Quiz 1 (Math 410, Fall 2023)

- There are three questions. Use *back* of sheets, and/or page 4, for scratch work. Write your final answer directly below the statement of the question.

- Justify all steps in your proofs. If you use a result from class, or from the text, provide a generic reference. E.g. "By [Artin, Ch 2], it follows that" or "The above equation follows by [Lectures on orders of elements]".

1. Let G be a group, and suppose an element $g \in G$ is of finite order. Take $k = \mathrm{ord}(g)$. Prove that

$$g^n = g^m \quad \text{if and only if} \quad k \mid (n-m).$$

Suppose $g^n = g^m$. Then

$$g^{n-m} = g^n g^{-m} = e.$$

Hence, by a result from the lectures, we have

$$n-m = r \cdot k \quad \text{for some integer } r,$$

giving $\quad k \mid (n-m)$.

Conversely, if $k \mid (n-m)$, then

$$(n-m) = r \cdot k \quad \text{for some } r,$$

and hence

$$e = g^{k \cdot r} = g^{n-m} = g^n g^{-m}.$$

Thus $\quad g^m = (g^n g^{-m}) \cdot g^m = g^n.$

2. Suppose that $f : G \to G'$ is a *surjective* group homomorphism, and let H be a subgroup in G. Let $f(H)$ denote the image of H in G',

$$f(H) = \{\ x \in G' : \text{there exists } g \in G \text{ with } x = f(g)\ \}.$$

Prove the following:

(a) $f(H)$ is a subgroup in G'.

(b) If H is normal in G,[1] then $f(H)$ is normal in G'.

[Hint: Did you actually use surjectivity of f?]

First note that the set f(H) is nonempty since H is nonempty.

(a) We need to check $x^{-1} \in f(H)$ whenever $x \in f(H)$, and $x \cdot y \in f(H)$ whenever $x, y \in f(H)$. In the first ~~case~~ property we have $x = f(g)$ for some $g \in H$ and thus

$$x^{-1} = f(g)^{-1} = f(g^{-1}) \in f(H),$$

since $g^{-1} \in H$ whenever $g \in H$.

For the second property, we have $x = f(g)$ and $y = f(g')$ for $g, g' \in H$, which gives

$$x \cdot y = f(g) \cdot f(g') = f(g \cdot g') \in H,$$

since $g \cdot g' \in H$. So, $f(H)$ is a subgroup.

[1] Recall that a subgroup N in a group M is normal if, for any elements $n \in N$ and $m \in M$, the element $m \cdot n \cdot m^{-1}$ is in N.

(b) Take $z \in G'$ and $x \in f(H)$. Then we can find $g \in G$ and $g' \in H$ with

$$z = f(g) \quad \text{and} \quad x = f(g').$$

Then

$$z \cdot x \cdot z^{-1} = f(g) \cdot f(g') \cdot f(g)^{-1}$$
$$= f(g \cdot g' \cdot g^{-1}).$$

Since $g \cdot g' \cdot g^{-1} \in H$ whenever $g' \in H$, via normality, we conclude $z \cdot x \cdot z^{-1} \in f(H)$ whenever $x \in f(H)$. So $f(H)$ is normal in G', by definition.

3. Consider 6 integers $a_1, \ldots, a_6 \in \mathbb{Z}$, and suppose that the subgroup generated by these 6 integers is all of $\mathbb{Z}$. Prove that there *does not exist* a prime number p which divides all of the a_i.

We have $\langle a_1, \ldots, a_6 \rangle = \mathbb{Z}$. Suppose, by way of contradiction, that some prime p divides all the a_i. Then

$$a_1, \ldots, a_6 \in p\mathbb{Z},$$

and hence $\langle a_1, \ldots, a_6 \rangle \subseteq p\mathbb{Z}$. This follows by the definition of $\langle a_1, \ldots, a_6 \rangle$ as the ~~[illegible]~~ minimal subgroup in $\mathbb{Z}$ which contains all the a_i. But now

$$1 \notin p\mathbb{Z}, \text{ which implies } 1 \notin \langle a_1, \ldots, a_6 \rangle.$$

So $\langle a_1, \ldots, a_6 \rangle \neq \mathbb{Z}$, contradicting our hypothesis. So it must be the case that ~~[illegible]~~ no prime divides all the a_i, as claimed.