System Identification
Lecture 7

Closed-loop identification

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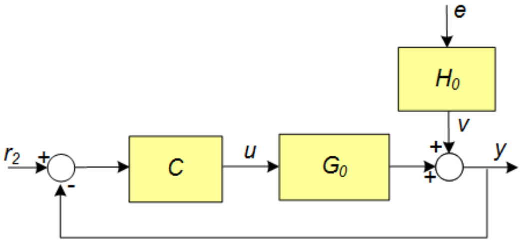
Introduction



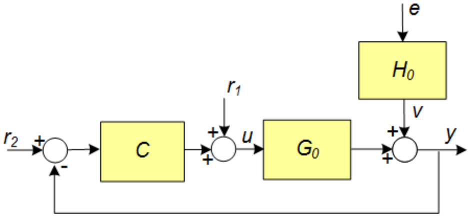
- Usual operation of many plants
- Particularly when processes are unstable
- Sometimes controller intrinsically present (biomedical, economic systems)
- Linearizing effect of controller

r_2 is reference or setpoint signal

Principle difference with the open-loop situation: u and v correlated.

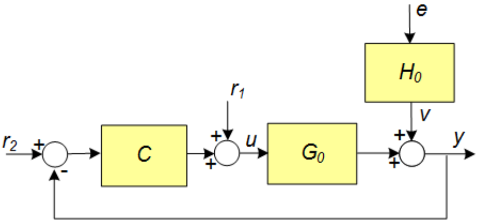
The closed-loop problem

System set-up:

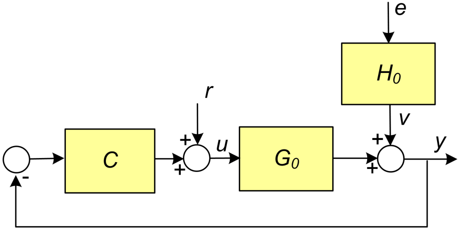


Available data:

- $(u(t), y(t)), \ t = 1, \cdots N$ are measured
- Additionally $r_1(t)$ and/or $r_2(t), \ t = 1, \cdots N$ might be present and/or measured
- Knowledge of $C(q)$ might be available / used
- r and v are assumed to be uncorrelated
- u and v are correlated



For ease of notation:
introduce $r(t) := r_1(t) + C(q)r_2(t)$.



System's equations:

$$\begin{aligned} u &= r - Cy \\ y &= G_0 u + v \end{aligned}$$

leading to:

$$\begin{aligned} y &= \frac{G_0}{1 + CG_0} r + \frac{1}{1 + CG_0} v \\ u &= \frac{1}{1 + CG_0} r - \frac{C}{1 + CG_0} v \end{aligned}$$

Using the sensitivity function: $S_0 := \dfrac{1}{1 + CG_0}$

the system relations become:

$$\begin{aligned} y &= G_0 S_0 \; r + S_0 \; v \\ u &= S_0 \; r - CS_0 \; v \end{aligned}$$

Example ("is there a problem?") – Spectral analysis

Suppose that we make a nonparametric spectral analysis/ETFE estimate on the basis of u and y only

$$\hat{G}(e^{i\omega}) = \frac{\Phi_{yu}(\omega)}{\Phi_u(\omega)};$$

$$\begin{aligned} y(t) &= S_0\left[G_0 r(t) + v(t)\right] \\ u(t) &= S_0\left[r(t) - Cv(t)\right] \end{aligned}$$

Then

$$\begin{aligned} \Phi_u(\omega) &= |S_0|^2[\Phi_r(\omega) + |C|^2\Phi_v(\omega)] \\ \Phi_{yu}(\omega) &= |S_0|^2[G_0\Phi_r(\omega) - C^*\Phi_v(\omega)] \end{aligned}$$

and so:

$$\hat{G} = \frac{\Phi_{yu}(\omega)}{\Phi_u(\omega)} = \frac{G_0\Phi_r(\omega) - C^*\Phi_v(\omega)}{\Phi_r(\omega) + |C|^2\Phi_v(\omega)}$$

$$\hat{G}(e^{i\omega}) = \frac{G_0\Phi_r(\omega) - C^*\Phi_v(\omega)}{\Phi_r(\omega) + |C|^2\Phi_v(\omega)}$$

Limit cases:

$$\begin{array}{llll} \Phi_v(\omega) = 0 & \text{no noise} & \Rightarrow & \hat{G} = G_0 \\ \Phi_r(\omega) = 0 & \text{no excitation} & \Rightarrow & \hat{G} = -1/C \end{array}$$

