ON COMPLETE TOPOLOGICAL SPACES*

BY

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Introduction

1. The notion of "completeness" is usually defined only for metric spaces (cf. for instance [1], p. 103). This seems reasonable, because this notion necessarily involves a certain "uniformity of the topology" of the space under consideration. Indeed, the definition of "completeness" is as follows:

Definition I. *If M is a space in which there is defined a metric* dist (f, g) *satisfying the usual postulates for distance* ([1], *p.* 94), *then a sequence F: $f_1, f_2, \cdots$ is fundamental if, for every $\delta > 0$, there exists an $n_1 = n_1(\delta)$ such that $m, n \geqq n_1$ imply* dist $(f_m, f_n) < \delta$; *and F is convergent if there exists an f such that, for every $\delta > 0$, there exists an $n_2 = n_2(\delta)$ such that $n \geqq n_2$ implies* dist $(f, f_n) < \delta$. *M is complete if every fundamental sequence is convergent.*

The need of uniformity in M arises from the fact that the elements of a fundamental sequence are postulated to be "near to each other," and not near to any fixed point. As a general topological space (cf. for instance [1], pp. 226–232) has no property which lends itself to the definition of such a "uniformity," it is improbable that a reasonable notion of "completeness" could be defined in it.

However, linear spaces (cf. [1], pp. 95–97, and Definition 1 in this paper), even if only topological, afford a possibility of "uniformization" for their topology: because of their homogeneity everything can be discussed in the neighborhood of 0. Thus one might introduce

Definition I′. *If L is a linear space (cf. above) with a topology (cf.* [1], *pp.* 226–232; *of course the "linear" operations αf [α a real number] and $f+g$ are supposed to be continuous), then a sequence $f_1, f_2, \cdots$ is fundamental if, for every neighborhood U of* 0 *(zero), there exists an $n_1 = n_1(U)$ such that $m, n \geqq n_1$ imply $f_m - f_n \epsilon U$;† and convergent if an f can be found so that for every neighborhood U of* 0 *there exists an $n_2 = n_2(U)$ such that $n \geqq n_2$ implies $f - f_n \epsilon U$. L is complete if every fundamental sequence is convergent.*

* Presented to the Society, December 28, 1934; received by the editors June 7, 1934.

† The fact that an element x belongs to a set S will be denoted by $x \epsilon S$ (not by $x \subset S$), while $T \subset S$ will mean that the set T is a subset of the set S. Other set-theoretical notations will be used: the sum of a set $(S, T, \cdots)$ of sets is $\mathfrak{S}(S, T, \cdots)$, the product (that is, the common part of the elements) of $(S, T, \cdots)$ is $\mathfrak{P}(S, T, \cdots)$, and the complementary set to S is $\mathfrak{C}S$. (These are not the notations of [1].)

This coincides with the previous definition if one defines, as usual, the neighborhoods of a point f_0 in the linear-metric case (in which dist $(f, g) \equiv$ dist $(f-g, 0) \equiv$ dist $(f-g)$) as spheres $S(f_0; \delta)$: dist $(f, f_0) =$ dist $(f-f_0) < \delta$, $\delta > 0$.

Another important notion is "total boundedness" (cf. [1], p. 108). We give the usual definition in the metric case and the generalization for the linear-topological case:

Definition II. *If M is a space in which a metric* dist (f, g) *is defined (cf. above), a set $S \subset M$ is totally bounded if, for every $\delta > 0$, there exist a finite number of spheres $S(f_1; \delta), \cdots, S(f_n; \delta)$ (of course $n, f_1, \cdots, f_n$ all depend on δ) such that $M \subset \mathfrak{S}(S(f_1; \delta), \cdots, S(f_n; \delta))$.*

Definition II′. *If L is a linear space with a topology (cf. above), a set $S \subset L$ is totally bounded if, for every neighborhood U of* 0, *a finite number of points $f_1, \cdots, f_n$ exist ($n, f_1, \cdots, f_n$ all depend on U) such that $M \subset \mathfrak{S}(f_1 + U, \cdots, f_n + U)$.**

Finally, we repeat the well known definition of "compactness" in a form which is particularly suited for our purposes.

Definition III. *If N is any topological space (cf. above) a set $S \subset N$ is compact if every infinite set $T \subset S$ has a condensation point† $f \epsilon S$. If we require only $f \epsilon N$, this expresses (at least if the countability axiom is satisfied) that S has a compact closure.*

An important fact connecting these notions is that I and II imply III (with $N = M$), the proof resulting from a simple application of the diagonal principle (cf. [1], pp. 108–109). The proof can be transferred immediately to the non-metric case: I′ and II′ imply III (with $N = L$), provided that the topology of L fulfills Hausdorff's first countability axiom (cf. [1], p. 229, axiom (9); for the proof, cf. Theorem 15 in this paper). That is: if L is complete and fulfills Hausdorff's first countability axiom, every totally bounded set $S \subset L$ has a compact closure.

2. It seems desirable, for various reasons, to get rid of the restriction represented by the countability axiom. Some important examples of linear spaces do not fulfill it.‡ Furthermore, the notions of total boundedness and closure-

* If L is a linear space, we use the following notation: (f, $g \epsilon L$; S, $T \subset L$; α, β real numbers): αS is the set of all αf, $f \epsilon S$; $f \pm S$ is the set of all $f \pm g$, $g \epsilon S$; $S \pm T$ is the set of all $f \pm g$, $f \epsilon S$ and $g \epsilon T$. Note that $\alpha(S+T) = \alpha S + \alpha T$, $(\alpha\beta)S = \alpha(\beta S)$, and $S+T = T+S$, $(S+T)+R = S+(T+R)$; but only the weakened conditions $\alpha S + \beta S \supset (\alpha+\beta)S$, $(S \pm T) \mp T \supset S$ are valid.

† f is a condensation point of T if $\mathfrak{P}(T, U)$ is infinite for every neighborhood U of f.

‡ For example, Hilbert space in its "weak" topology (cf. for instance [2], p. 379); the space of all bounded operators in Hilbert space, in its "strong" and in its "weak" topology (cf. [2], pp. 381–382; for the discussion of all these topologies, [2], pp. 378–388).

compactness play an important role in the general theory of almost periodic functions (cf. the following paper of S. Bochner and J. von Neumann on this subject), and their equivalence is necessary for the smooth working of this theory, which makes no other use of the countability axiom, and which therefore should be workable without its help. (This has actually been done, loc. cit., by the use of the results of this paper.) But if we use the definition of completeness given in I′, then II′ does not necessarily imply III (that is, total boundedness does not imply closure-compactness) if the countability axiom does not hold. Therefore we have to find another definition of completeness which leads to the desired implication.

The simplest thing is to postulate this directly:

DEFINITION IV. *If L is a linear space with a topology, it is topologically complete if every totally bounded set $S \subset L$ has a compact closure.*

For metric linear spaces, and even for every linear space satisfying the countability axiom, this is equivalent to the usual definition of completeness (I or I′; cf. Theorem 15). The various spaces mentioned at the beginning of this chapter are topologically complete (cf. Theorem 23). The most important property of this notion is, however, that if L is topologically complete, the linear space formed by the functions with a given domain D and with a range $\subset L$ is (if subjected to certain restrictions, like boundedness, etc., cf. Definition 11) topologically complete too. This is rather obvious for the Definitions I and I′, but not at all for IV; we will prove it in Theorem 18. All these properties make our notion of topological completeness just as useful for various applications (for instance in the generalized theory of almost periodic functions, as mentioned above), as the usual notion of (metric) completeness, while its range of generality is essentially wider.

We now pass on to the exact exposition of the subject.

I. DEFINITIONS

3. We define linear spaces in the usual way (cf. for instance [1], pp. 95–97):

DEFINITION 1. *The set L is a linear space, if, for $f, g \in L$ and any real number α,* αf and $f+g$ are in L and are defined so that*

(1) $f+g=g+f,$ (2) $(f+g)+h=f+(g+h),$

(3) $1\cdot f=f,$ (4) $\alpha(\beta f)=(\alpha\beta)f,$

(5) $(\alpha+\beta)f=\alpha f+\beta f,$ (6) $\alpha(f+g)=\alpha f+\alpha g,$

(7) $f+h=g+h$ *implies* $f=g.$

* It would be sufficient to admit only rational α's.

By (3) and (5), $f+0\cdot f=f$; by (1) and (2), $(f+g)+0\cdot f=f+g$; by interchanging f and g, $(f+g)+0\cdot g=f+g$; by (7), $0\cdot f=0\cdot g$. Thus $0\cdot f$ is independent of f; and we call it 0 (zero). We have $f+0=f$. Writing $-f$ for $(-1)\cdot f$, (5) gives $f+(-f)=0$; writing $f-g$ for $f+(-g)$, (1) and (2) give $(f-g)+g=f$; and by (7), $x=f-g$ is the only solution of $x+g=f$. Now all rules of computation for 0, $-f$, $f-g$ are easily deduced.

A metric (or an "absolute value") in L is defined in the usual way (cf. [1], p. 97):

DEFINITION 2a. *The linear set L is metric if, for every $f\epsilon L$, a real number $\|f\|$, its "absolute value," is defined, such that*

(1) $\|f\| > 0$ *if* $f \neq 0$, (2) $\|\alpha f\| = |\alpha|\cdot\|f\|$, (3) $\|f+g\| \leqq \|f\|+\|g\|$.

The metric is then defined by dist $(f, g)=\|f-g\|$.

This dist (f, g) possesses the characteristic properties of a distance and can be used to define a topology in L (cf. [1], p. 94; also the end of paragraph 1 of this paper). However, we shall not assume that L is metric, but only that it has a topology. This is done in the following definition, in which it was attempted to reduce the strength of the postulates to the necessary minimum.

