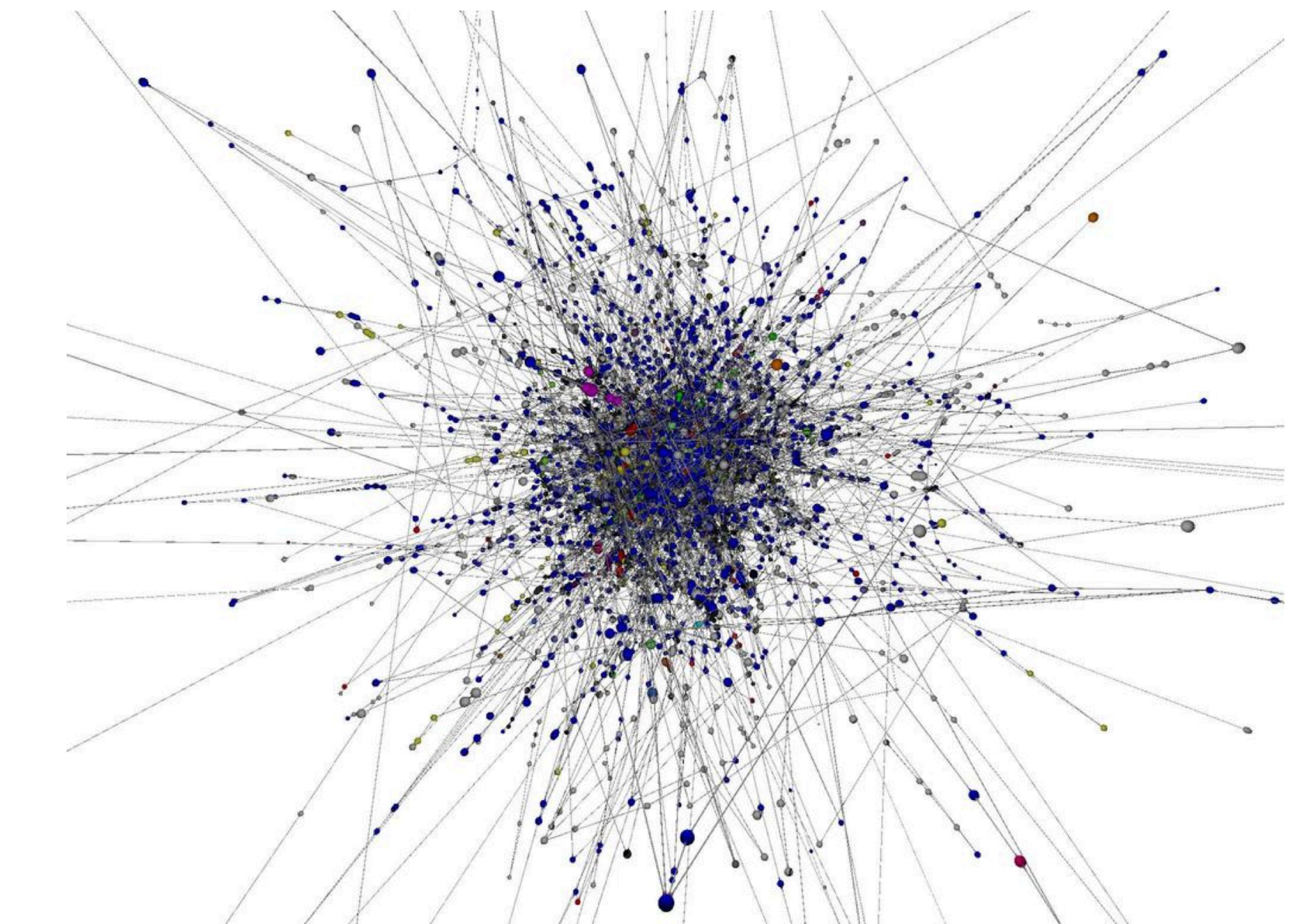


The Topology of Preferential Attachment

How Random Interaction Begets Holes

Chunyin Siu
Cornell University
cs2323@cornell.edu

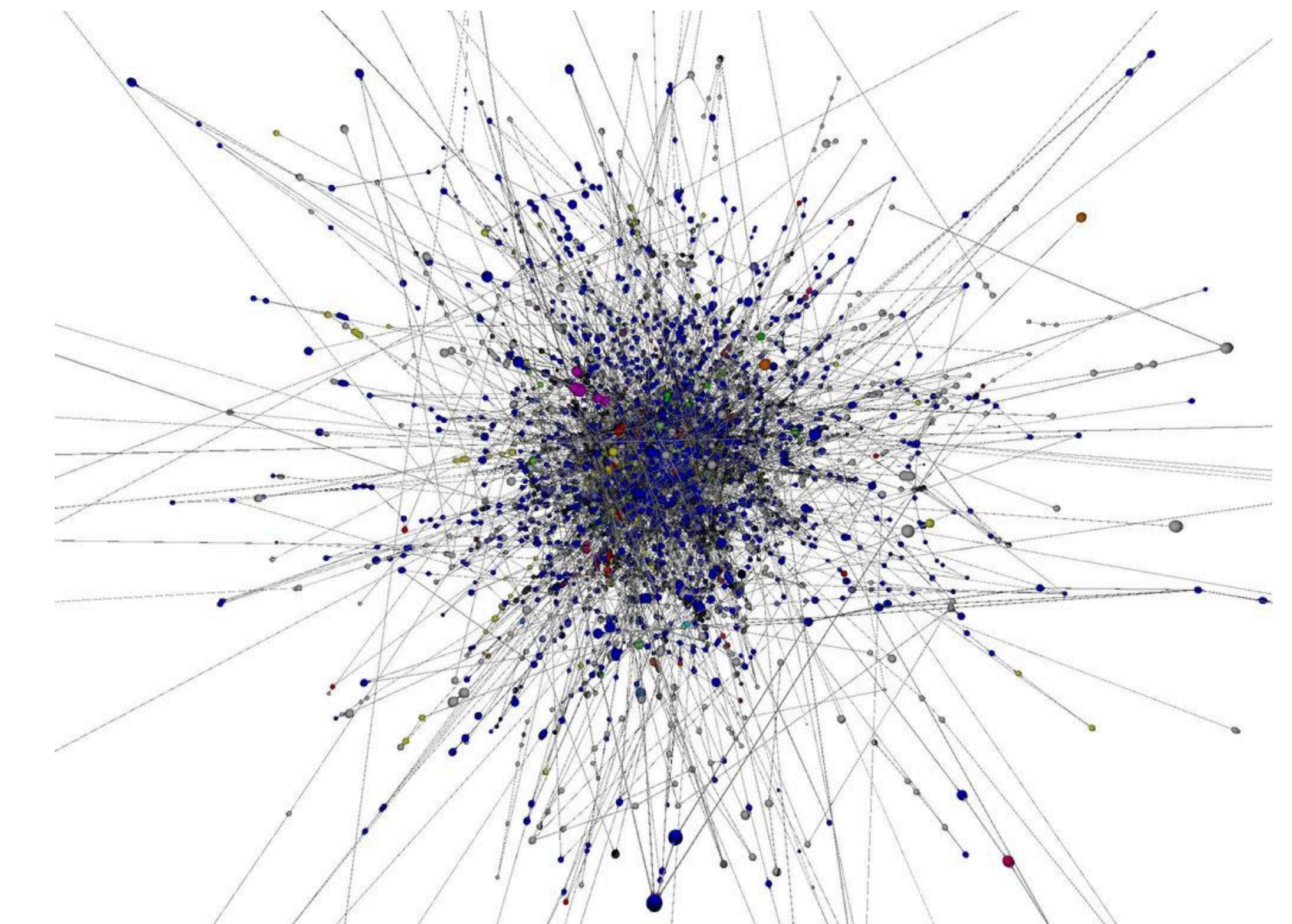
So, preferential attachment...



(Stephen Coast
<https://www.fractalus.com/steve/stuff/ipmap/>)

So, preferential attachment...

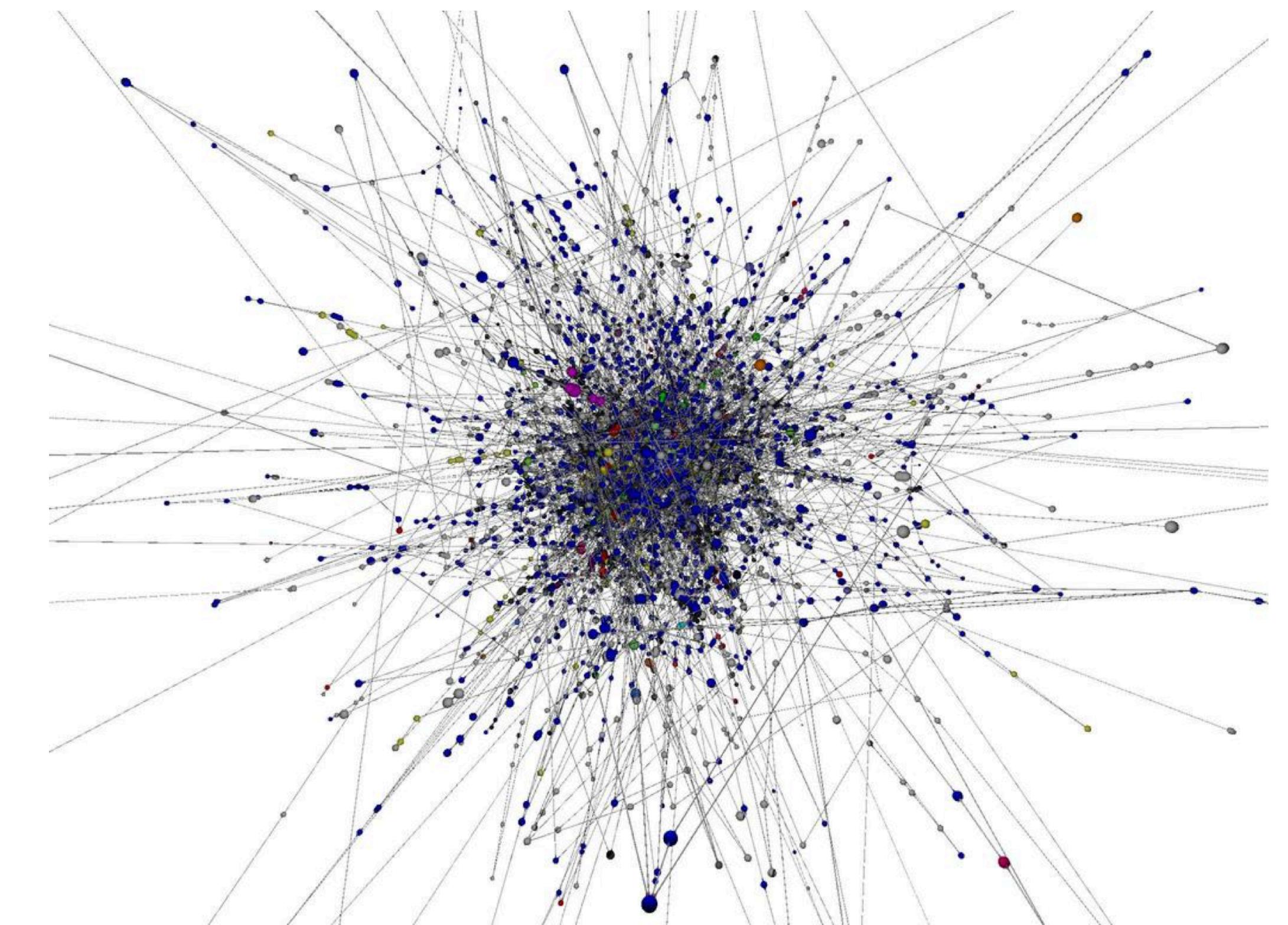
- Just a bouquet of circles?



(Stephen Coast
<https://www.fractalus.com/steve/stuff/ipmap/>)

So, preferential attachment...

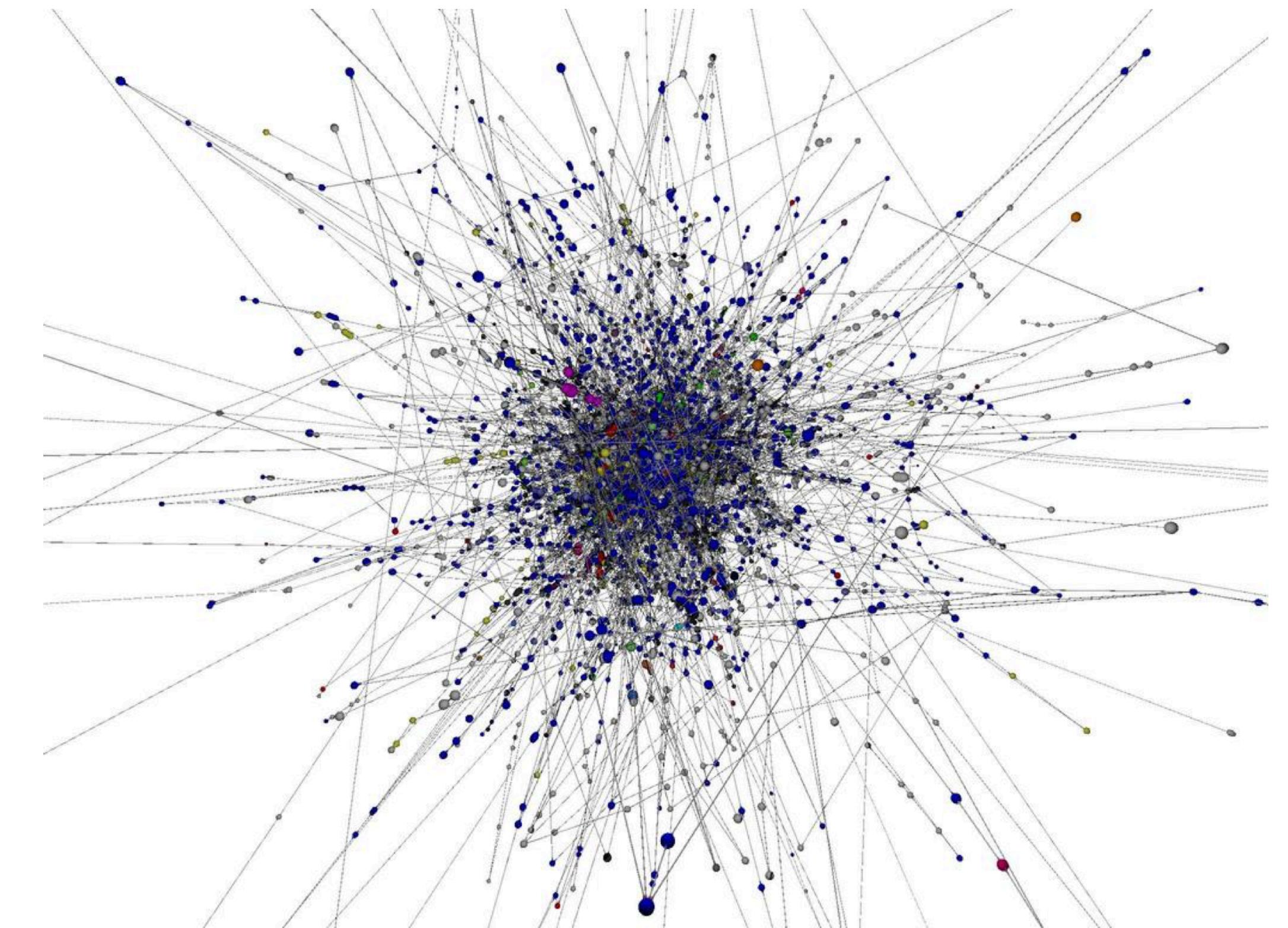
- Just a bouquet of circles?
- What is intrinsic and what is just random fluctuation?



(Stephen Coast
<https://www.fractalus.com/steve/stuff/ipmap/>)

So, preferential attachment...

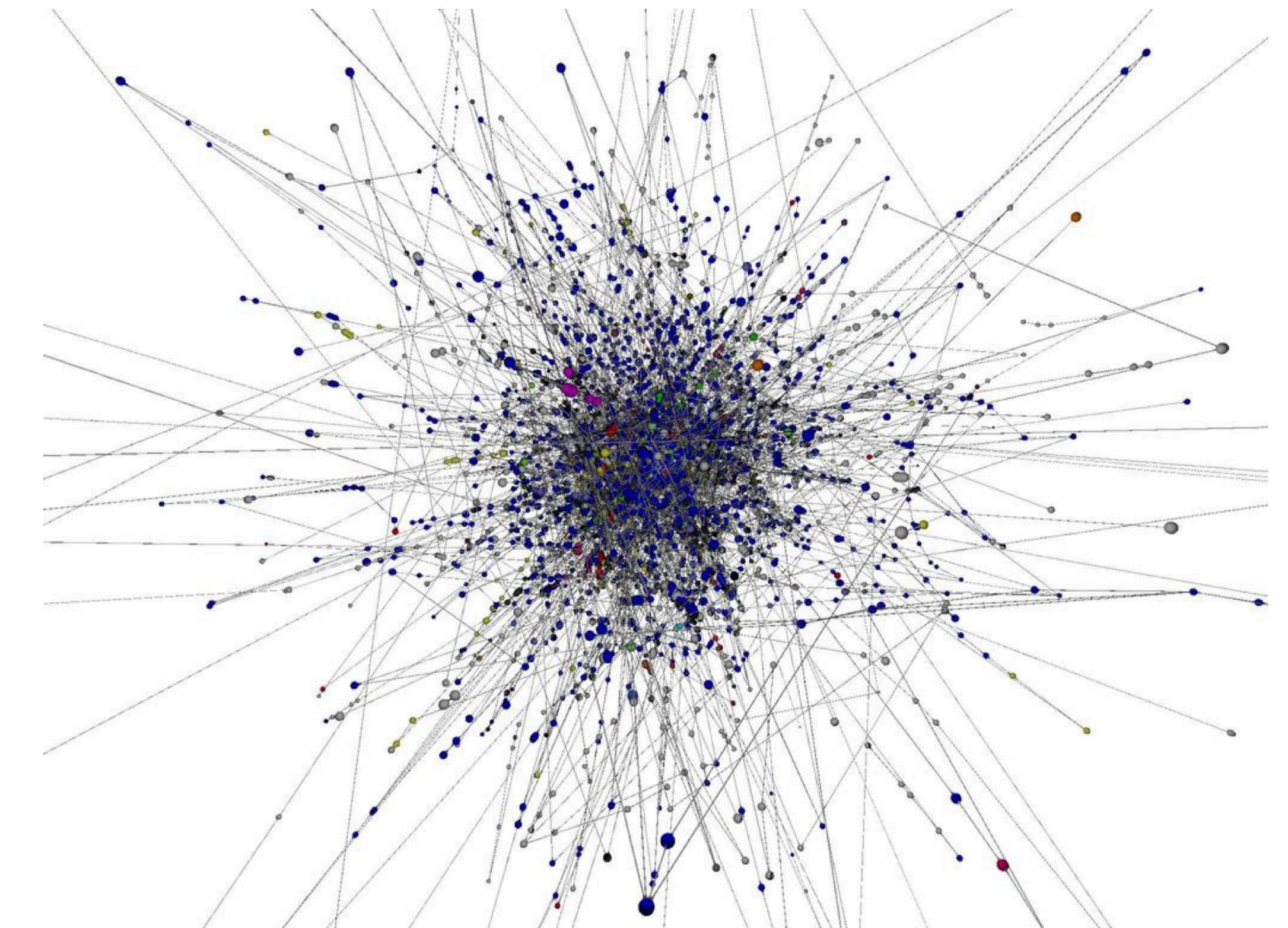
- Just a bouquet of circles?
- What is intrinsic and what is just random fluctuation?
- —> random topology



(Stephen Coast
<https://www.fractalus.com/steve/stuff/ipmap/>)

So, preferential attachment...

