DERIVED CATEGORIES, ARITHMETIC, AND GEOMETRY

1. Organizers

- **Matthew Ballard**, Professor at the University of South Carolina. Dr. Ballard's research focuses on derived categories in algebraic geometry with an emphasis on structural questions and connections to classical problems.

  Dr. Ballard has organized multiple summer workshops for early career researchers, including a PIMS Summer Workshop and a Fields Graduate Workshop.
- **Katrina Honigs**, Assistant Professor of Mathematics at Simon Fraser University. Dr. Honigs' research focuses on derived categories in arithmetic geometry and what they can tell us as an invariant attached to varieties.

  Dr. Honigs was a participant in an MRC during summer 2019. She hopes to replicate as well as build on the successes of that experience.
- **Daniel Krashen**, Presidential Professor of Mathematics at the University of Pennsylvania. Dr. Krashen studies noncommutative algebra and arithmetic geometry.

  Dr. Krashen has been a long term organizer of the annual Brauer Group Meetings in addition to other gatherings.
- **Alicia Lamarche**, NSF Postdoctoral Scholar in Mathematics at the University of Utah. Dr. Lamarche studies algebraic geometry in nonzero characteristic.

  She was a participant in the 2019 MRC on Explicit Methods in Arithmetic Geometry in Characteristic p, and has experience organizing the Utah Algebraic Geometry Seminar as well as the Graduate Colloquium at the University of South Carolina.
- **Emannuele Macrì**, Professor of Mathematics at Université Paris-Saclay. Dr. Macrì studies derived categories in algebraic geometry with an emphasis on stability and birational geometry.

  Dr. Macrì has organized multiple mathematical events including the International Derived Seminar. He assisted the organizers for the 2010 MRC on Birational Geometry and Moduli Spaces.

Derived categories have entered into our work in various ways, for example [AKW17, BDLM20, BLMNPS21, AAHF21]. We also have strong connections with other researchers in the area.

Informed by our experience with past MRC's and our organizational histories, we are all passionate about creating an inclusive and supportive environment for students and researchers to explore algebraic geometry.

To be absolutely clear:

All organizers have committed to in–person attendance for the full week period.

2. Mathematical focus

Derived categories are an important and often intimidating tool in algebraic geometry. Because derived categories allow novel connections both to other areas, like Symplectic Geometry, High Energy Physics, and Noncommutative Algebra, and within algebraic geometry itself, a mathematician with a working knowledge of the subject and its techniques can often discover deep and meaningful connections between their ideas and those from different domains. These connections also make derived categories an area in which mathematicians with other expertises can converge and complete meaningful work together.

Despite being decades old, derived categories are an area of rapid discovery. There are strong guiding conjectures, new tools, emerging novel perspectives, and many concrete questions.

The past decade has seen the development of important new tools to understand the structure of derived categories and to leverage them to understand other questions.

We propose to welcome a group of early career researchers and, through active problem solving and collaboration, help them develop a working knowledge of derived categories.

Given the proposed scope of the MRC, a 40 participant conference is preferable.

3. How derived categories fits in the MRC program

Derived categories have never been the research topic for an MRC. To best of our knowledge, it, in any form, has appeared twice: as the setting for a group's research question.

During the 2010 MRC on Birational Geometry and Moduli Spaces, Bridgeland stability was one of the possible research directions. Dr. Macrì assisted the MRC organizers.

In the 2019 MRC on Explicit Methods in Arithmetic Geometry in Characteristic p, one group, whose members include Drs Honigs and Lamarche, worked on a question relating rational points and derived categories.

Part of our motivation in proposing an MRC on derived categories is its absence from the past MRC lineup.

2020 MSC Codes

- **14F08** Derived categories of sheaves, dg categories, and related constructions in algebraic geometry.
- **18G80** Derived categories, triangulated categories.

