MOORE GRAPH WITH PARAMETERS (3250,57,0,1) DOES NOT EXIST *

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ABSTRACT

If a regular graph of degree k and diameter d has v vertices then

$$v \leq 1 + k + k(k-1) + \cdots + k(k-1)^{d-1}$$

Graphs with $v = 1+k+k(k-1)+ + k(k-1)^{d-1}$ are called Moore graphs. Damerell proved that a Moore graph of degree $k \geq 3$ has diameter 2. If Γ is a Moore graph of diameter 2, then $v = k^2+1$, Γ is strongly regular with $\lambda = 0$ and $\mu = 1$, and one of the following statements holds: $k = 2$ and Γ is the pentagon, $k = 3$ and Γ is the Petersen graph, $k = 7$ and Γ is the Hoffman-Singleton graph, or $k = 57$. The existence of a Moore graph of degree 57 was unknown.

Jurishich and Vidali have proved that the existence of a Moore graph of degree $k > 3$ is equivalent to the existence of a distance-regular graph with intersection array $\{k-2, k-3, 2; 1, 1, k-3\}$ (in the case $k = 57$ we have a distance-regular graph with intersection array $\{55, 54, 2; 1, 1, 54\}$).

In this paper we prove that a distance-regular graph with intersection array $\{55, 54, 2; 1, 1, 54\}$ does not exist. As a corollary, we prove that a Moore graph of degree 57 does not exist. K

***K*eywords** distance-regular graph · Moore graph

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1 Intro

We consider undirected graphs without loops and multi-edges. If a, b are vertices of a graph Γ then we denote by $d(a, b)$ the distance between a and b. We denote by $\Gamma_i(a)$ a subgraph of the graph Γ induced by a set of vertices with the distance i from the vertex a. A subgraph $\Gamma_1(a)$ is called a *neighbourhood* of a vertex a and denoted by $[a]$.

The definition of the *distance-regular graph* with the intersection array $\{b_0, ..., b_{d-1}; c_1, ..., c_d\}$ can be found in [1].

For a graph Γ with the diameter d and $i \in \{2, .., d\}$ let's denote by Γ_i a graph defined on the same set of vertices where any two vertices u, w are adjacent if the distance between u and w in the graph Γ is i.

If a regular graph of degree k and diameter d has k vertices then the following inequality holds:

$$v \leq 1 + k + k(k-1) + \cdots + k(k-1)^{d-1}$$

Graphs for which the non-strict inequality above turns into equality are called *Moore graphs* (1960). This definition belongs to Hoffman and Singleton who explored graphs with diameters of 2 and 3. The simplest example of Moore graph is a regular $(2d+1)$-polygon.

Damerell [3] (see also [4]) has proven that a graph of degree $k \geq 3$ has diameter 2. In this case $v = k^2+1$ the graph is strongly regular with $\lambda = 0$ and $\mu = 1$ and one of the following statements holds: $k = 2$ and the graph is the

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pentagon, $k = 3$ and the graph is the Petersen graph, $k = 7$ and the graph is the Hoffman-Singleton graph, or $k = 57$. The existence of a Moore graph of degree 57 was unknown.

Ashbacher [5] has proven that a Moore graph of degree 57 is not distance-transitive (the Moore graph of degree 57 sometimes called Ashbacher's graph). G. Higman ([6, Theorem 3.13]) has proven that a Moore graph of degree 57 is not a (vertex?)-symmetric graph. A.A. Makhnev and D.V. Paduchikh explored possible automorphisms of the Moore graph of degree 57.

Jurishich and Vidali [9] have noticed that the existence of a Moore graph of degree $k > 3$ is equivalent to the existence of a distance-regular graph with intersection array $\{k-2, k-3, 2; 1, 1, k-3\}$ (in the case k = 57 we have a distance-regular graph with intersection array $\{55, 54, 2; 1, 1, 54\}$).

In this paper we explore properties of the graph with intersection array $\{55, 54, 2; 1, 1, 54\}$ and claim:

Theorem 1. *A distance-regular graph with intersection array* $\{55, 54, 2; 1, 1, 54\}$ *does not exist.*

Corollary 1. *Strongly regular Moore graph with parameters* $(3250, 57, 0, 1)$ *does not exist.*

This result puts an end to 60 years of research on Moore graphs. The main methods used to prove the results are: the triple intersection number method applied to a graph without the Q-polynomial condition; method of symmetrization of an array of triple intersection numbers (proposed by A.A. Makhnev, the analogue of tensor symmetrization).

