as well.

We now replace the functor categories $\mathrm{Fun}(\mathscr{A},\mathscr{B})$ with their homotopy categories $\mathrm{h}\,\mathrm{Fun}(\mathscr{A},\mathscr{B})$ and work with the corresponding 2-category Cat^2_∞ obtained from the simplicial category $\underline{\mathrm{Cat}}_\infty$. At this level we consider the composites

$$F \overset{Fu}{\to} FGF \overset{\epsilon F}{\to} F \quad \text{and} \quad G \overset{uG}{\to} GFG \overset{G\epsilon}{\to} G.$$

The latter composite is the identity by the definition of ϵ. We consider now the composite $\beta = (\epsilon F)(Fu)$. By our assumptions, Fu is an isomorphism, so that β is an isomorphism as well. One now argues, using [16, 02CX], that β also satisfies $\beta^2 = \beta$ and hence $\beta = id_F$ necessarily. □

rem:1376 **Remark 8.9.** If a transformation $u : id_{\mathscr{C}} \to GF$ admits another transformation $\epsilon : FG \to id_{\mathscr{D}}$ which exhibits F as left adjoint to G, then the transformation ϵ is fixed up to the action of $\mathrm{Aut}_{\mathrm{Fun}(\mathscr{D},\mathscr{C})}(G)$ [16, 02D7]. In particular, *any* transformation $FG \to id_{\mathscr{D}}$ which pairs with u to realize F as left adjoint to G, in the situation of Lemma 8.8, must be a natural isomorphism.

prop:1380 **Proposition 8.10.** *Let $i : \mathscr{C}' \to \mathscr{C}$ be the inclusion of a full subcategory into an ∞-category $\mathscr{C}$. Consider any functor $L : \mathscr{C} \to \mathscr{C}'$ and transformation $u : id_{\mathscr{C}} \to iL$. The following are equivalent:*

(a) *The transformation u is part of an adjunction which exhibits L as left adjoint to the inclusion $i : \mathscr{C}' \to \mathscr{C}$.*

(b) *At each object x in $\mathscr{C}$, the map $u_x : x \to L(x)$ exhibits $L(x)$ as a $\mathscr{C}'$-reflection of x.*

(c) *At each x in $\mathscr{C}$ the map $L(u_x) : L(x) \to LL(x)$ is an isomorphism and, at each y in $\mathscr{C}'$, the map $u_y : y \to L(y)$ is an isomorphism.*

Furthermore, in this case, any transformation $\epsilon : Li \to id_{\mathscr{C}'}$ which pairs with u to realize L as left adjoint to i is a natural isomorphism.

Proof. Supposing (a) and (c) hold, the claim about ϵ follows by Lemma 8.8 and Remark 8.9. Now, suppose (a) holds. Then by Corollary I-14.3 the transformation u realizes the $\mathrm{h}\,\mathscr{K}an$-enriched functor $\pi L : \pi\mathscr{C} \to \pi\mathscr{C}'$ as left adjoint to the enriched embedding $\pi i : \pi\mathscr{C}' \to \pi\mathscr{C}$. Hence, at each z in $\mathscr{C}'$, u induces isomorphisms

$$u^* : \mathrm{Hom}_{\mathscr{C}}(L(x), z) \to \mathrm{Hom}_{\mathscr{C}}(x, z)$$

in $\mathrm{h}\,\mathscr{K}an$. Thus (b) holds.

Suppose now that (b) holds. When y is in $\mathscr{C}'$ applying u yields an isomorphism of sets

$$u^* : \pi_0 \mathrm{Hom}_{\mathscr{C}'}(L(y), z) \to \pi_0 \mathrm{Hom}_{\mathscr{C}'}(y, z)$$

which shows, via Yoneda, that $u_y : y \to L(y)$ is an isomorphism in $\mathrm{h}\,\mathscr{C}'$. By definition this implies that u_y is an isomorphism in $\mathscr{C}'$.

As for the transformation $L(u_x) : L(x) \to LL(x)$ at general x, we have the diagram

$$\begin{tikzcd}
x \arrow[r, "u_x"] \arrow[d, "u_x"'] & L(x) \arrow[d, "u_{L(x)}"] \\
L(x) \arrow[r, "L(u_x)"'] & LL(x)
\end{tikzcd}$$

in $\mathrm{h}\mathscr{C}$ and apply $\mathrm{Hom}_{\mathrm{h}\mathscr{C}}(-,z)$ at arbitrary z in $\mathscr{C}'$ to obtain a diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{h}\mathscr{C}}(L(x),z) \arrow[r, "\cong"] \arrow[d, "\cong"'] & \mathrm{Hom}_{\mathrm{h}\mathscr{C}'}(x,z) \arrow[d, "\cong"] \\
\mathrm{Hom}_{\mathrm{h}\mathscr{C}'}(LL(x),z) \arrow[r, "L(u_x)^*"'] & \mathrm{Hom}_{\mathrm{h}\mathscr{C}'}(L(x),z).
\end{tikzcd}$$

From this we conclude that $L(u_x)^*$ is an isomorphism at all z, and hence that $L(u_x)$ is an isomorphism in $\mathrm{h}\mathscr{C}$. It follows that $L(u_x)$ is an isomorphism in $\mathscr{C}$.

Finally, Lemma 8.8 tells us directly that (c) implies (a). This completes the proof. □

The following is an immediate consequence of Propositions 8.7 and 8.10.

thm:reflex_adj

Theorem 8.11. *Given a full ∞-subcategory $\mathscr{C}' \subseteq \mathscr{C}$ is reflective in $\mathscr{C}$ if and only if the inclusion $i : \mathscr{C}' \to \mathscr{C}$ admits a left adjoint $L : \mathscr{C} \to \mathscr{C}'$ whose unit and counit transformations have the properties outlined in Proposition 8.10. Similarly, $\mathscr{C}'$ is coreflective in $\mathscr{C}$ if and only if the inclusion admits a right adjoint $R : \mathscr{C} \to \mathscr{C}'$ with the prescribed properties.*

Proof. The claim about reflective subcategories is clear, and the claim about coreflective subcategories is obtain by applying opposites. □

ex:1454

Example 8.12. Let $\mathbb{A}$ be Grothendieck abelian. We saw in Example 8.2 that $\mathscr{D}_{\mathrm{Inj}}$ is reflective in $\mathscr{K}(\mathbb{A})$. Hence the inlclusion $\mathscr{D}_{\mathrm{Inj}} \to \mathscr{K}(\mathbb{A})$ admits a left adjoint $L : \mathscr{K}(\mathbb{A}) \to \mathscr{D}_{\mathrm{Inj}}$, by Theorem 8.11.

8.4. Adjoints via simultaneous fibrations.

lem:cocart_ref

Lemma 8.13. *Consider an inner fibration $q : \mathscr{E} \to \Delta^1$ along with its fibers $\mathscr{E}_i = \mathscr{E} \times_{\Delta^1} \{i\}$. The following hold:*

1. *$\mathscr{E}_1$ is a reflective subcategory in $\mathscr{E}$ if and only if q is a cocartesian fibration, and in this case a map $f : x \to y$ over $0 < 1$ is q-cocartesian if and only if it exhibits y as a $\mathscr{E}_1$-reflection for x.*
2. *$\mathscr{E}_0$ is a coreflective subcategory in $\mathscr{E}$ if and only if q is a cartesian fibration, and in this case a map $g : y \to x$ over $0 < 1$ is q-cartesian if and only if it exhibits y as a $\mathscr{E}_0$-coreflection for x.*

Proof. (1) Follows by Proposition 8.6 (d). (2) Follows from (1) by taking opposites. □

prop:adj_fibration

Proposition 8.14. *Let $q : \mathscr{E} \to \Delta^1$ be a cocartesian fibration, and $F : \mathscr{E}_0 \to \mathscr{E}_1$ be the functor given by covariant transport along q (Definiiton II-7.1). The functor F admits a right adjoint $G : \mathscr{E}_1 \to \mathscr{E}_0$ if and only if q is a cartesian fibration as well, and in this case G is given by contravariant transport along q.*

Our claim that G is "given by contravariant transport" should be interpreted in a strict sense. Namely, we claim that when F admits a right adjoint G, there is a cartesian transformation $\Delta^1 \times \mathscr{E}_1 \to \mathscr{E}$, i.e. a cartesian solution to the lifting problem

$$\begin{tikzcd} \{1\} \times \mathscr{E}_1 \arrow[r] \arrow[d] & \mathscr{E} \arrow[d, "q"] \\ \Delta^1 \times \mathscr{E}_1 \arrow[r] \arrow[ur, dotted] & \Delta^1 \end{tikzcd}$$

(see Proposition II-2.6), whose restriction to $\{0\} \times \mathscr{E}_1$ recovers G.

Proof. First suppose that F admits such a right adjoint $G : \mathscr{E}_1 \to \mathscr{E}_0$, and consider the unit and counit transformations

$$u : id_{\mathscr{E}_0} \to GF \quad \text{and} \quad \epsilon : FG \to id_{\mathscr{E}_1}.$$

By Proposition 8.6 (c) we understand that q is cartesian if and only if the subcategory $\mathscr{E}_0$ is coreflective in $\mathscr{E}$. So we seek to demonstrate $\mathscr{E}_0$-coreflections $g_x : y \to x$ at each x in $\mathscr{E}$. When x is in $\mathscr{E}_0$ we can just take $g_x = id_x$. So we may assume that x lies in the fiber $\mathscr{E}_1$.

First, let us consider the extended fibration $q^l : \mathscr{E}^l \to \Delta^1$ where $\mathscr{E}^l \subseteq \mathscr{E} \times \Delta^1$ is the full subcategory with objects $(\mathscr{E}[0] \times \{0\}) \cup (\mathscr{E}_1[0] \times \{1\})$. Here q^l is specifically the projection onto the second factor. The fact that q is a cocartesian fibration implies that q^l is a cocartesian fibration, and furthermore the apparent inclusion $\mathscr{E} \to \mathscr{E}^l$ preserves cocartesian edges.

The transport functor $L : \mathscr{E} \to \mathscr{E}_1$ along q^l comes equipped with a transformation $\eta : id_{\mathscr{E}} \to L$ which evaluates to a q^l-cocartesian morphism at each x in $\mathscr{E}$, by definition. By uniqueness of transport functors and the fact that the restriction functor

$$\mathrm{Fun}(\Delta^1 \times \mathscr{E}, \mathscr{E}) \to \mathrm{Fun}(\Delta^1 \times \mathscr{E}_0, \mathscr{E})$$

is an isofibration (Corollary I-6.14), we can choose L so that $L|_{\mathscr{E}_0} = F$. Note also that η exhibits L as left adjoint to the inclusion $i_1 : \mathscr{E}_1 \to \mathscr{E}$, by Lemma 8.8.

At each x in $\mathscr{E}_1$ define the map $g_x : G(x) \to x$ as a composite

$$G(x) \overset{\eta}{\to} LG(x) = FG(x) \overset{\epsilon}{\to} x.$$

We then have at each z in $\mathscr{E}_0$ the sequence of maps

$$\mathrm{Hom}_{\mathscr{E}_0}(z, G(x)) \overset{F}{\to} \mathrm{Hom}_{\mathscr{E}_1}(F(z), FG(x)) \overset{\epsilon_*}{\to} \mathrm{Hom}_{\mathscr{E}_1}(F(z), x) \overset{\eta^*}{\to} \mathrm{Hom}_{\mathscr{E}}(z, x)$$

in the homotopy category of spaces. As the first two maps compose to an isomorpihsm, and the third map is also an isomorphism, this composite is an isomorphism. By commutativity of the operations $\epsilon_* = \epsilon \circ -$ and $\eta^* = - \circ \eta$, i.e. by associativity of composition in $\mathrm{h}\mathscr{K}an$, and naturality of η (Lemma I-14.2), the above composite is equal to the map

$$(g_x)_* : \mathrm{Hom}_{\mathscr{E}_0}(z, G(x)) \to \mathrm{Hom}_{\mathscr{E}}(z, x),$$

which we conclude is an isomorphism. So each $g_x : G(x) \to x$ exhibits $G(x)$ as a $\mathscr{E}_0$-coreflecton by Lemma 8.13, $\mathscr{E}_0$ is seen to be coreflective in $\mathscr{E}$, and q is therefore a cartesian fibration.

Suppose conversely that $q : \mathscr{E} \to \Delta^1$ is cartesian. Then the subcategory $\mathscr{E}_0$ is coreflective in $\mathscr{E}$ by Lemma 8.13 and, by Theorem 8.11, the inclusion $i_0 : \mathscr{E}_0 \to \mathscr{E}$ admits a right adjoint $R : \mathscr{E} \to \mathscr{E}_0$ whose counit transformation $i_0 R \to id_{\mathscr{E}}$ evaluates to a cartesian edge at each object in $\mathscr{E}$ (Proposition 8.10). Hence the restriction of the counit to $\Delta^1 \times \mathscr{E}_1$ provides the unique cartesian solution to the lifting problem

$$\begin{tikzcd} \{1\} \times \mathscr{E}_1 \arrow[r] \arrow[d] & \mathscr{E} \arrow[d, "q"] \\ \Delta^1 \times \mathscr{E}_1 \arrow[r] \arrow[ur, dotted] & \Delta^1, \end{tikzcd}$$

and we see that $Ri_1 = R|_{\mathscr{E}_1}$ is recovered via contravariant transport along q.

Define $G : \mathscr{E}_1 \to \mathscr{E}_0$ to be the aforementioned restriction $G = Ri_1$. The inclusion $\mathscr{E}_1 \to \mathscr{E}$ has left adjoint $L : \mathscr{E} \to \mathscr{E}_1$, which restricts to F on $\mathscr{E}_0$, and we see that G is a composite of right adjoints. Hence G itself is right adjoint to the functor $Li_0 = F$, as desired (see [16, 02DT]).

As for the claim that *any* right adjoint to F is given by contravariant transport, consider $G : \mathscr{E}_1 \to \mathscr{E}_0$ a right adjoint to F with counit transformation $\epsilon : FG \to id_{\mathscr{E}_1}$. Consider the transformation $\widetilde{\epsilon} : \Delta^1 \times \mathscr{E}_1 \to \mathscr{E}$ which sends an n-simplex $(\alpha, \sigma) : \Delta^n \to \Delta^1 \times \mathscr{E}_1$ to the triple

$$(\alpha,\ G\sigma|_{\Delta^{\alpha^{-1}(0)}},\ \sigma) : \Delta^n \to \mathscr{E} = \mathrm{N}^F(\Delta^1).$$

At each $y : * \to \mathscr{E}_1$, $\widetilde{\epsilon}$ sends the edge $\Delta^1 \times \{y\}$ to the edge $(G(y), \epsilon_y : FG(y) \to y)$ in $\mathscr{E}$. Let us denote this edge $\widetilde{\epsilon}_y : \Delta^1 \to \mathscr{E}$.

By Corollary I-14.4, ϵ realizes the $\mathrm{h}\mathscr{K}an$-enriched functor πF and left adjoint to πG, so that the composite

$$\mathrm{Hom}_{\mathscr{E}_0}(z, G(y)) \xrightarrow{F} \mathrm{Hom}_{\mathscr{E}_1}(F(z), FG(y)) \xrightarrow{\epsilon_*} \mathrm{Hom}_{\mathscr{E}_1}(F(z), y)$$

is an isomorphism at arbitrary z in $\mathscr{E}_0$ and y in $\mathscr{E}_1$. One checks that this sequence recovers composition $(\widetilde{\epsilon}_y)_* : \mathrm{Hom}_{\mathscr{E}}(z, G(y)) \to \mathrm{Hom}_{\mathscr{E}}(z, y)$ in $\pi\mathscr{E}$ to see that $\widetilde{\epsilon}_y$ is an $\mathscr{E}_0$-coreflection in $\mathscr{E}$. By Lemma 8.13 we conclude that each $\widetilde{\epsilon}_y$ is q-cartesian in $\mathscr{E}$, and hence that $\widetilde{\epsilon}$ is a cartesian solution to the lifting problem

$$\begin{tikzcd} \{1\} \times \mathscr{E}_1 \arrow[r] \arrow[d] & \mathscr{E} \arrow[d, "q"] \\ \Delta^1 \times \mathscr{E}_1 \arrow[r] \arrow[ur, dotted] & \Delta^1. \end{tikzcd}$$

By construction $\widetilde{\epsilon}|_{\{0\} \times \mathscr{E}_1} = G$. □

We now obtain a characterization of adjunctions via fibrations over the 1-simplex.

thm:adj_fibration

Theorem 8.15. *Given a pair of functor $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$, the following are equivalent:*

(a) *The functors F and G admit transformations which exhibit F as left adjoint to G.*

(b) *There is a simultaneous cartesian and cocartesian fibration $q : \mathscr{E} \to \Delta^1$ with fixed isomorphisms at the fibers, $\mathscr{C} \cong \mathscr{E}_0$ and $\mathscr{D} \cong \mathscr{E}_1$, for which F and G are recovered as covariant and contravariant transport along q respectively.*

Proof. Note that F defines a functor $F : \Delta^1 \to \mathscr{C}at_\infty$ and consider the weighted nerve $q : \mathscr{E} = \mathrm{N}^F(\Delta^1) \to \Delta^1$. The fact that F is recovered by covariant transport along q is implicit in the claim that there is an isomorphism of fibrations $\mathscr{E} \cong \int_{\Delta^1} F$ (Theorem II-6.28). However, we can just observe this fact directly.

For an n-simplex $\sigma = (\sigma', \sigma'') : \Delta^n \to \Delta^1 \times \mathscr{C} = \Delta^1 \times \mathscr{E}_0$ take $\Delta^{n_0} = (\sigma')^{-1}(0)$. Define now

$$\widetilde{\sigma} : \Delta^n \to \mathscr{E}$$

as the triple consisting of the n-simplex $\sigma' : \Delta^n \to \Delta^1$ along with the pair of n-simplices $\sigma''|_{\Delta^{n_0}} : \Delta^{n_0} \to \mathscr{C}$ and $F\sigma'' : \Delta^n \to \mathscr{D}$. The assignment $\sigma \mapsto \widetilde{\sigma}$ defines a cocartesian lift of the inclusion $\{0\} \times \mathscr{C} \to \mathscr{E}$, and so recovers $F : \mathscr{C} \to \mathscr{E}$ as covariant transport along q.

In any case, we recover the claimed equivalence between (a) and (b) by applying Proposition 8.14 to the weighted nerve for F. □

8.5. **Local criterion for adjunction.**

thm:local_adj **Theorem 8.16.** *A functor between ∞-categories $F : \mathscr{C} \to \mathscr{D}$ admits a right adjoint if and only if, for each y in $\mathscr{D}$, there exists an object x in $\mathscr{C}$ and a morphism $g_y : F(x) \to y$ such that, at each z in $\mathscr{C}$, the sequence*

$$\mathrm{Hom}_{\mathscr{C}}(z, x) \overset{F}{\to} \mathrm{Hom}_{\mathscr{D}}(F(z), F(x)) \overset{(g_y)_*}{\to} \mathrm{Hom}_{\mathscr{D}}(F(z), y) \tag{31}$$ eq:1550

is an isomorphism in $\mathrm{h}\mathscr{K}an$.

Proof. If there exists an adjoint G then we can take $x = G(y)$ and g the counit morphism. Conversely, suppose we can find such g_y at each y in $\mathscr{D}$. Then for the weighted nerve $q : \mathscr{E} = \mathrm{N}^F(\Delta^1) \to \Delta^1$ covariant trasport along q recovers the functor F, as was argued in the proof of Theorem 8.15.

In the weighted nerve

$$\mathrm{Hom}_{\mathscr{E}}(z, x) = \mathrm{Hom}_{\mathscr{C}}(z, x) \quad \text{and} \quad \mathrm{Hom}_{\mathscr{E}}(z, y) = \mathrm{Hom}_{\mathscr{D}}(F(z), y),$$

and one can check directly that the composition function

$$\mathrm{Hom}_{\mathscr{E}}(x, y) \times \mathrm{Hom}_{\mathscr{E}}(z, x) \to \mathrm{Hom}_{\mathscr{E}}(z, y)$$

is obtained by applying $F : \mathrm{Hom}_{\mathscr{C}}(z, x) \to \mathrm{Hom}_{\mathscr{D}}(F(z), F(x))$ then composing in $\mathscr{D}$. Hence, if we view g_y as a morphism in $\mathscr{E}$, the operation

$$(g_y)_* : \mathrm{Hom}_{\mathscr{E}}(z, x) \to \mathrm{Hom}_{\mathscr{E}}(z, y)$$

is identified with the sequence (31). We conclude that the sequence (31) is an isomorphism at some g_y, for each y, if and only if the fiber $\mathscr{D} = \mathscr{E}_1$ is coreflective in $\mathscr{E}$, which then occurs if and only if $q : \mathscr{E} \to \Delta^1$ is a cartesian fibration by Proposition 8.6 (c). Apply Proposition 8.14 to see that F has a right adjoint in this case. □

Taking opposites, we observe the analagous local criterion for the existence of left adjoints.

Theorem 8.17. *A functor $G : \mathscr{D} \to \mathscr{C}$ admits a left adjoint if and only if, for each x in $\mathscr{C}$, there exists an object y in $\mathscr{D}$ and a morphism $f_x : x \to G(y)$ such that, at each z in $\mathscr{D}$, the sequence*

$$\mathrm{Hom}_{\mathscr{D}}(y, z) \overset{G}{\to} \mathrm{Hom}_{\mathscr{C}}(G(y), G(z)) \overset{f_x^*}{\to} \mathrm{Hom}_{\mathscr{C}}(x, G(z))$$

is an isomoprhism in $\mathrm{h}\mathscr{K}an$.

8.6. **Going halfsies on adjoints.**

prop:adj_half

Proposition 8.18. *For functors $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ between ∞-categories the following are equivalent:*

(a) *The functor F is left adjoint to G.*

(b) *There is a tranformation $\epsilon : FG \to id_{\mathscr{D}}$ for which, at each z in $\mathscr{C}$ and y in $\mathscr{D}$, the composite*

$$\mathrm{Hom}_{\mathscr{C}}(z, G(y)) \xrightarrow{F} \mathrm{Hom}_{\mathscr{D}}(F(z), FG(y)) \xrightarrow{(\epsilon_y)_*} \mathrm{Hom}_{\mathscr{D}}(F(z), y)$$

is an isomorphism in $\mathrm{h}\mathscr{K}an$.

(c) *There is a transformation $\eta : id_{\mathscr{C}} \to GF$ for which, at each x in $\mathscr{C}$ and z in $\mathscr{D}$, the composite*

$$\mathrm{Hom}_{\mathscr{D}}(F(x), z) \xrightarrow{G} \mathrm{Hom}_{\mathscr{C}}(GF(x), G(z)) \xrightarrow{\eta_x^*} \mathrm{Hom}_{\mathscr{C}}(x, G(z))$$

is an isomorphism in $\mathrm{h}\mathscr{K}an$.

Furthermore, in the case of (b) *the transformation ϵ is the counit of an adjunction between F and G, and in the case of* (c) *the transformation η is the unit of such an adjunction.*

Proof. The implications (a) $\Rightarrow$ (b) and (a) $\Rightarrow$ (c) are immediate. For the implication (b) $\Rightarrow$ (a), suppose we have a tranformation $\epsilon : FG \to id_{\mathscr{D}}$ as in (b). Consider the weighted nerve $q : \mathscr{E} = \mathrm{N}^F(\Delta^1) \to \Delta^1$ for F. As was argued in the proof of Theorem 8.16, $\mathscr{D} = \mathscr{E}_1$ is coreflective in $\mathscr{E}$ and by Lemma 8.13 the fibration q is both cartesian and cocartesian. Furthermore Lemma 8.13 tells us that the morphism $(G(y),\ \epsilon_y : FG(y) \to y)$ is q-cartesian in $\mathscr{E}$, at each y in $\mathscr{D}$.

Define the functor $\widetilde{\epsilon} : \Delta^1 \times \mathscr{D} = \Delta^1 \times \mathscr{E}_1 \to \mathscr{E}$ which takes a simplex $\widetilde{\sigma} = (\alpha, \sigma) : \Delta^n \to \Delta^1 \times \mathscr{D}$ to the triple

$$\big\{\alpha : \Delta^n \to \Delta^1,\ G\sigma|_{\Delta^{\alpha^{-1}(0)}} : \Delta^{\alpha^{-1}(0)} \to \mathscr{C},\ \epsilon\widetilde{\sigma} : \Delta^n \to \mathscr{D}\big\}.$$

By direct inspection $\widetilde{\epsilon}$ fits into a diagram

$$\begin{tikzcd} \{1\} \times \mathscr{D} \arrow[r, "\mathrm{incl}"] \arrow[d] & \mathscr{E} \arrow[d, "q"] \\ \Delta^1 \times \mathscr{D} \arrow[r, "\mathrm{proj}"'] \arrow[ur, "\widetilde{\epsilon}"] & \Delta^1 \end{tikzcd} \tag{32}$$ eq:2513

and at each y in $\mathscr{D}$ the edge $\widetilde{\epsilon}|_{\Delta^1 \times \{y\}} = (G(y),\ \epsilon_y : FG(y) \to y) : \Delta^1 \to \mathscr{E}$ is q-cartesian, as explained above. So $\widetilde{\epsilon}$ is the unique cartesian transformation which splits the above square (see Corollary II-2.8), and by Theorem 8.15 the functor $G = \widetilde{\epsilon}|_{\{0\} \times \mathscr{D}}$ is seen to be right adjoint to F.

