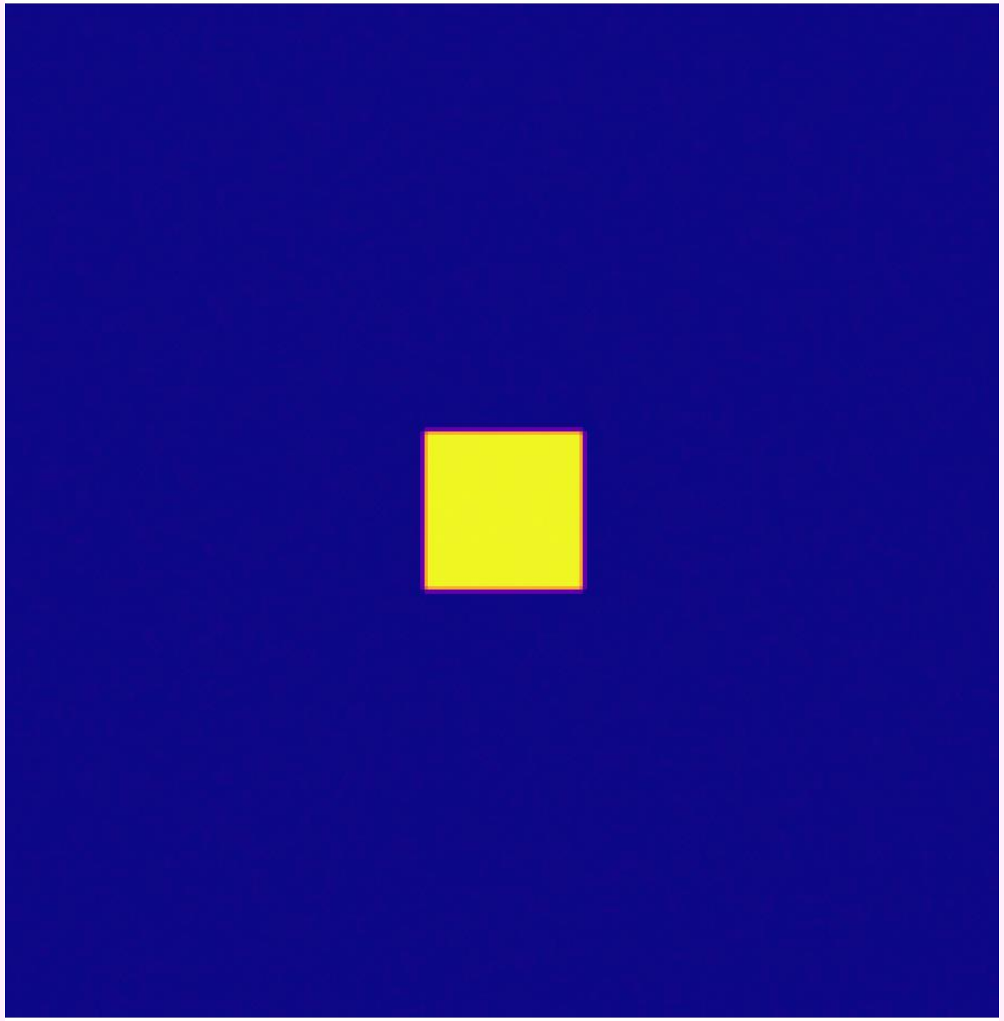
Turing patterns in reaction-diffusion systems



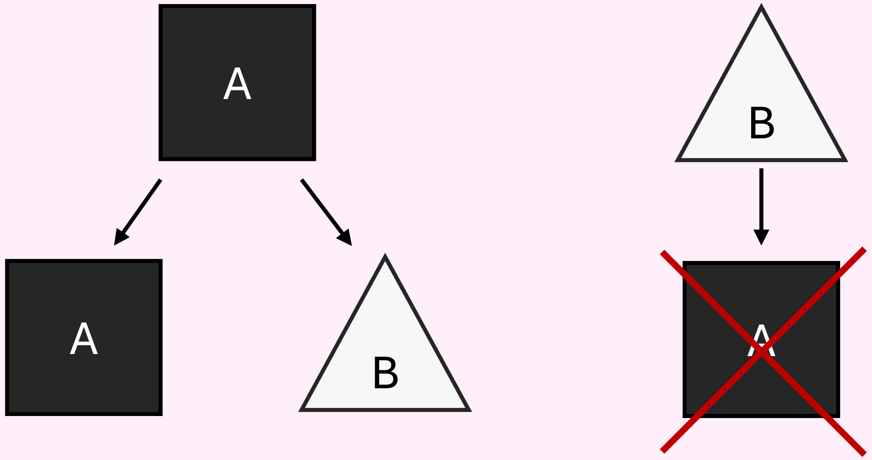
By Carolina and Sophia

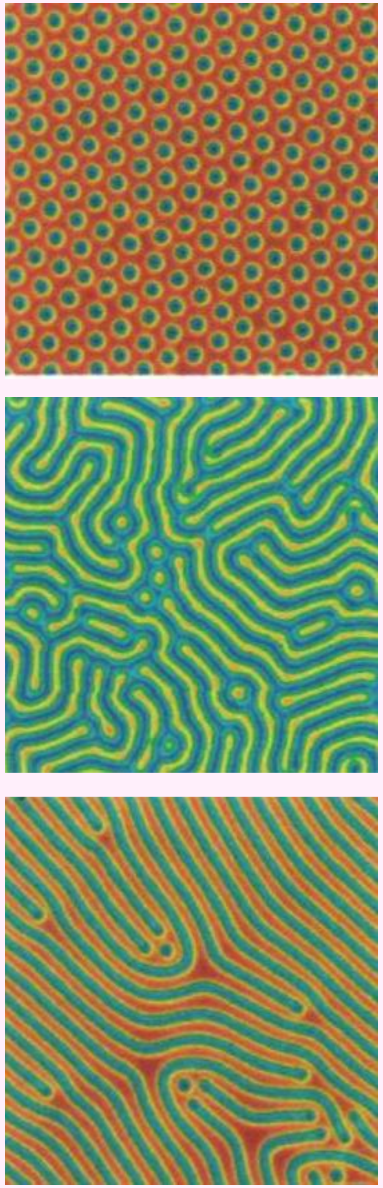
Overview

- Theory
- Research question & motivation
- Results: Pattern production
- Results: Minkowski measures and functionals
- Validity of results: Pearson comparison
- Performance evaluation

Theory: Turing Patterns

→ all due to reaction and diffusion interplay!

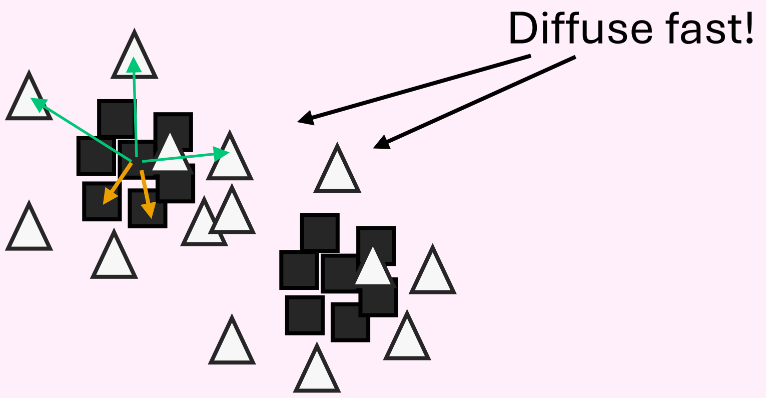


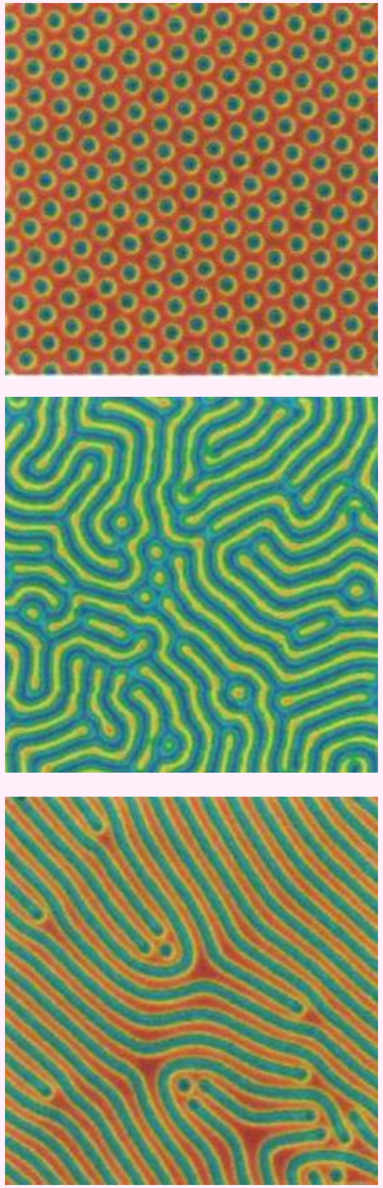


Images from Pearson's *Complex patterns in a Simple System* (1993)

Theory: Turing Patterns

→ all due to reaction and diffusion interplay!

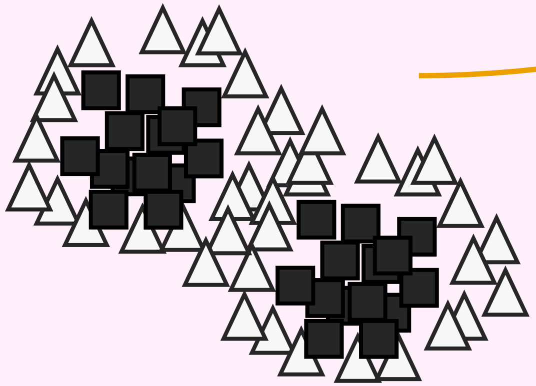


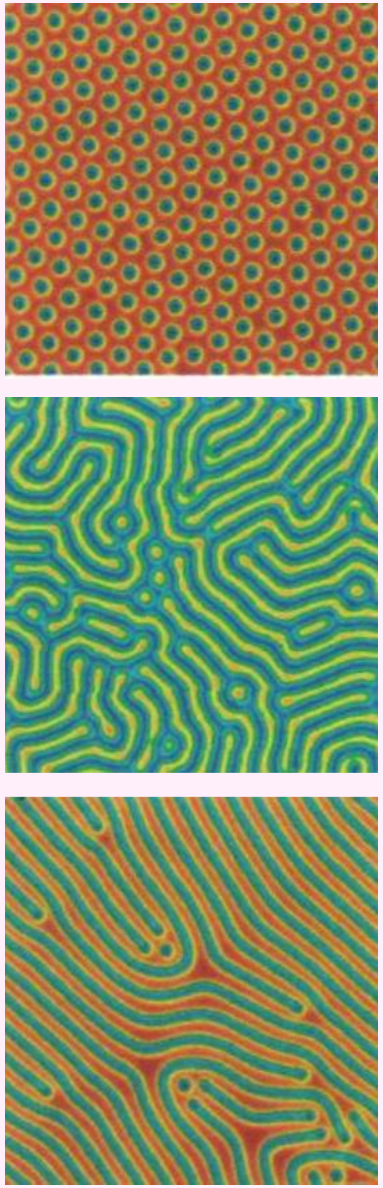


Images from Pearson's *Complex patterns in a Simple System* (1993)

Theory: Turing Patterns

→ all due to reaction and diffusion interplay!





Images from Pearson's *Complex patterns in a Simple System* (1993)

Theory: Grey-Scott model

$$\frac{\partial u}{\partial t} = D_u \nabla^2 u - uv^2 + F(1-u)$$

$$\frac{\partial v}{\partial t} = D_v \nabla^2 v + uv^2 - (F+k)v$$

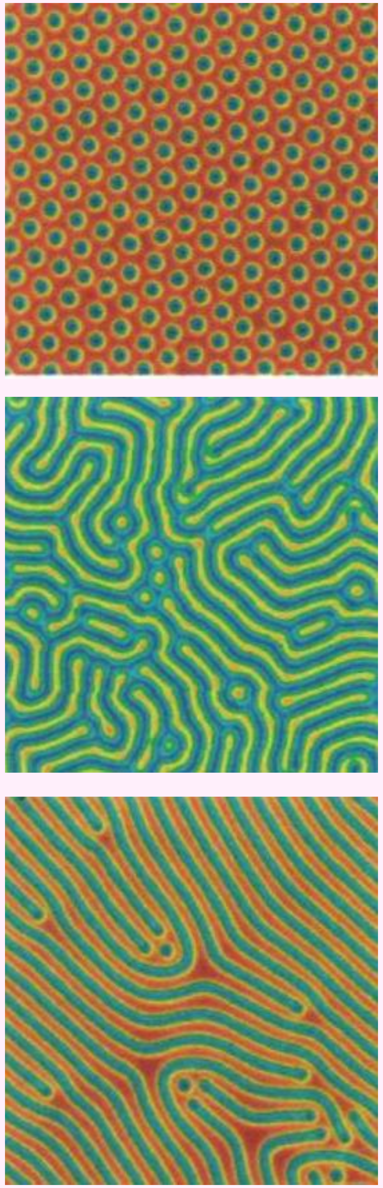
u - activator (A)

v - inhibitor (B)

F - feed rate

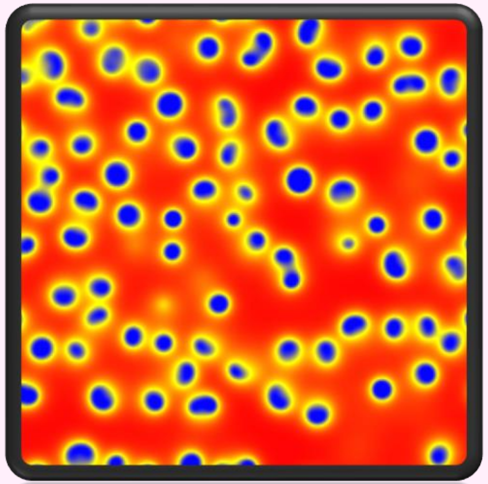
k - kill rate

Solve using 5-point stencil in space dimension & forward Euler in time dimension



Images from Pearson's *Complex patterns in a Simple System* (1993)

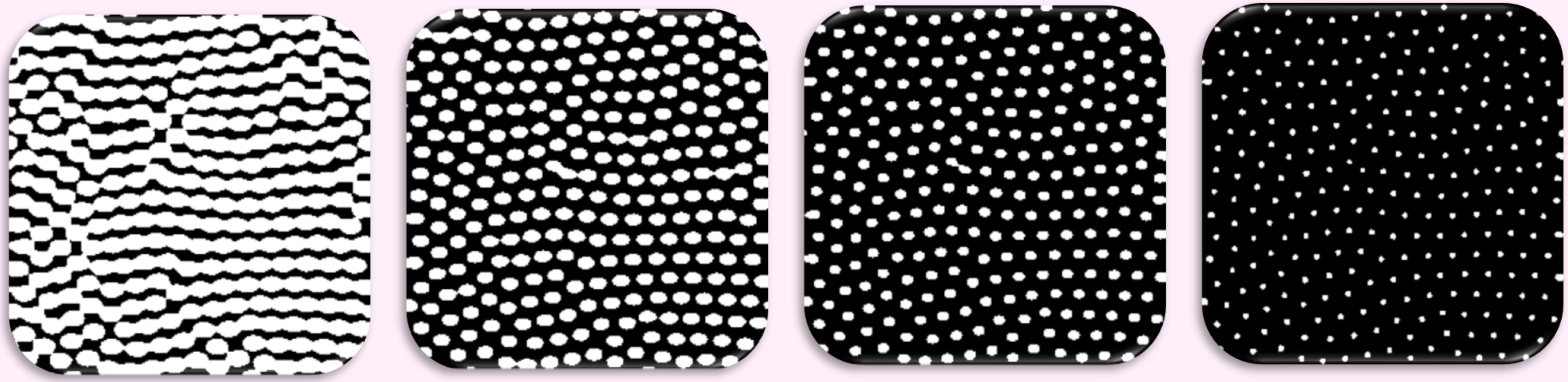
How can we quantify patterns?



