Chapter 8

Subspace Model Identification

In this chapter you learn:

- How to derive the data equation that is the key to subspace identification.
- To exploit the special structure of the data equation for impulse and step input signals to identify a state space model using subspace methods.
- To understand subspace identification for general input signals.
- How to use instrumental variables in subspace identification to deal with process and measurement noise.
- To derive subspace identification schemes for different noise models.
- How to use the RQ factorization for a computationally efficient implementation of subspace identification schemes.
- To relate different subspace identification schemes via the solution of a least squares problem.

8.1 Introduction

The problem of identifying an LTI state space model from input and output measurements of a dynamic system, analyzed in the previous chapter, is readdressed in this chapter via a completely different approach. The approach we take is indicated in the literature [1], [2], [3] as the class of subspace identification methods. These methods are based on the fact that by storing the input and output data into structured matrices, it is possible to retrieve a subspace of a certain matrix that is related to the system matrices of the signal generating state space model. In this chapter we explain how this subspace can be used to determine the system matrices up to a similarity transformation. Geometrically, the estimation of the system matrices follows from a relocation of a basis of the subspace of interest, which is the space spanned by the state sequence or the space spanned by the columns of the extended observability matrix.

Unlike the identification algorithms presented in the previous chapter, in subspace identification, there is no need to parameterize the model. Furthermore, the system model is obtained in a non-iterative fashion. In fact, computing the model boils down to computing an RQ factorization, an SVD and solving a least squares problem. Thus, the problem of performing a nonlinear optimization is circumvented. These properties of subspace identification make it an attractive alternative to the prediction error methods presented in the previous chapter. However, the statistical analysis of the subspace methods is much more complicated than the statistical analysis of the prediction error methods. This is because subspace identification methods do not explicitly minimize a cost function to obtain the system matrices. Although some results on the statistical analysis of subspace methods have been obtained, it remains a relevant research topic [4], [5], [6], [7], [8].

To explain subspace identification we need some theory from linear algebra. Therefore, in this chapter we rely on the matrix results reviewed in chapter 2.

In section 8.2 we describe the basics of subspace identification. In this section we only consider noise free systems. First, we describe subspace identification for impulse and step input signals and then switch to more general input sequences. Section 8.3 describes subspace identification in the presence of white measurement noise. To deal with more general noise disturbances, we introduce the concept of instrumental variables in section 8.4. The instrumental variables approach is used in section 8.5 to deal with colored measurement noise and in section 8.6 to deal with white process and white measurement noise simultaneously. In the latter section it is also shown that subspace identification for the case of white process and white measurement noise can be written as a least squares problem.

8.2 Subspace Model Identification for Deterministic Systems

In this section we describe subspace identification for deterministic LTI systems. i.e. LTI systems that are not disturbed by noise. Let such a system be given by

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) && (8.1)\\ y(k) &= Cx(k) + Du(k) && (8.2) \end{aligned}$$

where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$, and $y(k) \in \mathbb{R}^\ell$. It is assumed that this system is minimal, i.e. observable and controllable (see section 3.4.3). The goal is to determine the system matrices (A, B, C, D) up to a similarity transformation (see section 3.4.2) from a finite number of measurements of the input $u(k)$ and output $y(k)$. An important and critical step prior to the design (and use) of subspace identification algorithms is to find an appropriate relationship between the measured data sequences, on one hand, and the matrices that define the model, on the other hand. This relation will be derived in section 8.2.1. We proceed by describing subspace identification for the special cases where the input is an impulse sequence (section 8.2.2) and where the input is a step sequence (section 8.2.3). Finally, we describe subspace identification for more general input sequences (section 8.2.4).

8.2.1 The Data Equation

In section 3.4.2 we showed that the state of the system (8.1)–(8.2) with initial state $x(0)$ at time instant k is given by

$$x(k) = A^k x(0) + \sum_{i=0}^{k-1} A^{k-i-1} B u(i) \tag{8.3}$$

By invoking equation (8.2), we can specify the following relationship between the input data batch $\{u(k)\}_{k=0}^{s-1}$ and the output data batch $\{y(k)\}_{k=0}^{s-1}$,

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ \vdots \\ y(s-1) \end{bmatrix} = \underbrace{\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{s-1} \end{bmatrix}}_{\mathcal{O}_s} x(0) + \underbrace{\begin{bmatrix} D & 0 & 0 & \cdots & 0 \\ CB & D & 0 & \cdots & 0 \\ CAB & CB & D & & 0 \\ \vdots & & \ddots & \ddots & \\ CA^{s-2}B & CA^{s-3}B & \cdots & CB & D \end{bmatrix}}_{\mathcal{T}_s} \begin{bmatrix} u(0) \\ u(1) \\ u(2) \\ \vdots \\ u(s-1) \end{bmatrix} \tag{8.4}$$

where s is some arbitrary positive integer. To use this relation in subspace identification, it is necessary to take $s > n$, as will be explained below. In the sequel, the matrix $\mathcal{O}_s$ will be referred to as the *extended observability matrix*. Equation (8.4) relates vectors, derived from the input and output data sequences and the (unknown) initial condition $x(0)$, to the matrices $\mathcal{O}_s$ and $\mathcal{T}_s$, derived from the system matrices (A, B, C, D). Since the underlying system is time-invariant, we can relate the same matrices $\mathcal{O}_s$ and $\mathcal{T}_s$ to shifted versions of the input and output vectors used in (8.4). Consider for example a shift over k samples, we obtain:

$$\begin{bmatrix} y(k) \\ y(k+1) \\ \vdots \\ y(k+s-1) \end{bmatrix} = \mathcal{O}_s x(k) + \mathcal{T}_s \begin{bmatrix} u(k) \\ u(k+1) \\ \vdots \\ u(k+s-1) \end{bmatrix} \tag{8.5}$$

Now we can combine the relationships (8.4) and (8.5) for different shifts, as permitted by available input-output samples, to obtain:

data equation

$$\begin{aligned} \begin{bmatrix} y(0) & y(1) & \cdots & y(N-1) \\ y(1) & y(2) & \cdots & y(N) \\ \vdots & \vdots & \ddots & \vdots \\ y(s-1) & y(s) & \cdots & y(N+s-2) \end{bmatrix} &= \mathcal{O}_s \begin{bmatrix} x(0) & x(1) & \cdots & x(N-1) \end{bmatrix} \\ &+ \mathcal{T}_s \begin{bmatrix} u(0) & u(1) & \cdots & u(N-1) \\ u(1) & u(2) & \cdots & u(N) \\ \vdots & \vdots & \ddots & \vdots \\ u(s-1) & u(s) & & u(N+s-2) \end{bmatrix} \end{aligned} \tag{8.6}$$

where in general we have $s << N$. The above equation is referred to as the *data equation*. The matrices constructed from the input and output data are constant along the block anti-diagonals. A matrix with this property is called a block *Hankel matrix*. For ease of notation we define the block Hankel matrix constructed from $y(k)$ as follows

$$Y_{i,s,N} = \begin{bmatrix} y(i) & y(i+1) & \cdots & y(i+N-1) \\ y(i+1) & y(i+2) & \cdots & y(i+N) \\ \vdots & \vdots & \ddots & \vdots \\ y(i+s-1) & y(i+s) & \cdots & y(i+N+s-2) \end{bmatrix}$$

The first entry of the subscript of $Y_{i,s,N}$ refers to the time index of its top left entry, the second refers to the number of block-rows and the third refers to the number of columns. The block Hankel matrix constructed from $u(k)$ is defined in a similar way. We also define

$$X_{i,N} = \begin{bmatrix} x(i) & x(i+1) & \cdots & x(i+N-1) \end{bmatrix}$$

These definitions allow us to denote the data equation (8.6) in a compact way:

data equation in compact matrix form

$$Y_{0,s,N} = \mathcal{O}_s X_{0,N} + \mathcal{T}_s U_{0,s,N} \tag{8.7}$$

The data equation (8.7) relates matrices constructed from the data to matrices constructed from the system matrices. We will explain that this representation allows us to derive information on the system matrices (A, B, C, D) from data matrices, such as $Y_{0,s,N}, U_{0,s,N}$. This idea is explored first for the special case where the input is an impulse.

8.2.2 Identification using Impulse Input Sequences

The special case that the input is an impulse allows to explain the basic idea of subspace identification. The goal is to recover the system matrices (A, B, C, D) from the input and output data. The first step is to use the data equation (8.7) to estimate the column space of the extended observability matrix $\mathcal{O}_s$. From this subspace we can then estimate the matrices A and C up to a similarity transformation. The second step is to determine the matrix B up to a similarity transformation and to determine the matrix D. As we will see, the subspace identification method for impulse input sequences is very similar to the Ho-Kalman realization algorithm [9] based on lemma 3.4, described in section 3.4.4.

Deriving the Column Space of the Observability Matrix

We consider the system (8.1)–(8.2) with a single input, i.e, $m = 1$. This input signal equals an impulse sequence:

$$u(k) = \begin{cases} 1 & \text{for } k = 0 \\ 0 & \text{for } k \neq 0 \end{cases} \tag{8.8}$$

The data equation (8.7) for this special input sequence takes the form:

$$Y_{0,s,N+1} = \mathcal{O}_s X_{0,N+1} + \mathcal{T}_s \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \tag{8.9}$$

Therefore, we have

$$Y_{1,s,N} = \mathcal{O}_s X_{1,N} \tag{8.10}$$

This equation immediately shows that the columns of the matrix $Y_{1,s,N}$ are linear combinations of the columns of the matrix $\mathcal{O}_s$. This means that the column space of the matrix $Y_{1,s,N}$ is contained in that of $\mathcal{O}_s$, i.e. $\text{range}(Y_{1,s,N}) \subseteq \text{range}(\mathcal{O}_s)$. It is important to realize that from (8.9) we cannot conclude that the column spaces of $Y_{1,s,N}$ and $\mathcal{O}_s$ are equal, because the linear combinations of the columns of $\mathcal{O}_s$ can be such that the rank of $Y_{1,s,N}$ is lower than the rank of $\mathcal{O}_s$.

However, if $x(0) = 0$, $s > n$, and $N \geq s$ it can be shown that the column spaces of $Y_{1,s,N}$ and $\mathcal{O}_s$ are equal. To see this, observe that because $x(0) = 0$, the matrix $X_{1,N}$ can be written as

$$X_{1,N} = \begin{bmatrix} B & AB & A^2B & \cdots & A^{N-1}B \end{bmatrix} = \mathcal{C}_N$$

and therefore,

$$Y_{1,s,N} = \mathcal{O}_s \mathcal{C}_N \tag{8.11}$$

Since $s > n$, $N \geq s$ and the system is minimal, $\text{rank}(\mathcal{O}_s) = \text{rank}(\mathcal{C}_N) = n$. Application of Sylvester's inequality (lemma 2.1 on page 17) to equation (8.11) shows that $\text{rank}(Y_{1,s,N}) = n$ and thus $\text{range}(Y_{1,s,N}) = \text{range}(\mathcal{O}_s)$.

Computing the System Matrices

An SVD of the matrix $Y_{1,s,N}$ allows us to determine the column space of $Y_{1,s,N}$ (see section 2.5). And because the column space of $Y_{1,s,N}$ equals that of $\mathcal{O}_s$, it can be used to determine the system matrices A and C up to an unknown similarity transformation T, in a similar way as in the Ho-Kalman realization algorithm outlined in section 3.4.4. Denote the SVD of $Y_{1,s,N}$ by

$$Y_{1,s,N} = U_n \Sigma_n V_n^T \tag{8.12}$$

with $\Sigma_n \in \mathbb{R}^{n\times n}$ and $\text{rank}(\Sigma_n) = n$, then

$$U_n = \mathcal{O}_s T = \begin{bmatrix} CT \\ CTT^{-1}AT \\ \vdots \\ CT\left(T^{-1}AT\right)^{s-1} \end{bmatrix} = \begin{bmatrix} C_T \\ C_T A_T \\ \vdots \\ C_T A_T^{s-1} \end{bmatrix}$$

Hence, the matrix C_T equals the first ℓ rows of U_n, i.e. $C_T = U_n(1:\ell,:)$. The matrix A_T is computed by solving the following overdetermined equation, which due to the condition $s > n$ has a unique solution.

$$U_n(1:(s-1)\ell,:)A_T = U_n(\ell+1:s\ell,:)$$

To determine the matrix B_T observe that

$$\begin{aligned} \Sigma_n V_n^T &= T^{-1}\mathcal{C}_N \\ &= \begin{bmatrix} T^{-1}B & T^{-1}ATT^{-1}B & \cdots & (T^{-1}AT)^{N-1}T^{-1}B \end{bmatrix} \\ &= \begin{bmatrix} B_T & A_T B_T & \cdots & A_T^{N-1}B_T \end{bmatrix} \end{aligned}$$

So B_T equals the first column of the matrix $\Sigma_n V_n^T$. The matrix $D_T = D$ equals $y(0)$ as can be seen from equation (8.2), bearing in mind that $x(0) = 0$.

Example 8.1 (Impulse response subspace identification)

Consider the LTI system (8.1)–(8.2) with system matrices given by

$$\begin{aligned} A &= \begin{bmatrix} 1.69 & 1 \\ -0.96 & 0 \end{bmatrix} & B &= \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} \\ C &= \begin{bmatrix} 1 & 0 \end{bmatrix} & D &= 0 \end{aligned}$$

The first 100 data points of the impulse response of this system are shown in figure 8.1. These data points are used to construct the matrix $Y_{1,s,N}$ with $s = 3$. From the SVD of this matrix we determine the matrices A, B and C up to a similarity transformation T. The computed singular values are approximately equal to 15.9425, 6.9597 and 0, and the system matrices that we obtain up to four digits are

$$\begin{aligned} A_T &\approx \begin{bmatrix} 0.8529 & 1.0933 \\ -0.2250 & 0.8371 \end{bmatrix} & B_T &\approx \begin{bmatrix} 3.4155 \\ 1.2822 \end{bmatrix} \\ C_T &\approx \begin{bmatrix} 0.5573 & -0.7046 \end{bmatrix} \end{aligned}$$

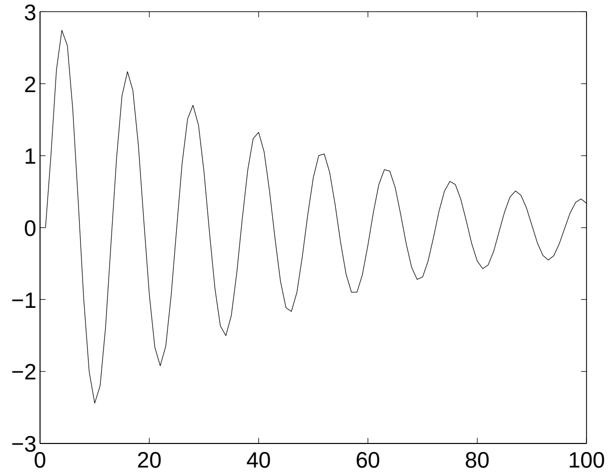

Figure 8.1: *Impulse response of the system used in example 8.1.*

It is easy to verify that A_T has the same eigenvalues as the matrix A, and that the system (A_T, B_T, C_T) has the same impulse response as the original one.

