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Ready-to-Blossom Bases in Chebyshev Spaces

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Abstract

This paper gives a survey on blossoms and Chebyshev spaces, with a number of new results and proofs. In particular, Extended Chebyshev spaces are characterised by the existence of a certain type of bases which are especially suited to enable us to prove both existence and properties of blossoms under the weakest possible differentiability assumptions. We also examine the case of piecewise spaces built from different Extended Chebyshev spaces and connection matrices.

Key words: Hermite interpolation, Taylor interpolation, extended Chebyshev space, W-space, extended Chebyshev piecewise space, W-piecewise space, Bernstein basis, B-spline basis, blossom, geometric design

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1. Introduction

In order to stress the main purpose of the present paper, we shall start with an elementary problem. Consider the function $\Phi : \mathbb{R} \to \mathbb{R}^n$ defined by $\Phi(x) := (\Phi_1(x), \dots, \Phi_n(x))^T := (x, x^2, \dots, x^n)^T$. Given pairwise distinct $a_1, \dots, a_r \in \mathbb{R}$ and

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positive integers $\mu_1, \ldots, \mu_r$, with $\mu_1 + \cdots + \mu_r = n$, how to prove that the r osculating flats $\mathrm{Osc}_{n-\mu_i}\Phi(a_i)$, $1 \le i \le r$, have a unique common point, given that, for any nonnegative integer k, the kth order osculating flat of Φ at $x \in \mathbb{R}$ is defined as the affine flat $\mathrm{Osc}_k\Phi(x)$ passing through $\Phi(x)$ and the direction of which is spanned by the first k derivatives of Φ at x, namely:

$$\mathrm{Osc}_k\Phi(x) := \left\{ \Phi(x) + \sum_{p=1}^{k} \lambda_p \Phi^{(p)}(x) \mid \lambda_1, \ldots, \lambda_k \in \mathbb{R} \right\} .$$

For the sake of simplicity let us first investigate the case $r = n$. We are then dealing with n osculating hyperplanes $\mathrm{Osc}_{n-1}\Phi(a_i)$, $1 \le i \le n$. One possible proof consists in using the directions orthogonal to the hyperplanes. Indeed, if $X \in \mathbb{R}^n$, we have the equivalence

$$X \in \bigcap_{i=1}^{n} \mathrm{Osc}_{n-1}\Phi(a_i) \Leftrightarrow \langle X, \Phi^\sharp(a_i)\rangle = \langle \Phi(a_i), \Phi^\sharp(a_i)\rangle,\ 1 \le i \le n, \tag{1}$$

where $\langle .,. \rangle$ denotes the inner product in $\mathbb{R}^n$ and where, for all $x \in \mathbb{R}$, $\Phi^\sharp(x) := \Phi'(x) \wedge \ldots \wedge \Phi^{(n-1)}(x)$ is the cross product of the first $(n-1)$ derivatives at x. One can check that, up to multiplication by some nonzero real number, we have $\Phi^\sharp = (\Phi_1^\sharp, \ldots, \Phi_n^\sharp)^T$, where the kth component $\Phi_k^\sharp$ is given by $\Phi_k^\sharp(x) := (-1)^{n-k}\binom{n}{k}x^{n-k}$. The main fact is that the space $\mathcal{E}^\sharp$ spanned by the functions $\Phi_1^\sharp, \ldots, \Phi_n^\sharp$ is the polynomial space $\mathcal{P}_{n-1}$ of degree $(n-1)$, and, because any nonzero element of $\mathcal{E}^\sharp$ cannot have n distinct zeros, the n vectors $\Phi^\sharp(a_i)$, $1 \le i \le n$, are linearly independent. This is the reason why the linear system appearing in (1) has a unique solution. In the general case, that is, when allowing $r < n$, the point X belongs to all osculating flats $\mathrm{Osc}_{n-\mu_i}\Phi(a_i)$, $1 \le i \le r$, iff it satisfies

$$\langle X, \Phi^{\sharp(k)}(a_i)\rangle = \langle \Phi(a_i), \Phi^{\sharp(k)}(a_i)\rangle,\ 0 \le k \le \mu_i - 1,\ 1 \le i \le r. \tag{2}$$

Again, existence and uniqueness of the solution to the system (2) is justified by the fact that any nonzero element of $\mathcal{E}^\sharp$ has at most $(n-1)$ zeros, but this time the zeros being counted with their multiplicities.

The unique solution to our system (1) is nothing but the point $\varphi(a_1, \ldots, a_n)$, where φ denotes the blossom of the degree n polynomial function Φ [20,16]. More generally, the unique solution to (2) is the point $\varphi(a_1{}^{[\mu_1]}, \ldots, a_r{}^{[\mu_r]})$, where the symbolic power notation $x^{[k]}$ stands for x repeated k times. Clearly the arguments developed above are not limited to polynomial spaces. They obviously remain valid whenever we deal with an $(n+1)$-dimensional space $\mathcal{E}$ of sufficiently differentiable functions which contains constants and for which the directions orthogonal to the osculating hyperplanes of some "mother" function Φ generate an n-dimensional space $\mathcal{E}^\sharp$ which possesses the same properties concerning (simple as well as multiple) zeros as $\mathcal{P}_{n-1}$. This means whenever $\mathcal{E}^\sharp$ is an Extended Chebyshev space. After H. Pottmann [19], it is now classical to define Chebyshevian blossoms this way, that is, by means of intersections of osculating flats.

The fact that $\mathcal{E}^\sharp$ is an Extended Chebyshev space implies that the space $\mathcal{E}$ itself is an Extended Chebyshev space. However, when it is of dimension $(n+1)$, such a space $\mathcal{E}$ is generally supposed to be composed of functions which are only C^n. Unfortunately, this does not guarantee a priori the elements of $\mathcal{E}^\sharp$ to be sufficiently differentiable to make it possible for it to be an Extended Chebyshev space. Our previous arguments thus implicitly assume too much differentiability and they may therefore be impossible to use. Nevertheless, blossoms do exist even in case we limit ourselves to the strict C^n differentiability in $\mathcal{E}$: their existence can be shown for instance using Theorem 3.1 of [5], but it is not a trivial proof.

Blossoms and EC-spaces are so intimately connected that it is a bit paradoxical that existence and uniqueness of a common point to relevant osculating flats are not somehow visible at first sight. This paradox is the actual motivation of the present work. Furthermore, if the blossom φ of our initial polynomial function Φ is known to be affine in each variable, in the Chebyshevian case, this property is replaced by a more general one, which we are used to refer to as pseudoaffinity in each variable. Again this property was initially proved using more differentiability than necessary. We established it under the strict C^n differentiability assumption in [14], but the latter proof was far from being direct. This difficulty too seemed somewhat paradoxical.

The paper brings an answer to the points mentioned above: it is indeed possible to make existence of blossoms visible at first sight, provided that we use relevant bases. To put the reader on the right track, we shall come back to the example of polynomial spaces in the first case we examined. Let us introduce the function $\Psi : \mathbb{R} \to \mathbb{R}^n$ as $\Psi := (\Psi_1, \ldots, \Psi_n)^T$, with $\Psi_i(x) := (x - a_i)^n$, $1 \le i \le n$. It is obvious that $x_i = 0$ is an equation of the osculating hyperplane $\mathrm{Osc}_{n-1}\Psi(a_i)$. Accordingly, the origin of $\mathbb{R}^n$ belongs to $\cap_{i=1}^n \mathrm{Osc}_{n-1}\Psi(a_i)$ and it is the only point in this intersection. Both $(\mathbf{1}, \Phi_1, \ldots, \Phi_n)$ and $(\mathbf{1}, \Psi_1, \ldots, \Psi_n)$ being bases of the polynomial space $\mathcal{P}_n$, there exists a unique affine map $h := \mathbb{R}^n \to \mathbb{R}^n$ such that $\Phi = h \circ \Psi$, and h is injective. Therefore

$$\bigcap_{i=1}^{n} \mathrm{Osc}_{n-1}\Phi(a_i) = h\Big(\bigcap_{i=1}^{n} \mathrm{Osc}_{n-1}\Psi(a_i)\Big) = \{h(0_{\mathbb{R}^n})\}.$$

Not only is this new proof very short, but it also has the advantage to make no use of the space $\mathcal{E}^\sharp$. What exactly made it possible? The answer is: the fact that the functions $\Psi_i'(x) = n(x - a_i)^{n-1}$, $1 \le i \le n$, form a basis of the space $\mathcal{P}_{n-1}$, each of them vanishing the appropriate number of times at the concerned point to make the corresponding osculating hyperplane as easy to express as possible. Similar arguments can be developed in the general polynomial case we considered later on. The simple example of polynomial spaces brings out the importance of choosing the appropriate basis to solve a given problem concerning blossoms: for other bases the result will follow by taking images under affine maps. This is exactly what we shall extend to the Chebyshevian framework.

This work was an excellent opportunity to revisit both Extended Chebyshev spaces and their links with blossoms. It is intended as a survey on the topic, even though it also presents new results and proofs. Section 2 gathers various characterisations of Extended Chebyshev spaces, either classical ones recalled in a way to prepare the rest of the article, or recent ones which can be considered our first step towards Section 3. In the latter section we characterise these spaces by the existence of bases generalising our previous polynomial basis $(\Psi'_1, \dots, \Psi'_n)$ (see Theorem 23). To achieve this, we first establish some interesting technical results emerging from Sylvester's identity for determinants. The fourth section shows the advantage of such bases for blossoms: not only do they make their existence obvious, but they are also the relevant bases to achieve their crucial pseudoaffinity property. They actually permit a direct proof of it, that is, unlike previous papers on the same subject, with no need to involve a "dual space", which required either more differentiability than necessary or complicated demonstrations. This section also makes a general survey of all important consequences of pseudoaffinity, along with the fundamental links between blossoms and Bernstein or B-spline bases, so far stated under more differentiability assumptions (see [13]). Finally the last two sections are devoted to the piecewise case, obtained by connecting different Extended Chebyshev spaces via lower triangular matrices with positive diagonal elements. We first investigate how to adapt the various properties reviewed in the second section to Extended Chebyshev Piecewise spaces. Then we show that, as in the nonpiecewise case, this provides us with bases tailor-made for blossoms.

2. Extended Chebyshev Spaces Revisited

In this section we survey both classical and recently obtained results about Chebyshev spaces. In particular we lay emphasis on various ways to characterise the possibility of doing Taylor and Hermite interpolation in a given functional space: number of zeros, nonvanishing determinants, or existence of special bases. The latter ones will be the starting point for the fundamental result presented in the next section. For further classical results on Chebyshev spaces, we refer to [6,21].

2.1. Taylor interpolation and W-spaces

Throughout the first three sections, I denotes a real interval with a nonempty interior. The possibility of solving interpolation problems in a unique way is classically characterised by bounding the number of zeros. This requires us to start with a few preliminary comments on how to count multiple zeros and on the notations we shall use. Given $k \leq n+1$, a function $U \in C^n(I)$ is said to ***vanish k times at a*** $\in I$, or a is said to be ***a zero of multiplicity k***, if $U(a) = U'(a) = \cdots = U^{(k-1)}(a) = 0$. Given $k \leq n$, U is said to ***vanish exactly k times at a***, or a is said to be ***a zero of exact multiplicity k***, if $U(a) = U'(a) = \cdots = U^{(k-1)}(a) = 0$ and $U^{(k)}(a) \neq 0$. We denote by $Z_{n+1}(U)$ the total number of zeros of U in I, counting multiplicities up to $n+1$. Similarly, if S is a subset of I, we denote by $Z^S_{n+1}(U)$ the total number of zeros of U on the points of S.

In the rest of the section, $\mathcal{E}$ denotes an $(n+1)$-dimensional subspace of $C^n(I)$. Selecting a basis $(F_0, \dots, F_n)$ of $\mathcal{E}$, we set $\mathbb{F} := (F_0, \dots, F_n)^T$. On the other hand, for $0 \leq k \leq n+1$, and $a \in I$, we introduce the subspace

$$\mathcal{E}_k(a) := \{F \in \mathcal{E} \mid F \text{ vanishes } k \text{ times at } a\}. \tag{3}$$

We can first address *Taylor interpolation problems in* $n+1$ *data in the space* $\mathcal{E}$, that is, problems of the following form:

$$\text{Find } F \in \mathcal{E} \text{ such that } F^{(i)}(a) = \alpha_i\ , \quad 0 \le i \le n, \tag{4}$$

where a is given in I, and $\alpha_0, \ldots, \alpha_n$ are any real numbers.

Definition 1. The $(n+1)$-dimensional space $\mathcal{E} \subset C^n(I)$ is said to be a *W-space on* I if it satisfies the following property:
(i) *Any Taylor interpolation problem* (4) *has a unique solution in* $\mathcal{E}$.

As recalled below, W-spaces can be characterised in terms of Wronskians, which justifies the name we adopted.

Proposition 2. *The* $(n+1)$*-dimensional subspace* $\mathcal{E}$ *of* $C^n(I)$ *is a W-space on* I *if and only if it meets any of the following equivalent requirements:*
(ii) *Any nonzero element* F *of* $\mathcal{E}$ *vanishes at most* n *times at any given point of* I*, i.e.,* $Z_{n+1}^{\{a\}}(F) \le n$ *for all* $a \in I$.
(iii) *The Wronskian*

$$W(F_0, \ldots, F_n)(x) := \det\big(\mathbb{F}(x), \mathbb{F}'(x), \ldots, \mathbb{F}^{(n)}(x)\big),$$

never vanishes on I.
(iv) *For any* $a \in I$*, there exists a basis* $(\Psi_0^a, \Psi_1^a, \ldots, \Psi_n^a)$ *in* $\mathcal{E}$ *such that, for* $0 \le i \le n$*,* Ψ_i^a *vanishes exactly* i *times at* a.
(v) *For any* $a \in I$*,* $\mathcal{E}_{n+1}(a) = \{0\}$.

Although classical, we insist on the following remarks because of their fundamental importance, in particular in this article.

Remark 3. (1) Given a function $\omega \in C^n(I)$, the Leibniz' rule to differentiate products leads to

$$W(\omega F_0, \ldots, \omega F_n) = \omega^{n+1} W(F_0, \ldots, F_n). \tag{5}$$

Hence, if ω never vanishes on I, the space $\mathcal{E}$ is a W-space on I if and only if the space $\omega\mathcal{E} := \{\omega F \mid F \in \mathcal{E}\}$ is a W-space on I.

(2) We denote by D the ordinary differentiation. On account of the obvious equality

$$W(\mathbb{1}, F_1, \ldots, F_n) = W({F_1}', \ldots, {F_n}'), \tag{6}$$

when the space $\mathcal{E}$ contains constants, it is a W-space on I if and only if the space $D\mathcal{E} := \{F' \mid F \in \mathcal{E}\}$ is a W-space on I (of dimension n).

(3) The latter two remarks provide an easy way to obtain W-spaces by successive integration of a given W-space, and with, at each step, possible multiplication by a sufficiently differentiable nonvanishing function. For instance, setting either

$U(x) := \cos x$ and $V(x) := \sin x$, or $U(x) := \cosh x$ and $V(x) := \sinh x$, the space these two functions span is a two-dimensional W-space on $I = \mathbb{R}$, because in either case they satisfy $W(U, V)(x) = 1$ for all $x \in \mathbb{R}$. It follows that, for all $n \geq 2$, the space $\mathcal{E}_n(U, V)$ spanned by the $(n+1)$ functions $1, x, \ldots, x^{n-2}, U(x), V(x)$ is an $(n+1)$-dimensional W-space on $\mathbb{R}$.

Remark 4. One can also say that the space $\mathcal{E}$ is a W-space on I if and only if, for all $a \in I$, the sequence

$$\{0\} = \mathcal{E}_{n+1}(a) \subset \mathcal{E}_n(a) \subset \cdots \subset \mathcal{E}_1(a) \subset \mathcal{E}_0(a) = \mathcal{E}$$

is a strictly increasing one, *i.e.*, for $0 \leq k \leq n+1$, $\mathcal{E}_k(a)$ is $(n-k+1)$-dimensional. In other words, it means that, given any $a \in I$, and any nonzero $U \in \mathcal{E}$, we are able to give the exact multiplicity k, $0 \leq k \leq n$, of a as a zero of U.

2.2. Hermite interpolation and EC-spaces

More generally one can consider *Hermite interpolation problems in $n+1$ data in the space $\mathcal{E}$*, that is, any problem of the following form:

$$\text{Find } F \in \mathcal{E} \text{ such that } F^{(j)}(\tau_i) = \alpha_{i,j}\ , \quad 1 \leq i \leq r,\ 0 \leq j \leq \mu_i - 1\ , \tag{7}$$

in which $\tau_1, \ldots, \tau_r$ are pairwise distinct points in I, $\mu_1, \ldots, \mu_r$ are positive numbers such that $\sum_{i=1}^r \mu_i = n+1$, and $\alpha_{i,j}$, $1 \leq i \leq r$, $0 \leq j \leq \mu_i - 1$, are any real numbers. We say that the problem (7) is *based on the r points $\tau_1, \ldots, \tau_r$*. Hermite interpolation problems based on one point are thus Taylor interpolation problems.

Definition 5. The $(n+1)$-dimensional space $\mathcal{E} \subset C^n(I)$ is said to be an *Extended Chebyshev space (in short, EC-space) on I* if it satisfies the following property:
(i)′ *Any Hermite interpolation problem* (7) *has a unique solution in $\mathcal{E}$.*

Below we recall some classical characterisations of EC-spaces.

Proposition 6. *The $(n+1)$-dimensional subspace $\mathcal{E}$ of $C^n(I)$ is an EC-space on I if and only if it meets any of the following equivalent requirements:*
(ii)′ *Any nonzero element F of $\mathcal{E}$ vanishes at most n times on I, counting multiplicities, i.e., $Z_{n+1}(F) \leq n$.*
(iii)′ *For any $r \geq 1$, any positive integers $\mu_1, \ldots, \mu_r$ such that $\sum_{i=1}^r \mu_i = n+1$, and any pairwise distinct $a_1, \ldots, a_r \in I$, the determinant*

$$\det\left(\mathbb{F}(a_1), \mathbb{F}'(a_1), \ldots, \mathbb{F}^{(\mu_1-1)}(a_1), \mathbb{F}(a_2), \ldots, \mathbb{F}(a_r), \ldots, \mathbb{F}^{(\mu_r-1)}(a_r)\right), \tag{8}$$

is not equal to zero.

