Topics in Multivariate Approximation and Interpolation
K. Jetter et al., Editors

Durrmeyer Operators and Their Natural Quasi-Interpolants

Elena Berdysheva [a], **Kurt Jetter** [a,1], **Joachim Stöckler** [b]

[a]Universität Hohenheim, Institut für Angewandte Mathematik und Statistik, D-70593 Stuttgart, Germany
[b]Universität Dortmund, Institut für Angewandte Mathematik, D-44221 Dortmund, Germany

Dedicated to Professor Charles K. Chui on the occasion of his 65th birthday.

Abstract

This paper provides a survey on spectral analysis and approximation order of our quasi-interpolants of Durrmeyer type on simplices, together with various new aspects and achievements. The latter include Bernstein type inequalities which are proved using a striking property of appropriately modified Durrmeyer operators, namely, their kernel functions are pointwise completely monotonic.

Key words: Bernstein basis polynomial, Bernstein inequality, completely monotonic sequence, Durrmeyer operator, hypergeometric series, Jackson-Favard estimate, Jacobi polynomial, K-functional, Laplace type integral, Legendre differential operator, positive operator, quasi-interpolants, Voronovskaja theorem

2000 MSC: 41A10, 41A17, 41A36, 41A63, 33C45

Email addresses: `berdyshe@uni-hohenheim.de` (Elena Berdysheva),
`kjetter@uni-hohenheim.de` (Kurt Jetter),
`joachim.stoeckler@math.uni-dortmund.de` (Joachim Stöckler).
[1] Corresponding author

1. Introduction

The construction of quasi-interpolant operators through linear combinations of (Bernstein-)Durrmeyer operators has a long history in Approximation Theory. Durrmeyer operators have several desirable properties such as positivity and stability, and their analysis can be performed using their elegant spectral properties. Their approximation order is low, however, and for this reason quasi-interpolants with better approximation properties are necessary for more efficient approximation. In our aim at constructing good quasi-interpolants on triangulated domains, the natural first step is to consider a single triangle - or a simplex in higher dimensions.

A comprehensive description of our previous results in this direction is included as part of this article. We emphasize the close relation of quasi-interpolants to certain partial differential operators on the simplex, which are generalizations of the Legendre differential operator and its Jacobi-type analogue

$$P_r^{\alpha,\beta} := w_{\alpha,\beta}^{-1}(x) \frac{d^r}{dx^r} \left[w_{\alpha,\beta}(x) x^r (1-x)^r \frac{d^r}{dx^r} \right],$$

where $w_{\alpha,\beta}(x) = x^\alpha(1-x)^\beta$ and $\alpha, \beta > -1$ define the Jacobi weight for the standard interval $[0,1]$. In addition to this survey we also present new results which lead to a Bernstein estimate for the aforementioned differential operators (Section 6) and to direct estimates of the error of approximation of our quasi-interpolants by newly defined K-functionals on the simplex (Section 7). The key result in order to prove the Bernstein inequality is a beautiful property of the sequence of appropriately modified Durrmeyer operators: their kernels constitute a pointwise completely monotonic sequence (Theorem 2). Here we employ methods of Koornwinder and Askey for the Laplace integral of Jacobi-polynomials and the characterization of completely monotonic sequences by Hausdorff's theorem (Section 4).

The structure of the paper is as follows. In Section 2 we give the definition of the Durrmeyer operators (with Jacobi weights), and in Section 3 we review their spectral properties, see Theorem 1. Section 4 deals with the kernel function of the appropriately modified Durrmeyer operator, and provides the striking result of Theorem 2 showing the pointwise complete monotonicity of the associated kernels. We then give the definition of our quasi-interpolants in Section 5, together with the adequate partial differential operators of Jacobi type. Their spectral analysis leads to a valuable representation of the quasi-interpolants as a linear combination of Durrmeyer operators, in Theorem 9. Section 6 is devoted to the proof of the Bernstein inequalities, which are stated in Theorem 11 and Theorem 12. This is the second key section of the paper, which contains new and unpublished material. Its application in Section 7 follows along the lines of classical Approximation Theory and provides a valuable and elegant extension of several properties of the Durrmeyer operator to our quasi-interpolants: The estimate of Jackson-Favard type, the Voronovskaja type theorem (including its 'strong' version) and the so-called direct estimate in terms of the proper K-functional. Rather than giving complete references for each result within the text, we conclude in Section 8 with historical remarks in order to point out the development of the main results.

2. The Bernstein Basis Functions

The standard simplex in $\mathbb{R}^d$ is given by

$$\mathbf{S}^d := \{ (x_1, \ldots, x_d) \in \mathbb{R}^d \mid 0 \le x_1, \ldots, x_d \le 1 \, , \; x_1 + \cdots + x_d \le 1 \} \, .$$

We shall use barycentric coordinates

$$\mathbf{x} = (x_0, x_1, \ldots, x_d) \, , \qquad x_0 := 1 - x_1 - \ldots - x_d \, ,$$

in order to define the d-variate Bernstein basis polynomials. Namely, for given $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^{d+1}$

$$B_\alpha(x_1, \ldots, x_d) := \binom{|\alpha|}{\alpha} \mathbf{x}^\alpha := \frac{|\alpha|!}{\alpha_0! \alpha_1! \cdots \alpha_d!} \, x_0^{\alpha_0} x_1^{\alpha_1} \cdots x_d^{\alpha_d} \, .$$

Here, we use standard multi-index notation. We also allow $\alpha \in \mathbb{Z}^d$ with $|\alpha| := \alpha_0 + \cdots + \alpha_d \in \mathbb{N}$. It is then convenient to put $B_\alpha \equiv 0$ if one α_i is negative.

For given $n \in \mathbb{N}$, the Bernstein basis polynomials $\{B_\alpha \mid \alpha \in \mathbb{N}_0^{d+1} \, , \; |\alpha| = n\}$ are a basis for $\mathbf{P}_n = \mathbf{P}_n^d$, the space of d-variate algebraic polynomials of (total) degree n. They are used for the definition of various polynomial operators. In this paper, we study quasi-interpolants based on the Bernstein-Durrmeyer operators with Jacobi weights. Here, the weight function is given by

$$\omega_\mu(x_1, \ldots, x_d) := \mathbf{x}^\mu = x_0^{\mu_0} x_1^{\mu_1} \cdots x_d^{\mu_d} \, ,$$

where $\mu = (\mu_0, \mu_1, \ldots, \mu_d) \in \mathbb{R}^{d+1}$ with $\mu_i > -1$, $i = 0, \ldots, d$. Whence, $|\mu| := \mu_0 + \mu_1 + \cdots + \mu_d > -d - 1$.

On the simplex, we use the (weighted) inner product

$$\langle f | g \rangle_\mu := \int_{\mathbf{S}^d} \omega_\mu \, f \, g \tag{1}$$

to define the (Bernstein-)Durrmeyer operator of degree n,

$$\mathbf{M}_{n,\mu} \; : \; f \mapsto \mathbf{M}_{n,\mu}(f) := \sum_{|\alpha| = n} \frac{\langle f | B_\alpha \rangle_\mu}{\langle \mathbf{1} | B_\alpha \rangle_\mu} \, B_\alpha \, . \tag{2}$$

Here, $\mathbf{1}$ denotes the function constant equal one. It is well-known that, for $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^{d+1}$, with $|\alpha| = n$,

$$\begin{aligned} \langle \mathbf{1} | B_\alpha \rangle_\mu &= \binom{|\alpha|}{\alpha} \frac{\Gamma(\alpha_0 + \mu_0 + 1)\Gamma(\alpha_1 + \mu_1 + 1) \cdots \Gamma(\alpha_d + \mu_d + 1)}{\Gamma(|\alpha| + |\mu| + d + 1)} \\ &= \frac{\Gamma(n+1)}{\Gamma(n + |\mu| + d + 1)} \prod_{i=0}^{d} \frac{\Gamma(\alpha_i + \mu_i + 1)}{\Gamma(\alpha_i + 1)} \, . \end{aligned} \tag{3}$$

For the unweighted case (where $\mu = (0,0,\dots,0)$) this recovers the formula

$$\int_{\mathbf{S}^d} B_\alpha = \frac{n!}{(n+d)!}\,, \quad \alpha \in \mathbb{N}_0^{d+1}\,,\ |\alpha| = n\,.$$

3. Spectral Properties of the Durrmeyer Operators

The Durrmeyer operator is usually considered on the domain $L^p_\mu(\mathbf{S}^d)$, $1 \le p < \infty$, which is the weighted L^p space consisting of all measurable functions on S^d such that

$$\|f\|_{p,\mu} := \Big(\int_{\mathbf{S}^d} \omega_\mu\, |f|^p\Big)^{1/p}$$

is finite. For $p = \infty$, the space $C(\mathbf{S}^d)$ of continuous functions is considered instead. In this setting the following properties are more or less obvious.