There are two dynamical relationships between u and y (forward and backward)

A linear combination of the two is estimated dependent on the signal to noise ratio

Note that the model structure used, does not have intrinsic causality
(as predictor models do)

Two principle approaches to identification in closed-loop:

- Direct identification (based on u and y only)
  Which part of the present PE theory can still be used?
- Indirect identification (based on u, y and (r and/or C))

Which results are we going to focus on:

(a) Consistency of $(\hat{G}, \hat{H})$ (situation $\mathcal{S} \in \mathcal{M}$)

(b) Consistency of $\hat{G}$ (situation $G_0 \in \mathcal{G}$)

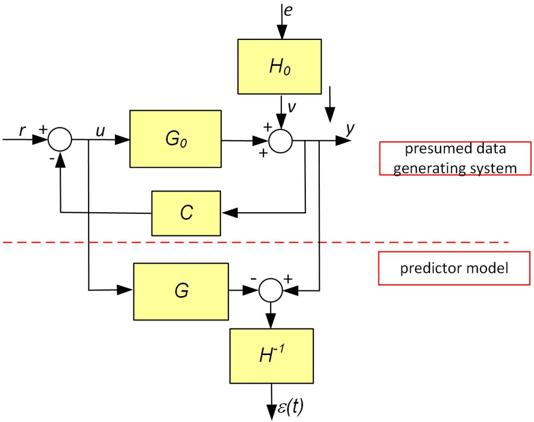
(c) Asymptotic variance

(d) Additional topics: model approximations, validation

Direct identification method

Leading principle:

Use measured y, u and identify a "standard" PE model, irrespective of the presence of C



Available results from Lecture 2:

- **Convergence result**

$$\hat{\theta}_N \xrightarrow[N\to\infty]{w.p.1} \theta^* = \arg\min_\theta \bar{V}(\theta) = \arg\min_\theta \bar{\mathbb{E}}\varepsilon^2(t,\theta)$$

- **Minimum of asymptotic cost function**
  If $C(q)G_0(q)$ and $C(q)G(q,\theta)$ are strictly proper $\forall\theta$
  then: $\bar{V}(\theta) \geq \sigma_e^2$, with equality for $\hat{\theta}$ if

$$\begin{aligned} G(q,\hat{\theta}) &= G_0(q) \\ H(q,\hat{\theta}) &= H_0(q) \end{aligned}$$

- **Consistency / uniqueness**
  If additionally $\mathcal{S} \in \mathcal{M}$ and Z^∞ is informative with respect to $\mathcal{M}$ then

$$\begin{aligned} G(q,\theta^*) &= G_0(q) \\ H(q,\theta^*) &= H_0(q) \end{aligned}$$

Justification of consistency / uniqueness:

Combining:

$$\begin{aligned} y(t) &= G_0 u(t) + H_0 e(t) \\ u(t) &= r(t) - Cy(t) \\ \varepsilon(t,\theta) &= H(\theta)^{-1}[y(t) - G(\theta)u(t)] \end{aligned}$$

delivers:

$$\varepsilon(t,\theta) = \underbrace{\frac{(G_0 - G(\theta))}{H(\theta)(1 + CG_0)}}_{T_{\varepsilon r}(q,\theta)} r(t) + \underbrace{\frac{H_0(1 + CG(\theta))}{H(\theta)(1 + CG_0)}}_{T_{\varepsilon e}(q,\theta)} e(t)$$

If CG_0 and $CG(\theta)$ are strictly proper (i.e. the products contain a delay), then

$T_{\varepsilon e}(q,\theta)$ is *monic*

This requires that there is no algebraic loop in the system: $\lim_{z\to\infty} C(z)G_0(z) = 0$.

$$\varepsilon(t,\theta) = \underbrace{\frac{(G_0 - G(\theta))}{H(\theta)(1+CG_0)}}_{T_{\varepsilon r}(q,\theta)} r(t) + \underbrace{\frac{H_0(1+CG(\theta))}{H(\theta)(1+CG_0)}}_{T_{\varepsilon e}(q,\theta)\ \textit{monic}} e(t)$$

Minimum of $\bar{E}\varepsilon(t,\theta)^2$ is achieved for

- $T_{\varepsilon r}(q,\theta^*) = 0$
- $T_{\varepsilon e}(q,\theta^*) = 1$

provided that r is persistently exciting, of a sufficiently high order.