Remark 7. (1) Due to Leibniz' formula, when multiplying each function F_i by $\omega \in C^n(I)$, the determinant (8) is multiplied by $\prod_{i=1}^r \omega(a_i)^{\mu_i}$. Therefore, as in the case of W-spaces, if ω is sufficiently differentiable and if it never vanishes on I, then the space $\mathcal{E}$ is an EC-space on I iff and only if the space $\omega\mathcal{E}$ is an EC-space on I.

(2) If the space $\mathcal{E}$ contains constants and if the space $D\mathcal{E}$ is an n-dimensional EC-space on I, then $\mathcal{E}$ is an $(n+1)$-dimensional EC-space on I. This results from the following inequality

$$Z_n(U') \geq Z_{n+1}(U) - 1, \tag{9}$$

which is valid for any $U \in C^n(I)$ as a direct consequence of Rolle's theorem. However, unlike the case of W-spaces, the converse is not true. Let us illustrate the latter observations by considering the same spaces $\mathcal{E}_n(U,V)$ as in Remark 3. In the hyperbolic case, the space $\mathcal{E}_1(U,V)$ spanned by U,V is a 2-dimensional EC-space on $\mathbb{R}$ because, in addition to their nonvanishing Wronskian, the two functions satisfy, for any distinct a_1, a_2, $\begin{vmatrix} U(a_1) & U(a_2) \\ V(a_1) & V(a_2) \end{vmatrix} = \sinh(a_2 - a_1) \neq 0$. Hence, for all $n \geq 2$, the space $\mathcal{E}_n(U,V)$ too is an EC-space on $\mathbb{R}$. In the trigonometric case, the latter determinant is equal to $\sin(a_2 - a_1)$. We can thus assert that it is nonzero for any distinct a_1, a_2 in I only if we require the interval I to be strictly contained in some $[\alpha, \alpha + \pi]$. Hence the space $\mathcal{E}_1(U,V) = D\mathcal{E}_2(U,V)$ is a 2-dimensional EC-space only on such an interval. Still, one can check that, for $n = 2$ (hence, for $n > 2$ too), the trigonometric space $\mathcal{E}_n(U,V)$ is an EC-space on any interval strictly contained in any $[\alpha, \alpha + 2\pi]$.

2.3. Complete W-spaces

While being an EC-space on I clearly implies being a W-space on I, the converse is not true as shown by the example of the trigonometric spaces mentioned in Remarks 3 and 7. However there exist further important links between W-spaces and EC-spaces, as recalled subsequently.

Definition 8. The $(n+1)$-dimensional space $\mathcal{E} \subset C^n(I)$ is said to be a *Complete W-space (in short, CW-space) on I* if there exists a nested sequence

$$\mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_{n-1} \subset \mathcal{E}_n := \mathcal{E}, \tag{10}$$

where, for $0 \leq i \leq n$, $\mathcal{E}_i$ is an $(i+1)$-dimensional W-space on I. A sequence $(U_0, \ldots, U_n)$ in $C^n(I)$ is said to be a CW-system on I if it satisfies

$$W(U_0, \ldots, U_k)(x) \neq 0 \quad x \in I\,, \quad 0 \leq k \leq n. \tag{11}$$

A sequence (10) automatically provides us with CW-systems by picking a function U_i in $\mathcal{E}_i \setminus \mathcal{E}_{i-1}$ for $0 \leq i \leq n$, with $\mathcal{E}_{-1} := \{0\}$. Accordingly, $\mathcal{E}$ is a CW-space on I iff it possesses a basis which is a CW-system on I. Another basis of $\mathcal{E}$ is not necessarily a CW-system on I.

The following lemma is a key-point in the proof of the well-known important result stated in Theorem 10 below.

Lemma 9. *Given a sequence $(U_0, \dots, U_n)$ in $C^n(I)$, the following two properties are equivalent:*

1- *$(U_0, \dots, U_n)$ is a CW-system on I.*
2- *U_0 does not vanish on I and $(DL_0U_1, \dots, DL_0U_n)$ is a CW-system on I, where $L_0F := F/U_0$.*

Proof. The equivalence readily follows from (5) and (6), □

Theorem 10. *If $\mathcal{E}$ is a CW-space on I, then it is an EC-space on I.*

Proof. Although classical, we need to give a brief proof of this theorem. It is done by induction on n. Being an EC-space or being a W-space is clearly the same for $n = 0$. Suppose that $n \geq 1$ and that the result has been proved for $n - 1$. Let $\mathcal{E}$ be a CW-space and let $(U_0, \dots, U_n)$ be a CW-system in it. According to Lemma 9, $(DL_0U_1, \dots, DL_0U_n)$ is a CW-system. The recursive assumption proves that the n-dimensional CW-space $DL_0\mathcal{E}$ is an EC-space on I. The fact that $\mathcal{E}$ is an $(n+1)$-dimensional EC-space on I follows by applying Remark 7, (1) and (2). □

Given a nested sequence (10) in $C^n(I)$, in which each $\mathcal{E}_i$ is $(i+1)$-dimensional, requiring all spaces $\mathcal{E}_i$ to be W-spaces on I is thus the same as requiring them all to be EC-spaces on I.

A given CW-system $(U_0, \dots, U_n)$ naturally generates differential operators $L_0, \dots, L_n$ on $C^n(I)$ by iteration of Lemma 9. This is a well-known fact [6] but it is desirable to recall it. We have already introduced L_0 in Lemma 9. Applying Lemma 9 to the CW-system $(DL_0U_1, \dots, DL_0U_n)$, we know that the function DL_0U_1 does not vanish on I and that $(DL_1U_2, \dots, DL_1U_n)$ is a CW-system on I, where $L_1F := DL_0F/DL_0U_1$, and so forth. The differential operators are thus defined on $C^n(I)$ in a recursive way as:

$$L_0F := \frac{F}{U_0}\ , \quad L_iF := \frac{DL_{i-1}F}{DL_{i-1}U_i}\ , \quad 1 \leq i \leq n\ . \tag{12}$$

The functions $w_0 := U_0$, $w_i := DL_{i-1}U_i$, $1 \leq i \leq n$, are classically called the *weight functions* associated with the CW-system $(U_0, \dots, U_n)$. Each w_i belongs to $C^{n-i}(I)$ and it does not vanish on I. The space $\mathcal{E}$ spanned by $U_0, \dots, U_n$ can then be described as

$$\mathcal{E} = \{F \in C^n(I) \mid L_nF \text{ is constant on } I\}.$$

Conversely, given non vanishing functions $w_0, \dots, w_n$, with $w_i \in C^{n-i}(I)$ for $0 \leq i \leq n$, one can consider the differential operators on $C^n(I)$ $L_0F := F/w_0$, $L_iF := DL_{i-1}F/w_i$, $1 \leq i \leq n$. It is well-known that the functions $F \in C^n(I)$ for which L_nF is constant on I form a CW-space $\mathcal{E}$: indeed, setting $\mathcal{E}_i := \mathrm{Ker}DL_i$ for $0 \leq i \leq n-1$ and $\mathcal{E}_n := \mathcal{E}$, one can show, via Remark 3, that these spaces form a strictly nested sequence of W-spaces on I. In such a case, we will write $\mathcal{E} = CW(w_0, \dots, w_n)$. This is a classical way to built EC-spaces from weight functions.

2.4. EC spaces and Hermite interpolation based on at most two points

As stated in (iv) of Proposition 2, W-spaces are characterised by the existence of certain type of bases with prescribed numbers of zeros at any given point in I. Among other things, in this subsection, we are interested in bases with prescribed numbers of zeros at any two distinct points in I, according to the definition below, obviously inspired by the polynomial Bernstein basis of degree n, $B_i^n(x) := \binom{n}{i}(1-x)^{n-i}x^i$, $0 \le i \le n$.

Definition 11. Given two distinct points $a, b \in I$, and $(n+1)$ elements $B_0, \dots, B_n \in \mathcal{E}$, we say that $(B_0, \dots, B_n)$ is a Bernstein-like basis relative to (a, b) if, for $0 \le i \le n$, the function B_i vanishes exactly i times at a and exactly $(n-i)$ times at b.

We recently pointed out [14] how intimately Bernstein-like bases are connected with EC-spaces, as stated in the following theorem. The proof we give here differs from [14] : it is intended to prepare for the next section along with the piecewise case examined in Sections 5 and 6.

Theorem 12. *The $(n+1)$-dimensional space $\mathcal{E} \subset C^n(I)$ is an EC-space on I if and only if it satisfies any of the following equivalent properties:*

(i)″ *Any Hermite interpolation problem based on any one or two points of I has a unique solution in $\mathcal{E}$.*

(ii)″ *For any distinct $a, b \in I$, any nonzero element $F \in \mathcal{E}$ vanishes at most n times on $\{a, b\}$, counting multiplicities, i.e., $Z_{n+1}^{\{a,b\}}(F) \le n$.*

(iii)″ *Given any distinct $a, b \in I$ and any integers $i, j \ge 0$ with $i + j = n + 1$, the determinant*

$$\det\left(\mathbb{F}(a), \mathbb{F}'(a), \dots, \mathbb{F}^{(i-1)}(a), \mathbb{F}(b), \mathbb{F}'(b) \dots, \mathbb{F}^{(j-1)}(b)\right) \tag{13}$$

is not equal to zero.

(iv)″ *$\mathcal{E}$ possesses a Bernstein-like basis relative to any pair of distinct points of I.*

(v)″ *Given any $a \in I$, for $0 \le k \le n$, the space $\mathcal{E}_k(a)$ is an $(n-k+1)$-dimensional W-space on any interval $J \subset I \setminus \{a\}$.*

Proof. The equivalence between the first four properties is easy to prove (see [14]). Suppose that (i)″ holds and let us prove that (v)″ is then satisfied. Given any $a \in I$, and any integer k, $0 \le k \le n$, the space $\mathcal{E}_k(a)$ is of dimension $(n-k+1)$ on any nontrivial subinterval of I (see Remark 4). Choose $b \in I \setminus \{a\}$, and any real numbers $\alpha_0 \dots, \alpha_{n-k}$. By (i)″, we know that there exists a unique $F \in \mathcal{E}$ such that

$$F(a) = \cdots = F^{(k-1)}(a) = 0, \quad F^{(j)}(b) = \alpha_j, \quad 0 \le j \le n-k,$$

which means the existence of a unique element $F \in \mathcal{E}_k(a)$ satisfying $F^{(j)}(b) = \alpha_j$ for $0 \le j \le n-k$. In other words, any Taylor interpolation problem in $(n-k+1)$ data at any $b \in I \setminus \{a\}$ has a unique solution. It follows that $\mathcal{E}_k(a)$ is a W-space on any J contained in $I \setminus \{a\}$.

Suppose now that (v)$''$ is satisfied and let us prove that $\mathcal{E}$ is an EC-space on I. Given any $a \in I$, consider the strictly nested sequence

$$\mathcal{E}_n(a) \subset \mathcal{E}_{n-1}(a) \subset \cdots \subset \mathcal{E}_1(a) \subset \mathcal{E}_0(a) := \mathcal{E}.$$

Taking the restrictions to a given interval $J \subset I \setminus \{a\}$ of all subspaces involved in the latter sequence shows that each $\mathcal{E}_k(a)$ is a CW-space on J, hence an $(n-k+1)$-dimensional EC-space on J (see Theorem 10). As a special case, it follows that the space $\mathcal{E}$ itself is an EC-space on any interval strictly contained in I. If the interval I is not closed and bounded, this means that $\mathcal{E}$ is an EC-space on the whole of I too. Suppose now that I is a closed bounded interval $[a,b]$. The space $\mathcal{E}$ being an EC-space on any interval strictly contained in I, it is a W-space on I. Let F be any nonzero element of $\mathcal{E}$. According to Remark 4, we can consider the exact multiplicity $i \in \{0,\ldots,n\}$ of a as a zero of U. Then, F belongs to $\mathcal{E}_i(a)$, which is an $(n-i+1)$-dimensional EC-space on $]a,b]$. Therefore, $Z_{n-i+1}^{]a,b]}(F) \le n-i$. The count of multiplicities implies that $Z_{n+1}^{]a,b]}(F) \le n-i$ as well. This proves that F satisfies $Z_{n+1}(F) = Z_{n+1}^{]a,b]}(F) + Z_{n+1}^{\{a\}}(F) \le n$. □

Although it is far from being the main point, Theorem 12 contains the fact that an EC-space on I is a CW-space on any strict subinterval of I. This gives us an opportunity to mention an important consequence of the latter fact ([19]).

Proposition 13. *Suppose that the interval I is closed and bounded. Then, as soon as $\mathcal{E}$ is an EC-space on I, it is a CW-space on I.*

Proof. We can assume that the space $\mathcal{E}$ is defined on the whole of $\mathbb{R}$, and that it is known to be an EC-space only on the closed bounded interval I. If so, let us set, for any $r \ge 0$, any positive integers $\mu_1,\ldots,\mu_r$ with $\sum_{i=1}^r \mu_i = n+1$, and any pairwise distinct $a_1,\ldots,a_r \in I$

$$\Lambda({a_1}^{[\mu_1]},\ldots,{a_r}^{[\mu_r]}) := \frac{\Gamma_r(\mu_1,\ldots,\mu_r)}{\prod_{1\le k<\ell\le r}(a_\ell - a_k)^{\mu_k\mu_\ell}}$$
$$\times \det\big(\mathbb{F}(a_1),\mathbb{F}'(a_1),\ldots,\mathbb{F}^{(\mu_1-1)}(a_1),\mathbb{F}(a_2),\ldots,\mathbb{F}(a_r),\ldots,\mathbb{F}^{(\mu_r-1)}(a_r)\big), \qquad (14)$$

with $\Gamma_r(\mu_1,\ldots,\mu_r) := 1/\prod_{\ell=1}^r 1!2!\ldots(\mu_\ell-1)!$. The function $\Lambda : \mathbb{R}^{n+1} \to \mathbb{R}$ so defined is symmetric and continuous on $\mathbb{R}^{n+1}$. As a matter of fact, this function Λ emerges as the continuous extension to the whole of $\mathbb{R}^n$ of the symmetric expression

$$\det\big(\mathbb{F}(a_1),\mathbb{F}(a_2),\ldots,\mathbb{F}(a_n)\big) / \prod_{1\le k<\ell\le n}(a_\ell - a_k),$$

well-defined only for pairwise distinct $a_1,\ldots,a_n$ (see [10]). Our assumption that $\mathcal{E}$ is an EC-space on I implies that Λ does not vanish on I^{n+1} (see (iii)$'$ of Proposition 6). Therefore, out of continuity, Λ does not vanish on J^{n+1}, where J is some interval strictly containing I. Hence, according to Proposition 6, (iii)$'$, $\mathcal{E}$ is an EC-space on the interval J. It follows from (v)$''$ that $\mathcal{E}$ is a CW-space on any interval strictly contained in J, so in particular on I. □

Therefore, whenever $\mathcal{E}$ is an $(n+1)$-dimensional EC-space on a closed bounded interval I, we can find nonvanishing weight functions $w_0, \dots, w_n$ such that $\mathcal{E} = CW(w_0, \dots, w_n)$. Since they all are continuous functions, if needed, we can assume them to be positive on I.

To conclude this section, we want to draw the reader's attention to the fact that each property (i)″, (ii)″, and (iii)″ of Theorem 12 is the "two point version" respectively of each property (i)′, (ii)′, and (iii)′ of Definition 5 and Proposition 6. It is thus natural to try and imagine what should be the "multiple point version" of either (iv)″ or (v)″. Below we state a natural (iv)′, for the proof of which we refer to [14].

Theorem 14. *The space $\mathcal{E}$ is an EC-space on I, if and only if it satisfies:*

(iv)′ *Given any sequence $x_0 \le x_1 \le x_2 \le \cdots \le x_n < x_{n+1} \le x_{2n} \le x_{2n+1}$ in I, there exists a basis $(V_0, \dots, V_n)$ of $\mathcal{E}$ such that, for $0 \le i \le n$, V_i vanishes on $(x_{i+1}, \dots, x_{i+n})$, but it vanishes neither on $(x_i, \dots, x_{i+n})$ nor on $(x_{i+1}, \dots, x_{i+n+1})$.*

In (iv)′, supposing that $(x_{i+1}, \dots, x_{i+n}) = (\xi_1{}^{[\mu_1]}, \dots, \xi_r{}^{[\mu_r]})$ with $\xi_1 < \xi_2 < \cdots < \xi_r$ and positive $\mu_1, \dots, \mu_r$, the expression "V_i vanishes on $(x_{i+1}, \dots, x_{i+n})$" means that the function V_i vanishes μ_k times at ξ_k for $1 \le k \le r$. If they generalise in a natural way the notion of Bernstein-like basis, the bases involved in (iv)′ are of no real use for blossoms. It is the "multiple point version" of (v)″, achieved in the next section, that will bring us closer to blossoming.

3. On the Way to Blossoms via Sylvester's Identity

The aim of this section is to characterise EC-spaces by the existence of a certain type of bases giving a generalisation of property (v)″ of Theorem 12 (see Theorem 23 below). In the next section the latter bases will prove to be especially suited to establish both existence and properties of blossoms. We shall achieve them via Sylvester's identity and a number of preliminary results proved in the first subsection.

3.1. Some consequences of Sylvester's identity

Let us start by recalling Sylvester's identity for determinants (see, for instance, [1]).

