Perfect graphs: definition, examples, characterizations

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Table of contents

Perfect Graph

Definition

A **perfect graph** is a graph G such that for every induced subgraph of G, the clique number equals the chromatic number, i.e., $\omega(G) = \chi(G)$. A graph that is not a perfect graph is called an imperfect graph.

Weakly Perfect Graph

Definition

A graph for which $\omega(G) = \chi(G)$ (without any requirement that this condition also hold on induced subgraphs) is called a **weakly perfect graph**. All perfects graphs are therefore by definition weakly perfect.

Strongly Perfect Graph

Definition

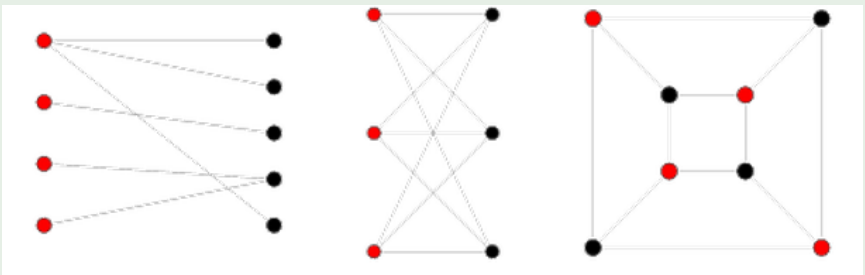
A graph is **strongly perfect** if every induced subgraph H has an independent set meeting all maximal cliques of H. While all strongly perfect graphs are perfect, the converse is not necessarily true. Since every P_4-free graph (where P_n is a path graph) is strongly perfect and every strongly perfect graph is perfect, if a graph is P_4-free, it is perfect.

Bipartite Graphs

Definition

A **bipartite graph**, also called a bigraph, is a set of graph vertices decomposed into two disjoint sets such that no two graph vertices within the same set are adjacent. A bipartite graph is a special case of a k-partite graph with $k = 2$.
The illustration above shows some bipartite graphs, with vertices in each graph colored based on to which of the two disjoint sets they belong.

Example

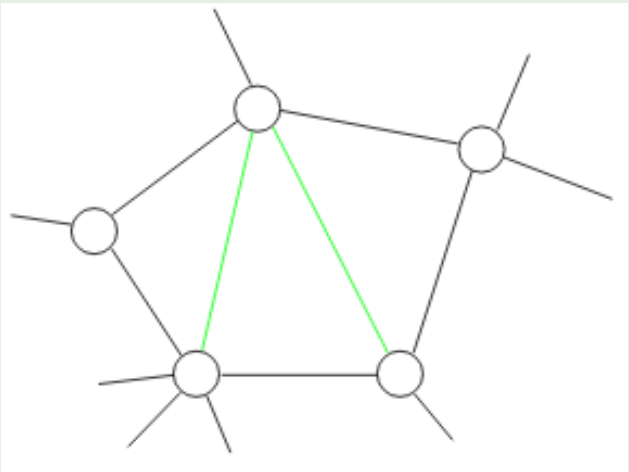


Chordal Graphs

Definition

A graph is called **chordal** if every cycle of length at least four contains a chord. These graphs are also called triangulated.

Example

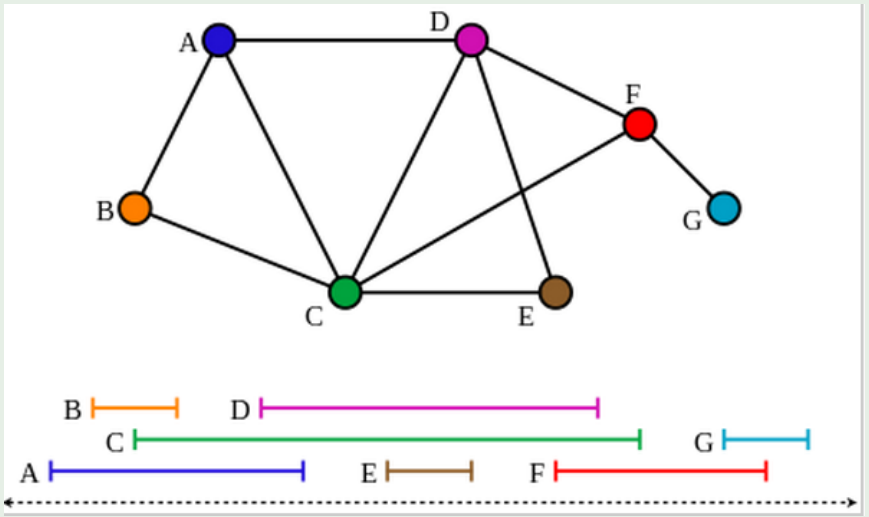


A cycle (black) with two chords (green).

Definition

A graph is an **interval graph** if each vertex can be represented by an interval on the real line in such a way that two vertices are adjacent if and only if their corresponding intervals intersect. These graphs are $\longrightarrow$ triangulated and therefore perfect.

Example



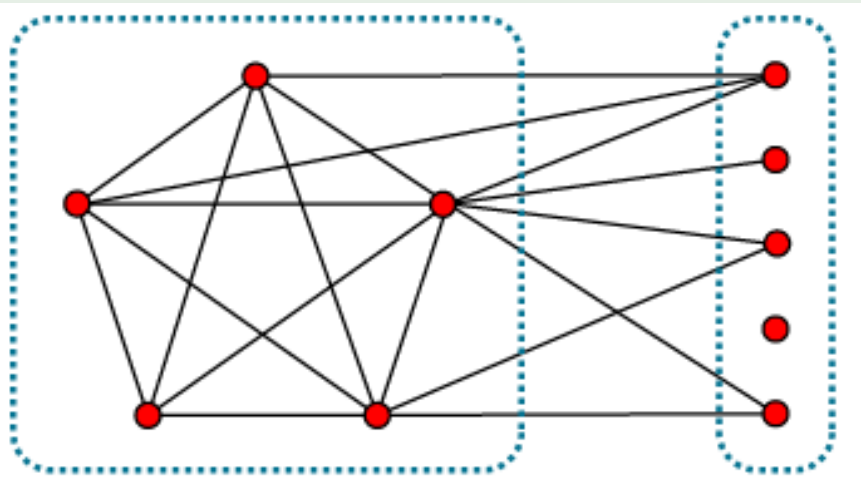
Seven intervals on the real line and the corresponding seven-vertex interval graph.

Split Graphs

Definition

A graph is called split if its vertex set can be partitioned into two sets V_1 and V_2 such that V_1 induces a stable set and V_2 induces a clique. Perfection of split graphs follows from the fact that they are triangulated.

Example



Strong Perfect Graph Theorem

The theorem, originally conjectured by Berge (1960, 1961), that a graph is perfect iff neither the graph nor its graph complement contains an odd graph cycle of length at least five as an induced subgraph became known as the strong perfect graph conjecture (Golumbic 1980; Skiena 1990, p. 221). The conjecture can be stated more simply as the assertion that a graph is perfect iff it contains no odd graph hole and no odd graph antihole. The proposition can be stated even more succinctly as "a graph is perfect iff it is a Berge graph."

Bibliography I

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[2] Stefan Hougardy, *Classes of Perfect Graphs*, July 2007

[3] Michele Alberti, *Small Survey on Perfect Graphs*, December 8, 2010

[4] Peter Ballen, *Perfect Graphs and the Perfect Graph Theorems*