The subspace identification method for a single impulse input signal, presented above, can easily be extended to multiple input signals. Let $Y^i_{1,s,N}$ denote the block Hankel matrix of the impulse response $y^i(k)$ with respect to the i-th input, then the system matrices (A, B, C, D) can be computed (up to an unknown similarity transformation) using an SVD as explained above, but now with the matrix $Y_{1,s,N}$ in equation 8.12 replaced by the matrix:

$$\begin{bmatrix} y^1(1) & \cdots & y^m(1) & y^1(2) & \cdots & y^m(2) & \cdots & y^1(N) & \cdots & y^m(N) \\ y^1(2) & \cdots & y^m(2) & \vdots & & \vdots & & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ y^1(s-1) & \cdots & y^m(s-1) & y^1(s) & \cdots & y^m(s) & \cdots & y^1(N+s-1) & \cdots & y^m(N+s-1) \end{bmatrix}$$

8.2.3 Identification using Step Input Sequences

In this section we explain subspace identification for the special case where the input is a step. As in the previous section the goal is to recover the system matrices (A, B, C, D) from the input and output data, and to achieve this goal we start by estimating the column space of the extended observability matrix.

Deriving the Column Space of the Observability Matrix

We consider the system (8.1)–(8.2) with only one input ($m = 1$), which equals a step sequence, given by:

$$u(k) = \begin{cases} 1 & \text{for } k \geq 0 \\ 0 & \text{for } k < 0 \end{cases} \tag{8.13}$$

For this special case, we can write the data equation (8.7) as

$$Y_{0,s,N} = \mathcal{O}_s X_{0,N} + \mathcal{T}_s(\mathbb{E}_s \mathbb{E}_N^T) \tag{8.14}$$

where $\mathbb{E}_N \in \mathbb{R}^N$ denotes the vector with all entries equal to one. The question now is: given only the matrices $Y_{0,s,N}$ and $\mathbb{E}_s\mathbb{E}_N^T$ can we retrieve the column space of $\mathcal{O}_s$? To solve this problem, we would like to decompose $Y_{0,s,N}$ into a sum of two components that each bear information on one of the two original components of the sum in equation (8.14). One such (orthogonal) decomposition is,

$$\begin{aligned} Y_{0,s,N} &= Y_{0,s,N}\left(I_N - \frac{\mathbb{E}_N\mathbb{E}_N^T}{N}\right) + Y_{0,s,N}\frac{\mathbb{E}_N\mathbb{E}_N^T}{N} \\ &= \mathcal{O}_s X_{0,N}\left(I_N - \frac{\mathbb{E}_N\mathbb{E}_N^T}{N}\right) + Y_{0,s,N}\frac{\mathbb{E}_N\mathbb{E}_N^T}{N} \end{aligned} \tag{8.15}$$

where the second equation follows from equation (8.14) and from the fact that $\mathbb{E}_N^T\mathbb{E}_N = N$ and therefore

$$\mathcal{T}_s(\mathbb{E}_s\mathbb{E}_N^T)\left(I_N - \frac{\mathbb{E}_N\mathbb{E}_N^T}{N}\right) = 0$$

Now we can conclude that

$$Y_{0,s,N}\left(I_N - \frac{\mathbb{E}_N{\mathbb{E}_N}^T}{N}\right) = \mathcal{O}_s X_{0,N}\left(I_N - \frac{\mathbb{E}_N\mathbb{E}_N^T}{N}\right) \tag{8.16}$$

In other words, the columns of the matrix on the left hand side of this equation are linear combinations of the columns of the extended observability matrix, and thus

$$\text{range}\left(Y_{0,s,N}\left(I_N - \frac{\mathbb{E}_N{\mathbb{E}_N}^T}{N}\right)\right) \subseteq \text{range}(\mathcal{O}_s)$$

However, to use the matrix $Y_{0,s,N}\left(I_N - \frac{\mathbb{E}_N{\mathbb{E}_N}^T}{N}\right)$ to determine the matrices A_T and C_T in a similar way as described for the impulse input case in section 8.2.2, we need to show that its column space equals that of the extended observability matrix. This can be shown asymptotically for $N \to \infty$ and is summarized in the next theorem.

Theorem 8.1 *Let the single input, multi output system given by* (8.1)–(8.2) *be minimal, A be asymptotically stable (a.s.), the input be step as defined in* (8.13)*, $x(0) = 0$, and $s > n$ then:*

1. $\displaystyle\lim_{N\to\infty} \frac{1}{N}\sum_{k=0}^{N-1} x(k) = (I_n - A)^{-1}B$

2. $\displaystyle\lim_{N\to\infty} Y_{0,s,N}\left(I_N - \frac{\mathbb{E}_N\mathbb{E}_N^T}{N}\right) = -\mathcal{O}_s(I_n - A)^{-1}\begin{bmatrix} B & AB & A^2B & \cdots \end{bmatrix}$

3. $\operatorname{rank}\left(\lim_{N\to\infty} Y_{0,s,N}\left(I_N - \frac{\mathbb{E}_N\mathbb{E}_N^T}{N}\right)\right) = \operatorname{rank}(\mathcal{O}_s)$

□

Proof: Since the system is asymptotically stable., the state converges to a stationary value $\overline{x}$, which satisfies: $\overline{x} = A\overline{x} + B$ and thus

$$\overline{x} = (I - A)^{-1}B$$

Because the state sequence is deterministic, its mean value also equals:

$$\overline{x} = \lim_{N\to\infty} \frac{1}{N}\sum_{k=0}^{N-1} x(k)$$

what concludes the proof of part 1. For part 2 note that since $(\mathbb{E}_N^T\mathbb{E}_N) = N$, we have

$$\begin{aligned}
\lim_{N\to\infty} Y_{0,s,N}\left(I_N - \frac{\mathbb{E}_N\mathbb{E}_N^T}{N}\right) &= \lim_{N\to\infty} \mathcal{O}_s X_{0,s,N}\left(I_N - \frac{\mathbb{E}_N\mathbb{E}_N^T}{N}\right) \\
&= \lim_{N\to\infty} \mathcal{O}_s\left(X_{0,s,N} - \frac{1}{N}X_{0,s,N}\mathbb{E}_N\mathbb{E}_N^T\right) \\
&= \lim_{N\to\infty} \mathcal{O}_s\left(X_{0,s,N} - \frac{1}{N}\sum_{k=0}^{N-1} x(k)\mathbb{E}_N^T\right)
\end{aligned}$$

and using the result of part 1 and the explicit expression for $X_{0,s,N}$ in terms of the system matrices (A, B), the right hand side of the last equation equals:

$$\begin{aligned}
&= \mathcal{O}_s\left(\left[\ 0 \mid B \mid AB + B \mid A^2B + AB + B \mid \cdots \quad \right] - (I - A)^{-1}B\mathbb{E}_{N'}^T\right) \\
&= \mathcal{O}_s\left[\ -(I - A)^{-1}B \mid B - (I - A)^{-1}B \mid AB + B - (I - A)^{-1}B \mid \cdots \quad \right] \\
&= \mathcal{O}_s\left[\ -(I - A)^{-1}B \mid B - B - AB - A^2B - \cdots \right. \\
&\qquad\qquad \left. \cdots \mid AB + B - B - AB - A^2B - \cdots\ \right] \\
&= -\mathcal{O}_s(I - A)^{-1}\left[\ B \mid AB \mid A^2B \mid \cdots \quad \right]
\end{aligned}$$

where we have made use of the series expansion,

$$(I - A)^{-1} = I + A + A^2 + A^3 + \cdots$$

which converges for A asymptotically stable. The proof of part 3 follows by application of Sylvester's inequality (lemma 2.1 on page 17) and keeping in mind that the system is minimal and that $s > n$. ■

Computing the System Matrices

Given the SVD

$$\lim_{N\to\infty} Y_{0,s,N}\left(I_N - \frac{\mathbb{E}_N\mathbb{E}_N^T}{N}\right) = U_n\Sigma_nV_n^T$$

we can determine the matrices A_T and C_T from U_n in a similar way as outlined in section 8.2.2.

We can determine the matrix B_T from $\Sigma_n V_n^T$. We have for an unknown similarity transformation T

$$\Sigma_n V_n^T = -T^{-1}(I_n - A)^{-1} \begin{bmatrix} B & AB & A^2B & \cdots \end{bmatrix}$$

Let $A_T = T^{-1}AT$ and $B_T = T^{-1}B$, then:

$$\Sigma_n V_n^T \quad = \quad (I_n - A_T)^{-1} \begin{bmatrix} B_T & A_T B_T & A_T^2 B_T & \cdots \end{bmatrix}$$

Therefore, an estimate of B_T can be formed by taking the m first columns of the matrix $\Sigma_n V_n^T$ pre-multiplied by $(I_n - A_T)$.

The matrix $D_T = D$ equals $y(0)$ as indicated by equation (8.2) on page 209, since $x(0) = 0$.

Example 8.2 (Step response subspace identification)

Consider the LTI system (8.1)–(8.2) with system matrices given in example 8.1 on page 212. The first 100 data points of the step response of this system are shown in figure 8.2. These data points are used to construct the matrix $Y_{0,s,N}$ with $s = 3$. From the SVD of the matrix

$$Y_{0,s,N} \left(I_N - \frac{\mathbb{E}_N \mathbb{E}_N^T}{N} \right)$$

we determine the matrices A, B and C up to a similarity transformation T. The computed singular values are approximately equal to 29.7884, 13.7837 and 0, and the system matrices that we obtain up to four digits are

$$\begin{aligned} A_T &\approx \begin{bmatrix} 0.8267 & 1.0930 \\ -0.2254 & 0.8633 \end{bmatrix} \quad B_T \approx \begin{bmatrix} 3.3753 \\ 1.3845 \end{bmatrix} \\ C_T &\approx \begin{bmatrix} 0.5783 & -0.6875 \end{bmatrix} \end{aligned}$$

It is easy to verify that A_T has the same eigenvalues as the matrix A, and that the system (A_T, B_T, C_T) has the same step response as the original one.

The subspace identification method for a single step input signal, presented above, can easily be extended to multiple input signals, in a similar way as has been done for impulse inputs at the end of section 8.2.2.

8.2.4 Identification using General Input Sequences

In the sections 8.2.2 and 8.2.3 we showed that when an impulse or step input is applied to the system, we can exploit the special structure of the block Hankel matrix $U_{0,s,N}$ in equation (8.7) on page 210, to get rid of the influence of the input and retrieve a matrix that has a column space equal to the column space of $\mathcal{O}_s$. In this section the retrieval of the column space of $\mathcal{O}_s$ is discussed for more general input sequences.

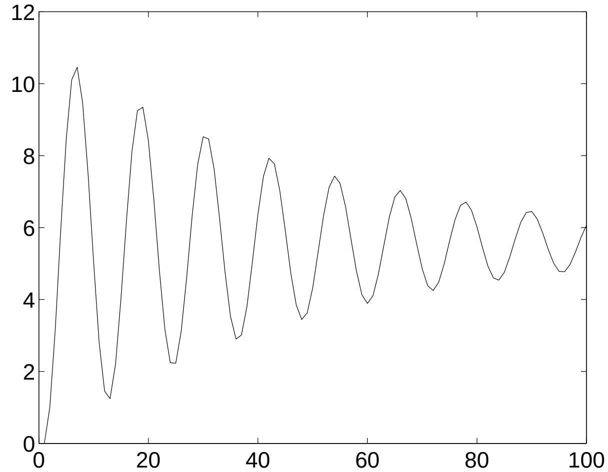

Figure 8.2: *Step response of the system used in example 8.2.*

Deriving the Column Space of the Observability Matrix

Consider the multi-variable system (8.1)–(8.2). We would like to find the column space of the extended observability matrix. If we know the matrix $\mathcal{T}_s$ we can obtain an estimate of this column space by subtracting $\mathcal{T}_s U_{0,s,N}$ from $Y_{0,s,N}$ followed by an SVD. But, since the system is unknown, $\mathcal{T}_s$ is also unknown and this trick is not appropriate. However, we can instead apply this trick using an estimate of the matrix $\mathcal{T}_s$. A possible estimate of $\mathcal{T}_s$ can be obtained from the following least squares problem [3]

$$\min_{\mathcal{T}_s} ||Y_{0,s,N} - \mathcal{T}_s U_{0,s,N}||_F^2$$

The solution is given by (see also exercise 8.4)

$$\hat{\mathcal{T}}_s = Y_{0,s,N} U_{0,s,N}^T (U_{0,s,N} U_{0,s,N}^T)^{-1}$$

Now we get

$$Y_{0,s,N} - \hat{\mathcal{T}}_s U_{0,s,N} = Y_{0,s,N}\Big(I_N - U_{0,s,N}^T (U_{0,s,N} U_{0,s,N}^T)^{-1} U_{0,s,N}\Big) = Y_{0,s,N} \Pi_{U_{0,s,N}}^{\perp}$$

where the matrix

$$\Pi_{U_{0,s,N}}^{\perp} = I_N - U_{0,s,N}^T (U_{0,s,N} U_{0,s,N}^T)^{-1} U_{0,s,N} \tag{8.17}$$

is a projection matrix referred to as the orthogonal projection onto the column space of $U_{0,s,N}$, because it has the property $U_{0,s,N}\Pi_{U_{0,s,N}}^{\perp} = 0$. To ensure that the inverse of $U_{0,s,N}U_{0,s,N}^T$ exists, we need enough samples of the input and output such that $N > sm$ holds, and we have to use a special input sequence that ensures that the matrix $U_{0,s,N}U_{0,s,N}^T$ is of full rank. This restricts the type of input sequences that we can use to identify the system. This means that not every input can be used to identify a system. In section 9.2.4 we will introduce the notion of persistency of excitation to characterize conditions the input has to satisfy to be able to identify the system under consideration. In this chapter we will see that in order to determine the system matrices using subspace identification some rank conditions on certain data matrices are necessary. In section 9.2.4 these rank conditions are related to the persistency of excitation of the input. For now, we assume that the input is such that the matrix $U_{0,s,N}U_{0,s,N}^T$ is of full rank.

