National University of Singapore

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Assignment for EE5101/ME5401 Linear Systems

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Assignment for EE5101/ME5401 Linear System

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October 28, 2022

Abstract

This is abstract

Plant description Control and observer design method description Design details Simulation results Possible comparison Comments and discussion Modification and refinements

Keywords: Keywords1; Keywords2;...

Contents

1 Introduction

Combining the high safty of car and low cost of motor, self-sustaining two-wheeled vehicle (Fig. 1) has drawn research interest in universities. Though most of study are still in experimental stage, different control methods have been tested on this mortorcycle-like vehicle and some exprimental results have been released online.

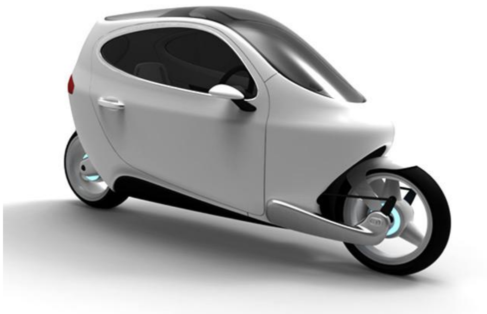

Figure 1: Two-wheeled self-balancing car

The two-wheeled vehicle is composed of a cart system, a steering system (front part) and a body (rear part). Driven by the DC servo motor, the cart and steering systems allow drivers to change cart position and handle angle. To simplify the modeling in this mini-project, the self-balance two-wheeled vehicle is assumed to be stationary and the mechanical structure is given in Fig. 2.

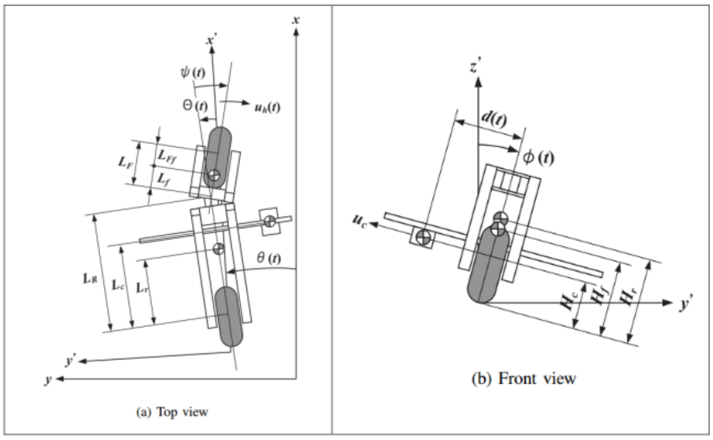

Figure 2: Simple two-wheeled vehicle model

Then, state space model is established

$$
\begin{aligned}
\dot{x} &= Ax + Bu \\
y &= Cx
\end{aligned}
\tag{1}
$$

where the state is designed according to the cart position, handle angle, bike angle and their corresponding velocities respectively

$$
x = \begin{bmatrix} d(t) & \phi(t) & \psi(t) & \dot{d}(t) & \dot{\phi}(t) & \dot{\psi}(t) \end{bmatrix}^T \tag{2}
$$

and the relative matrices and input vectors are

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 6.5 & -10 & -\alpha & 0 & 0 \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ 5 & -3.6 & 0 & 0 & 0 & -\gamma \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \beta & 11.2 \\ b_{51} & b_{52} \\ 40 & \delta \end{bmatrix} \tag{3}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, D = \begin{bmatrix} u_c(t) & u_h(t) \end{bmatrix}^T \tag{4}$$

The parameters in matrix A and B can be calculated as

$$\begin{aligned} a_{51} &= -\frac{M_c g}{den}, a_{52} = \frac{(M_f H_f + M_r H_r + M_c H_c)g}{den}, a_{53} = \frac{(M_r L_r L_F + M_c L_c L_F + M_f L_{Ff} L_R)g}{(L_R + L_F)den} \\ a_{54} &= -\frac{M_c H_c \alpha}{den}, a_{55} = -\frac{\mu_x}{den}, a_{56} = \frac{M_f H_f L_{Ff} \gamma}{den} \\ b_{51} &= \frac{M_c H_c \alpha}{den}, b_{52} = -\frac{M_f H_f L_{Ff} \delta}{den}, den = M_f H_f^2 + M_r H_r^2 + M_c H_c^2 + J_x \end{aligned} \tag{5}$$

The physical parameters appering in Eq. 4 and Eq. 5 are usually measured mannually by experiments and they are given in appendix.

