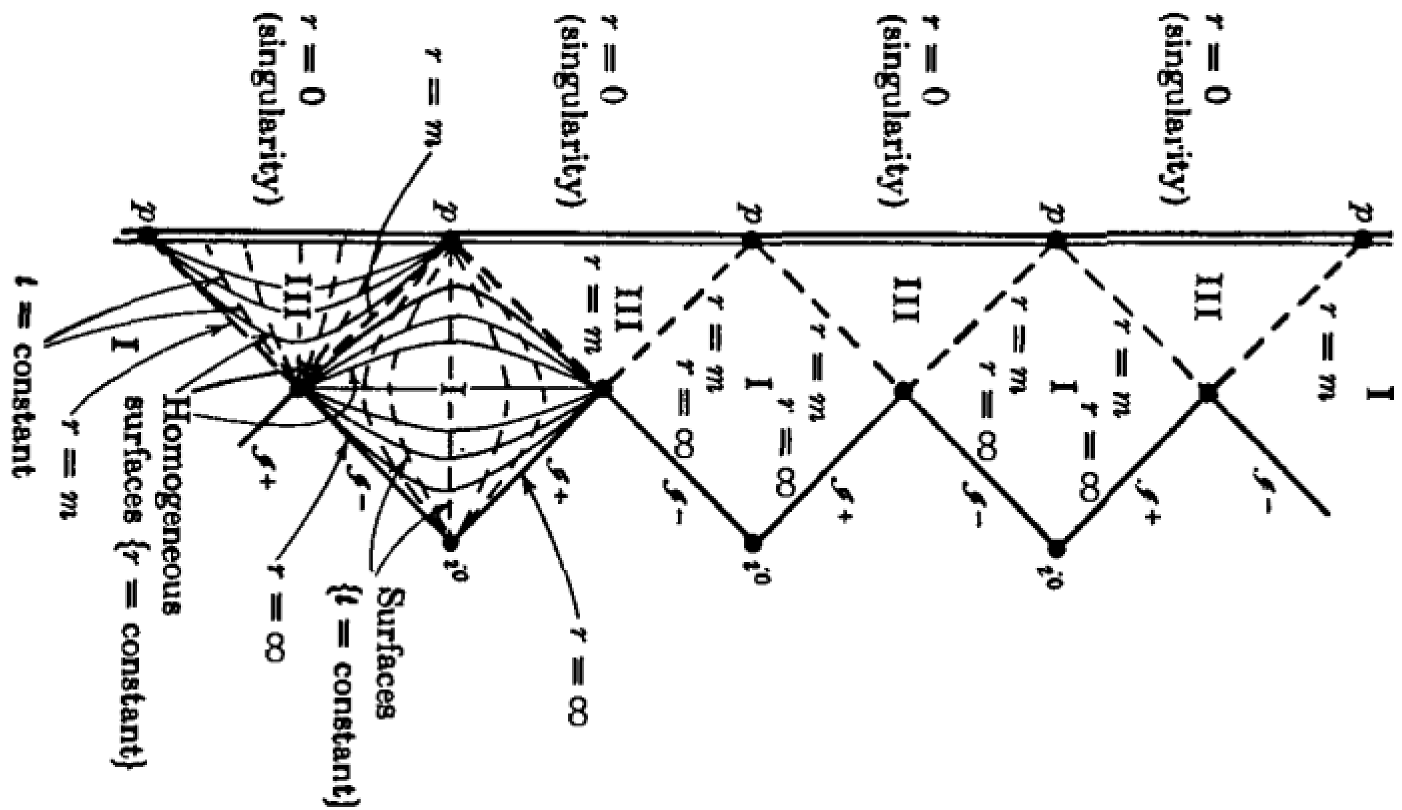
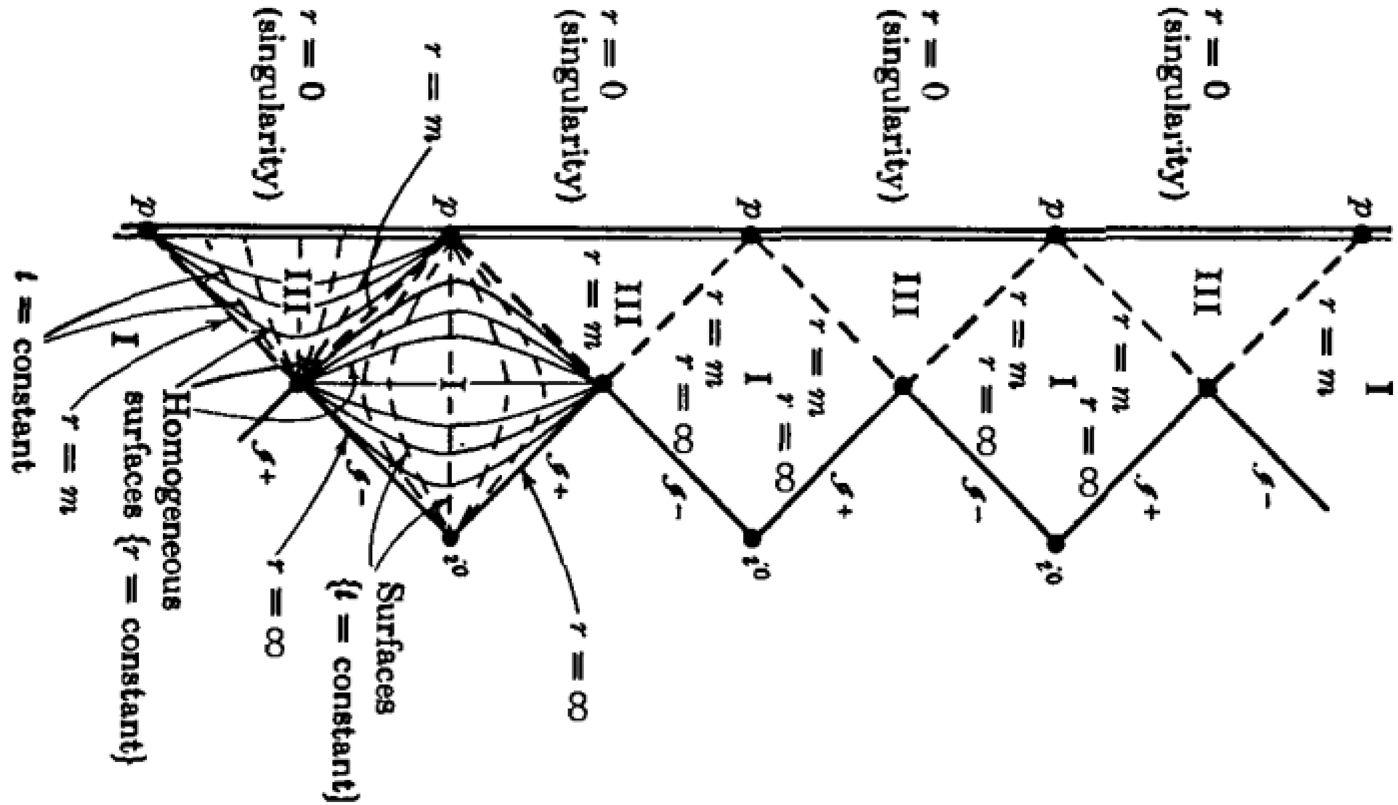
On and Off Extremal Black Branes



