Midterm 2 (Math 410, Fall 2023) Name:____________________

- There are four questions + extra credit. Use *back* of sheets, and/or page 6, for scratch work. Write your final answer directly below the statement of the question.

- Justify all steps in your proofs. If you use a result from class, or from the text, provide a generic reference. E.g. "By [Artin, Ch 2], it follows that" or "The above equation follows by [Lectures on orders of elements]".

1. [20 pt] Consider the orthogonal group $\mathrm{O}_n(\mathbb{R}) = \{A \in \mathrm{GL}_n(\mathbb{R}) : A^t = A^{-1}\}$.

(a) For $A \in \mathrm{O}_n(\mathbb{R})$, prove that $\det(A) \in \{\pm 1\}$.

(b) Prove that the subgroup $\mathrm{SO}_n(\mathbb{R}) = \{A \in \mathrm{O}_n(\mathbb{R}) : \det(A) = 1\}$ is normal in $\mathrm{O}_n(\mathbb{R})$.[1]

(c) Establish an isomorphism of groups $\alpha : \mathrm{O}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R}) \xrightarrow{\cong} \mathbb{Z}/2\mathbb{Z}$.

a) We have $\det(A^t) = \det(A)$ and $\det(A^{-1}) = \det(A)^{-1}$

so that the equation

$$1 = \det(I_n) = \det(A^t A) = \det(A^t)\det(A) = \det(A)^2$$

forces $\det(A) \in \{\pm 1\}$. // By the first isom. theorem.

b) $SO_n(\mathbb{R}) = \ker(\det : O_n(\mathbb{R}) \to \mathbb{R}^\times)$, and det is a group homomorphism. So, $SO_n(\mathbb{R})$ is normal.

c) The determinant homomorphism $\det : O_n(\mathbb{R}) \to \mathbb{R}^\times$ takes values in the subgroup $\{\pm 1\} \subseteq \mathbb{R}^\times$, and hence we can restrict the domain to get a group homomorphism $\det : O_n(\mathbb{R}) \to \{\pm 1\}$. Or, more directly, we simply have the set map $\det : O_n(\mathbb{R}) \to \{\pm 1\}$ by (a) and note the equality $\det(AB) = \det(A)\cdot\det(B)$ to see that det is a group homomorphism. We have the group isomorphism $\beta : \{\pm 1\} \to \mathbb{Z}/2\mathbb{Z}$, $\beta(1) = \bar{0}$, $\beta(-1) = \bar{1}$,

[1]You do not need to prove that it's a subgroup.

surjective group homomorphism

and get $\tilde{\alpha} := O_n(\mathbb{R}) \to \mathbb{Z}/2\mathbb{Z}$, $\tilde{\alpha} := \beta \circ \det$.

Since $\ker(\tilde{\alpha}) = SO_n(\mathbb{R})$ we then obtain an isom. $\alpha : O_n(\mathbb{R})/SO_n(\mathbb{R}) \to \mathbb{Z}/2\mathbb{Z}$

2. [25 pt] Consider the element $\tau = (42)$ in S_4.

(a) Explicitly calculate the orbit of τ under the conjugation action of S_4 on itself.

(b) How big is the stabilizer of τ? Prove that your anser is correct.

(c) Find two distinct, commuting, non-identity elements x and y in the stabilizer of τ.

(d) Construct an isomorphism of groups

$$\phi : \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \xrightarrow{\cong} \mathrm{Stab}_{S_4}(\tau).$$

a) $\text{Orbit}(\tau) = \{$ all cycles of the form $(i\,j)\}$

$= \{(12), (13), (14), (23), (24), (34)\}$.

b) Have $|\text{Orbit}(\tau)| = 6$ giving, by orbit stabilizer

$|\text{Stab}(\tau)| = |S_4| / |\text{Orbit}| = 4$.

c) $x = (42)$, $y = (13)$.

d) For $H = \langle x \rangle$ and $K = \langle y \rangle$ have $H, K \leq \text{Stab}$ and $H \cap K = \{e\}$, and these elements commute. We then get an injective group homomorphism from the product

$\phi : H \times K \longrightarrow \text{Stab}$, $\phi(x^n, y^m) = x^n y^m$.

Since, $H \cong \mathbb{Z}/2\mathbb{Z}$ and $K \cong \mathbb{Z}/2\mathbb{Z}$ then have

$\psi : \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \cong H \times K \longrightarrow \text{Stab}$.

injective group homomorphism. Since

$|\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}| = 4 = |\text{Stab}|$, ψ is an isomorphism

3. [20 pt] (a) Prove the following statement from Artin: Consider an element x in a finite group G, with conjugacy class $C(x)$. The order $|C(x)|$ divides $|G|$, and we have $|C(x)| = 1$ if and only if x is in the center of G.

(b) Suppose that a finite group G has prime power order, $|G| = p^n$. Use the class equation to show that the center of G is non-trivial.

(a) $C(x)$ is the orbit of x under the conj. action of G on itself. Hence, by orbit stabilizer

$$|G| = |\mathrm{Stab}|\,|C(x)|,$$

so that $|C(x)|$ divides $|G|$. The stabilizer of x is the centralizer $Z(x) = \{g \in G : gx = xg\}$. Hence, via orbit stabilizer $|C(x)| = 1 \Leftrightarrow |Z(x)| = |G|$

$\Leftrightarrow Z(x) = G \Leftrightarrow x$ is central in G.

