THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
2018-2019 semester 1 MATH4060
week 11 tutorial

Underlined contents were not included in the tutorial because of time constraint, but included here for completeness.

Materials in this note are well known, so no particular reference is given. No originality is implied.

1 Fractional Linear Transforms and Four-Point Ratio

Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $\det M \neq 0$, the fractional linear transformation, or Mobius transformation, φ_M associated with M is defined by $\varphi_M(z) = \frac{az+b}{cz+d}$.

Given four points z_j, $j = 1, 2, 3, 4$, on the Riemann sphere, the four-point ratio is defined by $(z_1, z_2, z_3, z_4) = \frac{z_1 - z_3}{z_1 - z_4} / \frac{z_2 - z_3}{z_2 - z_4}$.

Below are some of their elementary properties.

1. Fractional linear transformations can be written as compositions of translation, scaling, and $z \mapsto 1/z$.

   Proof. If $c = 0$, then the map is a scaling followed by translation; if not, $\frac{az+b}{cz+d} = \frac{1}{c} \frac{acz+ad+bc-ad}{cz+d} = \frac{1}{c}\left(a + \frac{bc-ad}{cz+d}\right)$. □

2. $M \mapsto \varphi_M$ is a homomorphism from $GL_2(\mathbb{C})$ to $\mathrm{Aut}(\hat{\mathbb{C}})$. In particular, $\varphi_{M^{-1}} = \varphi_M^{-1}$

   Proof. Since translation, scaling and inversion $z \mapsto 1/z$ are biholomorphic on $\hat{\mathbb{C}}$, the map is well-defined. Straight-forward computation shows it is a homomorphism. Alternatively, viewing $\hat{\mathbb{C}}$ as the space of complex lines in $\mathbb{C}^2$ gives a conceptual proof, as invertible matrices act on the space of lines. □

3. Given $z_j, w_j \in \hat{\mathbb{C}}$, $j = 1, 2, 3$, with distinct z_j's and distinct w_j's, there exists a unique fractional linear transformation that maps z_j to w_j.

   Proof. For existence, it suffices to consider $(w_1, w_2, w_3) = (1, 0, \infty)$ (as tuples), for which the four-point ratio $(\cdot, z_1, z_2, z_3)$ is such a function. For uniqueness, it suffices to consider $(z_1, z_2, z_3) = (w_1, w_2, w_3) = (1, 0, \infty)$. Then such a fractional linear transformation fixes ∞, and hence has no pole in $\mathbb{C}$, and hence is affine. The only linear map that fixes 0 and 1 is the identity map. Uniqueness then follows. □

4. Fractional linear transformations are all the biholomorphic functions on the Riemann sphere.

Proof. Postcomposing with a fractional linear transformation if necessary, assume the biholomorphic map f fixes ∞, we show that f is affine. By construction, $g(z) = f(1/z)$ has a unique pole at 0. By Rouche's theorem and injectivity of f, the pole is of at most order 1. Then $f(1/z) = g(z) = a_{-1}/z + a_0 + a_1 z + \ldots$, and hence $f(z) = a^{-1}z + a_0 + a_1/z + \ldots$. Since f is holomorphic, only a_{-1} and a_1 can be nonzero. The result then follows. □

Alternative Proof. Postcomposing with a fractional linear trasnformation if necessary, assume the pole a and zero b of the biholomorphic map f are finite. Then $f(z)(z-b)/(z-a)$ is holomorphic on $\hat{\mathbb{C}}$, and hence is constant, by maximum modulus principle. The result then follows. □

5. Fractional linear transformations preserve four-point ratios, circles (lines are infinitely large circles) and regions determined by circles.

   Proof. It suffices to check for $z \mapsto 1/z$. The case for four-point ratio is simple. For a circles $|z-a|^2 = r^2$, when z is replaced by $1/z$ and rearranging, the equation becomes
   $$(|a|^2 - r^2)|z|^2 - 2\Re az + 1 = 0.$$
   If $(|a|^2 - r^2) = 0$, then the equation describes a line; if not, further rearrangement gives
   $$|z - \frac{\bar{a}}{|a|^2 - r^2}|^2 = \left(\frac{r}{|a|^2 - r^2}\right)^2,$$
   which is a circle. For a line $\Re az = c$, when z is replaced by $1/z$, multiplying by $|z|^2$ gives $c|z|^2 - \Re \bar{a}\bar{z} = 0$, which is a line if $c = 0$, and a circle otherwise. Therefore, circles are preserved. For regions determined by circles, replace equations with inequalities in the argument above. □

6. Given distinct points z_2, z_3, z_4, the solution set to $\Im(z, z_2, z_3, z_4) = 0$ is the circle through z_2, z_3, z_4.

   Proof. The case is trivial for $(z_2, z_3, z_4) = (1, 0, \infty)$. The general case follows by applying the fractional linear transformation mapping $(1, 0, \infty)$ to (z_2, z_3, z_4). □

7. The inverse of a point z with respect to a circle through distinct points z_2, z_3, z_4 is the point z^* satisfying $(z, z_2, z_3, z_4) = \overline{(z^*, z_2, z_3, z_4)}$. Mobius transformations preserve inverses with respect to circles.

Exercise 1. Prove the following statements.

1. If two circles are normal (intersect at right angles) to each other, then inversion with respect to one circle maps the other circle to itself.

2. For any two disjoint circles on the plane, there exists a fractional linear transformation that maps them to two concentric circles.

3. Let L_1, L_2 be two straight lines through 0 and a, b be straight lines *not* through 0 that intersect L_j, $j = 1, 2$, at P_j and Q_j respectively. Let L be a line through 0 and intersect a and b and P and Q. There exists a fractional linear transformation, independent of L, that maps P_j to Q_j and P to Q.