- Just a bouquet of circles?
- What is intrinsic and what is just random fluctuation?
- —> random topology
 - the random process of preferential attachment



(Stephen Coast
<https://www.fractalus.com/steve/stuff/ipmap/>)

Agenda

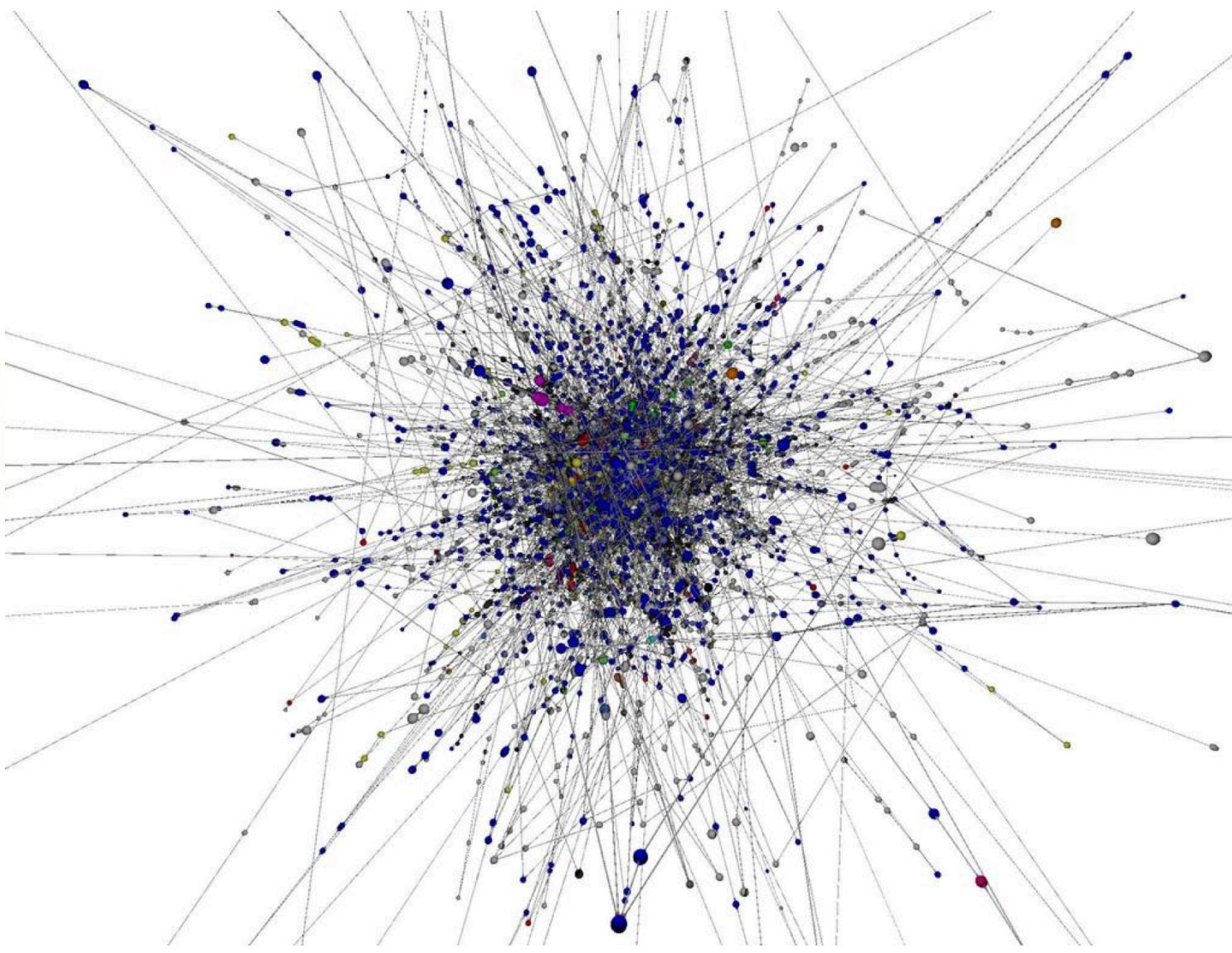


random topology

Agenda



random topology

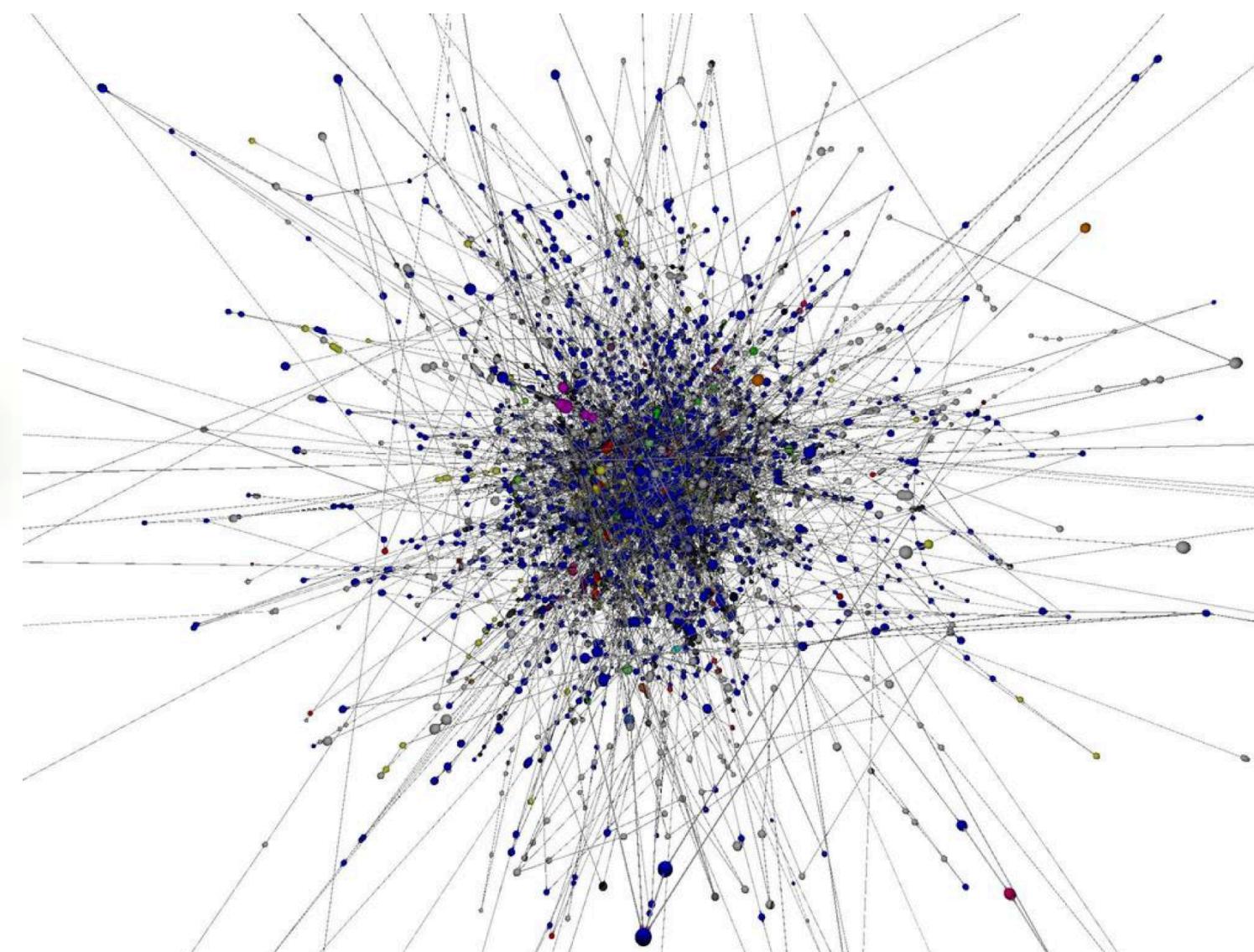


preferential attachment

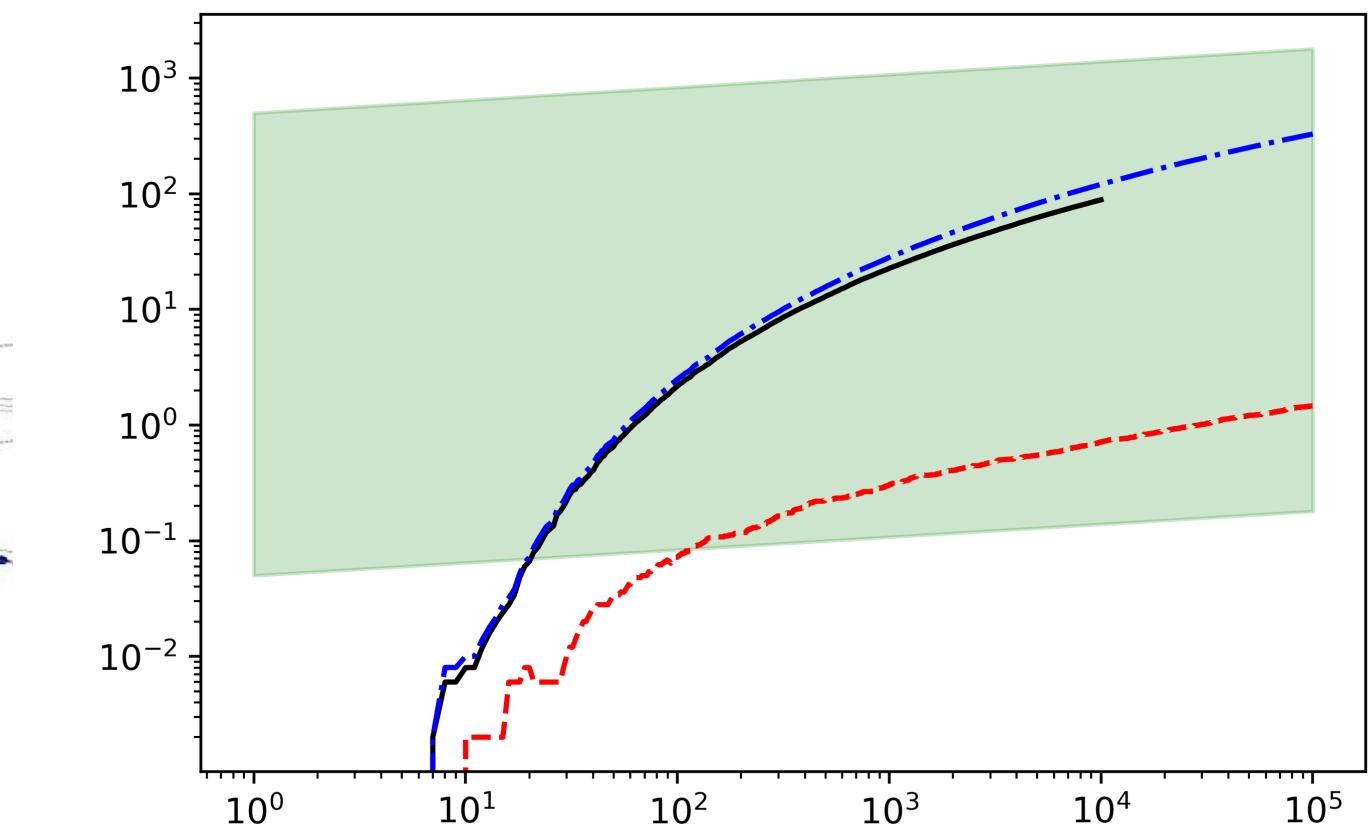
Agenda



random topology



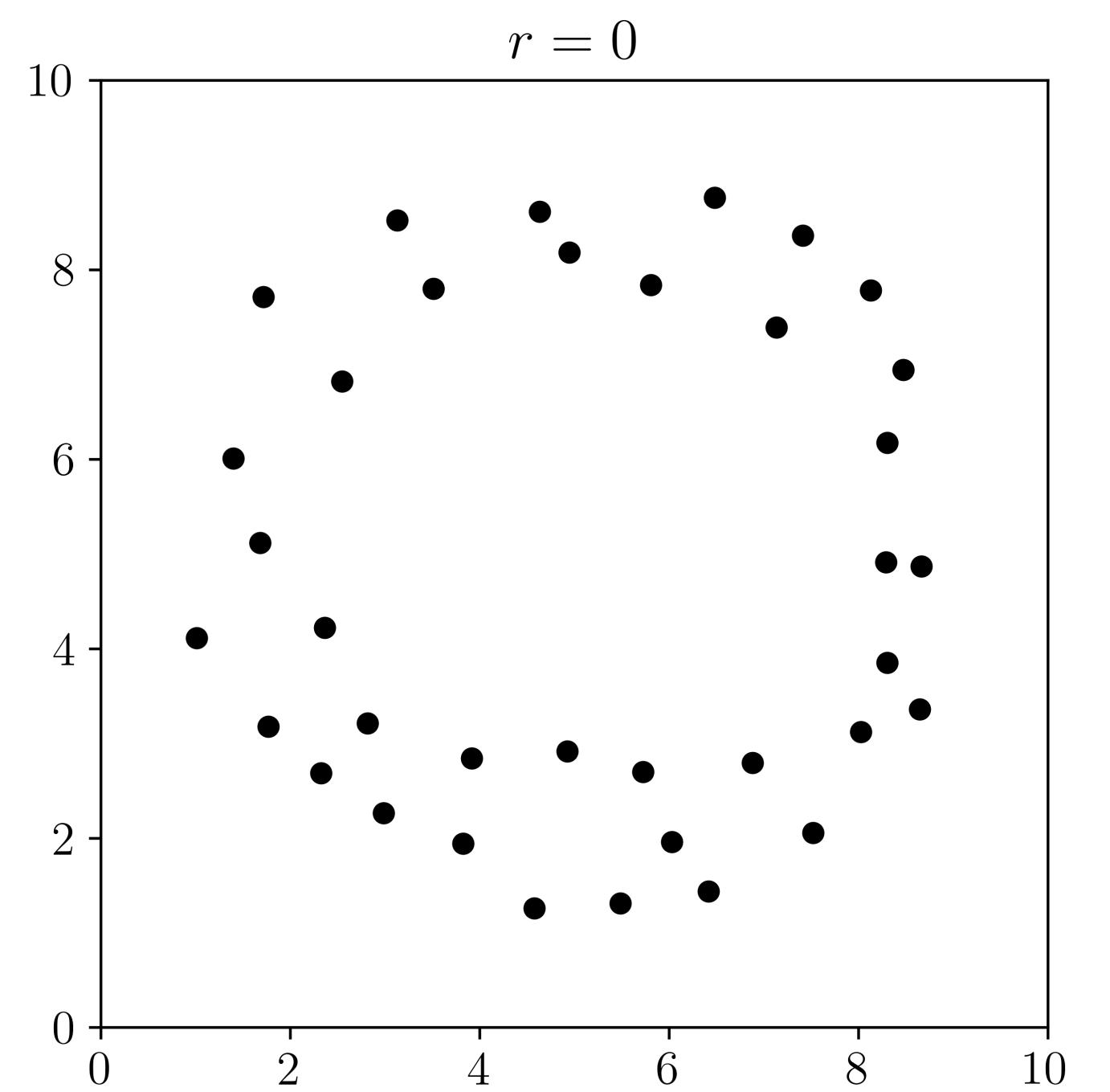
preferential attachment

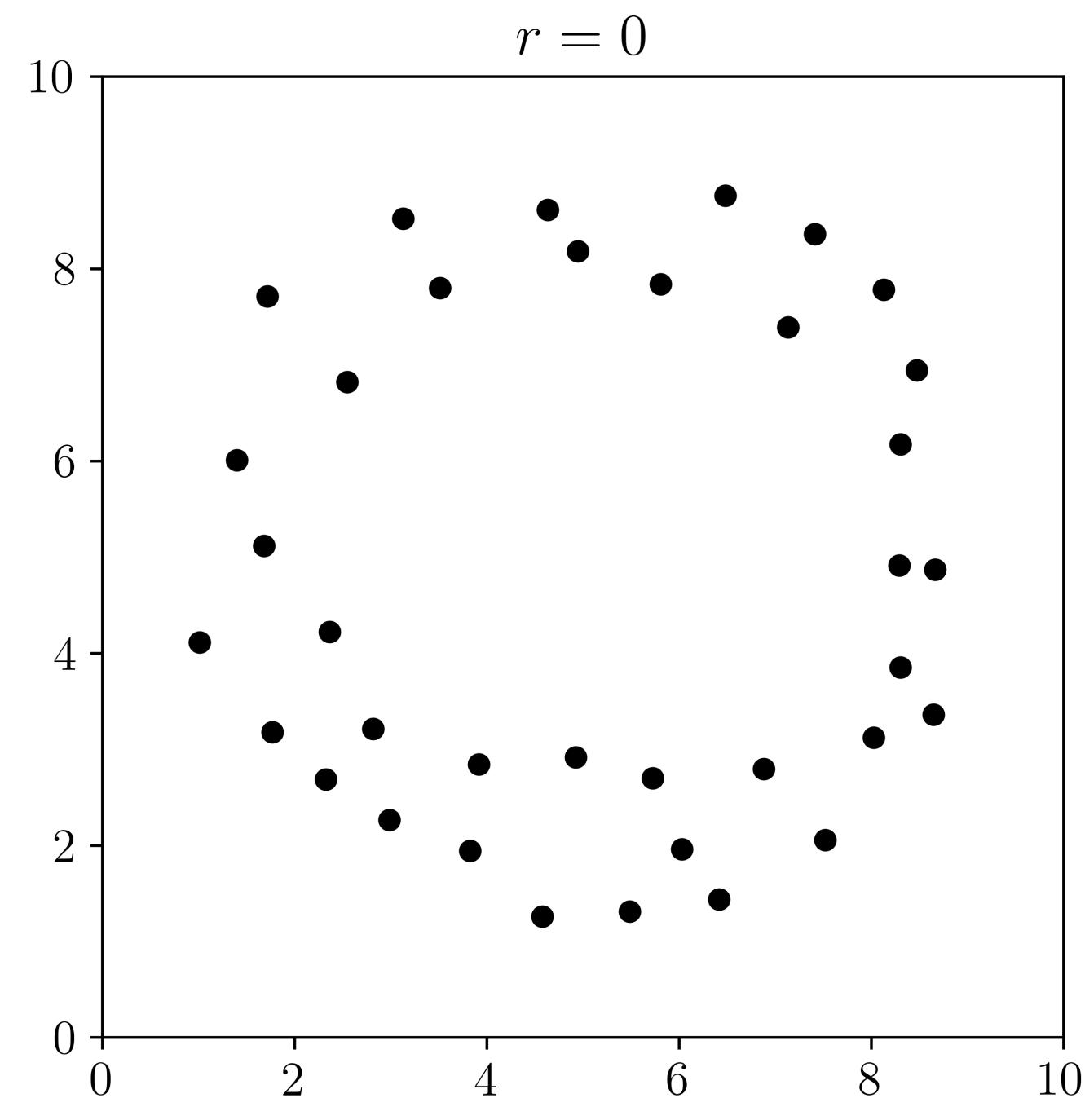
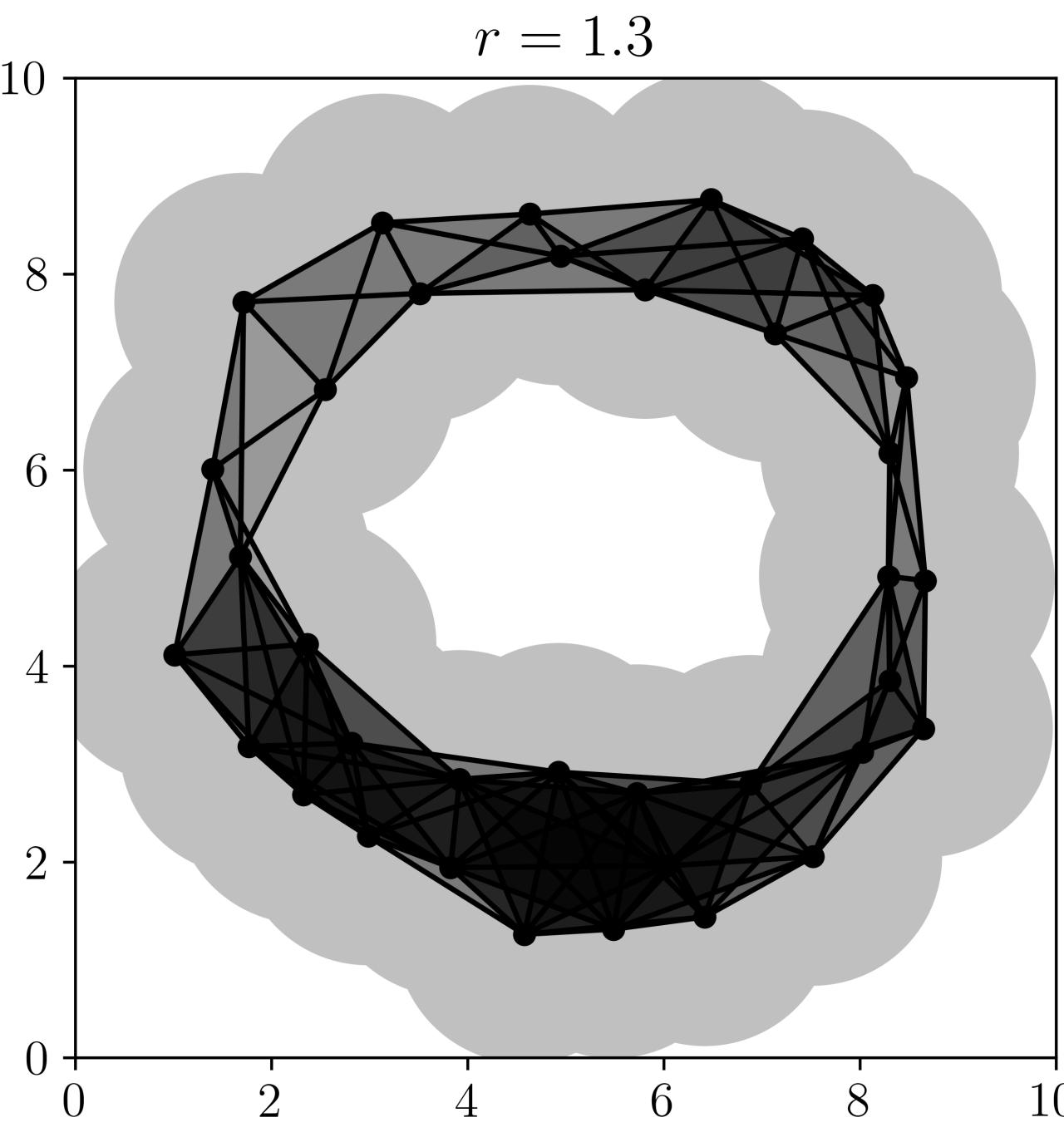
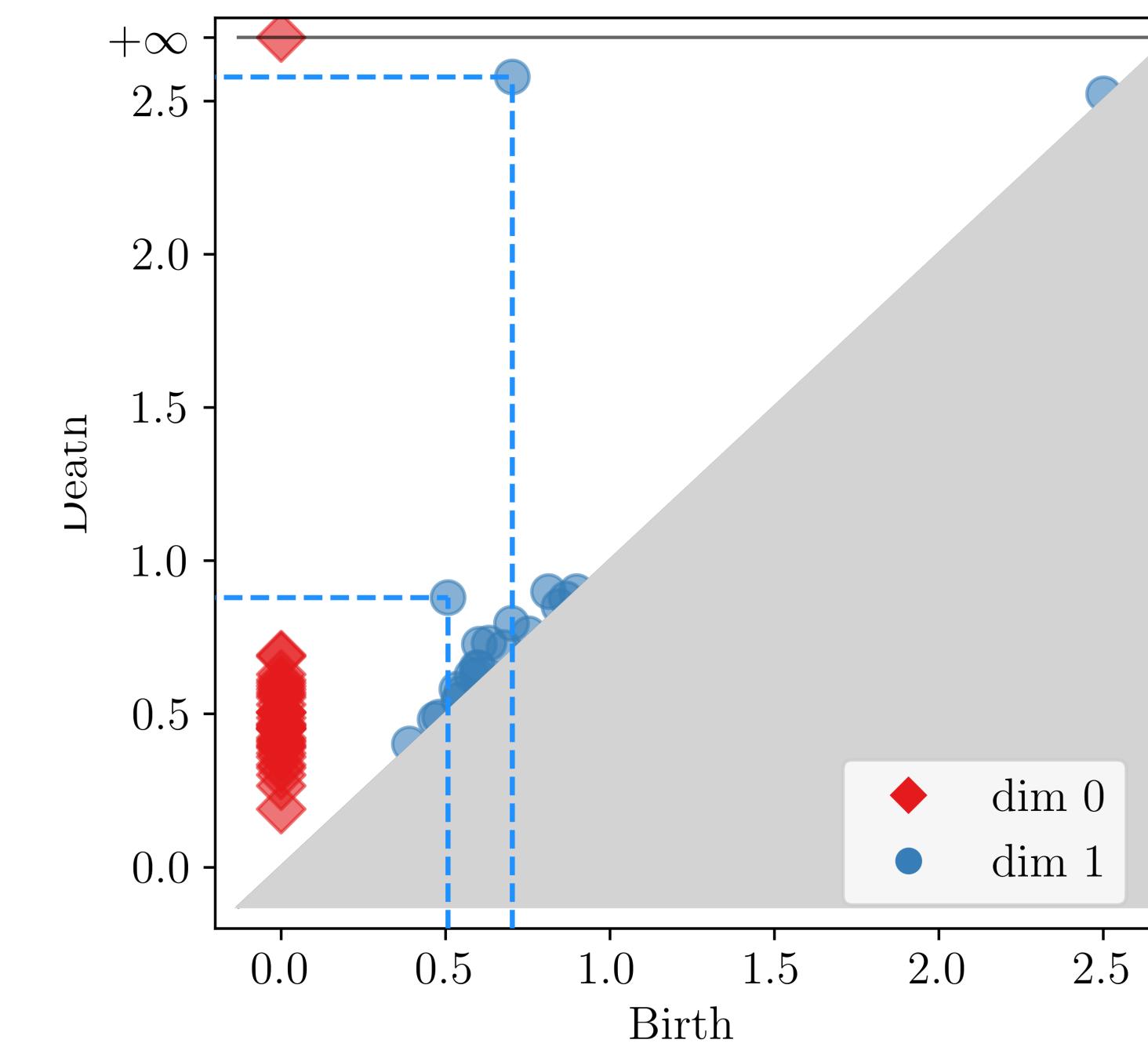