- *The operator is positive:* $\mathbf{M}_{n,\mu}(f) \ge 0$ for every $f \ge 0$.
- *It reproduces constant functions:* $\mathbf{M}_{n,\mu}(p) = p$ for $p \in \mathbf{P}_0$.
- *It is contractive:* $\|\mathbf{M}_{n,\mu}(f)\|_{p,\mu} \le \|f\|_{p,\mu}$ for every $f \in L^p_\mu(\mathbf{S}^d)$.

The most striking and useful property refers to the Hilbert space setting,

$$\mathbf{H} := L^2_\mu(\mathbf{S}^d)\,.$$

This space can be written as the sum of spaces of orthogonal polynomials,

$$L^2_\mu(\mathbf{S}^d) = \sum_{m=0}^{\infty} \mathbf{E}_{m,\mu}\,, \quad \text{where}$$

$$\mathbf{E}_{0,\mu} := \mathbf{P}_0 \quad \text{and} \quad \mathbf{E}_{m,\mu} := \mathbf{P}_m \cap \mathbf{P}_{m-1}^\perp \quad \text{for} \quad m > 0\,.$$

Here, orthogonality refers to the weighted inner product (1). It is clear that $\mathbf{M}_{n,\mu}$ is a bounded self-adjoint operator on $\mathbf{H}$. Its spectrum is given by the following result.

Theorem 1. *For all $n \in \mathbb{N}$, the spaces $\mathbf{E}_{m,\mu}$, $m \ge 0$, are eigenspaces of the Durrmeyer operator, and*

$$\mathbf{M}_{n,\mu}(p_m) = \gamma_{n,m,\mu}\, p_m \quad \textit{for} \quad p_m \in \mathbf{E}_{m,\mu}\,,$$

where

$$\gamma_{n,m,\mu} := \frac{n!}{(n-m)!}\,\frac{\Gamma(n+d+|\mu|+1)}{\Gamma(n+d+|\mu|+m+1)}\,, \quad \textit{for} \quad n \ge m\,,$$

while

$$\gamma_{n,m,\mu} = 0 \quad \textit{for} \quad n < m\,.$$

In particular,

$$\gamma_{n,0,\mu} = 1\,, \quad 0 \le \gamma_{n,m,\mu} < 1\,, \quad m > 0\,,$$

and

$$\lim_{n\to\infty} \gamma_{n,m,\mu} = 1 \quad \textit{for fixed } m\,. \tag{4}$$

Hence, for $f = \sum_{m=0}^{\infty} p_m$, with $p_m \in \mathbf{E}_{m,\mu}$, we find $\mathbf{M}_{n,\mu}(f) = \sum_{m=0}^{n} \gamma_{n,m,\mu}\, p_m$. In particular, the restrictions $\mathbf{M}_{n,\mu}|_{\mathbf{P}_k}$ act as isomorphisms on the spaces $\mathbf{P}_k$ as long as $k \le n$.

4. The Kernel Function

According to equations (2) and (3), the Durrmeyer operator of degree n can be written as

$$\{\mathbf{M}_{n,\mu}(f)\}(\mathbf{y}) = \int_{\mathbf{S}^d} \omega_\mu(\mathbf{x})\, f(\mathbf{x})\, K_{n,\mu}(\mathbf{x},\mathbf{y})\, d\mathbf{x}$$

with its kernel given by

$$K_{n,\mu}(\mathbf{x},\mathbf{y}) := \frac{\Gamma(n+|\mu|+d+1)}{\Gamma(n+1)} \sum_{|\alpha|=n} \Big(\prod_{i=0}^{d} \frac{\Gamma(\alpha_i+1)}{\Gamma(\alpha_i+\mu_i+1)} \Big)\, B_\alpha(\mathbf{x})\, B_\alpha(\mathbf{y}) .$$

Putting

$$\underline{\mu} := \min_i \mu_i ,$$

we are going to study properties of the modified kernel

$$\begin{aligned} T_{n,\mu}(\mathbf{x},\mathbf{y}) : &= \frac{\Gamma(n+\underline{\mu}+1)}{\Gamma(n+|\mu|+d+1)}\, K_{n,\mu}(\mathbf{x},\mathbf{y}) \\ &= \sum_{|\alpha|=n} \frac{\Gamma(n+\underline{\mu}+1)}{\prod_{i=0}^{d}\Gamma(\alpha_i+\mu_i+1)} \binom{n}{\alpha}\, \mathbf{x}^\alpha\, \mathbf{y}^\alpha , \qquad \mathbf{x},\mathbf{y} \in \mathbf{S}^d. \end{aligned} \tag{5}$$

This kernel is non-negative, but we are going to prove much more. Namely, under a slight restriction on the exponents of the Jacobi weight, the forward differences of the sequence $(T_{n,\mu}(\mathbf{x},\mathbf{y}))_{n\ge 0}$ alternate in sign. The result seems to be new even in the univariate setting.

Theorem 2. *Let $\mu = (\mu_0, \mu_1, \ldots, \mu_d)$ be such that $\mu_i \ge -1/2$, $i = 0, \ldots, d$. Then, for every $\mathbf{x},\mathbf{y} \in \mathbf{S}^d$, the sequence $(T_{n,\mu}(\mathbf{x},\mathbf{y}))_{n\ge 0}$ is bounded and completely monotonic; i.e., the inequalities*

$$T^{(r)}_{n,\mu}(\mathbf{x},\mathbf{y}) := (-1)^r \Delta^r T_{n,\mu}(\mathbf{x},\mathbf{y}) = \sum_{\ell=0}^{r} (-1)^\ell \binom{r}{\ell} T_{n+\ell,\mu}(\mathbf{x},\mathbf{y}) \ge 0$$

hold true for all $r, n \ge 0$.

Here, $(\Delta\nu_n)_{n\ge 0} = (\nu_{n+1} - \nu_n)_{n\ge 0}$ denotes the forward difference of the sequence $(\nu_n)_{n\ge 0}$. For the proof of this result, we make use of the characterization of completely monotonic sequences due to Hausdorff.

Lemma 3. *(see [28], Chapter III, Theorem 4a) A real sequence $(\nu_n)_{n\ge 0}$ is completely monotonic if and only if there exists a non-decreasing bounded function g on $[0,1]$ such that*

$$\nu_n = \int_0^1 t^n \, dg(t), \qquad n \in \mathbb{N}_0 .$$

Here, the integral is to be understood as a Lebesgue-Stieltjes integral.

Remarks. The following facts will be useful for our discussion of complete monotonicity.

(a) *The sequences $(q^n)_{n\geq 0}$, with $0 \leq q \leq 1$, are completely monotonic.* This result is obvious.

(b) *The sum and the product of two completely monotonic sequences are completely monotonic.* The first statement is again trivial, and the second statement follows from

$$-\Delta(c_n d_n) = (-\Delta c_n) d_n + c_{n+1}(-\Delta d_n)$$

by induction.

(c) *For given $\mu_0 \geq \mu_1 > -1$ the sequence*

$$c_n = \frac{\Gamma(n+\mu_1+1)}{\Gamma(n+\mu_0+1)}, \quad n \geq 0 ,$$

is completely monotonic. This follows from the formula

$$(-1)^k \Delta^k c_n = \frac{(\mu_0 - \mu_1)_k}{(n+1+\mu_0)_k} \, c_n , \quad k, n \geq 0 ,$$

with $(a)_0 := 1$ and $(a)_k := a(a+1)\cdots(a+k-1)$ for $k > 0$, the so-called shifted factorial or Pochhammer symbol.

(d) *The sequence of integrals $c_n = \int f_n \, dm$, $n \geq 0$, of a pointwise completely monotonic family $(f_n)_{n\geq 0}$ of functions which are integrable with respect to the non-negative measure dm, is completely monotonic.*

For the proof of Theorem 2 it is sufficient to consider the case

$$\mu_0 \geq \mu_1 \geq \cdots \geq \mu_d \geq -1/2 , \quad \text{whence } \underline{\mu} = \mu_d ,$$

since the kernel is invariant modulo a permutation of the variables. Under this assumption, we use induction on d, the number of variables. For $d = 1$, the statement is the special case $t = 1$, $(\alpha, \beta) = (\mu_0, \mu_1)$ of the following result.