Since $U_{0,s,N}\Pi_{U_{0,s,N}}^{\perp} = 0$ we can derive from equation (8.7) that

$$Y_{0,s,N}\Pi_{U_{0,s,N}}^{\perp} = \mathcal{O}_s X_{0,N}\Pi_{U_{0,s,N}}^{\perp} \tag{8.18}$$

Note the similarity to equation (8.16) on page 214. We have in fact removed the influence of the input on the output. What remains is the response of the system due to the state. Equation (8.18) shows that the column space of the matrix $Y_{0,s,N}\Pi_{U_{0,s,N}}^{\perp}$ is contained in the column space of the extended observability matrix. The next thing is, of course, to show that these column spaces are equal. This is equivalent to showing that $Y_{0,s,N}\Pi_{U_{0,s,N}}^{\perp}$ is of rank n. We have the following result.

Lemma 8.1 *Given the state space system (8.1)–(8.2). If the input $u(k)$ is such that*

$$\text{rank}\left(\begin{bmatrix} X_{0,N} \\ U_{0,s,N} \end{bmatrix}\right) = n + sm \tag{8.19}$$

then

$$\text{rank}\left(Y_{0,s,N}\Pi_{U_{0,s,N}}^{\perp}\right) = n$$

□

Proof: Equation (8.19) implies

$$\begin{bmatrix} X_{0,N}X_{0,N}^T & X_{0,N}U_{0,s,N}^T \\ U_{0,s,N}X_{0,N}^T & U_{0,s,N}U_{0,s,N}^T \end{bmatrix} > 0$$

With the Schur complement (lemma 2.2 on page 17) it follows that

$$\text{rank}\left(X_{0,N}X_{0,N}^T - X_{0,N}U_{0,s,N}^T(U_{0,s,N}U_{0,s,N}^T)^{-1}U_{0,s,N}X_{0,N}^T\right) = n \tag{8.20}$$

Using the fact that $\Pi_{U_{0,s,N}}^{\perp}(\Pi_{U_{0,s,N}}^{\perp})^T = \Pi_{U_{0,s,N}}^{\perp}$, we can write

$$Y_{0,s,N}\Pi_{U_{0,s,N}}^{\perp}(\Pi_{U_{0,s,N}}^{\perp})^T Y_{0,s,N}^T = Y_{0,s,N}\Pi_{U_{0,s,N}}^{\perp}Y_{0,s,N}^T$$

and with equation (8.18) also

$$\begin{aligned} Y_{0,s,N}\Pi_{U_{0,s,N}}^{\perp}Y_{0,s,N}^T &= \mathcal{O}_s X_{0,N}\Pi_{U_{0,s,N}}^{\perp}X_{0,N}^T\mathcal{O}_s^T \\ &= \mathcal{O}_s(X_{0,N}X_{0,N}^T - X_{0,N}U_{0,s,N}^T(U_{0,s,N}U_{0,s,N}^T)^{-1}U_{0,s,N}X_{0,N}^T)\mathcal{O}_s^T \end{aligned}$$

Because $\text{rank}(\mathcal{O}_s) = n$, an application of Sylvester's inequality (lemma 2.1 on page 17) shows that

$$\text{rank}\left(Y_{0,s,N}\Pi_{U_{0,s,N}}^{\perp}Y_{0,s,N}^T\right) = n$$

This completes the proof. ■

Hence, for inputs that satisfy (8.19) we have

$$\mathrm{range}\left(Y_{0,s,N}\Pi^{\perp}_{U_{0,s,N}}\right) = \mathrm{range}\left(\mathcal{O}_s\right) \tag{8.21}$$

Therefore, to be able to recover the column space of the extended observability matrix, we have to excite the system with an input sequence that satisfies the rank condition (8.19). It is easy to verify that this condition is satisfied if the input is a white noise sequence (see exercise 8.5). In section 9.2.4 we examine this rank condition in more detail.

Example 8.3 (Input and state rank condition)

Consider the state equation

$$x(k+1) = \frac{1}{2}x(k) + u(k)$$

If we take $x(0) = 0$, $u(0) = u(2) = 1$, and $u(1) = u(3) = 0$ then it is easy to see that

$$\begin{bmatrix} x(0) & x(1) & x(2) \\ u(0) & u(1) & u(2) \\ u(1) & u(2) & u(3) \end{bmatrix} = \begin{bmatrix} 0 & 1 & \frac{1}{2} \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Because this matrix has full row rank, adding columns to it will not change its rank. Therefore with $x(0) = 0$, the rank condition

$$\mathrm{rank}\left(\begin{bmatrix} X_{0,N} \\ U_{0,2,N} \end{bmatrix}\right) = 3$$

is satisfied for any finite sequence $u(k)$, $0 \leq k \leq N$ for which $u(0) = u(2) = 1$ and $u(1) = u(3) = 0$.

Efficient Implementation Using the RQ Factorization

We have seen above that the column space of the matrix $Y_{0,s,N}\Pi^{\perp}_{U_{0,s,N}}$ equals the column space of the extended observability matrix. Therefore, an SVD of this matrix can be used to determine the matrices A_T and C_T. However, this is not attractive from a computational point of view, because the matrix $Y_{0,s,N}\Pi^{\perp}_{U_{0,s,N}}$ has N columns and typically N is large, and because we need to construct the matrix $\Pi^{\perp}_{U_{0,s,N}}$ which is of size N and requires the computation of a matrix inverse, as shown by equation (8.17) on page 217. For a more efficient implementation both with respect to the number of flops and the required memory storage the explicit calculation of the product $Y_{0,s,N}\Pi^{\perp}_{U_{0,s,N}}$ can be avoided when using the following RQ factorization [10].

$$\begin{bmatrix} U_{0,s,N} \\ Y_{0,s,N} \end{bmatrix} = \begin{bmatrix} R_{11} & 0 & 0 \\ R_{21} & R_{22} & 0 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix} \tag{8.22}$$

where $R_{11} \in \mathbb{R}^{sm \times sm}$ and $R_{22} \in \mathbb{R}^{s\ell \times s\ell}$. The relation between this RQ factorization and the matrix $Y_{0,s,N}\Pi^{\perp}_{U_{0,s,N}}$ is given in the following lemma.

Lemma 8.2 *Given the RQ factorization (8.22), we have* $Y_{0,s,N}\Pi^{\perp}_{U_{0,s,N}} = R_{22}Q_2$. □

Proof: From the RQ factorization (8.22) we can express $Y_{0,s,N}$ as:

$$Y_{0,s,N} = R_{21}Q_1 + R_{22}Q_2$$

Further, it follows from the orthogonality of the matrix $\begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix}$ that,

$$\begin{bmatrix} Q_1^T & Q_2^T & Q_3^T \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix} = I_N \quad \Rightarrow \quad Q_1^TQ_1 + Q_2^TQ_2 + Q_3^TQ_3 = I_N$$

and

$$\begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix} \begin{bmatrix} Q_1^T & Q_2^T & Q_3^T \end{bmatrix} = I_N \quad \Rightarrow \quad \begin{cases} Q_iQ_j^T = 0 & i \neq j \\ Q_iQ_i^T = I \end{cases}$$

With $U_{0,s,N} = R_{11}Q_1$, and equation (8.17) on page 217 we can derive

$$\begin{aligned} \Pi^{\perp}_{U_{0,s,N}} &= I_N - Q_1^TR_{11}^T(R_{11}Q_1Q_1^TR_{11}^T)^{-1}R_{11}Q_1 \\ &= I_N - Q_1^TQ_1 \\ &= Q_2^TQ_2 + Q_3^TQ_3 \end{aligned}$$

And therefore

$$Y_{0,s,N}\Pi^{\perp}_{U_{0,s,N}} = R_{21}Q_1Q_2^TQ_2 + R_{22}Q_2Q_2^TQ_2 = R_{22}Q_2$$

which completes the proof. ■

From this lemma it follows that the column space of the matrix $Y_{0,s,N}\Pi^{\perp}_{U_{0,s,N}}$ equals the column space of the matrix $R_{22}Q_2$. Furthermore, we have:

Theorem 8.2 *Given the system (8.1)–(8.2) and the RQ factorization (8.22). If $u(k)$ is such that equation (8.19) holds, we have*

$$\textit{range}(\mathcal{O}_s) = \textit{range}(R_{22}) \tag{8.23}$$

□

Proof: From lemma 8.1 on page 218 we derived equation (8.21) on page 219. Combining this with the result of lemma 8.2 yields

$$\text{range}(Y_{0,s,N}\Pi^{\perp}_{U_{0,s,N}}) = \text{range}(R_{22}Q_2)$$

Using the fact that Q_2 has full row rank, an application of Sylvester's inequality (lemma 2.1 on page 17) completes the proof. ■

This shows that to compute the column space of $\mathcal{O}_s$, we do not need to store the matrix Q_2 which is much larger than the matrix R_{22}: $Q_2 \in \mathbb{R}^{s\ell \times N}$, $R_{22} \in \mathbb{R}^{s\ell \times s\ell}$ with typically $N >> s\ell$. Furthermore, to compute the matrices A_T and C_T we can instead of using an SVD of the matrix $Y_{0,s,N}\Pi^{\perp}_{U_{0,s,N}}$ having N columns, compute an SVD of the matrix R_{22} which has only $s\ell$ columns.

Computing the System Matrices

Theorem 8.2 shows that an SVD of the matrix R_{22} allows us to determine the column space of $\mathcal{O}_s$ provided that the input satisfies equation (8.19) on page 218. Hence, given the SVD

$$R_{22} = U_n \Sigma_n V_n^T$$

with $\Sigma_n \in \mathbb{R}^{n \times n}$ and $\text{rank}(\Sigma_n) = n$, we can compute A_T and C_T as outlined in section 8.2.2.

The matrices B_T and D_T, along with the initial state $x_T(0) = T^{-1}x(0)$ can be computed by solving a least squares problem. Given the matrices A_T and C_T, the output of the system (8.1)–(8.2) on page 209 depends linearly on the matrices B_T and D_T, as can be seen from the expression:

$$y(k) = C_T A_T^k x_T(0) + \left(\sum_{\tau=0}^{k-1} u(\tau)^T \otimes C_T A_T^{k-\tau-1} \right) \text{vec}(B_T) + \big(u(k)^T \otimes I_\ell\big) \text{vec}(D_T)$$

This equation is a direct result of theorem 7.1 on page 171. Let $\hat{A}_T$ and $\hat{C}_T$ denote the estimates of A_T and C_T computed in the previous step. Now taking

$$\phi(k)^T = \left[\hat{C}_T \hat{A}_T^k \quad \left(\sum_{\tau=0}^{k-1} u(\tau)^T \otimes \hat{C}_T \hat{A}_T^{k-\tau-1} \right) \quad \big(u(k)^T \otimes I_\ell\big) \right] \tag{8.24}$$

and

$$\theta = \begin{bmatrix} x_T(0) \\ \text{vec}(B_T) \\ \text{vec}(D_T) \end{bmatrix} \tag{8.25}$$

We can solve for θ in a least squares setting

$$\min_\theta \frac{1}{N} \sum_{k=0}^{N-1} ||y(k) - \phi(k)^T \theta||^2 \tag{8.26}$$

as described in example (7.6) on page 177.

8.3 Subspace Identification with White Measurement Noise

In the previous discussion the system was assumed to be noise free. In practice this of course rarely happens. Therefore, we now take a look at systems of which the output is perturbed by noise. Let this noise be denoted by $v(k)$ then the system that we consider can be written as

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) && (8.27) \\ y(k) &= Cx(k) + Du(k) + v(k) && (8.28) \end{aligned}$$

The data equation for this system is similar to (8.7) on page 210 and reads

$$Y_{i,s,N} = \mathcal{O}_s X_{i,N} + \mathcal{T}_s U_{i,s,N} + V_{i,s,N}$$

where $V_{i,s,N}$ is a block Hankel matrix constructed from the sequence $v(k)$.

In this section we consider the case where the noise $v(k)$ is a white noise sequence that is uncorrelated with the input $u(k)$. Later on, in section 8.5 we consider nonwhite noise sequences. It turns out that if $u(k)$ and $v(k)$ are ergodic sequences, and we take the limit for $N \to \infty$, despite the presence of the white noise at the output, the column space of the matrix $\frac{1}{\sqrt{N}} Y_{i,s,N} \Pi^{\perp}_{U_{i,s,N}}$ equals the column space of $\mathcal{O}_s$ provided that $u(k)$ satisfies equation (8.19) on page 218. This is summarized in the next lemma.