VS

Theory: Minkowski Measures

- Apply measures at each threshold grey level (1-256)



$\rho = 50$ $\rho = 100$ $\rho = 150$ $\rho = 200$

V *White pixel area*

S *Boundary length*

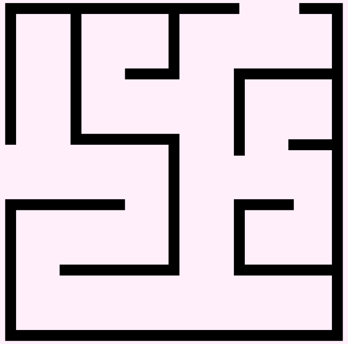
χ *Euler characteristic*

Research question:

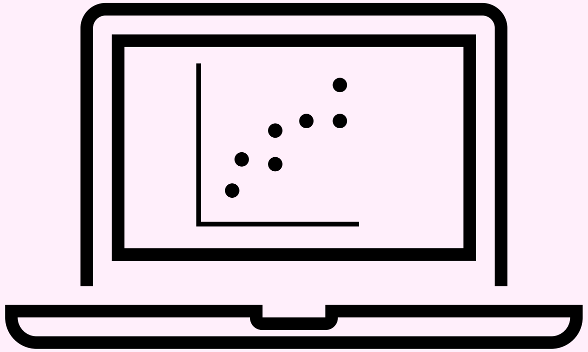
- Can we reproduce characteristic Turing patterns from the Gray–Scott model as reported in Pearson (1993), and

- Do our quantitative characterizations by Minkowski functionals correspond to experimental values established by Mecke (1996)?

Motivation



Turing patterns can explain pattern formation in nature!

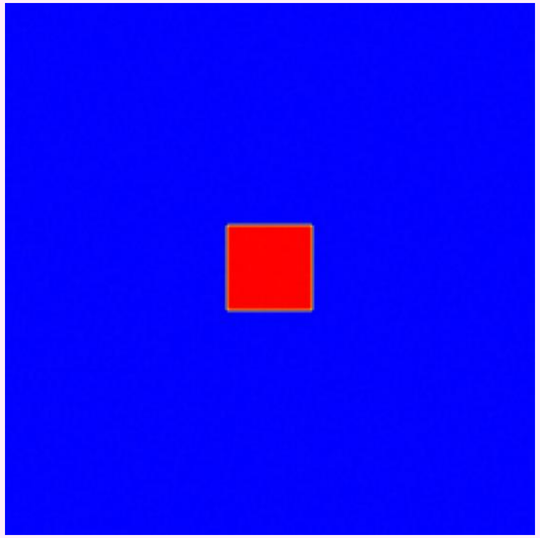


Little research on reproducing Mecke on modern simulations

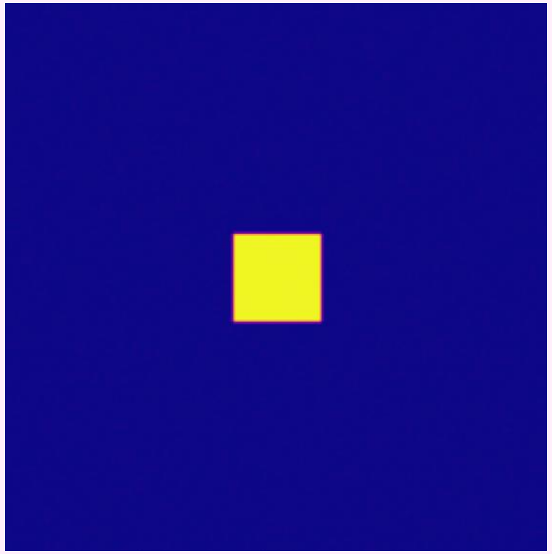
Results

The simulated system

Time-evolved simulations



Simulation for stripe pattern



Simulation for dotted pattern

Results

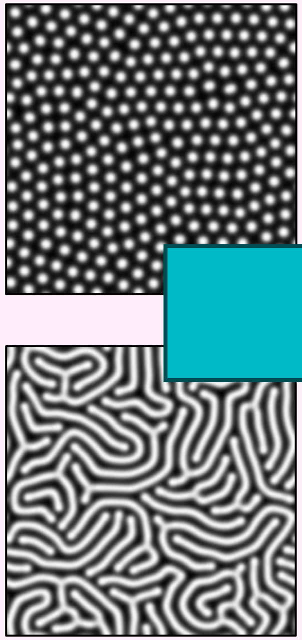
The Minkowski measures and functionals

Comparing to literature

Morphological characterization of patterns in reaction-diffusion systems

K. R. Mecke*
Center for Nonlinear Dynamics and Department of Physics, The University of Texas at Austin, Austin, Texas 78712
(Received 24 October 1995)

SIMULATION



v
s
χ

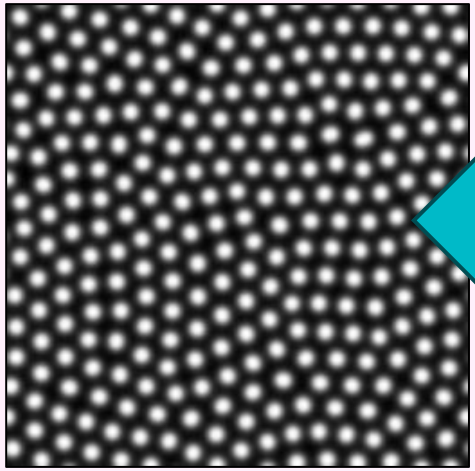
Transform into functional form

p_v
p_s
p_χ

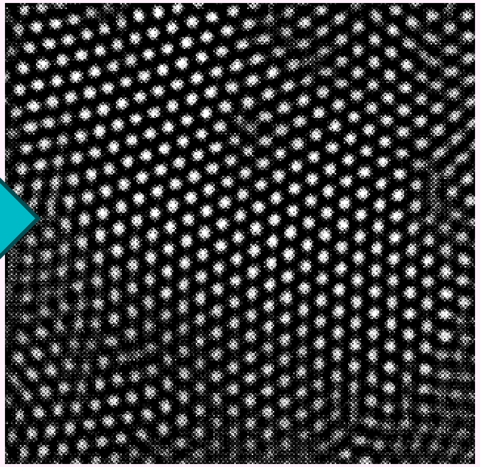
SIMULATION

EXPERIMENT

Dots

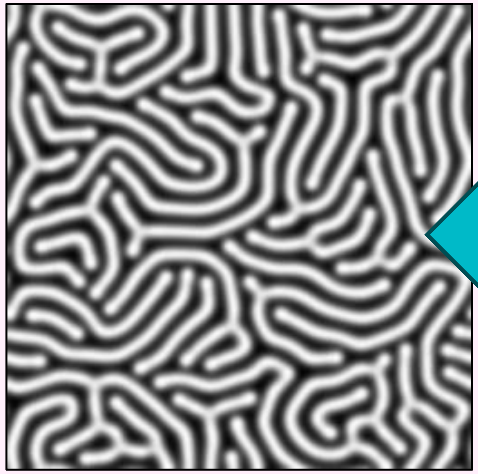


p_v

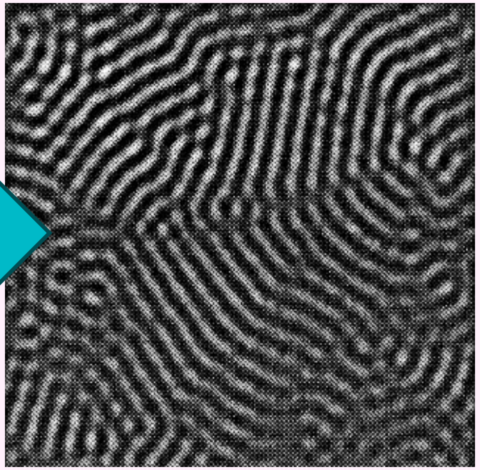


p_s

Stripes

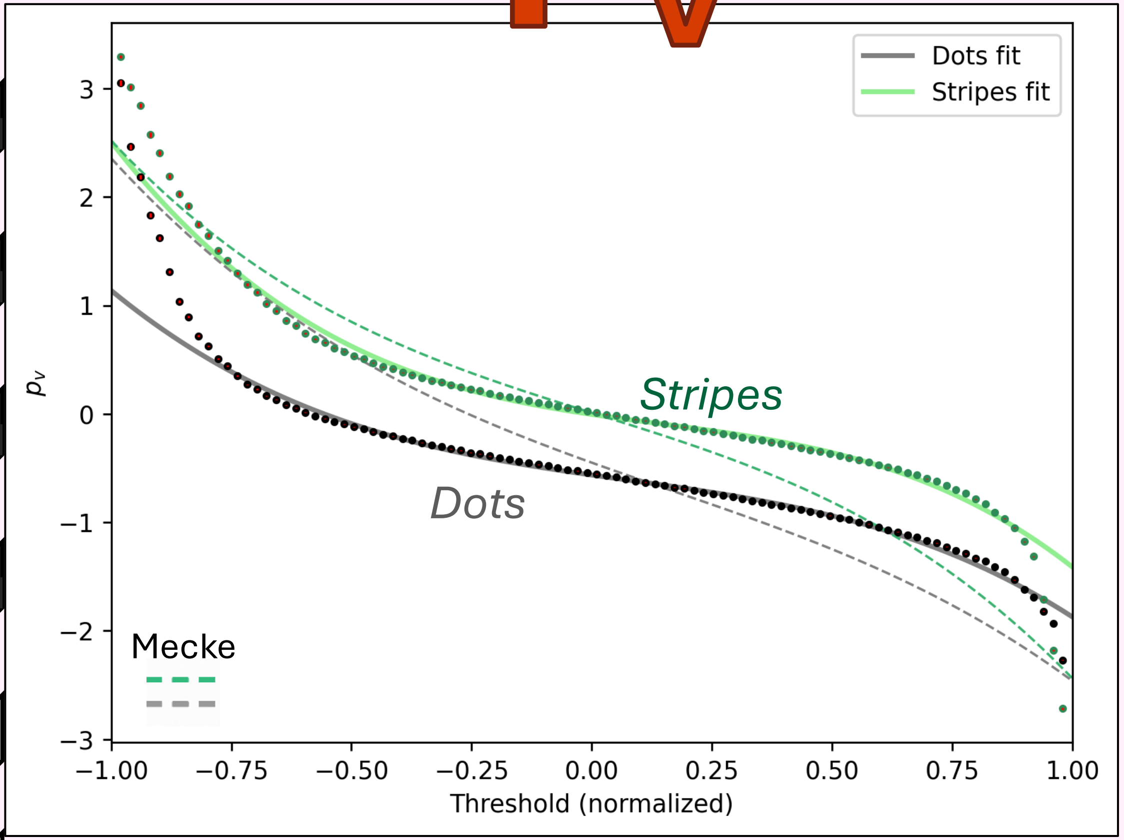


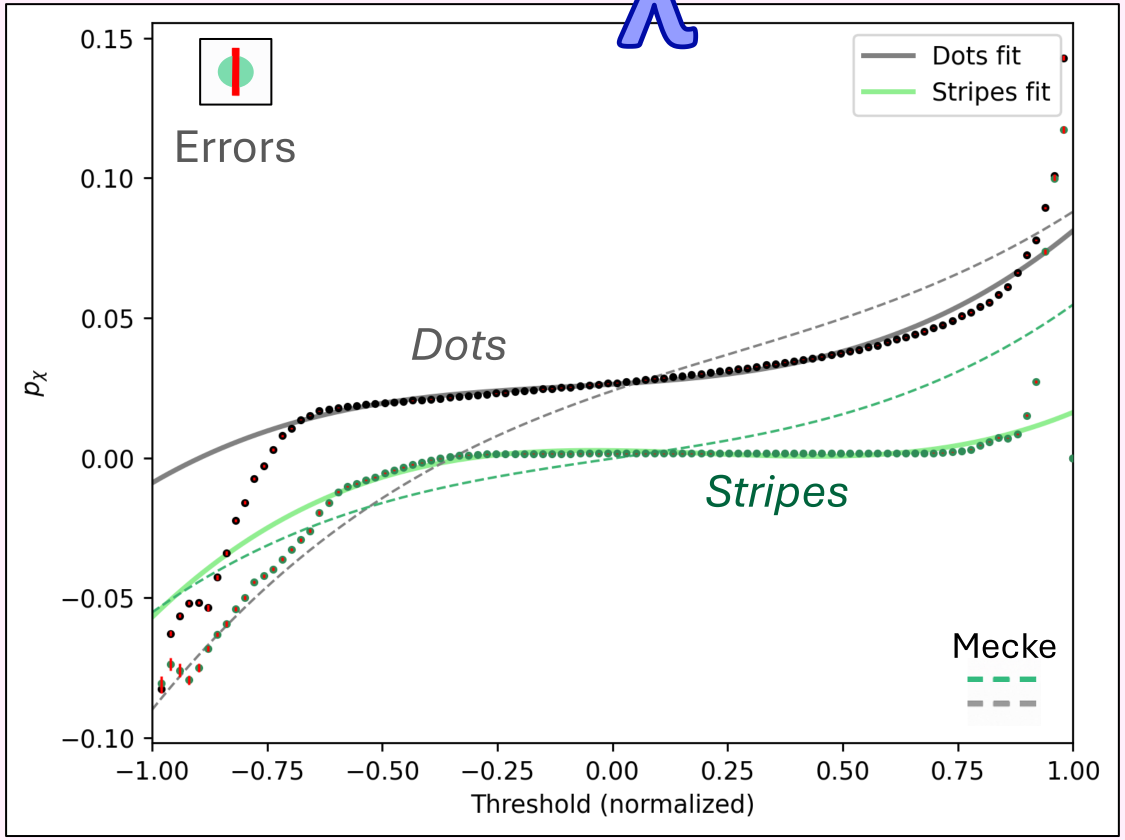
p_χ



Images from Mecke's *Morphological characterization of patterns in reaction-diffusion systems* (1996)