Lemma 4. *Let $\alpha \geq \beta \geq -1/2$. Then, for every $x, y, t \in [0,1]$, the sequence*

$$\begin{aligned} \nu_n(x,y;t) &= \nu_n^{(\alpha,\beta)}(x,y;t) \\ &= \sum_{k=0}^{n} \frac{\Gamma(n+\beta+1)}{\Gamma(k+\alpha+1)\Gamma(n-k+\beta+1)} \binom{n}{k} (xy)^k \big((1-x)(1-y)t\big)^{n-k} , \end{aligned}$$

$n \geq 0$, is bounded and completely monotonic.

Remark. The restriction on the Jacobi exponents is crucial. For example, a simple calculation shows that

$$-\Delta\nu_1^{(\beta,\beta)}\left(\frac{1}{2},\frac{1}{2};1\right) = \frac{1}{\Gamma(\beta+1)}\,\frac{2\beta+1}{8(\beta+1)}\,,$$

which is negative for $-\frac{1}{2} > \beta > -1$.

Taken the result of Lemma 4 for granted, the *proof of Theorem 2* is finished by the following induction step.

For $d \geq 2$, we write the kernel $T_{n,\mu} = T^d_{n,\mu}$ in equation (5) in terms of kernels of fewer variables (For clarity, we mark the number of variables as a superscript). We put $\mathbf{x} = (x_0, \mathbf{x}^*)$ with $\mathbf{x}^* = (x_1, \ldots, x_d)$ and $x_0 = (1 - x_1 - \cdots - x_d)$, and $\mathbf{y} = (y_0, \mathbf{y}^*)$ with analogous notation. Also, $\alpha = (\alpha_0, \alpha^*)$ with $\alpha^* = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$, and $\mu = (\mu_0, \mu^*)$ with $\mu^* = (\mu_1, \ldots, \mu_d) \in \mathbb{R}^d$.

If $x_0 = 1$, then $x_1 = \cdots = x_d = 0$ and

$$T^d_{n,\mu}(\mathbf{x},\mathbf{y}) = \frac{1}{\prod_{i=1}^d \Gamma(\mu_i+1)}\,\frac{\Gamma(n+\mu_d+1)}{\Gamma(n+\mu_0+1)}\,y_0^n\,, \qquad n \geq 0\,,$$

which is completely monotonic by an application of cases (a)-(c) of the remarks above. The same argument applies if $y_0 = 1$. So we may assume henceforth that

$$0 \leq x_0, y_0 < 1\,.$$

Here, both $\tilde{\mathbf{x}} := \frac{1}{1-x_0}\mathbf{x}^*$ and $\tilde{\mathbf{y}} := \frac{1}{1-y_0}\mathbf{y}^*$ are elements of $\mathbf{S}^{d-1}$. By simple computations and by letting $k := \alpha_0$, we obtain

$$\begin{aligned} T^d_{n,\mu}(\mathbf{x},\mathbf{y}) &= \sum_{k=0}^n \frac{\Gamma(n+\mu_d+1)}{\Gamma(k+\mu_0+1)}\binom{n}{k}(x_0y_0)^k\left((1-x_0)(1-y_0)\right)^{n-k} \\ &\qquad\qquad \sum_{|\alpha^*|=n-k}\frac{1}{\prod_{i=1}^d\Gamma(\alpha_i+\mu_i+1)}\binom{n-k}{\alpha^*}(\tilde{\mathbf{x}})^{\alpha^*}(\tilde{\mathbf{y}})^{\alpha^*} \\ &= \sum_{k=0}^n a_{k,n}(x_0,y_0)\,T^{d-1}_{n-k,\mu^*}(\tilde{\mathbf{x}},\tilde{\mathbf{y}})\,, \end{aligned} \tag{6}$$

with

$$a_{k,n}(x_0,y_0) := \frac{\Gamma(n+\mu_d+1)}{\Gamma(k+\mu_0+1)\Gamma(n-k+\mu_d+1)}\binom{n}{k}(x_0y_0)^k\left((1-x_0)(1-y_0)\right)^{n-k}.$$

Using the induction hypothesis we find - according to Lemma 3 - for each pair $(\tilde{\mathbf{x}},\tilde{\mathbf{y}})$ a bounded, nondecreasing function g^* such that

$$\nu_n^* := T^{d-1}_{n,\mu^*}(\tilde{\mathbf{x}},\tilde{\mathbf{y}}) = \int_0^1 t^n\,dg^*(t)\,, \qquad n \geq 0\,.$$

Inserting this in equation (6),

$$T^d_{n,\mu}(\mathbf{x},\mathbf{y}) = \int_0^1 \sum_{k=0}^n a_{k,n}(x_0,y_0)\,t^{n-k}\,dg^*(t)\,.$$

Here, the integrand is given by the sequence $\nu_n(x_0, y_0; t)$ considered in Lemma 4, by putting $(\alpha, \beta) = (\mu_0, \mu_d)$, and the induction step is completed by applying item (d) of the remarks above. This finishes the proof of Theorem 2.

For the *proof of Lemma 4*, it is sufficient to assume $\alpha > \beta > -\frac{1}{2}$, since the limit case $\alpha = \beta$ or $\beta = -\frac{1}{2}$ then follows by continuity. Here we make use of Koornwinder's integral representation (of Laplace type) for the normalized Jacobi polynomials which are given by a hypergeometric series as follows:

$$\begin{aligned} R_n^{(\alpha,\beta)}(x) := \frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(1)} &:= F_1^2\Big(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2}\Big) \\ &= \Big(\frac{x+1}{2}\Big)^n F_1^2\Big(-n, -n-\beta; \alpha+1; \frac{x-1}{x+1}\Big) \end{aligned} \tag{7}$$

for $\alpha, \beta > -1$. The latter identity follows from Euler's linear transformation,

$$F_1^2(a,b;c;z) = (1-z)^{-a} F_1^2\big(a, c-b; c; \frac{z}{z-1}\big) ,$$

(cf. [22], Section 2.4). Koornwinder's result (see [21], Section 3) reads as follows; for an easy analytic proof we refer to Askey [2].

Lemma 5. *For $\alpha > \beta > -\frac{1}{2}$ we have*

$$\begin{aligned} &R_n^{(\alpha,\beta)}(x) \\ &= \frac{2\Gamma(\alpha+1)}{\Gamma(\beta+\frac{1}{2})\Gamma(\frac{1}{2})\Gamma(\alpha-\beta)} \int_{u=0}^{1} \int_{\varphi=0}^{\pi} \Big[\frac{x+1}{2} + \frac{x-1}{2}u^2 + u\sqrt{x^2-1}\cos\varphi\Big]^n \\ &\qquad\qquad u^{2\beta+1} (1-u^2)^{\alpha-\beta-1} \big(\sin\varphi\big)^{2\beta} \, d\varphi \, du . \end{aligned}$$

With this result at hand, the proof is finished by a straightforward, but lengthy computation. Using (7), we find

$$\begin{aligned} \nu_n(x,y;t) &= \frac{[(1-x)(1-y)t]^n}{\Gamma(\alpha+1)} F_1^2\Big(-n, -n-\beta; \alpha+1; \frac{xy}{(1-x)(1-y)t}\Big) \\ &= \frac{1}{\Gamma(\alpha+1)} \big[(1-x)(1-y)t - xy\big]^n R_n^{(\alpha,\beta)}\Big(\frac{(1-x)(1-y)t+xy}{(1-x)(1-y)t-xy}\Big) , \end{aligned}$$

and Koornwinder's integral gives

$$\nu_n(x,y;t) = \int_{u=0}^{1} \int_{\varphi=0}^{\pi} \big[\Phi(x,y,t;u,\varphi)\big]^n \, dm_{\alpha,\beta}(u,\varphi) . \tag{8}$$

Here,

$$\Phi(x,y,t;u,\varphi) = (1-x)(1-y)t + xyu^2 + 2\varepsilon\sqrt{(1-x)(1-y)txy}\, u \, \cos\varphi \tag{9}$$

with $\varepsilon = \operatorname{sign}((1-x)(1-y)t - xy)$ and

$$dm_{\alpha,\beta}(u,\varphi) = \frac{2}{\Gamma(\beta+\frac{1}{2})\Gamma(\frac{1}{2})\Gamma(\alpha-\beta)} u^{2\beta+1} (1-u^2)^{\alpha-\beta-1} \big(\sin\varphi\big)^{2\beta} \, d\varphi \, du ,$$

which is a positive measure. From (9) we see that

$$\begin{aligned} 0 &\le \Big(\sqrt{(1-x)(1-y)t} - \sqrt{xyu} \Big)^2 \\ &\le \Phi(x,y,t;u,\varphi) \le \Big(\sqrt{(1-x)(1-y)} + \sqrt{xy} \Big)^2 \\ &\le \Big(\frac{1-x+1-y}{2} + \frac{x+y}{2} \Big)^2 = 1 \end{aligned}$$

for $0 \le x,y,u,t \le 1$. Whence - by remark (d) above - the sequence in (8) is completely monotonic. This finishes the proof of Lemma 4, and settles the proof of Theorem 2.