Calvin Y.-R. Chen
National Taiwan University

18th Jul 2025
GR24/Amaldi16

On and Off Extremal Black Branes



Based on 2507.12529 with A. D. Kovacs

Extremal Black Branes

Black (mem-)branes are **extended versions** of black holes

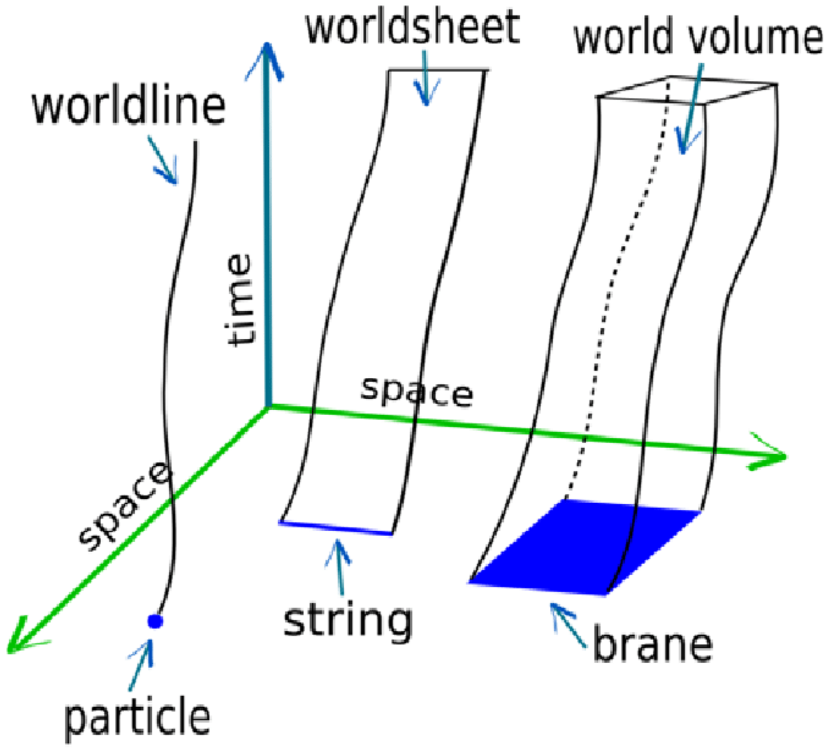
Black Branes

Consider a D-dimensional spacetime.

