ON RINGS OF OPERATORS. II*

BY

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Introduction. This paper is a continuation of one by the same authors: *On rings of operators*, Annals of Mathematics, (2), vol. 37 (1936), pp. 116–229. It contains the solution of certain problems which were left open there. We will prove the general additivity of trace $Tr_M(A)$, its weak continuity, and certain isomorphisms between $\mathfrak{H}$, $\boldsymbol{M}$, and $\boldsymbol{M}'$ (cf. the remarks (i)–(iv) at the end of the above quoted paper). All these considerations refer to "Case (II)" for $\boldsymbol{M}$ (cf. Theorem VIII, loc. cit.).

The properties of $Tr_M(A)$ are established by obtaining for it a representation

$$Tr_M(A) = \sum_{i=1}^{m} (Ag_i, g_i)$$

(with a fixed, finite $m=1, 2, \cdots$, and fixed $g_1, \cdots, g_m \epsilon \mathfrak{H}$). This representation is remarkable, because it is obviously a close analogue of the representation of $Tr_M(A)$ as a trace, that is, as the arithmetic mean of the diagonal matrix-elements of A in the cases (I_n), $n=1, 2, \cdots$, when $\boldsymbol{M}$ is essentially the full matrix ring of an n-(finite-) dimensional Euclidean space.

For certain cases (with the help of which the others are then mastered) we have even $m=1$.

In Part I the above representation of $Tr_M(A)$ is obtained approximately. The technically interested reader may find it worth observing that the exhaustion method we use there (§§1.2 and 1.3) is analogous to certain procedures which can be used advantageously in the theories of measures and integration too. On this basis we establish the main properties of $Tr_M(A)$ in Part II, and then obtain the exact representation of $Tr_M(A)$ in Part III. Here two maximum-problems, called (A) and (B), which seem to possess some independent interest too, play a decisive role.

Part IV is devoted to establishing an isomorphism between $\mathfrak{H}$, $\boldsymbol{M}$, and $\boldsymbol{M}'$. It turns out that a certain algebraic-topological extension $Q(\boldsymbol{M})$ of $\boldsymbol{M}$ is isomorphic to $\mathfrak{H}$ and that $\boldsymbol{M}$ and $\boldsymbol{M}'$ play in it the role of right- and left-multiplication. This leads to an interesting and entirely new type of infinite hypercomplex systems, which are at the same time Hilbert spaces. A subsequent paper will be devoted to their independent study.

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The appendix deals with the possibility of considering $\boldsymbol{M}$ (in the case (II_1)) as a system of matrices with continuously spread rows and columns.

We will use the notations, definitions, and results of our paper *On rings of operators*, quoted above, throughout this paper. We will quote it, whenever necessary, as R.O. All other quotations follow the bibliography of R.O. (pp. 125–126, Nos. (1)–(22)).

The isomorphism problems of different rings $\boldsymbol{M}$ of class (II_1) (cf. the remark (v) at the end of R.O.) are not discussed here. They will be dealt with in a subsequent publication.

Since the appearance of R.O. the second-named author has succeeded in finding new representations of case (II_1) in terms of infinite direct products, which throw new light on (II_1) as a limiting case of the (I_n), $n=1, 2, \cdots$, as well as a way of applying the present theory to quantum-mechanics. These subjects too will be discussed in papers which will follow soon.

CONTENTS

Chapter I. Approximate form of $Tr_{\boldsymbol{M}}(A)$ (for $\alpha \geqq 1$)

1.1. We assume that we have given a factor $\boldsymbol{M}$ in case II_1. Now we normalize $D_{\boldsymbol{M}}$ and $D_{\boldsymbol{M}'}$ so that $C=1$ (cf. R.O., pp. 179–182). This means (R.O., loc. cit.) that for every $f \epsilon \mathfrak{H}$, $D_{\boldsymbol{M}}(\mathfrak{M}_f^{\boldsymbol{M}'}) = D_{\boldsymbol{M}'}(\mathfrak{M}_f^{\boldsymbol{M}})$ and that the range of $D_{\boldsymbol{M}}$ is the interval $0 \leqq x \leqq 1$. Let the range of $D_{\boldsymbol{M}'}$ be the interval $0 \leqq x \leqq \alpha$, where $0 < \alpha \leqq \infty$. In this part, we assume that $\alpha \geqq 1$.