4. Research themes and mentor–organizer pairing

Derived categories in algebraic geometry is an expansive topic. To achieve appropriate scope for participants, we have identified 5 core topics. For each topic, we have/will recruit an external expert as a mentor. Each mentor will paired with an organizer. Together the mentor–organizer pair will be responsible for the research topic and leading the research group in–person during the week–long in–person period.

The mentor/organizer pair's main duties are the following.

- Record survey videos for all participants in advance of the MRC and answer questions on this content via chat prior to meeting in in-person,
- Develop, in consultation with the organizers, possible problems for their group, and ultimately guide their group to a final research question, and
- Lead research groups during the in-person component of the MRC.

Below we give an overview of each research topic, the (potential) external mentor, the organizer, and some sample problems.

4.1. **Bridgeland Stability.**

Perhaps the most spectacular recent bridge between derived categories and birational geometry in algebraic geometry is the topic of Bridgeland stability [Bri07]. We have recruited Dr. Laura Pertusi [1] to serve as the external mentor for Bridgeland Stability.

Dr. Pertusi is currently a Post-Doc in Algebraic Geometry at the Department of Mathematics of the Università degli Studi di Milano. She received her PhD in 2018 with the advisement of Dr. Paolo Stellari. Dr. Petrusi is herself an early-career researcher.

Dr. Petrusi is one of the leaders of the next generation of researchers in Bridgeland stability [LPZ18, FP21, PR21].

Dr. Macrì has agreed to serve as the paired organizer for the topic of Bridgeland Stability. Together they suggest the following list of research questions:

- Serre invariant stability conditions turned out to be very useful for studying geometric properties of moduli spaces of stable objects and for answering Categorical Torelli problems [APR19, BEFHMRS20]. On the other hand, there are examples of triangulated categories which do not have Serre-invariant stability conditions [KP21c]. Despite being the simplest of projective varieties, projective spaces are still mysterious in this regard: are there Serre-invariant stability conditions on $D^b(\mathbb{P}^2)$? And what about $D^b(\mathbb{P}^3)$?
- In [Li19] a stronger Bogomolov Inequality has been proved in the case of quintic threefolds, giving the first example of stability conditions on a strict (i.e. simply-connected) Calabi–Yau threefold. We look to push this method to the next logical example: is it possible to generalize the argument in

[1] Committed to attending the in-person period.

[Li19] to construct stability conditions on the bounded derived category of a complete intersection of a cubic and two quadric hypersurfaces in $\mathbb{P}^6$?

4.2. **Homological Projective Duality.** Since being introduced in [Kuz07], Homological Projective Duality, or HPD, has been a powerful machine for breaking derived categories into their constituient pieces via the main building blocks of the theory, semi-orthogonal decompositions. We have recruited Dr. Alexander Perry [2] to serve as the external mentor for HPD.

Dr. Perry is currently an Assistant Professor at the University of Michigan. Dr. Perry received his PhD from Harvard Univerity in 2016 under the advisement of Professor Joe Harris. Dr. Perry is currently a Sloan Fellow. Dr. Perry is himself an early-career researcher.

Dr. Perry, along with Dr. Alexander Kuznsetov, is one of today's two top experts in HPD [KP18, KP21a, KP21b].

Dr. Lamarche has agreed to serve as the paired organizer for the topic of HPD. Together they suggest the following research questions:

- Use HPD to do a "dimensional reduction" to embed semiorthogonal components of high-dimensional varieties into lower dimensions. This is useful for instance in the construction of Bridgeland stability conditions, e.g. it was crucial for the case of K3 categories of cubic fourfolds. There are still lots of interesting examples where we don't have stability conditions, but Dr. Perry suspects that similar methods could be used. For instance, the K3 categories of Debarre-Voisin varieties, or the Kuznetsov component of a quartic fourfold.
- Understand better when Kuznetsov components of complete intersections are equivalent to the derived category of a variety. In [KP21c] it was shown that geometricity of the Kuznetsov component imposes strong restrictions, but it would be interesting to understand exactly when the condition holds, and what its geometric significance is.

