Appendix A
Basics of Linear Algebra

A.1 Introduction

In this appendix we summarize some basic principles of linear algebra [1]–[4] that are needed to understand the derivation and analysis of the optimization algorithms and techniques presented in the book. We state these principles without derivations. However, a reader with an undergraduate-level linear-algebra background should be in a position to deduce most of them without much difficulty. Indeed, we encourage the reader to do so as the exercise will contribute to the understanding of the optimization methods described in this book.

In what follows, R^n denotes a vector space that consists of all column vectors with n real-valued components, and C^n denotes a vector space that consists of all column vectors with n complex-valued components. Likewise, $R^{m\times n}$ and $C^{m\times n}$ denote spaces consisting of all $m \times n$ matrices with real-valued and complex-valued components, respectively. Evidently, $R^{m\times 1} \equiv R^m$ and $C^{m\times 1} \equiv C^m$. Boldfaced uppercase letters, e.g., $\mathbf{A}$, $\mathbf{M}$, represent matrices, and boldfaced lowercase letters, e.g., $\mathbf{a}$, $\mathbf{x}$, represent column vectors. $\mathbf{A}^T$ and $\mathbf{A}^H = (\mathbf{A}^*)^T$ denote the transpose and complex-conjugate transpose of matrix $\mathbf{A}$, respectively. $\mathbf{A}^{-1}$ (if it exists) and $\det(\mathbf{A})$ denote the inverse and determinant of square matrix $\mathbf{A}$, respectively. The identity matrix of dimension n is denoted as $\mathbf{I}_n$. Column vectors will be referred to simply as vectors henceforth for the sake of brevity.

A.2 Linear Independence and Basis of a Span

A number of vectors $\mathbf{v}_1,\ \mathbf{v}_2,\ \ldots,\ \mathbf{v}_k$ in R^n are said to be *linearly independent* if

$$\sum_{i=1}^{k} \alpha_i \mathbf{v}_i = \mathbf{0} \tag{A.1}$$

only if $\alpha_i = 0$ for $i = 1,\ 2,\ \ldots,\ k$. Vectors $\mathbf{v}_1,\ \mathbf{v}_2,\ \ldots,\ \mathbf{v}_k$ are said to be *linearly dependent* if there exit real scalars α_i for $i = 1,\ 2,\ \ldots,\ k$, with at least one nonzero α_i, such that Eq. (A.1) holds.

A subspace $\mathcal{S}$ is a subset of R^n such that $\mathbf{x} \in \mathcal{S}$ and $\mathbf{y} \in \mathcal{S}$ imply that $\alpha\mathbf{x} + \beta\mathbf{y} \in \mathcal{S}$ for any real scalars α and β. The set of all linear combinations of vectors $\mathbf{v}_1,\ \mathbf{v}_2,\ \ldots,\ \mathbf{v}_k$ is a subspace called the *span* of $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \ldots,\ \mathbf{v}_k\}$ and is denoted as $\text{span}\{\mathbf{v}_1,\ \mathbf{v}_2,\ \ldots,\ \mathbf{v}_k\}$.

Given a set of vectors $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \ldots,\ \mathbf{v}_k\}$, a subset of r vectors $\{\mathbf{v}_{i_1},\ \mathbf{v}_{i_2},\ \ldots,\ \mathbf{v}_{i_r}\}$ is said to be a maximal linearly independent subset if (a) vectors $\mathbf{v}_{i_1},\ \mathbf{v}_{i_2},\ \ldots,\ \mathbf{v}_{i_r}$ are linearly independent, and (b) any vector in $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \ldots,\ \mathbf{v}_k\}$ can be expressed as a linear combination of $\mathbf{v}_{i_1},\ \mathbf{v}_{i_2},\ \ldots,\ \mathbf{v}_{i_r}$. In such a case, the vector set $\{\mathbf{v}_{i_1},\ \mathbf{v}_{i_2},\ \ldots,\ \mathbf{v}_{i_r}\}$ is called a *basis* for $\text{span}\{\mathbf{v}_1,\ \mathbf{v}_2,\ \ldots,\ \mathbf{v}_k\}$ and integer r is called the *dimension* of the subspace The dimension of a subspace $\mathcal{S}$ is denoted as $\dim(\mathcal{S})$.

Example A.1 Examine the linear dependence of vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 3 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ -7 \\ 7 \\ 2 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_4 = \begin{bmatrix} -1 \\ 5 \\ -1 \\ -2 \end{bmatrix}$$

and obtain a basis for $\text{span}\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \mathbf{v}_4\}$.

Solution We note that

$$3\mathbf{v}_1 + 2\mathbf{v}_2 - 2\mathbf{v}_3 - 3\mathbf{v}_4 = \mathbf{0} \tag{A.2}$$

Hence vectors $\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3$, and $\mathbf{v}_4$ are linearly dependent. If

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 = \mathbf{0}$$

then

$$\begin{bmatrix} \alpha_1 \\ -\alpha_1 + 2\alpha_2 \\ 3\alpha_1 \\ -\alpha_2 \end{bmatrix} = \mathbf{0}$$

which implies that $\alpha_1 = 0$ and $\alpha_2 = 0$. Hence $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent. We note that

$$\mathbf{v}_3 = 3\mathbf{v}_1 - 2\mathbf{v}_2 \tag{A.3}$$

and by substituting Eq. (A.3) into Eq. (A.2), we obtain

$$-3\mathbf{v}_1 + 6\mathbf{v}_2 - 3\mathbf{v}_4 = \mathbf{0}$$

i.e.,

$$\mathbf{v}_4 = -\mathbf{v}_1 + 2\mathbf{v}_2 \tag{A.4}$$

Thus vectors $\mathbf{v}_3$ and $\mathbf{v}_4$ can be expressed as linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$. Therefore, $\{\mathbf{v}_1,\ \mathbf{v}_2\}$ is a basis of span$\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \mathbf{v}_4\}$.

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A.3 Range, Null Space, and Rank

Consider a system of linear equations

$$\mathbf{A}\mathbf{x} = \mathbf{b} \tag{A.5}$$

where $\mathbf{A} \in R^{m\times n}$ and $\mathbf{b} \in R^{m\times 1}$. If we denote the ith column of matrix $\mathbf{A}$ as $\mathbf{a}_i \in R^{m\times 1}$, i.e.,

$$\mathbf{A} = [\mathbf{a}_1\ \mathbf{a}_2\ \cdots\ \mathbf{a}_n]$$

and let

$$\mathbf{x} = [x_1\ x_2\ \ldots\ x_n]^T$$

then Eq. (A.5) can be written as

$$\sum_{i=1}^{n} x_i \mathbf{a}_i = \mathbf{b}$$

It follows from the above expression that Eq. (A.5) is solvable if and only if

$$\mathbf{b} \in \text{span}\{\mathbf{a}_1,\ \mathbf{a}_2,\ \ldots,\ \mathbf{a}_n\}$$

The subspace span$\{\mathbf{a}_1,\ \mathbf{a}_2,\ \ldots,\ \mathbf{a}_n\}$ is called the *range* of $\mathbf{A}$ and is denoted as $\mathcal{R}(\mathbf{A})$. Thus, Eq. (A.5) has a solution if and only if vector $\mathbf{b}$ is in the range of $\mathbf{A}$.

The dimension of $\mathcal{R}(\mathbf{A})$ is called the *rank* of $\mathbf{A}$, i.e., $r = \text{rank}(\mathbf{A}) = \dim[\mathcal{R}(\mathbf{A})]$. Since $\mathbf{b} \in \text{span}\{\mathbf{a}_1,\ \mathbf{a}_2,\ \ldots,\ \mathbf{a}_n\}$ is equivalent to

$$\text{span}\{\mathbf{b},\ \mathbf{a}_1,\ \ldots,\ \mathbf{a}_n\} = \text{span}\{\mathbf{a}_1,\ \mathbf{a}_2,\ \ldots,\ \mathbf{a}_n\}$$

we conclude that Eq. (A.5) is solvable if and only if

$$\text{rank}(\mathbf{A}) = \text{rank}([\mathbf{A}\ \mathbf{b}]) \tag{A.6}$$

It can be shown that $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}^T)$. In other words, *the rank of a matrix is equal to the maximum number of linearly independent columns or rows*.

Another important concept associated with a matrix $\mathbf{A} \in R^{m\times n}$ is the *null space* of $\mathbf{A}$, which is defined as

$$\mathcal{N}(\mathbf{A}) = \{\mathbf{x} : \ \mathbf{x} \in R^n, \ \mathbf{A}\mathbf{x} = \mathbf{0}\}$$

It can be readily verified that $\mathcal{N}(\mathbf{A})$ is a subspace of R^n. If $\mathbf{x}$ is a solution of Eq. (A.5) then $\mathbf{x} + \mathbf{z}$ with $\mathbf{z} \in \mathcal{N}(\mathbf{A})$ also satisfies Eq. (A.5). Hence Eq. (A.5) has a unique solution only if $\mathcal{N}(\mathbf{A})$ contains just one component, namely, the zero vector in R^n. Furthermore, it can be shown that for $\mathbf{A} \in R^{m\times n}$

$$\text{rank}(\mathbf{A}) + \dim[\mathcal{N}(\mathbf{A})] = n \tag{A.7}$$

(see [2]). For the important special case where matrix $\mathbf{A}$ is square, i.e., $n = m$, the following statements are equivalent: (a) there exists a unique solution for Eq. (A.5); (b) $\mathcal{N}(\mathbf{A}) = \{\mathbf{0}\}$; (c) $\text{rank}(\mathbf{A}) = n$.

A matrix $\mathbf{A} \in R^{m\times n}$ is said to have full column rank if $\text{rank}(\mathbf{A}) = n$, i.e., the n column vectors of $\mathbf{A}$ are linearly independent, and $\mathbf{A}$ is said to have full row rank if $\text{rank}(\mathbf{A}) = m$, i.e., the m row vectors of $\mathbf{A}$ are linearly independent.

Example A.2 Find the rank and null space of matrix

$$\mathbf{V} = \begin{bmatrix} 1 & 0 & 3 & -1 \\ -1 & 2 & -7 & 5 \\ 3 & 1 & 7 & -1 \\ 0 & -1 & 2 & -2 \end{bmatrix}$$

Solution Note that the columns of $\mathbf{V}$ are the vectors $\mathbf{v}_i$ for $i = 1, 2, \ldots, 4$ in Example A.1. Since the maximum number of linearly independent columns is 2, we have $\text{rank}(\mathbf{V}) = 2$. To find $\mathcal{N}(\mathbf{V})$, we write $\mathbf{V} = [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3 \ \mathbf{v}_4]$; hence the equation $\mathbf{V}\mathbf{x} = \mathbf{0}$ becomes

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + x_4\mathbf{v}_4 = \mathbf{0} \tag{A.8}$$

Using Eqs. (A.3) and (A.4), Eq. (A.8) can be expressed as

$$(x_1 + 3x_3 - x_4)\mathbf{v}_1 + (x_2 - 2x_3 + 2x_4)\mathbf{v}_2 = \mathbf{0}$$

which implies that

$$\begin{aligned} x_1 + 3x_3 - x_4 &= 0 \\ x_2 - 2x_3 + 2x_4 &= 0 \end{aligned}$$

i.e.,

$$\begin{aligned} x_1 &= -3x_3 + x_4 \\ x_2 &= 2x_3 - 2x_4 \end{aligned}$$

Hence any vector $\mathbf{x}$ that can be expressed as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3x_3 + x_4 \\ 2x_3 - 2x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix} x_4$$

with arbitrary x_3 and x_4 satisfies $\mathbf{Ax} = \mathbf{0}$. Since the two vectors in the above expression, namely,

$$\mathbf{n}_1 = \begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{n}_2 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

are linearly independent, we have $\mathcal{N}(\mathbf{V}) = \text{span}\{\mathbf{n}_1,\ \mathbf{n}_2\}$.

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A.4 Sherman-Morrison Formula

The Sherman-Morrison formula [4] states that given matrices $\mathbf{A} \in C^{n\times n}$, $\mathbf{U} \in C^{n\times p}$, $\mathbf{W} \in C^{p\times p}$, and $\mathbf{V} \in C^{n\times p}$, such that $\mathbf{A}^{-1}$, $\mathbf{W}^{-1}$ and $(\mathbf{W}^{-1} + \mathbf{V}^H\mathbf{A}^{-1}\mathbf{U})^{-1}$ exist, then the inverse of $\mathbf{A} + \mathbf{UWV}^H$ exists and is given by

$$(\mathbf{A} + \mathbf{UWV}^H)^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{U}\mathbf{Y}^{-1}\mathbf{V}^H\mathbf{A}^{-1} \tag{A.9}$$

where

$$\mathbf{Y} = \mathbf{W}^{-1} + \mathbf{V}^H\mathbf{A}^{-1}\mathbf{U} \tag{A.10}$$

In particular, if $p = 1$ and $\mathbf{W} = 1$, then Eq. (A.9) assumes the form

$$(\mathbf{A} + \mathbf{uv}^H)^{-1} = \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}\mathbf{uv}^H\mathbf{A}^{-1}}{1 + \mathbf{v}^H\mathbf{A}^{-1}\mathbf{u}} \tag{A.11}$$

where $\mathbf{u}$ and $\mathbf{v}$ are vectors in $C^{n\times 1}$. Eq. (A.11) is useful for computing the inverse of a rank-one modification of $\mathbf{A}$, namely, $\mathbf{A}+\mathbf{uv}^H$, if $\mathbf{A}^{-1}$ is available.