2 Auxiliary Results

Our notation is conventional and can be found in [1]. In the proof of the theorem we use *triple intersection numbers* [11].

Let Γ be a distance-regular graph with the diameter d. For vertices u_1, u_2, u_3 and non-negative integers Γ and r_1, r_2, r_3 are non-greater then d define as $\left\{\begin{matrix} u_1u_2u_3 \\ r_1r_2r_3 \end{matrix}\right\}$ a set of vertices $w \in \Gamma$ such that $d(w, u_i) = r_i$.[2]

Let's denote as $\begin{bmatrix} u_1u_2u_3 \\ r_1r_2r_3 \end{bmatrix}$ the number of vertices in $\left\{\begin{matrix} u_1u_2u_3 \\ r_1r_2r_3 \end{matrix}\right\}$. Numbers $\begin{bmatrix} u_1u_2u_3 \\ r_1r_2r_3 \end{bmatrix}$ we refer to as *triple intersection numbers*. For the fixed riplet of vertices u_1, u_2, u_3 instead of $\begin{bmatrix} u_1u_2u_3 \\ r_1r_2r_3 \end{bmatrix}$ we write $[r_1r_2r_3]$. Unfortunately there is no general way to compute $[r_1r_2r_3]$. But in [11] we propose a technique to compute some of $[r_1r_2r_3]$ numbers.

Let u, v, w be vertices of a graph Γ and $W = d(u, v)$, $U = d(v, w)$, $V = d(u, w)$. There is only one vertex $x = u$ such that $d(x, u) = 0$. Therefore $[0jh]$ is equal to 0 or 1. Thus, $[0jh] = \delta_{jW}\delta_{hW}$, $[i0h] = \delta_{iW}\delta_{hU}$, $[ij0] = \delta_{iU}\delta_{jV}$ (??? it is unclear what is δ).

Another set of equations we can get by fixing the distance between any two vertices from $\{u, v, w\}$ and by counting the number of vertices with any distance to the third one (translation note: any distance $\leq d$?):

$$\sum_{l=1}^{d}[ljh] = p^U_{jh} - [0jh], \sum_{l=1}^{d}[ilh] = p^V_{ih} - [i0h], \sum_{l=1}^{d}[ijl] = p^W_{ij} - [ij0] \qquad (+)$$

Wherein some of the triplets disappear. When $|i - j| > W$ or $|i + j| < W$ we have $p^W_{ij} = 0$. Therefore $[ijh] = 0$ for all $h \in \{0, ..., d\}$.

Let $S_{ijh}(u, v, w) = \sum_{r,s,t=0}^{d} Q_{ri}Q_{sj}Q_{th}\begin{bmatrix} uvw \\ rst \end{bmatrix}$. If *Krein* parameter $q^h_{ij} = 0$ then $S_{ijh}(u, v, w) = 0$.

Let's fix vertices u, v, w of a distance-regular graph Γ with the diameter 3 and define $\{ijh\} = \left\{\begin{matrix} uvw \\ ijh \end{matrix}\right\}$, $[ijh] = \begin{bmatrix} uvw \\ ijh \end{bmatrix}$, $[ijh]' = \begin{bmatrix} uwv \\ ihj \end{bmatrix}$, $[ijh]^* = \begin{bmatrix} vuw \\ jih \end{bmatrix}$, $[ijh]^\sim = \begin{bmatrix} wvu \\ hji \end{bmatrix}$ In cases where $d(u, v) = d(u, w) = d(v, w) = 2$ or

[2] Translator's note: I found the definition from [11] to be more precise

$$\left\{\begin{matrix} u_1u_2u_3 \\ r_1r_2r_3 \end{matrix}\right\} := \{v \in \text{vertices}(\Gamma) \mid d(v, u_1) = r_1, d(v, u_2) = r_2, d(v, u_3) = r_3\}$$

$d(u,v) = d(u,w) = d(v,w) = 3$ computation of the parameters $[ijh]' = \begin{bmatrix} uwv \\ ihj \end{bmatrix}$, $[ijh]^* = \begin{bmatrix} vuw \\ jih \end{bmatrix}$, $[ijh]^\sim = \begin{bmatrix} wvu \\ hji \end{bmatrix}$ (symmetrization of the array of triple intersection numbers) can give new ratios to prove the non-existence of the graph.