Branes generalise point particles: A p-brane has p spatial dimensions, so break isometry group to

$$\mathrm{ISO}(1,p) \times \mathrm{SO}(D-p-1) \subset \mathrm{ISO}(1,D-1)$$

- *E.g.* a point particle is 0-brane, a string is a 1-brane, *etc.*



Black p-branes have singularities sourced by p-branes.

- Black hole singularity is sourced by a point: 0-brane!
- **Black string** sourced by a string: 1-brane, *etc.*

p-Form Electrodynamics

Would like to study **charged black branes**.

- Consider a d-form field $A \in \Omega^d(M)$ on a spacetime M with $\dim(M) = D$.
- Describe with **field strength**

$$F = \mathrm{d}A$$

  - Invariant under gauge transformations

$$A \mapsto A + \mathrm{d}\Lambda$$

- Minimal coupling to **gravity**:

$$S = \frac{1}{2\kappa}\int \mathrm{d}^D x\sqrt{-g}\left[R - \frac{1}{2(d+1)!}F^2_{(d+1)}\right]$$

→ Recover **Einstein-Maxwell theory** when $d = 1$.

Charged Black Branes

Idea: Charge the black branes under form field!

- Can couple **electrically** or **magnetically**:

$$d = p + 1 \qquad \text{or} \qquad \tilde{d} \equiv D - d - 2 = p + 1$$

- Charges satisfy inequalities (BPS bounds): **Extremal limit** when **saturated.**
  - *E.g.* electrically charged extremal black branes:

$$\mathrm{d}s^2 = H^{-2/d}\eta_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta + H^{2/\tilde{d}}\delta_{IJ}\mathrm{d}y^I \mathrm{d}y^J, \quad F_{I\alpha_1\ldots\alpha_d} = \sqrt{\frac{2(D-2)}{d\tilde{d}}}\varepsilon_{\alpha_1\ldots\alpha_d}\partial_I H^{-1}$$

$$H(\hat{\rho}) = 1 + \left(\frac{r_0}{\hat{\rho}}\right)^{\tilde{d}}, \quad \hat{\rho} = |\mathbf{y}|$$

→ Horizon located at $\hat{\rho} = r_0$.

Near-Horizon Geometry

Useful to look at the **near-horizon geometry**!

- Define **near-horizon coordinate**

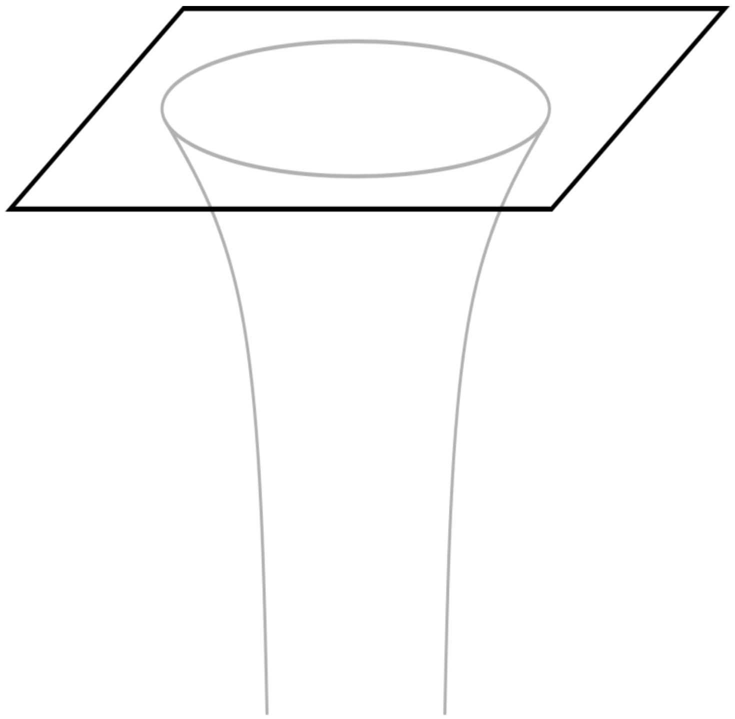
$$\hat{\rho} = r_0 \left(\rho/L\right)^{d/\tilde{d}}, \quad L = d r_0/\tilde{d}$$