After obtaining the linear system model and giving a step reference signal for each input channel, two basic response performance specifications are required to meet as follows

- The overshoot of system should below 10%.
- The 2% settling time should be less than 5 seconds.

The rest of the report is structured in the following manner: In section 2 a state feedback controller is designed using pole placement methods, followed by a discussion on effects of different pole choices. Section 3 concludes a Linear Quadratic Regulator controller design as well as the techique on choosing the weighting matrixes Q and R. Section 4 introduces a state observer based on the former LQR control system and its performance is evaluated by monitoring the state estimation error. Section 5 describes a decoupling controller for a 2-input-2-output system. Section 6 designes a controller enabling the output of plant to track the reference signal regardless of step disturbances in input channel. Section 7 tries the integral control method to maintain outputs at a constatnt set point with zero steady-state error. Section 8 concludes the work.

2 State feedback controller by pole placement method

2.1 Method description

Given the acquired six-order state space model, feedback controller is required to make the system stable. In this section, pole placement method will be used to obtain the feedback gain to meet control requirements. Firstly considering a second-order system, its overshoot and 2% settling time of a step response is given as follows

$$M_p = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}, t_s = \frac{4.0}{\zeta\omega_n} \tag{6}$$

where ζ and ω_n are damping ratio and natural frequency respectively. Now that the overshoot should below 10% and settling time less than 5s, we set $\zeta = 0.8$ and $\zeta\omega_n = 2.5$ in this case, and then the two pole of second-order system is calculated as

$$\lambda_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} \tag{7}$$

To assure the stability of system, the two dominant poles should have little smaller real parts, so we set them at $\lambda_{1,2} = -2.5 \pm j1.875$. The other 4 poles are chosen 4-5 times lefter than the two dominant poles. Then the desired characteristic polynomial becomes

$$\begin{aligned}\phi_d(s) &= (s-\lambda_1)(s-\lambda_2)(s-\lambda_3)(s-\lambda_4)(s-\lambda_5)(s-\lambda_6)\\ &= s^6 + a_5 s^5 + a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0\end{aligned} \tag{8}$$

The desired closed-loop matrix can be written in controllable canonical form

$$A_d = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -a_5 & -a_4 & -a_3 & -a_2 & -a_1 & -a_0 \end{bmatrix} \tag{9}$$

Then the remainning task is transform the system with state feedback also to the controllable canonical form. First compute the controllability matrix and check if it is full rank

$$W_c = \left\{B \quad AB \quad A^2B \quad A^3B \quad A^4B \quad A^5B\right\} \tag{10}$$

Then select 6 independent vectors from W_c in the strict order from left to right (the first six column vectors in this case) and group them in matrix X in the following form

$$X = \left\{b_1 \quad b_2 \quad Ab_1 \quad Ab_2 \quad A^2b_1 \quad A^2b_2\right\} \tag{11}$$

Also compute the inverse of X and write it as the following form

$$X^{-1} = \left[q_1^T \quad q_2^T \quad q_3^T \quad q_4^T \quad q_5^T \quad q_6^T\right]^T \tag{12}$$

Because that the system has multiple input channels, choose the $3^{th}(d_1 = 3)$ and $6^{th}(d_1 + d_2 = 6)$ row vectors of X^{-1} and construct transformation matrix T as

$$T = \left[q_3^T \quad q_3^T A \quad q_3^T A^2 \quad q_6^T \quad q_6^T A \quad q_6^T A^2\right]^T \tag{13}$$

Define the feedback gain matrix $\bar{K} \in \mathbb{R}^{2\times 6}$ with 12 unknown variables, we can solve it by establishing the equation

$$\bar{A} - \bar{B}\bar{K} = A_d \tag{14}$$

where $\bar{A} = TAT^{-1}$, $\bar{B} = TB$

After obtainning the matrix $\bar{K}$, we can finally get the original feedback gain matrix $K = \bar{K}T$. Substitute the control law $u = -Kx + r$ into Eq. 1 and the new closed-loop system becomes

$$\begin{aligned}\dot{x} &= (A - BK)x + Br\\ y &= Cx\end{aligned} \tag{15}$$

2.2 Simulation results and refinements

In this report, the 6 poles are selected as $\lambda_{1,2} = -2.5 \pm j1.875, \lambda_3 = -10, \lambda_4 = -10.75, \lambda_5 = -11.5$ and $\lambda_6 = -12.5$. The final feedback gain matrix K is calculated to be

$$K = \begin{bmatrix} -15.68 & 12.20 & 6.27 & 4.90 & -6.21 & -2.26 \\ 250960 & -21110 & -107660 & -845.58 & 9043.2 & 412.98 \end{bmatrix}$$

The number of elements differes a lot and are relatively weird. Temporarily we use the feedback matrix and simulate the closed-loop system with time domain from 0 to 10 seconds. Given a step signal for each input channel, the output response is showed in Fig. 3. Sadly though the system becomes stable with a short settling time, there are huge overshoots in both cases and the stable points are also impossible in real cases.