DEFINITION 2b. *The linear set L is topological if a set $\mathfrak{U}$ of sets $U\subset L$ is given such that*

(1) *if* $U\epsilon\mathfrak{U}$, *then* $0\epsilon U$,
(2) *there is a sequence* $U_1, U_2, \cdots \epsilon\mathfrak{U}$ *such that* $\mathfrak{P}(U_1, U_2, \cdots)=(0)$,†‡
(3) *if* $U, V\epsilon\mathfrak{U}$, *there is a* $W\epsilon\mathfrak{U}$ *with* $W\subset\mathfrak{P}(U, V)$,‡
(4) *if* $U\epsilon\mathfrak{U}$, *there is a* $V\epsilon\mathfrak{U}$ *such that, for every* α *with* $-1\leqq\alpha\leqq 1$, $\alpha V\subset U$,§
(5) *if* $U\epsilon\mathfrak{U}$, *there is a* $V\epsilon\mathfrak{U}$ *with* $V+V\subseteq U$,§
(6) *if* $f\epsilon L$, $U\epsilon\mathfrak{U}$, *there is an* α *with* $f\epsilon\alpha U$.§

L is "convex" if the further condition

(7) *if* $U\epsilon\mathfrak{U}$, *then* $U+U\subset 2U$§

is fulfilled.

† If Hausdorff's first countability axiom holds, (2) is fulfilled by choosing a complete system of neighborhoods of 0 for $U_1, U_2, \cdots$ (cf. [1], p. 229, axiom (9)), but the converse is not true: (2) is essentially weaker than the countability axiom. This is shown by the examples of Part IV, Theorem 23; cf. [3], p. 264.

‡ (2) and (3) could be replaced by two other postulates (2′) and (3′) which extend (2) and restrict (3):

(2′) there is an aleph $\aleph^*$ and a set $(U, V, \cdots)\subset\mathfrak{U}$ with this aleph $\aleph^*$, such that $\mathfrak{P}(U, V, \cdots)=(0)$;

(3′) for each set $(U, V, \cdots)\subset\mathfrak{U}$ with an aleph $<\aleph^*$ there is a $W\epsilon\mathfrak{U}$ with $W\subset\mathfrak{P}(U, V, \cdots)$.

All our discussions could be carried through, with little change, on this basis. In the present form of (2) and (3), $\aleph^*=\aleph_0$ (the set $(U, V, \cdots)$ is countable).

§ See first footnote on p. 2.

If L is metric, we consider the spheres $(\delta > 0)$

$$S^{o}(f_0;\,\delta)\text{: the set of all } f \text{ with } \|f - f_0\| < \delta,$$
$$S^{cl}(f_0;\,\delta)\text{: the set of all } f \text{ with } \|f - f_0\| \leqq \delta.$$

Choosing $\mathfrak{U}$ as the set of all $S^{o}(0;\,\delta)$ or as the set of all $S^{cl}(0;\,\delta)$ makes L topological and convex (one easily verifies Definition 2b, (1)–(7)), and the topology, which we will define with the aid of $\mathfrak{U}$ in Definition 4 and Theorem 6, coincides in this case with the usual metric topology of L.

II. General theorems

4. In the following discussion of Chapters II and III, L is assumed merely to be a topological space, that is, to fulfill Definition 2b, (1)–(6), except where the contrary is expressly stated.

Theorem 1. *If $U \epsilon \mathfrak{U}$, $A > 0$, $n = 1, 2, \cdots$, there is, for each value of n, a $V \epsilon \mathfrak{U}$ such that, for all sets $\alpha_1, \cdots, \alpha_n$ with $-A \leqq \alpha_1, \cdots, \alpha_n \leqq A$, $\alpha_1 V + \cdots + \alpha_n V \subset U$.*

Increasing A and n strengthens the statement, so we may assume $A = 2^p, n = 2^q, p, q = 0, 1, 2, \cdots$. It is sufficient to obtain $AV + \cdots + AV \subset U$ (n addends), because the W of Definition 2b, 4 (with $\alpha W \subset V$ for $-1 \leqq \alpha \leqq 1$) will then have the desired properties. As $AV \subset V + \cdots + V$ (A addends),* the statement is strengthened if we replace A, n by $1, An = 2^{p+q}$, respectively. Thus we may assume $A = 1$, $n = 2^r$, $r = 0, 1, 2, \cdots$. We have to find a $V \epsilon \mathfrak{U}$ with $V + \cdots + V \subset U$ (2^r addends).

For $r = 0$ we may choose $V = U$; if we have a V for any $r = 0, 1, 2, \cdots$, we can apply Definition 2b, (5), to it, and thus obtain a V for $r+1$. This completes the proof.

Definition 3. *If $S \subset L$, S_i is the set of all f for which a $U \epsilon \mathfrak{U}$ with $f + U \subset S$ exists.*†

Theorem 2. *$S_{ii} = S_i \subset S$.*

$0 \epsilon U$, $f \epsilon f + U$, so that $S_i \subset S$. Therefore $S_{ii} \subset S_i$. Now if $f \epsilon S_i$, that is, if $f + U \subset S$, choose a $V \epsilon \mathfrak{U}$ with $V + V \subset U$ (Definition 2b, (5)). Then for $g \epsilon f + V$, $g + V \subset f + V + V \subset f + U \subset S$, and $g \epsilon S_i$. Thus $f + V \subset S_i$, $f \epsilon S_{ii}$, and therefore $S_i \subset S_{ii}$. This completes the proof.

Theorem 3. *$0 \epsilon U_i$; $(f+S)_i = f + S_i$; if $\alpha \neq 0$, $(\alpha S)_i = \alpha S_i$; $S_i + T_i \subset (S+T)_i$.*

The first two statements are obvious. If $f \epsilon S_i$ and $f + U \subset S$, then $\alpha f + \alpha U \subset \alpha S$, and if V is chosen by Theorem 1 with $V/\alpha \subset U$, $\alpha f + V \subset \alpha S$ and

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* See first footnote on p. 2.

† S_i is the set of inner points of S (cf. Theorem 4).

$\alpha f \epsilon (\alpha S)_i$. Thus $\alpha S_i \subset (\alpha S)_i$; substitution of $1/\alpha$, αS for α, S leads to the result that $(\alpha S_i) \subset \alpha S_i$, proving the third statement. If $f \epsilon S_i$ and $g \epsilon T_i$, $f+g \epsilon f+T_i=(f+T)_i \subset (S+T)_i$, proving the fourth statement.

Definition 4. *S is open if $S=S_i$; S is closed if $\mathfrak{C}S$* is open. This means that if S is closed, $S=S_{cl}$, where we define $S_{cl}=\mathfrak{C}((\mathfrak{C}S)_i)$.*†

Theorem 4. *S_i is open and the greatest open set $\subset S$, S_{cl} is closed and the smallest closed set $\supset S$.*

The statements about S_i follow from Theorems 2 and 3; those about S_{cl} result from considering $\mathfrak{C}S$.

Theorem 5. *For every S, $S_{cl}=\mathfrak{P}(S+U)$, where U runs over all elements of $\mathfrak{U}$.*

If $U \epsilon \mathfrak{U}$, there is a $V \epsilon \mathfrak{U}$ with $-V \subset U$ (Definition 2b, (4)), and thus $V \subset -U$. For this reason $\mathfrak{P}(S+U)=\mathfrak{P}(S-V)$ (U, V run over all $\mathfrak{U}$). Now $f \epsilon \mathfrak{P}(S-V)$ means that for every $V \epsilon \mathfrak{U}$, $f \epsilon g-V$ for some $g \epsilon S$, that is, $g \epsilon S$, $g \epsilon f+V$. But this is equivalent to $f \epsilon S_{cl}$.

Theorem 6. *In the sense in which Hausdorff defined a topology, using the open sets as fundamental notions* (*cf.* [1], *p.* 228), *Definition* 4 *describes a regular Hausdorff topology, that is, one which fulfills Hausdorff's axioms* (1)–(3) *and* (6) (*cf.* [1], *pp.* 228–229; *thus all axioms* (1)–(6) *are fulfilled*).

Ad (1): 0 and L are obviously open. Ad (2): If S_1, S_2 are open, assume $f \epsilon \mathfrak{P}(S_1, S_2)$. Then $f+U_1 \subset S_1$, $f+U_2 \subset S_2$, and so with $V \epsilon \mathfrak{U}$, $V \subset \mathfrak{P}(U_1, U_2)$ (by Definition 2b, (3)) and $f+V \subset \mathfrak{P}(S_1, S_2)$, proving that $f \epsilon \mathfrak{P}(S_1, S_2)_i$. Thus $\mathfrak{P}(S_1, S_2)$ is open. Ad (3): If $S, T, \cdots$ are open, $\mathfrak{S}(S, T, \cdots)$ is obviously open. Ad (6): Let $\overline{S}$ be closed, f not an element of $\overline{S}$. Then $\mathfrak{C}\overline{S}$ is open, $f \epsilon \mathfrak{C}\overline{S}$, so that $U \epsilon \mathfrak{U}$, $f+U \subset \mathfrak{C}\overline{S}$. Choosing $V \epsilon \mathfrak{U}$ with $V+V \subset U$ (by Definition 2b, (3)) we have $(f+V_i)_{cl} \subset (f+V)_{cl} \subset f+V+V$ (by Theorem 5) $\subset f+U \subset \mathfrak{C}\overline{S}$, that is, for $T=f+V_i$, T is open, $f \epsilon T$, and $\mathfrak{P}(T_{cl}, \overline{S})$ is empty.

In the following discussions we shall always consider L as topologized by the topology of Definition 4 and Theorem 6, except where the contrary is explicitly stated. Thus we can use the whole topological terminology: we can speak of open and closed sets, which have already been defined in harmony with this by Definition 4, of continuous functions, limits of sequences, condensation points, etc.

Theorem 7. *αf and $f+g$ are continuous functions of α, f and f, g respectively.*

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* See second footnote on p. 1.