$$\{T_{\varepsilon r}(q,\theta^*) = 0\} \Rightarrow \left\{ \frac{(G_0 - G(\theta^*))}{H(\theta^*)(1+CG_0)} = 0 \right\} \quad \Rightarrow \quad \{G(q,\theta^*) = G_0(q)\}$$

This, together with

$$\{T_{\varepsilon e}(q,\theta^*) = 1\} \Rightarrow \left\{ \frac{H_0(1+CG(\theta^*))}{H(\theta^*)(1+CG_0)} = 1 \right\}$$

implies that $H(q,\theta^*) = H_0(q)$.

Consistency result for direct method

If $\mathcal{S} \in \mathcal{M}$, r is sufficiently exciting, and there are no algebraic loops in closed-loop system and parametrized models, then

$$G(q,\theta^*) = G_0(q); \qquad H(q,\theta^*) = H_0(q)$$

i.e. $G(q,\hat{\theta}_N)$ and $H(q,\hat{\theta}_N)$ are consistent estimates.

- For the situation $\mathcal{S} \in \mathcal{M}$ the existing consistency result is fully valid for the closed-loop case
- Two remaining questions:
  - What happens in the situation $G_0 \in \mathcal{G}$?
  - Relaxation of the excitation condition on r towards a condition for informative data with repect to $\mathcal{M}$.

The situation $G_0 \in \mathcal{G}$

$$\varepsilon(t,\theta) = \underbrace{\frac{(G_0 - G(\theta))}{H(\theta)(1+CG_0)}}_{T_{\varepsilon r}(q,\theta)} r(t) + \underbrace{\frac{H_0(1+CG(\theta))}{H(\theta)(1+CG_0)}}_{T_{\varepsilon e}(q,\theta)\ \textit{monic}} e(t)$$

- If $H(\theta) \neq H_0$, then bringing $T_{\varepsilon e}(q,\theta)$ "close to monic" needs to be comprised with making $T_{\varepsilon r}(q,\theta)$ "close to 0"; $\bar{V}(\theta) > \sigma_e^2$.
- This is due to the presence of $G(\theta)$ in $T_{\varepsilon e}(q,\theta)$.
- Conclusion that $G(q,\theta^*) = G_0(q)$ cannot be drawn anymore

Direct method - situation $G_0 \in \mathcal{G}$

In the closed-loop situation there is no consistency result for the situation $G_0 \in \mathcal{G}$ anymore.

Relaxation of the persistence of excitation condition on r

Rather than r being persistently exciting, it is sufficient to require that the data set is informative with respect to $\mathcal{M}$ (see lecture 2).

This is covered by the condition that

$$\Phi_z(\omega) > 0 \quad \text{for a sufficient number of frequencies}$$

where

$$\Phi_z(\omega) = \begin{bmatrix} \Phi_u(\omega) & \Phi_{uy}(\omega) \\ \Phi_{yu}(\omega) & \Phi_y(\omega) \end{bmatrix}$$

A data set is informative with respect to the set of all LTI models, if

$$\Phi_z(\omega) > 0 \quad \forall\, \omega$$

Relaxation of the persistence of excitation condition on r

The spectrum condition cannot be applied is r is absent

Note that $\begin{bmatrix} u \\ y \end{bmatrix} = \frac{1}{1+CG_0} \begin{bmatrix} 1 & C \\ G_0 & 1 \end{bmatrix} \begin{bmatrix} r \\ v \end{bmatrix}$.

If $r \equiv 0$ then excitation has to come from $\begin{bmatrix} C \\ 1 \end{bmatrix} v$, i.e. $\Phi_z \sim \begin{bmatrix} |C|^2 & C \\ C^* & 1 \end{bmatrix}$,

Φ_z is always rank deficient

Besides the spectrum condition there are other conditions to guarantee data-informativity:

- A sufficiently complex, nonlinear or time-varying controller[1]

[1] T. Söderström and P. Stoica, *System Identification*, Prentice-Hall, 1989; and M. Gevers et al., IEEE Trans. Autom. Control, *54*, 2828-2840.

Data informativity can be achieved by

- A persistently exciting input signal u in the open-loop situation;
- In the closed-loop situation:
  - Presence of a persistently exciting r, or
  - A controller of sufficiently high order, or
  - A time-varying or nonlinear controller

Question:

Can we use (frequency domain) identification methods that rely on a periodic input signal u?