our result

I. A Probabilist's Apology

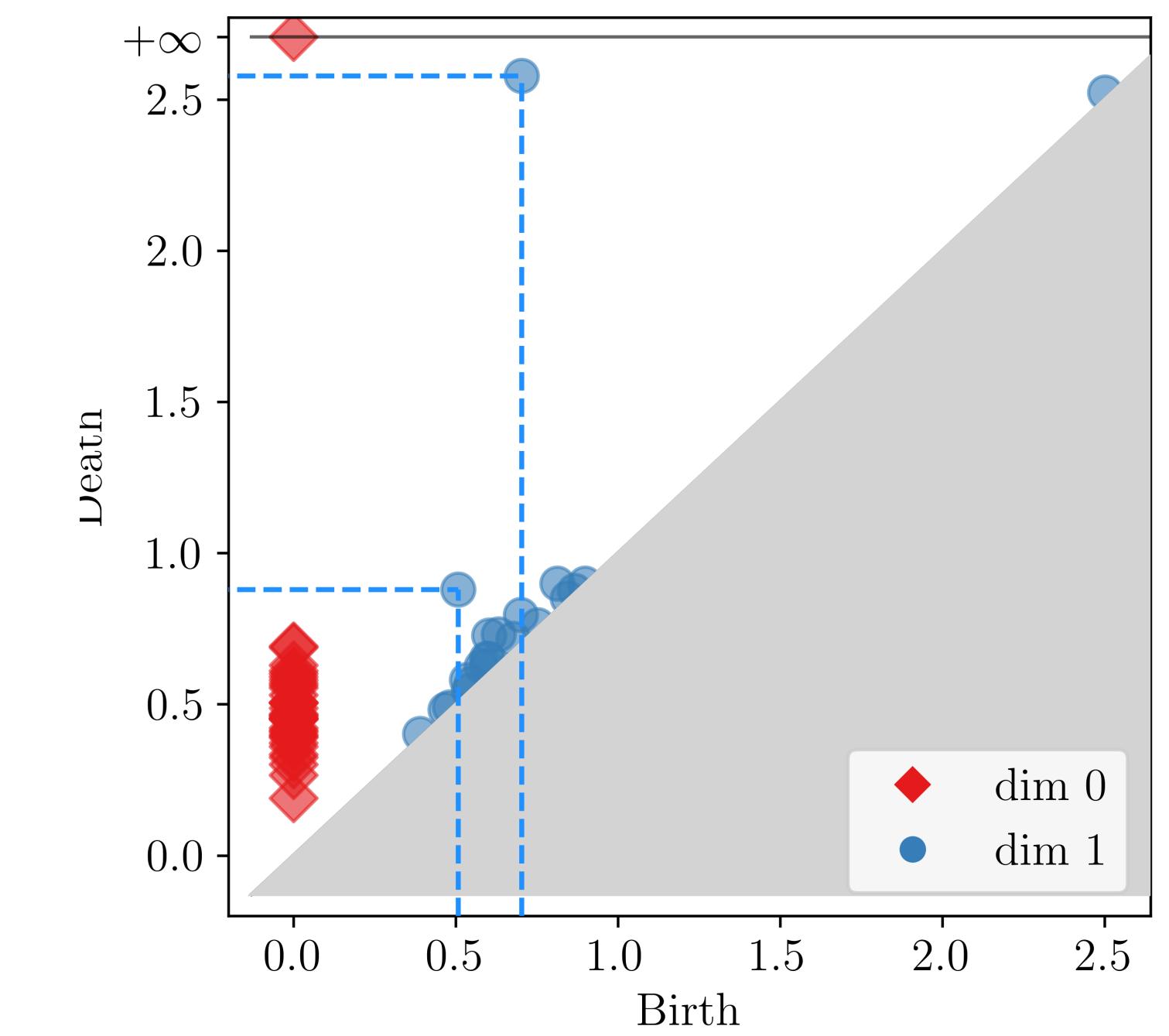
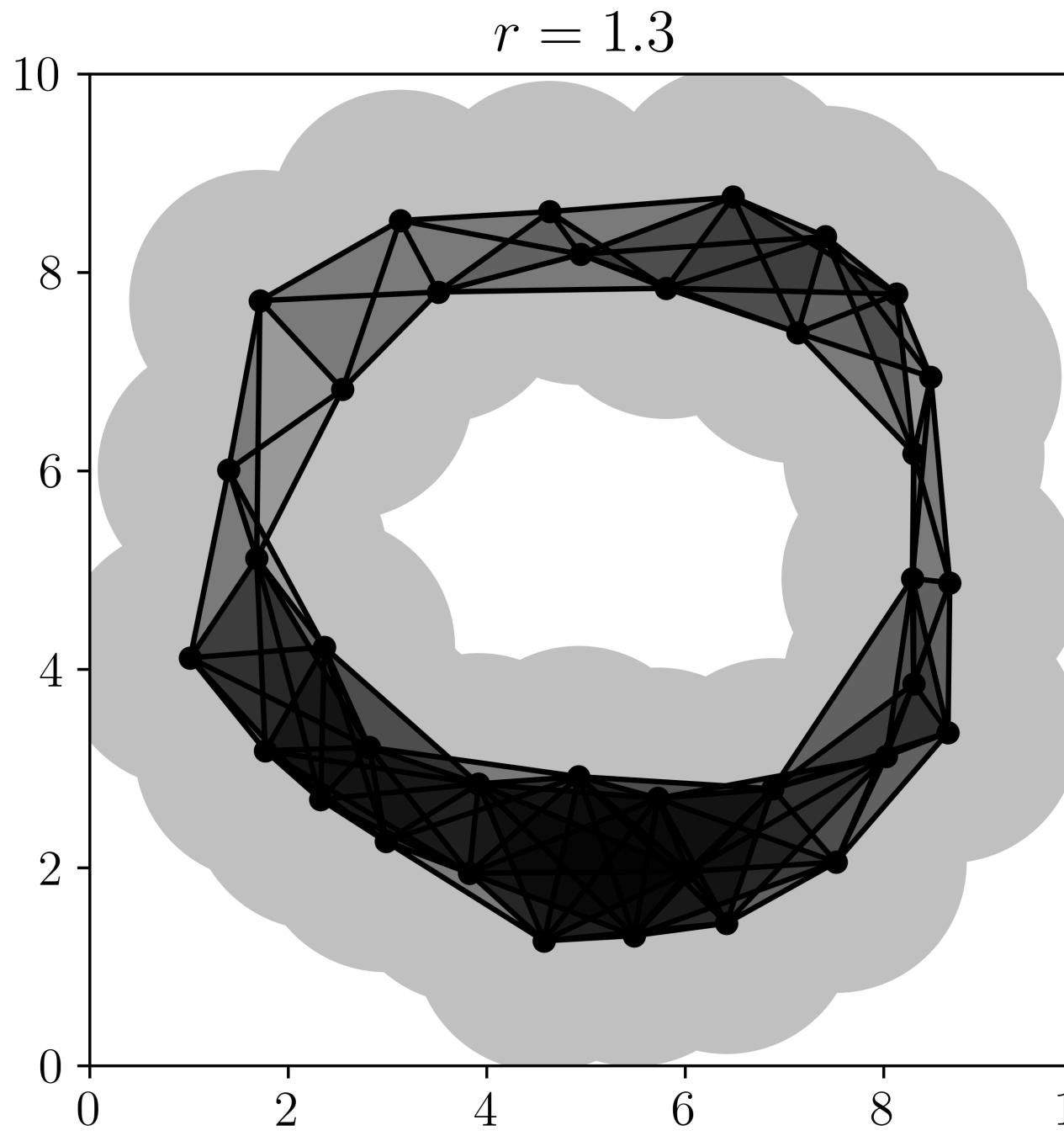
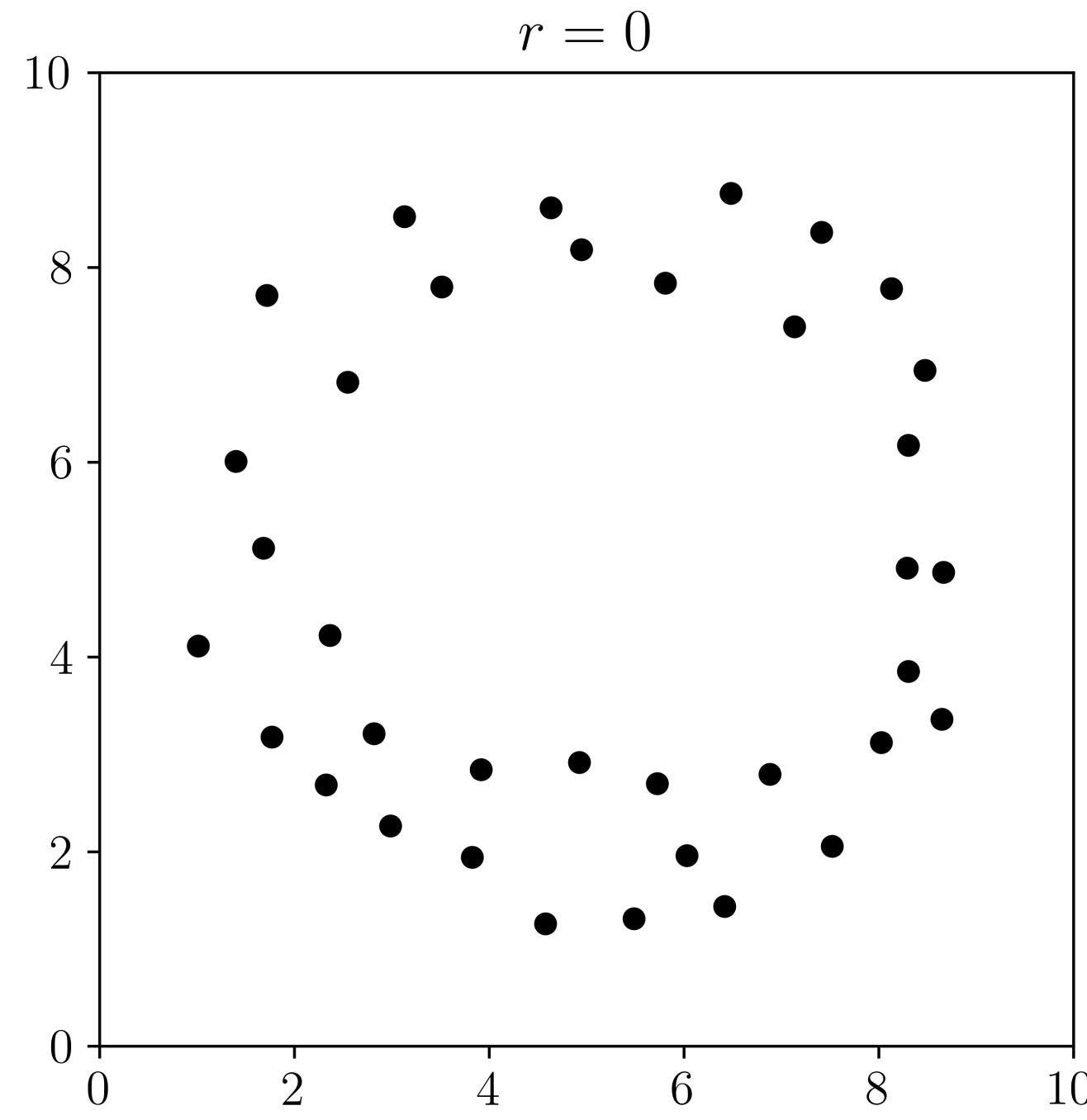
Why Random Topology and What we Know



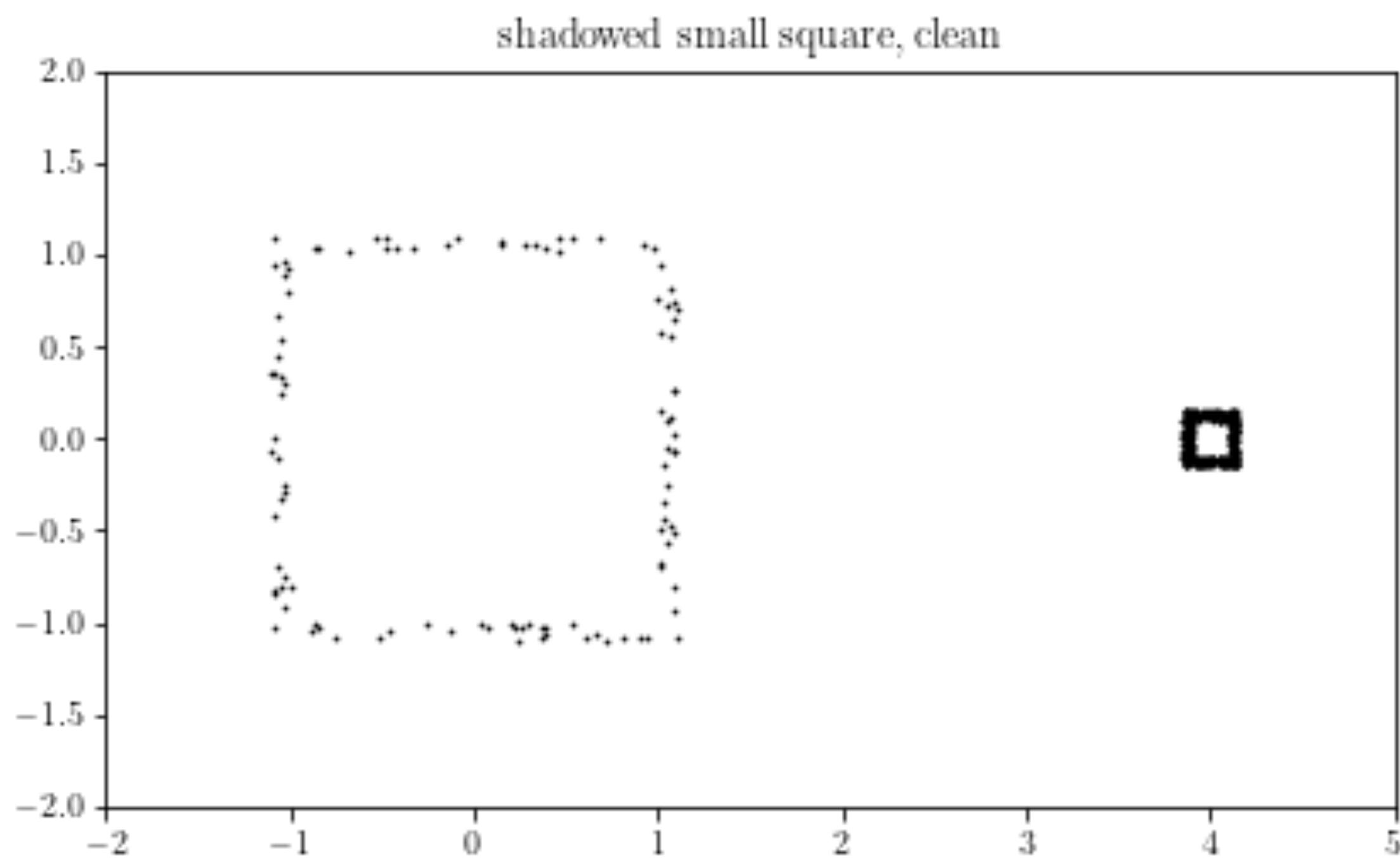

 $r = 0$

 $r = 1.3$


plots generated by Andrey Yao

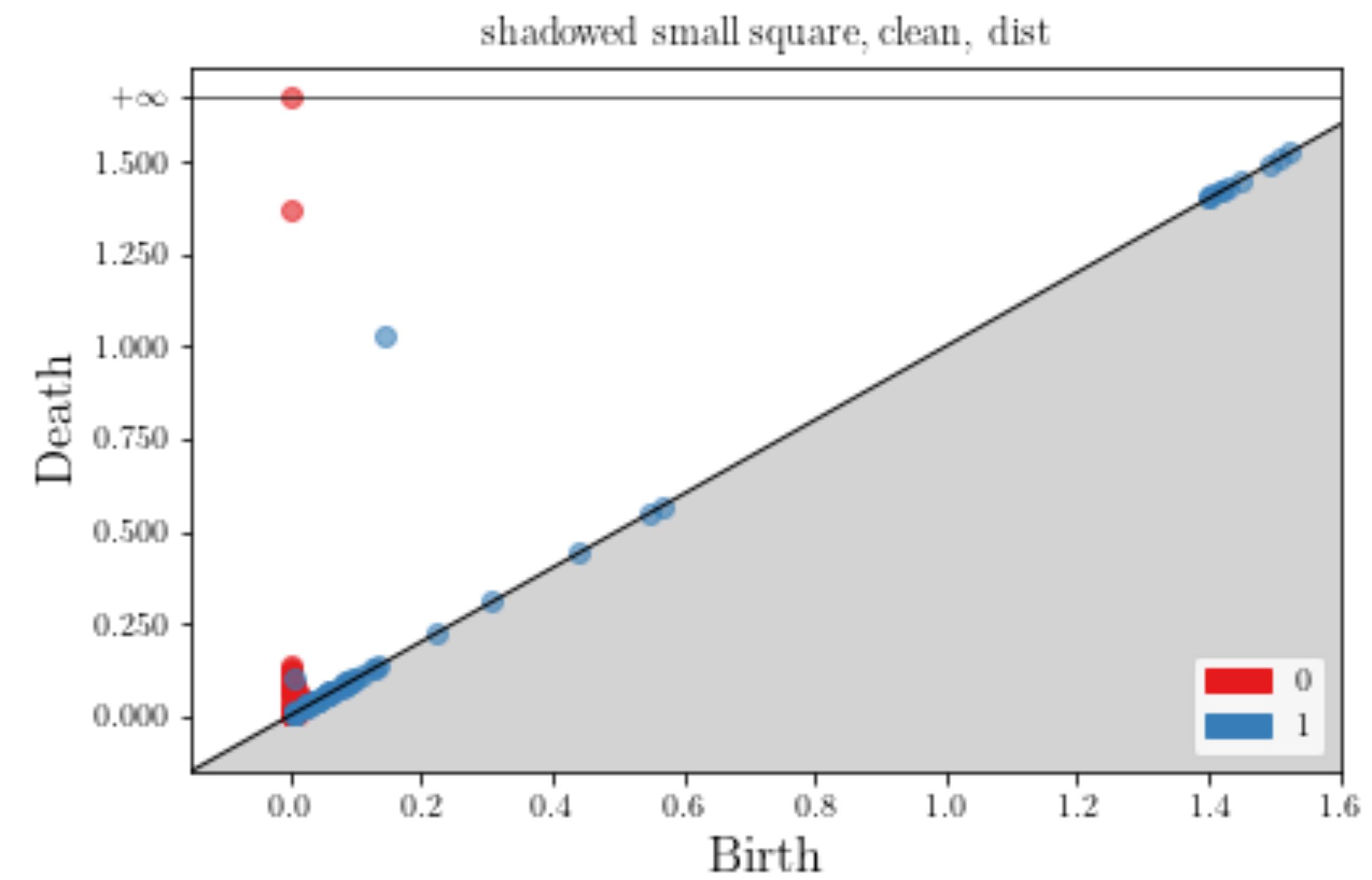
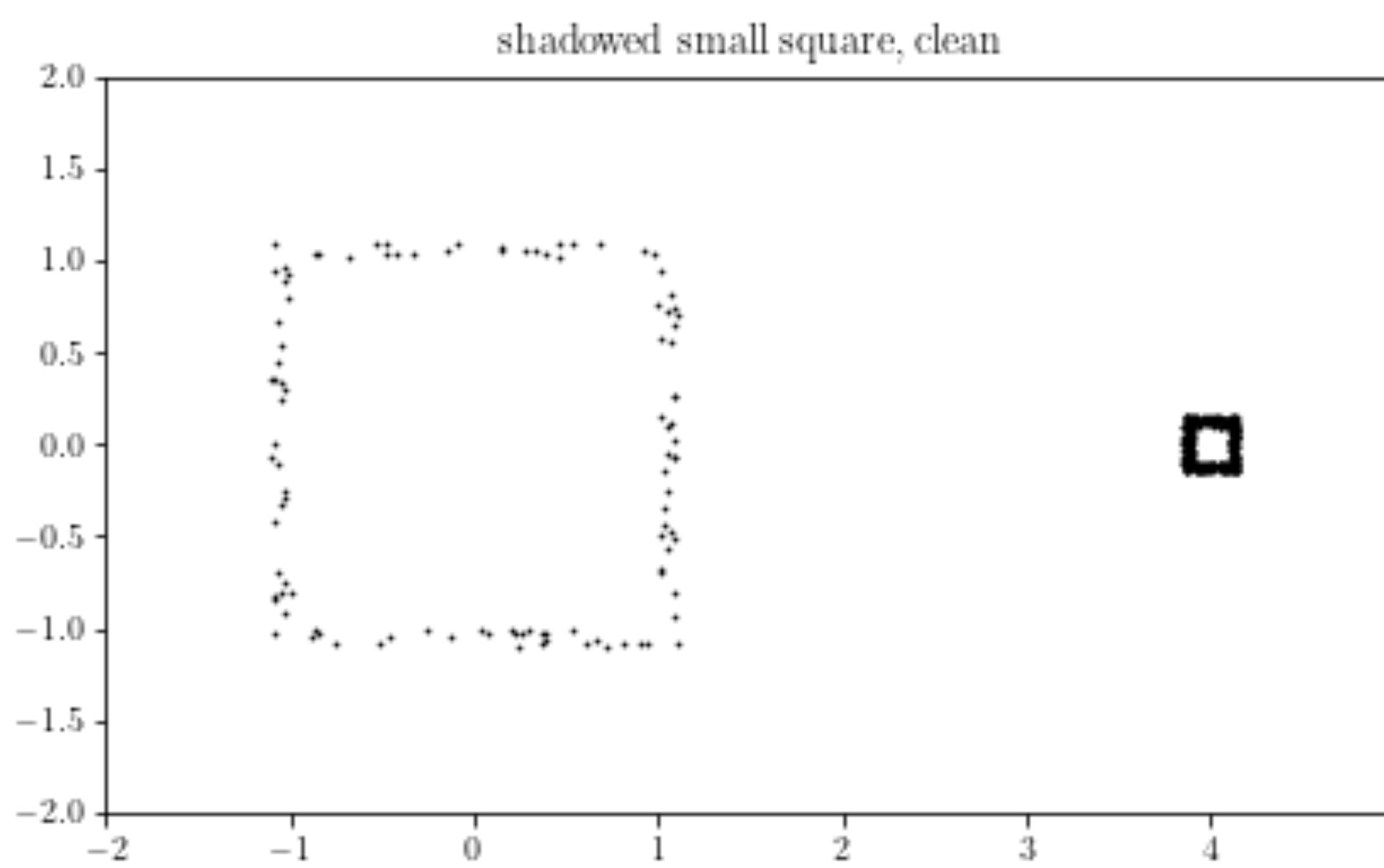
Size is Signal



Or is it?