4.3. **Noncommutative resolutions.** A surprising consequence of studying algebraic varieties through their derived categories is the interplay with noncommutative algebra. Since Beilinson described the structure of the derived categories of projective spaces in terms of path algebras of quivers with relations [Bei83], the dialog between algebraic geometers and noncommutative algebraists has been incredibly fruitful.

One specific highlight was the introduction of Noncommutative Creprant Resolutions, or NCCRS, [VdB04] which provided a window into understanding the structure of derived categories of singular algebraic varieties.

We have reached out to Dr. Michel Van den Bergh [3] to serve as a mentor for NCCRs. Dr. Van den Bergh is a Professor at Vrije Universiteit Brussel. He was awarded the Franqui Prize on Exact Sciences in 2003 and he will be a plenary speaker at the 2022 International Congress of Mathematicians.

Dr. Honigs has agreed to serve as the paired organizer for the topic of NCCRs.

The organizers suggests the following research questions:

- Extend the ideas of [FMS19] constructing explicit resolutions of simple modules for NCCRs of toric singularities to general reductive groups.
- Compute the global dimensions of the NCCRs arising in [SV17].

4.4. **Enhancements.** Higher order categorical structures are an essential tool for studying derived categories. Depending our your background, such structures go under the title of enhancements or ∞-categorical models. Despite the utility of enhancements, our understanding of when a triangulated category admits an enhancement remains fairly basic.

We have reached out to Dr. Alice Rizzardo [4] to serve as a mentor for Enhancements. Dr. Rizzardo is currently an Lecturer at the University of Liverpool. Dr. Rizzardo received her PhD from the Columbia in 2012 under the advisement of Professor Johan de Jong. Dr. Rizzardo is currently herself an early-career researcher.

[2]Committed to attending the in-person period.

[3]Contacted.

[4]Contacted.

Dr. Rizzardo is responsible for recent spectacular results informing our understanding [RVN19, RV20] on the lack of enhancements for both functors and triangulated categories.

Dr. Krashen has agreed to serve as the paired organizer for the topic of Enhancements. The organizers suggest the following research questions:

- Currently all examples of exact functors lacking an enhancement have as their source category varieties of dimension 3 or more. Construct non-enhancable functors whose source is a curve or surface.
- Does an exact functor compatible with base change necessarily have an enhancement? More generally, are there conditions, besides full and faithful, to guarantee enhancements?

4.5. **Computation and derived categories.** Questions that can be reduced to computer implemenation often provide a springboard to deeper understanding on a complicated topic. We have recruited Dr. Pieter Belmans [5] to serve as the external mentor for computational questions around derived categories.

Dr. Belmans is currently an Assistant Professor at the University of Luxembourg. Dr. Belmans received his PhD from the University of Antwerp in 2017 under the advisement of Professors Wendy Lowen and Michel Van den Bergh. Dr. Belmans is himself an early-career researcher.

Dr. Belmans is a technical master of derived categories who moonlights as a full-stack developer. His code makes the Stacks Project [Stacks] possible. Beyond maintaining the Stacks Project, Dr. Belmans is responsible for several vital mathematical websites including Dr. Jacob Lurie's `kerodon.net`.

Dr. Ballard has agreed to serve as the paired organizer for this topic. Together they suggest the following research questions:

- Develop efficient algorithms for computations of invariants like dimensions of Hochschild (co)homology and Hodge numbers of zero sections in homogeneous spaces G/P.
- Flesh out and refine the algorithm described in [SV17, Remark 11.3.2] for recognizing when algebras of covariants have finite global dimension. The authors remark that they have implemented it the case of two-dimensional tori.

4.6. **Additional possible topics.** As we have unconfirmed external mentors, we thought it prudent to include a short list of additional possible topics and mentors. The order is alphabetical by mentor last name.

