Total unimodular matrices (TU): examples, properties

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Total unimodular matrices

Definition

A Matrix A is **totally unimodular** (TU) if the determinant of every square submatrix of A (also called minor of A) has value 1, 0 or 1. Or mathematically

$$ATU \iff \forall \text{ square submatrix } K,\ det(K) \in \{-1, 0, 1\}$$

Example

$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ is a TUM, as its determinant is equal to 1.

Properties

Proposition 1

If $A \in TU, A_j$ be the j-th row of A, then

$$A^T \in TU$$

Proof

$det(M) = det(M^T)$

Properties

Proposition 2

If $A \in TU, A_j$ be the j-th row of A, then

A' is a submatrix of A, then $A' \in TU$

Proof

square submatrix of A' is also square submatrix of A

Properties

Proposition 3

If $A \in TU, A_j$ be the j-th row of A, then

$$\begin{bmatrix} 0 \\ A_1 \\ A_2 \\ . \\ . \\ . \\ A_m \end{bmatrix}$$

Proof

The submatrices that doesn't contain the first row has corresponding submatrix in A, ones that does contain the first row has determinant 0.

Properties

Proposition 4

If $A \in TU, A_j$ be the j-th row of A, then

If A is a square matrix, then $A^{-1} \in TU$

Proof

If $B = A^{-1}$, then we have an equality between minors:

$$B(I, J) = \pm \frac{A(J^c, I^c)}{det(A)},$$

for every subsets $i, J \subset [[1, n]]$ of same cardinals. This formula generalizes that giving the entries of $A1$ in terms of minors of A.

Properties

Proposition 5

If $A \in TU, A_j$ be the j-th row of A, then

If A' is formed by row swapping of A, then $A' \in TU$.

Proof

Consider we switched row i and j. Consider any submatrix, if it doesn't contain row i or j, then it still has determinant 1,0,1. If it contain both row i and j, then the determinant is just the negation when the rows are switched back. If it only contain one of row i and j, wlog let it be i, then it has the rows in order $a_1, ..., a_k, i, a_{k+1}, ..., a_l$, then there is a submatrix in A using the rows in sequence $a_1, ..., a_j, i, a_{j+1}, ..., a_l$, and the absolute value of their determinants are equal.

Properties

Proposition 6

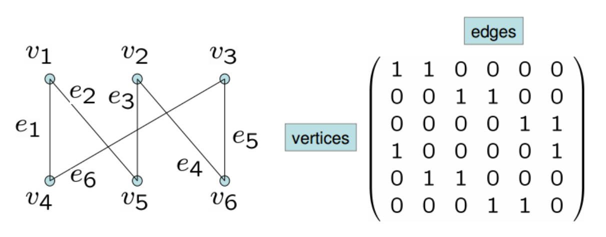
If $A \in TU, A_j$ be the j-th row of A, then
If A' id formed by multiplying by -1, 0 or 1 to a row, then $A' \in TU$

Proof

multiply by a constant to preserve the 1,0,1 properties, then use the argument similar as above, note the determinant of the submatrix can change only by sign.

Bipartite Graphs

Let A be the incidence matrix of a bipartite graph. Each row i represents a vertex $v(i)$, and each column j represents an edge $e(j)$. $A(ij) = 1$ if and only if edge $e(j)$ is incident to $v(i)$.



Bipartite Graphs

The incidence matrix A of a bipartite graph is totally unimodular.

Proof I

Consider an arbitrary square submatrix A' of A. Our goal is to show that A' has determinant 1,0, or $+1$.

Case 1. A' has a column with only 0.

$$\text{Then } det(A') = 0.$$

Case 2. A' has a column with only one 1.

$$A' = \begin{pmatrix} 1 & a^T \\ 0 & A'' \end{pmatrix}$$

By induction, A'' has determinant -1, 0, or $+1$. And so does A'

Bipartite Graphs

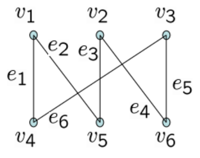
Proof II

Case 3. Each column of A' has exactly two 1.

$$\text{We can write } A' = \begin{pmatrix} A^{up} \\ A^{down} \end{pmatrix}$$

Since the graph is bipartite, each column has one 1 in A^{up} and one 1 in A^{down}.

So, by multiplying $+1$ on the rows in A^{up} and 1 on the columns in A^{down}, we get that the rows are linearly dependent, and thus $det(A') = 0$, and were done.



$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \begin{array}{l} \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} \boxed{+1} \\ \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} \boxed{-1} \end{array}$$

Directed Graphs

Let A be the incidence matrix of a directed graph. Each row i represents a vertex $v(i)$, and each column j represents an edge $e(j)$.

- $A(ij) = +1$ if vertex $v(i)$ is the tail of edge $e(j)$.
- $A(ij) = -1$ if vertex $v(i)$ is the head of edge $e(j)$.
- $A(ij) = 0$ otherwise.

The incidence matrix A of a directed graph is totally unimodular.

Other examples

Unimodular matrices form a subgroup of the general linear group under matrix multiplication, i.e. the following matrices are unimodular:

- **Identity matrix**
- **The inverse of a unimodular matrix**
- **The product of two unimodular matrices**

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