Lemma 8.3 *Given the system (8.27)–(8.28) with $u(k)$ an ergodic sequence such that equation (8.19) holds. Let the matrix M_X be defined as,*

$$M_X = \lim_{N\to\infty} \frac{1}{N} X_{i,N} \Pi^{\perp}_{U_{i,s,N}} X^T_{i,N}$$

Let $v(k)$ be an ergodic white noise sequence that is uncorrelated with $u(k)$, and that satisfies,

$$\lim_{N\to\infty} \frac{1}{N} V_{i,s,N} V^T_{i,s,N} = \sigma^2 I_{s\ell}$$

Let the non-zero singular values of the matrix $\mathcal{O}_s M_X \mathcal{O}_s$ be stored as the entries of the diagonal matrix Σ_n^2, then the SVD of the matrix $\lim_{N\to\infty} \frac{1}{N} Y_{i,s,N} \Pi^{\perp}_{U_{i,s,N}} Y^T_{i,s,N}$ is given by,

$$\lim_{N\to\infty} \frac{1}{N} Y_{i,s,N} \Pi^{\perp}_{U_{i,s,N}} Y^T_{i,s,N} = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_n^2 + \sigma^2 I_n & 0 \\ 0 & \sigma^2 I_{s\ell-n} \end{bmatrix} \begin{bmatrix} U_1^T \\ U_2^T \end{bmatrix} \tag{8.29}$$

and,

$$\text{range}(U_1) = \text{range}(\mathcal{O}_s) \tag{8.30}$$

□

Proof: From the data equation it follows that

$$Y_{i,s,N} \Pi^{\perp}_{U_{i,s,N}} = \mathcal{O}_s X_{i,N} \Pi^{\perp}_{U_{i,s,N}} + V_{i,s,N} \Pi^{\perp}_{U_{i,s,N}}$$

Using the fact that $\Pi^{\perp}_{U_{i,s,N}} (\Pi^{\perp}_{U_{i,s,N}})^T = \Pi^{\perp}_{U_{i,s,N}}$, we can write

$$Y_{i,s,N} \Pi^{\perp}_{U_{i,s,N}} (\Pi^{\perp}_{U_{i,s,N}})^T Y^T_{i,s,N} = Y_{i,s,N} \Pi^{\perp}_{U_{i,s,N}} Y^T_{i,s,N}$$

and also

$$\begin{aligned} Y_{i,s,N} \Pi^{\perp}_{U_{i,s,N}} Y^T_{i,s,N} &= \mathcal{O}_s X_{i,N} \Pi^{\perp}_{U_{i,s,N}} X^T_{i,N} \mathcal{O}_s^T + \mathcal{O}_s X_{i,N} \Pi^{\perp}_{U_{i,s,N}} V^T_{i,s,N} \\ &\quad + V_{i,s,N} \Pi^{\perp}_{U_{i,s,N}} X^T_{i,N} \mathcal{O}_s^T + V_{i,s,N} \Pi^{\perp}_{U_{i,s,N}} V^T_{i,s,N} \end{aligned}$$

Since $u(k)$ is uncorrelated with the white noise sequence $v(k)$ we have

$$\lim_{N\to\infty} \frac{1}{N} U_{i,s,N} V_{i,s,N}^T = 0 \qquad \lim_{N\to\infty} \frac{1}{N} X_{i,N} V_{i,s,N}^T = 0$$

which, using equation (8.17) on page 217, implies

$$\lim_{N\to\infty} \frac{1}{N} X_{i,N} \Pi_{U_{i,s,N}}^{\perp} V_{i,s,N}^T = 0$$

We also have

$$\lim_{N\to\infty} \frac{1}{N} V_{i,s,N} \Pi_{U_{i,s,N}}^{\perp} V_{i,s,N}^T = \lim_{N\to\infty} \frac{1}{N} V_{i,s,N} V_{i,s,N}^T$$

Using these special properties of $V_{i,s,N}$ yields

$$\lim_{N\to\infty} \frac{1}{N} Y_{i,s,N} \Pi_{U_{i,s,N}}^{\perp} Y_{i,s,N}^T = \lim_{N\to\infty} \frac{1}{N} \left(\mathcal{O}_s X_{i,N} \Pi_{U_{i,s,N}}^{\perp} X_{i,N}^T \mathcal{O}_s^T + V_{i,s,N} V_{i,s,N}^T \right) \tag{8.31}$$

With the property of the noise $v(k)$ and the definition of the matrix M_X, equation (8.31) becomes:

$$\lim_{N\to\infty} \frac{1}{N} Y_{i,s,N} \Pi_{U_{i,s,N}}^{\perp} Y_{i,s,N}^T = \mathcal{O}_s M_X \mathcal{O}_s^T + \sigma^2 I_{s\ell} \tag{8.32}$$

Since the input satisfies equation (8.19) on page 218, equation (8.20) on page 218 in the proof of lemma 8.1 holds, and therefore the matrix M_X has rank n. Given the following SVD

$$\mathcal{O}_s M_X \mathcal{O}_s^T = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_n^2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U_1^T \\ U_2^T \end{bmatrix}$$

we can also write equation (8.32) as equation (8.29). It is important to observe that equation (8.29) is a valid SVD of the matrix

$$\lim_{N\to\infty} \frac{1}{N} Y_{i,s,N} \Pi_{U_{i,s,N}}^{\perp} Y_{i,s,N}^T$$

since all diagonal entries of Σ_n differ from zero. Finally, application of Sylvester's inequality (lemma 2.1 on page 17) to equations (8.32) and (8.29) shows that the column space of $\mathcal{O}_s$ equals the column space of U_1. ■

From this lemma we conclude that the column space $\mathcal{O}_s$ does not change in the presence of white noise at the output. Therefore, we can still obtain estimates of the system matrices from an SVD of the matrix $Y_{i,s,N} \Pi_{U_{i,s,N}}^{\perp}$. We can use the algorithm described in section 8.2.4 to obtain estimates of the system matrices (A_T, B_T, C_T, D_T). The requirement $N \to \infty$ means that these estimates are asymptotically unbiased. In practice the system matrices will be estimated on the basis of a finite number of data samples; obviously the more data samples available, the better these estimates will be.

We can again use an RQ factorization for efficient implementation, because of the following theorem.

Theorem 8.3 *Given the system (8.27)–(8.28) and the RQ factorization (8.22). If $v(k)$ is an ergodic white noise sequence with variance σ^2 that is uncorrelated with $u(k)$, and $u(k)$ is an ergodic sequence such that equation (8.19) holds, and let the matrix M_X be defined as in lemma 8.3, then,*

$$\left(\lim_{N\to\infty} \frac{1}{N} R_{22} R_{22}^T\right) = \mathcal{O}_s M_X \mathcal{O}_s^T + \sigma^2 I_{s\ell}$$

□

The proof follows easily by combining lemma 8.3 and lemma 8.2.

In the noise free case, described in section 8.2.4, the number of nonzero singular values equals the order of the state space system. Equation (8.29) shows that this no longer holds if there is noise present. In this case the order can be determined from the singular values if we can distinguish the n disturbed singular values of the system, from the $s\ell - n$ remaining singular values that are due to the noise. Hence, the order can be determined, if the smallest singular value of $\Sigma_n^2 + \sigma^2 I_n$ is larger than σ^2. In other words, the ability to determine the order depends heavily on the signal to noise ratio. This is illustrated in the following example.

Example 8.4 (Noisy singular values)

Consider the LTI system (8.27)–(8.28) with system matrices given in example 8.1 on page 212. Let the input $u(k)$ be a unit-variance zero-mean white Gaussian noise sequence and the noise $v(k)$ a zero-mean white Gaussian noise with standard deviation σ. The input and the corresponding noise free output of the system are shown in figure 8.3. Figure 8.4 on page 225 shows the singular values of the matrix R_{22} from the RQ factorization (8.22) on page 219 for different values of the standard deviation σ of the noise $v(k)$. From this we see that all the singular values differ from zero, and that they become larger with increasing standard deviation σ. We clearly see that when the noise level increases, the gap between the two dominant singular values from the system and the spurious singular values from the noise, becomes smaller. This illustrates that when the noise level is higher, it becomes more difficult to determine the order of the system from the singular values. Observe that the magnitude of the singular values does not correspond to what we would expect from equation (8.29) on page 222, because we are not plotting the singular values of $Y_{i,s,N}\Pi^{\perp}_{U_{i,s,N}}$ but of R_{22} which differs from $Y_{i,s,N}\Pi^{\perp}_{U_{i,s,N}}$ up to a multiplication by the matrix Q_2.

As shown in this section, the subspace method presented in section 8.2.4 can be used for systems contaminated by white noise at the output. Because of this

MOESP

property, this subspace identification method is called the *MOESP* method [10], [11], where 'MOESP' stands for 'Multivariable Output Error State sPace'.

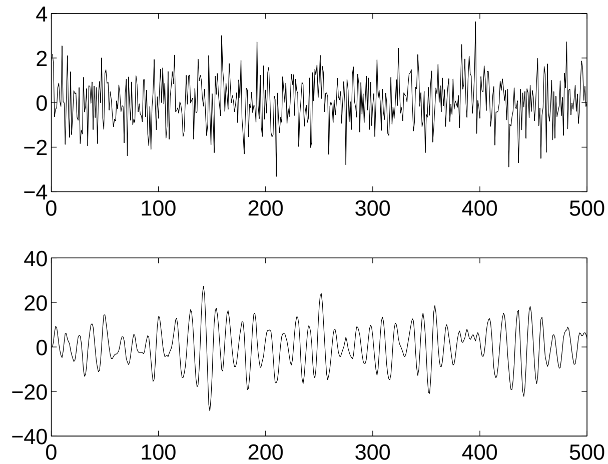

Figure 8.3: *Input and output data used in example 8.4.*

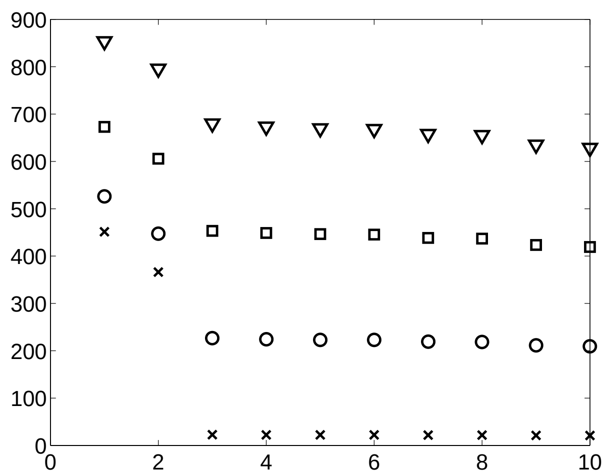

Figure 8.4: *Singular values of the matrix R_{22} corresponding to the system of example 8.4 for different values of σ; $\sigma = 1$ (crosses), $\sigma = 10$ (circles), $\sigma = 20$ (squares), $\sigma = 30$ (triangles).*

Summary MOESP:
Consider the system:

$$
\begin{aligned}
x(k+1) &= Ax(k) + Bu(k) \\
y(k) &= Cx(k) + Du(k) + v(k)
\end{aligned}
$$

with $v(k)$ an ergodic white noise sequence with variance σ^2, that is uncorrelated with $u(k)$, and $u(k)$ an ergodic sequence such that equation (8.19) on page 218 is satisfied.
From the RQ factorization

$$
\begin{bmatrix} U_{0,s,N} \\ Y_{0,s,N} \end{bmatrix} = \begin{bmatrix} R_{11} & 0 \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}
$$

and the SVD,

$$
\lim_{N\to\infty} \frac{1}{\sqrt{N}} R_{22} = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_n & 0 \\ 0 & \sigma I_{s\ell-n} \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}
$$

we have

$$
\text{range}\,(U_1) = \text{range}(\mathcal{O}_s)
$$

8.4 The Use of Instrumental Variables

When the noise v_k in equation (8.28) is not a white noise sequence but rather a colored noise, then the subspace method described in section 8.2.4 will give biased estimates of the system matrices. This is because the column space of the matrix $Y_{i,s,N}\Pi^{\perp}_{U_{i,s,N}}$ does no longer contain the column space of $\mathcal{O}_s$ due to the noise v_k, as can be seen from the derivation presented in the proof of lemma 8.3 on page 222. This is illustrated in the following example.

Example 8.5 (MOESP with colored noise)
Consider the system (8.27)–(8.28) on page 221 with system matrices

$$
\begin{aligned}
A &= \begin{bmatrix} 1.5 & 1 \\ -0.7 & 0 \end{bmatrix}, \quad & B &= \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} \\
C &= \begin{bmatrix} 1 & 0 \end{bmatrix}, & D &= 0
\end{aligned}
$$

We take the input u_k equal to a unit-variance zero-mean white noise sequence. The noise v_k is a colored sequence, generated as follows

$$
v_k = \frac{q^{-1} + 0.5q^{-2}}{1 - 1.69q^{-1} + 0.96q^{-2}} e_k
$$

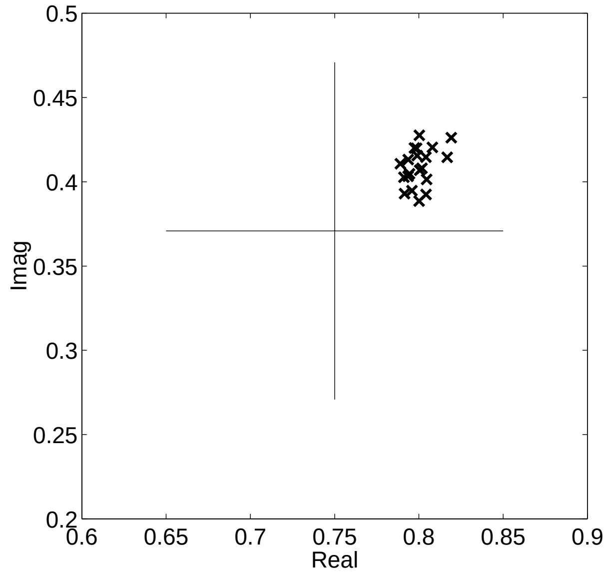

Figure 8.5: *One of the eigenvalues of the matrix A estimated by MOESP for 20 different realizations of colored measurement noise in example 8.5. The big cross corresponds to the real value.*

where e_k is a zero-mean white noise sequence with a variance equal to 0.2. We generate 1000 samples of the output signal, and use the MOESP method to identify the matrices A and C. To show that the MOESP method yields biased estimates, we look at the eigenvalues of the estimated A matrix. The real eigenvalues of this matrix are approximately equal to $0.75 \pm 0.3708j$. Figure 8.5 shows one of the eigenvalues of the estimated A matrix for 20 different realizations of the noise sequence e_k. We clearly see that these eigenvalues are biased.

It is possible to compute unbiased estimates of the system matrices by using so called *instrumental variables* [12], which is the topic of this section.