Example A.3 Find $\mathbf{A}^{-1}$ for

$$\mathbf{A} = \begin{bmatrix} 1.04 & 0.04 & \cdots & 0.04 \\ 0.04 & 1.04 & \cdots & 0.04 \\ \vdots & \vdots & & \vdots \\ 0.04 & 0.04 & \cdots & 1.04 \end{bmatrix} \in \mathcal{R}^{10\times 10}$$

Solution Matrix $\mathbf{A}$ can be treated as a rank-one perturbation of the identity matrix:

$$\mathbf{A} = \mathbf{I} + \mathbf{pp}^T$$

where $\mathbf{I}$ is the identity matrix and $\mathbf{p} = [0.2\ 0.2\ \cdots\ 0.2]^T$. Using Eq. (A.11), we can compute

$$\mathbf{A}^{-1} = (\mathbf{I} + \mathbf{p}\mathbf{p}^T)^{-1} = \mathbf{I} - \frac{\mathbf{p}\mathbf{p}^T}{1 + \mathbf{p}^T\mathbf{p}} = \mathbf{I} - \frac{1}{1.4}\mathbf{p}\mathbf{p}^T$$
$$= \begin{bmatrix} 0.9714 & -0.0286 & \cdots & -0.0286 \\ -0.0286 & 0.9714 & \cdots & -0.0286 \\ \vdots & \vdots & & \vdots \\ -0.0286 & -0.0286 & \ldots & 0.9714 \end{bmatrix}$$

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A.5 Eigenvalues and Eigenvectors

The *eigenvalues* of a matrix $\mathbf{A} \in C^{n\times n}$ are defined as the n roots of its so-called *characteristic equation*

$$\det(\lambda\mathbf{I} - \mathbf{A}) = 0 \tag{A.12}$$

If we denote the set of n eigenvalues $\{\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_n\}$ by $\lambda(\mathbf{A})$, then for a $\lambda_i \in \lambda(\mathbf{A})$, there exists a nonzero vector $\mathbf{x}_i \in C^{n\times 1}$ such that

$$\mathbf{A}\mathbf{x}_i = \lambda_i\mathbf{x}_i \tag{A.13}$$

Such a vector is called an *eigenvector* of $\mathbf{A}$ associated with eigenvalue λ_i.

Eigenvectors are not unique. For example, if $\mathbf{x}_i$ is an eigenvector of matrix $\mathbf{A}$ associated with eigenvalue λ_i and c is an arbitrary nonzero constant, then $c\mathbf{x}_i$ is also an eigenvector of $\mathbf{A}$ associated with eigenvalue λ_i.

If $\mathbf{A}$ has n distinct eigenvalues $\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_n$ with associated eigenvectors $\mathbf{x}_1,\ \mathbf{x}_2,\ \ldots,\ \mathbf{x}_n$, then these eigenvectors are linearly independent; hence we can write

$$\mathbf{A}[\mathbf{x}_1\ \mathbf{x}_2\ \cdots\ \mathbf{x}_n] = [\mathbf{A}\mathbf{x}_1\ \mathbf{A}\mathbf{x}_2\ \cdots\ \mathbf{A}\mathbf{x}_n] = [\lambda_1\mathbf{x}_1\ \lambda_2\mathbf{x}_2\ \cdots\ \lambda_n\mathbf{x}_n]$$
$$= [\mathbf{x}_1\ \mathbf{x}_2\ \cdots\ \mathbf{x}_n]\begin{bmatrix} \lambda_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \lambda_n \end{bmatrix}$$

In effect,

$$\mathbf{A}\mathbf{X} = \mathbf{X}\boldsymbol{\Lambda}$$

or

$$\mathbf{A} = \mathbf{X}\boldsymbol{\Lambda}\mathbf{X}^{-1} \tag{A.14}$$

with

$$\mathbf{X} = [\mathbf{x}_1\ \mathbf{x}_2\ \cdots\ \mathbf{x}_n] \qquad \text{and} \qquad \boldsymbol{\Lambda} = \text{diag}\{\lambda_1,\ \lambda_1,\ \ldots,\ \lambda_n\}$$

where diag$\{\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_n\}$ represents the diagonal matrix with components $\lambda_1,\ \lambda_2,\ \ \ldots,\ \lambda_n$ along its diagonal. The relation in (A.14) is often referred to as an *eigendecomposition* of $\mathbf{A}$.

A concept that is closely related to the eigendecomposition in Eq. (A.14) is that of similarity transformation. Two square matrices $\mathbf{A}$ and $\mathbf{B}$ are said to be *similar* if there exists a nonsingular $\mathbf{X}$, called a *similarity transformation*, such that

$$\mathbf{A} = \mathbf{X}\mathbf{B}\mathbf{X}^{-1} \tag{A.15}$$

From Eq. (A.14), it follows that if the eigenvalues of $\mathbf{A}$ are distinct, then $\mathbf{A}$ is similar to $\boldsymbol{\Lambda} = \text{diag}\{\lambda_1,\ \lambda_2,\ \ \ldots,\ \lambda_n\}$ and the similarity transformation involved, $\mathbf{X}$, is composed of the n eigenvectors of $\mathbf{A}$. For arbitrary matrices with repeated eigenvalues, the eigendecomposition becomes more complicated. The reader is referred to [1]–[3] for the theory and solution of the eigenvalue problem for the general case.

Example A.4 Find the diagonal matrix $\boldsymbol{\Lambda}$, if it exists, that is similar to matrix

$$\mathbf{A} = \begin{bmatrix} 4 & -3 & 1 & 1 \\ 2 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

Solution From Eq. (A.12), we have

$$\begin{aligned} \det(\lambda\mathbf{I} - \mathbf{A}) &= \det\begin{bmatrix} \lambda - 4 & 3 \\ -2 & \lambda + 1 \end{bmatrix} \cdot \det\begin{bmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{bmatrix} \\ &= (\lambda^2 - 3\lambda + 2)(\lambda^2 - 2\lambda - 3) \\ &= (\lambda - 1)(\lambda - 2)(\lambda + 1)(\lambda - 3) \end{aligned}$$

Hence the eigenvalues of $\mathbf{A}$ are $\lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = -1$, and $\lambda_4 = 3$. An eigenvector $\mathbf{x}_i$ associated with eigenvalue λ_i satisfies the relation

$$(\lambda_i\mathbf{I} - \mathbf{A})\mathbf{x}_i = \mathbf{0}$$

For $\lambda_1 = 1$, we have

$$\lambda_1\mathbf{I} - \mathbf{A} = \begin{bmatrix} -3 & 3 & -1 & -1 \\ -2 & 2 & -1 & -1 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & -2 & 0 \end{bmatrix}$$

It is easy to verify that $\mathbf{x}_1 = [1\ 1\ 0\ 0]^T$ satisfies the relation

$$(\lambda_1\mathbf{I} - \mathbf{A})\mathbf{x}_1 = \mathbf{0}$$

Similarly, $\mathbf{x}_2 = [3\ 2\ 0\ 0]^T$, $\mathbf{x}_3 = [0\ 0\ 1\ \ -1]^T$, and $\mathbf{x}_4 = [1\ 1\ 1\ 1]^T$ satisfy the relation

$$(\lambda_i \mathbf{I} - \mathbf{A})\mathbf{x}_i = \mathbf{0} \qquad \text{for } i = 2,\ 3,\ 4$$

If we let

$$\mathbf{X} = [\mathbf{x}_1\ \mathbf{x}_2\ \mathbf{x}_3\ \mathbf{x}_4] = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

then we have

$$\mathbf{AX} = \mathbf{\Lambda X}$$

where

$$\mathbf{\Lambda} = \text{diag}\{1,\ 2,\ -1,\ 3\}$$

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A.6 Symmetric Matrices

The matrices encountered most frequently in numerical optimization are symmetric. For these matrices, an elegant eigendecomposition theory and corresponding computation methods are available. If $\mathbf{A} = \{a_{ij}\} \in R^{n\times n}$ is a symmetric matrix, i.e., $a_{ij} = a_{ji}$, then there exists an orthogonal matrix $\mathbf{X} \in R^{n\times n}$, i.e., $\mathbf{XX}^T = \mathbf{X}^T\mathbf{X} = \mathbf{I}_n$, such that

$$\mathbf{A} = \mathbf{X\Lambda X}^T \tag{A.16}$$

where $\mathbf{\Lambda} = \text{diag}\{\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_n\}$. If $\mathbf{A} \in C^{n\times n}$ is such that $\mathbf{A} = \mathbf{A}^H$, then $\mathbf{A}$ is referred to as a *Hermitian matrix*. In such a case, there exists a so-called *unitary matrix* $\mathbf{U} \in C^{n\times n}$ for which $\mathbf{UU}^H = \mathbf{U}^H\mathbf{U} = \mathbf{I}_n$ such that

$$\mathbf{A} = \mathbf{U\Lambda U}^H \tag{A.17}$$

In Eqs. (A.16) and (A.17), the diagonal components of $\mathbf{\Lambda}$ are eigenvalues of $\mathbf{A}$, and the columns of $\mathbf{X}$ and $\mathbf{U}$ are corresponding eigenvectors of $\mathbf{A}$.

The following properties can be readily verified:

(a) A square matrix is nonsingular if and only if all its eigenvalues are nonzero.
(b) The magnitudes of the eigenvalues of an orthogonal or unitary matrix are always equal to unity.
(c) The eigenvalues of a symmetric or Hermitian matrix are always real.
(d) The determinant of a square matrix is equal to the product of its eigenvalues.

A symmetric matrix $\mathbf{A} \in R^{n\times n}$ is said to be *positive definite, positive semidefinite, negative semidefinite, negative definite* if $\mathbf{x}^T\mathbf{Ax} > 0,\ \mathbf{x}^T\mathbf{Ax} \geq 0,\ \mathbf{x}^T\mathbf{Ax} \leq 0,\ \mathbf{x}^T\mathbf{Ax} < 0$, respectively, for all nonzero $\mathbf{x} \in R^{n\times 1}$.

Using the decomposition in Eq. (A.16), it can be shown that matrix $\mathbf{A}$ is positive definite, positive semidefinite, negative semidefinite, negative definite, if and only if its eigenvalues are positive, nonnegative, nonpositive, negative, respectively. Otherwise, $\mathbf{A}$ is said to be indefinite. We use the shorthand notation $\mathbf{A} \succ, \succeq, \preceq, \prec \mathbf{0}$ to indicate that $\mathbf{A}$ is positive definite, positive semidefinite, negative semidefinite, negative definite throughout the book.

Another approach for the characterization of a square matrix $\mathbf{A}$ is based on the evaluation of the *leading principal minor determinants*. A *minor determinant*, which is usually referred to as a *minor*, is the determinant of a submatrix obtained by deleting a number of rows and an equal number of columns from the matrix. Specifically, a minor of order r of an $n \times n$ matrix $\mathbf{A}$ is obtained by deleting $n - r$ rows and $n - r$ columns. For example, if

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

then

$$\Delta_3^{(123,123)} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \quad \Delta_3^{(134,124)} = \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix}$$

and

$$\Delta_2^{(12,12)} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad \Delta_2^{(13,14)} = \begin{vmatrix} a_{11} & a_{14} \\ a_{31} & a_{34} \end{vmatrix}$$

$$\Delta_2^{(24,13)} = \begin{vmatrix} a_{21} & a_{23} \\ a_{41} & a_{43} \end{vmatrix}, \quad \Delta_2^{(34,34)} = \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix}$$

are third-order and second-order minors, respectively. An nth-order minor is the determinant of the matrix itself and a first-order minor, i.e., if $n-1$ rows and $n - 1$ columns are deleted, is simply the value of a single matrix component.[1]

If the indices of the deleted rows are the same as those of the deleted columns, then the minor is said to be a *principal minor*, e.g., $\Delta_3^{(123,123)}$, $\Delta_2^{(12,12)}$, and $\Delta_2^{(34,34)}$ in the above examples.

Principal minors $\Delta_3^{(123,123)}$ and $\Delta_2^{(12,12)}$ in the above examples can be represented by

$$\Delta_3^{(1,2,3)} = \det \mathbf{H}_3^{(1,2,3)}$$

and

$$\Delta_2^{(1,2)} = \det \mathbf{H}_2^{(1,2)}$$

[1]The zeroth-order minor is often defined to be unity.

respectively. An arbitrary principal minor of order i can be represented by

$$\Delta_i^{(l)} = \det \mathbf{H}_i^{(l)}$$

where

$$\mathbf{H}_i^{(l)} = \begin{bmatrix} a_{l_1 l_1} & a_{l_1 l_2} & \cdots & a_{l_1 l_i} \\ a_{l_2 l_1} & a_{l_2 l_2} & \cdots & a_{l_2 l_i} \\ \vdots & \vdots & & \vdots \\ a_{l_i l_1} & a_{l_i l_2} & \cdots & a_{l_i l_i} \end{bmatrix}$$

and $l \in \{l_1,\ l_2,\ \ldots,\ l_i\}$ with $1 \le l_1 < l_2 < \cdots < l_i \le n$ is the set of rows (and columns) retained in submatrix $\mathbf{H}_i^{(l)}$.

The specific principal minors

$$\Delta_r = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rr} \end{vmatrix} = \det \mathbf{H}_r$$

for $1 \le r \le n$ are said to be the *leading principal minors* of an $n \times n$ matrix. For a 4×4 matrix, the complete set of leading principal minors is as follows:

$$\Delta_1 = a_{11}, \quad \Delta_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$\Delta_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \quad \Delta_4 = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

The leading principal minors of a matrix $\mathbf{A}$ or its negative $-\mathbf{A}$ can be used to establish whether the matrix is positive or negative definite whereas the principal minors of $\mathbf{A}$ or $-\mathbf{A}$ can be used to establish whether the matrix is positive or negative semidefinite. These principles are stated in terms of Theorem 2.9 in Chap. 2 and are often used to establish the nature of the Hessian matrix in optimization algorithms.