Lemma 15. (Sylvester's identity). *Let M be a square matrix of order $n+1$, and let N be the square submatrix of order $n-1$ obtained by deleting the first and last rows and columns of M. Let A, B, C, D be the four square submatrices of order n obtained as follows:*

- *A: by deleting the last row and the last column of M.*
- *B: by deleting the last row and the first column of M.*
- *C: by deleting the first row and the last column of M.*
- *D: by deleting the first row and the first column of M.*

The following equality holds

$$\det M \ \det N = \det A \ \det D - \det B \ \det C \ . \tag{15}$$

Formula (15) is the exact generalisation of the rule to calculate determinants of order 2, namely $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$, with, by convention, $\det\emptyset = 1$. Sylvester's identity has plenty of interesting implications. The following lemma is an example for an application of (15).

Lemma 16. *Given $k \geq 0$, let $V_1, \dots, V_k, U, V$ be $(k+1)$ times differentiable functions on I. Then*

$$\begin{aligned} W(V_1,...,V_k)(x)W(V_1\ ,...,V_k,U,V)(x) = \\ W(V_1,\ ...,V_k,V)'(x)W(V_1,...,V_k,U)(x) \\ -W(V_1,...,V_k,V)(x)W(V_1,...,V_k,U)'(x) \end{aligned} \tag{16}$$

Proof. Let us set $\Gamma := (U, V_1, \dots, V_k, V)^T$. Then, for any $x \in I$, formula (16) is obtained by applying (15) to the determinant $D(x)$ of order $(k+2)$ defined by:

$$D(x) := \det\big(\Gamma^{(k+1)}(x), \Gamma(x), \Gamma'(x), \dots, \Gamma^{(k-1)}(x), \Gamma^{(k)}(x)\big) \ .$$

□

As a direct consequence of (16), we obtain the following result.

Lemma 17. *Given $k+2$ functions $V_1, \dots, V_k, U, V$ supposed to be $k+1$ times differentiable on the interval I, consider the function u defined by*

$$u := \frac{W(V_1, \dots, V_k, V)}{W(V_1, \dots, V_k, U)}$$

on the subset Ω of I where $W(V_1, \dots, V_k, U)$ does not vanish. Its derivative is then given by

$$u'(x) = \frac{W(V_1, \dots, V_k)(x)W(V_1, \dots, V_k, U, V)(x)}{[W(V_1, \dots, V_k, U)(x)]^2}, \quad x \in \Omega. \tag{17}$$

One important consequence of formula (17) is that it provides us with nice expressions in terms of Wronskians for the differential operators (12) associated with a given CW-system.

Lemma 18. *Let $(U_0,\dots,U_n)$ be a CW-system on I, and let $L_0,\dots,L_n$ denote the associated differential operators, defined according to* (12). *Then we have, for all $F \in C^n(I)$,*

$$L_iF = \frac{W(U_0,...,U_{i-1},F)}{W(U_0,...,U_{i-1},U_i)}, \quad 0 \le i \le n. \tag{18}$$

Proof. Given that $L_0F := F/U_0$, equality (18) is clearly satisfied for $i = 0$. Consider an integer i such that $0 \le i \le n-1$, and suppose that (18) holds for i. By application of (17), we can derive

$$DL_iF = \frac{W(U_0,...,U_{i-1})\ W(U_0,...,U_{i-1},U_i,F)}{W(U_0,...,U_i)^2}. \tag{19}$$

The equality $L_{i+1}F := (DL_iF)/(DL_iU_{i+1})$ then proves (18) for $i+1$. □

With the convention $W(\emptyset) = 𝟙$, equality (19) shows that the weight functions classically associated to a given CW-system are given by (see [6])

$$w_i := \frac{W(U_0,...,U_{i-2})\ W(U_0,...,U_i)}{W(U_0,...,U_{i-1})^2}\ , \quad 0 \le i \le n.$$

Using (18) and (1) of Remark 3, iteration of Lemma 9 yields the following result.

Lemma 19. *Given $(n+1)$ functions $U_0,\dots,U_n \in C^n(I)$, we fix an integer k, $0 \le k \le n$ and we consider the functions*

$$F_i^k := W(U_0,...,U_k,U_i)\ , \quad k+1 \le i \le n.$$

Then, $(U_0,\dots,U_n)$ is a CW-system on I if and only if both $(U_0,\dots,U_k)$ and $(F_{k+1}^k,\dots,F_n^k)$ are CW-systems on I.

The right-hand side of equality (16) is nothing but the Wronskian of the two functions $W(U_0,...,U_k,U)$ and $W(U_0,...,U_k,V)$. Formula (16) is thus a special case of the following result which will prove useful in Subsection 4.3 and with which we conclude the present subsection. Note that it also gives another proof of Lemma 19.

Lemma 20. *Given $(n+1)$ functions $U_0,\dots,U_n \in C^n(I)$, with the notations of the previous lemma, we have, for $0 \le k \le n$ and for all $x \in I$,*

$$W(F_{k+1}^k,...,F_n^k)(x) = \left[W(U_0,...,U_k)(x)\right]^{n-k-1} W(U_0,...,U_n)(x). \tag{20}$$

Proof. Note that formula (20) is independent of the order between $U_0,\dots,U_k$ on the one hand, and between $U_{k+1},\dots,U_n$ on the other. Furthermore, for any k, $1 \le k \le n-1$, for any x such that $F_k^{k-1}(x) \ne 0$, by application of (5) and (6), we obtain

$$W(F_k^{k-1},\dots,F_n^{k-1})(x) = \left[F_k^{k-1}(x)\right]^{n-k+1} W\left(\left(\frac{F_{k+1}^{k-1}}{F_k^{k-1}}\right)',\dots,\left(\frac{F_n^{k-1}}{F_k^{k-1}}\right)'\right)(x).$$

Using (17) to calculate the functions $\left(F_i^{k-1}/F_k^{k-1}\right)'$, we obtain

$$W(F_k^{k-1},\dots,F_n^{k-1})(x) = \frac{\left[W(U_0,\dots,U_{k-1})(x)\right]^{n-k}}{\left[W(U_0,\dots,U_k)(x)\right]^{n-k-1}} W(F_{k+1}^k,\dots,F_n^k)(x), \quad (21)$$

subject that $F_k^{k-1}(x) \neq 0$. We shall prove (20) by induction on $k \geq -1$. For $k=-1$, it is trivially satisfied independently of n. Suppose that $k \geq 0$ and that the result holds for any integer less than or equal to $k-1$.

1) Let us first consider a point $x_0 \in I$ such that $W(U_0,...,U_k)(x_0) = 0$.

We want to prove that $W(F_{k+1}^k,...,F_n^k)(x_0) = 0$. This is clearly satisfied as soon as $F_i^k(x_0) = 0$ for any integer i such that $k+1 \leq i \leq n$. Suppose that there exists an integer i, $k+1 \leq i \leq n$, such that $F_i^k(x_0) \neq 0$. We may as well assume that $i = k+1$. By continuity, F_{k+1}^k does not vanish on some neighbourhood I_0 of x_0 contained in I. Therefore, equality (21) in which we replace k by $k+1$ is valid for $x \in I_0$. This proves that $W(F_{k+1}^k,...,F_n^k)(x_0) = 0$.

2) Let us now consider a point $x_0 \in I$ such that $W(U_0,...,U_k)(x_0) \neq 0$.

In this case, we can find an integer i, $0 \leq i \leq k$, such that $W(U_0,...,U_{i-1},U_{i+1},...,U_k)(x_0) \neq 0$. Without loss of generality we may assume that $i = k$. Our recursive assumption enables us to write, for all $x \in I$,

$$W(F_k^{k-1},...,F_n^{k-1})(x) = \left[W(U_0,...,U_{k-1})(x)\right]^{n-k} W(U_0,...,U_n)(x). \quad (22)$$

Since $W(U_0,...,U_{k-1})(x_0) \neq 0$, the two equalities (21) and (22) prove that (20) is satisfied for $x = x_0$. □

3.2. A new characterisation of EC-spaces

The various consequences of Sylvester's identity developed in the previous subsection will provide us with interesting bases in the space $\mathcal{E}$, which later on will prove to be tailor-made for blossoms. Such bases will generalise Lemma 21 below which is nothing but a different way of stating property (v)″ of Theorem 12.

Lemma 21. *The $(n+1)$-dimensional space $\mathcal{E} \subset C^n(I)$ is an EC-space on I if and only if the following two properties hold simultaneously:*

(v)$_1''$ *$\mathcal{E}$ is a W-space on I, i.e., (according to Proposition 2) for any $a \in I$, there exists a basis $(\Psi_n^a,\dots,\Psi_0^a)$ in $\mathcal{E}$ such that, for $0 \leq i \leq n$, Ψ_i^a vanishes exactly i times at a.*

(v)$_2''$ *For any $a \in I$, $(\Psi_n^a,\dots,\Psi_0^a)$ is a CW-system on any interval contained in $I \setminus \{a\}$, i.e., it satisfies*

$$W(\Psi_n^a,\dots,\Psi_k^a)(x) \neq 0, \quad x \in I \setminus \{a\},\ 1 \leq k \leq n.$$

Note that part (v)$_1''$ of Lemma 21 guarantees that the Wronskian $W(\Psi_n^a,\dots,\Psi_0^a)$ never vanishes on the whole of I.

Example 22. In the polynomial case, take $\Psi_i^a(x) := (x-a)^i$, for $0 \le i \le n$. Then, choosing an integer k between 0 and n, by application of (5), it is easy to see that

$$\begin{aligned} W(\Psi_k^a, \dots, \Psi_n^a)(x) &= [\Psi_k^a(x)]^{n-k+1} W(\Psi_0^a, \dots, \Psi_{n-k}^a)(x) \\ &= \Big(\prod_{j=1}^{n-k} j!\Big)(x-a)^{k(n-k+1)}, \quad x \in \mathbb{R}. \end{aligned} \tag{23}$$

Below we state the "multiple point version" of Lemma 21, that is, of (v)″ of Theorem 12.

Theorem 23. *The space $\mathcal{E}$ is an EC-space on I if and only if it satisfies the following property:*

(v)′ *given any integer $r \ge 1$, any positive integers $\mu_1, \dots, \mu_r$ such that $\sum_{i=1}^r \mu_i = n+1$, and any pairwise distinct $a_1, \dots, a_r \in I$, $\mathcal{E}$ possesses a basis $\Psi_n^{a_1}, \dots, \Psi_{n-\mu_1+1}^{a_1}, \Psi_n^{a_2}, \dots, \Psi_n^{a_r}, \dots, \Psi_{n-\mu_r+1}^{a_r}$ such that, for $1 \le i \le r$ and $n-\mu_i+1 \le k \le n$, the function $\Psi_k^{a_i}$ vanishes exactly k times at a_i.*

Moreover, when $\mathcal{E}$ is an EC-space on I, each such basis is a CW-system on any interval J contained in $I \setminus \{a_1, \dots, a_r\}$, i.e., with the same data as in (v)′ *except that now we assume $\sum_{i=1}^r \mu_i \le n$, we have, for all $x \in I \setminus \{a_1, \dots, a_r\}$,*

$$W(\Psi_n^{a_1}, \dots, \Psi_{n-\mu_1+1}^{a_1}, \Psi_n^{a_2}, \dots, \Psi_n^{a_r}, \dots, \Psi_{n-\mu_r+1}^{a_r})(x) \neq 0. \tag{24}$$

Proof. Let us first suppose that (v)′ holds. The case $r=1$ proves that the space $\mathcal{E}$ satisfies (iv) of Proposition 2, hence it is a W-space on I. Let us now consider the case $r=2$, that is, consider distinct $a_1, a_2 \in I$ and positive integers μ_1, μ_2 such that $\mu_1+\mu_2 = n+1$. Since $\mathcal{E}$ is a W-space on I, the Wronskian of the corresponding basis $(\Psi_n^{a_1}, \dots, \Psi_{n-\mu_1+1}^{a_1}, \Psi_n^{a_2}, \dots, \Psi_{n-\mu_2+1}^{a_2})$ provided by (v)′ never vanishes on I. Now, from the zeros of these functions it is easy to derive

$$\begin{aligned} W(\Psi_n^{a_1}, \dots, \Psi_{n-\mu_1+1}^{a_1}, \Psi_n^{a_2}, \dots, \Psi_{n-\mu_2+1}^{a_2})(a_2) &= W(\Psi_n^{a_1}, \dots, \Psi_{n-\mu_1+1}^{a_1})(a_2) \\ &\times W(\Psi_n^{a_2(\mu_1)}, \dots, \Psi_{n-\mu_2+1}^{a_2\ (\mu_1)})(a_2). \end{aligned}$$

The latter relation enables us to conclude that, for any $a_1 \in I$ and any positive integer $\mu_1 \le n$,

$$W(\Psi_n^{a_1}, \dots, \Psi_{n-\mu_1+1}^{a_1})(a_2) \neq 0 \quad \text{for all } a_2 \in I \setminus \{a_1\}.$$

Lemma 21 says that $\mathcal{E}$ is an EC-space on I.

Conversely, let us now assume that $\mathcal{E}$ is an EC-space on I. Let us prove (v)′ along with (24) by induction on $r \ge 1$. For $r=1$, it is nothing but Lemma 21. Suppose that $r \ge 2$, and that both (v)′ and (24) hold for $r-1$ in any EC-space on any interval.

Consider positive integers $\mu_1, \dots, \mu_r$ such that $\mu := \sum_{i=1}^r \mu_i \le n+1$ and pairwise distinct $a_1, \dots, a_r \in I$. We may as well assume that $a_2, \dots, a_r$ all belong to some interval I_1 contained in $I \setminus \{a_1\}$. Let $\widetilde{\mathcal{E}}_1$ denote the restriction to I_1 of the space $\widetilde{\mathcal{E}}$ defined by

$$\widetilde{\mathcal{E}} := \{W(\Psi_n^{a_1}, \dots, \Psi_{n-\mu_1+1}^{a_1}, F) \mid F \in \mathcal{E}\}.$$

From Lemma 21, we know that $(\Psi_n^{a_1}, \dots, \Psi_0^{a_1})$ is a CW-system on I_1. Therefore, as a consequence of Lemma 19 and Theorem 10, we know that the space $\widetilde{\mathcal{E}}_1$ is an $(\widetilde{n}+1)$-dimensional EC-space on I_1, where $\widetilde{n} := n - \mu_1$. It is to this space $\widetilde{\mathcal{E}}_1$, and to $a_2, \dots, a_r$ and $\mu_2, \dots, \mu_r$, that we shall apply the recursive assumption. Given $b \in I \setminus \{a_1\}$ and $k \le \widetilde{n}$, we denote by $\widetilde{\Psi}_k^b$ any element of $\widetilde{\mathcal{E}}$ which vanishes exactly k times at b. Due to $(\mathrm{v})_2''$ of Lemma 21 and to Lemma 24 below, for any $b \in I_1$, we can choose

$$\widetilde{\Psi}_k^b := W(\Psi_n^{a_1}, \dots, \Psi_{n-\mu_1+1}^{a_1}, \Psi_{k+\mu_1}^b), \quad 0 \le k \le \widetilde{n}. \tag{25}$$

According to (20), such a choice yields:

$$\begin{aligned} W(\widetilde{\Psi}_{\widetilde{n}}^{a_2}, \dots, \widetilde{\Psi}_{\widetilde{n}-\mu_2+1}^{a_2}, \dots, \widetilde{\Psi}_{\widetilde{n}}^{a_r}, \dots, \widetilde{\Psi}_{\widetilde{n}-\mu_r+1}^{a_r})(x) = \\ \left[W(\Psi_n^{a_1}, \dots, \Psi_{n-\mu_1+1}^{a_1}(x)\right]^{\mu-\mu_1-1} \\ \times W(\Psi_n^{a_1}, \dots, \Psi_{n-\mu_1+1}^{a_1}, \Psi_n^{a_2}, \dots, \Psi_n^{a_r}, \dots, \Psi_{n-\mu_r+1}^{a_r})(x), \quad x \in I. \end{aligned}$$

On the other hand, the recursive assumption tells us that the left hand side of the latter equality does not vanish on $I_1 \setminus \{a_2, \dots, a_r\}$ if $\mu - \mu_1 \le \widetilde{n}$, and on the whole of I_1 when $\mu - \mu_1 = \widetilde{n}$. Accordingly, in any case, the Wronskian $W(\Psi_n^{a_1}, \dots, \Psi_{n-\mu_1+1}^{a_1}, \Psi_n^{a_2}, \dots, \Psi_n^{a_r}, \dots, \Psi_{n-\mu_r+1}^{a_r})$ never vanishes on $I_1 \setminus \{a_2, \dots, a_r\}$. For $\mu = n+1$, this implies the linear independence of the functions $\Psi_n^{a_1}, \dots, \Psi_{n-\mu_1+1}^{a_1}, \Psi_n^{a_2}, \dots, \Psi_n^{a_r}, \dots, \Psi_{n-\mu_r+1}^{a_r}$ on I, and for $\mu \le n$, it proves (24). $\square$

Theorem 23 says in particular that, when $\mu = n+1$, (24) is valid not only on $I \setminus \{a_1, \dots, a_r\}$, but even on the whole of I.