Now $\boldsymbol{M}'$ is a factor of class II_1 or II_∞ by R.O., Theorem X. In the first case the range of $D_{\boldsymbol{M}'}$ is in the normalization (loc. cit.) the interval, $0 \leqq x \leqq 1$, and $C = 1/\alpha$ (since $D_{\boldsymbol{M}'} = 1/\alpha$ times its value in the previous paragraph). Since $C \leqq 1$, we have in both cases by the discussion on page 182 of R.O., that $\Delta_0 = \Delta$. By R.O., Definition 10.1.1, this implies that there exists an f such that $D_{\boldsymbol{M}}(\mathfrak{M}_f^{\boldsymbol{M}'}) = 1$. Since we can pass from the normalization of this paragraph to that of the previous paragraph by multiplying $D_{\boldsymbol{M}'}$ by α and leaving $D_{\boldsymbol{M}}$ unchanged, we see that this holds even with our previous normalization.

Thus we have an f such that $D_{\boldsymbol{M}}(E_f^{\boldsymbol{M}'}) = 1 = D_{\boldsymbol{M}}(1)$ or $D_{\boldsymbol{M}}(1 - E_f^{\boldsymbol{M}'}) = 0$ which implies $1 - E_f^{\boldsymbol{M}'} = 0$; that is, $E_f^{\boldsymbol{M}'} = 1$. We now assume that such an f has been chosen and is held fixed for the following discussion.

1.2. With respect to this fixed f, we now define certain relations concerning a projection $E \epsilon \boldsymbol{M}$.

Definition 1.2.1. *Let $E \neq 0$ be a projection, $\epsilon \boldsymbol{M}$, θ a real number $\geqq 0$. The relation $E > \theta$ ($E \geqq \theta$, $E < \theta$, $E \leqq \theta$) is said to hold if $(Ef, f) > \theta D_{\boldsymbol{M}}(E)$ ($(Ef, f) \geqq \theta D_{\boldsymbol{M}}(E)$, $(Ef, f) < \theta D_{\boldsymbol{M}}(E)$, $(Ef, f) \leqq \theta D_{\boldsymbol{M}}(E)$ respectively).*

The relation $E \geqq {}_p\lambda$ ($E \leqq {}_p\lambda$) is said to hold for a projection $E \neq 0$ if for every $F \epsilon \boldsymbol{M}$ such that $F \leqq E$, $F \neq 0$, we have $F \geqq \lambda$ ($F \leqq \lambda$).

Lemma 1.2.1. *Let $\{E_i\}$ be a sequence of projections (finite or infinite) with $E_i \epsilon \boldsymbol{M}$, $E_i E_j = \delta_j^i E_i$. Then if for every i, $E_i \leqq \lambda$ ($E_i \geqq \lambda$), and if for some i, say i_0, $E_{i_0} < \lambda$ ($E_{i_0} > \lambda$) we have $\sum_i E_i < \lambda$ ($\sum_i E_i > \lambda$).*

We have, for every $i, \lambda D(E_i) \leqq (E_i f, f)$, hence $\lambda \sum_{i \neq i_0} D(E_i) \leqq \sum_{i \neq i_0} (E_i f, f)$ and $\lambda D(E_{i_0}) < (E_{i_0} f, f)$. These imply $\lambda \sum_i D(E_i) < \sum_i (E_i f, f)$ or $\lambda D(\sum_i E_i) < (\sum_i E_i f, f)$, which was to be shown.

LEMMA 1.2.2. *If* $E \geqq \lambda$ $(E \leqq \lambda)$, *then there exists an* F *such that* $E \geqq F$, $F \geqq_p \lambda$ $(F \leqq_p \lambda)$.