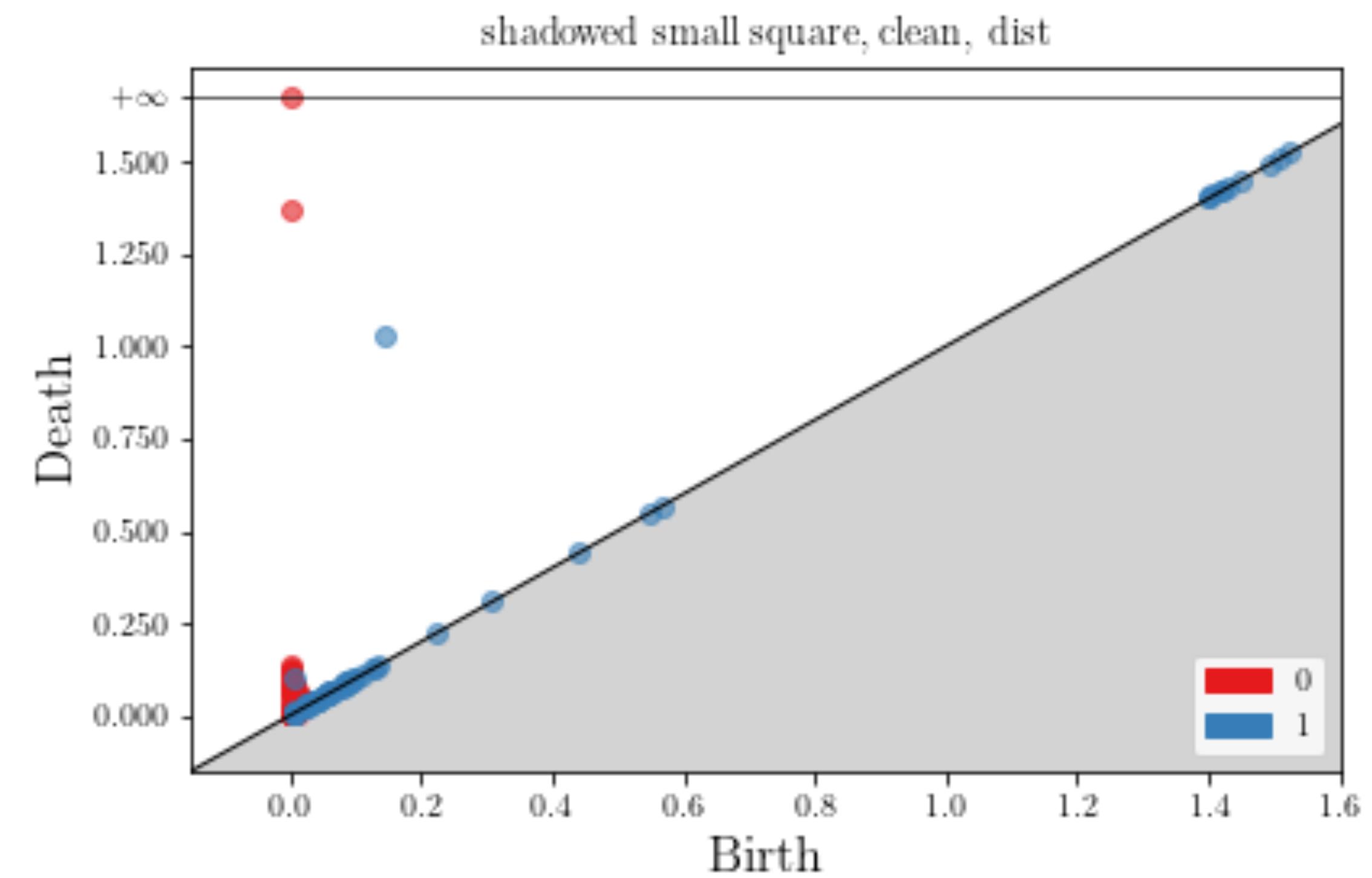
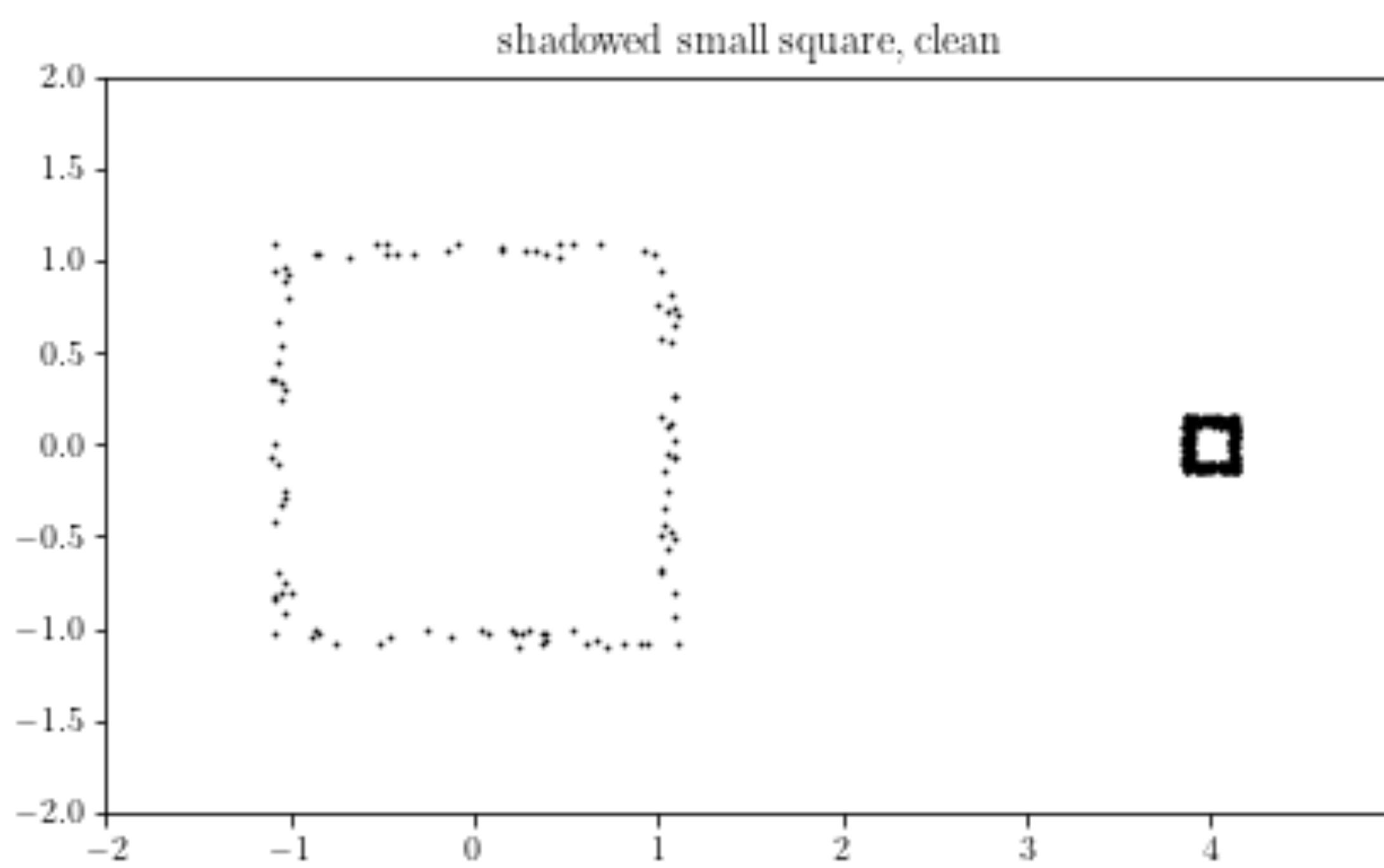
Or is it?



Size is Signal?

**Surprise
Size is Signal.**

Random points don't do that.



Signal is what is not random.

**Signal is what is not random.
So what is random?**

What we know

[not meant to be complete]

What we know

[not meant to be complete]

- Erdos-Renyi clique complexes

What we know

[not meant to be complete]

- Erdos-Renyi clique complexes
 - Kahle 2009, 2014
 - Kahle and Meckes 2013
 - Costa et al 2015
 - Malen 2013
 - etc

What we know

[not meant to be complete]

- Erdos-Renyi clique complexes
 - Kahle 2009, 2014
 - Kahle and Meckes 2013
 - Costa et al 2015
 - Malen 2013
 - etc
- random geometric complexes

What we know

[not meant to be complete]

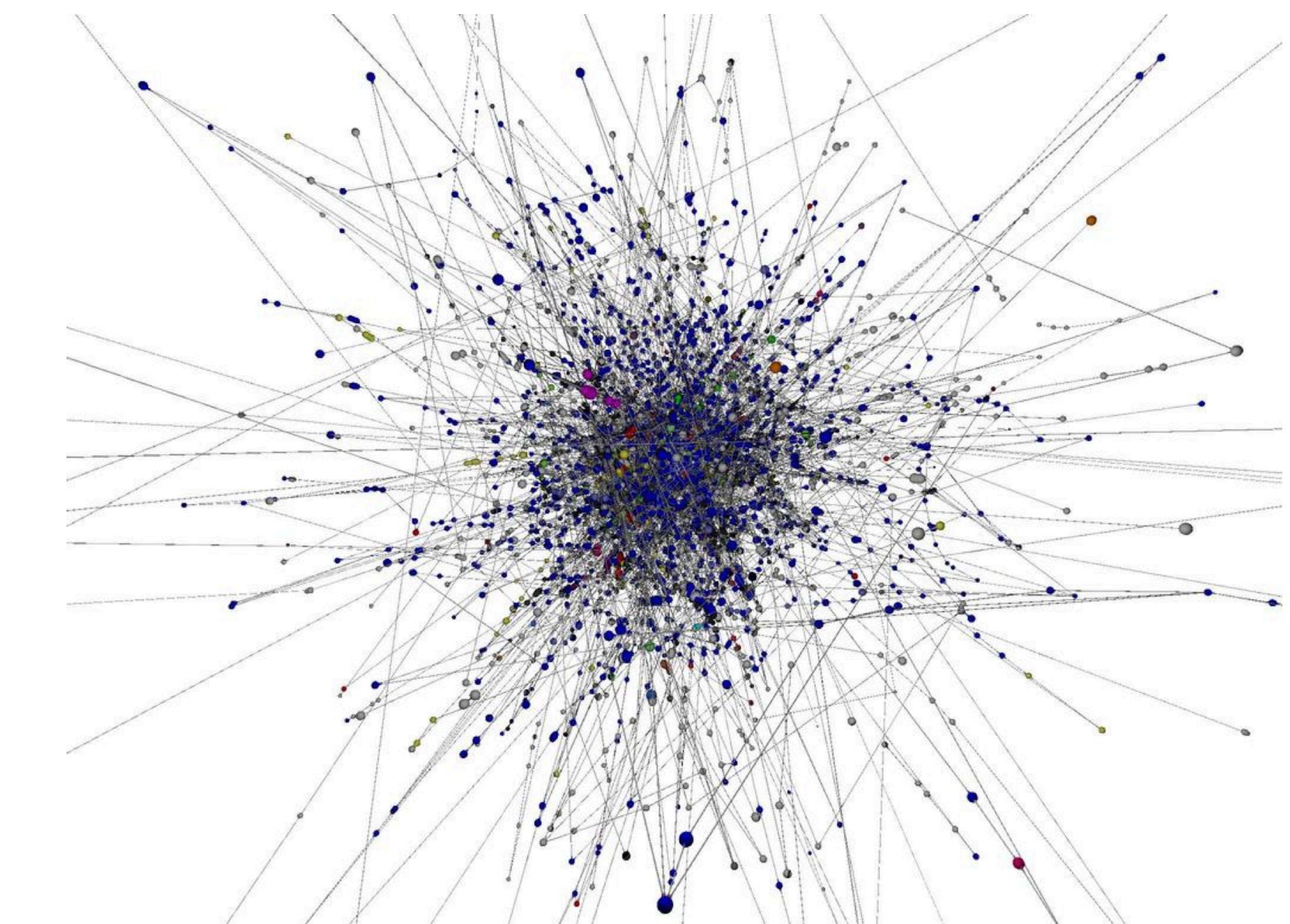
- Erdos-Renyi clique complexes
 - Kahle 2009, 2014
 - Kahle and Meckes 2013
 - Costa et al 2015
 - Malen 2013
 - etc
- random geometric complexes
 - Kahle 2011
 - Kahle and Meckes 2013
 - Yogeshwaran and Adler 2015
 - Bobrowski et al 2017
 - Hiraoka et al 2018
 - Thomas and Owada 2021a, b
 - Owada and Wei 2022
 - etc

II. Preferential Attachment

Beyond independence and homogeneity

Independent and identically distributed?

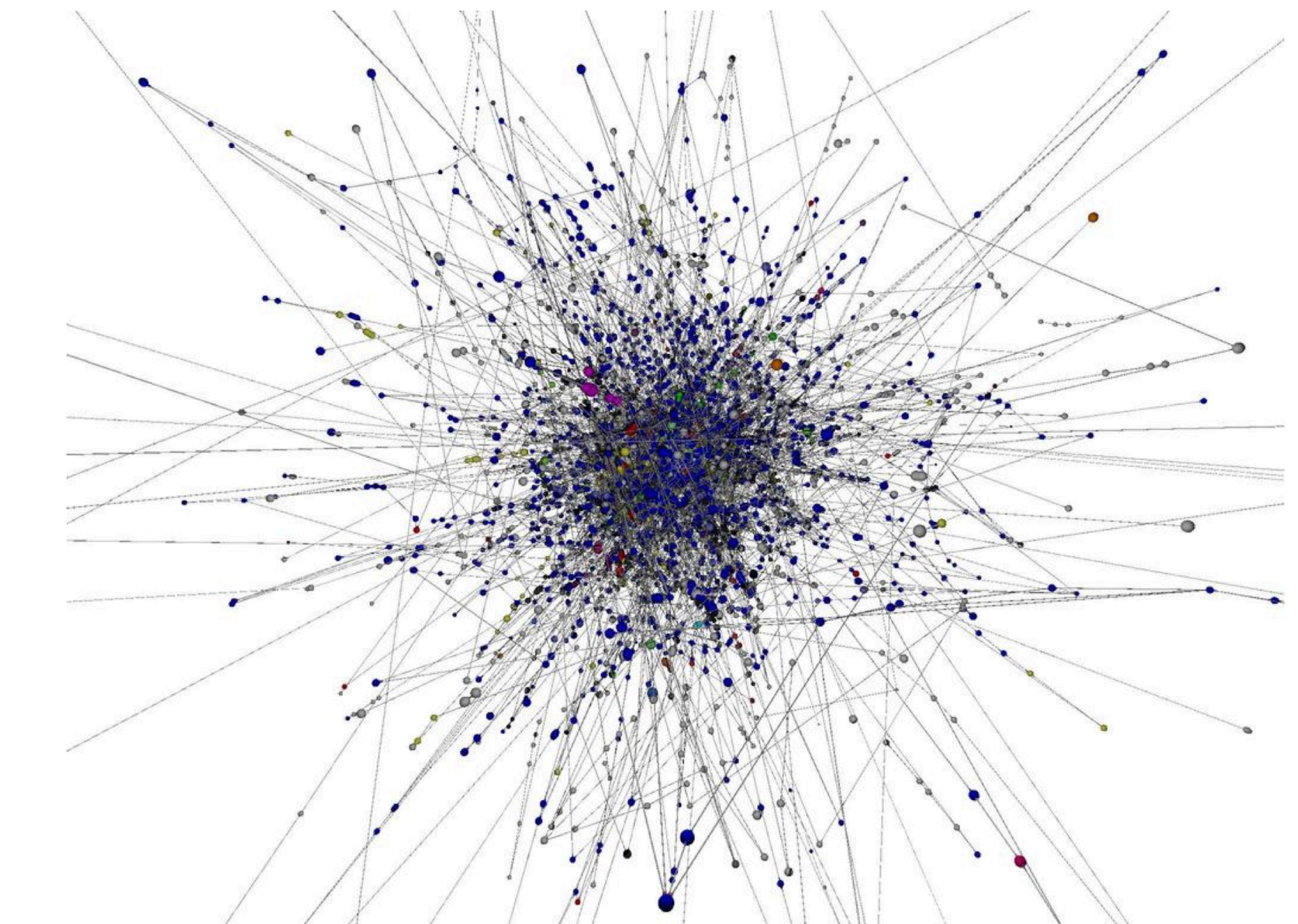
Independent and identically distributed?



(Stephen Coast
<https://www.fractalus.com/steve/stuff/ipmap/>)

Preferential Attachment

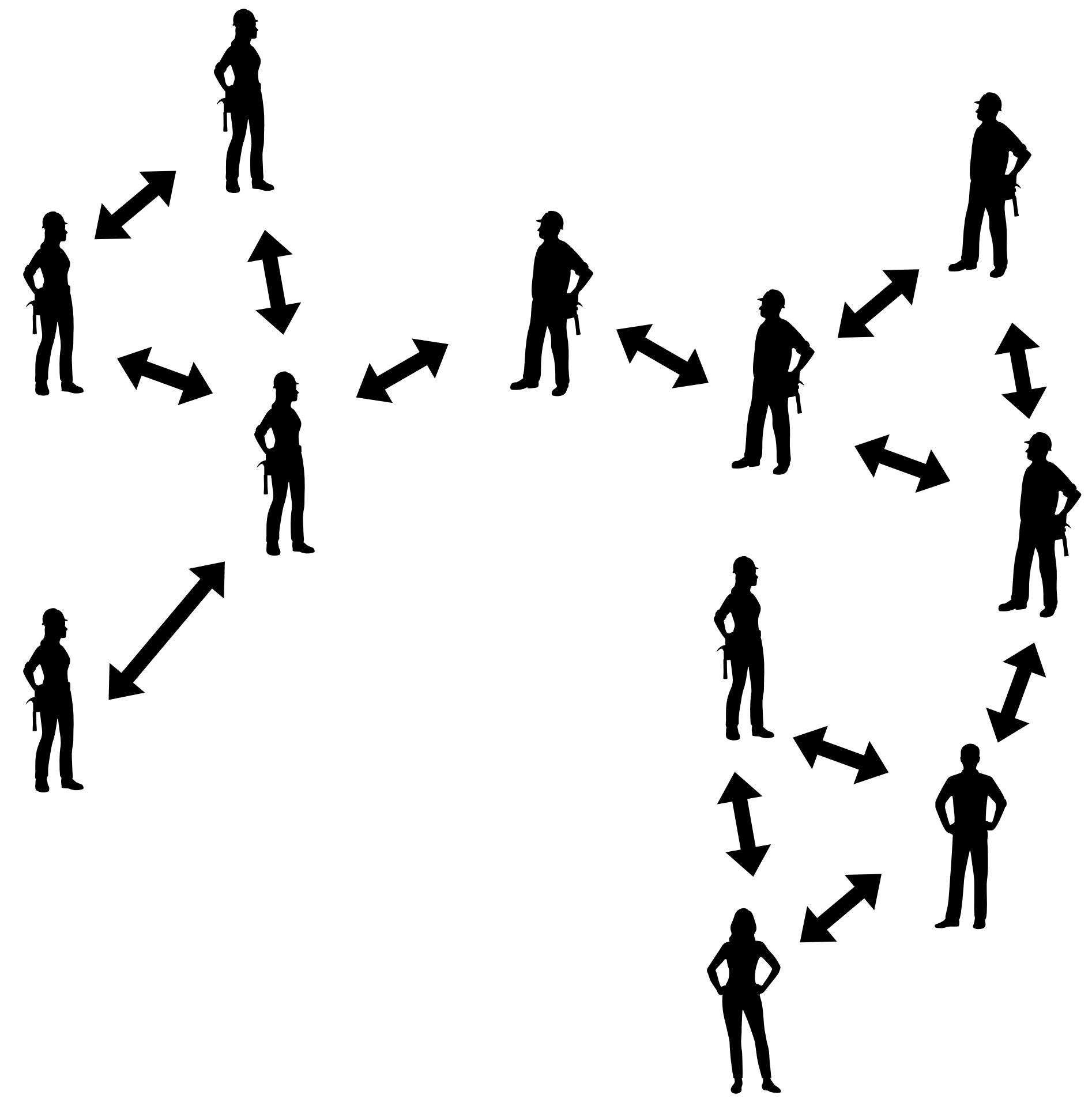
[Albert and Barabasi 1999]



(Stephen Coast
<https://www.fractalus.com/steve/stuff/ipmap/>)

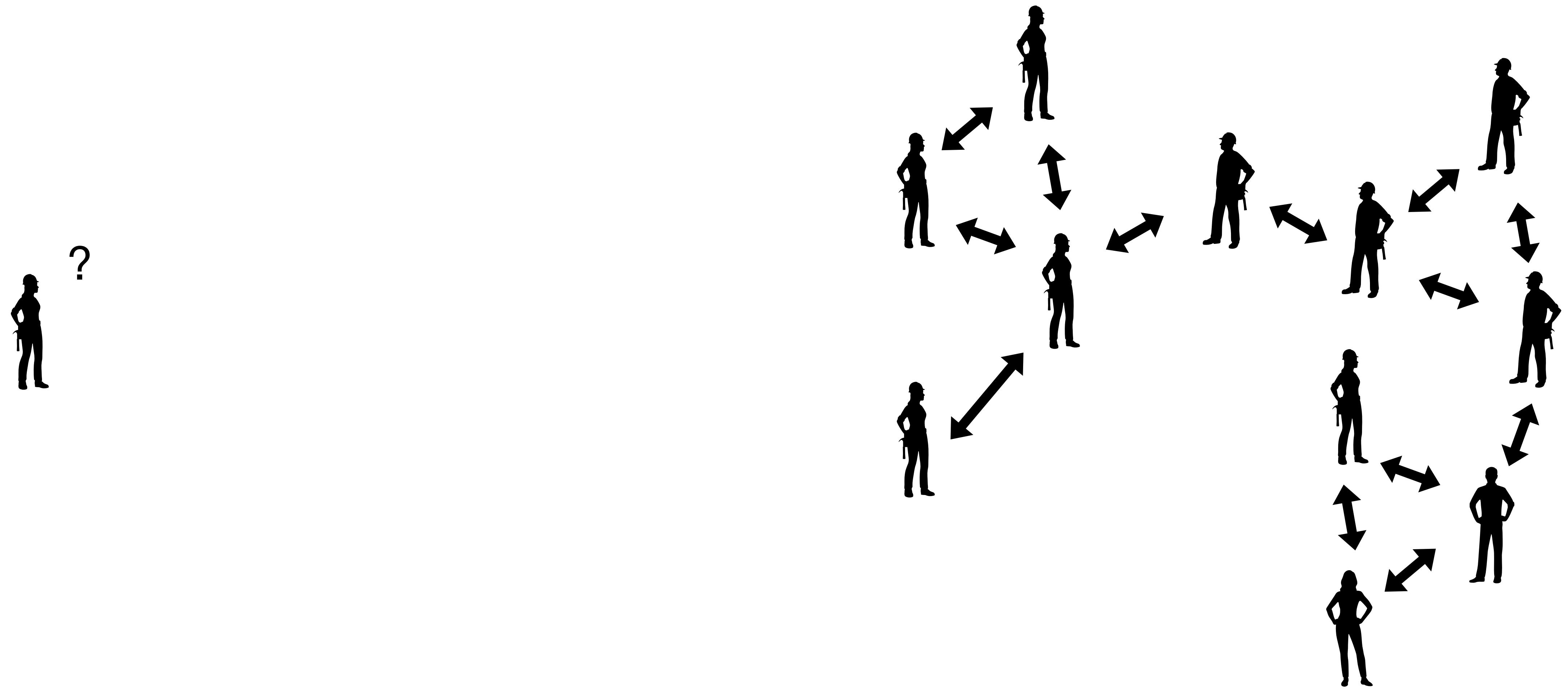
Preferential Attachment

[Albert and Barabasi 1999]