The fact that a nonnegative real number has positive and negative square roots can be extended to the class of positive semidefinite matrices. Assuming that matrix $\mathbf{A} \in R^{n\times n}$ is positive semidefinite, we can write its eigendecomposition in Eq. (A.16) as

$$\mathbf{A} = \mathbf{X}\boldsymbol{\Lambda}\mathbf{X}^T = \mathbf{X}\boldsymbol{\Lambda}^{1/2}\mathbf{W}\mathbf{W}^T\boldsymbol{\Lambda}^{1/2}\mathbf{X}^T$$

where $\boldsymbol{\Lambda}^{1/2} = \mathrm{diag}\{\lambda_1^{1/2},\ \lambda_2^{1/2},\ \ldots,\ \lambda_n^{1/2}\}$ and $\mathbf{W}$ is an arbitrary orthogonal matrix, which leads to

$$\mathbf{A} = \mathbf{A}^{1/2}(\mathbf{A}^{1/2})^T \tag{A.18}$$

where $\mathbf{A}^{1/2} = \mathbf{X}\mathbf{\Lambda}^{1/2}\mathbf{W}$ and is called an *asymmetric square root* of $\mathbf{A}$. Since matrix $\mathbf{W}$ can be an arbitrary orthogonal matrix, an infinite number of asymmetric square roots of $\mathbf{A}$ exist. Alternatively, since $\mathbf{X}$ is an orthogonal matrix, we can write

$$\mathbf{A} = (\alpha\mathbf{X}\mathbf{\Lambda}^{1/2}\mathbf{X}^T)(\alpha\mathbf{X}\mathbf{\Lambda}^{1/2}\mathbf{X}^T)$$

where α is either 1 or -1, which gives

$$\mathbf{A} = \mathbf{A}^{1/2}\mathbf{A}^{1/2} \tag{A.19}$$

where $\mathbf{A}^{1/2} = \alpha\mathbf{X}\mathbf{\Lambda}^{1/2}\mathbf{X}^T$ and is called a *symmetric square root* of $\mathbf{A}$. Again, because α can be either 1 or -1, more than one symmetric square roots exist. Obviously, the symmetric square roots $\mathbf{X}\mathbf{\Lambda}^{1/2}\mathbf{X}^T$ and $-\mathbf{X}\mathbf{\Lambda}^{1/2}\mathbf{X}^T$ are positive semidefinite and negative semidefinite, respectively.

If $\mathbf{A}$ is a complex-valued positive semidefinite matrix, then *non-Hermitian* and *Hermitian square roots* of $\mathbf{A}$ can be obtained using the eigendecomposition in Eq. (A.17). For example, we can write

$$\mathbf{A} = \mathbf{A}^{1/2}(\mathbf{A}^{1/2})^H$$

where $\mathbf{A}^{1/2} = \mathbf{U}\mathbf{\Lambda}^{1/2}\mathbf{W}$ is a non-Hermitian square root of $\mathbf{A}$ if $\mathbf{W}$ is unitary. On the other hand,

$$\mathbf{A} = \mathbf{A}^{1/2}\mathbf{A}^{1/2}$$

where $\mathbf{A}^{1/2} = \alpha\mathbf{U}\mathbf{\Lambda}^{1/2}\mathbf{U}^H$ is a Hermitian square root if $\alpha = 1$ or $\alpha = -1$.

Example A.5 Verify that

$$\mathbf{A} = \begin{bmatrix} 2.5 & 0 & 1.5 \\ 0 & \sqrt{2} & 0 \\ 1.5 & 0 & 2.5 \end{bmatrix}$$

is positive definite and compute a symmetric square root of $\mathbf{A}$.

Solution An eigendecomposition of matrix $\mathbf{A}$ is

$$\mathbf{A} = \mathbf{X}\mathbf{\Lambda}\mathbf{X}^T$$

with

$$\mathbf{\Lambda} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{X} = \begin{bmatrix} \sqrt{2}/2 & 0 & -\sqrt{2}/2 \\ 0 & -1 & 0 \\ \sqrt{2}/2 & 0 & \sqrt{2}/2 \end{bmatrix}$$

Since the eigenvalues of $\mathbf{A}$ are all positive, $\mathbf{A}$ is positive definite. A symmetric square root of $\mathbf{A}$ is given by

$$\mathbf{A}^{1/2} = \mathbf{X}\mathbf{\Lambda}^{1/2}\mathbf{X}^T = \begin{bmatrix} 1.5 & 0 & 0.5 \\ 0 & \sqrt{2} & 0 \\ 0.5 & 0 & 1.5 \end{bmatrix}$$

■

A.7 Trace

The trace of an $n \times n$ square matrix, $\mathbf{A} = \{a_{ij}\}$, is *the sum of its diagonal components*, i.e.,

$$\text{trace}(\mathbf{A}) = \sum_{i=1}^{n} a_{ii}$$

It can be verified that the trace of a square matrix $\mathbf{A}$ with eigenvalues $\lambda_1,\ \lambda_2,$ $\ldots,\ \lambda_n$ is equal to the sum of its eigenvalues, i.e.,

$$\text{trace}(\mathbf{A}) = \sum_{i=1}^{n} \lambda_i$$

A useful property pertaining to the product of two matrices is that the trace of a square matrix $\mathbf{AB}$ is equal to the trace of matrix $\mathbf{BA}$, i.e.,

$$\text{trace}(\mathbf{AB}) = \text{trace}(\mathbf{BA}) \tag{A.20}$$

By applying Eq. (A.20) to the quadratic form $\mathbf{x}^T\mathbf{Hx}$, we obtain

$$\mathbf{x}^T\mathbf{Hx} = \text{trace}(\mathbf{x}^T\mathbf{Hx}) = \text{trace}(\mathbf{Hxx}^T) = \text{trace}(\mathbf{HX})$$

where $\mathbf{X} = \mathbf{xx}^T$. Moreover, we can write a general quadratic function as

$$\mathbf{x}^T\mathbf{Hx} + 2\mathbf{p}^T\mathbf{x} + \kappa = \text{trace}(\hat{\mathbf{H}}\hat{\mathbf{X}}) \tag{A.21}$$

where

$$\hat{\mathbf{H}} = \begin{bmatrix} \mathbf{H} & \mathbf{p} \\ \mathbf{p}^T & \kappa \end{bmatrix} \quad \text{and} \quad \hat{\mathbf{X}} = \begin{bmatrix} \mathbf{xx}^T & \mathbf{x} \\ \mathbf{x}^T & 1 \end{bmatrix}$$

A.8 Vector Norms and Matrix Norms

A.8.1 Vector norms

The L_p norm of a vector $\mathbf{x} \in C^n$ for $p \geq 1$ is given by

$$\|\mathbf{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \tag{A.22}$$

where p is a positive integer and x_i is the ith component of $\mathbf{x}$. The most popular L_p norms are $\|\cdot\|_1$, $\|\cdot\|_2$, and $\|\cdot\|_\infty$, where the infinity norm $\|\cdot\|_\infty$ can easily be shown to satisfy the relation

$$\|\mathbf{x}\|_\infty = \lim_{p\to\infty} \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} = \max_i |x_i| \tag{A.23}$$

For example, if $\mathbf{x} = [1\ 2\ \cdots\ 100]^T$, then $\|\mathbf{x}\| = 581.68$, $\|\mathbf{x}\|_{10} = 125.38$, $\|\mathbf{x}\|_{50} = 101.85$, $\|\mathbf{x}\|_{100} = 100.45$, $\|\mathbf{x}\|_{200} = 100.07$ and, of course, $\|\mathbf{x}\|_\infty =$ 100.

The important point to note here is that for an even p, the L_p norm of a vector is a *differentiable* function of its components but the L_∞ norm is *not*. So when the L_∞ norm is used in a design problem, we can replace it by an L_p norm (with p even) so that powerful calculus-based tools can be used to solve the problem. Obviously, the results obtained can only be *approximate* with respect to the original design problem. However, as indicated by Eq. (9.23), the difference between the approximate and exact solutions becomes insignificant if p is sufficiently large.

The inner product of two vectors $\mathbf{x},\ \mathbf{y} \in C^n$ is a scalar given by

$$\mathbf{x}^H\mathbf{y} = \sum_{i=1}^{n} x_i^* y_i$$

where x_i^* denotes the complex-conjugate of x_i. Frequently, we need to estimate the absolute value of $\mathbf{x}^H\mathbf{y}$. There are two well-known inequalities that provide tight upper bounds for $|\mathbf{x}^H\mathbf{y}|$, namely, the *Hölder inequality*

$$|\mathbf{x}^H\mathbf{y}| \le \|\mathbf{x}\|_p \|\mathbf{y}\|_q \tag{A.24}$$

which holds for any $p \ge 1$ and $q \ge 1$ satisfying the equality

$$\frac{1}{p} + \frac{1}{q} = 1$$

and the *Cauchy-Schwartz inequality* which is the special case of the Hölder inequality with $p = q = 2$, i.e.,

$$|\mathbf{x}^H\mathbf{y}| \le \|\mathbf{x}\|_2 \|\mathbf{y}\|_2 \tag{A.25}$$

If vectors $\mathbf{x}$ and $\mathbf{y}$ have unity lengths, i.e., $\|\mathbf{x}\|_2 = \|\mathbf{y}\|_2 = 1$, then Eq. (A.25) becomes

$$|\mathbf{x}^H\mathbf{y}| \le 1 \tag{A.26}$$

A geometric interpretation of Eq. (A.26) is that for unit vectors $\mathbf{x}$ and $\mathbf{y}$, the inner product $\mathbf{x}^H\mathbf{y}$ is equal to $\cos\theta$, where θ denotes the angle between the two vectors, whose absolute value is always less than one.

Another property of the L_2 norm is its invariance under orthogonal or unitary transformation. That is, if $\mathbf{A}$ is an orthogonal or unitary matrix, then

$$\|\mathbf{A}\mathbf{x}\|_2 = \|\mathbf{x}\|_2 \tag{A.27}$$

The L_p norm of a vector $\mathbf{x}$, $\|\mathbf{x}\|_p$, is monotonically decreasing with respect to p for $p \ge 1$. For example, we can relate $\|\mathbf{x}\|_1$ and $\|\mathbf{x}\|_2$ as

$$\begin{aligned}
\|\mathbf{x}\|_1^2 &= \left(\sum_{i=1}^{n} |x_i|\right)^2 \\
&= |x_1|^2 + |x_2|^2 + \cdots + |x_n|^2 + 2|x_1x_2| + \cdots + 2|x_{n-1}x_n| \\
&\ge |x_1|^2 + |x_2|^2 + \cdots + |x_n|^2 = \|\mathbf{x}\|_2^2
\end{aligned}$$

which implies that

$$\|\mathbf{x}\|_1 \geq \|\mathbf{x}\|_2$$

Furthermore, if $\|\mathbf{x}\|_\infty$ is numerically equal to $|x_k|$ for some index k, i.e.,

$$\|\mathbf{x}\|_\infty = \max_i |x_i| = |x_k|$$

then we can write

$$\|\mathbf{x}\|_2 = (|x_1|^2 + \cdots + |x_n|^2)^{1/2} \geq (|x_k|^2)^{1/2} = |x_k| = \|\mathbf{x}\|_\infty$$

i.e.,

$$\|\mathbf{x}\|_2 \geq \|\mathbf{x}\|_\infty$$

Therefore, we have

$$\|\mathbf{x}\|_1 \geq \|\mathbf{x}\|_2 \geq \|\mathbf{x}\|_\infty$$

In general, it can be shown that

$$\|\mathbf{x}\|_1 \geq \|\mathbf{x}\|_2 \geq \|\mathbf{x}\|_3 \geq \cdots \geq \|\mathbf{x}\|_\infty$$

A.8.2 Matrix norms

The L_p norm of matrix $\mathbf{A} = \{a_{ij}\} \in C^{m\times n}$ is defined as

$$\|\mathbf{A}\|_p = \max_{\mathbf{x}\neq\mathbf{0}} \frac{\|\mathbf{A}\mathbf{x}\|_p}{\|\mathbf{x}\|_p} \qquad \text{for } p \geq 1 \tag{A.28}$$

The most useful matrix L_p norm is the L_2 norm

$$\|\mathbf{A}\|_2 = \max_{\mathbf{x}\neq\mathbf{0}} \frac{\|\mathbf{A}\mathbf{x}\|_2}{\|\mathbf{x}\|_2} = \left[\max_i \left|\lambda_i(\mathbf{A}^H\mathbf{A})\right|\right]^{1/2} = \left[\max_i \left|\lambda_i(\mathbf{A}\mathbf{A}^H)\right|\right]^{1/2} \tag{A.29}$$

which can be easily computed as the square root of the largest eigenvalue magnitude in $\mathbf{A}^H\mathbf{A}$ or $\mathbf{A}\mathbf{A}^H$. Some other frequently used matrix L_p norms are

$$\|\mathbf{A}\|_1 = \max_{\mathbf{x}\neq 0} \frac{\|\mathbf{A}\mathbf{x}\|_1}{\|\mathbf{x}\|_1} = \max_{1\leq j\leq n} \sum_{i=1}^{m} |a_{ij}|$$

and

$$\|\mathbf{A}\|_\infty = \max_{x\neq 0} \frac{\|\mathbf{A}\mathbf{x}\|_\infty}{\|\mathbf{x}\|_\infty} = \max_{1\leq i\leq m} \sum_{j=1}^{n} |a_{ij}|$$

Another popular matrix norm is the Frobenius norm which is defined as

$$\|\mathbf{A}\|_F = \left(\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2\right)^{1/2} \tag{A.30}$$

which can also be calculated as

$$\|\mathbf{A}\|_F = [\text{trace}(\mathbf{A}^H\mathbf{A})]^{1/2} = [\text{trace}(\mathbf{A}\mathbf{A}^H)]^{1/2} \tag{A.31}$$

