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Refinable Multivariate Spline Functions

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Abstract

We review recent developments concerning refinable, multivariate piecewise polynomials with compact support. We first consider uniform meshes, box splines, box spline wavelets and a generalisation of box splines called multi-box splines. Our next topic is spline functions on general triangulations, including continuous linear spline wavelets and hierarchical bases for C^1 splines based on macro-elements. Similar types of spline functions are then studied for meshes gained from triangulating a mesh of quadrilaterals.

Key words: splines, wavelets, refinable functions, triangulations

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1. Introduction

We shall consider spaces of spline functions on $\mathbb{R}^d$, $d \geq 2$, where by a spline function f we shall mean a piecewise polynomial. To make this more precise, the domain of f will be the union $D \subset \mathbb{R}^d$ of a collection T of regions, each of which is a union of a finite number of simplices. Distinct elements of T will intersect only in their common boundary, while any bounded subset of $\mathbb{R}^d$ will intersect only a finite

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number of elements of T. Then f is a spline function over T if it coincides on each element of T with an algebraic polynomial with values in $\mathbb{R}$. We say that f has degree n if these polynomial pieces each have degree at most n.

For simplicity, we refer to a collection T, as above, as a *mesh*. We say that a mesh T is a *refinement* of a mesh S, denoted $S \prec T$, if every element of T is a subset of an element of S and if the union D of the elements of T equals the union of the elements of S. We shall consider a sequence of meshes T_j, $j = 0, 1, 2, \ldots$, with

$$T_j \prec T_{j+1}, \qquad j \geq 0,$$

and we consider, for each $j \geq 0$, a space V_j of spline functions over T_j, where

$$V_j \subset V_{j+1}, \quad j \geq 0. \tag{1}$$

Now suppose that for $j \geq 0$, V_j is a subspace of $L^2(D)$. Then we denote by W_j the orthogonal complement of V_j in V_{j+1}, i.e.

$$V_{j+1} = V_j + W_j, \qquad V_j \perp W_j. \tag{2}$$

Thus for any $j \geq 1$, V_j has an orthogonal decomposition

$$V_j = W_{j-1} + W_{j-2} + \cdots + W_0 + V_0. \tag{3}$$

We shall refer to the spaces W_j as wavelet spaces and the decomposition (3) as a wavelet decomposition. Such decompositions of functions with orthogonal components at different levels are useful in many applications such as signal processing, data compression, and multi-scale methods in numerical analysis.

More generally, we can replace $L^2(D)$ by some Banach space X of functions on D and suppose that for $j \geq 0$, V_j is a subspace of X. We then define

$$W_j = \{f \in V_{j+1} \,:\, P_j f = 0\}, \quad j \geq 0, \tag{4}$$

where P_j is a projection from X onto V_j satisfying

$$P_j P_{j+1} = P_j, \quad j \geq 0.$$

Then we still have the decomposition (3) but the components in the direct sum need not be orthogonal. It is usual to require that $\bigcup_{j=0}^{\infty} V_j$ is dense in X.

We have made the above definitions very general in order to cover all cases considered, but we shall study only special types of meshes and spaces. Section 2 considers the uniform case, i.e. when $D = \mathbb{R}^d$ and

$$f \in V_0 \Rightarrow f(\cdot - k) \in V_0, \quad k \in \mathbb{Z}^d, \tag{5}$$

$$V_j = \{f(M^j \cdot) \,:\, f \in V_0\}, \quad j \geq 1, \tag{6}$$

where M is a $d \times d$ dilation matrix, i.e. it has integer coefficients and $M^{-n}x \to 0$ as $n \to \infty$ for all x in $\mathbb{R}^d$. Although other choices of M may be of interest, we shall focus on the case $M = 2I$, where I denotes the identity matrix. We shall study spaces spanned by box splines and corresponding wavelet spaces, and then consider a generalisation of box splines to multi-box splines.

In Section 3 we shall consider the case when T_j, $j \geq 0$, is a triangulation of $D \subset \mathbb{R}^2$, i.e. T_j comprises triangles, and distinct elements of T_j can intersect only in a common edge or common vertex. We suppose that for $j \geq 0$, T_{j+1} is obtained from T_j by ***mid-point subdivision***, i.e. each element of T_{j+1} is a sub-triangle of a triangle t in T_j and has as vertices either the mid-points of the edges of t, or a vertex v of t and the mid-points of the edges of t which meet in v. An extension to $\mathbb{R}^d$, $d \geq 3$, is also mentioned. We shall study continuous linear splines on these meshes and also C^1 quadratic splines gained through macro-elements. Also considered are C^1 cubic and quintic splines.

Finally, in Section 4, we consider the case where T_j, $j \geq 0$, is a triangulation of D gained from a mesh of quadrilaterals by inserting the diagonals of each quadrilateral. For $j \geq 0$, T_{j+1} is gained from T_j by mid-point subdivision of the quadrilateral, i.e. any quadrilateral for T_{j+1} lies in a quadrilateral for T_j and has as vertices the intersection of the diagonals of q, a vertex v of q, and the mid-points of the edges of q which meet in v. On these meshes we study continuous linear splines, C^1 quadratic splines and C^1 cubic splines.

2. Uniform Meshes

In this section we shall study some general constructions for spaces of spline functions over uniform meshes. Some further examples of spline functions over uniform meshes will be studied in the remaining two sections, where they appear as special cases of spaces over more general meshes. What we mean by our spaces being over uniform meshes is most easily described by saying that they are shift-invariant, as we proceed to describe.

A space V of real-valued functions on $\mathbb{R}^d$, $d \geq 1$, is ***shift-invariant*** if

$$f \in V \Rightarrow f(\cdot - j) \in V, \quad j \in \mathbb{Z}^d. \tag{7}$$

We shall say that V is ***refinable*** if

$$f \in V \Rightarrow f(M^{-1}\cdot) \in V, \tag{8}$$

where M is a dilation matrix, as described after (6). Defining

$$V_j := \{f(M^j\cdot) \,:\, f \in V\}, \quad j \geq 0, \tag{9}$$

we see that (8) is equivalent to

$$V_j \subset V_{j+1}, \quad j \geq 0,$$

as in (1). Now let W denote the orthogonal complement of V_0 in V_1. Then we see that for $j \geq 0$,

$$W_j := \{f(M^j\cdot) \,:\, f \in W\} \tag{10}$$

is the orthogonal complement of V_j in V_{j+1}, as in (2).

For integrable $\phi_1, \ldots, \phi_r \in V$ with compact support, $\phi = (\phi_1, \ldots, \phi_r)$ is called a *generator (of V)* if V comprises all linear combinations of shifts of ϕ, i.e. $f \in V$ if and only if

$$f = \sum_{j \in \mathbb{Z}^d} \phi(\cdot - j)c(j), \tag{11}$$

for a sequence c of $r \times 1$ matrices $c(j)$, $j \in \mathbb{Z}^d$. In this case we write $V = V(\phi)$ and call V a local finitely generated shift-invariant (local FSI) space. We shall be concerned with spline functions f of compact support and, for such functions, we wish the summation in (11) to comprise a finite number of terms. It will therefore be convenient to call ϕ a ***local generator*** of V if every f in V with compact support satisfies (11) for c with finite support.

Now suppose that ϕ is a local generator of a refinable local FSI space V. Since $\phi(M^{-1}\cdot) \in V$, it is a finite linear combination of shifts of ϕ, i.e.

$$\phi = \sum_{j \in \mathbb{Z}^d} \phi(M \cdot -j)a(j), \tag{12}$$

for a sequence a with finite support of $r \times r$ matrices $a(j)$, $j \in \mathbb{Z}^d$. Such an equation (12) is called a ***refinement equation*** (or two-scale equation) and a vector ϕ of functions satisfying such an equation is called ***refinable***. Conversely, if a generator ϕ of a local FSI space V is refinable, then for any $f \in V$, $f(M^{-1}\cdot)$ can be expressed as a linear combination of shifts of ϕ, i.e. V is refinable.

We say that a generator ϕ is ***linearly independent*** if its shifts are linearly independent, i.e.

$$\sum_{j \in \mathbb{Z}^d} \phi(\cdot - j)c(j) = 0 \ \Rightarrow \ c = 0.$$

This concept can be extended as follows. For a non-empty open subset U of $\mathbb{R}^d$, a generator ϕ is said to be ***linearly independent over*** U if

$$\sum_{j \in \mathbb{Z}^d} \phi(x - j)c(j) = 0, \qquad x \in U,$$

for $r \times 1$ matrices $c(j)$, $j \in \mathbb{Z}^d$, implies that $c(j)_i = 0$ whenever $\phi_i(\cdot - j) \not\equiv 0$ on U. Clearly if ϕ is linearly independent over $(0,1)^d$, then ϕ is linearly independent. The converse is not true; indeed it is shown in [42] that there is a local FSI space on $\mathbb{R}$ with a linearly independent generator $\phi = (\phi_1, \phi_2)$ of continuous functions, which has no generator which is linearly independent over $(0,1)$. We say that ϕ is ***locally linearly independent*** if it is linearly independent over any non-empty open subset in $\mathbb{R}^d$.

An elegant characterisation of linearly independent generators is given in [31], in terms of Fourier transforms. For a generator ϕ, its Fourier transform $\hat{\phi}$ is analytic in $\mathbb{R}^d$ and so can be extended to $\mathbb{C}^d$. It is shown in [31] that ϕ is linearly independent if and only if for each $z \in \mathbb{C}^d \setminus \{0\}$, there are $b_1, \ldots, b_r$ in $\mathbb{Z}^d$ for which the matrix

$$A := \left[\hat{\phi}_j(z + 2\pi b_k)\right]_{j,k=1}^{r} \tag{13}$$

is non-singular.

It is further shown in [31] that the shifts of ϕ form a Riesz basis in $L^2(\mathbb{R}^d)$ if the above condition holds for each $z \in \mathbb{R}^d \setminus \{0\}$. In this case we shall say that ϕ is ***stable***. Thus linear independence of ϕ implies its stability. The converse is not true: an example is given in [26] of a refinable local FSI space of univariate spline functions with a generator $\phi = (\phi_1, \phi_2)$ which is stable but not linearly independent.