Recall that after eliminating the influence of the input with the appropriate projection, the data equation becomes

$$Y_{i,s,N}\Pi^{\perp}_{U_{i,s,N}} = \mathcal{O}_s X_{i,N}\Pi^{\perp}_{U_{i,s,N}} + V_{i,s,N}\Pi^{\perp}_{U_{i,s,N}} \tag{8.33}$$

To retrieve the column space of $\mathcal{O}_s$ we have to eliminate the term $V_{i,s,N}\Pi^{\perp}_{U_{i,s,N}}$ which contains the influence of the noise. To do this, we search for a matrix $Z_N \in \mathbb{R}^{sz\times N}$ that has the following properties

Properties of instrumental variable matrix:

$$\lim_{N\to\infty} \frac{1}{N} V_{i,s,N} \Pi^{\perp}_{U_{i,s,N}} Z_N^T = 0 \quad (8.34)$$

$$\mathrm{rank}\left(\lim_{N\to\infty} \frac{1}{N} X_{i,N} \Pi^{\perp}_{U_{i,s,N}} Z_N^T\right) = n \quad (8.35)$$

Such a matrix Z_N is called an *instrumental variable matrix*.

Because of property (8.34) we can indeed get rid of the term $V_{i,s,N}\Pi^{\perp}_{U_{i,s,N}}$ in equation (8.33) by multiplying $Y_{i,s,N}\Pi^{\perp}_{U_{i,s,N}}$ on the right with Z_N and taking the limit for $N \to \infty$, i.e.

$$\lim_{N\to\infty} \frac{1}{N} Y_{i,s,N} \Pi^{\perp}_{U_{i,s,N}} Z_N^T = \lim_{N\to\infty} \frac{1}{N} \mathcal{O}_s X_{i,N} \Pi^{\perp}_{U_{i,s,N}} Z_N^T$$

Property (8.35) ensures that the multiplication by Z_N does not change the rank of the right hand side of the last equation, and therefore we have

$$\mathrm{range}\left(\lim_{N\to\infty} \frac{1}{N} Y_{i,s,N} \Pi^{\perp}_{U_{i,s,N}} Z_N^T\right) = \mathrm{range}(\mathcal{O}_s)$$

From this relation we immediately see that we can determine an asymptotically unbiased estimate of the column space of $\mathcal{O}_s$ from the SVD of the matrix $Y_{i,s,N}\Pi^{\perp}_{U_{i,s,N}} Z_N^T$.

For an efficient implementation of the instrumental variable method, we can again use an RQ factorization. This RQ factorization equals

$$\begin{bmatrix} U_{i,s,N} \\ Z_N \\ Y_{i,s,N} \end{bmatrix} = \begin{bmatrix} R_{11} & 0 & 0 & 0 \\ R_{21} & R_{22} & 0 & 0 \\ R_{31} & R_{32} & R_{33} & 0 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix} \quad (8.36)$$

with $R_{11} \in \mathbb{R}^{sm\times sm}$, $R_{22} \in \mathbb{R}^{sz\times sz}$ and $R_{33} \in \mathbb{R}^{s\ell\times s\ell}$. The next lemma shows the relation between this RQ factorization and the matrix $Y_{i,s,N}\Pi^{\perp}_{U_{i,s,N}} Z_N^T$.

Lemma 8.4 *Given the RQ factorization (8.36), we have* $Y_{i,s,N}\Pi^{\perp}_{U_{i,s,N}} Z_N^T = R_{32}R_{22}^T$. □

Proof: The proof is similar to the proof of lemma 8.2 on page 220. We can derive

$$\Pi^{\perp}_{U_{i,s,N}} = I_N - Q_1^T Q_1 = Q_2^T Q_2 + Q_3^T Q_3 + Q_4^T Q_4$$

And therefore

$$\begin{aligned} Y_{i,s,N}\Pi^{\perp}_{U_{i,s,N}} Z_N^T &= (R_{31}Q_1 + R_{32}Q_2 + R_{33}Q_3)(Q_2^T Q_2 + Q_3^T Q_3 + Q_4^T Q_4) Z_N^T \\ &= (R_{32}Q_2 + R_{33}Q_3)(Q_1^T R_{21}^T + Q_2^T R_{22}^T) \\ &= R_{32}R_{22}^T \end{aligned}$$

which completes the proof. ■

Because of this result, we see that if Z_N is such that equations (8.34) and (8.35) hold, we have

$$\text{range}\left(\lim_{N\to\infty}\frac{1}{N}R_{32}R_{22}^T\right)=\text{range}(\mathcal{O}_s) \tag{8.37}$$

Hence, the matrix $R_{32}R_{22}^T$ can be used to obtain asymptotically unbiased estimates of the matrices A_T and C_T. The question remains how to choose Z_N, this will be dealt with in the subsequent sections.

8.5 Subspace Identification with Colored Measurement Noise

In this section we consider subspace identification of the system (8.27)–(8.28) on page 221 that has an unknown nonwhite noise sequence v_k at its output. From section 8.4 we know that to deal with this case, we need to find an instrumental variable matrix Z_N that satisfies both equation (8.34) and equation (8.35) on page 228. If we take for example $Z_N=U_{i,s,N}$, equation (8.34) is satisfied, because u_k and v_k are uncorrelated, but equation (8.35) is clearly violated for all possible input sequences, since $X_{i,N}\Pi^{\perp}_{U_{i,s,N}}U^T_{i,s,N}=0$. Hence, $Z_N=U_{i,s,N}$ is not a good choice. However, if we take a shifted version of the input to construct Z_N with, like for example $Z_N=U_{0,s,N}$ with $i>0$, equation (8.34) holds, and as explained below there exist certain types of input sequences for which (8.35) also holds. Usually, to construct a suitable matrix Z_N the data available for identification are split up into two parts. Among the many choices possible for splitting the data into two parts [6], one that is often used is described below. The first part, from time instant 0 up to $N+s-2$ is used to construct the data matrix $U_{0,s,N}$, this can be thought of as the 'past input'. The second part, from time instant s up to $N+2s-2$ is used to construct the data matrices $U_{s,s,N}$ and $Y_{s,s,N}$, which can be thought of as the 'future input' and 'future output' respectively. With this terminology, we use the 'future' input and output to actually identify the system, and the 'past' input as the instrumental variable matrix Z_N to get rid of the influence of the noise. The next lemma shows that with this choice equation (8.34) is satisfied.

Lemma 8.5 *Consider the system (8.27)–(8.28). Take $Z_N=U_{0,s,N}$ and $i=s$, then equation (8.34) is satisfied if the input $u(k)$ is an ergodic sequence that is uncorrelated with the ergodic noise sequence $v(k)$.* □

Proof: Since $u(k)$ is uncorrelated with $v(k)$ we have

$$\lim_{N\to\infty}\frac{1}{N}V_{s,s,N}U^T_{0,s,N}=0,\quad \lim_{N\to\infty}\frac{1}{N}V_{s,s,N}U^T_{s,s,N}=0$$

This immediately implies that equation (8.34) holds. ■

With $Z_N=U_{0,s,N}$ and $i=s$, equation (8.35) is equivalent to

$$\text{rank}\left(\lim_{N\to\infty}\frac{1}{N}\left(X_{s,N}U^T_{0,s,N}-X_{s,N}U^T_{s,s,N}(U_{s,s,N}U^T_{s,s,N})^{-1}U_{s,s,N}U^T_{0,s,N}\right)\right)=n$$

Application of the Schur complement (lemma 2.2 on page 17) yields

$$\text{rank}\left(\lim_{N\to\infty}\frac{1}{N}\begin{bmatrix}X_{s,N}\\ U_{s,s,N}\end{bmatrix}\begin{bmatrix}U_{0,s,N}^T & U_{s,s,N}^T\end{bmatrix}\right) = n + sm \tag{8.38}$$

It is important to realize that this equation is only satisfied for certain types of input signals. One such input signal is a white noise sequence.

Lemma 8.6 *[7] Consider the system (8.27)–(8.28). Take $Z_N = U_{0,s,N}$ and $i = s$, then equation (8.35) is satisfied if the input $u(k)$ is a zero mean white noise sequence.* □

Proof: Because the input is a white noise sequence, we have

$$\lim_{N\to\infty}\frac{1}{N}U_{s,s,N}U_{0,s,N}^T = 0, \quad \lim_{N\to\infty}\frac{1}{N}U_{s,s,N}U_{s,s,N}^T = \Sigma_s$$

where $\Sigma_s \in \mathbb{R}^{ms\times ms}$ is a diagonal matrix containing only positive entries. We can write the state sequence $X_{s,N}$ as

$$X_{s,N} = A^s X_{0,N} + \mathcal{C}_s^r U_{0,s,N}$$

where $\mathcal{C}_s^r$ denotes the reversed controllability matrix, i.e.

$$\mathcal{C}_s^r = \begin{bmatrix}A^{s-1}B & A^{s-2}B & \cdots & B\end{bmatrix}$$

By the white noise property of the input $u(k)$, we have,

$$\lim_{N\to\infty}\frac{1}{N}X_{0,N}U_{0,s,N}^T = 0$$

Therefore,

$$\lim_{N\to\infty}\frac{1}{N}X_{s,N}U_{0,s,N}^T = \mathcal{C}_s^r\Sigma_s$$

With this we can write

$$\lim_{N\to\infty}\frac{1}{N}\begin{bmatrix}X_{s,N}\\ U_{s,s,N}\end{bmatrix}\begin{bmatrix}U_{0,s,N}^T & U_{s,s,N}^T\end{bmatrix} = \begin{bmatrix}\mathcal{C}_s^r\Sigma_s & 0\\ 0 & \Sigma_s\end{bmatrix}$$

Since equation (8.38) is equivalent to equation (8.35), this completes the proof. ■

In section 9.2.4 we discuss more general conditions on the input signal that are related to the satisfaction of equation (8.38). For now, we assume that the input signal is such that equation (8.38) holds. With this assumption we can use the RQ factorization (8.36) on page 228 with $Z_N = U_{0,s,N}$ and $i = s$ to compute unbiased estimates of the system matrices. We have the following important result

Theorem 8.4 *Consider the system (8.27)–(8.28). Given the RQ factorization (8.36) with $Z_N = U_{0,s,N}$ and $i = s$. If $u(k)$ is an ergodic sequence that is uncorrelated with the ergodic noise sequence $v(k)$, and is such that the matrix $U_{0,s,N}$ has full row rank and that equation (8.38) holds, we have*

$$\text{range}\left(\lim_{N\to\infty}\frac{1}{\sqrt{N}}R_{32}\right) = \text{range}(\mathcal{O}_s)$$

□

Proof: From lemmas 8.5 and 8.6 it follows that equations (8.34) and (8.35) hold. Therefore, lemma 8.4 allows us to derive equation (8.37) on page 229. Since the matrix $U_{0,s,N}$ has full row rank, the matrix R_{22} in the RQ factorization (8.36) on page 228 is invertible. Application of Sylvester's inequality to equation (8.37) yields the desired result. ∎

Theorem 8.4 shows that the matrices A_T and C_T can be estimated consistently from an SVD of the matrix R_{32} in a similar way as described in section 8.2.4. The matrices B_T and D_T and the initial state $x_T(0) = T^{-1}x(0)$ can be computed by solving a least squares problem. Using equations (8.24) and (8.25) on page 221, it is easy to see that

$$y(k) = \phi(k)^T\theta + v(k)$$

Because $v(k)$ is not correlated with $\phi(k)$, an unbiased estimate of θ can be obtained by solving

$$\min_\theta \frac{1}{N}\sum_{k=0}^{N-1} ||y(k) - \phi(k)^T\theta||^2$$

The subspace identification method presented in this section is called the *PI-MOESP* method [13], where 'PI' stands for 'Past Inputs' and refers to the fact that the past input data matrix is used as an instrumental variable. **PI-MOESP**

Summary PI-MOESP:
Consider the system:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + Du(k) + v(k) \end{aligned}$$

with $v(k)$ an ergodic noise sequence that is uncorrelated with $u(k)$, and $u(k)$ an ergodic sequence such that the matrix $U_{0,s,N}$ has full row rank and that equation (8.38) on page 230 is satisfied.
From the RQ factorization

$$\begin{bmatrix} U_{s,s,N} \\ U_{0,s,N} \\ Y_{s,s,N} \end{bmatrix} = \begin{bmatrix} R_{11} & 0 & 0 \\ R_{21} & R_{22} & 0 \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix}$$

we have

$$\text{range}\left(\lim_{N\to\infty} \frac{1}{\sqrt{N}} R_{32}\right) = \text{range}(\mathcal{O}_s)$$

Example 8.6 (PI-MOESP with colored noise)

To show that the PI-MOESP method yields unbiased estimates, we perform the same experiment as in example 8.5 on page 226. The eigenvalues of the

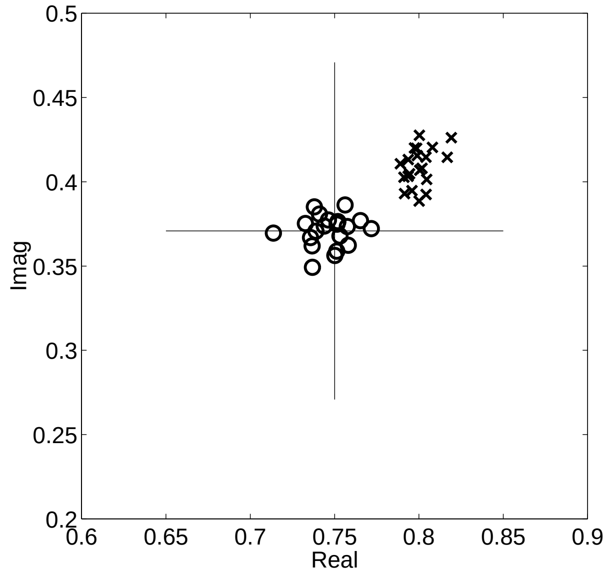

Figure 8.6: *One of the eigenvalues of the estimated A matrix for 20 different realizations of colored measurement noise in example 8.6. The crosses are the eigenvalues obtained by MOESP, the circles are the eigenvalues obtained by PI-MOESP. The big cross corresponds to the real value.*

estimated A matrix obtained by MOESP and by PI-MOESP are shown in figure 8.6 on page 232. We see that while the eigenvalues obtained from MOESP are biased, those obtained from PI-MOESP are unbiased.