Lemma 24. *Given $k+1$ functions $V_1, \dots, V_k, U$ supposed to be $N-1 \ge k$ times differentiable on I, we assume that U vanishes N times at a point $a \in I$. Then, the Wronskian $H := W(V_1, \dots, V_k, U)$ vanishes $N-k$ times at a. Moreover, if the functions are N times differentiable on I, H vanishes exactly $N-k$ times at a iff U vanishes exactly N times at a and $W(V_1, \dots, V_k)(a) \ne 0$.*

Proof. We denote by $\langle ., . \rangle$ the inner product in $\mathbb{R}^{k+1}$. Let us introduce the following notation $\Delta_k U := (U, U', \dots, U^{(k)})^T$, and let us set $\Theta(x) := \Delta_k V_1(x) \wedge \dots \wedge \Delta_k V_k(x)$ (cross product). For $x \in I$, we then have $H(x) = \langle \Theta(x), \Delta_k U(x) \rangle$, which gives, for $0 \le i \le N-k-1$,

$$H^{(i)}(x) = \sum_{\ell=0}^{i} \binom{i}{\ell} \langle \Theta^{(i-\ell)}(x), (\Delta_k U)^{(\ell)}(x) \rangle, \quad x \in I.$$

Now, $(\Delta_k U)^{(\ell)}(x) = (U^{(\ell)}(x), U^{(\ell+1)}, \dots, U^{(\ell+k)}(x))^T$. Therefore, on account of the assumption $U(a) = \dots = U^{(N-1)}(a) = 0$, we can derive from the previous equality that $H(a) = \cdots = H^{(N-1-k)}(a) = 0$. Moreover, if the functions are N times differentiable, the same equality is valid for $i = N - k$, which yields

$$H^{(N-k)}(a) = U^{(N)}(a)\ W(V_1, \dots, V_k)(a).$$

Whence the second part of the announced result. □

Example 25. Consider again the polynomial case, with, for all $a \in \mathbb{R}$, $\Psi_i^a(x) := (x-a)^i$. One can generalise formula (23), as follows: for any positive $\mu_1, \dots, \mu_r$ with $\mu := \sum_{i=1}^r \mu_i \le n+1$, and any $a_1, \dots, a_r, x \in \mathbb{R}$,

$$W(\Psi^{a_1}_{n-\mu_1+1}, \dots, \Psi^{a_1}_n, \dots \Psi^{a_r}_{n-\mu_r+1}, \dots, \Psi^{a_r}_n)(x) = \\ B\prod_{i=1}^r (x-a_i)^{\mu_i(n-\mu+1)} \prod_{1\le i<j\le r} (a_i - a_j)^{\mu_i\mu_j}, \qquad (26)$$

where $B := A/\prod_{i=1}^r \binom{n}{1}\binom{n}{2}\dots\binom{n}{\mu_i-1}$, and $A := 1!2!\dots(\mu-1)!\binom{n}{1}\binom{n}{2}\dots\binom{n}{\mu-1}$. Formula (26) can be obtained by applying the results of [10] (see also proof of Proposition 13) to the following easy to obtain equality:

$$W(\Psi^{a_1}_n, \Psi^{a_2}_n, \dots, \Psi^{a_\mu}_n)(x) = A\prod_{i=1}^{\mu}(x-a_i)^{n-\mu+1} \prod_{1\le i<j\le\mu}(a_j - a_i), \quad x \in \mathbb{R},$$

valid for any $\mu \le n+1$, and any pairwise distinct $a_1, \dots, a_\mu \in \mathbb{R}$ (see (14)). In particular, when $\mu = n+1$, we can verify that, as expected, (26) never vanishes on $\mathbb{R}$: indeed, it is a nonzero constant. Further interesting properties of such polynomial bases can be found in [18].

4. Blossoms in W-spaces

This section is devoted to blossoms: existence, properties, and consequences. As a matter of fact, the results stated in subsections 4.2 and 4.3 below, which concern existence and properties of blossoms, are not new, but so far they were achieved either under more differentiability assumptions [19,9], or via rather difficult (although interesting) proofs [5].

4.1. Blossoms and Bézier points

From now on, $\mathcal{E}$ denotes an $(n+1)$-dimensional subspace of $C^n(I)$, assumed to contain constants. Choosing n elements $\Phi_1, \dots, \Phi_n$ in $\mathcal{E}$ so that $(\mathbb{1}, \Phi_1, \dots, \Phi_n)$ is a basis of $\mathcal{E}$, we set

$$\Phi := (\Phi_1, \dots, \Phi_n)^T.$$

Such a function Φ play the rôle of a *mother-function*, in the sense that any $F \in \mathcal{E}^d$ $(d \ge 1)$ can be obtained as its image $F = h \circ \Phi$ under an affine map $h : \mathbb{R}^n \to \mathbb{R}^d$

and this affine map is uniquely determined. It is essential to be aware that, in the space $\mathcal{E}$, all results and all mathematical objects which we obtain subsequently from a given mother-function Φ will be independent of our choice of Φ.

Since $\mathcal{E} \subset C^n(I)$, for any $x \in \mathbb{R}$ and for $k \leq n$, we can consider osculating flats of Φ up to order n (see Section 1). Up to a permutation, a given n-tuple $(x_1, \ldots, x_n) \in I^n$ can always be written as $({a_1}^{[\mu_1]}, \ldots, {a_r}^{[\mu_r]})$, where $\mu_1, \ldots, \mu_r$ are positive integers such that $\sum_{i=1}^r \mu_i = n$, and where $a_1, \ldots, a_r$ are pairwise distinct points of I. We then set

$$\{ \varphi(x_1, \ldots, x_n) \} := \bigcap_{i=1}^{r} \mathrm{Osc}_{n-\mu_i} \Phi(a_i) \tag{27}$$

whenever the osculating flats $\mathrm{Osc}_{n-\mu_i} \Phi(a_i)$, $1 \leq i \leq r$, intersect at a single point. Let $\mathcal{D}(\varphi)$ denote the domain of definition of the function φ. It is a symmetric subset of I^n independent of our chosen mother-function Φ, on which φ is clearly symmetric. The set $\mathcal{D}(\varphi)$ obviously contains the diagonal of I^n and, by restriction to the latter diagonal, the function φ gives Φ in the sense that $\varphi(x^{[n]}) = \Phi(x)$ for all $x \in I$. With any $F \in \mathcal{E}^d$ we associate the function f defined on $\mathcal{D}(\varphi)$ by $f := h \circ \varphi$, where $h : \mathbb{R}^n \to \mathbb{R}^d$ is the unique affine map such that $F = h \circ \Phi$. Let us stress again that f does not depend on the mother-function Φ we start with.

Definition 26. We say that *Bézier points exist in the space* $\mathcal{E}$ when, for all distinct $a, b \in I$, the points

$$\Pi_i(a, b) := \varphi(a^{[n-i]}, b^{[i]}) \ , \quad 0 \leq i \leq n, \tag{28}$$

called *the Bézier points of* Φ *relative to* (a, b), are well-defined, that is, when the set $\mathcal{D}(\varphi)$ contains $\{a, b\}^n$ for all distinct $a, b \in I$. We say that *blossoms exist in the space* $\mathcal{E}$ when $\mathcal{D}(\varphi) = I^n$, in which case the function φ, defined on the whole of I^n, is called *the blossom of* Φ.

According to (28) and to (27), when they exist the Bézier points of Φ relative to (a, b) are thus defined by

$$\Pi_0(a, b) := \Phi(a), \quad \Pi_n(a, b) := \Phi(b) \ , \tag{29}$$

$$\{\Pi_i(a, b)\} := \mathrm{Osc}_i \Phi(a) \cap \mathrm{Osc}_{n-i} \Phi(b) \ , \ 1 \leq i \leq n-1 \ . \tag{30}$$

More generally, the Bézier points of $F \in \mathcal{E}^d$ relative to (a, b) are then defined as the points $f(a^{[n-i]}, b^{[i]})$, $0 \leq i \leq n$. If blossoms exist in $\mathcal{E}$, *i.e.*, if $\mathcal{D}(\varphi) = I^n$, the function f is called the blossom of F.

4.2. Existence of blossoms in EC-spaces

In this subsection we shall see that the choice of relevant bases makes the proof of existence of blossoms a child's play!

Theorem 27. *Let us suppose that the space $\mathcal{U} := D\mathcal{E}$ is an EC-space on I. Then, blossoms exist in the space $\mathcal{E}$. Moreover, for any positive integers $\mu_1, \ldots, \mu_r$ such that $\sum_{i=1}^{r} \mu_i = n$ and any pairwise distinct $a_1, \ldots, a_r \in I$, the value of the blossom f of any $F \in \mathcal{E}$ at $(a_1{}^{[\mu_1]}, \ldots, a_r{}^{[\mu_r]})$ is given by*

$$f(a_1{}^{[\mu_1]}, \ldots, a_r{}^{[\mu_r]}) = \frac{W(\Psi_n^{a_2}, \ldots, \Psi_{n-\mu_2+1}^{a_2}, \ldots, \Psi_n^{a_r}, \ldots, \Psi_{n-\mu_r+1}^{a_r}, F)(a_1)}{W(\Psi_n^{a_2}, \ldots, \Psi_{n-\mu_2+1}^{a_2}, \ldots, \Psi_n^{a_r}, \ldots, \Psi_{n-\mu_r+1}^{a_r}, \mathbb{1})(a_1)}, \tag{31}$$

and all other expressions obtained by permutation of $a_1, \ldots, a_r$.

Proof. Suppose that the space $\mathcal{U} := D\mathcal{E}$ is an EC-space on I. Then the space $\mathcal{E}$ itself is an EC space on I. As previously, given $a \in I$, we can thus denote by Ψ_k^a an element of $\mathcal{E}$ which vanishes exactly k times at a.

Consider positive $\mu_1, \ldots, \mu_r$ such that $\sum_{i=1}^{r} \mu_i = n$, and pairwise distinct $a_1, \ldots, a_r \in I$. In order first to prove that all osculating flats $\mathrm{Osc}_{n-\mu_i}\Phi(a_i)$, $1 \le i \le n$, have a unique common point, then to obtain the value (31) of the blossom of any $F \in \mathcal{E}$, we can use any suitable mother-function. Now, for $0 \le k \le n-1$, the function $U_k^a := D\Psi_{k+1}^a$ vanishes exactly k times at a. Applying Theorem 23 to the space $\mathcal{U}$, we thus know that the functions $U_{n-1}^{a_1}, \ldots, U_{n-\mu_1}^{a_1}, \ldots, U_{n-1}^{a_r}, \ldots, U_{n-\mu_r}^{a_r}$ form a basis of $\mathcal{U}$. This is the exact reason why, as the mother-function, we are allowed to choose

$$\Phi := (\Psi_n^{a_1}, \ldots, \Psi_{n-\mu_1+1}^{a_1}, \Psi_n^{a_2}, \ldots, \Psi_{n-\mu_2+1}^{a_2}, \ldots, \Psi_n^{a_r}, \ldots, \Psi_{n-\mu_r+1}^{a_r})^T.$$

1) Existence of blossoms.

The latter choice makes it obvious that, for $1 \le i \le r$, the $(n-\mu_i)$-dimensional osculating flat $\mathrm{Osc}_{n-\mu_i}\Phi(a_i)$ is composed of all points $X = (X_1, \ldots, X_n)^T$ such that $X_{\mu_1+\cdots+\mu_{i-1}+1} = \cdots = X_{\mu_1+\cdots+\mu_i} = 0$. Therefore the origin of $\mathbb{R}^n$ is the unique point in $\bigcap_{i=1}^{r} \mathrm{Osc}_{n-\mu_i}\Phi(a_i)$.

2) Expressions of blossoms.

Formula (31) being trivially satisfied for $r = 1$, we assume that $r \ge 2$ and we set $\mu := \sum_{i=2}^{r} \mu_i = n - \mu_1 \le n-1$. Now that we have proved existence of blossoms, we know that

$$\varphi(a_1{}^{[\mu_1]}, \ldots, a_r{}^{[\mu_r]}) = \Phi(a_1) + \sum_{\ell=1}^{\mu} \lambda_\ell \Phi^{(\ell)}(a_1), \tag{32}$$

where the real numbers $\lambda_1, \ldots, \lambda_\mu$ are uniquely determined by the following condition

$$\Phi(a_1) + \sum_{\ell=1}^{\mu} \lambda_\ell \Phi^{(\ell)}(a_1) \in \bigcap_{i=2}^{r} \mathrm{Osc}_{n-\mu_i}\Phi(a_i). \tag{33}$$

The value of the blossom f of any $F \in \mathcal{E}$ at the n-tuple $(a_1{}^{[\mu_1]}, \ldots, a_r{}^{[\mu_r]})$ is then obtained from (32) via affine maps. This yields:

$$f(a_1{}^{[\mu_1]}, \ldots, a_r{}^{[\mu_r]}) = F(a_1) + \sum_{\ell=1}^{\mu} \lambda_\ell F^{(\ell)}(a_1), \tag{34}$$

the λ_ℓ's being those which satisfy (33). Due to our choice for Φ, solving (33) just consists in solving the following linear system of μ equations in μ unknowns:

$$\sum_{\ell=1}^{\mu} \lambda_\ell \Psi_k^{a_i(\ell)}(a_1) = -\Psi_k^{a_i}(a_1), \quad n \geq k \geq n-\mu_i+1,\ i=2,\ldots,r, \tag{35}$$

the determinant of which is

$$W(U_{n-1}^{a_2},\ldots,U_{n-\mu_2}^{a_2},\ldots,U_{n-1}^{a_r},\ldots,U_{n-\mu_r}^{a_r})(a_1) = W(\mathbb{1},\Psi_n^{a_2},\ldots,\Psi_{n-\mu_r+1}^{a_r})(a_1).$$

Considering (35) and (34) as a linear system of $(\mu+1)$ equations in the μ unknowns $\lambda_1,\ldots,\lambda_\mu$ which possesses a unique solution yields the announced formula (31) □

Remark 28. We would like to draw the reader's attention to the fact that the previous arguments provide us with another interesting way to prove that, as soon as (v)$'$ of Theorem 23 holds, then (24) automatically holds too. Indeed, if a given n-dimensional space $\mathcal{U} \subset C^{n-1}(I)$ satisfies (v)$'$, then blossoms exist in the $(n+1)$-dimensional space $\mathcal{E} := \{F \in C^n(I) \mid F' \in \mathcal{U}\}$. This in turn guarantees that, for any pairwise distinct $a_2,\ldots,a_r$ and any positive $\mu_2\ldots\mu_r$ such that $\mu_2+\cdots+\mu_r < n$, all quantities $W(U_{n-1}^{a_2},\ldots,U_{n-\mu_2}^{a_2},\ldots,U_{n-1}^{a_r},\ldots,U_{n-\mu_r}^{a_r})(a_1)$, $a_1 \in I \setminus \{a_2,\ldots,a_r\}$, are not zero, as determinants of regular systems (35).

4.3. Pseudoaffinity

Their geometrical definition makes it obvious that, as soon as blossoms exist in the space $\mathcal{E}$, they are symmetric on I^n and they give the associated function by restriction to the diagonal. They possess another crucial property, which generalises the affinity in each variable satisfied by polynomial blossoms, and which we refer to as *pseudoaffinity* in each variable. Again Theorem 23 will prove to be the key-point to achieve the latter property.

Theorem 29. *Suppose that $\mathcal{U} := D\mathcal{E}$ is an EC-space on I. Then, blossoms are pseudoaffine with respect to each variable, in the sense that, for any $x_1,\ldots,x_{n-1}$, $\gamma,\delta \in I$ with $\gamma<\delta$, for all $x \in I$, and any $F \in \mathcal{E}^d$, we have*

$$\begin{aligned} f(x_1,\ldots,x_{n-1},x) = \big[1-\beta(x_1,\ldots,x_{n-1};\gamma,\delta;x)\big] f(x_1,\ldots,x_{n-1},\gamma) \\ +\beta(x_1,\ldots,x_{n-1};\gamma,\delta;x) f(x_1,\ldots,x_{n-1},\delta), \end{aligned} \tag{36}$$

where $\beta(x_1,\ldots,x_{n-1};\gamma,\delta;.) : I \to \mathbb{R}$ is a C^1 strictly increasing function (depending on $x_1,\ldots,x_{n-1},\gamma,\delta$ but independent of F) with $\beta(x_1,\ldots,x_{n-1};\gamma,\delta;\gamma)=0$ and $\beta(x_1,\ldots,x_{n-1};\gamma,\delta;\delta)=1$.

Proof. Given $(x_1,\ldots,x_{n-1}) \in I^{n-1}$, in order to prove (36) for any $F \in \mathcal{E}^d$, it is sufficient to prove it when F is the mother-function Φ, with the announced properties for the function $\beta(x_1, \ldots, x_{n-1}; \gamma, \delta; .)$. This actually amounts to showing that the function

$$\widehat{\Phi}(x) := \varphi(x_1, \ldots, x_{n-1}, x), \quad x \in I,$$

is one-to-one and C^1 on I with values in an affine line.

1) $\widehat{\Phi}$ is C^1 on I, with values in an affine line. This can be proved using any mother-function Φ. In order to facilitate the proof, we shall adapt our choice to the $(n-1)$-tuple $(x_1, \ldots, x_{n-1})$. Suppose that, up to a permutation, $(x_1, \ldots, x_{n-1}) = (a_1{}^{[\mu_1]}, \ldots, a_r{}^{[\mu_r]})$, with $a_1 < a_2 < \cdots < a_r$ and with positive $\mu_1, \ldots, \mu_r$. Then, we choose our mother-function as

$$\Phi := (\Psi_n^{a_1}, \ldots, \Psi_{n-\mu_1+1}^{a_1}, \ldots, \Psi_n^{a_r}, \ldots, \Psi_{n-\mu_r+1}^{a_r}, \Phi_n)^T, \tag{37}$$

where Φ_n is any element of $\mathcal{E}$ which does not belong to $\mathrm{span}(\Psi_n^{a_1}, \ldots, \Psi_{n-\mu_1+1}^{a_1}, \ldots, \Psi_{n-\mu_r+1}^{a_r}, \mathbb{1})$. According to Theorem 23, this choice is indeed allowed because $\mathcal{U}$ is assumed to be an EC-space on I.

Due to (27), all points $\widehat{\Phi}(x)$, $x \in I$, belong to $\cap_{i=1}^r \mathrm{Osc}_{n-\mu_i} \Phi(a_i)$. This is why any such choice leads to

$$\widehat{\Phi}(x) = \big(\underbrace{0, \ \ldots \ , 0}_{(n-1) \text{ times}}, \ \widehat{\varphi}_n(x)\big)^T, \quad x \in I,$$

where $\widehat{\varphi}_n := \varphi_n(a_1{}^{[\mu_1]}, \ldots, a_r{}^{[\mu_r]}, .)$. Accordingly, we just have to prove that the function $\widehat{\varphi}_n$ is C^1 on I.