Note that the matrix L_2 norm and the Frobenius norm are *invariant* under orthogonal or unitary transformation, i.e., if $\mathbf{U} \in C^{n\times n}$ and $\mathbf{V} \in C^{m\times m}$ are unitary or orthogonal matrices, then

$$\|\mathbf{UAV}\|_2 = \|\mathbf{A}\|_2 \tag{A.32}$$

and

$$\|\mathbf{UAV}\|_F = \|\mathbf{A}\|_F \tag{A.33}$$

Example A.6 Evaluate matrix norms $||\mathbf{A}||_1,\ ||\mathbf{A}||_2,\ ||\mathbf{A}||_\infty$, and $||\mathbf{A}||_F$ for

$$\mathbf{A} = \begin{bmatrix} 1 & 5 & 6 & 3 \\ 0 & 4 & -7 & 0 \\ 3 & 1 & 4 & 1 \\ -1 & 1 & 0 & 1 \end{bmatrix}$$

Solution

$$\begin{aligned} ||\mathbf{A}||_1 &= \max_{1\le j\le 4}\left(\sum_{i=1}^{4}|a_{ij}|\right) = \max\{5,\ 11,\ 17,\ 5\} = 17 \\ ||\mathbf{A}||_\infty &= \max_{1\le i\le 4}\left(\sum_{j=1}^{4}|a_{ij}|\right) = \max\{15,\ 11,\ 9,\ 3\} = 15 \\ ||\mathbf{A}||_F &= \left(\sum_{i=1}^{4}\sum_{j=1}^{4}|a_{ij}|^2\right)^{1/2} = \sqrt{166} = 12.8841 \end{aligned}$$

To obtain $||\mathbf{A}||_2$, we compute the eigenvalues of $\mathbf{A}^T\mathbf{A}$ as

$$\lambda(\mathbf{A}^T\mathbf{A}) = \{0.2099,\ 6.9877,\ 47.4010,\ 111.4014\}$$

Hence

$$||\mathbf{A}||_2 = [\max_i |\lambda_i(\mathbf{A}^T\mathbf{A})|]^{1/2} = \sqrt{111.4014} = 10.5547$$

■

A.9 Singular-Value Decomposition

Given a matrix $\mathbf{A} \in C^{m\times n}$ of rank r, there exist unitary matrices $\mathbf{U} \in C^{m\times m}$ and $\mathbf{V} \in C^{n\times n}$ such that

$$\mathbf{A} = \mathbf{U\Sigma V}^H \tag{A.34}$$

where

$$\boldsymbol{\Sigma} = \begin{bmatrix} \mathbf{S} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}_{m\times n} \tag{A.34}$$

and

$$\mathbf{S} = \text{diag}\{\sigma_1,\ \sigma_2,\ \ldots,\ \sigma_r\} \tag{A.34}$$

with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$.

The matrix decomposition in Eq. (A.34a) is known as the *singular-value* decomposition (SVD) of $\mathbf{A}$. It has many applications in optimization and elsewhere. If $\mathbf{A}$ is a real-valued matrix, then $\mathbf{U}$ and $\mathbf{V}$ in Eq. (A.34a) become orthogonal matrices and $\mathbf{V}^H$ becomes $\mathbf{V}^T$. The positive scalars σ_i for $i = 1,\ 2,\ \ldots,\ r$ in Eq. (A.34c) are called the *singular values* of $\mathbf{A}$. If $\mathbf{U} = [\mathbf{u}_1\ \mathbf{u}_2\ \cdots\ \mathbf{u}_m]$ and $\mathbf{V} = [\mathbf{v}_1\ \mathbf{v}_2\ \cdots\ \mathbf{v}_n]$, vectors $\mathbf{u}_i$ and $\mathbf{v}_i$ are called the *left* and *right singular vectors* of $\mathbf{A}$, respectively. From Eq. (A.34), it follows that

$$\mathbf{AA}^H = \mathbf{U} \begin{bmatrix} \mathbf{S}^2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}_{m\times m} \mathbf{U}^H \tag{A.35}$$

and

$$\mathbf{A}^H\mathbf{A} = \mathbf{V} \begin{bmatrix} \mathbf{S}^2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}_{n\times n} \mathbf{V}^H \tag{A.35}$$

Therefore, the singular values of $\mathbf{A}$ are the positive square roots of the nonzero eigenvalues of $\mathbf{AA}^H$ (or $\mathbf{A}^H\mathbf{A}$), the ith left singular vector $\mathbf{u}_i$ is the ith eigenvector of $\mathbf{AA}^H$, and the ith right singular vector $\mathbf{v}_i$ is the ith eigenvector of $\mathbf{A}^H\mathbf{A}$.

Several important applications of the SVD are as follows:

(a) The L_2 norm and Frobenius norm of a matrix $\mathbf{A} \in C^{m\times n}$ of rank r are given, respectively, by

$$\|\mathbf{A}\|_2 = \sigma_1 \tag{A.36}$$

and

$$\|\mathbf{A}\|_F = \left(\sum_{i=1}^{r} \sigma_i^2\right)^{1/2} \tag{A.37}$$

(b) The *condition number* of a nonsingular matrix $\mathbf{A} \in C^{n\times n}$ is defined as

$$\text{cond}(\mathbf{A}) = \|\mathbf{A}\|_2 \|\mathbf{A}^{-1}\|_2 = \frac{\sigma_1}{\sigma_n} \tag{A.38}$$

(c) The range and null space of a matrix $\mathbf{A} \in C^{m\times n}$ of rank r assume the forms

$$\mathcal{R}(\mathbf{A}) = \text{span}\{\mathbf{u}_1,\ \mathbf{u}_2,\ \ldots,\ \mathbf{u}_r\} \tag{A.39}$$
$$\mathcal{N}(\mathbf{A}) = \text{span}\{\mathbf{v}_{r+1},\ \mathbf{v}_{r+2},\ \ldots,\ \mathbf{v}_n\} \tag{A.40}$$

(d) Properties and computation of Moore-Penrose pseudo-inverse:

The Moore-Penrose pseudo-inverse of a matrix $\mathbf{A} \in C^{m\times n}$ is defined as the matrix $\mathbf{A}^+ \in C^{n\times m}$ that satisfies the following four conditions:

(i) $\mathbf{A}\mathbf{A}^+\mathbf{A} = \mathbf{A}$

(ii) $\mathbf{A}^+\mathbf{A}\mathbf{A}^+ = \mathbf{A}^+$

(iii) $(\mathbf{A}\mathbf{A}^+)^H = \mathbf{A}\mathbf{A}^+$

(iv) $(\mathbf{A}^+\mathbf{A})^H = \mathbf{A}^+\mathbf{A}$

Using the SVD of $\mathbf{A}$ in Eq. (A.34), the Moore-Penrose pseudo-inverse of $\mathbf{A}$ can be obtained as

$$\mathbf{A}^+ = \mathbf{V}\boldsymbol{\Sigma}^+\mathbf{U}^H \tag{A.41}$$

where

$$\boldsymbol{\Sigma}^+ = \begin{bmatrix} \mathbf{S}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}_{n\times m} \tag{A.41}$$

and

$$\mathbf{S}^{-1} = \text{diag}\{\sigma_1^{-1},\ \sigma_2^{-1},\ \ldots,\ \sigma_r^{-1}\} \tag{A.41}$$

Consequently, we have

$$\mathbf{A}^+ = \sum_{i=1}^{r} \frac{\mathbf{v}_i\mathbf{u}_i^H}{\sigma_i} \tag{A.42}$$

(e) For an underdetermined system of linear equations

$$\mathbf{A}\mathbf{x} = \mathbf{b} \tag{A.43}$$

where $\mathbf{A} \in C^{m\times n}$, $\mathbf{b} \in C^{m\times 1}$ with $m < n$, and $\mathbf{b} \in \mathcal{R}(\mathbf{A})$, all the solutions of Eq. (A.43) are characterized by

$$\mathbf{x} = \mathbf{A}^+\mathbf{b} + \mathbf{V}_r\boldsymbol{\phi} \tag{A.44}$$

where $\mathbf{A}^+$ is the Moore-Penrose pseudo-inverse of $\mathbf{A}$,

$$\mathbf{V}_r = [\mathbf{v}_{r+1}\ \mathbf{v}_{r+2}\ \cdots\ \mathbf{v}_n] \tag{A.44}$$

is a matrix of dimension $n\times(n-r)$ composed of the last $n-r$ columns of matrix $\mathbf{V}$ which is obtained by constructing the SVD of $\mathbf{A}$ in Eq. (A.34), and $\phi \in C^{(n-r)\times 1}$ is an *arbitrary* $(n-r)$-dimensional vector. Note that the first term in Eq. (A.44a), i.e., $\mathbf{A}^+\mathbf{b}$, is a solution of Eq. (A.43) while the second term, $\mathbf{V}_r\phi$, belongs to the null space of $\mathbf{A}$ (see Eq. (A.40)). Through vector ϕ, the expression in Eq. (A.44) parameterizes all the solutions of an underdetermined system of linear equations.

Example A.7 Perform the SVD of matrix

$$\mathbf{A} = \begin{bmatrix} 2.8284 & -1 & 1 \\ 2.8284 & 1 & -1 \end{bmatrix}$$

and compute $||\mathbf{A}||_2$, $||\mathbf{A}||_F$, and $\mathbf{A}^+$.

Solution To compute matrix $\mathbf{V}$ in Eq. (A.34a), from Eq. (A.35b) we obtain

$$\mathbf{A}^T\mathbf{A} = \begin{bmatrix} 16 & 0 & 0 \\ 0 & 2 & -2 \\ 0 & -2 & 2 \end{bmatrix} = \mathbf{V} \begin{bmatrix} 16 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{V}^T$$

where

$$\mathbf{V} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.7071 & -0.7071 \\ 0 & -0.7071 & -0.7071 \end{bmatrix} = [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3]$$

Hence the nonzero singular values of $\mathbf{A}$ are $\sigma_1 = \sqrt{16} = 4$ and $\sigma_2 = \sqrt{4} = 2$. Now we can write (A.34a) as $\mathbf{U\Sigma} = \mathbf{AV}$, where

$$\mathbf{U\Sigma} = [\sigma_1\mathbf{u}_1 \ \ \sigma_2\mathbf{u}_2 \ \ \mathbf{0}] = [4\mathbf{u}_1 \ \ 2\mathbf{u}_2 \ \ \mathbf{0}]$$

and

$$\mathbf{AV} = \begin{bmatrix} 2.8284 & -1.4142 & 0 \\ 2.8284 & 1.4142 & 0 \end{bmatrix}$$

Hence

$$\mathbf{u}_1 = \frac{1}{4}\begin{bmatrix} 2.8284 \\ 2.8284 \end{bmatrix} = \begin{bmatrix} 0.7071 \\ 0.7071 \end{bmatrix}, \quad \mathbf{u}_2 = \frac{1}{2}\begin{bmatrix} -1.4142 \\ 1.4142 \end{bmatrix} = \begin{bmatrix} -0.7071 \\ 0.7071 \end{bmatrix}$$

and

$$\mathbf{U} = [\mathbf{u}_1 \ \mathbf{u}_2] = \begin{bmatrix} 0.7071 & -0.7071 \\ 0.7071 & 0.7071 \end{bmatrix}$$

On using Eqs. (A.36) and (A.37), we have

$$||\mathbf{A}||_2 = \sigma_1 = 4 \quad \text{and} \quad ||\mathbf{A}||_F = (\sigma_1^2 + \sigma_2^2)^{1/2} = \sqrt{20} = 4.4721$$

Now from Eq. (A.42), we obtain

$$\mathbf{A}^+ = \frac{\mathbf{v}_1\mathbf{u}_1^T}{\sigma_1} + \frac{\mathbf{v}_2\mathbf{u}_2^T}{\sigma_2} = \begin{bmatrix} 0.1768 & 0.1768 \\ -0.2500 & 0.2500 \\ 0.2500 & -0.2500 \end{bmatrix}$$

■

A.10 Orthogonal Projections

Let $\mathcal{S}$ be a subspace in C^n. Matrix $\mathbf{P} \in C^{n\times n}$ is said to be an orthogonal projection matrix onto $\mathcal{S}$ if $\mathcal{R}(\mathbf{P}) = \mathcal{S},\ \mathbf{P}^2 = \mathbf{P}$, and $\mathbf{P}^H = \mathbf{P}$, where $\mathcal{R}(\mathbf{P})$ denotes the range of transformation $\mathbf{P}$ (see Sec. A.3), i.e., $\mathcal{R}(\mathbf{P}) = \{\mathbf{y} :\ \mathbf{y} = \mathbf{Px},\ \mathbf{x} \in C^n\}$. The term 'orthogonal projection' originates from the fact that if $\mathbf{x} \in C^n$ is a vector outside $\mathcal{S}$, then $\mathbf{Px}$ is a vector in $\mathcal{S}$ such that $\mathbf{x} - \mathbf{Px}$ is orthogonal to every vector in $\mathcal{S}$ and $\|\mathbf{x}-\mathbf{Px}\|$ is the minimum distance between $\mathbf{x}$ and $\mathbf{s}$, i.e., $\min \|\mathbf{x} - \mathbf{s}\|$, for $\mathbf{s} \in \mathcal{S}$, as illustrated in Fig. A.1.