Unstable plant G_0

For all presented results is required:

- Predictor

$$\varepsilon(t,\theta) = H(\theta)^{-1}G(\theta)u(t) + (1 - H(\theta)^{-1})y(t)$$

  is (uniformly) stable.

For unstable G_0 this can be satisfied if system can be modelled in an ARX or ARMAX structure.
Then unstable dynamics in $G(q,\theta)$ is cancelled out in $H(q,\theta)^{-1}G(q,\theta)$:

$$\varepsilon(t,\theta) = C(\theta)^{-1}B(\theta)u(t) + (1 - \frac{A(\theta)}{C(\theta)}y(t)$$

and predictor filters remain stable for $A(z,\theta)$ having unstable roots.

Summary direct identification method

- Consistent estimates in the situation $\mathcal{S} \in \mathcal{M}$, under excitation conditions
- Excitation conditions can be realized by either presence of an exciting r-signal, or by excitation with the noise through a sufficiently complex controller.
- No consistency when only $G_0 \in \mathcal{G}$
- No "free" excitation of input u; periodic excitation of u is not feasible
- Unstable plants can be modelled only with particular model structures (ARX,ARMAX)
- In situation of consistency, maximum likelihood results remain valid (Cramer Rao lower bound)
- But noise models need to be accurately estimated!
- Results remain valid for nonlinear and/or time-varying controllers

Indirect identification methods

Main step with respect to direct methods:

- Additional use of measured r, and
- Possibly use knowledge of C
- Utilizing the linearity of the closed loop system (linear controller)

Several indirect methods are all closely related.

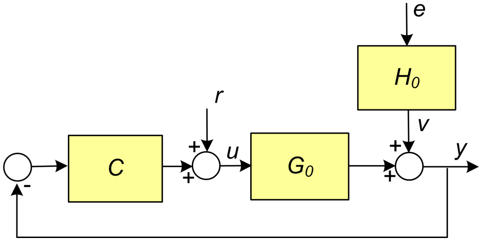
- Explanation of the principle
- Consistency results
- Variation of approaches and algorithms

1. Indirect method (coprime factor approach)

System equations:

$$\begin{aligned} y &= G_0 S_0 \; r + S_0 \; v \\ u &= S_0 \; r - C S_0 \; v \end{aligned}$$

with $S_0 = \dfrac{1}{1 + CG_0}$



The transfers $r \rightarrow \begin{bmatrix} y \\ u \end{bmatrix}$ can be estimated with open-loop methods.

Predictor model: $\varepsilon(t,\theta) = H_{ind}(q,\eta)^{-1} \left[\begin{bmatrix} y \\ u \end{bmatrix} - \begin{bmatrix} G_{yr}(q,\theta) \\ G_{ur}(q,\theta) \end{bmatrix} r(t) \right]$

Two separate SISO problems when $H_{ind}(q,\eta)$ chosen diagonal.

Indirect method (coprime factor approach)

If G_{yr} and G_{ur} are estimated consistently, it follows from

$$\begin{aligned} G_{yr}(q, \theta_0) &= G_0 S_0 \\ G_{ur}(q, \theta_0) &= S_0 \end{aligned}$$

that

$$\hat{G} := \frac{\hat{G}_{yr}(q, \hat{\theta}_N)}{\hat{G}_{ur}(q, \hat{\theta}_N)}$$

is a consistent estimate of G_0.

Consistency result

The prediction error analysis results (Lecture 2) lead to the following statement:

Consistency for indirect method in closed-loop

If model structures $G_{yr}(q,\theta)$ and $G_{ur}(q,\theta)$ are chosen such that they contain the underlying true system dynamics, and $H_{ind}(q,\eta)$ is parametrized independently from θ, then

$$\hat{G} \text{ is a consistent estimate of } G_0$$

provided that r is persistently exciting of a sufficiently high order.

Note: We do not need to estimate a noise model to obtain a consistent estimate of G_0.