Remark. For ultraspherical polynomials, the representation formula of Lemma 5 has the following limit as $\alpha \to \beta = \lambda - \frac{1}{2}$,

$$\frac{C_n^{(\lambda)}(x)}{C_n^{(\lambda)}(1)} = \frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\lambda)} \int_0^\pi \Big[x + \sqrt{x^2-1}\ \cos\varphi\Big]^n \big(\sin\varphi\big)^{2\lambda-1}\, d\varphi\,, \quad \lambda > 0\,.$$

In particular, for the Legendre polynomials P_n normalized by $P_n(1) = 1$ (case $\lambda = \frac{1}{2}$), we recover the Laplace integral

$$P_n(x) = \frac{1}{\pi} \int_0^\pi \Big[x + \sqrt{x^2-1}\ \cos\varphi\Big]^n d\varphi\,.$$

For details, see again [2] and [21].

5. The Quasi-Interpolants

The following second order differential operator $\mathbf{U}_\mu$ plays a prominent role in our analysis,

$$\begin{aligned} -\mathbf{U}_\mu := &\sum_{i=1}^{d} \big(\omega_\mu(\mathbf{x})\big)^{-1} \frac{\partial}{\partial x_i}\Big\{ \omega_\mu(\mathbf{x})\, x_0 x_i\, \frac{\partial}{\partial x_i} \Big\} \\ &+ \sum_{1\le i<j\le d} \big(\omega_\mu(\mathbf{x})\big)^{-1} \Big(\frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_i}\Big)\Big\{ \omega_\mu(\mathbf{x})\, x_i x_j \Big(\frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_i}\Big) \Big\}\,. \end{aligned} \tag{10}$$

Here, as before, $\mathbf{x} = (x_0, x_1, \ldots, x_d)$ with $x_0 = 1 - x_1 - \cdots - x_d$. In the definition of the operator, we take the negative sign in order to have a positive spectrum.

Lemma 6. *The differential operator $\mathbf{U}_\mu$ is densely defined on the Hilbert space $\mathbf{H}$, and symmetric. We have*

$$\mathbf{U}_\mu(p_m) = m(m+d+|\mu|)\, p_m\,, \quad m \ge 0\,, \quad p_m \in \mathbf{E}_{m,\mu}\,,$$

i.e., the spaces $\mathbf{E}_{m,\mu}$ are also eigenspaces of $\mathbf{U}_\mu$. In particular,

$$\mathbf{U}_\mu(\mathbf{M}_{n,\mu}(f)) = \mathbf{M}_{n,\mu}(\mathbf{U}_\mu(f))$$

for $f \in C^2(\mathbf{S}^d)$.

We put $\mathbf{U}_{0,\mu} := \mathbf{I}$ and define the differential operators $\mathbf{U}_{\ell,\mu}$ of order 2ℓ by

$$\mathbf{U}_{\ell+1,\mu} := \frac{1}{(\ell+1)^2}\Big(\mathbf{U}_\mu - \ell(\ell+d+|\mu|)\mathbf{I} \Big) \, \mathbf{U}_{\ell,\mu} \, , \qquad \ell \geq 0 \, . \tag{11}$$

Thus, $\mathbf{U}_\mu = \mathbf{U}_{1,\mu}$. The operators are again densely defined and symmetric, with spectral properties as follows.

$$\mathbf{U}_{\ell,\mu}(p_m) = \frac{\sigma_{\ell,m,\mu}}{\ell!} \, p_m \, , \quad m \geq 0 \, , \quad p_m \in \mathbf{E}_{m,\mu} \, ,$$

with

$$\sigma_{0,m,\mu} = 1 \quad \text{and}$$
$$\sigma_{\ell+1,m,\mu} = \frac{m(m+d+|\mu|) - \ell(\ell+d+|\mu|)}{\ell+1} \, \sigma_{\ell,m,\mu} \, , \qquad \ell \geq 0 \, .$$

In particular, $\sigma_{\ell,m,\mu} = 0$ for $\ell > m$.

The relation to the spectrum of the Durrmeyer operators is explained by the following identity.

$$\gamma_{n,m,\mu} \sum_{\ell=0}^{m} \frac{\sigma_{\ell,m,\mu}}{n(n-1)\cdots(n-\ell+1)} = 1 \, , \qquad 0 \leq m \leq n \, . \tag{12}$$

From this,

$$\sum_{\ell=0}^{r} \frac{1}{\binom{n}{\ell}} \mathbf{U}_{\ell,\mu}(\mathbf{M}_{n,\mu} p) = p \, , \qquad p \in \mathbf{P}_r \, , \qquad 0 \leq r \leq n \, ,$$

and in particular

$$\left(\sum_{\ell=0}^{n} \frac{1}{\binom{n}{\ell}} \mathbf{U}_{\ell,\mu}(\mathbf{M}_{n,\mu}) \right)\Bigg|_{\mathbf{P}_n} = \mathbf{I}\Big|_{\mathbf{P}_n} \, , \qquad n \in \mathbb{N} \, .$$

These properties show that the following natural definition of (Bernstein-)Durrmeyer quasi-interpolant operators of order (r,n) arises,

$$\mathbf{M}_{n,\mu}^{(r)}(f) := \sum_{\ell=0}^{r} \frac{1}{\binom{n}{\ell}} \mathbf{U}_{\ell,\mu}(\mathbf{M}_{n,\mu}(f)) \, , \quad 0 \leq r \leq n \, . \tag{13}$$

These are polynomial operators with range $\mathbf{P}_n$, and they show the reproduction property

$$\mathbf{M}_{n,\mu}^{(r)}(p) = p \, , \qquad p \in \mathbf{P}_r \, .$$

In particular, $\mathbf{M}_{n,\mu}^{(0)} = \mathbf{M}_{n,\mu}$ and $\mathbf{M}_{n,\mu}^{(n)}|_{\mathbf{P}_n} = \mathbf{I}|_{\mathbf{P}_n}$. The spectral properties of these operators are given by

Theorem 7. *For all $n, m, r \in \mathbb{N}_0$, $0 \leq r \leq n$, the spaces $\mathbf{E}_{m,\mu}$ are eigenspaces of the operator $\mathbf{M}_{n,\mu}^{(r)}$. Namely, for $p_m \in \mathbf{E}_{m,\mu}$ we have*

$$\mathbf{M}_{n,\mu}^{(r)}(p_m) = \lambda_{n,m,\mu}^{(r)} \, p_m$$

with the eigenvalues

$$\lambda^{(r)}_{n,m,\mu} = \gamma_{n,m,\mu} \sum_{\ell=0}^{r} \frac{1}{\binom{n}{\ell}} \frac{\sigma_{\ell,m,\mu}}{\ell!} .$$

In particular, $\lambda^{(r)}_{n,m,\mu} = 1$ *for* $m \le r \le n$, *while* $\lambda^{(r)}_{n,m,\mu} = 0$ *for* $m > n$. *Also,*

$$\lambda^{(r)}_{n,m,\mu} = 1 - \gamma_{n,m,\mu} \sum_{\ell=r+1}^{m} \frac{1}{\binom{n}{\ell}} \frac{\sigma_{\ell,m,\mu}}{\ell!} , \qquad r < m \le n ,$$

and, for fixed m,

$$\lim_{n\to\infty} \lambda^{(r)}_{n,m,\mu} = 1 .$$

The latter properties follow from (12) and (4). Some further identities will be used in the sequel. The basic lemma here is

Lemma 8. *For* $n, m, r \in \mathbb{N}_0$, $0 \le r \le n$, *the following difference equation holds true:*

$$\lambda^{(r)}_{n,m,\mu} - \lambda^{(r)}_{n-1,m,\mu} = \frac{r+1}{n+d+|\mu|} \frac{1}{\binom{n}{r+1}} \frac{\sigma_{r+1,m,\mu}}{(r+1)!} \gamma_{n,m,\mu} .$$