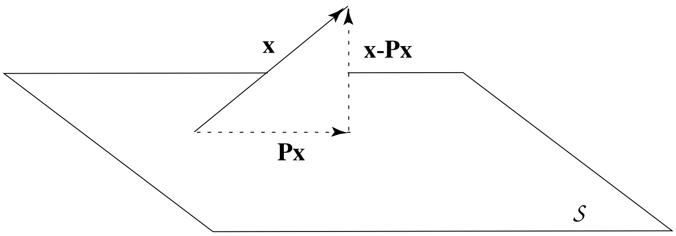

Figure A.1. Orthogonal projection of $\mathbf{x}$ onto subspace $\mathcal{S}$.

Let $\{\mathbf{s}_1,\ \mathbf{s}_2,\ \ldots,\ \mathbf{s}_k\}$ be a basis of a subspace $\mathcal{S}$ of dimension k (see Sec. A.2) such that $||\mathbf{s}_i|| = 1$ and $\mathbf{s}_i^T\mathbf{s}_j = 0$ for $i,\ j = 1,\ 2,\ \ldots, k$ and $i \neq j$. Such a basis is called *orthonormal*. It can be readily verified that an orthogonal projection matrix onto $\mathcal{S}$ can be explicitly constructed in terms of an orthonormal basis as

$$\mathbf{P} = \mathbf{SS}^H \tag{A.45}$$

where

$$\mathbf{S} = [\mathbf{s}_1\ \mathbf{s}_2\ \cdots\ \mathbf{s}_k] \tag{A.45}$$

It follows from Eqs. (A.39), (A.40), and (A.45) that $[\mathbf{u}_1\ \mathbf{u}_2\ \cdots\ \mathbf{u}_r]\cdot[\mathbf{u}_1\ \mathbf{u}_2\ \cdots\ \mathbf{u}_r]^H$ is the orthogonal projection onto $\mathcal{R}(\mathbf{A})$ and $[\mathbf{v}_{r+1}\ \mathbf{v}_{r+2}\ \cdots\ \mathbf{v}_n]\cdot[\mathbf{v}_{r+1}\ \mathbf{v}_{r+2}\ \cdots\ \mathbf{v}_n]^H$ is the orthogonal projection onto $\mathcal{N}(\mathbf{A})$.

Example A.8 Let $\mathcal{S} = \text{span}\{\mathbf{v}_1,\ \mathbf{v}_2\}$ where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Find the orthogonal projection onto $\mathcal{S}$.

Solution First, we need to find an orthonormal basis $\{\mathbf{s}_1,\ \mathbf{s}_2\}$ of subspace $\mathcal{S}$. To this end, we take

$$\mathbf{s}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

Then we try to find vector $\hat{\mathbf{s}}_2$ such that $\hat{\mathbf{s}}_2 \in \mathcal{S}$ and $\hat{\mathbf{s}}_2$ is orthogonal to $\mathbf{s}_1$. Such an $\hat{\mathbf{s}}_2$ must satisfy the relation

$$\hat{\mathbf{s}}_2 = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2$$

for some $\alpha_1,\ \alpha_2$ and

$$\hat{\mathbf{s}}_2^T \mathbf{s}_1 = 0$$

Hence we have

$$(\alpha_1 \mathbf{v}_1^T + \alpha_2 \mathbf{v}_2^T)\mathbf{s}_1 = \alpha_1 \mathbf{v}_1^T \mathbf{s}_1 + \alpha_2 \mathbf{v}_2^T \mathbf{s}_1 = \sqrt{3}\alpha_1 + \frac{1}{\sqrt{3}}\alpha_2 = 0$$

i.e., $\alpha_2 = -3\alpha_1$. Thus

$$\hat{\mathbf{s}}_2 = \alpha_1 \mathbf{v}_1 - 3\alpha_1 \mathbf{v}_2 = \alpha_1 \begin{bmatrix} 4 \\ -2 \\ -2 \end{bmatrix}$$

where α_1 is a parameter that can assume an arbitrary nonzero value.

By normalizing vector $\hat{\mathbf{s}}_2$, we obtain

$$\mathbf{s}_2 = \frac{\hat{\mathbf{s}}_2}{\|\hat{\mathbf{s}}_2\|} = \frac{1}{\sqrt{4^2 + (-2)^2 + (-2)^2}} \begin{bmatrix} 4 \\ -2 \\ -2 \end{bmatrix} = \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$$

It now follows from Eq. (A.45) that the orthogonal projection onto $\mathcal{S}$ can be characterized by

$$\mathbf{P} = [\mathbf{s}_1\ \mathbf{s}_2][\mathbf{s}_1\ \mathbf{s}_2]^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix}$$

■

A.11 Householder Transformations and Givens Rotations

A.11.1 Householder transformations

The *Householder transformation* associated with a nonzero vector $\mathbf{u} \in R^{n\times 1}$ is characterized by the symmetric orthogonal matrix

$$\mathbf{H} = \mathbf{I} - 2\frac{\mathbf{u}\mathbf{u}^T}{\|\mathbf{u}\|^2} \tag{A.46}$$

If

$$\mathbf{u} = \mathbf{x} - \|\mathbf{x}\|\mathbf{e}_1 \tag{A.47}$$

where $\mathbf{e}_1 = [1 \;\; 0 \;\; \cdots \;\; 0]^T$, then the Householder transformation will convert vector $\mathbf{x}$ to coordinate vector $\mathbf{e}_1$ to within a scale factor $\|\mathbf{x}\|$, i.e.,

$$\mathbf{Hx} = \|\mathbf{x}\| \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{A.48}$$

Alternatively, if vector $\mathbf{u}$ in Eq. (A.46) is chosen as

$$\mathbf{u} = \mathbf{x} + \|\mathbf{x}\|\mathbf{e}_1 \tag{A.49}$$

then

$$\mathbf{Hx} = -\|\mathbf{x}\| \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{A.50}$$

From Eqs. (A.47) and (A.49), we see that the transformed vector $\mathbf{Hx}$ contains $n-1$ zeros. Furthermore, since $\mathbf{H}$ is an orthogonal matrix, we have

$$\|\mathbf{Hx}\|^2 = (\mathbf{Hx})^T\mathbf{Hx} = \mathbf{x}^T\mathbf{H}^T\mathbf{Hx} = \mathbf{x}^T\mathbf{x} = \|\mathbf{x}\|^2$$

Therefore, $\mathbf{Hx}$ preserves the length of $\mathbf{x}$. For the sake of numerical robustness, a good choice of vector $\mathbf{u}$ between Eqs. (A.47) and (A.49) is

$$\mathbf{u} = \mathbf{x} + \text{sign}(x_1)\|\mathbf{x}\|\mathbf{e}_1 \tag{A.51}$$

because the alternative choice, $\mathbf{u} = \mathbf{x} - \text{sign}(x_1)\|\mathbf{x}\|\mathbf{e}_1$, may yield a vector $\mathbf{u}$ whose magnitude becomes too small when $\mathbf{x}$ is close to a multiple of $\mathbf{e}_1$.

Given a matrix $\mathbf{A} \in R^{n \times n}$, the matrix product $\mathbf{HA}$ is called a *Householder update* of $\mathbf{A}$ and it can be evaluated as

$$\mathbf{HA} = \left(\mathbf{I} - \frac{2\mathbf{uu}^T}{\|\mathbf{u}\|^2}\right)\mathbf{A} = \mathbf{A} - \mathbf{uv}^T \tag{A.52}$$

where

$$\mathbf{v} = \alpha\mathbf{A}^T\mathbf{u}, \quad \alpha = -\frac{2}{\|\mathbf{u}\|^2} \tag{A.52}$$

We see that a Household update of $\mathbf{A}$ is actually a rank-one correction of $\mathbf{A}$, which can be obtained by using a matrix-vector multiplication and then an outer product update. In this way, a Householder update can be carried out efficiently without requiring matrix multiplication explicitly.

By successively applying the Householder update with appropriate values of $\mathbf{u}$, a given matrix $\mathbf{A}$ can be transformed to an upper triangular matrix. To see this, consider a matrix $\mathbf{A} \in R^{n\times n}$ and let $\mathbf{H}_i$ be the ith Householder update such that after $k-1$ successive applications of $\mathbf{H}_i$ for $i = 1, 2, \ldots, k-1$ the transformed matrix becomes

$$\mathbf{A}^{(k-1)} = \mathbf{H}_{k-1} \cdots \mathbf{H}_1\mathbf{A} = \left[\begin{array}{cccc:ccc} * & * & \cdots & * & & & \\ 0 & * & & * & & * & \\ & & \ddots & & & & \\ 0 & 0 & & * & & & \\ \hdashline 0 & 0 & \cdots & 0 & & & \\ \vdots & \vdots & \cdots & \vdots & \mathbf{a}_k^{(k-1)} & \cdots & \mathbf{a}_n^{(k-1)} \\ 0 & 0 & \cdots & 0 & & & \end{array}\right] \tag{A.53}$$

The next Householder update is characterized by

$$\mathbf{H}_k = \mathbf{I} - 2\frac{\mathbf{u}_k\mathbf{u}_k^T}{\|\mathbf{u}_k\|^2} = \begin{bmatrix} \mathbf{I}_{k-1} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{H}}_k \end{bmatrix} \tag{A.54}$$

where

$$\mathbf{u}_k = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \\ \mathbf{u}_k^{(k-1)} \end{bmatrix}\!\!\begin{array}{l} \left.\vphantom{\begin{matrix}0\\ \vdots\\ 0\end{matrix}}\right\}k-1 \\ \\ \\ \end{array}, \quad \mathbf{u}_k^{(k-1)} = \mathbf{a}_k^{(k-1)} + \text{sign}[\mathbf{a}_k^{(k-1)}(1)]||\mathbf{a}_k^{(k-1)}||\mathbf{e}_1$$

$$\tilde{\mathbf{H}}_k = \mathbf{I}_{n-k+1} - 2\frac{\mathbf{u}_k^{(k-1)}(\mathbf{u}_k^{(k-1)})^T}{\|\mathbf{u}_k^{(k-1)}\|^2}$$

and $\mathbf{a}_k^{(k-1)}(1)$ represents the first component of vector $\mathbf{a}_k^{(k-1)}$.

Evidently, premultiplying $\mathbf{A}^{(k-1)}$ by $\mathbf{H}_k$ alters only the lower right block of $\mathbf{A}^{(k-1)}$ in Eq. (A.53) thereby converting its first column $\mathbf{a}_k^{(k-1)}$ to $[* \; 0 \; \cdots \; 0]^T$. Proceeding in this way, all the entries in the lower triangle will become zero.

Example A.9 Applying a series of Householder transformations, reduce matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 3 & -1 \\ -1 & 2 & -7 & 5 \\ 3 & 1 & 7 & -1 \\ 0 & -1 & 2 & -2 \end{bmatrix}$$

to an upper triangular matrix.

Solution Using Eq. (A.51), we compute vector $\mathbf{u}_1$ as

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 3 \\ 0 \end{bmatrix} + \sqrt{11} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+\sqrt{11} \\ -1 \\ 3 \\ 0 \end{bmatrix}$$

The associated Householder transformation is given by

$$\mathbf{H} = \mathbf{I} - \frac{2\mathbf{u}_1\mathbf{u}_1^T}{\|\mathbf{u}_1\|^2} = \begin{bmatrix} -0.3015 & 0.3015 & -0.9045 & 0 \\ 0.3015 & 0.9302 & 0.2095 & 0 \\ -0.9045 & 0.2095 & 0.3714 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The first Householder update is found to be

$$\mathbf{A}^{(1)} = \mathbf{H}_1\mathbf{A} = \begin{bmatrix} -3.3166 & -0.3015 & -9.3469 & 2.7136 \\ 0 & 2.0698 & -4.1397 & 4.1397 \\ 0 & 0.7905 & -1.5809 & 1.5809 \\ 0 & -1 & 2 & -2 \end{bmatrix}$$

From Eq. (A.53), we obtain

$$\mathbf{a}_2^{(1)} = \begin{bmatrix} 2.0698 \\ 0.7905 \\ -1 \end{bmatrix}$$

Using Eq. (A.54), we can compute

$$\mathbf{u}_2^{(1)} = \begin{bmatrix} 4.5007 \\ 0.7905 \\ -1 \end{bmatrix}$$

$$\tilde{\mathbf{H}}_2 = \begin{bmatrix} -0.8515 & -0.3252 & 0.4114 \\ -0.3252 & 0.9429 & 0.0722 \\ 0.4114 & 0.0722 & 0.9086 \end{bmatrix}$$

and

$$\mathbf{H}_2 = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{H}}_2 \end{bmatrix}$$

By premultiplying matrix $\mathbf{H}_1\mathbf{A}$ by $\mathbf{H}_2$, we obtain the required upper triangular matrix in terms of the second Householder update as

$$\mathbf{H}_2\mathbf{H}_1\mathbf{A} = \begin{bmatrix} -3.3166 & -0.3015 & -9.3469 & 2.7136 \\ 0 & -2.4309 & 4.8617 & -4.8617 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

■

A.11.2 Givens rotations

Givens rotations are rank-two corrections of the identity matrix and are characterized by

$$\mathbf{G}_{ik}(\theta) = \begin{array}{cc} \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ 0 & \cdots & c & \cdots & s & \cdots & 0 \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ 0 & \cdots & -s & \cdots & c & \cdots & 0 \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} & \begin{array}{c} \\ \\ i \\ \\ k \\ \\ \\ \end{array} \\ \begin{array}{ccccccc} & & i & & k & & \end{array} & \end{array}$$

for $1 \leq i,\ k \leq n$, where $c = \cos\theta$ and $s = \sin\theta$ for some θ. It can be verified that $\mathbf{G}_{ik}(\theta)$ is an orthogonal matrix and $\mathbf{G}_{ik}^T(\theta)\mathbf{x}$ only affects the ith and kth components of vector $\mathbf{x}$, i.e.,

$$\mathbf{y} = \mathbf{G}_{ik}^T(\theta)\mathbf{x} \quad \text{with} \quad y_l = \begin{cases} cx_i - sx_k & \text{for } \ l = i \\ sx_i + cx_k & \text{for } \ l = k \\ x_l & \text{otherwise} \end{cases}$$

By choosing an appropriate θ such that

$$sx_i + cx_k = 0 \tag{A.55}$$

the kth component of vector $\mathbf{y}$ is forced to zero. A numerically stable method for determining suitable values for s and c in Eq. (A.55) is described below, where we denote x_i and x_k as a and b, respectively.