By Definition 1.2.1 we have $E \neq 0$. Suppose there is no such F. Then there must be an E_1, such that $E \geqq E_1$, and $E_1 < \lambda$. (Otherwise E itself would be such an F.) Now let Ω be the first ordinal number which belongs to a cardinal number $\aleph > \aleph_0$. Let α be an ordinal number such that $\alpha < \Omega$. Let us suppose that for all $\beta < \alpha$ an E_β has been defined in such a way that $E_\beta \neq 0$, $E_\beta \leqq E$, $E_\beta < \lambda$, $E_{\beta_1} E_{\beta_2} = \delta_{\beta_1, \beta_2} E_{\beta_1}$. Now if $E - \sum_{\beta < \alpha} E_\beta = 0$, let $E_{\alpha'}$ be undefined for $\alpha' \geqq \alpha$. If, however, $E - \sum_{\beta < \alpha} E_\beta \neq 0$, by hypothesis, $E - \sum_{\beta < \alpha} E_\beta$ is not $\geqq_p \lambda$, hence there exists an E_0, with $E_0 \neq 0$, $E - \sum_{\beta < \alpha} E_\beta \geqq E_0$, $E_0 < \lambda$, $E \geqq E - \sum_{\beta < \alpha} E_\beta \geqq E_0$. Thus if we let $E_\alpha = E_0$, we have, if $E - \sum_{\beta < \alpha} E_\beta \neq 0$, an E_α which is orthogonal to all previous E_β and $E_\alpha < \lambda$.

Now $D(E) \geqq \sum_{\beta < \alpha} D(E_\beta)$ for every α. Since $D(E_\beta) > 0$, this implies that there is only denumerably many of the numbers $D(E_\beta)$. Hence for some $\alpha < \Omega$, $E - \sum_{\beta < \alpha} E_\beta = 0$. Inasmuch as $\alpha < \Omega$, we can re-index the E_β's into a finite or a simply infinite sequence. Now $E_1 < \lambda$, $E_i \leqq \lambda$, and since $E = \sum_i E_i$, Lemma 1.1 implies that $E < \lambda$, a contradiction.

LEMMA 1.2.3. *If* $E \geqq_p \lambda$ $(E \leqq_p \lambda)$, *then if* $E \geqq F$, $F \neq 0$, $F \epsilon \boldsymbol{M}$, *we have* $F \geqq_p \lambda$ $(F \leqq_p \lambda)$.

If F_1 is such that $F \geqq F_1$, $F_1 \neq 0$ then $E \geqq F \geqq F_1$ and hence by Definition 1.2.1, $F_1 \geqq \lambda$. This statement implies that $F \geqq_p \lambda$.

1.3. Now $(1f, f) = \|f\|^2 = \|f\|^2 D(1)$. Now let $\theta_0 = \|f\|^2$ which is of course not zero for $\mathfrak{M}_f{}^{\boldsymbol{M}} = \mathfrak{H}$ and this precludes $f = 0$. Thus 1 (the projection, not the number) is $\geqq \theta_0$. By Lemma 1.2.2, there exists an $E \epsilon \boldsymbol{M}$, such that $E \geqq_p \theta_0$. Now let λ be the least upper bound of the numbers θ such that $E \geqq_p \theta$. λ must be less than $(Ef, f)/D_{\boldsymbol{M}}(E)$ and since $D_{\boldsymbol{M}}(E) > 0$, it must be finite. Let ϵ be any number > 0. E is not $\geqq_p \lambda + \epsilon$. Hence there is an E_1, such that $E \geqq E_1$, and $E_1 \leqq \lambda + \epsilon$. Lemma 1.2.2 now implies that there is an E_2 with $E_1 \geqq E_2$ and $E_2 \leqq_p \lambda + \epsilon$.

Now if F is such that $E \geqq F$, $F \epsilon \boldsymbol{M}$, $F \neq 0$, we have $F \geqq \theta$ for all $\theta < \lambda$, which means of course that $F \geqq \lambda$ and $E \geqq_p \lambda$. Now since $E \geqq E_1 \geqq E_2$, Lemma 1.2.3 implies that $E_2 \geqq_p \lambda$. Thus for some fixed $\lambda > 0$, given any $\epsilon > 0$, we can find a non-zero $E_2 \epsilon \boldsymbol{M}$, such that for all $F \epsilon \boldsymbol{M}$ with the property $E_2 \geqq F$, we have

$$(\lambda + \epsilon) D_{\boldsymbol{M}}(F) \geqq (Ff, f) \geqq \lambda D_{\boldsymbol{M}}(F).$$

Now if we let f be $\lambda^{1/2} f$ above, $\epsilon = \lambda(K-1)E_2 = E$, we see that