8.6 Subspace Identification with Process and Measurement Noise

Now we consider the case where both the output and the state are contaminated by noise. The noise on the output, which is called the measurement noise is denoted by $v(k)$ and the noise on the state, which is called the process noise is denoted by $w(k)$. The noisy system that we consider equals

$$\begin{aligned} \tilde{x}(k+1) &= A\tilde{x}(k) + Bu(k) + w(k) \quad (8.39)\\ y(k) &= C\tilde{x}(k) + Du(k) + v(k) \quad (8.40) \end{aligned}$$

Throughout this section we assume that the process noise $w(k)$ and the measurement noise $v(k)$ are zero mean white noise sequences that are uncorrelated with the input $u(k)$. Note that what we consider here is in fact a special case of colored measurement noise at the output, because we can write the system (8.39)–(8.40) as

$$\begin{aligned} \bar{x}(k+1) &= A\bar{x}(k) + Bu(k) \\ y(k) &= C\bar{x}(k) + Du(k) + \bar{v}(k) \end{aligned}$$

where $\bar{v}(k)$ is given by

$$\begin{aligned} \xi(k+1) &= A\xi(k) + w(k) \\ \bar{v}(k) &= C\xi(k) + v(k) \end{aligned}$$

with $\xi(k) = \tilde{x}(k) - \bar{x}(k)$. Hence,

$$\bar{v}(k) = CA^k\Big(\tilde{x}(0) - \bar{x}(0)\Big) + \sum_{i=0}^{k-1} CA^{k-i-1}Bw(i) + v(k)$$

From which we clearly see that $\bar{v}(k)$ is a colored sequence.

Another representation for the system (8.39)–(8.40) was given in section 5.6.3 where we have seen that such a system can be written in innovations form as follows

$$x(k+1) = Ax(k) + Bu(k) + Ke(k) \tag{8.41}$$
$$y(k) = Cx(k) + Du(k) + e(k) \tag{8.42}$$

where the innovation $e(k)$ is a white noise sequence and K is the Kalman gain. The Kalman gain allows us to come up with a model for the noise and construct a one-step ahead predictor as outlined in section 7.4.

In the remaining part of this section we will use the system representation (8.41)–(8.42). The data equation for this system is given by

$$Y_{i,s,N} = \mathcal{O}_s X_{i,N} + \mathcal{T}_s U_{i,s,N} + \mathcal{S}_s E_{i,s,N} \tag{8.43}$$

where $E_{i,s,N}$ is a block Hankel matrix constructed from the sequence $e(k)$, and

$$\mathcal{S}_s = \begin{bmatrix} I_\ell & 0 & 0 & \cdots & 0 \\ CK & I_\ell & 0 & \cdots & 0 \\ CAK & CK & I_\ell & & 0 \\ \vdots & & \ddots & \ddots & \\ CA^{s-2}K & CA^{s-3}K & \cdots & CK & I_\ell \end{bmatrix}$$

describes the influence of the innovation sequence $e(k)$ on the output.

The goal is again to retrieve the column space of the matrix $\mathcal{O}_s$. To achieve this, first the influence of the input is removed as follows

$$Y_{i,s,N}\Pi^\perp_{U_{i,s,N}} = \mathcal{O}_s X_{i,N}\Pi^\perp_{U_{i,s,N}} + \mathcal{S}_s E_{i,s,N}\Pi^\perp_{U_{i,s,N}}$$

Next, we have to get rid of the term $\mathcal{S}_s E_{i,s,N}\Pi^\perp_{U_{i,s,N}}$. As in section 8.5 we will use the instrumental variable approach from section 8.4. We search for a matrix Z_N that has the following properties

$$\lim_{N\to\infty} \frac{1}{N} E_{i,s,N}\Pi^\perp_{U_{i,s,N}} Z_N^T = 0 \tag{8.44}$$
$$\text{rank}\left(\lim_{N\to\infty} \frac{1}{N} X_{i,N}\Pi^\perp_{U_{i,s,N}} Z_N^T\right) = n \tag{8.45}$$

If we take $i = s$ as in section 8.5 we can use $U_{0,s,N}$ as an instrumental variable. In addition we can also use the past output $Y_{0,s,N}$ as an instrumental variable. Using both $U_{0,s,N}$ and $Y_{0,s,N}$ instead of only $U_{0,s,N}$ will result in better models when a finite number of data points is used. This is intuitively clear by looking at equations (8.44) and (8.45) and keeping in mind that in practice we have only a finite number of data points available. This will be illustrated by an example.

Example 8.7 (Instrumental variables)
Consider the system given by

$$\begin{aligned} A &= \begin{bmatrix} 1.5 & 1 \\ -0.7 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}, \quad K = \begin{bmatrix} 2.5 \\ -0.5 \end{bmatrix} \\ C &= \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad\quad D = 0 \end{aligned}$$

This system is simulated for 2000 samples with an input signal $u(k)$ and a noise signal $e(k)$, that are both white noise zero-mean unit-variance sequences which are uncorrelated. Figure 8.7 on page 235 compares the singular values of the matrix R_{32} from the RQ factorization (8.36) on page 228 with $i = s$, for the case that $Z_N = U_{0,s,N}$ and the case that Z_N contains both $U_{0,s,N}$ and $Y_{0,s,N}$. We see that in the latter case the first two singular values are larger than the ones obtained for $Z_N = U_{0,s,N}$, and that the singular values corresponding to the noise remain almost the same. This means that for the case that Z_N contains both $U_{0,s,N}$ and $Y_{0,s,N}$ the directions corresponding to the state of the system become stronger with respect to the noise, compared to the case where $Z_N = U_{0,s,N}$. Thus, we conclude that using both $U_{0,s,N}$ and $Y_{0,s,N}$ results in better models than using only $U_{0,s,N}$.

The previous example motivates the choice

$$Z_N = \begin{bmatrix} U_{0,s,N} \\ Y_{0,s,N} \end{bmatrix} \tag{8.46}$$

The next lemma shows that with this choice equation (8.44) is satisfied.

Lemma 8.7 *Consider the system (8.41)–(8.42). Take $i = s$ and Z_N as in equation (8.46), then equation (8.44) is satisfied if the input $u(k)$ is an ergodic sequence that is uncorrelated with the ergodic white noise sequence $e(k)$.* □

Proof: Since $u(k)$ is uncorrelated with $e(k)$ we have

$$\lim_{N\to\infty} \frac{1}{N} E_{s,s,N} U_{0,s,N}^T = 0 \quad \lim_{N\to\infty} \frac{1}{N} E_{s,s,N} U_{s,s,N}^T = 0$$

and also

$$\begin{aligned} \lim_{N\to\infty} \frac{1}{N} Y_{0,s,N} E_{s,s,N}^T &= \lim_{N\to\infty} \frac{1}{N} \Big(\mathcal{O}_s X_{0,N} E_{s,s,N}^T + \mathcal{T}_s U_{0,s,N} E_{s,s,N}^T \\ &\qquad + \mathcal{S}_s E_{0,s,N} E_{s,s,N}^T \Big) \\ &= 0 \end{aligned}$$

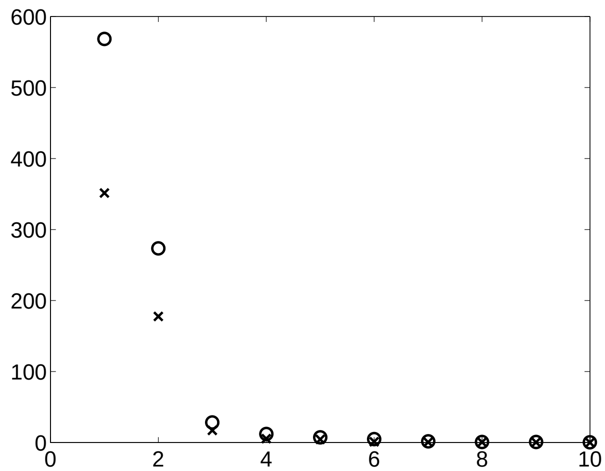

Figure 8.7: *Singular values of the matrix R_{32} from the RQ factorization (8.36) with $i = s$, for example 8.7. The crosses correspond to the case that $Z_N = U_{0,s,N}$ and the circles correspond to the case that Z_N contains both $U_{0,s,N}$ and $Y_{0,s,N}$.*

where we have used the fact that $e(k)$ is a white noise sequence that is uncorrelated with the input for all time instances and because of equation (8.41) on page 233 is only correlated with the state at time instant $k+1$. This completes the proof. ∎

In a similar way as done in section 8.5, the Schur complement (lemma 2.2 on page 17) can be used to transform equation (8.45) into

$$\operatorname{rank}\left(\lim_{N\to\infty}\frac{1}{N}\begin{bmatrix}X_{s,N}\\U_{s,s,N}\end{bmatrix}\begin{bmatrix}Y_{0,s,N}^T & U_{0,s,N}^T & U_{s,s,N}^T\end{bmatrix}\right) = n + sm \tag{8.47}$$

Note that we have switched the positions of $Y_{0,s,N}$ and $U_{0,s,N}$, which of course does not change the rank condition. In general, equation (8.47) is almost always satisfied. However, only for certain types of input signals $u(k)$ and noise signals $e(k)$ it is easy to formally proof the above rank condition. An example of such an input signal is a zero mean white noise sequence. For this signal we can state the following lemma, which you are requested to proof in exercise 8.8 on page 246.

Lemma 8.8 *[7] Consider the system (8.41)–(8.42). Take $i = s$ and Z_N as in equation (8.46), then equation (8.45) is satisfied if the input $u(k)$ is a zero mean white noise sequence.* □

We refer to section 9.2.4 for a more elaborate discussion on the relationship between conditions on the input signal and the rank condition (8.47).

8.6.1 The PO-MOESP Method

PO-MOESP

Based on the following lemma, we can derive a subspace identification method similar to the one presented in section 8.5. This method is called the *PO-MOESP* method [2], where 'PO' stands for 'Past Outputs' and refers to the fact that the instrumental variables also contain the past output data.

Lemma 8.9 *Let the system (8.41)–(8.42) be given. Let the input $u(k)$ and the noise $e(k)$ be ergodic sequences such that*

$$\text{rank}\left(\lim_{N\to\infty}\frac{1}{\sqrt{N}}\begin{bmatrix}X_{0,N}\\U_{0,s,N}\end{bmatrix}\right) = n + sm \tag{8.48}$$

then,

$$\text{rank}\left(\lim_{N\to\infty}\frac{1}{\sqrt{N}}\begin{bmatrix}Y_{0,s,N}\\U_{0,s,N}\end{bmatrix}\right) = s(\ell + m)$$

□

Proof: We can write

$$\begin{bmatrix}Y_{0,s,N}\\U_{0,s,N}\end{bmatrix} = \underbrace{\begin{bmatrix}\mathcal{O}_s & \mathcal{T}_s & \mathcal{S}_s\\ 0 & I_{sm} & 0\end{bmatrix}}\begin{bmatrix}X_{0,N}\\U_{0,s,N}\\E_{0,s,N}\end{bmatrix}$$

Since $\mathcal{S}_s$ is square and has full rank, the underbraced matrix has full row rank and an application of Sylvester's inequality shows that the lemma is proven if the matrix

$$\frac{1}{\sqrt{N}}\begin{bmatrix}X_{0,N}\\U_{0,s,N}\\E_{0,s,N}\end{bmatrix} \tag{8.49}$$

has full rank for $N \to \infty$. Observe that because of the white noise properties of $e(k)$ we have

$$\lim_{N\to\infty}\frac{1}{N}\begin{bmatrix}X_{0,N}\\U_{0,s,N}\\E_{0,s,N}\end{bmatrix}\begin{bmatrix}X_{0,N}^T & U_{0,s,N}^T & E_{0,s,N}^T\end{bmatrix} =$$

$$\lim_{N\to\infty}\frac{1}{N}\begin{bmatrix}X_{0,N}X_{0,N}^T & X_{0,N}U_{0,s,N}^T & 0\\ U_{0,s,N}X_{0,N}^T & U_{0,s,N}U_{0,s,N}^T & 0\\ 0 & 0 & E_{0,s,N}E_{0,s,N}^T\end{bmatrix}$$

and

$$\text{rank}\left(\lim_{N\to\infty}\frac{1}{N}E_{0,s,N}E_{0,s,N}^T\right) = I_{\ell s}$$

Since,

$$\text{rank}\left(\begin{bmatrix}X_{0,N}X_{0,N}^T & X_{0,N}U_{0,s,N}^T\\ U_{0,s,N}X_{0,N}^T & U_{0,s,N}U_{0,s,N}^T\end{bmatrix}\right) = n + ms$$

it follows that the matrix (8.49) indeed has full rank, for $N \to \infty$. ■

The main result on which the PO-MOESP method is based is presented in the following theorem.