(a) If $b = 0$, set $c = 1,\ s = 0$.

(b) If $b \neq 0$, then

(i) if $|b| > |a|$, set

$$\tau = -\frac{a}{b}, \quad s = \frac{1}{\sqrt{1+\tau^2}}, \quad c = \tau s$$

(ii) otherwise, if $|b| \leq |a|$, set

$$\tau = -\frac{b}{a}, \quad c = \frac{1}{\sqrt{1+\tau^2}}, \quad s = \tau c$$

Note that when premultiplying matrix $\mathbf{A}$ by $\mathbf{G}_{ik}^T(\theta)$, matrix $\mathbf{G}_{ik}^T(\theta)\mathbf{A}$ alters only the ith and kth rows of $\mathbf{A}$. The application of Givens rotations is illustrated by the following example.

Example A.10 Convert matrix $\mathbf{A}$ given by

$$\mathbf{A} = \begin{bmatrix} 3 & -1 \\ -3 & 5 \\ 2 & 1 \end{bmatrix}$$

into an upper triangular matrix by premultiplying it by an orthogonal transformation matrix that can be obtained using Givens rotations.

Solution To handle the first column, we first use $\mathbf{G}_{2,3}^T(\theta)$ to force its last component to zero. In this case, $a = -3$ and $b = 2$, hence

$$\tau = \frac{2}{3}, \quad c = \frac{1}{\sqrt{1+\tau^2}} = 0.8321, \quad \text{and} \quad s = \tau c = 0.5547$$

Therefore, matrix $\mathbf{G}_{2,3}(\theta_1)$ is given by

$$\mathbf{G}_{2,3}(\theta_1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.8321 & 0.5547 \\ 0 & -0.5547 & 0.8321 \end{bmatrix}$$

which leads to

$$\mathbf{G}_{2,3}^T(\theta_1)\mathbf{A} = \begin{bmatrix} 3 & -1 \\ -3.6056 & 3.6056 \\ 0 & 3.6056 \end{bmatrix}$$

In order to apply $\mathbf{G}_{1,2}^T(\theta_2)$ to the resulting matrix to force the second component of its first column to zero, we note that $a = 3$ and $b = -3.6056$; hence

$$\tau = \frac{3}{3.6056}, \quad s = \frac{1}{\sqrt{1+\tau^2}} = 0.7687, \quad \text{and} \quad c = \tau s = 0.6396$$

Therefore, matrix $\mathbf{G}_{1,2}(\theta_2)$ is given by

$$\mathbf{G}_{1,2}(\theta_2) = \begin{bmatrix} 0.6396 & 0.7687 & 0 \\ -0.7687 & 0.6396 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\mathbf{G}_{1,2}^T(\theta_2)\mathbf{G}_{2,3}^T(\theta_1)\mathbf{A} = \begin{bmatrix} 4.6904 & -3.4112 \\ 0 & 1.5374 \\ 0 & 3.6056 \end{bmatrix}$$

Now we can force the last component of the second column of the resulting matrix to zero by applying $\mathbf{G}_{2,3}^T(\theta_3)$. With $a = 1.5374$ and $b = 3.6056$, we compute

$$\tau = \frac{1.5374}{3.6056}, \quad s = \frac{1}{\sqrt{1+\tau^2}} = 0.9199, \quad \text{and} \quad c = \tau s = 0.3922$$

Therefore, matrix $\mathbf{G}_{2,3}(\theta_3)$ is given by

$$\mathbf{G}_{2,3}(\theta_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -0.3922 & 0.9199 \\ 0 & -0.9199 & -0.3922 \end{bmatrix}$$

which yields

$$\mathbf{G}_{2,3}^T(\theta_3)\mathbf{G}_{1,2}^T(\theta_2)\mathbf{G}_{2,3}^T(\theta_1)\mathbf{A} = \begin{bmatrix} 4.6904 & -3.4112 \\ 0 & -3.9196 \\ 0 & 0 \end{bmatrix}$$

■

A.12 QR Decomposition

A.12.1 Full-rank case

A QR decomposition of a matrix $\mathbf{A} \in R^{m\times n}$ is given by

$$\mathbf{A} = \mathbf{QR} \tag{A.56}$$

where $\mathbf{Q} \in R^{m\times m}$ is an orthogonal matrix and $\mathbf{R} \in R^{m\times n}$ is an upper triangular matrix.

In general, more than one QR decompositions exist. For example, if $\mathbf{A} = \mathbf{QR}$ is a QR decomposition of $\mathbf{A}$, then $\mathbf{A} = \tilde{\mathbf{Q}}\tilde{\mathbf{R}}$ is also a QR decomposition of $\mathbf{A}$ if $\tilde{\mathbf{Q}} = \mathbf{Q}\tilde{\mathbf{I}}$ and $\tilde{\mathbf{R}} = \tilde{\mathbf{I}}\mathbf{R}$ and $\tilde{\mathbf{I}}$ is a diagonal matrix whose diagonal comprises a mixture of 1's and -1's. Obviously, $\tilde{\mathbf{Q}}$ remains orthogonal and $\tilde{\mathbf{R}}$ is a triangular matrix but the signs of the rows in $\tilde{\mathbf{R}}$ corresponding to the -1's in $\tilde{\mathbf{I}}$ are changed compared with those in $\mathbf{R}$.

For the sake of convenience, we assume in the rest of this section that $m \geq n$. This assumption implies that $\mathbf{R}$ has the form

$$\mathbf{R} = \underbrace{\begin{bmatrix} \hat{\mathbf{R}} \\ \mathbf{0} \end{bmatrix}}_{n \text{ columns}} \begin{matrix} \}n \text{ rows} \\ \}m-n \text{ rows} \end{matrix}$$

where $\hat{\mathbf{R}}$ is an upper triangular square matrix of dimension n, and that Eq. (A.56) can be expressed as

$$\mathbf{A} = \hat{\mathbf{Q}}\hat{\mathbf{R}} \tag{A.57}$$

where $\hat{\mathbf{Q}}$ is the matrix formed by the first n columns of $\mathbf{Q}$. Now if we let $\hat{\mathbf{Q}} = [\mathbf{q}_1\ \mathbf{q}_2\ \cdots\ \mathbf{q}_n]$, Eq. (A.57) yields

$$\mathbf{Ax} = \hat{\mathbf{Q}}\hat{\mathbf{R}}\mathbf{x} = \hat{\mathbf{Q}}\hat{\mathbf{x}} = \sum_{i=1}^{n} \hat{x}_i\mathbf{q}_i$$

In other words, if $\mathbf{A}$ has full column rank n, then the first n columns in $\mathbf{Q}$ form an orthogonal basis for the range of $\mathbf{A}$, i.e., $\mathcal{R}(\mathbf{A})$.

As discussed in Sec. A.11.1, a total of n successive applications of the Householder transformation can convert matrix $\mathbf{A}$ into an upper triangular matrix, $\mathbf{R}$, i.e.,

$$\mathbf{H}_n \cdots \mathbf{H}_2 \mathbf{H}_1 \mathbf{A} = \mathbf{R} \tag{A.58}$$

Since each $\mathbf{H}_i$ in Eq. (A.58) is orthogonal, we obtain

$$\mathbf{A} = (\mathbf{H}_n \cdots \mathbf{H}_2 \mathbf{H}_1)^T \mathbf{R} = \mathbf{QR} \tag{A.59}$$

where $\mathbf{Q} = (\mathbf{H}_n \cdots \mathbf{H}_2 \mathbf{H}_1)^T$ is an orthogonal matrix and, therefore, Eqs. (A.58) and (A.59) yield a QR decomposition of $\mathbf{A}$. This method requires $n^2(m - n/3)$ multiplications [3].

An alternative approach for obtaining a QR decomposition is to apply Givens rotations as illustrated in Sec. A.11.2. For a general matrix $\mathbf{A} \in R^{m\times n}$ with $m \geq n$, a total of $mn - n(n+1)/2$ Givens rotations are required to convert $\mathbf{A}$ into an upper triangular matrix and this Givens-rotation-based algorithm requires $1.5n^2(m - n/3)$ multiplications [3].

A.12.2 QR decomposition for rank-deficient matrices

If the rank of a matrix $\mathbf{A} \in R^{m\times n}$ where $m \geq n$ is less than n, then there is at least one zero component in the diagonal of $\mathbf{R}$ in Eq. (A.56). In such a case, the conventional QR decomposition discussed in Sec. A.12.1 does not always produce an orthogonal basis for $\mathcal{R}(\mathbf{A})$. For such rank-deficient matrices, however, the Householder-transformation-based QR decomposition described in Sec. A.12.1 can be modified as

$$\mathbf{AP} = \mathbf{QR} \tag{A.60}$$

where $\text{rank}(\mathbf{A}) = r < n$, $\mathbf{Q} \in R^{m\times m}$ is an orthogonal matrix,

$$\mathbf{R} = \begin{bmatrix} \mathbf{R}_{11} & \mathbf{R}_{12} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{A.60}$$

where $\mathbf{R}_{11} \in R^{r\times r}$ is a triangular and nonsingular matrix, and $\mathbf{P} \in R^{n\times n}$ assumes the form

$$\mathbf{P} = [\mathbf{e}_{s_1}\ \mathbf{e}_{s_2}\ \cdots\ \mathbf{e}_{s_n}]$$

where $\mathbf{e}_{s_i}$ denotes the s_ith column of the $n \times n$ identity matrix and index set $\{s_1,\ s_2,\ \ldots,\ s_n\}$ is a permutation of $\{1, 2,\ \ldots,\ n\}$. Such a matrix is said to be *a permutation matrix* [1].

To illustrate how Eq. (A.60) is obtained, assume that $k-1$ (with $k-1 < r$) Householder transformations and permutations have been applied to $\mathbf{A}$ to obtain

$$\mathbf{R}^{(k-1)} = (\mathbf{H}_{k-1} \cdots \mathbf{H}_2 \mathbf{H}_1)\mathbf{A}(\mathbf{P}_1 \mathbf{P}_2 \cdots \mathbf{P}_{k-1})$$

$$= \left[\begin{array}{cc} \mathbf{R}_{11}^{(k-1)} & \mathbf{R}_{12}^{(k-1)} \\ \underbrace{\mathbf{0}}_{k-1} & \underbrace{\mathbf{R}_{22}^{(k-1)}}_{n-k+1} \end{array}\right] \begin{array}{l} \}k-1 \\ \}m-k+1 \end{array} \tag{A.61}$$

where $\mathbf{R}_{11}^{(k-1)} \in R^{(k-1)\times(k-1)}$ is upper triangular and $\text{rank}(\mathbf{R}_{11}^{(k-1)}) = k-1$. Since $\text{rank}(\mathbf{A}) = r$, block $\mathbf{R}_{22}^{(k-1)}$ is nonzero. Now we postmultiply Eq. (A.61) by a permutation matrix $\mathbf{P}_k$ which rearranges the last $n-k+1$ columns of $\mathbf{R}^{(k-1)}$ such that the column in $\mathbf{R}_{22}^{(k-1)}$ with the largest L_2 norm becomes its first column. A Householder matrix $\mathbf{H}_k$ is then applied to obtain

$$\mathbf{H}_k\mathbf{R}^{(k-1)}\mathbf{P}_k = \left[\begin{array}{cc} \mathbf{R}_{11}^{(k)} & \mathbf{R}_{12}^{(k)} \\ \underbrace{\mathbf{0}}_{k} & \underbrace{\mathbf{R}_{22}^{(k)}}_{n-k} \end{array}\right] \begin{array}{l} \}k \\ \}m-k \end{array}$$

where $\mathbf{R}_{11}^{(k)} \in R^{k\times k}$ is an upper triangular nonsingular matrix. If $r = k$, then $\mathbf{R}_{22}^{(k)}$ must be a zero matrix since $\text{rank}(\mathbf{A}) = r$; otherwise, $\mathbf{R}_{22}^{(k)}$ is a nonzero block, and we proceed with postmultiplying $\mathbf{R}^{(k)}$ by a new permutation matrix $\mathbf{P}_{k+1}$ and then premultiplying by a Householder matrix $\mathbf{H}_{k+1}$. This procedure is continued until the modified QR decomposition in Eq. (A.60) is obtained where

$$\mathbf{Q} = (\mathbf{H}_r\mathbf{H}_{r-1} \cdots \mathbf{H}_1)^T \qquad \text{and} \qquad \mathbf{P} = \mathbf{P}_1\mathbf{P}_2 \cdots \mathbf{P}_r$$

The decomposition in Eq. (A.60) is called the *QR decomposition of matrix* $\mathbf{A}$ with *column pivoting*. It follows from Eq. (A.60) that the first r columns of matrix $\mathbf{Q}$ form an orthogonal basis for the range of $\mathbf{A}$.