From this we find two important identities. The first one is

$$\mathbf{M}^{(r)}_{n,\mu} - \mathbf{M}^{(r)}_{n-1,\mu} = \frac{r+1}{n+d+|\mu|} \frac{1}{\binom{n}{r+1}} \mathbf{U}_{r+1,\mu}(\mathbf{M}_{n,\mu}) . \tag{14}$$

This is a difference equation with respect to the index n. The second identity follows from this,

$$\begin{aligned} \mathbf{M}^{(r)}_{n,\mu} &= \mathbf{M}^{(r-1)}_{n,\mu} + \frac{1}{\binom{n}{r}} \mathbf{U}_{r,\mu}(\mathbf{M}_{n,\mu}) \\ &= \frac{1}{r} \left\{ (n+d+|\mu|+r)\, \mathbf{M}^{(r-1)}_{n,\mu} - (n+d+|\mu|)\, \mathbf{M}^{(r-1)}_{n-1,\mu} \right\} . \end{aligned}$$

This is a recurrence relation with respect to the superscript r showing that higher order quasi-interpolants are built from lower order ones through successive 'linear interpolation'. It can be used in order to show the following identity which writes the quasi-interpolants as a linear combination of Durrmeyer operators, viz. using induction with respect to r we find

Theorem 9. *For* $0 \le r \le n$ *the following identity holds true:*

$$\mathbf{M}^{(r)}_{n,\mu} = \sum_{\ell=0}^{r} (-1)^{\ell} \binom{r}{\ell} \binom{n+d+|\mu|+r-\ell}{r} \mathbf{M}_{n-\ell,\mu} .$$

6. The Bernstein Inequality

In order to find an upper bound for the norm of the operator $\mathbf{U}_{r,\mu}(\mathbf{M}_{n,\mu})$, we make use of the identity

$$\frac{1}{\binom{n}{r}}\mathbf{U}_{r,\mu}(\mathbf{M}_{n,\mu}) = \frac{n+d+|\mu|}{r}\sum_{\ell=0}^{r}(-1)^{\ell}\binom{r}{\ell}\binom{n+d+|\mu|+r-\ell-1}{r-1}\mathbf{M}_{n-\ell,\mu}\,, \tag{15}$$

which follows from equation (14) and Theorem 9 by straightforward calculation. We relate this to the modified operators

$$\{\mathbf{T}_{n,\mu}(f)\}(\mathbf{y}) = \int_{\mathbf{S}^d}\omega_\mu(\mathbf{x})\; f(\mathbf{x})\; T_{n,\mu}(\mathbf{x},\mathbf{y})\; d\mathbf{x}\,,$$

with the kernel $T_{n,\mu}$ defined in (5), and to the 'difference' operators

$$\mathbf{T}^{(r)}_{n,\mu} := (-1)^r\Delta^r\mathbf{T}_{n,\mu} = \sum_{\ell=0}^{r}(-1)^{\ell}\binom{r}{\ell}\mathbf{T}_{n+\ell,\mu}\,, \quad r\geq 0\,. \tag{16}$$

By equation (5)

$$\mathbf{T}_{n,\mu} = \frac{\Gamma(n+\underline{\mu}+1)}{\Gamma(n+d+|\mu|+1)}\,\mathbf{M}_{n,\mu}\,,$$

with $\underline{\mu} := \min_i \mu_i$, and according to Theorem 2, the operators $\mathbf{T}^{(r)}_{n,\mu}$ are positive.

Lemma 10. *For $0 \leq r \leq n$ we have*

$$\begin{aligned}&\frac{1}{\binom{n}{r}}\mathbf{U}_{r,\mu}(\mathbf{M}_{n,\mu})\\ &\quad= \frac{\Gamma(n+d+|\mu|+1)}{\Gamma(n+\underline{\mu}+1)}\sum_{\ell=0}^{r}(-1)^{\ell}\binom{n+\underline{\mu}}{\ell}\binom{d+|\mu|-\underline{\mu}+r-1}{r-\ell}\mathbf{T}^{(\ell)}_{n-\ell,\mu}\,.\end{aligned}$$

Proof. We evaluate the right-hand side by inserting

$$\mathbf{T}^{(\ell)}_{n-\ell,\mu} = \sum_{\lambda=0}^{\ell}(-1)^{\ell-\lambda}\binom{\ell}{\lambda}\frac{\Gamma(n-\lambda+\underline{\mu}+1)}{\Gamma(n-\lambda+d+|\mu|+1)}\,\mathbf{M}_{n-\lambda,\mu}\,,$$

which follows from (16). The result is

$$\begin{aligned}&\frac{\Gamma(n+d+|\mu|+1)}{\Gamma(n+\underline{\mu}+1)}\sum_{\ell=0}^{r}(-1)^{\ell}\binom{n+\underline{\mu}}{\ell}\binom{d+|\mu|-\underline{\mu}+r-1}{r-\ell}\mathbf{T}^{(\ell)}_{n-\ell,\mu}\\ &\qquad\qquad= \frac{n+d+|\mu|}{r}\sum_{\lambda=0}^{r}(-1)^{\lambda}c^{(r)}_{\lambda}\,\mathbf{M}_{n-\lambda,\mu}\end{aligned}$$

with

$$\begin{aligned}
c_\lambda^{(r)} &= \frac{r}{n+d+|\mu|}\binom{n+d+|\mu|}{\lambda}\sum_{\ell=\lambda}^{r}\binom{n-\lambda+\underline{\mu}}{\ell-\lambda}\binom{d+|\mu|-\underline{\mu}+r-1}{r-\ell} \\
&= \frac{r}{n+d+|\mu|}\binom{n+d+|\mu|}{\lambda}\sum_{\ell=0}^{r-\lambda}\binom{n-\lambda+\underline{\mu}}{\ell}\binom{d+|\mu|-\underline{\mu}+r-1}{r-\lambda-\ell} \\
&= \frac{r}{n+d+|\mu|}\binom{n+d+|\mu|}{\lambda}\binom{n-\lambda+d+|\mu|+r-1}{r-\lambda} \\
&= \binom{r}{\lambda}\binom{n+d+|\mu|+r-\lambda-1}{r-1} .
\end{aligned}$$

This proves the lemma via identity (15). □

We are now prepared for our main result in this section.

Theorem 11. *Let $\mu = (\mu_0, \mu_1, \ldots, \mu_d) \in \mathbb{R}^{d+1}$ with $\underline{\mu} := \min \mu_i \geq -\frac{1}{2}$. Then, for $1 \leq p \leq \infty$ and $n, r \in \mathbb{N}_0$, $0 \leq r \leq n$, we have*

$$\left\|\mathbf{U}_{r,\mu}(\mathbf{M}_{n,\mu}(f))\right\|_{p,\mu} \leq 2^r \binom{n}{r}\binom{d+|\mu|-\underline{\mu}+r-1}{r}\|f\|_{p,\mu} .$$

Proof. Following the standard approach in the literature, we look at the dual pairing $\left(L^p_\mu(\mathbf{S}^d), L^q_\mu(\mathbf{S}^d)\right)$, with $\frac{1}{p}+\frac{1}{q}=1$, and the corresponding bilinear form

$$\langle\, f \mid g\,\rangle = \int_{\mathbf{S}^d} \omega_\mu(\mathbf{x})\; f(\mathbf{x})\; g(\mathbf{x})\; d\mathbf{x} .$$

Since the operators $\mathbf{U}_{r,\mu}$ and $\mathbf{M}_{n,\mu}$ commute, and are symmetric, the result for $p=\infty$ implies the result for $p=1$, and the Riesz-Thorin interpolation theorem then provides the full result for all p. Thus, it is sufficient to consider $p=\infty$.