We claim now that $\epsilon : FG \to id_{\mathscr{D}}$ is the counit transformation in a pair of transformation which (ϵ, η) which realize G as right adjoint to F. However, this follows by uniqueness of the cartesian transformation which splits the diagram (32). Specifically, the counit $\epsilon' : FG \to id_{\mathscr{D}}$ can be used to define a cocartesian transformation $\widetilde{\epsilon}' : \Delta^1 \times \mathscr{D} \to \mathscr{E}$ exactly as above, so that we obtain an isomorphism $\widetilde{\epsilon} \cong \widetilde{\epsilon}'$ in $\mathrm{Fun}(\Delta^1 \times \mathscr{D}, \mathscr{E})$.

We have the transformation $k : F \to id_{\mathscr{D}}$ in $\mathrm{Fun}(\Delta^1, \mathscr{C}at_\infty)$ which one simply observes by the existence of the strictly commuting diagram

$$\begin{tikzcd} \mathscr{C} \arrow[r, "F"] \arrow[d, "F"'] & \mathscr{D} \arrow[d, "id_{\mathscr{D}}"] \\ \mathscr{D} \arrow[r, "id_{\mathscr{D}}"'] & \mathscr{D}. \end{tikzcd}$$

This tranformation defines a functor $\mathrm{N}^k : \mathscr{E} = \mathrm{N}^F(\Delta^1) \to \mathrm{N}^{id_{\mathscr{D}}}(\Delta^1) = \Delta^1 \times \mathscr{D}$, and we compose with the projection $\Delta^1 \times \mathscr{D} \to \mathscr{D}$ to obtain a functor

$$\pi : \mathscr{E} \to \mathscr{D}.$$

We recover our original transformations as $\pi\widetilde{\epsilon} = \epsilon$ and $\pi\widetilde{\epsilon}' = \epsilon'$, so that we obtain an isomorphism $\epsilon \cong \epsilon'$ in $\mathrm{Fun}(\Delta^1 \times \mathscr{D}, \mathscr{D})$. Since compositions of morphisms are stable under isomorphisms, in any ∞-category, the fact that ϵ' can be paired with a transformation $\eta : id_{\mathscr{C}} \to GF$ with witnesses G as right adjoint to F implies that the pair (ϵ, η) also witnesses G as right adjoint to F.

We now obtain the converse implication (b) $\Rightarrow$ (a), and hence the equivalence (a) $\Leftrightarrow$ (b). The equivalence (a) $\Leftrightarrow$ (c) follows by taking opposites. $\square$

8.7. **Universal properties of adjoints.**

prop:univprop_right

Proposition 8.19. *Suppose a functor $F : \mathscr{C} \to \mathscr{D}$ admits a right adjoint $G : \mathscr{D} \to \mathscr{C}$, and let $\epsilon : FG \to id_{\mathscr{D}}$ be the counit transformation for this adjunction. Suppose that we have another functor $G' : \mathscr{D} \to \mathscr{C}$ and a transformation $\epsilon' : FG' \to id_{\mathscr{D}}$. Then the following hold:*

1. *There exists a transformation $\zeta : G' \to G$ for which ϵ' is a composite $\epsilon' = \epsilon(F\zeta)$ in $\mathrm{Fun}(\Delta^1 \times \mathscr{D}, \mathscr{D})$.*
2. *A transformation ζ as in (1) is an isomorphism if and only if ϵ' realizes G' as a(nother) right adjoint F.*

Proof. (1) We have the cartesian (and cocartesian) fibration $q : \mathscr{E} = \mathrm{N}^F(\Delta^1) \to \Delta^1$ and the cartesian lift

$$\begin{tikzcd} \{1\} \arrow[r, "\mathrm{incl}"] \arrow[d] & \mathrm{Fun}(\mathscr{D}, \mathscr{E}) \arrow[d, "q_*"] \\ \Delta^1 \arrow[r, "\mathrm{proj}"'] \arrow[ur, "\widetilde{\epsilon}"] & \mathrm{Fun}(\mathscr{D}, \Delta^1) \end{tikzcd}$$

as in the proof of Proposition 8.18. (Here we recall that q_* is a cartesian fibration, by Proposition II-2.6.) The map $\widetilde{\epsilon} : \Delta^1 \times \mathscr{D} \to \mathscr{E}$ has restrictions

$$\widetilde{\epsilon}|_0 = (\mathscr{D} \overset{G}{\to} \mathscr{C} = \mathscr{E}_0 \to \mathscr{E}) \ \text{ and } \ \widetilde{\epsilon}|_1 = (\mathscr{D} = \mathscr{E}_1 \to \mathscr{E}).$$

Let us denote these maps $i_0 G$ and i_1 respectively.

Pulling back q_* along the projection, we obtain a cartesian (and cocartesian) fibration $\mathscr{F} \to \Delta^1$ and a cartesian lift of the morphism $0 < 1$, which we also denote by $\widetilde{\epsilon} : \Delta^1 \to \mathscr{F}$ by an abuse of notation. From ϵ' we obtain a not-necessarily-cartesian lift $\widetilde{\epsilon}' : \Delta^1 \to \mathscr{F}$ which is defined in exactly the same manner.

By Lemma 8.13 the map $\widetilde{\epsilon}$ induces an isomorphism

$$\widetilde{\epsilon}_* : \mathrm{Hom}_{\mathrm{Fun}(\mathscr{D}, \mathscr{C})}(G', G) \to \mathrm{Hom}_{\mathrm{Fun}(\mathscr{D}, \mathscr{E})}(i_0 G', i_1)$$

in $\mathrm{h}\,\mathscr{K}an$, so that the transformation $\widetilde{\epsilon}' : i_0G' \to i_1$ lifts uniquely to a map $\zeta : G' \to G$ with $\widetilde{\epsilon}\zeta = \widetilde{\epsilon}'$ in $\mathrm{Fun}(\Delta^1, \mathrm{Fun}(\mathscr{D}, \mathscr{E})) = \mathrm{Fun}(\Delta^1 \times \mathscr{D}, \mathscr{E})$.

Again we consider the transformation $k : \Delta^1 \to \mathrm{Fun}(\Delta^1, \mathscr{C}at_\infty)$ between F and $id_{\mathscr{D}}$ determined by the strictly commuting diagram

$$\begin{tikzcd} \mathscr{C} \arrow[r, "F"] \arrow[d, "F"'] & \mathscr{D} \arrow[d, "id"] \\ \mathscr{D} \arrow[r, "id"'] & \mathscr{D} \end{tikzcd}$$

to obtain a projection $\pi : \mathscr{E} \to \mathscr{D}$ via the composite

$$\mathscr{E} = \mathrm{N}^F(\Delta^1) \xrightarrow{\mathrm{N}^k} \mathrm{N}^{id_{\mathscr{D}}}(\Delta^1) = \Delta^1 \times \mathscr{D} \xrightarrow{p_2} \mathscr{D}.$$

Composing with π provides the claimed identification $\epsilon(F\zeta) = \pi\widetilde{\epsilon}\zeta = \pi\widetilde{\epsilon}' = \epsilon'$.

The proof of (2) is left to the interested reader. $\square$

We have the analogous universal property for the left adjoint, which one obtains by taking opposites.

prop:univprop_left

Proposition 8.20. *Suppose a functor $G : \mathscr{D} \to \mathscr{C}$ admits a left adjoint $F : \mathscr{C} \to \mathscr{D}$, and let $\eta : id_{\mathscr{C}} \to GF$ be the unit transformation for this adjunction. Suppose that we have another functor $F' : \mathscr{D} \to \mathscr{C}$ and a transformation $\eta' : id_{\mathscr{C}} \to GF'$. Then the following hold:*

(1) *There exists a transformation $\zeta : F \to F'$ for which η' is a composite $\eta' = (G\zeta)\eta$ in $\mathrm{Fun}(\Delta^1 \times \mathscr{C}, \mathscr{C})$.*

(2) *A transformation ζ as in (1) is an isomorphism if and only if η' realizes F' as a(nother) left adjoint F.*

sect:D_loc

9. The derived ∞-category via localization

We show that the derived ∞-category $\mathscr{D}(\mathbb{A})$ can be recovered as a localization $\mathscr{D}(\mathbb{A}) = \mathscr{K}(\mathbb{A})[\mathrm{Qiso}^{-1}]$ of the homotopy ∞-category relative to the class of quasi-isomorphisms. After applying some fundamental results from [15], we observe a further identification of the derived ∞-category as a localization of the *discrete* category of cochains

$$\mathscr{D}(\mathbb{A}) = \mathrm{Ch}(\mathbb{A})[\mathrm{Qiso}^{-1}]. \tag{33}$$ eq:ch_loc

This latter characterization is, from the "classical" perspective of triangulated categories, completely unanticipated. The formulation (33) is extremely useful in that it allows one to transfer structures up from the descrete setting to the ∞-setting.

9.1. **The setup.** Throughout this section $\mathbb{A}$ is a Grothendieck abelian category. We recall that $\mathbb{A}$ has enough injectives in this case, and that at the level of cochains every complex V admits a quasi-isomorphism $V \to I$ to a K-injective [22, Theorem 3.13]. From this we conclude that the full subcategory

$$\mathscr{K}(\mathbb{A}) \supseteq \mathscr{D}_{\mathrm{Inj}} = \left\{ \begin{array}{c} \text{The full subcategory of } K\text{-injective} \\ \text{complexes in } \mathscr{K}(\mathbb{A}). \end{array} \right\}$$

is reflective (Example 8.2).

We are also interested in the cases where $\mathbb{A}$ admits enough projectives. For some examples, one might consider the following:

- $\mathbb{A} = A$-Mod for a ring A.

- $\mathbb{A} = \mathrm{QCoh}(\mathfrak{X})$ where $\mathfrak{X} = [X/G]$ is the quotient stack of an affine scheme by the action of a reductive algebraic group G, in characteristic 0. In this case the compact projective objects are identified with equivariant vector bundles on X, under the pullback equivalence from $\mathrm{QCoh}(\mathfrak{X})$ the the category of equivariant quasi-coherent sheaves on X.
- $\mathbb{A} = \mathrm{Rep}_q(G)$, the category of quantum group representations for a semisimple algebraic group G at an arbitrary complex parameter $q \in \mathbb{C}$.

In such cases the full subcategory of K-projectives

$$\mathscr{K}(\mathbb{A}) \supseteq \mathscr{D}_{\mathrm{Proj}} = \left\{ \begin{array}{c} \text{The full } \infty\text{-subcategory of } K\text{-projective} \\ \text{complexes in } \mathscr{K}(\mathbb{A}) \end{array} \right.$$

forms a *co*reflective subcategory in $\mathscr{K}(\mathbb{A})$ (Example 8.3).

Remark 9.1. Recall from Part I, Theorem I-13.5, that the two "models" for the derived ∞-category are identified, $\mathscr{D}_{\mathrm{Proj}} \cong \mathscr{D}_{\mathrm{Inj}}$, whenever they both exist.

9.2. **Reflections and coreflections as reoslutions.**

lem:injres_refl

Lemma 9.2. *For a morphism $f : V \to X$ in $\mathscr{K}(\mathbb{A})$, the following are equivalent:*

(a) *The object X is K-injective and f is a quasi-isomorpihsm.*
(b) *The morphism f is a $\mathscr{D}_{\mathrm{Inj}}$-reflection.*

Proof. The implication (a) $\Rightarrow$ (b) follows from the fact that maps into any K-injective $\mathrm{Hom}^*_{\mathbb{A}}(-, Z)$ preserve quasi-isomoprphisms. Hence the corepresentable functor

$$K\,\mathrm{Hom}^*_{\mathbb{A}}(-, Z) : \mathscr{K}(\mathbb{A})^{\mathrm{op}} \to \mathrm{h}\,\mathscr{K}an$$

sends quasi-isomorphisms to isomorphisms.

For (b) $\Rightarrow$ (a), suppose f is a $\mathscr{D}_{\mathrm{Inj}}$-reflection. Then X is K-injective, by definition. Suppose, by way of contradiction, that f is not a quasi-isomorphism. Then the mapping cone $\mathrm{cone}(f)$ is not acyclic, and there is some integer i so that

$$H^i(\mathrm{cone}(f)) \neq 0.$$

Let $\alpha'' : H^i(\mathrm{cone}(f)) \to I^0$ be an inclusion into an injective object (which exists since $\mathbb{A}$ has enough injectives), $\alpha' : Z^0(\mathrm{cone}(f)) \to I^0$ be the restriction along the projection from the cocycles, and $\alpha : (\mathrm{cone}(f))^i \to I^0$ be an arbitrary lift to degree i cochains. We note that such a lift exists via injectivity of I^0.

Take now $I = \Sigma^{-i} I^0$, considered as a complex. The map α now defines a map of cochains

$$\alpha : \mathrm{cone}(f) \to I$$

which recovers α'' on cohomology. In particular, α is not homotopically trivial, and hence realizes a nonzero class in cohomology

$$\bar{\alpha} \in H^0\,\mathrm{Hom}^*_{\mathbb{A}}(\mathrm{cone}(f), I) \cong H^0\left(\Sigma^{-1}\,\mathrm{cone}(f^*)\right).$$

It follows that the induced map $f^* : \mathrm{Hom}^*_{\mathbb{A}}(X, I) \to \mathrm{Hom}^*_{\mathbb{A}}(V, I)$ is not a quasi-isomorphism.

We have in particular

$$\mathrm{gr}\,H^0\left(\Sigma^{-1}\,\mathrm{cone}(f^*)\right) = \begin{array}{c} \ker\left(H^0\,\mathrm{Hom}^*_{\mathbb{A}}(X, I) \to H^0\,\mathrm{Hom}^*_{\mathbb{A}}(V, I)\right) \\ \oplus \\ \mathrm{coker}\left(H^{-1}\,\mathrm{Hom}^*_{\mathbb{A}}(X, I) \to H^{-1}\,\mathrm{Hom}^*_{\mathbb{A}}(V, I)\right) \end{array}$$

under the apparent filtation on the mapping cone so that the above arguments show that at least one of the maps

$$H^{\varepsilon}(f^*) : H^{\varepsilon}\operatorname{Hom}^*_{\mathbb{A}}(X, I) \to H^{\varepsilon}\operatorname{Hom}^*_{\mathbb{A}}(V, I)$$

at $\varepsilon = 0, -1$, is not an isomorphism. Thus the induced map of simplicial abelian groups

$$f^* : K\operatorname{Hom}^*_{\mathbb{A}}(X, I) \to K\operatorname{Hom}^*_{\mathbb{A}}(V, I)$$

is not a isomorphism in $\mathrm{h}\,\mathscr{K}an$, by Theorem I-11.13. Since the complex I is K-injective, this contradicts the assumption that f is a $\mathscr{D}_{\mathrm{Inj}}$-reflection, and we conclude that reflection-ness of f forces f to be a quasi-isomorphism. □

Completely similar arguments apply in the projective situation.

lem:projres_corefl **Lemma 9.3.** *Suppose that $\mathbb{A}$ has enough projectives. Then for a morphism $g : X \to V$ in $\mathscr{K}(\mathbb{A})$ the following are equivalent:*

(a) *The object X is K-projective and g is a quasi-isomorphism.*
(b) *The morphism g is a $\mathscr{D}_{\mathrm{Proj}}$-coreflection.*

9.3. **Precomposition and natural isomorphisms in functor categories.** The following general lemma will prove useful in a moment.

lem:1696 **Lemma 9.4.** *Let $\zeta : F_0 \to F_1$ be a natural transformation between functors $F_i : \mathscr{K} \to \mathscr{K}'$.*

(1) *For each ∞-category $\mathscr{C}$ the functors ζ induces a natural transformation $\zeta^* : F_0^* \to F_1^*$ between the corresponding functors $F_i^* : \operatorname{Fun}(\mathscr{K}', \mathscr{C}) \to \operatorname{Fun}(\mathscr{K}, \mathscr{C})$.*
(2) *If ζ is an isomorphism, then ζ^* is an isomorphism as well.*

The proof is similar to that of Proposition II-13.21.

Construction 9.4. The transformation ζ is a 1-simplex $\zeta : \Delta^1 \to \operatorname{Fun}(\mathscr{K}, \mathscr{K}')$ which restricts to F_i at $\{i\}$, for $i = 0, 1$. So composition in the simplicial category $\underline{\mathrm{Cat}}_\infty$ provides us with a map

$$\zeta^* : \operatorname{Fun}(\mathscr{K}', \mathscr{C}) \times \Delta^1 \to \operatorname{Fun}(\mathscr{K}', \mathscr{C}) \times \operatorname{Fun}(\mathscr{K}, \mathscr{K}') \xrightarrow{\circ} \operatorname{Fun}(\mathscr{K}, \mathscr{C})$$

whose restrictions to $\{i\} \subseteq \Delta^1$ recover the maps F_i^*. Thus ζ^* is a transformation $\zeta^* : F_0^* \to F_1^*$.

Similarly, if we have an n-simplex $\sigma : \Delta^n \to \operatorname{Fun}(\mathscr{K}, \mathscr{K}')$ with vertices $G_i : \mathscr{K} \to \mathscr{K}'$ we get an n-simplex $\sigma^* : \operatorname{Fun}(\mathscr{K}', \mathscr{C}) \times \Delta^n \to \operatorname{Fun}(\mathscr{K}, \mathscr{C})$ with vertices G_i^*, and one sees that σ^* is degenerate whenever σ is degenerate. Hence diagrams of the form

$$\begin{tikzcd} & F_1 \arrow[dr, "\eta"] & \\ F_0 \arrow[ur, "\zeta"] \arrow[rr, "id_{F_0}"'] & & F_0 \end{tikzcd} \qquad \begin{tikzcd} & F_0 \arrow[dr, "\zeta"] & \\ F_1 \arrow[ur, "\eta"] \arrow[rr, "id_{F_1}"'] & & F_1 \end{tikzcd}$$

in $\mathrm{Fun}(\mathscr{K},\mathscr{K}')$ imply diagrams of the form

$$\begin{tikzcd} & F_1^* \arrow[dr, "\eta^*"] & \\ F_0^* \arrow[ur, "\zeta^*"] \arrow[rr, "id_{F_0^*}"'] & & F_0^* \end{tikzcd} \qquad \begin{tikzcd} & F_0^* \arrow[dr, "\zeta^*"] & \\ F_1^* \arrow[ur, "\eta^*"] \arrow[rr, "id_{F_1^*}"'] & & F_1^* \end{tikzcd}$$

in $\mathrm{Fun}(\mathrm{Fun}(\mathscr{K}',\mathscr{C}),\mathrm{Fun}(\mathscr{K},\mathscr{C}))$. So we see directly that ζ^* is isomorphism whenever ζ is an isomorphism. $\square$

sect:loclocloc

9.4. **Localizating the homotopy ∞-category against quasi-isomorphisms.**

prop:pre_loc

Proposition 9.5. *Let $L:\mathscr{K}(\mathbb{A})\to\mathscr{D}_{\mathrm{Inj}}$ be the left adjoint to the inclusion $i:\mathscr{D}_{\mathrm{Inj}}\to\mathscr{K}(\mathbb{A})$, along with the unit and counit transformations*

$$u: id_{\mathscr{K}(\mathbb{A})}\to iL \quad and \quad \epsilon: Li\to id_{\mathscr{D}_{\mathrm{Inj}}}$$

as in Proposition 8.10.

1. *At each x in $\mathscr{K}(\mathbb{A})$ the unit transformation $u_x: x\to L(x)$ is a quasi-isomorphism.*
2. *A map $\alpha: x\to y$ in $\mathscr{K}(\mathbb{A})$ is a quasi-isomorphism if and only if $L(\alpha): L(x)\to L(y)$ is an isomoprhism.*
3. *The counit transformation is a natural isomoprhism.*

Proof. Statement (1) follows from the characterization of $\mathscr{D}_{\mathrm{Inj}}$-reflections provided by Lemma 9.2 and Theorem 8.11. For (2), naturality of u implies, for each morphism $\alpha: x\to y$, the existence of a diagram

$$\begin{tikzcd} L(x) \arrow[r, "L(\alpha)"] & L(y) \\ x \arrow[u, "u_x"] \arrow[r, "\alpha"'] & y \arrow[u, "u_y"'] \end{tikzcd}$$

in the discrete homotopy category $K(\mathbb{A})=\mathrm{h}\mathscr{K}(\mathbb{A})$. The vertical maps in this diagram are quasi-isomorpihsms by (1), so that α is a quasi-isomorphism if and only if $L(\alpha)$ is a quasi-isomorphism. However, a map between K-injectives is a quasi-isomorphism if and only if it is a homotopy equivalence, i.e. an isomoprhism in $\mathscr{D}_{\mathrm{Inj}}$.

Statement (3) is implied directly by the generic description of ϵ given in Proposition 8.10. $\square$

Let us recall that the localization $\mathscr{C}[W^{-1}]$ of an ∞-category at a class of morphisms $W\subseteq\mathscr{C}[1]$, with all degenerate 1-simplices in W, is any ∞-category $\mathscr{D}$ equipped with a functor $F:\mathscr{C}\to\mathscr{D}$ which induces, at all $\mathscr{E}$, a fully faithful functor

$$F^*:\mathrm{Fun}(\mathscr{D},\mathscr{E})\to\mathrm{Fun}(\mathscr{C},\mathscr{E})$$

whose essential image is the full subcategory spanned by those functors $\mathscr{C}\to\mathscr{E}$ which send all maps in W to isomorphisms in $\mathscr{E}$ (Definition II-14.17). In this case we write, somewhat ambiguously, $\mathscr{D}=\mathscr{C}[W^{-1}]$.

thm:DInj_as_loc

Theorem 9.6. *Let $\mathbb{A}$ be a Grothendieck abelian category and $L:\mathscr{K}(\mathbb{A})\to\mathscr{D}_{\mathrm{Inj}}$ be left adjoint to the inclusion.*

(1) *Given any ∞-category $\mathscr{E}$, restriction along L provides a fully faithful functor*
$$L^* : \mathrm{Fun}(\mathscr{D}_{\mathrm{Inj}}, \mathscr{E}) \to \mathrm{Fun}(\mathscr{K}(\mathbb{A}), \mathscr{E})$$
which is an equivalence onto the full subcategory spanned by those functors which send quasi-isomoprihsms in $\mathscr{K}(\mathbb{A})$ to isomoprhisms in $\mathscr{E}$.

(2) *For $\mathrm{Fun}(\mathscr{K}(\mathbb{A}), \mathscr{E})^{\mathrm{Qiso}}$ the full subcategory spanned by those functors which send quasi-isomorphisms in $\mathscr{K}(\mathbb{A})$ to isomorphisms in $\mathscr{E}$, the inverse to the equivalence L^* is given by restiction along the inclusion $i : \mathscr{D}_{\mathrm{Inj}} \to \mathscr{K}(\mathbb{A})$,*
$$i^* : \mathrm{Fun}(\mathscr{K}(\mathbb{A}), \mathscr{E})^{\mathrm{Qiso}} \to \mathrm{Fun}(\mathscr{D}_{\mathrm{inj}}, \mathscr{E})$$

(3) *The functor $L : \mathscr{K}(\mathbb{A}) \to \mathscr{D}_{\mathrm{Inj}}$ exhibits $\mathscr{D}_{\mathrm{Inj}}$ as a localization $\mathscr{D}_{\mathrm{Inj}} = \mathscr{K}(\mathbb{A})[\mathrm{Qiso}^{-1}]$.*

Proof. (3) Follows from (1), simply by the definition of a localization. We prove (1) and (2). Take $\mathscr{K} = \mathscr{K}(\mathbb{A})$ and $\mathscr{D} = \mathscr{D}_{\mathrm{Inj}}$, and let $\mathscr{E}$ be arbitrary. Take $\mathrm{Fun}(\mathscr{K}, \mathscr{E})^{\mathrm{Qiso}}$ as in (2). By Proposition 9.5 (2), the functor L^* has image in $\mathrm{Fun}(\mathscr{K}, \mathscr{E})^{\mathrm{Qiso}}$ and so restricts to a map
$$L^* : \mathrm{Fun}(\mathscr{D}, \mathscr{E}) \to \mathrm{Fun}(\mathscr{K}, \mathscr{E})^{\mathrm{Qiso}}.$$
We also have the functor $i^* : \mathrm{Fun}(\mathscr{K}, \mathscr{E})^{\mathrm{Qiso}} \to \mathrm{Fun}(\mathscr{D}, \mathscr{E})$. We claim that these functors are mutually inverse, and hence realize the claimed equivalence. More prescisely, we claim that the counit and unit transformations ϵ and u induce isomorphisms
$$\epsilon^* : i^* L^* \to id_{\mathrm{Fun}(\mathscr{D}, \mathscr{E})} \quad \text{and} \quad u^* : id_{\mathrm{Fun}(\mathscr{K}, \mathscr{E})^{\mathrm{Qiso}}} \to L^* i^*.$$
The fact that ϵ^* is an isomorphism just follows from the fact that ϵ itself is a isomorphism. See Proposition 9.5 and Lemma 9.4. So we need only address the transformation u^*.