Example A.11 Find a QR decomposition of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 3 & -1 \\ -1 & 2 & -7 & 5 \\ 3 & 1 & 7 & -1 \\ 0 & -1 & 2 & 2 \end{bmatrix}$$

Solution In Example A.9, two Householder transformation matrices

$$\mathbf{H}_1 = \begin{bmatrix} -0.3015 & 0.3015 & -0.9045 & 0 \\ 0.3015 & 0.9302 & 0.2095 & 0 \\ -0.9045 & 0.2095 & 0.3714 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{H}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -0.8515 & -0.3252 & 0.4114 \\ 0 & -0.3252 & 0.9429 & 0.0722 \\ 0 & 0.4114 & 0.0722 & 0.9086 \end{bmatrix}$$

were obtained that reduce matrix $\mathbf{A}$ to the upper triangular matrix

$$\mathbf{R} = \mathbf{H}_2\mathbf{H}_1\mathbf{A} = \begin{bmatrix} -3.3166 & -0.3015 & -9.3469 & 2.7136 \\ 0 & -2.4309 & 4.8617 & -4.8617 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore, a QR decomposition of $\mathbf{A}$ can be obtained as $\mathbf{A} = \mathbf{QR}$ where $\mathbf{R}$ is the above upper triangular matrix.

$$\mathbf{Q} = (\mathbf{H}_2\mathbf{H}_1)^{-1} = \mathbf{H}_1^T\mathbf{H}_2^T = \begin{bmatrix} -0.3015 & 0.0374 & -0.9509 & 0.0587 \\ 0.3015 & -0.8602 & -0.1049 & 0.3978 \\ -0.9045 & -0.2992 & 0.2820 & 0.1130 \\ 0 & 0.4114 & 0.0722 & 0.9086 \end{bmatrix}$$

■

A.13 Cholesky Decomposition

For a symmetric positive-definite matrix $\mathbf{A} \in \mathcal{R}^{n\times n}$, there exists a unique lower triangular matrix $\mathbf{G} \in R^{n\times n}$ with positive diagonal components such that

$$\mathbf{A} = \mathbf{G}\mathbf{G}^T \tag{A.62}$$

The decomposition in Eq. (A.62) is known as the *Cholesky decomposition* and matrix $\mathbf{G}$ as the *Cholesky triangle*.

One of the methods that can be used to obtain the Cholesky decomposition of a given positive-definite matrix is based on the use of the outer-product updates [1] as illustrated below.

A positive-definite matrix $\mathbf{A} \in R^{n\times n}$ can be expressed as

$$\mathbf{A} = \begin{bmatrix} a_{11} & \mathbf{u}^T \\ \mathbf{u} & \mathbf{B} \end{bmatrix} \tag{A.63}$$

where a_{11} is a positive number. It can be readily verified that with

$$\mathbf{T} = \begin{bmatrix} \frac{1}{\sqrt{a_{11}}} & \mathbf{0} \\ -\mathbf{u}/a_{11} & \mathbf{I}_{n-1} \end{bmatrix} \tag{A.64}$$

we have

$$\mathbf{T}\mathbf{A}\mathbf{T}^T = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{B} - \mathbf{u}\mathbf{u}^T/a_{11} \end{bmatrix} \equiv \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_1 \end{bmatrix} \tag{A.65}$$

which implies that

$$\mathbf{A} = \begin{bmatrix} \sqrt{a_{11}} & \mathbf{0} \\ \mathbf{u}/\sqrt{a_{11}} & \mathbf{I}_{n-1} \end{bmatrix} \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{B} - \mathbf{u}\mathbf{u}^T/a_{11} \end{bmatrix} \begin{bmatrix} \sqrt{a_{11}} & \mathbf{u}/\sqrt{a_{11}} \\ \mathbf{0} & \mathbf{I}_{n-1} \end{bmatrix}$$
$$\equiv \mathbf{G}_1 \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_1 \end{bmatrix} \mathbf{G}_1^T \tag{A.66}$$

where $\mathbf{G}_1$ is a lower triangular matrix and $\mathbf{A}_1 = \mathbf{B} - \mathbf{u}\mathbf{u}^T/a_{11}$ is an $(n-1) \times (n-1)$ symmetric matrix. Since $\mathbf{A}$ is positive definite and $\mathbf{T}$ is nonsingular, it follows from Eq. (A.65) that matrix $\mathbf{A}_1$ is positive definite; hence the above procedure can be applied to matrix $\mathbf{A}_1$. In other words, we can find an $(n-1) \times (n-1)$ lower triangular matrix $\mathbf{G}_2$ such that

$$\mathbf{A}_1 = \mathbf{G}_2 \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_2 \end{bmatrix} \mathbf{G}_2^T \tag{A.67}$$

where $\mathbf{A}_2$ is an $(n-2) \times (n-2)$ positive-definite matrix. By combining Eqs. (A.66) and (A.67), we obtain

$$\mathbf{A} = \begin{bmatrix} \sqrt{a_{11}} & \mathbf{0} \\ \mathbf{u}/\sqrt{a_{11}} & \mathbf{G}_2 \end{bmatrix} \begin{bmatrix} \mathbf{I}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_2 \end{bmatrix} \begin{bmatrix} \sqrt{a_{11}} & \mathbf{u}^T/\sqrt{a_{11}} \\ \mathbf{0} & \mathbf{G}_2^T \end{bmatrix}$$
$$\equiv \mathbf{G}_{12} \begin{bmatrix} \mathbf{I}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_2 \end{bmatrix} \mathbf{G}_{12}^T \tag{A.68}$$

where $\mathbf{I}_2$ is the 2×2 identity matrix and $\mathbf{G}_{12}$ is lower triangular. The above procedure is repeated until the second matrix at the right-hand side of Eq. (A.68) is reduced to the identity matrix $\mathbf{I}_n$. The Cholesky decomposition of $\mathbf{A}$ is then obtained.

Example A.12 Compute the Cholesky triangle of the positive-definite matrix

$$\mathbf{A} = \begin{bmatrix} 4 & -2 & 1 \\ -2 & 7 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

Solution From Eq. (A.66), we obtain

$$\mathbf{G}_1 = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 0.5 & 0 & 1 \end{bmatrix}$$

and

$$\mathbf{A}_1 = \begin{bmatrix} 7 & -1 \\ -1 & 1 \end{bmatrix} - \frac{1}{4} \begin{bmatrix} -2 \\ 1 \end{bmatrix} [2\ 1] = \begin{bmatrix} 6 & -0.50 \\ -0.50 & 0.75 \end{bmatrix}$$

Now working on matrix $\mathbf{A}_1$, we get

$$\mathbf{G}_2 = \begin{bmatrix} \sqrt{6} & 0 \\ -0.5/\sqrt{6} & 1 \end{bmatrix}$$

and

$$\mathbf{A}_2 = 0.75 - (-0.5)^2/6 = 0.7083$$

In this case, Eq. (A.66) becomes

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 \\ -1 & \sqrt{6} & 0 \\ 0.5 & -0.5/\sqrt{6} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.7083 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ -1 & \sqrt{6} & 0 \\ 0.5 & -0.5/\sqrt{6} & 1 \end{bmatrix}^T$$

Finally, we use

$$\mathbf{G}_3 = \sqrt{0.7083} \approx 0.8416$$

to reduce $\mathbf{A}_2$ to $\mathbf{A}_3 = 1$, which leads to the Cholesky triangle

$$\mathbf{G} = \begin{bmatrix} 2 & 0 & 0 \\ -1 & \sqrt{6} & 0 \\ 0.5 & -0.5/\sqrt{6} & \sqrt{0.7083} \end{bmatrix} \approx \begin{bmatrix} 2 & 0 & 0 \\ -1 & 2.4495 & 0 \\ 0.5 & -0.2041 & 0.8416 \end{bmatrix}$$

■

A.14 Kronecker Product

Let $\mathbf{A} \in R^{p\times m}$ and $\mathbf{B} \in R^{q\times n}$. The Kronecker product of $\mathbf{A}$ and $\mathbf{B}$, denoted as $\mathbf{A} \otimes \mathbf{B}$, is a $pq \times mn$ matrix defined by

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & \cdots & a_{1m}\mathbf{B} \\ \vdots & & \vdots \\ a_{p1}\mathbf{B} & \cdots & a_{pm}\mathbf{B} \end{bmatrix} \tag{A.69}$$

where a_{ij} denotes the $(i,\ j)$th component of $\mathbf{A}$ [5]. It can be verified that

(i) $(\mathbf{A} \otimes \mathbf{B})^T = \mathbf{A}^T \otimes \mathbf{B}^T$

(ii) $(\mathbf{A} \otimes \mathbf{B}) \cdot (\mathbf{C} \otimes \mathbf{D}) = \mathbf{AC} \otimes \mathbf{BD}$ where $\mathbf{C} \in R^{m\times r}$ and $\mathbf{D} \in R^{n\times s}$

(iii) If $p = m$, $q = n$, and $\mathbf{A},\ \mathbf{B}$ are nonsingular, then

$$(\mathbf{A} \otimes \mathbf{B})^{-1} = \mathbf{A}^{-1} \otimes \mathbf{B}^{-1}$$

(iv) If $\mathbf{A} \in R^{m\times m}$ and $\mathbf{B} \in R^{n\times n}$, then the eigenvalues of $\mathbf{A} \otimes \mathbf{B}$ and $\mathbf{A} \otimes \mathbf{I}_n + \mathbf{I}_m \otimes \mathbf{B}$ are $\lambda_i\mu_j$ and $\lambda_i + \mu_j$, respectively, for $i = 1,\ 2,\ \ldots,\ m$ and $j = 1,\ 2,\ \ldots,\ n$, where λ_i and μ_j are the ith and jth eigenvalues of $\mathbf{A}$ and $\mathbf{B}$, respectively.

The Kronecker product is useful when we are dealing with matrix variables. If we use nvec($\mathbf{X}$) to denote the column vector obtained by stacking the column vectors of matrix $\mathbf{X}$, then it is easy to verify that for $\mathbf{M} \in R^{p\times m}$, $\mathbf{N} \in R^{q\times n}$ and $\mathbf{X} \in R^{n\times m}$, we have

$$\text{nvec}(\mathbf{N}\mathbf{X}\mathbf{M}^T) = (\mathbf{M} \otimes \mathbf{N})\text{nvec}(\mathbf{X}) \tag{A.70}$$

In particular, if $p = m = q = n$, $\mathbf{N} = \mathbf{A}^T$, and $\mathbf{M} = \mathbf{I}_n$, then Eq. (A.70) becomes

$$\text{nvec}(\mathbf{A}^T\mathbf{X}) = (\mathbf{I}_n \otimes \mathbf{A}^T)\text{nvec}(\mathbf{X}) \tag{A.71}$$

Similarly, we have

$$\text{nvec}(\mathbf{X}\mathbf{A}) = (\mathbf{A}^T \otimes \mathbf{I}_n)\text{nvec}(\mathbf{X}) \tag{A.71}$$

For example, we can apply Eq. (A.71) to the Lyapunov equation [5]

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} = -\mathbf{Q} \tag{A.72}$$

where matrices $\mathbf{A}$ and $\mathbf{Q}$ are given and $\mathbf{Q}$ is positive definite. First, we write Eq. (A.72) in vector form as

$$\text{nvec}(\mathbf{A}^T\mathbf{P}) + \text{nvec}(\mathbf{P}\mathbf{A}) = -\text{nvec}(\mathbf{Q}) \tag{A.73}$$

Using Eq. (A.71), Eq. (A.73) becomes

$$(\mathbf{I}_n \otimes \mathbf{A}^T)\text{nvec}(\mathbf{P}) + (\mathbf{A}^T \otimes \mathbf{I}_n)\text{nvec}(\mathbf{P}) = -\text{nvec}(\mathbf{Q})$$

which can be solved to obtain nvec($\mathbf{P}$) as

$$\text{nvec}(\mathbf{P}) = -(\mathbf{I}_n \otimes \mathbf{A}^T + \mathbf{A}^T \otimes \mathbf{I}_n)^{-1}\text{nvec}(\mathbf{Q}) \tag{A.74}$$

Example A.13 Solve the Lyapunov equation

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} = -\mathbf{Q}$$

for matrix $\mathbf{P}$ where

$$\mathbf{A} = \begin{bmatrix} -2 & -2 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{Q} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

Solution From Eq. (A.69), we compute

$$\mathbf{I}_2 \otimes \mathbf{A}^T + \mathbf{A}^T \otimes \mathbf{I}_2 = \begin{bmatrix} -4 & 1 & 1 & 0 \\ -2 & -2 & 0 & 1 \\ -2 & 0 & -2 & 1 \\ 0 & -2 & -2 & 0 \end{bmatrix}$$

Since

$$\text{nvec}(\mathbf{Q}) = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 2 \end{bmatrix}$$

Eq. (A.74) gives

$$\begin{aligned} \text{nvec}(\mathbf{P}) &= -(\mathbf{I}_2 \otimes \mathbf{A}^T + \mathbf{A}^T \otimes \mathbf{I}_2)^{-1}\text{nvec}(\mathbf{Q}) \\ &= -\begin{bmatrix} -4 & 1 & 1 & 0 \\ -2 & -2 & 0 & 1 \\ -2 & 0 & -2 & 1 \\ 0 & -2 & -2 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 3 \end{bmatrix} \end{aligned}$$

from which we obtain

$$\mathbf{P} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 3 \end{bmatrix}$$

■

A.15 Vector Spaces of Symmetric Matrices

Let $\mathcal{S}^n$ be the vector space of real symmetric $n \times n$ matrices. As in the n-dimensional Euclidean space where the inner product is defined for two vectors, the *inner product* for matrices $\mathbf{A}$ and $\mathbf{B}$ in $\mathcal{S}^n$ is defined as

$$\mathbf{A} \cdot \mathbf{B} = \text{trace}(\mathbf{AB})$$

If $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$, then we have

$$\mathbf{A} \cdot \mathbf{B} = \text{trace}(\mathbf{AB}) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_{ij} \tag{A.75}$$

The norm $\|\mathbf{A}\|_{\mathcal{S}^n}$ associated to this inner product is

$$\|\mathbf{A}\|_{\mathcal{S}^n} = \sqrt{\mathbf{A} \cdot \mathbf{A}} = \left[\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2 \right]^{1/2} = \|\mathbf{A}\|_F \tag{A.76}$$

where $\|\mathbf{A}\|_F$ denotes the Frobenius norm of $\mathbf{A}$ (see Sec. A.8.2).