Since the operators $\mathbf{T}^{(\ell)}_{n-\ell,\mu}$ are positive,

$$\|\mathbf{T}^{(\ell)}_{n-\ell,\mu}\|_{C(\mathbf{S}^d)\to C(\mathbf{S}^d)} = \|\mathbf{T}^{(\ell)}_{n-\ell,\mu}(\mathbf{1})\|_\infty . \tag{17}$$

For $\ell=0$, we find

$$\mathbf{T}^{(0)}_{n,\mu}(\mathbf{1}) = \frac{\Gamma(n+\underline{\mu}+1)}{\Gamma(n+d+|\mu|+1)}\,\mathbf{M}_{n,\mu}(\mathbf{1}) = \frac{\Gamma(n+\underline{\mu}+1)}{\Gamma(n+d+|\mu|+1)}\,\mathbf{1} ,$$

while for $\ell > 0$,

$$\begin{aligned}
\mathbf{T}^{(\ell)}_{n-\ell,\mu}(\mathbf{1}) &= \sum_{\lambda=0}^{\ell}(-1)^{\ell-\lambda}\binom{\ell}{\lambda}\mathbf{T}^{(0)}_{n-\lambda,\mu}(\mathbf{1}) \\
&= \sum_{\lambda=0}^{\ell}(-1)^{\ell-\lambda}\binom{\ell}{\lambda}\frac{\Gamma(n-\lambda+\underline{\mu}+1)}{\Gamma(n-\lambda+d+|\mu|+1)}\,\mathbf{1} .
\end{aligned}$$

Now

$$\sum_{\lambda=0}^{\ell}(-1)^{\ell-\lambda}\binom{\ell}{\lambda}\frac{\Gamma(n-\lambda+\underline{\mu}+1)}{\Gamma(n-\lambda+d+|\mu|+1)}$$

$$=\frac{\ell!\,\Gamma(n-\ell+\underline{\mu}+1)}{\Gamma(n+d+|\mu|+1)}\sum_{\lambda=0}^{\ell}(-1)^{\ell-\lambda}\binom{n-\lambda+\underline{\mu}}{\ell-\lambda}\binom{n+d+|\mu|}{\lambda}$$

$$=\frac{\ell!\,\Gamma(n-\ell+\underline{\mu}+1)}{\Gamma(n+d+|\mu|+1)}\sum_{\lambda=0}^{\ell}\binom{-n+\ell-\underline{\mu}-1}{\ell-\lambda}\binom{n+d+|\mu|}{\lambda}$$

$$=\frac{\ell!\,\Gamma(n-\ell+\underline{\mu}+1)}{\Gamma(n+d+|\mu|+1)}\binom{d+|\mu|-\underline{\mu}+\ell-1}{\ell},$$

whence the operator norm in (17) is given by

$$\|\mathbf{T}^{(\ell)}_{n-\ell,\mu}\|_{C(\mathbb{S}^d)\to C(\mathbb{S}^d)}=\frac{\ell!\,\Gamma(n-\ell+\underline{\mu}+1)}{\Gamma(n+d+|\mu|+1)}\binom{d+|\mu|-\underline{\mu}+\ell-1}{\ell},\quad 0\le\ell\le n. \tag{18}$$

The proof is finished by an application of Lemma 10. We have

$$\frac{1}{\binom{n}{r}}\|\mathbf{U}_{r,\mu}(\mathbf{M}_{n,\mu})\|_{C(\mathbb{S}^d)\to C(\mathbb{S}^d)}$$

$$\le\frac{\Gamma(n+d+|\mu|+1)}{\Gamma(n+\underline{\mu}+1)}\sum_{\ell=0}^{r}\binom{n+\underline{\mu}}{\ell}\binom{d+|\mu|-\underline{\mu}+r-1}{r-\ell}\|\mathbf{T}^{(\ell)}_{n-\ell,\mu}\|_{C(\mathbb{S}^d)\to C(\mathbb{S}^d)}$$

and the bound, by (18), takes the form

$$\sum_{\ell=0}^{r}\binom{n+\underline{\mu}}{\ell}\binom{d+|\mu|-\underline{\mu}+r-1}{r-\ell}\frac{\ell!\,\Gamma(n-\ell+\underline{\mu}+1)}{\Gamma(n+\underline{\mu}+1)}\binom{d+|\mu|-\underline{\mu}+\ell-1}{\ell}$$

$$=\sum_{\ell=0}^{r}\binom{d+|\mu|-\underline{\mu}+r-1}{r-\ell}\binom{d+|\mu|-\underline{\mu}+\ell-1}{\ell}$$

$$=\binom{d+|\mu|-\underline{\mu}+r-1}{r}\sum_{\ell=0}^{r}\binom{r}{\ell}$$

$$=2^r\binom{d+|\mu|-\underline{\mu}+r-1}{r}.$$

This proves the theorem and prepares us to state a few implications thereof. □

Theorem 12. *For given $n,r,\rho\in\mathbb{N}_0$, $0\le r,\rho\le n$, and $1\le p\le\infty$, we have*

$$\|\mathbf{U}_{\rho,\mu}(\mathbf{M}^{(r)}_{n,\mu}(f))\|_{p,\mu}\le c_{r,\rho}\binom{n}{\rho}\|f\|_{p,\mu},\quad f\in L^p_\mu(\mathbb{S}^d),$$

where the constant $c_{r,\rho}=c_{r,\rho,\mu,d}$ depends only on r,ρ,μ and d. Here, $\mu\in\mathbb{R}^{d+1}$ with $\underline{\mu}\ge-\frac{1}{2}$.

Proof. From equation (11) we find a representation

$$\mathbf{U}_{\rho,\mu}\mathbf{U}_{\ell,\mu} = \sum_{k=0}^{\rho+\ell} \alpha_{\ell,\rho}^{(k)}\, \mathbf{U}_{k,\mu} \,.$$

We insert this in the definition (13) of the quasi-interpolant to give

$$\mathbf{U}_{\rho,\mu}(\mathbf{M}_{n,\mu}^{(r)}(f)) = \sum_{\ell=0}^{r} \frac{1}{\binom{n}{\ell}} \sum_{k=0}^{\rho+\ell} \alpha_{\ell,\rho}^{(k)}\, \mathbf{U}_{k,\mu}(\mathbf{M}_{n,\mu}(f)) \,.$$

An application of Theorem 11 finishes the proof. □

The special case $\rho = 0$ is of particular importance, since this shows that our quasi-interpolants are uniformly bounded.

Theorem 13. *For given $n, r \in \mathbb{N}_0$, $0 \le r \le n$, $\mu \in \mathbb{R}^{d+1}$ with $\underline{\mu} \ge -\frac{1}{2}$ and $1 \le p \le \infty$, we have*

$$\left\| \mathbf{M}_{n,\mu}^{(r)}(f)) \right\|_{p,\mu} \le c_{r,0} \left\| f \right\|_{p,\mu} , \quad f \in L_{\mu}^{p}(\mathbf{S}^d) \,, \tag{19}$$

where

$$c_{r,0} := \sum_{\ell=0}^{r} 2^{\ell} \binom{d+|\mu| - \underline{\mu} + \ell - 1}{\ell} \,. \tag{20}$$

Another way to get this result is via an extension of Lemma 10 by writing the quasi-interpolants as linear combinations of the positive operators $\mathbf{T}_{n-\ell,\mu}^{(\ell)}$. We state this important identity as a separate result.

Theorem 14. *For $0 \le r \le n$ we have*

$$\mathbf{M}_{n,\mu}^{(r)} = \frac{\Gamma(n+d+|\mu|+1)}{\Gamma(n+\underline{\mu}+1)} \sum_{\ell=0}^{r} (-1)^{\ell} \binom{n+\underline{\mu}}{\ell} \binom{d+|\mu|-\underline{\mu}+r}{r-\ell} \mathbf{T}_{n-\ell,\mu}^{(\ell)} \,.$$

This follows from Lemma 10 via (13), and provides the following expression for the constant in (20).

$$\begin{aligned} c_{r,0} &= \frac{\Gamma(n+d+|\mu|+1)}{\Gamma(n+\underline{\mu}+1)} \sum_{\ell=0}^{r} \binom{n+\underline{\mu}}{\ell} \binom{d+|\mu|-\underline{\mu}+r}{r-\ell} \mathbf{T}_{n-\ell,\mu}^{(\ell)}(\mathbf{1}) \\ &= \binom{d+|\mu|-\underline{\mu}+r-1}{r} \sum_{\ell=0}^{r} \binom{r}{\ell} \frac{d+|\mu|-\underline{\mu}+r}{d+|\mu|-\underline{\mu}+\ell} \,. \end{aligned}$$

7. Convergence and Direct Results

The results of the previous section allow us to prove convergence. The first statement is a direct consequence of the uniform bound (19) for the quasi-interpolants. Namely,

$$\lim_{n\to\infty} \|f - \mathbf{M}_{n,\mu}^{(r)}(f)\|_{p,\mu} \to 0 , \quad n \to \infty ,$$

for $f \in L^p_\mu(\mathbf{S}^d)$ or $f \in C(\mathbf{S}^d)$, respectively, since the convergence holds for polynomials, due to Theorem 7. Inserting this in (14) leads to the following expansion, via a telescoping argument.