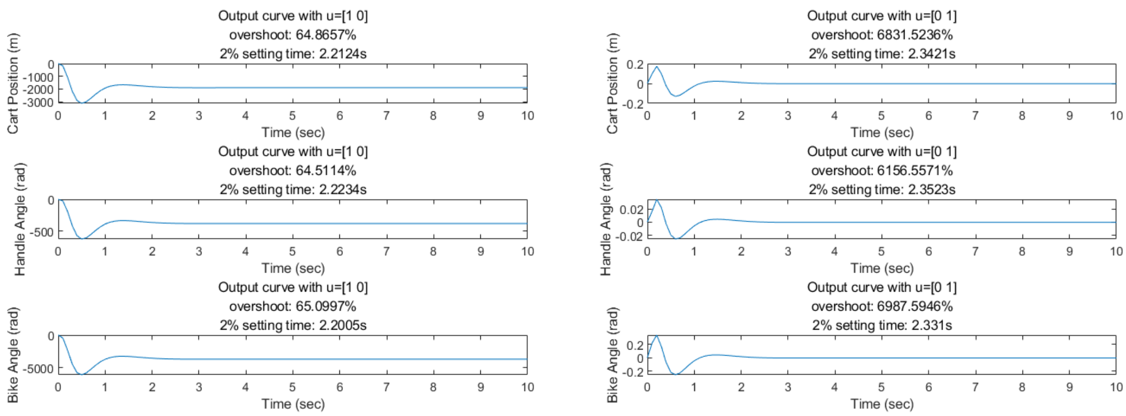

Figure 3: Step response for the closed-loop system

The results are unacceptable and need further refinements. Obviously, the bad results are closely realted to the feedback gain matrix K, which influnces the system matrix $A - BK$ of closed-loop system. In another point of view, we can find infinite K to place the same poles for a plant as there are infinite system matrixes with the same eigenvalues.

Actuallly when tackling pole placement problems for high order system, some pole choices will result in sensitivity problems, which leads to a large gain as in our former method. One robust algorithm was proposed in [1] and has been applied in matlab function. The aim of this algorithm is to find the feedback gain matrix K that makes $A - BK$ the most robust. One method to measure the robustness of a matrix is to use condition number

$$c_j = \frac{1}{s_j} = \frac{\|y_j\|_2 \|x_j\|_2}{|y_j^T x_j|} \tag{16}$$

where x_j and y_j are the left and right eigenvalues of $A - BK$ respectively. It can be proved that c_j has a upper bound i.e. $max_j c_j \leq \kappa_2(X) \equiv \|X\|_2 \left\|X^{-1}\right\|_2$, where $X = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}$ is the eigenvector matrix. The maximum value $v = max(c_j)$ of all condition numbers is chosen and the system is said to be the most robust if v is the smallest. Then the problem becomes how to find the minimum upper bound of condition numbers.

The minimum upper bound can ben found by cauchy inequality and the result is given as follows

$$min(\kappa_2(X)) \leq \frac{\kappa_2(S)}{\sqrt{k}} \tag{17}$$

S is orthogonal normal basis of the null space of $N(U_1^T(A - \lambda I))$, where U_1 is the orthogonal basis of null space of input matrix B, λ are eigenvalues of matrix $A - BK$ and k is the dimensions of S.

QR decomposition or singular value decomposition (SVD) can be used to calculate the orthogonal basis. Finally, feedback gain matrix K can be obtained by

$$K = Z^{-1}U_0^T(A - X\Lambda X^{-1}) \tag{18}$$

where $\Lambda = diag(\lambda_1, \lambda_2, ..., \lambda_n)$, Z is the Q matrix of B and U_0 is also orthogonal basis of B.