† S_{cl}, the closure of S, is the set of all points and all condensation points of S (cf. Theorem 4).

Ad $f+g$: Assume $f_0+g_0 \epsilon S$, S open. Choose $U\epsilon\mathfrak{U}$, $f_0+g_0+U\subset S$, $V\epsilon\mathfrak{U}$, $V+V\subset U$. Then

$$(f_0+V_i)+(g_0+V_i)\subset f_0+g_0+V_i+V_i\subset f_0+g_0+V+V\subset f_0+g_0+U\subset S,$$

that is, the sets $T_1=f_0+V_i$, $T_2=g_0+V_i$ are open, $f_0\epsilon T_1$, $g_0\epsilon T_2$, $T_1+T_2\subset S$. Ad αf: Due to the continuity of $f+g$ (and Definition 1, (5)–(6)) we need consider only $\alpha_0=0$ or $f_0=0$. Then, in any event, $\alpha_0 f_0=0$, so that we have the situation $0\epsilon S$, S open. Ad $\alpha_0=0$: Choose $U\epsilon\mathfrak{U}$, $U\subset S$, and $V\epsilon\mathfrak{U}$ by Theorem 1 with $n=2$, $A=1$. Now choose a β with $f_0\epsilon\beta V$ (by Definition 2b, (6)); then for

$$|\alpha|<\frac{1}{\max(1,\ |\beta|)}$$

we have $\alpha(f_0+V)\subset\alpha\beta V+\alpha V\subset U\subset S$, that is, the sets

$$A:\qquad |\alpha|<\frac{1}{\max(1,\ |\beta|)}$$

and $T=f_0+V_i$ are open, $0\epsilon A$, $f_0\epsilon T$, $AT\subset S$. Ad $f_0=0$: Choose $U\epsilon\mathfrak{U}$, $U\subset S$ and $V\epsilon\mathfrak{U}$ by Theorem 1 with $n=1$, $A=|\alpha_0|+1$. Then $|\alpha-\alpha_0|<1$ implies $|\alpha|\leqq A$, $\alpha V\subset U\subset S$, that is, the sets A: $|\alpha-\alpha_0|<1$ and $T=V_i$ are open, $\alpha_0\epsilon A$, $0\epsilon T$, $AT\subset S$.

5. We now introduce the notion of boundedness, which is usually considered as a metric notion. One sees at once, remembering the remarks made at the end of §3, that our general definition coincides in the case of a metric L with the usual definition.

Definition 5. *S is bounded if, for every $U\epsilon\mathfrak{U}$, there is an α such that $S\subset\alpha U$.*

Remark. We could replace herein the $U\epsilon\mathfrak{U}$ by the open sets T with $0\epsilon T$ for if T is such a set, a $U\epsilon\mathfrak{U}$, $U\subset T$, exists, and for $U\epsilon\mathfrak{U}$ an open T, $0\epsilon T$, $T\subset U$ exists: $T=U_i$.

Theorem 8. *Every finite set is bounded. If $S,\cdots,T$ are a finite number of bounded sets, $\mathfrak{S}(S,\cdots,T)$ is bounded. If S, T are bounded sets, αS, $f+S$, $S+T$ are bounded.*

It follows by Definition 2b, (6), that a one-element set (f) is bounded; therefore every finite set is bounded if the second statement is true. If the second statement holds for two addends, it holds by induction for any finite number. Let S, T be bounded, $U\epsilon\mathfrak{U}$, choose $V\epsilon\mathfrak{U}$ by Definition 2b, (4), and α, β with $S\subset\alpha V$, $T\subset\beta V$. Then for $\gamma=\max(|\alpha|,\ |\beta|)$,

$$S\subset\alpha V=\gamma\left(\frac{\alpha}{\gamma}V\right)\subset\gamma U;$$

similarly, $T \subset \gamma U$, $\mathfrak{S}(S, T) \subset \gamma U$. Thus the first two statements are proved.

In the last statement the first part (concerning αS) is obvious; the second follows from the third by putting $T=(f)$, so that we have to prove only the third part. Assume $U \epsilon \mathfrak{U}$, choose $V \epsilon \mathfrak{U}$ by Theorem 1 with $n=2$, $A=1$. Choose β, γ with $S \subset \beta V$, $T \subset \gamma V$. Then, for $\delta = \max(|\beta|, |\gamma|)$,

$$S + T \subset \beta V + \gamma V = \delta\left(\frac{\beta}{\delta} V\right) + \delta\left(\frac{\gamma}{\delta} V\right) \subset \delta U.$$

This completes the proof.

The next definition is a repetition of Definition II′ in §1 (cf. the comment given there).

Definition 6. *S is totally bounded if, for every $U \epsilon \mathfrak{U}$, there is a finite number of elements $f_1, \cdots, f_n$ of L, such that $S \subset \mathfrak{S}(f_1 + U, \cdots, f_n + U)$.*

Remark 1. The $f_1, \cdots, f_n$ could be restricted to S. In this form the condition is obviously sufficient; but it is also necessary: Choose $V \epsilon \mathfrak{U}$ by Theorem 1, with $n=2$, $A=1$; then $V - V \subset U$. Apply Definition 6 to V: $S \subset \mathfrak{S}(g_1 + V, \cdots, g_n + V)$. As the sets $g_\nu + V$ which contain no point of S can be omitted, we may assume that every $g_\nu + V$ contains a point of S, say f_ν. Then $g_\nu \epsilon f_\nu - V$, $S \subset \mathfrak{S}(f_1 - V + V, \cdots, f_n - V + V) \subset \mathfrak{S}(f_1 + U, \cdots, f_n + U)$.

Remark 2. For the reasons given in the Remark after Definition 5, we could replace the $U \epsilon \mathfrak{U}$ by the open sets T with $0 \epsilon T$ in all these considerations.

Remark 3. The set of all real numbers is a particularly simple linear space. It is clear that its customary metric, $\|x\|$ = absolute value of x, is a metric in the sense of Definition 2a, and a topology in the sense of Definition 2b (cf. the end of Part I). The boundedness in the sense of Definition 5, and the total boundedness in the sense of Definition 6, obviously coincide with the customary notion of boundedness for real numbers in this case.

Theorem 9. *Every finite set is totally bounded. The set of all αf, $-1 \leqq \alpha \leqq 1$ (f fixed), is totally bounded. If S, T are totally bounded, αS, $f+S$, $S+T$ are also totally bounded, and so is $\mathfrak{S}(S, T)$.*

If $L_1, \cdots, L_k$, L are topological linear spaces, if $S_\kappa \subset L_\kappa$ and is totally bounded, $\kappa = 1, \cdots, k$, if $\mathfrak{F}(f_1, \cdots, f_k)$ is a function with the domain $f_\kappa \epsilon S_\kappa$, $\kappa = 1, \cdots, k$, uniformly continuous in this domain, and with a range $\subset L$, then the set of all $\mathfrak{F}(f_1, \cdots, f_k)$, $f_\kappa \epsilon S_\kappa$, $\kappa = 1, \cdots, k$, is totally bounded.

The statements for finite sets, $f+S$, $\mathfrak{S}(S, T)$, are obvious, and the statement for αS follows from Theorem 1. The statements concerning the sets αf, $-1 \leqq \alpha \leqq 1$, and $S+T$ are both special cases of the last statement ($k=1$, L = set of all real numbers, $L_1 = L$, f_1 is to be replaced by α, $\mathfrak{F}(\alpha) = \alpha f$; and $k=2$, $L_1 = L_2 = L$, $\mathfrak{F}(f_1, f_2) = f_1 + f_2$ respectively; cf. Theorem 7 and Remark

3 after Definition 6). So our only task is to prove that statement. Denote the $\mathfrak{U}$ of $L_1, \cdots, L_k, L$ by $\mathfrak{U}_1, \cdots, \mathfrak{U}_k, \mathfrak{U}$ respectively. If a $U \epsilon \mathfrak{U}$ is given, the uniform continuity of $\mathfrak{F}$ means that we can choose $U_1 \epsilon \mathfrak{U}_1, \cdots, U_k \epsilon \mathfrak{U}_k$ so that the conditions $f_\kappa - g_\kappa \epsilon U_\kappa$, f_κ, $g_\kappa \epsilon S_\kappa$, $\kappa = 1, \cdots, k$, imply that $\mathfrak{F}(f_1, \cdots, f_k) - \mathfrak{F}(g_1, \cdots, g_k) \epsilon U$. Now apply Definition 6 and its Remark 1 to S_κ: $S_\kappa \subset \mathfrak{S}(g_{\kappa,1} + U_\kappa, \cdots, g_{\kappa,n_\kappa} + U_\kappa)$, $g_{\kappa,1}, \cdots, g_{\kappa,n_\kappa} \epsilon S_\kappa$. If a system $f_\kappa \epsilon S_\kappa$, $\kappa = 1, \cdots, k$, is given, we have $f_\kappa \epsilon g_{\kappa,\nu_\kappa} + U_\kappa$, with a $\nu_\kappa = 1, \cdots, n_\kappa$ for each $\kappa = 1, \cdots, k$, and thus $\mathfrak{F}(f_1, \cdots, f_k) \epsilon \mathfrak{F}(g_{1,\nu_1}, \cdots, g_{k,\nu_k}) + U$. So if we put $n = n_1 \cdots n_k$, and arrange the $\mathfrak{F}(g_{1,\nu_1}, \cdots, g_{k,\nu_k})$ $(\nu_\kappa = 1, \cdots, n_\kappa$, $\kappa = 1, \cdots, k)$ in some order $\mathfrak{F}_1, \cdots, \mathfrak{F}_n$, then our set is contained in $\mathfrak{S}(\mathfrak{F}_1 + U, \cdots, \mathfrak{F}_n + U)$.