Discussion - indirect method

- The consistency results for $G_0 \in \mathcal{G}$ carry over to the closed-loop case
- Effective use is made of external signal r which needs to be measured and persistently exciting of a sufficiently high order
- Because of the postprocessing
$$\hat{G}(q, \hat{\theta}_N) := \frac{\hat{G}_{yr}(q, \hat{\theta}_N)}{\hat{G}_{ur}(q, \hat{\theta}_N)}$$
  it is hard to prespecify the model order of $\hat{G}(q, \hat{\theta}_N)$
  (result of taking quotient of two estimates)
- The method can also be applied on the basis of non-parametric frequency response estimates of G_{yr} and G_{ur}
- Any desired excitation signal can be used for r (e.g. periodic)

Algorithms for indirect methods

There are different variations of indirect methods,

- Focussing on controlling the model set and model order of the final model $\hat{G}$:
  - Projection methods (two-stage method / instrumental variable (IV) method)
- Focussing on constraining the models to be stabilized by C
  - Dual-Youla parametrization

1. Projection methods (two-stage / IV)

Approach

- Decompose the input signal u into

$$u(t) = u^r(t) + u^e(t)$$

  i.e. the components of u that result from r and e respectively.
  Since r and e are uncorrelated this implies that both u-components are uncorrelated too.

- Then

$$y(t) = G_0 u^r(t) + \underbrace{G_0 u^e(t) + H_0 e(t)}_{\text{disturbance terms}}.$$

- Identify G_0 (and possibly a noise model), based on input u^r and output y. This is basically an open-loop problem.

1. Projection methods (two-stage / IV)

$$y(t) = G_0 u^r(t) + \underbrace{G_0 u^e(t) + H_0 e(t)}_{\text{disturbance terms}}.$$

- Typically G_0 is identified with a parametric model
- The model order of $\hat{G}$ can directly be prespecified
- By "shifting" part of signal u to "noise", the SNR will decrease $\rightarrow$ higher variance

How to construct (an estimate of) u^r?

- Identify the transfer function G_{ur} on the basis of r and u (open-loop problem)
- Simulate:

$$\hat{u}^r(t) = \hat{G}_{ur}(q) r(t)$$

- Use $\hat{u}^r(t)$ as input signal in the identification of G_0.

1. Projection methods (two-stage / IV)

Instrumental variable (IV) method:

Alternative identification criterion:

$$\hat{\theta}_N = \mathrm{sol}_\theta \{ \sum_{t=1}^{N} \zeta(t)\varepsilon(t,\theta) = 0 \}$$

with $\zeta(t) \in \mathbb{R}^d$, the instrument vector.

LS-ARX method is IV with $\zeta(t) = \varphi(t) =$

$$[-y(t-1) \;\; -y(t-2) \;\; \cdots -y(t-n_a) \;\; u(t) \;\; u(t-1) \;\; \cdots u(t-n_b+1)]^T$$

Indeed for LS-ARX:

$$\sum_t \varphi(t)\varepsilon(t,\hat{\theta}_N) = \sum_t \varphi(t)[y(t) - \varphi^T(t)\hat{\theta}_N] = 0$$

1. Projection methods (two-stage / IV)

For consistency of $G(, q, \hat{\theta}_N)$, ζ should typically satisfy:

- ζ is correlated to the input and output signals of the system to be modelled;
- ζ is uncorrelated to the output noise

If ζ chosen as delayed versions of the reference signal (uncorrelated with v), then it satisfies the conditions.

So e.g.:

$$\zeta(t) = [r(t)\ r(t-1)\ r(t-2)\ \cdots\ r(t-d)]^T$$

1. Projection methods (two-stage / IV)

IV is usually applied with an ARX (linear regression) model structure:

$$\varepsilon(t,\theta) = y(t) - \varphi^T(t)\theta$$

Then

$$\hat{\theta}_N^{IV} = \left[\frac{1}{N}\sum_{t=1}^{N}\zeta(t)\varphi^T(t)\right]^{-1}\left[\frac{1}{N}\sum_{t=1}^{N}\zeta(t)y(t)\right].$$

If the real system satisfies: $y(t) = \varphi^T(t)\theta_0 + w(t)$, then

$$\hat{\theta}_N^{IV} = \theta_0 + \left[\frac{1}{N}\sum_{t=1}^{N}\zeta(t)\varphi^T(t)\right]^{-1}\left[\frac{1}{N}\sum_{t=1}^{N}\zeta(t)w(t)\right]$$

1. Projection methods (two-stage / IV)

Summary

- Open-loop methods applied to a closed-loop situation
- Consistency for $G_0 \in \mathcal{G}$
- Less emphasis on noise modeling
- Higher variance than the direct method

2. Dual Youla method

Main principle:

Parametrize the plant model $G(\theta)$ within the class of all linear plant models that are stabilized by a given (and known) controller C.