The same poles are chosen but we use the new method to calculate feedback matrix K', the result is given below

$$K' = \begin{bmatrix} 8.7165 & -3.6746 & -1.8207 & 0.5685 & -0.4985 & -0.0880 \\ -11.2971 & 10.6474 & 5.4543 & -0.8739 & 1.4368 & 0.3794 \end{bmatrix}$$

The step response for the new closed-loop system is shown in Fig. 4 and both the overshoot and settling time meet the requirements. Now assume that all six states can be observed, their response with non-zero initial condition x_0 to zero external inputs is given in Fig. 5. It is clearly shown that all states are stable and no weired values apper.

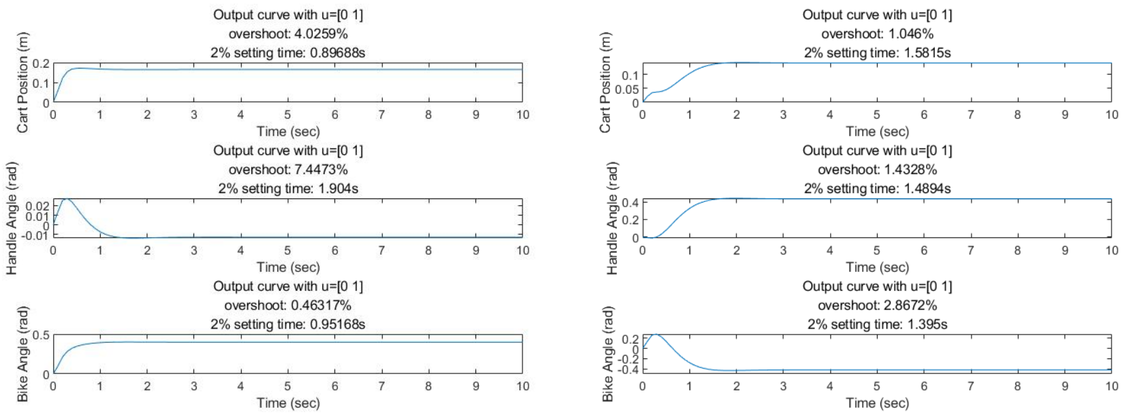

Figure 4: Step response for the new closed-loop system

2.3 Discussion

With refinements on feedback gain matrix K, the overshoot of the closed-loop system is largely cut down as well as its sensitivity to noise. On the other side, the system performance is also largely affected by different pole choosing strategies. Although one can always stablize the system and make system response faster by designing poles on the very left plain, the input cost will increase in this case. To investigate the influence of poles, two more groups of poles are used. The new poles are listed as followings

$$p' = \begin{bmatrix} -1.6 + j0.96 & -1.6 - j0.96 & -6.4 & -6.88 & -7.36 & -8 \end{bmatrix}$$
$$p'' = \begin{bmatrix} -4 + j3 & -4 - j3 & -16 & -17.2 & -18.4 & -20 \end{bmatrix}$$

It is clearly shown in Fig. 6, 7 and 8 that the larger negative real parts the pole has, the more control effect is required. However if the designed poles are too right, large overshoot or long settling time may be expected to appear (as is shown in Fig. 9).

Pole placement methods have the magic of easily stabilizing the system and ensuring the expected design performance by using state feedback controller. On the other hand, poles of systems are hard to select and have no strict standards. In next section, linear quadratic regulator will be introduced to solve the trade-off problem between system performance and control cost.