Theorem 8.5 *Let the system (8.41)–(8.42) be given. Let the RQ factorization (8.36) with Z_N as in equation (8.46) and $i = s$ be given. Let $u(k)$ be an ergodic sequence that is uncorrelated with the ergodic white noise sequence $e(k)$, and let $u(k)$ and $e(k)$ be such that equations (8.47) and (8.48) hold, then,*

$$\text{range}\left(\lim_{N\to\infty} \frac{1}{\sqrt{N}} R_{32}\right) = \text{range}(\mathcal{O}_s)$$

□

Proof: From lemma 8.7 on page 234 it follows that equation (8.44) on page 233 holds. The assumption that equation (8.47) on page 235 holds, implies that equation (8.45) on page 233 also holds. Therefore, lemma 8.4 on page 228 allows us to derive equation (8.37) on page 229. From lemma 8.9 on page 236 we conclude that the choice of Z_N given by equation (8.46) on page 234 implies that the matrix $\frac{1}{\sqrt{N}} Z_N$ has full row rank for $N \to \infty$. It follows from the RQ factorization (8.36) on page 228 that for $N \to \infty$ the matrix $\frac{1}{\sqrt{N}} R_{22}$ is invertible. An application of Sylvester's inequality to equation (8.37) yields the desired result. ■

So the matrices A_T and C_T can be estimated consistently from an SVD of the matrix R_{32} in a similar way as described in section 8.2.4. The matrices B_T and D_T and the initial state $x_T(0) = T^{-1}x(0)$ can be computed by solving a least squares problem. Using equations (8.24) and (8.25) on page 221, it is easy to see that

$$y(k) = \phi(k)^T \theta + \left(\sum_{\tau=0}^{k-1} C_T A_T^{k-\tau-1} K_T e(\tau) \right) + e(k)$$

Because $e(k)$ is not correlated with $\phi(k)$, an unbiased estimate of θ can be obtained by solving

$$\min_\theta \frac{1}{N} \sum_{k=0}^{N-1} ||y(k) - \phi(k)^T \theta||^2$$

Summary PO-MOESP:
Consider the system:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + Ke(k) \\ y(k) &= Cx(k) + Du(k) + e(k) \end{aligned}$$

with $e(k)$ an ergodic white noise sequence that is uncorrelated with the ergodic sequence $u(k)$, and $u(k)$ and $e(k)$ are such that equation (8.47) and equation (8.48) on page 236 are satisfied.
From the RQ factorization

$$\begin{bmatrix} U_{s,s,N} \\ \begin{bmatrix} U_{0,s,N} \\ Y_{0,s,N} \end{bmatrix} \\ Y_{s,s,N} \end{bmatrix} = \begin{bmatrix} R_{11} & 0 & 0 \\ R_{21} & R_{22} & 0 \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix}$$

we have

$$\text{range}\left(\lim_{N\to\infty} \frac{1}{\sqrt{N}} R_{32} \right) = \text{range}(\mathcal{O}_s)$$

8.6.2 Subspace Identification as a Least Squares Problem

In this section we reveal a close link between the RQ factorization (8.36) on page 228 used in the PO-MOESP scheme and the solution to a least squares problem. First, it will be shown in the next theorem, that the extended observability matrix $\mathcal{O}_s$ can also be derived from the solution of a least squares problem. Second, it will be shown that this solution enables the approximation of the state sequence of a Kalman filter. The latter idea was exploited in [1] to derive another subspace identification method.

Theorem 8.6 *[4] Let the system (8.41)–(8.42) be given and let Z_N be as in equation (8.46). Let the input $u(k)$ be an ergodic sequence that is uncorrelated with the ergodic white noise sequence $e(k)$, and let $u(k)$ and $e(k)$ be such that equation (8.47) holds. Let in addition the following rank condition be satisfied,*

$$\text{rank}\left(\lim_{N\to\infty} \frac{1}{\sqrt{N}} \begin{bmatrix} X_{0,N} \\ U_{0,s,N} \\ U_{s,s,N} \end{bmatrix} \right) = n + 2ms \tag{8.50}$$

Define the bounded matrices $\Delta_u \in \mathbb{R}^{n\times s\ell}$ and $\Delta_z \in \mathbb{R}^{n\times s(\ell+m)}$ as,

$$\lim_{N\to\infty} \frac{1}{N} X_{0,N} \begin{bmatrix} U_{s,s,N}^T & Z_N^T \end{bmatrix} \left(\frac{1}{N} \begin{bmatrix} U_{s,s,N} \\ Z_N \end{bmatrix} \begin{bmatrix} U_{s,s,N}^T & Z_N^T \end{bmatrix} \right)^{-1} = \begin{bmatrix} \Delta_u & \Delta_z \end{bmatrix}$$

and consider the following least squares problem,

$$\begin{bmatrix} \hat{L}_N^u & \hat{L}_N^z \end{bmatrix} = \arg \min_{L^u, L^z} \left\| Y_{s,s,N} - \begin{bmatrix} L^u & L^z \end{bmatrix} \begin{bmatrix} U_{s,s,N} \\ Z_N \end{bmatrix} \right\|_F \tag{8.51}$$

Define the matrix $\mathcal{L}_s$ as,

$$\begin{aligned}
\mathcal{L}_s &= \begin{bmatrix} \mathcal{L}_s^u & \mathcal{L}_s^y \end{bmatrix} \qquad (8.52)\\
\mathcal{L}_s^u &= \begin{bmatrix} (A-KC)^{s-1}(B-KD) & (A-KC)^{s-2}(B-KD) & \cdots & (B-KD) \end{bmatrix}\\
\mathcal{L}_s^y &= \begin{bmatrix} (A-KC)^{s-1}K & (A-KC)^{s-2}K & \cdots & K \end{bmatrix}
\end{aligned}$$

then,

$$\lim_{N\to\infty} \hat{L}_N^z = \mathcal{O}_s\mathcal{L}_s + \mathcal{O}_s(A-KC)^s\Delta_z \qquad (8.53)$$

□

Proof: Substitution of equation (8.42) on page 233 into equation (8.41) yields

$$x(k+1) = (A-KC)x(k) + (B-KD)u(k) + Ky(k)$$

therefore,

$$X_{s,N} = (A-KC)^s X_{0,N} + \mathcal{L}_s \begin{bmatrix} U_{0,s,N} \\ Y_{0,s,N} \end{bmatrix} \qquad (8.54)$$

and the data equation for $i = s$ can be written as,

$$Y_{s,s,N} = \mathcal{O}_s\mathcal{L}_s Z_N + \mathcal{T}_s U_{s,s,N} + \mathcal{S}_s E_{s,s,N} + \mathcal{O}_s(A-KC)^s X_{0,N}$$

The normal equations corresponding to the given least squares problem (8.51) read,

$$\begin{aligned}
\begin{bmatrix} L^u & L^z \end{bmatrix} \begin{bmatrix} U_{s,s,N} \\ Z_N \end{bmatrix} \begin{bmatrix} U_{s,s,N}^T & Z_N^T \end{bmatrix} &= Y_{s,s,N} \begin{bmatrix} U_{s,s,N}^T & Z_N^T \end{bmatrix}\\
&= \begin{bmatrix} \mathcal{T}_s & \mathcal{O}_s\mathcal{L}_s \end{bmatrix} \begin{bmatrix} U_{s,s,N} \\ Z_N \end{bmatrix} \begin{bmatrix} U_{s,s,N}^T & Z_N^T \end{bmatrix}\\
&\quad + \mathcal{S}_s E_{s,s,N} \begin{bmatrix} U_{s,s,N}^T & Z_N^T \end{bmatrix}\\
&\quad + \mathcal{O}_s(A-KC)^s X_{0,N} \begin{bmatrix} U_{s,s,N}^T & Z_N^T \end{bmatrix}
\end{aligned}$$

The white noise property of $e(k)$ gives,

$$\lim_{N\to\infty} \frac{1}{N} E_{s,s,N} \begin{bmatrix} U_{s,s,N}^T & Z_N^T \end{bmatrix} = 0$$

Since $\begin{bmatrix} U_{s,s,N} \\ Z_N \end{bmatrix}$ is full rank for $N \to \infty$,

$$\lim_{N\to\infty} \begin{bmatrix} \hat{L}_N^u & \hat{L}_N^z \end{bmatrix} = \begin{bmatrix} \mathcal{T}_s & \mathcal{O}_s\mathcal{L}_s \end{bmatrix} + \mathcal{O}_s(A-KC)^s \begin{bmatrix} \Delta_u & \Delta_z \end{bmatrix}$$

and the proof is completed. ■

With the RQ factorization (8.36) on page 228, with Z_N as in equation (8.46) on page 234 and $i = s$, the solution $\hat{L}_N^z$ of the least squares problem (8.51) can be written as,

$$\hat{L}_N^z = R_{32}R_{22}^{-1} \qquad (8.55)$$

You are requested to verify this result in exercise 8.9 on page 246. Because of this result we have that,

$$\lim_{N\to\infty} R_{32}R_{22}^{-1} = \mathcal{O}_s\Big(\mathcal{L}_s + (A-KC)^s\Delta_z\Big)$$

The rank of the matrix R_{32} in the limit $\lim_{N\to\infty}$ is established from the following conditions:

$$\begin{aligned}
\text{rank}\left(\lim_{N\to\infty}\frac{1}{\sqrt{N}}R_{32}\right) &= \text{rank}\left(\lim_{N\to\infty}\frac{1}{N}R_{32}R_{22}^T\right)\\
&= \text{rank}\left(\lim_{N\to\infty}\frac{1}{N}Y_{s,s,N}\Pi^{\perp}_{U_{s,s,N}}Z_N^T\right)\\
&= \text{rank}\left(\lim_{N\to\infty}\frac{1}{N}X_{s,N}\Pi^{\perp}_{U_{s,s,N}}Z_N^T\right)
\end{aligned}$$

The latter rank condition has already been discussed at the beginning of this section.

Based on the above theorem and representation of the least squares solution in terms of the quantities computed in the RQ factorization of the PO-MOESP scheme, we have shown in another way that in the limit $N\to\infty$ the range spaces of the matrices R_{32} of extended observability matrix $\mathcal{O}_s$ are equal.

Using the result of theorem 8.6, the definition of the matrix Z_N in this theorem and the expression (8.54) for $X_{s,N}$ given in the proof of the theorem, we have the following relationship,

$$\begin{aligned}
\left(\lim_{N\to\infty}\hat{L}_N^z\right)Z_N &= \left(\lim_{N\to\infty}R_{32}R_{22}^{-1}\right)Z_N\\
&= \mathcal{O}_s\mathcal{L}_sZ_N + \mathcal{O}_s(A-KC)^s\Delta_zZ_N\\
&= \mathcal{O}_sX_{s,N} + \underbrace{\mathcal{O}_s(A-KC)^s\left(\Delta_zZ_N - X_{0,N}\right)}
\end{aligned}$$

Based on this expression, it was argued in [1] that the asymptotic stability of the matrix $(A-KC)$ assures that for large enough s, the underbraced term in the above relationship is small. Hence an approximation of the state sequence can be obtained from a SVD of the matrix,

$$R_{32}R_{22}^{-1}Z_N = U_n\Sigma_nV_n^T$$

and is given by,

$$\hat{X}_{s,N} = \Sigma_n^{\frac{1}{2}}V_n^T$$

The system matrices A_T, B_T, C_T and D_T can now be estimated by solving the least squares problem

$$\min_{A_T,B_T,C_T,D_T}\left\|\begin{bmatrix}\hat{X}_{s+1,N}\\ Y_{s,1,N-1}\end{bmatrix} - \begin{bmatrix}A_T & B_T\\ C_T & D_T\end{bmatrix}\begin{bmatrix}\hat{X}_{s,N-1}\\ U_{s,1,N-1}\end{bmatrix}\right\|_2^2 \tag{8.56}$$

This subspace method, based on a reconstruction of the state sequence is called

N4SID the *N4SID* method [1], where 'N4SID' stands for 'Numerical algorithm for Subspace IDentification'. In section 8.6.2 we show that the first step of the N4SID and PO-MOESP methods, in which the SVD of a certain matrix is computed, only differ up to certain nonsingular weighting matrices.

Summary N4SID:
Consider the system:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + Ke(k) \\ y(k) &= Cx(k) + Du(k) + e(k) \end{aligned}$$

with $e(k)$ a white noise sequence that is uncorrelated with $u(k)$, and $u(k)$ and $e(k)$ such that equations (8.47) on page 235 and (8.50) on page 238 are satisfied. From the RQ factorization

$$\begin{bmatrix} U_{s,s,N} \\ \begin{bmatrix} U_{0,s,N} \\ Y_{0,s,N} \end{bmatrix} \\ Y_{s,s,N} \end{bmatrix} = \begin{bmatrix} R_{11} & 0 & 0 \\ R_{21} & R_{22} & 0 \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix}$$

we have

$$\lim_{N\to\infty} \frac{1}{\sqrt{N}} \mathcal{O}_s X_{s,N} \approx \lim_{N\to\infty} \frac{1}{\sqrt{N}} R_{32} R_{22}^{-1} Z_N$$

and

$$\text{rank}\left(\lim_{N\to\infty} \frac{1}{\sqrt{N}} R_{32} R_{22}^{-1} Z_N \right) = n$$

8.6.3 Estimating the Kalman Gain K_T

The estimated state sequence in the matrix $\hat{X}_{s,N}$ is an approximation of the state sequence of the innovation model (8.41)–(8.42). Estimates of the covariance matrices that are related to this innovation model can be obtained from the state estimate, together with the estimated system matrices from the least squares problem (8.56). Let the estimated system matrices be denoted by $\hat{A}_T$, $\hat{B}_T$, $\hat{C}_T$ and $\hat{D}_T$, then the residuals of the least squares problem (8.56) are given by

$$\begin{bmatrix} \hat{W}_{s,1,N-1} \\ \hat{V}_{s,1,N-1} \end{bmatrix} = \begin{bmatrix} \hat{X}_{s+1,N} \\ Y_{s,1,N-1} \end{bmatrix} - \begin{bmatrix} \hat{A}_T & \hat{B}_T \\ \hat{C}_T & \hat{D}_T \end{bmatrix} \begin{bmatrix} \hat{X}_{s,N-1} \\ U_{s,1,N-1} \end{bmatrix} \tag{8.57}$$

These residuals can be used to estimate the covariance matrices as follows,

$$\begin{bmatrix} \hat{Q} & \hat{S} \\ \hat{S}^T & \hat{R} \end{bmatrix} = \lim_{N\to\infty} \frac{1}{N} \begin{bmatrix} \hat{W}_{s,1,N} \\ \hat{V}_{s,1,N} \end{bmatrix} \begin{bmatrix} \hat{W}_{s,1,N}^T & \hat{V}_{s,1,N}^T \end{bmatrix} \tag{8.58}$$

The solution P of the following Riccati equation, that was derived in section 5.6.4,

$$\hat{P} = \hat{A}_T \hat{P} \hat{A}_T^T + \hat{Q} - (\hat{S} + \hat{A}_T \hat{P} \hat{C}_T^T)(\hat{C}_T \hat{P} \hat{C}_T^T + \hat{R})^{-1} (\hat{S} + \hat{A}_T \hat{P} \hat{C}_T^T)^T$$

can be used to obtain an estimate of the Kalman gain, denoted by $\hat{K}_T$,

$$\hat{K}_T = (\hat{S} + \hat{A}_T \hat{P} \hat{C}_T^T)(\hat{R} + \hat{C}_T \hat{P} \hat{C}_T^T)^{-1}$$

8.6.4 Relationship Between Different Subspace Identification Schemes

The least squares formulation in theorem 8.6 can be used to relate different subspace identification schemes for the estimation of the system matrices in (8.41)–(8.42). To show these relations, we present in the next theorem the solution to the least squares problem (8.51) on page 238 in another alternative manner.