THEOREM 10. *Every totally bounded set S is bounded.*

Let S be totally bounded, $U \epsilon \mathfrak{U}$. Choose $V \epsilon \mathfrak{U}$ by Theorem 1, with $n = 2$, $A = 1$, and then $f_1, \cdots, f_n$ with $S \subset \mathfrak{S}(f_1 + V, \cdots, f_n + V)$. Select $\alpha_1, \cdots, \alpha_n$ with $f_\nu \epsilon \alpha_\nu V$ and put $\beta = \max(1, |\alpha_1|, \cdots, |\alpha_n|)$; then

$$f_\nu + V \subset \alpha_\nu V + V = \beta\left(\frac{\alpha_\nu}{\beta} V\right) + \beta\left(\frac{1}{\beta} V\right) \subset \beta U.$$

THEOREM 11. *The boundedness (total boundedness) of S is a necessary and sufficient condition for that of S_{cl}.*

As $S \subset S_{\text{cl}}$, the condition is necessary. If $U \epsilon \mathfrak{U}$, choose $V \epsilon \mathfrak{U}$, $V + V \epsilon \mathfrak{U}$; then by Theorem 5, $V_{\text{cl}} \subset V + V \subset U$, and thus $S \subset \alpha V$ (or $S \subset \mathfrak{S}(f_1 + V, \cdots, f_n + V)$) implies $S_{\text{cl}} \subset \alpha U$ (or $S_{\text{cl}} \subset \mathfrak{S}(f_1 + U, \cdots, f_n + U)$). Thus the condition is also sufficient.

We now investigate convexity.

THEOREM 12. *If L is convex (Definition 2b, (7)) then, for $U \epsilon \mathfrak{U}$ and $\alpha_1, \cdots, \alpha_n \geqq 0$, $\alpha_1 U_{\text{cl}} + \cdots + \alpha_n U_{\text{cl}} = (\alpha_1 + \cdots + \alpha_n) U_{\text{cl}}$.*

Induction proves this theorem for all $n = 1, 2, \cdots$, if it holds for $n = 2$. For $\alpha_1 = \alpha_2 = 0$ it is obvious, therefore we may assume $\alpha_1 + \alpha_2 > 0$. Division by $\alpha_1 + \alpha_2$ then gives

$$\alpha U_{\text{cl}} + (1 - \alpha) U_{\text{cl}} = U_{\text{cl}} \qquad \left(\alpha = \frac{\alpha_1}{\alpha_1 + \alpha_2},\ 0 \leqq \alpha \leqq 1\right).$$

As $\supset$ is obvious, we need to prove only $\subset$.

Now Definition 2b, (7), states that $U + U \subset 2U$; iterated n times, this becomes $U + \cdots + U$ (2^n addends) $\subset 2^n U$, and, a fortiori, $kU + (2^n - k) U \subset 2^n U$, $k = 0, 1, \cdots, 2^n$. Thus $\alpha U + (1 - \alpha) U \subset U$ if $0 \leqq \alpha \leqq 1$ with α dyadic-rational; from this, consideration of continuity leads to $\alpha U_{\text{cl}} + (1 - \alpha) U_{\text{cl}} \subset U_{\text{cl}}$ for all $0 \leqq \alpha \leqq 1$, completing the proof.

Definition 7. *S_{conv} is the set of all $\alpha_1 f_1 + \cdots + \alpha_n f_n$, with $n = 1, 2, \cdots$, $\alpha_1, \cdots, \alpha_n \geqq 0$, $\alpha_1 + \cdots + \alpha_n = 1$, $f_1, \cdots, f_n \in S$.*

Theorem 13. *If L is convex, then, for all $U \in \mathfrak{U}$, $(U_{cl})_{conv} = U_{cl}$.*

This follows immediately from Theorem 12.

Theorem 14. *The boundedness (total boundedness) of S is a necessary condition for that of S_{conv}; if L is convex, it is also sufficient.*

As $S \subset S_{conv}$, the condition is necessary. For the sufficiency we have to assume that L is convex. If $U \in \mathfrak{U}$, choose $V \in \mathfrak{U}$, $V + V \subset U$; then $V_{conv} \subset (V_{cl})_{conv} = V_{cl} \subset V + V \subset U$ (by Theorems 5, 13). Thus $S \subset \alpha V$ implies $S_{conv} \subset \alpha V_{conv} \subset \alpha U$, settling the case for boundedness. $S \subset \mathfrak{S}(f_1 + V, \cdots, f_n + V)$ implies $S_{conv} \subset \mathfrak{S}(\beta_1 f_1 + \cdots + \beta_n f_n + V_{conv}) \subset \mathfrak{S}(\beta_1 f_1 + \cdots + \beta_n f_n + U)$, where the $\beta_1, \cdots, \beta_n$ run over all combinations with $\beta_1, \cdots, \beta_n \geqq 0$, $\beta_1 + \cdots + \beta_n = 1$. If the set of these $\beta_1 f_1 + \cdots + \beta_n f_n$ is totally bounded, then this set is contained in $\mathfrak{S}(g_1 + U, \cdots, g_p + U)$, and thus $S_{conv} \subset \mathfrak{S}(g_1 + U + U, \cdots, g_p + U + U)$. We could replace U by V, and we would then have $S_{conv} \subset \mathfrak{S}(g_1 + U, \cdots, g_p + U)$. This settles the case for total boundedness too, provided that the set of the $\beta_1 f_1 + \cdots + \beta_n f_n$ (n and $f_1, \cdots, f_n$ fixed) is totally bounded.

This is a subset of the $(\beta_1 f_1 + \cdots + \beta_n f_n)$-set with $0 \leqq \beta_1 \leqq 1, \cdots, 0 \leqq \beta_n \leqq 1$. This set is disposed of by Theorem 9, if each set $\beta_\nu f_\nu$, $0 \leqq \beta_\nu \leqq 1$, is totally bounded, but this again follows from Theorem 9.

Finally we repeat Definition III in §1.

Definition 8. *S is compact if every infinite set $T \subset S$ has a condensation point $f \in S$.*

Note that we did not assume that any of Hausdorff's countability axioms hold (cf. [1], p. 229, axioms (9) and (10)), and that we still are considering "compactness" and not the Alexandroff-Urysohn "bicompactness" (cf. [3], [4], in particular [3], pp. 259–260), although the latter is specially adapted to these cases. The reason is that for the totally bounded sets S compactness implies the countability axioms (although they need not hold for L, cf. Theorem 16) and thus bicompactness.

III. Topological completeness

6. The two definitions of completeness which we discussed in §1, I′, and IV, are the following:

Definition 9. *L is sequentially complete if every "fundamental sequence" $f_1, f_2, \cdots$ in L (i.e., every sequence such that for each $U \in \mathfrak{U}$ there is an $n_1 = n_1(U)$, such that $m, n \geqq n_1$ imply $f_m - f_n \in U$) is "convergent" (i.e., an f exists such that for each $U \in \mathfrak{U}$ there is an $n_2 = n_2(U)$, such that $n \geqq n_2$ implies $f_n - f \in U$).*

DEFINITION 10. *L is topologically complete if every closed and totally bounded set $S \subset L$ is compact.*

To characterize the relationship between these two notions, we prove

THEOREM 15. *Sequential completeness is a necessary condition for topological completeness; if L satisfies Hausdorff's first countability axiom (cf. above), it is also sufficient.*

Assume first that L is topologically complete, and let $f_1, f_2, \cdots$ be a fundamental sequence. If $U \epsilon \mathfrak{U}$, $f_m \epsilon f_{N_1} + U$ for $m \geqq N_1 = N_1(U)$; thus $(f_1, f_2, \cdots) \subset \mathfrak{S}(f_1 + U, \cdots, f_{N_1} + U)$. So $(f_1, f_2, \cdots)$ is totally bounded; $(f_1, f_2, \cdots)_{cl}$ is also totally bounded (by Theorem 11) and closed, and thus compact. If infinitely many $f_1, f_2, \cdots$ are different, $(f_1, f_2, \cdots)$ is an infinite set $\subset (f_1, f_2, \cdots)_{cl}$, and has a condensation point f, that is, an f such that, for each $U \epsilon \mathfrak{U}$, there are infinitely many n for which $f_n \epsilon f + U$. If only a finite number of $f_1, f_2, \cdots$ are different, infinitely many f_n must coincide, and their common value f has the above property. So such an f exists in any event. Corresponding to $U \epsilon \mathfrak{U}$, choose $V \epsilon \mathfrak{U}$, $V + V \subset U$, $n_1 = n_1(V)$, and the above f with respect to V. Then $f_n - f \epsilon V$ occurs for some $n \geqq n_1$, and $f_m - f_n \epsilon V$ for every $m, n \geqq n_1$; thus, for every $m \geqq n_1$, $f_m - f = (f_m - f_n) + (f_n - f) \epsilon V + V \subset U$. Putting $n_2(U) = n_1(V)$ we see that $f_1, f_2, \cdots$ is convergent, and thus L is sequentially complete.

Now consider the converse situation. Let L be sequentially complete, let $U_1, U_2, \cdots$ be a complete system of neighborhoods for 0 (cf. [1], p. 229; this means that for each $U \epsilon \mathfrak{U}$ some $U_n \subset U$), and assume S to be totally bounded and closed. As every infinite $T \subset S$ contains a sequence $f_1, f_2, \cdots$ of distinct elements, we may assume $T = (f_1, f_2, \cdots)$. As every fundamental sequence is convergent and thus has a condensation point f, we need only exhibit a fundamental subsequence $f^{(1)}, f^{(2)}, \cdots$ in a sequence $(f_1, f_2, \cdots) \subset S$. Now we can apply the well known diagonal process.

