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Coherent States from Nonunitary Representations

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For Ina
with thanks for her patience
and moral support.

Abstract

We try to obtain a transformation of wavelet-type with a reproducing property on the unit circle using the group of Möbius transformations. Since the natural unitary representations of this group are not square integrable, we use a nonunitary representation together with its contragredient representation. We also present conditions under which this approach works in general.

Key words: coherent states, nonunitary representation, Möbius transformation, reproducing property

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1. Introduction

1.1. Coherent states from unitary representations

In recent years, wavelets had enormous success in a broad variety of disciplines, practically in every area where large quantities of data need to be denoised or compressed. Wavelet bases turned out to be a versatile tool, and not only in applications, but also for mathematical problems. The central idea is the use of a single shape, described by a function $\varphi : \mathbb{R} \to \mathbb{R}$, and to vary its position and scale. In mathematical terms, we apply the affine group of translation and dilation operators, given by

$$\left(\mathrm{T}_b \, \mathrm{D}_a \, \varphi\right)(x) = \frac{1}{\sqrt{a}} \, \varphi\left(\frac{x-b}{a}\right), \quad (a,b) \in \mathbb{R}^+ \times \mathbb{R}.$$

This enables us to study given data at various positions (by translation), like using a microscope at various resolutions (by dilation).

The importance of the continuous wavelet transformation lies in the reproducing equation

$$\iint\limits_{\mathbb{R}^+ \times \mathbb{R}} \left\langle f(t), \frac{1}{\sqrt{a}} \, \varphi\left(\frac{t-b}{a}\right) \right\rangle \frac{1}{\sqrt{a}} \, \varphi\left(\frac{x-b}{a}\right) \frac{da\, db}{a^2} = C_\varphi \, f(x),$$

where C_φ is a constant that can be made equal one by simply replacing φ by an appropriate multiple, which we now want to assume to be done.

To get a better understanding of the principle behind this equation, we compose the left hand side in two steps. First, there is the *analysis operator* which maps a function $f \in \boldsymbol{L}^2(\mathbb{R})$ to its wavelet transform given by

$$\left(\mathcal{W}_\varphi f\right)(a,b) = \left\langle f, \mathrm{T}_b \, \mathrm{D}_a \, \varphi \right\rangle = \int\limits_{\mathbb{R}} f(x) \, \overline{\frac{1}{\sqrt{a}} \, \varphi\left(\frac{x-b}{a}\right)} \, dx.$$

This is followed by the *synthesis operator*, the wavelet reconstruction

$$\iint\limits_{\mathbb{R}^+ \times \mathbb{R}} \left(\mathcal{W}_\varphi f\right)(a,b) \, \frac{1}{\sqrt{a}} \, \varphi\left(\frac{x-b}{a}\right) \frac{da\, db}{a^2} = f(x).$$

In a sense, we use the family $\left(\mathrm{T}_b \, \mathrm{D}_a \, \varphi\right)_{(a,b) \in \mathbb{R}^+ \times \mathbb{R}}$ as building blocks to generate the (arbitrary) function f, and the wavelet transform $\mathcal{W}_\varphi f$ tells us the appropriate weight for each block. The important point is that this family of building blocks shows a lot of structure, since it is generated from a single function φ by applying a group of operators. This is usually referred to as a family of *coherent states.*

There is a general mathematical principle behind this phenomenon, of which we want to give a brief overview. Details can be found, *e.g.*, in [18], [21], [23], and [24]. Given a group G and a Hilbert space $\boldsymbol{H}$, a *representation* of G on $\boldsymbol{H}$ is a group homomorphism $\rho : G \to \boldsymbol{GL}(\boldsymbol{H})$. The representation is *unitary*, if $\rho(g)$ is unitary for each $g \in G$, *i.e.*, if we have $\rho : G \to \boldsymbol{U}(\boldsymbol{H})$. For such a unitary representation ρ, we can define the analysis operator $T_{\rho,\varphi}$ via

$$\big(T_{\rho,\varphi} f\big)(g) = \langle f, \rho(g)\varphi \rangle ,$$

for some appropriate φ. Note that $T_{\rho,\varphi}$ maps each vector $f \in \boldsymbol{H}$ to a function on G, the "ρ-transform" of f, also denoted a *matrix coefficient* of ρ. Unitarity of ρ implies that $\|\rho(g)\varphi\|_{\boldsymbol{H}} = \|\varphi\|_{\boldsymbol{H}}$ for all $g \in G$, and therefore $|\langle f, \rho(g)\varphi\rangle| \le \|f\|_{\boldsymbol{H}} \|\varphi\|_{\boldsymbol{H}}$, so the function $T_{\rho,\varphi} f$ is bounded on G. If $T_{\rho,\varphi} f$ is even continuous for all φ and f, then we call ρ a ***continuous representation.***

A subspace $\boldsymbol{H}_1$ of $\boldsymbol{H}$ is ***ρ-invariant***, if $\rho(g)\,\boldsymbol{H}_1 \subseteq \boldsymbol{H}_1$ for all $g \in G$. The representation ρ is ***irreducible***, if the only ρ-invariant closed subspaces of $\boldsymbol{H}$ are the trivial ones, *i.e.*, $\{0\}$ and $\boldsymbol{H}$ itself. Otherwise, ρ is ***reducible.***

The (formal) adjoint of $T_{\rho,\varphi}$ is the synthesis operator $T^*_{\rho,\varphi}$, which maps an (integrable) function F on the group G back into our space $\boldsymbol{H}$ via

$$T^*_{\rho,\varphi} F = \int_G F(g)\, \rho(g)\varphi \, dm(g) .$$

For this integral to make sense at all, the group G has to be a topological group with a locally compact topology, since then, there exists a (left) Haar measure with respect to which we can integrate. Details on locally compact groups, left and right Haar measure, the modular function, and unimodularity can be found, *e.g.*, in [15], [19], or [25].

If we can choose φ in such a way that $T_{\rho,\varphi}$ maps $\boldsymbol{H}$ into $\boldsymbol{L}^2(G)$, *i.e.*, that $T_{\rho,\varphi} f$ is a square integrable function for each $f \in \boldsymbol{H}$, then we say that ρ is a ***square integrable representation***, and φ is ***admissible*** for ρ. In that case, we have

$$T_{\rho,\varphi} : \boldsymbol{H} \to \boldsymbol{L}^2(G) \qquad \text{and thus} \qquad T^*_{\rho,\varphi} : \boldsymbol{L}^2(G) \to \boldsymbol{H} ,$$

so we can concatenate the two operators to obtain the linear map $T^*_{\rho,\varphi} T_{\rho,\varphi} : \boldsymbol{H} \to \boldsymbol{H}$. If, furthermore, the representation ρ is irreducible, then $T_{\rho,\varphi}$ can be shown to be an isometry (after proper normalization of φ), *i.e.*, we have

$$\|f\|^2_{\boldsymbol{H}} = \int_G |\langle f, \rho(g)\varphi\rangle|^2 \, dm(g) .$$

This implies that $T_{\rho,\varphi}$ is a unitary operator, and thus we have the reproducing property

$$T^*_{\rho,\varphi} T_{\rho,\varphi} = \mathrm{Id}_{\boldsymbol{H}} ,$$

or, explicitly,

$$\int_G \langle f, \rho(g)\varphi\rangle \, \rho(g)\varphi \, dm(g) = f .$$

This means that we can reconstruct any element of $\boldsymbol{H}$ using the building blocks $(\rho(g)\varphi)_{g\in G}$, a family of ***coherent states from a unitary representation.***

It might help the intuitive understanding of what follows if we look at this construction as a generalization of an orthonormal basis. Consider a family of vectors $(v_i)_{i\in I}$ in a Hilbert space $\boldsymbol{H}$, and the associated analysis operator

$$T_v : \boldsymbol{H} \to \mathbb{C}^I\,, \quad x \mapsto \big(\langle x, v_i\rangle\big)_{i\in I}\,.$$

Its formal adjoint is the synthesis operator

$$T_v^* : \mathbb{C}^I \to \boldsymbol{H}\,, \quad (c_i)_{i\in I} \mapsto \textstyle\sum_{i\in I} c_i\, v_i\,,$$

which a priori is only defined for finitely supported sequences. If $(v_i)_{i\in I}$ is a Bessel family, though, which means that T_v is a bounded map into $\ell^2(I)$, then T_v^* is well-defined on all of $\ell^2(I)$. In particular, this is the case if $(v_i)_{i\in I}$ is an orthonormal system. If, furthermore, this system is complete in $\boldsymbol{H}$, *i.e.*, an orthonormal basis, then T_v is an isometry between $\boldsymbol{H}$ and $\ell^2(I)$, since we have

$$\|x\|_{\boldsymbol{H}}^2 = \textstyle\sum_{i\in I}\big|\langle x, v_i\rangle\big|^2\,.$$

Again, this implies that T_v is unitary, and we have the reproducing property

$$T_v^*\, T_v = \mathrm{Id}_{\boldsymbol{H}}\,,$$

or, explicitly,

$$\textstyle\sum_{i\in I}\langle x, v_i\rangle\, v_i = x\,.$$

This simple setup exhibits a number of parallels to the one above: the Bessel property corresponds to ρ being square integrable; completeness somehow reflects irreducibility, since the latter ensures that the orbit of every nonzero element spans all of $\boldsymbol{H}$; and orthonormality corresponds to ρ being unitary, since together with the other properties, after an appropriate renormalization, this makes $T_{\rho,\varphi}$ a unitary operator.

1.2. Schur's Lemma and the reproducing property

The central principle behind the whole construction is given by Schur's Lemma, which in its basic form states the following.

Proposition 1 (Schur's Lemma). [23, Prop. 1.5] *A unitary representation $\rho : G \to \mathcal{U}(\boldsymbol{H})$ is irreducible if and only if the only bounded linear operators on $\boldsymbol{H}$ which commute with all $\rho(g)$, $g \in G$, are the multiples of the identity.*

A generalization of Schur's Lemma considers two unitary representations and their intertwining operators.

Proposition 2. [18, Prop. A.1] (i) *Consider an irreducible unitary representation $\rho_1 : G \to \mathcal{U}(\boldsymbol{H}_1)$ and another (not necessarily irreducible) unitary representation $\rho_2 : G \to \mathcal{U}(\boldsymbol{H}_2)$. Suppose that T is a closed linear operator from $\boldsymbol{H}_1$ to $\boldsymbol{H}_2$ with* *domain $\mathbb{D}$ dense in $\boldsymbol{H}_1$ and invariant under ρ_1, and assume that T intertwines ρ_1 with ρ_2. Then $\mathbb{D} = \boldsymbol{H}_1$, and T is a multiple of an isometry.*

(ii) *If, in addition, $\boldsymbol{H}_1 = \boldsymbol{H}_2$ and $\rho_1 = \rho_2$, then T is a multiple of the identity.*

For our purposes, the importance of Schur's Lemma lies in the so-called *orthogonality relations* for the matrix coefficients, since these result in the reproducing property we are interested in. The following results actually go back to [5], [8].

Lemma 3. *Let $\rho : G \to \mathcal{U}(\boldsymbol{H})$ be a square integrable, irreducible unitary representation. Then the following hold.*

(i) *If $\varphi \in \boldsymbol{H}$ is admissible, then $\langle f, \rho(g)\varphi\rangle \in \boldsymbol{L}^2(G)$ for all $f \in \boldsymbol{H}$, and the map*

$$T_{\rho,\varphi} : \boldsymbol{H} \to \boldsymbol{L}^2(G)\,, \quad f \mapsto \Big(g \mapsto \langle f, \rho(g)\varphi\rangle\Big)\,,$$

is a multiple of an isometry.

(ii) *The set of admissible vectors is invariant under $\rho(G)$ and hence dense in $\boldsymbol{H}$.*

Proposition 4. [18, Thm. 3.1] *Let $\rho : G \to \mathcal{U}(\boldsymbol{H})$ be a square integrable, irreducible unitary representation. Then there exists a unique self-adjoint positive operator C such that the following hold.*

(i) *The domain of C is the set of admissible vectors.*

(ii) *For any vectors f_1, $f_2 \in \boldsymbol{H}$ and any admissible vectors φ_1, $\varphi_2 \in \boldsymbol{H}$, we have*

$$\int\limits_G \langle f_1, \rho(g)\varphi_1\rangle\,\overline{\langle f_2, \rho(g)\varphi_2\rangle}\,dm(g) = \langle f_1, f_2\rangle\,\overline{\langle C\varphi_1, C\varphi_2\rangle}\,. \tag{1}$$

(iii) *If G is unimodular, then C is a multiple of the identity, and thus all vectors are admissible.*

The Bessel property of a sequence of vectors in a Hilbert space has two aspects: on the one hand, the range of the coefficient map lies in $\boldsymbol{\ell}^2$, and on the other hand, the evaluation map (which is simply the adjoint of the former) is well defined from $\boldsymbol{\ell}^2$ into the Hilbert space.

Lemma 3.(i) states that for an admissible φ, the family $\big(\rho(g)\varphi\big)_{g\in G}$ has a generalized Bessel property, and thus, by considering the adjoint operator, obtain the following result.

Corollary 5. *Let $\rho : G \to \mathcal{U}(\boldsymbol{H})$ be a square integrable, irreducible unitary representation. Then the following hold.*

(i) *If $\varphi \in \boldsymbol{H}$ is admissible, then the map*

$$T^*_{\rho,\varphi} : F \mapsto \int\limits_G F(g)\,\rho(g)\varphi\,dm(g)$$

is a bounded linear operator from $\boldsymbol{L}^2(G)$ onto $\boldsymbol{H}$, where the integral converges weakly in $\boldsymbol{H}$.

(ii) **(Reproducing Property)**
indexreproducing property If $\varphi_{1,2} \in \boldsymbol{H}$ are admissible, then $T^*_{\rho,\varphi_2} T_{\rho,\varphi_1} = \langle C\varphi_2, C\varphi_1\rangle \, \mathrm{Id}_{\boldsymbol{H}}$, i.e.,

$$\int_G \langle f, \rho(g)\varphi_1\rangle \, \rho(g)\varphi_2 \, dm(g) = \langle C\varphi_2, C\varphi_1\rangle \, f \quad \textit{for all } f \in \boldsymbol{H}, \tag{2}$$

where the integral converges weakly in $\boldsymbol{H}$, and C is the operator from Proposition 4.

1.3. The reducible case

In case the representation ρ is not irreducible, we call it *completely reducible*, if we can decompose the underlying Hilbert space $\boldsymbol{H}$ into irreducible, pairwise orthogonal closed subspaces

$$\boldsymbol{H} = \bigoplus_i \boldsymbol{H}^{(i)} . \tag{3}$$

Accordingly, the representation ρ decomposes into its irreducible components $\rho^{(i)} : G \to \boldsymbol{\mathcal{U}}(\boldsymbol{H}^{(i)})$ with $\rho^{(i)}(g) = \rho(g)\Big|_{\boldsymbol{H}^{(i)}}$.

In case the $\rho^{(i)}$ are pairwise nonequivalent, the orthogonality relation (1) becomes

$$\int_G \langle f_1, \rho(g)\varphi_1\rangle \, \overline{\langle f_2, \rho(g)\varphi_2\rangle} \, dm(g) = \sum_i \langle f_1^{(i)}, f_2^{(i)}\rangle \, \overline{\langle C^{(i)}\varphi_1^{(i)}, C^{(i)}\varphi_2^{(i)}\rangle} , \tag{1'}$$

and (2) changes accordingly. Nevertheless, the reproducing property does still hold if we can choose the $\varphi_{1,2}^{(i)}$ in such a way that the constants $\langle C^{(i)}\varphi_1^{(i)}, C^{(i)}\varphi_2^{(i)}\rangle$ do not depend on i. In case the decomposition (3) is finite, this is certainly possible.

1.4. Affine wavelets as an example

Let us have a closer look at affine wavelets from the aspect of the theory described above. Here, the group G is the affine group $\mathbb{R}^* \ltimes \mathbb{R}$ with $(a, b) \circ (a', b') = (a\,a', b + a\,b')$, also known as the $a\,x + b$-group. The left Haar measure on this group is given by

$$dm(a, b) = \frac{da\,db}{a^2} .$$

The Hilbert space $\boldsymbol{H}$ is $\boldsymbol{L}^2(\mathbb{R})$, and the representation is given by

$$\big(\rho(a, b)\varphi\big)(x) = \frac{1}{\sqrt{|a|}} \, \varphi\Big(\frac{x - b}{a}\Big) .$$

This representation is unitary, irreducible, and square integrable. The admissibility condition is given by

$$\int_{\widehat{\mathbb{R}}} \frac{\big|\widehat{\varphi}(\xi)\big|^2}{|\xi|} \, d\xi < \infty ,$$

and the operator C can be written as

$$C\varphi = \left(\frac{\widehat{\varphi}}{\sqrt{|\xi|}}\right)^{\vee}.$$

Consequently, we have the reproducing formula

$$\iint\limits_{\mathbb{R}^2} \left\langle f(t), \frac{1}{\sqrt{|a|}}\varphi_1\left(\frac{t-b}{a}\right)\right\rangle \frac{1}{\sqrt{|a|}}\varphi_2\left(\frac{x-b}{a}\right)\frac{da\,db}{|a|^2} = C_{\varphi_2,\varphi_1} f(x),$$

where

$$C_{\varphi_2,\varphi_1} = \langle C\varphi_2, C\varphi_1\rangle = \int\limits_{\widehat{\mathbb{R}}} \frac{\widehat{\varphi}_2(\xi)\,\overline{\widehat{\varphi}_1(\xi)}}{|\xi|}\,d\xi.$$

In applications, it is common to use for G the subgroup $\mathbb{R}^+ \ltimes \mathbb{R}$ of the affine group. For this G, the representation ρ given above is not irreducible anymore; instead, $\boldsymbol{H} = \boldsymbol{L}^2(\mathbb{R})$ is the orthogonal sum of the two irreducible subspaces $\boldsymbol{H}^+$ and $\boldsymbol{H}^-$, where

$$\boldsymbol{H}^+ = \{f \in \boldsymbol{L}^2(\mathbb{R}) : \operatorname{supp}\widehat{f} \subseteq [0,\infty[\,\}$$

is the Hardy space, and, analogously, $\boldsymbol{H}^-$ consists of the functions in $\boldsymbol{L}^2(\mathbb{R})$ whose Fourier transforms are supported on the left half axis. In this setup, the reproducing property still holds if

$$\langle C\varphi_2^-, C\varphi_1^-\rangle = \langle C\varphi_2^+, C\varphi_1^+\rangle,$$

which amounts to

$$\int\limits_{-\infty}^{0} \frac{\widehat{\varphi}_2(\xi)\,\overline{\widehat{\varphi}_1(\xi)}}{|\xi|}\,d\xi = \int\limits_{0}^{+\infty} \frac{\widehat{\varphi}_2(\xi)\,\overline{\widehat{\varphi}_1(\xi)}}{|\xi|}\,d\xi.$$

(The equality of the two integrals can be guaranteed, *e.g.*, by choosing $\varphi_{1,2}$ to be equal and real-valued.) If we normalize this constant to equal 1, we can reconstruct f in $\boldsymbol{L}^2(\mathbb{R})$ via

$$\int\limits_{0}^{\infty}\int\limits_{-\infty}^{+\infty} \left\langle f(t), \frac{1}{\sqrt{a}}\varphi_1\left(\frac{t-b}{a}\right)\right\rangle \frac{1}{\sqrt{a}}\varphi_2\left(\frac{x-b}{a}\right) db\,\frac{da}{a^2} = f(x). \tag{4}$$

1.5. Möbius wavelets on the unit circle

Since wavelets have been so successful, several approaches have been made to transfer the basic concept to other domains than just the real line or Euclidean space. We want to concentrate on the unit circle, where, as on any topological group, there exists a natural group of translation operators, which in this particular case are simply rotations of the circle. The problem is, however, as stated in [20], "On the circle it is difficult to define a good dilation operator."

Our approach to this problem is to use the group of Möbius transformations mapping the unit circle to itself while preserving the orientation, *i.e.*, mapping the unit disk onto itself. We shall see at the beginning of Section 2 that the construction described above does not work, since the natural unitary representations we would like to use are not square integrable. Therefore, we try a different approach and give up the unitarity condition on the representation. To make up for that, we have to use a different representation, the *contragredient representation* ρ^*, for the synthesis operator.

We show that these two representations are square integrable on the Korobov or Sobolev type spaces $\boldsymbol{H}_{-1/2}(\mathbb{T})$ and $\boldsymbol{H}_{1/2}(\mathbb{T})$, respectively (Proposition 7 and Corollary 8), and that their action is irreducible on the respective subspaces of holomorphic and antiholomorphic functions (Proposition 9 and Corollary 10), which is very much comparable to the situation in the case of affine wavelets on the real line. Finally, we show in Corollary 12 that, again after proper renormalization, we indeed have the reproducing property

$$\int_G \langle f, \rho^*(g)\varphi_1\rangle\, \rho(g)\varphi_2\, dm(g) = f\,,$$

where f, $\varphi_2 \in \boldsymbol{H}_{-1/2}(\mathbb{T})$ and $\varphi_1 \in \boldsymbol{H}_{1/2}(\mathbb{T})$.

It turns out, though, that behind the scenes, there nevertheless is a unitary representation at work. The strong advantage of our approach lies in the fact that we work with the standard inner product from $\boldsymbol{L}^2(\mathbb{T})$, which is certainly highly desirable for applications.

We finish Section 2 with a few graphical examples for the action of the group.