Now, formula (31) yields, for any $x \in I \setminus \{a_1, \ldots, a_r\}$:

$$\widehat{\varphi}_n(x) = \frac{W(\Psi_n^{a_1}, \ldots, \Psi_n^{a_r}, \ldots, \Psi_{n-\mu_r+1}^{a_r}, \Phi_n)(x)}{W(\Psi_n^{a_1}, \ldots, \Psi_n^{a_r}, \ldots, \Psi_{n-\mu_r+1}^{a_r}, \mathbb{1})(x)}. \tag{38}$$

Both the numerator and the denominator in (38) are C^1 on I. Moreover, due to (24), the denominator of (38) never vanishes on $I \setminus \{a_1, \ldots, a_r\}$ since, up to the sign, it is equal to $W(U_{n-1}^{a_1}, \ldots, U_{n-1}^{a_r}, \ldots, U_{n-\mu_r}^{a_r})(x)$. Accordingly, $\widehat{\varphi}_n$ is C^1 on $I \setminus \{a_1, \ldots, a_r\}$. Let us now examine what happens at any of the a_i's. Again, we shall adapt the choice of Φ_n to the point a_i. For instance, consider the case $i = r$. Then, formula (31) gives

$$\widehat{\varphi}_n(a_r) = \frac{W(\Psi_n^{a_1}, \ldots, \Psi_n^{a_{r-1}}, \ldots, \Psi_{n-\mu_{r-1}+1}^{a_{r-1}}, \Phi_n)(a_r)}{W(\Psi_n^{a_1}, \ldots, \Psi_n^{a_{r-1}}, \ldots, \Psi_{n-\mu_{r-1}+1}^{a_{r-1}}, \mathbb{1})(a_r)}. \tag{39}$$

Let I_r denote the largest interval which contains a_r and which is contained in $I \setminus \{a_1, \ldots, a_{r-1}\}$, and let us choose $\Phi_n := \Psi_{n-\mu_r}^{a_r}$. This is allowed by the fact that $\mathcal{U}$ is an EC-space on I. With this choice, by application of (20) to (38) and (39), the values of $\widehat{\varphi}_n$ are given by

$$\widehat{\varphi}_n(x) = \frac{N(x)}{D(x)} \quad \text{for } x \in I_r \setminus \{a_r\}, \quad \widehat{\varphi}_n(a_r) = \frac{F_1^{a_r}(a_r)}{F_0^{a_r}(a_r)}, \tag{40}$$

the functions N and D being defined on the whole of I_r by

$$N(x) := W(F_1^{a_r}, F_2^{a_r}, \ldots, F_{\mu_r+1}^{a_r})(x), \qquad D(x) := W(F_0^{a_r}, F_2^{a_r}, \ldots, F_{\mu_r+1}^{a_r})(x),$$

where

$$F_0^{a_r} := W(\Psi_n^{a_1}, \ldots, \Psi_n^{a_{r-1}}, \ldots, \Psi_{n-\mu_{r-1}+1}^{a_{r-1}}, \mathbb{1}) ,$$
$$F_i^{a_r} := W(\Psi_n^{a_1}, \ldots, \Psi_n^{a_{r-1}}, \ldots, \Psi_{n-\mu_{r-1}+1}^{a_{r-1}}, \Psi_{i+n-\mu_r-1}^{a_r}) , \quad 1 \le i \le \mu_r + 1.$$

On account of Theorem 23 applied to both spaces $\mathcal{E}$ and $\mathcal{U}$ (that is, in particular, on account of (24)), and of the equality $\sum_{i=1}^r \mu_i = n-1$, Lemma 24 ensures that, for $0 \le i \le \mu_r + 1$, the function $F_i^{a_r}$ vanishes exactly i times at a_r. Lemma 31 below thus guarantees the existence of two nonzero real numbers A, B, such that

$$N(x) \sim A(x-a_r)^{\mu_r+1} , \quad D(x) \sim B(x-a_r)^{\mu_r} \quad \text{for } x \in I_r \setminus \{a_r\} \text{ close to } a_r.$$

Therefore, the left equality in (40) shows that

$$\widehat{\varphi}_n(x) \sim \frac{A}{B}(x-a_r) \quad \text{for } x \in I_r \setminus \{a_r\} \text{ close to } a_r. \tag{41}$$

On the other hand, the right equality in (40) proves that $\widehat{\varphi}_n(a_r) = 0$. Accordingly, the latter relation shows that $\widehat{\varphi}_n$ is differentiable (hence continuous) at a_r. Differentiating $\widehat{\varphi}_n$ on $I_r \setminus \{a_r\}$ through (17) gives

$$\widehat{\varphi}_n'(x) = \frac{W(F_2^{a_r}, F_3^{a_r}, \ldots, F_{\mu_r+1}^{a_r})(x) W(F_0^{a_r}, F_1^{a_r}, \ldots, F_{\mu_r+1}^{a_r})(x)}{[W(F_1^{a_r}, F_2^{a_r}, \ldots, F_{\mu_r+1}^{a_r})(x)]^2}, \quad x \in I_r \setminus \{a_r\}.$$

Using Lemma 31 as previously shows that $\lim_{x \to a_r} \widehat{\varphi}_n'(x)$ exists, which proves that $\widehat{\varphi}_n$ is C^1 at a_r.

2) injectivity of $\widehat{\Phi}$.

For any $F \in \mathcal{E}$ and any $x \in I$, $f(x_1, \ldots, x_{n-1}, x)$ is obtained as the image of $\widehat{\Phi}(x)$ under an affine map. Therefore, in order to prove the injectivity of $\widehat{\Phi}$, it is sufficient to check that, for any $r \ge 2$, any nonnegative μ_1, μ_r, and any positive $\mu_2, \ldots, \mu_{r-1}$ such that $\sum_{i=1}^r \mu_i = n-1$, it is possible to find a function $F \in \mathcal{E}$ such that

$$f({a_1}^{[\mu_1+1]}, {a_2}^{[\mu_2]}, \ldots, {a_{r-1}}^{[\mu_{r-1}]}, {a_r}^{[\mu_r]}) \neq \\ f({a_1}^{[\mu_1]}, {a_2}^{[\mu_2]}, \ldots, {a_{r-1}}^{[\mu_{r-1}]}, {a_r}^{[\mu_r+1]}). \tag{42}$$

By application of (31), we know that, whether $\mu_1, \mu_r = 0$ or not, for any $F \in \mathcal{E}$, $f({a_1}^{[\mu_1+1]}, {a_2}^{[\mu_2]}, \ldots, {a_r}^{[\mu_r]})$ is equal to

$$\frac{W(\Psi_n^{a_2}, \ldots, \Psi_{n-\mu_2+1}^{a_2}, \ldots, \Psi_n^{a_r}, \ldots, \Psi_{n-\mu_r+1}^{a_r}, F)(a_1)}{W(\Psi_n^{a_2}, \ldots, \Psi_{n-\mu_2}^{a_2}, \ldots, \Psi_n^{a_r}, \ldots, \Psi_{n-\mu_r}^{a_r}, \mathbb{1})(a_1)}, \tag{43}$$

while $f({a_1}^{[\mu_1]}, \ldots, {a_{r-1}}^{[\mu_{r-1}]}, {a_r}^{[\mu_r+1]})$ is equal to

$$\frac{W(\Psi_n^{a_1}, \ldots, \Psi_{n-\mu_1+1}^{a_1}, \ldots, \Psi_n^{a_{r-1}}, \ldots, \Psi_{n-\mu_{r-1}+1}^{a_{r-1}}, F)(a_r)}{W(\Psi_n^{a_1}, \ldots, \Psi_{n-\mu_1}^{a_1}, \ldots, \Psi_n^{a_{r-1}}, \ldots, \Psi_{n-\mu_{r-1}}^{a_{r-1}}, \mathbb{1})(a_r)}. \tag{44}$$

Choose $F := \Psi_{n-\mu_r}^{a_r}$. In (44) the numerator is then equal to 0 (see part 1 of the proof). To the contrary, Theorem 23 guarantees that, in (43), it is not equal to 0. Therefore, with this choice of F, (42) is proven. $\square$

Example 30. Consider the polynomial case $\mathcal{E} = \mathcal{P}_n$, with, as usual, $\Psi_k^a(x) := (x-a)^k$. We choose $\Phi_n := \Psi_{n-\mu_r}^{a_r}$. Then, according to (26), there exist two constants Γ_1, Γ_2 such that, for any $x \in \mathbb{R}$,

$$W(\Psi_n^{a_1}, ..., \Psi_{n-\mu_1+1}^{a_1}, ..., \Psi_n^{a_r}, ..., \Psi_{n-\mu_r+1}^{a_r}, \Phi_n)(x) = \Gamma_1 (x - a_r)^{\mu_r+1} \prod_{i=1}^{r-1} (x - a_i)^{\mu_i}$$

and

$$W(U_{n-1}^{a_1}, ..., U_{n-\mu_1}^{a_1}, ..., U_{n-1}^{a_r}, ..., U_{n-\mu_r}^{a_r}, \mathbb{1})(x) = \Gamma_2 \prod_{i=1}^{r} (x - a_i)^{\mu_i}.$$

Take the mother-function Φ as in (37) with our previous choice for Φ_n. Then the previous equalities show that, instead of (41), formulae (38) and (39) lead to

$$\widehat{\phi}_n(x) = \Gamma(x - a_r), \quad x \in I_r,$$

for some non-zero constant Γ. Blossoms for polynomials, defined geometrically as we did, are thus proved to be affine in each variable, which fits in with the algebraic definition by L. Ramshaw [20].

The previous proof made use of the following result an illustration of which is given by formula (26) in the polynomial case.

Lemma 31. *Consider $(n+1)$ integers $k_0, \ldots, k_n$ such that $k_0 \geq 0$, $k_i \geq k_{i-1}+1$ for $1 \leq i \leq n$, and $(n+1)$ functions $U_0, \ldots, U_n \in C^{k_n}(I)$. Suppose that, for $0 \leq i \leq n$, U_i vanishes exactly k_i times at $a \in I$. Then we have, for $x \in I$ close to a and with $N := \sum_{i=0}^{n}(k_i - i)$,*

$$W(U_0, \ldots, U_n)(x) \sim (x - a)^N \prod_{i=0}^{n} \frac{U_i^{(k_i)}(a)}{k_i!} \prod_{0 \leq i < j \leq n} (k_j - k_i). \tag{45}$$

Proof. The result trivially holds for $n = 0$. Suppose that it is proven for $n - 1 \geq 0$ and let us prove it for n. Since U_0 vanishes exactly $k_0 \geq 0$ times at a, we can choose a nontrivial interval I_0 contained in I and containing a such that $U_0(x) \neq 0$ for any $x \in I_0 \setminus \{a\}$. For $1 \leq i \leq n$, we define the function $\widetilde{U}_i$ on $I_0 \setminus \{a\}$ by

$$\widetilde{U}_i(x) := \left(\frac{U_i}{U_0}\right)'(x), \quad x \in I_0 \setminus \{a\}.$$

Each function U_i/U_0 is clearly C^{k_n} on $I_0 \setminus \{a\}$, and according to (5) and (6), we can write

$$H(x) := W(U_0, \ldots, U_n)(x) = [U_0(x)]^{n+1}\, W(\widetilde{U}_1, \ldots, \widetilde{U}_n)(x), \quad x \in I_0 \setminus \{a\}. \tag{46}$$

Now, one can actually prove that, for $0 \leq i \leq n$, the function $\overline{U}_i(x)$ defined by

$$\overline{U}_i(x) := \frac{U_i(x)}{(x - a)^{k_0}} \quad \text{if } x \in I_0 \setminus \{a\}, \quad \overline{U}_i(a) := \frac{U_i^{(k_0)}(a)}{k_0!},$$

is $C^{k_n-k_0}$ on I_0. The fact that $\overline{U}_i$ is (k_n-k_0)-times continuously differentiable at a requires a long technical proof that can found in [8]. Clearly, $\overline{U}_i$ vanishes exactly k_i-k_0 times at a. For $1\le i\le n$, we can thus extend $\widetilde{U}_i$ by continuity on the whole of I_0 by setting

$$\widetilde{U}_i(x) := \left(\frac{\overline{U}_i}{\overline{U}_0}\right)'(x), \quad x\in I_0.$$

The function $\widetilde{U}_i$ thus belongs to $C^{k_n-k_0-1}(I_0)$. We can therefore introduce the function

$$\widetilde{H}(x) := W(\widetilde{U}_1,\dots,\widetilde{U}_n)(x)\ , \quad x\in I_0.$$

It is continuous on I_0 and equality (46) can now be extended by continuity at a, so that

$$H(x) = [U_0(x)]^{n+1}\ \widetilde{H}(x), \quad x\in I_0. \tag{47}$$

Moreover, for $1\le i\le n$, the function $\widetilde{U}_i$ vanishes exactly (k_i-k_0-1) times at a. We can apply the recursive assumption to $\widetilde{H}$. It guarantees that

$$\widetilde{H}(x)\sim (x-a)^{\widetilde{N}}\widetilde{A} \quad \text{for } x\in I_0 \text{ close to } a, \tag{48}$$

with $\widetilde{N} := \sum_{i=1}^n (k_i-k_0-i)$ and with

$$\widetilde{A} := \prod_{i=1}^n \frac{\widetilde{U}_i^{(k_i-k_0-1)}(a)}{(k_i-k_0-1)!}\prod_{1\le i<j\le n}(k_j-k_i)$$

The two relations (47) and (48) yield (45) after observing that

$$\frac{\widetilde{U}_i^{(k_i-k_0-1)}(a)}{(k_i-k_0-1)!} = (k_i-k_0)\frac{U_i{}^{(k_i)}(a)}{k_i!}\frac{k_0!}{U_0{}^{(k_0)}(a)}\ , \quad 1\le i\le n,$$

which easily follows from the definition of the $\widetilde{U}_i$'s. □

Remark 32. The arguments used in the proof of Lemma 31 actually state that any sufficiently differentiable functions $U_0,\dots,U_n$ on I which vanish exactly $k_0,\dots,k_n$ times at $a\in I$, respectively, with $k_0\le k_1\le\cdots\le k_n$, form a CW-system on $]a,a+\eta[$ and/or $[a-\eta,a[$ for some sufficiently small positive η. Note that relation (45) is consistent with the particular case $U_i(x)=(x-a)^{k_i}$, $0\le i\le n$, for which we have $W(U_0,\dots,U_n)(x)=(x-a)^N\prod_{0\le i<j\le n}(k_j-k_i)$ for all $x\in\mathbb{R}$.

4.4. Blossoms, Bézier points, and Bernstein bases

Clearly, if blossoms exist in $\mathcal{E}$, then Bézier points exist in $\mathcal{E}$. As a consequence of Theorem 12, Theorem 34 below states the converse, along with the equivalence between existence of blossoms and existence of Bernstein bases [14], according to the following definition.

Definition 33. In $\mathcal{E}$, a Bernstein-like basis $(B_0, \ldots, B_n)$ relative to $(a, b) \in I^2$, $a \neq b$, is said to be the *Bernstein basis relative to* (a, b) if it is normalized (*i.e.*, if it satisfies $\sum_{i=0}^{n} B_i = \mathbb{1}$), and if each B_i is positive strictly between a and b.

As is classical, when writing any $F \in \mathcal{E}^d$ as $F(x) = \sum_{i=0}^{n} F_i(x) P_i$, $x \in I$, where $(F_0, \ldots, F_n)$ is a normalised basis of $\mathcal{E}$, we call the points $P_0, \ldots, P_n \in \mathbb{R}^d$ *the control points* of F relative to $(F_0, \ldots, F_n)$.

Theorem 34. *Let $\mathcal{E} \subset C^n(I)$ be an $(n+1)$-dimensional W-space on I containing constants, and let $\mathcal{U}$ denote the space $D\mathcal{E}$. The following four properties are then equivalent:*

(a) *The space $\mathcal{U}$ is an EC-space on I.*
(b) *Blossoms exist in the space $\mathcal{E}$.*
(c) *Bézier points exist in the space $\mathcal{E}$.*
(d) *The space $\mathcal{E}$ possesses a normalised Bernstein-like basis relative to any pair of distinct points of I.*

Moreover, if the latter properties are satisfied, then:

(e) *Relative to any given $(a, b) \in I^2$, $a \neq b$, the normalised Bernstein-like basis $(B_0, \ldots, B_n)$ is automatically positive strictly between a and b, i.e., it is the Bernstein basis. The control points of any $F \in \mathcal{E}^d$ relative to $(B_0, \ldots, B_n)$ are its Bézier points relative to (a, b).*

Proof. Suppose that $\mathcal{E}$ is a W-space on I, or, equivalently, that $\mathcal{U}$ is a W-space on I (see Remark 3,(2)). Then, for any given $(a, b) \in I^2$, $a \neq b$, and for $1 \leq i \leq n-1$, the Bézier point $\varphi(a^{[n-i]}, b^{[i]})$ exists iff there exit unique real numbers $\lambda_1, \ldots, \lambda_i$, $\varrho_1, \ldots, \varrho_{n-i}$ such that

$$\Phi(b) - \Phi(a) = \sum_{k=1}^{i} \lambda_k \Phi^{(k)}(a) + \sum_{\ell=1}^{n-i} \varrho_\ell \Phi^{(\ell)}(b).$$

that is, iff the n vectors $\Phi'(a), \ldots, \Phi^{(i)}(a), \Phi'(b), \ldots, \Phi^{(n-i)}(b)$ are linearly independent. Accordingly, the equivalence between (a) and (c) results from applying Theorem 12 to the space $\mathcal{U}$. Taking account of Theorem 27, and of the implication (b) $\Rightarrow$ (c) being obvious, we thus have proved that (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c).