Choose $g_{n1}, \cdots, g_{nm_n}$ with $S \subset (g_{n1} + U_n, \cdots, g_{nm_n} + U_n)$. Some $g_{1\nu} + U_1$, say $\nu = \nu_1$, contains infinitely many elements $f_1^{(1)}, f_2^{(1)}, \cdots$ of the sequence $f_1, f_2, \cdots$. Some $g_{2\nu} + U_2$, say $\nu = \nu_2$, contains infinitely many elements $f_1^{(2)}, f_2^{(2)}, \cdots$ of the sequence $f_1^{(1)}, f_2^{(1)}, \cdots$. And so on. Now put $f^{(1)} = f_1^{(1)}$, $f^{(2)} = f_2^{(2)}, \cdots$. If $U \epsilon \mathfrak{U}$, choose $V \epsilon \mathfrak{U}$, $V - V \subset U$ (cf. Definition 2b, (4), (5)) *and* $U_p \subset V$. For $m \geqq p$, $f^{(m)} = f_m^{(m)}$ belongs to $(f_1^{(m)}, f_2^{(m)}, \cdots)$, thus to $(f_1^{(p)}, f_2^{(p)}, \cdots)$, and to $g_{p\nu_p} + U_p$. Thus, for $m, n \geqq p$, $f^{(m)} - f^{(n)} \epsilon U_p - U_p \subset V - V \subset U$. Putting $n_1(U) = p$ we see that $f^{(1)}, f^{(2)}, \cdots$ is fundamental, thus completing the proof of the topological completeness of L.

As our main interest belongs to the cases in which L violates the countability axiom, and as the equivalence of sequential and of topological completeness has not been established for them, we continue by investigating the properties of topological completeness.

THEOREM 16. *If L is topologically complete and S totally bounded, then if we consider S as a space with the topology of L, this is a normal and separable Hausdorff topology in S, that is, one which fulfills Hausdorff's axioms* (1)–(3), (8), *and* (10) (*cf.* [1], *pp.* 228–229; *thus all axioms* (1)–(10) *are fulfilled*).

Remark. For this reason we can replace condensation points in S by limits of convergent sequences.—

As $S \subset S_{cl}$ and as S_{cl} is also totally bounded, we may consider S_{cl} instead of S, that is, we can restrict ourselves to closed sets S. Then S is compact. The topology of L fulfilled axioms (1)–(3), (6) in L, therefore it also fulfills them in S. Let us therefore consider the other axioms.

Ad (9): Choose $U_1, U_2, \cdots$ by Definition 2b, (2), with $\mathfrak{P}(U_1, U_2, \cdots) = (0)$. Now choose (by Definition 2b, (3), (5)) $V_n \epsilon \mathfrak{U}$, $V_n + V_n \subset U_n$ and $W_n \epsilon \mathfrak{U}$, $W_n \subset \mathfrak{P}(V_1, \cdots, V_n)$. Let T be an open set, $0 \epsilon T$. Assume that, for $n = 1, 2, \cdots, \mathfrak{P}(W_n, S)$ is not a subset of $\mathfrak{P}(T, S)$. Then there exists an f_n with $f_n \epsilon \mathfrak{P}(W_n, S)$, f_n not an element of $\mathfrak{P}(T, S)$. If infinitely many $f_1, f_2, \cdots$ are different, $(f_1, f_2, \cdots)$ is an infinite set $\subset S$, and has a condensation point f, that is, an f such that, for each $U \epsilon \mathfrak{U}$, there are infinitely many m for which $f_m \epsilon f + U$.If only a finite number of $f_1, f_2, \cdots$ are different, infinitely many f_n must coincide, and their common value f has the above property. So such an f exists in any event. Choose $m \geqq n$ and $f_m \epsilon f + U$. $f_m \epsilon W_m \subset V_n$, $f_m \epsilon \mathfrak{C}T$, so that f is a point or a condensation point of V_n and of $\mathfrak{C}T$; hence $f \epsilon (V_n)_{cl} \subset V_n + V_n \subset U_n$ (by Theorem 5), and $f \epsilon \mathfrak{C}T$ ($\mathfrak{C}T$ is closed). Thus $f \epsilon \mathfrak{P}(U_1, U_2, \cdots) = (0)$, $f = 0 \epsilon T$, contradicting the condition that $f \epsilon \mathfrak{C}T$. This proves that an $n = 1, 2, \cdots$ with $\mathfrak{P}(W_n, S) \subset \mathfrak{P}(T, S)$ must exist, so that $(W_1)_i, (W_2)_i, \cdots$ form a complete system of neighborhoods of 0 in S, and $f + (W_1)_i, f + (W_2)_i, \cdots$ form a complete system of neighborhoods of f in S (consider 0 in $-f + S$).

Ad (10): Take the $W_1, W_2, \cdots$ constructed above and choose $X_n \epsilon \mathfrak{U}$, $X_n - X_n \subset W_n$ (by Theorem 1, $n = 2$, $A = 1$, and then $g_{n1}, \cdots, g_{nm_n}$ with $S \subset \mathfrak{S}(g_{n1} + (X_n)_i, \cdots, g_{nm_n} + (X_n)_i)$. If an open set T and an $f \epsilon \mathfrak{P}(S, T)$ are given, there is an n for which $\mathfrak{P}(S, f + W_n) \subset \mathfrak{P}(S, T)$, and a ν with $f \epsilon g_{n\nu} + (X_n)_i$. Then $g_{n\nu} + (X_n)_i \subset f - (X_n)_i + (X_n)_i \subset f + (X_n - X_n)_i$ (by Theorem 3) $\subset f + W_n$, and $\mathfrak{P}(S, g_{n\nu} + (X_n)_i) \subset \mathfrak{P}(S, f + W_n) \subset \mathfrak{P}(S, T)$. So in the sequence of open sets $g_{n\nu} + (X_n)_i$, $n = 1, 2, \cdots, \nu = 1, \cdots, m_n$, we can find, for every open set T and every $f \epsilon \mathfrak{P}(S, T)$, a T' with $f \epsilon \mathfrak{P}(S, T') \subset \mathfrak{P}(S, T)$. Therefore they form a complete system of neighborhoods in S.

Ad (8): In separable spaces regularity implies normality (cf. [5]), that is, (8) follows from (6) and (10).†

7. We are now in a position to formulate and prove the fundamental Theorem 18.

DEFINITION 11. *If D is an arbitrary set, L^D is the set of all functions $F(a)$ with the domain D and a range $\subset L$ (that is, $a\epsilon D$, $F(a)\epsilon L$). A function $F(a)$ is "bounded" if its range (the set of all $F(a)$, $a\epsilon D$) is bounded (see Definition* 5, *this set being $\subset L$). The set of all bounded $F\epsilon L^D$ is $L_b{}^D$. If $U\epsilon\mathfrak{U}$, define a set $U'\subset L_b{}^D$ as the set of all $F\epsilon L_b{}^D$ with a range $\subset U$. The set of all U' is $\mathfrak{U}'$.*

Remark. If L is metric, this boundedness means that the (real-numerical) function $\|F(a)\|$ should be bounded. Our $\mathfrak{U}'$ corresponds to the metric $\|F\|'$ of $L_b{}^D$ defined by $\|F\|'=\text{l.u.b.}\|F(a)\|$.

THEOREM 17. *L_b^D forms with $\mathfrak{U}'$ a topological linear space, that is, it satisfies Definition* 2b, (1)–(6). *It is convex, that is, it also satisfies Definition* 2b, (7), *provided that L is convex.*

All parts of this theorem are obvious.

THEOREM 18. *If L is topologically complete, so is L_b^D.*

Remark. It is easily seen that this statement holds for sequential completeness instead of for topological completeness. But, in view of the application to the generalized theory of almost periodic functions, and because we believe that this notion of completeness is the natural one, it is important to prove our theorem in its present form.—

Let $S^*\subset L_b^D$ be a closed and totally bounded set; we have to prove that it is compact. As every infinite $T^*\subset S^*$ contains a sequence $\mathfrak{F}_1, \mathfrak{F}_2, \cdots$ of distinct elements, we may assume $T^*=(\mathfrak{F}_1, \mathfrak{F}_2, \cdots)$.

For a fixed $a\epsilon D$ denote the set of all $\mathfrak{F}(a)$, $\mathfrak{F}\epsilon S^*$, by R_a. As $S^*\subset\mathfrak{S}(\mathfrak{F}_1+U', \cdots, \mathfrak{F}_n+U')$ implies that $R_a\subset\mathfrak{S}(\mathfrak{F}_1(a)+U, \cdots, \mathfrak{F}_n(a)+U)$, R_a is totally bounded, and, with it, R_a-R_a and $(R_a-R_a)_{\text{cl}}$ (by Theorems 9, 11). The latter set is also closed, and as it is contained in L, it is compact. Thus Theorem 16 applies to it; and, in particular, the open sets $(W_1)_i$, $(W_2)_i$, $\cdots$ constructed in the part "Ad (9)" of its proof form a complete system of neighborhoods of 0 in $(R_a-R_a)_{\text{cl}}$. (Note that the $(W_n)_i$ do not depend on $(R_a-R_a)_{\text{cl}}$, that is, on a; but if we choose for an open set T with $0\epsilon T$ the $p=p(T, a)$ with $\mathfrak{P}((R_a-R_a)_{\text{cl}}, W_p)\subset\mathfrak{P}((R_a-R_a)_{\text{cl}}, T)$ does depend on T and on a.)

Now apply the diagonal process used in the last paragraph of the proof of Theorem 15 to $\mathfrak{F}_1, \mathfrak{F}_2, \cdots$ and $W_1', W_2', \cdots$ in L_b^D (instead of $f_1, f_2, \cdots$ and

† Tychonoff proves, loc. cit., only normality, that is, Hausdorff's axiom (7) ([1], p. 229). But if we replace in his result ([5], p. 140) the closed sets F, ϕ by two sets F, ϕ without common condensation points, and the space R by $F+\phi$ (in which F, ϕ are relatively closed), (8) obtains. (Hausdorff's notations for F, ϕ, R are F_1, F_2, E.)

