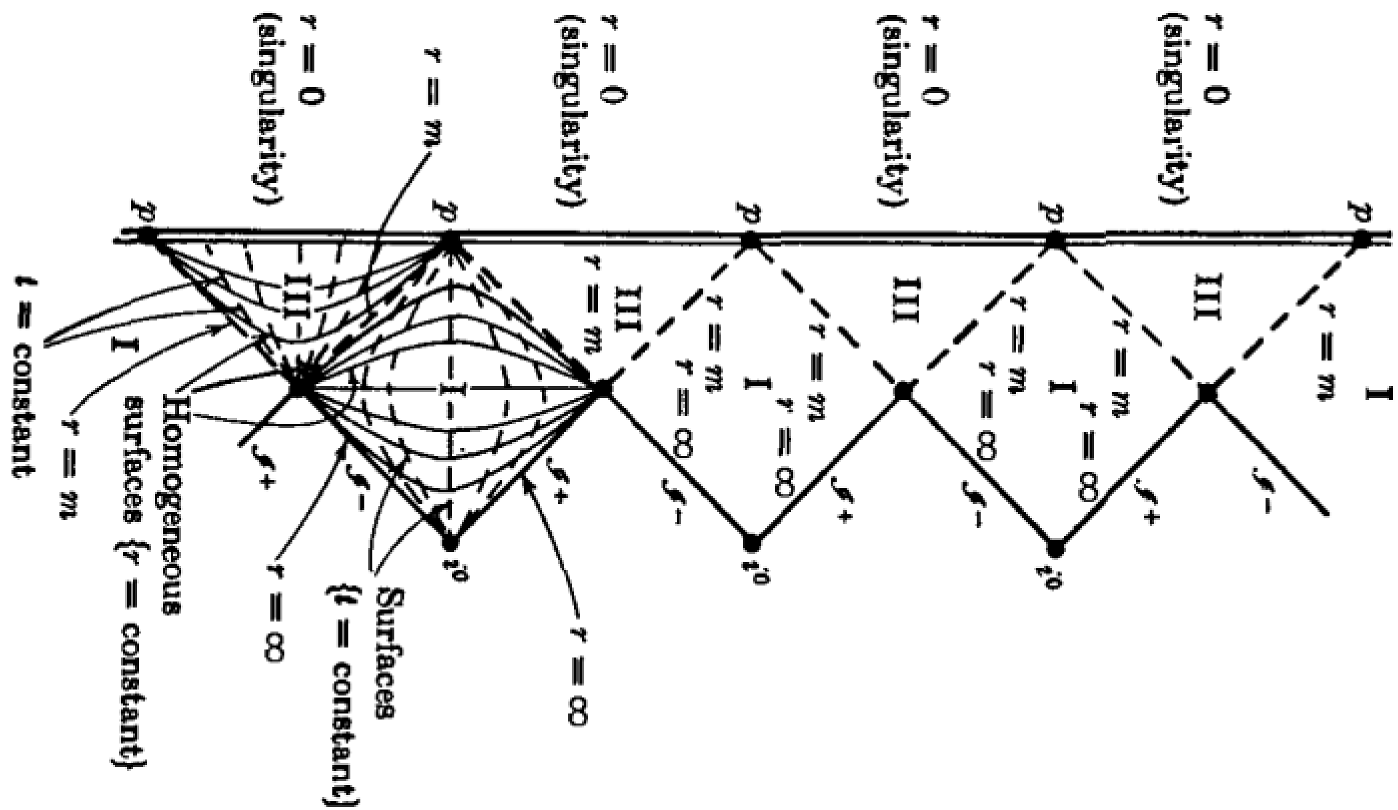
Extremal Black Holes and UV Sensitivity



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30th Jun 2025
GbRP

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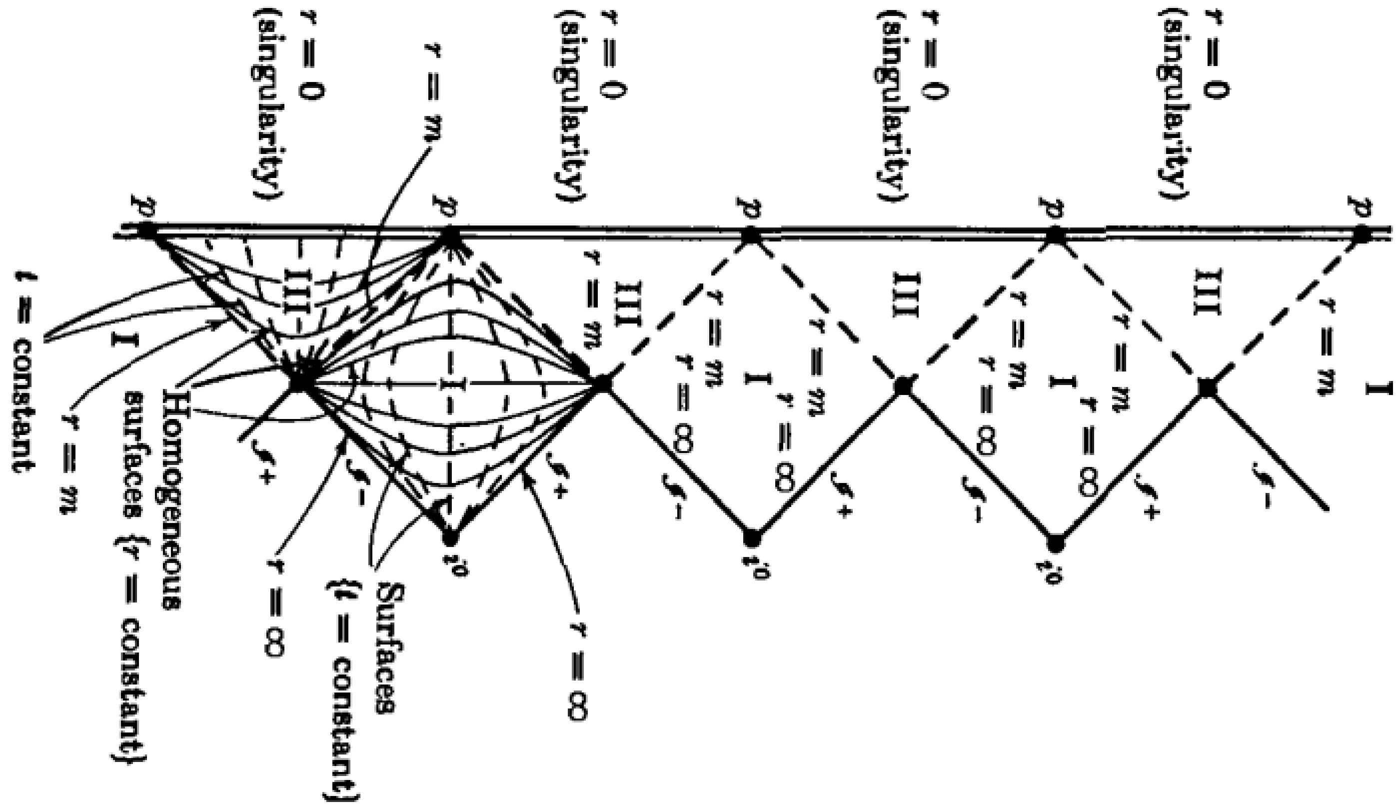


mostly based on 2408.05549 with C. de Rham and A. J. Tolley and 2507.XXXXX with A. D. Kovacs

Charged Black Holes

Charged Black Holes

Interested in charged black holes in GR.