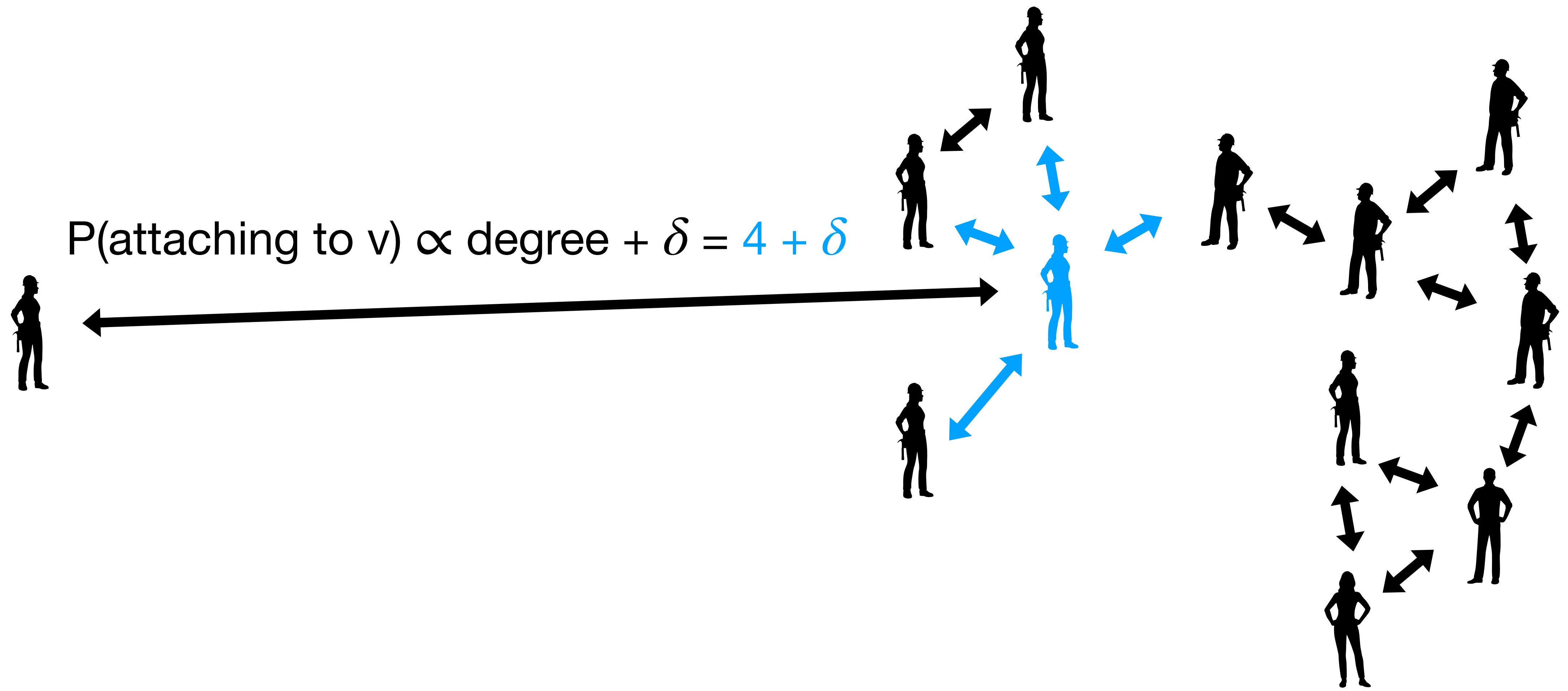
Preferential Attachment

[Albert and Barabasi 1999]



Preferential Attachment

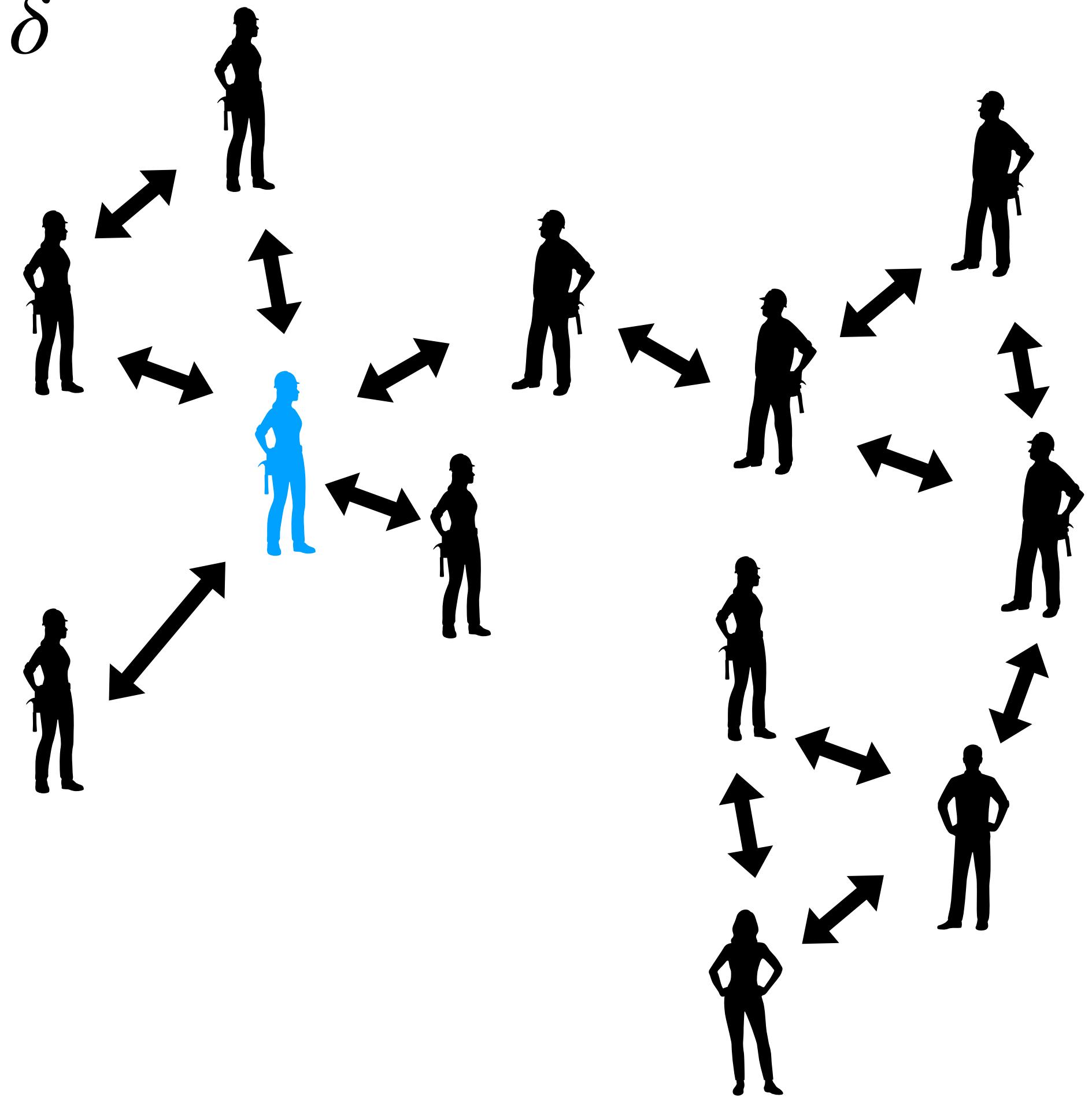
[Albert and Barabasi 1999]



Preferential Attachment

[Albert and Barabasi 1999]

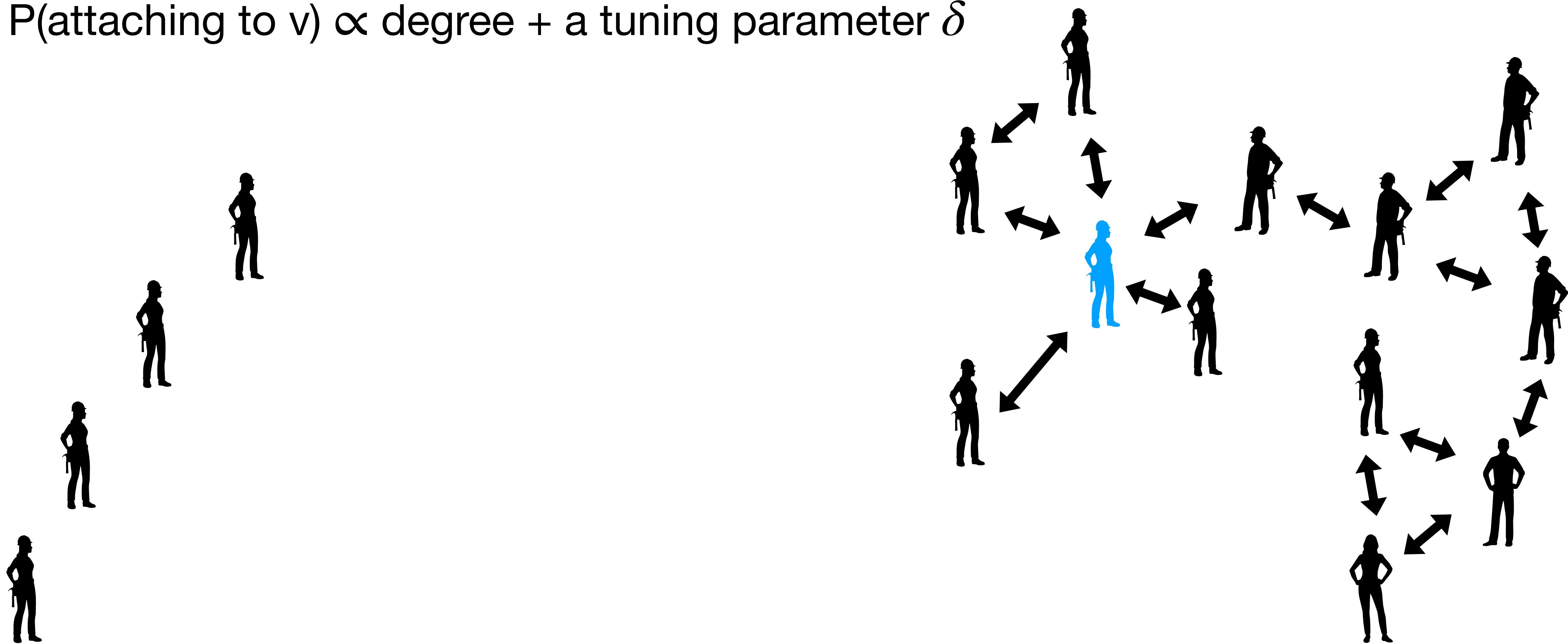
$P(\text{attaching to } v) \propto \text{degree} + \text{a tuning parameter } \delta$



Preferential Attachment

[Albert and Barabasi 1999]

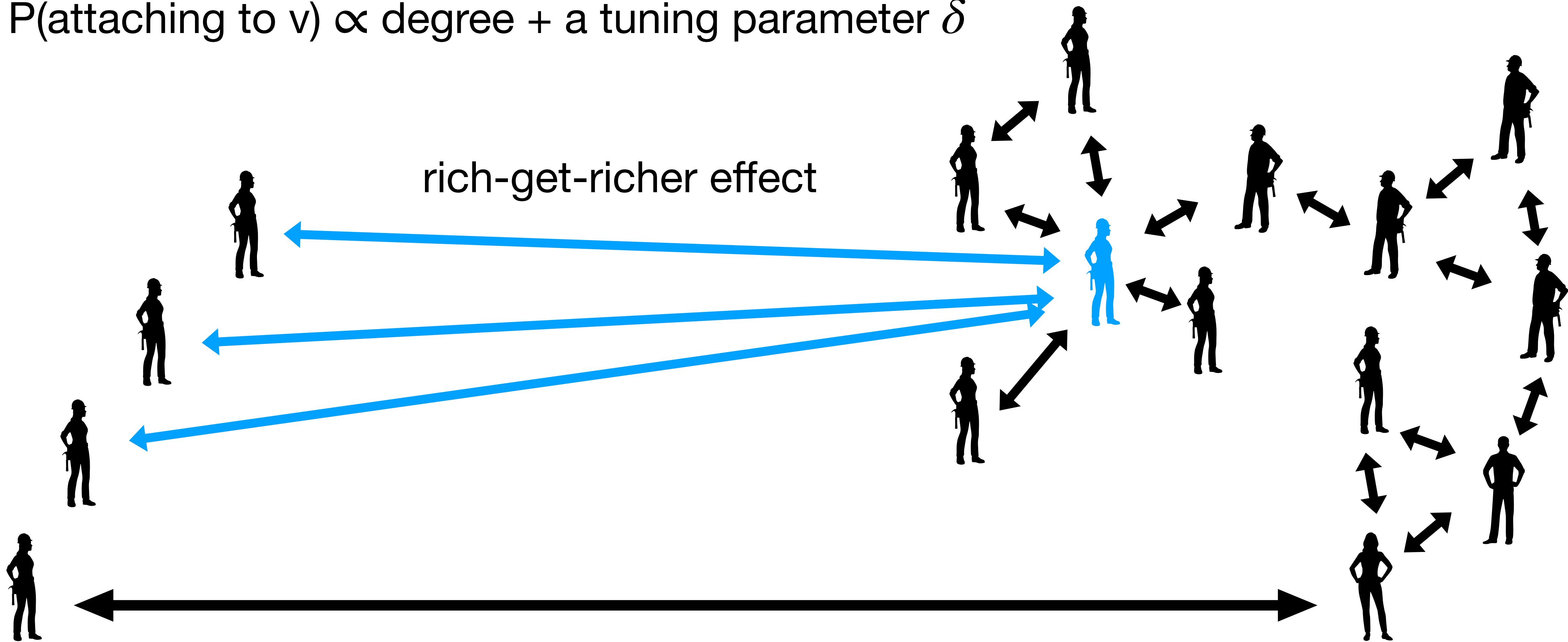
$$P(\text{attaching to } v) \propto \text{degree} + \text{a tuning parameter } \delta$$



Preferential Attachment

[Albert and Barabasi 1999]

$$P(\text{attaching to } v) \propto \text{degree} + \text{a tuning parameter } \delta$$



What do we know?

What do we know?

- triangle counts and clustering coefficient [Bollobas and Ridden 2002, Prokhorenkova et al 2013]

What do we know?

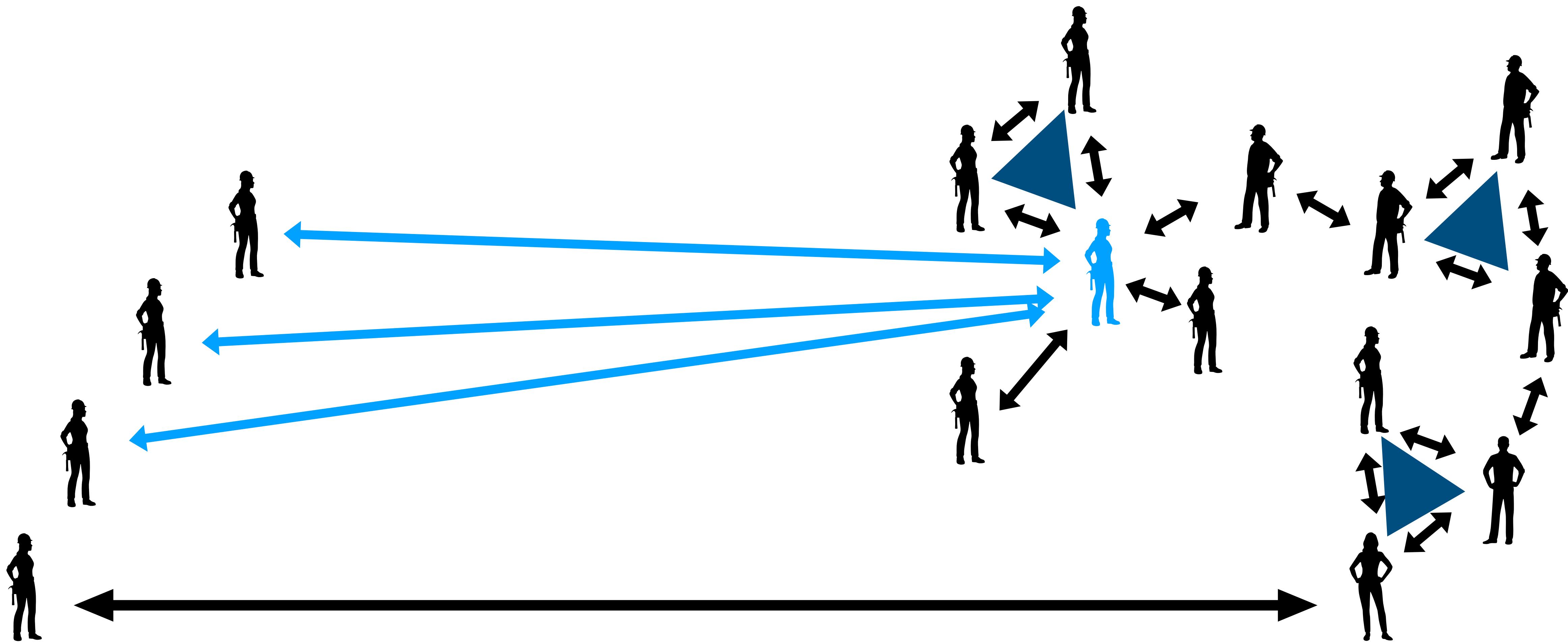
- triangle counts and clustering coefficient [Bollobas and Ridden 2002, Prokhorenkova et al 2013]
- subgraph counts [Garavaglia and Steghuis 2019]

What do we know?

- triangle counts and clustering coefficient [Bollobas and Ridden 2002, Prokhorenkova et al 2013]
- subgraph counts [Garavaglia and Steghuis 2019]
- and more...

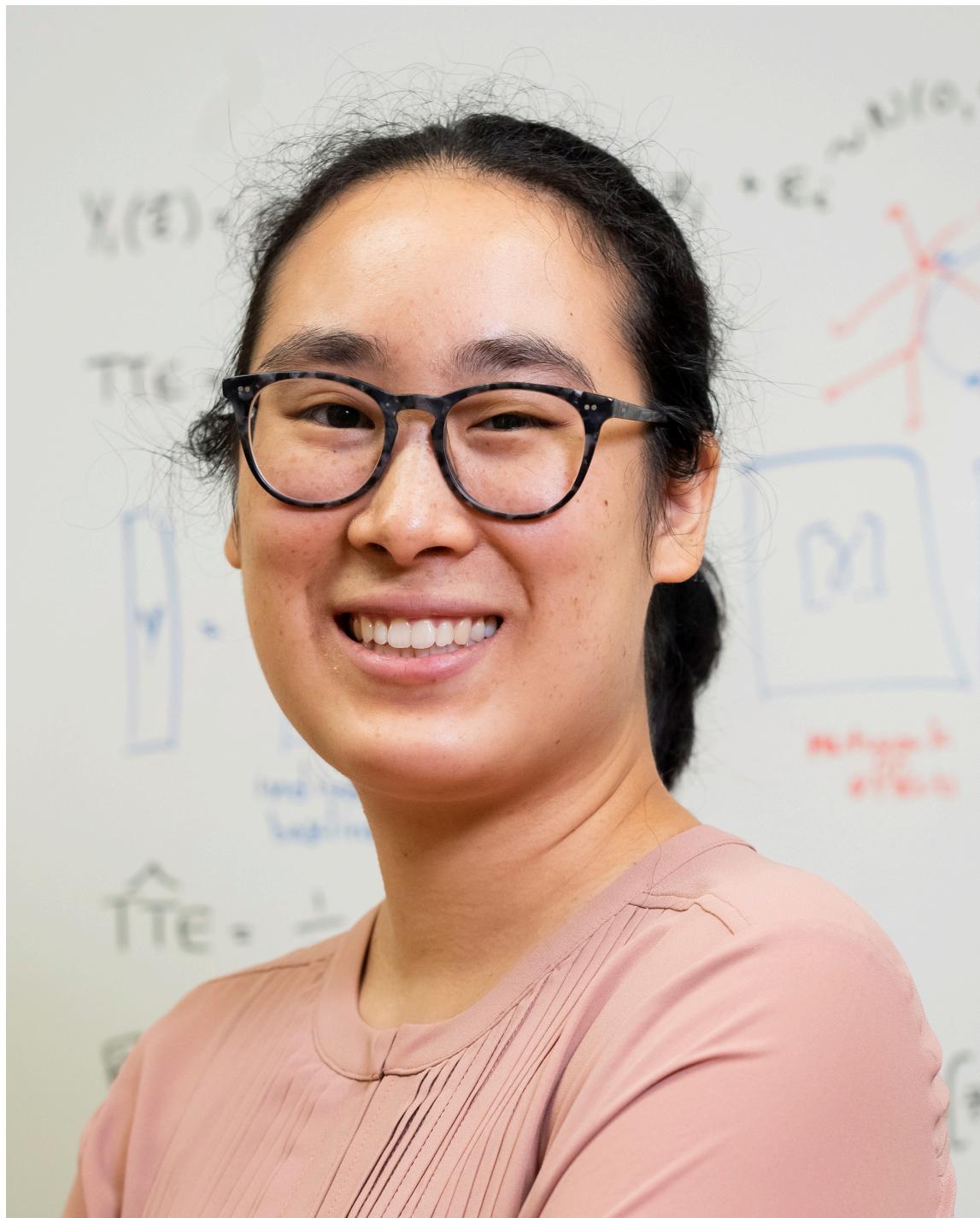
Clique Complex

aka Flag Complex



III Topology of Preferential Attachment

My Lovely Collaborators



Christina Lee Yu



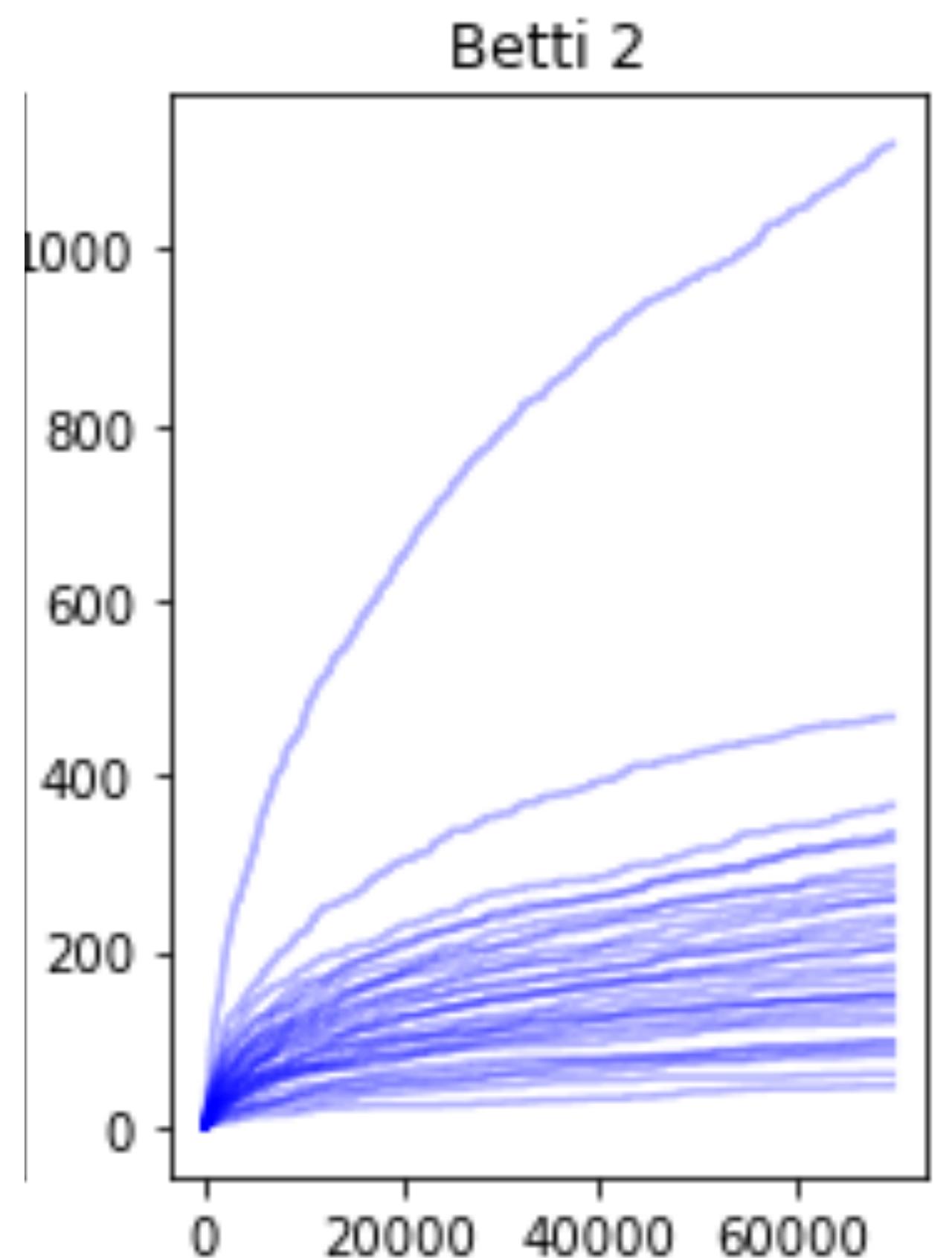
Gennady Samorodnitsky



Rongyi He (Caroline)