$U_1, U_2, \cdots$ in L, which we considered there). As S^* is totally bounded (as S was there) we obtain a subsequence $\mathfrak{F}^{(1)}, \mathfrak{F}^{(2)}, \cdots$, such that, for each W_p', an $n_1' = n_1'(p)$ exists so that $m, n \geqq n_1'$ imply $\mathfrak{F}^{(m)} - \mathfrak{F}^{(n)} \epsilon W_p'$. Thus $\mathfrak{F}^{(m)}(a) - \mathfrak{F}^{(n)}(a) \epsilon W_p$. If any open set $T \subset L$ with $0 \epsilon T$ is given, there is a $p = p(T, a)$ with $\mathfrak{P}((R_a - R_a)_{\mathrm{cl}}, W_p) \subset \mathfrak{P}((R_a - R_a)_{\mathrm{cl}}, T)$. As $\mathfrak{F}^{(m)}(a) - \mathfrak{F}^{(n)}(a) \epsilon R_a - R_a \subset (R_a - R_a)_{\mathrm{cl}}$, we thus have $\mathfrak{F}^{(m)}(a) - \mathfrak{F}^{(n)}(a) \epsilon T$. Putting $n_1(T, a) = n_1'(p(T, a))$, we see that $\mathfrak{F}^{(1)}(a), \mathfrak{F}^{(2)}(a), \cdots$ is a fundamental sequence (in L) in the sense of Definition 9 ($m, n \geqq n_1$ implying $\mathfrak{F}^{(m)}(a) - \mathfrak{F}^{(n)}(a) \epsilon T$). As L is topologically complete, it is also sequentially complete (by Theorem 15), and so $\mathfrak{F}^{(1)}(a), \mathfrak{F}^{(2)}(a), \cdots$ is convergent. Denote its limit by $\mathfrak{F}(a)$ (we know so far only that $\mathfrak{F} \epsilon L^D$).

We constructed above an $n_1' = n_1'(p)$, independent of a, such that, for $m, n \geqq n_1'$, $\mathfrak{F}^{(m)} - \mathfrak{F}^{(n)} \epsilon W_p'$. (Note that, for an arbitrary open set T, $0 \epsilon T$, in the place occupied by W_p, the corresponding $n_1 = n_1(T, a)$ would have depended on a.) Thus $\mathfrak{F}^{(m)}(a) - \mathfrak{F}^{(n)}(a) \epsilon W_p$, $\mathfrak{F}^{(m)}(a) - \mathfrak{F}(a) \epsilon (W_p)_{\mathrm{cl}}$.

Now assume that there exists a $U' \epsilon \mathfrak{U}'$, such that, for infinitely many n, $\mathfrak{F}^{(n)} - \mathfrak{F}$ is not $\epsilon U'$. Then we can select a subsequence $\mathfrak{F}_1^{(1)}, \mathfrak{F}_2^{(1)}, \cdots$ from $\mathfrak{F}^{(1)}, \mathfrak{F}^{(2)}, \cdots$ such that, for all n, $\mathfrak{F}_1^{(n)} - \mathfrak{F}$ is not $\epsilon U'$. Now choose $V \epsilon \mathfrak{U}$, $V + V \subset U$, and repeat the preceding construction with $\mathfrak{F}_1^{(1)}, \mathfrak{F}_2^{(1)}, \cdots$ and $V, W_1, W_2, \cdots$ instead of with $\mathfrak{F}_1, \mathfrak{F}_2, \cdots$ and $W_1, W_2, \cdots$. We obtain a subsequence $\mathfrak{F}_2^{(1)}, \mathfrak{F}_2^{(2)}, \cdots$ of $\mathfrak{F}_1^{(1)}, \mathfrak{F}_1^{(2)}, \cdots$ and a $\mathfrak{G}$ such that, for every $a \epsilon D$, $\mathfrak{G}(a)$ is the limit of $\mathfrak{F}_2^{(1)}(a), \mathfrak{F}_2^{(2)}(a), \cdots$. But this is a subsequence of $\mathfrak{F}^{(1)}(a), \mathfrak{F}^{(2)}(a), \cdots$ and therefore has the limit $\mathfrak{F}(a)$; thus $\mathfrak{F}(a) = \mathfrak{G}(a)$ for all $a \epsilon D$, and $\mathfrak{F} = \mathfrak{G}$. Furthermore, we have, from a certain n on (it is $n_1''(1)$, if $n_1''(p)$ is the analogue of $n_1'(p)$ in the present construction, and thus independent of a) $\mathfrak{F}_2^{(n)}(a) - \mathfrak{F}(a) \epsilon V_{\mathrm{cl}} \subset V + V \subset U$, $\mathfrak{F}_2^{(n)} - \mathfrak{F} \epsilon U'$. But as $\mathfrak{F}_2^{(1)}, \mathfrak{F}_2^{(2)}, \cdots$ is a subsequence of $\mathfrak{F}_1^{(1)}, \mathfrak{F}_1^{(2)}, \cdots$ we should always have $\mathfrak{F}_2^{(n)} - \mathfrak{F}$ is not $\epsilon U'$. This is a contradiction. Thus there is an $n_2' = n_2'(U')$ for every $U' \epsilon \mathfrak{U}'$ such that $n \geqq n_2'$ implies $\mathfrak{F}^{(n)} - \mathfrak{F} \epsilon U'$. (Remember that the sequence $\mathfrak{F}^{(1)}, \mathfrak{F}^{(2)}, \cdots$ is independent of U'.) So if $\mathfrak{F} \epsilon L_b^D$, it is the limit of the sequence $\mathfrak{F}^{(1)}, \mathfrak{F}^{(2)}, \cdots$ and therefore a condensation point of the original sequence $\mathfrak{F}_1, \mathfrak{F}_2, \cdots$. Thus the compactness of S^* would have been established.

Assume $U' \epsilon \mathfrak{U}'$. We proceed as in the last paragraph of the proof of Theorem 8. Choose $V \epsilon \mathfrak{U}$, $V + V \subset U$ and $W \epsilon \mathfrak{U}$, $\alpha W \subset V$ if $-1 \leqq \alpha \leqq 1$. There is an n such that $\mathfrak{F}_n - \mathfrak{F} \epsilon W'$, that is, all $\mathfrak{F}_n(a) - \mathfrak{F}(a) \epsilon W$. The set of all $\mathfrak{F}_n(a)$, $a \epsilon D$, for this n, is bounded, say $\subset \beta W$. Thus, with $\gamma = \max(1, |\beta|)$,

$$\begin{aligned} \mathfrak{F}(a) &= \mathfrak{F}_n(a) - (\mathfrak{F}_n(a) - \mathfrak{F}(a)) \epsilon \beta W - W \\ &= \gamma\left(\frac{\beta}{\gamma} W\right) + \gamma\left(-\frac{1}{\gamma} W\right) \subset \gamma(V + V) \subset \gamma U. \end{aligned}$$

So the set of all $\mathfrak{F}(a)$, $a \epsilon D$, is bounded, and $\mathfrak{F} \epsilon L_b^D$. This completes the proof.

IV. Non-metric examples

8. Many metric and complete linear spaces are known, so that it is not necessary to point out such examples. By Theorem 15, our notion of topological completeness gives nothing new as long as L satisfies Hausdorff's first countability axiom. For this reason those examples will be of particular interest which violate this axiom. We mentioned three such spaces in the last footnote on page 2, and we shall discuss them now in detail.

Definition 12. *Denote Hilbert space by $\mathfrak{H}$ and the space of all bounded linear operators in $\mathfrak{H}$ by $\mathfrak{B}$ (cf. for instance* [6], *Chapter* I; [7], *Chapter* I, *paragraph* 2; [2], *pp.* 372–373). *There are various ways to define sets $\mathfrak{U}$ for $L = \mathfrak{H}$ or $\mathfrak{B}$, of which we shall consider the following.* (*Cf.* [2], *pp.* 378–388, *where the discussion of all these topologies is to be found. The notations $\|f\|$, (f, g), etc. are explained in each of the above references.*)

(a) *For any $\delta > 0$ define $U_1(\delta)$ as the set of all $f \epsilon \mathfrak{H}$ with $\|f\| \leqq \delta$; $\mathfrak{U}_1$ is the set of all $U_1(\delta)$.*

(b) *For any $n = 1, 2, \cdots, \phi_1, \cdots, \phi_n \epsilon \mathfrak{H}$, $\delta > 0$, define $U_2(\phi_1, \cdots, \phi_n; \delta)$ as the set of all $f \epsilon \mathfrak{H}$ with $|(f, \phi_\nu)| \leqq \delta$ for $\nu = 1, \cdots, n$; $\mathfrak{U}_2$ is the set of all $U_2(\phi_1, \cdots, \phi_n; \delta)$.*

(c) *For any $\delta > 0$ define $U_3(\delta)$ as the set of all $A \epsilon \mathfrak{B}$ with $|||\, A \,||| \leqq \delta$, where*

$$|||\, A \,||| = \operatorname*{l.u.b.}_{f \epsilon \mathfrak{H},\, f \neq 0} \frac{\|Af\|}{\|f\|}$$

(*that is, the set of all $A \epsilon \mathfrak{B}$ with $\|Af\| \leqq \delta \|f\|$ identically*); *$\mathfrak{U}_3$ is the set of all $U_3(\delta)$.*

(d) *For any $n = 1, 2, \cdots, \phi_1, \cdots, \phi_n \epsilon \mathfrak{H}$, $\delta > 0$, define $U_4(\phi_1, \cdots, \phi_n; \delta)$ as the set of all $A \epsilon \mathfrak{B}$ with $\|A\phi_\nu\| \leqq \delta$ for $\nu = 1, \cdots, n$; $\mathfrak{U}_4$ is the set of all $U_4(\phi_1, \cdots, \phi_n; \delta)$.*

(e) *For any $n = 1, 2, \cdots, \phi_1, \psi_1, \cdots, \phi_n, \psi_n \epsilon \mathfrak{H}$, $\delta > 0$, define $U_5(\phi_1, \psi_1, \cdots, \phi_n, \psi_n; \delta)$ as the set of all $A \epsilon \mathfrak{B}$ with $|(A\phi_\nu, \psi_\nu)| \leqq \delta$ for $\nu = 1, \cdots, n$; $\mathfrak{U}_5$ is the set of all $U_5(\phi_1, \psi_1, \cdots, \phi_n, \psi_n; \delta)$.*

$\mathfrak{U}_1$ describes the strong, $\mathfrak{U}_2$ the weak topology of $\mathfrak{H}$; $\mathfrak{U}_3$ describes the uniform, $\mathfrak{U}_4$ the strong, $\mathfrak{U}_5$ the weak topology of $\mathfrak{B}$.