p_v p_χ

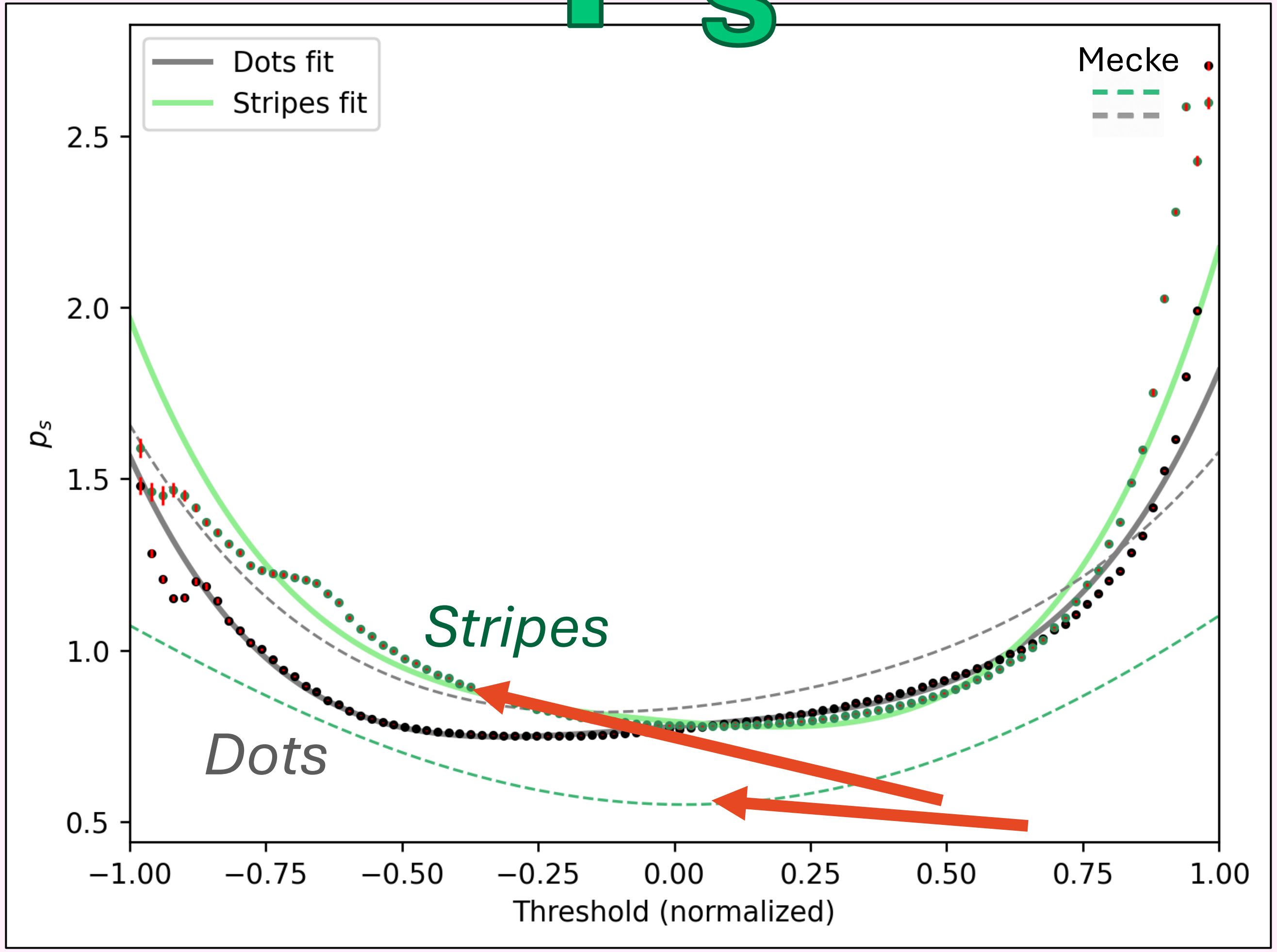




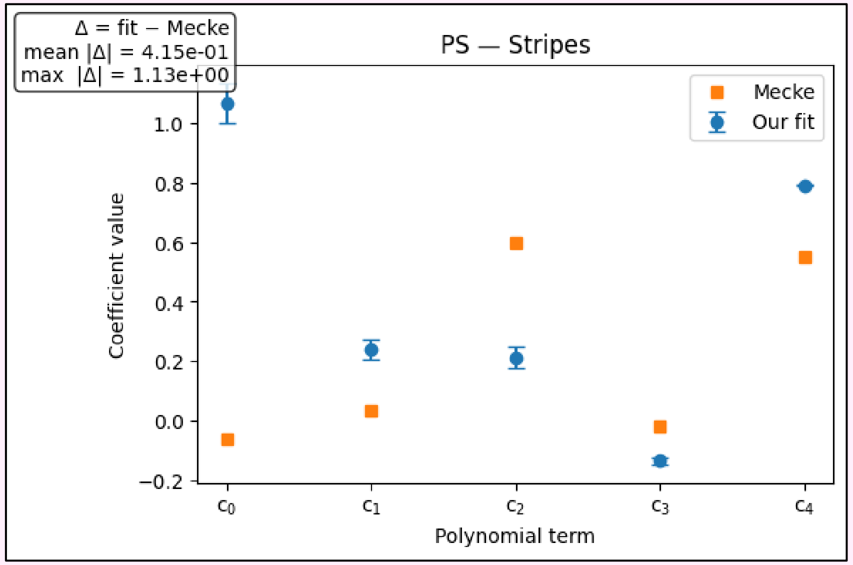
RMS	Dots	Stripes
	0.118	0.084

RMS	Dots	Stripes
	0.176	0.137

p_s



RMS vs Mecke	
Dots	*Stripes*
0.131	0.720



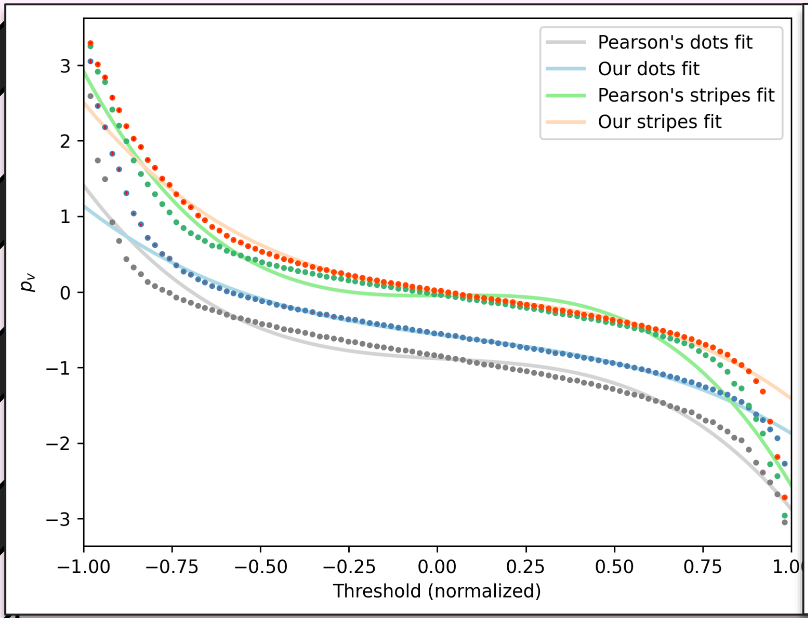
*Coefficient errors for all fits in appendix

Validity of simulation results

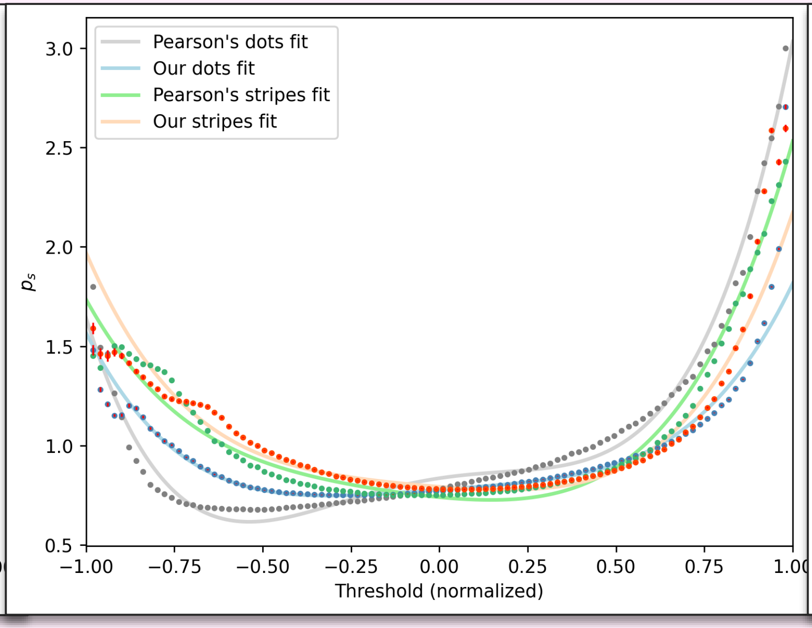
Comparing our functionals to Pearson’s (1993)

[redacted]

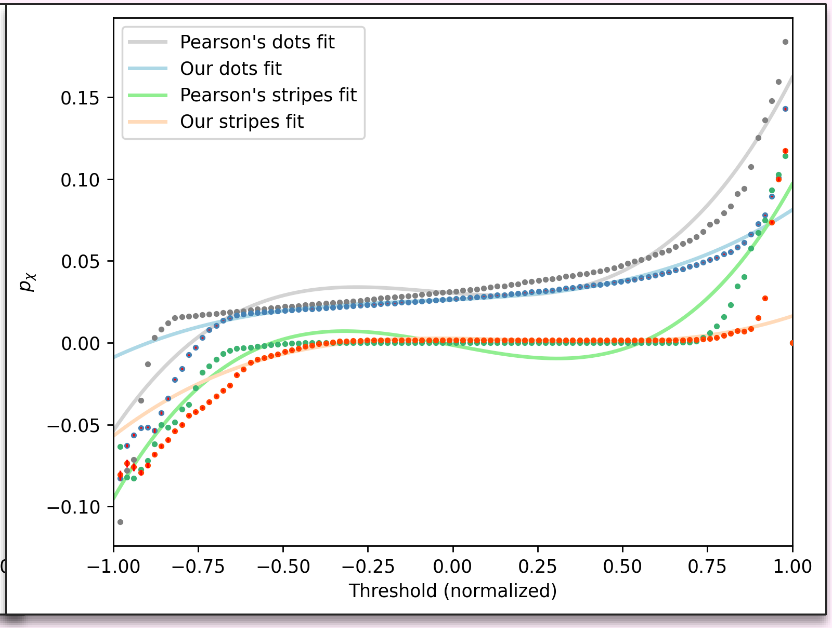
p_v



p_s



p_χ



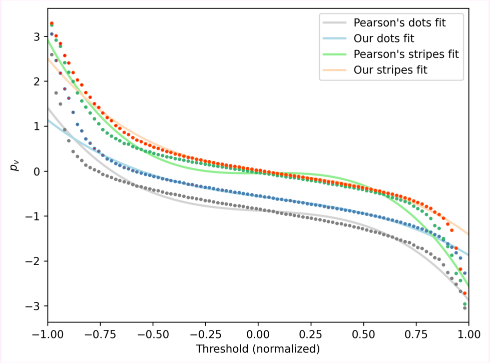
Statistical analysis:

Pearson’s vs. our functionals

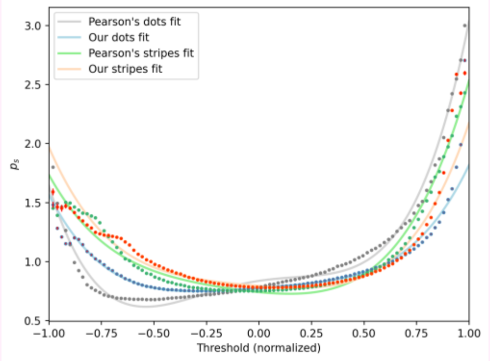
RMS	
Dots	*Stripes*
0.131	0.080
0.283	0.083
0.258	0.290

Orientation agreement	
Same-sign	*Pearson corr. coeff.*
1.000	0.851
0.698	0.531
1.000	0.893