- **Gravity + Maxwell action** in D dimensions

$$S_{\mathrm{EM}} = \int \mathrm{d}^D x \sqrt{-g} \left[\frac{R}{2\kappa} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right]$$

- Spherically symmetric and static **background solution:** **Reissner-Nordström**

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2 \mathrm{d}\Omega_d^2, \qquad F = \Psi'(r)\mathrm{d}t \wedge \mathrm{d}r$$

$d = D - 2$

$$Q^2 = \frac{D-3}{D-2}\kappa q^2$$

$$f(r) := 1 - \frac{M}{r^{D-3}} + \frac{Q^2}{r^{2(D-3)}}, \qquad \Psi(r) = \frac{q}{r^{D-3}}$$

Extremal Black Holes

- Solution possesses two real **horizons.** For AF RN:

$$r_\pm = \frac{M}{2} - \sqrt{\left(\frac{M}{2}\right)^2 - Q^2}$$

→ Degenerate to extremal horizon $r_H := r_+ = r_-$ in **extremal limit**.

- **Near-horizon geometry**: Define $\rho = r - r_+$. At leading-order in near-horizon expansion:

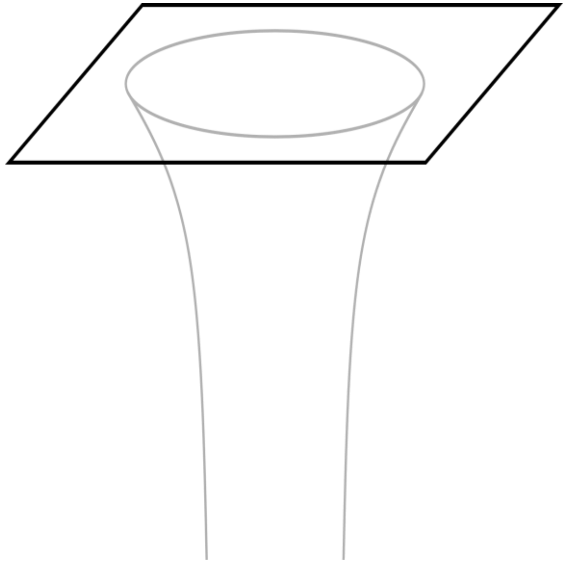
$$\mathrm{d}s^2 = -\frac{\rho^2}{2/f''(r_H)}\mathrm{d}t^2 + \frac{2/f''(r_H)}{\rho^2}\mathrm{d}\rho^2 + r_H^2 \mathrm{d}\Omega_d^2$$

i.e. $\mathrm{AdS}_2 \times S^d$!

[Bertotti ‘59; Robinson ‘59]

→ **Generic** to extremal black holes!

[Kunduri, Lucietti, Reall ‘07]



Why?

- **Supersymmetry** leads to a lot of analytic control: *E.g.* microstate counting of $D=5$ RN via the $D1-D5$ system in type-IIB string theory.

[Strominger & Vafa ‘96]

- The third “law” of **BH thermodynamics** does not hold generally: **Extremal** black holes can form in **finite (advanced) time**.

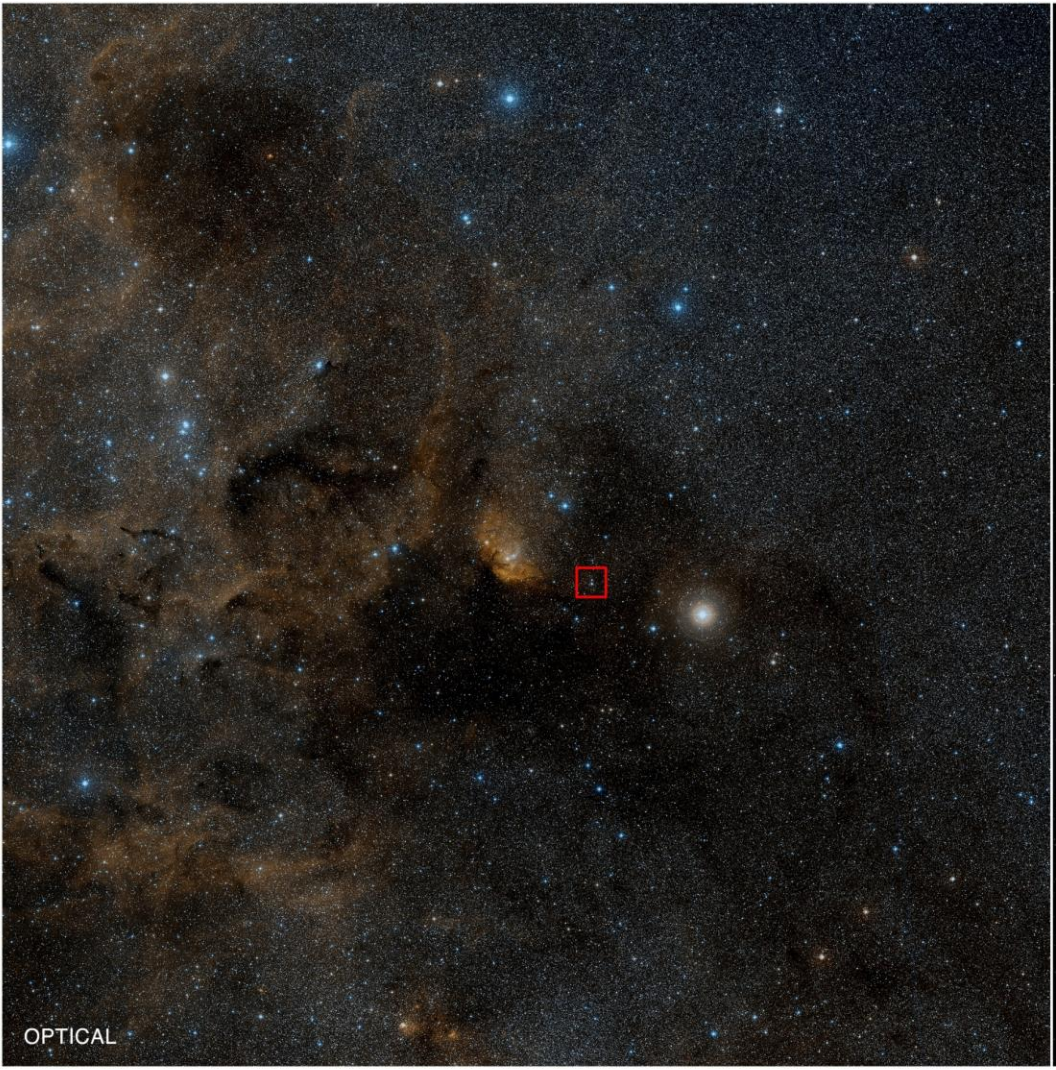
[Kehle & Unger ‘22]

