THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
2018-2019 semester 1 MATH4060
week 7 tutorial

Underlined contents were not included in the tutorial because of time constraint, but included here for completeness.

Some properties of function order were discussed. The order of Θ was computed as in exercise 3 of Chapter 5 of Stein and Shakarchi's *Complex Analysis*. An equivalent definition of function order is given in Proposition 1 as in Chapter IX.2 of Conway's *Functions of One Complex Variable*. The order of a general Taylor series is computed in Corollary 3 as in problems 3 and 4 of Chapter 5 of Stein and Shakarchi's *Complex Analysis*.

For the function order of $\Theta(\cdot|\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 \tau} e^{2\pi i n z}$, see solution to Homework 3.

Proposition 1. Let f be an entire function of order ρ. Let $M(r) = \sup_{|z|=r} |f(z)|$. Then

$$\rho = \limsup \frac{\log\log M(r)}{\log r}$$

Proof. Denote the limit superior by λ. Recall by definition

$$\rho = \inf\{\sigma : |f(z)| \leq \exp(A|z|^{\sigma}) \text{ for some positive constants } A, B\}.$$

For $\lambda \leq \rho$,

$$\begin{aligned}
|f(z)| &\leq A\exp(B|z|^{\sigma}) \\
|f(z)| &\leq \exp((B+1)|z|^{\sigma}) \text{..............(if } |z| \text{ large enough)} \\
M(r) &\leq \exp((B+1)r^{\sigma}) \\
\log\log M(r) &\leq \log B + \sigma \log r \\
\frac{\log\log M(r)}{\log r} &\leq \frac{\log B}{\log r} + \sigma
\end{aligned}$$

The inequality then follows by taking limit superior on both sides and letting $\sigma \to \rho$.

For $\lambda \leq \rho$,

For r large enough,

$$\begin{aligned}
\frac{\log\log M(r)}{\log r} &\leq \lambda + \varepsilon \\
M(r) &\leq \exp(r^{\lambda+\varepsilon}) \\
|f(z)| &\leq \exp(|z|^{\lambda+\varepsilon})
\end{aligned}$$

Then by definition, $\lambda + \varepsilon \geq \rho$. The result follows by letting $\varepsilon \to 0$.

$\square$

Proposition 2. Let $f(z) = \sum a_n z^n$ be entire. Let $\rho < +\infty$. Then the order of f is at most ρ iff $|a_n|^{1/n} = O(\frac{1}{n^{1/\rho}})$.

Proof. Suppose $|f(z)| \leq Ae^{B|z|^\sigma}$. By Cauchy integral formula, $|a_n| \leq \frac{Ae^{BR^\sigma}}{R^n}$, where differentiating shows the optimal R is $\left(\frac{n}{B\sigma}\right)^{1/\sigma}$. This gives

$$|a_n|^{1/n} \leq A^{1/n} \left(\frac{eB\sigma}{n}\right)^{1/\sigma}$$

Necessity then follows by letting $\sigma \to \rho$.

Conversely, suppose $|a_n|^{1/n} = O(\frac{1}{n^{1/\rho}})$. Then $|f(z)| \leq \sum \left(\frac{C}{n^{1/\rho}}\right)^n |z|^n = \sum \left(\frac{C|z|}{n^{1/\rho}}\right)^n$, and hence

$$\begin{aligned}
|f(z)| &\leq \sum \left(\frac{C|z|}{n^{1/\rho}}\right)^n \\
&\leq \sum_{n^{1/\rho} \leq 2C|z|} \left(\frac{C|z|}{n^{1/\rho}}\right)^n + \sum_{n^{1/\rho} > 2C|z|} \left(\frac{C|z|}{n^{1/\rho}}\right)^n \\
&\leq (2C|z|)^\rho (C|z|)^{(2C|z|)^\rho} + \sum (1/2)^n \\
&\leq \exp(\rho \log(2C|z|) + 2C|z|^\rho \log(C|z|)) + 2 \\
&\leq \exp(2C|z|^{\rho+\varepsilon})
\end{aligned}$$

The result then follows by letting $\varepsilon \to 0$. □

Corollary 3. Let $f(z) = \sum a_n z^n$ be entire and of order ρ, not necessarily finite. The order of f is given by

$$\limsup -\frac{n \log n}{\log |a_n|}$$