Taking Fourier transforms of (11) shows that for $f \in S(\phi)$,

$$\hat{f}(u) = \hat{\phi}(u)P(e^{-iu}), \qquad u \in \mathbb{R}^d,$$

where P is the $p \times 1$ matrix of Laurent polynomials,

$$P(z) = \sum_{j \in \mathbb{Z}^d} c(j) z^j, \qquad z \in (\mathbb{C} \setminus \{0\})^d.$$

It follows that if $\phi = (\phi_1, \ldots, \phi_r)$ is a stable, local generator of S, then $\psi = (\psi_1, \ldots, \psi_r)$ is also a stable, local generator of S if and only if $s = r$ and

$$\hat{\psi}(u) = \hat{\phi}(u)A(e^{-iu}), \qquad u \in \mathbb{R}^d,$$

where A is an $r \times r$ matrix of Laurent polynomials which is *unimodular*, i.e. $\det A(z)$ is a non-trivial monomial. This allows us to define the following analogy of dimension of a vector space.

If V is a local FSI space with a stable, local generator $\phi = (\phi_1, \ldots, \phi_r)$, then we say V has *multiplicity* r.

Now suppose that the wavelet space W has a stable generator $\psi = (\psi_1, \ldots, \psi_r)$. Then there are constants $A, B > 0$ such that for any $f \in W$,

$$f = \sum_{j=1}^{r} \sum_{k \in \mathbb{Z}^d} a_{j,k} \psi_j(\cdot - k),$$

the following estimates hold true:

$$A \sum_{j=1}^{r} \sum_{k \in \mathbb{Z}^d} a_{j,k}^2 \leq \|f\|_2^2 \leq B \sum_{j=1}^{r} \sum_{k \in \mathbb{Z}^d} a_{j,k}^2. \tag{14}$$

Next suppose that $f \in L^2(\mathbb{R}^d)$ has a wavelet decomposition

$$f = \sum_{i=0}^{\infty} \sum_{j=1}^{r} \sum_{k \in \mathbb{Z}^d} a_{i,j,k} |\det M|^{i/2} \psi_j(M^i \cdot - k).$$

Then by the orthogonality between levels (2), (10), we have the same stability constants as in (14):

$$A \sum_{i=0}^{\infty} \sum_{j=1}^{r} \sum_{k \in \mathbb{Z}^d} a_{i,j,k}^2 \leq \|f\|_2^2 \leq B \sum_{i=0}^{\infty} \sum_{j=1}^{r} \sum_{k \in \mathbb{Z}^d} a_{i,j,k}^2. \tag{15}$$

We shall shortly state certain characterisations of local FSI spaces of spline functions with multiplicity 1, but in order to do this we must first introduce box splines. These were defined by de Boor and Höllig in [1] and have since been studied and applied by many authors. For a comprehensive study of box splines, see [2].

Take $n \geq 0$ and non-trivial vectors $v_1, \ldots, v_{n+d}$ in $\mathbb{Z}^d$ which span $\mathbb{R}^d$, $d \geq 1$, where for $j = 1, \ldots, n+d$, the components of v_j are coprime. One elegant way to define the *box spline* $B_n = B_n(\cdot | v_1, \ldots, v_{n+d})$ is by its Fourier transform

$$\hat{B}_n(u) = \prod_{j=1}^{n+d} \frac{1 - e^{-iuv_j}}{iuv_j}, \qquad u \in \mathbb{R}^d, \tag{16}$$

where for $u, v \in \mathbb{R}^d$, uv denotes their scalar product uv^t.

Putting $d = 1$, $v_1 = \ldots = v_{n+1}$, (16) reduces to

$$\hat{B}_n(u) = \left(\frac{1 - e^{-iu}}{iu}\right)^{n+1}, \qquad u \in \mathbb{R}^d,$$

which is a definition of the well-known B-spline of degree n with support on $[0, n+1]$ and simple knots at $0, 1, \ldots, n+1$, which was introduced by Schoenberg [51].

In order to describe the structure of the box spline B_n, $d \geq 2$, as a spline function, we define $H = H(v_1, \ldots, v_{n+d})$ as the set of all hyperplanes in $\mathbb{R}^d$ of the form

$$\left\{ k + \sum_{j=1}^{d-1} t_j u_j \, : \, t_j \in \mathbb{R}, \, j = 1, \ldots, d-1 \right\}, \tag{17}$$

for $k \in \mathbb{Z}^d$ and linearly independent elements $u_1, \ldots, u_{d-1}$ of $\{v_1, \ldots, v_{n+d}\}$. Then we define the mesh $T = T(v_1, \ldots, v_{n+d})$ as the collection of all regions which are bounded by but whose interiors are not intersected by elements of H. As an example we take $d = 2$ and $\{v_1, \ldots, v_{n+2}\} = \{(1,0), (0,1), (1,1)\}$. Then H comprises all lines through points in $\mathbb{Z}^2$ in the directions of $(1,0)$, $(0,1)$, and $(1,1)$, and T comprises all triangles with vertices $\{k, k+(1,0), k+(1,1)\}$ or $\{k, k+(0,1), k+(1,1)\}$, for $k \in \mathbb{Z}^2$. This choice of T is called a three-direction mesh or *type-1 triangulation*; and we shall consider this further in Section 3.

Theorem 1 ([1]). *The box spline $B(\cdot|v_1, \ldots, v_{n+d})$ is a spline function of degree n over $T(v_1, \ldots, v_{n+d})$ with support $\{\sum_{j=1}^{n+d} t_j v_j \, : \, 0 \leq t_j \leq 1, j = 1, \ldots, n+d\}$. Across any element of $H(v_1, \ldots, v_{n+d})$ of form (17) it has continuous derivatives of order $n+d-2-|J|$, where $J := \{1 \leq j \leq n+d \, : \, v_j$ lies in the span of $u_1, \ldots, u_{d-1}\}$.*

Now for $m \in \mathbb{Z}$, $m \geq 2$, we see from (16) that

$$\hat{B}_n(mu) = \prod_{j=1}^{n+d} \left(\frac{1 + e^{-iuv_j} + \cdots + e^{-i(m-1)uv_j}}{m} \right) \hat{B}_n(u), \qquad u \in \mathbb{R}^d,$$

and thus B_n is refinable with dilation matrix $M = mI$. The final property which we mention concerns the linear independence of the generator (B_n), i.e. the linear independence of the shifts of B_n.

Theorem 2 ([10,30]). *The following are equivalent.*

(a) (B_n) is linearly independent.

(b) (B_n) is locally linearly independent.

(c) (B_n) is stable.

(d) For any elements $u_1, \ldots, u_d$ in $\{v_1, \ldots, v_{n+d}\}$ which are linearly independent, the $d \times d$ matrix $[u_1^t, \ldots, u_d^t]$ has determinant 1 or -1.

We remark that for $d = 2$, condition (d) is equivalent to requiring that lines in $H(v_1, \ldots, v_{n+2})$ intersect only in points of $\mathbb{Z}^2$. In [54] it is shown that if a refinable local FSI space of spline functions is generated by a single function, then that function must be a homogeneous differential operator acting on a certain linear combination of translates of a box spline. This result did not consider stability, but stability was later studied in [27], which gives the following result. This result and the work in [27] are generalizations of results in [34] for the univariate case.

Theorem 3 ([27]). *If V is a local FSI space of spline functions on $\mathbb{R}^d$ which is refinable with respect to $M = mI$, $m \geq 2$, and has multiplicity one, then it has a generator of form*

$$\phi = B_n \left(\cdot - \frac{k}{m-1} \right),$$

for a box spline B_n and $k \in \mathbb{Z}^d$.

Henceforward we shall assume $M = 2I$. Suppose that V is as in Theorem 3, so that it is generated by a box spline B_n satisfying the conditions of Theorem 2. Let $V_0 = V \cap L^2(\mathbb{R}^d)$. As in (6) we define

$$V_j = \{f(2^j \cdot) \, : \, f \in V_0\}, \qquad j \geq 1, \tag{18}$$

and, as before, W denotes the orthogonal complement of V_0 in V_1. Let E denote the set of vertices of $[0,1]^d$. Then V_1 has as generator $(\phi(2 \cdot - j) \, : \, j \in E)$ and so V_1 has multiplicity 2^d. Then W has multiplicity $2^d - 1$, and a generator of W is said to comprise *prewavelets* (the term wavelets being reserved for a generator whose shifts are orthogonal).

We now consider a construction of prewavelets due to Riemenschneider and Shen [46] and found independently in [5]. The construction depends on a function $\eta : E \to E$ satisfying $\eta(0) = 0$, $(\eta(\mu) + \eta(\nu))(\mu + \nu)$ is odd for $\mu \neq \nu$. For $d = 1$, such a map is given by $\eta(0) = 0$, $\eta(1) = 1$, while for $d = 2$ it can be given by $\eta(0) = 0$, $\eta(0,1) = (0,1)$, $\eta(1,0) = (1,1)$, $\eta(1,1) = (1,0)$. Such a mapping for $d = 3$ is given in [45] but, as remarked there, no such maps exist for $d > 3$.

We note that any element ψ of V_1 with compact support satisfies

$$\hat{\psi}(u) = Q(e^{-iu/2}) \hat{B}_n \left(\frac{u}{2} \right), \quad u \in \mathbb{R}^d,$$

for some Laurent polynomial Q in $\mathbb{C}^d$. Now define Laurent polynomials P and H on $\mathbb{C}^d$ by

$$P(z) := \sum_{j \in \mathbb{Z}^d} z^j \int B_n B_n(\cdot - j), \tag{19}$$

$$H(z) := \prod_{k=1}^{n+d} (1 + z^{v_k}). \tag{20}$$

It is shown in [46] that a stable generator of W is given by $(\psi_j \, : \, j \in E \setminus \{0\})$,

$$\hat{\psi}_j(u) := H_j(e^{-iu/2}) \hat{B}_n \left(\frac{u}{2} \right), \quad u \in \mathbb{R}^d, \tag{21}$$

where for $z = e^{-iu/2} = (e^{-iu_1/2}, \ldots, e^{-iu_d/2})$,

$$H_j(z) := z^{\eta(j)} P((-1)^j z) H((-1)^j z). \tag{22}$$

The above construction, as we have said, does not work for $d \geq 4$. Constructions for prewavelets from box splines in general dimensions are given in [53]. Further results on box spline prewavelets appear in [35,36,55].