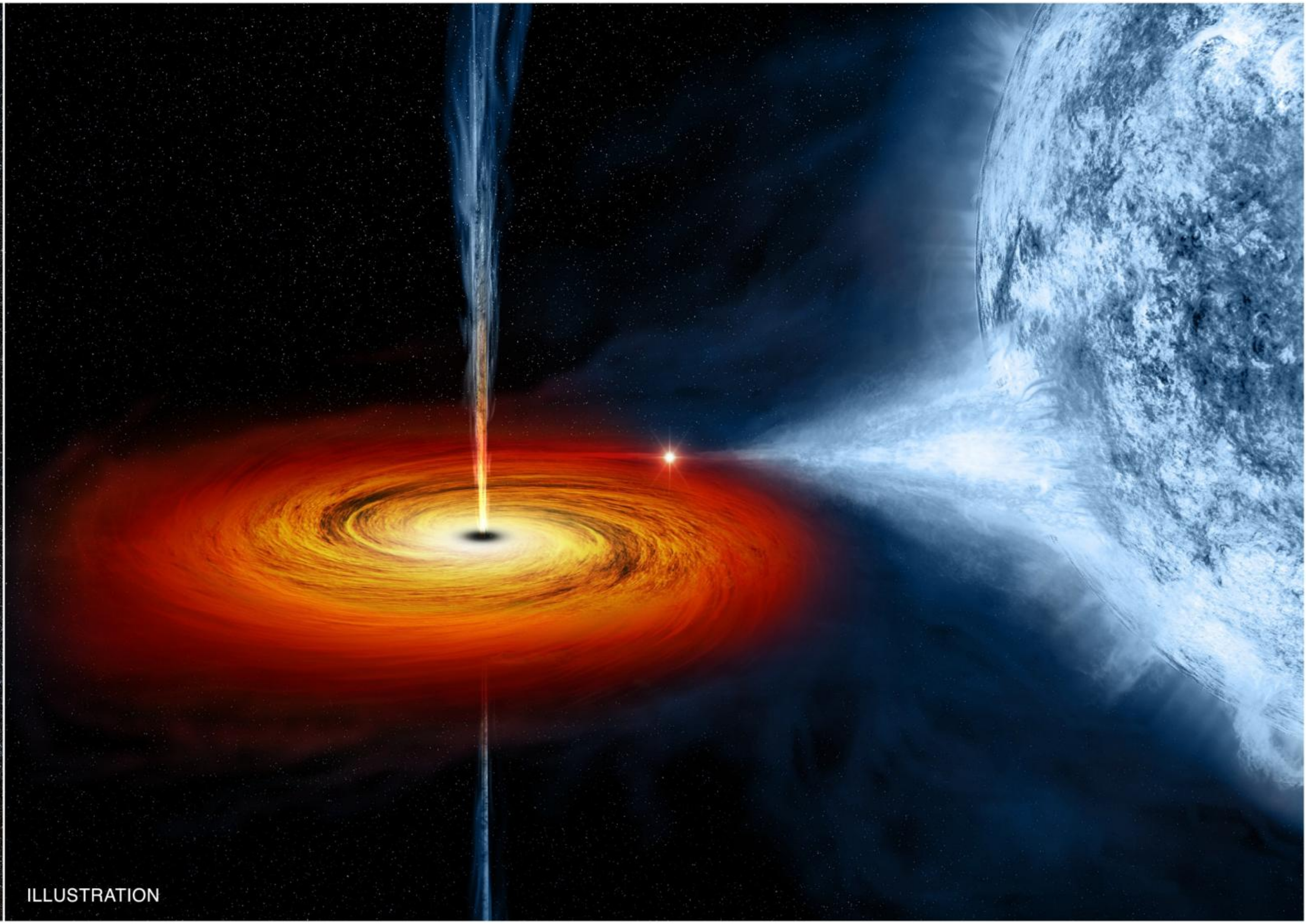
- **Astrophysical** black holes typically not charged, but rotate...
  - ...Often quite **rapidly** (possibly biased measurements)!

[e.g. Bambi ’19 for review]

Object	a_* (Iron)
IRAS 13224-3809	> 0.99
Mrk 110	> 0.99
NGC 4051	> 0.99
Mrk 509	> 0.99
1H0707-495	> 0.98
RBS 1124	> 0.98
NGC 3783	> 0.98
1H0419-577	> 0.98
Fairall 9	> 0.97
NGC 1365	$0.97^{+0.01}_{-0.04}$
Swift J0501-3239	> 0.96

→ (My perspective: Do not expect real black holes to be **exactly extremal**).





[Chandra X-Ray Observatory]

The Aretakis Instability

Key Insight: Extremal black holes suffer from instability

→ Window into UV physics?

Aretakis Instability

Classical instability of extremal black holes.

[Aretakis ‘11 & ’12]
[Lucietti & Reall ’12]

For simplicity, consider massless scalar field in $D=4$ RN.

$$\mathrm{d}s^2 = -f(r)\mathrm{d}v^2 + 2\mathrm{d}v\,\mathrm{d}r + r^2\mathrm{d}\Omega^2, \quad f(r) = \left(1-\frac{r_H}{r}\right)^2$$

- Expanding in **spherical harmonics**

$$\phi(v,r,\Omega) = \sum_{\ell,m}\phi_{\ell,m}(v,r)Y_{\ell,m}(\Omega)$$

→ Individual modes decouple, obey wave equation

$$r^2\,\Box\phi_\ell = 2r\partial_v\partial_r\left(r\phi_\ell\right) + \partial_r\left[r^2 f(r)^2\partial_r\phi_\ell\right] - \ell(\ell+1)\phi_\ell = 0$$

- Take n-th derivative and **evaluate on horizon**:

$$2\partial_v\partial_r^n\left[r\partial_r\left(r\phi_\ell\right)\right]\Big|_H + \left[n(n+1)-\ell(\ell+1)\right]\partial_r^n\phi_\ell\Big|_H = 0$$

Aretakis Instability

- Implies conserved quantity on horizon (Aretakis constant):

$$H_\ell = r_H^{\ell-1}\partial_r^\ell \left[r\partial_r\left(r\phi_\ell\right)\right]\Big|_H$$

- More detailed analysis shows ϕ_k decays at late time for $k \leq \ell$, so

$$\partial_r^n \phi_\ell\Big|_H \sim r_H^{-2n+\ell+1} H_\ell v^{n-\ell-1}$$

→ **Blow-up** at late times!

In **near-horizon** limit,

$$\Box\phi_\ell = \Box_{\mathrm{AdS}_2}\phi_\ell - \frac{\ell(\ell+1)}{r_H^2}\phi_\ell$$

And we can identify

$$\partial_r^n \phi_\ell\Big|_H \sim v^{n-\Delta}, \quad \Delta = \ell + 1$$

Near-Horizon Perspective

Not a coincidence!

[Lucietti, Murata, Reall, & Tanahashi ’12]

For generic **field** obeying near-horizon equation

$$\Box_{\mathrm{AdS}}\phi - m_{\mathrm{eff}}^2\phi = 0$$

can show

$$\partial_r^n \phi\big|_H \sim v^{n-\Delta}$$

→ Δ Is scaling dimension in AdS_2

$$\Delta = \frac{1}{2}\left(1 + \sqrt{1 + 4m_{\mathrm{eff}}^2 L_2^2}\right), \quad L_2 = \sqrt{2/f''(r_H)}$$

At least $\lceil \Delta \rceil$ or $\lfloor \Delta \rfloor + 1$ transverse derivatives necessary to see **non-decay** or **blow-up** in null time.

Severity of Aretakis instability determined by <u>integer</u> part of scaling dimensions!

Gravitational Perturbations

Is this a feature of just the fields propagating on the fixed **background geometry**?