1.6. Reproducing properties from nonunitary representations

The approach that proves successful in this special example is again based on a general principle, which we describe in Section 3. To this end, we have to give up the Hilbert space setup and consider a Banach space $\boldsymbol{V}$ instead. In this case, we find that associated with a representation $\rho : G \to \boldsymbol{GL}(\boldsymbol{V})$, there exists a contragredient representation $\rho^* : G \to \boldsymbol{GL}(\boldsymbol{V}^*)$, given by

$$\rho^*(g) = \big(\rho(g^{-1})\big)^* = \big(\rho(g)^{-1}\big)^*\,.$$

Note that for a unitary representation on a Hilbert space, the definition of ρ^* yields $\rho^* = \rho$, and thus our construction will turn out to be a true generalization of the unitary approach described above.

In our analogy to bases discussed above, this approach corresponds to considering two families $(v_i)_{i\in I} \subset \boldsymbol{V}$ and $(w_i)_{i\in I} \subset \boldsymbol{V}^*$, and associated with these, the analysis and synthesis operators

$$T_w : \boldsymbol{V} \to \mathbb{C}^I\,,\ x \mapsto \big(\langle x, w_i\rangle\big)_{i\in I}\,,\quad \text{with}\quad T_w^* : \mathbb{C}^I \to \boldsymbol{V}^*\,,\ (c_i)_{i\in I} \to \textstyle\sum_{i\in I} c_i\, w_i\,,$$

and

$$T_v : \boldsymbol{V}^* \to \mathbb{C}^I\,, y \mapsto \big(\overline{\langle v_i, y\rangle}\big)_{i\in I}\,,\quad \text{with}\quad T_v^* : \mathbb{C}^I \to \boldsymbol{V}^{**}\,, (c_i)_{i\in I} \to \textstyle\sum_{i\in I} c_i\, v_i\,.$$

If $(v_i, w_i)_{i\in I}$ is a Schauder basis for $\boldsymbol{V}$ (and thus in particular, a complete biorthogonal system), then we have two reproducing properties for $\boldsymbol{V}$ and $\boldsymbol{V}^*$, respectively, given by

$$T_v^* T_w = \mathrm{Id}_{\boldsymbol{V}} \qquad \text{and} \qquad T_w^* T_v = \mathrm{Id}_{\boldsymbol{V}^*} \ .$$

This is due to the fact that $(v_i, w_i)_{i\in I}$ being a Schauder basis for $\boldsymbol{V}$ implies that $(w_i, v_i)_{i\in I}$ is a Schauder basis for $\boldsymbol{V}^*$. If we actually are considering Riesz bases, then the coefficient spaces are again $\ell^2(I)$, and $\boldsymbol{V}$ (and thus $\boldsymbol{V}^*$ also) is topologically isomorphic to a Hilbert space.

We can weaken these conditions somewhat by considering a pair of dual frames rather than bases, which in the above simply means that the operators T_v and T_w are injective, but not surjective, and thus have T_w^* and T_v^* as left inverses only.

Our results in Section 3 show that this analogy indeed can be carried over to a nonunitary representation ρ on a Banach space $\boldsymbol{V}$ together with its contragredient representation ρ^* on $\boldsymbol{V}^*$.

All in all, we show that in the standard method of generating reproducing identities from square integrable, irreducible unitary representations, it is possible to give up the unitarity condition, and to use a pair of mutually contragredient representations instead. As an example, we work with the group of Möbius transformations on the unit circle.

Ideas for further research include the application of the developed theory to a wavelet-type transformation on the sphere using the group of all Möbius transformations. These include all rotations of the sphere, which form the natural group of translations, and the maps $z \mapsto a\,z$ for $a > 0$ are a good family of generalized dilations. Again, we encounter the problem that the natural unitary representations are not square integrable. In [1], this problem has been approached differently, but is seems promising to try and apply our methods.

2. Möbius Wavelets on the Unit Circle

We want to study the following setup. For the Hilbert space, we want to use $\boldsymbol{H} = \boldsymbol{L}^2(\mathbb{T})$, where we use the unit circle $\mathbb{S}^1$ in $\mathbb{C}$ as model for $\mathbb{T}$, and for the group G the group of Möbius transformations of $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ which map $\mathbb{S}^1$ to itself, preserving its orientation.

2.1. The group and its Haar measure

Möbius transformations are complex functions of the form $z \mapsto \dfrac{a\,z+b}{c\,z+d}$, where a, b, c, $d \in \mathbb{C}$ with $a\,d - b\,c \neq 0$. They are the biholomorphic maps $\overline{\mathbb{C}} \to \overline{\mathbb{C}}$ and form a group under concatenation. A peculiarity of this group is the fact that the mapping

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left(z \mapsto \frac{a\,z+b}{c\,z+d} \right)$$

is a group homomorphism from $\boldsymbol{GL}_2(\mathbb{C})$. Since $\frac{a\,z+b}{c\,z+d} \equiv \frac{a'\,z+b'}{c'\,z+d'}$ if and only if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \lambda \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ for some $\lambda \in \mathbb{C}^*$, the group of Möbius transformations is isomorphic to the quotient of $\boldsymbol{GL}_2(\mathbb{C})$ or $\boldsymbol{SL}_2(\mathbb{C})$ by their respective center, usually referred to as $\boldsymbol{PGL}_2(\mathbb{C})$ or $\boldsymbol{PSL}_2(\mathbb{C})$.

To determine the subgroup G which maps the unit circle to itself and preserves the orientation, note that since $|z| = 1$ is equivalent to $z = 1/\overline{z}$, an element $g \in G$ has to satisfy

$$g(z) = \frac{a\,z+b}{c\,z+d} = 1\Big/\overline{g(z)} = \frac{\overline{c}\,\overline{z}+\overline{d}}{\overline{a}\,\overline{z}+\overline{b}} = \frac{\overline{c}+\overline{d}\,z}{\overline{a}+\overline{b}\,z} \qquad \text{for all } z \text{ with } z = 1/\overline{z}.$$

Since a holomorphic function is determined by its values on the set $\{|z| = 1\}$, we must have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \lambda \begin{pmatrix} \overline{d} & \overline{c} \\ \overline{b} & \overline{a} \end{pmatrix}$$

which implies $\lambda\overline{\lambda} = 1$. Fixing a value for $\sqrt{\lambda}$, we define $\alpha = a/\sqrt{\lambda}$ and $\beta = b/\sqrt{\lambda}$ which yields $c/\sqrt{\lambda} = \sqrt{\lambda}\,\overline{b} = \overline{\beta}$ and $d/\sqrt{\lambda} = \sqrt{\lambda}\,\overline{a} = \overline{\alpha}$, *i.e.*, we have

$$g(z) = \frac{\alpha\,z+\beta}{\overline{\beta}\,z+\overline{\alpha}} \quad \text{for some } \alpha,\, \beta \in \mathbb{C} \text{ with } |\alpha|^2 - |\beta|^2 \neq 0.$$

Since we want g to preserve the orientation of the unit circle, it has to map the unit disk onto itself, so $|g(0)| = |\beta/\overline{\alpha}| < 1$, *i.e.*, $|\alpha| > |\beta|$. Thus we may impose the normalization $|\alpha|^2 - |\beta|^2 = 1$, which determines the pair (α, β) up to a factor of ± 1.

In other words, G is isomorphic to the quotient group of

$$\boldsymbol{SU}_{1,1} = \left\{ \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} : |\alpha|^2 - |\beta|^2 = 1 \right\}$$

by its center $\{\pm I_2\}$, usually referred to as $\boldsymbol{PSU}_{1,1}$, and we can use $\boldsymbol{SU}_{1,1}$ as a double cover for G.

In particular, Haar measure on G is equivalent to Haar measure on $\boldsymbol{SU}_{1,1}$, and to integrate over G we may integrate over $\boldsymbol{SU}_{1,1}$ (the extra factor of two simply means choosing a different normalization of the Haar measure). To be able to integrate over $\boldsymbol{SU}_{1,1}$, we use the map

$$\mathbb{R}^+ \times \mathbb{T} \times \mathbb{T} \to \boldsymbol{SU}_{1,1}\,, \quad (s, \phi, \theta) \mapsto \begin{pmatrix} e^{2\pi i \phi} \cosh(s) & e^{2\pi i \theta} \sinh(s) \\ e^{-2\pi i \theta} \sinh(s) & e^{-2\pi i \phi} \cosh(s) \end{pmatrix}.$$

This map covers $\boldsymbol{SU}_{1,1}$ except for the set $\left\{ \begin{pmatrix} e^{2\pi i \phi} & 0 \\ 0 & e^{-2\pi i \phi} \end{pmatrix} : \phi \in \mathbb{T} \right\}$ which is a submanifold of codimension 2 and as such a set of measure zero. We leave it as an exercise to the reader to show that both the left and the right Haar measure for $\boldsymbol{SU}_{1,1}$ on this map are given by

$$dm(s, \phi, \theta) = \sinh 2s \, ds \, d\phi \, d\theta\,,$$

so in particular, $\boldsymbol{SU}_{1,1}$ and thus G is unimodular. We will frequently use the substitution

$$(u,v) = (e^{2\pi i\phi}, e^{2\pi i\theta}) \quad \text{with} \quad d\phi\, d\theta = \frac{du}{2\pi i\, u}\,\frac{dv}{2\pi i\, v}\,.$$

To simplify notation, we shall write (α,β) for $\left(\begin{smallmatrix}\alpha & \beta\\ \overline{\beta} & \overline{\alpha}\end{smallmatrix}\right)$, and use the abbreviation

$$m_{(\alpha,\beta)}(z) := \frac{\alpha\, z+\beta}{\overline{\beta}\, z+\overline{\alpha}} \quad \text{for any } \alpha,\ \beta \in \mathbb{C} \text{ with } |\alpha|^2-|\beta|^2 \neq 0.$$

The following isomorphism should at least be mentioned here. We have $\boldsymbol{SL}_2(\mathbb{R}) \cong \boldsymbol{SU}_{1,1}$ via

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix} \longleftrightarrow \tfrac{1}{2}\begin{pmatrix} 1 & -i\\ -i & 1\end{pmatrix}\begin{pmatrix} a & b\\ c & d\end{pmatrix}\begin{pmatrix} 1 & i\\ i & 1\end{pmatrix} = \tfrac{1}{2}\begin{pmatrix} (a+d)+i(b-c) & (b+c)+i(a-d)\\ (b+c)-i(a-d) & (a+d)-i(b-c)\end{pmatrix},$$

i.e., they are conjugate subgroups of $\boldsymbol{SL}_2(\mathbb{C})$. This carries over to $\boldsymbol{PSL}_2(\mathbb{R}) \cong \boldsymbol{PSU}_{1,1}$, which geometrically means the following. $\boldsymbol{PSL}_2(\mathbb{R})$ represents the subgroup of Möbius transformations mapping the real line to itself, preserving orientation. Since the Möbius transformation

$$z \mapsto \frac{z+i}{i\,z+1}$$

maps the unit circle to the real line and the unit disk to the upper half plane, it intertwines the two groups.

2.2. Unitary representation of G on $L^2(\mathbb{T})$

We want to work with a representation of G on $L^2(\mathbb{T})$ of the form

$$\big(\rho(\alpha,\beta)\varphi\big)(z) = \omega(\alpha,\beta;z)\,\varphi\big(m_{(\alpha,-\overline{\beta})}(z)\big) = \omega(\alpha,\beta;z)\,\varphi\Big(\frac{\alpha\, z-\overline{\beta}}{-\beta\, z+\overline{\alpha}}\Big)\,,$$

where ω is an appropriate weight function. The reason for this Ansatz instead of using $\varphi\big(m_{(\alpha,\beta)}(z)\big)$ is the fact that the latter yields

$$\rho\big((\alpha,\beta)\circ(\alpha',\beta')\big) = \rho(\alpha',\beta')\,\rho(\alpha,\beta)$$

instead of the correct group law (compare the case of affine wavelets, where for the very same reason we have to use $\varphi\big((x-b)/a\big)$ instead of $\varphi(a\,x+b)$). It might seem more natural to use $\varphi\big(m_{(\alpha,\beta)^{-1}}(z)\big) = \varphi\big(m_{(\overline{\alpha},-\beta)}(z)\big)$, but the above approach has become standard in the literature (*e.g.*, [23]).

To determine $\omega(\alpha,\beta;z)$, note that the substitution $z = \frac{\alpha w-\overline{\beta}}{-\beta\, w+\overline{\alpha}}$ with $dz = \frac{dw}{(-\beta\, w+\overline{\alpha})^2}$ yields

$$\oint_{|z|=1} |\varphi(z)|^2\,\frac{dz}{2\pi i\, z} = \oint_{|w|=1} \Big|\varphi\Big(\frac{\alpha\, w-\overline{\beta}}{-\beta\, w+\overline{\alpha}}\Big)\Big|^2\,\frac{w}{(\alpha\, w-\overline{\beta})(-\beta\, w+\overline{\alpha})}\,\frac{dw}{2\pi i\, w}\,.$$

So ρ being unitary is equivalent to

$$\left|\omega(\alpha,\beta;z)\right|^2 = \frac{z}{(\alpha\, z-\overline{\beta})(-\beta\, z+\overline{\alpha})} = \frac{1}{|-\beta\, z+\overline{\alpha}|^2}\,.$$

To ensure that $\rho(1,0) = I$, we must have $\omega(1,0;z) \equiv 1$, so natural candidates are $1/|-\beta\, z+\overline{\alpha}|$ and $1/(-\beta\, z+\overline{\alpha})$. These lead to the following two representations:

$$\left(\mathcal{P}^{+,0}(\alpha,\beta)\,\varphi\right)(z) = \frac{1}{|-\beta\, z+\overline{\alpha}|}\,\varphi\Big(\frac{\alpha\, z-\overline{\beta}}{-\beta\, z+\overline{\alpha}}\Big)\,,$$

which acts irreducibly on $\boldsymbol{L}^2(\mathbb{T})$, but is not square integrable; and

$$\left(\mathcal{P}^{-,0}(\alpha,\beta)\,\varphi\right)(z) = \frac{1}{(-\beta\, z+\overline{\alpha})}\,\varphi\Big(\frac{\alpha\, z-\overline{\beta}}{-\beta\, z+\overline{\alpha}}\Big)\,.$$

This representation is reducible, it acts irreducibly on the space of analytic functions on the disk and on the space of their conjugate complex functions, and it is not square integrable, either. Note, though, that $\mathcal{P}^{-,0}(-\alpha,-\beta) = -\mathcal{P}^{-,0}(\alpha,\beta)$, so $\mathcal{P}^{-,0}$ is not even a representation of G, but only of $\boldsymbol{SU}_{1,1}$; but that by itself would not make much difference for our purposes. For details on $\mathcal{P}^{\pm,0}$ and unitary representations of $\boldsymbol{SU}_{1,1}$ on other spaces, we refer to [23, § II.5&6].

The fact that these representations are not square integrable can be interpreted as saying that the constant C_{φ_1,φ_2} is always infinity. What can be done to nevertheless get useful results? One possibility is to restrict the representation to a subset of the group, where one has to ensure that the action is still irreducible. This has been used in [1] for the case $G = \boldsymbol{PSL}_2(\mathbb{C})$ and $\boldsymbol{H} = \boldsymbol{L}^2(\mathbb{S}^2)$. In this approach, technical problems arise from the loss of the group structure, unless there exists an appropriate subgroup.

We should also mention that in [4], the approach is totally different, since there, the authors consider the complete discrete series of representations of $\boldsymbol{SU}_{1,1}$.

2.3. A nonunitary representation of G

We want to take a different approach, namely, to give up the unitarity of the representation. This idea is similar to giving up the concept of an orthonormal basis and using a general basis instead; consequently, one has to work with a dual basis. In analogy, we should expect having to use something like a dual representation, called the ***contragredient*** representation, for the dual side to preserve the reproducing property. (This notion should not be confused with the ***adjoint*** representation.)

Instead of the unitary representation described above, we start from the $\boldsymbol{L}^1$-normalized version

$$\begin{aligned}\left(\rho(\alpha,\beta)\varphi\right)(z) &= \frac{z}{(\alpha\, z-\overline{\beta})(-\beta\, z+\overline{\alpha})}\,\varphi\Big(\frac{\alpha\, z-\overline{\beta}}{-\beta\, z+\overline{\alpha}}\Big)\\ &= \frac{1}{|-\beta\, z+\overline{\alpha}|^2}\,\varphi\Big(\frac{\alpha\, z-\overline{\beta}}{-\beta\, z+\overline{\alpha}}\Big)\,,\end{aligned}$$

which by the same calculation as in Section 2.2 can be shown to be isometric on $L^1(\mathbb{T})$. Therefore, it seems natural to use on the dual side the L^∞-normalized version

$$\big(\rho^*(\alpha,\beta)\varphi\big)(z) = \varphi\Big(\frac{\alpha\, z - \overline{\beta}}{-\beta\, z + \overline{\alpha}}\Big)\,,$$

and we shall show that this yields the orthogonality relation

$$\int_G \big\langle f_1\,,\,\rho^*(\alpha,\beta)\varphi_1\big\rangle\,\big\langle \rho(\alpha,\beta)\varphi_2\,,\,f_2\big\rangle\,dm(\alpha,\beta) = \langle f_1, f_2\rangle\,\langle\varphi_2,\varphi_1\rangle$$

for appropriate $f_{1,2}$ and $\varphi_{1,2}$.

Note that by substituting $w = \frac{\alpha z - \overline{\beta}}{-\beta z + \overline{\alpha}}$, we have

$$\begin{aligned}
\big\langle \varphi_1\,,\,\rho^*(\alpha,\beta)\varphi_2\big\rangle &= \oint_{|z|=1} \varphi_1(z)\,\overline{\varphi_2\Big(\frac{\alpha\, z - \overline{\beta}}{-\beta\, z + \overline{\alpha}}\Big)}\,\frac{dz}{2\pi i\, z}\\
&= \oint_{|w|=1} \varphi_1\Big(\frac{\overline{\alpha}\, w + \overline{\beta}}{\beta\, w + \alpha}\Big)\,\overline{\varphi_2(w)}\,\frac{w}{(\overline{\alpha}\, w + \overline{\beta})(\beta\, w + \alpha)}\,\frac{dw}{2\pi i\, w}\\
&= \big\langle \rho(\overline{\alpha}, -\beta)\varphi_1\,,\,\varphi_2\big\rangle = \big\langle \rho((\alpha,\beta)^{-1})\varphi_1\,,\,\varphi_2\big\rangle\,, \qquad (5)
\end{aligned}$$

i.e., $\rho^*(\alpha,\beta) = \big(\rho((\alpha,\beta)^{-1})\big)^*$.

2.4. The matrix coefficients

Let us calculate the matrix coefficients of ρ^* and ρ for $f(z) = z^n$ and $\varphi(z) = z^m$, *i.e.*,

$$\big\langle f\,,\,\rho^*(\alpha,\beta)\varphi\big\rangle = \oint_{|z|=1} z^n\,\overline{\Big(\frac{\alpha\, z - \overline{\beta}}{-\beta\, z + \overline{\alpha}}\Big)^m}\,\frac{dz}{2\pi i\, z} =: A^*_{n,m}(\alpha,\beta)$$

and

$$\big\langle \rho(\alpha,\beta)\varphi\,,\,f\big\rangle = \oint_{|z|=1} \frac{z}{(\alpha\, z - \overline{\beta})(-\beta\, z + \overline{\alpha})}\,\Big(\frac{\alpha\, z - \overline{\beta}}{-\beta\, z + \overline{\alpha}}\Big)^m\,\overline{z^n}\,\frac{dz}{2\pi i\, z} =: A_{m,n}(\alpha,\beta)\,.$$

Lemma 6. (i) *The matrix coefficients of ρ^* are given as follows.*

$$a)\ m > 0 : A^*_{n,m}(\alpha,\beta) = \begin{cases} \dfrac{1}{\alpha^{n+m}} \displaystyle\sum_{j=1}^{\min\{n,m\}} \binom{m}{m-j}(-\beta)^{m-j}\binom{n-1}{n-j}\overline{\beta}^{\,n-j}\,, & n > 0\,,\\ \Big(-\dfrac{\beta}{\alpha}\Big)^m\,, & n = 0\,,\\ 0\,, & n < 0\,;\end{cases}$$

$b)\ m = 0 : A^*_{n,0}(\alpha,\beta) = \delta_{0,n}\ ;$

$c)\ m < 0 : A^*_{n,m}(\alpha,\beta) = \overline{A^*_{-n,|m|}(\alpha,\beta)}\ .$

(ii) *The matrix coefficients of ρ satisfy*

$$A_{m,n}(\alpha,\beta) = A^*_{m,n}(\overline{\alpha},-\beta)\,.$$

Proof. Part (ii) follows from (5). In order to verify (i), we first consider the case $m>0$ in three steps. For $n>0$, we have

$$\begin{aligned}
A^*_{n,m}(\alpha,\beta) &= \oint_{|z|=1} z^n\,\overline{\left(\frac{\alpha\,z-\overline{\beta}}{-\beta\,z+\overline{\alpha}}\right)^m}\,\frac{dz}{2\pi i\,z} = \frac{1}{2\pi i}\oint_{|z|=1} z^{n-1}\left(\frac{-\overline{\beta}\,z+\overline{\alpha}}{\alpha\,z-\overline{\beta}}\right)^m dz \\
&= \frac{1}{2\pi i}\oint_{|z|=1} H(z)\,\frac{1}{(z-\overline{\beta}/\alpha)^m}\,dz \\
&= \frac{1}{(m-1)!}\,H^{(m-1)}\left(\frac{\overline{\beta}}{\alpha}\right),
\end{aligned}$$

where $H(z) = \left(-\frac{\overline{\beta}}{\alpha}\right)^m z^{n-1}\left(z-\frac{\overline{\alpha}}{\overline{\beta}}\right)^m$. By the Leibniz' product rule,

$$\begin{aligned}
&\frac{1}{(m-1)!}\,H^{(m-1)}(z) = \\
&= \left(-\frac{\overline{\beta}}{\alpha}\right)^m \frac{1}{(m-1)!}\sum_{\substack{\ell=0\\ \ell\le n-1}}^{m-1}\binom{m-1}{\ell}\frac{(n-1)!}{(n-1-\ell)!}\,z^{n-1-\ell}\,\cdot \\
&\qquad\qquad \cdot\frac{m!}{(m-(m-1-\ell))!}\left(z-\frac{\overline{\alpha}}{\overline{\beta}}\right)^{m-(m-1-\ell)} \\
&\underset{(j=\ell+1)}{=}\left(-\frac{\overline{\beta}}{\alpha}\right)^m \sum_{j=1}^{\min\{n,m\}}\binom{n-1}{n-j} z^{n-j}\binom{m}{j}\left(z-\frac{\overline{\alpha}}{\overline{\beta}}\right)^j
\end{aligned}$$

and thus

$$\begin{aligned}
A^*_{n,m}(\alpha,\beta) &= \left(-\frac{\overline{\beta}}{\alpha}\right)^m \sum_{j=1}^{\min\{n,m\}}\binom{n-1}{n-j}\left(\frac{\overline{\beta}}{\alpha}\right)^{n-j}\binom{m}{j}\left(\frac{\overline{\beta}}{\alpha}-\frac{\overline{\alpha}}{\overline{\beta}}\right)^j \\
&= \frac{1}{\alpha^{m+n}}\sum_{j=1}^{\min\{n,m\}}\binom{m}{m-j}(-\beta)^{m-j}\binom{n-1}{n-j}\,\overline{\beta}^{\,n-j}\,.
\end{aligned}$$