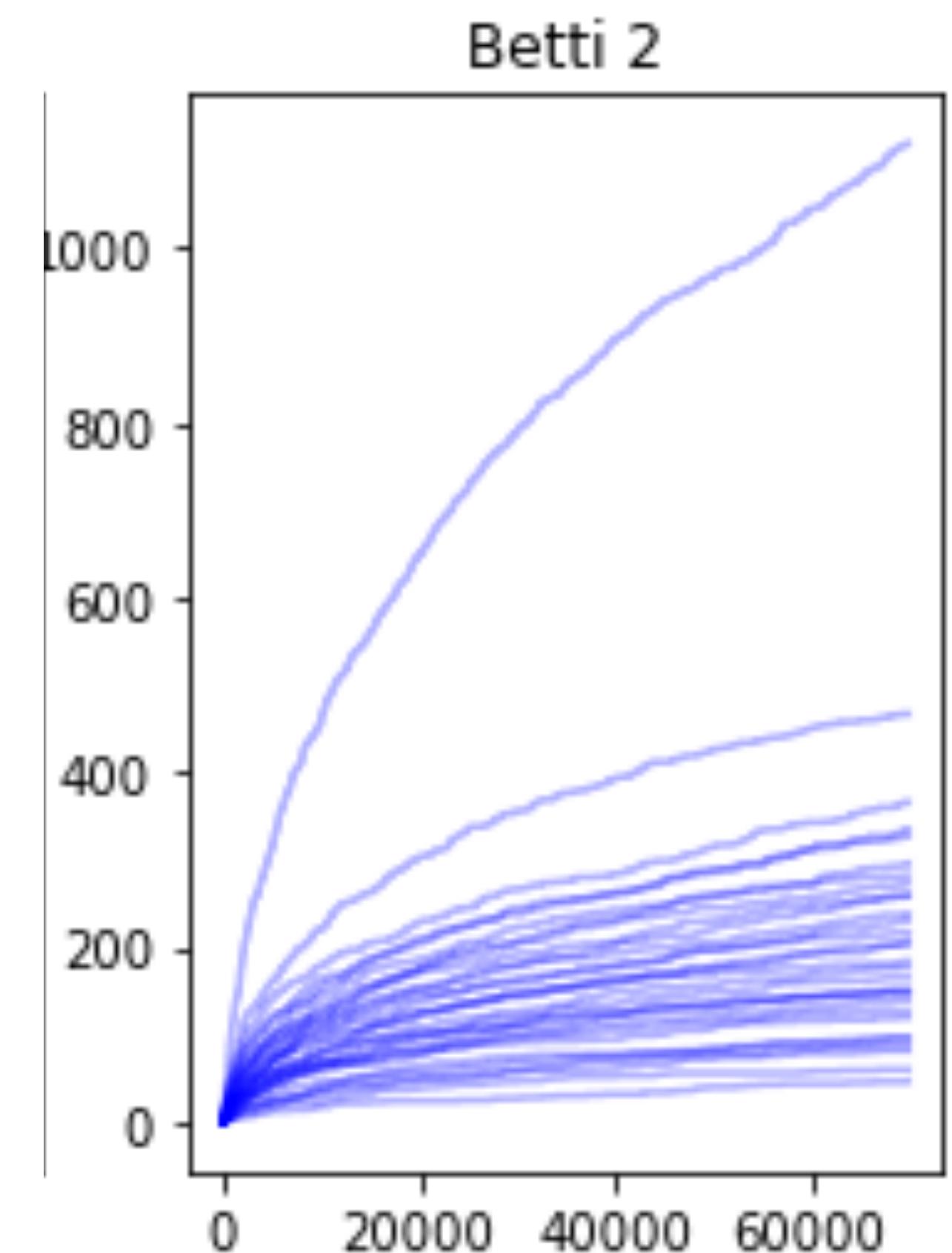
Expected Betti Number $E[\beta_q]$

Expected Betti Number $E[\beta_q]$



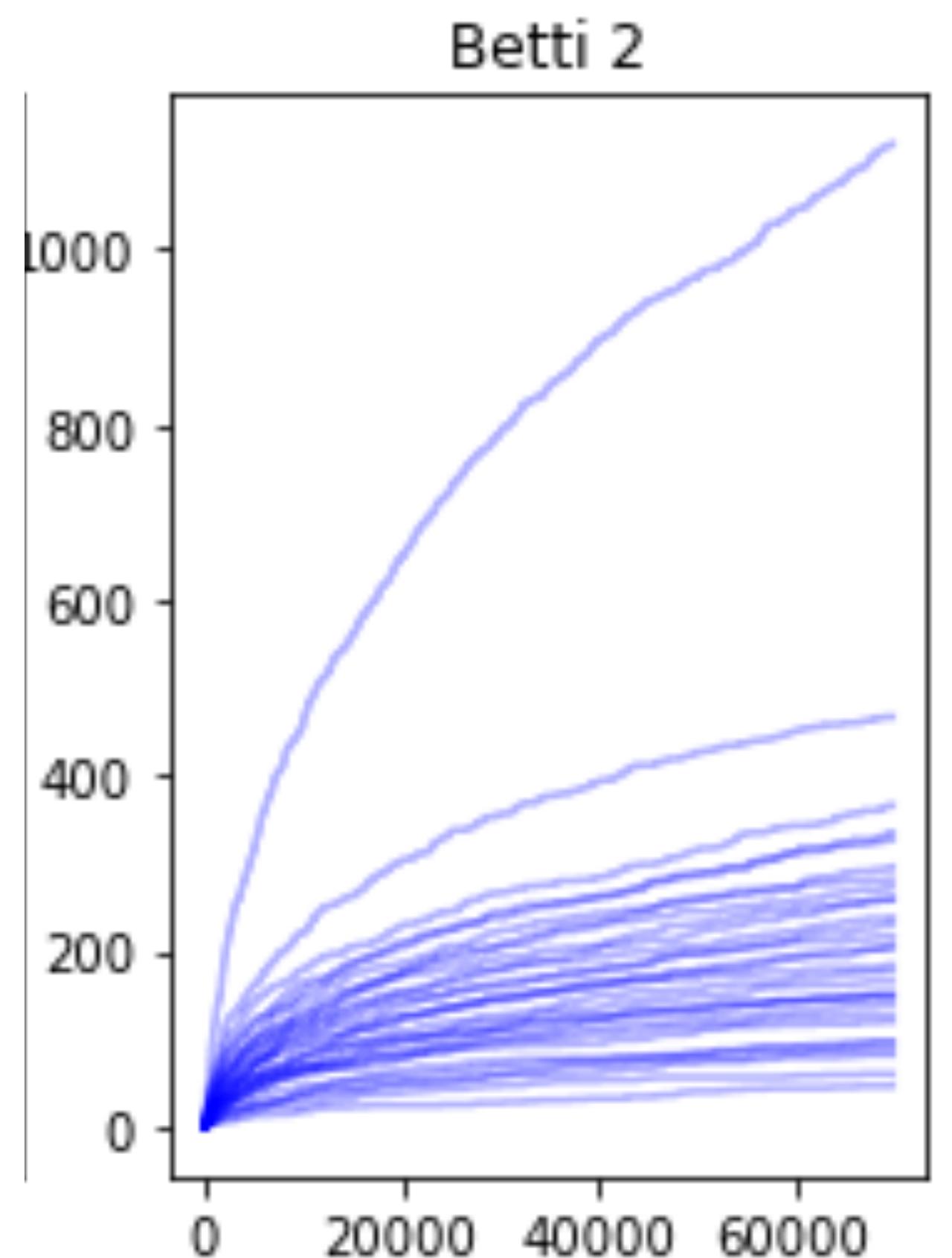
Expected Betti Number $E[\beta_q]$

- $c(\text{num of nodes}^{1-4\chi}) \leq E[\beta_2] \leq C(\text{num of nodes}^{1-4\chi})$
under mild assumptions
- $\chi \in (0, 1/2)$ depends on model parameters



Expected Betti Number $E[\beta_q]$

- $c(\text{num of nodes}^{1-4\chi}) \leq E[\beta_2] \leq C(\text{num of nodes}^{1-4\chi})$
under mild assumptions
- $\chi \in (0, 1/2)$ depends on model parameters
- $c(\text{num of nodes}^{1-2q\chi}) \leq E[\beta_q] \leq C(\text{num of nodes}^{1-2q\chi})$
for $q \geq 2$

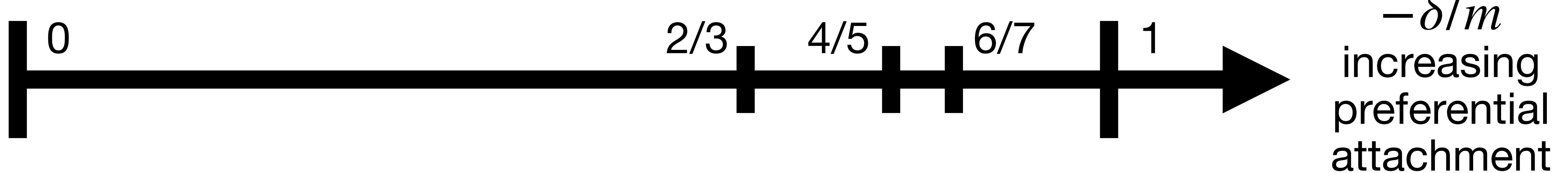


Phase transition

Recall

$P(\text{attaching to } v) \propto \text{degree} + \delta$

$m = \text{number of edges per new node}$

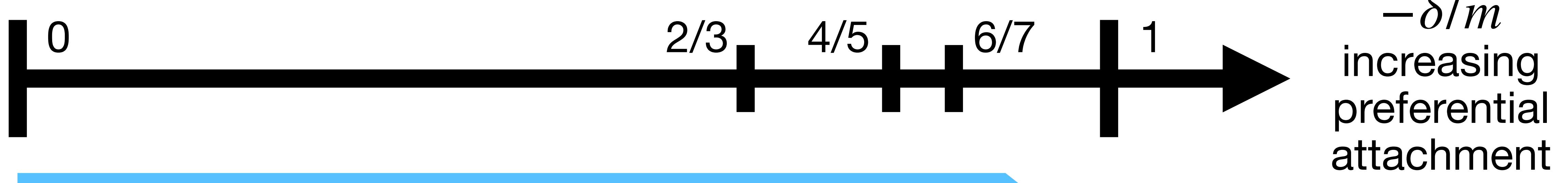


Phase transition

Recall

$$P(\text{attaching to } v) \propto \text{degree} + \delta$$

m = number of edges per new node



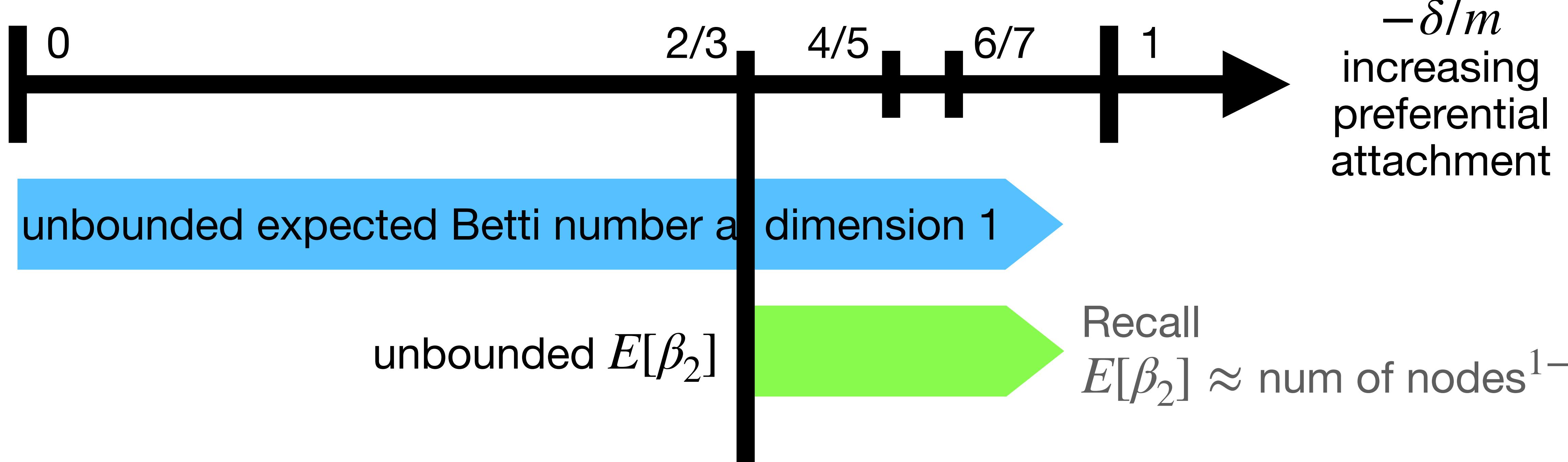
unbounded expected Betti number at dimension 1

Phase transition

Recall

$$P(\text{attaching to } v) \propto \text{degree} + \delta$$

m = number of edges per new node

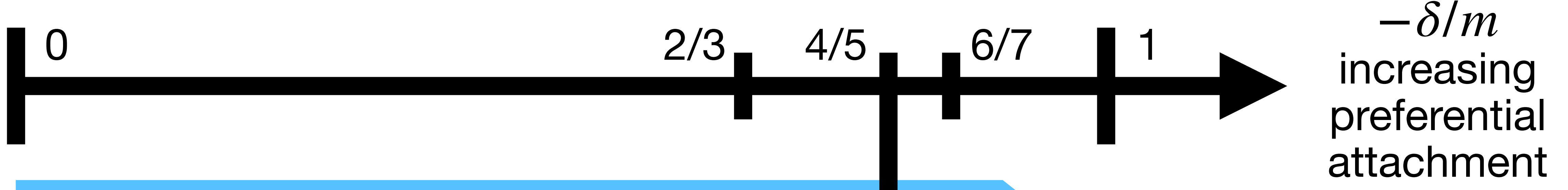


Phase transition

Recall

$P(\text{attaching to } v) \propto \text{degree} + \delta$

$m = \text{number of edges per new node}$



unbounded expected Betti number at dimension 1

unbounded $E[\beta_2]$

unbounded $E[\beta_3]$

Recall

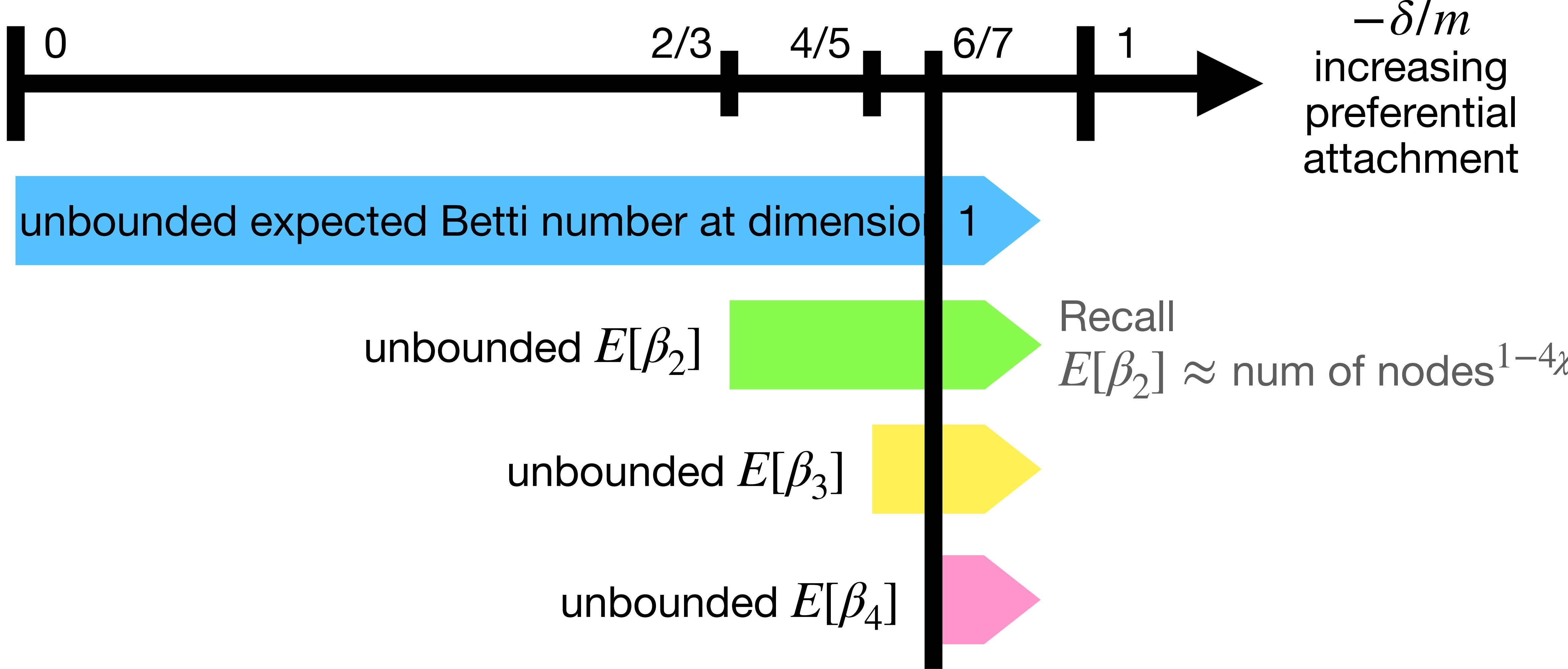
$E[\beta_2] \approx \text{num of nodes}^{1-4\chi}$