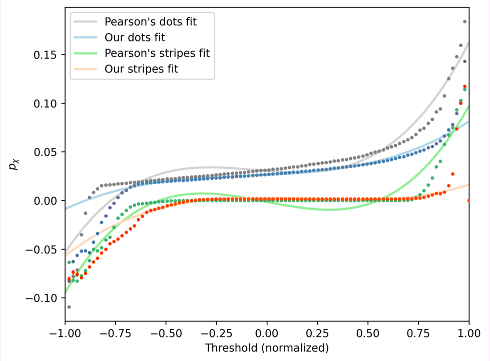
p_v



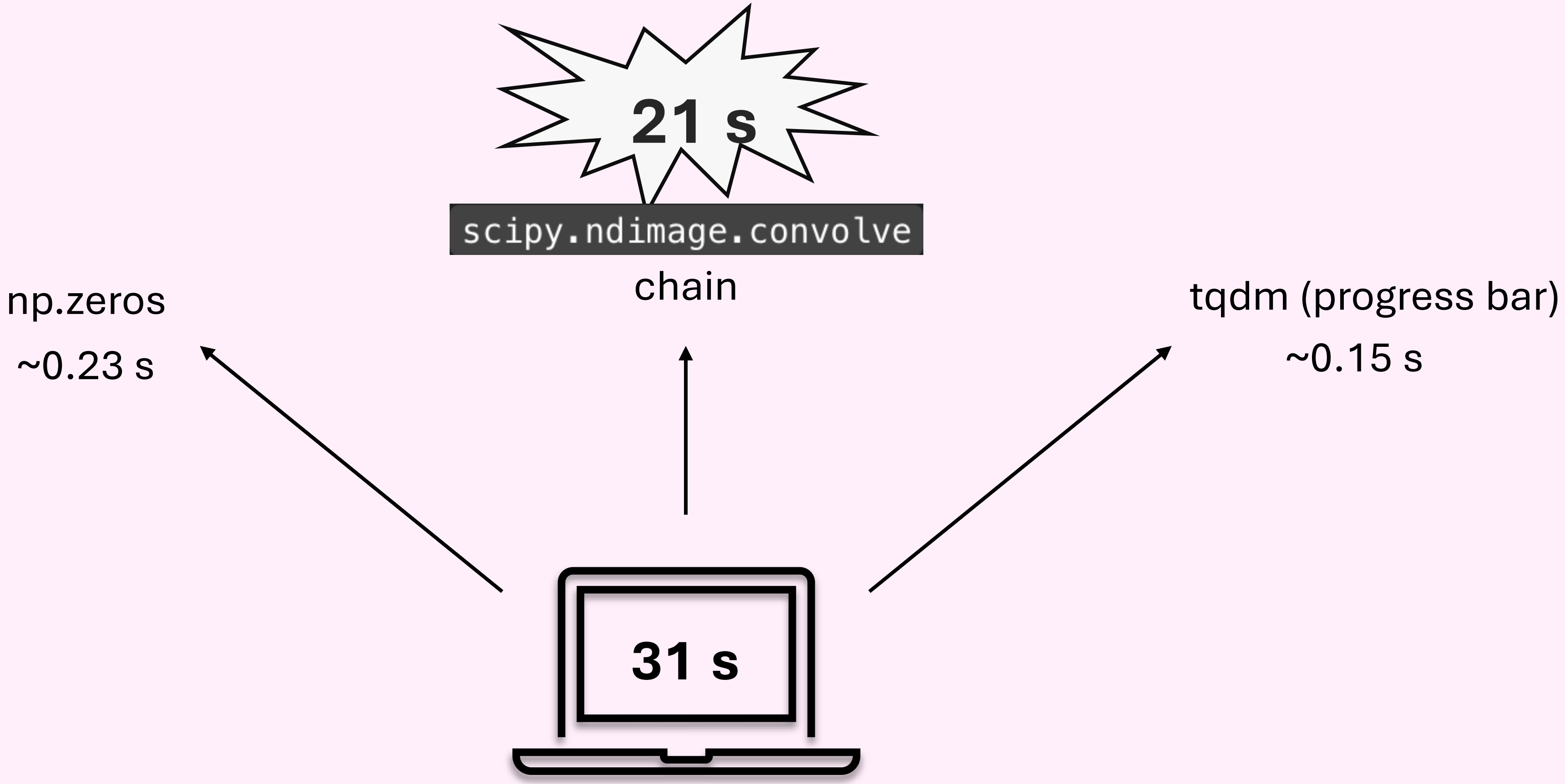
p_s



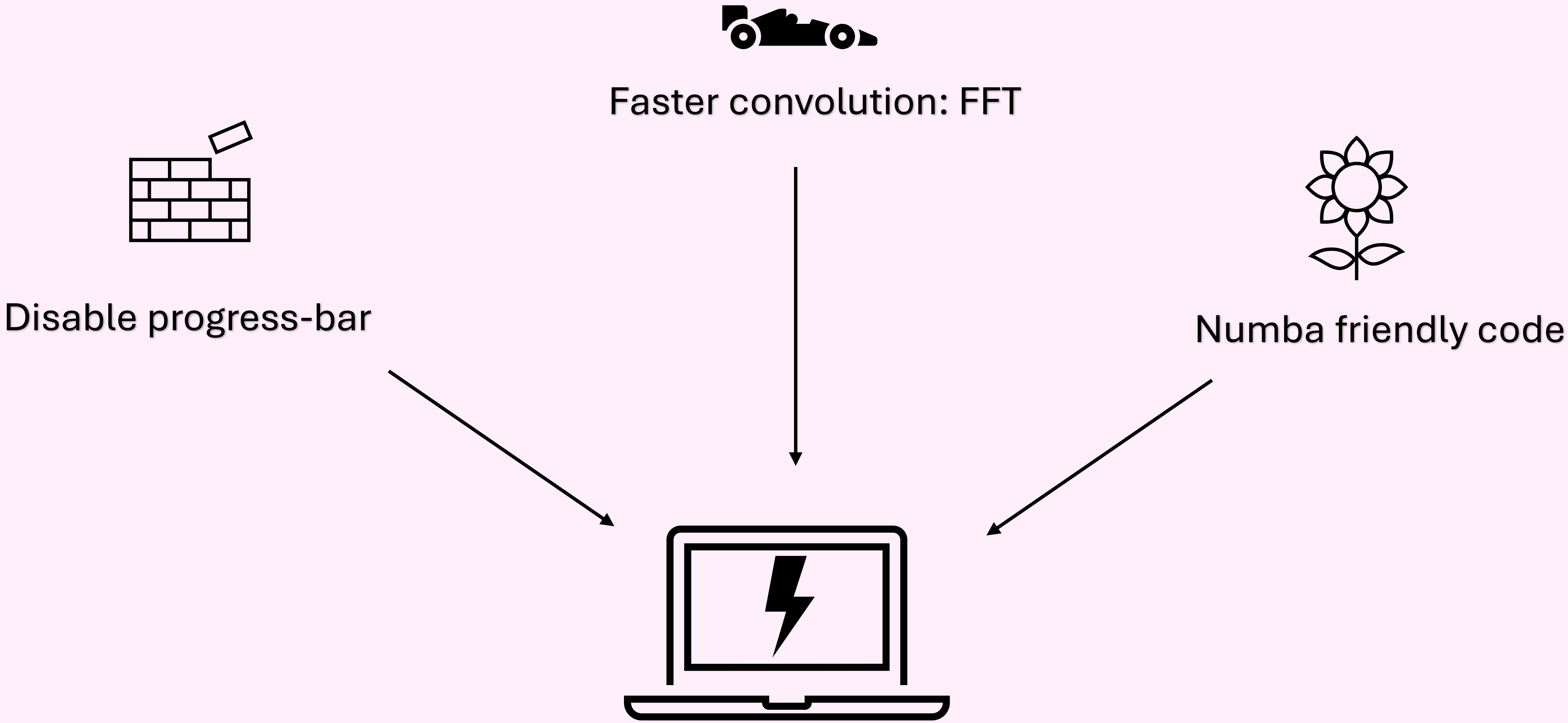
p_χ



Performance evaluation

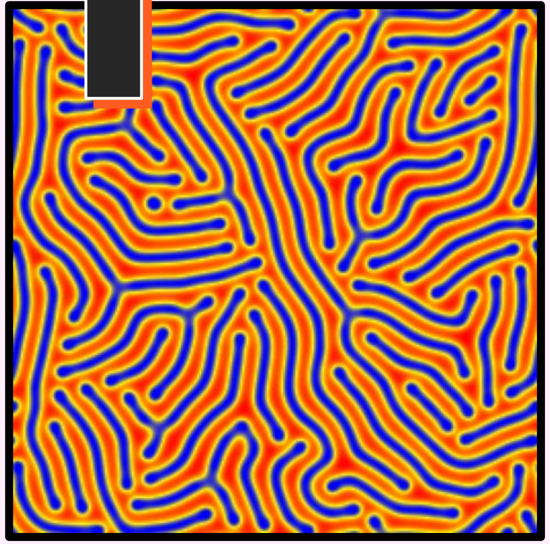


Performance evaluation



Conclusion

1



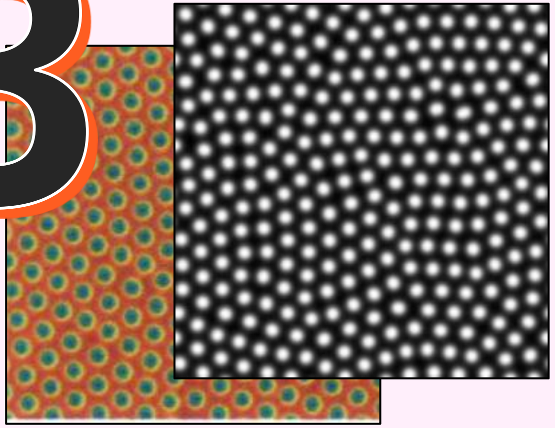
2

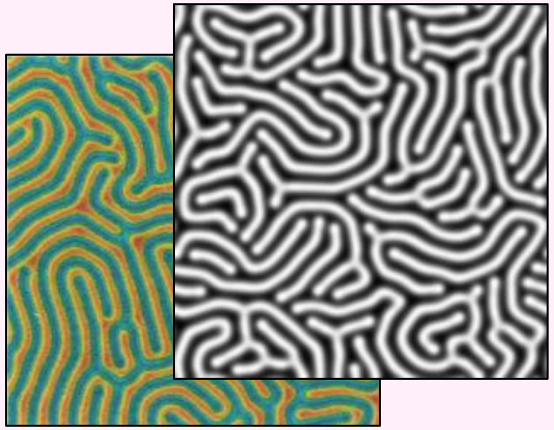
p_v

p_s

p_χ

3





Thank you for listening!

Extra slides: Errors

Different RNG seeds for each run – 20 runs, mean and std error for p, v and chi, as well as p_v, p_s, p_chi

Polynomial analysis

$$\mathrm{NRMS}_{range} = \frac{\sqrt{\frac{1}{N}\sum_{i=1}^{N}\left[P_1(r_i) - P_2(r_i)\right]^2}}{\max_r P_2(r) - \min_r P_2(r)}$$

Normalised RMS to compare p_chi with the others;
a dimensionless ‘fraction-of-full-height’ error

$$\rho = \frac{\sum_{i=1}^{N}(\Delta_{ours}(r_i) - \overline{\Delta}_{ours})(\Delta_{Mecke}(r_i) - \overline{\Delta}_{Mecke})}{\sqrt{\sum_{i=1}^{N}(\Delta_{ours}(r_i) - \overline{\Delta}_{ours})^2}\sqrt{\sum_{i=1}^{N}(\Delta_{Mecke}(r_i) - \overline{\Delta}_{Mecke})^2}},$$

Pearson coefficient – measures linear relationship between two variables; shows us global consistency of fluctuations around the mean (mean-centered, d/n take into account vertical offsets).
Cosine angle between two sampled difference vectors

$$\Delta_{ours}(r) = P_{dots}(r) - P_{stripes}(r), \Delta_{Mecke}(r) = M_{dots}(r) - M_{stripes}(r)$$

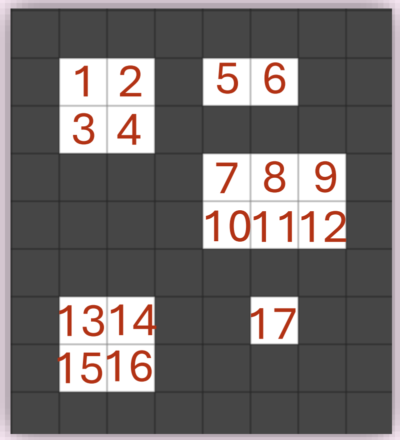
$$f_{same} = \frac{1}{N}\sum_{i=1}^{N}\mathbf{1}[\Delta_{ours}(r_i)\,\Delta_{Mecke}(r_i) > 0],$$