- At leading order in $\hat{\rho} \to 0$

$$\mathrm{d}s^2_{\mathrm{NH}} = \frac{\rho^2}{L^2}\eta_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta + \frac{L^2}{\rho^2}\mathrm{d}\rho^2 + r_0^2 \mathrm{d}\Omega^2_{D-p-2}$$

$$F_{\mu_1\ldots\mu_{d+1}} = f\varepsilon_{\mu_1\ldots\mu_{d+1}}, \quad f^2 = 2(D-2)\frac{d}{\tilde{d}}\frac{1}{L^2}$$

  - **Freund-Rubin** compactification: $\mathrm{AdS}_{d+1} \times S^{D-d-1}$



→ For extremal black holes ($d = 1$): **Rigidity**!

[Kunduri, Lucietti, Reall '07]

Why should I care?

- **Analytic control**: *E.g.* microstate counting of $D=5$ RN via the $D1-D5$ system in type-IIB string theory.

[Strominger & Vafa '96]

- The third "law" of **BH thermodynamics** does not hold generally: **Extremal** black holes can form in **finite (advanced) time**.

[Kehle & Unger '22]

- **Astrophysical** black holes typically not charged, but rotate...
  - ...Often quite **rapidly** (possibly biased measurements)!

[e.g. Bambi '19 for review]

- **Gregory-Laflamme instability:** Instability of subextremal branes switches off at extremality.

[Gregory and Laflamme '94]

Object	a_* (Iron)
IRAS 13224-3809	> 0.99
Mrk 110	> 0.99
NGC 4051	> 0.99
Mrk 509	> 0.99
1H0707-495	> 0.98
RBS 1124	> 0.98
NGC 3783	> 0.98
1H0419-577	> 0.98
Fairall 9	> 0.97
NGC 1365	$0.97^{+0.01}_{-0.04}$
Swift J0501-3239	> 0.96

Key Insight: Extremal black holes suffer from classical instability

The Aretakis Instability

Aretakis Instability

For simplicity, consider **toy model**: Massless scalar field propagating on **fixed background** for $(D, d) = (4, 1)$ (Reissner-Nordström)

[Aretakis '11 & '12]
[Lucietti & Reall '12]

$$\mathrm{d}s^2 = -f(r)\mathrm{d}v^2 + 2\mathrm{d}v\,\mathrm{d}r + r^2\mathrm{d}\Omega^2, \quad f(r) = \left(1 - \frac{r_H}{r}\right)^2$$

- Expanding in **spherical harmonics**

$$\phi(v, r, \Omega) = \sum_{\ell,m} \phi_{\ell,m}(v, r) Y_{\ell,m}(\Omega)$$

→ Individual modes decouple, obey wave equation

$$r^2\,\Box\phi_\ell = 2r\partial_v\partial_r\left(r\phi_\ell\right) + \partial_r\left[r^2 f(r)^2\partial_r\phi_\ell\right] - \ell(\ell+1)\phi_\ell = 0$$

- Take n-th derivative and **evaluate on horizon**:

$$2\partial_v\partial_r^n\left[r\partial_r\left(r\phi_\ell\right)\right]\Big|_H + \left[n(n+1) - \ell(\ell+1)\right]\partial_r^n\phi_\ell\Big|_H = 0$$

Aretakis Instability

- Implies conserved quantity on horizon (Aretakis constant):

$$H_\ell = r_H^{\ell-1} \partial_r^\ell \left[r \partial_r \left(r \phi_\ell \right) \right] \Big|_H$$

- More detailed analysis shows $\partial_k \phi_\ell$ decays at late time for $k \leq \ell$, so

$$\partial_r^n \phi_\ell \Big|_H \sim r_H^{-2n+\ell+1} H_\ell v^{n-\ell-1}$$

→ **Blow-up** at late times!

In **near-horizon** limit, wave equation:

$$\Box \phi_\ell = \Box_{\mathrm{AdS}_2} \phi_\ell - \frac{\ell(\ell+1)}{r_H^2} \phi_\ell$$

And we can identify

$$\partial_r^n \phi_\ell \Big|_H \sim v^{n-\Delta}, \quad \Delta = \ell + 1$$

Near-Horizon Perspective

Not a coincidence!

[Lucietti, Murata, Reall, & Tanahashi ’12]

For generic **field** obeying near-horizon equation

$$\Box_{\mathrm{AdS}_2}\phi - m_{\mathrm{eff}}^2\phi = 0$$

- Define **scaling dimension** with respect to AdS_2 is

$$\Delta = \frac{1}{2}\left(1 + \sqrt{1 + 4m_{\mathrm{eff}}^2 L^2}\right)$$

→ Generically non-integer!