LEMMA 1.3.1. *To every $K>1$ there exists an f and $E\epsilon M$, $E\neq 0$, such that for every $F\epsilon M$ with the property $E\geqq F$,*

$$KD_M(F) \geqq (Ff,f) \geqq D_M(F).$$

1.4. We can now show

LEMMA 1.4.1. *Let K, f, and E be as in Lemma 1.3.1. Let $A\epsilon M$ be a positive definite self-adjoint operator such that $EA=AE=A$. Then*

$$KTr_M(A) \geqq (Af,f) \geqq Tr_M(A).$$

Let c be the bound of A, $E(\lambda)$ the resolution of the identity corresponding to A. Now by R.O., pp. 212–213,

$$Tr_M(A) = \int_0^c \lambda dD(E(\lambda)); \qquad (Af,f) = \int_0^c \lambda d(E(\lambda)f,f).$$

Now since $AE=A$, $E(0)\geqq 1-E$, and since A commutes with E, E commutes with all $E(\lambda)$ (cf. (15), Theorem 8.2). Now $E(\lambda)=E(\lambda)E+E(\lambda)(1-E)$. Since $E(\lambda)$ and E commute, $E(\lambda)E=F(\lambda)$ is a projection and similarly $E(\lambda)(1-E)$ is a projection. For $\lambda\geqq 0$, we have $E(\lambda)(1-E)\leqq 1-E$ and also $E(\lambda)(1-E)\geqq E(0)(1-E)\geqq(1-E)^2=1-E$ and thus $E(\lambda)(1-E)=1-E$. So for $\lambda\geqq 0$, $E(\lambda)=F(\lambda)+(1-E)$. Now $F(\lambda)(1-E)=E(\lambda)E(1-E)=0$, hence by R.O., Definition 8.2.1, $D(E(\lambda))=D(F(\lambda))+D(1-E)$. Thus we have

$$Tr_M(A) = \int_0^c \lambda dD(E(\lambda)) = \int_0^c \lambda dD(F(\lambda)) + \int_0^c \lambda dD(1-E) = \int_0^c \lambda dD(F(\lambda))$$

and

$$\begin{aligned}(Af,f) &= \int_0^c \lambda d(E(\lambda)f,f) = \int_0^c \lambda d(F(\lambda)f,f) + \int_0^c \lambda d((1-E)f,f)\\ &= \int_0^c \lambda d(F(\lambda)f,f).\end{aligned}$$

But Lemma 1.3.1 now implies that if $0\leqq\alpha<\beta\leqq c$, then

$$KD_M(F(\beta)-F(\alpha)) \geqq ((F(\beta)-F(\alpha))f,f) \geqq D_M(F(\beta)-F(\alpha))$$

which by the definition of the Riemann-Stieltjes integral yields that

$$KTr_M(A) \geqq (Af,f) \geqq Tr_M(A).$$

LEMMA 1.4.2. *For a projection $E\neq 0$, $E\geqq_p\theta$ ($\leqq_p\theta$) is equivalent to the statement that for all positive definite $A\epsilon M$ such that $EA=AE=A$,*

$$(Af,f) \geqq \theta Tr_M(A) \; (\leqq \theta Tr_M(A)).$$

Since $Tr_M(F)=D_M(F)$, F a projection, the last statement implies the first. The converse is shown by a proof similar to that of Lemma 1.4.1.

LEMMA 1.4.3. *Let $A\epsilon M$ be such that $EA=AE=A$, then $\|A^*f\|^2 \leqq K\|Af\|^2$, if E, f, K are as in Lemma* 1.3.1.

A^*A is positive definite and ϵM. Furthermore $AE=EA=A$ implies $EA^*=A^*E=A^*$, and thus $EA^*A=A^*A=A^*AE$. Hence Lemma 1.4.1 applies to A^*A and we have

$$(\alpha) \qquad KTr_M(A^*A) \geqq (A^*Af, f) \geqq Tr_M(A^*A).$$

Using the canonical decomposition (R.O., Definition 4.4.1) for A^* we have $A^*=UB$, where U may be taken as unitary. (In the finite cases, we see (cf. R.O., Lemma 16.1.1) that there exists a partially isometric V with initial set $(f; A^*f=0)$ and final set $(f; Af=0)$. Now if W is as in R.O., Definition 4.4.1, for $A=A^*$, let $U=W+V$.) B is self-adjoint and equals $(AA^*)^{1/2}$. Now $A=BU^*=BU^{-1}$ and hence $A^*A=UBBU^{-1}=UB^2U^{-1}=UAA^*U^{-1}$. Hence