Fraction of thresholds whose orientation matches

Theory: Minkowski Measures

- Geometric measures for morphological differences

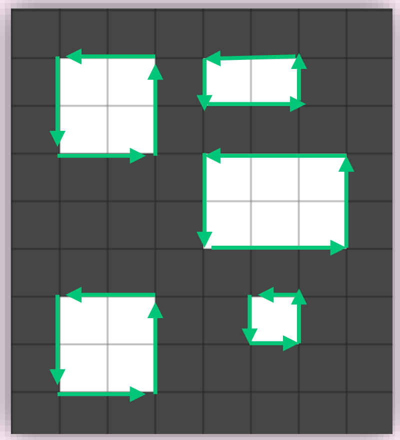
$$\frac{N_{wp}}{N}$$



V

White pixel area

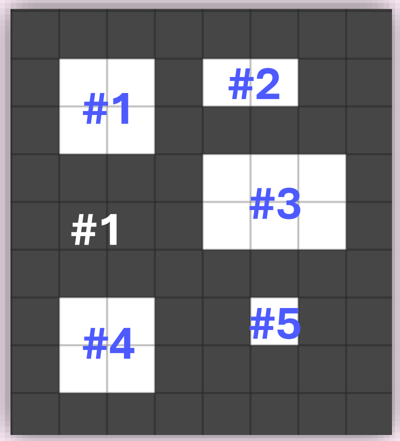
$$\frac{B_l}{N}$$



S

Boundary length

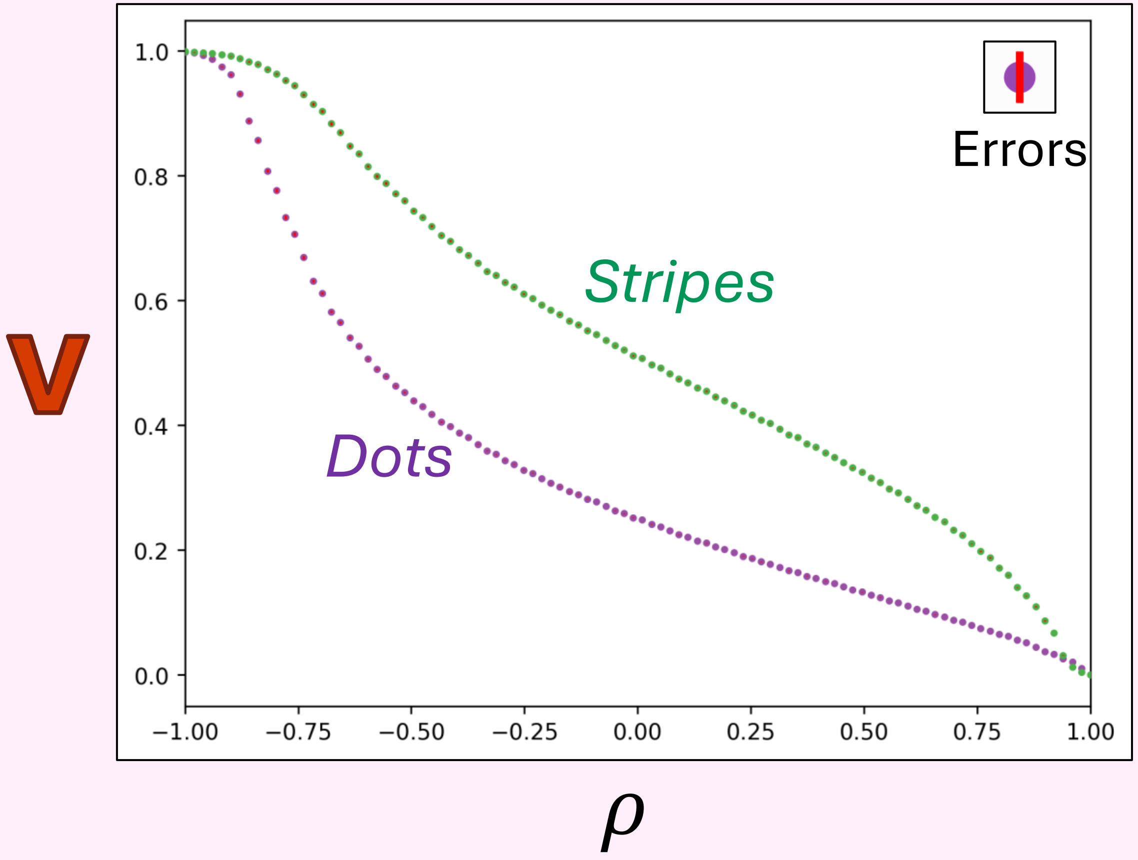
$$\frac{C_w - C_B}{N}$$



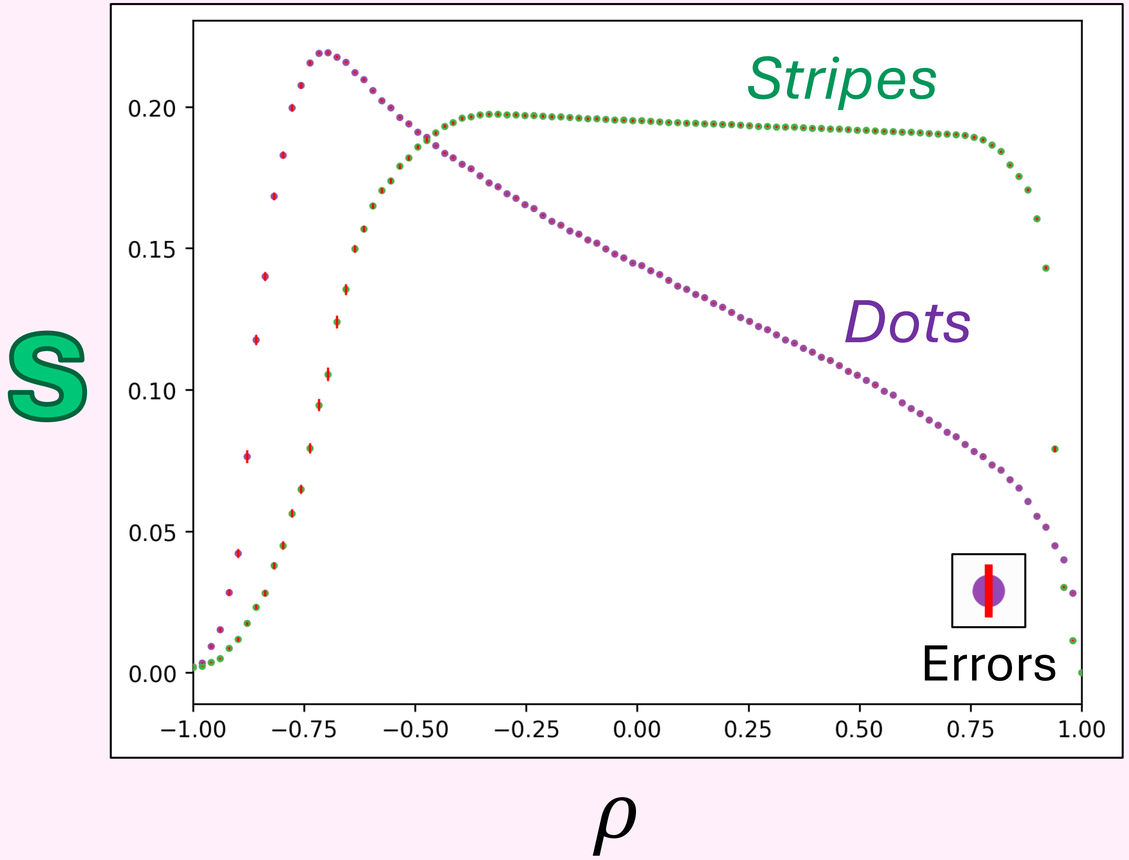
χ

Euler characteristic

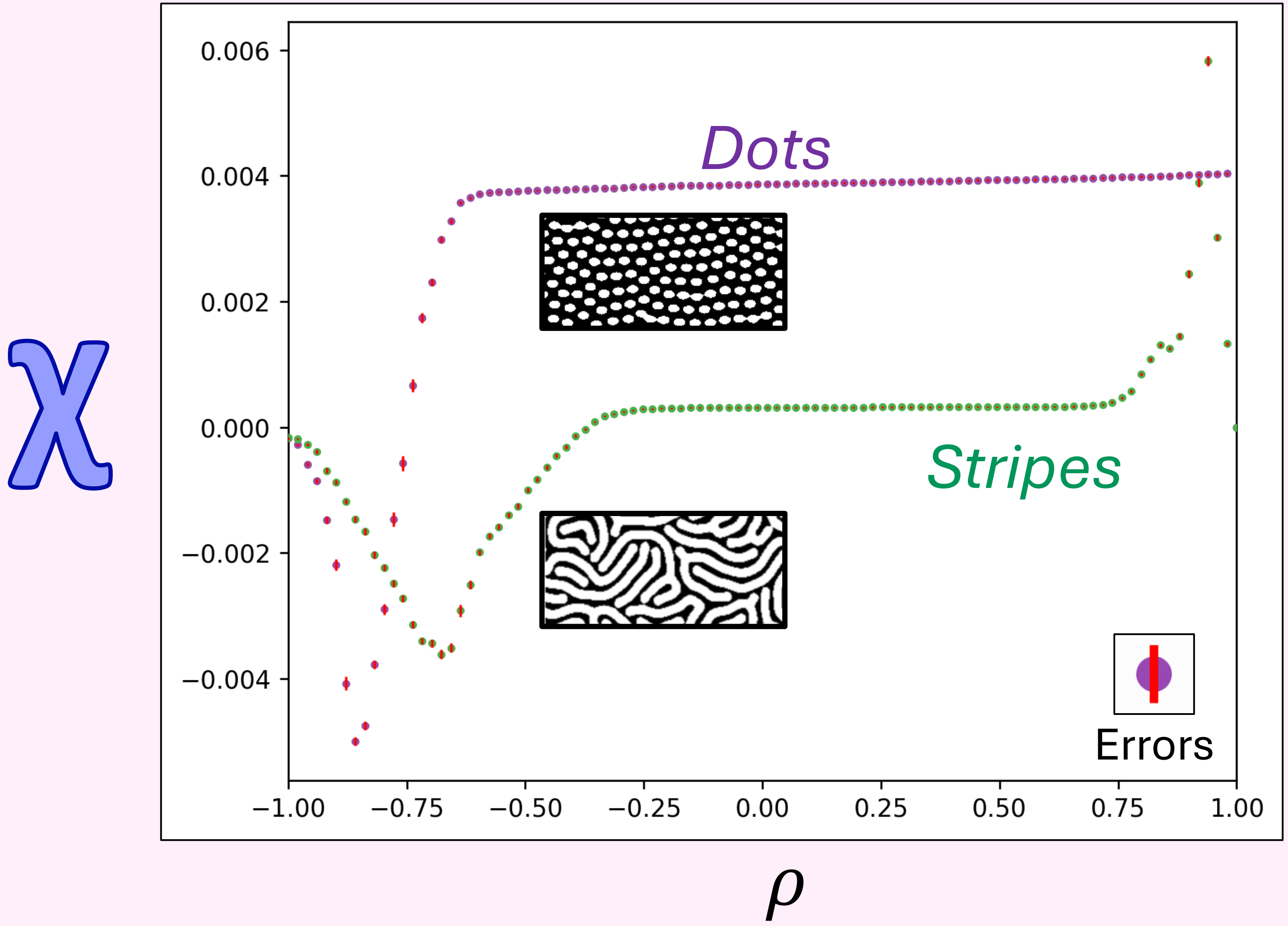
White pixel area



Boundary length



Difference in # of components



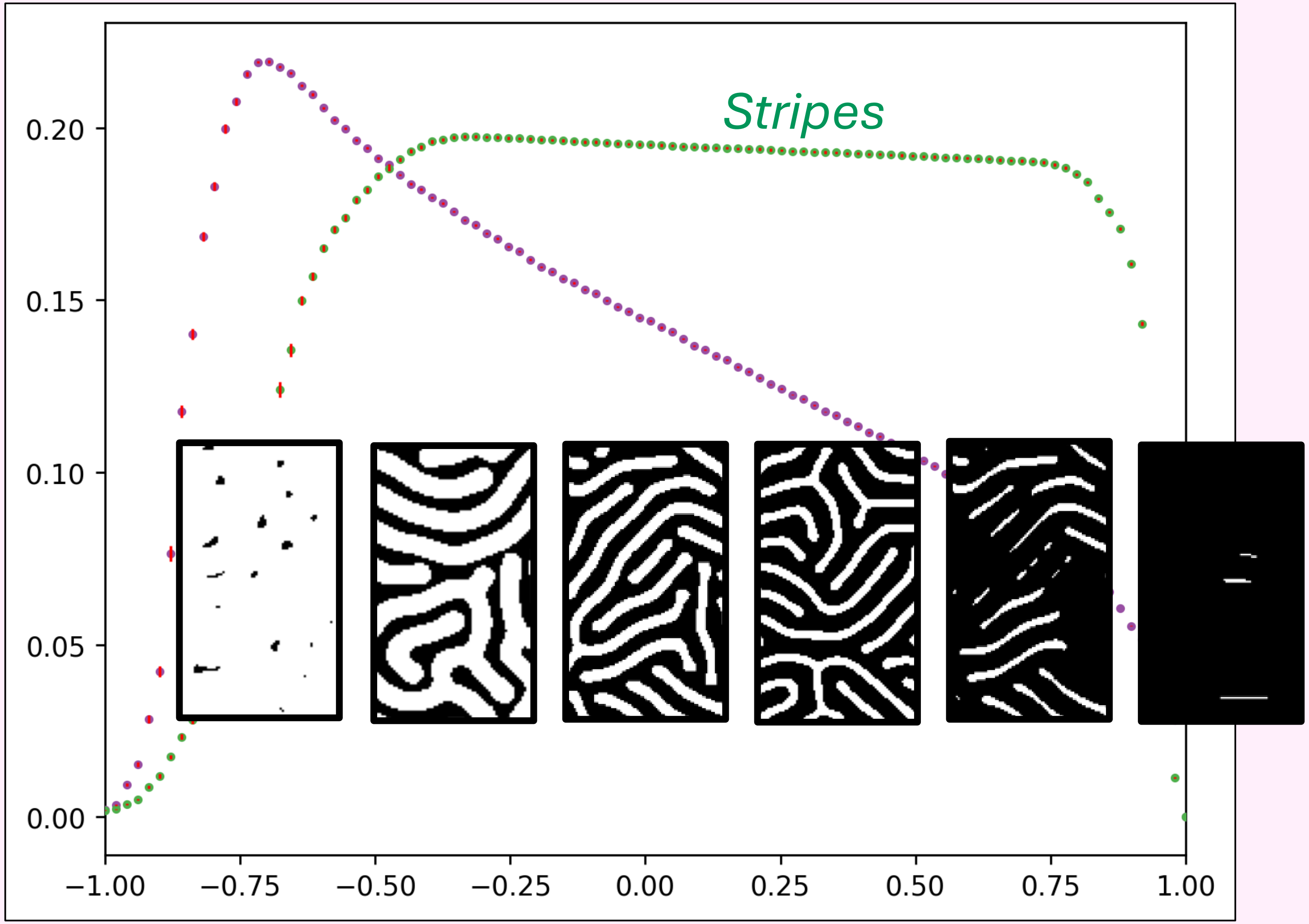
Extra slides: Functional forms

$$p_v(\rho) = \tanh^{-1}(2v - 1),$$

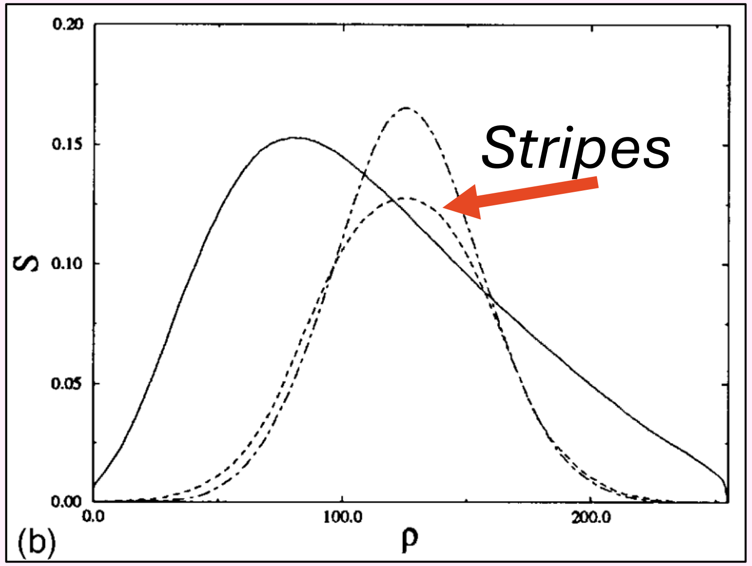
$$p_s(\rho) = \frac{s}{v(1-v)} = 4s\ \cosh^2[p_v(\rho)],$$

$$p_\chi(\rho) = \frac{\chi}{s},$$

Chosen by Mecke because the functionals occur in many fields of physics. Direct connection to reaction-diffusion systems unclear.
Allows for polynomial fitting – and coefficients that depend on experimental conditions