Now suppose $d = 1, 2$, or 3, as above, and $v_1, \ldots, v_{n+d}$ lie in $\{-1, 0, 1\}^d$. For this case [3] gives a construction of prewavelets giving smaller support (and hence less computational cost in applications) than those above. Without loss of generality we may suppose

$$v_j \neq -v_k, \quad j, k = 1, \ldots, n + d.$$

We suppose that the distinct elements of $\{v_1, \ldots, v_{n+d}\}$ are $w_1, \ldots, w_\ell$, occurring with multiplicities $n_1, \ldots, n_\ell$, so that $n_1 + \cdots + n_l = n + d$ and (16) becomes

$$\hat{B}_n(u) = \prod_{k=1}^{\ell} \left(\frac{1 - e^{-iuw_k}}{iuw_k} \right)^{n_k}, \qquad u \in \mathbb{R}^d.$$

We note that, for $d = 2$, we may suppose $w_1, \ldots, w_\ell$ lie in $\{(1,0), (0,1), (1,1), (1,-1)\}$ and the condition (d) of Theorem 2 implies that $\ell = 2$ or 3.

It is shown in [3] that a stable generator of W is given by $(\psi_j : j \in E \setminus \{0\})$, where

$$\hat{\psi}_j(u) := G_j(e^{-iu/2}) \hat{B}_n \left(\frac{u}{2} \right), \quad u \in \mathbb{R}^d, \tag{23}$$

where for $z = e^{-iu}$,

$$\overline{G_j(z)} := z^{\eta(j)} P((-1)^j z) \prod_{\substack{k=1 \\ w_k j \text{ even}}}^{\ell} (1 - z^{w_k})^{n_k} \prod_{\substack{k=1 \\ w_k j \text{ odd}}}^{\ell} S_k(z^{w_k}), \tag{24}$$

and for $k = 1, \ldots, \ell$, $w \in \mathbb{C}$, $r \in \mathbb{Z}$,

$$S_k(w) = \begin{cases} w^{-r}, & n_k = 2r, \\ w^{-r}(1 + w), & n_k = 2r - 1. \end{cases} \tag{25}$$

As an example of the above constructions, consider the case $d = 2$, $w_1 = (1,0)$, $w_2 = (0,1)$, $w_3 = (1,1)$ with multiplicities $n_1 = n_2 = n_3 = 2$. So $n = 4$ and B_4 is a C^2 quartic spline function. Here

$$H(z) = (1 + z_1)^2 (1 + z_2)^2 (1 + z_1 z_2)^2, \quad z \in \mathbb{C}^2.$$

Choosing q as above we have by (22),

$$H_j(z) = P((-1)^j z) \begin{cases} z_1 z_2 (1 - z_1)^2 (1 + z_2)^2 (1 - z_1 z_2)^2, & j = (1,0), \\ z_2 (1 + z_1)^2 (1 - z_2)^2 (1 - z_1 z_2)^2, & j = (0,1), \\ z_1 (1 - z_1)^2 (1 - z_2)^2 (1 + z_1 z_2)^2, & j = (1,1), \end{cases}$$

while by (24), (25),

$$\overline{G_j(z)} = P((-1)^j z) \begin{cases} z_1^{-1}(1-z_2)^2 & j=(1,0), \\ z_1^{-1}z_2^{-1}(1-z_1)^2, & j=(0,1), \\ z_2^{-1}(1-z_1z_2)^2, & j=(1,1). \end{cases}$$

The coefficients of polynomials H_j or G_j are referred to as the *mask* of the corresponding prewavelet ψ_j. Here the masks for the first construction each have 91 non-zero coefficients, while those for the second construction each have 51 non-zero coefficients.

For the case of the continuous linear box spline B_1 on $\mathbb{R}^2$ with $v_1 = (1,0)$, $v_2 = (0,1)$, $v_3 = (1,1)$, the above two constructions give the same prewavelets, each with 19 non-zero coefficients in its mask. For this special case a construction of prewavelets with only 10 non-zero coefficients in each mask is given in [32]. We shall consider this further in Section 3, when we study extensions to non-uniform triangulations.

We do not know of any extension of Theorem 3 characterising refinable local FSI spline functions with multiplicity more than one, however we shall now consider a family of spaces of this form which give a partial generalisation of the spaces V in Theorem 3 generated by box splines B_n. Despite their many elegant properties, such spaces generated by box splines fail to generalise some of the basic properties of the space of all univariate C^{n-1} spline functions of degree n with knots in $\mathbb{Z}$, which is generated by a uniform B-spline. Except for the cases $n = 0$ and 1, the elements of V do not have maximal continuity C^{n-1}. Moreover V does not comprise all piecewise polynomials of degree n subject to some continuity conditions. Indeed the restriction of V to any region in $T(v_1, \ldots, v_{n+d})$ coincides with a space P which is a proper subspace of polynomials of degree n comprising the common null space of certain differential operators [1,10]. Moreover V does not comprise all piecewise polynomials with pieces coinciding with such spaces P subject to certain continuity conditions. In contrast, we now consider spaces of the following form.

Take integers $n \geq 0$, $r \geq 1$, and pairwise linearly independent vectors $v_0, \ldots, v_{n+r}$ in $\mathbb{Z}^2$. Let $S_n = S_n(v_0, \ldots, v_{n+r})$ comprise all functions f with continuous Fourier transforms of form

$$\hat{f}(u) = \frac{\sum_{|\alpha|=r-1} P_\alpha(e^{-iu})u^\alpha}{(uv_0)\cdots(uv_{n+r})}, \quad u \in \mathbb{R}^2,$$

where P_α is a Laurent polynomial for $\alpha \in \mathbb{Z}^2$, $|\alpha| = r - 1$. It follows easily from the definition that the space S_n is refinable. Our next result characterises elements of S_n as spline functions. As in (17), $H = H(v_0, \ldots, v_{n+r})$ denotes the set of lines in $\mathbb{R}^2$ of the form $\{k + tv_j : t \in \mathbb{R}\}$, for $k \in \mathbb{Z}^2$ and $0 \leq j \leq n + r$. The mesh $T = T(v_0, \ldots, v_{n+r})$ comprises all regions which are bounded by, but whose interiors are not intersected by, lines in H.

Theorem 4 (**[25]**). *The space S_n comprises all C^{n-1} spline functions of degree n over the mesh $T(v_0, \ldots, v_{n+r})$ with compact support, such that the jump of any derivative of order n across any line in $H(v_0, \ldots, v_{n+r})$ changes only at points in $\mathbb{Z}^2$.*

The condition on the discontinuities of the derivatives of a function f in S_n can be thought of as f having knots in $\mathbb{Z}^2$. Thus S_n can be thought of as a generalization of the space of all univariate C^{n-1} spline functions of degree n with knots in $\mathbb{Z}$ and compact support. Next we consider generators for S_n.

Theorem 5 (**[25]**). *There is a generator $\phi = (\phi_1, \ldots, \phi_r)$ such that any element of S_n is a finite linear combination of shifts of ϕ. Moreover ϕ is such a generator if and only if*

$$\hat{\phi}(u) = \frac{\tilde{u}M(e^{-iu})}{(uv_0)\cdots(uv_{n+r})}, \qquad u \in \mathbb{R}^2, \tag{26}$$

where $\tilde{u} := (u_1^{r-1}, u_1^{r-2}u_2, \ldots, u_2^{r-1})$ and M is an $r \times r$ matrix of Laurent polynomials with

$$\det M(z) = cz^k \prod_{j=0}^{n+r}(1 - z^{v_j}), \quad z \in \mathbb{C}^2, \tag{27}$$

for some $k \in \mathbb{Z}^2$, $c \in \mathbb{R}$, $c \neq 0$.

We note that when $r = 1$, (26) and (27) show that any generating function ϕ satisfies

$$\hat{\phi}(u) = ce^{-iku}\frac{\prod_{j=0}^{n+r}(1 - e^{-iuv_j})}{(uv_0)\cdots(uv_{n+r})},$$

and so by (16), ϕ is a multiple of a shift of a box spline

$$\phi = cB_n(\cdot - k|v_0, \ldots, v_{n+r}).$$

The case $r = 2$ was introduced in [23], where the generator was called a multi-box spline, and this case was studied further in [24]. Our next result analyses stability of the generators ϕ in Theorem 5.

Theorem 6 (**[25]**). *For any generator ϕ as in Theorem 5, ϕ is stable if and only if at most r lines in $H(v_0, \ldots, v_{n+r})$ intersect except at points in $\mathbb{Z}^2$.*

By the remark after Theorem 2, we see that the conditions of Theorem 6 generalise condition (d) of Theorem 2 for the case $d = 2$, $r = 1$.

Now consider the line $L = \{tu : t \in \mathbb{R}\}$ for $u = (u_1, u_2) \in \mathbb{Z}^2$. If u_1 and u_2 are odd, then putting $t = \frac{1}{2}$ show that L passes through a point equal to $(\frac{1}{2}, \frac{1}{2}) \bmod \mathbb{Z}^2$. Thus for stability of ϕ, the condition of Theorem 6 requires that at most r vectors in $\{v_0, \ldots, v_{n+r}\}$ have both components odd. Similarly there are at most r vectors in $\{v_0, \ldots, v_{n+r}\}$ with components of form (odd, even) and at most r vectors with components of form (even, odd). Thus stability of ϕ implies that $n \le 2r-1$. It can be shown that for any $r \ge 1$, there is a choice of vectors $\{v_0, \dots, v_{n+r}\}$ for which at most r lines in $H(v_0, \dots, v_{n+r})$ intersect other than in points of $\mathbb{Z}^2$, and hence we have the following result.

Theorem 7 ([25]). *For any positive integer r, there is a space $S_{2r-1}(v_0, \dots, v_{3r-1})$, as in Theorem 4, with a stable generator $\phi = (\phi_1, \dots, \phi_r)$.*

Thus by choosing large enough r, we can construct stable multi-box splines of arbitrarily high degree n and smoothness C^{n-1}. Before giving some examples, we discuss the possible symmetry of multi-box splines ϕ for all $r \ge 1$. We say that $\phi = (\phi_1, \dots, \phi_r)$ is *symmetric* if for $j = 1, \dots, r$, there are $\sigma_j = \pm 1$, $\alpha_j \in \{0,1\}^2$, with

$$\phi_j(-\cdot) = \sigma_j \phi_j(\cdot + \alpha_j), \tag{28}$$

i.e. ϕ_j is even or odd about $\frac{1}{2}\alpha_j$ as $\sigma_j = 1$ or -1.