2. Dual Youla method

Coprime factorization over $\mathbb{R}\mathcal{H}_\infty$ (Vidyasagar, 1985).
Let G_0 be a (possibly unstable) system, and let N, D be stable rational transfer functions in $\mathbb{R}\mathcal{H}_\infty$. Then the pair (N, D) is a (right) coprime factorization (rcf) of G_0 over $\mathbb{R}\mathcal{H}_\infty$, if

(a) $G_0 = ND^{-1}$, and

(b) there exist stable transfer functions $X, Y \in \mathbb{R}\mathcal{H}_\infty$ such that $XN + YD = I$ $\square$

Coprime factors do not have unstable zeros that cancel in the quotient.

2. Dual-Youla method

Dual-Youla Parametrization

Hansen, Franklin (1989), Lee et al. (1993), Schrama (1991)

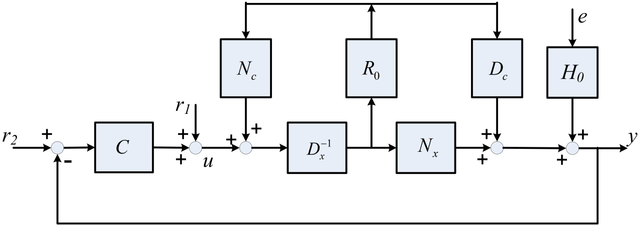
Dual-Youla parametrization

Let $C = N_c/D_c$ be a coprime factorization of an LTI controller C, and let N_x/D_x be a coprime factorization of any plant G_x that is stabilized by C. Then an LTI plant G_0 is stabilized by C if and only if there exists a stable transfer function R_0 such that

$$G_0 = \frac{N_x + D_c R_0}{D_x - N_c R_0}$$

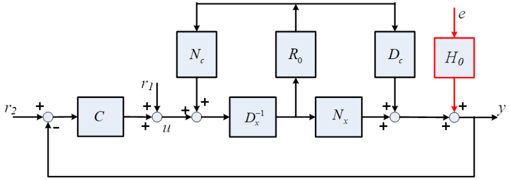
$R(\theta)$: parametrization of all LTI plants that are stabilized by a given controller.

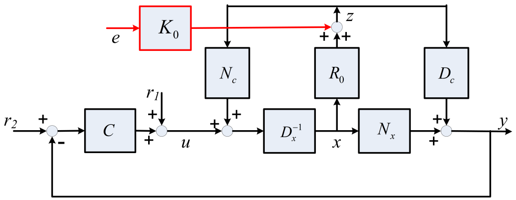
2. Dual Youla method



G_0 is replaced by $$\frac{N_x + D_c R_0}{D_x - N_c R_0}$$

2. Dual Youla method





Equivalent in view of signals u and y, if $K_0 = \frac{H_0}{D_c(1+CG_0)}$.

2. Dual Youla method

- With $K_0 = \frac{H_0}{D_c(1+CG_0)}$ it appears that the block diagram is equivalent to the standard one, in view of the signals y, u, r and e.
- The indicated signals z and x satisfy

$$\begin{aligned} x(t) &= (D_x + CN_x)^{-1}[u(t) + C(q)y(t)] \\ z(t) &= (D_c + G_x N_c)^{-1}[y(t) - G_x(q)u(t)] \end{aligned}$$

  and therefore can be calculated on the basis of measured data y, u, and knowledge of C and a choice of G_x.
- Since $u + Cy = r_1 + Cr_2 = r$, x and e are uncorrelated.

2. Dual Youla method

As a result we can write "new" system equations, taking the form (see block diagram):

$$z(t) = R_0(q)x(t) + K_0(q)e(t)$$

with x and e uncorrelated, and

$$R_0 = \frac{(G_0 - G_x)D_x}{D_c(1 + CG_0)} \text{ (see next slide)} \qquad K_0 = \frac{H_0}{D_c(1 + CG_0)}.$$

R_0 and K_0 can be identified as in an open-loop problem.