For $n=0$,

$$\begin{aligned}
A^*_{0,m}(\alpha,\beta) &= \oint_{|z|=1} 1\,\overline{\left(\frac{\alpha\,z-\overline{\beta}}{-\beta\,z+\overline{\alpha}}\right)^m}\,\frac{dz}{2\pi i\,z} = \frac{1}{2\pi i}\oint_{|z|=1}\frac{1}{z}\left(\frac{-\overline{\beta}\,z+\overline{\alpha}}{\alpha\,z-\overline{\beta}}\right)^m dz \\
&\qquad \text{(use } w=\tfrac{1}{z},\ z=\tfrac{1}{w},\ dz=-\tfrac{1}{w^2}\,dw\text{; this reverses the orientation:)} \\
&= -\frac{1}{2\pi i}\oint_{|w|=1} w\left(\frac{-\overline{\beta}/w+\overline{\alpha}}{\alpha/w-\overline{\beta}}\right)^m\frac{dw}{-w^2} \\
&= \frac{1}{2\pi i}\oint_{|w|=1}\frac{1}{w}\left(\frac{-\overline{\beta}+\overline{\alpha}\,w}{\alpha-\overline{\beta}\,w}\right)^m dw = \left(-\frac{\overline{\beta}}{\alpha}\right)^m,
\end{aligned}$$

and for $n < 0$,

$$A^*_{n,m}(\alpha,\beta) = \oint\limits_{|z|=1} z^n \overline{\left(\frac{\alpha z - \overline{\beta}}{-\beta z + \overline{\alpha}}\right)^m} \frac{dz}{2\pi i z} = \frac{1}{2\pi i} \oint\limits_{|z|=1} z^{n-1} \left(\frac{-\overline{\beta} z + \overline{\alpha}}{\alpha z - \overline{\beta}}\right)^m dz$$

(use $w = \frac{1}{z}$, $z = \frac{1}{w}$, $dz = -\frac{1}{w^2}\,dw$; this reverses the orientation:)

$$= -\frac{1}{2\pi i} \oint\limits_{|w|=1} w^{1-n} \left(\frac{-\overline{\beta}/w + \overline{\alpha}}{\alpha/w - \overline{\beta}}\right)^m \frac{dw}{-w^2}$$

$$= \frac{1}{2\pi i} \oint\limits_{|w|=1} w^{|n|-1} \left(\frac{-\overline{\beta} + \overline{\alpha} w}{\alpha - \overline{\beta} w}\right)^m dw = 0\,.$$

For $m = 0$, we obtain

$$A^*_{n,0}(\alpha,\beta) = \oint\limits_{|z|=1} z^n\, 1 \frac{dz}{2\pi i z} = \frac{1}{2\pi i} \oint\limits_{|z|=1} z^{n-1}\, dz = \delta_{0,n}\,.$$

Finally, for $m < 0$, we note that we have

$$\overline{\oint\limits_{|z|=1} f(z) \frac{dz}{2\pi i z}} = \overline{\int_0^1 f(e^{2\pi i \theta})\, d\theta}$$

$$= \int_0^1 \overline{f(e^{2\pi i \theta})}\, d\theta = \oint\limits_{|z|=1} \overline{f(z)} \frac{dz}{2\pi i z}\,.$$

Therefore,

$$A^*_{n,m}(\alpha,\beta) = \oint\limits_{|z|=1} z^n \overline{\left(\frac{\alpha z - \overline{\beta}}{-\beta z + \overline{\alpha}}\right)^m} \frac{dz}{2\pi i z} = \overline{\oint\limits_{|z|=1} \overline{z^n} \left(\frac{\alpha z - \overline{\beta}}{-\beta z + \overline{\alpha}}\right)^m \frac{dz}{2\pi i z}}$$

$$= \overline{\oint\limits_{|z|=1} z^{-n} \overline{\left(\frac{\alpha z - \overline{\beta}}{-\beta z + \overline{\alpha}}\right)^{-m}} \frac{dz}{2\pi i z}} = \overline{A^*_{-n,-m}(\alpha,\beta)}\,.$$

This completes the proof. □

2.5. Square integrability

We saw in Lemma 6 that the matrix coefficients of ρ^* for the constant function $f(z) \equiv 1$ are given by

$$A^*_{0,m}(\alpha,\beta) = \begin{cases} \left(-\frac{\beta}{\alpha}\right)^m, & m > 0\,, \\ 1\,, & m = 0\,, \\ \left(-\frac{\overline{\beta}}{\overline{\alpha}}\right)^{|m|}, & m < 0\,. \end{cases}$$

Since these are all not square integrable on G, it only makes sense to consider for f functions with zero mean.

Proposition 7. *For $f(z) = \sum_{n\neq 0} a_n z^n$ and $\varphi(z) = \sum_{m\in\mathbb{Z}} b_m z^m$, we have*

$$\int_G \left|\langle f\,,\, \rho^*(\alpha,\beta)\varphi\rangle\right|^2 dm(\alpha,\beta) =$$

$$= \Big(\sum_{n>0} |a_n|^2 \frac{1}{n}\Big)\Big(\sum_{m>0} |b_m|^2\, m\Big) + \Big(\sum_{n<0} |a_n|^2 \frac{1}{|n|}\Big)\Big(\sum_{m<0} |b_m|^2\, |m|\Big)\,. \qquad (6)$$

Proof. We begin by evaluating the integral

$$\int_G A^*_{n,m}(\alpha,\beta)\, \overline{A^*_{p,q}(\alpha,\beta)}\, dm(\alpha,\beta) =: B^*_{n,m,p,q}$$

for $n\,p \neq 0$. If $n\,m \leq 0$ or $p\,q \leq 0$, then $A^*_{n,m} \equiv 0$ or $A^*_{p,q} \equiv 0$ and thus $B^*_{n,m,p,q} = 0$.

For n, $m > 0$ and p, $q > 0$, we have by Lemma 6

$$\begin{aligned} B^*_{n,m,p,q} &= \int_G \Big(\frac{1}{\alpha^{n+m}} \sum_{j=1}^{\min\{n,m\}} \binom{m}{m-j}(-\beta)^{m-j}\binom{n-1}{n-j}\overline{\beta}^{n-j}\Big) \\ &\qquad \Big(\frac{1}{\overline{\alpha}^{p+q}} \sum_{\ell=1}^{\min\{p,q\}} \binom{q}{q-\ell}(-\overline{\beta})^{q-\ell}\binom{p-1}{p-\ell}\beta^{p-\ell}\Big)\, dm(\alpha,\beta) \\ &= \sum_{j=1}^{\min\{n,m\}} \sum_{\ell=1}^{\min\{p,q\}} \binom{m}{m-j}\binom{n-1}{n-j}\binom{q}{q-\ell}\binom{p-1}{p-\ell}(-1)^{m-j+q-\ell} \\ &\qquad \int_0^\infty \oint_{|u|=1} \oint_{|v|=1} \frac{v^{m-n-q+p}}{u^{n+m-p-q}}\, \frac{(\sinh s)^{n+m-2j+p+q-2\ell}}{(\cosh s)^{n+m+p+q}}\, \frac{du}{2\pi i\, u}\, \frac{dv}{2\pi i\, v}\, \sinh 2s\, ds\,. \end{aligned}$$

Now integration with respect to u and v yields 0 unless we have both $m+n-q-p = 0$ and $m-n-q+p = 0$, *i.e.*, $(n,m) = (p,q)$. In this case, we obtain

$$\begin{aligned} B^*_{n,m,n,m} &= \sum_{j,\ell=1}^{\min\{n,m\}} \binom{m}{m-j}\binom{n-1}{n-j}\binom{m}{m-\ell}\binom{n-1}{n-\ell}(-1)^{j+\ell} \int_0^\infty \frac{(\sinh s)^{2m+2n-2j-2\ell+1}}{(\cosh s)^{2m+2n-1}}\, 2\, ds \\ &\underset{\text{(Lemma A.2)}}{=} \sum_{j,\ell=1}^{\min\{n,m\}} (-1)^{j+\ell}\binom{m}{m-j}\binom{n-1}{n-j}\binom{m}{m-\ell}\binom{n-1}{n-\ell}\frac{(m+n-j-\ell)!\,(j+\ell-2)!}{(m+n-1)!} \\ &= \frac{1}{n\binom{m+n-1}{m}} \sum_{j=1}^{\min\{n,m\}} (-1)^{n-j}\binom{m}{j} \sum_{\ell=1}^{\min\{n,m\}} (-1)^{n-\ell}\binom{n}{\ell}\binom{m+n-j-\ell}{n-j}\binom{j+\ell-2}{j-1} \\ &\underset{\text{(Lemma A.1)}}{=} \frac{1}{n\binom{m+n-1}{m}}(-1)^{n-1}\binom{m}{1}(-1)^{n+1}\binom{m+n-1}{n-1} = \frac{m}{n}\,. \end{aligned}$$

For n, $m > 0$ and p, $q < 0$, we have

$$
\begin{aligned}
B^*_{n,m,p,q} &= \int_G A^*_{n,m}(\alpha,\beta)\,\overline{A^*_{p,q}(\alpha,\beta)}\,dm(\alpha,\beta) = \int_G A^*_{n,m}(\alpha,\beta)\,A^*_{|p|,|q|}(\alpha,\beta)\,dm(\alpha,\beta) \\
&= \int_G \left(\frac{1}{\alpha^{n+m}} \sum_{j=1}^{\min\{n,m\}} \binom{m}{m-j}(-\beta)^{m-j}\binom{n-1}{n-j}\overline{\beta}^{n-j}\right) \\
&\qquad \left(\frac{1}{\alpha^{|p|+|q|}} \sum_{\ell=1}^{\min\{|p|,|q|\}} \binom{|q|}{|q|-\ell}(-\beta)^{|q|-\ell}\binom{|p|-1}{|p|-\ell}\overline{\beta}^{|p|-\ell}\right) dm(\alpha,\beta) \\
&= \sum_{j=1}^{\min\{n,m\}} \sum_{\ell=1}^{\min\{|p|,|q|\}} \binom{m}{m-j}\binom{n-1}{n-j}\binom{|q|}{|q|-\ell}\binom{|p|-1}{|p|-\ell}(-1)^{m-j+|q|-\ell} \\
&\qquad \int_0^\infty \oint_{|u|=1} \oint_{|v|=1} \frac{v^{m-n+|q|-|p|}}{u^{n+m+|p|+|q|}}\,\frac{(\sinh s)^{n+m-2j+|p|+|q|-2\ell}}{(\cosh s)^{n+m+|p|+|q|}}\,\frac{du}{2\pi i\,u}\,\frac{dv}{2\pi i\,v}\,\sinh 2s\,ds\,,
\end{aligned}
$$

and integration with respect to u yields 0.

For n, $m < 0$, we have

$$
B^*_{n,m,p,q} = \int_G A^*_{n,m}\,\overline{A^*_{p,q}}\,dm = \int_G \overline{A^*_{-n,-m}\,A^*_{-p,-q}}\,dm = \overline{B^*_{-n,-m,-p,-q}}\,.
$$

All in all, we find for n, $p \neq 0$

$$
B^*_{n,m,p,q} = \begin{cases} \delta_{(n,m),(p,q)}\,\dfrac{m}{n}\,, & n\,m > 0\,, \\ 0\,, & n\,m \leq 0\,. \end{cases}
$$

From this, we obtain

$$
\begin{aligned}
&\int_G \left|\langle f\,,\,\rho^*(\alpha,\beta)\varphi\rangle\right|^2 dm(\alpha,\beta) = \\
&\quad = \int_G \left\langle \sum_{n\neq 0} a_n\,z^n\,,\,\rho^*(\alpha,\beta)\sum_{m\in\mathbf{Z}} b_m\,z^m\right\rangle \overline{\left\langle \sum_{p\neq 0} a_p\,z^p\,,\,\rho^*(\alpha,\beta)\sum_{q\in\mathbf{Z}} b_q\,z^q\right\rangle}\,dm(\alpha,\beta) \\
&\quad = \sum_{n\neq 0} a_n \sum_{m\in\mathbf{Z}} \overline{b_m} \sum_{p\neq 0} \overline{a_p} \sum_{q\in\mathbf{Z}} b_q\,B^*_{n,m,p,q} \\
&\quad = \sum_{n,m>0} |a_n|^2\,|b_m|^2\,\frac{m}{n} + \sum_{n,m<0} |a_n|^2\,|b_m|^2\,\frac{m}{n}\,,
\end{aligned}
$$

which yields the claim. □

Corollary 8. *For $\varphi(z) = \sum_{p\neq 0} c_p\, z^p$ and $f(z) = \sum_{q\in\mathbb{Z}} d_q\, z^q$, we have*

$$\int_G \left|\left\langle \rho(\alpha,\beta)\varphi\,,\, f\right\rangle\right|^2 dm(\alpha,\beta) =$$
$$= \Big(\sum_{p>0} |c_p|^2\, \frac{1}{p}\Big)\Big(\sum_{q>0} |d_q|^2\, q\Big) + \Big(\sum_{p<0} |c_p|^2\, \frac{1}{|p|}\Big)\Big(\sum_{q<0} |d_q|^2\, |q|\Big)\,. \qquad (7)$$

Proof. We have

$$\int_G \left|\left\langle \rho(g)\varphi\,,\, f\right\rangle\right|^2 dm(g) \underset{(5)}{=} \int_G \left|\left\langle \varphi\,,\, \rho^*(g^{-1})f\right\rangle\right|^2 dm(g) = \int_G \left|\left\langle \varphi\,,\, \rho^*(g)f\right\rangle\right|^2 dm(g)\,,$$

where the second equality holds because of the unimodularity of G, and the claim follows by Proposition 7. □

2.6. Irreducibility

Proposition 7 and Corollary 8 suggest the following approach. Consider the Korobov or Sobolev type spaces of periodic distributions with vanishing mean

$$\boldsymbol{H}_s(\mathbb{T}) = \left\{ f \in \mathcal{D}'(\mathbb{T}) : \widehat{f}[0] = 0\,,\, \|f\|_{\boldsymbol{H}_s} := \Big(\sum_{n\neq 0} \big|\widehat{f}[n]\big|^2\, |n|^{2s}\Big)^{1/2} < \infty \right\}.$$

These spaces are Hilbert spaces, but we can also interpret $\boldsymbol{H}_{-s}$ as the dual of $\boldsymbol{H}_s$ via the natural sesquilinear pairing

$$\langle f, \varphi\rangle = \sum_{n\neq 0} \widehat{f}[n]\,\overline{\widehat{\varphi}[n]}\,, \quad f \in \boldsymbol{H}_s\,,\, \varphi \in \boldsymbol{H}_{-s}\,.$$

These spaces decompose naturally into a holomorphic and an antiholomorphic part, *i.e.*, $\boldsymbol{H}_s = \boldsymbol{H}_s^+ \oplus \boldsymbol{H}_s^-$ where

$$\boldsymbol{H}_s^+(\mathbb{T}) = \left\{ f \in \boldsymbol{H}_s : \widehat{f}[n] = 0 \text{ for } n \leq 0 \right\}$$

and $\boldsymbol{H}_s^- = \overline{\boldsymbol{H}_s^+}$. Consequently, every element $f \in \boldsymbol{H}_s$ decomposes uniquely as

$$f = f^+ + f^- \quad \text{where } f^+ \in \boldsymbol{H}_s^+\,,\, f^- \in \boldsymbol{H}_s^-\,.$$

Thus we can rewrite Proposition 7 and Corollary 8 as

$$\|T_{\rho^*,\varphi}f\|^2_{\boldsymbol{L}^2(G)} = \|f^+\|^2_{\boldsymbol{H}_{-1/2}}\,\|\varphi^+\|^2_{\boldsymbol{H}_{1/2}} + \|f^-\|^2_{\boldsymbol{H}_{-1/2}}\,\|\varphi^-\|^2_{\boldsymbol{H}_{1/2}}$$

and

$$\|T_{\rho,\varphi}f\|^2_{\boldsymbol{L}^2(G)} = \|\varphi^+\|^2_{\boldsymbol{H}_{-1/2}}\,\|f^+\|^2_{\boldsymbol{H}_{1/2}} + \|\varphi^-\|^2_{\boldsymbol{H}_{-1/2}}\,\|f^-\|^2_{\boldsymbol{H}_{1/2}}\,.$$

Proposition 9. *$\rho(G)$ acts isometrically and irreducibly on $\boldsymbol{H}^+_{-1/2}(\mathbb{T})$ and on $\boldsymbol{H}^-_{-1/2}(\mathbb{T})$.*

Proof. (i) First, we need to show that $\rho(G)$ acts on $\boldsymbol{H}^+_{-1/2}$. To this end, note that $\varphi \in \boldsymbol{H}^+_{-1/2}$ implies that φ is holomorphic on the open unit disk $\mathbb{D}$ with $\varphi(0) = 0$, so we can write $\varphi(z) = z\,\psi(z)$ for some ψ holomorphic on $\mathbb{D}$. Then we have

$$\begin{aligned}\big(\rho(\alpha,\beta)\varphi\big)(z) &= \frac{z}{(\alpha z - \overline{\beta})(-\beta z + \overline{\alpha})}\,\varphi\Big(\frac{\alpha z - \overline{\beta}}{-\beta z + \overline{\alpha}}\Big)\\ &= \frac{z}{(-\beta z + \overline{\alpha})^2}\,\psi\Big(\frac{\alpha z - \overline{\beta}}{-\beta z + \overline{\alpha}}\Big)\\ &= z\left(\frac{1}{(-\beta z + \overline{\alpha})^2}\,\psi\Big(\frac{\alpha z - \overline{\beta}}{-\beta z + \overline{\alpha}}\Big)\right),\end{aligned}$$

i.e., we again have a function that is holomorphic on $\mathbb{D}$, multiplied by z. That $\rho(\alpha,\beta)\varphi$ is indeed in $\boldsymbol{H}^+_{-1/2}$ now follows from the isometry property.