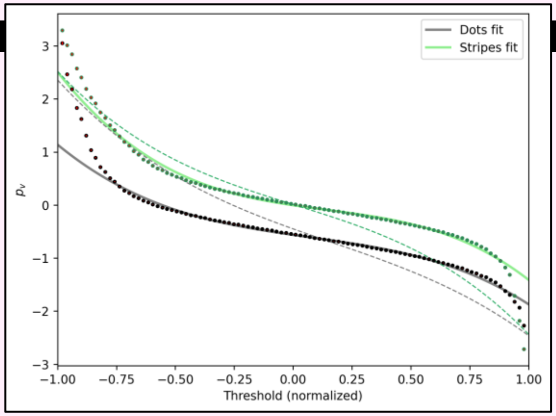


$$p_s(\rho) = \frac{s}{v(1-v)}$$

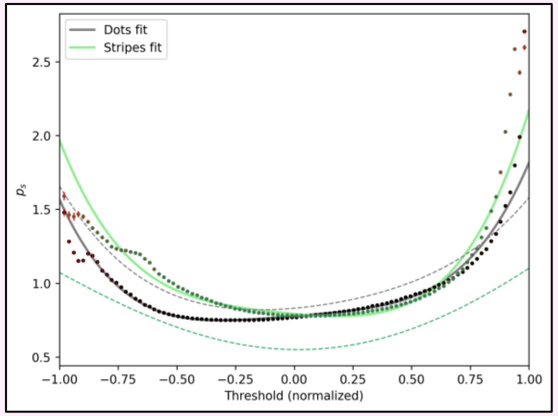


Shape due to inhomogeneities in experimental photos?

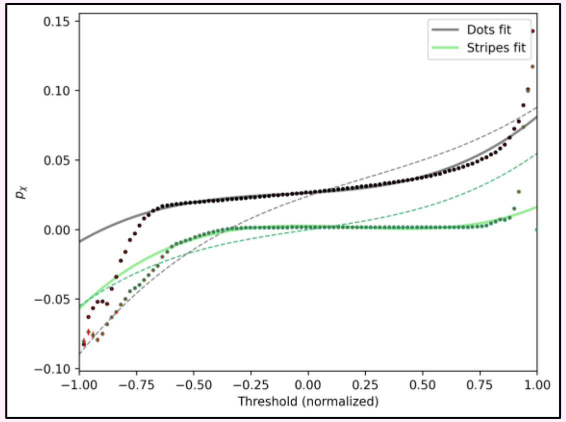
p_v



p_s



p_χ

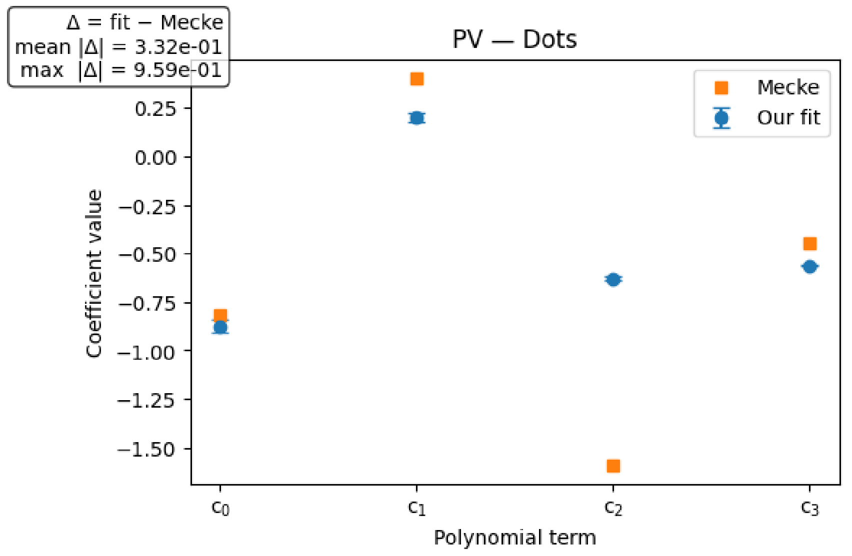


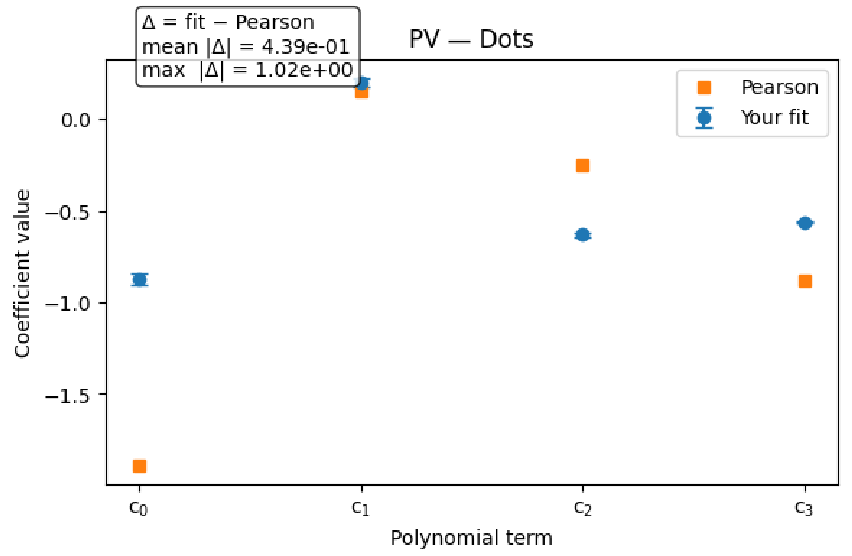
RMS	Dots	Stripes
	0.118	0.084
RMS	Dots	Stripes
	0.131	0.720
RMS	Dots	Stripes
	0.176	0.137

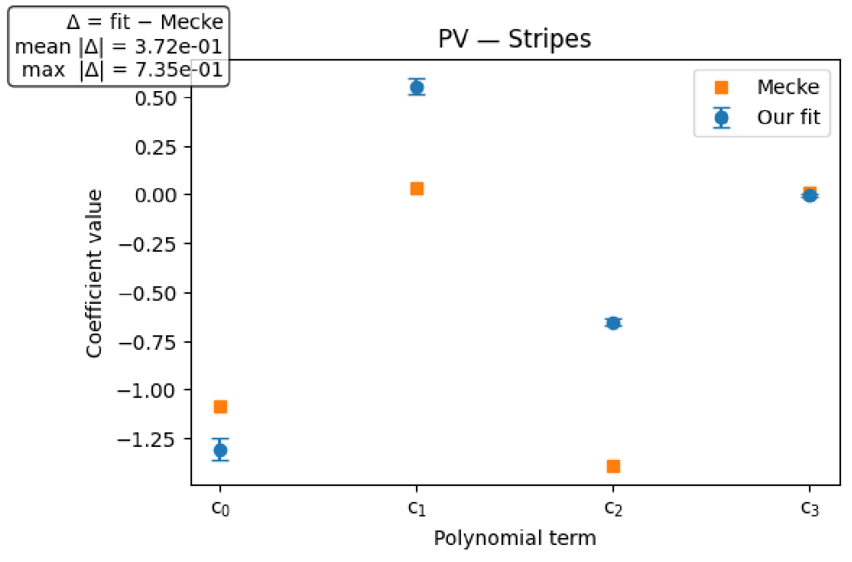
Orientation agreement	
Same-sign	*Pearson corr. coeff.*
1.000	-0.510
0.278	-0.390
0.764	0.151