Gravitational/EM Perturbations

To answer, consider perturbations

$$\sqrt{\kappa}h := g - \bar{g}, \quad \delta F := F - \bar{F}$$

of the background geometry itself!

- Dynamics governed by set of **decoupled** wave equations.

  → Technically challenging: Gravitational and electromagnetic perturbations mix.

- **Kaluza-Klein** set-up: Focus on **near-horizon geometry**, and integrate over internal space (sphere).

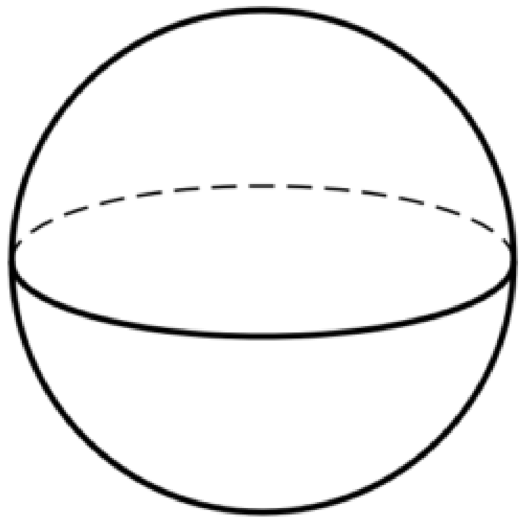
- Equivalent lower-dim. theory has tower of massive states, governed by AdS_2 wave equations

$$\Box_{\mathrm{AdS}}\phi - m_{\mathrm{eff}}^2\phi = 0$$

Gravitational/EM Perturbations

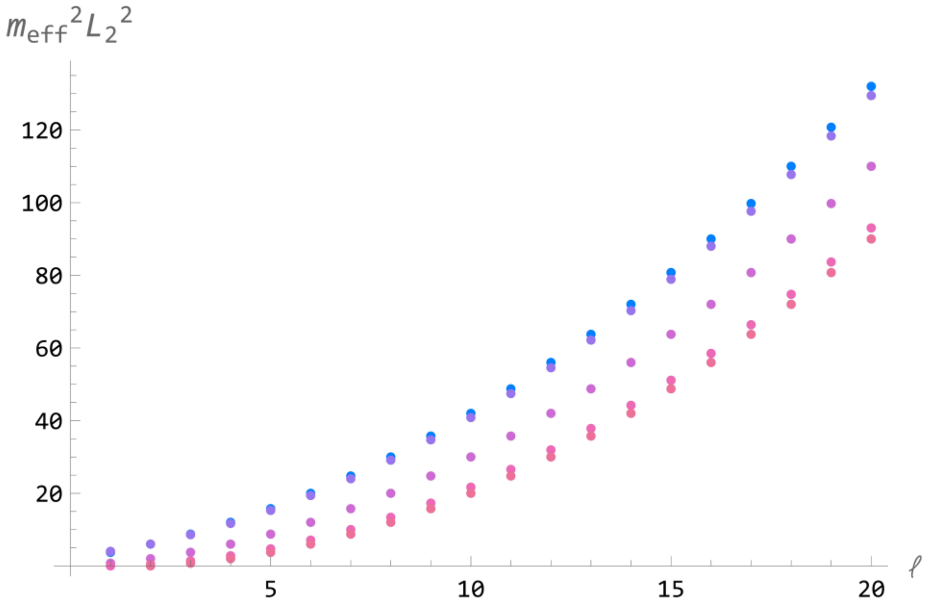
Perturbations organised into representations on the **sphere**.

[Kodama & Ishibashi '04]

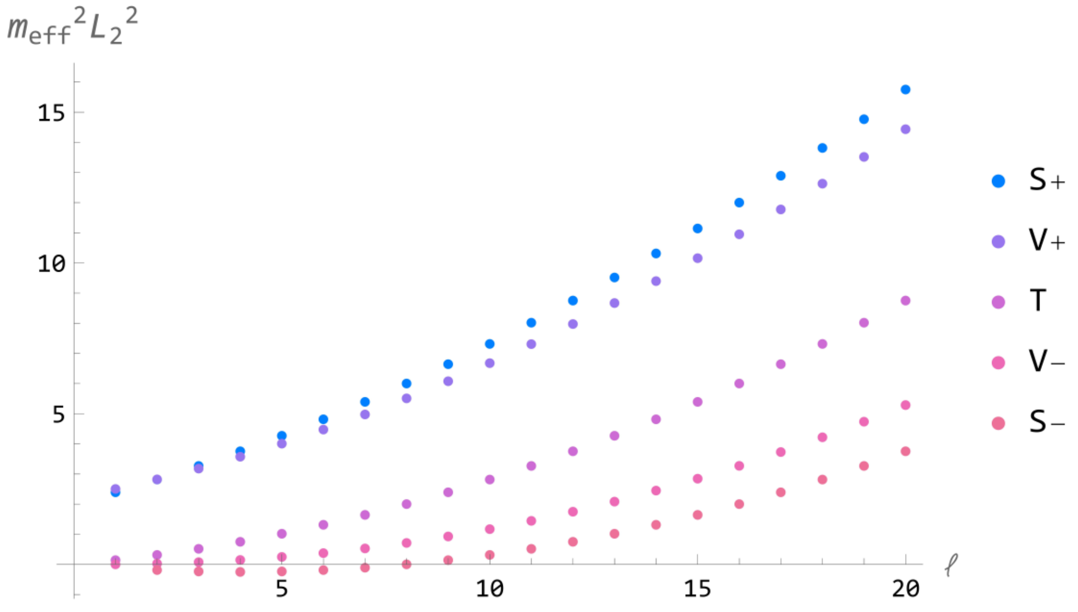


Mixing between...

- EM **perturbations** (+): Vector and scalar
- Gravitational **perturbations** (-): Tensor, vector, and scalar



$D = 5$



$D = 11$

Generic Features of Kaluza-Klein Spectrum

Generally

$$m_{S-}^2 < m_{V-}^2 < m_T^2 < m_{S+/V+}^2, \quad m_{T/V+/S+}^2 > 0$$

with everything above **BF bound** (perturbative stability).

- Tensor, and electromagnetic vector and scalar modes **positive-definite.**

→ Gravitational scalar and vector modes **negative mass squared!**

Aretakis instability is dominated by least massive mode, i.e. gravitational scalar mode, for which

$$\Delta = \frac{1}{2}\left(1 + \left|1 - \frac{2\ell}{D-3}\right|\right), \quad \ell \in \{2, 3, \dots\}$$

Aretakis Instability of Extremal Black Holes

Dominant mode depends on dimension.

- For odd $D \geq 6$, we have

$$\ell = \frac{D-3}{2} \quad \longrightarrow \quad \Delta = \frac{1}{2}$$

- For even $D \geq 6$, minimised for

$$\ell = \left\lfloor \frac{D-3}{2} \right\rceil \quad \longrightarrow \quad \Delta = \frac{1}{4}\left(-1 + \frac{1}{(D-3)^2}\right)$$

- For $D = 4, 5$, minimised for

$$\ell = 2 \quad \longrightarrow \quad \Delta = \frac{2}{D-3}$$

→ **Integers!** **<u>Crucial</u>:** Severity of Aretakis instability depends on integer part, so now sensitive to small corrections!

Cosmological Constant

Let's briefly switch gear: What happens when **cosmological constant** non-zero?

