Farkas Lemma

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Farkas Lemma, Version I

Lemma 1.1. (Farkas Lemma, Version I)

Given any $d \times n$ real matrix, A, and any vector, $z \in \mathbb{R}^d$, exactly one of the following alternatives occurs:
(a) The linear system, $Ax = z$, has a solution, $x = (x_1, ..., x_n)$ such that $x \geq 0$ and $x_1 + ... + x_n = 1$, or
(b) There is some $c \in \mathbb{R}^d$ and some $\alpha \in]mathbbR$ such that $c^T z < \alpha$ and $c^T A \geq \alpha$.

Farkas Lemma, Version I

Remark

If we relax the requirements on solution of $Ax = z$ and only require $x \geq 0$ ($x_1 + ... + x_n = 1$ is no longer required) then, in condition (b), we can take $\alpha = 0$. This is another verion of Farkas Lemma.

In this case, instead of considering the convex hull of $\{A_1, ...A_n\}$ we are considering the convex cone,

$$cone(A_1, ..., A_n) = \{\lambda A_1 + ... + \lambda_n A_n | \lambda_i \geq 0, 1 \leq i \leq n\},$$

that is, we are dropping the condition $\lambda_1 + ... + \lambda_n = 1$.

Farkas Lemma, Version I

For this version of Farkas Lemma we need the following separation lemma:

Proposition 1.1.

Let $C \subseteq \mathbb{E}^d$ be any closed convex cone with vertex O. Then, for every point, a, not in C, there is a hyperplane, H, passing through O separating a and C with $a \notin H$.

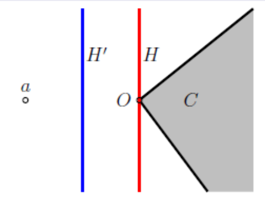

Figure 1. Illustration for Proposition 1.1

Farkas Lemma, Version II

Lemma 1.2. (Farkas Lemma, Version II)

Given any $d \times n$ real matrix, A, and any vector, $z \in \mathbb{R}^d$, exactly one of the following alternatives occurs:
(a) The linear system, $Ax = z$, has a solution, x, such that $x \geq 0$, or
(b) There is some $c \in \mathbb{R}^d$ such that $c^T z < 0$ and $c^T A \geq 0$.

Farkas Lemma, Version III

Lemma 1.3. (Farkas Lemma, Version III)

Given any $d \times n$ real matrix, A, and any vector, $z \in \mathbb{R}^d$, exactly one of the following alternatives occurs:
(a) The system of inequalities, $Ax \leq z$, has a solution, x, or
(b) There is some $c \in \mathbb{R}^d$ such that $c \geq 0$, $c^T z < 0$ and $c^T A = 0$.

Farkas Lemma, Version III

Proof I

The proof uses two tricks from linear programming:
1. We convert the system of inequalities, $Ax \leq z$, into a system of equations by introducing a vector of **slack variables**, $= ({}_1, \ldots, {}_d)$, where the system of equations is

$$(A, I) \begin{pmatrix} x \\ \gamma \end{pmatrix} = z,$$

with $\gamma \geq 0$.
2. We replace each "uncontrained variable", x_i, by $x_i = X_i - Y_i$, with $X_i, Y_I \geq 0$.

Farkas Lemma, Version III

Proof II

Then, the original system $Ax \leq z$ has a solution, x (unconstrained), iff the systen of equations

$$(A, -A, I) \begin{pmatrix} X \\ Y \\ \gamma \end{pmatrix} = z$$

has a solution with $X, Y, \gamma \geq 0$.

By Farkas II, this system has no solution iff there exists some $c \in \mathbb{R}^d$ with $c^T z < 0$ and

$$c^T (A, -A, I) \geq 0,$$

that is, $c^T A \geq 0$. $-c^T A \geq 0$, and $c \geq 0$.
However, these four conditions reduce to $c^T z < 0$, $c^T A = 0$ and $c \geq 0$.

Bibliography I

[1] Rudi Pendavingh, *Optimization in* $\mathbb{R}^n$*, lecture 2*, Eindhoven Technical University

[2] University of Pennsylvania, School of Engineering and Applied Science, *Basic Properties of Convex Sets*, Advanced Geometric Methods in Computer Science Course, 2014-2015

[3] L. Vandenberghe, *Alternatives Lecture*, Linear Programming Course, University of California, Los Angeles