To be able to show the isometry property, we first want to express the norm in $\boldsymbol{H}^+_{-1/2}$ in terms of $\varphi(z)$. To this end, note that for $\varphi(z) = \sum_{n\geq 1} a_n z^n$, we have

$$\int_0^z \frac{\varphi(\zeta)}{\zeta}\,d\zeta = \int_0^z \sum_{n=1}^{\infty} a_n \zeta^{n-1}\,d\zeta = \sum_{n=1}^{\infty} a_n \frac{\zeta^n}{n}\bigg|_0^z = \sum_{n=1}^{\infty} \frac{a_n}{n} z^n .$$

Therefore, we can write by invoking Parseval-Plancherel

$$\|\varphi\|^2_{\boldsymbol{H}^+_{-1/2}} = \sum_{n=1}^{\infty}\Big(\frac{a_n}{n}\Big)\,\overline{a_n} = \oint_{|z|=1}\Big(\int_0^z \frac{\varphi(\zeta)}{\zeta}\,d\zeta\Big)\,\overline{\varphi(z)}\,\frac{dz}{2\pi i\,z},$$

and thus

$$\begin{aligned}\Big\|\rho(\alpha,\beta)\varphi\Big\|^2_{\boldsymbol{H}^+_{-1/2}} &=\\ &= \oint_{|z|=1}\left(\int_0^z \frac{\varphi\Big(\frac{\alpha\zeta-\overline{\beta}}{-\beta\zeta+\overline{\alpha}}\Big)}{(\alpha\zeta-\overline{\beta})(-\beta\zeta+\overline{\alpha})}\,d\zeta\right)\frac{\overline{z}}{(\overline{\alpha}\,\overline{z}-\beta)(-\overline{\beta}\,\overline{z}+\alpha)}\,\overline{\varphi\Big(\frac{\alpha z-\overline{\beta}}{-\beta z+\overline{\alpha}}\Big)}\,\frac{dz}{2\pi i\,z}\\ &= \oint_{|z|=1}\left(\int_{-\frac{\overline{\beta}}{\overline{\alpha}}}^{\frac{\alpha z-\overline{\beta}}{-\beta z+\overline{\alpha}}} \frac{\varphi(\omega)}{\omega}\,d\omega\right)\frac{\overline{z}}{(\overline{\alpha}\,\overline{z}-\beta)(-\overline{\beta}\,\overline{z}+\alpha)}\,\overline{\varphi\Big(\frac{\alpha z-\overline{\beta}}{-\beta z+\overline{\alpha}}\Big)}\,\frac{dz}{2\pi i\,z}\\ &= \oint_{|w|=1}\left(\int_{-\overline{\beta}/\overline{\alpha}}^{w} \frac{\varphi(\omega)}{\omega}\,d\omega\right)\overline{\varphi(w)}\,\frac{dw}{2\pi i\,w}\\ &= \|\varphi\|^2_{\boldsymbol{H}^+_{-1/2}} + \oint_{|w|=1}\left(\int_{-\overline{\beta}/\overline{\alpha}}^{0} \frac{\varphi(\omega)}{\omega}\,d\omega\right)\overline{\varphi(w)}\,\frac{dw}{2\pi i\,w},\end{aligned}$$

where the extraneous term on the right vanishes, since we have

$$\oint\limits_{|w|=1} \Big(\int\limits_{-\overline{\beta}/\overline{\alpha}}^{0} \frac{\varphi(\omega)}{\omega}\, d\omega \Big) \overline{\varphi(w)}\, \frac{dw}{2\pi i\, w} = \Big(\int\limits_{-\overline{\beta}/\overline{\alpha}}^{0} \frac{\varphi(\omega)}{\omega}\, d\omega \Big) \overline{\oint\limits_{|w|=1} \varphi(w)\, \frac{dw}{2\pi i\, w}}$$
$$= \Big(\int\limits_{-\overline{\beta}/\overline{\alpha}}^{0} \frac{\varphi(\omega)}{\omega}\, d\omega \Big) \overline{\varphi(0)}\,.$$

For the proof of the irreducibility, we follow [23]. Assume that $\boldsymbol{V} \subseteq \boldsymbol{H}^+_{-1/2}$ is a nonzero, closed, ρ-invariant subspace. Choose $\varphi \in \boldsymbol{V} \setminus \{0\}$. Since $\big(\rho(\alpha,\beta)\varphi\big)'(0) = -\varphi(-\overline{\beta}/\overline{\alpha})/(\overline{\alpha}\overline{\beta})$ whenever $\beta \neq 0$, we can choose $\lambda \in \mathbb{C}$ and (α,β) such that $\varphi_0 = \lambda\, \rho(\alpha,\beta)\varphi \in \boldsymbol{V}$ satisfies $\varphi_0'(0) = 1$. Now since $\big(\rho(e^{i\theta},0)\varphi\big)(z) = \varphi(e^{2i\theta} z)$ and $\boldsymbol{V}$ is closed, the function φ_1 given by

$$\varphi_1(z) = \frac{1}{2\pi} \int\limits_0^{2\pi} e^{-2i\theta}\, \varphi_0(e^{2i\theta} z)\, d\theta = \sum_{n=1}^{\infty} \frac{\varphi_0^{(n)}(0)}{n!}\, \frac{1}{2\pi} \int\limits_0^{2\pi} e^{-2i\theta}\, e^{2in\theta}\, z^n\, d\theta = z$$

is also in $\boldsymbol{V}$. For this function, we have

$$\big(\rho(\alpha,\beta)\varphi_1\big)(z) = \frac{z}{(-\beta\, z + \overline{\alpha})^2} = \sum_{n=1}^{\infty} \frac{n\, \beta^{n-1}}{\overline{\alpha}^{n+1}}\, z^n\,,$$

so fixing (α,β) with $\beta \neq 0$, we have $\varphi_2 = \rho(\alpha,\beta)\varphi_1 \in \boldsymbol{V}$ with $\varphi_2^{(n)}(0) \neq 0$ for all $n \in \mathbb{N}$. By the same reasoning as before, the function

$$\frac{1}{2\pi} \int\limits_0^{2\pi} e^{-2in\theta}\, \varphi_2(e^{2i\theta} z)\, d\theta = \sum_{k=1}^{\infty} \frac{\varphi_2^{(k)}(0)}{k!}\, \frac{1}{2\pi} \int\limits_0^{2\pi} e^{-2in\theta}\, e^{2ik\theta}\, z^k\, d\theta = \frac{\varphi_2^{(n)}(0)}{n!}\, z^n$$

is also in $\boldsymbol{V}$, so $z^n \in \boldsymbol{V}$ for all $n \in \mathbb{N}$, and thus $\boldsymbol{V} = \boldsymbol{H}^+_{-1/2}$ since $\boldsymbol{V}$ is closed.

(ii) To show the same for $\boldsymbol{H}^-_{-1/2}$, we consider the complex conjugation operator

$$CC : \boldsymbol{H}^+_{-1/2} \to \boldsymbol{H}^-_{-1/2}\,, \quad \varphi \mapsto \overline{\varphi}\,,$$

i.e., we have

$$\varphi(z) = \sum_{n=1}^{\infty} a_n\, z^n \quad \Longrightarrow \quad \big(CC\, \varphi\big)(z) = \sum_{n=1}^{\infty} \overline{a_n}\, z^{-n}\,.$$

Consequently,

$$\big(CC\, \rho(\alpha,\beta)\, \varphi\big)(z) = \overline{\frac{z}{(-\beta\, z + \overline{\alpha})(\alpha\, z - \overline{\beta})}\, \varphi\Big(\frac{\alpha\, z - \overline{\beta}}{-\beta\, z + \overline{\alpha}}\Big)}$$
$$= \frac{\overline{z}}{(-\overline{\beta}\, \overline{z} + \alpha)(\overline{\alpha}\, \overline{z} - \beta)} \sum_{n=1}^{\infty} \overline{a_n} \Big(\frac{\overline{\alpha}\, \overline{z} - \beta}{-\overline{\beta}\, \overline{z} + \alpha}\Big)^n$$

$$= \frac{z}{(-\overline{\beta}+\alpha z)(\overline{\alpha}-\beta z)} \sum_{n=1}^{\infty} \overline{a_n} \left(\frac{\overline{\alpha}-\beta z}{-\overline{\beta}+\alpha z}\right)^n$$
$$= \frac{z}{(-\beta z+\overline{\alpha})(\alpha z-\overline{\beta})} \sum_{n=1}^{\infty} \overline{a_n} \left(\frac{\alpha z-\overline{\beta}}{-\beta z+\overline{\alpha}}\right)^{-n}$$
$$= \Big(\rho(\alpha,\beta)\, CC\, \varphi\Big)(z),$$

i.e., $CC : \boldsymbol{H}^{+}_{-1/2} \to \boldsymbol{H}^{-}_{-1/2}$ is an antilinear isometry which commutes with all $\rho(\alpha,\beta)$. Therefore, $\rho(\alpha,\beta)$ is also an isometry on $\boldsymbol{H}^{-}_{-1/2}$, and the irreducibility of the action of ρ on $\boldsymbol{H}^{+}_{-1/2}$ is equivalent to that of its action on $\boldsymbol{H}^{-}_{-1/2}$. □

The contragredient representation ρ^*, on the other hand, creates a minor formal problem, since $\varphi^+(0) = \varphi^-(\infty) = 0$ does not necessarily imply $\big(\rho^*(\alpha,\beta)\varphi^+\big)(0) = \big(\rho^*(\alpha,\beta)\varphi^-\big)(\infty) = 0$. One possibility is to modify the definition of ρ^* by letting

$$\begin{aligned} &\big(\widetilde{\rho^*}(\alpha,\beta)\varphi^+\big)(z) = \varphi^+\Big(\frac{\alpha z-\overline{\beta}}{-\beta z+\overline{\alpha}}\Big) - \varphi^+\Big(\frac{-\overline{\beta}}{\overline{\alpha}}\Big) \\ \text{and}\quad &\big(\widetilde{\rho^*}(\alpha,\beta)\varphi^-\big)(z) = \varphi^-\Big(\frac{\alpha z-\overline{\beta}}{-\beta z+\overline{\alpha}}\Big) - \varphi^-\Big(\frac{\alpha}{-\beta}\Big), \end{aligned} \tag{8}$$

and we only have to check that $\widetilde{\rho^*}$ is still a group homomorphism.

A somewhat more elegant solution is the introduction of equivalence classes, as the norm on $\boldsymbol{H}_{1/2}$ already suggests. Consider the space

$$\boldsymbol{F} := \left\{ \sum_{n\in\mathbf{Z}} a_n z^n : \Big\| \sum_{n\in\mathbf{Z}} a_n z^n \Big\|_{\boldsymbol{F}} = \Big(\sum_{n\in\mathbf{Z}} |n|\, |a_n|^2 \Big)^{1/2} < \infty \right\}.$$

Obviously, $\|\cdot\|_{\boldsymbol{F}}$ is only a seminorm on $\boldsymbol{F}$ whose kernel $\boldsymbol{K}$ is just the space of constant functions. The constant functions, on the other hand, represent just those elements of $\boldsymbol{F}$ which, as functionals, vanish identically on $\boldsymbol{H}_{-1/2}$, and thus it is natural to define

$$\boldsymbol{H}_{1/2} := \boldsymbol{F}/\boldsymbol{K}$$

as dual space of $\boldsymbol{H}_{-1/2}$. We still have the natural decomposition

$$\boldsymbol{H}_{1/2} = \boldsymbol{H}^{+}_{1/2} \oplus \boldsymbol{H}^{-}_{1/2}.$$

Corollary 10. *$\rho^*(G)$ acts isometrically and irreducibly on the spaces $\boldsymbol{H}^{+}_{1/2}(\mathbb{T})$ and $\boldsymbol{H}^{-}_{1/2}(\mathbb{T})$.*

Proof. An element $\varphi \in \boldsymbol{H}^{+}_{1/2}$ represents an equivalence class of holomorphic functions in the unit disk, differing from each other by constants. Then

$$\big(\rho^*(\alpha,\beta)\varphi\big)(z) = \varphi\Big(\frac{\alpha z-\overline{\beta}}{-\beta z+\overline{\alpha}}\Big)$$

also represents holomorphic functions on the unit disk, since $z \mapsto \dfrac{\alpha z-\overline{\beta}}{-\beta z+\overline{\alpha}}$ maps the unit circle onto itself. That we indeed have $\rho^*(\alpha,\beta)\varphi \in \boldsymbol{H}^{+}_{1/2}$ follows once more from the isometry property, which again is an immediate consequence of $\rho^*(\alpha,\beta) = \big(\rho(\overline{\alpha},-\beta)\big)^*$ and the isometry property of ρ. Also, the irreducibility of the action of ρ^* follows from that of ρ by Lemma 18 and the reflexivity of our setup.

The same may be concluded for $\boldsymbol{H}^-_{1/2}$ since $CC : \boldsymbol{H}^+_{1/2} \to \boldsymbol{H}^-_{1/2}$ is again an antilinear isometry commuting with all $\rho^*(\alpha,\beta)$. $\square$

Remark. The appearance of the space $\boldsymbol{H}^+_{1/2}(\mathbb{T})$ in this connection comes as no surprise. In [3], Möbius-invariant function spaces have been defined as spaces of analytic functions on the unit disk on which ρ^* acts isometrically, and in [2], it is shown that under mild technical assumptions, the space $\boldsymbol{H}^+_{1/2}(\mathbb{T})$, known as the *Dirichlet space*, is the unique Hilbert space with this property.

2.7. The orthogonality relation

Theorem 11. *For f_1, $\varphi_2 \in \boldsymbol{H}_{-1/2}$ and f_2, $\varphi_1 \in \boldsymbol{H}_{1/2}$, we have*

$$\int_G \langle f_1\,,\, \rho^*(\alpha,\beta)\varphi_1\rangle\,\langle \rho(\alpha,\beta)\varphi_2\,,\, f_2\rangle\, dm(\alpha,\beta) = \\ = \langle f_1^+, f_2^+\rangle\,\langle \varphi_2^+,\varphi_1^+\rangle + \langle f_1^-, f_2^-\rangle\,\langle \varphi_2^-,\varphi_1^-\rangle\,. \quad (9)$$

Proof. In analogy to the proof of Proposition 7, we begin by evaluating the integral

$$\int_G A^*_{n,m}(\alpha,\beta)\, A_{p,q}(\alpha,\beta)\, dm(\alpha,\beta) =: C_{n,m,p,q}$$

for $n\,p \neq 0$. If $n\,m \leq 0$ or $p\,q \leq 0$, then $A^*_{n,m} \equiv 0$ or $A_{p,q} \equiv 0$ and thus $C_{n,m,p,q} = 0$.

For n, m, p, $q > 0$, we have by Lemma 6

$$\begin{aligned} C_{n,m,p,q} &= \int_G \left(\frac{1}{\alpha^{n+m}} \sum_{j=1}^{\min\{n,m\}} \binom{m}{m-j}(-\beta)^{m-j}\binom{n-1}{n-j}\overline{\beta}^{n-j}\right) \\ &\qquad \left(\frac{1}{\overline{\alpha}^{p+q}} \sum_{\ell=1}^{\min\{p,q\}} \binom{q}{q-\ell}\beta^{q-\ell}\binom{p-1}{p-\ell}(-\overline{\beta})^{p-\ell}\right) dm(\alpha,\beta) \\ &= \sum_{j=1}^{\min\{n,m\}} \sum_{\ell=1}^{\min\{p,q\}} \binom{m}{m-j}\binom{n-1}{n-j}\binom{q}{q-\ell}\binom{p-1}{p-\ell}(-1)^{m-j+p-\ell} \\ &\qquad \int_0^\infty \oint_{|u|=1} \oint_{|v|=1} \frac{v^{m-n+q-p}}{u^{n+m-p-q}}\, \frac{(\sinh s)^{m+n-2j+q+p-2\ell}}{(\cosh s)^{n+m+p+q}}\, \frac{du}{2\pi i\, u}\, \frac{dv}{2\pi i\, v}\, \sinh 2s\, ds\,. \end{aligned}$$

Now integration with respect to u and v yields 0 unless we have both $m-n+q-p=0$ and $n+m-p-q=0$, *i.e.*, $(n,m)=(q,p)$. In this case, we obtain

$$\begin{aligned}
C_{n,m,m,n} &= \sum_{j,\ell=1}^{\min\{n,m\}} \binom{m}{m-j}\binom{n-1}{n-j}\binom{n}{n-\ell}\binom{m-1}{m-\ell}(-1)^{j+\ell}\int_0^\infty \frac{(\sinh s)^{2m+2n-2j-2\ell+1}}{(\cosh s)^{2m+2n-1}}\,2\,ds \\
&\underset{(\text{Lemma A.2})}{=} \sum_{j,\ell=1}^{\min\{n,m\}} (-1)^{j+\ell}\binom{m}{m-j}\binom{n-1}{n-j}\binom{n}{n-\ell}\binom{m-1}{m-\ell}\frac{(m+n-j-\ell)!\,(j+\ell-2)!}{(m+n-1)!} \\
&= \frac{1}{n\binom{m+n-1}{m-1}} \sum_{j=1}^{\min\{n,m\}} (-1)^{n-j}\binom{m}{j} \sum_{\ell=1}^{\min\{n,m\}} (-1)^{n-\ell}\binom{n}{\ell}\binom{m+n-j-\ell}{n-j}\binom{j+\ell-2}{j-1} \\
&\underset{(\text{Lemma A.1})}{=} \frac{1}{n\binom{m+n-1}{m-1}}(-1)^{n-1}\binom{m}{1}(-1)^{n+1}\binom{m+n-1}{n-1} = 1\,.
\end{aligned}$$

For $n,\ m>0$ and $p,\ q<0$, we have

$$\begin{aligned}
C_{n,m,p,q} &= \int_G A^*_{n,m}(\alpha,\beta)\,A_{p,q}(\alpha,\beta)\,dm(\alpha,\beta) = \int_G A^*_{n,m}(\alpha,\beta)\,\overline{A_{|p|,|q|}(\alpha,\beta)}\,dm(\alpha,\beta) \\
&= \int_G \left(\frac{1}{\alpha^{n+m}} \sum_{j=1}^{\min\{n,m\}} \binom{m}{m-j}(-\beta)^{m-j}\binom{n-1}{n-j}\bar{\beta}^{n-j}\right) \\
&\qquad \left(\frac{1}{\bar{\alpha}^{|p|+|q|}} \sum_{\ell=1}^{\min\{|p|,|q|\}} \binom{|q|}{|q|-\ell}(-\bar{\beta})^{|q|-\ell}\binom{|p|-1}{|p|-\ell}\beta^{|p|-\ell}\right) dm(\alpha,\beta) \\
&= \sum_{j=1}^{\min\{n,m\}} \sum_{\ell=1}^{\min\{|p|,|q|\}} \binom{m}{m-j}\binom{n-1}{n-j}\binom{|q|}{|q|-\ell}\binom{|p|-1}{|p|-\ell}(-1)^{m-j+|q|-\ell} \\
&\qquad \int_0^\infty \oint_{|u|=1} \oint_{|v|=1} \frac{v^{m-n-|q|+|p|}}{u^{n+m+|p|+|q|}}\,\frac{(\sinh s)^{n+m-2j+|p|+|q|-2\ell}}{(\cosh s)^{n+m+|p|+|q|}}\,\frac{du}{2\pi i\,u}\,\frac{dv}{2\pi i\,v}\,\sinh 2s\,ds\,,
\end{aligned}$$

and integration with respect to u yields 0.

For $n,\ m<0$, we have

$$C_{n,m,p,q} = \int_G A^*_{n,m}\,A_{p,q}\,dm = \int_G \overline{A^*_{-n,-m}}\,\overline{A_{-p,-q}}\,dm = \overline{C_{-n,-m,-p,-q}}\,.$$

All in all, we find for $n,\ p\neq 0$

$$C_{n,m,p,q} = \begin{cases} \delta_{(n,m),(q,p)}\,, & nm>0\,,\\ 0\,, & nm\le 0\,. \end{cases}$$

From this, we obtain

$$\int_G \langle f_1\,,\, \rho^*(\alpha,\beta)\varphi_1\rangle\,\langle \rho(\alpha,\beta)\varphi_2\,,\, f_2\rangle\, dm(\alpha,\beta) =$$

$$\begin{aligned}
&= \int_G \Big\langle \sum_{n\neq 0} a_n\, z^n\,,\, \rho^*(\alpha,\beta) \sum_{m\in\mathbf{Z}} b_m\, z^m \Big\rangle \Big\langle \rho(\alpha,\beta) \sum_{p\neq 0} c_p\, z^p\,,\, \sum_{q\in\mathbf{Z}} d_q\, z^q \Big\rangle\, dm(\alpha,\beta) \\
&= \sum_{n\neq 0} a_n \sum_{m\neq 0} \overline{b_m} \sum_{p\neq 0} c_p \sum_{q\neq 0} \overline{d_q}\, C_{n,m,p,q} \\
&= \sum_{n,m>0} a_n\, \overline{b_m}\, c_m\, \overline{d_n} + \sum_{n,m<0} a_n\, \overline{b_m}\, c_m\, \overline{d_n} \\
&= \Big(\sum_{n>0} a_n\, \overline{d_n}\Big) \Big(\sum_{m>0} c_m\, \overline{b_m}\Big) + \Big(\sum_{n<0} a_n\, \overline{d_n}\Big) \Big(\sum_{m<0} c_m\, \overline{b_m}\Big)\,,
\end{aligned}$$

which yields the claim. □

Corollary 12 (Reproducing Property). *Choose $\varphi_1 \in \boldsymbol{H}_{1/2}$ and $\varphi_2 \in \boldsymbol{H}_{-1/2}$ with*

$$\langle \varphi_2^+, \varphi_1^+\rangle = \langle \varphi_2^-, \varphi_1^-\rangle = 1\,.$$

Then we have for all $f \in \boldsymbol{H}_{-1/2}$ that

$$\int_G \langle f\,,\, \rho^*(\alpha,\beta)\varphi_1\rangle\, \rho(\alpha,\beta)\varphi_2\, dm(\alpha,\beta) = f \tag{10}$$

in the weak sense.

2.8. A different perspective

As we have shown in Proposition 9, $\rho(G)$ acts isometrically on $\boldsymbol{H}_{-1/2}(\mathbb{T})$. If we give up the idea of treating $\boldsymbol{H}_{1/2}$ as dual space of $\boldsymbol{H}_{-1/2}$ and switch back to the Hilbert space point of view instead, Proposition 9 says that ρ is a unitary representation on $\boldsymbol{H}_{-1/2}$. This leads to a different orthogonality relation and thus also to another reproducing formula. We want to show how these are related to the results above.

To this end, we consider the canonical isometry

$$S:\, \boldsymbol{H}_{1/2} \to \boldsymbol{H}_{-1/2}\,, \quad \sum_{n\neq 0} a_n\, z^n \mapsto \sum_{n\neq 0} |n|\, a_n\, z^n\,.$$

Obviously, we have

$$S:\, \boldsymbol{H}_{1/2}^+ \to \boldsymbol{H}_{-1/2}^+ \quad \text{and} \quad S:\, \boldsymbol{H}_{1/2}^- \to \boldsymbol{H}_{-1/2}^-$$

and thus

$$(Sf)^+ = S(f^+) \quad \text{and} \quad (Sf)^- = S(f^-)\,.$$

Also, note that for $f = \sum a_n z^n \in \boldsymbol{H}_{-1/2}$ and $\varphi = \sum b_n z^n \in \boldsymbol{H}_{1/2}$, we have

$$\begin{aligned}\langle f\,,\,\varphi\rangle &= \sum_{n\neq 0} a_n\,\overline{b_n} \\ &= \sum_{n\neq 0} a_n\,\overline{|n|\,b_n}\,\frac{1}{|n|} = \langle f\,,\,S\varphi\rangle_{\boldsymbol{H}_{-1/2}}\,.\end{aligned}$$

Furthermore, S intertwines ρ and ρ^*, *i.e.*, we have the commutative diagram

$$\begin{tikzcd} \boldsymbol{H}_{1/2} \arrow[r, "\rho^*(g)"] \arrow[d, "S"'] & \boldsymbol{H}_{1/2} \arrow[d, "S"] \\ \boldsymbol{H}_{-1/2} \arrow[r, "\rho(g)"] & \boldsymbol{H}_{-1/2} \end{tikzcd}\,.$$

To see this, let $f \in \boldsymbol{H}_{-1/2}$ and $\varphi \in \boldsymbol{H}_{1/2}$. Then we have for all $g \in G$ that

$$\begin{aligned}\langle f\,,\,S\rho^*(g)\varphi\rangle_{\boldsymbol{H}_{-1/2}} &= \langle f\,,\,\rho^*(g)\varphi\rangle \\ &= \langle \rho(g^{-1})f\,,\,\varphi\rangle \\ &= \langle \rho(g^{-1})f\,,\,S\varphi\rangle_{\boldsymbol{H}_{-1/2}} = \langle f\,,\,\rho(g)S\varphi\rangle_{\boldsymbol{H}_{-1/2}}\,,\end{aligned}$$

where the last equality holds since ρ acts unitarily on $\boldsymbol{H}_{-1/2}$.