First note that
$$u^* : \mathrm{Fun}(\mathscr{K}, \mathscr{E}) \times \Delta^1 \to \mathrm{Fun}(\mathscr{K}, \mathscr{E})$$
sends each object in the subcategory $\mathrm{Fun}(\mathscr{K}, \mathscr{E})^{\mathrm{Qiso}} \times \Delta^1$ to an object in $\mathrm{Fun}(\mathscr{K}, \mathscr{E})^{\mathrm{Qiso}}$, since u^* is a transformation between $id_{\mathrm{Fun}(\mathscr{K}, \mathscr{E})}$ and $L^* i^*$ and these endofunctors preserve the subcategory $\mathrm{Fun}(\mathscr{K}, \mathscr{E})^{\mathrm{Qiso}}$. Since $\mathrm{Fun}(\mathscr{K}, \mathscr{E})^{\mathrm{Qiso}}$ is full in $\mathrm{Fun}(\mathscr{K}, \mathscr{E})$ it follows that u^* does in fact restrict to a transformation
$$u^* : \mathrm{Fun}(\mathscr{K}, \mathscr{E})^{\mathrm{Qiso}} \times \Delta^1 \to \mathrm{Fun}(\mathscr{K}, \mathscr{E})^{\mathrm{Qiso}}$$
between the identity and $L^* i^*$.

We need to show that u^* is a natural isomorphism. By Proposition I-7.9 it suffices to show that u^* evaluates to an isomorphism in $\mathrm{Fun}(\mathscr{K}, \mathscr{E})^{\mathrm{Qiso}}$ at each functor $T : \mathscr{K} \to \mathscr{E}$ in $\mathrm{Fun}(\mathscr{K}, \mathscr{E})^{\mathrm{Qiso}}$. By the definition of u^* from Construction 9.4 we have at each T
$$u^*_T : \Delta^1 \xrightarrow{u} \mathrm{Fun}(\mathscr{K}, \mathscr{K}) \xrightarrow{T_*} \mathrm{Fun}(\mathscr{K}, \mathscr{E})^{\mathrm{Qiso}} \subseteq \mathrm{Fun}(\mathscr{K}, \mathscr{E}),$$
and to see that u^*_T is an isomoprhism it again suffices to show that u^*_T evaluates to an isomorphism at each x in $\mathscr{K}$. At any such x we have
$$(u^*_T)_x = T(u_x) : T(x) \to TL(x).$$

By Proposition 9.5 (1) each map u_x is a quasi-isomorphism, and since T sends quasi-isomorphisms to isomorphism we have that $(u_T^*)_x$ is an isomoprhism in $\mathscr{E}$, as desired. So we see that u^* is in fact a natural isomorpihsm.

We now have natural isomorphisms $\epsilon^* : i^*L^* \to id_{\mathrm{Fun}(\mathscr{D},\mathscr{E})}$ and $(u^*)^{-1} : L^*i^* \to id_{\mathrm{Fun}(\mathscr{K},\mathscr{E})^{\mathrm{Qiso}}}$. By the definition of $\mathscr{C}at_\infty$ as the homotopy coherent nerve of the simplicial category $\underline{\mathrm{Cat}}_\infty^+$, these natural isomorphsms provide 2-simplices

$$\begin{tikzcd} & \mathrm{Fun}(\mathscr{K},\mathscr{E})^{\mathrm{Qiso}} \arrow[dr, "i^*"] & \\ \mathrm{Fun}(\mathscr{D},\mathscr{E}) \arrow[ur, "L^*"] \arrow[rr, "id_{\mathrm{Fun}(\mathscr{K},\mathscr{D})}"'] & & \mathrm{Fun}(\mathscr{D},\mathscr{E}) \end{tikzcd}$$

and

$$\begin{tikzcd} & \mathrm{Fun}(\mathscr{D},\mathscr{E}) \arrow[dr, "L^*"] & \\ \mathrm{Fun}(\mathscr{K},\mathscr{E})^{\mathrm{Qiso}} \arrow[ur, "i^*"] \arrow[rr, "id_{\mathrm{Fun}(\mathscr{K},\mathscr{E})^{\mathrm{Qiso}}}"'] & & \mathrm{Fun}(\mathscr{K},\mathscr{E})^{\mathrm{Qiso}} \end{tikzcd}$$

in $\mathscr{C}at_\infty$ which realize L^* and i^* as mutually inverse. □

In the event that $\mathbb{A}$ has enough projectives, we can consider the right adjoint $R : \mathscr{K}(\mathbb{A}) \to \mathscr{D}_{\mathrm{Proj}}$ along with its unit and counit transformations $u : id_{\mathscr{D}_{\mathrm{Proj}}} \to Ri$ and $\epsilon : iR \to id_{\mathscr{K}(\mathbb{A})}$. We have that u is a natural isomorphism, that ϵ evaluates to a quasi-isomorphism $\epsilon_x : R(x) \to x$ at each x in $\mathscr{K}(\mathbb{A})$, and that a map $\alpha : x \to y$ in $\mathscr{K}(\mathbb{A})$ is a quasi-isomorphism if and only if $R(\alpha) : R(x) \to R(y)$ is an isomorphism. To observe these properties one argues exactly as in the proof of Proposition 9.5. We can therefore argue as in the proof of Theorem 9.6 to realize the projective construction of the derived ∞-category as a localization as well.

thm:DProj_as_loc

Theorem 9.7. *Let $\mathbb{A}$ be an abelian category with enough projectives, and $R : \mathscr{K}(\mathbb{A}) \to \mathscr{D}_{\mathrm{Proj}}$ be right adjoint to the inclusion.*

(1) *For any ∞-category $\mathscr{E}$, restriction along R provides a fully faithful functor*

$$R^* : \mathrm{Fun}(\mathscr{D}_{\mathrm{Proj}},\mathscr{E}) \to \mathrm{Fun}(\mathscr{K}(\mathbb{A}),\mathscr{E})$$

which is an equivalence onto the full subcategory spanned by those functors $T : \mathscr{K}(\mathbb{A}) \to \mathscr{E}$ which send quasi-isomorphisms in $\mathscr{K}(\mathbb{A})$ to isomorphisms in $\mathscr{E}$.

(2) *For $\mathrm{Fun}(\mathscr{K}(\mathbb{A}),\mathscr{E})^{\mathrm{Qiso}}$ the full subcategory spanned by those functors which send quasi-isomorphisms in $\mathscr{K}(\mathbb{A})$ to isomorphisms in $\mathscr{E}$, the inverse to R^* is given by restriction $i^* : \mathrm{Fun}(\mathscr{K}(\mathbb{A}),\mathscr{E})^{\mathrm{Qiso}} \to \mathrm{Fun}(\mathscr{D}_{\mathrm{Proj}},\mathscr{E})$ along the inclusion $i : \mathscr{D}_{\mathrm{Proj}} \to \mathscr{K}(\mathbb{A})$.*

(3) *The functor $R : \mathscr{K}(\mathbb{A}) \to \mathscr{D}_{\mathrm{Proj}}$ realizes $\mathscr{D}_{\mathrm{Proj}}$ as a localization $\mathscr{D}_{\mathrm{Proj}} = \mathscr{K}(\mathbb{A})[\mathrm{Qiso}^{-1}]$.*

As in the case of $\mathscr{D}_{\mathrm{Inj}}$, one sees that the inverse to R^* is given by restriction along the inclusion $\mathscr{D}_{\mathrm{Proj}} \to \mathscr{K}(\mathbb{A})$.

cor:1844 **Corollary 9.8.** *For any Grothendieck abelian category $\mathbb{A}$ which has enough projectives, there is a unique equivalence $\psi : \mathscr{D}_{\mathrm{Inj}} \xrightarrow{\sim} \mathscr{D}_{\mathrm{Proj}}$ which fits into a diagram*

$$\begin{tikzcd} & \mathscr{K}(\mathbb{A}) \arrow[dl, "L"'] \arrow[dr, "R"] & \\ \mathscr{D}_{\mathrm{Inj}} \arrow[rr, "\sim", "\psi"'] & & \mathscr{D}_{\mathrm{Proj}} \end{tikzcd}$$

in $\mathscr{C}at_\infty$.

Proof. This is just uniqueness of localization. □

We leave the following exercise to the reader.

Exercise 9.9. Prove that the equivalence $\psi : \mathscr{D}_{\mathrm{Inj}} \to \mathscr{D}_{\mathrm{Proj}}$ of Corollary 9.8 is precisely the equivalence realized previously in Section I-13. (Here of course we accept that the functor ψ is only defined up to a contractible space.)

9.5. **Re-definining the derived ∞-category.**

Definition 9.10. Given a Grothendieck abelian category $\mathbb{A}$, the derived ∞-category $\mathscr{D}(\mathbb{A})$ is the localization of the homotopy ∞-category against to the class of quasi-isomorphisms

$$\mathscr{D}(\mathbb{A}) := \mathscr{K}(\mathbb{A})[\mathrm{Qiso}^{-1}].$$

Theorems 9.6 and 9.7 say that we can construct the derived ∞-category via K-injectives in $\mathrm{Ch}(\mathbb{A})$, or via K-projectives when they exist.

9.6. **Stable localization.** We present a stable analog of Verdier localization. This stable localization will be used to study derived categories under various finiteness constraints below. We first recall how localization works in the triangulated setting.

For a triangulated subcategory T in a triangulated category C we have the Verdier localization

$$C/T := C[W_T^{-1}],$$

where W_T is the collection of all morphisms whose mapping cones lie in T. We have the canonical functor $C \to C/T$ whose kernel is T, and which is universal amongst all exact functors which annihilate T.

Definition 9.11. Given a stable subcategory $\mathscr{T}$ in a stable ∞-category $\mathscr{C}$, let $W_{\mathscr{T}}$ be the collection of all morphisms $\alpha : x \to y$ in $\mathscr{C}$ whose cofiber $\mathrm{cofib}(\alpha)$ is isomorphic to an object in $\mathscr{T}$. (Note that identity morphisms are all in $W_{\mathscr{T}}$ since $\mathscr{T}$ contains a zero object in $\mathscr{C}$.) The Verdier quotient of $\mathscr{C}$ by $\mathscr{T}$ is the localization

$$\mathscr{C}/\mathscr{T} := \mathscr{C}[W_{\mathscr{T}}^{-1}].$$

We recall also the construction of "discrete localizations" in the non-∞ context. Given a discrete category $\mathbb{A}$ and a class of morphisms S in $\mathbb{A}$ which contains all identity maps, we can consider the discrete localization $\mathbb{A}[S^{-1}]_{\mathrm{disc}}$. This is any discrete category equipped with a functor $\mathbb{A} \to \mathbb{A}[S^{-1}]_{\mathrm{disc}}$ which is universal amongst all functors to a discrete target which invert all morphisms in S. One can construct such a discrete localization by taking the homotopy category of the ∞-categorical (Dwyer-Kan) localization, for example.

Lemma 9.12. *Consider an ∞-category $\mathscr{C}$ and class of morphisms $W \subseteq \mathscr{C}[1]$ which contains all degenerate edges. Let $F : \mathscr{C} \to \mathscr{C}[W^{-1}]$ be a localization functor. Then the unique map*

$$\bar{F} : (\mathrm{h}\,\mathscr{C})[W^{-1}]_{\mathrm{disc}} \to \mathrm{h}(\mathscr{C}[W^{-1}])$$

induced by $\mathrm{h}\,F$ is an equivalence of categories.

Proof. Take $C = \mathrm{h}\,\mathscr{C}$ and consider the discrete localization $C[W^{-1}]_{\mathrm{disc}}$. For any discrete category E let $\mathrm{Fun}(C, E)^W$ and $\mathrm{Fun}(C, E)^W$ denote the full subcategories of functors which send maps in W to isomorphisms in E. If follows that, for any discrete category E, restriction along the localization map $C \to C[W^{-1}]$, which exists and is produced via a calculus of fractions [25, Theorem 10.3.7], provides an equivalence

$$\mathrm{Fun}(C[W^{-1}]_{\mathrm{disc}}, E) \overset{\sim}{\to} \mathrm{Fun}(C, E)^W \overset{\sim}{\to} \mathrm{Fun}(\mathscr{C}, E)^W.$$

Here the final equivalence is ensured since the homotopy category functor is left adjoint to the inclusion $\mathrm{Cat} \to \mathrm{Cat}_\infty$.

Similarly, we have an equivalence

$$\mathrm{Fun}(\mathrm{h}(\mathscr{C}[W^{-1}]), E) \overset{\sim}{\to} \mathrm{Fun}(\mathscr{C}[W^{-1}], E) \overset{\sim}{\to} \mathrm{Fun}(\mathscr{C}, E)^W.$$

These two equivalences fit into a diagram

$$\begin{tikzcd}
\mathrm{Fun}(\mathscr{C}, E)^W & \\
\mathrm{Fun}(C[W^{-1}]_{\mathrm{disc}}, E) \arrow[u, "\sim"] & \mathrm{Fun}(\mathrm{h}(\mathscr{C}[W^{-1}]), E), \arrow[l, "\bar{F}^*"'] \arrow[ul, "\sim"']
\end{tikzcd}$$

at arbitrary discrete E, from which we conclude that $\bar{F}^*$ is an equivalence at all E. Hence $\bar{F}$ is an equivalence, by Yoneda. $\square$

As a particular example, we find that the homotopy category of the Verdier localization $\mathscr{C}/\mathscr{T}$, where $\mathscr{C}$ is stable and $\mathscr{T}$ is stable in $\mathscr{C}$, recovers the discrete Verdier localization of the homotopy category,

$$\mathrm{h}\,\mathscr{C}/\,\mathrm{h}\,\mathscr{T} \overset{\sim}{\to} \mathrm{h}(\mathscr{C}/\mathscr{T}).$$

See for example [13, Lemma 4.6.1, Proposition 4.6.2].

From this observation one sees that the homotopy category of the localization carries a unique triangulated structure so that the localization map $F : \mathscr{C} \to \mathscr{C}/\mathscr{T}$ induces an exact functor $\mathrm{h}\,F : \mathrm{h}\,\mathscr{C} \to \mathrm{h}\,\mathscr{C}/\mathscr{T}$. One can show that this discrete triangulated structure lifts, in the most advantageous fashion, to the ∞-level.

thm:stable_loc **Theorem 9.13** ([20, Theorem I.3.3])**.** *Given a stable subcategory $\mathscr{T}$ in a stable ∞-category $\mathscr{C}$, the following hold:*

1. *The Verdier quotient $\mathscr{C}/\mathscr{T}$ is stable.*
2. *The localization functor $l : \mathscr{C} \to \mathscr{C}/\mathscr{T}$ is exact.*
3. *The induced map on the homotopy category $\bar{l} : \mathrm{h}\,\mathscr{C}/\,\mathrm{h}\,\mathscr{T} \to \mathrm{h}(\mathscr{C}/\mathscr{T})$ is an equivalence of triangulated categories.*
4. *For any stable ∞-category $\mathscr{D}$, and exact functor $\mathscr{C} \to \mathscr{D}$ which sends $\mathscr{T}$ to the subcategory $\mathscr{D}_{\mathrm{Zero}}$ of zero objects in $\mathscr{D}$, the induced functor $F' : \mathscr{C}/\mathscr{T} \to \mathscr{D}$ is also exact.*

Proof. All is covered in [20] save for (4). For (4), since F and l preserve zero objects, F' preserves zero objects as well. So we need only show that F' preserves cofiber sequences. Any morphism $\alpha' : x \to y'$ in $\mathscr{C}/\mathscr{T}$ fits into a diagram

$$\begin{tikzcd} x \arrow[r, "\alpha"] \arrow[d, "="'] \arrow[dr] & y \arrow[d, "\sim"] \\ x \arrow[r, "\alpha'"'] & y' \end{tikzcd}$$

where α is in the image of the localization map. This just follows from the fact that morphisms in the homotopy category $\mathrm{h}\,\mathscr{C}/\,\mathrm{h}\,\mathscr{T}$ are determined by a calculus of fractions.

Taking cofibers we obtain an isomorphism of pushout diagrams

$$\begin{tikzcd} & y \arrow[dr] & & & & y' \arrow[dr] & \\ x \arrow[ur, "\alpha"] \arrow[rr] \arrow[dr] & & \mathrm{cofib}(\alpha) & \xrightarrow{\sim} & x \arrow[ur, "\alpha'"] \arrow[rr] \arrow[dr] & & \mathrm{cofib}(\alpha') \\ & 0 \arrow[ur] & & & & 0 \arrow[ur] & \end{tikzcd}$$

by Proposition II-13.20. Applying F we obtain an another isomorphism of diagrams

$$\begin{tikzcd} & Fy \arrow[dr] & & & & F'y' \arrow[dr] & \\ Fx \arrow[ur, "\alpha"] \arrow[rr] \arrow[dr] & & F\,\mathrm{cofib}(\alpha) & \xrightarrow{\sim} & F'x \arrow[ur, "\alpha'"] \arrow[rr] \arrow[dr] & & F'\,\mathrm{cofib}(\alpha') \\ & F0 \arrow[ur] & & & & F'0 \arrow[ur] & \end{tikzcd}$$

in $\mathscr{D}$. Since F is exact the left hand diagram is a cofiber sequence, and hence that the right hand diagram is a cofiber sequence as well. $\square$

sect:Dsmall_loc

9.7. **Small derived categories via localization.**

prop:Dsmall_loc

Proposition 9.14. *Let $\mathbb{A}$ be a Grothendieck abelian category, $\mathscr{K} \subseteq \mathscr{K}(\mathbb{A})$ be a stable subcategory in the homotopy ∞-category, and take $\mathrm{K} = \mathrm{h}\,\mathscr{K}$. Let* Acyc *and* $\mathrm{Acyc}_{\mathrm{K}}$ *be the triangulated subcategories of acyclic complexes in $K(\mathbb{A})$ and* K *respectively, and suppose that the induced map on discrete Veridier localizations*

$$\mathrm{K}\,/\,\mathrm{Acyc}_{\mathrm{K}} \to K(\mathbb{A})/\,\mathrm{Acyc} = D(\mathbb{A})$$

is fully faithful. Then for $\mathscr{D} \subseteq \mathscr{D}(\mathbb{A})$ the full subcategory spanned by the image of $\mathscr{K}$ under localization, the induced map

$$\mathscr{K}[\mathrm{Qiso}^{-1}] \to \mathscr{D}$$

is an equivalence.

Proof. We are considering the unique map $F : \mathscr{K}[\mathrm{Qiso}^{-1}] \to \mathscr{D}$ induced by the functor $\mathscr{K} \to \mathscr{D} \subseteq \mathscr{D}(\mathbb{A})$. According to Theorem 9.13, the localization $\mathscr{K}[\mathrm{Qiso}^{-1}]$ is stable and the functor F is exact. Hence, by Proposition 7.15, F is fully faithful, and thus an equivalence, if and only if the induced map on homotopy categories

$$\mathrm{h}\,F : \mathrm{h}\,\mathscr{K}[\mathrm{Qiso}^{-1}] \to \mathrm{h}\,\mathscr{D} \subseteq D(\mathbb{A})$$

is fully faithful. But now, again by Theorem 9.13, the map $\mathrm{h}\,F$ is identified with the map $\mathrm{K}\,/\,\mathrm{Acyc}_{\mathrm{K}} \to D(\mathbb{A})$ induced by the discrete localization $K \subseteq K(\mathbb{A}) \to D(\mathbb{A})$. So $\mathrm{h}\,F$ is fully faithful by assumption, and if follows that F is fully faithful. $\square$

In each of the following examples one employs standard arguments to prove that the relevant map $\mathrm{K}/\mathrm{Acyc}_{\mathrm{K}} \to D(\mathbb{A})$ is fully faithful. Since I cannot find a reference for such arguments, we record all of the details for (only) the first example.

ex:D_bounded **Example 9.15** (The bounded derived category)**.** Let $\mathbb{A}$ be any Grothendieck abelian category, $\mathscr{K}^b(\mathbb{A})$ be the homotopy ∞-category of bounded complexes in $\mathbb{A}$, and $\mathscr{D}^b(\mathbb{A})$ be the full subcategory in $\mathscr{D}(\mathbb{A})$ spanned by complexes with bounded cohomology. We also consider the ∞-category $\mathscr{K}^-(\mathbb{A})$ of bounded above complexes. We claim that the functor $\mathscr{K}^b(\mathbb{A})[\mathrm{Qiso}^{-1}] \to \mathscr{D}^b(\mathbb{A})$ is an equivalence. For this it suffices to show, via Proposition 9.14, that the functor $K(\mathbb{A})/\mathrm{Acyc}^b \to D(\mathbb{A})$ is fully faithful. Below we express morphisms in $D(\mathbb{A})$ by either left or right fractions, following [13, Section 3].

Since any cochain complex X which has bounded above cohomology admits a quasi-isomorphism $X' \to X$ from a bounded above complex, we see that every morphism $V \leftarrow X \to W$ between bounded above complexes in $D(\mathbb{A})$ is equivalent to a morphism $V \leftarrow X' \to W$ which only involves bounded above complexes. This shows, from the perspective of the calculus of left fractions, that the functor $K^-(\mathbb{A})/\mathrm{Acyc}^- \to D(\mathbb{A})$ is full. This same fact, and a direct consideration of the equivalence relation on morphisms $V \leftarrow X \to W$ in the calculus of left fractions, also tells us that the given functor is faithful. We conclude that the functor $K^-(\mathbb{A})/\mathrm{Acyc}^- \to D(\mathbb{A})$ is fully faithful.