- Nonetheless, can show

$$\partial_r^n \phi\big|_H \sim v^{n-\Delta}$$

Black Branes

Consider **generic wave equation** in near-horizon geometry of **extremal black brane**:

$$\Box_{\mathrm{AdS}_{d+1}}\phi - m_{\mathrm{eff}}^2\phi = 0$$

- Again: Useful to define AdS_{d+1} **scaling dimension**

$$\Delta = \frac{d}{2}\left(1+\sqrt{1+\frac{4m_{\mathrm{eff}}^2L^2}{d^2}}\right)$$

- Can show that (need to work a bit harder...)

[Cvetic, Porfirio, and Satz ‘20]

$$\partial_r^n\phi\big|_H \sim v^{n-\Delta}$$

→ See paper for new perspective based on **symmetries** of near-horizon geometry.

At least $\lceil\Delta\rceil$ or $\lfloor\Delta\rfloor+1$ transverse derivatives necessary to see **non-decay** or **blow-up** in null time.

Severity of Aretakis instability determined by integer part of scaling dimensions!

Is this a feature of just the fields propagating on the fixed **background geometry**?

Gravitational Perturbations

Gravitational/EM Perturbations

To answer, consider **perturbations**

$$\sqrt{\kappa}h := g - \bar{g}, \quad \delta F := F - \bar{F}$$

of the background geometry itself!

- Organise according to representations on **sphere**: **SVT decomposition** for gravity and **Hodge decomposition** for anti-symmetric fields.

[Kodama & Ishibashi ‘04]

[Rubin & Ordonez ‘84; Pilch & Schellekens ‘84; Camporesi & Higuchi ’94]

- Focus on **mixed** perturbations: Technically challenging...
    - **EM perturbations** (+): Vector and scalar
    - **Gravitational perturbations** (-): Tensor, vector, and scalar

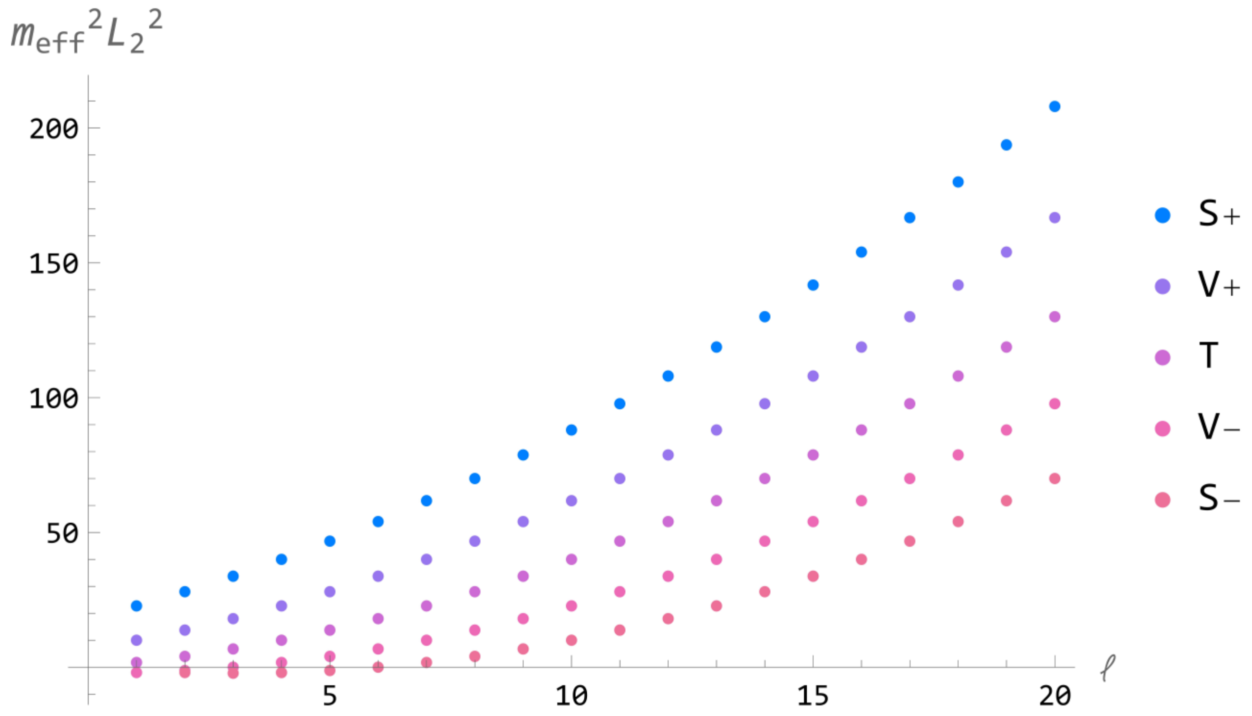
Kaluza-Klein Spectrum

Equivalent description in **near-horizon geometry**: Kaluza-Klein towers of fields with **decoupled** wave equations

$$\Box_{\mathrm{AdS}_{d+1}}\phi - m_{\mathrm{eff}}^2\phi = 0$$

- Now **Aretakis instability** determined by **KK spectrum**: Set of m_{eff}!