Consequently, we can prove Theorem 11 with the general methods described in Section 1 by the following argument.

Alternative proof of Theorem 11. With the aid of the map S, we want to deduce (9) from (1′).

We saw in Proposition 9 that ρ acts unitarily and irreducibly on $\boldsymbol{H}^+_{-1/2}$ and on $\boldsymbol{H}^-_{-1/2}$. Denoting the restrictions of ρ to these two spaces by ρ^+ and ρ^-, respectively, we need to show that ρ^+ and ρ^- are not unitarily equivalent. To this end, we shall show to which standard representations of $\boldsymbol{SU}_{1,1}$ they are equivalent. Defining the operator

$$(Z\psi)(z) := z\,\psi(z)$$

yields

$$\begin{aligned}\big(\rho(\alpha,\beta)Z\psi\big)(z) &= \frac{z}{(\alpha z-\overline{\beta})(-\beta z+\overline{\alpha})}\,\frac{\alpha z-\overline{\beta}}{-\beta z+\overline{\alpha}}\,\psi\Big(\frac{\alpha z-\overline{\beta}}{-\beta z+\overline{\alpha}}\Big) \\ &= z\,\frac{1}{(-\beta z+\overline{\alpha})^2}\,\psi\Big(\frac{\alpha z-\overline{\beta}}{-\beta z+\overline{\alpha}}\Big) = \big(Z\mathcal{D}^+_2(\alpha,\beta)\psi\big)(z)\,,\end{aligned}$$

i.e., ρ^+ is equivalent to $\mathcal{D}^+_2$. Since ρ^- is obtained from ρ^+ by complex conjugation, it follows that ρ^- is equivalent to $\mathcal{D}^-_2$, and it is well known that the two representations $\mathcal{D}^+_2$ and $\mathcal{D}^-_2$ are not equivalent (*e.g.*, see [23]).

So everything is prepared for the application of Proposition 4 with (1′) instead of (1). Since G is unimodular, we know that $C^+ = \lambda^+\,\mathrm{Id}_{\boldsymbol{H}^+_{-1/2}}$ and $C^- = \lambda^-\,\mathrm{Id}_{\boldsymbol{H}^-_{-1/2}}$. Finally, since ρ^+ and ρ^- are related via complex conjugation, we have

$\lambda^+ = \lambda^-$, and we may assume — after proper normalization of the Haar measure — that $\lambda^+ = \lambda^- = 1$.

Consequently, we have for $f_1, \varphi_2 \in \boldsymbol{H}_{-1/2}$ and $\varphi_1, f_2 \in \boldsymbol{H}_{1/2}$

$$\int_G \langle f_1 \,,\, \rho^*(\alpha,\beta)\varphi_1 \rangle \, \langle \rho(\alpha,\beta)\varphi_2 \,,\, f_2 \rangle \, dm(\alpha,\beta) =$$

$$\begin{aligned}
&= \int_G \langle f_1 \,,\, S\rho^*(\alpha,\beta)\varphi_1 \rangle_{\boldsymbol{H}_{-1/2}} \langle \rho(\alpha,\beta)\varphi_2 \,,\, Sf_2 \rangle_{\boldsymbol{H}_{-1/2}} \, dm(\alpha,\beta) \\
&= \int_G \langle f_1 \,,\, \rho(\alpha,\beta)S\varphi_1 \rangle_{\boldsymbol{H}_{-1/2}} \overline{\langle Sf_2 \,,\, \rho(\alpha,\beta)\varphi_2 \rangle_{\boldsymbol{H}_{-1/2}}} \, dm(\alpha,\beta) \\
&\underset{(1')}{=} \langle f_1^+ \,,\, (Sf_2)^+ \rangle_{\boldsymbol{H}_{-1/2}} \overline{\langle (S\varphi_1)^+ \,,\, \varphi_2^+ \rangle_{\boldsymbol{H}_{-1/2}}} \\
&\qquad + \langle f_1^- \,,\, (Sf_2)^- \rangle_{\boldsymbol{H}_{-1/2}} \overline{\langle (S\varphi_1)^- \,,\, \varphi_2^- \rangle_{\boldsymbol{H}_{-1/2}}} \\
&= \langle f_1^+ \,,\, f_2^+ \rangle \, \langle \varphi_2^+ \,,\, \varphi_1^+ \rangle + \langle f_1^- \,,\, f_2^- \rangle \, \langle \varphi_2^- \,,\, \varphi_1^- \rangle
\end{aligned}$$

as claimed. □

After realizing these equivalences, we should ask ourselves, why not work with the direct sum of $\mathcal{D}_2^+$ and $\mathcal{D}_2^-$ right away. This would cause technical problems since the two spaces on which these act are not disjoint, since both contain the constant functions. Also, the strong advantage of our approach is the usage of the standard inner product on the unit circle, which is certainly highly desirable for applications. Furthermore, we have found an example of a reproducing formula containing a pair of contragredient representations, and we shall study this phenomenon in general in Section 3. But first, we have a look at some graphical examples for the above setup.

2.9. Generators for $\rho(G)$

In order to get a visual impression of the action of the group under this representation, we describe two basic types of operators which generate $\rho(G)$. Note that for $\beta = 0$, we have $|\alpha| = 1$ and thus may write $\alpha = e^{-i\pi\omega}$, so we obtain

$$\big(\rho(\alpha,\beta)\varphi\big)(z) = \big(\rho(e^{-i\pi\omega},0)\varphi\big)(z) = \frac{z}{(e^{-i\pi\omega}\,z)(e^{i\pi\omega})}\,\varphi\Big(\frac{e^{-i\pi\omega}\,z}{e^{i\pi\omega}}\Big) = \varphi(e^{-2\pi i\omega}\,z)\,.$$

So $\rho(e^{-i\pi\omega},0)$ induces a rotation of $\mathbb{S}^1$ by the angle $2\pi\omega$; in other words, it is the translation operator T_ω on $\mathbb{T}$ by ω. On the other hand, if both α and β are nonnegative, the Möbius transformation

$$m_{(\alpha,-\overline{\beta})}(z) = m_{(\cosh(s),-\sinh(s))}(z) = \frac{\cosh(s)\,z - \sinh(s)}{-\sinh(s)\,z + \cosh(s)}$$

satisfies

$$m_{(\cosh(s),-\sinh(s))}(1) = 1 \quad \text{and} \quad m_{(\cosh(s),-\sinh(s))}(-1) = -1\,,$$

and locally around $\xi = 0$ behaves like a dilation on $\mathbb{T}$, which we want to show. To this end, let $t = e^{2s}$, so $\cosh(s) = \frac{t+1}{2\sqrt{t}}$ and $\sinh(s) = \frac{t-1}{2\sqrt{t}}$.

Lemma 13. *The mapping*

$$m_{(t+1,-(t-1))} : \mathbb{S}^1 \to \mathbb{S}^1, \quad z \mapsto \frac{(t+1)z-(t-1)}{-(t-1)z+(t+1)}$$

corresponds to the mapping

$$d_t : \mathbb{T} \to \mathbb{T}, \quad \xi \mapsto \frac{1}{\pi} \arctan\big(t \tan(\pi\xi)\big) .$$

Proof.

$$\begin{aligned} d_t(\xi) &= \frac{1}{2\pi} \arg\Big(\frac{(t+1)\,e^{2\pi i\,\xi}-(t-1)}{-(t-1)\,e^{2\pi i\,\xi}+(t+1)}\Big) \\ &= \frac{1}{2\pi} \arg\Big(\frac{(t+1)\,e^{i\pi\xi}-(t-1)\,e^{-i\pi\xi}}{-(t-1)\,e^{i\pi\xi}+(t+1)\,e^{-i\pi\xi}}\Big) \\ &= \frac{1}{2\pi} \arg\Big(\frac{2\cos(\pi\xi)+2\,i\,t\,\sin(\pi\xi)}{2\cos(\pi\xi)-2\,i\,t\,\sin(\pi\xi)}\Big) \\ &= \frac{1}{2\pi}\, 2 \arg\big(\cos(\pi\xi)+i\,t\,\sin(\pi\xi)\big) = \frac{1}{\pi} \arctan\big(t \tan(\pi\xi)\big) , \end{aligned}$$

where we used $\arg(w/\overline{w}) = 2\arg(w)$. □

In Figure 1, we show the graphs of d_t for $t = 2$, 4, and 8.

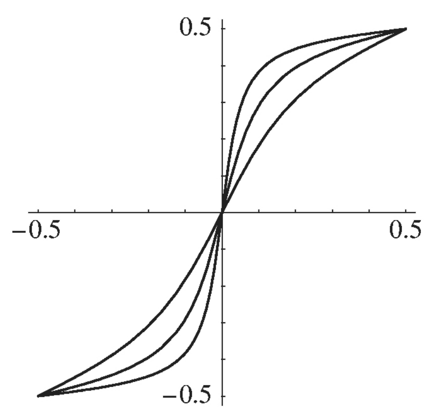

Fig. 1. Graphs of d_2, d_4, and d_8.

It is worth noting that if we transfer the stereographic projection $\mathbb{S}^1 \to \overline{\mathbb{R}}$ (with $1 \mapsto 0$) to $\mathbb{T}$, we obtain the mapping $\mathbb{T} \to \overline{\mathbb{R}}$, $\xi \mapsto \tan(\pi\xi)$. Thus the above shows that the transformations $d_t : \mathbb{T} \to \mathbb{T}$ and $m_{(t+1,-t+1)} : \mathbb{S}^1 \to \mathbb{S}^1$ correspond to the usual dilation $\mathbb{R} \to \mathbb{R}$, $x \mapsto t\,x$ in the sense that the stereographic projection intertwines $m_{(t+1,-t+1)}$ with this dilation on $\mathbb{R}$. Consequently, $\{d_t : t \in \mathbb{R}^+\}$ is a group with $d_{t_1}\, d_{t_2} = d_{t_1 t_2}$, which of course can also be seen directly from

$$\begin{pmatrix} \frac{t_1+1}{2\sqrt{t_1}} & \frac{t_1-1}{2\sqrt{t_1}} \\ \frac{t_1-1}{2\sqrt{t_1}} & \frac{t_1+1}{2\sqrt{t_1}} \end{pmatrix} \begin{pmatrix} \frac{t_2+1}{2\sqrt{t_2}} & \frac{t_2-1}{2\sqrt{t_2}} \\ \frac{t_2-1}{2\sqrt{t_2}} & \frac{t_2+1}{2\sqrt{t_2}} \end{pmatrix} = \begin{pmatrix} \frac{t_1 t_2+1}{2\sqrt{t_1 t_2}} & \frac{t_1 t_2-1}{2\sqrt{t_1 t_2}} \\ \frac{t_1 t_2-1}{2\sqrt{t_1 t_2}} & \frac{t_1 t_2+1}{2\sqrt{t_1 t_2}} \end{pmatrix}.$$

Therefore it makes sense to refer to

$$D_t := \rho\Big(\frac{t+1}{2\sqrt{t}}, \frac{t-1}{2\sqrt{t}}\Big)$$

as *(generalized) dilation operator* on $\mathbb{S}^1$ or $\mathbb{T}$.

From these two classes of operators, $\{T_\omega : \omega \in \mathbb{T}\}$ and $\{D_t : t \geq 1\}$, we can generate all $\rho(\alpha,\beta)$. Since

$$\begin{pmatrix} e^{2\pi i\phi}\,\frac{t+1}{2\sqrt{t}} & e^{2\pi i\theta}\,\frac{t-1}{2\sqrt{t}} \\ e^{-2\pi i\theta}\,\frac{t-1}{2\sqrt{t}} & e^{-2\pi i\phi}\,\frac{t+1}{2\sqrt{t}} \end{pmatrix} =$$

$$= \begin{pmatrix} e^{-i\pi(-\phi-\theta)} & 0 \\ 0 & e^{i\pi(-\phi-\theta)} \end{pmatrix} \begin{pmatrix} \frac{t+1}{2\sqrt{t}} & \frac{t-1}{2\sqrt{t}} \\ \frac{t-1}{2\sqrt{t}} & \frac{t+1}{2\sqrt{t}} \end{pmatrix} \begin{pmatrix} e^{-i\pi(\theta-\phi)} & 0 \\ 0 & e^{i\pi(\theta-\phi)} \end{pmatrix},$$

we have

$$\rho\Big(e^{2\pi i\phi}\,\frac{t+1}{2\sqrt{t}}, e^{2\pi i\theta}\,\frac{t-1}{2\sqrt{t}}\Big) = T_{-\phi-\theta}\, D_t\, T_{-\phi+\theta}\,,$$

so each operator is a combination of a translation, a dilation, and another translation.

In order to allow the generalized dilation operator to be centered at some other point $\xi \in \mathbb{T}$, we define

$$D_{\xi,t} := T_\xi\, D_t\, T_{-\xi} = \rho\Big(\frac{t+1}{2\sqrt{t}}, e^{-2\pi i\xi}\,\frac{t-1}{2\sqrt{t}}\Big).$$

Then we can write

$$\rho\Big(e^{2\pi i\phi}\,\frac{t+1}{2\sqrt{t}}, e^{2\pi i\theta}\,\frac{t-1}{2\sqrt{t}}\Big) = T_{-2\phi}\, T_{\phi-\theta}\, D_t\, T_{-(\phi-\theta)} = T_{-2\phi}\, D_{\phi-\theta,t}\,,$$

i.e., a dilation centered at the point $\xi = \phi - \theta$ followed by a translation.

This corresponds to a well-known parameterization of G given by

$$e^{i\gamma}\,\frac{z-a}{-\bar{a}\,z+1}, \quad \gamma \in [0, 2\pi[\,,\ a \in \mathbb{D},$$

where we have $\gamma = 4\pi\phi$ and $a = e^{2\pi i(\theta-\phi)}\tanh(s) = e^{2\pi i(\theta-\phi)}\,\frac{t-1}{t+1}$. Note that the Möbius transformation $m_{(1,-a)}(z) = \frac{z-a}{-\bar{a}\,z+1}$ satisfies

$$m_{(1,-a)}(a) = 0\,, \quad m_{(1,-a)}(1/\bar{a}) = \infty\,, \quad \text{and} \quad m_{(1,-a)}(\pm\operatorname{sign}(a)) = \pm\operatorname{sign}(a)\,.$$

So $m_{(1,-a)}$ indeed corresponds to a generalized dilation on $\mathbb{S}^1$ centered at $\operatorname{sign}(a)$.

2.10. Graphical examples for the action of ρ and ρ^*

To visualize the action of $\rho(G)$ and $\rho^*(G)$, we use

$$\varphi(z) = z + z^{-1} \quad \bigl(= 2\cos(2\pi\xi) \text{ for } z = e^{2\pi i \xi} \bigr)\,.$$

In Figures 2 and 3, we show some examples for $\rho(\alpha,\beta)\varphi$ and $\rho^*(\alpha,\beta)\varphi$.

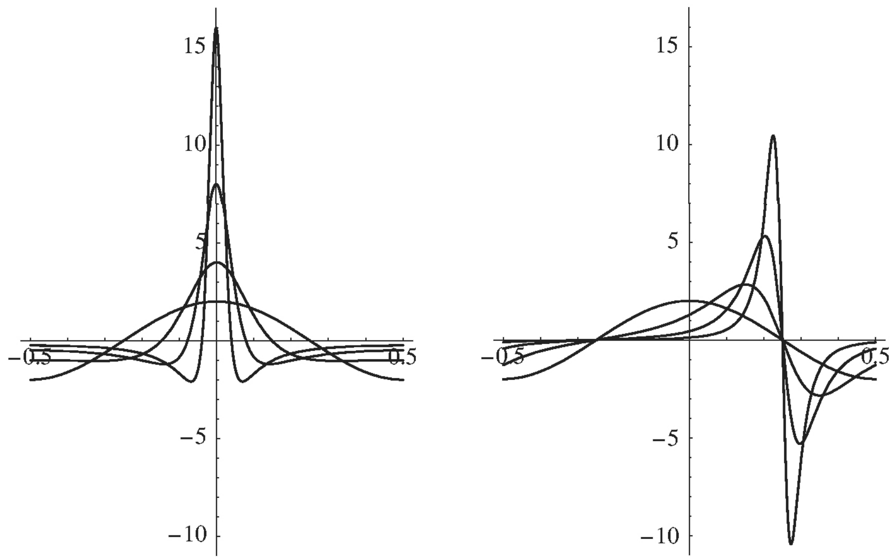

Fig. 2. Graphs of $\rho\bigl(\frac{t+1}{2\sqrt{t}}, \frac{t-1}{2\sqrt{t}}\bigr)\varphi$ and $\rho\bigl(\frac{t+1}{2\sqrt{t}}, e^{-2\pi i/4}\,\frac{t-1}{2\sqrt{t}}\bigr)\varphi$ for $t = 1, 2, 4,$ and 8.

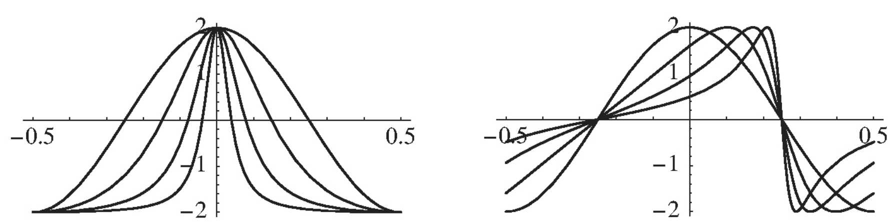

Fig. 3. Graphs of $\rho^*\bigl(\frac{t+1}{2\sqrt{t}}, \frac{t-1}{2\sqrt{t}}\bigr)\varphi$ and $\rho^*\bigl(\frac{t+1}{2\sqrt{t}}, e^{-2\pi i/4}\,\frac{t-1}{2\sqrt{t}}\bigr)\varphi$ for $t = 1, 2, 4,$ and 8.

On the left, we apply D_t for various values of t, and on the right, we show the effect of choosing $\xi = \frac{1}{4}$ as center of dilation, *i.e.*, we apply $D_{\frac{1}{4},t}$. Note that since φ exhibits even symmetry about $\xi = 0$ and odd symmetry about $\xi = \frac{1}{4}$, we obtain functions of both symmetry types from the same generating function. This is highly desirable; in [6], it was shown how this can be achieved for the case of affine wavelets on $\mathbb{R}$. Using $k = 3$ instead of $k = 2$ as scaling factor in the multiresolution analysis leads to a wavelet basis generated from two basic functions, and the authors exhibit conditions making it possible to choose one of these to be even and the other one odd. Note, however, that this actually requires two generating functions, while due to the richness of operators in our setup, we obtain functions of both symmetry types from the same generator.

Allowing arbitrary points as center of dilation yields functions without any symmetry, though, as Figure 4 shows.

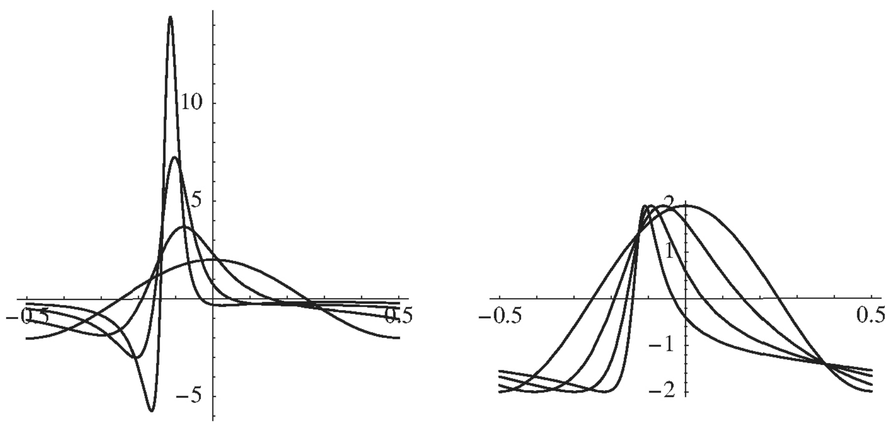

Fig. 4. Graphs of $\rho\left(\frac{t+1}{2\sqrt{t}}, e^{2\pi i/8}\,\frac{t-1}{2\sqrt{t}}\right)\varphi$ and $\rho^*\left(\frac{t+1}{2\sqrt{t}}, e^{2\pi i/8}\,\frac{t-1}{2\sqrt{t}}\right)\varphi$ for $t = 1, 2, 4$, and 8.