It now suffices to show that the functor $K^b(\mathbb{A})/\mathrm{Acyc}^b \to K^-(\mathbb{A})/\mathrm{Acyc}^-$ is fully faithful. For this one notes that any bounded above complex X which has bounded cohomology admits a quasi-isomorphism $X \to X''$ onto a bounded complex. From this fact, and a direct consideration of the calculus of right fractions which describes morphisms in the latter category, we see that the given functor is in fact fully faithful. In total, this recovers the well-known fact that the functor

$$\mathrm{K}^b(\mathbb{A})/\mathrm{Acyc}^b \to D(\mathbb{A})$$

is fully faithful, and hence that the restriction $\mathscr{K}^b(\mathbb{A}) \to \mathscr{D}^b(\mathbb{A})$ of the localization functor $L : \mathscr{K}(\mathbb{A}) \to \mathscr{D}(\mathbb{A})$ induces an equivalence of ∞-categories

$$\mathscr{K}^b(\mathbb{A})[\mathrm{Qiso}^{-1}] \overset{\sim}{\to} \mathscr{D}^b(\mathbb{A}).$$

ex:D_finite **Example 9.16** (The finite-dimensional derived category)**.** Let G be an affine algebraic group in arbitrary characteristic, say, and consider the representation category $\mathrm{Rep}(G)$. Take $\mathscr{K}(G)_{fin}$ the homotopy ∞-category of bounded complexes of finite-dimensional representations. Take $\mathscr{D}(G)_{fin}$ the full subcategory in $\mathscr{D}(G) = \mathscr{D}(\mathrm{Rep}(G))$ consisting of all complexes with finite-dimensional cohomology. By Proposition 9.14, the functor $L : \mathscr{K}(G)_{fin} \to \mathscr{D}(G)_{fin}$ given by restricted localization functor on $\mathscr{K}(G)$ induces an equivalence of ∞-categories

$$\mathscr{K}(G)_{fin}[\mathrm{Qiso}^{-1}] \to \mathscr{D}(G)_{fin}.$$

ex:D_coh **Example 9.17** (The coherent derived category)**.** Let X be a Noetherian scheme and $\mathscr{K}(X)_{coh}$ denote the homotopy ∞-category of bounded complexes of coherent sheaves on X. Let $\mathscr{D}(X)_{coh}$ be the full subcategory of complexes in $\mathscr{D}(X) = \mathscr{D}(\mathrm{QCoh}(X))$ with coherent total cohomology. The restriction of the localization functor $L : \mathscr{K}(X)_{coh} \to \mathscr{D}(X)_{coh}$ induces an equivalence

$$\mathscr{K}(X)_{coh}[\mathrm{Qiso}^{-1}] \overset{\sim}{\to} \mathscr{D}(X)_{coh}$$

ex:D_flat

Example 9.18 (The derived category via flat sheaves)**.** Let X be a quasi-compact quasi-separated scheme and $\mathscr{K}(X)_{flat}$ be the homotopy ∞-category of K-flat quasi-coherent sheaves on X. The restriction of the localization functor $L : \mathscr{K}(X)_{flat} \to \mathscr{D}(X)$ induces an equivalence

$$\mathscr{K}(X)_{flat}[\mathrm{Qiso}^{-1}] \xrightarrow{\sim} \mathscr{D}(X)$$

[23, Lemma 3.3].

ex:D_perf

Example 9.19 (The perfect derived category)**.** Let X be any Noetherian algebraic stack with the resolution property, i.e. a stack for which every coherent sheaf M admits a surjection $V \to M$ from a finite rank vector bundle. Let $\mathscr{D}(X)_{perf}$ be the full ∞-subcategory of perfect sheaves in $\mathscr{D}(X) = \mathscr{D}(\mathrm{QCoh}(X))$, i.e. sheaves whose image in $D(X)$ is dualizable for the product $\otimes^{\mathrm{L}}_{\mathscr{O}_X}$, and $\mathscr{K}(X)_{vec}$ be the homotopy ∞-category of bounded complexes of finite rank vector bundles on X. The restricted localization functor $L : \mathscr{K}(X)_{vec} \to \mathscr{D}(X)_{perf}$ induces an equivalence

$$\mathscr{K}(X)_{vec}[\mathrm{Qiso}^{-1}] \xrightarrow{\sim} \mathscr{D}(X)_{perf}.$$

9.8. Localizing directly from cochains.

thm:K_loc_chain

Theorem 9.20 ([15, Proposition 1.3.4.5])**.** *Consider an additive category $\mathfrak{A}$ and let $\mathrm{Ch}^\star(\mathfrak{A}) \subseteq \mathrm{Ch}(\mathfrak{A})$ be any full subcategory which is closed under finite sums and the formation of mapping cones. Then the inclusion $\mathrm{Ch}^\star(\mathfrak{A}) \to \mathscr{K}(\mathfrak{A})$ induces an fully faithful functor from the localization of $\mathrm{Ch}^\star(\mathfrak{A})$ relative to the class of homotopy equivalences*

$$\mathrm{Ch}^\star(\mathfrak{A})[\mathrm{Htop}^{-1}] \to \mathscr{K}(\mathfrak{A})$$

which is an equivalence onto the full subcategory in $\mathscr{K}(\mathfrak{A})$ spanned by the complexes in $\mathrm{Ch}^\star(\mathfrak{A})$.

cor:D_loc_chain

Corollary 9.21. *Let $\mathbb{A}$ be a Grothendieck abelian category and $\mathrm{Ch}^\star(\mathbb{A}) \subseteq \mathrm{Ch}(\mathbb{A})$ be a full subcategory which is stable under suspension, desuspension, and the formation of mapping cones. Suppose also that, for the corresponding homotopy ∞-category $\mathscr{K}^\star(\mathbb{A})$, the induced map $\mathrm{h}\,\mathscr{K}^\star(\mathbb{A})/\,\mathrm{Acyc}^\star \to D(\mathbb{A})$ is fully faithful. Then the inclusion of simplicial sets $\mathrm{Ch}^\star(\mathbb{A}) \to \mathscr{D}(\mathbb{A})$ induces a fully faithful functor*

$$\mathrm{Ch}^\star(\mathbb{A})[\mathrm{Qiso}^{-1}] \to \mathscr{D}(\mathbb{A})$$

which is an equivalence onto the full subcategory in $\mathscr{D}(\mathbb{A})$ spanned by the (images of the) complexes in $\mathrm{Ch}^\star(\mathbb{A})$.

Proof. Combine Proposition 9.14 and Theorem 9.20. □

Of course, in the absolute setting we obtain an equivalence

$$\mathrm{Ch}(\mathbb{A})[\mathrm{Qiso}^{-1}] \xrightarrow{\sim} \mathscr{D}(\mathbb{A}).$$

Corollary 9.21 applies to all of the examples discussed in Section 9.7 above. We employ the notations from Examples 9.15–9.19 and recall a few instances here.

Example 9.22 (The bounded derived category)**.** For $\mathbb{A}$ Grothendieck abelian, the canonical functor

$$\mathrm{Ch}^b(\mathbb{A})[\mathrm{Qiso}^{-1}] \to \mathscr{D}^b(\mathbb{A})$$

is an equivalence.

Example 9.23 (The finite-dimensional derived category)**.** For any algebraic group G, the canonical functor

$$\mathrm{Ch}(G)_{fin}[\mathrm{Qiso}^{-1}] \to \mathscr{D}(G)_{fin}$$

is an equivalence.

Example 9.24 (The perfect derived category)**.** For any Noetherian algebraic stack X with the resolution property, the canonical functor

$$\mathrm{Ch}(X)_{vec}[\mathrm{Qiso}^{-1}] \to \mathscr{D}(X)_{perf}$$

is an equivalence.

sect:deriving_fun

10. Left and right derived functors

We define the left and right derived functors

$$\mathrm{L}\,F : \mathscr{D}(\mathbb{A}) \to \mathscr{D}(\mathbb{B}) \quad \text{and} \quad \mathrm{R}\,G : \mathscr{D}(\mathbb{B}) \to \mathscr{D}(\mathbb{A})$$

for pairs of adjoint functors $F : \mathbb{A} \to \mathbb{B}$ and $G : \mathbb{B} \to \mathbb{A}$ between Grothendieck abelian categories. We show, in particular, that the left derived functor can be computed by taking F-acyclic resolutions in the domain. This approach mirrors precisely the approach taken in the discrete derived setting.

10.1. **Derived functors in ideal situations.** Consider a Grothendieck abelian category $\mathbb{A}$ with enough projectives, and $\mathbb{B}$ arbitrary Grothendieck abelian. Let

$$\mathbf{F} : \mathbf{Ch}(\mathbb{A}) \to \mathbf{Ch}(\mathbb{B}) \quad \text{and} \quad \mathbf{G} : \mathbf{Ch}(\mathbb{B}) \to \mathbf{Ch}(\mathbb{A})$$

be dg functors with $\bar{F}$ left adjoint to $\bar{G}$. For example, we can consider the case where $\bar{F}$ and $\bar{G}$ are induced by adjoint functors between $\mathbb{A}$ and $\mathbb{G}$. In this situation, by Theorem I-14.9, the induced functors on homotopy ∞-categories

$$F : \mathscr{K}(\mathbb{A}) \to \mathscr{K}(\mathbb{B}) \quad \text{and} \quad G : \mathscr{K}(\mathbb{B}) \to \mathscr{K}(\mathbb{A})$$

are such that F is left adjoint to G.

In this situation, we define the left and right derived functors in the expected ways.

def:der_fun1

Definition 10.1. Let $\mathbb{A}$ and $\mathbb{B}$ be Grothendieck abelian categories. Take $\mathrm{R} : \mathscr{D}(\mathbb{B}) \to \mathscr{K}(\mathbb{B})$ the right adjoint to the localization functor for $\mathscr{D}(\mathbb{B}) = \mathscr{K}(\mathbb{B})[\mathrm{Qiso}^{-1}]$.

For any continuous functor $G : \mathscr{K}(\mathbb{B}) \to \mathscr{K}(\mathbb{A})$ we define the right derived functor $\mathrm{R}\,G : \mathscr{D}(\mathbb{B}) \to \mathscr{D}(\mathbb{A})$ as the composite

$$\mathrm{R}\,G := \big(\mathscr{D}(\mathbb{B}) \xrightarrow{\mathrm{R}} \mathscr{K}(\mathbb{B}) \xrightarrow{G} \mathscr{K}(\mathbb{A}) \xrightarrow{\mathrm{loc}} \mathscr{D}(\mathbb{A})\big).$$

We note that G is continuous whenever it admits a left adjoint $F : \mathscr{D}(\mathbb{A}) \to \mathscr{D}(\mathbb{B})$.

def:der_fun2

Definition 10.2. Let $\mathbb{A}$ and $\mathbb{B}$ be Grothendieck abelian categories, and suppose the localization functor $\mathscr{K}(\mathbb{A}) \to \mathscr{D}(\mathbb{A})$ has a left adjoint $\mathrm{L} : \mathscr{D}(\mathbb{A}) \to \mathscr{K}(\mathbb{A})$.

For any cocontinuous functor $F : \mathscr{K}(\mathbb{A}) \to \mathscr{K}(\mathbb{B})$ we define the left derived functor $\mathrm{L}\,F : \mathscr{D}(\mathbb{A}) \to \mathscr{D}(\mathbb{B})$ as the composite

$$\mathrm{L}\,F : \mathscr{D}(\mathbb{A}) \xrightarrow{\mathrm{L}} \mathscr{K}(\mathbb{A}) \xrightarrow{F} \mathscr{K}(\mathbb{B}) \xrightarrow{\mathrm{loc}} \mathscr{D}(\mathbb{B}).$$

Proposition 10.3. *Take $\mathbb{A}$ and $\mathbb{B}$ Grothendieck abelian and let $F : \mathscr{K}(\mathbb{A}) \to \mathscr{K}(\mathbb{B})$ be left adjoint to a functor $G : \mathscr{K}(\mathbb{B}) \to \mathscr{K}(\mathbb{A})$. Suppose that the localization functor $\mathscr{K}(\mathbb{A}) \to \mathscr{D}(\mathbb{A})$ admits a left adjoint. Then the functor $\mathrm{L}\,F : \mathscr{K}(\mathbb{A}) \to \mathscr{K}(\mathbb{B})$ is left adjoint to $\mathrm{R}\,G : \mathscr{D}(\mathbb{B}) \to \mathscr{D}(\mathbb{A})$.*

Proof. This just follows from the fact that a composite of left adjoints is left adjoint to the respective composite of right adjoints. See [16, 02DT]. □

10.2. **The candidate left derived functor.** In a general setting we do not have enough projective in $\mathbb{A}$, so that we cannot define the left derived functor as in the ideal setting discussed above. (One might consider, for example, the case of sheaves $\mathbb{A} = \mathrm{QCoh}(X)$ on a projective variety X.) However, we can also approach the sitation via complexes which are acyclic relative to a given functor.

def:enough_F

Definition 10.4. Let $\mathbb{A}$ and $\mathbb{B}$ be Grothendieck abelian categories, and $F : \mathscr{K}(\mathbb{A}) \to \mathscr{K}(\mathbb{B})$ be any cocontinuous functor. We say $\mathscr{K}(\mathbb{A})$ has enough F-acyclic objects if there is a stable subcategory $\mathscr{K}(\mathbb{A})_{F\text{-ac}}$ in $\mathscr{K}(\mathbb{A})$ which satisfies the following:

(a) $\mathscr{K}(\mathbb{A})_{F\text{-ac}}$ admits all small colimits and the inclusion $\mathscr{K}(\mathbb{A})_{F\text{-ac}} \to \mathscr{K}(\mathbb{A})$ is cocontinuous.

(b) Every object V in $\mathscr{K}(\mathbb{A})$ admits a quasi-isomorphism $W \to V$ from an object W in $\mathscr{K}(\mathbb{A})_{F\text{-ac}}$.

(c) If an object W in $\mathscr{K}(\mathbb{A})_{F\text{-ac}}$ is acyclic then the image $F(W)$ in $\mathscr{K}(\mathbb{B})$ is acyclic as well.

We note that, by condition (a), any such subcategory $\mathscr{K}(\mathbb{A})_{F\text{-ac}}$ is a stable subcategory in $\mathscr{K}(\mathbb{A})$.

Lemma 10.5. *For $\mathscr{K}(\mathbb{A})_{F\text{-}ac}$ as in Defintion 10.4, the functor $\mathscr{K}(\mathbb{A})_{F\text{-}ac} \to \mathscr{D}(\mathbb{A})$ induces an equivalence*

$$\mathscr{K}(\mathbb{A})_{F\text{-}ac}[\mathrm{Qiso}^{-1}] \overset{\sim}{\to} \mathscr{D}(\mathbb{A}).$$

Proof. Since all objects in $\mathscr{K}(\mathbb{A})$ admit resolutions by F-acyclics, by assumption, the result follows just as in the examples from Section 9.7. □

The following is more-or-less apparent.

lem:LF_candidate

Lemma 10.6. *Let $\mathbb{A}$ and $\mathbb{B}$ be Grothendieck abelian, $F : \mathscr{K}(\mathbb{A}) \to \mathscr{K}(\mathbb{B})$ be a cocontinuous functor, and suppose that $\mathscr{K}(\mathbb{A})$ has enough F-acyclic objects. Let $\mathscr{K}(\mathbb{A})_{F\text{-}ac} \subseteq \mathscr{K}(\mathbb{A})$ be a full subcategory of F-acyclics as in Definition 10.4.*

There is a unique functor $\mathrm{L}\,F : \mathscr{D}(\mathbb{A}) \to \mathscr{D}(\mathbb{B})$ which fits into a diagram

$$\begin{tikzcd}
\mathscr{K}(\mathbb{A})_{F\text{-}ac} \arrow[r, "F"] \arrow[d, "\mathrm{loc}"'] \arrow[dr] & \mathscr{K}(\mathbb{B}) \arrow[d, "\mathrm{loc}"] \\
\mathscr{D}(\mathbb{A}) \arrow[r, "\mathrm{L}\,F"'] & \mathscr{D}(\mathbb{B})
\end{tikzcd}$$

in $\mathscr{C}at_\infty$.

Proof. The identification $\mathscr{K}(\mathbb{A})_{F\text{-ac}}[\mathrm{Qiso}^{-1}] = \mathscr{D}(\mathbb{A})$ tells us that the restriction functor

$$\mathrm{Fun}(\mathscr{D}(\mathbb{A}), \mathscr{D}(\mathbb{B})) \to \mathrm{Fun}(\mathscr{K}(\mathbb{A})_{F\text{-ac}}, \mathscr{D}(\mathbb{B}))$$

is an equivalence onto the full subcategory of functors which send quasi-isomorphisms to isomorphisms. Existence and uniqueness of the functor $\mathrm{L}\,F$ now follows by contractibility of the homotopy fiber

$$\mathrm{Fun}(\mathscr{D}(\mathbb{A}), \mathscr{D}(\mathbb{B})) \times^{\mathrm{htop}}_{\mathrm{Fun}(\mathscr{K}(\mathbb{A})_{F\text{-ac}}, \mathscr{D}(\mathbb{B}))} \{\mathrm{loc}\, F|_{F\text{-ac}}\}.$$

□

10.3. **Checking adjoints at the homotopy level.**

prop:adj_htop

Proposition 10.7 ([16, 02EY]). *Suppose $G : \mathscr{D} \to \mathscr{C}$ is a functor between ∞-categories which admits a left adjoint. Let $F : \mathscr{C} \to \mathscr{D}$ be any functor which is paired with a tranformation $\eta : id_{\mathscr{C}} \to GF$. Then the following are equivalent:*

(1) *The transformation η exhibits F as left adjoint to G.*

(2) *The induced transformation $\mathrm{h}\,\eta : id_{\mathrm{h}\,\mathscr{C}} \to \mathrm{h}\,G\,\mathrm{h}\,F$ exhibits $\mathrm{h}\,F$ as left adjoint to $\mathrm{h}\,G$.*

Proof. The implication (1) $\Rightarrow$ (2) is trivial. So we deal with the converse claim.

Let $F' : \mathscr{C} \to \mathscr{D}$ be left adjoint to G with unit transformation $\eta' : id \to GF$. Then by Proposition 8.20 there is a transformation $\zeta : F' \to F$ with $\eta = (G\zeta)\eta'$, and η is a unit transformation which realizes F as left adjoint to G if and only if ζ is a natural isomorphism. Taking the homotopy categories, we see that $\mathrm{h}\,\zeta : \mathrm{h}\,F' \to \mathrm{h}\,F$ is an isomorphism since $\mathrm{h}\,F$ is left adjoint to $\mathrm{h}\,G$.

Recall that ζ is a natural isomorphism if and only if, at each x in $\mathscr{C}$, the map $\zeta_x : F'(x) \to F(x)$ is an isomorphism in $\mathscr{C}$ (Theorem I-7.6). This property can be checked at the level of the homotopy category. So we see that ζ is in fact a natural isomorphism since $\mathrm{h}\,\zeta$ is a natural isomorphism. $\square$

The obvious analog of Proposition 10.7 holds for right adjoints, simply by taking opposites. In the stable setting we can forgo the presupposition that a left adjoint to G exists.

prop:adj_htop_stab

Proposition 10.8. *Let $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ be exact functors between stable ∞-categories, and consider a transformation $\eta : id_{\mathscr{C}} \to GF$. Then the following are equivalent:*

(a) *The transformation η exhibits F as left adjoint to G.*

(b) *The induced transformation $\mathrm{h}\,\eta$ exhibits $\mathrm{h}\,F$ as left adjoint to $\mathrm{h}\,G$.*

Proof. We consider the composite

$$\mathrm{Hom}_{\mathscr{D}}(Fx, y') \overset{G}{\to} \mathrm{Hom}_{\mathscr{C}}(GFx, Gy') \overset{\eta^*}{\to} \mathrm{Hom}_{\mathscr{C}}(x, Gy')$$

in $\mathrm{h}\,\mathscr{K}an_*$. By exactness of G and Proposition 6.26, the fact that the composite

$$\pi_0\,\mathrm{Hom}_{\mathscr{D}}(Fx, y') \overset{G}{\to} \pi_0\,\mathrm{Hom}_{\mathscr{C}}(GFx, Gy') \overset{\eta^*}{\to} \pi_0\,\mathrm{Hom}_{\mathscr{C}}(x, Gy')$$

is an isomorphism at arbitrary y' implies that at all higher homotopy groups, based at 0, the composite

$$\pi_n\,\mathrm{Hom}_{\mathscr{D}}(Fx, y') \overset{G}{\to} \pi_n\,\mathrm{Hom}_{\mathscr{C}}(GFx, Gy') \overset{\eta^*}{\to} \pi_n\,\mathrm{Hom}_{\mathscr{C}}(x, Gy')$$

is an isomorphisms. Hence the induced map on loop spaces $\Omega\,\mathrm{Hom}_{\mathscr{D}}(Fx, y') \to \Omega\,\mathrm{Hom}_{\mathscr{C}}(x, Gy')$ is an equivalence. We considering the case $y' = \Sigma y$, and apply the isomorphism Proposition 6.26, so see that the composite map $\mathrm{Hom}_{\mathscr{D}}(Fx, y) \to \mathrm{Hom}_{\mathscr{C}}(x, Gy)$ is an equivalence at all objects $x, y : * \to \mathscr{D}$. By Proposition 8.18 it follows that η realizes F as left adjoint to G. $\square$

10.4. Left adjoints to right derived functors.

prop:left_derived **Proposition 10.9.** *Let $\mathbb{A}$ and $\mathbb{B}$ be Grothendieck abelian categories and $F : \mathscr{K}(\mathbb{A}) \to \mathscr{K}(\mathbb{B})$ be left adjoint to a functor $G : \mathscr{K}(\mathbb{B}) \to \mathscr{K}(\mathbb{A})$. Suppose that $\mathscr{K}(\mathbb{A})$ has enough F-acyclic objects, and let $\mathrm{L}\,F : \mathscr{D}(\mathbb{A}) \to \mathscr{D}(\mathbb{B})$ be as in the statement of Lemma 10.6. Then $\mathrm{L}\,F$ is left adjoint to the right derived functor $\mathrm{R}\,G : \mathscr{D}(\mathbb{B}) \to \mathscr{D}(\mathbb{A})$.*

Before giving the proof, let us consider the situation at the level of homotopy categories. We have the left adjoint $i : D(\mathbb{B}) \to K(\mathbb{B})$ to the localization functor, and recall that the composite $i\,\mathrm{loc} : K(\mathbb{B}) \to K(\mathbb{B})$, along with the unit transformation $u_V : V \to i\,\mathrm{loc}(V)$, takes K-injective resolutions of complexes. We now have the commuting diagram

$$\begin{tikzcd}
K(\mathbb{A})_{F\text{-ac}} \arrow[r, "\mathbb{F}"] \arrow[d, "\mathrm{loc}"'] & K(\mathbb{B}) \arrow[r, "\mathbb{G}i\,\mathrm{loc}"] \arrow[d, "\mathrm{loc}"'] & K(\mathbb{B}) \arrow[d, "\mathrm{loc}"'] \\
D(\mathbb{A}) \arrow[r, "\mathrm{L}\,\mathbb{F}"'] & D(\mathbb{B}) \arrow[r, "\mathrm{R}\,\mathbb{G}"'] & D(\mathbb{A}),
\end{tikzcd}$$

where $\mathbb{F} = \mathrm{h}\,F$ and $\mathbb{G} = \mathrm{h}\,G$, and the unit transformation $id_{K(\mathbb{A})} \to \mathbb{G}\mathbb{F}$ induces a unique transformation

$$\mathrm{loc}\,|_{F\text{-ac}} \to \mathrm{loc}\,\mathbb{G}\mathbb{F}|_{F\text{-ac}} \xrightarrow{u} \mathrm{loc}\,\mathbb{G}i\,\mathrm{loc}\,\mathbb{F}|_{F\text{-ac}} \cong \mathrm{R}\,\mathbb{G}\,\mathrm{loc}\,\mathbb{F}|_{F\text{-ac}} \cong \mathrm{R}\,\mathbb{G}\,\mathrm{L}\,\mathbb{F}\,\mathrm{loc}\,|_{F\text{-ac}}.$$

The above transformation induces a unique transformation at the level of the discrete derived category

$$\eta : id_{D(\mathbb{A})} \to \mathrm{R}\,\mathbb{G}\,\mathrm{L}\,\mathbb{F}. \qquad (34)$$ eq:3454

lem:3628 **Lemma 10.10.** *The transformation $\eta : id_{D(\mathbb{A})} \to \mathrm{R}\,\mathbb{G}\,\mathrm{L}\,\mathbb{F}$ realizes $\mathrm{L}\,\mathbb{F}$ as left adjoint to $\mathrm{R}\,\mathbb{G}$, at the level of the discrete derived category.*

Proof. At the discrete level the derived category is realizable via a calculus of fractions, so that the localization functor $\mathrm{loc} : K(\mathbb{A}) \to D(\mathbb{A})$ does nothing on objects and morphisms–though some maps in $K(\mathbb{A})$ are identified under localization. Let $u : id_{K(\mathbb{B})} \to i(= i\,\mathrm{loc})$ denote the unit transformation for the (loc, i)-adjunction and $\bar{\eta} : id_{K(\mathbb{B})} \to \mathbb{G}\mathbb{F}$ denote the unit of the $(\mathbb{F}, \mathbb{G})$-adjunction.

For $\mathbb{F}$-acyclic M and K-injective N we have the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{K(\mathbb{B})}(\mathbb{F}M, N) \arrow[r, "i"] & \mathrm{Hom}_{K(\mathbb{B})}(i\mathbb{F}M, iN) \arrow[r, "\mathbb{G}"] \arrow[d, "u^*"'] & \mathrm{Hom}_{K(\mathbb{A})}(\mathbb{G}i\mathbb{F}M, \mathbb{G}iN) \arrow[d, "\mathbb{G}u^*"] \\
& \mathrm{Hom}_{K(\mathbb{B})}(\mathbb{F}M, iN) \arrow[r, "\mathbb{G}"'] & \mathrm{Hom}_{K(\mathbb{A})}(\mathbb{G}\mathbb{F}M, \mathbb{G}iN) \arrow[d, "\bar{\eta}^*"] \\
& & \mathrm{Hom}_{K(\mathbb{A})}(M, \mathbb{G}iN)
\end{tikzcd}$$

in which the maps i and u^* are isomorphisms, since N is K-injective. Hence the outer sequence is an isomorphism if and only if the composite

$$\mathrm{Hom}_{K(\mathbb{B})}(\mathbb{F}M, iN) \xrightarrow{\mathbb{G}} \mathrm{Hom}_{K(\mathbb{A})}(\mathbb{G}\mathbb{F}M, \mathbb{G}iN) \xrightarrow{\bar{\eta}^*} \mathrm{Hom}_{K(\mathbb{A})}(M, \mathbb{G}iN)$$

is an isomorphism. However, this holds since $\bar{\eta}$ is the unit of the relevant adjunction.