3 Properties of $\Gamma_3(u)$ graph

In this section the graph Γ is a distance-regular graph with intersection array $\{55, 54, 2; 1, 1, 54\}$. Then Γ has the spectrum $55^1, 7^{1617}, -1^{110}, -8^{1408}, 1 + 55 + 2970 + 110 = 3136$ vertices and dual eigenvalue matrix (translator's note: dual eigenmatrix?):

$$\begin{pmatrix} 1 & 1617 & 110 & 1408 \\ 1 & \frac{1029}{5} & -2 & -\frac{1024}{5} \\ 1 & -\frac{49}{15} & -2 & \frac{64}{15} \\ 1 & \frac{147}{5} & 54 & -\frac{128}{5} \end{pmatrix}$$

Then the graph Γ_3 is 56×56 lattice graph and the neighborhood of the vertex in Γ_3 is a union of two isolated 55-cliques.

Lemma 1. *The intersection numbers of the graph Γ are:*

1. $p^1_{11} = 0, p^1_{12} = 54, p^1_{22} = 2808, p^1_{23} = 108, p^1_{33} = 2$
2. $p^2_{11} = 1, p^2_{12} = 52, p^2_{13} = 2, p^2_{22} = 2811, p^2_{23} = 106, p^2_{33} = 2$
3. $p^3_{12} = 54, p^3_{13} = 1, p^3_{22} = 2862, p^3_{23} = 54, p^3_{33} = 54$

Proof. Direct computation according to [1, Lemma 4.1.7]. Let's fix the vertices u, v, w of the graph Γ and define $\{ijh\} = \begin{Bmatrix} uvw \\ ijh \end{Bmatrix}$, $[ijh] = \begin{bmatrix} uvw \\ ijh \end{bmatrix}$.

Let $\Delta = \Gamma_2(u)$ and $\Lambda = \Delta_2$. Then Λ is a regular graph of degree $p^2_{22} = 2811$ defined on $k_2 = 2970$ vertices. □

Lemma 2. *Let $d(u,v) = 2, d(u,w) = d(v,w) = 1$. Then triple intersection numbers are:*

1. $[122] = 52, [132] = 2$
2. $[212] = 52, [221] = 53, [222] = r_1 + 2650, [223] = -r_1 + 108, [232] = -r_1 + 106, [233] = r_1$
3. $[312] = 2, [322] = -r_1 + 106, [323] = [332] = r_1, [333] = -r_1 + 2$

where $r_1 \in \{0, 1, 2\}$

Proof. Simplification of the set of equations **??**. □

Lemma 3. *Let $d(u,v) = d(u,w) = 2, d(v,w) = 3$. Then triple intersection numbers are:*

1. $[112] = [121] = -r_6 + 1, [113] = [131] = r_6, [122] = r_7, [123] = [132] = r_6 - r_7 + 51, [133] = -2r_6 + r_7 - 49$
2. $[212] = [221] = 52, [213] = [231] = 0, [222] = r_5 + r_6 + 2756 - r_7 = r_9, [223] = [232] = -2r_6 + r_7 + 3, [233] = 2r_6 - r_7 + 102$
3. $[312] = [321] = r_6 + 1, [313] = [331] = 1 - r_6, [322] = 5 - 2r_6, [323] = [332] = r_6, [333] = 1$

where $r_6 \in \{0,1\}$*,* $r_7 \in \{49,...,52\}$*. Then one of the followings is true:*

$$[222] = 2706 \text{ and } [222] = 2r_6 + 50 = r_7$$

or

$$[222] = 2707 \text{ and } [222] = 2r_6 + 51 = r_7 \text{ and } r_6 = 0$$

Proof. Using computer aided simplification of the set of equations **??** we have

