# Supporting Hyperplane Theorem

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# Hyperplanes

## Introduction

Hyperplanes dominate the entire theory of optimization; appearing in Lagrange multipliers, duality theory, gradient calculations, etc. The most natural definition for a hyperplane is the generalization of a plane in $\mathbb{R}^3$.

# Hyperplanes

## Linear variety

A set $V$ in $\mathbb{R}^n$ is said to be **linear variety**, if, given any $x_1, x_2 \in V$, we have $\alpha x_1 + (1 - \alpha) x_2 \in V, \forall \alpha \in \mathbb{R}$.
The only difference between a linear variety and a convex set is that a linear variety is the entire line passing through any two points, rather than a simple line segment.

## Definition - Hyperplane

A **hyperplane** in $\mathbb{R}^n$ is an $(n-1)$-dimensional linear variety. It can be regarded as the largest linear variety in a space other than the entire space itself.

# Hyperplanes

## Proposition 1.1

Let $a \in \mathbb{R}^n, a \neq \Theta$ and $b \in \mathbb{R}$. The set
$$H = \{x \in \mathbb{R}^n : a^T x = b\}$$
is a *hyperplane* in $\mathbb{R}^n$.

## Proof

Let $x_1 \in H$. Translate $H$ by $-x_1$, we obtain the set
$$M = H - x_1 = \{y \in \mathbb{R}^n : \exists x \in H \ni y = x - x_1\},$$
which is a linear subspace of $\mathbb{R}^n$. $M = \{y \in \mathbb{R}^n : a^T y = 0\}$ is also the set of all orthogonal vectors to $a \in \mathbb{R}^n$, which is clearly $(n-1)$ dimensional.

# Hyperplanes

## Proposition 1.2

Let $x_1 \in H$ be an hyperplane in $\mathbb{R}^n$. Then,
$\exists a \in \mathbb{R}^n \ni H = \{x \in \mathbb{R} : a^T x = b\}$.

## Proof

Let $x_1 \in H$, and translate $-x_1$ obtaining $M = H - x_1$. Since $H$ is a hyperplane, $M$ is an $(n-1)$ dimensional space. Let $a$ be any orthogonal to $M$, i.e. $a \in M^{\perp}$. Thus, $M = \{y \in \mathbb{R}^n : a^T y = 0\}$. Let $b = a^T x_1$; we see that if $x_2 \in H, x_2 - x_1 \in M$ and therefore $a^T x_2 - a^T x_1 = 0 \Rightarrow a^T x_2 = b$. Hence, $H \subset x \in \mathbb{R} : a^T x = b$. Since $H$ is, by definition, of $(n-1)$ dimension, and $\{x \in \mathbb{R} : a^T x = b\}$ is of dimension $(n-1)$ by the above proposition, these two sets must be equal.

# Half Space

## Definition

Let $a \in \mathbb{R}^n, b \in \mathbb{R}$. Corresponding to the hyperplane $H = \{x : a^T x = b\}$, there are **positive** and **negative closed half spaces**:

$$H_+ = \{x : a^T x \geq b\}, H_- = \{x : a^T x \leq b\}$$

and

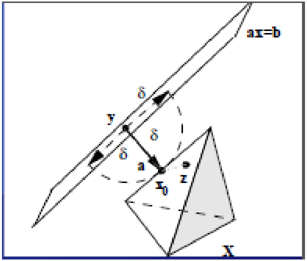
$$\dot{H}_+ = \{x : a^T x > b\}, \dot{H}_- = \{x : a^T x < b\}.$$

Half spaces are convex sets and $H_+ \bigcup H_- = \mathbb{R}^n$.

# Separating Hyperplane Theorem

## Separating Hyperplane Theorem

Let $X$ be a convex set and $y$ be a point exterior to the closure of $X$. Then, there exists a vector $a \in \mathbb{R}^n \ni a^T y < inf_{x \in X} a^T x$. (Geometrically, a given point $y$ outside $X$, a **separating** hyperplane can be passed through the point $y$ that does not touch $X$).



# Separating Hyperplane Theorem

## Proof (I) [1]

Let $\delta = inf_{x \in X}|x - y| > 0$.
Then, there is an $x_0$ on the boundary of $X$ such that $|x_0 - y| = \delta$.
Let $z \in X$. Then, $\forall \alpha, 0 \leq \alpha \leq 1, x_0 + \alpha(z - x_0)$ is the line segment between $x_0$ and $z$.
Thus, by definition of $x_0$, $|x_0 + \alpha(z - x_0) - y|^2 \geq |x_0 - y|^2 \Leftrightarrow$
$(x_0 - y)^T(x_0 - y) + 2\alpha(x_0 - y)^T(z - x_0) + \alpha^2(z - x_0)^T(z - x_0) \geq (x_0 - y)^T(x_0 - y) \Leftrightarrow$
$2\alpha(x_0 - y)^T(z - x_0) + \alpha^2|z - x_0|^2 \geq 0$
Let $\alpha \to 0^+$, then $\alpha^2$ tends to 0 more rapidly than $2\alpha$.

# Separating Hyperplane Theorem

## Proof (II)

Thus, $(x_0 - y)^T(z - x_0) \geq 0 \Leftrightarrow$
$(x_0 - y)^T z - (x_0 - y)^T x_0 \geq 0 \Leftrightarrow$
$(x_0 - y)^T z \geq (x_0 - y)^T x_0 = (x_0 - y)^T y + (x_0 + y)^T(x_0 - y) =$
$(x_0 - y)^T y + \delta^2 \Leftrightarrow$
$(x_0 - y)^T y < (x_0 - y)^T x_0 \geq (x_0 - y)^T z, \forall z \in X$ (Since $\delta > 0$).
Let $a = (x_0 - y)$, then
$a^T y < a^T x_0 = inf_{z \in X} a^T z.$

$\square$

# Supporting Hyperplane Theorem

**Supporting Hyperplane Theorem**

Let $X$ be a convex set, and let $y$ be a boundary point of $X$. Then, there is a hyperplane containing $y$ and containing $X$ in one of its closed half spaces.

# Supporting Hyperplane Theorem

## Proof [1]

Let $\{y_k\}$ be sequence of vectors, exterior to the closure of $X$, converging to $y$.
Let $\{a_k\}$ be a sequence of corresponding vectors constructed according to the previous theorem, normalized so that $|a_k| = 1$, such that $a_k^T y_k < inf_{x \in X}$.
Since $\{a_k\}$ is a boundary sequence, it converges to $a$.
For this vector, we have $a^T y = lim a_k^T y_k \leq ax$.

□

# Supporting Hyperplane Theorem

## Definition

A hyperplane containing a convex set $X$ in one of its closed half spaces and containing a boundary point of $X$ is said to be **supporting hyperplane** of $X$.

# Bibliography I

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[3] Stephen Boyd, Lieven Vandenberghe *Convex Optimization*, Cambridge University Press 2009.