**1.7** We first show $\left|\frac{w-z}{1-\bar{w}z}\right| = 1$ if either variable lies on the unit circle. Note the conjugation and reciprocation are the same on the unit circle. Suppose $|w| = 1$. Then

$$\left|\frac{w-z}{1-\bar{w}z}\right| = \left|\frac{w-z}{1-z/w}\right| = |w|\left|\frac{w-z}{w-z}\right| = |w| = 1.$$

The other case is similar. Direct computation shows fraction is 0 if $z = w$.

Now, the second part of (a), the whole (bii) and one part of (biii) have been proven. Maximum principle implies (bi) ($F$ is clearly holomorphic on the disc, because the only pole is $1/\bar{w}$, which lies outside the unit disc when $w$ lies inside), and hence the remaining part of (a). Direct computation shows $F \circ F = \mathrm{id}$, and hence (iv) holds (since $F$ is its own inverse) and the remaining of (bii) follows.

**2.7** Suppose $d$ is finite, for there is nothing to prove otherwise. Use the equation in the hint. The numerator is bounded above by $d$ and the denominator is bounded below by $r^2$. The length of contour is bounded above by $2\pi r$, and hence the right-hand side is bounded by $\frac{1}{2\pi}\frac{d}{r}(2\pi r) = \frac{d}{r}$. Therefore, $2|f'(0)| \le d/r$. The result follows from letting $r \to 1$.

**8.1** Recall from Tutorial note 3, $\det Df = |\partial_z f|^2 - |\partial_{\bar{z}} f|^2$, which is nonzero if $f$ is holomorphic and $f' \ne 0$. Then local bijection follows from inverse function theorem.

**8.4** Let $f(z) = -i\frac{z+1}{z-1}$. Then $f$ maps the unit disc to the upper half-plane, because $f$ is bijective on the Riemann sphere and its inverse $f^{-1}(z) = \frac{z-i}{z+i}$ maps the upper half-plane to the unit disc, where $|z - i| < |z + i|$ (points in upper half-plane are closer to $i$ than $-i$). Then letting $g(z) = (z-i)^2$, $g \circ f$ is the desired map.

**8.5** $f(z) = -\frac{1}{2}(z + 1/z)$. $f$ maps into upper half-plane because $\Im f(re^{i\theta}) = -\frac{1}{2}(r - 1/r)\sin\theta > 0$ if $0 < r < 1$ and $0 < \theta < \pi$. $f'(z) = -\frac{1}{2}(1 - 1/z^2)$, which vanishes only at $z = \pm 1$, which does not lie in the half-disc. It suffices to show $f$ is bijective.

Fix $w$ in the upper half-plane and consider the equation $z^2 + 2wz + 1 = 0$ as hinted. Then if a solution $z$ lies in the half-disc, $f(z) = w$. Let $z_1, z_2$ be the roots of the equation, Since the product of roots is $z_2 = 1/z_1 = \bar{z}_1/|z_1|^2$, and hence

- one root has modulus $\le 1$, the other $\ge 1$; and
- one has imaginary part $\ge 0$, the other $\le 0$.

Since the mean of roots is $-w$, which is in the lower half-plane, neither root lies on the unit circle or the real axis, because the otherwise suggests the mean lies on the real axis. Then the mean being the in lower half-plane implies the root with larger modulus is on the lower half-plane because it dominates in the mean, and hence the root with modulus $< 1$ lies in the half-disc. Since there is exactly one root in the half-disc, the map is bijective.

**8.10** Let $G(z) = -i\frac{z+1}{z-1}$. By the same argument in the solution of (8.4), $F \circ G$ maps the disc to the disc and fixes 0. By Schwarz lemma, $(F \circ G)(w) \leq |w|$. Direct computation shows $\frac{z-i}{z+i}$ is the inverse of $G$. The result then follows.

**8.12**   a. Suppose $f$ fixes $a$ and $b$, which are distinct. Let $\varphi_a$ be the Blaschke factor in (1.7) with $w = a$. Then $\varphi_a \circ f \circ \varphi_a$ fixes 0 and $\varphi_a(b)$. By Schwarz lemma, $\varphi_a \circ f \circ \varphi_a$ is a rotation with a nonzero fixed point, and hence is the identity. The result then follows.

      b. Define $G$ as in (8.10) and $H(z) = z + 1$. Then $G^{-1}HG$ has no fixed point.