(b) Class equation says $|G| = \sum_i |C_i|$
conj classes

$$= \sum_{x \in Z(G)} |C(x)| + \underbrace{\sum_j |C_j|}_{\text{divisible by } p}$$

classes for non-central elements

$$= |Z(G)| + p \cdot r.$$

So $|Z(G)| = p \cdot r - |G|$
$= p \cdot r - p^n$

implying $|Z(G)|$ is divisible by p, and in particular $Z(G)$ is not just $\{e\}$.

(d) ~~We have a map from~~ By a Theorem from lecture there is a unique group hom. $f: H \to S_n$ with $f(x_i) = \tau_i$ for all i, since the τ_i satisfy the relations for H (via (a)–(b) and the fact $\tau_i^2 = e$). Surjectivity follows by (c), since $\mathrm{im}(f) = \langle \tau_i \mid 1 \le i < n \rangle$.

4. [25 pt] Consider the elements $\tau_i = (i\; i+1)$ in S_n, and $\tau_{kl} = (k\; l)$ for $l > k$.

(a) Prove that $\tau_i\tau_j = \tau_j\tau_i$ whenever $|j-i| > 1$.

(b) Prove that $\tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1}$ for each $i < n$.

(c) Prove, by induction on the difference $|l-k|$, that each τ_{kl} is in the subgroup generated by the τ_i in S_n.

(d) Take

$$H = \langle x_1, \ldots, x_{n-1} \mid x_i^2 = 1,\ x_ix_j = x_jx_i \text{ when } |j-i| > 1,\ x_ix_{i+1}x_i = x_{i+1}x_ix_{i+1} \text{ when } i < n\rangle.$$

Prove that there is a *surjective* group homomorphism $f : H \to S_4$.

(a) $\tau_i\, (j\;\; j+1)\, \tau_i = (\tau_i(j)\;\; \tau_i(j+1))$

$\tau_i\, \tau_j\, \tau_i = (j\;\; j+1)$

since $\{j, j+1\} \cap \{i, i+1\} = \emptyset$ in this case.

So $\tau_i \tau_j \tau_i = \tau_j \Rightarrow \tau_i \tau_j = \tau_j \tau_i^{-1} = \tau_j \tau_i$.

(b) $\tau_i \tau_{i+1} \tau_i = (i\;\; i+2) = \tau_{i+1} \tau_i \tau_{i+1}$.

(c) ~~We have~~ $\cancel{\tau_l \tau_{kl} \tau_l = (\tau_l(k)\;\; \tau_l(l)) = (k\;\; l+1) = \tau_{k\, l+1}}$

~~Have~~ $\cancel{\tau_{kl} \in \langle \tau_i \mid 1 \le i < n \rangle}$ ~~when~~ $\cancel{|k-l| = 1}$, ~~so that the above equation implies~~ $\cancel{\tau_{kl} \in \langle \tau_i \mid 1 \le i < n \rangle}$ ~~by induction on the difference~~

We have $\tau_{kl} = \tau_k$ when $l-k = 1$.

Now, supposing $\tau_{kl} \in \langle \tau_i \mid 1 \le i < n \rangle$, we get

$$\tau_{k\, l+1} = (\tau_l(k)\;\; \tau_l(l)) = \tau_l\, \tau_{kl}\, \tau_l \in \langle \tau_i \mid 1 \le i < n \rangle.$$

Hence by induction on $l-k$ we see that all $\tau_{kl} \in \langle \tau_i \mid 1 \le i < n \rangle$.

EC [4 pt] Consider the set $\mathbb{P}^1$ of lines in the plane which pass through 0,

$$\mathbb{P}^1 := \{V \subseteq \mathbb{R}^2 : V \text{ is an } \mathbb{R}\text{-subspace of dimension } 1\},$$

and note that $\mathrm{GL}_2(\mathbb{R})$ acts on $\mathbb{P}^1$ by translating lines. Explicitly, $A * V = \{A \cdot v : v \in V\}$. Let $B \subseteq \mathrm{GL}_2(\mathbb{R})$ denote the subgroup of (non-strictly) upper triangular matrices.

(a) Prove that $\mathrm{GL}_2(\mathbb{R})$ acts transitively on $\mathbb{P}^1$, i.e. that $\mathbb{P}^1$ has a single orbit.

(b) Prove that there is a natural bijection of sets $\mathrm{GL}_2(\mathbb{R})/B \xrightarrow{\cong} \mathbb{P}^1$.

a) For any nonzero vector $v \in \mathbb{R}^2$ ~~we have~~, and $A = [v \; v']$ for v' not in $\mathrm{Span}(v)$, we have $A \in \mathrm{GL}_2(\mathbb{R})$ and $A \cdot \mathrm{Span}(e_1) = \mathrm{Span}(v)$.

Since all 1-dim subspaces are spanned by a single nonzero vector, this gives

$$\mathbb{P}^1 = \mathrm{Orbit}_{GL_2}(e_1).$$

b) For any G-action on a set X ~~we have~~, and $x \in X$, we have a set bijection

$$\alpha : G/\mathrm{Stab}_G(x) \xrightarrow{\cong} \mathrm{Orbit}(x), \quad \alpha(\bar{g}) := g \cdot x.$$

(Theorem from Class.) In this case

$$\mathrm{Stab}_{GL_2}(e_1) = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, c \in \mathbb{R}^\times, b \in \mathbb{R} \right\}$$
$$= B,$$

giving

$$GL_2(\mathbb{R})/B \xrightarrow{\cong} \mathbb{P}^1.$$