Theorem 19. *$\mathfrak{H}$ and $\mathfrak{B}$ are convex topological linear spaces* (*that is, they fulfill Definition* 2b, (1)–(7)) *in all five topologies of Definition* 11.

The proof is immediate.

THEOREM 20. *$\mathfrak{H}$, $\mathfrak{U}_1$ and $\mathfrak{B}$, $\mathfrak{U}_3$ are originated by metrics (in the sense of the remark after Definition 2b), with the absolute values $\|f\|$ and $|||A|||$ respectively. Both are sequentially, and thus topologically, complete.*

The metric properties are verified immediately. Sequential completeness is one of the fundamental properties of $\mathfrak{H}$, $\mathfrak{U}_1$ (cf. [6], pp. 66 and 111), and extends from it immediately to $\mathfrak{B}$, $\mathfrak{U}_3$. Topological completeness follows by Theorem 15.

We now investigate the non-metric topologies $\mathfrak{H}$, $\mathfrak{U}_2$ and $\mathfrak{B}$, $\mathfrak{U}_4$ or $\mathfrak{U}_5$. Theorem 22 has some independent interest.

THEOREM 21. *A set $S \subset \mathfrak{H}$ or $\mathfrak{B}$ is totally bounded in the topology $\mathfrak{U}_2$ or $\mathfrak{U}_5$ (these are the weak topologies), if and only if the (real-numerical) sets of all $|(f, \phi)|$, $f \in S$, or $|(A\phi, \psi)|$, $A \in S$, are bounded for every choice of $\phi \in \mathfrak{H}$ or ϕ, $\psi \in \mathfrak{H}$ (no uniformity is required).*

Consider $\mathfrak{H}$, $\mathfrak{U}_2$. If S is totally bounded, take $U = U_2(\phi; 1)$, $S \subset \mathfrak{S}(f_1 + U, \cdots, f_n + U)$. Then each $f \in S$ is such that $|(f - f_\nu, \phi)| \leqq 1$ for some $\nu = 1, \cdots, n$, and thus

$$|(f, \phi)| = \max_{\nu=1,\cdots,n} |(f_\nu, \phi)| + 1,$$

proving the necessity of our condition. To prove its sufficiency, consider a $U \in \mathfrak{U}_2$, that is, $U = U_2(\phi_1, \cdots, \phi_n; \delta)$. Choose $c \geqq |(f, \phi_\nu)|$ for all $f \in S$, $\nu = 1, \cdots, n$, and choose $N = 1, 2, \cdots$ with $2^{3/2}c/N \leqq \delta$. Consider all N^{2n} systems of $2n$ integers $\rho_1, \sigma_1, \cdots, \rho_n, \sigma_n = -N+1, -N+3, \cdots, N-1$. If for a combination $\rho_1, \sigma_1, \cdots, \rho_n, \sigma_n$ there exists an f with

$$\frac{\rho_\nu - 1}{N} c \leqq \Re(f, \phi_\nu) \leqq \frac{\rho_\nu + 1}{N} c,$$

$$\frac{\sigma_\nu - 1}{N} c \leqq \Im(f, \phi_\nu) \leqq \frac{\sigma_\nu + 1}{N} c \quad \text{for all } \nu = 1, \cdots, n,$$

choose one and call it $f_{\rho_1\sigma_1\cdots\rho_n\sigma_n}$. The number of these $f_{\rho_1\sigma_1\cdots\rho_n\sigma_n}$ is finite, say $M \leqq N^{2n}$; let us denote them by $f^{(1)}, \cdots, f^{(M)}$. One easily sees that, for each $f \in S$, there exists some $f^{(\mu)}$ with

$$|\Re(f - f^{(\mu)}, \phi_\nu)| \leqq \frac{2c}{N}, \quad |\Im(f - f^{(\mu)}, \phi_\nu)| \leqq \frac{2c}{N} \text{ for all } \nu = 1, \cdots, n,$$

implying that

$$|(f - f^{(\mu)}, \phi_\nu)| \leqq \frac{2^{3/2}c}{N} \leqq \delta.$$

Thus $S \subset \mathfrak{S}(f^{(1)}+U, \cdots, f^{(M)}+U)$, proving the total boundedness of our set S, and the sufficiency of our condition.

$\mathfrak{B}$, $\mathfrak{U}_5$ are discussed in the same way.

THEOREM 22. *A set $S \subset \mathfrak{H}$ or $\mathfrak{B}$ is totally bounded in the topology $\mathfrak{U}_2$ or $\mathfrak{U}_5$, if and only if it is bounded in the topology $\mathfrak{U}_1$ or $\mathfrak{U}_3$ (these are the metric topologies), that is, if the (real-numerical) set of all $\|f\|$, $f \epsilon S$, or $|||A|||$, $A \epsilon S$, is bounded.*

Consider $\mathfrak{H}$, $\mathfrak{U}_2$. If $\|f\| \leqq c$, then $|(f, \phi)| \leqq \|f\| \cdot \|\phi\| \leqq c\|\phi\|$, proving the total boundedness by Theorem 21, and thus the sufficiency of our condition. To prove the necessity, assume S totally bounded and the absolute values $\|f\|$, $f \epsilon S$, not bounded. For each $n=1, 2, \cdots$ choose $f_n \epsilon S$, $\|f_n\| \geqq n^2$. For each ϕ, $|(f_n, \phi)|$ are bounded (by Theorem 21). Thus $\lim_{n\to\infty} \|f_n/n\| = \infty$ and, for each ϕ, $\lim_{n\to\infty} (f_n/n, \phi) = 0$, contradicting a basic property of "weak convergence" (cf. [2], p. 380, footnote 32, those considerations being a special case of a result of S. Banach, cf. [8], Theorem 5, pp. 157–160). Thus our condition is necessary.

$\mathfrak{B}$, $\mathfrak{U}_5$ are discussed in the same way (using [2], p. 382, footnote 35).

THEOREM 23. *Each of the three topologies $\mathfrak{U}_2$, for $\mathfrak{H}$, $\mathfrak{U}_4$ and $\mathfrak{U}_5$ for $\mathfrak{B}$ violate both countability axioms of Hausdorff, but all three are topologically complete.*

Hausdorff's first countability axiom is not fulfilled, by [2], p. 380 and pp. 382–383; hence the second is not fulfilled either. Every $\mathfrak{U}_2$-totally bounded set $S \subset \mathfrak{H}$ is a subset of some set $U_1(c)$: $\|f\| \leqq c$. Now the latter sets are all compact (cf. [2], p. 381, footnote 34), and S, being a closed subset of a compact set, is also compact. Therefore $\mathfrak{H}$ with $\mathfrak{U}_2$ is topologically complete. $\mathfrak{B}$ with $\mathfrak{U}_5$ is discussed in the same way.

It remains to discuss $\mathfrak{B}$ with $\mathfrak{U}_4$. We may replace every $A \epsilon \mathfrak{B}$ by its adjoint A^*; then $A \epsilon \mathfrak{U}_4(\phi_1, \cdots, \phi_n, \delta)$ means that $\|A^*\phi_\nu\| \leqq \delta$ for $\nu=1, \cdots, n$. This means that, for every $f \epsilon \mathfrak{H}$ with $\|f\|=1$, $|(A^*\phi_\nu, f)| \leqq \delta$ (the sufficiency follows from Schwarz's inequality, the necessity from the substitution $f = A^*\phi_\nu/\|A^*\phi_\nu\|$), that is, $|(Af, \phi_\nu)| \leqq \delta$. The bounded linear operators A are characterized by the properties $A(\alpha f) = \alpha A f$ (α any complex number), $A(f+g) = Af + Ag$ within the set $\mathfrak{B}_1$ of all operators A, subject only to the restriction $A(\alpha f) = \alpha A f (\alpha > 0)$. The boundedness condition $|(Af, g)| \leqq c \cdot \|f\| \cdot \|g\|$ (or $\|Af\| \leqq c \cdot \|f\|$, cf. the remark made above), with some fixed $c > 0$, is required in any case. If we use the topology analogous to $\mathfrak{U}_4$ in $\mathfrak{B}_1$, $\mathfrak{B}$ is a closed subset of $\mathfrak{B}_1$, therefore topologically complete if $\mathfrak{B}_1$ is topologically complete. For the operators A of $\mathfrak{B}_1$ we may restrict the definition domain to the set S_1: $\|f\|=1$, as $A(\alpha f) = \alpha A f (\alpha > 0)$ then allows a unique extension to $\mathfrak{H}$. Now in this interpretation $\mathfrak{B}_1$, $\mathfrak{U}_4$ coincides exactly with the $\mathfrak{H}_b^{S_1}$ of $\mathfrak{H}$, $\mathfrak{U}_2$, from

Definition 11, and thus it is topologically complete by Theorem 18. This completes the proof.