**8.13**   a. Schwarz lemma applied to $\psi_{f(w)} \circ f \circ \psi_w$ as hinted $(\psi_w = \psi_w^{-1})$ gives $\rho(f(z), f(w)) = |\psi_{f(w)}(f(z))| \leq |\psi_w(z)| = \rho(z, w)$. The desired inequality then follows by unravelling the definitions. The equation follows by applying the inequality on $f^{-1}$.

      b. Since $\psi_z'(0) = 1 - |z|^2$, the result follows from Schwarz lemma (derivative-at-zero part) applied $\psi_{f(z)} \circ f \circ \psi_z$.

## Additional exercise

1.  a. The image is the open sector of angle 1 in the first quadrant with the positive real axis as part of the boundary. Horizontal and vertical lines are mapped to rays and arcs centred at 0.

    b. $e^{\pi z}$

    c. $\frac{1}{\pi} \log z$

2. Let $f(z) = \frac{1+z}{1-z}$, $g(z) = z^2$ and $h(z) = \frac{z-i}{z+i}$. It is claimed $f$ maps the upper half-disc to the first quadrant, and hence $h \circ g \circ f$ is the desired map, as $g$ maps the first quadrant to the upper half-plane and $h$, by Theorem 1.2 of the book, maps the upper half-plane to the unit disc.

   Direct computation shows $f$ maps $-1, 0, 1, i, i/2$ to $0, 1, \infty, i, (3 + 4i)/5$ respectively, and since Mobius transformations preserve circles, $f$ maps the upper half-plane and the unit disc to the upper half-plane and the right-half-plane respectively. Since $f$ is bijective on $\hat{\mathbb{C}}$, the $f$ maps the upper half-disc, which is the intersection of the upper half-plane and the unit disc, to the first quadrant, which is the intersection of the upper half-plane and the right half-plane. Bijectivity follows from that of $f$ on $\hat{\mathbb{C}}$.

3. let $g(z) = \exp(i(z + \pi/2))$ and $f$ be the map defined in 8.5. Then $f \circ g$ is the desired map as $g$ maps the half-strip to the upper half-disc.

4. The hint follows from the last equation in 1.3 of Tutorial note 3 by putting $g = \mathrm{id}$. Then

$$\int_{B(0,R)} |f'(z)|^2 dxdy = \int_{B(0,R)} f'(z)\overline{f'(z)}dxdy$$

$$= \int_0^R \int_0^{2\pi} |f'(re^{it})|^2 r\,dt\,dr$$

$$= \int_0^R \int_0^{2\pi} \sum_{n,m\geq 0} nma_n\bar{a}_m r^{n-1} r^{m-1} e^{i(n-m)t} r\,dt\,dr$$

$$= \sum_{n,m\geq 0} nma_n\bar{a}_m \int_0^R r^{n+m-1}dr \int_0^{2\pi} e^{i(n-m)t}dt$$

$$= \sum_{n,m\geq 0} nma_n\bar{a}_m \frac{R^{n+m}}{n+m}(2\pi\delta_{n,m})$$

$$= \sum_{n\geq 0} nm|a_n|^2 \frac{R^{2n}}{2n}(2\pi)$$

$$= \pi \sum_{n\geq 0} n|a_n|^2 R^{2n}.$$

The interchange of order of integral and summation is justified by absolute uniform convergence of the Taylor series of $f'$ on $\overline{B(0,R)}$. Now, the final expression in the above chain of equations is a power series in $R^2$, which, by Theorem 8.2 of Rudin's *Principles of Mathematical Analysis*, converges to $\pi \sum_{n\geq 0 n|a_n|^2}$. (It is actually not difficult to prove the convergence with bare hands. If $\pi \sum_{n\geq 0} n|a_n|^2 = \infty$, the convergence still holds by monotonicity.) The left-hand side converges to the integral on $\mathbb{D}$, by monotonicity. The result then follows.

remark: Alternatively, appealing to Parseval's theorem applied on $t \mapsto f'(re^{it})$ bypasses the convergence argument, and hence removes the need to integrate on a smaller ball.