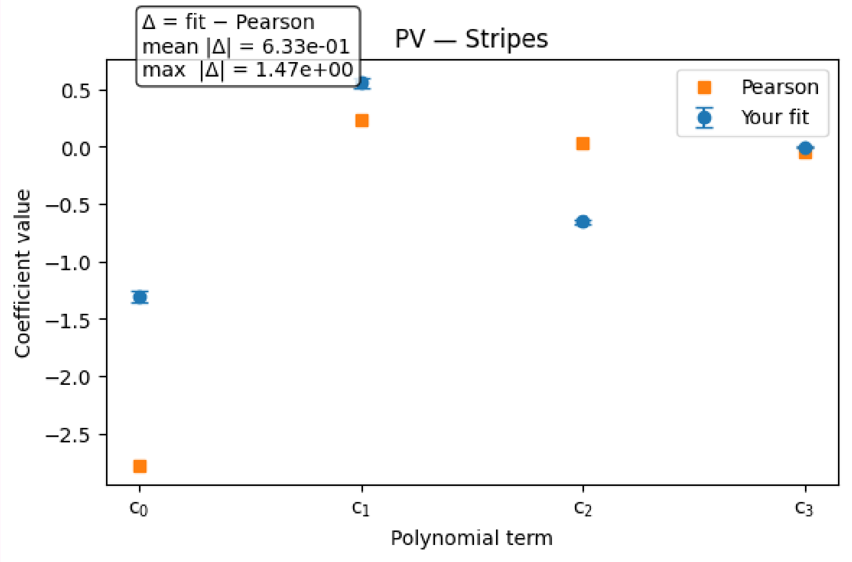
Polynomial coefficient errors

p_v



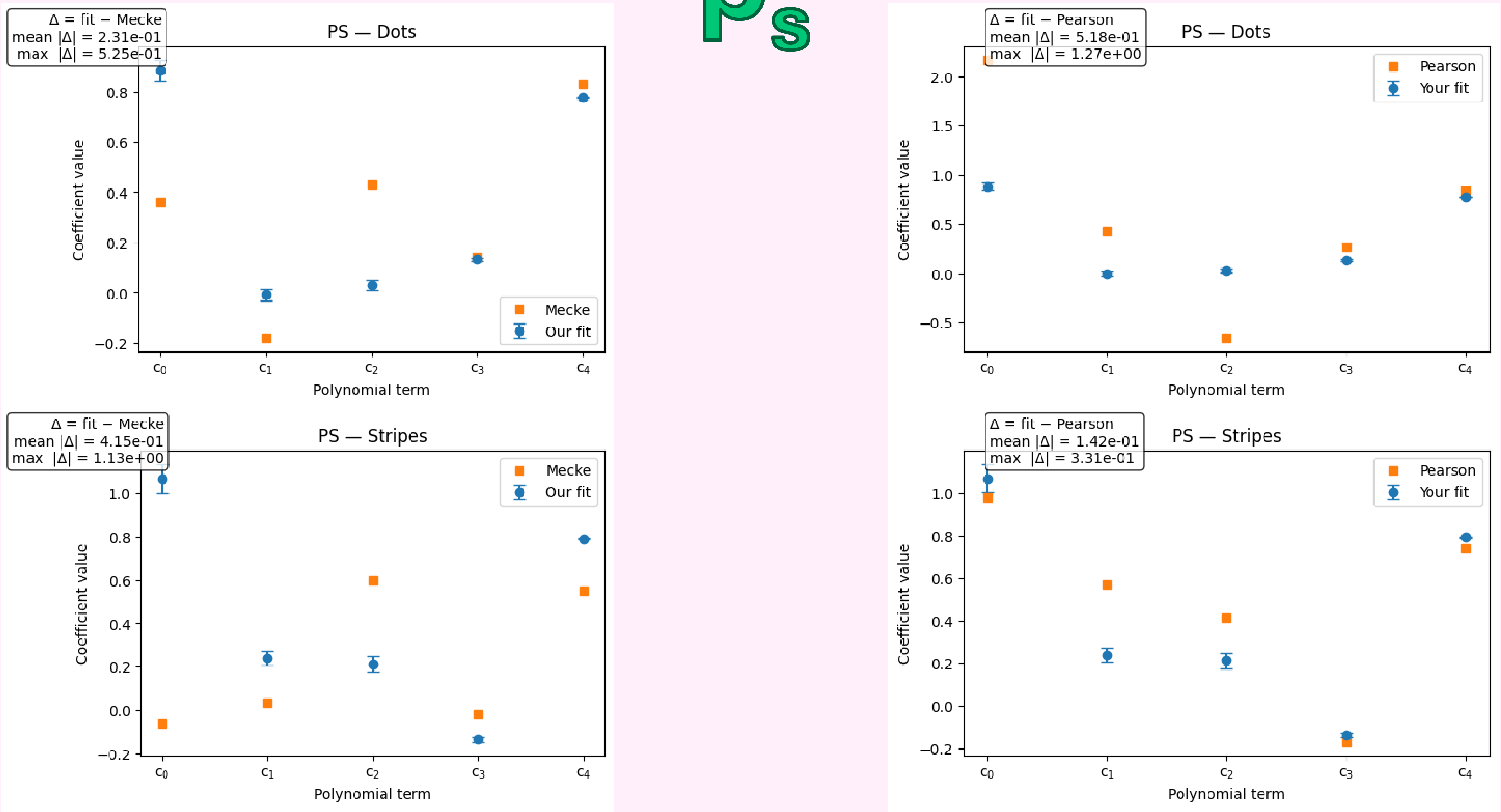






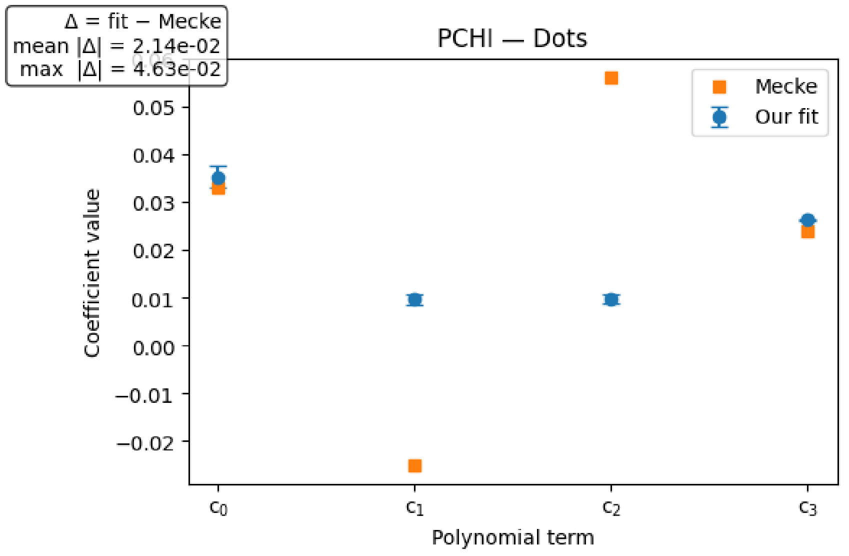
Polynomial coefficient errors

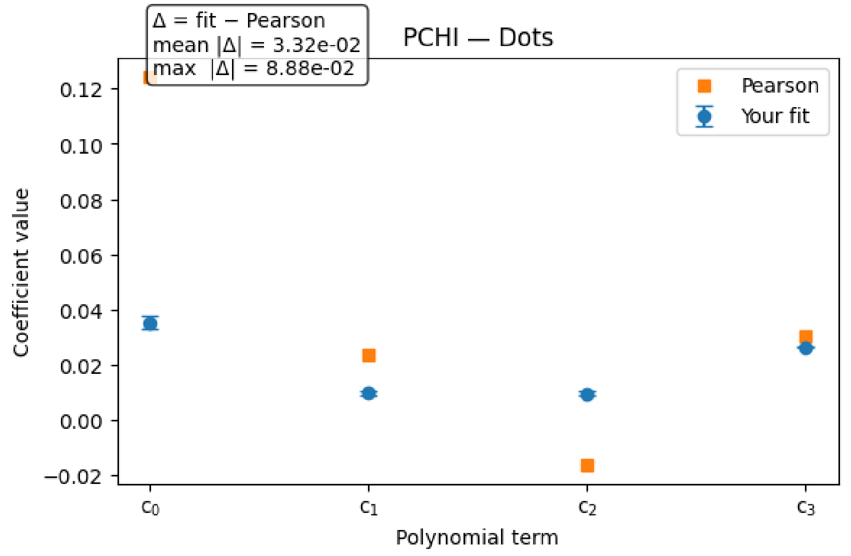
p_s

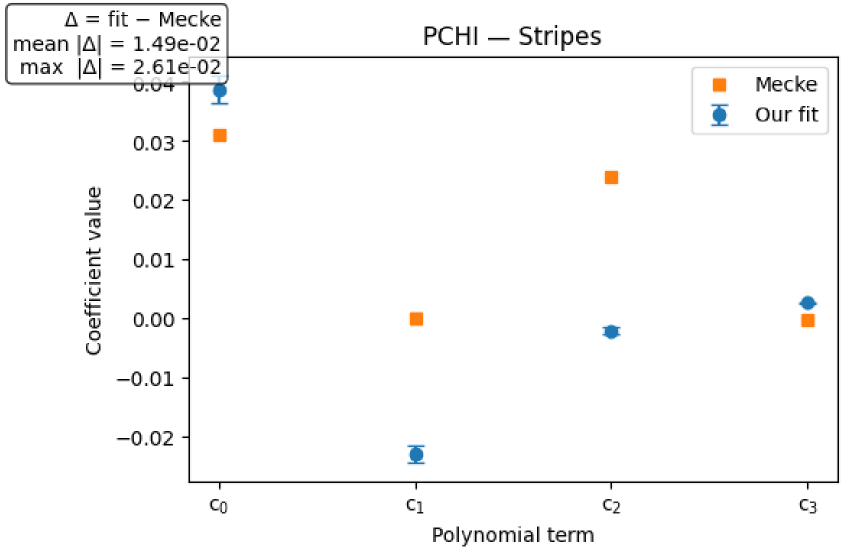


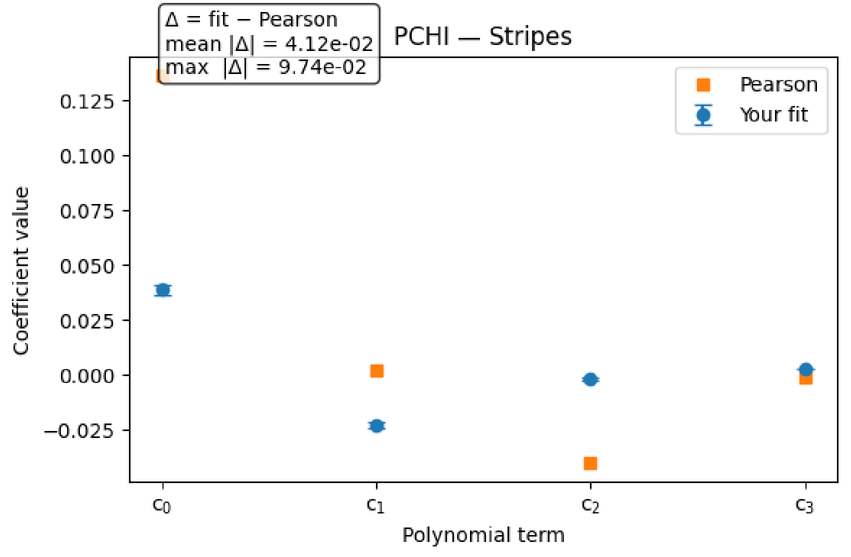
Polynomial coefficient errors

p_χ

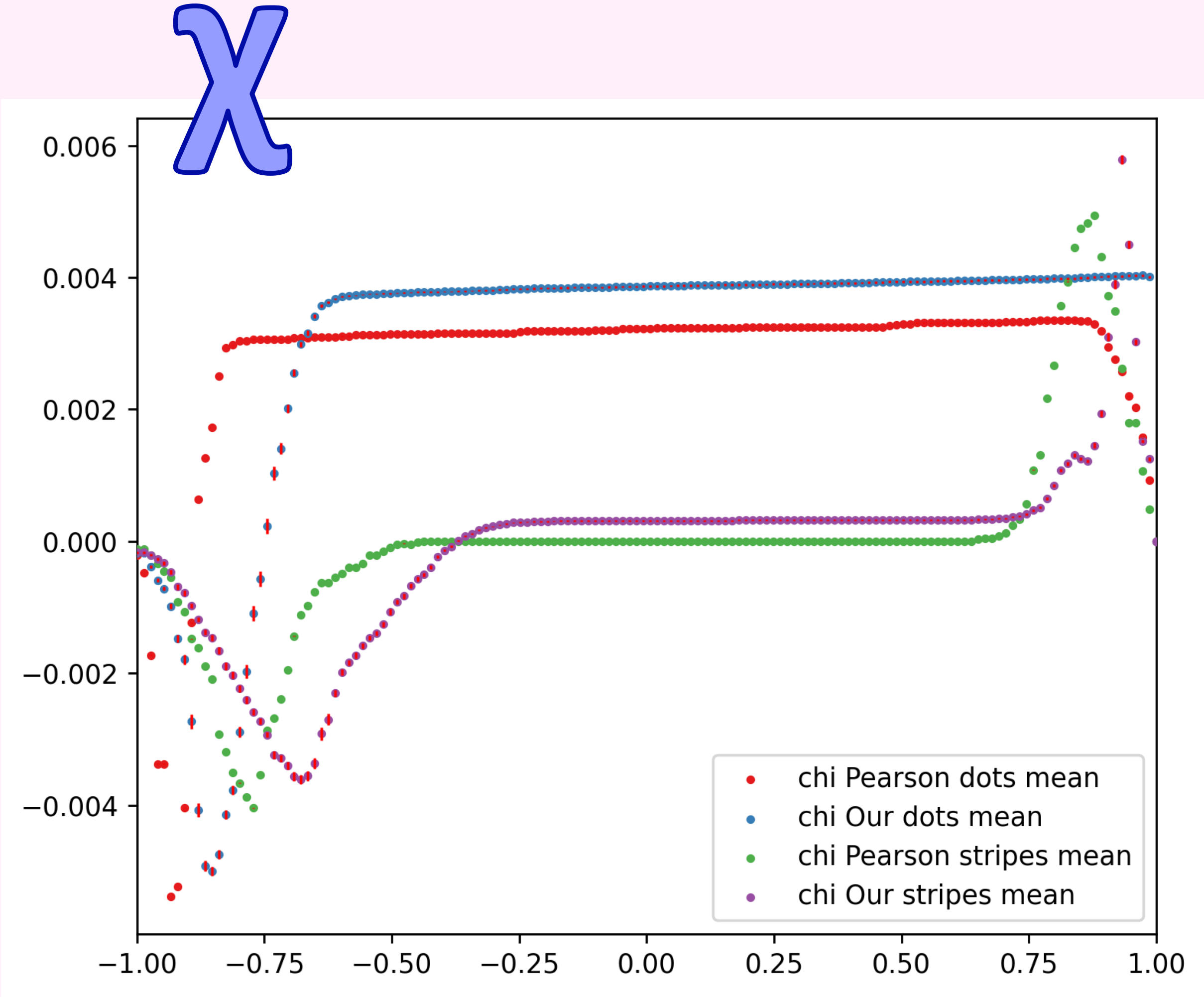




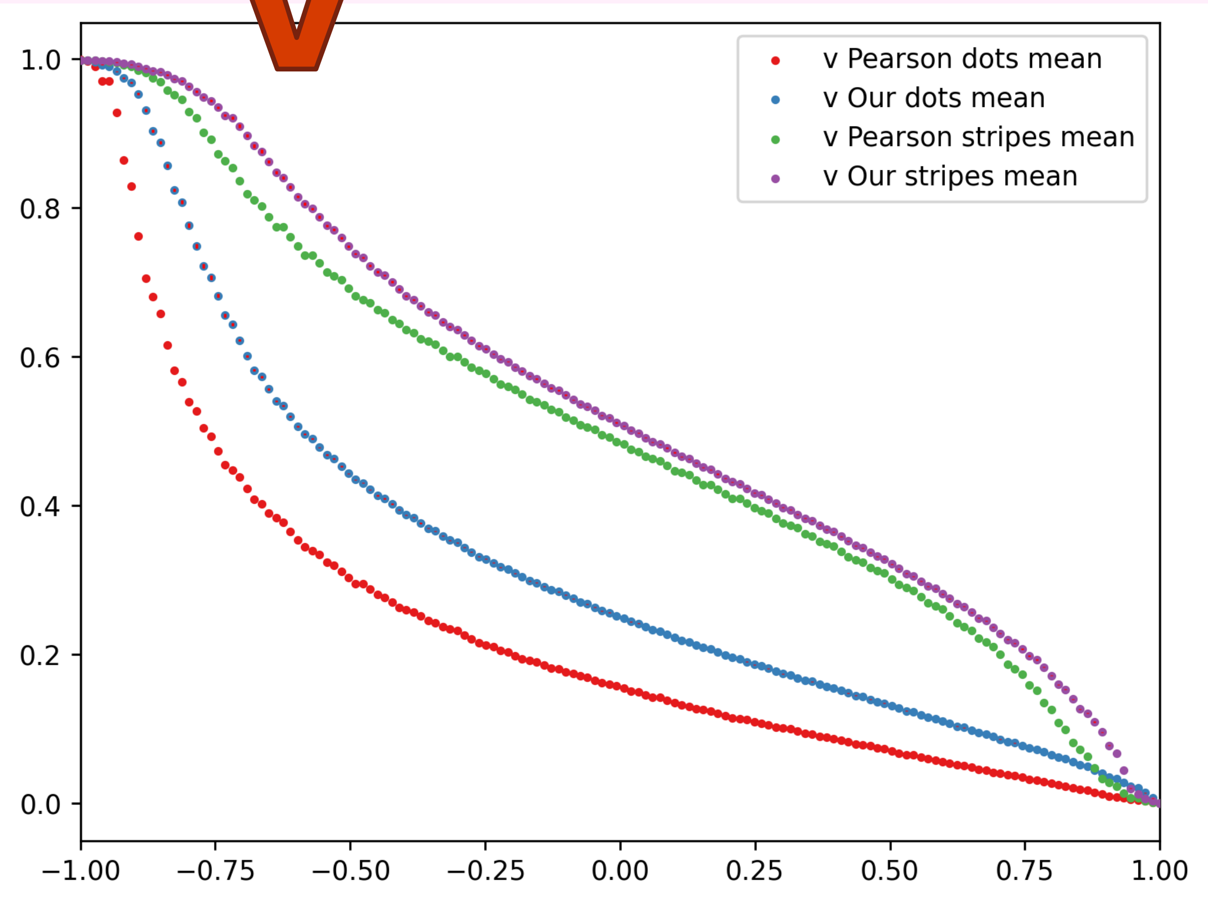




Extra slides: All Pearson plots

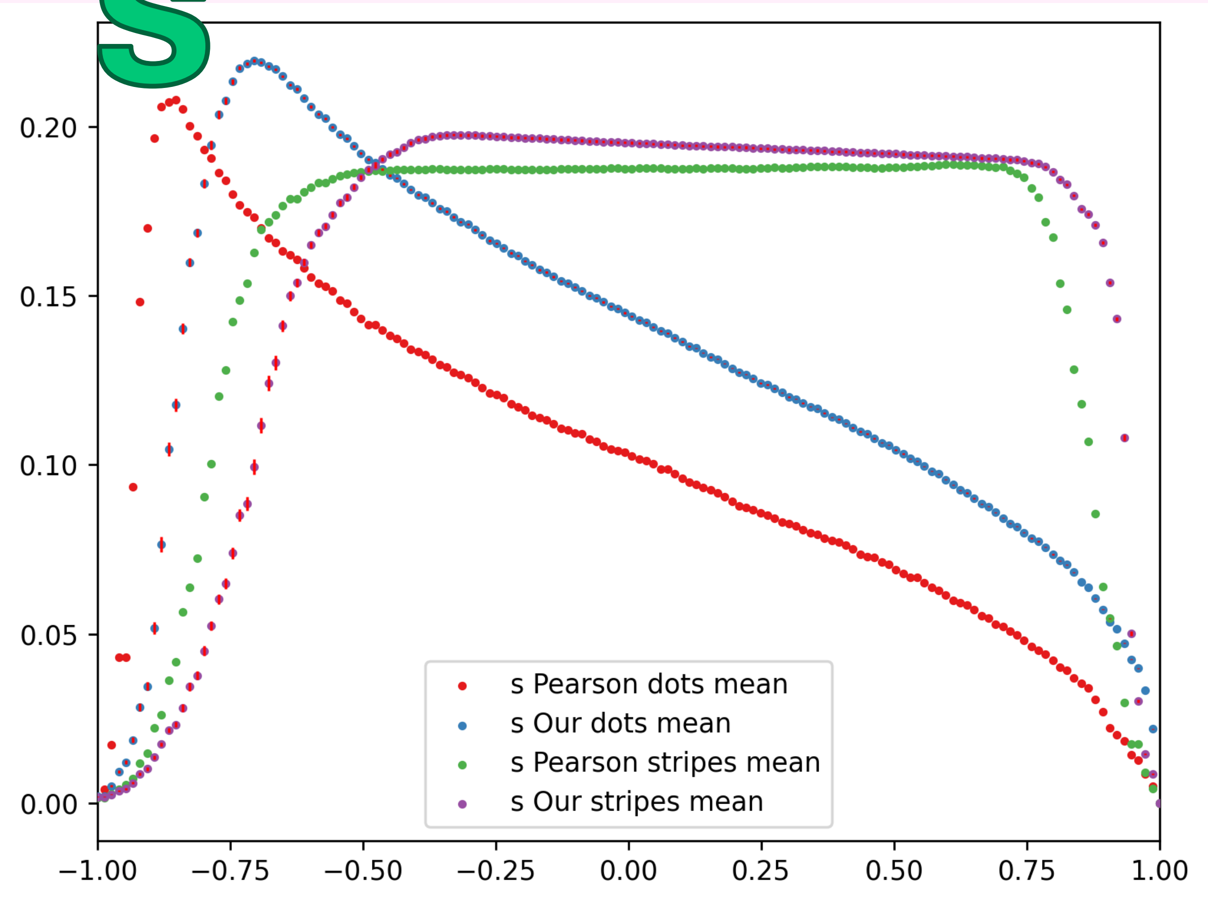

χ
0.006
0.004
0.002
0.000
−0.002
−0.004
−1.00
−0.75
−0.50
−0.25
0.00
0.25
0.50
0.75
1.00
chi Pearson dots mean
chi Our dots mean
chi Pearson stripes mean
chi Our stripes mean

V

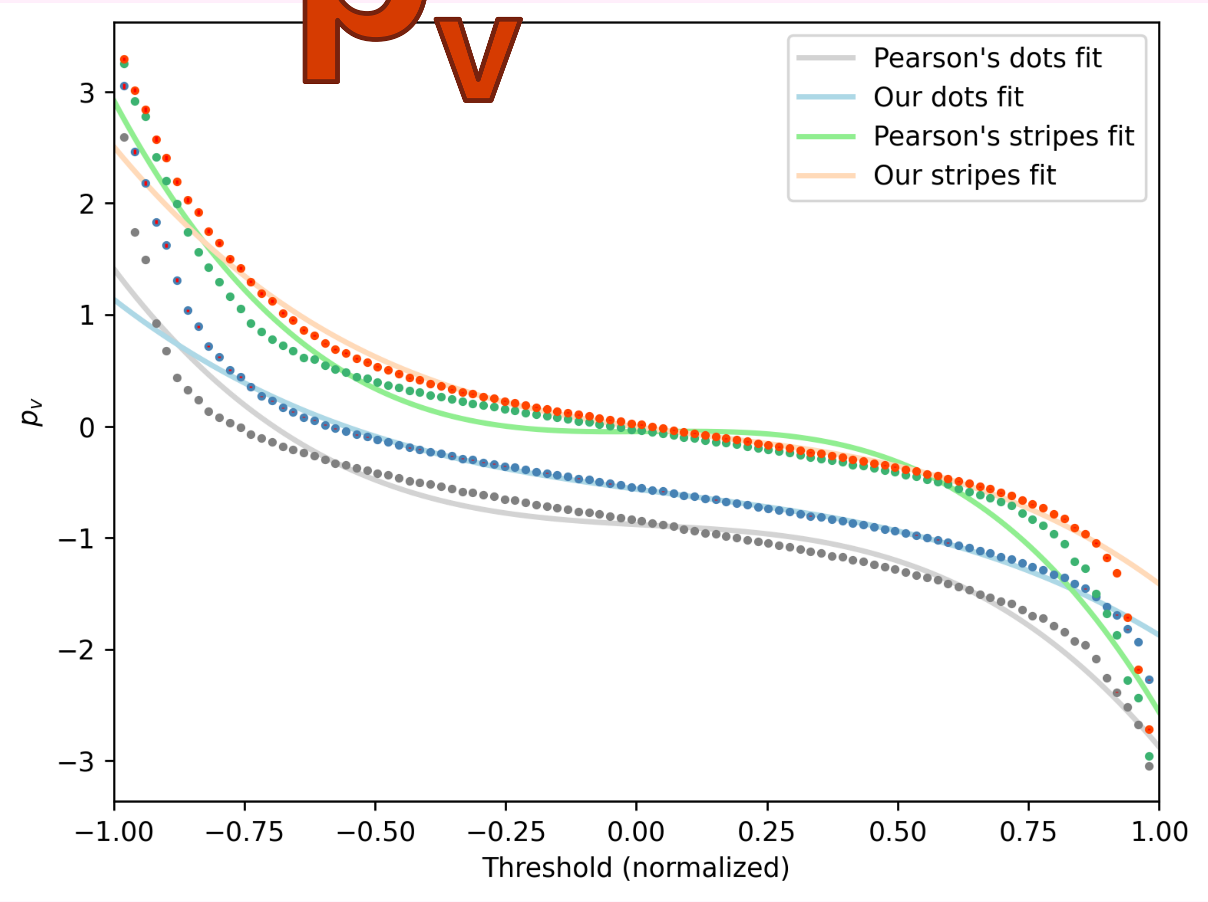

v Pearson dots mean
v Our dots mean
v Pearson stripes mean
v Our stripes mean
1.0
0.8
0.6