Lemma 15. *For $n, r \in \mathbb{N}_0$, $0 \le r \le n$, and $\underline{\mu} \ge -\frac{1}{2}$ we have*

$$f - \mathbf{M}_{n,\mu}^{(r)}(f) = \sum_{\ell=n+1}^{\infty} \frac{r+1}{\ell + d + |\mu|} \frac{1}{\binom{\ell}{r+1}} \mathbf{U}_{r+1,\mu}(\mathbf{M}_{\ell,\mu}(f)) \tag{21}$$

for $f \in L^p_\mu(\mathbf{S}^d)$ or $f \in C(\mathbf{S}^d)$, respectively, with convergence in the norm.

Remark. For smooth functions $f \in C^{2r+2}(\mathbf{S}^d)$ the error expansion (21) holds true for general μ with the usual restriction $\underline{\mu} > -1$, see [4], Theorem 5, where a proof is given with no recourse to the Bernstein inequality.

This error expansion is the basis for several quantitative convergence results. The arguments are more or less standard. We start with the Jackson-Favard type estimate.

Theorem 16. *Let $n, r \in \mathbb{N}_0$, $0 \le r \le n$, and $\mu \in \mathbb{R}^{d+1}$ with $\underline{\mu} > -1$. Then, for $f \in C^{2r+2}(\mathbf{S}^d)$,*

$$\left\|f - \mathbf{M}_{n,\mu}^{(r)}(f)\right\|_{p,\mu} \le \frac{C_{n,r,d,\mu}}{\binom{n}{r+1}} \left\|\mathbf{U}_{r+1,\mu}(f)\right\|_{p,\mu}$$

with

$$C_{n,r,d,\mu} := \sum_{\ell=n+1}^{\infty} \frac{r+1}{\ell + d + |\mu|} \frac{\binom{n}{r+1}}{\binom{\ell}{r+1}} \to 1 , \quad n \to \infty . \tag{22}$$

Proof. Under the smoothness assumption for f the differential operator commutes with the Durrmeyer operator, and we can estimate the series in (21) by using

$$\left\|\mathbf{U}_{r+1,\mu}(\mathbf{M}_{\ell,\mu}(f))\right\|_{p,\mu} = \left\|\mathbf{M}_{\ell,\mu}(\mathbf{U}_{r+1,\mu}(f))\right\|_{p,\mu} \le \left\|\mathbf{U}_{r+1,\mu}(f)\right\|_{p,\mu} .$$

□

The second quantitative result is of Voronovskaja type.

Theorem 17. *Let $r \in \mathbb{N}$ and $\mu \in \mathbb{R}^{d+1}$ with $\underline{\mu} > -1$. Then, for $f \in C^{2r+2}(\mathbf{S}^d)$,*

$$\begin{aligned}\lim_{n\to\infty} \binom{n}{r+1} \left\{f - \mathbf{M}_{n,\mu}^{(r)}(f)\right\}(\mathbf{x}) &= \lim_{n\to\infty} C_{n,r,d,\mu} \left\{\mathbf{U}_{r+1,\mu}(f)\right\}(\mathbf{x}) \\ &= \left\{\mathbf{U}_{r+1,\mu}(f)\right\}(\mathbf{x}) ,\end{aligned}$$

where the convergence is uniform in $\mathbf{S}^d$.

Proof. Using Lemma 15 and (22), we find

$$\begin{aligned}\left\{\binom{n}{r+1}\{f - \mathbf{M}^{(r)}_{n,\mu}(f)\} - C_{n,r,d,\mu}\,\mathbf{U}_{r+1,\mu}(f)\right\} \\ = \sum_{\ell=n+1}^{\infty} \frac{r+1}{\ell+d+|\mu|}\frac{\binom{n}{r+1}}{\binom{\ell}{r+1}}\{\mathbf{U}_{r+1,\mu}(\mathbf{M}_{\ell,\mu}(f)) - \mathbf{U}_{r+1,\mu}(f)\}\,.\end{aligned} \tag{23}$$

The result follows from

$$\lim_{\ell\to\infty}\{\mathbf{U}_{r+1,\mu}(\mathbf{M}_{\ell,\mu}(f)) - \mathbf{U}_{r+1,\mu}(f)\} = \lim_{\ell\to\infty}\{\mathbf{M}_{\ell,\mu}(\mathbf{U}_{r+1,\mu}(f)) - \mathbf{U}_{r+1,\mu}(f)\} = 0$$

in the uniform norm. □

Our third quantitative convergence result is of 'strong' Voronovskaja type. It follows from another application of Lemma 15 (case $r = 0$) on the right-hand side of (23).

Theorem 18. *Let $n, r \in \mathbb{N}_0$, $0 \le r \le n$, and $\mu \in \mathbb{R}^{d+1}$ with $\underline{\mu} > -1$. Then, for $f \in C^{2r+2}(\mathbf{S}^d)$,*

$$\begin{aligned}&\binom{n}{r+1}\{f - \mathbf{M}^{(r)}_{n,\mu}(f)\} - C_{n,r,d,\mu}\mathbf{U}_{r+1,\mu}(f) \\ &= -\sum_{\ell_0=n+1}^{\infty} \frac{r+1}{\ell_0+d+|\mu|}\frac{\binom{n}{r+1}}{\binom{\ell_0}{r+1}} \sum_{\ell_1=\ell_0+1}^{\infty} \frac{1}{\ell_1(\ell_1+d+|\mu|)}\,\mathbf{U}_{r+1,\mu}\{\mathbf{U}_{1,\mu}(\mathbf{M}_{\ell_1,\mu}(f))\}\,,\end{aligned}$$

with convergence in the uniform norm. In particular, for $f \in C^{2r+4}(\mathbf{S}^d)$,

$$\binom{n}{r+1}\{f - \mathbf{M}^{(r)}_{n,\mu}(f)\} - C_{n,r,d,\mu}\mathbf{U}_{r+1,\mu}(f) = O(n^{-1}) \quad as \quad n \to \infty\,.$$

The idea can be iterated in order to find an error expansion in powers of n^{-1}. We omit the details but rather give a final result in terms of the K-functional

$$K(f,t) := K_{\ell,\mu,p}(f,t) := \inf_{g\in C^{2\ell}(\mathbf{S}^d)}\left\{\,\|f-g\|_{p,\mu} + t\,\|\mathbf{U}_{\ell,\mu}(g)\|_{p,\mu}\,\right\}, \quad \ell > 0\,.$$

Theorem 19. *Let $n, r \in \mathbb{N}_0$, $0 \le r \le n$, and $\mu \in \mathbb{R}^{d+1}$ with $\underline{\mu} \ge -\frac{1}{2}$. Then, for $f \in L^p_\mu(\mathbf{S}^d)$ or $f \in C(\mathbf{S}^d)$, respectively,*

$$\|f - \mathbf{M}^{(r)}_{n,\mu}(f)\|_{p,\mu} \le c_r\, K_{r+1,\mu,p}\left(f, \binom{n}{r+1}^{-1}\right).$$

The *proof* is again standard. Since

$$f - \mathbf{M}_{n,\mu}^{(r)}(f) = (f-g) - \mathbf{M}_{n,\mu}^{(r)}(f-g) + (g - \mathbf{M}_{n,\mu}^{(r)}(g)) ,$$

we can apply Theorems 13 and 16 to find

$$\|f - \mathbf{M}_{n,\mu}^{(r)}(f)\|_{p,\mu} \le (1 + c_{r,0})\|f - g\|_{p,\mu} + \frac{C_{n,r,d,\mu}}{\binom{n}{r+1}}\|\mathbf{U}_{r+1,\mu}(g)\|_{p,\mu}$$

for arbitrary $g \in C^{2r+2}(\mathbf{S}^d)$. Taking the infimum with respect to g proves the result with constant

$$c_r := \max(\, 1 + c_{r,0} \;,\; \sup_n \; C_{n,r,d,\mu} \,) \, .$$

8. Additional Notes

The Durrmeyer operators (2) were introduced by Durrmeyer in his thesis [15]. The study of their approximation properties was initiated by Derriennic in several papers, and later studied by many authors. The spectral properties of Theorem 1 appear in [10], [11] and [12] for the unweighted case, in [6] and [7] for the weighted univariate case, and in [14] for the weighted multivariate case. See also Chapter 5.2 in Paltanea's recent book [23], and the references given there.

The statement of Theorem 2 is original, as is its application in Section 6. For the properties and formulas for hypergeometric functions, we refer to standard tablework, such as [1] and [22]. For the Laplace type integrals, a direct approach can be taken from Koornwinder [21] and, in particular, from Askey's [2] elegant work related to this. It is also worthwhile to consult Szegö's [27] chapter on Jacobi polynomials.