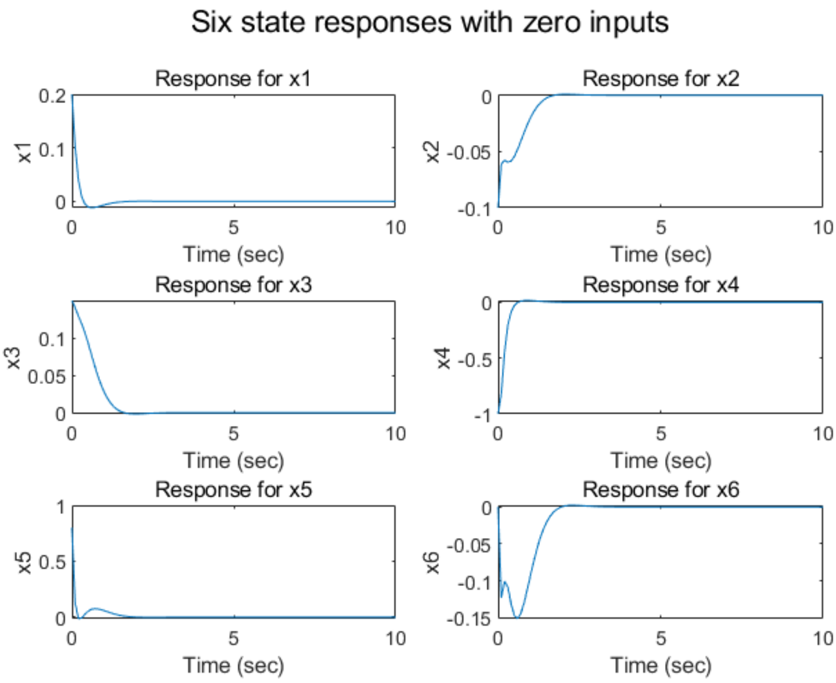

Figure 5: All state responses with zero inputs

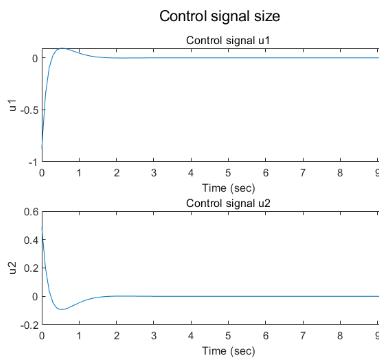

Figure 6: Control signal size with p

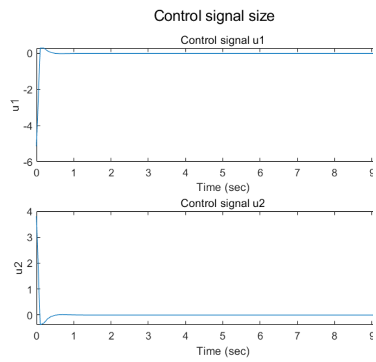

Figure 7: Control signal size with p'

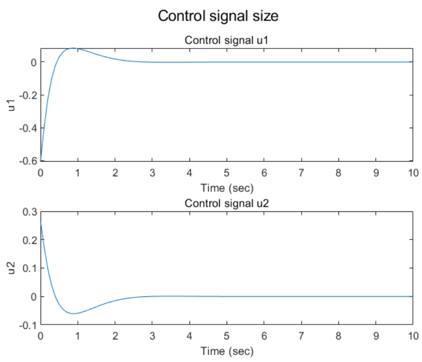

Figure 8: Control signal size with p''

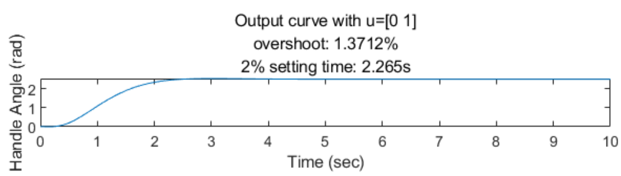

Figure 9: Large overshoot show when given p'' with small negative real parts

3 Linear quadratic regulator controller

3.1 Method description

Linear quadratic resgulator (LQR) follows the theory of optimal control, which is aimed to minimized the cost function. Given a multi-input-multi-output system and consider both control performance and input cost, the cost function can be defined as

$$J = \frac{1}{2}\int_0^{\infty}(x^T Q x + u^T R u)dt \tag{19}$$

The Q and R in Eq. 19 are weighting matrixes representing cost on response speed and energy efficiency respectively. After selecting the appropriate weighting matrix, a state feedback controller $u = -Kx + r$ can be designed to minimize the cost function, where the gain matrix $K = R^{-1}B^T P$ and matrix P is given by solving the Riccati equation

$$A^T P + PA + Q - PBR^{-1}B^T P = 0 \tag{20}$$

Then the problem becomes solving the algebraic matrix riccati equation. One method is to use an eigenvalue-eigenvector based algorithm. First a $2n \times 2n$ matrix is derived

$$\Gamma = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \tag{21}$$

Then n stable eigenvalues of Γ are found as well as their corresponding eigenvectors. Write the eigenvectors in the form of $\begin{pmatrix} v_i & u_i \end{pmatrix}^T, i = 1, 2, \cdots, n$. Then the P matrix in riccati equation can be calculated by

$$P = \begin{pmatrix} \mu_1 & \cdots & \mu_n \end{pmatrix} \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}^{-1} \tag{22}$$

3.2 Simulation results

3.3 Discussion

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4 Conclusion

Acknowledgments

References

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