V. The pseudo-metrics in convex spaces

9. If L is topological and convex, that is, if L fulfills Definition 2b, (1)–(7), we can define a family of notions, each of which is an analogue to the absolute value, and which together describe the topology of L. In various applications (for instance, in the theory of almost periodic functions) this can be used to replace the metric, even when the countability axioms are violated.

Theorem 24. *If L is topological and convex,* $(\alpha U)_{cl} \subset (\beta U)_i$ *for* $U \in \mathfrak{U}$ *and* $0<\alpha<\beta$.

By Definition 2b, (7), $U+U \subset 2U$; repeating this n times, $U+ \cdots +U$ (2^n times) $\subset 2^n U$, and thus, a fortiori, $(2^n-2)U+U+U \subset 2^n U$. Therefore $(2^n-2)U_{cl} \subset (2^n-2)U+U$ (by Theorem 5) $\subset (2^n U)_i = 2^n U_i$, $((2^n-2)/2^n)U_{cl} \subset U_i$. By Theorem 12, $0<\alpha<1$ implies $\alpha U_{cl} \subset \alpha U_{cl} + (1-\alpha)U_{cl} = U_{cl}$, and, upon replacing α by α/β and multiplying by β, $0<\alpha<\beta$, $\alpha U_{cl} \subset \beta U_{cl}$. Now for $0<\alpha<\beta$ we can find an integer n for which $\alpha/\beta<(2^n-2)/2^n<1$, and then

$$\alpha U_{cl} \subset \beta\left(\frac{2^n-2}{2^n} U_{cl}\right) \subset \beta U_i.$$

Definition 13. *For $f \in L$ and $U \in \mathfrak{U}$ consider the set of all $\alpha>0$ with $f \in \alpha U$. Its* g.l.b. *is denoted by* $\|f\|_U^+$.

Theorem 25. $\|f\|_U^+$ *is finite,* $\geqq 0$, *and continuous. If* $\alpha \geqq 0$, $\|\alpha f\|_U^+ = \alpha \|f\|_U^+$; *furthermore,* $\|f+g\|_U^+ \leqq \|f\|_U^+ + \|g\|_U^+$. $\|f\|_U^+=0$ *means that* $f \in \mathfrak{P}$ (αU *over all* $\alpha>0$) $=(+0)U$.†

As βf is continuous, for small $\beta>0$, $\beta f \in U$, $f \in U/\beta$; thus the α-set is not empty and $\|f\|_U^+$ is finite; it is obviously non-negative. $\|\alpha f\|_U^+ = \alpha\|f\|_U^+$ is obvious for $\alpha>0$; for $\alpha=0$ it states merely that $\|0\|_U^+=0$. If $f \in \alpha U$, $g \in \beta U$, Theorems 12 and 24 imply that $f+g \in (\alpha+\beta)U_{cl} \subset (\alpha+\beta+\delta)U$ for every $\delta>0$; from this it follows that $\|f+g\|_U^+ \leqq \|f\|_U^+ + \|g\|_U^+$. The last statement is obvious. It remains to prove the continuity of $\|f\|_U^+$.

Assume $0<\alpha<\|f_0\|_U^+<\beta$. (If $\|f_0\|_U^+=0$ then omit the α.) Choose α', β' with $\alpha<\alpha'<\|f_0\|_U^+<\beta'<\beta$. Then $f_0 \in \beta' U \subset \beta U_i$, f_0 is not $\in \alpha' U \supset \alpha U_{cl}$, $f_0 \in \mathfrak{P}(\beta U_i, \mathfrak{C}(\alpha U_{cl}))$. If an f belongs to this set, we have $f \in \beta U_i \subset \beta U$, f is not $\in \alpha U_{cl} \supset \alpha U$, and thus $\alpha \leqq \|f\|_U^+ \leqq \beta$. Furthermore, $\mathfrak{P}(\beta U_i, \mathfrak{C}(\alpha U_{cl}))$ is obviously open, proving the continuity of $\|f\|_U^+$.

† We write $(+0)U$ in order to distinguish this set from $0U$, which of course is (0).

THEOREM 26. *The sets* $\|f-f_0\|_U^+<\delta$, $f_0\epsilon L$, $U\epsilon\mathfrak{U}$, $\delta>0$, *form a complete system of neighborhoods in* L. (*It would be sufficient to consider* $\delta=1$.)

As $\|f-f_0\|_U^+$ is continuous in f, these sets are all open. If S is an open set, $f_0\epsilon S$, there is a $U\epsilon\mathfrak{U}$ with $f_0+U\subset S$, and $\|f-f_0\|_U^+<1$ implies $f-f_0\epsilon U$, $f\epsilon f_0+U\subset S$. Thus the set $\|f-f_0\|_U^+<1$ contains f_0 and is contained in S.

We could extend $\|\alpha f\|_U^+=\alpha\|f\|_U^+$ from the non-negative α's to all real α by introducing $\|f\|_U=\max\ (\|f\|_U^+,\ \|-f\|_U^+)$, but we shall not discuss this further here.

Note that in metric spaces L (cf. the remark after Definition 2b), the functions $\|f\|_U^+$ with $U=S^o(0;\delta)$ or with $U=S^{cl}(0;\delta)$ coincide with each other and with their $\|f\|_U$, all of them being equal to $\|f\|/\delta$. If L is only topological and if we form L_b^D by Definition 11, then obviously

$$\|\mathfrak{F}\|_{U'}^+ = \underset{x\,\epsilon\, D}{\mathrm{l.u.b.}}\ \|\mathfrak{F}(x)\|_U^+.$$

APPENDIX I

10. We wish to make two remarks which are useful for some applications.

Remark 1. The linear space L, as defined in Definition 1, may allow complex numbers α (instead of the real ones alone) as factors. We then call L complex linear and we change Definition 1 so as to admit in its conditions (4), (5), (6) also complex α and β. Then Definition 2a for the metric should be formulated with complex α's in its condition (2). But the important change is the one in Definition 2b for topological L's: here condition (4) must include all complex α's with $|\alpha|\leqq 1$ instead of only the real α's with $-1\leqq\alpha\leqq 1$. Then Theorem 7, stating the continuity of αf as a function of α and f, can be proved without any changes. (Note that the definition of convexity, Definition 2b, (7), is unaffected, and that Definition 7, Theorems 12 and 13 remain restricted to real coefficients.)

This stronger form of Definitions 2b, (4), which we call (4'), can be replaced by the following two conditions:

($4_1'$) if $U\epsilon\mathfrak{U}$, there is a $V\epsilon\mathfrak{U}$, such that, for every real α with $0\leqq\alpha\leqq 1$, $\alpha V\subset U$,

($4_2'$) if $U\epsilon\mathfrak{U}$, there is a $V\epsilon\mathfrak{U}$ with $iV\subset U$.

($4_1'$) and ($4_2'$), with the help of (3) and (5), include (4'): Choose $V+V\subset U$, W with $\alpha W\subset V$ for $0\leqq\alpha\leqq 1$, X with $iX\subset W$, Y with $iY\subset X$, Z with $iZ\subset Y$, $\Xi\subset\mathfrak{P}(W,X,Y,Z)$, where V, W, X, Y, Z, Ξ are all $\epsilon\mathfrak{U}$. Thus $i^n\Xi\subset W$ for $n=0, 1, 2, 3$. Now if a complex α is such that $|\alpha|\leqq 1$, we can write $\alpha=i^m\beta+i^n\gamma$, $m=0, 2$; $n=1, 3$; β, γ real; $0\leqq\beta, \gamma\leqq 1$. Thus $\alpha\Xi\subset\beta i^m\Xi+\gamma i^n\Xi\subset\beta W+\gamma W\subset V+V\subset U$, proving (4') as we stated it.

Remark 2. If L is convex and $S \subset L$, then

$$((S_{\mathrm{conv}})_{\mathrm{cl}})_{\mathrm{conv}} = (S_{\mathrm{conv}})_{\mathrm{cl}}.$$

As the left side obviously $\supset$ the right side, we need only prove $\subset$. And as $(S_{\mathrm{conv}})_{\mathrm{conv}} = S_{\mathrm{conv}}$, we may write T for S_{conv} and prove $(T_{\mathrm{cl}})_{\mathrm{conv}} \subset (T_{\mathrm{conv}})_{\mathrm{cl}}$. Assume $U \in \mathfrak{U}$, choose $V \in \mathfrak{U}$, $V+V \subset U$. Then $V_{\mathrm{conv}} \subset U$ (cf. the beginning of the proof of Theorem 14), and $T_{\mathrm{cl}} \subset T+V$, $(T_{\mathrm{cl}})_{\mathrm{conv}} \subset T_{\mathrm{conv}} + V_{\mathrm{conv}} = T_{\mathrm{conv}} + U$. Thus $(T_{\mathrm{cl}})_{\mathrm{conv}} \subset \mathfrak{P}(T_{\mathrm{conv}} + U)$, where U runs over all elements of $\mathfrak{U}$. By Theorem 5, the right side is $= (T_{\mathrm{conv}})_{\mathrm{cl}}$, thus $(T_{\mathrm{cl}})_{\mathrm{conv}} \subset (T_{\mathrm{conv}})_{\mathrm{cl}}$, completing the proof.

Appendix II

11. The coefficients α, which occur in Definition 1, play an important role in the applications of this theory, but the theory itself could be developed without them. That is, we could work on the basis of Definition 1, parts (1), (2), (7) alone; in other words, the theory could be extended from linear spaces to Abelian groups (except, of course, for the statements about convexity). Definition 2a has then to be restricted to (1), (3), and Definition 2b to (1), (2), (3), (4) with $\alpha = -1$ only, and (5). Note that (1), (2), (3) are general topological axioms, while (4), (5) express that $-f$ and $f+g$ (that is, the "group operations") are continuous.

After these changes are made all our considerations remain almost unaltered, except that the notion of "boundedness" must be avoided (because Definition 2b, (6), on which it is based, has been omitted); it has to be replaced by "total boundedness."

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