0.4
0.2
0.0
−1.00
−0.75
−0.50
−0.25
0.00
0.25
0.50
0.75
1.00

[redacted]

S

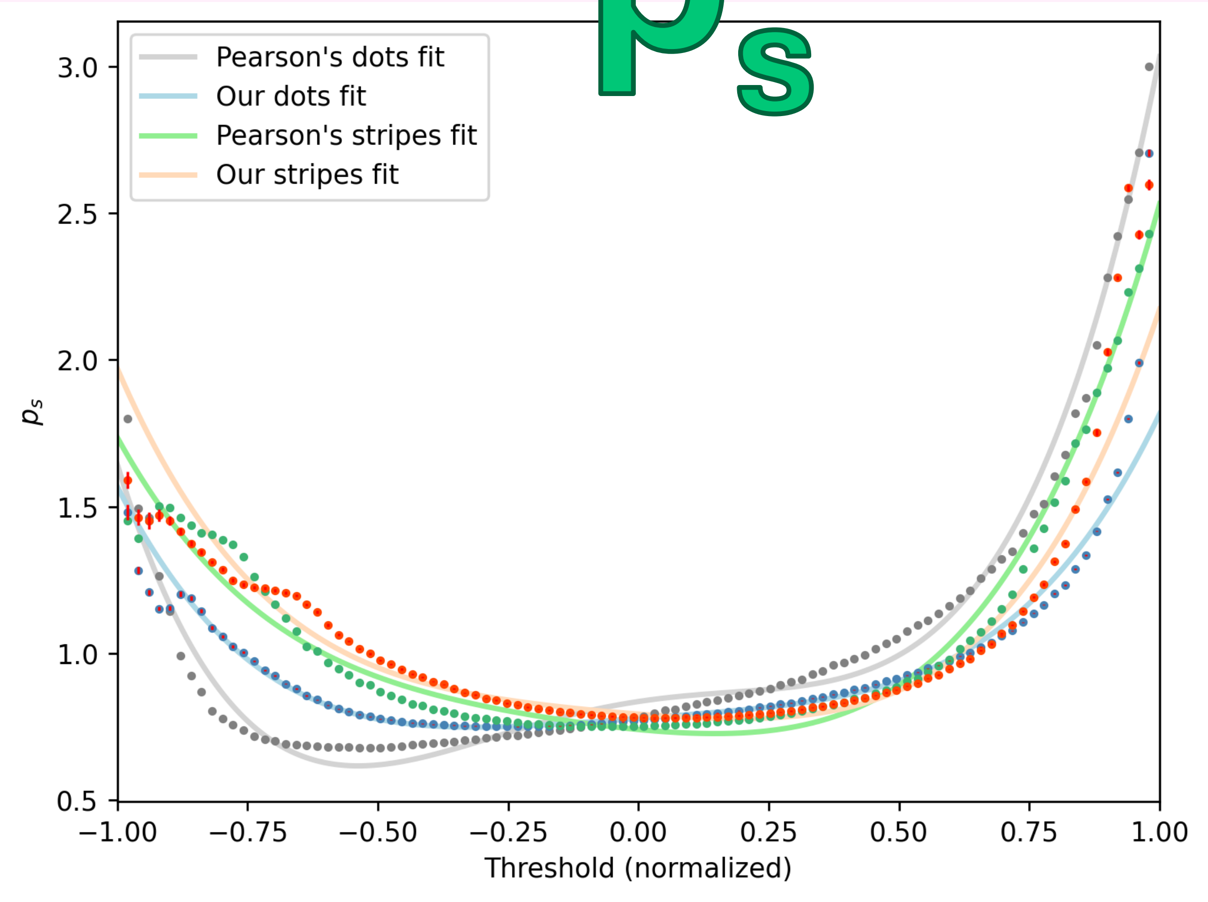

0.20
0.15
0.10
0.05
0.00
−1.00
−0.75
−0.50
−0.25
0.00
0.25
0.50
0.75
1.00
s Pearson dots mean
s Our dots mean
s Pearson stripes mean
s Our stripes mean

p_v

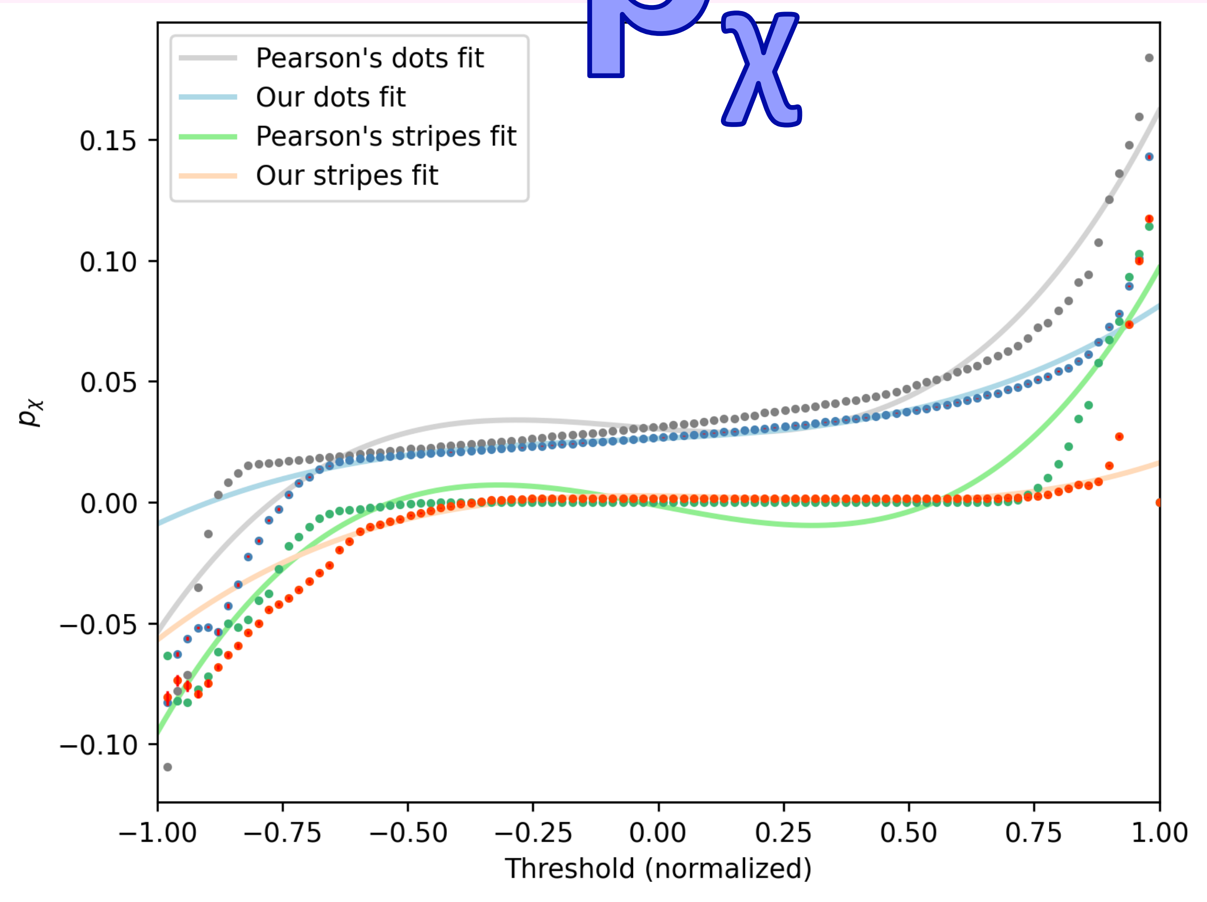


Pearson's dots fit
Our dots fit
Pearson's stripes fit
Our stripes fit
p_v
3
2
1
0
−1
−2
−3
−1.00
−0.75
−0.50
−0.25
0.00
0.25
0.50
0.75
1.00
Threshold (normalized)

p_s


Pearson's dots fit
Our dots fit
Pearson's stripes fit
Our stripes fit
3.0
2.5
2.0
1.5
1.0
0.5
p_s
−1.00
−0.75
−0.50
−0.25
0.00
0.25
0.50
0.75
1.00
Threshold (normalized)

p_χ


Pearson's dots fit
Our dots fit
Pearson's stripes fit
Our stripes fit
0.15
0.10
0.05
0.00
−0.05
−0.10
p_χ
−1.00
−0.75
−0.50
−0.25
0.00
0.25
0.50
0.75
1.00
Threshold (normalized)