Supposing that (a) holds, let us now briefly recall why both (d) and (e) are then satisfied. We know not only that blossoms exist, but also that they are symmetric and pseudoaffine in each variable (Theorem 29). Given $a, b, x \in I$, with $a < b$, applying (36) with $(x_1, \ldots, x_{n-1}) := (a^{[n-i-k]}, b^{[i]}, x^{[k-1]})$ for some integers k, i such that $1 \leq k \leq n$ and $0 \leq i \leq n-k$, provides us with the existence of a one-to-one function $\alpha_{i,k} : I \to \mathbb{R}$ (depending on a, b, x), satisfying

$$\alpha_{i,k}(a) = 0, \ \alpha_{i,k}(b) = 1, \ \alpha_{i,k}(y) > 0 \quad \text{for } y \in]a, b[, \tag{49}$$

and such that, for all $F \in \mathcal{E}^d$, we have in particular:

$$f(a^{[n-i-k]}, b^{[i]}, x^{[k]}) = (1 - \alpha_{i,k}(x))\ f(a^{[n-i-k+1]}, b^{[i]}, x^{[k-1]}) \\ +\alpha_{i,k}(x)\ f(a^{[n-i-k]}, b^{[i+1]}, x^{[k-1]})\ , \quad x \in I. \quad (50)$$

Equality (50) describes the well-known *de Casteljau algorithm* relative to $[a, b]$. For $k = n$, it gives, for any $x \in I$, the point $f(x^{[n]}) = F(x)$ as an affine combination of the $(n + 1)$ points Bézier points of F relative to (a, b), i.e.,

$$F(x) = \sum_{i=0}^{n} B_i(x)\ f(a^{[n-i]}, b^{[i]})\ , \quad \sum_{i=0}^{n} B_i(x) = 1\ , \quad x \in I\ , \tag{51}$$

the coefficients being independent of F. Relations (51) prove that the functions $B_0, \ldots, B_n$ provided by the de Casteljau algorithm form a normalised basis of the space $\mathcal{E}$. On the other hand, from (49) we can deduce that each affine combination (50) is a strictly convex one when $x \in]a, b[$, which proves the positivity of $B_0, \ldots, B_n$ on $]a, b[$. Applying (51) to a mother-function Φ, the geometrical meaning (29) and (30) of its Bézier points guarantees that $(B_0, \ldots, B_n)$ is a Bernstein-like basis relative to (a, b). It is thus the Bernstein basis relative to any (a, b). Hence, both (d) and (e) are satisfied.

Given distinct $a, b \in I$, suppose there exists of a normalised Bernstein-like basis $(B_0, \ldots, B_n)$ in the space $\mathcal{E}$ relative to (a, b). Due to the normalisation, the functions $\widetilde{B}_i := B'_0 + \cdots B'_i$, $0 \le i \le n-1$, form a Bernstein-like basis relative to (a, b) in the space $\mathcal{U}$. Accordingly, when (d) holds, the space $\mathcal{U}$ possesses Bernstein-like bases relative to any pair of distinct points of I. Theorem 12 ensures that (a) holds, and the proof of Theorem 34 is now complete. □

4.5. Blossoms, B-spline bases, and Total Positivity

Throughout this subsection we assume that $\mathcal{E}$ is a W-space as in Theorem 34. In the previous subsection we saw the equivalence between existence of blossoms and existence of Bernstein bases in the space $\mathcal{E}$. For the sake of completeness, we shall now briefly see the links between blossoms and B-spline bases of which Bernstein bases can actually be considered a special case. We shall also recall why these bases have nice shape preserving properties.

Given a sequence of *knots* $\eta_0 < \eta_1 < \cdots < \eta_q < \eta_{q+1}$ in the interval I, and an associated sequence of *multiplicities* $m_0, \ldots, m_{q+1}$, with $1 \le m_i \le n$ for $1 \le i \le q$ and $m_0 = m_{q+1} = n+1$, we define the corresponding *knot vector* as the $(m+2n+2)$-tuple (where $m := \sum_{i=1}^{q} m_i$ is the sum of the multiplicities of all interior knots)

$$\mathcal{K} := (\xi_{-n}, \ldots, \xi_{m+n+1}) := ({\eta_0}^{[m_0]}, {\eta_1}^{[m_1]}, \ldots, {\eta_q}^{[m_q]}, {\eta_{q+1}}^{[m_{q+1}]})\ . \tag{52}$$

Associated with the knot vector $\mathcal{K}$, we define the *spline space based on* $\mathcal{E}$, as the $(n+m+1)$-dimensional space $\mathcal{S}(\mathcal{E}, \mathcal{K})$ composed of all functions $S : [\eta_0, \eta_{q+1}] \to \mathbb{R}$ such that

(S)$_1$ For $1 \leq i \leq q$, S is C^{n-m_i} at η_i;

(S)$_2$ For $0 \leq i \leq q$, there exists a function $F_i \in \mathcal{E}$ such that

$$S(x) = F_i(x) , \quad x \in [\eta_i, \eta_{i+1}] .$$

With the knot vector $\mathcal{K}$ we also associate the set $\mathcal{A}(\mathcal{K})$ composed of all n-tuples $(x_1, \ldots, x_n) \in [\eta_0, \eta_{q+1}]^n$ which are *admissible relative to* $\mathcal{K}$, that is, such that, whenever $\min(x_1, \ldots, x_n) < \eta_\ell < \max(x_1, \ldots, x_n)$ for some integer ℓ, $1 \leq \ell \leq q$, then at least m_ℓ among the points $x_1, \ldots, x_n$ are equal to η_ℓ. The set $\mathcal{A}(\mathcal{K})$ is symmetric and it contains the diagonal of $[\eta_0, \eta_{q+1}]^n$. In the particular case $q = 0$, $\mathcal{A}(\mathcal{K}) = [\eta_0, \eta_1]^n$. Exactly as we did with the space $\mathcal{E}$, we select a mother-spline function $\Sigma := (\Sigma_1, \ldots, \Sigma_{n+m})^T$, meaning that $(\mathbb{1}, \Sigma_1, \ldots, \Sigma_{n+m})$ is a basis of $\mathcal{S}(\mathcal{E}, \mathcal{K})$. Then, any spline $S \in \mathcal{S}(\mathcal{E}, \mathcal{K})^d$ is the image of Σ under a unique affine map on $\mathbb{R}^{n+m}$. For $0 \leq \ell \leq q$, denote by Φ_ℓ the unique element of $\mathcal{E}^{n+m}$ such that

$$\Sigma(x) = \Phi_\ell(x), \quad x \in [\eta_\ell, \eta_{\ell+1}],\ 0 \leq \ell \leq q. \tag{53}$$

Given an admissible n-tuple $(x_1, \ldots, x_n)$, equal, up to a permutation, to $({\tau_1}^{[\mu_1]}, \ldots, {\tau_r}^{[\mu_r]})$, with $\tau_1 < \cdots < \tau_r$ and positive μ_i's, we then have (see [9])

$$\bigcap_{i=1}^{r} \mathrm{Osc}_{n-\mu_i} \Sigma(\tau_i) = \bigcap_{i=1}^{r} \mathrm{Osc}_{n-\mu_i} \Phi_j(\tau_i) , \quad j \in \mathcal{J}(x_1, \ldots, x_n) , \tag{54}$$

where $\mathcal{J}(x_1, \ldots, x_n)$ is some nonempty set composed of consecutive integers. More precisely, in case there exists at least one integer $\ell \in \{1, \ldots, q\}$ such that η_ℓ appears at least m_ℓ times in the sequence $(x_1, \ldots, x_n)$, denoting by ℓ_1 and ℓ_2 the smallest and greatest such integer, respectively, we have $\mathcal{J}(x_1, \ldots, x_n) := \{\ell_1 - 1, \ldots, \ell_2\}$. Otherwise, $x_1, \ldots, x_n$ all belong to the same interval $[t_{\ell_0}, t_{\ell_0+1}]$ and we then have $\mathcal{J}(x_1, \ldots, x_n) := \{\ell_0\}$. Due to the admissibility, all osculating flats $\mathrm{Osc}_{n-\mu_i} \Sigma(\tau_i)$ involved in (54) are well defined, except possibly for the first and last ones, which, if necessary, must be interpreted as $\mathrm{Osc}_{n-\mu_1} \Sigma(\tau_1^+)$ and $\mathrm{Osc}_{n-\mu_r} \Sigma(\tau_r^-)$, respectively.

Suppose that blossoms exist in the space $\mathcal{E}$. Then, equality (54) shows that its left-hand side consists of a single point which is labeled as $\sigma(x_1, \ldots, x_n)$. The function σ, called the blossom of the mother spline function Σ, is then defined on the set $\mathcal{A}(\mathcal{K})$. Blossoms for other spline functions are defined from σ via affine maps. Due to (54), except that instead of being defined on I^n they are defined only on $\mathcal{A}(\mathcal{K})$, blossoms for splines possess the same properties as blossoms in the space $\mathcal{E}$: they are symmetric, pseudoaffine within $\mathcal{A}(\mathcal{K})$, and possess the diagonal property: $\Sigma(x) = \sigma(x^{[n]})$ for all $x \in [\eta_0, \eta_{q+1}]$. Classically, given x in some $[\eta_\ell, \eta_{\ell+1}]$, $0 \leq \ell \leq q$, for any spline $S \in \mathcal{S}(\mathcal{E}, \mathcal{K})^d$, the latter properties make it possible to develop an n-step *de Boor algorithm* to evaluate $S(x)$ as a strictly convex combination of at most $(n+1)$ consecutive *poles of* S, given that the $(n+m+1)$ poles of S are defined as the points

$$s(\xi_{i+1}, \ldots, x_{i+n}), \quad -n \leq i \leq m. \tag{55}$$

where s is the blossoms of S. This generalises the de Casteljau algorithm (50) and it eventually leads to

$$S(x) = \sum_{i=-n}^{m} N_i(x) s(\xi_{i+1}, \ldots, x_{i+n}), \quad \sum_{i=-n}^{m} N_i(x) = 1, \quad x \in [\eta_0, \eta_{q+1}]. \tag{56}$$

Due to (56), the functions N_i, $-n \le j \le m$, so obtained form a normalised basis of the spline space $\mathcal{S}(\mathcal{E}, \mathcal{K})$. Taking account of the geometrical definition of the blossom σ of the mother-spline function Σ through (54) and (27), they form *the B-spline basis* of $\mathcal{S}(\mathcal{E}, \mathcal{K})$, according to the definition below.

Definition 35. Given $(n + m + 1)$ functions $N_{-n}, \ldots, N_m$ in the spline space $\mathcal{S}(\mathcal{E}, \mathcal{K})$, we say that they form *the B-spline basis* if they satisfy the following four properties:

(i) $(\mathrm{BSB})_1$ *support property:* For all $j \in \{-n, \ldots, m\}$, the support of N_j is $[\xi_j, \xi_{j+n+1}]$.

(ii) $(\mathrm{BSB})_2$ *end point property:* For all $j \in \{-n, \ldots, m\}$, the function N_j vanishes exactly $(n - m_k + p + 1)$ times at the left end point $\xi_j = \eta_k$ of its support, and exactly $(n - m_{k'} + p' + 1)$ times at its right end point $\xi_{j+n+1} = \eta_{k'}$, where

$$p := \sharp\{\ell < j \mid \xi_\ell = \xi_j\}\ , \quad p' := \sharp\{\ell > j + n + 1 \mid \xi_\ell = \xi_{j+n+1}\}\ .$$

(iii) $(\mathrm{BSB})_3$ *normalisation property:* $\sum_{j=-n}^{m} N_j(x) = 1$ for all $x \in I$;

(iv) $(\mathrm{BSB})_4$ *positivity property:* for all $j \in \{-n, \ldots, m\}$, N_j is positive on the interior of its support.

If $N_{-n}, \ldots, N_m$ satisfy only the first two properties $(\mathrm{BSB})_1$ and $(\mathrm{BSB})_2$, we say that they form a *B-spline-like basis.*

When the knot vector is $\mathcal{K} = (\eta_0{}^{[m_0]}, \eta_1{}^{[m_1]})$, *i.e.*, when $q = 0$, the spline space $\mathcal{S}(\mathcal{E}, \mathcal{K})$ is just the space $\mathcal{E}$ restricted to $[\eta_0, \eta_1]$, and in the latter space being a B-spline-like basis (resp. the B-spline basis) is the same as being a Bernstein-like basis (resp. the Bernstein basis) relative to (η_0, η_1).

On account of the previous observation and of Theorem 34, we can thus state the equivalence between existence of blossoms and existence of B-spline bases as follows:

Theorem 36. *The space $\mathcal{U}$ is an EC space on I if and only if B-spline bases exist in the sense that*

(f) *for any knot vector $\mathcal{K}$, the space $\mathcal{S}(\mathcal{E}, \mathcal{K})$ possesses a normalised B-spline-like basis.*

Moreover, when (f) *holds, all normalised B-spline-like bases are B-spline bases, that is, they satisfy the positivity property* $(\mathrm{BSB})_4$.

For the sake of completeness, we shall briefly mention further essential properties of the bases obtained in Theorems 34 and 36. Let us recall that a basis $(U_0, \ldots, U_n)$ of a given functional space on some interval J is said to be ***Totally Positive*** when, for any sequence $x_0 < x_1 < \cdots < x_{n-1} < x_n$ in J, the matrix $\big(U_i(x_j)\big)_{0 \le i,j \le n}$ is totally positive, *i.e.*, all its minors are nonnegative [7]. The utmost importance of Normalised Totally Positive (NTP) bases in geometric design, due to the fact that they guarantee good shape preserving properties, is clearly pointed out in [4,3] for instance. Let us just recall that when a functional space on $[a,b]$, $a < b$, possesses an NTP basis, it automatically possesses an optimal one, that is, a unique NTP basis $(U_0, \ldots, U_n)$ from which any other NTP basis $(V_0, \ldots, V_n)$ can be obtained via multiplication by a totally positive matrix, *i.e.*, such that the matrix M of the equality $(V_0, \ldots, V_n) = (U_0, \ldots, U_n)M$ is totally positive. The optimal NTP basis has optimal shape preserving properties (see [3]).

Let us get back to the de Casteljau algorithm (50) relative to $(a,b) \in I^2$, $a < b$. We have already mentioned that each affine combination (50) is a strictly convex one whenever x belongs to $]a,b[$. This implies that the de Casteljau algorithm is a so-called *corner cutting algorithm* on $[a,b]$. The importance of this remark is not limited to the fact that it proves the positivity on $]a,b[$ of the corresponding Bernstein basis: indeed, it also implies its total positivity on $[a,b]$. Hence, it is an NTP basis on $[a,b]$. As a matter of fact, the number of zeros of its elements at the end points even guarantees its optimality. The same arguments are more generally valid for de Boor algorithms and B-spline bases, as stated in the following theorem as a conclusion of this section (see details in [11]).

Theorem 37. *Suppose that any of the properties* (a)-(d) *of Theorem* 34 *is satisfied. Then, for any knot vector $\mathcal{K}$, the B-spline basis is the optimal NTP basis of the space $\mathcal{S}(\mathcal{E}, \mathcal{K})$. In particular, the Bernstein basis relative to any $(a,b) \in I^2$, $a < b$, is the optimal NTP basis of the space $\mathcal{E}$ restricted to* $[a,b]$.

5. Extended Chebyshev Piecewise Spaces

From a design point of view, in order to take full advantage of the richness of shape effects EC-spaces provide, it is desirable to be able to consider curves with sections in different such spaces while maintaining existence of blossoms. This is why the last two sections are devoted to the extension of the results described so far to the piecewise framework.

5.1. Counting zeros for piecewise functions

Let $\mathcal{Z}$ denote any (finite, infinite, or bi-infinite) subset of $\mathbb{Z}$ composed of consecutive integers (at least two). From now on, we consider a fixed sequence of intervals, $\mathcal{T} = ([t_\ell, t_{\ell+1}])_{\ell \in \mathcal{Z}}$, with $t_\ell < t_{\ell+1}$, and the interval $I := \cup_{\ell \in \mathcal{Z}} [t_\ell, t_{\ell+1}]$. Let $\mathcal{Z}_0$ denote the set of all integers $\ell \in \mathcal{Z}$ such that t_ℓ is in the interior of I. The word "piecewise" will always refer to the sequence $\mathcal{T}$. We shall say that F is a *piecewise function on I* if F is defined separately on each interval $[t_\ell, t_{\ell+1}]$, implying in particular that, for any $\ell \in \mathcal{Z}_0$, both $F(t_\ell^-)$ and $F(t_\ell^+)$ are defined while a priori $F(t_\ell)$ is not. In such a case, unless explicitly mentioned, F is not a function on I. We shall deliberately use the abusive notation $F : \cup_{\ell\in\mathcal{Z}}[t_\ell, t_{\ell+1}] \to \mathbb{R}$ to stress this fact. We shall say that F is a *piecewise C^n function on I* if F is C^n on each interval $[t_\ell, t_{\ell+1}]$. Given two piecewise functions F and G, the equality $F = G$ thus means that $F(x) = G(x)$ for all $x \in I \setminus \{t_\ell\,, \ell \in \mathcal{Z}_0\}$ and both $F(t_\ell^-) = G(t_\ell^-)$ and $F(t_\ell^+) = G(t_\ell^+)$ for all $\ell \in \mathcal{Z}_0$, which we shall sometimes sum up as $F(x^\varepsilon) = G(x^\varepsilon)$ for all $x \in I$ and for $\varepsilon \in \{-, +\}$. Of course in case x is an end point of I, then ε must be the appropriate sign.

Using notations similar to those used in Lemma 24, let us set, for any piecewise C^n function F, any $x \in I$ and for $\varepsilon \in \{-, +\}$,

$$\Delta_n F(x^\varepsilon) := \big(F(x^\varepsilon), F'(x^\varepsilon), \ldots, F^{(n)}(x^\varepsilon)\big)^T.$$

Given a sequence $\mathcal{M} := \big(\mathcal{M}_\ell\big)_{\ell\in\mathcal{Z}_0}$, where each $\mathcal{M}_\ell := \big(m_{i,j}^\ell\big)_{0\le i,j\le n}$ is a lower triangular matrix with positive diagonal elements, we introduce the space $PC^n(\mathcal{T}, \mathcal{M})$ of all piecewise C^n functions $F : \cup_{\ell\in\mathcal{Z}}[t_\ell, t_{\ell+1}] \to \mathbb{R}$ satisfying the connection conditions

$$\Delta_n F(t_\ell^+) = \mathcal{M}_\ell \,.\, \Delta_n F(t_\ell^-)\ , \quad \ell \in \mathcal{Z}_0. \tag{57}$$

In this section the space $PC^n(\mathcal{T}, \mathcal{M})$ will play the same rôle as the space $C^n(I)$ in the previous ones. The interest of considering such lower triangular matrices with positive diagonal elements will be justified geometrically in Subsection 6.2. At this stage, let us just point out that the regular lower triangular structure enables us to count zeros in the space $PC^n(\mathcal{T}, \mathcal{M})$, for, if $\ell \in \mathcal{Z}_0$, t_ℓ^+ is a zero of (exact) multiplicity k iff so is t_ℓ^-. Hence, for any $F \in PC^n(\mathcal{T}, \mathcal{M})$, we can introduce the number $Z_{n+1}(F)$ with the same meaning as in subsection 2.1. Moreover, for any $a \in I$, all subspaces $\mathcal{E}_k(a)$ introduced in (3) are well-defined. The interest of the additional positivity assumption on the diagonal elements lies in the fact that it makes possible piecewise versions of Rolle's theorem and (9) (see Lemma 38 below).