Theorem 8 ([25]). *If ϕ as in Theorem 5 is symmetric, then*

$$\sigma_1 \cdots \sigma_r = (-1)^{(n+1)(r+1)}, \ \alpha_1 + \cdots + \alpha_r = v_0 + \cdots + v_{n+r} \bmod \mathbb{Z}^2.$$

For $r = 1$, this corresponds to the well-known fact that the box spline $B_n(\cdot|v_0, \dots, v_{n+r})$ is even about $\frac{1}{2}(v_0 + \cdots + v_{n+1})$. Next we consider symmetry under the operator $T : \mathbb{R}^2 \to \mathbb{R}^2$, $Tx = (x_2, x_1)$. If we have

$$\{Tv_0, \dots, Tv_{n+r}\} = \{t_0 v_0, \dots, t_{n+r} v_{n+r}\}, \tag{29}$$

where $t_j = \pm 1$, $j = 0, \dots, n+r$, then the space $S_n(v_0, \dots, v_{n+r})$ is invariant under T and it is natural to require a corresponding multi-box spline to be symmetric under T, i.e. for $j = 1, \dots, r$,

$$\phi_j(T\cdot) = \tau_j \phi_j, \quad \tau_j = \pm 1. \tag{30}$$

Theorem 9 ([25]). *Suppose that (29) holds and ϕ as in Theorem 5 satisfies (30). Then*

$$\tau_1 \cdots \tau_r = (-1)^s (t_0 \cdots t_{n+r})^{r-1}, \tag{31}$$

where $r = 2s$ or $2s+1$.

It may be that (30) is not satisfied for $j = k, l$, some $k \ne l$, but instead $\phi_k = \phi_l(T\cdot)$. In this case (31) holds with $\tau_k \tau_l$ replaced by -1.

Another possible symmetry is reflection in the x_2-axis, i.e. $R : \mathbb{R}^2 \to \mathbb{R}^2$, $Rx = (-x_1, x_2)$. If we have

$$\{Rv_0, \dots, Rv_{n+r}\} = \{r_0 v_0, \dots, r_{n+r} v_{n+r}\}, \tag{32}$$

where $r_j = \pm 1$, $j = 0, \dots, n+r$, then $S_n(v_0, \dots, v_{n+r})$ is invariant under R and it is natural to require that for $j = 1, \dots, r$,

$$\phi_j(Rx) = p_j \phi_j(x_1 + (\alpha_j)_1, x_2), \quad x \in \mathbb{R}^2, \quad \rho_j = \pm 1. \tag{33}$$

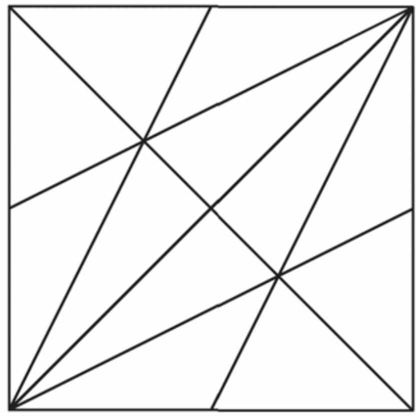

Fig. 1. $T(v_0, \dots, v_5)$ on $[0,1]^2$.

Theorem 10 ([25]). *Suppose that (32) holds and ϕ as in Theorem 8 satisfies (33). Then for s as in Theorem 9*

$$\rho_1 \cdots \rho_r = (-1)^s (r_0 \cdots r_{n+r})^{r-1}. \tag{34}$$

As for Theorem 9, we replace $\rho_k \rho_l$ in (34) by -1 if $\phi_k = \phi_l(R\cdot)$. Clearly a corresponding result to Theorem 10 holds for reflection in the x_1-axis.

It is suggested in [25], from examples considered there, that there is always a symmetric multi-box spline which satisfies the above symmetry conditions where appropriate, but this is not proved in general.

We finish by considering some examples of multi-box splines. First we take $n = 1$, $r = 2$, $v_0 = (1,0)$, $v_1 = (0,1)$, $v_2 = (1,1)$, $v_3 = (1,-1)$. The mesh $T = T(v_0, \dots, v_3)$ is called a four-direction mesh or *type-2 triangulation* and we shall consider this further in Section 4. Here the space S_1 comprises *all* continuous linear splines over T. There is a choice of symmetric generator $\phi = (\phi_1, \phi_2)$, where ϕ_1 has support $[0,1]^2$ and

$$\phi_2(x) = \phi_1\left(\frac{x_1 + x_2 + 1}{2}, \frac{x_1 - x_2 + 1}{2}\right), \quad x \in \mathbb{R}^2.$$

Since at most two lines in $H(v_0, \dots, v_3)$ intersect other than in $\mathbb{Z}^2$, ϕ is stable. Equations (28), (30), and (33) hold with

$$\alpha_1 = (1,1), \quad \alpha_2 = (0,0), \quad \sigma_j = \tau_j = \rho_j = 1, \quad j = 1,2.$$

Next we take $n = 2$, $r = 3$, $v_0, \dots, v_3$ as above, and $v_4 = (2,1)$, $v_5 = (1,2)$. The mesh $T(v_0, \dots, v_5)$ restricted to the triangle with vertices $(0,0)$, $(1,0)$, $(1,1)$ gives the Powell-Sabin 6-split of this triangle, [44], and we have the corresponding split for the triangle with vertices $(0,0)$, $(1,0)$, $(1,1)$. This is illustrated in Figure 1. Here the space S_2 comprises *all* C^1 quadratic splines over T. A choice of symmetric generator for S_2 is given by the finite element basis $\phi = (\phi_1, \phi_2, \phi_3)$, which is defined as follows. For $k \in \mathbb{Z}^2$, define linear functionals on S_2 by

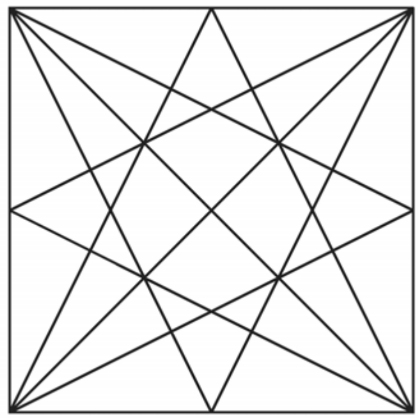

Fig. 2. $T(v_0, \ldots, v_7)$ on $[0,1]^2$.

$$\sigma_{1k} f = f(k), \quad \sigma_{2k} f = D_{v_2} f(k), \quad \sigma_{3k} f = D_{v_3} f(k), \tag{35}$$

where for $v \in \mathbb{R}^2$, D_v denotes the directional derivative in direction v. Then for $i = 1, 2, 3$, ϕ_i is defined as the unique function in S_2 satisfying

$$\sigma_{jk} \phi_i = \delta_{ij} \delta_{k0}, \quad j = 1, 2, 3, \quad k \in \mathbb{Z}^2. \tag{36}$$

These functions are considered in [9] (under a linear transformation of $\mathbb{R}^2$), where they are defined explicitly in terms of Bézier coefficients. Further properties of ϕ are given in [9] and will be considered in in Section 3. The support of ϕ_i, $i = 1, 2, 3$, is the hexagon with vertices $(-1,-1)$, $(-1,0)$, $(0,-1)$, $(1,0)$, $(0,1)$, $(1,1)$. Equations (28) and (30) hold with

$$\sigma_1 = 1, \quad \sigma_2 = \sigma_3 = -1, \quad \alpha_1 = \alpha_2 = \alpha_3 = (0,0),$$

$$\tau_1 = \tau_2 = 1, \quad \tau_3 = -1.$$

Since at most three lines in $H(v_0, \ldots, v_5)$ intersect other than in $\mathbb{Z}^2$, the above generator ϕ is stable. A more symmetric mesh $T(v_0, \ldots, v_7)$ is gained by taking $v_0, \ldots, v_5$ as before, and $v_6 = (2,-1)$, $v_7 = (1,-2)$, see Figure 2. There are still at most three lines in $H(v_0, \ldots, v_7)$ which intersect other than in $\mathbb{Z}^2$. Thus there is a stable generator for the case $n = 3$, $r = 4$, and for the case $n = 4$, $r = 3$.

3. General Triangulations

Let T denote a triangulation whose union D is a simply connected region in $\mathbb{R}^2$. Let $V(T)$ denote the set of all vertices of triangles in T and $E(T)$ the set of all edges of triangles in T. We suppose that any boundary vertex (i.e. vertex in the boundary of D) is the intersection of exactly two boundary edges. We shall first consider the space $L(T)$ of all continuous linear spline functions over T. This space has a natural basis of ***nodal functions***, defined as follows. For each v in $V(T)$ we let ϕ_v be the unique element of $L(T)$ satisfying

$$\phi_v(v) = 1, \quad \phi_v(w) = 0, \quad w \in V(T), \quad w \neq v.$$

Then for any element f of $L(T)$,

$$f = \sum_{v \in V(T)} f(v)\phi_v.$$

The support of ϕ_v is the union of all triangles in T which have v as a vertex.

Now put $T_0 = T$ and let T_1 denote the triangulation gained from T_0 by mid-point subdivision as described in Section 1. Thus T_1 is gained by subdividing each element t of T into four congruent sub-triangles, each similar to t. For $j \geq 1$, we recursively obtain T_{j+1} from T_j by mid-point subdivision. Then we define

$$V_j = L(T_j) \cap L^2(D), \quad j \geq 0. \tag{37}$$

First we shall consider the uniform case where $D = \mathbb{R}^2$, T denotes the type-1 triangulation $T((1,0),(0,1),(1,1))$ and $L(T) = S_1((1,0),(0,1),(1,1))$. In this case $\phi_{(0,0)}$ is the box spline $B_1(\cdot|(1,0),(0,1),(1,1))$ which generates S_1. Now $L(T_1) = \{f(2\cdot) : f \in L(T)\}$ and so V_j in (37) satisfies (18). As before, we let W denote the orthogonal complement of V_0 in V_1.