Phase transition

Recall

$P(\text{attaching to } v) \propto \text{degree} + \delta$

$m = \text{number of edges per new node}$

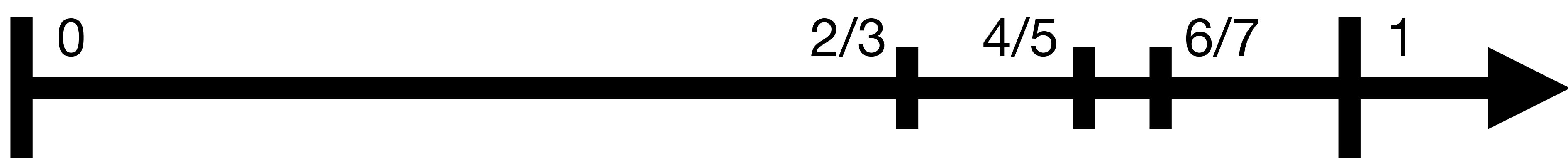


Phase transition

Recall

$P(\text{attaching to } v) \propto \text{degree} + \delta$

$m = \text{number of edges per new node}$



$-\delta/m$
increasing
preferential
attachment

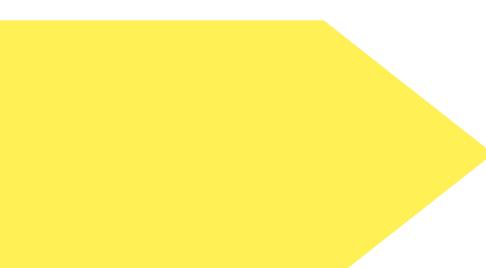
unbounded expected Betti number at dimension 1

unbounded $E[\beta_2]$

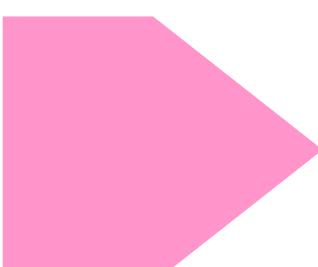


Recall
 $E[\beta_2] \approx \text{num of nodes}^{1-4\chi}$

unbounded $E[\beta_3]$



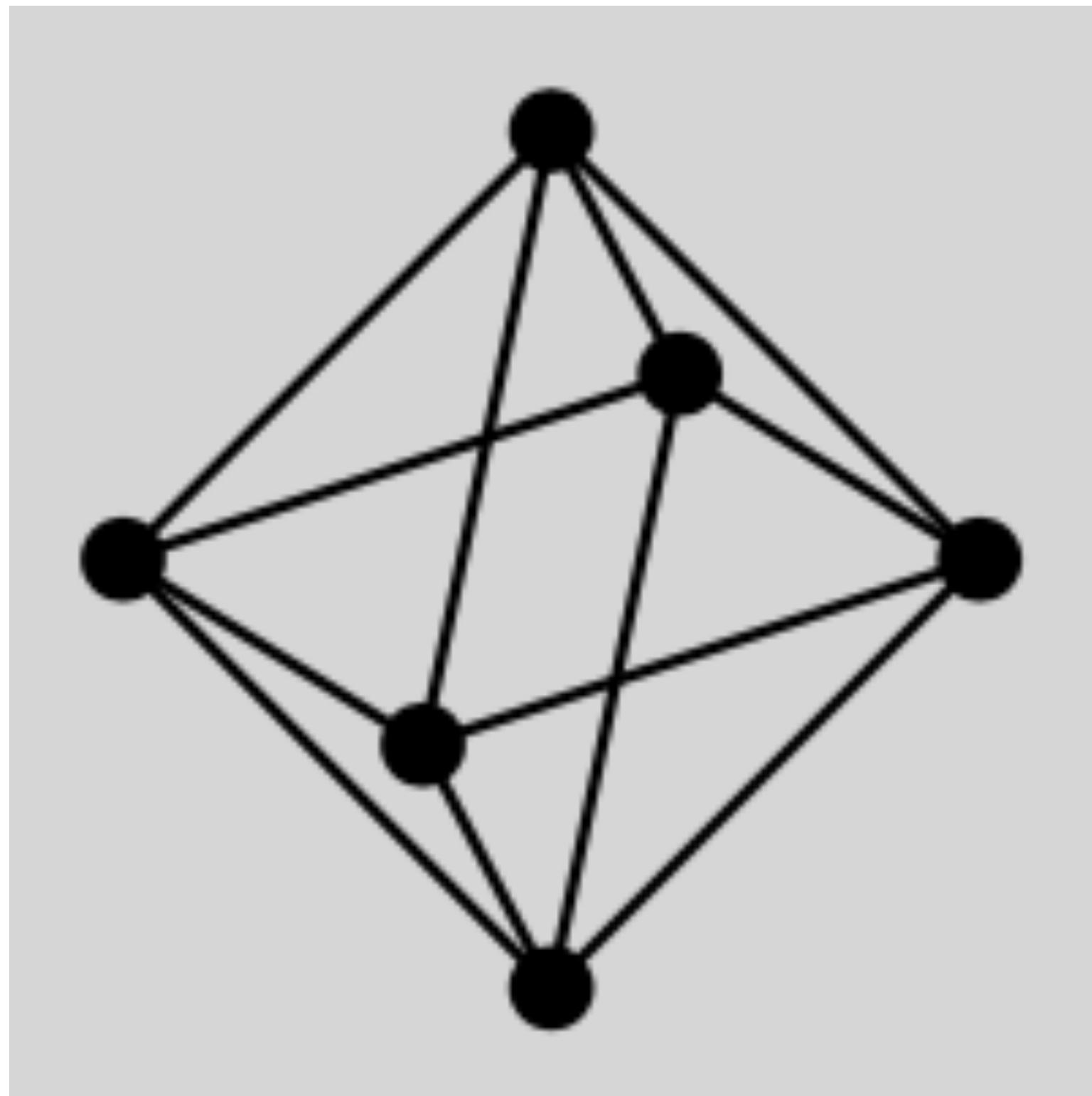
unbounded $E[\beta_4]$



:

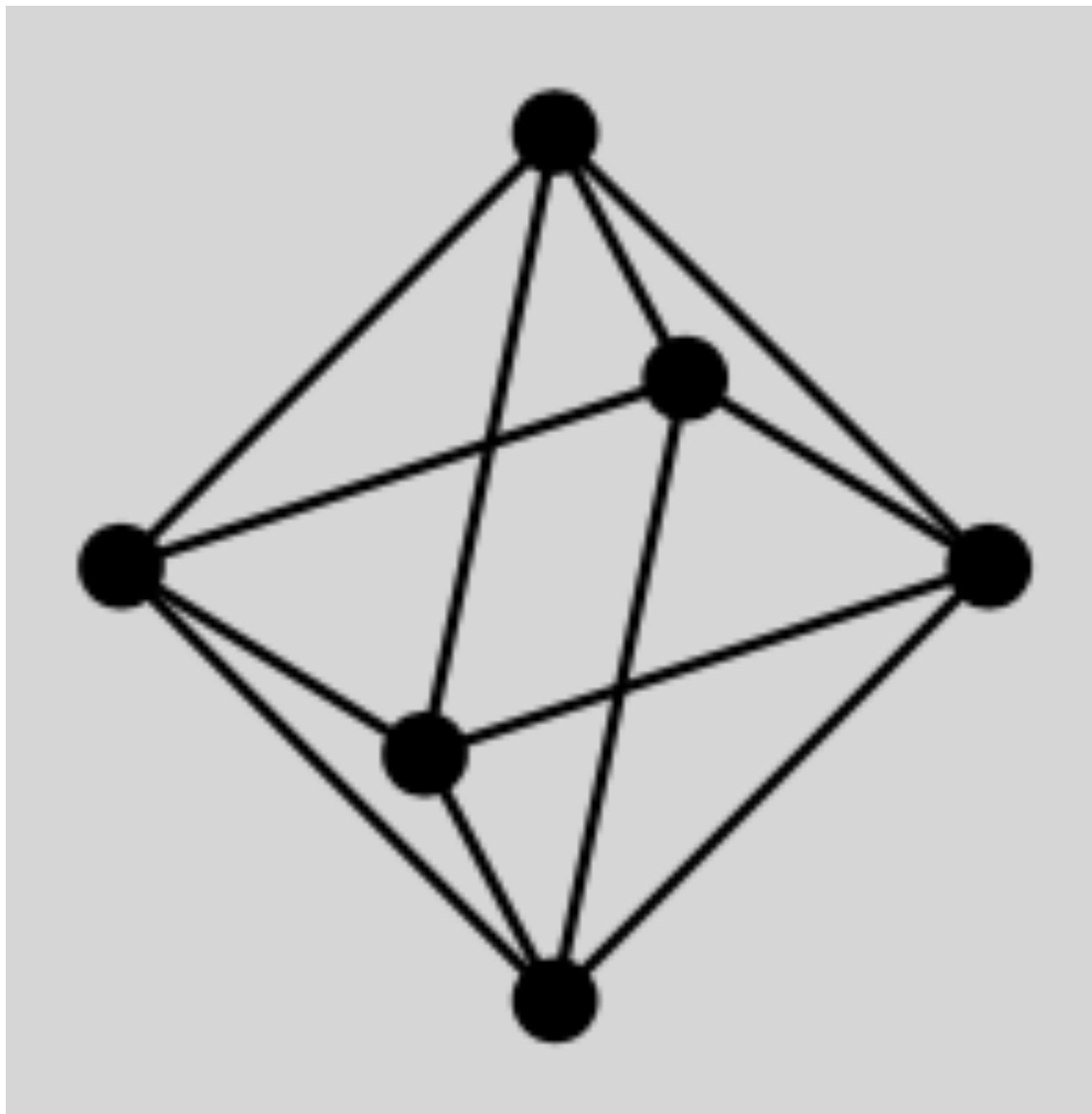
Theorem: $E[\beta_2] \approx \text{num of nodes}^{1-4\chi}$
Proof?

Proof of $E[\beta_2] \approx \text{num of nodes}^{1-4\chi}$

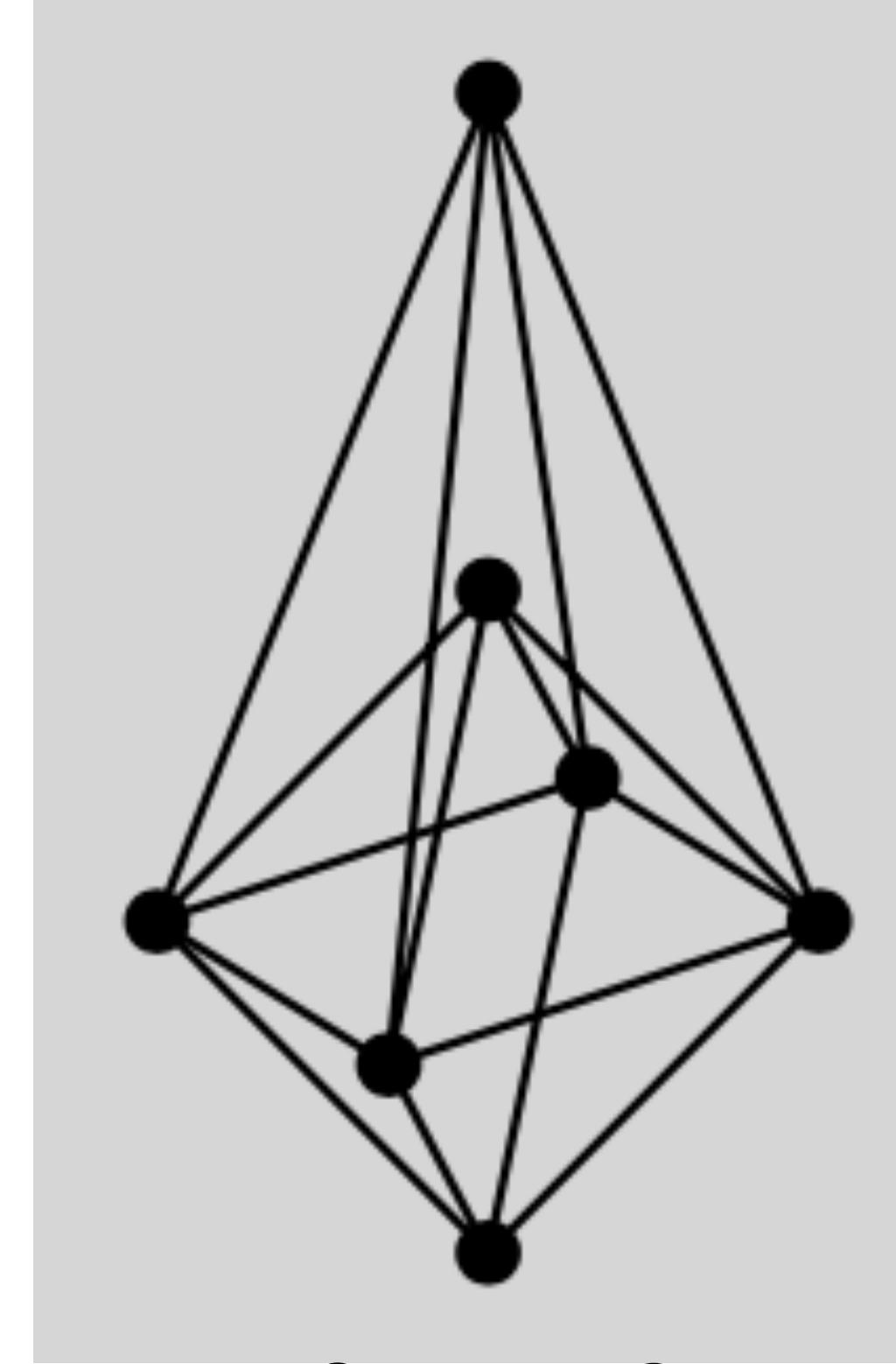


$$\beta_2 = 1$$

Proof of $E[\beta_2] \approx \text{num of nodes}^{1-4\chi}$

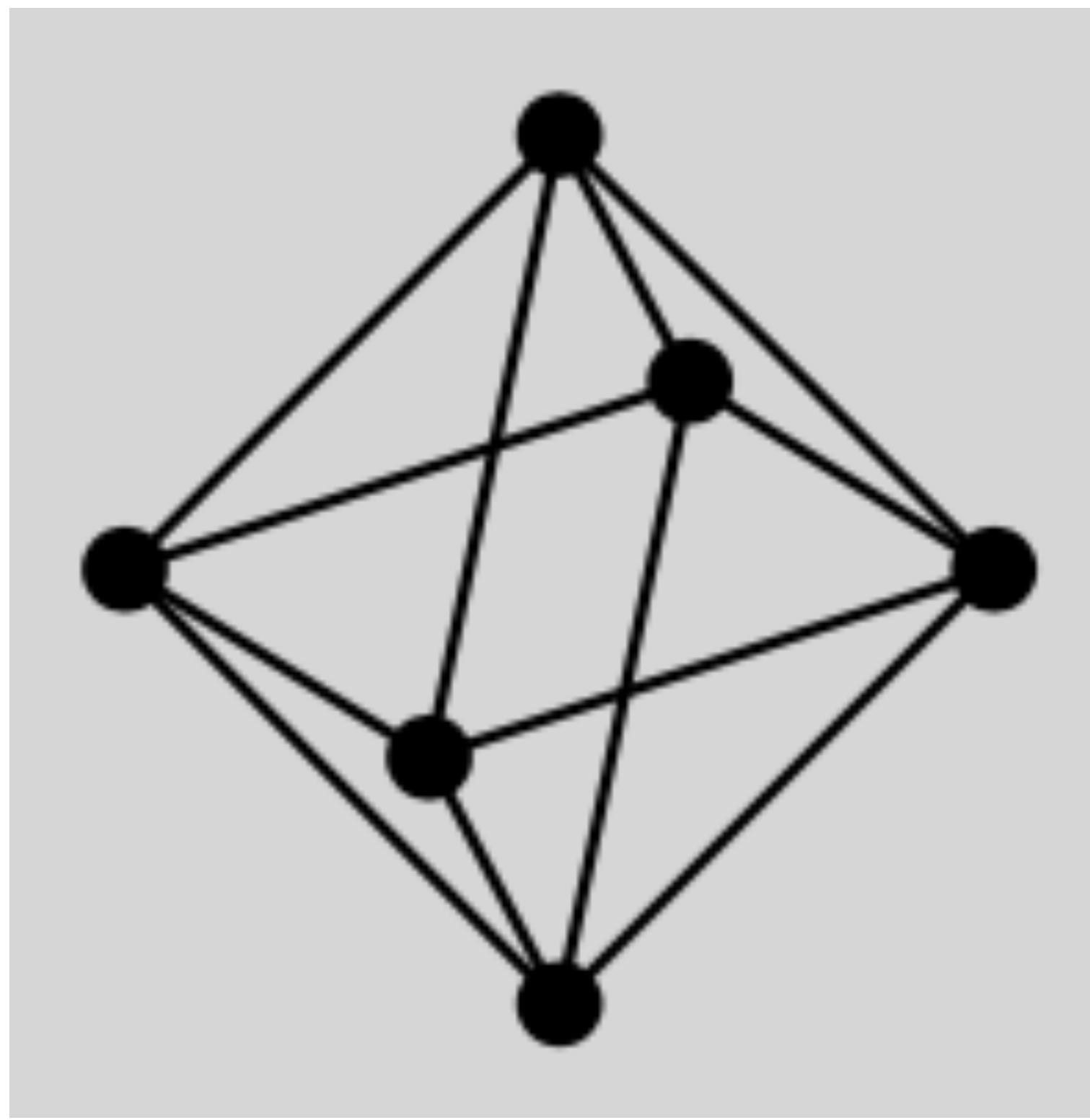


$$\beta_2 = 1$$

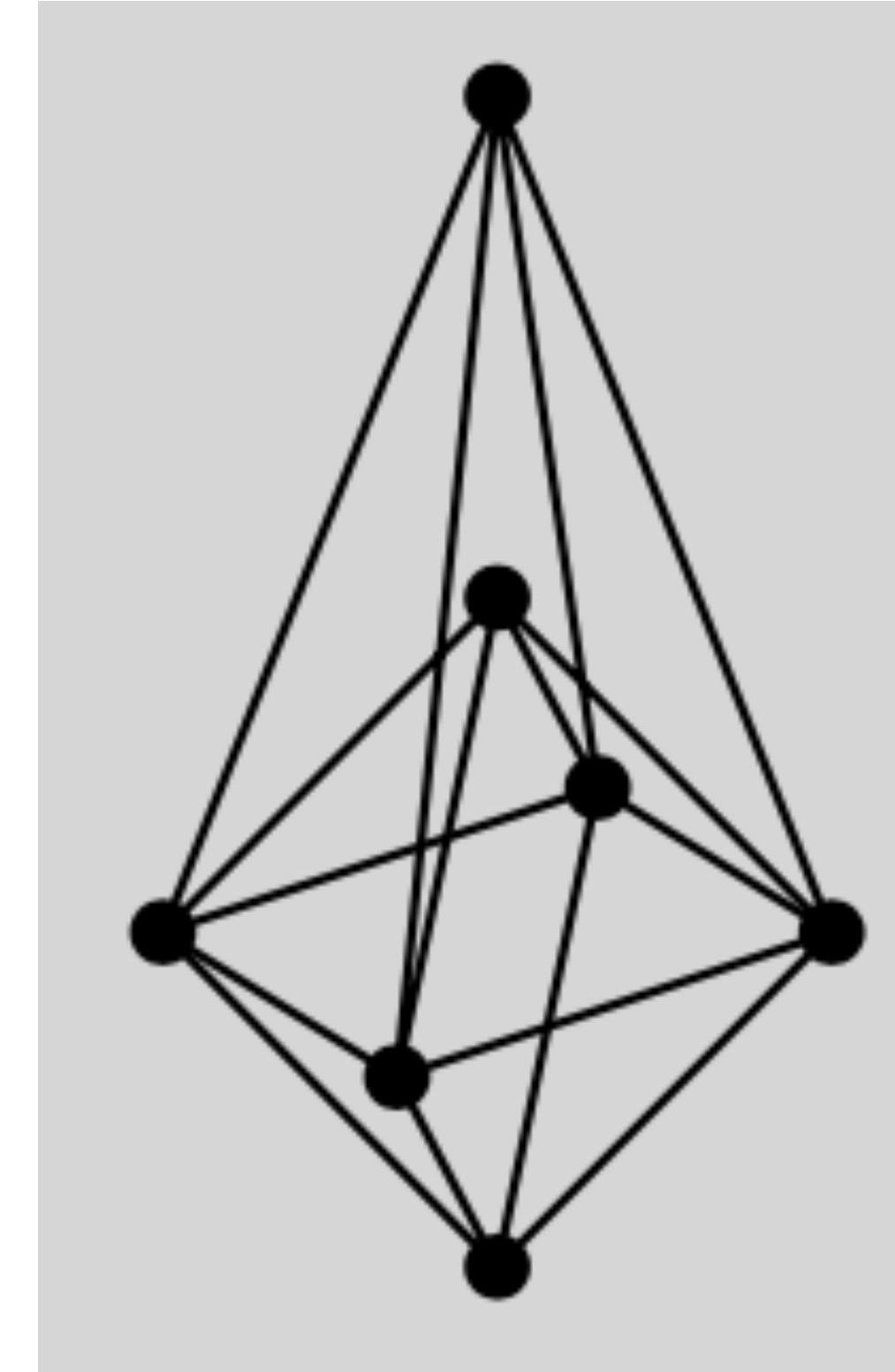


$$\beta_2 = 2$$

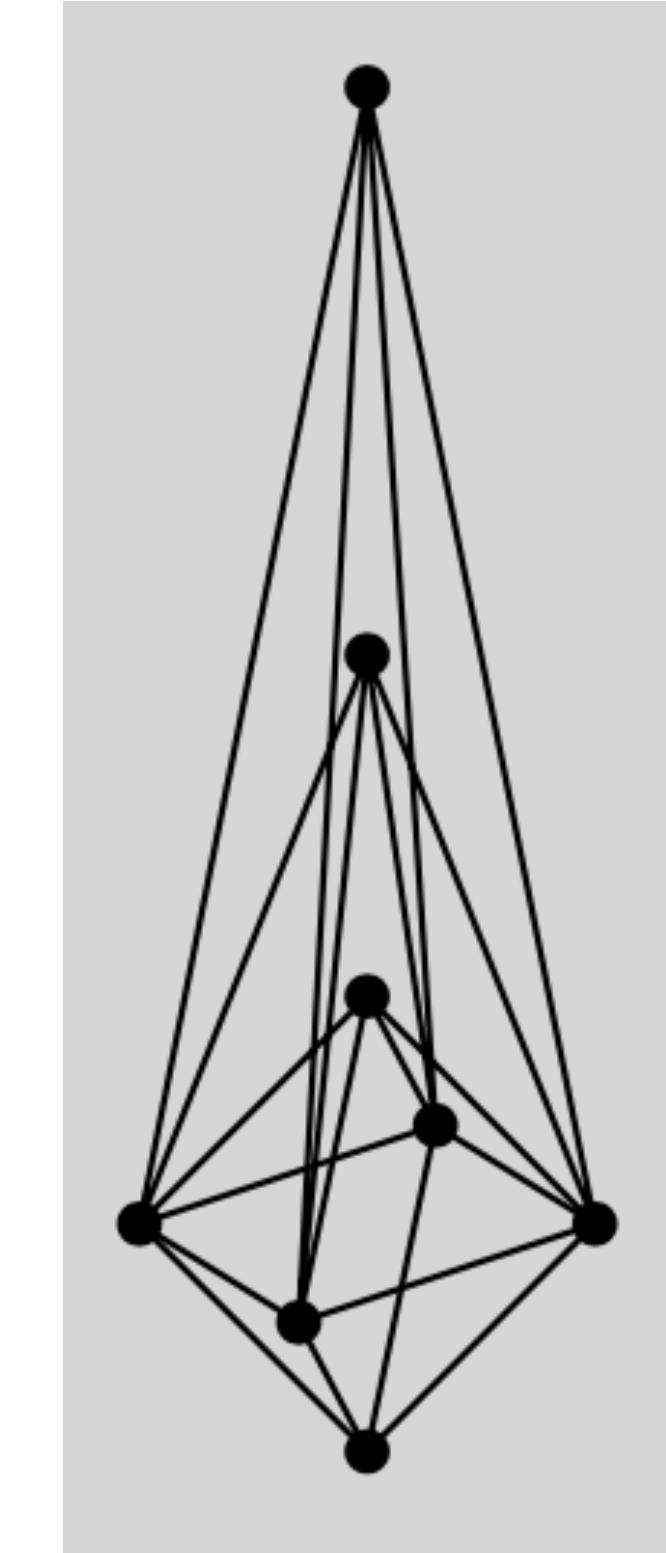
Proof of $E[\beta_2] \approx \text{num of nodes}^{1-4\chi}$



$$\beta_2 = 1$$



$$\beta_2 = 2$$



$$\beta_2 = 3$$

Subtleties

- Need homological algebra to relate Betti numbers with counts

Subtleties

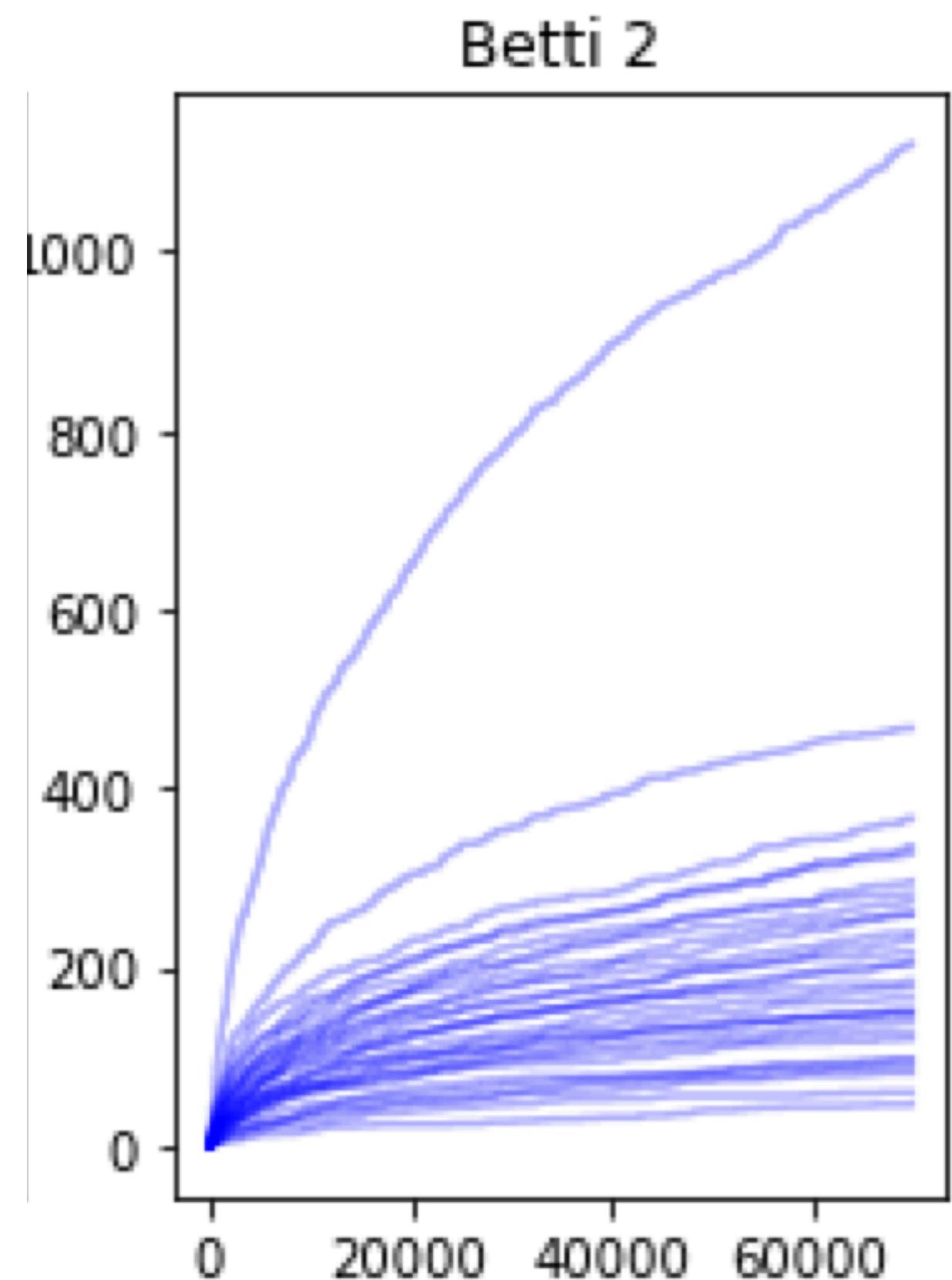
- Need homological algebra to relate Betti numbers with counts
- Generalize minimal cycle results in [Gal 2005] and [Kahle 2009]
-