Theorem 8.7 *Let the conditions stated in theorem 8.6 be satisfied, then the solution* $\hat{L}_N^z$ *to* (8.51) *can be formulated as:*

$$\lim_{N\to\infty} \hat{L}_N^z = \lim_{N\to\infty} (\frac{1}{N} Y_{s,s,N} \Pi^{\perp}_{U_{s,s,N}} Z_N^T)(\frac{1}{N} Z_N \Pi^{\perp}_{U_{s,s,N}} Z_N^T)^{-1}$$

□

Proof: We first show that the matrix $\frac{1}{N} Z_N \Pi^{\perp}_{U_{s,s,N}} Z_N^T$ is invertible. Since,

$$\lim_{N\to\infty} \frac{1}{N} Z_N \Pi^{\perp}_{U_{s,s,N}} Z_N^T = \lim_{N\to\infty} \frac{1}{N} \left(Z_N Z_N^T - Z_N U_{s,s,N}^T (U_{s,s,N} U_{s,s,N}^T)^{-1} U_{s,s,N} Z_N^T \right)$$

Application of the Schur complement (lemma 2.2 on page 17) shows that this is equivalent to

$$\lim_{N\to\infty} \frac{1}{N} \begin{bmatrix} Z_N \\ U_{s,s,N} \end{bmatrix} \begin{bmatrix} Z_N^T & U_{s,s,N}^T \end{bmatrix}$$

having full rank. This is one of the conditions in theorem 8.6 on page 238.

The RQ factorization (8.36) on page 228, with Z_N as in equation (8.46) on page 234 and $i = s$, allows us to derive

$$\begin{aligned} Z_N \Pi^{\perp}_{U_{s,s,N}} Z_N^T &= (R_{21} Q_1 + R_{22} Q_2)(Q_2^T Q_2)(Q_1^T R_{21}^T + Q_2^T R_{22}^T) \\ &= R_{22} R_{22}^T \end{aligned} \tag{8.59}$$

From lemma 8.4 on page 228 we have

$$Y_{s,s,N} \Pi^{\perp}_{U_{s,s,N}} Z_N^T = R_{32} R_{22}^T$$

Thus, we arrive at equation (8.55) on page 239, which was derived from the expression (8.51) on page 238. This completes the proof. ■

Theorem 8.7 can be used to obtain an approximation of $\mathcal{O}_s \mathcal{L}_s$ and subsequently of the state sequence of the innovation model (8.41)–(8.42). Given the following weighted SVD

$$W_1 \Big((Y_{s,s,N} \Pi^{\perp}_{U_{s,s,N}} Z_N^T)(Z_N \Pi^{\perp}_{U_{s,s,N}} Z_N^T)^{-1} \Big) W_2 = U_n \Sigma_n V_n^T$$

where W_1 and W_2 are nonsingular weighting matrices, we can estimate $\mathcal{O}_s$ as

$$\hat{\mathcal{O}}_s = W_1^{-1} U_n \Sigma_n^{\frac{1}{2}} \tag{8.60}$$

and $\mathcal{L}_s$ as

$$\hat{\mathcal{L}}_s = \Sigma_n^{\frac{1}{2}} V_n^T W_2^{-1} \tag{8.61}$$

We can use this estimate of $\mathcal{L}_s$ to reconstruct the state sequence, because

$$\hat{X}_{s,N} = \hat{\mathcal{L}}_s Z_N$$

The different possibilities to choose the weighting matrices induces a whole set of subspace identification methods. The PO-MOESP method has weighting matrices

$$W_1 = I_{s\ell} \qquad W_2 = (Z_N \Pi^{\perp}_{U_{s,s,N}} Z_N^T)^{\frac{1}{2}}$$

To see this, note that the PO-MOESP scheme is based on computing the SVD of the matrix R_{32} which by lemma 8.4 on page 228 is equal to

$$R_{32} = Y_{s,s,N} \Pi^{\perp}_{U_{s,s,N}} Z_N^T (R_{22}^T)^{-1}$$

The matrix R_{22}^T is the matrix square root of $Z_N \Pi^{\perp}_{U_{s,s,N}} Z_N^T$, because of equation (8.59).

The N4SID method has weighting matrices

$$W_1 = I_{s\ell} \qquad W_2 = (Z_N Z_N^T)^{\frac{1}{2}}$$

because it is based on computing the SVD of the matrix $\mathcal{O}_s \mathcal{L}_s Z_N$. To see this denote the SVD of the matrix $\mathcal{O}_s \mathcal{L}_s Z_N$ by

$$\mathcal{O}_s \mathcal{L}_s Z_N = \bar{U}_n \bar{\Sigma}_n \bar{V}^T$$

Taking $W_1 = I_{s\ell}$, equation (8.60) on page 242 yields $\bar{U}_n = U_n$ and $\bar{\Sigma}_n = \Sigma$. Now, because of equation (8.61), the matrix W_2 must satisfy $V_n^T W_2^{-1} Z_N = \bar{V}_n^T$. Since $\bar{V}_n^T \bar{V}_n = I_n$ we have $W_2^{-1} Z_N Z_N^T W_2^{-T} = I_{s(m+\ell)}$, which is satisfied for $W_2 = (Z_N Z_N^T)^{\frac{1}{2}}$.

We see that both the PO-MOESP and N4SID method have a weighting matrix W_1 equal to the identity matrix. There exists methods where the W_1 weighting matrix differs from the identity matrix. An example of such a method is the CVA (Canonical Variate Analysis) method described in [14]. This method is obtained by taking the weighting matrices equal to

$$W_1 = (Y_{s,s,N} \Pi^{\perp}_{U_{s,s,N}} Y_{s,s,N}^T)^{-\frac{1}{2}} \qquad W_2 = (Z_N \Pi^{\perp}_{U_{s,s,N}} Z_N^T)^{\frac{1}{2}}$$

8.7 Summary

In this chapter we described several subspace identification methods. These methods are based on deriving a certain subspace that contains information about the system, from structured matrices constructed from the input and output data. To estimate this subspace the singular value decomposition is used.

The singular values obtained from this decomposition can be used to estimate the order of the system.

First, we described subspace identification methods for the special cases where the input equals an impulse sequence and a step sequence. In these cases, it is possible to exploit the special structure of the data matrices, to get an estimate of the extended observability matrix. From this estimate it is then easy to derive the system matrices (A, B, C, D) up to a similarity transformation.

Next, we described how to deal with more general input sequences. Again we showed that it is possible to get an estimate of the extended observability matrix. From this estimate we computed the system matrices A and C up to a similarity transformation. The corresponding matrices B and D can then be found by solving a linear least squares problem. We showed that the RQ factorization can be used for a computationally efficient implementation of this subspace identification method.

We continued by describing how to deal with noise. It was shown that in the presence of white noise at the output, the subspace identification method for general inputs yields asymptotically unbiased estimates of the system matrices. This subspace method is therefore called the MOESP (Multivariable Output Error State sPace) method. To deal with colored noise at the output, the concept of instrumental variables was introduced. Different choices of instrumental variables lead to the PI-MOESP and PO-MOESP methods. The PI-MOESP method handles arbitrarily colored measurement noise, while the PO-MOESP method deals with white process and white measurement noise. Again we showed how to use the RQ factorization for an efficient implementation. It has been shown that different alternative subspace identification schemes can be derived from a least squares problem formulation based on the structured data matrices treated in the PO-MOESP scheme. The least squares approach enables the approximation of the state sequence of the innovation model, as was originally proposed in the N4SID subspace identification method. Based on this approximation of the state sequence we explained how to approximate the Kalman gain for the PO-MOESP and N4SID methods, to be able to construct an approximate of the one-step ahead predictor. Though a theoretical foundation on the accuracy of this approximation is not given, experimental evidence has shown that it has proven its practical relevance. For an illustration we refer to chapter 9 (see example 9.13).

Exercises

Exercise 8.1 Consider the subspace identification method for step input signals, described in section 8.2.3. Show that when the output has reached its stationary value at a certain time instant r the matrix

$$Y_{r,s,N}\left(I_N - \frac{\mathbb{E}_N \mathbb{E}_N{}^T}{N}\right)$$

does not contain any information that can be used to estimate the matrices A_T and C_T.

Exercise 8.2 Consider the subspace identification methods for impulse and step input signals, described in sections 8.2.2 and 8.2.3, respectively. **CE**

(a) Write Matlab programs to determine the system matrices A, B, and C up to a similarity transformation, for both impulse and step input signals.

(b) Test these algorithms using 20 data points obtained from the following system:

$$\begin{aligned} A &= \begin{bmatrix} -0.5 & 1 \\ 0 & -0.5 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ C &= \begin{bmatrix} 1 & 0 \end{bmatrix} \end{aligned}$$

Check the eigenvalues of the estimated A matrices, and compare the outputs from the models with the real output of the system.

Exercise 8.3 When the input to the state space system (8.1)–(8.2) is periodic, the output will also be periodic. In this case there is no need to build a block Hankel matrix from the output, one period of the output already contains all the information. Assume that the system (8.1)–(8.2) is minimal, asymptotically stable and that $D = 0$. Let the input be periodic with period N_0, i.e. $u(k) = u(k + N_0)$.

(a) Show that

$$\begin{bmatrix} y(0) & y(1) & \cdots & y(N_0-1) \end{bmatrix} = \begin{bmatrix} \bar{g}(N_0) & \bar{g}(N_0-1) & \cdots & \bar{g}(1) \end{bmatrix} U_{0,N_0,N_0}$$

where $\bar{g}(i) = CA^{i-1}(I_n - A^{N_0})^{-1}B$.

(b) What condition must the input $u(k)$ satisfy in order to determine the sequence $\bar{g}(i)$, $i = 1, 2, \ldots, N_0$ from this equation?

(c) Explain how the sequence $\bar{g}(i)$ can be used to determine the system matrices (A, B, C) up to a similarity transformation if $N_0 > 2n + 1$.

Exercise 8.4 Show that if the matrix $U_{0,s,N}$ is of full rank, and $N > s * \ell$, $N > sm$, the solution to

$$\min_{\mathcal{T}_s} ||Y_{0,s,N} - \mathcal{T}_s U_{0,s,N}||_F^2$$

is given by

$$\hat{\mathcal{T}}_s = Y_{0,s,N} U_{0,s,N}^T (U_{0,s,N} U_{0,s,N}^T)^{-1}$$

Exercise 8.5 Show that if we apply a white noise input signal to the system (8.1)–(8.2) the following rank condition is satisfied

$$\operatorname{rank}\left(\begin{bmatrix} X_{0,N} \\ U_{0,s,N} \end{bmatrix}\right) = n + sm$$

Exercise 8.6 Consider the MOESP subspace identification method summarized on page 226. CE

(a) Write a Matlab program to determine the matrices A and C based on the RQ factorization (8.22) on page 219 as described in section 8.2.4.

(b) Test the program on the system used in exercise 8.2 with a white noise input sequence. Check the eigenvalues of the estimated A matrix.

Exercise 8.7 Consider the system (8.27)–(8.28) on page 221, with $v(k)$ a colored noise sequence. Show that when unbiased estimates $\hat{A}_T$ and $\hat{C}_T$ are used in equation (8.26) on page 221 to estimate B_T and D_T, these estimates are also unbiased.

Exercise 8.8 Prove lemma 8.8 on page 235.

Exercise 8.9 Derive equation (8.55) on page 239.

Exercise 8.10 Subspace identification with white process and measurement noise is a special case of subspace identification with colored measurement noise. Explain why the instrumental variables used in the white process and measurement case cannot be used in the general colored measurement noise case.

References

[1] P. Van Overschee and B. De Moor, "N4SID: Subspace algorithms for the identification of combined deterministic and stochastic systems," *Automatica*, vol. 30, no. 1, pp. 75–93, 1994.

[2] M. Verhaegen, "Identification of the deterministic part of MIMO state space models given in innovations form from input-output data," *Automatica*, vol. 30, no. 1, pp. 61–74, 1994.

[3] M. Viberg, "Subspace-based methods for the identification of linear time-invariant systems," *Automatica*, vol. 31, no. 12, pp. 1835–1851, 1995.

[4] K. Peternell, W. Scherrer, and M. Deistler, "Statistical analysis of novel subspace identification methods," *Signal Processing*, vol. 52, pp. 161–177, 1996.

[5] D. Bauer, *Some Asymptotic Theory For the Estimation of Linear Systems Using Maximum Likelihood Methods or Subspace Algorithms*. PhD thesis, Technischen Universität Wien, Vienna, Austria, 1998.

[6] M. Jansson, *On Subspace Methods in System Identification and Sensor Array Signal Processing*. PhD thesis, Royal Institute of Technology (KTH), Stockholm, Sweden, 1997.

[7] M. Jansson and B. Wahlberg, "On consistency of subspace methods for system identification," *Automatica*, vol. 34, no. 12, pp. 1507–1519, 1998.

[8] D. Bauer and M. Jansson, "Analysis of the asymptotic properties of the moesp type of subspace algorithms," *Automatica*, vol. 36, pp. 497–509, 2000.

[9] B. L. Ho and R. E. Kalman, "Effective construction of linear state-variable models from input-output functions," *Regelungstechnik*, vol. 14, pp. 545–548, 1966.

[10] M. Verhaegen and P. Dewilde, "Subspace model identification part 1. The output-error state-space model identification class of algorithms," *International Journal of Control*, vol. 56, no. 5, pp. 1187–1210, 1992.

[11] M. Verhaegen and P. Dewilde, "Subspace model identification part 2. Analysis of the elementary output-error state-space model identification algorithm," *International Journal of Control*, vol. 56, no. 5, pp. 1211–1241, 1992.

[12] S. Söderström and P. Stoica, *Instrumental Variable Methods for System Identification*. New York: Springer-Verlag, 1983.

[13] M. Verhaegen, "Subspace model identification part 3. Analysis of the ordinary output-error state-space model identification algorithm," *International Journal of Control*, vol. 56, no. 3, pp. 555–586, 1993.

[14] W. E. Larimore, "Canonical variate analysis in identification, filtering and adaptive control," in *Proceedings of the 29th Conference on Decision and Control*, (Honolulu, Hawaii), pp. 596–604, 1990.