An important set in space $\mathcal{S}^n$ is the set of all positive-semidefinite matrices given by

$$\mathcal{P} = \{\mathbf{X} : \ \mathbf{X} \in \mathcal{S}^n \text{ and } \mathbf{X} \succeq \mathbf{0}\} \tag{A.77}$$

A set $\mathcal{K}$ in a vector space is said to be a *convex cone* if $\mathcal{K}$ is a convex set such that $\mathbf{v} \in \mathcal{K}$ implies $\alpha\mathbf{v} \in \mathcal{K}$ for any nonnegative scalar α. It is easy to verify that set $\mathcal{P}$ forms a convex cone in space $\mathcal{S}^n$.

Let matrices $\mathbf{X}$ and $\mathbf{S}$ be two components of $\mathcal{P}$, i.e., $\mathbf{X} \succeq \mathbf{0}$ and $\mathbf{S} \succeq \mathbf{0}$. The eigendecomposition of $\mathbf{X}$ gives

$$\mathbf{X} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^T \tag{A.78}$$

where $\mathbf{U} \in R^{n\times n}$ is orthogonal and $\mathbf{\Lambda} = \text{diag}\{\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_n\}$. The decomposition in Eq. (A.78) can be expressed as

$$\mathbf{X} = \sum_{i=1}^{n} \lambda_i \mathbf{u}_i \mathbf{u}_i^T$$

where $\mathbf{u}_i$ denotes the ith column of $\mathbf{U}$. By using the property that

$$\text{trace}(\mathbf{AB}) = \text{trace}(\mathbf{BA})$$

(see Eq. (A.20)), we can compute the inner product $\mathbf{X} \cdot \mathbf{S}$ as

$$\begin{aligned}
\mathbf{X} \cdot \mathbf{S} &= \text{trace}(\mathbf{XS}) = \text{trace}\left(\sum_{i=1}^{n} \lambda_i \mathbf{u}_i \mathbf{u}_i^T \mathbf{S}\right) = \sum_{i=1}^{n} \lambda_i \ \text{trace}(\mathbf{u}_i \mathbf{u}_i^T \mathbf{S}) \\
&= \sum_{i=1}^{n} \lambda_i \ \text{trace}(\mathbf{u}_i^T \mathbf{S} \mathbf{u}_i) = \sum_{i=1}^{n} \lambda_i \mu_i
\end{aligned} \tag{A.79}$$

where $\mu_i = \mathbf{u}_i^T \mathbf{S} \mathbf{u}_i$. Since both $\mathbf{X}$ and $\mathbf{S}$ are positive semidefinite, we have $\lambda_i \geq 0$ and $\mu_i \geq 0$ for $i = 1,\ 2,\ \ldots,\ n$. Therefore, Eq. (A.79) implies that

$$\mathbf{X} \cdot \mathbf{S} \geq 0 \tag{A.80}$$

In other words, the inner product of two positive-semidefinite matrices is always nonnegative.

A further property of the inner product on set $\mathcal{P}$ is that if $\mathbf{X}$ and $\mathbf{S}$ are positive semidefinite and $\mathbf{X} \cdot \mathbf{S} = 0$, then the product matrix $\mathbf{XS}$ must be the zero matrix, i.e.,

$$\mathbf{XS} = \mathbf{0} \tag{A.81}$$

To show this, we can write

$$\mathbf{u}_i^T \mathbf{XS} \mathbf{u}_j = \mathbf{u}_i^T \left(\sum_{k=1}^{n} \lambda_k \mathbf{u}_k \mathbf{u}_k^T\right) \mathbf{S} \mathbf{u}_j = \lambda_i \mathbf{u}_i^T \mathbf{S} \mathbf{u}_j \tag{A.82}$$

Using the Cauchy-Schwartz inequality (see Eq. (A.25)), we have

$$|\mathbf{u}_i^T \mathbf{S} \mathbf{u}_j|^2 = |(\mathbf{S}^{1/2}\mathbf{u}_i)^T(\mathbf{S}^{1/2}\mathbf{u}_j)|^2 \leq \|\mathbf{S}^{1/2}\mathbf{u}_i\|^2 \|\mathbf{S}^{1/2}\mathbf{u}_j\|^2 = \mu_i \mu_j \tag{A.83}$$

Now if $\mathbf{X} \cdot \mathbf{S} = 0$, then Eq. (A.79) implies that

$$\sum_{i=0}^{n} \lambda_i \mu_i = 0 \tag{A.84}$$

Since λ_i and μ_i are all nonnegative, Eq. (A.84) implies that $\lambda_i \mu_i = 0$ for $i = 1, 2, \ldots, n$; hence for each index i, either $\lambda_i = 0$ or $\mu_i = 0$. If $\lambda_i = 0$, Eq. (A.82) gives

$$\mathbf{u}_i^T \mathbf{XSu}_j = 0 \tag{A.85}$$

If $\lambda_i \neq 0$, then μ_i must be zero and Eq. (A.83) implies that $\mathbf{u}_i^T \mathbf{Su}_j = 0$ which, in conjunction with Eq. (A.82) also leads to Eq. (A.85). Since Eq. (A.85) holds for any i and j, we conclude that

$$\mathbf{U}^T \mathbf{XSU} = \mathbf{0} \tag{A.86}$$

Since $\mathbf{U}$ is nonsingular, Eq. (A.86) implies Eq. (A.81).

Given $p + 1$ symmetric matrices $\mathbf{F}_0, \mathbf{F}_1, \ldots, \mathbf{F}_p$ in space $\mathcal{S}^n$, and a p-dimensional vector $\mathbf{x} = [x_1 \; x_2 \; \cdots \; x_p]^T$, we can generate a symmetric matrix

$$\mathbf{F}(\mathbf{x}) = \mathbf{F}_0 + x_1\mathbf{F}_1 + \cdots + x_p\mathbf{F}_p = \mathbf{F}_0 + \sum_{i=1}^{p} x_i \mathbf{F}_i \tag{A.87}$$

which is said to be *affine* with respect to $\mathbf{x}$. Note that if the constant term $\mathbf{F_0}$ were a zero matrix, then $\mathbf{F}(\mathbf{x})$ would be a *linear* function of vector $\mathbf{x}$, i.e., $\mathbf{F}(\mathbf{x})$ would satisfy the condition $\mathbf{F}(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha\mathbf{F}(\mathbf{x}) + \beta\mathbf{F}(\mathbf{y})$ for any vectors $\mathbf{x}, \mathbf{y} \in R^p$ and any scalars α and β. However, because of the presence of $\mathbf{F}_0$, $\mathbf{F}(\mathbf{x})$ in Eq. (A.87) is *not* linear with respect to $\mathbf{x}$ in a strict sense and the term 'affine' is often used in the literature to describe such a class of matrices. In effect, the affine property is a somewhat relaxed version of the linearity property.

In the context of linear programming, the concept of an *affine manifold* is sometimes encountered. A manifold is a subset of the Euclidean space that satisfies a certain structural property of interest, for example, a set of vectors satisfying the relation $\mathbf{x}^T\mathbf{c} = \beta$. Such a set of vectors may possess the affine property, as illustrated in the following example.

Example A.14 Describe the set of n-dimensional vectors $\{\mathbf{x} : \mathbf{x}^T\mathbf{c} = \beta\}$ for a given vector $\mathbf{c} \in R^{n\times 1}$ and a scalar β as an affine manifold in the n-dimensional Euclidean space E^n.

Solution Obviously, the set of vectors $\{\mathbf{x} : \mathbf{x}^T\mathbf{c} = \beta\}$ is a subset in E^n. If we denote $\mathbf{x} = [x_1 \; x_2 \; \cdots \; x_n]^T$ and $\mathbf{c} = [c_1 \; c_2 \; \cdots \; c_n]^T$, then equation $\mathbf{x}^T\mathbf{c} = \beta$ can be expressed as $F(\mathbf{x}) = 0$ where

$$F(\mathbf{x}) = -\beta + x_1c_1 + x_2c_2 + \cdots + x_nc_n \tag{A.88}$$

By viewing $-\beta, c_1, c_2, \ldots, c_n$ as one-dimensional symmetric matrices, $F(\mathbf{x})$ in Eq. (A.88) assumes the form in Eq. (A.87), which is affine with respect to $\mathbf{x}$. Therefore the set $\{\mathbf{x} : \mathbf{x}^T\mathbf{c} = \beta\}$ is an affine manifold in E^n.

■

Example A.15 Convert the following constraints

$$\mathbf{X} = (x_{ij}) \succeq \mathbf{0} \qquad \text{for } i,\ j = 1,\ 2,\ 3 \tag{A.89}$$

and

$$x_{ii} = 1 \qquad \text{for } i = 1,\ 2,\ 3 \tag{A.89}$$

into a constraint of the type

$$\mathbf{F}(\mathbf{x}) \succeq \mathbf{0} \tag{A.90}$$

for some vector variable $\mathbf{x}$ where $\mathbf{F}(\mathbf{x})$ assumes the form in Eq. (A.87).

Solution The constraints in Eqs. (A.89a) and (A.89b) can be combined into

$$\mathbf{X} = \begin{bmatrix} 1 & x_{12} & x_{13} \\ x_{12} & 1 & x_{23} \\ x_{13} & x_{23} & 1 \end{bmatrix} \succeq \mathbf{0} \tag{A.91}$$

Next we write matrix $\mathbf{X}$ in (A.91) as

$$\mathbf{X} = \mathbf{F}_0 + x_{12}\mathbf{F}_1 + x_{13}\mathbf{F}_2 + x_{23}\mathbf{F}_3$$

where $\mathbf{F}_0 = \mathbf{I}_3$ and

$$\mathbf{F}_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{F}_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{F}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Hence the constraint in Eq. (A.89) can be expressed in terms of Eq. (A.90) with $\mathbf{F}_i$ given by the above equations and $\mathbf{x} = [x_{12}\ x_{13}\ x_{23}]^T$.

■

A.16 Polygon, Polyhedron, Polytope, and Convex Hull

A *polygon* is a closed plane figure with an arbitrary number of sides. A polygon is said to be convex if the region inside the polygon is a convex set (see Def. 2.7). A convex polygon with m sides can be described in terms of m linear inequalities which can be expressed in matrix form as

$$\mathcal{P}_y = \{\mathbf{x} : \ \mathbf{A}\mathbf{x} \geq \mathbf{b}\} \tag{A.92}$$

where $\mathbf{A} \in R^{m\times 2}$, $\mathbf{x} \in R^{2\times 1}$, and $\mathbf{b} \in R^{m\times 1}$.

A *convex polyhedron* is an n-dimensional extension of a convex polygon. A convex polyhedron can be described by the equation

$$\mathcal{P}_h = \{\mathbf{x} : \ \mathbf{A}\mathbf{x} \geq \mathbf{b}\} \tag{A.93}$$

where $\mathbf{A} \in R^{m\times n}$, $\mathbf{x} \in R^{n\times 1}$, and $\mathbf{b} \in R^{m\times 1}$. For example, a 3-dimensional convex polyhedron is a 3-dimensional solid which consists of several polygons, usually joined at their edges such as that shown Fig. 11.4.

A polyhedron may or may not be bounded depending on the numerical values of $\mathbf{A}$ and $\mathbf{b}$ in Eq. (A.93). A bounded polyhedron is called a *polytope*.

Given a set of points $\mathcal{S} = \{p_1,\ p_2,\ \ldots,\ p_L\}$ in an n-dimensional space, the *convex hull* spanned by $\mathcal{S}$ is defined as the smallest convex set that contains $\mathcal{S}$. It can be verified that the convex hull is characterized by

$$\text{Co}\{p_1,\ p_2,\ \ldots,\ p_L\} = \{p:\ p = \sum_{i=1}^{L} \lambda_i p_i,\ \lambda_i \geq 0, \sum_{i=1}^{L} \lambda_i = 1\} \tag{A.94}$$

In the above definition, each point p_i represents an abstract n-dimensional point. For example, if point p_i is represented by an n-dimensional vector, say, $\mathbf{v}_i$, then the convex hull spanned by the L vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_L\}$ is given by

$$\text{Co}\{\mathbf{v}_1,\ \mathbf{v}_2,\ \ldots,\ \mathbf{v}_L\} = \{\mathbf{v}:\ \mathbf{v} = \sum_{i=1}^{L} \lambda_i \mathbf{v}_i,\ \lambda_i \geq 0,\ \sum_{i=1}^{L} \lambda_i = 1\} \tag{A.95}$$

Alternatively, if point p_i is represented by a pair of matrices $[\mathbf{A}_i\ \ \mathbf{B}_i]$ with $\mathbf{A}_i \in R^{n\times n}$ and $\mathbf{B}_i \in R^{n\times m}$, then the convex hull spanned by $\{[\mathbf{A}_i\ \ \mathbf{B}_i]$ for $i = 1,\ 2,\ \ldots,\ L\}$ is given by

$$\begin{aligned} &\text{Co}\{[\mathbf{A}_1\ \mathbf{B}_1],\ [\mathbf{A}_2\ \mathbf{B}_2],\ldots,\ [\mathbf{A}_L\ \mathbf{B}_L]\} = \\ &\quad \{[\mathbf{A}\ \mathbf{B}]:\ [\mathbf{A}\ \mathbf{B}] = \sum_{i=1}^{L} \lambda_i [\mathbf{A}_i\ \mathbf{B}_i],\ \lambda_i \geq 0,\ \sum_{i=1}^{L} \lambda_i = 1\} \end{aligned}$$

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