$[112] = -r_6+1, [113] = r_6, [121] = -r_4-r_8+54, [122] = r_7, [123] = r_4-r_7+r_8-2, [131] = r_4+r_8-53, [132] = r_6 - r_7 + 51, [133] = -r_4 - r_6 + r_7 - r_8 + 4;$
$[212] = r_5 + r_6 - r_7 - r_9 + 2808, [213] = -r_5 - r_6 + r_7 + r_9 - 2756, [221] = r_4, [222] = r_9, [223] = -r_4 - r_9 + 2811, [231] = -r_4 + 52, [232] = -r_5 - r_6 + r_7 + 3, [233] = r_4 + r_5 + r_6 - r_7 + 50;$
$[312] = -r_5 + r_7 + r_9 - 2755, [313] = r_5 - r_7 - r_9 + 2757, [321] = r_8, [322] = -r_7 - r_9 + 2861, [323] = r_7 - r_8 + r_9 - 2755, [331] = -r_8 + 2, [332] = r_5, [333] = -r_5 + r_8,$

where $r_4 \in \{51,52\}, r_5 \in \{0,1,2\}, r_6 \in \{0,1\}, r_7 \in \{49,...,52\}, r_8 \in \{0,1,2\}, r_9 \in \{2705,...,2710\}$. Thus $2705 \le [222] = r_9 \le 2710$.

In the 56×56 lattice Γ_3 a subgraph $\Gamma_3(u) \cap \Gamma_3(v)$ is 2-co-clique which has intersection with $\Gamma_3(w)$ exactly in one vertex. Therefore $[333] = -r_5 + r_8 = 1$ and $r_5 \in \{0,1\}$.

Same, $\Gamma_3(v) \cap \Gamma_3(w)$ is 54-clique which has intersection just with $\Gamma_3(u)$ in no more then one vertex. Therefore $[133] = -r_4 - r_6 + r_7 - r_5 + 3 \le 1$.

Symmetrization. For the vertex triplet (u,v,w) we have $[122] = r'_7 = r'_7, -r_6 + 1 = [112] = [121]' = -r'_4 - r'_5 + 53$ and $r_4 + r_5 = r'_6 + 52$. Then, $[222] = r_9 = r'_9$, $r_6 = [323] = [332]' = r'_5$ and $r_4 = 52$. Therefore $[133] = -49 - r_6 + r_7 - r_5 \le 1$ and $r_7 \le r_6 + r_5 + 50$.

Then we have $[313] = r_5 - r_7 - r_9 + 2757$, therefore $r_7 + r_9 \le r_5 + 2757$ and $[213] = -r_5 - r_6 + r_7 + r_9 - 2756 \le 1 - r_6$.

Assume $[213] = 1$. Then $r_6 = 0, [233] = r_5 - r_7 + 102$ and $r_5 = r'_5 = 0$, thus $r_7 + r_9 = 2757$.

Because $[133] = r_7 - r_8 - 48 = r'_7 - r'_8 - 48$ then $r_8 = r'_8$. Further, $1 = [112] = [121]' = r'_8$ and $r_8 = 1$. Same, $[233] = r_4 - r_7 + 50$, therefore, $r_4 = r'_4$. We have a contradiction because $51 = [212] = [221]' = r'_4 = 52$. Therefore $[213] = 0$ and $r_7 + r_9 = r_5 + r_6 + 2756$. If $r_6 = 0$, then $[233] = r_5 - r_7 + 102$ and $r_5 = r'_5 = 1$.

In case if $r_6 = 1$, then $[233] = r_5 - r_7 + 103$ and $r_5 = r'_5 = 1$. Thus in either cases we have $r_5 = r_6$ and we have symmetrized array:

$[112] = [121] = -r_6 + 1, [113] = [131] = r_6, [122] = r_7, [123] = [132] = r_6 - r_7 + 51, [133] = -2r_6 + r_7 - 49;$

$[212] = [221] = 52, [213] = [231] = 0, [222] = r_5 + r_6 + 2756 - r_7 = r_9, [223] = [232] = -2r_6 + r_7 + 3, [233] = 2r_6 - r_7 + 102;$

$[312] = [321] = r_6 + 1, [313] = [331] = 1 - r_6, [322] = 5 - 2r_6, [323] = [332] = r_6, [333] = 1$

where $r_6 \in \{0,1\}, r_7 \in \{49,...,52\}$.