3. Reproducing Properties from Nonunitary Representations

In the previous section, we presented a pair of representations which together gave rise to a reproducing property. Now we want to try and exhibit a general principle behind this phenomenon. To this end, we have to give up the Hilbert space setup and consider a Banach space $\boldsymbol{V}$ instead. Since we want to maintain the square integrability property, we have to discuss operators $\boldsymbol{V} \to \boldsymbol{L}^2(G)$ together with their adjoints. This turns out to be formally much easier if we do not work with the space $\boldsymbol{V}'$ of bounded linear functionals, but instead use $\boldsymbol{V}^*$, the space of bounded antilinear functionals, as dual space for $\boldsymbol{V}$. The two spaces $\boldsymbol{V}'$ and $\boldsymbol{V}^*$ are easily seen to be (antilinearly!) isometrically isomorphic, so essentially nothing changes; but the pairing between $\boldsymbol{V}$ and $\boldsymbol{V}^*$ is (not bilinear, but) sesquilinear, and thus a generalization of the inner product in a Hilbert space. Consequently, we use Hilbert space notation and define

$$\langle v, w\rangle_{(\boldsymbol{V},\boldsymbol{V}^*)} := \overline{w(v)}$$

for $v \in \boldsymbol{V}$ and $w \in \boldsymbol{V}^*$. We include some more details on $\boldsymbol{V}^*$ in Section A.1.

3.1. Representations and continuity

Definition 14. Let G be a locally compact group and $\boldsymbol{V}$ a Banach space. A *representation* of G on $\boldsymbol{V}$ is a group homomorphism $G \to \boldsymbol{GL}(\boldsymbol{V})$. The representation ρ is *bounded*, if

$$\|\rho\| := \sup_{g\in G} \|\rho(g)\|_{\mathcal{L}(\boldsymbol{V})} < \infty .$$

The representation ρ is *continuous*, if the mapping

$$G \times \boldsymbol{V} \to \boldsymbol{V}, \quad (g, v) \mapsto \rho(g)\, v$$

is continuous. ρ is *separately continuous*, if it is continuous with the strong operator topology on $\mathcal{L}(\boldsymbol{V})$, *i.e.*, if for each $v \in \boldsymbol{V}$, the mapping

$$G \to \boldsymbol{V}, \quad g \mapsto \rho(g)\, v$$

is continuous. (Note that for fixed $g \in G$, the mapping $v \mapsto \rho(g)\, v$ is continuous by definition.)

Furthermore, we say that ρ is *weakly continuous*, if it is continuous with the weak operator topology on $\mathcal{L}(\boldsymbol{V})$, *i.e.*, if the mapping

$$G \to \mathbb{C}, \quad g \mapsto \langle \rho(g)v, w\rangle$$

is continuous for all $v \in \boldsymbol{V}$ and $w \in \boldsymbol{V}^*$. Analogously, ρ is *weak*-continuous*, if $\boldsymbol{V}$ is a dual space and ρ is continuous with the weak* operator topology on $\mathcal{L}(\boldsymbol{V})$.

The following result shows that there is no need to distinguish between continuity and separate continuity. For related statements, see also [23] and [24].

Proposition 15. [22] *A separately continuous representation ρ of a locally compact group G on a Banach space $\boldsymbol{V}$ is continuous.*

Proof. Let $g \in G$ and $v \in \boldsymbol{V}$, and $\varepsilon > 0$ be given. By separate continuity of ρ, there is a neighborhood U of g such that $\|\rho(h)v - \rho(g)v\|_{\boldsymbol{V}} < \varepsilon/2$ for all $h \in U$, and since G is locally compact, we may choose U to be compact. For each $w \in \boldsymbol{V}$, the mapping $h \mapsto \rho(h)w$ is continuous and thus bounded on U. Therefore, by the uniform boundedness principle, there exists $C > 0$ such that $\|\rho(h)\|_{\mathcal{L}(\boldsymbol{V})} \le C$ for all $h \in U$. Letting $\delta = \varepsilon/(2C) > 0$, we have for all $(h, w) \in U \times U_\delta(v)$ that

$$\begin{aligned} \|\rho(h)w - \rho(g)v\|_{\boldsymbol{V}} &\le \|\rho(h)w - \rho(h)v\|_{\boldsymbol{V}} + \|\rho(h)v - \rho(g)v\|_{\boldsymbol{V}} \\ &\le C\, \|w - v\|_{\boldsymbol{V}} + \|\rho(h)v - \rho(g)v\|_{\boldsymbol{V}} \\ &< C\, \frac{\varepsilon}{2C} + \frac{\varepsilon}{2} = \varepsilon , \end{aligned}$$

so ρ is continuous. □

3.2. The contragredient representation

Lemma 16. *Let $\rho : G \to \boldsymbol{GL}(\boldsymbol{V})$ be a representation of a locally compact group G on a Banach space $\boldsymbol{V}$. Then the map $\rho^* : G \to \boldsymbol{GL}(\boldsymbol{V}^*)$ given by*

$$\rho^*(g) := \big(\rho(g^{-1})\big)^*$$

is a representation of G on the dual space $\boldsymbol{V}^*$, the contragredient representation *of ρ.*

ρ^* is bounded if and only if ρ is bounded.

If ρ is continuous, then ρ^* is weak-continuous.*

Proof. The definition of ρ^* immediately implies that $\rho^* : G \to \mathcal{L}(\boldsymbol{V}^*)$, and we have

$$\begin{aligned}
\rho^*(g_1\, g_2) &= \big(\rho((g_1\, g_2)^{-1})\big)^* \\
&= \big(\rho(g_2^{-1} g_1^{-1})\big)^* \\
&= \big(\rho(g_2^{-1})\rho(g_1^{-1})\big)^* \\
&= \big(\rho(g_1^{-1})\big)^* \big(\rho(g_2^{-1})\big)^* = \rho^*(g_1)\, \rho^*(g_2)\,,
\end{aligned}$$

so ρ^* is a group homomorphism, and thus $\rho^* : G \to \boldsymbol{GL}(\boldsymbol{V}^*)$.

Since $\|T^*\|_{\mathcal{L}(\boldsymbol{V}^*)} = \|T\|_{\mathcal{L}(\boldsymbol{V})}$, boundedness of ρ is equivalent to that of ρ^*.

ρ being continuous means that for each $v \in \boldsymbol{V}$, the mapping

$$G \to \boldsymbol{V}\,, \quad g \mapsto \rho(g)\, v\,,$$

is continuous. Consequently, for each $v \in \boldsymbol{V}$ and $w \in \boldsymbol{V}^*$, the mapping

$$G \to \mathbb{C}, \quad g \mapsto \langle v, \rho^*(g) w\rangle = \langle \rho(g^{-1})\, v, w\rangle\,,$$

is continuous, since $g \mapsto g^{-1}$ is continuous on the topological group G. So ρ^* is weak*-continuous as claimed. □

Remark. It is worth noting that for a unitary representation on a Hilbert space, we have $\rho^* = \rho$, since $\rho^*(g) = \big(\rho(g^{-1})\big)^* = \big(\rho(g^{-1})\big)^{-1} = \rho(g)$. Unitarity is imitated by the pair ρ and ρ^* in the sense that for all $g \in G$, we have

$$\langle \rho(g)v, \rho^*(g)w\rangle = \langle v, w\rangle \tag{11}$$

for $v \in \boldsymbol{V}$ and $w \in \boldsymbol{V}^*$.

In case $\boldsymbol{V}$ is a reflexive Banach space, we can strengthen the above statement somewhat (compare [15] and [19]).

Proposition 17. *Let $\rho : G \to \boldsymbol{GL}(\boldsymbol{V})$ be a continuous, bounded representation of a locally compact group G on a reflexive Banach space $\boldsymbol{V}$. Then the contragredient representation ρ^* is continuous.*

Proof. We know from Lemma 16 that ρ^* is weak*-continuous, which in a reflexive Banach space is the same as weakly continuous. Furthermore, under the assumption that $\boldsymbol{V}$ is reflexive, we may apply [19], Thm. (22.8) which states that in that case, a bounded, weakly continuous representation is strongly continuous. □

This result can also be found in [15] for unitary representations in a Hilbert space. We would like to extend it to the non-reflexive case, and to this end, we need a few preliminary results. We begin with comparing irreducibility of ρ and ρ^*.

Lemma 18. *Let ρ be a representation on a Banach space $\boldsymbol{V}$ and ρ^* its contragredient representation. If ρ^* is irreducible, then so is ρ.*

Proof. Assume that ρ is reducible, *i.e.*, that there exists a closed ρ-invariant subspace $\boldsymbol{W}$ of $\boldsymbol{V}$ with $\{0\} \neq \boldsymbol{W} \neq \boldsymbol{V}$. Then its orthogonal complement

$$\boldsymbol{W}^{\perp} = \{w \in \boldsymbol{V}^* : \langle v, w\rangle = 0 \text{ for all } v \in \boldsymbol{W}\}$$

is a closed subspace of $\boldsymbol{V}^*$ with $\{0\} \neq \boldsymbol{W}^{\perp} \neq \boldsymbol{V}^*$, which follows from the Hahn-Banach-extension theorem. Namely, for $v \in \boldsymbol{W} \setminus \{0\}$, there exists $w \in \boldsymbol{V}^*$ with $\langle v, w\rangle \neq 0$, so $\boldsymbol{W}^{\perp} \neq \boldsymbol{V}^*$; and on the other hand, for $v \in \boldsymbol{V} \setminus \boldsymbol{W}$, there exists $w \in \boldsymbol{W}^{\perp}$ with $\langle v, w\rangle \neq 0$, so $\boldsymbol{W}^{\perp} \neq \{0\}$. Furthermore, $\boldsymbol{W}^{\perp}$ is ρ^*-invariant. To see this, consider $g \in G$ and $w \in \boldsymbol{W}^{\perp}$. Then we have for all $v \in \boldsymbol{W}$ that

$$\langle v, \rho^*(g)w\rangle = \langle \rho(g^{-1})v, w\rangle = 0$$

by the ρ-invariance of $\boldsymbol{W}$, so $\rho^*(g)w \in \boldsymbol{W}^{\perp}$. Thus ρ^* is reducible also. □

Remark. Lemma 18 immediately implies that on a reflexive Banach space, irreducibility of ρ is equivalent to that of ρ^*, since $\rho^{**} = \rho$. If $\boldsymbol{V}$ is non-reflexive, though, we have that $\boldsymbol{V}$ is a nontrivial, closed, ρ^{**}-invariant subspace of $\boldsymbol{V}^{**}$, since $\rho^{**}(g)\big|_{\boldsymbol{V}} = \rho(g)$ for all $g \in G$. So even if ρ is irreducible, ρ^{**} is reducible, and the converse of Lemma 18 becomes false for at least one of the pairs (ρ, ρ^*) and (ρ^*, ρ^{**}).

The essential tool both in [15] and in [19] for the results corresponding to Proposition 17 is the algebra representation of $\boldsymbol{L}^1(G)$ induced from a bounded representation of G, which we now define.

Theorem 19. *Let ρ be a bounded, continuous representation of a locally compact group G in a Banach space $\boldsymbol{V}$, with contragredient representation ρ^*. Then*

$$\sigma^* : \boldsymbol{L}^1(G) \to \boldsymbol{\mathcal{L}}(\boldsymbol{V}^*)\,, \quad f \mapsto \int_G f(g)\, \rho^*(g)\, dm(g)\,,$$

is a bounded algebra representation.

Proof. For $f \in \boldsymbol{L}^1(G)$ and $w \in \boldsymbol{V}^*$, we define $\sigma^*(f)\, w \in \boldsymbol{V}^*$ via

$$\langle v, \sigma^*(f)\, w\rangle = \int_G \overline{f(g)}\, \langle v, \rho^*(g)\, w\rangle\, dm(g)\,.$$

Since, as we saw before, the map $g \mapsto \langle v, \rho^*(g)\, w\rangle$ is continuous, the integrand is measurable. Boundedness of ρ^* implies

$$\left|\overline{f(g)}\,\langle v, \rho^*(g)\, w\rangle\right| \leq |f(g)|\, \|v\|_{V}\, \|\rho^*\|\, \|w\|_{V^*},$$

and thus the integral is well-defined. $\sigma^*(f)\, w$ is antilinear in v (recall that $w(v) = \overline{\langle v, w\rangle}$) and satisfies

$$\left|\langle v, \sigma^*(f)\, w\rangle\right| \leq \|f\|_{L^1(G)}\, \|v\|_{V}\, \|\rho^*\|\, \|w\|_{V^*},$$

i.e., $\sigma^*(f)\, w \in V^*$ with

$$\left\|\sigma^*(f)\, w\right\|_{V^*} \leq \|f\|_{L^1(G)}\, \|\rho^*\|\, \|w\|_{V^*}.$$

Furthermore, the mapping

$$w \mapsto \sigma^*(f)\, w = \int\limits_G f(g)\, \rho^*(g)\, w\, dm(g)$$

is linear in w, and thus $\sigma^*(f) \in \mathcal{L}(V^*)$ with $\left\|\sigma^*(f)\right\|_{\mathcal{L}(V^*)} \leq \|f\|_{L^1(G)}\, \|\rho^*\|$, so σ^* is bounded. Obviously, σ^* is linear in f, and it remains to show that $\sigma^*(f * k) = \sigma^*(f)\, \sigma^*(k)$. Indeed, we have for $v \in V$ and $w \in V^*$ that

$$\begin{aligned}
\left\langle v, \sigma^*(f * k)\, w\right\rangle &= \int\limits_G \int\limits_G \overline{f(h)}\, \overline{k(h^{-1}g)}\, dm(h)\, \left\langle v, \rho^*(g)\, w\right\rangle dm(g) \\
&= \int\limits_G \int\limits_G \overline{f(h)}\, \overline{k(g)}\, \left\langle v, \rho^*(h\, g)\, w\right\rangle dm(g)\, dm(h) \\
&= \int\limits_G \int\limits_G \overline{f(h)}\, \overline{k(g)}\, \left\langle \rho(h^{-1})\, v, \rho^*(g)\, w\right\rangle dm(g)\, dm(h) \\
&= \int\limits_G \overline{f(h)}\, \left\langle \rho(h^{-1})\, v, \sigma^*(k)\, w\right\rangle dm(h) \\
&= \int\limits_G \overline{f(h)}\, \left\langle v, \rho^*(h)\, \sigma^*(k)\, w\right\rangle dm(h) = \left\langle v, \sigma^*(f)\, \sigma^*(k)\, w\right\rangle,
\end{aligned}$$

where we may apply Fubini's theorem several times, since all integrals are absolutely convergent. □

Remark. Similar statements can be found in [15], Thm. V.5.2 and also in [19], Thm. (22.3) for the reflexive Banach space case. We should note that in [19], we find a footnote stating that V be reflexive is needed in the computation corresponding to the above, since otherwise the adjoint of an operator on V^* could not be regarded as an operator on V. Here, however, we do not encounter this problem since we are dealing with adjoints of operators to begin with.

The following lemma states the interaction between ρ^* and σ^*.

Lemma 20. *For $g \in G$ and $f \in \boldsymbol{L}^1(G)$, we have*

$$\rho^*(g)\,\sigma^*(f) = \sigma^*(L_g f)\,, \tag{12}$$

where L_g is left translation, i.e., $(L_g f)(h) = f(g^{-1}h)$.

Proof. The result follows from

$$\begin{aligned}\rho^*(g)\,\sigma^*(f) &= \int_G f(h)\,\rho^*(g\,h)\,dm(h)\\ &= \int_G f(g^{-1}h)\,\rho^*(h)\,dm(h) = \sigma^*(L_g f)\,.\end{aligned}$$

□

This property is the main ingredient for the proof of the following result, since it enables us to essentially derive continuity of ρ^* from the continuity of the left regular representation L on $\boldsymbol{L}^1(G)$.

Theorem 21. *Let $\rho : G \to \boldsymbol{GL}(\boldsymbol{V})$ be a bounded, continuous representation of a locally compact group G on a Banach space $\boldsymbol{V}$, with contragredient representation ρ^*. If ρ^* is irreducible, then it is continuous.*

Proof. We have to show that for all $w \in \boldsymbol{V}^*$ and $h \in G$, we have

$$\left\|\rho^*(g)w - \rho^*(h)w\right\|_{\boldsymbol{V}^*} \to 0 \quad (g \to h)\,.$$

If $w = 0$, there is nothing to prove. So assume $w \in \boldsymbol{V}^* \setminus \{0\}$, and let $h \in G$ and $\varepsilon > 0$ be given. Consider the space

$$\boldsymbol{W} = \overline{\left\{\sigma^*(f)w : f \in \boldsymbol{L}^1(G)\right\}}\,.$$

$\boldsymbol{W}$ is a closed subspace of $\boldsymbol{V}^*$, and we want to show that $\boldsymbol{W} = \boldsymbol{V}^*$. Because of (12), $\boldsymbol{W}$ is ρ^*-invariant, since $L_g\big(\boldsymbol{L}^1(G)\big) = \boldsymbol{L}^1(G)$ for all $g \in G$. So by the irreducibility of ρ^*, it suffices to show that $\boldsymbol{W}$ contains a nonzero element.

To this end, note that $w \neq 0$ means that there is $v \in \boldsymbol{V}$ with $\langle v, w\rangle \neq 0$, and after replacing v by an appropriate multiple we have

$$1 = \langle v, w\rangle = \big\langle v, \rho^*(e)w\big\rangle\,.$$

Since by Lemma 16, ρ^* is weakly continuous, there exists a neighborhood U of e in G such that

$$\mathrm{Re}\Big(\big\langle v, \rho^*(g)w\big\rangle\Big) > \frac{1}{2} \quad \text{for } g \in U,$$

and we may assume $0 < m(U) < \infty$, since G is locally compact. Therefore, the characteristic function χ_U of U is an element of $\boldsymbol{L}^1(G)$, and it satisfies

$$\mathrm{Re}\Big(\big\langle v, \sigma^*(\chi_U)w\big\rangle\Big) = \mathrm{Re}\Big(\int_U \big\langle v, \rho^*(g)w\big\rangle\,dm(g)\Big) > \frac{1}{2}\,m(U)\,,$$

hence we have $0 \neq \sigma^*(\chi_U)w \in \boldsymbol{W}$. So we may conclude that indeed $\boldsymbol{W} = \boldsymbol{V}^*$.

Consequently, we can choose $f \in \boldsymbol{L}^1(G)$ such that

$$\left\|\sigma^*(f)w - w\right\|_{\boldsymbol{V}^*} < \frac{\varepsilon}{2\|\rho^*\|\left(\|\rho^*\|+1\right)} .$$

Then we have for arbitrary $g \in G$ that

$$\begin{aligned}\left\|\rho^*(g)\sigma^*(f)w - \sigma^*(f)w\right\| &= \left\|\sigma^*(L_g f - f)w\right\| \\ &\le \|\rho^*\| \left\|L_g f - f\right\|_{\boldsymbol{L}^1(G)} \|w\|_{\boldsymbol{V}^*} .\end{aligned}$$

By continuity of the left regular representation L, we have $\|L_g f - f\|_{\boldsymbol{L}^1(G)} \to 0$ as $g \to e$, so there exists a neighborhood U of e in G with

$$\left\|L_g f - f\right\|_{\boldsymbol{L}^1(G)} < \frac{\varepsilon}{2\|\rho^*\|^2 \|w\|_{\boldsymbol{V}^*}} \quad \text{for } g \in U.$$

Then hU is a neighborhood of h, and we have for all $g \in hU$ that $h^{-1}g \in U$ and thus

$$\begin{aligned}&\left\|\rho^*(g)w - \rho^*(h)w\right\|_{\boldsymbol{V}^*} \le \\ &\quad \le \|\rho^*(h)\|_{\mathcal{L}(\boldsymbol{V}^*)} \left\|\rho^*(h^{-1}g)w - w\right\|_{\boldsymbol{V}^*} \\ &\quad \le \|\rho^*\| \Big(\left\|\rho^*(h^{-1}g)w - \rho^*(h^{-1}g)\sigma^*(f)w\right\|_{\boldsymbol{V}^*} + \\ &\qquad\qquad + \left\|\rho^*(h^{-1}g)\sigma^*(f)w - \sigma^*(f)w\right\|_{\boldsymbol{V}^*} + \left\|\sigma^*(f)w - w\right\|_{\boldsymbol{V}^*} \Big) \\ &\quad \le \|\rho^*\| \Big(\left(\|\rho^*(h^{-1}g)\|_{\mathcal{L}(\boldsymbol{V}^*)} + 1\right) \left\|\sigma^*(f)w - w\right\|_{\boldsymbol{V}^*} + \\ &\qquad\qquad + \|\rho^*\| \left\|L_{h^{-1}g} f - f\right\|_{\boldsymbol{L}^1(G)} \|w\|_{\boldsymbol{V}^*} \Big) \\ &\quad < \|\rho^*\| \Big(\left(\|\rho^*\|+1\right) \frac{\varepsilon}{2\|\rho^*\|\left(\|\rho^*\|+1\right)} + \|\rho^*\| \frac{\varepsilon}{2\|\rho^*\|^2 \|w\|_{\boldsymbol{V}^*}} \|w\|_{\boldsymbol{V}^*} \Big) = \varepsilon ,\end{aligned}$$

which proves our claim. □

3.3. Square integrability and reproducing properties

Recall that for an irreducible, unitary representation ρ on a Hilbert space, we defined square integrability by the existence of a nonzero vector φ with $\langle \varphi, \rho(\cdot)\varphi \rangle \in \boldsymbol{L}^2(G)$, and called such a φ admissible. The reason for this approach was stated in Lemma 3.(i), namely, that as a consequence, we have $\langle v, \rho(\cdot)\varphi \rangle \in \boldsymbol{L}^2(G)$ for all $v \in \boldsymbol{H}$. In a Banach space, this approach does not make sense, and we have to modify the definitions accordingly.

Definition 22. A representation ρ on a Banach space $\boldsymbol{V}$ is *square integrable*, if there exists a nonzero vector $\varphi \in \boldsymbol{V}$ such that $\langle \rho(\cdot)\varphi, w \rangle \in \boldsymbol{L}^2(G)$ for all $w \in \boldsymbol{V}^*$, *i.e.*,

$$\int_G \left|\langle \rho(g)\,\varphi, w \rangle\right|^2 dm(g) < \infty \quad \text{for all } w \in \boldsymbol{V}^*. \tag{13}$$

A vector $\varphi \in \boldsymbol{V} \setminus \{0\}$ is *admissible* for ρ, if it satisfies (13).