- *E.g.* for $(D, p) = (11, 2)$



Kaluza-Klein Spectrum

- For all $(d, \tilde{d}, \ell)$

$$m^2_{\mathrm{BF}} \leq m^2_{S-/V-} < m^2_T < m^2_{S+/V+}, \quad m^2_{T/V+/S+} > 0$$

→ **BF bound** satisfied (perturbative stability)

- Similar results for magnetically charged black branes.

[Copeland & Toms ’84; Kinoshita & Mukohyama ’09; Brown & Dahlen ’13; Hinterbichler, Levin, Zukowski ‘13]

  - **Electromagnetic duality** $F \mapsto \star F$ relates some sectors, but background generically breaks this.

- **Aretakis instability** dominated by least massive modes: *E.g.* gravitational scalar modes saturate **BF bound** for

$$\ell = \frac{\tilde{d}}{2} \quad \longrightarrow \quad \Delta = \frac{d}{2}, \qquad \tilde{d} \text{ even}$$

Worst case: Need $\lfloor d/2 \rfloor + 1$ derivatives to see blow-up.

→ Black branes suffer still from Aretakis instability,
but milder than for black holes!

Conclusion

Concluding Remarks

<u>Punchline</u>: Extremal black holes/branes suffer from classical instability, determined by the scaling dimensions (*w.r.t.* AdS in near-horizon geometry).

- Aretakis instability depends on **integer part** of scaling dimensions: Modes with **integer scaling** dimensions are **UV sensitive**.

  → Suggests breakdown of EFT, but no: **Non-linearities** kick in!

- Better understanding of non-linearities, end-point of instability? Use **numerics**?

- In **Anti-de Sitter**: Holographic picture? Relationship with shift-symmetric fields or scale separation?

- Scaling dimensions also describe **deformations of branes**: Admissible curvature singularities on horizon.

Bonus Slides

Symmetry Argument for Aretakis Scaling

Simple **symmetry argument.**

[Gralla & Zimmermann '18; Chen & Stein '17 & '18]

Consider generic field with not necessarily integer Δ

$$\Box_{\mathrm{AdS}}\phi - m_{\mathrm{eff}}^2\phi = 0, \quad L_2^2 m_{\mathrm{eff}}^2 = \Delta(\Delta - 1)$$

- AdS_2 **Killing vector fields**

$$L_0 = v\partial_v - \rho\partial_\rho, \qquad L_+ = \partial_v, \qquad L_- = v^2\partial_v - 2(\rho v + 1)\partial_\rho$$

obey standard $\mathfrak{sl}(2,\mathbb{R})$ commutation relations

$$[L_+, L_-] = 2L_0, \qquad [L_\pm, L_0] = \pm L_\pm$$

→ Casimir coincides with **wave operator**!

$$\mathcal{C} \equiv \pounds_{L_0}(\pounds_{L_0} - 1) - \pounds_{L_-}\pounds_{L_+} = \Box_{\mathrm{AdS}_2}$$

Symmetry Argument for Aretakis Scaling

- For solutions of wave equation take as basis functions **simultaneous eigenfunctions** of $\mathcal{C}$ and one of the generators, say L_0

$$\mathcal{C}\psi_{\ell,h} = \Delta(\Delta-1)\psi_{\ell,h}$$
$$L_0\psi_{\ell,h} = h\psi_{\ell,h}$$

→ General solution is $\phi_{\Delta,h} = v^{-h}F_{\Delta,h}(v\rho)$, where

$$(z^2+2z)\,F''_{\ell,h}(z) + 2(z+1-h)\,F'_{\ell,h}(z) - \Delta(\Delta-1)\,F_{\ell,h}(z) = 0, \qquad z = v\rho$$

- **Boundedness** at $z=0$ (horizon) and decay at $z\to\infty$ (infinity) enforces $h\geq\Delta$ so

$$F_{\Delta,h}(z) = c_{\Delta,h}\,z^{-\Delta}\,{}_2F_1\left(h+\Delta,\Delta;2\Delta;-\frac{2}{z}\right)$$

- **Late-time behaviour** dominated by $h=\Delta$, for which, as promised

$$\partial_\rho^n\phi_{\Delta,\Delta}\big|_H = v^{n-\Delta}F^{(n)}(v\rho)\big|_H \sim v^{n-\Delta}$$

Deformations of Extremal Black Holes/Branes

Related to **deformations** of extremal black branes.