In [32] there is given a construction of a generator $\psi = (\psi_1, \psi_2, \psi_3)$ for W where ψ_1 has support as in Figure 3 and

$$\psi_2(x,y) = \psi_1(y,x), \quad \psi_3(x,y) = \psi_1(y, y-x), \quad x,y \in \mathbb{R}. \tag{38}$$

Also ψ_1 is even about $(\frac{1}{2}, 0)$, i.e.

$$\psi_1(-x,-y) = \psi_1(x+1,y), \quad x,y \in \mathbb{R}. \tag{39}$$

The function ψ_1 is non-zero at all 13 points in $\frac{1}{2}\mathbb{Z}^2$ which lie inside its support. Equivalently, when ψ_1 is written in the form

$$\psi_1 = \sum_{j \in \mathbb{Z}^2} \psi_1\left(\frac{j}{2}\right) B_1(2 \cdot - j),$$

then the mask of ψ_1, i.e. the coefficient in this linear combination, has 13 non-zero terms. At the expense of the symmetry (39), [32] also constructs a generator $\tilde{\psi} = (\tilde{\psi}_1, \tilde{\psi}_2, \tilde{\psi}_3)$ of W which satisfies the analogue of (38) but such that the mask of $\tilde{\psi}_1$ has only 10 non-zero coefficients. In [29] a characterisation is given of prewavelets with the minimum of 10 non-zero coefficients in their masks for spaces comprising those elements of $L(T_1)$ with supports on given bounded regions.

In a series of papers [18–20], Floater and Quak generalise the construction of the prewavelet ψ above to the case of a general triangulation on a bounded domain D, as considered earlier. In order to gain an elegant construction, they define the space W_j, as in (2), as the orthogonal complement of V_j in V_{j+1} with respect to the inner product

$$\langle f, g \rangle = \sum_{t \in T} \frac{1}{2a(t)} \int_t fg, \qquad f, g \in C(D), \tag{40}$$

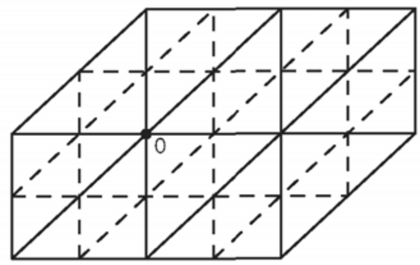

Fig. 3. Support of ψ_1.

where $a(t)$ denotes the area of triangle t. Thus W_j is defined by (4), where P_j denotes the orthogonal projection with respect to (40). For the uniform case above, (40) is the usual L^2 inner product.

Floater and Quak construct a basis $\{\psi_v : v \in V(T_1) \setminus V(T)\}$ for W_0. Take any $v \in V(T_1) \setminus V(T)$. Then $v = \frac{1}{2}(u_1 + u_2)$ for $u_1, u_2 \in V(T)$, and the support of ψ_v is the union of all triangles in T having u_1 or u_2 as a vertex. The prewavelets ψ_v satisfy the elegant symmetry property

$$\psi_v(u) = \psi_u(v), \quad u, v \in V(T_1) \setminus V(T).$$

For the case when T is the restriction of a type-1 triangulation to a domain D and the vertex $v \in V(T_1) \setminus V(T)$ is the centre of an edge whose end-points are interior vertices, the function ψ_v coincides with $\psi_j(\cdot - k)$, as above, for some $k \in \mathbb{Z}^2$, $1 \le j \le 3$.

By replacing T above by T_j, $j \ge 0$, there is a corresponding basis $\{\psi_v^j : v \in V_*^j\}$ for W_j, where $V_*^j := V(T_{j+1}) \setminus V(T_j)$, and we normalise by requiring $\|\psi_v^j\|_2 = 1$. For the uniform case [32] derives estimates for the stability constants as in (14). In [20] such estimates are derived for the constants A, B in the following expansion for the non-uniform case:

$$f = \sum_{v \in V(T)} a_v \phi_v + \sum_{j=0}^{\infty} \sum_{v \in V_*^j} b_{j,v} \psi_v^j,$$

$$A \left(\sum_{v \in V(T)} a_v^2 + \sum_{j=0}^{\infty} \sum_{v \in V_*^j} b_{j,v}^2 \right) \le \|f\|_2^2 \le B \left(\sum_{v \in V(T)} a_v^2 + \sum_{j=0}^{\infty} \sum_{v \in V_*^j} b_{j,v}^2 \right).$$

In [52] a construction of linear prewavelets is given which holds in $\mathbb{R}^d$ for any $d \ge 2$. In this case the meshes T_j, $j \ge 0$, comprise simplices in $\mathbb{R}^d$. Any such simplex has $d+1$ faces which are simplices of dimension $d-1$. These in turn have faces of dimension $d-2$, and proceeding recursively we have a collection $F(s)$ of simplices of dimension $0, \ldots, d-1$ in the boundary of s. We assume that distinct elements s, t of T_j can intersect only in an element of $F(s) \cap F(t)$. We suppose that for $j \ge 0$, T_{j+1} is gained from T_j by subdividing any simplex t in T_j into 2^d congruent sub-simplices, each similar to t, though for $d \ge 3$ there is no canonical way to do this.

As before, we denote by $L(T_j)$ the space of all continuous linear spline functions over T_j. We define V_j, $j \geq 0$, by (37) and define W_j, $j \geq 0$, by (2), this time with the usual inner product in $L^2(D)$. Then [52] gives a construction for a basis $\{\psi_v : v \in V(T_{j+1}) \setminus V(T_j)\}$ for W_j. For the case of a type-1 triangulation of $\mathbb{R}^2$ this leads to a generator $\bar{\psi} = (\bar{\psi}_1, \bar{\psi}_2, \bar{\psi}_3)$, which satisfies the analogues of (38) and (39). However, while ψ_1 has support of area 5 and mask with 13 non-zero coefficients, $\bar{\psi}_1$ has support of area 8 and mask with 23 non-zero coefficients.

Next we shall consider refinable spaces of C^1 quadratic splines constructed from Powell-Sabin macro-elements. First we study the uniform case where each triangle in the type-1 triangulation $T(v_0, v_1, v_2)$, for $v_0 = (1,0)$, $v_1 = (0,1)$, $v_2 = (1,1)$, is divided into six sub-triangles by inserting the medians, thus producing the triangulation $T = T(v_0, \ldots, v_5)$ for $v_3 = (1,-1)$, $v_1 = (2,1)$, $v_2 = (1,2)$, as discussed near the end of Section 2. As described there, the space $S_2 = S_2(v_0, \ldots, v_5)$ of all C^1 quadratic splines over T is generated by the fundamental functions for Hermite interpolation of values and first-order derivatives at $\mathbb{Z}^2$, i.e. the shifts of the multi-box splines $\phi = (\phi_1, \phi_2, \phi_3)$ defined by the conditions (35) and (36).

Now let $V_0 = S_2 \cap L^\infty(\mathbb{R}^2)$ and, as in (18),

$$V_j = \{f(2^j \cdot) : f \in V_j\}, \quad j \geq 1.$$

For $j \geq 0$, we let P_j denote the projection from $C^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ onto V_j given by

$$P_j f(2^{-j}k) = f(2^{-j}k), \quad D_{v_i} P_j f(2^{-j}k) = D_{v_i} f(2^{-j}k), \quad i = 0,1, \quad k \in \mathbb{Z}^2.$$

Then $P_j P_{j+1} = P_j$, $j \geq 0$, and we define, as in (4),

$$W_j = \{f \in V_{j+1} : P_j f = 0\}, \quad j \geq 0.$$

Thus W_0 comprises all elements of V_1 whose values and first order derivatives vanish on $\mathbb{Z}^2$, and

$$W_j = \{f(2^j \cdot) : f \in W_0\}, \quad j \geq 0.$$

Then for $j \geq 0$, a basis for W_j is given by $\psi_{i,j,k} := \phi_i(2^{j+1} \cdot -k)$, $i = 1,2,3$, $k \in \mathbb{Z}^2 \setminus 2\mathbb{Z}^2$. Such functions comprise what is called a *hierarchical basis*. In [9] it is shown that this basis is 'weakly stable' in the sense that there are constants $A, B > 0$ such that for any $n \geq 0$ and $f \in V_{n+1}$ of the form

$$f = \sum_{j=0}^{n} \sum_{i=1}^{3} \sum_{k \in \mathbb{Z}^2} a_{i,j,k} \psi_{i,j,k},$$

$$A\|a\|_\infty \leq \|f\|_\infty \leq Bn\|a\|_\infty,$$

where $\|a\|_\infty = \sup\{|a_{i,j,k}| : j = 0, \ldots, n,\ i = 1,2,3,\ k \in \mathbb{Z}^2\}$. It is also shown in [9] that the operators P_j are uniformly bounded and that for any $f \in C^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, $\lim_{j \to \infty} P_j f = f$ uniformly on bounded subsets of $\mathbb{R}^2$.

In [41] hierarchical bases are considered on general triangulations using Powell-Sabin elements based on splitting each triangle t into a mesh $T(t)$ of 12 sub-triangles formed by joining the mid-points of each edge with each other and with the opposite vertex, see Figure 4. Each C^1 quadratic spline over $T(t)$ is defined uniquely by the following 12 values: the values and first derivatives at the vertices A_1, A_2, A_3, and

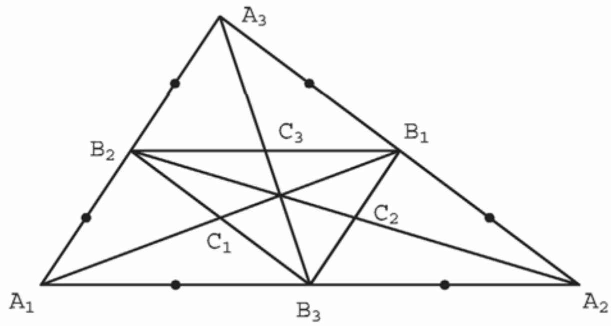

Fig. 4. The Powell-Sabin 12-split.

the normal derivatives at the mid-points of the edges B_1, B_2, B_3. Let T denote a triangulation with union D as described at the beginning of this section. For any such triangulation T we denote by T' the triangulation gained by splitting each element of T into 12 sub-triangles, as above. The space of all C^1 quadratic splines over T' will be denoted by $Q(T')$.