Subtleties

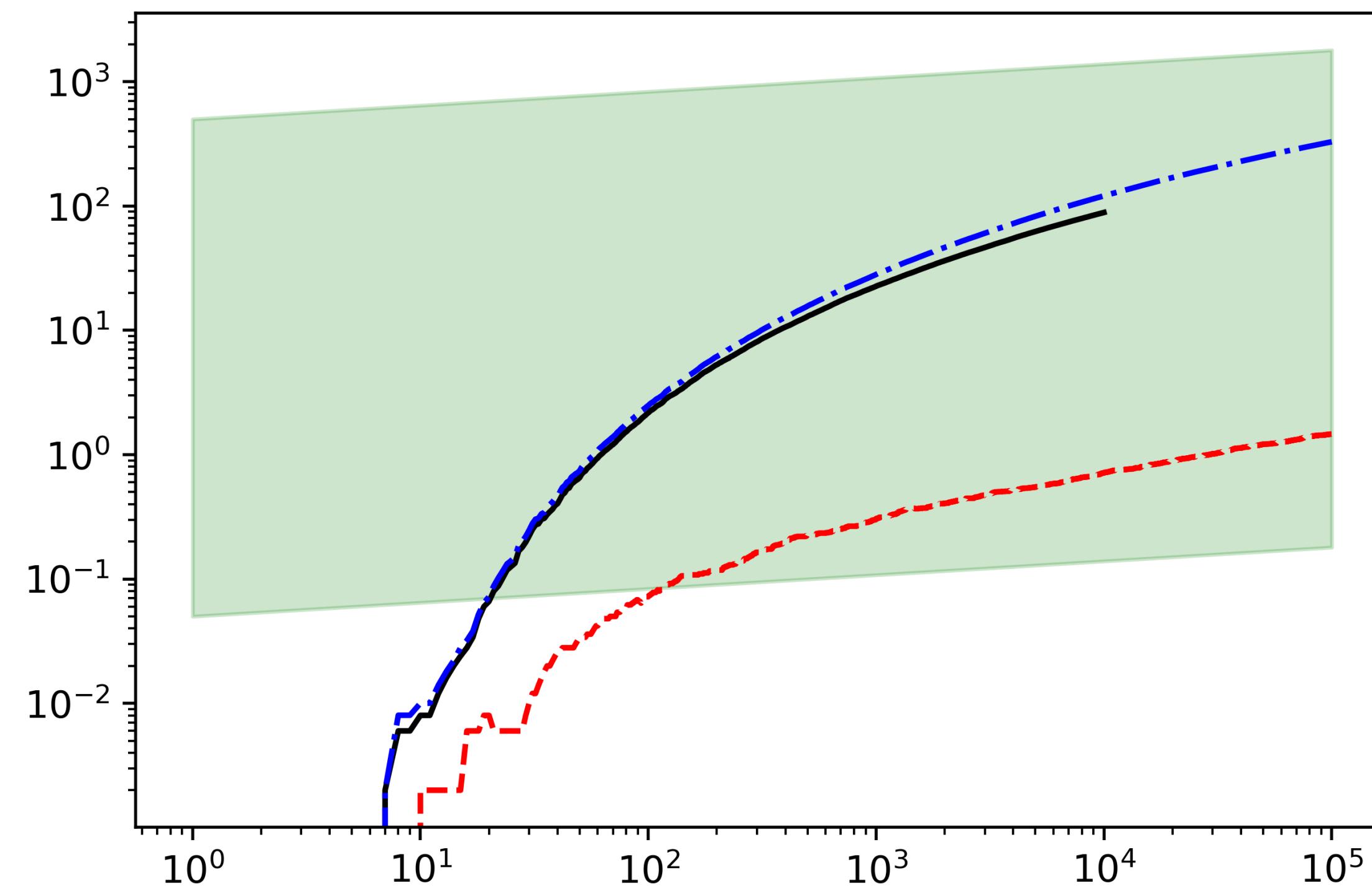
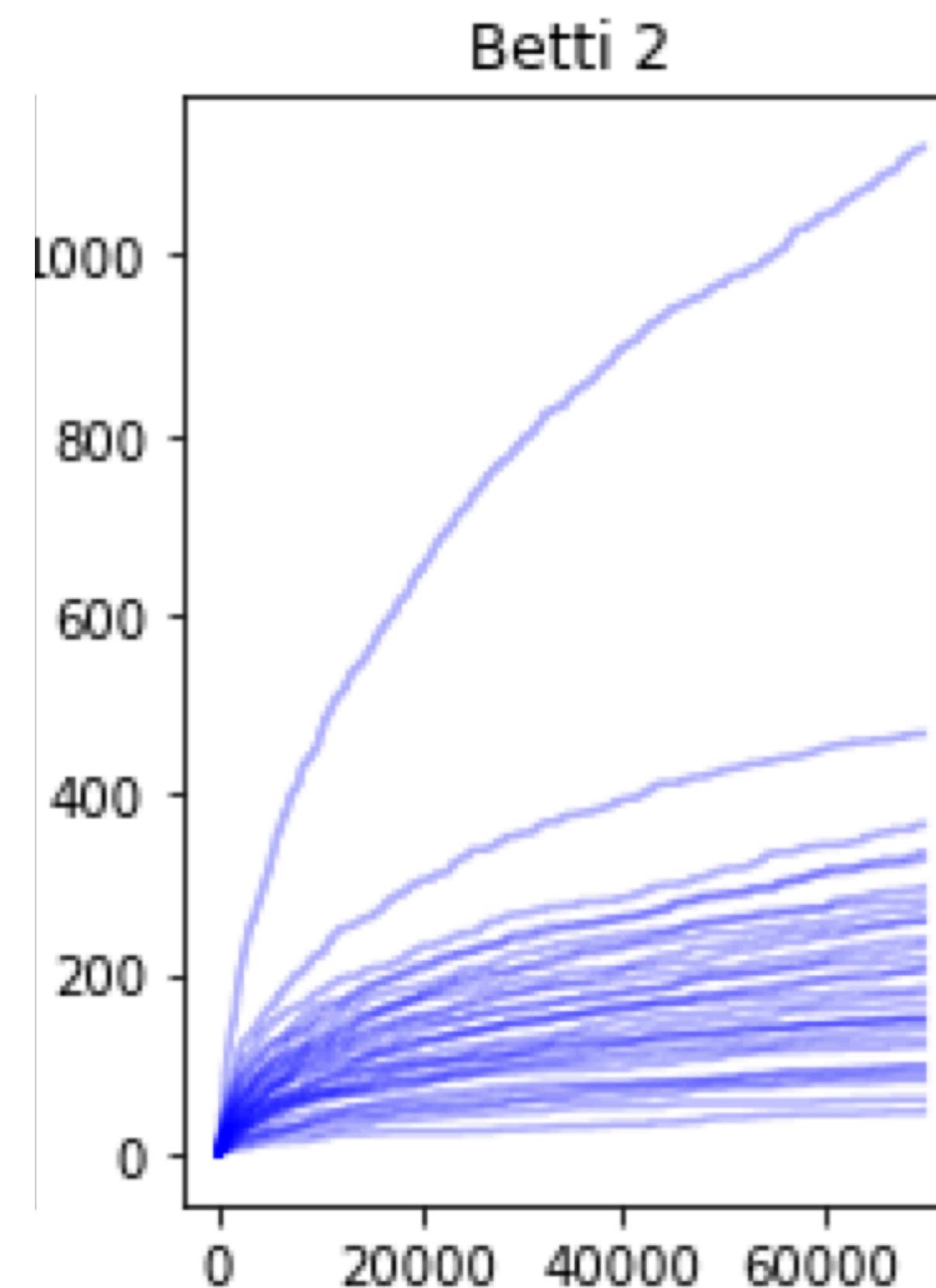
- Need homological algebra to relate Betti numbers with counts
- Generalize minimal cycle results in [Gal 2005] and [Kahle 2009]
- Apply graph counting result in [Garavaglia and Stegehuis 2019] on a large class of subgraphs
-

Theorem: $E[\beta_2] \approx \text{num of nodes}^{1-4\chi}$
In practice???

$$E[\beta_2] \approx \text{num of nodes}^{1-4\chi}$$



$$E[\beta_2] \approx \text{num of nodes}^{1-4\chi}$$



IV. What lies ahead

order of magnitude of
expected Betti numbers

homotopy
connectedness?

order of magnitude of
expected Betti numbers

homotopy
connectedness?

order of magnitude of
expected Betti numbers

parameter estimation?

homotopy
connectedness?

order of magnitude of
expected Betti numbers

parameter estimation?

simplicial preferential
attachment?

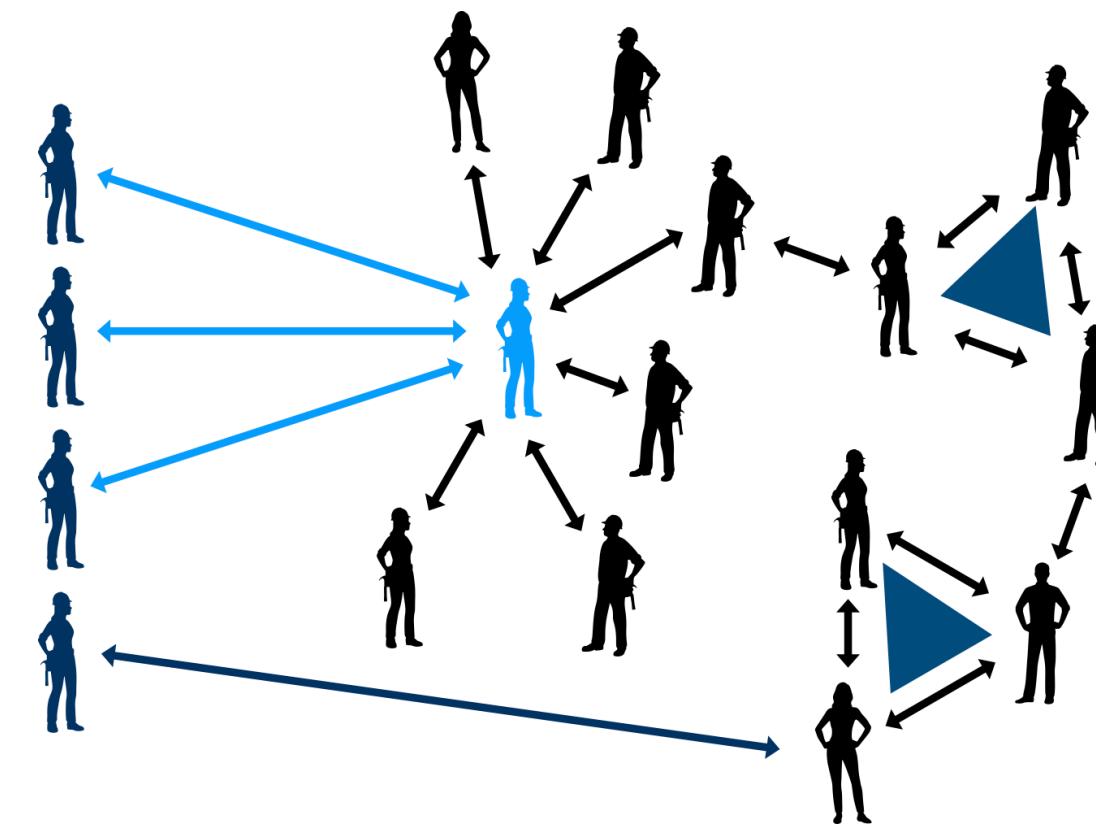
What did we learn today?

- Random topology is cool.
- Preferential attachment graph has interesting topology.
- More interesting things are waiting to be discovered.

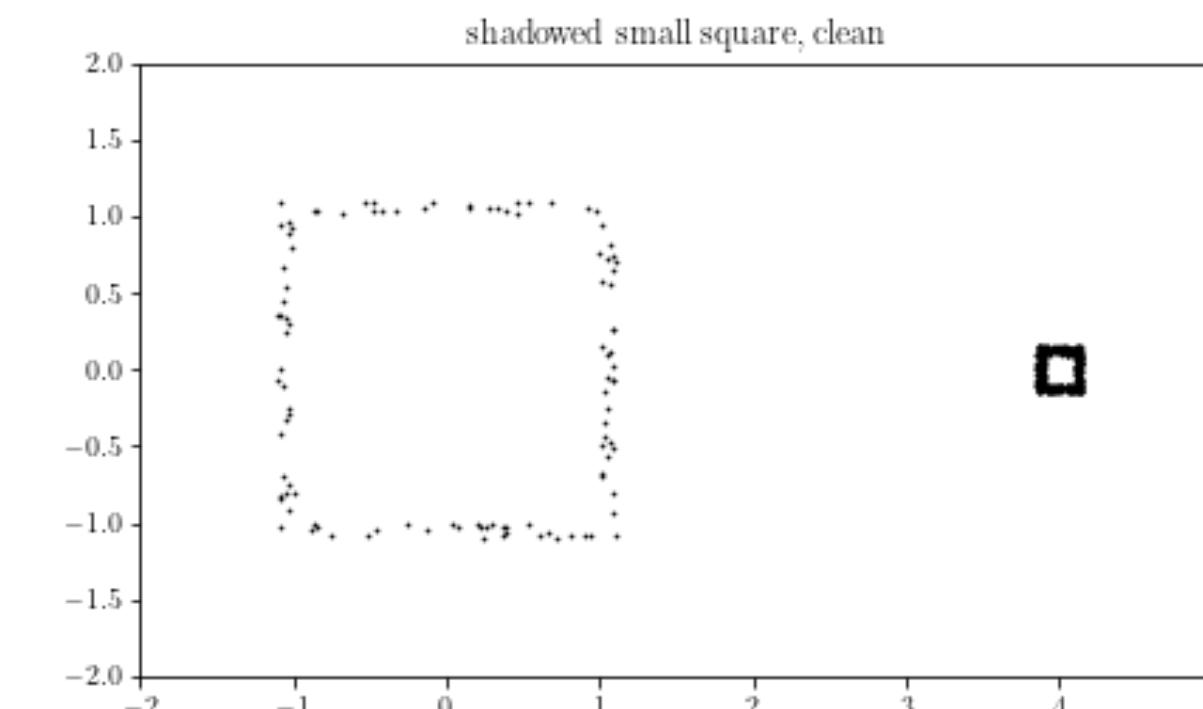
Chunyin Siu

cs2323@cornell.edu

Cornell University



arxiv paper



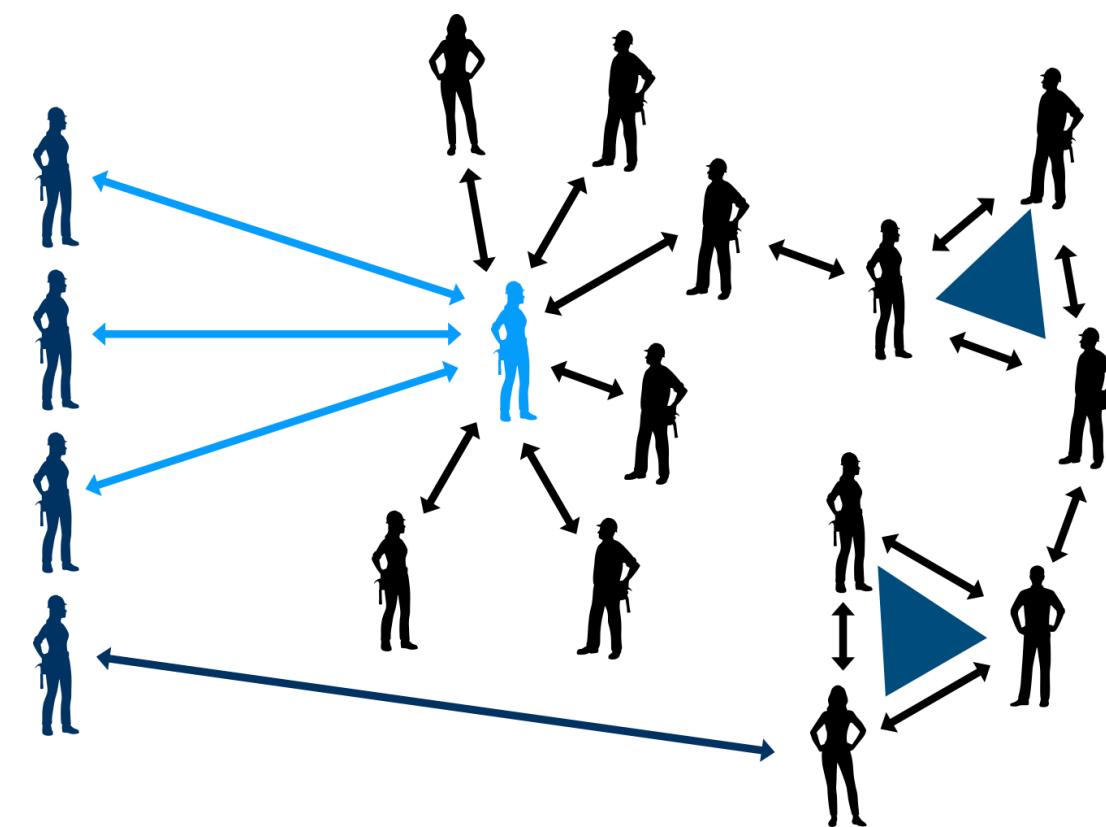
my video about small holes

Thank you!

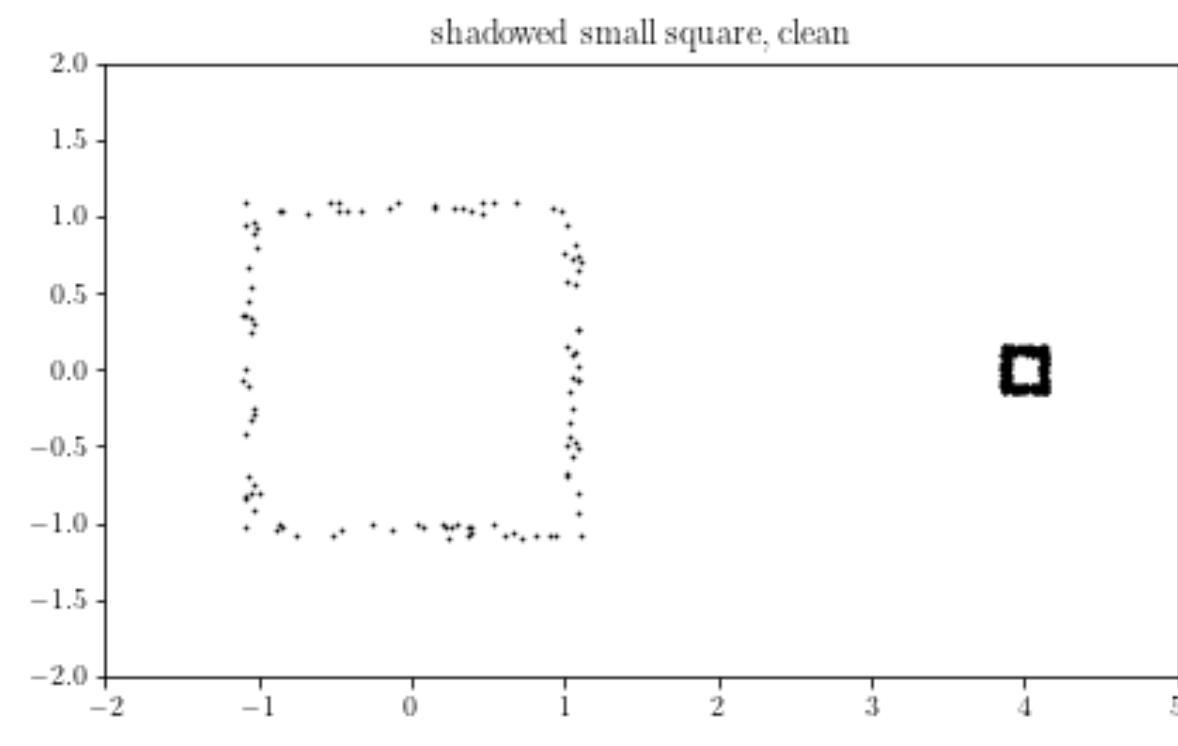
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arxiv paper



my video about small holes