[Horowitz, Kolanowski, & Santos ’21, ‘22, & ‘23]

Recall in near-horizon limit, perturbations obey

$$\Box_{\mathrm{AdS}_{p+2}}\phi - m_{\text{eff}}^2\phi = 0$$

- Restrict to static deformations to near-horizon geometry

$$\partial_\alpha \phi = 0 \quad \longrightarrow \quad \frac{\rho^2}{L^2}\left(\partial_\rho^2\phi + \frac{p+2}{\rho}\partial_\rho\phi\right) - m_{\text{eff}}^2 = 0$$

  - Solutions scale near horizon:

$$\phi \sim \rho^{-\Delta} + \rho^{\Delta - d}$$

→ Describes near-horizon behaviour tidal **deformation**, e.g. from **multi-centred black branes**.

Multi-Centred Black Branes

Multi-centred black brane solutions have

[cf. Majumdar '47; Papapetrou '47]

$$H(\mathbf{y}) = 1 + \sum_{i=0}^{N} \frac{M_i}{|\mathbf{y} - \mathbf{y}_i|^{\tilde{d}}}$$

- Reference brane at $\mathbf{y}_0 = \mathbf{0}$ with $M_0 := M$, and define

$$\hat{\rho}_i = |\mathbf{y}_i|, \quad |\mathbf{y}| = \hat{\rho}, \quad \mathbf{y} \cdot \mathbf{y}_i = \hat{\rho}\, \hat{\rho}_i \cos\theta_i$$

- Near horizon of reference brane $\hat{\rho}/\hat{\rho}_i \ll 1$:

$$\mathrm{d}s^2 = \left(\frac{\rho}{L}\right)^2 \eta_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta + \left(\frac{L}{\rho}\right)^2 \left[1 + \tilde{d}h(\rho) + \frac{\tilde{d}^2}{d}\frac{\mathrm{d}h(\rho)}{\mathrm{d}\log\rho}\right] \mathrm{d}\rho^2 + r_0^2\left[1 + h(\rho)\right]^{2\tilde{d}} \mathrm{d}\Omega^2_{\tilde{d}+1}$$

with **perturbation**

$$h(\rho) \sim \sum_{i=1}^{N} \frac{M_i}{M} \sum_{j=0}^{\infty} C_j^{(\tilde{d}/2)}(\cos\theta_i) \left(\frac{\rho}{\rho_i}\right)^{d/\tilde{d}+d} \sim \rho^{jd/\tilde{d}+d} + \dots$$

The Extremal Limit

Sub-extremal charged black brane metric is

$$ds^2 = f_+ f_-^{\frac{2-d}{d}} dt^2 + f_-^{2/d} \delta_{ab} dx^a dx^b + \frac{1}{f_+ f_-} dr^2 + r^2 d\Omega^2$$

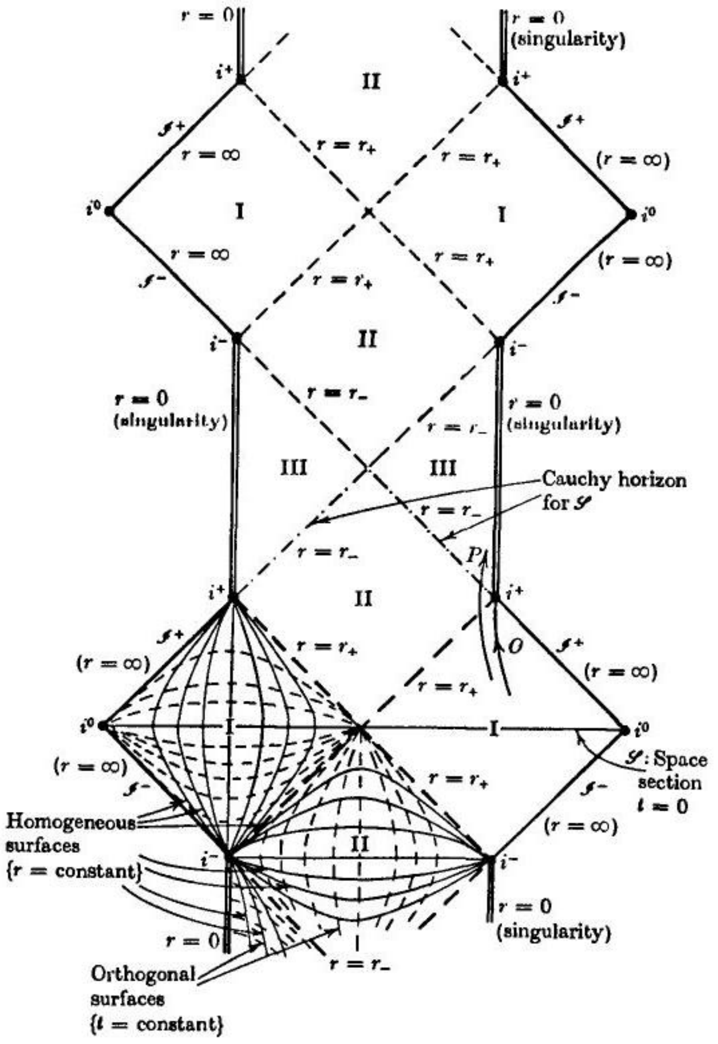
$$f_\pm = 1 - \left(\frac{r_\pm}{r}\right)^{\tilde{d}}$$