Now for any point $x \in \mathbb{R}^2$ we define the linear functionals on $C^1(D)$:

$$\sigma_{x1} f = f(x), \quad \sigma_{x2} f = D_{(1,0)} f(x), \quad \sigma_{x3} f = D_{(0,1)} f(x). \tag{41}$$

Also for any finite line segment e in $\mathbb{R}^2$, we denote by $\sigma_{et} f$ and $\sigma_{en} f$ derivatives of f at the mid-point of e in directions tangential and normal to e, respectively. We denote by ϕ_{vj}, $j = 1,2,3$, $v \in V(T)$, and ϕ_e, $e \in E(T)$, the unique fundamental functions in $Q(T')$ satisfying

$$\sigma_{ui}\phi_{vj} = \delta_{uv}\delta_{ij}, \quad \sigma_{ui}\phi_e = 0, \quad u \in V(T), \quad i = 1,2,3,$$

$$\sigma_{fn}\phi_{vj} = 0, \quad \sigma_{fn}\sigma_e = \delta_{ef}, \quad e \in E(T).$$

Then a nodal basis for $Q(T')$ is given by

$$\{\phi_{vj},\, \phi_e \,:\, v \in V(T),\, j = 1,2,3,\, e \in E(T)\}\,.$$

The support of ϕ_{vj}, $j = 1,2,3$, is the union of all triangles in T having v as a vertex, and the support of ϕ_e is the union of the triangles in T having e as an edge.

The triangulation T' is refinable under mid-point subdivision of T, as we now describe. Put $T_0 = T$, $T_0' = T'$, and let T_1 denote the triangulation gained from T_0 by mid-point subdivision. Then $T_0' \prec T_1'$. More generally, for $j \geq 1$, we may recursively define T_{j+1} by mid-point subdivision of T_j, and we have

$$T_j' \prec T_{j+1}', \quad j \geq 0,$$

and hence

$$Q(T_j') \subset Q(T_{j+1}'), \quad j \geq 0.$$

For $j \geq 0$, we let P_j denote the projection from $C^1(D)$ onto $Q(T_j')$ defined by

$$\sigma_{vi} P_j f = \sigma_{vi} f, \quad v \in V(T_j), \quad i = 1,2,3,$$

$$\sigma_{en} P_j f = \sigma_{en} f, \quad e \in E(T_j).$$

As before we have $P_j P_{j+1} = P_j$, $j \geq 0$, and define

$$W_j := \left\{ f \in Q(T'_{j+1}) \,:\, P_j f = 0 \right\}.$$

A basis for W_0 is given by the fundamental functions

$$\left\{ \psi_e, \tilde{\psi}_e, \psi_f \,:\, e \in E(T_0),\, f \in E(T_1) \right\},$$

where, with $\sigma_e \psi$ denoting the value of ψ at the mid-point of edge e,

$$\sigma_g \psi_e = \delta_{ge}, \quad \sigma_g \tilde{\psi}_e = 0, \quad \sigma_g \psi_f = 0, \quad g \in E(T_0),$$

$$\sigma_{gt} \psi_e = 0, \quad \sigma_{gt} \tilde{\psi}_e = \delta_{ge}, \quad \sigma_{gt} \psi_f = 0, \quad g \in E(T_0),$$

$$\sigma_{hn} \psi_e = 0, \quad \sigma_{hn} \tilde{\psi}_e = 0, \quad \sigma_{hn} \psi_f = \delta_{hf}, \quad h \in E(T_1).$$

Similarly, we may define a hierarchical basis for all W_j, $j \geq 0$.

In [41] there is also considered a hierarchical basis of C^1 cubic splines. The well-known Clough-Tocher element [8] is not refinable under mid-point subdivision and so they consider instead the space of all C^1 cubic splines on the Powell-Sabin 12-split of a triangle, as in Figure 4. Each such function is determined uniquely by the following 30 values: the values and first order derivatives at the points A_1, A_2, A_3, B_1, B_2, B_3, the values and tangential derivatives at C_1, C_2, C_3, and the normal derivatives at the six mid-points of the edges of sub-triangles, denoted by dots in Figure 4. For any triangulation T and refinement T', as for the previous example, [41] then considers the space of all C^1 cubic spline function on T' and studies the nodal basis and corresponding hierarchical basis derived from the above Hermite interpolation functionals.

The above hierarchical bases studied in [41] are introduced for preconditioning the finite element equations for fourth-order elliptic boundary value problems when using the conjugate gradient method. Motivated by the same problem, [11] constructs hierarchical bases of C^1 quintic splines, as we now briefly discuss.

As before we let T_0 denote a general triangulation with union D and define the triangulation T_j, $j \geq 1$, recursively by mid-point subdivision. For $j \geq 0$, let $S_5(T_j)$ denote the space of all C^1 quintic spline functions over T_j. In [37] a nodal basis for $S_5(T_j)$ is constructed comprising fundamental functions for Hermite interpolation functionals involving values, first-order and certain second-order directional derivatives at vertices in $V(T_j)$ and normal derivatives at mid-points of edges in $E(T_j)$. For the applications considered in [11], the authors study the subspace S^0 of $S_5(T_j)$ comprising all functions whose values and first-order derivatives vanish on the boundary of D. Since S^0 is not spanned by a subset of the above nodal basis for $S_5(T_j)$, [11] constructs a nodal basis for S^0 by keeping those of the above nodal functions whose supports lie in the interior of D but modifying the remaining elements of the above nodal basis.

4. Triangulated Quadrangulations

Let Q denote a mesh of quadrilaterals with union $D \subset \mathbb{R}^2$, each with interior angles less than π, such that distinct elements of Q intersect only in a common edge or a common vertex. We now divide each quadrilateral q in Q into four triangles, by inserting the diagonals of q, to produce a triangulation T. As in Section 3, we assume that D is a simply connected region in $\mathbb{R}^2$ and any boundary vertex of W is the intersection of exactly two boundary edges.

We now construct a mesh Q_1 of quadrilaterals from Q by mid-point subdivision, i.e. each element q of Q is divided into four elements of Q_1 by joining the mid-points of the edges of q to the intersection of the diagonals of q. We then define a triangulation T_1, which is a refinement of T, by inserting the diagonals of all the elements of Q_1. This is illustrated in Figure 5. We shall first consider the space $L(T)$ of all continuous linear spline functions over T.

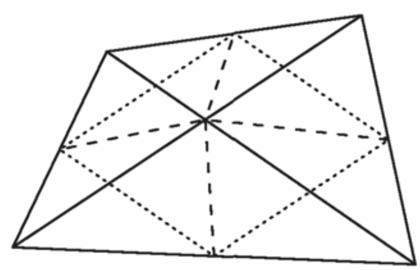

Fig. 5. Subdivision of T on a quadrilateral.

Now consider the uniform case when $D = \mathbb{R}^2$ and the vertices of Q comprise $\mathbb{Z}^2$. Then T is the type-2 triangulation $T(v_0, v_1, v_2, v_3)$, where $v_0 = (1,0)$, $v_1 = (0,1)$, $v_2 = (1,1)$, $v_3 = (1,-1)$, and T_1 is gained from T by shrinking by a factor of two, as in Section 2. In this case $L(T) = S_1(v_0, v_1, v_2, v_3)$, which is generated by the multi-box spline $\phi = (\phi_1, \phi_2)$, as in Section 2, where ϕ_1, ϕ_2 are the nodal functions (see Section 3),

$$\phi_1 = \phi_{(\frac{1}{2},\frac{1}{2})}, \quad \phi_2 = \phi_{(0,0)}.$$

Since the triangulation T_1 is not gained by mid-point subdivision of T, the general constructions of prewavelets of [20] and [52], as described in Section 3, do not hold in this case. In [21], Floater and Quak construct prewavelets for this case using the same approach as in [20], (i.e. each prewavelet is the sum of two 'semi-wavelets'). To be more precise, let T_0 denote the above type-2 triangulation restricted to a rectangle R with vertices in $\mathbb{Z}^2$ and sides of length at least three. Let T_1 denote the refinement of T_0, as above. As before, $V(T_0)$ and $V(T_1)$ denote the sets of vertices of all triangles in T_0 and T_1 respectively, while W denotes the orthogonal complement of $L(T_0)$ in $L(T_1)$. Then [21] constructs a basis $\{\psi_v : v \in V(T_1) \setminus V(T_0)\}$ for W. As for the previous construction in [20], the support of the prewavelet ψ_v is the union of all triangles in the T_0 having u_1 or u_2 as a vertex, where $v = \frac{1}{2}(u_1 + u_2)$, $u_1, u_2 \in V(T_0)$. There are two types of interior prewavelets, one with a mask of 17 non-zero coefficients and one with a mask of 13 coefficients. As in the prewavelet construction given in [29] on type-1 meshes, interior prewavelets with masks containing 11 and 17 terms, respectively, may also be constructed (see [28]).

Now consider the corresponding uniform case, where

$$V_0 = S_1(v_0, v_1, v_2, v_3) \cap L^2(\mathbb{R}^2), \tag{42}$$

$$V_1 = \{f(2\cdot) : f \in V_0\}, \tag{43}$$

and W denotes the orthogonal complement of V_0 in V_1. Since V_0 has multiplicity 2, W has multiplicity 6. A generator of W is given by

$$\psi = \left(\psi_v : v = (\frac{1}{2}, 0), (0, \frac{1}{2}), (\frac{1}{4}, \frac{1}{4}), (\frac{1}{4}, \frac{3}{4}), (\frac{3}{4}, \frac{1}{4}), (\frac{3}{4}, \frac{3}{4})\right).$$

Two examples of refinable, piecewise linear, orthogonal generators were constructed in [16] where by orthogonal we mean that the shifts of the components of these generators form orthogonal systems. The first generator is piecewise linear on a type-2 triangulation and the second on type-1 triangulation. Associated piecewise linear orthogonal wavelets were constructed in [14] (also see [17]), but here we restrict our discussion to the refinable generators. The construction relies on techniques introduced in [15] for constructing univariate orthogonal spline wavelets. The main observation is that if V is a refinable FSI space and $\tilde{V}$ is a local FSI space such that $V_0 \subset \tilde{V} \subset V_1$ for some $k \in \mathbb{Z}$ (where $(V_j)_{j\in\mathbb{Z}}$ is defined by (9) with dilation matrix $M = 2I$) then $\tilde{V}$ is also refinable since if $f \in \tilde{V}_0$ then $f \in V_1$ and so $f(\cdot/2) \in V_0 \subset \tilde{V}$. In this case the sequences of spaces $(V_j)_{j\in\mathbb{Z}}$ and $(\tilde{V}_j)_{j\in\mathbb{Z}}$ are said to form ***intertwining multiresolution analyses*** because of the relations

$$\cdots \subset \tilde{V}_{-1} \subset V_0 \subset \tilde{V}_0 := \tilde{V} \subset V_1 \subset \tilde{V}_1 \subset \cdots.$$

Note that $\tilde{V}$ inherits many of the properties of V. For example, if V is a spline space then $\tilde{V}$ is also a spline space with the same polynomial degree, smoothness, and approximation order.