The differential operator $\mathbf{U}_\mu = \mathbf{U}_{1,\mu}$ in (10) and its powers play a prominent role in the study of direct and inverse theorems for the Durrmeyer operator; see again the papers by Derriennic, Berens and Xu, and Ditzian, where also the spectral properties of Lemma 6 can be found. The higher order case $\mathbf{U}_{\ell,\mu}$ was first investigated in [4], with different notation. In the univariate (unweighted) case, these operators are called Legendre differential operators in Heilmann's Habilitationsschrift [16]. In the present paper, we have chosen the recursive definition (11) leading to a product representation for $\mathbf{U}_{\ell,\mu}$ which was communicated to us by Michael Felten.

The quasi-interpolants (13) were introduced in [17], for the unweighted case. The weighted case was considered in [18] and [4], where also the statements of Theorem 7 and Lemma 8 can be found. The expression of the quasi-interpolants in terms of Durrmeyer operators in Theorem 9, however, is new. This latter result embeds our operators into the class of quasi-interpolants constructed as linear combinations of Durrmeyer operators. However, this approach is usually less direct and less explicit than ours. We refer to work of Derriennic [13] again, and to Sablonnière [24], [25] and Heilmann [16]. Sablonnière's recent paper [26] presents a good account on these constructions.

The Bernstein inequality, Theorem 11, is again original. It confirms our conjecture posed in [3] where a different proof was given for the special case $d = 1$ and $\mu = (0,0)$. We expect that the representation of the quasi-interpolants as a linear combination of the positive operators $\mathbf{T}^{(\ell)}_{n-\ell,\mu}$ as given in Theorem 14 will have far-reaching applications. Last not least, the case $\mathbf{M}^{(n)}_{n,\mu}$ is expected to lead to a new representation of polynomial reproducing kernels of full order on the simplex $\mathbf{S}^d$, with connections to addition theorems for orthogonal polynomials.

The direct results in Section 7 are more or less immediate consequences of the Bernstein inequality. We refer to our paper [4], and again to earlier work by Derriennic [13], Berens et al. [5]-[7] and Ditzian [14]. However, we still do not settle the natural question of 'converse' or even 'strong converse' theorems for our quasi-interpolants. Concerning this, the paper of Chen, Ditzian and Ivanov [8], the refined techniques of Knoop and Zhou in [19], [20] and Zhou's Habilitationsschrift [29] might be helpful.

The initial motivation for our studies came from the article by Chui et al. [9], in which univariate quasi-interpolants on irregular partitions of a bounded interval $I \subset \mathbb{R}$ were constructed as linear combinations of B-splines. In their approach, the quasi-interpolants $M_n^{(r)}$ (for the unweighted case) are the starting point for an inductive method of knot insertion. The quasi-interpolants in [9] give rise to the definition of "approximate duals" of B-splines, which are the centerpiece for their construction of nonstationary wavelet frames.

References

1. Andrews, G. E., Askey, R. and Roy, R., *Special Functions*, Cambridge University Press, Cambridge, 2000.
2. Askey, R., Jacobi polynomials, I. New proofs of Koornwinder's Laplace type integral representation and Bateman's bilinear sum, *SIAM J. Math. Anal.* **5** (1974), 119–124.
3. Berdysheva, E., Jetter, K. and Stöckler, J., Bernstein-Durrmeyer type quasi-interpolants on intervals, in: *Approximation Theory* (D. K. Dimitrov, G. Nikolov, R. Uluchev, Eds.), Marin Drinov Academic Publishing House, Sofia, 2004, pp. 32–42.
4. Berdysheva, E., Jetter, K. and Stöckler, J., New polynomial preserving operators on the simplex: direct results, *J. Approx. Theory* **131** (2004), 59–73.
5. Berens, H., Schmid, H. J. and Xu, Y., Bernstein-Durrmeyer polynomials on a simplex, *J. Approx. Theory* **68** (1992), 247–261.
6. Berens, H. and Xu, Y., On Bernstein-Durrmeyer polynomials with Jacobi-weights, in: *Approximation Theory and Functional Analysis* (C. K. Chui, Ed.), Academic Press, Boston, 1991, pp. 25–46.

7. Berens, H. and Xu, Y., On Bernstein-Durrmeyer polynomials with Jacobi-weights: the cases $p = 1$ and $p = \infty$, *Israel Math. Conf. Proc.* **4** (1991), 51–62.
8. Chen, W., Ditzian, Z. and Ivanov, K., Strong converse inequality for the Bernstein-Durrmeyer operator, *J. Approx. Theory* **75** (1993), 25–43.
9. Chui, C. K., He, W. and Stöckler, J., Nonstationary tight wavelet frames, I: bounded intervals, *Applied Comp. Harmonic Analysis* **17** (2004), 141–197.
10. Derriennic, M.-M., Sur l'approximation de fonctions intégrables sur $[0, 1]$ par des polynômes de Bernstein modifiés, *J. Approx. Theory* **31** (1981), 325–343.
11. Derriennic, M.-M., Polynômes de Bernstein modifiés sur un simplex T de R^l, problème des moments, in: *Proc. Polynômes Orthogonaux et Applications*, Bar-le-Duc, LNM 1171, Springer-Verlag, 1984, pp. 296–301.
12. Derriennic, M.-M., On multivariate approximation by Bernstein-type polynomials, *J. Approx. Theory* **45** (1985), 155–166.
13. Derriennic, M.-M., Linear combinations of derivatives of Bernstein-type polynomials on a simplex, *Coll. Math. Soc. Janos Bolyai* **58** (1990), 197–220.
14. Ditzian, Z., Multidimensional Jacobi-type Bernstein-Durrmeyer operators, *Acta Sci. Math. (Szeged)* **60** (1995), 225–243.
15. Durrmeyer, J.-L., Une formule d'inversion de la transformée de Laplace: Applications à la théorie des moments, Thèse de 3e cycle, Faculté des Sciences de l'Université de Paris, 1967.
16. Heilmann, M., *Erhöhung der Konvergenzgeschwindigkeit bei der Approximation von Funktionen mit Hilfe von Linearkombinationen spezieller positiver linearer Operatoren*, Habilitationsschrift, Universität Dortmund, 1992.
17. Jetter, K. and Stöckler, J., An identity for multivariate Bernstein polynomials, *Computer Aided Geometric Design* **20** (2003), 563–577.
18. Jetter, K. and Stöckler, J., New polynomial preserving operators on the simplex, Ergebnisberichte Angewandte Mathematik, Nr. 242, Universität Dortmund (November 2003).
19. Knoop, H.-B. and Zhou, X. L., The lower estimate for linear positive operators, I, *Constr. Approximation* **11** (1995), 53–66.
20. Knoop, H.-B. and Zhou, X. L., The lower estimate for linear positive operators, II, *Results Math.* **25** (1994), 315–330.
21. Koornwinder, T., Jacobi polynomials, II. An analytic proof of the product formula, *SIAM J. Math. Anal.* **5** (1974), 125–137.
22. Magnus, W., Oberhettinger, F. and Soni, R. P., *Formulas and Theorems for the Special Functions of Mathematical Physics*, 3rd ed., Springer-Verlag, Berlin, 1966.
23. Paltanea, R. , *Approximation Theory Using Positive Linear Operators*, Birkhäuser-Verlag, Boston, 2004.
24. Sablonnière, P., Bernstein-type quasi-interpolants, in: *Curves and Surfaces* (P.-J. Laurent, A. Le Méhauté, L. L. Schumaker, Eds.), Academic Press, Boston, 1991, pp. 421-426.
25. Sablonnière, P., Representation of quasi-interpolants as differential operators and applications, in: *New Developments in Approximation Theory*

(M. D. Buhmann, M. Felten, D. H. Mache and M. W. Müller, Eds.), Birkhäuser-Verlag, Basel, 1999, pp. 233–253.

26. Sablonnière, P., Recent progress on univariate and multivariate polynomial and spline quasi-interpolants, Prépublication 04-20, INSA Rennes, March 2004.
27. Szegö, G., *Orthogonal Polynomials*, 4th ed., Amer. Math. Soc., Providence, RI, 1975.
28. Widder, D. V., *The Laplace Transform*, 8th printing, Princeton University Press, Princeton, 1972.
29. Zhou, X., *Approximationsordnung und Regularität von Differentialoperatoren*, Habilitationsschrift, Gerhard-Mercator-Universität Duisburg, 1996.