Corresponds to background solution with

$$f(r) := 1 + \frac{r^2}{L^2} - \frac{M}{r^{D-3}} + \frac{Q^2}{r^{2(D-3)}}, \qquad \Lambda = -\frac{(D-1)(D-2)}{2L^2}$$

- **Extremal horizon** at r_H defined by

$$f(r_H) = f'(r_H) = 0, \quad L_2^2 = \frac{2}{f''(r_H)}$$

- Leading behaviour still captured by near-horizon limit.

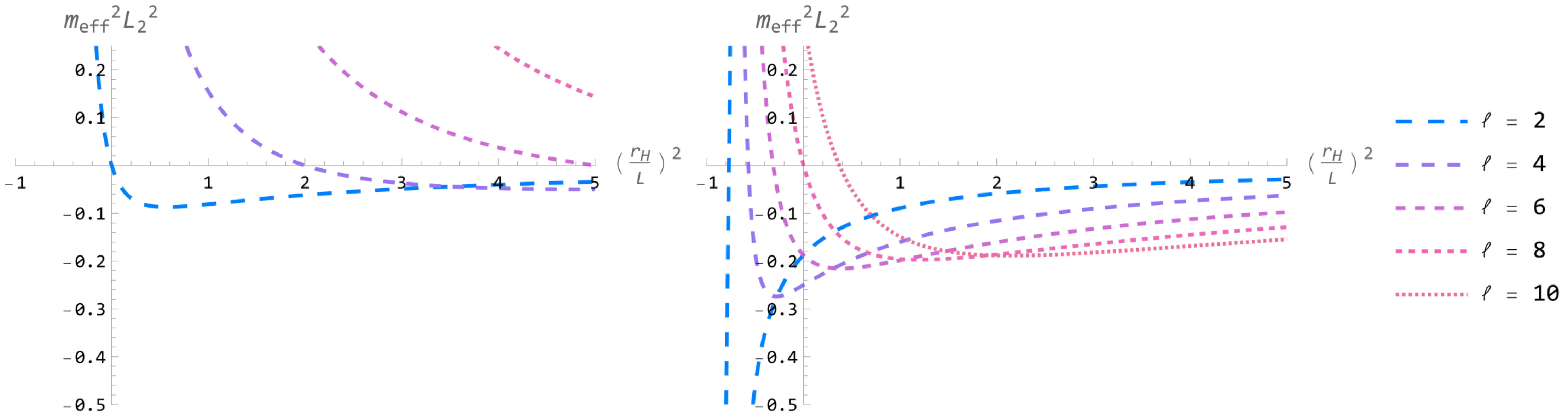
[Gajic & de Moortel '24]

- **Scaling dimensions** receive **corrections** through effective masses and effective AdS_2 scale.

→ Masses obey same inequalities, so Aretakis instability still dominated by **gravitational scalar modes**.

Cosmological Constant

- In (A)dS modes with (higher) lower ℓ dominate.



$D = 5$ $\qquad$ $D = 11$

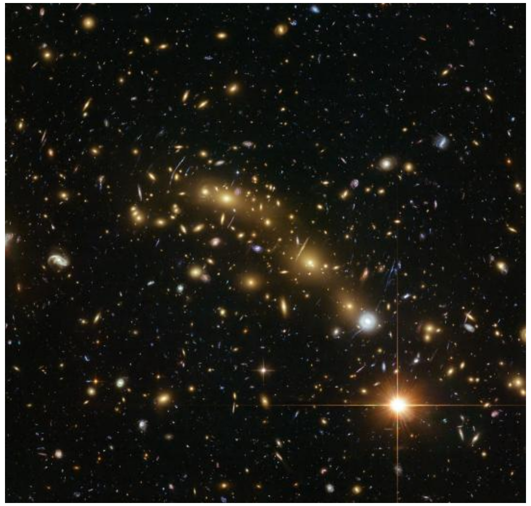
→ Generally **non-integer**, unless we allow **isolated modes**. For example,

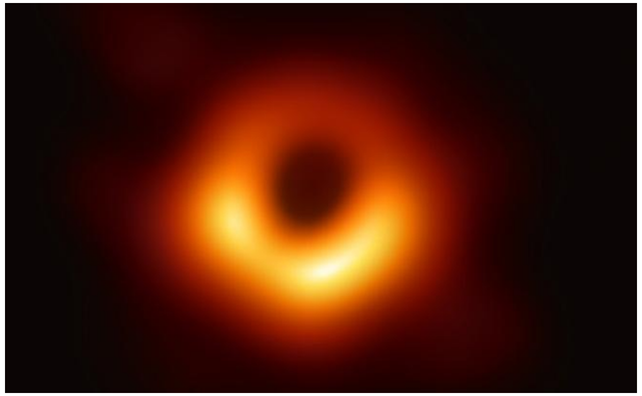
$$\frac{r_H^2}{L^2} = \frac{D-2}{D-4}\left[-1 + \frac{1}{2(D-3)^2}\ell(\ell + D - 3)\right] \quad \longrightarrow \quad \Delta = 1$$

What is the significance of modes with integer scaling dimensions?

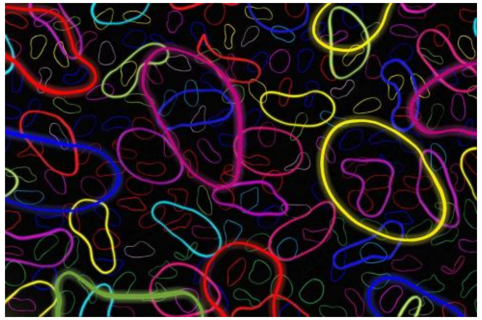
EFT Corrections

General relativity accurately describes gravity across various scales...



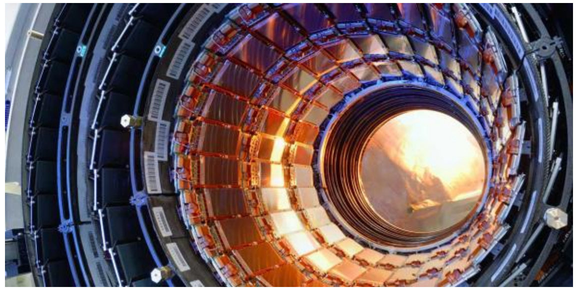


Λ



Energy

M_{Pl}



?

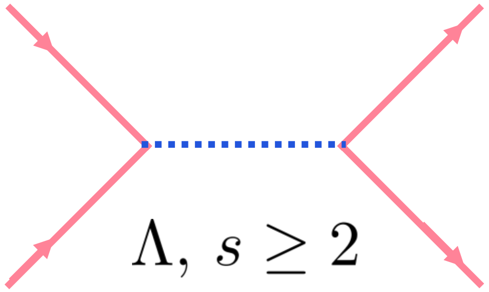
...but predicts its own breakdown: Need **UV completion**!