We apply the localization functor to obtain a diagram

$$\begin{tikzcd}
\mathrm{Hom}_{K(\mathbb{B})}(\mathbb{F}M, N) \arrow[r, "\mathbb{G}"] \arrow[d, "\mathrm{loc}"', "\cong"] & \mathrm{Hom}_{K(\mathbb{A})}(\mathbb{G}i\mathbb{F}M, \mathbb{G}iN) \arrow[r, "\bar{\eta}^*"] \arrow[d, "\mathrm{loc}"'] & \mathrm{Hom}_{K(\mathbb{A})}(M, \mathbb{G}iN) \arrow[d, "\mathrm{loc}"'] \\
\mathrm{Hom}_{D(\mathbb{B})}(\mathrm{L}\,\mathbb{F}M, N) \arrow[r, "\mathrm{R}\,\mathbb{G}"'] & \mathrm{Hom}_{D(\mathbb{A})}(\mathrm{R}\,\mathbb{G}\,\mathrm{L}\,\mathbb{F}M, \mathrm{R}\,\mathbb{G}N) \arrow[r, "\eta^*"'] & \mathrm{Hom}_{D(\mathbb{A})}(M, \mathrm{R}\,\mathbb{G}N).
\end{tikzcd}$$

To see that the bottom sequence is an isomorphism it now suffices to prove that the localization functor induces an isomorphism

$$\mathrm{loc}:\mathrm{Hom}_{K(\mathbb{A})}(M, \mathbb{G}N') \to \mathrm{Hom}_{D(\mathbb{A})}(M, \mathbb{G}N') \tag{35}$$ eq:3489

whenever N' is K-injective and M is $\mathbb{F}$-acyclic.

In general, maps in the derived category are recovered as the colimit

$$\mathrm{colim}_\beta\,\mathrm{Hom}_{K(\mathbb{A})}(M_\beta, \mathbb{G}N') = \mathrm{Hom}_{D(\mathbb{A})}(M_\beta, \mathbb{G}N')$$

over all quasi-isomorphisms $\beta : M_\beta \to M$. Since $K(\mathbb{A})$ has enough $\mathbb{F}$-acyclics we can reduce this expression to a colimit

$$\mathrm{colim}_\alpha\,\mathrm{Hom}_{K(\mathbb{A})}(M_\alpha, \mathbb{G}N') = \mathrm{Hom}_{D(\mathbb{A})}(M, \mathbb{G}N')$$

over quasi-isomorphisms $\alpha : M_\alpha \to M$ between $\mathbb{F}$-acyclics, and under this identification the localization map

$$\mathrm{loc}:\mathrm{Hom}_{K(\mathbb{A})}(M, \mathbb{G}N') \to \mathrm{colim}_\alpha\,\mathrm{Hom}_{K(\mathbb{A})}(M_\alpha, \mathbb{G}N')$$

is just the structure map at $\alpha = id_M$. But now, via $(\mathbb{F},\mathbb{G})$-adjunction, $\mathbb{F}$-acyclicity of the M_α, and K-injectivity of N' each transition function

$$\alpha^*:\mathrm{Hom}_{K(\mathbb{A})}(M, \mathbb{G}N') \to \mathrm{Hom}_{K(\mathbb{A})}(M_\alpha, \mathbb{G}N')$$

is an isomorphism, so that this colimit diagram is essentially constant. Hence the map (35) is seen to be an isomorphism, and that the composite

$$\mathrm{Hom}_{D(\mathbb{B})}(\mathbb{F}M, N) \overset{\mathrm{R}\,\mathbb{G}}{\to} \mathrm{Hom}_{D(\mathbb{A})}(\mathrm{R}\,\mathbb{G}\,\mathrm{L}\,\mathbb{F}M, \mathrm{R}\,\mathbb{G}N) \overset{\eta^*}{\to} \mathrm{Hom}_{D(\mathbb{A})}(M, \mathrm{R}\,\mathbb{G}N) \tag{36}$$ eq:3508

is an isomorphism at all $\mathbb{F}$-acyclic M and K-injective N. Since we have enough $\mathbb{F}$-acyclics and enough K-injectives, it follows that the sequence (36) is an isomorphism at all M and N in $D(\mathbb{A})$ and $D(\mathbb{B})$ respectively. Hence η realizes the claimed adjunction. □

We now prove Proposition 10.9.

Proof of Proposition 10.9. Let $\eta_{\mathscr{K}} : id_{\mathscr{K}(\mathbb{A})} \to GF$ denote the unit of the adjunction. We obtain a transformation $\eta' : \Delta^1 \times \mathscr{K}(\mathbb{A}) \to \mathscr{D}(\mathbb{A})$ as the composite

$$\mathrm{loc}\,|_{F\text{-ac}} \overset{\eta_{\mathscr{K}}}{\to} \mathrm{loc}\,GF|_{F\text{-ac}} \overset{u}{\to} \mathrm{loc}\,Gi\,\mathrm{loc}\,F|_{F\text{-ac}} \cong \mathrm{R}\,G\,\mathrm{loc}\,F|_{F\text{-ac}} \cong \mathrm{R}\,G\,\mathrm{L}\,F\,\mathrm{loc}\,|_{F\text{-ac}}, \tag{37}$$ eq:3316

where $i : \mathscr{D}(\mathbb{B}) \to \mathscr{K}(\mathbb{B})$ is the right adjoint to localization. (Recall that the composite $i\,\mathrm{loc}$ takes functorial injective resolutions of objects.) We claim that this transformation induces a unique transformation at the level of derived ∞-categories

$\eta : id_{\mathscr{D}(\mathbb{A})} \to \mathrm{R}\,G\,\mathrm{L}\,F$ which fits into a diagram

$$\begin{tikzcd} \Delta^1 \times \mathscr{K}(\mathbb{A}) \arrow[d, "\mathrm{loc}"'] \arrow[drr, "\eta'"] & & \\ \Delta^1 \times \mathscr{D}(\mathbb{A}) \arrow[rr, "\eta"'] & & \mathscr{D}(\mathbb{A}) \end{tikzcd} \tag{38}$$

eq:3320

in $\mathscr{C}at_\infty$.

Let us take $\mathscr{D} = \mathscr{D}(\mathbb{A})$ and $\mathscr{K} = \mathscr{K}(\mathbb{A})$. First note that the restriction functor

$$\mathrm{Fun}(\Delta^1 \times \mathscr{D}, \mathscr{D}) \to \mathrm{Fun}(\Delta^1 \times \mathscr{K}, \mathscr{D})^{\mathrm{Qiso}}$$

is an equivalence, so that we can find some functor $\eta'' : \Delta^1 \times \mathscr{D} \to \mathscr{D}$ which fits into a 2-simplex as in (38). On the boundary $\partial\Delta^1 \times \mathscr{D}$ we obtain unique functors F_i (up to a contractible space of choices) which fit into diagrams

$$\begin{tikzcd} & \mathscr{K} \arrow[dl, "\mathrm{loc}"'] \arrow[d, "\mathrm{loc}"] \\ \mathscr{D} & \mathscr{D} \arrow[l, "F_0"] \end{tikzcd} \qquad \begin{tikzcd} \mathscr{K} \arrow[d, "\mathrm{loc}"'] \arrow[dr, "\mathrm{R}\,f_*\,\mathrm{L}\,f^*\,\mathrm{loc}"] & \\ \mathscr{D} \arrow[r, "F_1"'] & \mathscr{D} \end{tikzcd} \tag{39}$$

eq:3334

in $\mathscr{C}at_\infty$. Via uniqueness we have $F_0 \cong id_{\mathscr{D}}$ and $F_1 \cong \mathrm{R}\,G\,\mathrm{L}\,F$.

Since the restriction functor

$$\mathrm{Fun}(\Delta^1 \times \mathscr{D}, \mathscr{D}) \to \mathrm{Fun}(\partial\Delta^1 \times \mathscr{D}, \mathscr{D})$$

is an isofibration (Corollary I-6.14) we can replace η'' with an isomorphic map $\eta : \Delta^1 \times \mathscr{D} \to \mathscr{D}$ which completes the proposed diagram (38) and has the desired restrictions $\eta|_{\{0\}\times\mathscr{D}} = id_{\mathscr{D}}$ and $\eta|_{\{1\}\times\mathscr{D}} = \mathrm{R}\,G\mathbb{L}F$. By considering the isofibration

$$\mathrm{Fun}(\Delta^1 \times \Delta^1 \times \mathscr{K}, \mathscr{D}) \to \mathrm{Fun}(\Delta^1 \times \partial\Delta^1 \times \mathscr{K}, \mathscr{D})$$

we can also assume that the diagram (38) restricts to the boundary to produce diagrams (39) in which the left simplex in (39) is degenerate and the right simplex is given as the composite (37).

Finally, Proposition 10.8 tells us that the transformation $\eta : id_{\mathscr{D}} \to \mathrm{R}\,G\,\mathrm{L}\,F$ is the unit of an adjunction between $\mathrm{L}\,F$ and $\mathrm{R}\,G$ if and only if the induced transformation on the homotopy category exhibits $\mathrm{h}\,\mathrm{L}\,F$ as left adjoint to $\mathrm{h}\,\mathrm{R}\,G$. However, this was already argued in Lemma 10.10. $\square$

10.5. **Left derived functors.**

Definition 10.11. Let $\mathbb{A}$ and $\mathbb{B}$ be Grothendieck abelian categories. Suppose that an exact functor $F : \mathscr{K}(\mathbb{A}) \to \mathscr{K}(\mathbb{B})$ is left adjoint to an exact functor $G : \mathscr{K}(\mathbb{B}) \to \mathscr{K}(\mathbb{A})$, and that $\mathscr{K}(\mathbb{A})$ admits enough F-acyclic complexes.

In this setting, the left derived functor $\mathrm{L}\,F : \mathscr{D}(\mathbb{A}) \to \mathscr{D}(\mathbb{B})$ for F is defined as the left adjoint to the right derived functor $\mathrm{R}\,G : \mathscr{D}(\mathbb{B}) \to \mathscr{D}(\mathbb{A})$.

We note that, by Proposition 10.9, such a left adjoint exists under the given hypothesis. Furthermore, by uniqueness of adjoints, the functor $\mathrm{L}\,F$ can be computed via any sufficiently large collection of F-acyclic complexes $\mathscr{K}(\mathbb{A})_{F\text{-ac}} \subseteq \mathscr{K}(\mathbb{A})$, as in Lemma 10.6.

Example 10.12 (The push-pull adjunction)**.** Let $f : X \to Y$ be a map of quasi-compact and quasi-separated schemes. Then we have the push-pull adjunction on quasi-coherent sheaves, which provides an adjunction between functors

$$f^* : \mathscr{K}(\mathrm{QCoh}(Y)) \to \mathscr{K}(\mathrm{QCoh}(X)) \quad \text{and} \quad f_* : \mathscr{K}(\mathrm{QCoh}(X)) \to \mathscr{K}(\mathrm{QCoh}(Y))$$

at the level of homotopy ∞-categories. This follows by Theorem I-14.9, for example.

The category $\mathscr{K}(\mathrm{QCoh}(Y))$ admits enough K-flat complexes, so that we can define the derived pullback functor

$$\mathrm{L}\, f^* : \mathscr{D}(\mathrm{QCoh}(Y)) \to \mathscr{D}(\mathrm{QCoh}(X)),$$

loosely speaking, by taking K-flat resolutions over Y and applying the underived pullback functor. This is precisely what one expects from standard algebraic practices, as outlined in [25] for example.

Remark 10.13. This example generalizes to the setting where X and Y are geometric stacks. Here we note that the categories of quasi-coherent sheaves remains Grothendieck abelian [24, Corollary 5.10] and that the categories of complexes of quasi-coherent sheaves admit enough K-flat objects by [8, Theorem 3.5.5].

Remark 10.14. For a comparison with the topological perspective see [5, Section 7.5.23] and [21, Section 2.1].

sect:pr

11. Presentability of the derived ∞-category

In our penultimate section we provide an abridged discussion of presentability for ∞-categories. The main conclusion here is that the derived ∞-category of a Grothendieck abelian category $\mathscr{D}(\mathbb{A})$ is presentable.

Presentability is a kind of sub-compact generation condition which, under its imposition, allows one to effectively "do algebra" with categories. In this context we can begin to speak of linear categories [17, Section D.1.2], sheaves of categories [17, Section 10.2], etc. (see also [15, Section 4.8]).

Remark 11.1. One should consult works of Kelley, Chirvasitu, and Johnson-Fryed for thorough analyses of a "categorical algebra" for presentable discrete categories which precedes the corresponding algebra in the ∞-setting [12, Section 6.5] [4].

11.1. **Filtering and compactness at a regular cardinal.** We begin with some technical shenanigans regarding filtered ∞-categories.

By a regular cardinal we mean, formally, a cardinal κ which satisfies the following: If Λ is an indexing set of cardinality $|\Lambda| < \kappa$, and $\{S_\lambda : \lambda \in \Lambda\}$ is a collection of sets with $|S_\lambda| < \kappa$ at all λ, then the union $\cup_{\lambda\in\Lambda} S_\lambda$ also has cardinality less that κ. For some examples, $\aleph_0$, which characterizes countably infinite sets, is regular. Under the continuum hypothesis, $2^{\aleph_0}$ is also regular, and in general any successor cardinal is regular. Informally, regular cardinals are just mechanism through which one imposes functional size constraints on sets and categories.

We say a simplicial set A is κ-small, for a regular cardinal κ, if the collection of non-degenerate simplices in A has cardinality $< \kappa$.

def:filtered

Definition 11.2. An ∞-category $\mathscr{K}$ is called κ-filtered if, for any κ-small simplicial set A and map $i : A \to \mathscr{K}$, there is a map $i^+ : A \star \Delta^0 \to \mathscr{K}$ with $i^+|_A = i$. By a filtered ∞-category we mean a $\aleph_0$-filtered ∞-category.

We note that, via functoriality of the join, all maps $i : A \to \mathscr{K}$ from κ-small A extend to the join $A \star \Delta^0$ if and only if all injective maps from κ-small K admit such an extension. Hence $\mathscr{K}$ is filtered if and only if each finite simplicial subset in $\mathscr{K}$ admits a cone point in $\mathscr{K}$. In particular, we recover the familiar notion of a filtered category in the discrete setting.

Remark 11.3. Though we won't use the notion here, in [14, Remark 5.3.1.11] Lurie defines a simplicial set K to be κ-filtered if it admits a categorical equivalence $K \to \mathscr{K}$ to a κ-filtered ∞-category $\mathscr{K}$.

Definition 11.4. We say an ∞-category $\mathscr{C}$ is κ-cocomplete if each diagram $p : \mathscr{K} \to \mathscr{C}$ from a κ-filtered ∞-category admits a colimit in $\mathscr{C}$. We call a functor $F : \mathscr{C} \to \mathscr{D}$ between κ-cocomplete ∞-categories κ-cocontinuous if F preserves κ-filtered colimits.

def:k_compact **Definition 11.5.** Let $\mathscr{C}$ be a κ-cocomplete ∞-category. We call an object x in $\mathscr{C}$ κ-compact if each functor $h^x : \mathscr{C} \to \mathscr{K}an$ which is corepresented by x is κ-cocontinuous.

In comparing with Definition 7.8, we find that an object is compact if and only if it is $\aleph_0$-compact. The following provides a sanity check for the discrete setting.

Proposition 11.6 ([16])**.** *For an ∞-category $\mathscr{C}$, the following are equivalent:*

(a) *$\mathscr{C}$ is κ-cocomplete.*

(b) *For each κ-filtered partially ordered set A, and diagram $p : A \to \mathscr{C}$, p admits a colimit in $\mathscr{C}$.*

Supposing $\mathscr{C}$ is κ-cocomplete, and $F : \mathscr{C} \to \mathscr{D}$ is any functor, the following are equivalent:

(a′) *F is κ-cocontinuous.*

(b′) *For each κ-filtered partially ordered set A, F preserves A-filtered colimits.*

Proof. See [16, 02QA, 02NU]. □

11.2. **Compact generation and presentability of derived categories.**

Definition 11.7. A cocomplete ∞-category $\mathscr{C}$ is called compactly generated if $\mathscr{C}$ is generated under small colimits by an essentially small subcategory of compact objects.

Recall that an ∞-category $\mathscr{C}$ is called locally small if all of its mapping spaces $\mathrm{Hom}_{\mathscr{C}}(x, y)$, at arbitrary x and y, are essentially small.

lem:compgen_viah **Lemma 11.8.** *A locally small stable ∞-category $\mathscr{C}$ is compactly generated if and only if its homotopy category $\mathrm{h}\mathscr{C}$ is compactly generated (in the triangulated sense of the term). More precisely, if $C_0 \subseteq \mathrm{h}\mathscr{C}$ is a small subcategory which generates $\mathrm{h}\mathscr{C}$, and $\mathscr{C}_0$ is the full ∞-subcategory in $\mathscr{C}$ spanned by the objects of C_0, then*

(1) *C_0 consists of compacts in $\mathrm{h}\mathscr{C}$ if and only if $\mathscr{C}_0$ consists of compacts in $\mathscr{C}$, and*

(2) *C_0 generates $\mathrm{h}\mathscr{C}$ under small coproducts and the formation of mapping cones if and only if $\mathscr{C}_0$ generates $\mathscr{C}$ under small colimits.*

Proof. Let $C_0 \subseteq \mathrm{h}\mathscr{C}$ be any full subcategory, and $\mathscr{C}_0$ be the full subcategory in $\mathscr{C}$ spanned by the objects in C_0. Since $\mathscr{C}$ is locally small, we understand that $\mathscr{C}_0$ is essentially small if and only if C_0 is essentially small. Also, by Corollary 7.14, $\mathscr{C}_0$ consists entirely of compact objects in $\mathscr{C}$ if and only if C_0 consists entirely of compact objects in $\mathrm{h}\mathscr{C}$. Let us suppose that $\mathscr{C}_0$ is in fact essentially small and consists of compact objects.

Take now $\mathrm{Loc}(C_0)$ the smallest subcategory in $\mathrm{h}\mathscr{C}$ which is stable under the formation of triangles and small sums, as well as isomorphisms, and $\mathrm{Loc}(\mathscr{C}_0)$ the corresponding lift to a full subcategory in $\mathscr{C}$. Then $\mathrm{Loc}(\mathscr{C}_0)$ is the smallest full subcategory in $\mathscr{C}$ which is stable under small sums, the formation of cofibers, and isomorphisms in $\mathscr{C}$. It follows by Corollary 7.5 that $\mathrm{Loc}(\mathscr{C}_0)$ is the smallest subcategory in $\mathscr{C}$ which contains $\mathscr{C}_0$, is stable under the formation of small colimits, and is closed under isomorphism. Clearly we have

$$\mathrm{Loc}(C_0) = \mathrm{h}\mathscr{C} \;\Leftrightarrow\; \mathrm{Loc}(\mathscr{C}_0) = \mathscr{C}.$$

By definition, $\mathrm{h}\mathscr{C}$ is compactly generated if we can find such an essentially small subcategory of compacts C' with $\mathrm{Loc}(C') = \mathrm{h}\mathscr{C}$. By the information above we also see that $\mathscr{C}$ is compactly generated if and only if we can find such $\mathscr{C}'$ with $\mathrm{Loc}(\mathscr{C}') = \mathscr{C}$. So, considering $C' = C_0$ and $\mathscr{C}' = \mathscr{C}_0$, we observe that $\mathrm{h}\mathscr{C}$ is compactly generated if and only if $\mathscr{C}$ is compactly generated. $\square$

We apply Lemma 11.8 to observe a number of familiar examples.

Example 11.9. Let R be any ring and take $\mathscr{D}(R) = \mathscr{D}(R\text{-Mod})$. All bounded complexes of projectives are compact in the discrete derived category $D(R)$, and $D(R)$ is generated by this subcategory under small sums and the formation of cones. Hence $D(R)$ is compactly generated, and we conclude that $\mathscr{D}(R)$ is compactly generated.

Example 11.10. If X is a quasi-projective scheme, or more generally quasi-compact and separated, then the discrete derived category $D(X) = D(\mathrm{QCoh}(X))$ is compactly generated [3, Theorems 3.1.1, 3.1.3]. It is, in particular, generated by the full subcategory $D(X)_{perf}$ of perfect sheaves. It follows that the derived ∞-category $\mathscr{D}(X)$ is generated by the essentially small subcategory of compacts $\mathscr{D}(X)_{perf}$, and in particular is compactly generated.

We consider the following generalization of compact generation via κ-filtered simplicial sets and κ-compact objects.

Definition 11.11 ([14, Theorem 5.5.1.1])**.** An ∞-category $\mathscr{C}$ is called presentable if it satisfies the following:

(a) $\mathscr{C}$ is cocomplete.

(b) For some fixed regular cardinal κ, $\mathscr{C}$ is generated under small colimits by a small, finitely cocomplete, full subcategory $\mathscr{C}_0$ of κ-small objects.

In many situations the derived category $\mathscr{D}(\mathbb{A})$ is straight up compactly generated, and hence satisfies (b) for $\kappa = \aleph_0$. However, there are very reasonable settings where compact generation fails for $\mathscr{D}(\mathbb{A})$. For example, if we take $\mathbb{A} = \mathrm{Rep}\,\mathrm{GL}_2$ in finite characteristic, then the discrete derived category $D(\mathrm{Rep}\,\mathrm{GL}_2)$ is not compactly generated [9, Theorem 1.1]. Hence the derived ∞-category $\mathscr{D}(\mathrm{Rep}\,\mathrm{GL}_2)$ is not compactly generated either. It is the case, however, that presentability always holds.

Theorem 11.12 ([15, Proposition 1.3.5.21]). *For any Grothendieck abelian category $\mathbb{A}$, the derived ∞-category $\mathscr{D}(\mathbb{A})$ is presentable.*

For the proof–which we omit–one finds that the derived ∞-category is the "underlying ∞-category" [15, Definition 1.3.4.15] associated to a combinatorial model structure on the discrete category of cochains $\mathrm{Ch}(\mathbb{A})$, at which point presentability follows by a general result [15, Proposition 1.3.4.22].

11.3. **Presentability and the adjoint functor theorem.** As an example application of presentability, we record below the adjoint functor theorem.

def:acc_F **Definition 11.13.** Let $\mathscr{C}$ be a presentable ∞-category. A functor $F : \mathscr{C} \to \mathscr{D}$ is called accessible if there exists a regular cardinal for which F preserves all κ-filtered colimits.

Remark 11.14. One generally defines accessibility of a functor $F : \mathscr{C} \to \mathscr{D}$ in the situation where $\mathscr{C}$ is only accessible [14, Definition 5.4.2.5], not necessarily presentable.

The easiest way for a functor to be accessible if for it to be cocontinuous. We have the following fundamental characterization of left and right adjoints.

thm:AFT **Theorem 11.15** (Adjoint functor theorem, [14, Corollary 5.5.2.9]). *For a functor $F : \mathscr{C} \to \mathscr{D}$ between presentable ∞-categories,*

(1) *F admits a right adjoint if and only if it is cocontinuous.*

(2) *F admits a left adjoint if and only if it is accessible and continuous.*

sect:ind_deriver

12. Ind-completion and renormalized derived categories

To conclude, we discuss the ind-completion functor $\mathrm{Ind} : \mathscr{C}at_\infty^{\mathrm{sm}} \to \mathscr{C}at_\infty$. The main point here is fairly benign; Namely, ind-completion provides an alternate means of "compactifying" small derived categories of interest. This compactifying process produces, from any essentially small ∞-category $\mathscr{C}$ with all finite colimits, a universal cocompletion $\mathrm{Ind}(\mathscr{C})$ of $\mathscr{C}$ which is presentable and which recovers $\mathscr{C}$ as its subcategory of compact objects.

Such alternate cocompletions for the derived category actually appear constantly in the discrete setting, though we've not recognized them as such. For example, Koszul duality provides an equivalence between the discrete derived category $D(\Lambda)_{fin}$ of finite-dimensional representations over the exterior algebra and the derived category $D(S)_{coh}$ of coherent dg modules over the polynomial ring. This equivalence, however, does not extend to the unbounded setting as there is a disagreement between the compact objects in $D(\Lambda)$ and in $D(S)$. Instead, as one sees clearly from the indization perspective, Koszul duality calculates the ind-completion of $\mathscr{D}(\Lambda)_{fin}$ as the unbounded category of dg modules over S,

$$\mathrm{Ind}\,\mathscr{D}(\Lambda)_{fin} \cong \mathscr{D}(S).$$

This example is discussed, along with a few others, in the final subsection below.

Remark 12.1. As with Section 11, our presentation here is fairly coarse, as our main goal is simply to collect results from [14, 15] and present them in a linear ordering which allows one to understand how ind-completion functions in familiar siuations from algebra and algebraic geometry. In order to understand many arguments in complete detail, the reader will need to follow the specific references given herein, and independently consume the sub-arguments provided in higher topos theory and/or higher algebra.