- Careful about double expansion in

$$\varepsilon = 2\frac{r_+ - r_-}{r_+ + r_-}, \quad \rho_* = r - r_+$$

- At leading order, **wave equation:**

$$\partial_{\rho_*}\left[\left(\frac{\rho_*}{r_+} + \epsilon\right)\frac{\rho_*}{r_+}\tilde{d}^2 \partial_{\rho_*}\phi\right] - m_{\text{eff}}^2 \phi = 0$$



The Extremal Limit

- Fortunately, solutions are known:

$$\phi \sim A P_{\gamma_+/d}(z) + B Q_{\gamma_+/d}(z), \qquad z = 1 + \frac{\rho_*}{r_+ \varepsilon}$$

  - Near the horizon

$$\phi \sim A\left[1 + \frac{1}{2}\frac{\gamma_+}{d}\left(\frac{\gamma_+}{d} - 1\right)(z-1) + \dots\right] + B\left[c - \frac{1}{2}\log(z-1) + \dots\right]$$

→ **Regularity** requires $B = 0$.

- Leading order solution in **extremal limit**

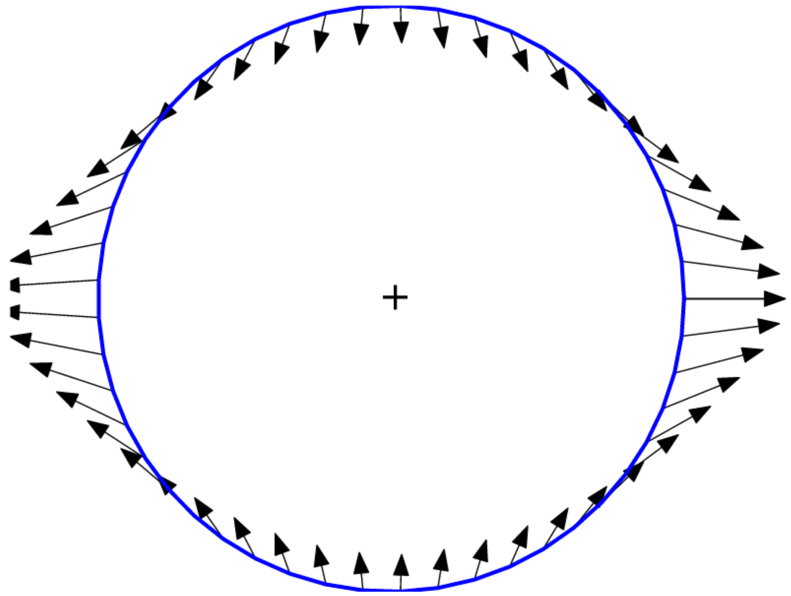
$$\phi \sim \rho_*^{\gamma_+/d} + \dots$$

→ Regular radial coordinate is

$$\rho \sim L(\tilde{d}\rho_*/r_0)^{1/d}$$

so **extremal limit** picks out the γ_+-branch of solutions!

Singularities



What does **smoothness** mean physically?

For **metric perturbation** which scales as

$$h_{\underline{AB}} \sim \rho^{\gamma}$$

Backreacted geometry will have:

- **Scalar invariants** such as

$$S \in \{R_{ABCD}R^{ABCD}, R_{AB}R^{AB}, \ldots\}$$

scaling as

$$S \sim \rho^{n\gamma}, \quad n \in \mathbb{N}^{+}$$

→ Scalar polynomial (s.p.) **singularity** for

$$\gamma < 0$$

- Perturbation to the **Weyl tensor** scales as

$$\delta C_{\underline{ABCD}} \sim \rho^{\gamma-2}$$

This *e.g.* enters the Raychaudhuri equation, describes **tidal forces**

→ Parallel-propagated (p.p.) **singularity** when

$$\gamma < 2$$