Effective Field Theory perspective: Use most general **local** action

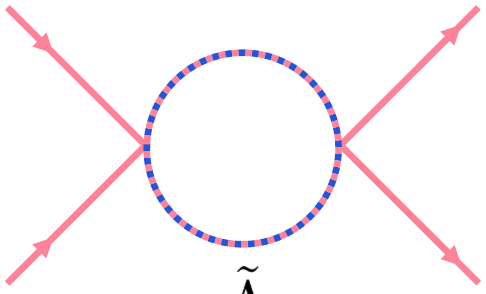
1) consistent with **symmetries**,

2) organised in **derivative expansion**, and

3) with **coefficients** fixed by dimensional analysis.



$$S_{\text{Gravity}} = M_{\text{Pl}}^{D-2} \int d^D x \sqrt{-g} \left[\frac{1}{2} R + \Lambda^2 \sum_{m\geq 0, n\geq 2} c_{mn} \left(\frac{\nabla}{\Lambda}\right)^m \left(\frac{\text{Riemann}}{\Lambda^2}\right)^n \right]$$



Suppressed by loops

→ Treat as **standard QFT** (careful about breakdown!)

EFT Corrections to Scaling Dimensions

Parameterise corrections from UV with **EFT corrections**

$$S = S_{\rm GR} + S_{\rm EFT}$$

- Due to rigidity of near-horizon geometry:

$$m_{\rm eff}^2 = m_{\rm GR}^2 + m_{\rm EFT}^2 \quad \longrightarrow \quad \Delta = \Delta_{\rm GR} + \Delta_{\rm EFT}$$

Recall that $\lfloor \Delta \rfloor + 1$ derivatives needed to see **blow-up of perturbations** at late times.

- Indicates breakdown of **derivative expansion**?

[Hadar & Reall '17]

- **Marginal case**: When $\Delta_{\rm GR} = k \in \mathbb{N}$, severity of Aretakis instability determined by sign of $\Delta_{\rm EFT}$! Explicitly,

$$\Delta_{\rm EFT} < 0 \quad \longrightarrow \quad \lfloor \Delta \rfloor = \lfloor \Delta_{\rm GR} \rfloor - 1$$
$$\Delta_{\rm EFT} > 0 \quad \longrightarrow \quad \lceil \Delta \rceil = \lceil \Delta_{\rm GR} \rceil + 1$$

→ For $D = 4, 5$, we had $\Delta_{\rm GR} = 2, 1$ respectively!

Is this a “breakdown” of EFT?

Breakdown of Breakdowns

Estimate ranges of validity for different approximations used.

EFT expansion

(quadratic in metric perturbation)

$$\kappa\mathcal{L} \supset \sum_{p=0,q=0}^{\infty} \left(\sqrt{\kappa}h\right)^2 \left(\frac{\nabla}{\Lambda}\right)^p \left(\frac{\mathrm{Rie}}{\Lambda^2}\right)^q$$

Metric perturbation series

(in two-derivative theory)

$$\kappa\mathcal{L} \supset \sum_{m=2}^{\infty} \nabla^2 \left(\sqrt{\kappa}h\right)^m$$

...under control when...

$$r_H\Lambda \gg 1, \quad h \ll (\Lambda r_H)^2$$

$$h \ll 1$$

Metric perturbation theory out of control <u>before</u> EFT breaks down!

Example: EFT Correction with Partial UV Completion

Consider specific (yet generic) EFT correction

$$S = S_{\mathrm{EM}} + \frac{\kappa c}{\Lambda^2} \int \mathrm{d}^5 x \sqrt{-g} (F_{\mu\nu} F^{\mu\nu})^2$$

- This is UV completed (at leading-order, tree-level) by

$$S_{\mathrm{EMD}} = \int \mathrm{d}^D x \sqrt{-g} \left(\frac{1}{2\kappa}(R - 2\Lambda) - \frac{1}{4} e^{\alpha\phi} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi - \frac{1}{2} m^2 \phi^2 \right)$$

  if $m^2 r_H^2 \gg 1$, identifying

$$c = \frac{\alpha^2}{32\kappa}, \quad \Lambda = m$$

- As expected, leading scaling dimension match across scales

$$\Delta = 1 - \frac{3k_S^2 (k_S^2 - 4)^2 \alpha^2 / \kappa}{4(15k_S^4 - 128k_S^2 + 256)} \left(\frac{4}{r_H^2 m^2 + k_S^2} - \frac{1}{r_H^2 m^2} \right) = 1 - \frac{c}{\Lambda^2 r_H^2} \frac{72 k_S^2 (k_S^2 - 4)^2}{15k_S^4 - 128k_S^2 + 256} + \ldots$$

Complete agreement: EFT does not break down!
→ Still accurately describes UV theory!

Exactly when metric perturbations go out of control is UV sensitive.

Extremal Black Branes

WIP in collaboration with A. D. Kovacs

Charged Black Branes

Seems like this can be generalised to **extremal black branes**.

- Consider Gravity + form field $F_{(n)} = \mathrm{d}A_{(n-1)}$:

$$S = \frac{1}{2\kappa} \int \mathrm{d}^D x \sqrt{-g} \left[R - \frac{1}{2n!} F_{(n)}^2 \right]$$

- Branes are solutions describing $d = p + 1$-dimensional world volumes.

- Interested in **non-dilatonic** extremal electric black branes. $\{\alpha, \beta, \ldots, \rho\} \in \{0, \ldots, d\}$

$$\mathrm{d}s^2 = H^{-2/d} \eta_{\alpha\beta} \mathrm{d}x^\alpha \mathrm{d}x^\beta + H^{2/\tilde{d}} \left(\mathrm{d}r^2 + r^2 \mathrm{d}\Omega_{\tilde{d}+1}^2 \right), \quad H(r) = 1 + \left(\frac{r_0}{r} \right)^{\tilde{d}}$$

$\tilde{d} = D - d - 2$

$$F_{I\alpha_1 \ldots \alpha_{n-1}} = \sqrt{\frac{2(D-2)}{d\tilde{d}}} \varepsilon_{\alpha_1 \ldots \alpha_{n-1}} \partial_I H^{-1}, \quad n = d + 1$$

→ At background level, duality with magnetically charged background.

Near-Horizon Geometry

- Near-horizon geometry is $\mathrm{AdS}_{d+1} \times S^{\tilde{d}+1}$!

→ **Freund-Rubin compactifications**: Exact solutions to background equations of motion!

- Perturbation equations: Equivalent description in terms of KK fields obeying decoupled **wave equations** on AdS_{d+1}

$$\Box_{\mathrm{AdS}_{d+1}} \phi - m_{\mathrm{eff}}^2 \phi = 0$$

- Similar (more involved) arguments show extremal black branes also suffer from **Aretakis instability.** In particular

[Cvetic, Porfirio, & Satz '20]

$$\partial_r^n \phi \sim v^{n-\Delta}$$

$$\Delta = \frac{d}{2}\left(1 + \sqrt{1 + \frac{4m_{\mathrm{eff}}^2 L_{d+1}^2}{d^2}}\right), \quad L_{d+1} = \sqrt{2/f''(r_H)}$$