12.1. **Ind-completion of small ∞-categories.**

Notation 12.2. For ∞-categories $\mathscr{A}$ and $\mathscr{B}$ which admit small ($\aleph_0$-)filtered colimits, we let $\mathrm{Fun}^{\aleph_0}(\mathscr{A}, \mathscr{B})$ denote the full ∞-subcategory of functors in $\mathrm{Fun}(\mathscr{A}, \mathscr{B})$ which preserve small filtered colimits.

thm:ind_C **Theorem 12.3** ([14, Corollary 5.3.5.4 & Proposition 5.3.5.11])**.** *For any essentially small ∞-category $\mathscr{C}$, there is an ∞-category $\mathscr{C}'$ and a functor $i : \mathscr{C} \to \mathscr{C}'$ which has the following properties:*

(1) *$\mathscr{C}'$ admits all small filtered colimits.*
(2) *i is fully faithful, and for each x in $\mathscr{C}$ the image $F(x)$ is compact in $\mathscr{C}'$.*
(3) *Every object z in $\mathscr{C}'$ admits a filtered diagram $p : \mathscr{K} \to \mathscr{C}$ under which z is recovered as a colimit of the corresponding diagram $ip : \mathscr{K} \to \mathscr{C}'$.*

Furthermore, the ∞-category $\mathscr{C}'$ is uniquely determined, as an object in the under-category $(\mathscr{C}at_\infty)_{\mathscr{C}/}$, up to equivalence.

Let us outline how this theorem works, according to the fundamentals laid out in [14].

Proof outline. Consider the full subcategory $\mathscr{C}' = \mathrm{Ind}(\mathscr{C}) \subseteq \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathscr{K}an)$ spanned by all those functors $\xi : \mathscr{C}^{\mathrm{op}} \to \mathscr{K}an$ which classify a cartesian fibration $\mathscr{E} \to \mathscr{C}$ for which $\mathscr{E}$ is filtered, as an ∞-category. We note that all representable functors lie in $\mathrm{Ind}(\mathscr{C})$ since each fibration $\mathscr{C}_{/x} \to \mathscr{C}$, for x in $\mathscr{C}$, has terminal object $id_x : x \to x$ (Proposition II-9.11). Hence the Yoneda embedding $i : \mathscr{C} \to \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathscr{K}an)$ has image in $\mathrm{Ind}(\mathscr{C})$. Furthermore, i is fully faithful since the Yoneda embedding is fully faithful, by Theorem II-16.1.

The fact that $\mathrm{Ind}(\mathscr{C})$ is stable under the formation of filtered colimits in $\mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathscr{K}an)$ is covered in [14, Proposition 5.3.5.3], and the fact that $\mathrm{Ind}(\mathscr{C})$ is generated by the image of $\mathscr{C}$ under small filtered colimits is covered in [14, Corollary 5.3.5.4]. Compactness of the image $i(x)$ for each x in $\mathscr{C}$ is covered in [14, Proposition 5.3.5.5].

As for the uniqueness claim, for any ∞-category $\mathscr{A}$ which admits small filtered limits, and any functor $f : \mathscr{C} \to \mathscr{A}$ with compact image, f admits a unique extension to a $\aleph_0$-cocontinuous functor $F : \mathrm{Ind}(\mathscr{C}) \to \mathscr{A}$ which is obtained via left Kan extension [14, Lemma 5.3.5.8]. This functor is an equivalence if and only if f is fully faithful and $\mathscr{A}$ is generated by the image of $\mathscr{C}$ under filtered limits [14, Proposition 5.3.5.11]. $\square$

Remark 12.4. We note that essential smallness of $\mathscr{C}$ is required to ensure the existence of the Kan extension $F : \mathrm{Ind}(\mathscr{C}) \to \mathscr{A}$ which we've employed in the proof of Theorem 12.3.

def:ind_completion **Definition 12.5.** Given an essentially small ∞-category $\mathscr{C}$, any $\aleph_0$-cocomplete ∞-category $\mathscr{C}'$ equipped with a fully faithful functor $i : \mathscr{C} \to \mathscr{C}'$ as in Theorem 12.3 is called an ind-completion of $\mathscr{C}$.

We have the following universal property for the ind-completion.

thm:ind_univ

Theorem 12.6 ([14, Proposition 5.3.5.10]). *Let $\mathscr{C}$ be a small ∞-category and $i : \mathscr{C} \to \mathscr{C}'$ be any ind-completion of $\mathscr{C}$. Then for any ∞-category $\mathscr{A}$ which admits small filtered colimits, restriction along i provides an equivalence*

$$i^* : \mathrm{Fun}^{\aleph_0}(\mathscr{C}', \mathscr{A}) \xrightarrow{\sim} \mathrm{Fun}(\mathscr{C}, \mathscr{A}).$$

We can now be more precise about our uniqueness claim from Theorem 12.3.

thm:ind_equiv

Theorem 12.7 ([14, Proposition 5.3.5.11]). *Let $\mathscr{A}$ be an ∞-category which admits all small filtered colimits, and $f : \mathscr{C} \to \mathscr{A}$ be a functor from an essentially small ∞-category. Consider any ind-completion $i : \mathscr{C} \to \mathscr{C}'$ and the unique extension $F : \mathscr{C}' \to \mathscr{A}$ of f to an $\aleph_0$-cocontinuous map from $\mathscr{C}'$. Then*

(1) *F is fully faithful provided f is fully faithful and has compact image.*

(2) *F is an equivalence if and only if f is fully faithful with compact image, and $\mathscr{A}$ is generated by the image of $\mathscr{C}$ under small filtered colimits.*

We consider some examples. Below we take for granted that, for any ring R, the compact objects in the discrete derived category $D(R) = \mathscr{D}(R\text{-mod})$ are precisely those objects which are quasi-isomorphic to a bounded complex of projectives, and for any quasi-compact quasi-separated scheme X the compacts in the discrete derived category $D(X) = \mathscr{D}(\mathrm{QCoh}(X))$ are the perfect complexes [3], i.e. those complexes which are isomorphic over any affine open to a bounded complex of locally free sheaves.

Example 12.8 (Ind-finite representations I). Let R be a finite-dimensional algebra and $\mathscr{D}(R)$ be the derived ∞-category. Let $\mathscr{D}(R)_{fin}$ be the full subcategory of complexes with finite-dimensional cohomology. As $\mathscr{D}(R)$ is cocomplete it admits all small filtered colimits, and by Corollary 7.14 the compact objects in $\mathscr{D}(R)$ are precisely those complexes of finite projective dimension, i.e. which are quasi-isomorphic to a bounded complexe of projectives. Hence, if R is of finite global dimension, then all objects in $\mathscr{D}(R)_{fin}$ are compact in $\mathscr{D}(R)$ and the inclusion $\mathscr{D}(R)_{fin} \to \mathscr{D}(R)$ identifies $\mathscr{D}(R)$ as an ind-completion of $\mathscr{D}(R)_{fin}$.

Example 12.9 (Ind-coherent sheaves I). Let X be a complex variety with a singular point $x : \mathrm{Spec}(\mathbb{C}) \to X$. Then the residue field $k(x)$ is a coherent sheaf on X which is non-compact in the derived ∞-category $\mathscr{D}(X) = \mathscr{D}(\mathrm{QCoh}(X))$. Hence, the inclusion $\mathscr{D}(X)_{coh} \to \mathscr{D}(X)$ does *not* identify the derived ∞-category as an ind-completion of $\mathscr{D}(X)_{coh}$.

We note, however, that in this case we can still consider the ind-completion $\mathrm{Ind}\,\mathscr{D}(X)_{coh}$ and we have the unique extension $F : \mathrm{Ind}\,\mathscr{D}(X)_{coh} \to \mathscr{D}(X)$ of the inclusion. To say that $\mathscr{D}(X)$ is not an ind-completion of $\mathscr{D}(X)$ is, more precisely, to say that the functor F is not an equivalence.

Example 12.10 (Ind-coherent sheaves II). For X a smooth Noetherian scheme we have $\mathscr{D}(X)_{perf} = \mathscr{D}(X)_{coh}$, and $\mathscr{D}(X)$ is an ind-completion of $\mathscr{D}(X)_{coh}$.

Example 12.11 (Ind-finite representations II). Let G be a finite group with p dividing the order of G, for some prime p. Let k be a field of characteristic p. Then the trivial representation k is not compact in $\mathscr{D}(G) = \mathscr{D}(\mathrm{Rep}_k(G))$, and the unique extension

$$F : \mathrm{Ind}\,\mathscr{D}(G)_{fin} \to \mathscr{D}(G)$$

of the inclusion $\mathscr{D}(G)_{fin} \to \mathscr{D}(G)$ is not an equivalence.

sect:ind_fun

12.2. **Ind-completion as a functor.** Let $\mathscr{C}at_\infty^{\aleph_0}$ denote the non-full ∞-subcategory in $\mathscr{C}at_\infty$ whose objects are ∞-categories which admit all small filtered colimits, and whose maps are those functors $F : \mathscr{A} \to \mathscr{B}$ which commute with small filtered colimits. Consider also the non-full subcategory

$$\mathscr{M}_{\mathrm{Ind}} \subseteq \mathscr{C}at_\infty \times \Delta^1$$

whose objects are the union

$$\mathscr{M}_{\mathrm{Ind}}[0] = (\mathscr{C}at_\infty^{\mathrm{sm}}[0] \times \{0\}) \cup \left(\mathscr{C}at_\infty^{\aleph_0}[0] \times \{1\}\right).$$

The morphisms (edges) over $\{0\}$ and over $0 < 1$, under the projection $\mathscr{C}at_\infty \times \Delta^1 \to \Delta^1$, are arbitrary maps between essentially small ∞-categories, and from essentially small ∞-categories to $\aleph_0$-cocomplete categories. The morphisms over 1 are $\aleph_0$-cocontinuous functors. The n-simplices $\sigma : \Delta^n \to \mathscr{M}_{\mathrm{Ind}}$ are precisely those n-simplices in $\mathscr{C}at_\infty \times \Delta^1$ whose vertices $\sigma|_{\Delta^{\{i\}}}$ and edges $\sigma|_{\Delta^{\{i,i+1\}}}$ satisfy the above restrictions.

One sees that $\mathscr{M}_{\mathrm{Ind}}$ is an ∞-category since composites of κ-cocontinuous functors are κ-cocontinuous. The fibers over Δ^1 under the projection $q : \mathscr{M}_{\mathrm{Ind}} \to \Delta^1$ are precisely

$$(\mathscr{M}_{\mathrm{Ind}})_0 = \mathscr{C}at_\infty^{\mathrm{sm}} \ \text{ and } \ (\mathscr{M}_{\mathrm{Ind}})_1 = \mathscr{C}at_\infty^{\aleph_0}.$$

One sees that the projection q is an inner fibration, as it is a composite of the inner fibrations provided by the inclusion $\mathscr{M}_{\mathrm{Ind}} \to \mathscr{C}at_\infty \times \Delta^1$ and the projection $\mathscr{C}at_\infty \times \Delta^1 \to \Delta^1$.

lem:3693

Lemma 12.12. *The projection $q : \mathscr{M}_{\mathrm{Ind}} \to \Delta^1$ is a cocartesian fibration, and for any ∞-category $\mathscr{C}$ in the fiber over 0, i.e. any essentially small ∞-category, an edge $i : \mathscr{C} \to \mathscr{C}'$ over $0 < 1$ is q-cocartesian if and only if i realizes $\mathscr{C}'$ as an ind-completion of $\mathscr{C}$.*

Proof. The category $\mathscr{C}at_\infty \times \Delta^1$ is the homotopy coherent nerve of the simplicial category $\underline{\mathrm{Cat}}_\infty^+ \times \{0 < 1\}$, and $\mathscr{M}_{\mathrm{Ind}}$ is the homotopy coherent nerve of the simplicial category $\underline{\mathrm{M}}$ with prescribed objects and morphisms given by

$$\underline{\mathrm{Hom}}_{\underline{\mathrm{M}}}(\mathscr{C}, \mathscr{A}) = \begin{cases} \mathrm{Fun}(\mathscr{C}, \mathscr{A}) & \text{if } \mathscr{C} \text{ lies over } 0 \\ \mathrm{Fun}^{\aleph_0}(\mathscr{C}, \mathscr{A}) & \text{if } \mathscr{C} \text{ and } \mathscr{A} \text{ lie over } 1 \\ \emptyset & \text{otherwise.} \end{cases}$$

We therefore identify the $\mathrm{h}\mathscr{K}an$-enriched category $\pi\mathscr{M}_{\mathrm{Ind}}$ with the category $\pi\underline{\mathrm{M}}$ obtained by applying the symmetric monoidal functor $\pi : \mathscr{K}an \to \mathrm{h}\mathscr{K}an$ to the morphism complexes (Proposition II-7.6). By Theorem 12.6 we conclude that the subcategory

$$\mathscr{C}at_\infty^{\aleph_0} = (\mathscr{M}_{\mathrm{Ind}})_1 \ \subseteq \ \mathscr{M}_{\mathrm{Ind}}$$

is reflective in $\mathscr{M}_{\mathrm{Ind}}$. By Lemma 8.13 it follows that the projection $q : \mathscr{M}_{\mathrm{Ind}} \to \Delta^1$ is a cocartesian fibration, and a map $i : \mathscr{C} \to \mathscr{C}'$ is a q-cocartesian edge in $\mathscr{M}_{\mathrm{Ind}}$, over $0 < 1$, if and only if i realizes $\mathscr{C}'$ as an ind-completion of $\mathscr{C}$. $\square$

We now have the unique cocartesian transformation $I : \Delta^1 \times \mathscr{C}at_\infty^{\mathrm{sm}} \to \mathscr{M}_{\mathrm{Ind}}$ which solves the lifting problem

$$\begin{tikzcd} \{0\} \times \mathscr{C}at_\infty^{\mathrm{sm}} \arrow[r] \arrow[d] & \mathscr{M}_{\mathrm{Ind}} \arrow[d, "q"] \\ \Delta^1 \times \mathscr{C}at_\infty^{\mathrm{sm}} \arrow[r] \arrow[ur, dotted, "I"] & \Delta^1, \end{tikzcd} \tag{40}$$

eq:indize

i.e. the unique functor which solves the lifting problem and evaluates to a q-cocartesian edge in $\mathscr{M}_{\mathrm{Ind}}$ at each edge $\Delta^1 \times \{\mathscr{C}\}$. By Lemma 12.12 this cocartesian edge is an ind-completion $i : \mathscr{C} \to \mathscr{C}'$ of $\mathscr{C}$.

We now consider the other projection

$$p : \mathscr{M}_{\mathrm{Ind}} \subseteq \mathscr{C}at_\infty \times \Delta^1 \to \mathscr{C}at_\infty$$

and the composition $\widetilde{\mathrm{Ind}} := pI : \Delta^1 \times \mathscr{C}at_\infty^{\mathrm{sm}} \to \mathscr{C}at_\infty$.

Definition 12.13. The indization functor is the pairing $(\mathrm{Ind}, \widetilde{\mathrm{Ind}})$ of the restriction

$$\mathrm{Ind} := \widetilde{\mathrm{Ind}}|_{\{1\} \times \mathscr{C}at_\infty^{\mathrm{sm}}} : \mathscr{C}at_\infty^{\mathrm{sm}} \to \mathscr{C}at_\infty^{\aleph_0}$$

along with the transformation $\widetilde{\mathrm{Ind}} : \Delta^1 \times \mathscr{C}at_\infty^{\mathrm{sm}} \to \mathscr{C}at_\infty$ constructed above.

We note that the cocartesian solution from (40) is unique up to a contractible space of choices, by Theorem II-2.7. Hence the indization functor is determined up to a contractible space of choices.

Remark 12.14. By an abuse of notation we often refer to the functor $\mathrm{Ind} : \mathscr{C}at_\infty^{\mathrm{sm}} \to \mathscr{C}at_\infty$ itself as the indization functor. We also generally let i denote the structural transformation $i = \widetilde{\mathrm{Ind}}$ in order to ease notation.

Remark 12.15. Despite the apparent differences in the constructions, our indization functor is the same as the one introduced in [14, Proposition 5.4.2.19]. See Proposition A.21 below.

To be very clear, the functor $\widetilde{\mathrm{Ind}}$ picks, at each object $(0, \mathscr{C})$ in $\Delta^1 \times \mathscr{C}at_\infty^{\mathrm{sm}}$, an ind-completion $\mathrm{Ind}(\mathscr{C})$ and $\widetilde{\mathrm{Ind}}$ itself is a tranformation between the inclusion $\mathscr{C}at_\infty^{\mathrm{sm}} \to \mathscr{C}at_\infty$ and the functor $\mathrm{Ind} : \mathscr{C}at_\infty^{\mathrm{sm}} \to \mathscr{C}at_\infty^{\aleph_0} \subseteq \mathscr{C}at_\infty$. This transformation evaluates at each essentially small ∞-category $\mathscr{C}$ to provide a functor $i_{\mathscr{C}} : \mathscr{C} \to \mathrm{Ind}(\mathscr{C})$ which specifically witnesses $\mathrm{Ind}(\mathscr{C})$ as an ind-completion of $\mathscr{C}$.

12.3. **Accessibility and idempotent completion: the rundown.** In the remaining portions of the section we make various arguments which reference accessibility and idempotent completion. Since we are not directly invested in accessibility, we avoid a serious deviation into this topic at the moment.

However, in short, an accessible ∞-category is one of the form $\mathscr{A} = \mathrm{Ind}(\mathscr{C})$ for essentially small $\mathscr{C}$. Or, to be more precise, an accessible ∞-category is one which admits small filtered colimits, and admits a functor $\mathscr{C} \to \mathscr{A}$ from an essentially small ∞-category which induces an equivalence $\mathrm{Ind}(\mathscr{C}) \xrightarrow{\sim} \mathscr{A}$. In this case the original category $\mathscr{C}$ lands in the subcategory of compact objects in $\mathscr{A}$, and the corresponding map $\mathscr{C} \to \mathscr{A}^c$ identifies $\mathscr{A}^c$ as an idempotent completion of $\mathscr{C}$ (see Definition A.8).

We have the ∞-category $\mathrm{Acc}_{\aleph_0}$ of accessible ∞-categories and functors which preserve small filetered colimits and compact objects, and one finds that the ind-completion functor $\mathrm{Ind} : \mathscr{C}at_\infty^{\mathrm{sm}} \to \mathscr{C}at_\infty$ restricts to an equivalence between the ∞-category of essentially small, idempotent complete ∞-categories and accessible ∞-categories. The inverse is provided by taking the compacts.

For now, we take the above points for granted and continue with our analysis of ind-completion, specifically in the stable setting. We return to the topic of accessibility, and provide a bare-bones accounting of the subject in Appendix A.

12.4. Ind-completion and products.

Proposition 12.16. *The indization functor* $\mathrm{Ind} : \mathscr{C}at_\infty^{\mathrm{sm}} \to \mathscr{C}at_\infty$ *commutes with arbitrary products.*

Proof. By [14, Proposition 5.4.7.3] the category $\mathrm{Acc}_{\aleph_0}$ admits all small limits and the inclusion $\mathrm{Acc}_{\aleph_0} \to \mathscr{C}at_\infty$ is continuous. Since the restriction to idempotent complete categories

$$\mathrm{Ind} : (\mathscr{C}at_\infty^{\mathrm{sm}})_{\mathrm{split}} \overset{\sim}{\to} \mathrm{Acc}_{\aleph_0}$$

is an equivalence by Corollary A.20, we conclude that the category of idempotent split ∞-categories is complete as well and that indization from $(\mathscr{C}at_\infty^{\mathrm{sm}})_{\mathrm{split}}$ is continuous. From the factorization of Proposition A.21 it now suffices to provide that the idempotent completion functor $(-)^\vee : \mathscr{C}at_\infty^{\mathrm{sm}} \to (\mathscr{C}at_\infty^{\mathrm{sm}})_{\mathrm{split}}$ preserves products.

For any simplicial set K and small collection of categories $\{\mathscr{C}_\lambda : \lambda \in \Lambda\}$ we have the natural isomorphism

$$\mathrm{Fun}(K, \prod_\lambda \mathscr{C}_\lambda) \overset{\sim}{\to} \prod_\lambda \mathrm{Fun}(K, \mathscr{C}_\lambda)$$

so that, by the characterization of idempotent complete categories provided in Proposition A.5, $(\mathscr{C}at_\infty^{\mathrm{sm}})_{\mathrm{split}}$ is stable under small products in $\mathscr{C}at_\infty$, and hence products in the complete category $(\mathscr{C}at_\infty^{\mathrm{sm}})_{\mathrm{split}}$ are the usual cartesian product.

We consider a small collection $\{\mathscr{C}_\lambda : \lambda \in \Lambda\}$ of arbitrary essentially small ∞-categories and consider the idempotent completions $i_\lambda : \mathscr{C}_\lambda \to \mathscr{C}_\lambda^\vee$. Consider also the idempotent completion of the product $i : \prod_\lambda \mathscr{C}_\lambda \to (\prod_\lambda \mathscr{C}_\lambda)^\vee$. Via naturality of the transformation $i : id_{\mathscr{C}at_\infty} \to (-)^\vee$, each projection $p_\mu^\vee : (\prod_\lambda \mathscr{C}_\lambda)^\vee \to \mathscr{C}_\mu^\vee$ fits into a diagram

$$\begin{tikzcd}
(\prod_\lambda \mathscr{C}_\lambda)^\vee \arrow[r, "p_\mu^\vee"] & \mathscr{C}_\mu^\vee \\
\prod_\lambda \mathscr{C}_\lambda \arrow[u, "i"] \arrow[r, "p_\mu"'] & \mathscr{C}_\mu . \arrow[u, "i_\mu"']
\end{tikzcd}$$

Hence the induced maps to the product fit into a diagram

$$\begin{tikzcd}
(\prod_\lambda \mathscr{C}_\lambda)^\vee \arrow[rr, "\mathrm{comp}"] & & \prod_\lambda \mathscr{C}_\mu^\vee \\
& \prod_\lambda \mathscr{C}_\lambda \arrow[ul, "i"] \arrow[ur, "\prod i_\lambda"'] &
\end{tikzcd} .$$

To show that the comparison map above is an equivalence it suffices to show that the product $\prod_\lambda \mathscr{C}_\lambda^\vee$ is an idempotent completion of $\prod_\lambda \mathscr{C}_\lambda$.

For this final point, since each functor $i_\lambda : \mathscr{C}_\lambda \to \mathscr{C}_\lambda^\vee$ is fully faithful, the product map $\prod_\lambda i_\lambda$ is also fully faithful. Furthermore, for a tuple of objects $y = (y_\lambda : \lambda \in \Lambda)$ in $\prod_\lambda \mathscr{C}_\lambda^\vee$ any collection of retract diagrams $r_\lambda : \mathrm{Ret} \to \mathscr{C}_\lambda^\vee$ which realize each y_λ as a retract of some x_λ in $\mathscr{C}_\lambda$, the uniquely associated diagram

$$r = [r_\lambda : \lambda \in \Lambda]^t : \mathrm{Ret} \to \prod_\lambda \mathscr{C}_\lambda^\vee$$

expresses y as a retract of the object $(x_\lambda : \lambda \in \Lambda)$ in $\prod_\lambda \mathscr{C}_\lambda$. Therefore, by definition, the product map

$$\prod_\lambda i_\lambda : \prod_\lambda \mathscr{C}_\lambda \to \prod_\lambda \mathscr{C}_\lambda^\vee$$

exhibits the target category as an idempotent completion of the product $\prod_\lambda \mathscr{C}_\lambda$, as desired. □

Remark 12.17. Preservation of products is important when considering the transfer of monoidal structures, or actions, through ind-completion. See for example [15, Corollary 2.4.1.8].

12.5. **Ind-completion in $\aleph_0$-cocomplete and stable settings.**

thm:ind_present

Theorem 12.18. *Consider an essentially small ∞-category $\mathscr{C}$. If $\mathscr{C}$ admits all finite colimits then the following hold:*

(1) *The ind-completion $\mathrm{Ind}(\mathscr{C})$ is presentable and the structure map $\mathscr{C} \to \mathrm{Ind}(\mathscr{C})$ preserves finite colimits.*
(2) *For any cocomplete ∞-category $\mathscr{A}$, and functor $\mathscr{C} \to \mathscr{A}$ which commutes with finite colimits, the induced functor $\mathrm{Ind}(\mathscr{C}) \to \mathscr{A}$ is cocontinuous.*

We first offer a supporting lemma regarding idempotent completion. (See Appendix A.)

lem:3814

Lemma 12.19. *If $i : \mathscr{C} \to \mathscr{C}^\vee$ is an idempotent completion in which $\mathscr{C}$ and $\mathscr{D}$ both admit κ-small limits (resp. colimits), then i commutes with κ-small limits (resp. colimits).*

Proof. We argue the point for limits. The case of colimits follows by taking opposites.