A contragredient representation ρ^* on $\boldsymbol{V}^*$ is **-square integrable*, if there exists a nonzero vector $\psi \in \boldsymbol{V}^*$ such that $\langle v, \rho^*(\cdot)\psi\rangle \in \boldsymbol{L}^2(G)$ for all $v \in \boldsymbol{V}$, *i.e.*,

$$\int_G \left|\langle v, \rho^*(g)\,\psi\rangle\right|^2 dm(g) < \infty \quad \text{for all } v \in \boldsymbol{V}. \tag{14}$$

A vector $\psi \in \boldsymbol{V}^* \setminus \{0\}$ is **-admissible* for ρ^*, if it satisfies (14).

Lemma 23. *For a square integrable representation ρ, the set of admissible vectors is ρ-invariant.*

*For a *-square integrable contragredient representation ρ^*, the set of *-admissible vectors is ρ^*-invariant.*

Proof. Let φ be admissible for ρ. Then we have for $h \in G$ and for all $w \in \boldsymbol{V}^*$

$$\begin{aligned}\int_G \left|\langle \rho(g)(\rho(h)\varphi), w\rangle\right|^2 dm(g) &= \int_G \left|\langle \rho(g\,h)\varphi, w\rangle\right|^2 dm(g)\\ &= \int_G \left|\langle \rho(g)\varphi, w)\rangle\right|^2 \Delta(h^{-1})\, dm(g) < \infty,\end{aligned}$$

so $\rho(h)\varphi$ is admissible also.

The proof for ρ^* is analogous. □

Now we are ready to define the analysis and synthesis operators induced by the representations, and then we can discuss the reproducing property.

Lemma 24.

(i) *Let ρ be a square integrable representation of a locally compact group G on a Banach space $\boldsymbol{V}$, and $\varphi \in \boldsymbol{V}$ an admissible vector for ρ. Then the* analysis operator *associated with ρ and φ is the bounded linear transformation*

$$T_{\rho,\varphi} : \boldsymbol{V}^* \to \boldsymbol{L}^2(G), \qquad (T_{\rho,\varphi}w)(g) = \overline{\langle \rho(g)\varphi, w\rangle}. \tag{15}$$

*We refer to $T_{\rho,\varphi}w$ as the ρ-*transform *of w with window φ. The* synthesis operator *is the adjoint transformation*

$$T^*_{\rho,\varphi} : \boldsymbol{L}^2(G) \to \boldsymbol{V}^{**}, \qquad k \mapsto \int_G k(g)\,\rho(g)\varphi\, dm(g),$$

where the integral is defined in the weak sense.

(ii) *For $g \in G$, we have*

$$T_{\rho,\varphi}\,\rho^*(g) = L_g\,T_{\rho,\varphi} \quad \textit{and thus} \quad \rho^{**}(g)\,T^*_{\rho,\varphi} = T^*_{\rho,\varphi}\,L_g\,.$$

(iii) *If ρ^* is *-square integrable with *-admissible vector $\psi \in \boldsymbol{V}^*$, then the analysis operator associated with ρ^* and ψ is the bounded linear transformation*

$$T_{\rho^*,\psi} : \boldsymbol{V} \to \boldsymbol{L}^2(G), \qquad (T_{\rho^*,\psi}v)(g) = \langle v, \rho^*(g)\psi\rangle\,.$$

We refer to $T_{\rho^*,\psi}v$ *as the* ρ^*-transform *of* v *with window* ψ. *The synthesis operator is the adjoint transformation*

$$T^*_{\rho^*,\psi} : \boldsymbol{L}^2(G) \to \boldsymbol{V}^*, \qquad f \mapsto \int_G f(g)\,\rho^*(g)\psi\,dm(g),$$

where the integral is defined in the weak sense.*

(iv) *For* $g \in G$, *we have*

$$T_{\rho^*,\psi}\,\rho(g) = L_g\,T_{\rho^*,\psi} \quad \textit{and thus} \quad \rho^*(g)\,T^*_{\rho^*,\psi} = T^*_{\rho^*,\psi}\,L_g\,.$$

Proof. (i) Under the assumptions made, $T_{\rho,\varphi}$ obviously is a well-defined linear map from $\boldsymbol{V}^*$ to $\boldsymbol{L}^2(G)$. In order to show that $T_{\rho,\varphi}$ is bounded, it suffices to show that it is closed, since then boundedness follows by the closed graph theorem. So assume that $w_n \to w$ in $\boldsymbol{V}^*$ with $T_{\rho,\varphi}w_n \to f$ in $\boldsymbol{L}^2(G)$. Then $(T_{\rho,\varphi}w_n)(g) = \overline{\langle \rho(g)\varphi, w_n\rangle} \to \overline{\langle \rho(g)\varphi, w\rangle} = (T_{\rho,\varphi}w)(g)$ for all $g \in G$, *i.e.*, $T_{\rho,\varphi}w_n \to T_{\rho,\varphi}w$ pointwise, and thus $T_{\rho,\varphi}w = f$.

Given $k \in \boldsymbol{L}^2(G)$, we have for all $w \in \boldsymbol{V}^*$ that

$$\begin{aligned}
\langle T_{\rho,\varphi}w, k\rangle_{\boldsymbol{L}^2(G)} &= \int_G \overline{\langle \rho(g)\varphi, w\rangle_{(\boldsymbol{V},\boldsymbol{V}^*)}}\,\overline{k(g)}\,dm(g) \\
&= \int_G \langle w, k(g)\,\rho(g)\varphi\rangle_{(\boldsymbol{V}^*,\boldsymbol{V}^{**})}\,dm(g) \\
&= \Big\langle w, \int_G k(g)\,\rho(g)\varphi\,dm(g)\Big\rangle_{(\boldsymbol{V}^*,\boldsymbol{V}^{**})},
\end{aligned}$$

so $T^*_{\rho,\varphi}$ does indeed have the form stated above.

(ii) For $h \in G$ and $w \in \boldsymbol{W}$, we have

$$\begin{aligned}
\big(T_{\rho,\varphi}\,\rho^*(g)w\big)(h) &= \overline{\langle \rho(h)\varphi, \rho^*(g)w\rangle} \\
&= \overline{\langle \rho(g^{-1}h)\varphi, w\rangle} \\
&= \big(T_{\rho,\varphi}w\big)(g^{-1}h) = \big(L_g\,T_{\rho,\varphi}w\big)(h)\,.
\end{aligned}$$

The second identity follows from $(L_g)^* = L_{g^{-1}}$ and $(\rho^*(g))^* = \rho^{**}(g^{-1})$.

(iii), (iv) The corresponding properties of $T_{\rho^*,\psi}$ and $T^*_{\rho^*,\psi}$ are shown analogously. □

Remark. It is worth noting that $T^*_{\rho^*,\psi}$ and σ^* (*cf.*, Thm. 19) are related via

$$T^*_{\rho^*,\psi}f = \sigma^*(f)\psi \quad \text{for } f \in \boldsymbol{L}^1 \cap \boldsymbol{L}^2(G).$$

Lemma 25. *If, in addition to the assumptions of Lemma 24,* ρ *is continuous, then we have*

$$T^*_{\rho,\varphi} : \boldsymbol{L}^2(G) \to \boldsymbol{V}\,.$$

Proof. For $k \in C_c \subseteq L^2(G)$, we know that

$$T^*_{\rho,\varphi} k = \int_G k(g)\, \rho(g)\varphi \, dm(g) \in V \, ,$$

since the right hand side is a convergent Bochner integral. Furthermore, since $C_c(G)$ is dense in $L^2(G)$ and V is norm-closed in V^{**}, the claim follows from the boundedness of the operator $T^*_{\rho,\varphi}$. □

The important assumption for Schur's Lemma is the commuting property, which we may conclude directly from Lemma 24.

Corollary 26.
(i) *Under the assumptions of Lemmata 24 and 25, the operators* $T^*_{\rho,\varphi} T_{\rho^*,\psi}$ *and* $T^*_{\rho^*,\psi} T_{\rho,\varphi}$ *commute with* $\rho(G)$ *and* $\rho^*(G)$, *respectively.*
(ii) *If furthermore* ρ^* *is irreducible, then the two operators either are both injective, or both vanish identically.*

Proof. (i) By Lemma 24, we know that $T^*_{\rho^*,\psi} T_{\rho,\varphi}$ is a bounded linear operator from V^* into itself, and it satisfies

$$T^*_{\rho^*,\psi}\, T_{\rho,\varphi}\, \rho^*(g) = T^*_{\rho^*,\psi}\, L_g\, T_{\rho,\varphi} = \rho^*(g)\, T^*_{\rho^*,\psi}\, T_{\rho,\varphi} \tag{16}$$

for all $g \in G$. Also, by Lemmas 24 and 25, we know that $T^*_{\rho,\varphi} T_{\rho^*,\psi}$ is a bounded linear operator from V into itself, and it satisfies

$$T^*_{\rho,\varphi}\, T_{\rho^*,\psi}\, \rho(g) = T^*_{\rho,\varphi}\, L_g\, T_{\rho^*,\psi} = \rho(g)\, T^*_{\rho,\varphi}\, T_{\rho^*,\psi} \tag{17}$$

for all $g \in G$.

(ii) Because of (16), the kernel of $T^*_{\rho^*,\psi} T_{\rho,\varphi}$ is invariant under $\rho^*(G)$ and hence can only be $\{0\}$ or all of V^*, *i.e.*, the operator is injective or identically zero. By Lemma 18, ρ is also irreducible, and we may apply the analogous argument to $T^*_{\rho,\varphi} T_{\rho^*,\psi}$. Since $T^*_{\rho^*,\psi} T_{\rho,\varphi} = (T^*_{\rho,\varphi} T_{\rho^*,\psi})^*$, the two operators can only vanish simultaneously. □

In case V actually is (topologically isomorphic to) a Hilbert space, we can apply Schur's Lemma to obtain the desired reproducing property. In order to allow for the situation from Section 2, where we treated the Hilbert space $H_{-1/2}$ as a Banach space with dual $H_{1/2}$, we formulate the following theorem for this somewhat more general case.

Theorem 27 (Reproducing Property). *Let* ρ *be a representation of a locally compact group* G *on a Banach space* V, *with contragredient representation* ρ^* *on* V^*. *Assume that* ρ *is square integrable with admissible vector* φ, *and that* ρ^* *is irreducible and *-square integrable with *-admissible vector* ψ. *If* V *is topologically isomorphic to a Hilbert space, the following hold.*

(i) *The operator*

$$T^*_{\rho,\varphi} T_{\rho^*,\psi} : v \mapsto \int_G \langle v, \rho^*(g)\psi \rangle \, \rho(g)\varphi \, dm(g)$$

is a multiple of the identity on $\boldsymbol{V}$.

(ii) *The operator*

$$T^*_{\rho^*,\psi} T_{\rho,\varphi} : w \mapsto \int_G \overline{\langle \rho(g)\varphi, w \rangle} \, \rho^*(g)\psi \, dm(g)$$

is a multiple of the identity on $\boldsymbol{V}^*$.

Proof. (ii) If $\boldsymbol{V}$ is topologically isomorphic to a Hilbert space, then so is $\boldsymbol{V}^*$. So because of (16) and the irreducibility of ρ^*, the claim follows by Schur's Lemma.

(i) $\boldsymbol{V}$ being isomorphic to a Hilbert space implies that it is reflexive. Consequently, we have $T^*_{\rho,\varphi} T_{\rho^*,\psi} = (T^*_{\rho^*,\psi} T_{\rho,\varphi})^*$, and the claim follows from (ii). □

The multiplicative factor of the above operators can be determined as follows. Letting

$$T^*_{\rho,\varphi} T_{\rho^*,\psi} =: C_{\varphi,\psi} \, I_{\boldsymbol{V}} ,$$

we have

$$\langle C_{\varphi,\psi}\varphi, \psi \rangle = \langle T^*_{\rho,\varphi} T_{\rho^*,\psi}\varphi, \psi \rangle = \langle T_{\rho^*,\psi}\varphi, T_{\rho,\varphi}\psi \rangle_{\boldsymbol{L}^2(G)}$$

and thus

$$C_{\varphi,\psi} = \frac{\langle T_{\rho^*,\psi}\varphi, T_{\rho,\varphi}\psi \rangle_{\boldsymbol{L}^2(G)}}{\langle \varphi, \psi \rangle} \quad \text{if } \langle \varphi, \psi \rangle \neq 0 .$$

Consequently,

$$T^*_{\rho^*,\psi} T_{\rho,\varphi} = (T^*_{\rho,\varphi} T_{\rho^*,\psi})^* = \overline{C_{\varphi,\psi}} \, I_{\boldsymbol{V}^*} .$$

In the general case, however, we encounter technical difficulties. The proof of Schur's Lemma relies heavily on the spectral decomposition for operators in Hilbert spaces and thus can not be transferred to the Banach space case. On the other hand, the assumption of square integrability implies a strong interrelation between the spaces $\boldsymbol{V}$ and $\boldsymbol{V}^*$ on the one hand and an appropriate subspace of $\boldsymbol{L}^2(G)$ on the other. Indeed, the converse of Theorem 27 holds in the sense that the reproducing property implies that $\boldsymbol{V}$ (and thus $\boldsymbol{V}^*$) is isomorphic to a Hilbert space.

Theorem 28. *Let* ρ *be a continuous representation of a locally compact group* G *on a Banach space* $\boldsymbol{V}$, *with contragredient representation* ρ^* *on* $\boldsymbol{V}^*$. *Assume that* ρ *is square integrable with admissible vector* φ, *and that* ρ^* *is irreducible and *-square integrable with *-admissible vector* ψ. *If the operator* $T^*_{\rho,\varphi} T_{\rho^*,\psi}$ *does not vanish identically, then the following are equivalent.*

(i) $T^*_{\rho,\varphi} T_{\rho^*,\psi}$ *is a multiple of the identity on* $\boldsymbol{V}$.

(ii) $T^*_{\rho^*,\psi} T_{\rho,\varphi}$ *is a multiple of the identity on* $\boldsymbol{V}^*$.

(iii) $T_{\rho^*,\psi}(\boldsymbol{V})$ *is closed in* $\boldsymbol{L}^2(G)$.

(iv) $T_{\rho,\varphi}(V^*)$ *is closed in* $L^2(G)$.
(v) V *is topologically isomorphic to a Hilbert space.*
(vi) V^* *is topologically isomorphic to a Hilbert space.*
(vii) *The left regular representation acts irreducibly on* $\overline{T_{\rho^*,\psi}(V)} \subseteq L^2(G)$.
(viii) *The left regular representation acts irreducibly on* $\overline{T_{\rho,\varphi}(V^*)} \subseteq L^2(G)$.

Proof. (i)$\Rightarrow$(ii). If $T^*_{\rho,\varphi}T_{\rho^*,\psi} = C\,I_V$, then $T^*_{\rho^*,\psi}T_{\rho,\varphi} = (T^*_{\rho,\varphi}T_{\rho^*,\psi})^* = \overline{C}\,I_{V^*}$.

(ii)$\Rightarrow$(i). As usual, consider V to be embedded in V^{**}. Then $T^*_{\rho^*,\psi}T_{\rho,\varphi} = C\,I_{V^*}$ implies $T^*_{\rho,\varphi}T^{**}_{\rho^*,\psi} = (T^*_{\rho^*,\psi}T_{\rho,\varphi})^* = \overline{C}\,I_{V^{**}}$. Replacing $T^{**}_{\rho^*,\psi}$ by $T_{\rho^*,\psi}$ simply restricts this operator to V which yields the claim.

(i)$\Rightarrow$(iii). $T^*_{\rho,\varphi}T_{\rho^*,\psi} = C\,I_V$ with, by assumption, $C \neq 0$ means $T^{-1}_{\rho^*,\psi} = C^{-1}\,T^*_{\rho,\varphi}$, so $T_{\rho^*,\psi}$ is boundedly invertible and thus a topological isomorphism between V and its image. Hence $T_{\rho^*,\psi}(V)$ is complete and therefore closed in $L^2(G)$.

(ii)$\Rightarrow$(iv). Analogously, since $T^*_{\rho,\varphi}T_{\rho^*,\psi} \not\equiv 0$ implies $T^*_{\rho^*,\psi}T_{\rho,\varphi} = (T^*_{\rho,\varphi}T_{\rho^*,\psi})^* \not\equiv 0$.

(iii)$\Rightarrow$(v). As a closed subspace of $L^2(G)$, the space $T_{\rho^*,\psi}(V)$ is a Hilbert space itself. From Corollary 26(ii), we may conclude that $T_{\rho^*,\psi}$ is injective. So by the open mapping theorem, the continuous bijection $T_{\rho^*,\psi}$ is a topological isomorphism between V and its range.

(iv)$\Rightarrow$(vi). Analogously.

(v)$\Rightarrow$(i). See Theorem 27.

(vi)$\Rightarrow$(v). Let $F : V^* \to H$ be a topological isomorphism between V^* and some Hilbert space H. Then V^* is reflexive, and thus so is V (*e.g.*, see [7]). Consequently, we have the topological isomorpism $F^* : H \to V$.

(iii)$\Rightarrow$(vii). Since $T_{\rho^*,\psi}$ intertwines the actions of ρ and L, irreducibility of L on $T_{\rho^*,\psi}(V) = \overline{T_{\rho^*,\psi}(V)}$ is equivalent to that of ρ on V.

(iv)$\Rightarrow$(viii). Analogously.

(vii)$\Rightarrow$(i). Let H be the closure of $T_{\rho^*,\psi}(V)$ in $L^2(G)$. Then we can treat the operator

$$T_{\rho^*,\psi}T^*_{\rho,\varphi} : L^2(G) \to T_{\rho^*,\psi}(V)$$

as an element of $\mathcal{L}(H)$. By Lemma 24, this operator commutes with the left regular representation L of G, so by the assumed irreducibility and Schur's Lemma, we have $T_{\rho^*,\psi}T^*_{\rho,\varphi} = C\,I_H$ and thus

$$T^*_{\rho,\varphi}T_{\rho^*,\psi}T^*_{\rho,\varphi}T_{\rho^*,\psi} = C\,T^*_{\rho,\varphi}T_{\rho^*,\psi}\,,$$

so $T^*_{\rho,\varphi}T_{\rho^*,\psi}$ is a multiple of a projection. But from Corollary 26(ii), we know that $T^*_{\rho,\varphi}T_{\rho^*,\psi}$ is injective, and therefore $T^*_{\rho,\varphi}T_{\rho^*,\psi} = C\,I_V$.

(viii)$\Rightarrow$(ii). Analogously. □

It is important to realize that in the proof of Theorem 28, we have explicitly described the Hilbert spaces for part (v) and (vi): namely, closed subspaces of $L^2(G)$. Thus, if we do have the reproducing properties (i) and (ii), we can replace the norms on V and V^* by norms such that $T_{\rho^*,\psi}$ and $T_{\rho,\varphi}$ become isometries. Then ρ and ρ^* act isometrically, and thus we end up with unitary representations. This explains our observations from Section 2.8.

3.4. Other integrability properties

The only possibility to avoid this lies in giving up square integrability. What we do need for the explicit definition of the synthesis operators is the existence of the integral

$$\int_G \langle v, \rho^*(g)\psi\rangle \langle \rho(g)\varphi, w\rangle \, dm(g),$$

which we can also achieve by ensuring that the two factors are p- and q-integrable, respectively, for conjugate exponents p and q. This motivates the following definitions.

Definition 29. A representation ρ on a Banach space $\boldsymbol{V}$ is *p-integrable* for some $p \in [1,\infty[$, if there exists a nonzero vector $\varphi \in \boldsymbol{V}$ such that $\langle \rho(\cdot)\varphi, w\rangle \in \boldsymbol{L}^p(G)$ for all $w \in \boldsymbol{V}^*$, *i.e.*,

$$\int_G \left|\langle \rho(g)\varphi, w\rangle\right|^p dm(g) < \infty \quad \text{for all } w \in \boldsymbol{V}^*. \tag{18}$$

A vector $\varphi \in \boldsymbol{V} \setminus \{0\}$ is *p-admissible* for ρ, if it satisfies (18).

For the term 1-integrable, we also use *integrable* for short.

A contragredient representation ρ^* on $\boldsymbol{V}^*$ is **-q-integrable* for some $q \in [1,\infty[$, if there exists a nonzero vector $\psi \in \boldsymbol{V}^*$ such that $\langle v, \rho^*(\cdot)\psi\rangle \in \boldsymbol{L}^q(G)$ for all $v \in \boldsymbol{V}$, *i.e.*,

$$\int_G \left|\langle v, \rho^*(g)\psi\rangle\right|^q dm(g) < \infty \quad \text{for all } v \in \boldsymbol{V}. \tag{19}$$

A vector $\psi \in \boldsymbol{V}^* \setminus \{0\}$ is **-q-admissible* for ρ^*, if it satisfies (19).

Straightforward modifications of the proofs of Lemma 23–25 and Corollary 26 yield the following analogous statements.

Corollary 30. *For a p-integrable representation ρ, the set of p-admissible vectors is ρ-invariant.*

*For a *-q-integrable contragredient representation ρ^*, the set of *-q-admissible vectors is ρ^*-invariant.*

Corollary 31. *Let $p \in [1,\infty[$ and $\frac{1}{p} + \frac{1}{q} = 1$. Consider a continuous p-integrable representation ρ of a locally compact group G on a Banach space $\boldsymbol{V}$, and let $\varphi \in \boldsymbol{V}$ be a p-admissible vector for ρ. Then the analysis operator $T_{\rho,\varphi} : \boldsymbol{V}^* \to \boldsymbol{L}^p(G)$ and the synthesis operator $T^*_{\rho,\varphi} : \boldsymbol{L}^q(G) \to \boldsymbol{V}$ are bounded linear transformations. They satisfy*

$$T_{\rho,\varphi}\, \rho^*(g) = L_g\, T_{\rho,\varphi} \quad \textit{and thus} \quad \rho(g)\, T^*_{\rho,\varphi} = T^*_{\rho,\varphi}\, L_g$$

for all $g \in G$, where ρ^* is the contragredient representation of ρ, and L_g is left translation in $L^p(G)$ and $L^q(G)$, respectively. Analogous statements hold for $T_{\rho^*,\psi}$ and $T^*_{\rho^*,\psi}$.