If $[222] \ge 2708$ then $r_7 \le 48 + 2r_6$ and the equation $[133] = -2r_6 + r_7 - 49$ gives a contradiction. If $[222] = 2705$, then $r_7 = 51 + 2r_6$, we have contradiction with the fact $[133] = -2r_6 + r_7 - 49 \le 1$. So, either $[222] = 2706$ and $[222] = r_5 + r_6 + 50 = r_7$ or $[222] = 2707, r_5 + r_6 + 51 = r_7$ and $r_5 + r_6 = 0$.

□

Lemma 4. *Let* $d(u,v) = d(u,w) = 2$*,* $d(v,w) = 1$*. Then triple intersection numbers are:*

1. $[122] = 51, [123] = [132] = 0, [133] = 2$*;*
2. $[212] = [221] = 51, [222] = 2654, [223] = [232] = 106, [233] = 0$*;*
3. $[312] = [321] = 2, [322] = 102, [323] = [332] = 2, [333] = 0.$

Further, $[222] = 2654.$

Proof. Using simplification of the formula set **??** we get:

$[122]=r_3, [123]=[132]=-r_3+51, [133]=r_3-49;$

$[212]=[221]=51, [222]=-r_2-r_3+2705, [223]=[232]=r_2+r_3+55, [233]=-r_2-r_3+51;$

$[312]=[321]=2, [322]=r_2+102, [323]=[332]=-r_2+2, [333]=r_2$

where $r_2\in\{0,1,2\}, r_3\in\{49,50,51\}$.

In the 56×56-lattice Γ_3 a subgraph $\Gamma_3(u)\cap\Gamma_3(w)$ is 2-co-clique. It has no intersection with $\Gamma_3(v)$. Therefore $[333]=r_2=0$.

Let's compute a number d of pairs of vertices (y,z) with the distance 3 in the graph Λ where $y\in\begin{Bmatrix}uv\\21\end{Bmatrix}, z\in\begin{Bmatrix}uv\\23\end{Bmatrix}$.

On the one hand we have $[213]=0$ and $d=0$ by the Lemma **??**. On the other hand we have $d=52[233]=0$, therefore $[233]=0$ and $r_3=51$. Thereby we have the equations from the lemma statement. ☐

Lemma 5. *Let $d(u,v)=d(u,w)=(v,w)=2$. Then triple intersection numbers are:*
$[111]=-r_{16}-r_{11}+1, [112]=r_{11}, [113]=r_{16}, [121]=r_{14}, [122]=-r_{12}-r_{14}+52, [123]=r_{12}, [131]=-r_{14}+r_{16}+r_{11}, [132]=r_{12}+r_{14}-r_{11}, [133]=-r_{12}-r_{16}+2$*;*

$[211]=r_{15}+r_{16}-r_{17}+50, [212]=r_{17}, [213]=-r_{15}-r_{16}+2, [221]=-r_{15}-r_{16}+r_{17}-r_{10}+2, [222]=r_{12}+r_{16}-r_{17}+r_{10}+2704, [223]=-r_{12}+r_{15}+104, [231]=r_{10}, [232]=-r_{12}-r_{16}-r_{10}+106, [233]=r_{12}+r_{16}$*;*

$[311]=-r_{15}+r_{17}+r_{11}-50, [312]=-r_{17}-r_{11}+52, [313]=r_{15}, [321]=-r_{14}+r_{15}+r_{16}-r_{17}+r_{10}+50, [322]=r_{14}-r_{16}+r_{17}-r_{10}+54, [323]=-r_{15}+2, [331]=r_{14}-r_{16}-r_{10}-r_{11}+2, [332]=-r_{14}+r_{16}+r_{10}+r_{11}, [333]=0$

where $r_{10},r_{12},r_{15}\in\{0,1,2\}, r_{11},r_{14},r_{16}\in\{0,1\}, r_{17}\in\{49,...,52\}$.

Further, $2652\le[222]=r_{12}+r_{16}-r_{17}+r_{10}+2704\le 2659$.