By identifying $\mathscr{C}$ with its essential image in $\mathscr{C}^\vee$ we may assume that i is an inclusion of simplicial sets. For any diagram $p : K \to \mathscr{C}$ the corresponding inclusion $\mathscr{C}_{/p} \to \mathscr{C}^\vee_{/p}$ is an idempotent completion of $\mathscr{C}_{/p}$ by [14, Lemma 5.1.4.4]. So it suffices to prove that idempotent completion preserves terminal objects. So, let us suppose that an object z is terminal in $\mathscr{C}$. For any y in $\mathscr{C}^\vee$, express y as a retract $y \to x \to y$ of an object x in $\mathscr{C}$ to obtain a retract diagram

$$\begin{tikzcd} & \mathrm{Hom}_{\mathscr{C}}(x,z) \arrow[dr] & \\ \mathrm{Hom}_{\mathscr{C}^\vee}(y,z) \arrow[ur] \arrow[rr, "id"'] & & \mathrm{Hom}_{\mathscr{C}^\vee}(y,z) \end{tikzcd}$$

in $\mathrm{h}\mathscr{K}an$. Contractibility of $\mathrm{Hom}_{\mathscr{C}}(x,z)$ now implies contractibility of $\mathrm{Hom}_{\mathscr{C}^\vee}(y,z)$. Since y was chosen arbitrarily we see that, by definition, z is terminal in $\mathscr{C}^\vee$. □

Proof of Theorem 12.18. (1) By [14, Theorem 5.5.1.1] the category $\mathrm{Ind}(\mathscr{C})$ is presentable, and by [14, Corollary 5.3.4.15] the subcategory of compacts $\mathrm{Ind}(\mathscr{C})^c$ is stable under finite colimits in $\mathrm{Ind}(\mathscr{C})$. By Lemma A.16 the fully faithful functor $\mathscr{C} \to \mathrm{Idem}(\mathscr{C})^c$ exhibits $\mathrm{Idem}(\mathscr{C})^c$ as an idempotent completion of $\mathscr{C}$. By Lemma 12.19 the map $\mathscr{C} \to \mathrm{Idem}(\mathscr{C})$ preserves finite colimits, so that the composite

$$\mathscr{C} \to \mathrm{Ind}(\mathscr{C})^c \subseteq \mathrm{Ind}(\mathscr{C})$$

preserves finite colimits as well. Claim (2) follows by [14, Proposition 5.5.1.9]. □

rem:idem_colim

Remark 12.20. Since $\mathrm{Ind}(\mathscr{C})^c$ is an idempotent completion of $\mathscr{C}$, one can employ [14, Corollary 5.3.4.15] and Lemma 12.19 to see that the idempotent completion $\mathscr{C}^\vee$ of any finitely cocomplete ∞-category $\mathscr{C}$ admits all finite colimits, and the structure map $i : \mathscr{C} \to \mathscr{C}^\vee$ commutes with finite colimits. One can similarly argue, by replacing $\mathrm{Ind} = \mathrm{Ind}_{\aleph_0}$ with Ind_κ at a generic regular cardinal, that the idempotent completion $\mathscr{C}^\vee$ admits κ-small colimits whenever $\mathscr{C}$ admits κ-small colimits and that the structure map $i : \mathscr{C} \to \mathscr{C}^\vee$ respects such colimits. By taking opposite categories the analogous results are seen to hold for $(\kappa-)$small limits as well.

Remark 12.21. As in the accessible situation, represented in Corollary A.20, one observes that the indization functor Ind provides an equivalence between the category of idempotent complete, essentially small, and finitely cocomplete ∞-categories to a category $\mathrm{Pr}_{\aleph_0}$ of compactly generated presentable ∞-categories. See [15, Lemma 5.3.2.9].

As one expects, stability is preserved under ind-completion as well.

prop:ind_stable

Proposition 12.22 ([15, Proposition 1.1.3.6])**.** *If $\mathscr{C}$ is essentially small and stable, then the ind-completion $\mathrm{Ind}(\mathscr{C})$ is presentable and stable. Furthermore, the structure map $i : \mathscr{C} \to \mathrm{Ind}(\mathscr{C})$ is exact.*

Proof. Stability is covered in [15], and exactness of i follows by Theorem 12.18 (2). □

12.6. **Renormalized derived categories.** One can employ ind-completion to produce presentable and stable alternatives to the unbounded derived category. The fundamental point here is the following.

Proposition 12.23. *If $\mathscr{C}$ is an idempotent complete essentially small ∞-category, then the structure map $i : \mathscr{C} \to \mathrm{Ind}(\mathscr{C})$ restricts to an equivalence*

$$\mathscr{C} \overset{\sim}{\to} \mathrm{Ind}(\mathscr{C})^c.$$

Proof. This is Lemma A.17 (2). □

Hence, for $\mathscr{D}(\mathbb{A})_{fav}$ some essentially small, idempotent complete, stable subcategory of "favored" objects in $\mathscr{D}(\mathbb{A})$, the ind-completion $\mathrm{Ind}\,\mathscr{D}(\mathbb{A})_{fav}$ is a stable cocompletion of $\mathscr{D}(\mathbb{A})_{fav}$ which is freely generated by $\mathscr{D}(\mathbb{A})_{fav}$, and from which one recovers $\mathscr{D}(\mathbb{A})_{fav}$ as the subcategory of compacts. Taking the homotopy category, we obtain a triangulated category $D = \mathrm{h}\,\mathrm{Ind}\,\mathscr{D}(\mathbb{A})_{fav}$ which contains the discrtete derived category $D(\mathbb{A})_{fav}$ and which satisfies

$$\mathrm{Loc}\, D(\mathbb{A})_{fav} = D.$$

We note furthermore that we have, via the universal property of ind-completion, a unique cocontinuous and exact functor $\operatorname{Ind}\mathscr{D}(\mathbb{A})_{fin} \to \mathscr{D}(\mathbb{A})$ which fits into a diagram

$$\begin{tikzcd} \operatorname{Ind}\mathscr{D}(\mathbb{A})_{fin} \arrow[rr] & & \mathscr{D}(\mathbb{A}) \\ & \mathscr{D}(\mathbb{A})_{fav} \arrow[ul] \arrow[ur] & \end{tikzcd}.$$

One can view the construction $\operatorname{Ind}\mathscr{D}(\mathbb{A})_{fin}$ as a kind of "renormalization" of the derived ∞-category around this subcategory of favored objects. (Such terminology is oft-employed in the geometric Langlands literature.)

We do not attempt to convince the reader that this kind of construction is of any relevance to them, but cover two examples which already appear in the literature.

Example 12.24 (Ind-coherent sheaves)**.** Let X be a reasonable scheme or stack, and $\mathscr{D}(X)_{\mathrm{coh}}$ be the derived category of complexes with (bounded) coherent cohomology. Take

$$\mathbf{IndCoh}(X) := \operatorname{Ind}\mathscr{D}(X)_{coh},$$

where the derived-ness is implicit in the left-hand notation. This is the category of so-called ind-coherent sheaves on X.

Ind-coherent sheaves play an essential role in the geometric Langlands program [6, 7], for example.

Example 12.25 (Ind-finite representations)**.** Let $\mathbb{G}$ be an affine algebraic group in, say, finite characteristic. Take $\mathscr{D}(\mathbb{G})_{fin}$ the derived category of dg $\mathbb{G}$-representations with (bounded) finite-dimensional cohomology, and take

$$\mathbf{Rep}(\mathbb{G}) = \operatorname{Ind}\mathscr{D}(\mathbb{G})_{fin}.$$

This is a compactly generated stable ∞-category with compact objects $\mathbf{Rep}(\mathbb{G})^c := \mathscr{D}(\mathbb{G})_{fin}$.

In comparing with the derived ∞-category, we have that $\mathscr{D}(\mathbb{G})$ is not even compactly generated, except under very specific circumstances [9]. Furthermore, if we consider $\mathscr{D}(\mathbb{G})$ as a symmetric monoidal ∞-category, it has dualizable objects $\mathscr{D}(\mathbb{G})_{fin}$, so that its dualizable and compact objects disagree horribly. From the perspective of ind-coherent sheaves, we have $\mathbf{Rep}(\mathbb{G}) = \mathbf{IndCoh}(*/\mathbb{G})$, where the quotient $*/\mathbb{G}$ is specifically the stack quotient.

The following example demonstrates that ind-constructions appear very naturally, and should be expected to appear generically, when considering equivalences of derived categories.

ex:kos_dual

Example 12.26 (Koszul duality)**.** Let V be a finite dimensional vector space. Take $A = \wedge^*(V)$ and $S = k[\Sigma^{-1}V^*]$, where we consider S as a dg algebra generated in cohomological degree 1. Koszul duality provides an equivalence

$$\mathrm{Kos} = \mathrm{RHom}_A(k, -) : \mathscr{D}(A)_{fin} \to \mathscr{D}(S)_{coh}.$$

To be clear, $\mathscr{D}(S)$ is obtained as the dg nerve of the dg category S-dgmod of arbitrary dg S-modules, and $\mathscr{D}(S)_{coh}$ is the full subcategory of complexes with finitely generated cohomology.

The equivalence Kos sends the trivial representation k to $\mathrm{Kos}(k) = S$, and we note that the trivial representation is non-compact in $\mathscr{D}(A)$, since A is Frobenius.

Specifically, an A-module M is compact in $\mathscr{D}(A)$ if and only if M is projective. So we see that Kos does *not* extend to an equivalence from $\mathscr{D}(A)$ to $\mathscr{D}(S)$. However, trivially, Kos does extend to an equivalence

$$\operatorname{Ind}\mathrm{Kos} : \operatorname{Ind}\mathscr{D}(A)_{fin} \overset{\sim}{\to} \operatorname{Ind}\mathscr{D}(S)_{coh} \cong \mathscr{D}(S).$$

In the other direction, the subcategory of bounded complexes of projectives in $\mathscr{D}(A)$, i.e. compact objects in $\mathscr{D}(A)$, are sent under Kos to the subcategory $\mathscr{D}(S)_{tors}$ of coherent dg sheaves with torsion cohomology. Hence the inverse equivalence $\mathrm{Kos}^{-1} = k \otimes^{\mathrm{L}}_S - : \mathscr{D}(S)_{coh} \to \mathscr{D}(A)_{fin}$ induces an equivalence

$$\operatorname{Ind}\mathrm{Kos}^{-1} : \operatorname{Ind}\mathscr{D}(S)_{tors} \to \mathscr{D}(A)$$

which identifies the unbounded derived category of A-complexes with ind-torsion dg sheaves on shifted affine space.

sect:idem

Appendix A. Idempotents and idempotent completion

A.1. **Idempotents and retracts.** By a *linear equivalence relation* on an ordered set I we mean an equivalence relation $\sim$ which satisfies the following:

$$\text{if } a \sim c \text{ and } a \leq b \leq c, \text{ then } a \sim b \sim c.$$

Take Idem the simplicial set with n-simplices

$$\mathrm{Idem}[n] = \big\{\text{the set of linear equivalence relations} \sim \text{on } [n]\big\}$$

The structure maps are the apparent ones, i.e. for $\alpha : [m] \to [n]$ and relation $\sim$ on $[n]$, $\alpha^*(\sim)$ is the relation $\sim_\alpha$ on $[m]$ with $a \sim_\alpha b$ is and only if $\alpha(a) \sim \alpha(b)$. Note that Idem contains a single nondegenerate simplex in each dimension, which is given specifically by the equivalence relation with $a \sim b$ if and only if $a = b$. Furthermore, each face map $d_i^* : \mathrm{Idem}[n] \to \mathrm{Idem}[n-1]$ sends the unique nondegenerate simplex in dimension n to the nondegenerate simplex in dimension $n-1$.

A diagram in an ∞-category $F : \mathrm{Idem} \to \mathscr{C}$ can be seen as an "infinitely coherent idempotent" in $\mathscr{C}$, and consists of a choice of object x, endomorphism $e : x \to x$, 2-simplex

$$\begin{tikzcd} & x \arrow[dr, "e"] & \\ x \arrow[ur, "e"] \arrow[rr, "e"'] & & x \end{tikzcd}$$

and furthermore n-simplices $F_n : \Delta^n \to \mathscr{C}$ at all $n \geq 0$ with all faces equal in $\mathscr{C}$.

Definition A.1. An idempotent in and ∞-category $\mathscr{C}$ is a diagram $e : \mathrm{Idem} \to \mathscr{C}$.

We consider also the simplicial set $\mathrm{Ret} = \Delta^2/\Delta^{\{0,2\}}$ whose diagrams $\mathrm{Ret} \to \mathscr{C}$ classify retracts in $\mathscr{C}$. Explicitly, a diagram $\mathrm{Ret} \to \mathscr{C}$ is a choice of a 2-simplex of the form

$$\begin{tikzcd} & x \arrow[dr, "\pi"] & \\ y \arrow[ur, "l"] \arrow[rr, "id_y"'] & & y. \end{tikzcd}$$

We claim that, as in the discrete setting, any retract diagram $r : \mathrm{Ret} \to \mathscr{C}$ determines a uniquely associated idempotent $e : \mathrm{Idem} \to \mathscr{C}$. In establishing this relation we factor through an intermediate construction Idem^+.

We define the complex Idem^+ whose n-simplices are

$$\mathrm{Idem}^+[n] = \big\{\text{subsets } I \subseteq [n] \text{ equipped with a linear equivalence relation } \sim\big\}.$$

Given a map $\alpha : [m] \to [n]$ the function $\alpha^* : \mathrm{Idem}^+[n] \to \mathrm{Idem}^+[m]$ sends the pair $(I, \sim)$ to $\alpha^{-1}I \subseteq [m]$ paired with the relation $\sim_\alpha$ where $a \sim_\alpha b$ in $\alpha^{-1}I$ if and only if $\alpha(a) \sim \alpha(b)$.

We have the inclusion $\mathrm{Idem} \hookrightarrow \mathrm{Idem}^+$ and in low dimensions the non-degenerate vertices in Idem^+ appear as

$$\mathrm{Idem}^+[0] = \{[0], \emptyset\}, \quad \mathrm{Idem}^+[1]_{\mathrm{nd}} = \big\{[1], \{0\}, \{1\}\big\},$$

$$\mathrm{Idem}^+[2]_{\mathrm{nd}} = \big\{[2], \{0,1\}, \{0,2\}, \{1,2\}, \{1\}\big\},$$

where in each case above we give the set I the equivalence relation $a \sim b$ if and only if $a = b$. We have additionally the inclusion $j : \mathrm{Ret} = \Delta^2/\Delta^{\{0,2\}} \to \mathrm{Idem}^+$ which has

$$j(0) = j(2) = \emptyset,\ j(1) = [1],\ j(0<1) = \{1\},\ j(1<2) = \{0\},\ j(0<2) = \emptyset$$

and $j(0<1<2) = \{1\}$.

We see now that a diagram $e^+ : \mathrm{Idem}^+ \to \mathscr{C}$ specifies an idempotent $e : \mathrm{Idem} \to \mathscr{C}$ and a retract $r : \mathrm{Ret} \to \mathscr{C}$, which one might display as above, along with additional simplices which validate an equation "$e = \pi l : x \to y \to x$" up to all higher levels of compatibility. For example e^+ specifies diagrams of the form

$$e^+(\{0,1\} \subseteq [2]) = \begin{tikzcd} & x \arrow[dr, "\pi"] & \\ x \arrow[ur, "e"] \arrow[rr, "\pi"'] & & y \end{tikzcd} \qquad e^+(\{0,2\} \subseteq [2]) = \begin{tikzcd} & y \arrow[dr, "l"] & \\ x \arrow[ur, "\pi"] \arrow[rr, "e"'] & & x \end{tikzcd}$$

$$e^+(\{1,2\} \subseteq [2]) = \begin{tikzcd} & x \arrow[dr, "e"] & \\ y \arrow[ur, "l"] \arrow[rr, "l"'] & & x \end{tikzcd}$$

in $\mathscr{C}$, in dimension 2, and also diagrams of the form

$$e^+(\{0,1,2\} \subseteq [3]) = \begin{tikzcd} & x \arrow[d, "e"] \arrow[ddl, "e"'] \arrow[ddr, "\pi"] & \\ & x \arrow[dr, "\pi"'] & \\ x \arrow[ur, "e"] \arrow[rr, "\pi"'] & & y \end{tikzcd} \qquad e^+(\{1,2,3\} \subseteq [3]) = \begin{tikzcd} & y \arrow[d, "l"] \arrow[ddl, "l"'] \arrow[ddr, "l"] & \\ & x \arrow[dr, "e"'] & \\ x \arrow[ur, "e"] \arrow[rr, "e"'] & & x \end{tikzcd}$$

$$e^+(\{0,1,3\} \subseteq [3]) = \begin{tikzcd} & x \arrow[d, "\pi"] \arrow[ddl, "e"'] \arrow[ddr, "e"] & \\ & y \arrow[dr, "l"'] & \\ x \arrow[ur, "e"] \arrow[rr, "e"'] & & x \end{tikzcd} \qquad e^+(\{0,2,3\} \subseteq [3]) = \begin{tikzcd} & x \arrow[d, "e"] \arrow[ddl, "e"'] \arrow[ddr, "\pi"] & \\ & x \arrow[dr, "e"'] & \\ y \arrow[ur, "l"] \arrow[rr, "l"'] & & x \end{tikzcd}$$

in dimension 3.

Proposition A.2 ([14, Proposition 4.4.5.6])**.** *The inclusion* $\mathrm{Ret} \to \mathrm{Idem}^+$ *is inner anodyne.*

We can now reasonably speak of split idempotents.

cor:split

Corollary A.3. *At any* ∞*-category* $\mathscr{C}$ *there is a functor* $\mathrm{split} : \mathrm{Fun}(\mathrm{Ret}, \mathscr{C}) \to \mathrm{Fun}(\mathrm{Idem}, \mathscr{C})$ *which into a* 2*-simplex*

$$\begin{tikzcd} & \mathrm{Fun}(\mathrm{Idem}^+, \mathscr{C}) \arrow[dl, "\mathrm{restrict}"'] \arrow[dr, "\mathrm{restrict}"] & \\ \mathrm{Fun}(\mathrm{Ret}, \mathscr{C}) \arrow[rr, "\mathrm{split}"'] & & \mathrm{Fun}(\mathrm{Idem}, \mathscr{C}). \end{tikzcd}$$

in $\mathscr{C}at_\infty$*. Furthermore, the functor* split *is uniquely determined up to a contractible space of choices.*

Proof. Since $\mathrm{Ret} \to \mathrm{Idem}^+$ is inner anodyne, the map $\mathrm{Fun}(\mathrm{Idem}^+, \mathscr{C}) \to \mathrm{Fun}(\mathrm{Ret}, \mathscr{C})$ is a trivial Kan fibration, by Proposition I-5.7. Hence the functor

$$\mathrm{Fun}(\mathrm{Fun}(\mathrm{Ret}, \mathscr{C}), \mathrm{Fun}(\mathrm{Idem}, \mathscr{C})) \to \mathrm{Fun}(\mathrm{Fun}(\mathrm{Idem}^+, \mathscr{C}), \mathrm{Fun}(\mathrm{Idem}, \mathscr{C}))$$

is an equivalence, and the space

$$\{\mathrm{restrict}\} \times^{\mathrm{htop}}_{\mathrm{Fun}(\mathrm{Fun}(\mathrm{Idem}^+, \mathscr{C}), \mathrm{Fun}(\mathrm{Idem}, \mathscr{C}))} \mathrm{Fun}(\mathrm{Fun}(\mathrm{Ret}, \mathscr{C}), \mathrm{Fun}(\mathrm{Idem}, \mathscr{C}))$$

of functors completing the given diagram is contractible. □

Definition A.4. An idempotent $e : \mathrm{Idem} \to \mathscr{C}$ is called split if e is in the essential image of the functor $\mathrm{split} : \mathrm{Fun}(\mathrm{Ret}, \mathscr{C}) \to \mathrm{Fun}(\mathrm{Idem}, \mathscr{C})$.

prop:idem_split

Proposition A.5 ([14, Corollary 4.4.5.14])**.** *For a given* ∞*-category* $\mathscr{C}$*, the following are equivalent:*

(a) *Every idempotent in* $\mathscr{C}$ *is split.*

(b) *The restriction functor* $\mathrm{Fun}(\mathrm{Idem}^+, \mathscr{C}) \to \mathrm{Fun}(\mathrm{Idem}, \mathscr{C})$ *is a trivial Kan fibration.*

(c) *The functor* $\mathrm{split} : \mathrm{Fun}(\mathrm{Ret}, \mathscr{C}) \to \mathrm{Fun}(\mathrm{Idem}, \mathscr{C})$ *is an equivalence.*

Proof. The equivalence between (a) and (b) is covered in [14, Corollary 4.4.5.14]. For the equivalence between (b) and (c), the diagram from Corollary A.3 and the fact that the restriction $\mathrm{Fun}(\mathrm{Idem}^+, \mathscr{C}) \to \mathrm{Fun}(\mathrm{Ret}, \mathscr{C})$ is an equivalence tells us that the map split is an equivalence if and only if the restriction functor $\mathrm{Fun}(\mathrm{Idem}^+, \mathscr{C}) \to \mathrm{Fun}(\mathrm{Idem}, \mathscr{C})$ is an equivalence.

Since the map $\mathrm{Idem} \to \mathrm{Idem}^+$ is injective, the restriction functor

$$\mathrm{Fun}(\mathrm{Idem}^+, \mathscr{C}) \to \mathrm{Fun}(\mathrm{Idem}, \mathscr{C})$$

is an isofibration (Corollary I-6.14). By Proposition II-16.12, an isofibration between ∞-categories is a trivial Kan fibration if and only if it is an equivalence, giving (b) $\Leftrightarrow$ (c). □

Definition A.6. We call an ∞-category $\mathscr{C}$ idempotent complete if any of the equivalent conditions from Proposition A.5 hold.

The following tells us that any cocomplete ∞-category is idempotent complete.

prop:cocomp_split

Proposition A.7 ([14, Corollary 4.4.5.16])**.** *Let* κ *be any regular cardinal. If* $\mathscr{A}$ *is a* κ*-cocomplete* ∞*-category then* $\mathscr{A}$ *is idempotent complete.*

A.2. **Idempotent completions.**

def:idem_comp

Definition A.8. An idempotent completion of an ∞-category $\mathscr{C}$ is a functor $i : \mathscr{C} \to \mathscr{C}^\vee$ to an ∞-category $\mathscr{C}^\vee$ for which:

(a) The functor i is fully faithful.

(b) $\mathscr{C}^\vee$ is idempotent complete.

(c) Every object in $\mathscr{C}^\vee$ is a retract of an object in the image of $\mathscr{C}$.

By a retract of an object x in $\mathscr{C}^\vee$ we mean, of course, an object y which admits a retract diagram $r : \mathrm{Ret} \to \mathscr{C}^\vee$ with $r(0) = r(2) = y$ and $r(1) = x$. The following tells us that the collection of objects which appear as retracts of a subset $O \subset \mathscr{C}^\vee[0]$ is stable under all expected operations.

lem:3745

Lemma A.9. *Let $\mathscr{C}$ be any ∞-category, and $x, x', x'' : * \to \mathscr{C}$ be arbitrary objects. If x' is a retract of x, and x'' is a retract of x', then x'' is a retract of x.*

Proof. Follows by a consideration of the diagram

$$\begin{tikzcd}
 & & x \arrow[dr] & & \\
 & x' \arrow[ur] \arrow[rr, "id"] \arrow[dr] & & x' \arrow[dr] & \\
x'' \arrow[ur] \arrow[rr, "id"'] & & x'' \arrow[ur] \arrow[rr, "id"'] & & x''
\end{tikzcd}$$

in $\mathrm{h}\mathscr{C}$. □

We note that idempotent completions always exist.

lem:idem_exist

Lemma A.10. (1) *For any ∞-category $\mathscr{C}$, an idempotent completion $i : \mathscr{C} \to \mathscr{C}^\vee$ exists.*

(2) *If $\mathscr{C}$ is essentially small, then there is an idempotent completion $i : \mathscr{C} \to \mathscr{C}^\vee$ for which $\mathscr{C}^\vee$ is also essentially small.*

Proof. We will prove that any essentially small ∞-category $\mathscr{C}$ admits an essentially small idempotent completion. For (1) one just repeats the same arguments without keeping track of size constraints.

Suppose that $\mathscr{C}$ is essentially small. Since the category $\mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathscr{K}an^{\mathrm{sm}})$ is cocomplete it is idempotent split. Let $\mathscr{C}' \subseteq \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathscr{K}an^{\mathrm{sm}})$ denote the essential image of the Yoneda embedding $\mathscr{C} \to \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathscr{K}an^{\mathrm{sm}})$. Now, the collection of idempotents $\mathrm{Fun}(\mathrm{Idem}, \mathscr{C}')$ is also essentially small, as the simplicial set Idem is small (Example II-15.5).