Corollary 32. *Let $p \in]1,\infty[$ and $\frac{1}{p}+\frac{1}{q}=1$. Let ρ be a p-integrable representation of a locally compact group G on a Banach space V with p-admissible vector φ, and assume that ρ^* is *-q-integrable with *-q-admissible vector ψ. Then the operators $T^*_{\rho,\varphi}T_{\rho^*,\psi}$ and $T^*_{\rho^*,\psi}T_{\rho,\varphi}$ commute with $\rho(G)$ and $\rho^*(G)$, respectively.*
The same holds if ρ is integrable and ρ^* is bounded, or vice versa.

We encounter the problem, though, that we can not apply Schur's Lemma to prove a reproducing property. Assuming Hilbert space structure for V and V^* does not make sense in this situation, since then a reproducing property would impose Hilbert space structure to closed subspaces of $L^p(G)$ and $L^q(G)$.

Integrable representations and their discretizations have been used for so-called atomic decompositions of Banach spaces in [10] and [11], where the authors also introduce and discuss the concept of ***coorbit spaces***. They start from an integrable, irreducible, unitary, continuous representation ρ and define the coorbit of Y under ρ to be the space

$$\mathcal{C}o(Y) = \{v : T_{\rho,\varphi}v \in Y\}$$

for an appropriate φ, where v is allowed to range in some space of generalized distributions. In case $Y = L^p(G)$, this can be seen as a somehow reverse approach compared to the idea of a p-integrable representation.

For the special case of the wavelet transformation, this leads to characterizations, *e.g.*, of Besov spaces ([14], [27]). Applied to the Gabor transformation, this approach gives rise to the so-called modulation spaces (*e.g.*, see [9], [12], [13], [17], [26]).

Appendix

A.1. Antilinear functionals

As dual space of a Banach space V, we usually consider the space of bounded linear functionals V'. This space has vector space structure, since for $w_1, w_2 \in V'$ and $\lambda \in \mathbb{C}$, we can define

$$(w_1+w_2)(v) := w_1(v)+w_2(v) \quad \text{and} \quad (\lambda\, w_1)(v) := \lambda\, w_1(v) \quad \text{for all } v \in V.$$

Also, for a bounded linear operator $T : V_1 \to V_2$ between two Banach spaces, we usually define the adjoint operator $T^* : V_2' \to V_1'$ via

$$(T^*w)(v) := w(T\,v)\,,$$

and the mapping $\mathcal{L}(V_1, V_2) \to \mathcal{L}(V_2', V_1')$, $T \mapsto T^*$, is linear.

When dealing with a Hilbert space $\boldsymbol{H}$, though, we encounter a formal problem. $\boldsymbol{H}$ is self-dual in the sense that for each bounded linear functional w on $\boldsymbol{H}$, there exists $v' \in \boldsymbol{H}$ such that $w(v) = \langle v, v' \rangle$ for all $v \in \boldsymbol{H}$, but the correspondence between $\boldsymbol{H}'$ and $\boldsymbol{H}$ given by $w \longleftrightarrow v'$ is antilinear in the sense that $\lambda\, w \longleftrightarrow \overline{\lambda}\, v'$.

Furthermore, for a bounded linear operator $T : \boldsymbol{H}_1 \to \boldsymbol{H}_2$ between Hilbert spaces, we define the Hilbert space adjoint or Hermitian adjoint operator $T^* : \boldsymbol{H}_2 \to \boldsymbol{H}_1$ via

$$\langle v, T^* w \rangle_{\boldsymbol{H}_1} := \langle T\, v, w \rangle_{\boldsymbol{H}_2}, \tag{A.1}$$

and here, the mapping $\mathcal{L}(\boldsymbol{H}_1, \boldsymbol{H}_2) \to \mathcal{L}(\boldsymbol{H}_2, \boldsymbol{H}_1)$, $T \mapsto T^*$, is antilinear.

As natural as these conventions are, they will create havoc as soon as we consider adjoints of operators between a Banach space and a Hilbert space. To avoid this problem, we follow the convention in [19] and replace $\boldsymbol{V}'$, the space of bounded linear functionals, by $\boldsymbol{V}^*$, the space of bounded antilinear functionals on $\boldsymbol{V}$, which satisfy

$$w(\lambda_1\, v_1 + \lambda_2\, v_2) = \overline{\lambda_1}\, w(v_1) + \overline{\lambda_2}\, w(v_2)\,.$$

This space also has vector space structure over $\mathbb{C}$, if we define

$$(w_1 + w_2)(v) := w_1(v) + w_2(v) \quad \text{and} \quad (\lambda\, w_1)(v) := \lambda\, w_1(v) \quad \text{for all } v \in \boldsymbol{V},$$

as before. Then we can introduce a sesquilinear pairing between $\boldsymbol{V}$ and $\boldsymbol{V}^*$ via

$$\big\langle v, w \big\rangle_{(\boldsymbol{V}, \boldsymbol{V}^*)} := \overline{w(v)}\,.$$

(Here we deviate somewhat from [19], where the authors define $\langle w, v \rangle_{(\boldsymbol{V}^*, \boldsymbol{V})} = w(v)$.) This pairing is linear in the first and antilinear in the second argument, since we have

$$\begin{aligned} \big\langle \lambda_1\, v_1 + \lambda_2\, v_2, w \big\rangle &= \overline{w(\lambda_1\, v_1 + \lambda_2\, v_2)} \\ &= \lambda_1\, \overline{w(v_1)} + \lambda_2\, \overline{w(v_2)} = \lambda_1 \langle v_1, w \rangle + \lambda_2 \langle v_2, w \rangle \\ \text{and} \qquad \big\langle v, \mu_1\, w_1 + \mu_2\, w_2 \big\rangle &= \overline{(\mu_1\, w_1 + \mu_2\, w_2)(v)} \\ &= \overline{\mu_1\, w_1(v) + \mu_2\, w_2(v)} = \overline{\mu_1} \langle v, w_1 \rangle + \overline{\mu_2} \langle v, w_2 \rangle\,. \end{aligned}$$

The space $\boldsymbol{V}^*$ is antilinearly isomorphic to $\boldsymbol{V}'$ via

$$w : v \mapsto w(v) \qquad \longleftrightarrow \qquad \widetilde{w} : v \mapsto \overline{w(v)}\,,$$

since we have

$$\begin{aligned} \widetilde{w}(\lambda_1\, v_1 + \lambda_2\, v_2) &= \overline{w(\lambda_1\, v_1 + \lambda_2\, v_2)} \\ &= \overline{\overline{\lambda_1}\, w(v_1) + \overline{\lambda_2}\, w(v_2)} = \lambda_1\, \widetilde{w}(v_1) + \lambda_2\, \widetilde{w}(v_2)\,, \end{aligned}$$

i.e., $\widetilde{w}$ is a linear functional, and this correspondence is obviously isometric, so $\widetilde{w} \in \boldsymbol{V}'$.

We still have the canonical embedding $\phi : \boldsymbol{V} \hookrightarrow \boldsymbol{V}^{**}$ via

$$\langle w, \phi(v) \rangle_{(\boldsymbol{V}^*, \boldsymbol{V}^{**})} = \overline{\langle v, w \rangle_{(\boldsymbol{V}, \boldsymbol{V}^*)}}\,.$$

This embedding is again linear, and $\boldsymbol{V}$ being reflexive in the usual sense is equivalent to ϕ being an isomorphism.

Consequently, we define for a bounded linear operator $T : \boldsymbol{V}_1 \to \boldsymbol{V}_2$ the adjoint operator $T^* : \boldsymbol{V}_2^* \to \boldsymbol{V}_1^*$ by

$$\langle v, T^* w\rangle_{(\boldsymbol{V}_1,\boldsymbol{V}_1^*)} := \langle T\,v, w\rangle_{(\boldsymbol{V}_2,\boldsymbol{V}_2^*)}$$

which includes (A.1) as a special case; in particular, the mapping $\mathcal{L}(\boldsymbol{V}_1, \boldsymbol{V}_2) \to \mathcal{L}(\boldsymbol{V}_2', \boldsymbol{V}_1')$, $T \mapsto T^*$, is antilinear. This definition still makes T^* a linear (and not antilinear) operator, since

$$\langle v, T^* \lambda\, w\rangle = \langle T\,v, \lambda\, w\rangle = \overline{\lambda}\,\langle T\,v, w\rangle = \overline{\lambda}\,\langle v, T^* w\rangle = \langle v, \lambda\, T^* w\rangle\,.$$

The advantage of this approach is that we actually have the natural *linear* isomorphism $\boldsymbol{H} \cong \boldsymbol{H}^*$ (instead of the *antilinear* isomorphism $\boldsymbol{H} \cong \boldsymbol{H}'$ mentioned above), and this avoids the ambiguity in the definition of T^*.

A.2. Two technical lemmata

Lemma A.1. For m, $n \in \mathbb{N}$ and $1 \le j \le \min\{m, n\}$, we have

$$\sum_{\substack{\ell=0\\ \ell\ge m-n}}^{m-1} (-1)^{m-1-\ell} \binom{m-1}{\ell}\binom{n+\ell-j}{m-j} = \delta_{1,j}\,,$$

and

$$\sum_{\ell=1}^{\min\{m,n\}} (-1)^{n-\ell} \binom{n}{\ell}\binom{m+n-j-\ell}{n-j}\binom{j+\ell-2}{j-1} = (-1)^{n+1} \binom{m+n-1}{n-1}\delta_{1,j}\,.$$

Proof. For a function f on $\mathbb{R}$, consider iterated forward differences on knots with integer distances:

$$\begin{aligned} f^{[0]}(x) &= f(x)\,,\\ f^{[1]}(x) &= f(x+1) - f(x)\,,\\ f^{[k]}(x) &= f^{[k-1]}(x+k) - f^{[k-1]}(x)\,. \end{aligned}$$

This yields

$$f^{[k]}(x) = \sum_{\ell=0}^{k} (-1)^{k-\ell} \binom{k}{\ell} f(x+\ell)\,,$$

which we show by induction on k.

For $k = 0$, we have $f^{[0]}(x) = f(x)$ by definition. For the induction step, consider

$$\begin{aligned} f^{[k+1]}(x) &= f^{[k]}(x+k+1) - f^{[k]}(x)\\ &\underset{\text{(ind. hyp.)}}{=} \sum_{\ell=0}^{k} (-1)^{k-\ell} \binom{k}{\ell} f(x+1+\ell) - \sum_{\ell=0}^{k} (-1)^{k-\ell} \binom{k}{\ell} f(x+\ell) \end{aligned}$$

$$\underset{(j=\ell+1)}{=} f(x+1+k) + \sum_{j=1}^{k}(-1)^{k-(j-1)}\binom{k}{j-1} f(x+j) +$$

$$+ \sum_{\ell=1}^{k}(-1)^{k-\ell+1}\binom{k}{\ell} f(x+\ell) + (-1)^{k+1} f(x)$$

$$= f(x+k+1) + \sum_{\ell=1}^{k}(-1)^{k+1-\ell}\left(\binom{k}{\ell-1}+\binom{k}{\ell}\right) f(x+\ell) + (-1)^{k+1} f(x)$$

$$= \sum_{\ell=0}^{k+1}(-1)^{k+1-\ell}\binom{k+1}{\ell} f(x+\ell),$$

which completes the induction.

In case the function under consideration is a polynomial $p(x)$ of degree r with leading coefficient a_r, it is easy to see that for $k \le r$, the iterated difference $p^{[k]}(x)$ is a polynomial of degree $r-k$ with leading coefficient $\frac{r!}{(r-k)!}\,a_r$, and $p^{[k]}(x) = 0$ for $k > r$.

To prove our first claim, let $p(x) = \binom{n+x-j}{m-j} = \frac{(n+x-j)(n+x-j-1)\cdots(n+x-m+1)}{(m-j)!}$, which is a polynomial of degree $m-j$ with leading coefficient $a_{m-j} = \frac{1}{(m-j)!}$. Note that if $m > n$, we have for $\ell \in \mathbb{N}_0$ with $0 \le \ell < m-n$ that $p(\ell) = 0$. Therefore,

$$\sum_{\substack{\ell=0\\ \ell \ge m-n}}^{m-1} (-1)^{m-1-\ell}\binom{m-1}{\ell}\binom{n+\ell-j}{m-j} = \sum_{\ell=0}^{m-1}(-1)^{m-1-\ell}\binom{m-1}{\ell} p(\ell) = p^{[m-1]}(0),$$

which equals 0 for $m-1 > m-j$, *i.e.*, $j > 1$, and $(m-1)!\,a_{m-1} = 1$ for $j = 1$, as claimed.

For the second claim, let $p(x) = \binom{m+n-j-x}{n-j}\binom{j+x-2}{j-1}$, which is a polynomial in x of degree $(n-j)+(j-1) = n-1$, so $p^{[n]}(x) \equiv 0$. Note that $p(\ell) = 0$ for $\ell > m$, and for $j > 1$, also $p(0) = 0$, hence

$$\sum_{\ell=1}^{\min\{m,n\}} (-1)^{n-\ell}\binom{n}{\ell}\binom{m+n-j-\ell}{n-j}\binom{j+\ell-2}{j-1} = \sum_{\ell=0}^{n}(-1)^{n-\ell}\binom{n}{\ell} p(\ell) = p^{[n]}(0) = 0.$$

For $j = 1$, though, we have $p(0) = \binom{m+n-1}{n-1}$, so

$$\sum_{\ell=1}^{\min\{m,n\}} (-1)^{n-\ell}\binom{n}{\ell}\binom{m+n-j-\ell}{n-j}\binom{j+\ell-2}{j-1} = p^{[n]}(0) - (-1)^n p(0) = (-1)^{n+1}\binom{m+n-1}{n-1},$$

which yields the claim. □

Lemma A.2. For $p, q \in \mathbb{N}$ with $p < q$, we have

$$\int_0^\infty \frac{(\sinh s)^{2p+1}}{(\cosh s)^{2q+1}}\,ds = \frac{1}{2}\,\frac{p!\,(q-p-1)!}{q!}.$$

Proof.

$$\int_0^\infty \frac{(\sinh s)^{2p+1}}{(\cosh s)^{2q+1}}\,ds \underset{(t=\cosh s)}{=} \int_1^\infty \frac{(t^2-1)^p}{t^{2q+1}}\,dt$$
$$= \sum_{r=0}^{p} \binom{p}{r}(-1)^r \int_1^\infty t^{2(p-r)-(2q+1)}\,dt$$
$$= \sum_{r=0}^{p} \binom{p}{r}(-1)^r \left[\frac{t^{2(p-r-q)}}{2(p-r-q)}\right]_1^\infty$$
$$= \frac{1}{2}\sum_{r=0}^{p} \binom{p}{r} \frac{(-1)^r}{q-p+r}\,.$$

So it suffices to show that

$$\sum_{r=0}^{p} \binom{p}{r} \frac{(-1)^r}{q-p+r} = \frac{p!\,(q-p-1)!}{q!} \qquad \text{for } q > p, \tag{A.2}$$

for which we use induction on p.

For $p = 0$, (A.2) becomes

$$\frac{1}{q} = \frac{0!\,(q-1)!}{q!} \qquad \text{for } q \in \mathbb{N},$$

which is obviously true.

For the induction step, we find that for $q > p{+}1$,

$$\sum_{r=0}^{p+1} \binom{p+1}{r} \frac{(-1)^r}{q-(p+1)+r} = \frac{1}{q-p-1} + \sum_{r=1}^{p}\left(\binom{p}{r} + \binom{p}{r-1}\right)\frac{(-1)^r}{q-(p+1)+r} + \frac{(-1)^{p+1}}{q}$$
$$= \frac{1}{(q-1)-p} + \sum_{r=1}^{p}\binom{p}{r}\frac{(-1)^r}{(q-1)-p+r} + \sum_{s=0}^{p-1}\binom{p}{s}\frac{(-1)^{s+1}}{q-p+s} + \frac{(-1)^{p+1}}{q}$$
$$= \sum_{r=0}^{p}\binom{p}{r}\frac{(-1)^r}{(q-1)-p+r} - \sum_{s=0}^{p}\binom{p}{s}\frac{(-1)^s}{q-p+s}$$
$$\underset{\text{(ind. hyp.)}}{=} \frac{p!\,(q-1-p-1)!}{(q-1)!} - \frac{p!\,(q-p-1)!}{q!}$$
$$= \frac{p!\,(q-p-2)!}{q!}\big(q-(q-p-1)\big) = \frac{(p+1)!\,(q-(p+1)-1)!}{q!}\,,$$

which proves the claim. □

It is worth noting that the left hand side of (A.2) is just the p^{th} order forward difference $f^{[p]}(q-p)$ of the function $f(x) = \frac{1}{x}$ at the point $x = q-p$, and this is what we actually make use of in the proof. For more details on iterated differences, *e.g.*, see [16].

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References

1. Ali, S. T., Antoine, J.-P. and Gazeau, J.-P., *Coherent States, Wavelets and Their Generalizations*, Graduate Texts in Contemporary Physics, Springer, New York, 2000.
2. Arazy, J. and Fisher, S. D., The uniqueness of the Dirichlet space among Möbius-invariant Hilbert spaces, *Illinois J. Math.* **29** (1985), 449–462.
3. Arazy, J., Fisher, S. D. and Peetre, J., Möbius-invariant function spaces, *J. Reine Angew. Math.* **363** (1985), 110–145.
4. Barut, A. O. and Girardello, L., New "coherent" states associated with non-compact groups, *Comm. Math. Phys.* **21** (1971), 41–55.
5. Carey, A. L., Square-integrable representations of non-unimodular groups, *Bull. Austral. Math. Soc.* **15** (1976), 1–12.
6. Chui, C. K. and Lian, J. A., A study of orthonormal wavelets, *Appl. Numer. Math.* **20** (1996), 273–298.
7. Day, M. M., *Normed Linear Spaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 21, Springer, Berlin, 1962.
8. Duflo, M. and Moore, C. C., On the regular representation of a nonunimodular locally compact group, *J. Funct. Anal.* **21** (1976), 209–243.
9. Feichtinger, H. G., Minimal Banach spaces and atomic representations, *Publ. Math. Debrecen* **34** (1987), 231–240.
10. Feichtinger, H. G. and Gröchenig, K., A unified approach to atomic characterizations via integrable group representations, in: *Function Spaces and Applications*, Proc. US-Swed. Semin. Lund/Swed. (M. Cwikel, J. Peetre, Y. Sagher and H. Wallin, Eds.), Lecture Notes Math., Vol. 1302, Springer, Berlin, 1988, pp. 52–73.
11. Feichtinger, H. G. and Gröchenig, K., Banach spaces related to integrable group representations and their atomic decompositions, I, *J. Funct. Anal.* **86** (1989), 307–340.
12. Feichtinger, H. G. and Gröchenig, K., Gabor wavelets and the Heisenberg group: Gabor expansions and short time Fourier transform from the group theoretical point of view, in: *Wavelets: A Tutorial in Theory and Applications* (C. K. Chui, Ed.), Academic Press, Boston, 1992, pp. 359–397.
13. Feichtinger, H. G. and Zimmermann, G., A Banach space of test functions for Gabor analysis, in: *Gabor Analysis and Algorithms: Theory and Applications* (H. G. Feichtinger and T. Strohmer, Eds.), Birkhäuser, Boston, 1998, pp. 123–170.

14. Frazier, M. and Jawerth, B., Decomposition of Besov spaces, *Indiana Univ. Math. J.* **34** (1985), 777–799.
15. Gaal, S. A., *Linear Analysis and Representation Theory*, Grundlehren Math. Wiss., Vol 198, Springer, Berlin, 1973.
16. Gould, H. W., *Combinatorial Identities*, Gould, Morgantown, 1972.
17. Gröchenig, K., *Foundation of Time–Frequency Analysis*, Birkhäuser, Boston, 2001.
18. Grossmann, A., Morlet, J. and Paul, T., Transforms associated to square integrable group representations. I. General results, *J. Math. Phys.* **26** (1985), 2473–2479.
19. Hewitt, E. and Ross, K. A., *Abstract Harmonic Analysis I*, Grundlehren Math. Wiss., Vol. 115, Springer, New York, 1979.
20. Holschneider, M., Wavelet analysis on the circle, *J. Math. Phys.* **31** (1990), 39–44.
21. Holschneider, M., *Wavelets: An Analysis Tool*, Oxford Univ. Press, New York, 1995.
22. Kirillov, A. A., *Elements of the Theory of Representations*, Grundlehren Math. Wiss., Vol. 220, Springer, Berlin, 1976 (Translated from the Russian by E. Hewitt).
23. Knapp, A., *Representation Theory of Semisimple Groups. An Overview Based on Examples*, Princeton Math. Ser., Vol. 36, Princeton University Press, Princeton, 1986.
24. Lang, S., $SL_2(\mathbb{R})$, Grad. Texts in Math., Vol. 105, Springer, New York, 1985.
25. Loomis, L. H., *An Introduction to Abstract Harmonic Analysis*, Univ. Ser. Higher Math., Van Nostrand, Princeton, 1953.
26. Triebel, H., Modulation spaces on the Euclidean n-space, *Z. Anal. Anw.* **2** (1983), 443–457.
27. Triebel, H., *Theory of Function Spaces*, Birkhäuser, Basel, 1983.