After estimating $\hat{R}$, a plant model is obtained according to Youla: $\hat{G} = \dfrac{N_x + D_c\hat{R}}{D_x - N_c\hat{R}}$

2. Dual Youla method

Expression of R_0:

From the (dual) Youla parametrization:

$$G_0 = \frac{N_x + D_c R_0}{D_x - N_c R_0}$$

follows

$$\begin{aligned} (D_x - N_c R_0) G_0 &= N_x + D_c R_0 \\ D_x G_0 - N_x &= (D_c + N_c G_0) R_0 \\ R_0 &= \frac{(G_0 - G_x) D_x}{D_c (1 + C G_0)} \end{aligned}$$

2. Dual Youla method

Summary

- Any stable $\hat{R}$ leads to a model $\hat{G}$ that is stabilized by C
- Explicit prior knowledge of the controller C is used
- Estimation properties of indirect method are kept (if R_0 is the model set for $R(\theta)$)
- Model order of $\hat{G}$ is hard to control
- Method allows handling of unstable plants and controllers

2. Dual Youla method

Particular example

If C is a stable controller, a possible choice is:

$$C = C/1; \qquad G_x = 0/1$$

$N_c = C$, $D_c = 1$, $N_x = 0$, $D_x = 1$, and consequently

$$R_0 = \frac{G_0}{1 + CG_0}$$
$$z(t) = y(t); \quad x(t) = r(t)$$

This leads to:

$$\hat{G} = \frac{\hat{R}}{1 - C\hat{R}}$$

(called the "classical" indirect method of closed-loop ID).

Consistency

Direct method

Consistency of (G, H) can be obtained in the situation $\mathcal{S} \in \mathcal{M}$.

- Condition on excitaton of $\begin{bmatrix} u \\ y \end{bmatrix}$

Indirect methods

Consistency of G can be obtained in the situation $G_0 \in \mathcal{G}$.

- Condition on excitation of r

Asymptotic variance

Direct method

- The variance results of the open-loop situation remain valid, provided that we have consistency ($\mathcal{S} \in \mathcal{M}$).
- This includes the Maximum Likelihood properties of the estimates (minimum variance asymptotically)
- Asymptotic-in-order-of-G result for $n, N \to \infty$, and noise model in model set:

$$var\,\hat{G}(e^{i\omega}) \sim \frac{n}{N}\frac{\Phi_v(\omega)}{\Phi_{u^r}(\omega)}$$

- In the case $\mathcal{S} \in \mathcal{M}$ and a fixed and correct noise model, the full input signal u is used for variance reduction.
  (For $n \to \infty$ all noise excitation is required to estimate H)

Asymptotic variance

Indirect methods

- Typically the reference signal r is used as input for identification;
- Typical variance result (asymptotic in model order n and in N):

$$var\,\hat{G}(e^{i\omega}) \sim \frac{n}{N}\frac{\Phi_v(\omega)}{\Phi_{u^r}(\omega)}$$

  valid in the situation $G_0 \in \mathcal{G}$.

- Only the reference-part of the input signal contributes to variance reduction.
- For finite model orders:
  neglecting u^e as input signal contributes to a worse SN-ratio.

Asymptotic variance

Reasoning behind asymptotic variance result

Asymptotic $(n, N \to \infty)$ result is:

$$E \begin{pmatrix} \hat{G}(e^{i\omega}) - G_0(e^{i\omega}) \\ \hat{H}(e^{i\omega}) - H_0(e^{i\omega}) \end{pmatrix} \begin{pmatrix} \\ \end{pmatrix}^*$$

$$\sim \frac{n}{N} \Phi_v(\omega) \cdot \begin{bmatrix} \Phi_u(\omega) & \Phi_{eu}(\omega) \\ \Phi_{ue}(\omega) & \sigma_e^2 \end{bmatrix}^{-1}.$$

Using

$$\Phi_u = \Phi_u^r + \Phi_u^e$$

and direct use of the system's equations delivers:

$$var(\hat{G}(e^{i\omega})) \quad \sim \quad \frac{n}{N} \frac{\Phi_v(\omega)}{\Phi_u^r(\omega)}$$

Bias expressions - Approximations

Bias in direct closed-loop identification

$$\hat{\theta}_N \to \theta^* = \arg\min_{\theta} \bar{V}(\theta); \qquad \bar{V}(\theta) = \bar{\mathbb{E}}\varepsilon_f^2(t,\theta)$$

By Parsseval, $\bar{V}(\theta) =$ (see slide 13)

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left\{\left|\frac{G_0 - G(\theta)}{1+CG_0}\right|^2\Phi_r + \left|\frac{1+CG(\theta)}{1+CG_0}\right|^2\Phi_v\right\}\frac{|L|^2}{|H(\theta)|^2}d\omega$$

No explicit (tunable) approximation criterion for $G(\theta)$,
since $G(\theta)$ appears in both terms of the integrand