Let us take

$$\mathscr{Z} = \mathrm{Fun}(\mathrm{Idem}^+ \times \mathscr{C}^{\mathrm{op}}, \mathscr{K}an^{\mathrm{sm}}) \times_{\mathrm{Fun}(\mathrm{Idem} \times \mathscr{C}^{\mathrm{op}}, \mathscr{K}an^{\mathrm{sm}})} \mathrm{Fun}(\mathrm{Idem}, \mathscr{C}').$$

As the restriction functor $\mathrm{Fun}(\mathrm{Idem}^+ \times \mathscr{C}^{\mathrm{op}}, \mathscr{K}an^{\mathrm{sm}}) \to \mathrm{Fun}(\mathrm{Idem} \times \mathscr{C}^{\mathrm{op}}, \mathscr{K}an^{\mathrm{sm}})$ is a trivial Kan fibration, the projection $p : \mathscr{Z} \to \mathrm{Fun}(\mathrm{Idem}, \mathscr{C}')$ is a trivial Kan fibration as well.

We note that $\mathscr{Z}$ is a fully subcategory in $\mathrm{Fun}(\mathrm{Idem}^+ \times \mathscr{C}^{\mathrm{op}}, \mathscr{K}an^{\mathrm{sm}})$. The restriction functor $\mathscr{Z} \to \mathrm{Fun}(\mathrm{Ret} \times \mathscr{C}^{\mathrm{op}}, \mathscr{K}an^{\mathrm{sm}})$ is a trivial Kan fibration onto its image as well [14, Proposition 4.4.5.6]. (Here we mean strict image, not essential image.) This image, call it $\mathscr{Y}$, is precisely the collection of diagrams $F : \mathrm{Ret} \to \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathscr{K}an^{\mathrm{sm}})$ with $F(1)$ in $\mathscr{C}'$, and we deduce an equivalence $\mathscr{Y} \overset{\sim}{\to} \mathrm{Fun}(\mathrm{Idem}, \mathscr{C}')$. In particular $\mathscr{Y}$ is essentially small.

Evaluating at $0 = 2$ we obtain a functor

$$ev_0 : \mathscr{Y} \to \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathscr{K}an^{\mathrm{sm}})$$

and let $\mathscr{C}^\vee \subseteq \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathscr{K}an^{\mathrm{sm}})$ denote the full ∞-subcategory spanned by the image of $\mathscr{Y}$. Since $\mathscr{Y}$ is essentially small and $\mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathscr{K}an^{\mathrm{sm}})$ is locally small (Lemma II-15.7), we conclude that $\mathscr{C}^\vee$ is essentially small. From a consideration of the degenerate diagram

$$\mathrm{Ret} \to * \overset{x}{\to} \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathscr{K}an^{\mathrm{sm}})$$

at any x in $\mathscr{C}'$ it is clear that $\mathscr{C}^\vee$ contains $\mathscr{C}'$, and we see directly that $\mathscr{C}^\vee$ is spanned by all objects which are retracts of objects in $\mathscr{C}'$. Hence the Yoneda embedding provides a fully faithful functor $i : \mathscr{C} \to \mathscr{C}^\vee$ under which all objects in $\mathscr{C}^\vee$ are obtained by taking retracts of objects in the image of i. Since the functor category $\mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathscr{K}an^{\mathrm{sm}})$ is idempotent split, $\mathscr{C}^\vee$ is seen to be idempotent complete by Lemma A.9. So the Yoneda embedding realizes $\mathscr{C}^\vee$ as an idempotent completion of $\mathscr{C}$. □

We have the following universal property for idempotent completion.

prop:idemcomp_univ

Proposition A.11 ([14, Proposition 5.1.4.9])**.** *If $i : \mathscr{C} \to \mathscr{C}^\vee$ is an idempotent completion of an ∞-category $\mathscr{C}$, then for any idempotent complete ∞-category $\mathscr{A}$ restricting along i provides an equivalence of ∞-categories*

$$i^* : \mathrm{Fun}(\mathscr{C}^\vee, \mathscr{A}) \overset{\sim}{\to} \mathrm{Fun}(\mathscr{C}, \mathscr{A}).$$

Via the above universal property, one observes uniqueness of idempotent completions. We leave the details of the following to the interested reader.

Corollary A.12. *For any ∞-category $\mathscr{C}$, the idempotent completion $i : \mathscr{C} \to \mathscr{C}^\vee$ is uniquely determined up to isomorphism in the undercategory $(\mathscr{C}at_\infty)_{\mathscr{C}/}$.*

We combine uniqueness with the smallness assertion form Lemma A.10 to observe that idempotent completion preserves essentially small ∞-categories.

cor:split_small

Corollary A.13. *If $\mathscr{C}$ is essentially small, then any idempotent completion $\mathscr{C}^\vee$ of $\mathscr{C}$ is also essentially small.*

Using Proposition A.11 one can also establish existence and uniqueness of an idempotent completion functor

$$(-)^\vee : \mathscr{C}at_\infty \to \mathscr{C}at_\infty.$$

This functor comes equipped with a transformation $i : id_{\mathscr{C}at_\infty} \to (-)^\vee$ which evaluates to an idempotent completion $i : \mathscr{C} \to \mathscr{C}^\vee$ at each ∞-category $\mathscr{C}$. One realizes the above functor, and transformation, by constructing a cocartesian fibration $q : \mathscr{M}_{\mathrm{Idem}} \to \Delta^1$ for idempotent completions and proceeds exactly as in Section 12.2.

Below we let $(\mathscr{C}at_\infty)_{\mathrm{split}}$ denote the full subcategory of idempotent split ∞-categories in $\mathscr{C}at_\infty$. We leave the proof of the following as an exercise for the interested reader.

prop:idem_functr

Proposition A.14. *There is a functor $(-)^\vee : \mathscr{C}at_\infty \to (\mathscr{C}at_\infty)_{\mathrm{split}}$ is right adjoint to the inclusion $(\mathscr{C}at_\infty)_{\mathrm{split}} \to \mathscr{C}at_\infty$.*

A.3. $\aleph_0$**-accessibility and idempotent completion.**

Definition A.15. An ∞-category $\mathscr{A}$ is called $\aleph_0$-accessible if $\mathscr{A}$ admits a functor $i : \mathscr{C} \to \mathscr{A}$ from an essentially small ∞-category $\mathscr{C}$ which exhibits $\mathscr{A}$ as an ind-completion of $\mathscr{C}$ (Definition 12.5).

The category $\mathrm{Acc}_{\aleph_0}$ of $\aleph_0$-accessible categories is the non-full subcategory in $\mathscr{C}at_\infty$ whose n-simplices $\sigma : \Delta^n \to \mathrm{Acc}_{\aleph_0}$ are those simplices $\sigma : \Delta^n \to \mathscr{C}at_\infty$ for which each vertex $\mathscr{A}_i = \sigma(\{i\})$ is $\aleph_0$-accessible, and for which each edge $F_{ij} = \sigma|_{\Delta^{\{i,j\}}} : \mathscr{A}_i \to \mathscr{A}_j$ is $\aleph_0$-cocontinuous and preserves ($\aleph_0-$)compact objects.

We can construct the category $\mathrm{Acc}_{\aleph_0}$ as the homotopy coherent nerve of the simplicial subcategory $\underline{\mathrm{Acc}}_{\aleph_0} \subseteq \underline{\mathrm{Cat}}^+_\infty$ whose objects are $\aleph_0$-accessible categories and whose mapping complexes are the full subcategories

$$\mathrm{Fun}_{\aleph_0}(\mathscr{A}, \mathscr{B}) \ \subseteq \ \mathrm{Fun}(\mathscr{A}, \mathscr{B})^{\mathrm{Kan}}$$

spanned by accessible functors.

Taking compacts provides a simplicial functor $\underline{\mathrm{Acc}}_{\aleph_0} \to \underline{\mathrm{Cat}}^+_\infty$ and we apply the homotopy coherent nerve to obtain a functor $(-)^c : \mathrm{Acc}_{\aleph_0} \to \mathscr{C}at_\infty$. The following gives essential information about the image of the compacts functor.

lem:indc_idem

Lemma A.16 ([14, Lemma 5.4.2.4])**.** *If $\mathscr{C}$ is essentially small, then the subcategory $\mathrm{Ind}(\mathscr{C})^c$ of compact objects in $\mathrm{Ind}(\mathscr{C})$ is an idempotent completion of $\mathscr{C}$.*

Applying Corollary A.13, we see that the full subcategory of compacts $\mathscr{A}^c$ in any accessible ∞-category forms an essentially small ∞-category.

lem:aleph_prequiv

Lemma A.17. (1) *If $\mathscr{A}$ is an $\aleph_0$-accessible ∞-category then the full subcategory of compacts $\mathscr{A}^c$ is essentially small, and the inclusion $\mathscr{A}^c \to \mathscr{A}$ induces an equivalence $\mathrm{Ind}(\mathscr{A}^c) \to \mathscr{A}$.*

(2) *If $\mathscr{C}$ is essentially small and idempotent complete, then the inclusion $i : \mathscr{C} \to \mathrm{Ind}(\mathscr{C})^c$ is an equivalence.*

Proof. Statement (2) is clear, by uniqueness of idempotent completion. For (1) we have an equivalence $\mathrm{Ind}(\mathscr{C}) \xrightarrow{\sim} \mathscr{A}$ for some essentially small $\mathscr{C}$, so that the inclusion $\mathscr{C} \to \mathrm{Ind}(\mathscr{C}) \to \mathscr{A}^c$ exhibits $\mathscr{A}^c$ as an idempotent completion of $\mathscr{C}$, by Lemma A.16. By Corollary A.13 it follows that $\mathscr{A}^c$ is essentially small. Furthermore, since $\mathscr{A}$ is generated by the image of $\mathscr{C}$ under filtered colimits, we see that $\mathscr{A}$ is generated by $\mathscr{A}^c$ under filtered colimits. Hence the induced map $\mathrm{Ind}(\mathscr{A}^c) \to \mathscr{A}$ is an equivalence by Theorem 12.7. $\square$

We note that, via the structural transformation $i : \mathrm{incl} \to \mathrm{Ind}$ we have a diagram

$$\begin{tikzcd}
\mathrm{Ind}(\mathscr{C}) \arrow[r, "\mathrm{Ind}(f)"] & \mathrm{Ind}(\mathscr{C}') \\
\mathscr{C} \arrow[u, "i"] \arrow[ur] \arrow[r, "f"'] & \mathscr{C}' \arrow[u, "i"']
\end{tikzcd}$$

at any map $f : \mathscr{C} \to \mathscr{C}'$ between essentially small ∞-categories. It follows that the functor $\mathrm{Ind}(f)$ preserves compact objects, and is therefore $\aleph_0$-accesible. We can therefore restrict the codomain of the ind-completion functor to obtain a functor $\mathrm{Ind} : \mathscr{C}at_\infty^{\mathrm{sm}} \to \mathrm{Acc}_{\aleph_0}$.

Let us take

$$(\mathscr{C}at_\infty^{\mathrm{sm}})_{\mathrm{split}} := \text{the full subcategory of idempotent complete } \infty\text{-categories in } \mathscr{C}at_\infty^{\mathrm{sm}}.$$

We have functors

$$(-)^c : \mathrm{Acc}_{\aleph_0} \to (\mathscr{C}at_\infty^{\mathrm{sm}})_{\mathrm{split}} \quad \text{and} \quad \mathrm{Ind} : (\mathscr{C}at_\infty^{\mathrm{sm}})_{\mathrm{split}} \to \mathrm{Acc}_{\aleph_0},$$

and by Lemma A.17 the functor $(-)^c$, taken now to have image in idempotent complete categories, is essentially surjective. Furthermore, for the simplicial category $\underline{\mathrm{Acc}}_{\aleph_0}$, Lemma A.17 and Theorem 12.6 tell us that restricting to the compacts provides a fully faithful functor

$$\mathrm{Fun}_{\aleph_0}(\mathscr{A}, \mathscr{B}) \to \mathrm{Fun}(\mathscr{A}^c, \mathscr{B})$$

whose essential image is precisely the subcategory $\mathrm{Fun}(\mathscr{A}^c, \mathscr{B}^c)$. So, at the level of simplicial categories, the functor given by restricting to the compacts

$$\underline{\mathrm{Acc}}_{\aleph_0} \to (\underline{\mathrm{Cat}}_\infty^{\mathrm{sm}})_{\mathrm{split}}$$

is fully faithful. By Proposition II-7.6 it follows that the functor $(-)^c : \mathrm{Acc}_{\aleph_0} \to (\mathscr{C}at_\infty^{\mathrm{sm}})_{\mathrm{split}}$ is fully faithful as well.

thm:compacts_equiv

Theorem A.18. *The functor* $(-)^c : \mathrm{Acc}_{\aleph_0} \to (\mathscr{C}at_\infty^{\mathrm{sm}})_{\mathrm{split}}$ *is an equivalence.*

Proof. By the information above, this functor is fully faithful and essentially surjective, and hence an equivalence by Theorem I-8.2. □

We claim finally that the inverse to $(-)^c$ is provided by the indization functor. To prove this it suffices to prove that indization is left adjoint to the compact objects functor.

prop:ind_compacts_adj

Proposition A.19. *The functor* $(-)^c : \mathrm{Acc}_{\aleph_0} \to \mathscr{C}at_\infty^{\mathrm{sm}}$ *is right adjoint to the indization functor* $\mathrm{Ind} : \mathscr{C}at_\infty^{\mathrm{sm}} \to \mathrm{Acc}_{\aleph_0}$.

Proof. Let $\mathscr{M}_{\aleph_0} \subseteq \Delta^1 \times \mathscr{C}at_\infty$ denote the non-full subcategory whose objects are

$$\mathscr{M}_{\aleph_0}[0] = (\{0\} \times \mathscr{C}at_\infty^{\mathrm{sm}}[0]) \amalg (\{1\} \times \mathrm{Acc}_{\aleph_0}[0])$$

and whose edges $(i, \mathscr{C}) \to (j, \mathscr{A})$ are arbitrary maps of essentially small ∞-categories when $i = j = 0$, maps in $\mathrm{Acc}_{\aleph_0}$ when $i = j = 1$, and maps $\mathscr{C} \to \mathscr{A}$ which have compact image when $i < j$. We place no additional restrictions on n-simplices in $\mathscr{M}_{\aleph_0}$. One checks directly that the inclusion $\mathscr{M}_{\aleph_0} \to \Delta^1 \times \mathscr{C}at_\infty$ is an inner fibration, so that the composte $q : \mathscr{M}_{\aleph_0} \to \Delta^1$ of the inclusion with the projection $\Delta^1 \times \mathscr{C}at_\infty \to \Delta^1$ is also an inner fibration. One argues exactly as in the proof of Lemma 12.12 to see that the projection $q : \mathscr{M}_{\aleph_0} \to \Delta^1$ is a cocartesian fibration with q-cocartesian edges over $0 < 1$ provided by those functors $\mathscr{C} \to \mathscr{A}$ which realize $\mathscr{A}$ as an ind-completion of $\mathscr{C}$.

(To recall, we can identify $\mathscr{M}_{\aleph_0}$ with the homotopy coherent nerve of the apparent simplicial subcategory $\underline{\mathrm{M}}_{\aleph_0} \subseteq \{0 < 1\} \times \underline{\mathrm{Cat}}_\infty^{\mathrm{sm}}$, and use this simplicial construction along with Proposition II-7.6 to check reflexivity of $\mathrm{Acc}_{\aleph_0}$ in $\mathscr{M}_{\aleph_0}$.)

From this characterization of q-cocartesian edges we see that the indization functor, as defined in Section 12.2, provides the unique cocartesian solution to the lifting problem

$$\begin{tikzcd} \{0\}\times \mathscr{C}at^{\mathrm{sm}}_\infty \arrow[r] \arrow[d] & \mathscr{M}_{\aleph_0} \arrow[d, "q"] \\ \Delta^1 \times \mathscr{C}at^{\mathrm{sm}}_\infty \arrow[r] \arrow[ur, dotted, "I"] & \Delta^1, \end{tikzcd}$$

and hence that the functor $\mathrm{Ind} : \mathscr{C}at^{\mathrm{sm}}_\infty \to \mathrm{Acc}_{\aleph_0}$ is realized as homotopy transport functor along q.

We claim now that $q : \mathscr{M}_{\aleph_0} \to \Delta^1$ is also cartesian. For this it suffices to show that $\mathscr{C}at^{\mathrm{sm}}_\infty \to \mathscr{M}_{\aleph_0}$ is coreflexive. However, this is clear since, for any $\aleph_0$-accessible ∞-category $\mathscr{A}$, the inclusion $f : \mathscr{A}^c \to \mathscr{A}$ provides an equivalence $f_* : \mathrm{Hom}_{\mathscr{M}}(\mathscr{C}, \mathscr{A}^c) \to \mathrm{Hom}_{\mathscr{M}}(\mathscr{C}, \mathscr{A})$. One observes this fact precisely by considering the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathscr{M}}(\mathscr{C}, \mathscr{A}^c) \arrow[r, "f_*"] & \mathrm{Hom}_{\mathscr{M}}(\mathscr{C}, \mathscr{A}) \\ \mathrm{Fun}(\mathscr{C}, \mathscr{A}^c) \arrow[r, "\sim", "f_*"'] \arrow[u, "\sim"] & \mathrm{Fun}_c(\mathscr{C}, \mathscr{A}) \arrow[u, "\sim"'] \end{tikzcd}$$

in $\mathrm{h}\,\mathscr{K}an$, which exists by Proposition II-7.6.

Using Lemma 8.13 we furthermore characterize q-cartesian edges over $0 < 1$ as those map $k : \mathscr{A}' \to \mathscr{A}$ which are an equivalence onto the compacts in $\mathscr{A}$. At the level of simplicial categories, the inclusions $\mathscr{A}^c \to \mathscr{A}$ provide a transformation $(-)^c \to \mathrm{incl}$ between the simplicial functor $(-)^c : \underline{\mathrm{Acc}_{\aleph_0}} \to \underline{\mathrm{Cat}_\infty}$ and the inclusion $\mathrm{incl} : \underline{\mathrm{Acc}_{\aleph_0}} \to \underline{\mathrm{Cat}^+_\infty}$, and we obtain an induced transformation $k : (-)^c \to \mathrm{incl}$ between objects in $\mathrm{Fun}(\mathrm{Acc}_{\aleph_0}, \mathscr{C}at_\infty)$. This transformation provides the unique cartesian solution to the lifting problem

$$\begin{tikzcd} \{1\}\times \mathrm{Acc}_{\aleph_0} \arrow[r] \arrow[d] & \mathscr{M}_{\aleph_0} \arrow[d, "q"] \\ \Delta^1 \times \mathrm{Acc}_{\aleph_0} \arrow[r] \arrow[ur, dotted, "\mathrm{K}"] & \Delta^1, \end{tikzcd}$$

which is given explicitly by the sequence

$$\Delta^1 \times \mathrm{Acc}_{\aleph_0} \xrightarrow{\delta\times 1} \Delta^1 \times \Delta^1 \times \mathrm{Acc}_{\aleph_0} \xrightarrow{1\times k} \mathscr{M}_{\aleph_0} \subseteq \Delta^1 \times \mathscr{C}at_\infty.$$

Evaluating at 0 recovers the functor which takes compacts

$$\mathrm{K}\,|_{\{0\}\times \mathrm{Acc}_{\aleph_0}} = (-)^c : \mathrm{Acc}_{\aleph_0} \to \mathscr{C}at^{\mathrm{sm}}_\infty$$

so that this functor is realized as homotopy contraviant transport along q. It follows by Theorem 8.15 that $(-)^c$ is right adjoint to the indization functor. $\square$

Since taking the compacts in $\mathrm{Acc}_{\aleph_0}$ has image in $(\mathscr{C}at^{\mathrm{sm}}_\infty)_{\mathrm{split}}$, we see that the restricted functor

$$\mathrm{Ind} : (\mathscr{C}at^{\mathrm{sm}}_\infty)_{\mathrm{split}} \to \mathrm{Acc}_{\aleph_0}$$

remains left adjoint to the functor $(-)^c$. By uniqueness of adjoints, we conclude that Ind is in fact inverse to the equivalence $(-)^c$.

cor:ind_inverse **Corollary A.20.** *The functors*

$$\mathrm{Ind} : (\mathscr{C}at^{\mathrm{sm}}_\infty)_{\mathrm{split}} \to \mathrm{Acc}_{\aleph_0} \quad \textit{and} \quad (-)^c : \mathrm{Acc}_{\aleph_0} \to (\mathscr{C}at^{\mathrm{sm}}_\infty)_{\mathrm{split}}$$

are mutually inverse equivalences.

The following identifies our indization functor with the corresponding functor appearing in [14, Proposition 5.4.2.19].

prop:ind_composite

Proposition A.21. *In* $\mathrm{Fun}(\mathscr{C}at_\infty^{\mathrm{sm}}, \mathscr{C}at_\infty)$*, there is a natural isomorphism of functors*

$$\mathrm{Ind} \cong (\mathrm{Ind}|_{(\mathscr{C}at_\infty^{\mathrm{sm}})_{\mathrm{split}}}) \circ (-)^\vee.$$

Proof. If functors F_i are left adjoint to some G_i then the composite F_1F_0 is left adjoint to G_0G_1 [16, 02ES]. Hence, by Proposition A.14 and Corollary A.20, the composite $(\mathrm{Ind}|_{(\mathscr{C}at_\infty^{\mathrm{sm}})_{\mathrm{split}}}) \circ (-)^\vee$ is left adjoint to the composite

$$(-)^c = \mathrm{incl} \circ (-)^c : \mathrm{Acc}_{\aleph_0} \to (\mathscr{C}at_\infty^{\mathrm{sm}})_{\mathrm{split}} \to \mathscr{C}at_\infty^{\mathrm{sm}}.$$

By uniqueness of left adjoints, and Proposition A.19, we obtain the claimed natural isomorphism. □

A.4. **Idempotent splitting in the stable setting.**

Proposition A.22. *A stable* ∞*-category* $\mathscr{C}$ *is idempotent complete if and only if its homotopy category* $\mathrm{h}\,\mathscr{C}$ *is idempotent complete.*

Proof. By enlarging our universe if necessary, we may assume $\mathscr{C}$ is essentially small. Taking $\mathscr{A} = \mathrm{Ind}(\mathscr{C})$, we have the exact embedding $\mathscr{C} \to \mathscr{A}$ into a presentable stable idempotent complete ∞-category, by Proposition 12.22. As $\mathscr{A}^c$ is an idempotent completion of $\mathscr{C}$, we understand that $\mathscr{C}$ is idempotent complete if and only if the map $\mathscr{C} \to \mathscr{A}^c$ is an equivalence.

Note that $\mathrm{h}\,\mathscr{A}$ is idempotent complete, as it is triangulated with all small coproducts [19, Proposition 1.6.8]. So we have a bijection between isoclasses of retract diagrams in $\mathrm{h}\,\mathscr{A}$ and idempotents in $\mathrm{h}\,\mathscr{A}$. Since every retract diagram in $\mathrm{h}\,\mathscr{A}$ lifts to a retract diagram in $\mathscr{A}$, we obtain a bijection between isoclasses of retract diagrams in $\mathscr{A}$ and idempotents in $\mathrm{h}\,\mathscr{A}$. This bijection simply sends a retract $r = (y \to x \to y)$ to the idempotent $e_r : x \to y \to x$ in $\mathrm{h}\,\mathscr{A}$.

Since $\mathrm{h}\,\mathscr{A}$ is idempotent complete, we have that the subcategory of compacts $(\mathrm{h}\,\mathscr{A})^c$ is idempotent complete. Now, at the level of the homotopy category we have $\mathrm{h}(\mathscr{A}^c) = (\mathrm{h}\,\mathscr{A})^c$, by Corollary 7.14. So if $\mathscr{C}$ is idempotent complete we have that the homotopy category $\mathrm{h}\,\mathscr{C} \cong (\mathrm{h}\,\mathscr{A})^c$ is idempotent complete as well.

Suppose conversely that $\mathrm{h}\,\mathscr{C}$ is idempotent complete. By the definition of idempotent completion we have that every object in $\mathrm{h}(\mathscr{A}^c) = (\mathrm{h}\,\mathscr{A})^c$ is a retract of an object in the image of $\mathrm{h}\,\mathscr{C}$. By idempotent completeness of $\mathrm{h}(\mathscr{A}^c)$ it follows that every object is obtained by splitting an idempotent in the image of $\mathrm{h}\,\mathscr{C}$. Since $\mathrm{h}\,\mathscr{C}$ is idempotent complete, it follows that every object in $\mathrm{h}(\mathscr{A}^c)$ is in the essential image of $\mathrm{h}\,\mathscr{C}$, and hence that the map $\mathrm{h}\,\mathscr{C} \to \mathrm{h}(\mathscr{A}^c)$ is an equivalence. So we see that the original map $\mathscr{C} \to \mathscr{A}^c$ is an equivalence, by Proposition 7.15, and hence that $\mathscr{C}$ is idempotent complete. □

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