Proof. With computer-aided simplification of **??** we have:

$[111]=-r_{16}-r_{11}+1, [112]=r_{11}, [113]=r_{16}, [121]=r_{14}, [122]=-r_{12}-r_{14}+52, [123]=r_{12}, [131]=-r_{14}+r_{16}+r_{11}, [132]=r_{12}+r_{14}-r_{11}, [133]=-r_{12}-r_{16}+2;$

$[211]=r_{15}+r_{16}-r_{17}+50, [212]=r_{17}, [213]=-r_{15}-r_{16}+2, [221]=-r_{15}-r_{16}+r_{17}-r_{10}+2, [222]=r_{12}-r_{13}+r_{16}-r_{17}+r_{10}+2704, [223]=-r_{12}+r_{13}+r_{15}+104, [231]=r_{10}, [232]=-r_{12}+r_{13}-r_{16}-r_{10}+106, [233]=r_{12}-r_{13}+r_{16};$

$[311]=-r_{15}+r_{17}+r_{11}-50, [312]=-r_{17}-r_{11}+52, [313]=r_{15}, [321]=-r_{14}+r_{15}+r_{16}-r_{17}+r_{10}+50, [322]=r_{13}+r_{14}-r_{16}+r_{17}-r_{10}+54, [323]=-r_{13}-r_{15}+2, [331]=r_{14}-r_{16}-r_{10}-r_{11}+2, [332]=-r_{13}-r_{14}+r_{16}+r_{10}+r_{11}, [333]=r_{13}$

where $r_{10},r_{12},r_{13},r_{15}\in\{0,1,2\}, r_{11},r_{14},r_{16}\in\{0,1\}, r_{17}\in\{49,...,52\}$.

In the 56×56-lattice Γ_3 a subgraph $\Gamma_3(u)\cap\Gamma_3(w)$ is 2-co-clique. It has no intersection with $\Gamma_3(v)$. Therefore $[333]=r_{13}=0$. Thereby we have the required equations.

Let's note that $r_{12}+r_{16}\le 2$, therefore $2652\le[222]=r_{12}+r_{16}-r_{17}+r_{10}+2704\le 2659$. Lemma is proved. ☐

For number e of edges between $\Lambda(v)$ and $\Lambda_2(v)$ in a graph Λ the following inequalities hold: $424844=52\cdot 2654+106\cdot 2706\le e\le 52\cdot 2654+106\cdot 2707=424950$. Therefore $151.136\le 2810-\lambda\le 151.174$ and $2658.826\le\lambda\le 2658.864$, where λ is a mean value of the parameter $\lambda(\Lambda)$.

Let's note that the mean value $\lambda(\Lambda)$ is very close to the upper bound 2659.

4 Proof of the theorem

Let Γ be a distance-regular graph with an intersection array $\{55, 54, 2; 1, 1, 54\}$. Here we will prove the Theorem **??**. Let's fix (u, v, w) vertices of the graph Γ and define $\{ijh\} = \begin{Bmatrix} uvw \\ ijh \end{Bmatrix}$, $[ijh] = \begin{bmatrix} uvw \\ ijh \end{bmatrix}$. Let $\Delta = \Gamma_2(u)$ and $\Lambda = \Delta_2$. Then Λ is a regular graph with the degree $p^2_{22} = 2811$ defined on $k_2 = 2970$ vertices.

Lemma 6. *Let $d(u, v) = d(u, w) = (v, w) = 2$. Then the following statements hold:*

1. $r_{11} = r'_{14}, r'_{10} = [213] = r^*_{12}, r_{15} + r_{16} + r'_{10} = 2$ *and* $-r_{17} + 52 = r^*_{12} + r^*_{14}$
2. *if* $r'_{10} = 2$ *then* $r_{10} = 2$ *and one of the following cases is true:*

   either $r_{11} = 0$ *and*
   $[111] = 1, [112] = [113] = [121] = [131] = 0, [122] = 50, [123] = [132] = 2, [133] = 0;$
   $[211] = 0, [212] = r_{17} = 50, [213] = [231] = 2, [221] = 50, [222] = 2658, [223] = [232] = 102, [233] = 2;$
   $[311] = 0, [312] = [321] = 2, [313] = 0, [322] = 102, [323] = [332] = 2, [331] = [333] = 0,$

   or $r_{11} = 1$ *and*
   $[111] = 0, [112] = [121] = 1, [113] = [131] = 0, [122] = 49, [123] = [132] = 2, [133] = 0;$
   $[211] = 1, [212] = [221] = r_{17} = 49, [213] = [231] = 2, [222] = 2659, [223] = [232] = 102, [233] = 2;$
   $[311] = 0, [312] = [321] = 2, [313] = [331] = 0, [322] = 102, [323] = [332] = 2, [333] = 0.$