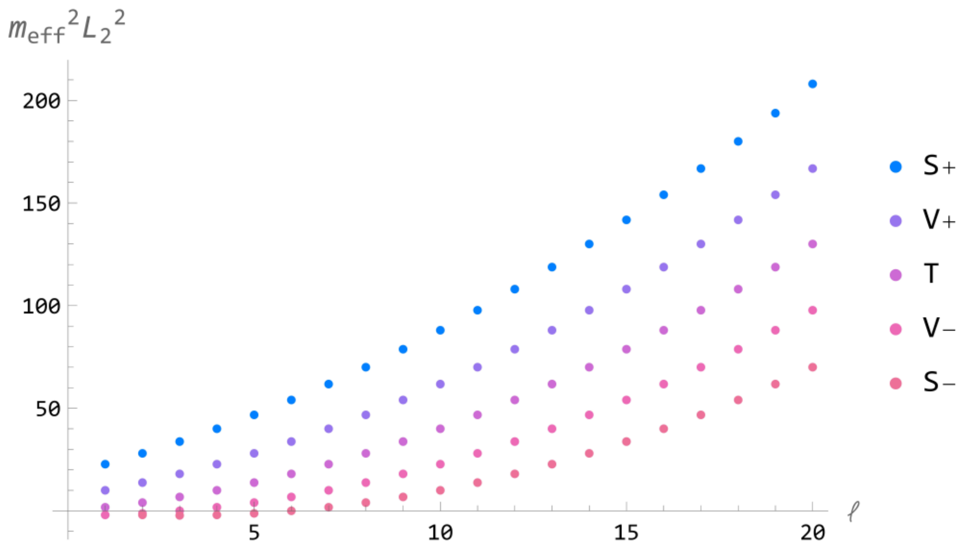
Kaluza-Klein Reduction

To show instability persists without additional fields: Consider **background perturbations.**

- **Hodge decomposition** for anti-symmetric fields (focus on mixed perturbations).

[Rubin & Ordonez '84; Pilch & Schellekens '84; Camporesi & Higuchi '94]

→ Qualitatively similar to $p = 0$! E.g. for $(D, p) = (11, 2)$



- For all values of $(d, \tilde{d}, \ell)$

$$m^2_{\text{BF}} \leq m^2_{S-/V-} < m^2_T < m^2_{S+/V+}, \quad m^2_{T/V+/S+} > 0$$

Kaluza-Klein Spectrum

→ **BF bound** satisfied (perturbative stability)

- Similar results for **magnetically charged** black branes.

[Copeland & Toms ’84]
[Kinoshita & Mukohyama ’09; Brown & Dahlen ’13; Hinterbichler, Levin, Zukowski ‘13]

  - **Electromagnetic duality** $F \mapsto \star F$ relates some sectors, but background generically breaks this.

- **BF bound saturated** e.g. for gravitational modes for

$$\ell = \frac{\tilde{d}}{2} \quad \longrightarrow \quad \Delta = \frac{d}{2}$$

In the worst case, still need $\lfloor d/2 \rfloor + 1$ derivatives to see blow-up.

Black branes suffer still from Aretakis instability, but milder than for black holes!

Conclusion

Deformations of Extremal Black Holes/Branes

Original motivation for calculation was **deformations** of extremal black holes.

[Horowitz, Kolanowski, & Santos '21, '22, & '23]
[Cano & David '24]

Recall in near-horizon limit, perturbations obey

$$\Box_{\mathrm{AdS}_{p+2}}\phi - m_{\mathrm{eff}}^2\phi = 0$$

- Restrict to static deformations to near-horizon geometry

$$\{\alpha, \beta, \ldots,\} \in \{0, \ldots, p\}$$

$$\partial_\alpha \phi = 0 \quad \longrightarrow \quad \frac{\rho^2}{L^2}\left(\partial_\rho^2\phi + \frac{p+2}{\rho}\partial_\rho\phi\right) - m_{\mathrm{eff}}^2 = 0$$

  - Solutions scale near horizon:

$$\phi \sim \rho^{-\Delta} + \rho^{\Delta - d}$$

→ Describes near-horizon behaviour tidal **deformation**, e.g. from **multi-centred black branes**.

Concluding Remarks

Extremal black holes/branes suffer from classical instability, determined by the scaling dimensions (w.r.t. AdS in near-horizon geometry).

- Modes with **integer scaling** dimensions are **UV sensitive**.

  → Suggests breakdown of EFT, but no: **Non-linearities** kick in!

- Other observational signatures?
- Better understanding of non-linearities? Use **numerics**?
- In **Anti-de Sitter**: Holographic picture? Relationship with shift symmetric fields or scale separation??
- Deformations of black holes/branes: What happens to set-ups with **stacked branes** (e.g. ABJM)?

Bonus Slides

Symmetry Argument for Aretakis Scaling

Simple **symmetry argument.**

[Gralla & Zimmermann ‘18; Chen & Stein ’17 & ’18]

Consider generic field with not necessarily integer Δ

$$\Box_{\mathrm{AdS}}\phi - m_{\mathrm{eff}}^2\phi = 0, \quad L_2^2 m_{\mathrm{eff}}^2 = \Delta(\Delta - 1)$$

- AdS_2 **Killing vector fields**

$$L_0 = v\partial_v - \rho\partial_\rho, \qquad L_+ = \partial_v, \qquad L_- = v^2\partial_v - 2(\rho v + 1)\partial_\rho$$

obey standard $\mathfrak{sl}(2,\mathbb{R})$ commutation relations

$$[L_+, L_-] = 2L_0, \qquad [L_\pm, L_0] = \pm L_\pm$$

→ Casimir coincides with **wave operator**!

$$\mathcal{C} \equiv \pounds_{L_0}(\pounds_{L_0} - 1) - \pounds_{L_-}\pounds_{L_+} = \Box_{\mathrm{AdS}_2}$$

Symmetry Argument for Aretakis Scaling

- For solutions of wave equation take as basis functions **simultaneous eigenfunctions** of $\mathcal{C}$ and one of the generators, say L_0

$$\mathcal{C}\psi_{\ell,h} = \Delta(\Delta-1)\psi_{\ell,h}$$
$$L_0\psi_{\ell,h} = h\psi_{\ell,h}$$

→ General solution is $\phi_{\Delta,h} = v^{-h}F_{\Delta,h}(v\rho)$, where

$$(z^2+2z)\,F''_{\ell,h}(z) + 2(z+1-h)\,F'_{\ell,h}(z) - \Delta(\Delta-1)\,F_{\ell,h}(z) = 0, \qquad z = v\rho$$

- **Boundedness** at $z=0$ (horizon) and decay at $z\to\infty$ (infinity) enforces $h\geq\Delta$ so

$$F_{\Delta,h}(z) = c_{\Delta,h}\,z^{-\Delta}\,{}_2F_1\left(h+\Delta,\Delta;2\Delta;-\frac{2}{z}\right)$$

- Late-time behaviour dominated by $h=\Delta$, for which, as promised

$$\partial_\rho^n\phi_{\Delta,\Delta}\big|_H = v^{n-\Delta}F^{(n)}(v\rho)\big|_H \sim v^{n-\Delta}$$

Multi-Centred Black Branes

Multi-centred black brane solutions have

[cf. Majumdar '47; Papapetrou '47]

$$H\left(\mathbf{y}\right) = 1 + \sum_{i=0}^{N} \frac{M_i}{|\mathbf{y} - \mathbf{y}_i|^{\tilde{d}}}$$

- Reference brane at $\mathbf{y}_0 = \mathbf{0}$ with $M_0 := M$, and define

$$\hat{\rho}_i = |\mathbf{y}_i|, \quad |\mathbf{y}| = \hat{\rho}, \quad \mathbf{y} \cdot \mathbf{y}_i = \hat{\rho}\,\hat{\rho}_i \cos\theta_i$$