- Dr. Asher Auel: Arithmetic, rationality, and derived categories
- Dr. Eleonore Faber: Noncommutative crepant resolutions
- Dr. Sarah Frei: Arithmetic aspects of moduli
- Dr. Michael Weymss: Noncommutative crepant resolutions

5. Participant background and selection

In order to spread awareness of our MRC, we intend to use existing conference mailing lists, listservs, and websites to ask established researchers to disseminate the call for applications to their students, postdocs, and any interested parties. We also plan to use social media platforms– facebook and twitter in particular, to advertise.

Beyond advertising, by far the most impactful method of recruiting a diverse applicant pool is to directly reach out to early-career mathematicians (from a variety of institutional backgrounds, gender, race, and ethnicity groups) to welcome and encourage them to apply, as well as give them the opportunity to ask any questions that they might have. In addition to this, we plan on promoting the MRC via talks oriented towards graduate students or general mathematical audiences (for example, the well-attended online seminar "*What is a... seminar?*", modeled after the similarly named column in the notices of the AMS) in order to pique interest and encourage applications.

We are setting the general guideline that participants should have at a minimum reached candidacy or equivalent in the pursuit of a PhD in algebraic geometry and are within five years of the receipt of their PhD. We strove to achieve a balance of research topics in the proposal with the intention that all participants in this range will be able to engage wholly with the MRC content.

[5] Committed to attending the in-person period.

6. Conference structure and schedule

6.1. **Timeline.**

Fall 2022–Mid February 2023. Mentors and organizers develop materials. MRC is advertised and applications are solicited.

Mid February 2023–Mid May 2023. Selected participants engage with materials, mentors, organizers, and each other as a single group via Zulip and Zoom.

Mid May 2023. A survey is sent to the participants asking for their preferences on research topics. Particpants are sorted into groups based on their declared preferences. Mutual-agreed swapping of groups is allowed at any time.

Mid May 2023–MRC Week. Groups get acquainted through their own dedicated topics channels on Zulip. Together with the mentor and organizer, they focus on a research problem and identify resources for consumption before the in-person week.

MRC Week. Groups spend the majority of the week working together. Some professional development and social activities are interspersed in the schedule.

Post MRC Week–Jan 2024. Groups continue working on questions.

Jan 2024. Groups present at the Joint Meetings special session.

6.2. **Prior to the meeting.** The mentors would record survey talks aimed to providing a "bird's eye view" of their particular area of focus. Emphasis would be on conveying the big picture: what are the tools, what are the guiding questions, what currently are hot topics. Guidance to further resources would be provided.

The videos would be distributed through a Zulip chat community asynchronously. Participants will be encouraged to ask questions and discuss the videos on Zulip, and organizers would schedule times for the mentors to field questions synchronously from participants via the chat.

The mentors would provide some ideas for their group project and participants would submit a ranked order list of their preference. The organizers would assemble groups based on participant preference, a diversity in career stage, and a diversity in demographic composition.

A few weeks before the in-person component, the groups would be created with the intention of maximizing the amount of research time they have in-person. Group members would meet online together with their mentors to select their problem, ask questions, and to get to know one another.

6.3. **At the meeting.** With much of the logistics taken care of in advance of the MRC week, the groups will be able to dive directly into collaborative investigations.

Tuesday and Thursday afternoons would be reserved for update presentations on the status of each group's work. The remainder of "work time" is allotted for active research work.

Complementing the research experience, specific events focused on preparing and developing holistic mathematicians will occur in the evening. These events are optional.

- Monday: developing a research program
- Tuesday: grants and grant writing
- Wednesday: giving a good talk
- Thursday: applying for jobs

We would structure these events a panels and group discussions. Guest appearances by outside experts would be possible.

Social events to encourage networking and bonding would also be part of the in-person meeting.

6.4. **After the event.** It is our hope that, after having spent multiple weeks (both online and offline) invested in their groups, participants will already have an infrastructure to continue working on their projects. We would provide collaboration resources to foster continuing work on the MRC projects, including Zulip, Discord, and GitHub.

All participants would be given the chance to blog about their experience, research, social, or other, on the Derived Categories MRC webpage.

We would also encourage groups to present their work at the Joint Meetings the following winter.

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