From now on, D denotes the piecewise differentiation. Given a piecewise function $F \in PC^n(\mathcal{T}, \mathcal{M})$, its piecewise derivative DF is obviously a piecewise C^{n-1} function. However, we generally cannot count its zeros as we can for F unless we are in the special case examined in the lemma below.

Lemma 38. *Given a sequence $\mathcal{M}$ of lower triangular matrices with positive diagonal elements, we suppose that each $\mathcal{M}_\ell$, $\ell \in \mathcal{Z}_0$, has $(m_{0,0}^\ell, 0\ldots, 0)^T$ as its first column and we denote by $\widetilde{\mathcal{M}}_\ell$ the square matrix of order $n-1$ obtained by deleting the first row and column of $\mathcal{M}_\ell$. Then, the piecewise derivative DF of any $F \in PC^n(\mathcal{T}, \mathcal{M})$ belongs to $PC^{n-1}(\mathcal{T}, \widetilde{\mathcal{M}})$, where $\widetilde{\mathcal{M}} := \big(\widetilde{\mathcal{M}}_\ell\big)_{\ell\in\mathcal{Z}_0}$, and we have*

$$Z_n(DF) \ge Z_{n+1}(F) - 1\ .$$

Proof. The first statement is obvious. Due to our observations above, if F vanishes k times at a given $a \in \mathbb{R}$, $1 \le k \le n+1$, DF vanishes $k-1$ times at a. On the other hand, for each $\ell \in \mathcal{Z}_0$, we have $DF(t_\ell^+) = m_{1,1}^\ell DF(t_\ell^-)$. All m_{11}^ℓ's being positive, Rolle's theorem is valid for F: indeed, whenever $F(a) = F(b) = 0$ with $a < b$, this implies the existence of some $\xi \in]a,b[$ such that $DF(\xi) = 0$, the latter equality meaning $DF(t_\ell^-) = DF(t_\ell^+) = 0$ in case $\xi = t_\ell$ for some $\ell \in \mathcal{Z}_0$. $\square$

Lemma 39. *Let $w : \cup_{\ell \in \mathcal{Z}}[t_\ell, t_{\ell+1}] \to \mathbb{R}$ be a piecewise C^n function satisfying $w(t_\ell^-)w(t_\ell^+) > 0$ for all $\ell \in \mathcal{Z}_0$, and let $\mathcal{M} := (\mathcal{M}_\ell)_{\ell \in \mathcal{Z}_0}$ be a sequence of lower triangular matrices with positive diagonal elements. Then, for all $F \in PC^n(\mathcal{T}, \mathcal{M})$, the piecewise function wF belongs to $PC^n(\mathcal{T}, \mathcal{R})$, where $\mathcal{R}$ stands for the sequence composed of all matrices $\mathcal{R}_\ell$ (lower triangular with positive diagonal elements) defined by*

$$\mathcal{R}_\ell := \mathcal{C}_n(w, t_\ell^+) \,.\, \mathcal{M}_\ell \,.\, \mathcal{C}_n(w, t_\ell^-)^{-1}, \quad \ell \in \mathcal{Z}_0, \tag{58}$$

where, for $x \in I$ and $\varepsilon \in \{-,+\}$, $\mathcal{C}_n(w, x^\varepsilon) = \big(\mathcal{C}_n(w, x^\varepsilon)_{p,q}\big)_{0 \le p,q \le n}$ stands for the lower triangular square matrix defined by $\mathcal{C}_n(w, x^\varepsilon)_{p,q} := \binom{p}{q}\, w^{(p-q)}(x^\varepsilon)$ for $0 \le q \le p \le n$.

Proof. Equality (58) is a straightforward consequence of both (57) and of the Leibniz' formula to differentiate wF up to order n at t_ℓ^- and t_ℓ^+, $\ell \in \mathcal{Z}_0$. The positivity of the diagonal elements of each lower triangular matrix $\mathcal{R}_\ell$ is due to our assumption $w(t_\ell^-)w(t_\ell^+) > 0$. $\square$

Lemma 40. *With the same notations as in Lemma 39, suppose that a given piecewise function $U_0 \in PC^n(\mathcal{T}, \mathcal{M})$ never vanishes on I. Then for any $F \in PC^n(\mathcal{T}, \mathcal{M})$, the piecewise function $D(F/U_0)$ belongs to $PC^{n-1}(\mathcal{T}, \mathcal{N})$, where each matrix $\mathcal{N}_\ell$, $\ell \in \mathcal{Z}_0$, (lower triangular with positive diagonal elements) is obtained from the matrix*

$$\mathcal{R}_\ell := \mathcal{C}_n(U_0, t_\ell^+)^{-1} .\, \mathcal{M}_\ell \,.\, \mathcal{C}_n(U_0, t_\ell^-)$$

by deleting its first row and column. Moreover, any $F \in PC^n(\mathcal{T}, \mathcal{M})$ satisfies

$$Z_n\big(D(F/U_0)\big) \ge Z_{n+1}(F) - 1 \,. \tag{59}$$

Proof. First note that, due to the positivity of all diagonal elements $m_{0,0}^\ell$, the piecewise function U_0 keeps the same strict sign on $\cup_{\ell \in \mathcal{Z}}[t_\ell, t_{\ell+1}]$. Let F belong to $PC^n(\mathcal{T}, \mathcal{M})$. From Lemma 39 we know that F/U_0 belongs to $PC^n(\mathcal{T}, \mathcal{R})$, where each matrix $\mathcal{R}_\ell$ is given by (58), with $w := 1/U_0$. The announced equality follows, taking account of the equality

$$\mathcal{C}_n(w_1, x^\varepsilon) \,.\, \mathcal{C}_n(w_2, x^\varepsilon) = \mathcal{C}_n(w_1 w_2, x^\varepsilon),$$

valid for any piecewise C^n functions w_1, w_2. The special case $f := U_0$ yields $(1, 0, \ldots, 0)^T = \mathcal{R}_\ell \,.\, (1, 0, \ldots, 0)^T$ for all $\ell \in \mathcal{Z}_0$, which means that the first column of each $\mathcal{R}_\ell$ is equal to $(1, 0, \ldots, 0)^T$. The fact that $D(F/U_0)$ belongs to $PC^{n-1}(\mathcal{T}, \mathcal{N})$ results from Lemma 38, and so does relation (59). $\square$

5.2. W-Piecewise spaces, EC-Piecewise spaces

We shall use the expression *piecewise space* for a linear space composed of piecewise functions. In this subsection, we suppose that $\mathcal{E}$ is an $(n+1)$-dimensional subspace of $PC^n(\mathcal{T},\mathcal{M})$. We now consider *piecewise Hermite interpolation problems in* $n+1$ *data in the piecewise space* $\mathcal{E}$, that is, problems of the following form:

$$\text{Find } F \in \mathcal{E} \text{ such that } F^{(j)}(\tau_i^{\varepsilon_i}) = \alpha_{i,j}\ , \quad 1 \le i \le r,\ 0 \le j \le \mu_i - 1\ , \qquad (60)$$

for given pairwise distinct $\tau_1,\dots,\tau_r \in \mathbb{R}$, given $\varepsilon_1,\dots\varepsilon_r \in \{-,+\}$, given positive integers $\mu_1,\dots,\mu_r$ such that $\sum_{i=1}^r \mu_i = n+1$, and given real numbers $\alpha_{i,j}$, $1 \le i \le r$, $0 \le j \le \mu_i - 1$. Of course, whenever τ_i is not one of the interior t_ℓ's we can omit ε_i in (60). Piecewise Taylor interpolation problems correspond to the case $r=1$. We can state piecewise versions of Definitions 1 and 5 and Propositions 2 and 6.

Proposition and Definition 41. The $(n+1)$-dimensional piecewise space $\mathcal{E} \subset PC^n(\mathcal{T},\mathcal{M})$ is said to be a *W-Piecewise space* (W-Pspace) on $I = \cup_{\ell\in\mathcal{Z}}[t_\ell,t_{\ell+1}]$ if it meets any of the following equivalent requirements:

(Pi) Any piecewise Taylor interpolation problem has a unique solution in $\mathcal{E}$.

(Pii) Any nonzero element $F \in \mathcal{E}$ vanishes at most n times at any given point of I, *i.e.*, $Z_{n+1}^{\{a\}}(F) \le n$ for all $a \in I$.

(Piii) The (possibly left or right) Wronskian $W(F_0,\dots,F_n)$ never vanishes on I, *i.e.*, for all $x \in I$ and for $\varepsilon \in \{-,+\}$,

$$W(F_0,\dots,F_n)(x^\varepsilon) := \det\big(\mathbb{F}(x^\varepsilon), \mathbb{F}'(x^\varepsilon), \dots, \mathbb{F}^{(n)}(x^\varepsilon)\big) \ne 0\ .$$

(Piv) For any $a \in I$, there exists a basis $(\Psi_0^a, \Psi_1^a, \dots, \Psi_n^a)$ in $\mathcal{E}$ such that, for $0 \le i \le n$, the piecewise function Ψ_i^a vanishes exactly i times at a.

(Pv) For any $a \in I$, $\mathcal{E}_{n+1}(a) = \{0\}$.

Proposition and Definition 42. The $(n+1)$-dimensional piecewise space $\mathcal{E} \subset PC^n(\mathcal{T},\mathcal{M})$ is said to be an *EC-Piecewise space* (EC-Pspace) on $I = \cup_{\ell\in\mathcal{Z}}[t_\ell,t_{\ell+1}]$ if it meets any of the following equivalent requirements:

(Pi)′ Any piecewise Hermite interpolation problem (60) has a unique solution in $\mathcal{E}$.

(Pii)′ Any nonzero element $F \in \mathcal{E}$ vanishes at most n times on I, counting multiplicities, *i.e.*, $Z_{n+1}(F) \le n$.

(Piii)′ For any $r \ge 1$, any positive integers $\mu_1,\dots,\mu_r$ such that $\sum_{i=1}^r \mu_i = n+1$, and any pairwise distinct $a_1,\dots,a_r \in I$, and any $\varepsilon_1,\dots,\varepsilon_r \in \{-,+\}$, the determinant

$$\det\big(\mathbb{F}({a_1}^{\varepsilon_1}), \mathbb{F}'({a_1}^{\varepsilon_1}), \dots, \mathbb{F}^{(\mu_1-1)}({a_1}^{\varepsilon_1}), \dots, \mathbb{F}({a_r}^{\varepsilon_r}), \dots, \mathbb{F}^{(\mu_r-1)}({a_r}^{\varepsilon_r})\big), \qquad (61)$$

is not equal to zero.

Remark 43. (1) The $(n+1)$-dimensional subspace $\mathcal{E}$ of $PC^n(\mathcal{T},\mathcal{M})$ is a W-Pspace on I if and only, for each $\ell \in \mathcal{Z}$, its restriction $\mathcal{E}_\ell$ to $[t_\ell,t_{\ell+1}]$ is a W-space on $[t_\ell,t_{\ell+1}]$.

(2) If $\mathcal{E}$ is an EC-Pspace on I, then for each $\ell \in \mathcal{Z}$, $\mathcal{E}_\ell$ is an EC-space on $[t_\ell, t_{\ell+1}]$. The converse is not true (see example below). Supposing that, for all $\ell \in \mathcal{Z}$, $\mathcal{E}_\ell$ is an EC-space on $[t_\ell, t_{\ell+1}]$, according to Proposition 13, $\mathcal{E}_\ell$ is an CW-space on $[t_\ell, t_{\ell+1}]$, hence there exists positive weight functions $w_0^\ell, \dots, w_n^\ell$ (with $w_i^\ell \in C^{n-i}([t_\ell, t_{\ell+1}])$ such that $\mathcal{E}_\ell = CW(w_0^\ell, \dots, w_n^\ell)$. Denoting by $L_0^\ell, \dots, L_n^\ell$ the associated differential operators on $C^n([t_\ell, t_{\ell+1}])$, at an interior t_ℓ the connection condition (57) can also be expressed in terms of the differential operators as

$$\left(L_0^\ell F(t_\ell^+), \dots, L_n^\ell F(t_\ell^+)\right)^T = \mathcal{Q}_\ell \,.\, \left(L_0^{\ell-1} F(t_\ell^-), \dots, L_n^{\ell-1} F(t_\ell^-)\right)^T, \qquad (62)$$

where the matrix $\mathcal{Q}_\ell$ is lower triangular and has positive diagonal elements $q_{i,i}^\ell = m_{i,i}^\ell \prod_{0\leq k\leq i} w_k^{\ell-1}(t_\ell^-)/w_k^\ell(t_\ell^+)$, $0 \leq i \leq n$. After P.J. Barry's work ([2], see also [9]) it is known that, if all matrices $\mathcal{Q}_\ell$, $\ell \in \mathcal{Z}_0$, are totally positive, then the space $\mathcal{E}$ is an EC-Pspace on I. However this is only a sufficient condition as pointed out by the following elementary example. Consider the space $\mathcal{E}$ composed of all piecewise affine functions F on $I := [-1, 0] \cup [0, 1]$ with values in $\mathbb{R}$ which satisfy the connection condition

$$\left(F(0^+), F'(0^+)\right)^T = \mathcal{M}_0 \,.\, \left(F(0^-), F'(0^-)\right)^T, \quad \mathcal{M}_0 := \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}.$$

In this example $\mathcal{Q}_0 = \mathcal{M}_0$, so the total positivity sufficient condition is satisfied if and only if $\alpha \geq 0$. Still, using (Piii)$'$ of Proposition 42, one can check that for $\mathcal{E}$ to be an EC-Pspace on I it is necessary and sufficient to take $\alpha > -2$.

As consequences of the previous propositions and of the lemmas established in Subsection 5.1, we obtain the piecewise versions of Remarks 3 and 7.

Proposition 44. *Given a piecewise function $U_0 \in PC^n(\mathcal{T}, \mathcal{M})$ ($n > 0$) supposed to never vanish on I, we set $L_0 F := F/U_0$ for any $F \in PC^n(\mathcal{T}, \mathcal{M})$. Let $\mathcal{E}$ be an $(n+1)$-dimensional subspace of $PC^n(\mathcal{T}, \mathcal{M})$ containing U_0. Then*

1) *$\mathcal{E}$ is a W-Pspace on I if and only if the piecewise space*

$$DL_0\mathcal{E} := \{DL_0 F \;,\; F \in \mathcal{E}\}$$

is a W-Pspace on I.

2) *If $DL_0\mathcal{E}$ is an EC-Pspace on I, then $\mathcal{E}$ is an EC-Pspace on I.*

Proof. Let us first observe that, in our present piecewise context, each of the two equalities (5) and (6) holds as a piecewise equality, given that the function $\mathbb{1}$ belongs to some piecewise space $PC^n(\mathcal{T}, \mathcal{R})$ iff the first columns of each matrix $\mathcal{R}_\ell$ is equal to $(1, 0, \dots, 0)^T$. Our first statement thus follows from (Piii). The second one results from (Pii)$'$ and Lemma 40. □

5.3. Complete W-Pspaces

Definition 45. A sequence $(U_0,\ldots,U_n)$ in $PC^n(\mathcal{T},\mathcal{M})$ is said to be a *Complete W-Psystem* (CW-Psystem) on I if it satisfies, for $\varepsilon \in \{-,+\}$,

$$W(U_0,\ldots,U_k)(x^\varepsilon) \neq 0 \quad x \in I\,, \quad 0 \leq k \leq n. \tag{63}$$

If (63) is satisfied, the $(n+1)$-dimensional piecewise space spanned by $U_0,\ldots,U_n$ is said to be *a Complete W-Piecewise space* (in short, CW-Pspace) on I.

With a given CW-Psystem $(U_0,\ldots,U_n)$ in $PC^n(\mathcal{T},\mathcal{M})$, it is possible to associate differential operators $L_0,\ldots,L_n$, but of course piecewise ones. This fact presents a theoretical interest for specialists. It is why we give some details about it.