- Near horizon of reference brane $\hat{\rho}/\hat{\rho}_i \ll 1$:

$$\mathrm{d}s^2 = \left(\frac{\rho}{L}\right)^2 \eta_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta + \left(\frac{L}{\rho}\right)^2 \left[1 + \tilde{d}h(\rho) + \frac{\tilde{d}^2}{d}\frac{\mathrm{d}h(\rho)}{\mathrm{d}\log\rho}\right]\mathrm{d}\rho^2 + r_0^2\left[1 + h(\rho)\right]^{2\tilde{d}}\mathrm{d}\Omega^2_{\tilde{d}+1}$$

with **perturbation**

$$h(\rho) \sim \sum_{i=1}^{N} \frac{M_i}{M} \sum_{j=0}^{\infty} C_j^{(\tilde{d}/2)}\left(\cos\theta_i\right)\left(\frac{\rho}{\rho_i}\right)^{d/\tilde{d}+d} \sim \rho^{jd/\tilde{d}+d} + \dots$$

The Extremal Limit

Sub-extremal charged black brane metric is

$$\mathrm{d}s^2 = f_+ f_-^{\frac{2-d}{d}} \mathrm{d}t^2 + f_-^{2/d} \delta_{ab} \mathrm{d}x^a \mathrm{d}x^b + \frac{1}{f_+ f_-} \mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2$$

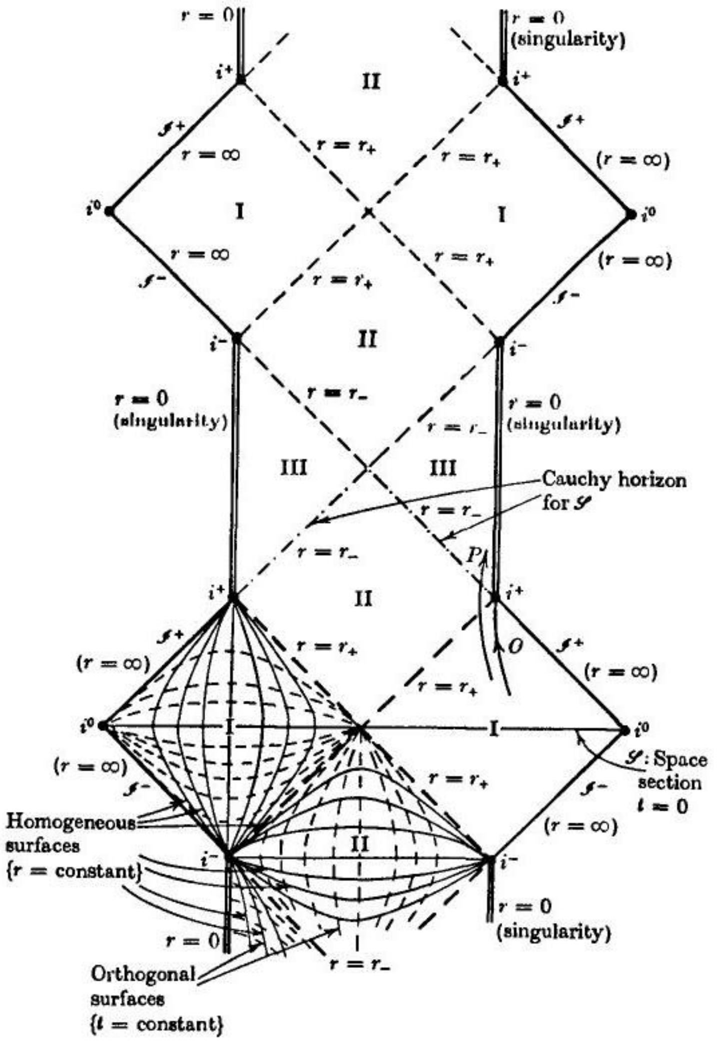
$$f_\pm = 1 - \left(\frac{r_\pm}{r}\right)^{\tilde{d}}$$

- Careful about double expansion in

$$\varepsilon = 2\frac{r_+ - r_-}{r_+ + r_-}, \quad \rho_* = r - r_+$$

- At leading order, **wave equation:**

$$\partial_{\rho_*}\left[\left(\frac{\rho_*}{r_+} + \epsilon\right)\frac{\rho_*}{r_+}\tilde{d}^2 \partial_{\rho_*}\phi\right] - m_{\text{eff}}^2 \phi = 0$$



The Extremal Limit

- Fortunately, solutions are known:

$$\phi \sim AP_{\gamma_+/d}(z) + BQ_{\gamma_+/d}(z), \quad z = 1 + \frac{\rho_*}{r_+\varepsilon}$$

  - Near the horizon

$$\phi \sim A\left[1 + \frac{1}{2}\frac{\gamma_+}{d}\left(\frac{\gamma_+}{d} - 1\right)(z-1) + \dots\right] + B\left[c - \frac{1}{2}\log(z-1) + \dots\right]$$

→ **Regularity** requires $B = 0$.

- Leading order solution in **extremal limit**

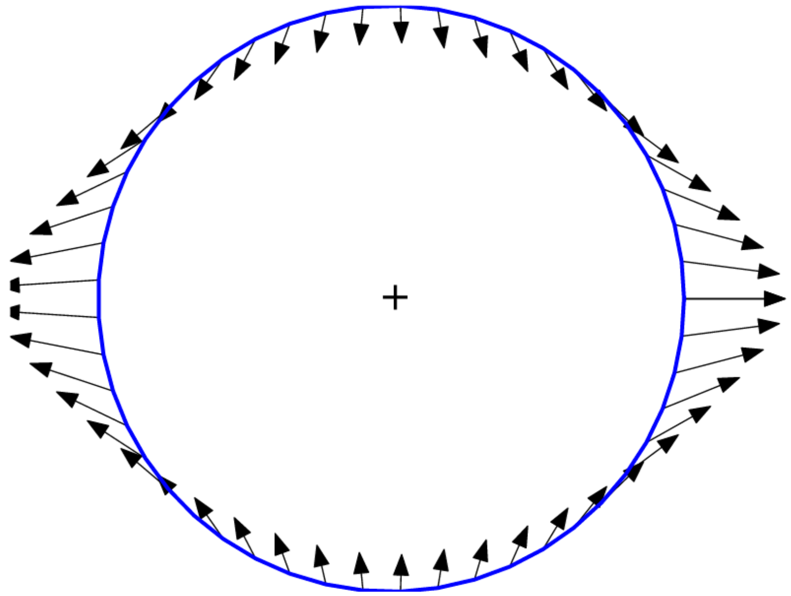
$$\phi \sim \rho_*^{\gamma_+/d} + \dots$$

→ Regular radial coordinate is

$$\rho \sim L(\tilde{d}\rho_*/r_0)^{1/d}$$

so **extremal limit** picks out the γ_+-branch of solutions!

Singularities



What does **smoothness** mean physically?

For **metric perturbation** which scales as

$$h_{\underline{AB}} \sim \rho^{\gamma}$$

Backreacted geometry will have:

- **Scalar invariants** such as

$$S \in \{R_{ABCD}R^{ABCD}, R_{AB}R^{AB}, \dots\}$$

scaling as

$$S \sim \rho^{n\gamma}, \quad n \in \mathbb{N}^+$$

→ Scalar polynomial (s.p.) **singularity** for

$$\gamma < 0$$

- Perturbation to the **Weyl tensor** scales as

$$\delta C_{\underline{ABCD}} \sim \rho^{\gamma-2}$$

This *e.g.* enters the Raychaudhuri equation, describes **tidal forces**

→ Parallel-propagated (p.p.) **singularity** when

$$\gamma < 2$$