Matroids: equivalent definitions, examples

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April 4, 2015

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Matroids

Definition - Matroid

A matroid $M = (S, \mathcal{I})$ is a finite ground set S together with a collection of sets $\mathcal{I} \subseteq 2^S$, known as the independent sets, satisfying the following axioms:
(I_1) If $I \in \mathcal{I}$ and $J \subseteq I$ then $J \in \mathcal{I}$.
(I_2) If $I, J \in \mathcal{I}$ and $|J| > |I|$, then there exists an element $z \in J \backslash I$ such that $I \cup \{z\} \in \mathcal{I}$.

Uniform Matroid

One trivial example of a matroid $M = (E, \mathcal{I})$ is a **uniform matroid** in which

$$\mathcal{I} = \{X \subseteq E : |X| \leq k\},$$

for a given k. It is usually denoted as $U_{k,n}$ where $|E| = n$. A base is any set of cardinality k (unless $k > |E|$ in which case the only base is $|E|$).
A **free matroid** is one in which all sets are independent; it is $U_{n,n}$.

Partition Matroid

Another example is the **partition matroid** in which E is partitioned into (disjoint) sets $E_1, E_2, ..., E_l$ and

$$\mathcal{I} = \{X \subseteq E : |X \cap E_i| \leq k_i \text{ for all } i = 1, ..., l\},$$

for some given parameters $k_1, ..., K_l$. As an exercise, let us check that (I_2) is satisfied. If $X, Y \in \mathcal{I}$ and $|Y| > |X|$, there must exist i such that $|Y \cap E_i| > |X \cap E_i|$ and this means that adding any element e in $E_i \cap (Y \subseteq X)$ to X will maintain independence.

Observe that M would not be a matroid if the sets E_i were not disjoint. For example, if $E_1 = \{1, 2\}$ and $E_2 = \{2, 3\}$ with $k_1 = 1$ and $k_2 = 1$ then both $Y = \{1, 3\}$ and $X = \{2\}$ have at most one element of each E_i, but one can't find an element of Y to add to X.

Linear Matroid

Linear matroids (or representable matroids) are defined from a matrix A, and this is where the term matroid comes from. Let E denote the index set of the columns of A. For a subset X of E, let A_X denote the submatrix of A consisting only of those columns indexed by X. Now, define

$$\mathcal{I} = \{X \subseteq E : rank(A_X) = |X|\},$$

i.e. a set X is independent if the corresponding columns are linearly independent. A base B corresponds to a linearly independent set of columns of cardinality $rank(A)$.

Matching Matroid

Here is an example of something that is not a matroid. Take a graph $G = (V, E)$, and let $\mathcal{I} = \{F \subseteq E : F \text{ is a matching}\}$. This is not a matroid since (I_2) is not necessarily satisfied ((I_1) is satisfied, however). Consider, for example, a graph on 4 vertices and let $X = \{(2, 3)\}$ and $Y = \{(1, 2), (3, 4)\}$. Both X and Y are matchings, but one cannot add an edge of Y to X and still have a matching.

There is, however, another matroid associated with matchings in a (general, not necessarily bipartite) graph $G = (V, E)$, but this time the ground set of M corresponds to V . In the **matching matroid**, $\mathcal{I} = \{S \subseteq V : S \text{ is covered by some matching } M\}$. In this definition, the matching does not need to cover precisely S; other vertices can be covered as well.

Graphic Matroid

A very important class of matroids in combinatorial optimization is the class of **graphic matroids** (also called cycle matroids). Given a graph $G = (V, E)$, we define independent sets to be those subsets of edges which are forests, i.e. do not contain any cycles. This is called the graphic matroid $M = (E, \mathcal{I})$, or $M(G)$.

(I_1) is clearly satisfied. To check (I_2), first notice that if F is a forest then the number of connected components of the graph (V, F) is given by $K(V, F) = |V||F|$. Therefore, if X and Y are 2 forests and $|Y| > |X|$ then $K(V, Y) < K(V, X)$ and therefore there must exist an edge of $Y \subseteq X$ which connects two different connected components of X; adding this edge to X results in a larger forest. This shows (I_2).

Bibliography I

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[2] Michel X. Goemans, *Lecture 8: Matroids* Advanced Combinatorial Optimization, Massachusetts Institute of Technology, 2009