As previously, for any piecewise C^n function F, we set $L_0F(x^\varepsilon) = F(x^\varepsilon)/U(x^\varepsilon)$ for all $x \in \mathbb{R}$ and for $\varepsilon \in \{-,+\}$. Using the piecewise version of (5) and (6) along with Lemmas 39 and 40, we can state that:
- on the one hand, $(\mathbb{1}, L_0U_1,\ldots,L_0U_n)$ is a continuous CW-Psystem in $PC^n(\mathcal{T},\mathcal{R})$, where the matrices $\mathcal{R}_\ell$ are defined as in Lemma 40;
- on the other hand, $(DL_0U_1,\ldots,DL_0U_n)$ is a CW-Psystem in $PC^{n-1}(\mathcal{T},\mathcal{M}^{\{1\}})$, each matrix $\mathcal{M}_\ell^{\{1\}}$ being obtained by deleting the first row and column in $\mathcal{R}_\ell$. Iteration of the process leads to (18), but the latter formulae are now meant piecewisely, defining piecewise differential operators $L_0,\ldots,L_n$ on the set of all piecewise C^n functions. This provides us with two sequences of sequences of matrices: $\mathcal{R}^{\{i\}} = \big(\mathcal{R}_\ell^{\{i\}}\big)_{\ell\in\mathcal{Z}_0}$ and $\mathcal{M}^{\{i\}} = \big(\mathcal{M}_\ell^{\{i\}}\big)_{\ell\in\mathcal{Z}_0}$, $i \geq 0$, where the upper index $^{\{i\}}$ means that we are dealing with square matrices of order $(n+1-i)$. The two sequences are defined recursively by $\mathcal{M}^{\{0\}} := \mathcal{M}$, $\mathcal{R}^{\{0\}} := \mathcal{R}$ and, for $1 \leq i \leq n$,

$$\mathcal{R}_\ell^{\{i\}} := \mathcal{C}_{n-i}(DL_{i-1}U_i, t_\ell^+)^{-1}\ .\ M_\ell^{\{i\}}\ .\ \mathcal{C}_{n-i}(DL_{i-1}U_i,, t_\ell^-), \quad \ell \in \mathcal{Z}_0, \tag{64}$$

while $\mathcal{M}_\ell^{\{i\}} := \big(m_{i,j}^{\ell\{i\}}\big)_{0\leq i,j\leq n-i}$ is obtained by deleting the first row and column in $\mathcal{R}_\ell^{\{i-1\}}$. All matrices $\mathcal{R}_\ell^{\{i\}}$ are lower triangular with positive diagonal elements and their first columns are equal to $(1,0,\ldots,0)^T$. From (64), for instance, we can easily derive the diagonal elements of each matrix $\mathcal{M}_\ell^{\{i\}}$:

$$m_{k,k}^{\ell\{i\}} := m_{k+1,k+1}^{\ell\{i-1\}}/m_{0,0}^{\ell\{i-1\}} = m_{k+i,k+i}^{\ell}/m_{i-1,i-1}^{\ell}\ , \quad 0 \leq k \leq n-i,\ 1 \leq i \leq n.$$

Each space $PC^{n-i}(\mathcal{T},\mathcal{R}^{\{i\}})$ is thus composed of functions which are continuous and piecewise C^{n-i} on I and, for $0 \leq i \leq n$, we have

$$L_iPC^n(\mathcal{T},\mathcal{M}) \subset PC^{n-i}(\mathcal{T},\mathcal{R}^{\{i\}}),\ DL_iPC^n(\mathcal{T},\mathcal{M}) \subset PC^{n-i-1}(\mathcal{T},\mathcal{M}^{\{i+1\}}).$$

Moreover, $(\mathbb{1}, L_iU_{i+1},\ldots,L_iU_n)$ is a continuous CW-Psystem in $PC^{n-i}(\mathcal{T},\mathcal{R}^{\{i\}})$ and $(DL_iU_{i+1},\ldots,DL_iU_n)$ is a CW-Psystem in $PC^{n-i-1}(\mathcal{T},\mathcal{M}^{\{i+1\}})$. The space $\mathcal{E}$ is the set of all piecewise functions $F \in PC^n(\mathcal{T},\mathcal{M})$ such that L_nF is a constant function on the whole of I.

As is classical in CW-spaces, we can thus also associate weights with the CW-Psystem $(U_0,\ldots,U_n)$. The weights are not functions, but piecewise functions, defined by the piecewise equalities $w_0 := U_0$, $w_i := DL_{i-1}U_i$ for $1 \leq i \leq n$. The weight piecewise functions do not vanish on I and they satisfy:

$$w_i \in PC^{n-i}(\mathcal{T}, \mathcal{M}^{\{i\}}), \quad 0 \leq i \leq n.$$

Conversely, it would be interesting to find how to choose the matrices of the initial sequence $\mathcal{M}$ so as to provide us with a non vanishing function w_i in $PC^{n-i}(\mathcal{T}, \mathcal{M}^{\{i\}})$ at each stage i. The same arguments as in the nonpiecewise case would then lead to a CW-Pspace on I, hence to an EC-Pspace according to Theorem 46 below.

Theorem 46. *If $\mathcal{E}$ is a CW-Pspace on I, then it is an EC-Pspace on I.*

Proof. As in the nonpiecewise case, for a 1-dimensional piecewise space, being an EC-Pspace is equivalent to being a W-Pspace on I since both mean that any nonzero element never vanishes on I. Let $\mathcal{E}$ be the CW-Psubspace of $PC^n(\mathcal{T}, \mathcal{M})$ spanned by some CW-Psystem $(U_0, \ldots, U_n)$. Then, part 1 of Proposition 44 shows that $DL_0\mathcal{E}$ is a CW-Pspace on I. Supposing that the result holds for $n - 1 \geq 0$, we can thus assert that the piecewise space $DL_0\mathcal{E}$ is an EC-Pspace on I. Part 2 of Proposition 44 implies that $\mathcal{E}$ itself is an EC-Pspace on I. □

Consider a nested sequence (10) of subspaces in $PC^n(\mathcal{T}, \mathcal{M})$, in which each $\mathcal{E}_i$ is $(i + 1)$-dimensional. Similarly to the nonpiecewise case, Theorem 46 says that requiring all $\mathcal{E}_i$'s to be W-Pspaces on I is thus the same as requiring them all to be EC-Pspaces on I.

6. EC-Pspaces and Blossoms

Blossoms in piecewise spaces with connection matrices were first considered by H.-P. Seidel in the case of geometrically continuous polynomial splines [22]. In the more general framework of W-Pspaces, existence of blossoms was proved to be equivalent to existence of B-spline bases in [12], and to existence of Bézier points in [15]. However, in either case the spaces were supposed to be piecewise C^∞. Prior to that, in [2], P. J. Barry was the first to point out existence of B-spline bases with sections in different EC-spaces and with connections by means of totally positive regular lower triangular matrices. As we recalled in Remark 43, (2), the total positivity assumption concerned the differential operators associated with each section. He, too, supposed more differentiability than really needed since his proof relied on the fact that the dual spaces were connected so as to form an EC-Pspace (with the meaning of our present paper).

In this subsection as in the nonpiecewise context, existence and properties of blossoms will be derived from using relevant bases in EC-Pspaces, to which the first subsection is devoted. The result of it is that the equivalences stated in [12] and [15] will be valid with no extra differentiability assumption. However, this does not provide new practical conditions ensuring existence of blossoms.

6.1. Various ways to characterise EC-Pspaces

All relations obtained in Subsection 3.1 as consequences of Sylvester's identity are still valid in the piecewise context, apart from the fact that, of course, they are now valid piecewisely, that is, they give rise to two different equalities at each interior t_ℓ, one at t_ℓ^-, the other at t_ℓ^+. This, in particular, will enable us to extend Theorem 23 to EC-Pspaces. To achieve this, we need to first show that the various two point characterisations of EC-spaces given in Theorem 12 are valid in the piecewise context too. All these extensions are gathered in Theorem 47 below. We do not give details of the proofs, we just point out the exact facts which make the extensions possible.

Given piecewise functions $B_0, \dots, B_n$ in some $PC^n(\mathcal{T}, \mathcal{M})$, we say that they form a *piecewise Bernstein-like basis* of the space they span, relative to $(a, b) \in I^2$, $a \neq b$, if they satisfy the same conditions of zeros at a and b as Bernstein-like bases.

Theorem 47. *Let $\mathcal{E}$ be an $(n+1)$-dimensional subspace of $PC^n(\mathcal{T}, \mathcal{M})$. Then, $\mathcal{E}$ is an EC-Pspace on I if and only if it meets any of the following equivalent requirements:*

(Pi)″ *Any piecewise Hermite interpolation problem based on any one or two points of I has a unique solution in $\mathcal{E}$.*

(Pii)″ *For any distinct $a, b \in I$, any nonzero element $F \in \mathcal{E}$ vanishes at most n times on $\{a, b\}$, counting multiplicities, i.e., $Z_{n+1}^{\{a,b\}}(F) \leq n$.*

(Piii)″ *Given any distinct $a, b \in I$, any integers $i, j \geq 0$ with $i + j = n + 1$, and any $\varepsilon, \varepsilon' \in \{-, +\}$, the determinant*

$$\det\big(\mathbb{F}(a^\varepsilon), \mathbb{F}'(a^\varepsilon), \dots, \mathbb{F}^{(i-1)}(a^\varepsilon), \mathbb{F}(b^{\varepsilon'}), \mathbb{F}'(b^{\varepsilon'}) \dots, \mathbb{F}^{(j-1)}(b^{\varepsilon'})\big), \qquad (65)$$

is not equal to zero.

(Piv)″ *$\mathcal{E}$ possesses a piecewise Bernstein-like basis relative to any pair of distinct points of I.*

(Pv)″ *Given any $a \in I$, for $0 \leq k \leq n$, the space $\mathcal{E}_k(a)$ is an $(n-k+1)$-dimensional W-Pspace on any interval $J \subset I \setminus \{a\}$.*

(Pv)′ *Given any integer $r \geq 1$, any positive integers $\mu_1, \dots, \mu_r$ such that $\sum_{i=1}^r \mu_i = n + 1$, and any pairwise distinct $a_1, \dots, a_r \in I$, $\mathcal{E}$ possesses a basis*

$$\Psi_n^{a_1}, \dots, \Psi_{n-\mu_1+1}^{a_1}, \Psi_n^{a_2}, \dots, \Psi_n^{a_r}, \dots, \Psi_{n-\mu_r+1}^{a_r}$$

such that, for $1 \leq i \leq r$, and $n - \mu_i + 1 \leq k \leq n$, the piecewise function $\Psi_k^{a_i}$ vanishes exactly k times at a_i.

Proof. On account of Theorem 46, of Propositions 41 and 42, and of Remark 43, the fact that any among the properties (Pi)″ – (Pv)″ is equivalent to $\mathcal{E}$ being an EC-Pspace on I can be proved by following the same arguments we used for Theorem 12, including the case where I is a closed bounded interval, that is when the set $\mathcal{Z}$ is finite.

A piecewise version of Lemma 19 does exist, as a consequence of the (now piecewise) equality (20). This makes it possible to also extend to the present piecewise context the proof by induction of Theorem 23, starting the induction from (Pv)″. This enables us to show that if $\mathcal{E}$ is an EC-Pspace on I, it satisfies (Pv)′. The proof of the converse is similar to that of Theorem 23. □

6.2. Blossoms in EC-Pspaces

In this subsection we start with a sequence $\mathcal{M} = \left(\mathcal{M}_\ell\right)_{\ell\in\mathcal{Z}}$ of lower triangular matrices of order n with positive diagonals, and with a sequence of $(n+1)$-dimensional spaces $\mathcal{E}_\ell$, $\ell \in \mathcal{Z}$, each $\mathcal{E}_\ell$ being assumed both to be an EC-space on $[t_\ell, t_{\ell+1}]$ and to contain constants. We define the $(n+1)$-dimensional space $\mathcal{E}$ as the space of all continuous functions F on I the restrictions of which to each $[t_\ell, t_{\ell+1}]$ belong to $\mathcal{E}_\ell$ and which satisfy the connection conditions

$$\left(F'(t_\ell^+), \dots, F^{(n)}(t_\ell^+)\right)^T = \mathcal{M}_\ell \cdot \left(F'(t_\ell^-), \dots, F^{(n)}(t_\ell^-)\right)^T, \quad \ell \in \mathcal{Z}_0.$$

In other words, the space $\mathcal{E}$ itself contains constants and the space $\mathcal{U} := D\mathcal{E}$ is contained in $PC^{n-1}(\mathcal{T}, \mathcal{M})$.

We can choose a mother-function $\Phi \in \mathcal{E}^n$ the same way as in Section 4. Our requirements on the connection matrices $\mathcal{M}_\ell$ are exactly what is needed to ensure the minimum geometric continuity to be wished for, namely the fact that the parametric curve produced by Φ has the same left and right Frénet frames at each point t_ℓ, $\ell \in \mathcal{Z}_0$. Note that this also guarantees existence of osculating flats of any order $k \leq n$ at each point t_ℓ, $\ell \in \mathcal{Z}_0$, defined as $\mathrm{Osc}\Phi_k(t_\ell) := \mathrm{Osc}\Phi_k(t_\ell^-) = \mathrm{Osc}\Phi_k(t_\ell^+)$. As a consequence, we can still associate with Φ the function φ defined by (27). In particular, the definitions of blossoms and Bézier points do not differ from Subsection 4.1.

We need to say how to define the spline spaces based on $\mathcal{E}$. First of all, if $\mathcal{T}'$ is a new sequence of intervals obtained from $\mathcal{T}$ by splitting the interval $[t_{\ell_0}, t_{\ell_0+1}]$ into two nontrivial subintervals $[t_{\ell_0}, \tau]$ and $[\tau, t_{\ell_0+1}]$, we observe that $PC^{n-1}(\mathcal{T}', \mathcal{M}') = PC^{n-1}(\mathcal{T}, \mathcal{M})$ provided that $\mathcal{M}'$ is obtained by inserting at the right place the identity matrix of order n in the sequence of matrices $\mathcal{M} = \left(\mathcal{M}_\ell\right)_{\ell\in\mathcal{Z}_0}$. As in Subsection 4.5, we consider a finite sequence of knots $\eta_0 < \eta_1 < \cdots < \eta_q < \eta_{q+1}$ in the interval I, and an associated sequence of multiplicities $m_0, \dots, m_{q+1}$. We assume that $m_0 = m_{q+1} = n+1$ as previously, but now, for $1 \leq i \leq q$, we suppose that $0 \leq m_i \leq n$. Due to our observation above, in order to describe the spline space $\mathcal{S}(\mathcal{E}, \mathcal{K})$ associated with the corresponding knot vector (52), we may assume the existence of an integer ℓ such that $\eta_i = t_{\ell+i}$ for $0 \leq i \leq q+1$. If so, $\mathcal{S}(\mathcal{E}, \mathcal{K})$ is composed of all continuous functions $S : [\eta_0, \eta_{q+1}] \to \mathbb{R}$ satisfying property $(\mathrm{S})_2$ of Subsection 4.5 along with the connection conditions:

$$\left(S'(\eta_i^+), \dots, S^{(n-m_i)}(\eta_i^+)\right)^T = \mathcal{C}_i \, . \, \left(S'(\eta_i^-), \dots, S^{(n-m_i)}(\eta_i^-)\right)^T, \quad 1 \leq i \leq q,$$

where $\mathcal{C}_i$ is the square matrix of order $(n - m_i)$ obtained by deleting the m_i last rows and columns in $\mathcal{M}_{\ell+i}$. Subject to existence, the B-spline basis of the space $\mathcal{S}(\mathcal{E}, \mathcal{K})$

is the unique basis satisfying the support, end point, positivity, and normalisation properties of Definition 35. As a special case, Bernstein bases have the meaning of Definition 33.

Below is a simplified version of the extensions of Theorem 34 and 36 to our present piecewise context.

Theorem 48. *Let $\mathcal{E}$ be the $(n+1)$-dimensional space described at the beginning of the present subsection and let $\mathcal{U}$ denote the piecewise space $D\mathcal{E}$. The following five properties are equivalent:*

(Pa) *The space $\mathcal{U}$ is an EC-Pspace on I.*
(Pb) *Blossoms exist in the space $\mathcal{E}$.*
(Pc) *Bézier points exist in the space $\mathcal{E}$.*
(Pd) *The space $\mathcal{E}$ possesses a normalised Bernstein basis relative to any pair of distinct points of I.*
(Pf) *For any knot vector $\mathcal{K}$, the space $\mathcal{S}(\mathcal{E},\mathcal{K})$ possesses a B-spline basis.*

Proof. Suppose that (Pa) holds. Existence of blossoms can easily be obtained using the relevant bases presented in (Pv)′, exactly as in the proof of Theorem 27. Moreover formula (31) is valid, except that we have to replace a_1 by ${a_1}^{\varepsilon_1}$ in both the numerator and the denominator. Existence of Bernstein bases, or more generally, of B-spline bases, will follow from the pseudoaffinity property of blossoms. In our piecewise context, equality (36) is still valid, except that $\beta(x_1,\dots,x_{n-1};\gamma,\delta;.)$ is now a continuous strictly increasing function on I which is piecewise C^1 on I. To achieve this, we need a piecewise version of Lemma 31, which we give below.

Taking account of Theorem 47, the rest of the proof results from the same arguments as in the nonpiecewise case. □

Lemma 49. *Let a be an interior point of a given interval I, let $(n+1)$ integers $k_0,\dots,k_n$ satisfy $k_0 \geq 0$, $k_i \geq k_{i-1}+1$ for $1 \leq i \leq n$, and let Q denote a lower triangular matrix of order (k_n+1), with positive diagonal elements. We consider $(n+1)$ piecewise C^{k_n} functions $U_0,\dots,U_n$ on I (here, in the sense that they are C^{k_n} functions separately on $I^- := \{x \in I, x \leq a\}$ and $I^+ := \{x \in I, x \geq a\}$) assumed to satisfy*

$$\Delta_{k_n} U_i(a^+) = Q \,.\, \Delta_{k_n} U_i(a^-), \quad 0 \leq i \leq n. \tag{66}$$

Suppose that, for $0 \leq i \leq n$, U_i vanishes exactly k_i times at a. Then, we have, for $\varepsilon \in \{-,+\}$,

$$W(U_0,\dots,U_n)(x) \sim (x-a)^N A_\varepsilon \quad \text{for } x \in I^\varepsilon \text{ close to } a \tag{67}$$

where $N := \sum_{i=0}^n (k_i - i)$ and where the two real numbers A_-, A_+ have the same strict sign.

Proof. We can apply Lemma 31 separately on I^- and I^+. This yields (67) with:

$$A_\varepsilon := \prod_{i=0}^{n} \frac{U_i^{(k_i)}(a^\varepsilon)}{k_i!} \prod_{0\le i<j\le n} (k_j - k_i). \tag{68}$$

Denote by $q_{0,0},\dots,q_{k_n,k_n}$ the (positive) diagonal elements of Q. Taking account of relation (66) along with the fact that each U_i vanishes exactly k_i times at a, we have

$$U_i^{(k_i)}(a^+) = q_{k_i,k_i} U_i^{(k_i)}(a^-)\ , \quad 0 \le i \le n.$$

As a consequence $A_+ = A_- \prod_{i=0}^{n} q_{k_i,k_i}$, and the proof is complete. □

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