We next outline the construction of the generator on the type-2 triangulation $T = T(v_0, v_1, v_2, v_3)$. Let $V_0 = L(T) \cap L^2(\mathbb{R}^2)$ and let ϕ_1 denote the continuous function that is piecewise linear on the triangulation of $[-1,1]^2$ consisting of the four triangles formed by the boundary of this square and its two diagonals and satisfying $\phi_1(0,0) = 1$ and $\phi(v) = 0$ for $v \in \{-1,1\}^2$. Furthermore, let $\phi_2 = \phi_1(2\cdot -(1,1))$. It is not difficult to verify that (ϕ_1, ϕ_2) is a generator for V_0. Next, for $i = 3, \ldots, 6$, let $\phi_i = \phi_2(2\cdot -k_i)$ where $\{k_3, k_4, k_5, k_6\} = \{0,1\}^2$ and let $\hat{V} = V(\phi_1, \ldots, \phi_6)$. Since $V_0 \subset \hat{V} \subset V_1$ it follows that $\hat{V}$ is refinable. Then it is shown that there exist functions $\phi_7, \phi_8 \in \hat{V}_1$ with support $[-1,1] \times [0,1]$ and $[0,1] \times [-1,1]$, respectively, and satisfying the conditions

(a) $\phi_7(-x, y) = \phi_7(x, y)$ and $\phi_8(x, y) = \phi_7(y, x)$ for $(x, y) \in \mathbb{R}^2$,
(b) $\phi_7 \perp V(\phi_2, \ldots \phi_6)$,
(c) $\phi_7 \perp V(\phi_8)$,
(d) $P_{V(\phi_2,\ldots,\phi_8)^\perp}\phi_1 \perp \phi_1(\cdot - k)$, for $k = (1,0)$ and $(0,1)$.

Condition (a) ensures symmetry of $V(\phi_1, \ldots, \phi_8)$ with respect to the symmetry group of T. The conditions in (b) are linear and, because the supports of the functions $\phi_2, \ldots \phi_6$ are contained in $[0,1]$, are easy to characterise. The conditions (c) and (d) are the critical conditions and reduce to solving three homogeneous quadratic equations in 8 variables. Similarly functions ϕ_9 and ϕ_{10} in V_1 are found that satisfy the conditions

(e) $\phi_9(y,x) = \phi_9(x,y)$, $\phi_9(1-y, 1-x) = -\phi_9(x,y)$, and $\phi_{10}(x,y) = \phi_9(1-x,y)$ for $(x,y) \in \mathbb{R}^2$,
(f) $\phi_9 \perp V(\phi_2, \ldots, \phi_8)$,
(g) $\phi_9 \perp V(\phi_{10})$,
(h) $P_{V(\phi_2,\ldots,\phi_{10})^\perp}\phi_1 \perp \phi_1(\cdot - k)$, for $k = (1,1)$ and $(-1,-1)$.

Finally, an orthogonal generator $\tilde{\phi} = (\tilde{\phi}_1, \ldots, \tilde{\phi}_{10})$ may be chosen for the space $\tilde{V} = V(\phi_1, \ldots, \phi_{10})$ in the following way. Let $\tilde{\phi}_1 = P_{V(\phi_2,\ldots,\phi_{10})^\perp}\phi_1$, choose $(\tilde{\phi}_2, \ldots, \tilde{\phi}_6)$ so that it forms an orthogonal basis for the span of $(\phi_2, \ldots, \phi_6)$ and let $\tilde{\phi}_j = \phi_j$ for $j = 7, 8, 9, 10$. Then it follows from the above conditions (a–h) as well as properties of the supports of these functions that $\tilde{\phi}$ is an orthogonal generator. Since $\tilde{V}$ is refinable, then so is $\tilde{\phi}$.

The second piecewise linear orthogonal refinable generator $\tilde{\phi} = (\tilde{\phi}_1, \ldots, \tilde{\phi}_{14})$ given in [16] is on the three direction 'hexagonal' triangulation given by $v_1 = (1,0)$, $v_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$ and $v_2 = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$. Then $V(\tilde{\phi})$ is invariant under the symmetry group $S(T)$ associated with the triangulation $T = T(v_1, v_2, v_3)$. In this case $\tilde{\phi}$ has 14 components. However, as shown in [28], one may generate $V(\tilde{\phi})$ from a 'macro-element' $\gamma = (\gamma_1, \ldots, \gamma_5)$ consisting of 5 functions supported on the triangle $(0,0), v_1, v_2$. Let $V_{S(T)}(\gamma)$ denote the algebraic span of the collection of functions $\{\gamma \circ \lambda : \lambda \in S(T)\}$. Then

$$V(\tilde{\phi}) = V_{S(T)}(\gamma) \cap C(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$$

and

$$\tilde{\phi}_i = \sum_{\lambda \in S(T)} a(\lambda)_{i,j} \gamma_j \circ \lambda$$

for some finitely supported collection of 14×5 matrices $(a(\lambda))_{\lambda \in S(T)}$. Because the essential orthogonality relations of $\tilde{\phi}$ occur triangle by triangle and are reflected in γ, the macro-element point of view provides a construction of continuous orthogonal wavelets on semi-regular sequence of triangulations generated by successively applying mid-point subdivision to an arbitrary initial triangulation (cf. [17]).

Next we consider the space $S_2 = S_2(v_0, v_1, v_2, v_3)$ of all C^1 quadratic splines on the type-2 triangulation $T = T(v_0, v_1, v_2, v_3)$. This is generated by the box spline $B_2 = B_2(\cdot | v_0, v_1, v_2, v_3)$, which is called the Zwart-Powell element, because it was first considered in [56,43]. This function has support in the octagon with vertices (0,-1), (1,-1), (2,0), (2,1), (1,2), (0,2), (-1,1), (-1,0). It shares all the symmetries of T, i.e. it satisfies (28), (30), (33) with

$$\sigma_1 = \tau_1 = \rho_1 = 1, \quad \alpha_1 = (1,1).$$

Since two lines in $H(v_0, v_1, v_2, v_3)$ intersect in $(\frac{1}{2}, \frac{1}{2})$, the generator (B_2) is *not* stable. Nevertheless it has been much studied and generalized to a triangulation of a non-uniform rectangular mesh [6,7,47–49]. To be more precise we let $D = [a, b] \times [c, d]$ and let Q denote the mesh of rectangles $\{[x_{i-1}, x_i] \times [y_{i-1}, y_i] : 1 \leq i \leq m,\ 1 \leq j \leq n\}$, where $a = x_0 < x_1 < \ldots < x_m = b$, $c = y_0 < y_1 < \ldots < y_m = d$. Let T denote the triangulation formed from Q by inserting the diagonals of all rectangles in Q, and let S denote the space of all C^1 quadratic splines over T. Then S is generated by $\{B_{ij} : 0 \leq i \leq m+1,\ 0 \leq j \leq n+1\}$ for locally supported functions B_{ij}. These functions generalise B_2 in the sense that if $2 \leq i \leq m-1$, $2 \leq j \leq n-1$ and

$$x_{i-2+k} = x_{i-2} + kh,\ y_{j-2+k} = y_{j-2} + kh,\ k = 1, 2, 3,\ h > 0,$$

then

$$B_{ij}(x, y) = B_2\left(\frac{1}{h}(x - x_{i-1}), \frac{1}{h}(y - y_{j-1})\right), \quad (x, y) \in D.$$

Now consider again a triangulation T formed by inserting the diagonals into a general mesh Q of quadrilaterals, as in the beginning of this section. We let $C(T)$ denote the space of all C^1 cubic splines over T. To consider a basis for $C(T)$ we first consider the mesh T_1 comprising the triangulation of a single quadrilateral q divided by its diagonals. Then each element F of $C(T_2)$ is determined uniquely by the value and first order derivatives of f at each vertex of q, and the first order normal derivative of f at the mid-point of each edge of q. Thus $C(T)$ has a nodal basis, defined as follows. Let $V(Q)$ and $E(Q)$ denote the sets of all vertices and all edges of elements of Q respectively. For $v \in V(Q)$ we define the linear functionals σ_{v_1}, σ_{v_2}, σ_{v_3} as in (41), while for $e \in E(Q)$, $\sigma_{en} f$ denotes, as before, the normal derivative of f at the mid-point of e. We denote by ϕ_{vj}, $v \in V(Q)$, $j = 1, 2, 3$, and ϕ_e, $e \in E(Q)$, the corresponding fundamental functions in $C(T)$ satisfying

$$\sigma_{ui}\phi_{vj} = \delta_{uv}\delta_{ij},\ \sigma_{ui}\phi_e = 0,\ u \in V(Q),\ i = 1, 2, 3,$$

$$\sigma_{fn}\phi_{vj} = 0,\ \sigma_{fn}\phi_e = \delta_{ef},\ e \in E(Q).$$

Then $\{\phi_{vj}, \phi_e : v \in V(Q),\ j = 1, 2, 3,\ e \in E(Q)\}$ forms a nodal basis for $C(T)$. This basis was first considered by Fraeijs de Veubeke [22] and Sander [50] and for this reason the type of triangulation T considered above is often called a Fraeijs de Veubeke–Sander (FVS) triangulation. The support of ϕ_{vj}, $j = 1, 2, 3$, is the union of all quadrilaterals in Q having v as a vertex, and the support of ϕ_e is the union of the quadrilaterals in Q having e as an edge.