Proof. Symmetrization. From $[111] = -r_{16} - r_{11} + 1$ we have $r_{16} + r_{11} = r'_{16} + r'_{11} = r^*_{16} + r^*_{11} = r^{\sim}_{16} + r^{\sim}_{11}$. Then $[112] = r_{11} = r{*}_{11}, [113] = r_{16} = r^*_{16}, [121] = r_{14} = r^{\sim}_{14}, [212] = r_{17} = r^{\sim}_{17}, [313] = r_{15} = r^{\sim}_{15}, r_{11} = [112] = [121]' = r'_{14}, r'_{10} = [213] = r^*_{12}$.

From $[122] = -r_{12} - r_{14} + 52$ follows $r_{12} + r_{14} = r'_{12} + r'_{14}, [233] = r_{12} + r_{16} = r'_{12} + r'_{16}, r_{10} = [231] = [213]' = -r'_{15} - r'_{16} + 2$ and $r'_{10} + r_{15} + r_{16} = 2$.

Similarly, $-r_{17} - r_{11} + 52 = [312] = [132]^* = r^*_{12} + r^*_{14} - r^*_{11}$ and $-r_{17} + 52 = r^*_{12} + r^*_{14}$. The statement (1) is proved.

Let's assume $r'_{10} = 2$. Then $r^*_{12} = 2, r_{15} + r_{16} = 0$ and $r_{17} + r^*_{14} = 50$. Because $[131] = -r_{14} + r_{11}$ then in the case of $r_{11} = 0$ we have $r_{14} = 0, [132] = r_{12} + r_{14} - r_{11} = r_{12} = r'_{12}, [211] = -r_{17} + 50, [311] = r_{17} - 50$. Now $-r_{12} - r_{10} + 106 = [232] = [223]' = -r'_{12} + 104 = -r_{12} + 104$, therefore $r_{10} = 2, [212] = [221] = r_{17} = 50, r_{17} = [122] = -r_{12} + 52, r_{12} = 2$ and

$[111] = 1, [112] = [113] = [121] = [131] = 0, [122] = 50, [123] = [132] = 2, [133] = 0;$
$[211] = 0, [212] = r_{17} = 50, [213] = [231] = 2, [221] = 50, [222] = 2658, [223] = [232] = 102, [233] = 2;$
$[311] = 0, [312] = [321] = 2, [313] = 0, [322] = 102, [323] = [332] = 2, [331] = [333] = 0.$

In case where $r_{11} = 1$ we have $[233] = r_{12} = r'_{12}$, thereby $r_{12} = [123] = [132] = [132] = r_{12} + r_{14} - 1, r_{14} = 1, -r_{17} + 51 = [312] = [321]' = -r'_{17} + r'_{10} + 49, r_{17} = r'_{17}, r_{10} = 2$ and $-r_{17} + 51 = [312] = [132]^* = r^*_{12}$.

We have $[222] = r_{12} + 2657$, thereby $r_{12} = r^*_{12}, r_{12} + r_{17} = 51$ and
$[111] = 0, [112] = [121] = 1, [113] = [131] = 0, [122] = 49, [123] = [132] = 2, [133] = 0;$
$[211] = 1, [212] = [221] = r_{17} = 49, [213] = [231] = 2, [222] = 2659, [223] = [232] = 102, [233] = 2;$
$[311] = 0, [312] = [321] = 2, [313] = [331] = 0, [322] = 102, [323] = [332] = 2, [333] = 0.$

Now $[212] = [221] = r_{17} = r^*_{17}, r_{17} = [122] = -r_{12} + 51$ and $r_{12} = 2$. Lemma is proved. □

The following idea from Vidali can be used to simplify the proof of the theorem. Let $d(u, v) = 2$. Then $[u]$ has one element in each row and each column of the 55×55-lattice $\Gamma_3 - (\{u\} \cap \Gamma_3(u))$ Let $\{x, y\} = [u] \cap \Gamma_3(v)$. Then there is only one vertex in the subgraph $\Gamma_3(x) \cap \Gamma_3(y) - \{v\}$. We have a contradiction with the fact $[133] = 2]$ (see Lemma **??**) for any three vertices (u, v, w) with $d(u, v) = d(u, w) = 2, d(v, w) = 1$.

Theorem **??** is proved.

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