Bias in indirect closed-loop identification

Bias expressions for “all” indirect alternatives

$$\theta^* = \arg\min_{\theta} \int_{-\pi}^{\pi} \left| \frac{G_0}{1+CG_0} - \frac{G(\theta)}{1+CG(\theta)} \right|^2 \frac{\Phi_r |L|^2}{|K|^2} d\omega$$

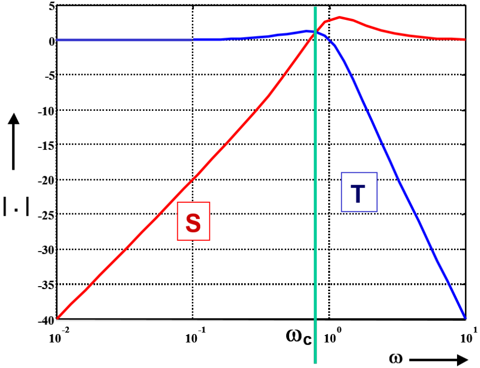
(with slight variations) and K a fixed noise model.

Closed-loop properties of the plant ($G_0 S_0$) are best approximated.

Note

$$\frac{G_0}{1+CG_0} - \frac{G(\theta)}{1+CG(\theta)} = \frac{G_0 - G(\theta)}{(1+CG_0)(1+CG(\theta))}$$

“Additive error” is weighted with sensitivity of plant and model.



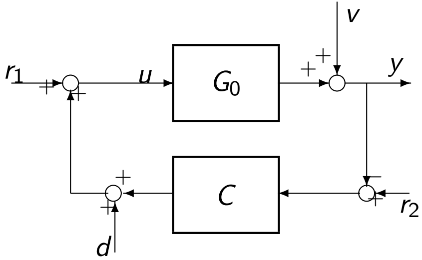
Typical curve for Bode magnitude plot of sensitivity function S_0 and related complementary sensitivity $G_0 C/(1 + CG_0)$.

Model errors are highly weighted around the cross-over frequency of the closed-loop.

Main properties of the different methods

	Direct	Indirect		
		Indirect (CF)	Projection	dual Youla
Consistency $(\hat{G}, \hat{H})$	$+$	$+$	$+$	$+$
Consistency $\hat{G}$	$-$	$+$	$+$	$+$
Tunable bias	$-$	$+$	$+$	$+$
Fixed model order	$+$	$-$	$+$	$-$
Variance	$+$	$-$	$-$	$-$
C assumed known	no	no	no	yes
C assumed linear	no	yes	yes	yes
$G(\hat{\theta}_N)$, C stable	no	no	no	yes

For noise disturbed controller output:



- If C known: use $u + Cy$ as external signal
  $\rightarrow$ d can effectively be used as external signal reducing the variance
- If C unknown, r measured: d acts as additional disturbance

Knowledge of C is more informative than knowledge of r

Model validation in closed-loop

- For all indirect methods:
  validation with correlation tests as in open-loop
- For direct method: Careful with test on $R_{\varepsilon u}(\tau)$.

$$
\begin{aligned}
\varepsilon(t,\theta) &= H(\theta)^{-1}[(G_0 - G(\theta))u(t) + H_0 e(t)] \\
R_{\varepsilon u}(\tau) &= H(\theta)^{-1}[G_0 - G(\theta)]R_u(\tau) + H(\theta)^{-1}H_0 R_{eu}(\tau).
\end{aligned}
$$

Can the second term influence the cross-correlation test?

$R_{eu}(\tau)$ will have a contribution for $\tau < 0$ only.
The second term can then have a contribution for $\tau > 0$ if the filter $H(\theta)^{-1}H_0$ has dynamics, i.e. when the noise model is incorrect.

For the direct method the residual tests should not be interpreted independently (validation of $\hat{G}$ and $\hat{H}$ simultaneously).

Summary - Closed-loop identification

- Parametric models can be consistently identified with a direct method
- but only through modelling G and H simultaneously ($\mathcal{S} \in \mathcal{M}$)
- Indirect methods can provide consistent estimates in the situation $G_0 \in \mathcal{G}$
- Direct methods lead to smaller variance (ML-properties)
- In closed-loop identification, the frequency area around the cross-over frequency of the closed-loop system, typically is most dominantly present in the plant data/models.
- Indirect methods rely on linearity of the closed-loop system, while direct methods can handel nonlinear/time-varying controllers