We first consider the uniform case when $D = \mathbb{R}^2$ and $T = T(v_0, v_1, v_2, v_3)$, as before. Here the space $C(T)$ is shift-invariant with multiplicity 5 and generator $\phi = (\phi_1, \ldots, \phi_5)$, where

$$\phi_j = \phi_{(0,0),j},\ j = 1, 2, 3,\ \phi_4 = \phi_e,\ \phi_5 = \phi_f,$$

where e denotes the edge from $(0, 0)$ to $(0, 1)$ and f the edge from $(0, 0)$ to $(1, 0)$. The generator ϕ is linearly independent over $[0, 1]^2$ and satisfies (28), (30), and (33) with

$$\sigma_1 = 1,\ \sigma_j = -1,\ j = 2, \dots, 5,$$
$$\alpha_j = (0,0),\ j = 1,2,3,\ \alpha_4 = (0,1),\ \alpha_5 = (1,0),$$
$$\tau_1 = 1\ \rho_1 = \rho_3 = \rho_5 = 1,\ \rho_2 = \rho_4 = -1.$$

Moreover, with T as in (30),

$$\phi_3 = \phi_2(T\cdot),\ \phi_5 = \phi_4(T\cdot).$$

For $j = 1,2,3$, ϕ_j has support $[-1,1]^2$ and ϕ_4 has support $[-1,1] \times [0,1]$. The Bernstein-Bézier representation of ϕ is given in [33]. As noted in [4], we can form a generator of $C(T)$ with smaller support by replacing ϕ_2 and ϕ_3 by $\tilde{\phi}_2$ and $\tilde{\phi}_3$ where

$$\tilde{\phi}_2 = \phi_2 - \phi_3 - \frac{1}{4}(\phi_4 + \phi_4(\cdot + (0,1)) - \phi_5 - \phi_5(\cdot + (1,0)),$$
$$\tilde{\phi}_3(x,y) = \tilde{\phi}_2(-x,y), \quad x, y \in \mathbb{R}.$$

The support of $\tilde{\phi}_2$ is the hexagon with vertices $(-1,-1)$, $(-1,0)$, $(0,-1)$, $(1,0)$, $(0,1)$, $(1,1)$.

Now, in a similar manner to (42), (43), we define

$$V_0 = C(T) \cap L^2(\mathbb{R}^2), \quad V_1 = \{f(2\cdot) : f \in V_0\},$$

and let W denote the orthogonal complement of V_0 in V_1. In [4] a Riesz basis is constructed for W, as we now describe. First a Riesz basis $B = \{t_{i,k} : i = 1, \dots, 5,\ k \in \mathbb{Z}^2\}$ for V_0 is defined by

$$t_{1,k} = \phi_1(\cdot - k),\ t_{j,k} = \tilde{\phi}_j(\cdot - k),\ j = 2,3,$$
$$t_{4,k} = \begin{cases} \phi_4(\cdot - k) + \phi_4(\cdot - k + (0,1)),\ k \in \mathbb{Z} \times 2\mathbb{Z}, \\ \phi_4(\cdot - k) - \phi_4(\cdot - k - (0,1)),\ k \in \mathbb{Z} \times (2\mathbb{Z} - 1), \end{cases}$$
$$t_{5,k} = t_{4,Tk}(T\cdot).$$

Next a set $\tilde{B} = \{\tilde{t}_{i,k} : i = 1, \dots, 5,\ k \in \mathbb{Z}^2\}$ in V_1 is constructed which is defined uniquely by each function $\tilde{t}_{i,k}$ having the same support and symmetry as $t_{i,k}$, and $\tilde{B}$ being biorthogonal to B, i.e.

$$\langle t_{i,k}, \tilde{t}_{j,\ell} \rangle = \|t_{i,k}\|_2^2 \delta_{ij} \delta_{k\ell},$$

where $\langle\, , \rangle$ denotes the inner product on $L^2(\mathbb{R}^2)$. Finally a Riesz basis $\{\psi_{i,k} : i = 1, \dots, 5,\ k \in \mathbb{Z}^2 \setminus 2\mathbb{Z}^2\}$ for W is defined by

$$\psi_{i,k} = t_{I,k}(2\cdot) - \sum_{j=1}^{5} \sum_{\ell \in \mathbb{Z}^2} \frac{\langle t_{i,k}(2\cdot), t_{j,\ell} \rangle}{\|t_{j,\ell}\|^2} \tilde{t}_{j,\ell}.$$

We now return to the non-uniform case and describe a generalisation of the FVS triangulation defined in [11]. Let Q_0 be a mesh comprising triangles and quadrilaterals. We define a triangulation T_0 by dividing each quadrilateral in Q_0 into four triangles by inserting the diagonals, as before, and dividing each triangle T in Q_0 into three sub-triangles by joining the vertices of t to its barycentre. We now construct a mesh Q_1 of quadrilaterals by dividing each quadrilateral in Q_0 into four sub-quadrilaterals, as before, and by dividing each triangle t in Q_0 into three quadrilaterals by joining the mid-points of the edges of t to its barycentre.

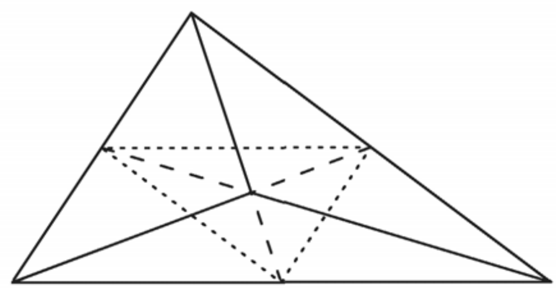

Fig. 6. Subdivision of a triangle in Q_0.

We then define a triangulation T_1, $T_0 \prec T_1$, by inserting the diagonals of all elements of Q_1. This process on a triangle in Q_0 is illustrated in Figure 6.

For $j \geq 1$, we may recursively define Q_{j+1} by mid-point subdivision of Q_j, and T_{j+1} with $T_j \prec T_{j+1}$, by inserting the diagonals of all elements of Q_{j+1}. It is noted in [11] that for $j \geq 2$, each quadrilateral in Q_j is either a parallelogram or similar to a quadrilateral in Q_1.

For T_j as above, $j \geq 0$, we denote by $C(T_j)$ the space of all C^1 cubic splines over T_j. Then

$$C(T_j) \subset C(T_{j+1}), \quad j \geq 0.$$

Now the restriction of any function f in $C(T_0)$ to a triangle t in Q_0 is determined uniquely, as in the Clough-Tocher element [8], by the value and first-order derivatives of f at each vertex of t and the normal derivative of f at the mid-points of each edge of t. Thus there is a nodal basis for $C(T_0)$ of fundamental functions for Hermite interpolation of value and first-order derivatives at each vertex $v \in V(Q_0)$ and normal derivatives at the mid-point of each edge $e \in E(Q_0)$. A corresponding hierarchical basis can be defined as before.

All C^1 nodal bases considered so far, in both this and the previous section, are based on Hermite interpolation. To finish, we mention some nodal bases of C^1 cubic splines, and some corresponding hierarchical bases, based on Lagrange interpolation. As before, let Q denote a mesh of quadrilaterals with union D, let T denote the triangulation formed by inserting the diagonals of the elements of Q, and let $C(T)$ denote the space of all C^1 cubic splines over T. In [38] it is assumed that Q is *checkerboard*, i.e. all interior vertices are of degree 4 and elements of Q can be coloured black and white in such a way that any two quadrilaterals sharing an edge have opposite colours. Then [38] constructs a basis of $C(T)$ with local support of fundamental functions for Lagrange interpolation at certain points. (Further study of bases of this type is given in [39,40]. A similar construction for a nodal basis of C^2 splines of degree 6 is given in [13].)

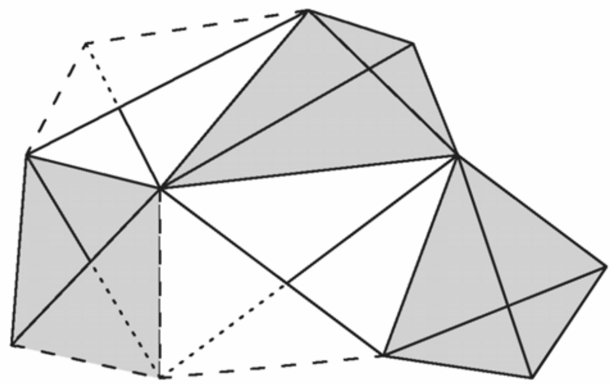

Fig. 7. Triangulation T_0.

Similar nodal bases, and corresponding hierarchical bases, are studied in the recent paper [12]. Here more general triangulations T are allowed, as we now describe. For certain boundary quadrilaterals q in Q, divide q into two triangles by inserting a diagonal and denote one of the triangles, which has at least one edge in the boundary of D, by $t(q)$. Then let Ω be the region gained by removing all triangles $t(q)$ from D. Let T_0 denote the triangulation of Ω comprising all triangles in T which lie in Ω, see Figure 7. It is shown in [12] that any domain with Lipschitz continuous piecewise linear boundary can be triangulated by such a triangulation T_0.

Unlike our previous constructions, we form a refinement Q_1 of Q by dividing each element of Q into *nine* sub-quadrilaterals by trisecting each edge and sub-diagonal. This is illustrated in Figure 8, where we have shown the colouring of the sub-quadrilaterals for the case of a 'white quadrilateral.' For a 'black quadrilateral,' the colours are reversed. By inserting the diagonals of all elements of Q_1 we obtain a triangulation T_1 which is a refinement of T, each triangle in T being divided into nine sub-triangles, see Figure 8.

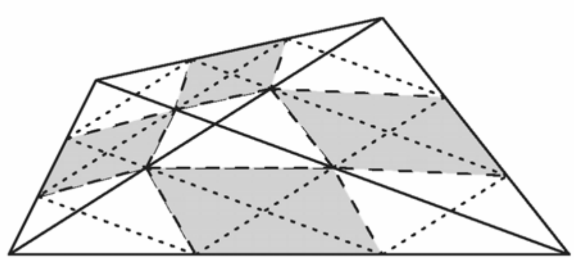

Fig. 8. Refinement of a 'white quadrilateral.'

A basis of $C(T_0)$ is given in [12] comprising fundamental functions for Lagrange interpolation at certain points $P(T_0)$, which depend on the colouring of the quadrilaterals. These points are chosen so that corresponding points $P(T_1)$ contain $P(T_0)$ and hence a corresponding hierarchical basis can be constructed. It is furthermore shown that the hierarchical basis is a Riesz basis for $H^s(\Omega)$, $1 < s < 5/2$, thus improving the stability of formerly constructed C^1 hierarchical bases, though further discussion of stability of hierarchical bases is beyond the scope of this paper.

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