$$(\beta) \qquad Tr_M(A^*A) = Tr_M(UAA^*U^{-1}) = Tr_M(AA^*).$$

Substituting A^* for A in our previous result we have

$$(\gamma) \qquad KTr_M(AA^*) \geqq (AA^*f, f) \geqq Tr_M(AA^*).$$

Combining (α), (β), and (ϕ), we obtain

$$K(A^*Af, f) \geqq KTr_M(A^*A) = KTr_M(AA^*) \geqq (AA^*f, f).$$

Since $(A^*Af, f)=(Af, Af)=\|Af\|^2$, $(AA^*f, f)=(A^*f, A^*f)=\|A^*f\|^2$; this is the desired inequality.

1.5. Now if E is as in Lemma 1.3.1, let n be the smallest integer such that $1/n \leqq D_M(E)$. Then if $E^0 \leqq E$ is such that $D_M(E^0)=1/n$, Lemma 1.3.1 will hold with E^0 in place of E. Now f was chosen in such a manner that $\mathfrak{M}_f^{M'}=\mathfrak{H}$. This implies that $\mathfrak{M}_{E^0f}^{M'}$ is the range of E^0, because since the set $(A'f; A'\epsilon M')$ is dense in $\mathfrak{H}$, the set $(E^0A'f; A'\epsilon M')=(A'E^0f; A'\epsilon M')$ must be dense in the range of E^0, or $E^0=E_{E^0f}^{M'}$. Since $C=1$, $D_{M'}(E_{E^0f}^{M})=D_M(E_{E^0f}^{M'})=D_M(E^0)=1/n$.

Now let $E^0=E_1$, $E_{E^0f}^{M}=E_1'$. Since $D_M(E_1)=1/n$, $D_M(1)=1$, we can find n projections $\{E_j\}$, $j=1, \cdots, n$ with the first equal to E_1 such that $E_i \epsilon M$, $\sum_{j=1}E_j=1$, $E_iE_j=0$ if $i\neq j$, $D_M(E_j)=1/n$. Since $D_{M'}(E_1')=1/n$, $D_{M'}(1)=\alpha \geqq 1$, there exist n projections $E_j'\ \epsilon M'$, $j=1, \cdots, n$, with the first equal to E_1' and such that $E_i'E_j'=0$ if $i\neq j$ and $D_{M'}(E_j')=1/n$. Since $D_M(E_1)=D_M(E_j)$, there is a partially isometric operator $W_j \epsilon M$ with initial set the range of E_1 and final set the range of E_j (cf. R.O., Definition 4.3.1). Similarly we have a $W_j'\ \epsilon M'$ with initial set the range of E_1' and final set E_j'. The relations

$W_j^*W_j=E_1$, $W_jW_j^*=E_j$, $W_j'^*W_j'=E_1'$ and $W_j'W_j'^*=E_j'$ hold, also $W_jE_1=W_j$, $E_jW_j=W_j$, $E_1W_j^*=W_j^*$, $W_j^*E_j=W_j^*$, $W_j'E_1'=W_j'$, $E_j'W_j'=W_j'$, $E_1'W_j'^*=W_j'^*$, $W_j'^*E_j'=W_j'^*$ (cf. R.O., Lemma 4.3.1).

Let $W_j'W_jE_1f=f_j$, $E_1f=f_1$. Then $E_jf_j=E_jW_j'W_jE_1f=W_j'E_jW_jE_1f=W_j'W_jE_1f=f_j$ and $E_j'f_j=E_j'W_j'W_jE_1f=W_j'W_jE_1f=f_j$. Thus $E_jf_j=f_j=E_j'f_j$.

Let $g=\sum_{j=1}^n f_j$, then $E_ig=E_i\sum_{j=1}^n f_j=E_i\sum_{j=1}^n E_jf_j=\sum_{j=1}^n E_iE_jf_j=E_if_i=f_i$. Similarly $E_i'g=f_i$. Now for any $A\epsilon\boldsymbol{M}$ if either $i\neq j$ or $k\neq l$, then E_iAE_kg is orthogonal to E_jAE_lg. For inasmuch as $E_kg=f_k=E_k'g$, we have,

$$\begin{aligned}(E_iAE_kg,\ E_jAE_lg) &= (E_jE_iAE_kg,\ AE_lg) &&= \delta_j^{\,i}(E_iAE_kg,\ AE_lg)\\ &= \delta_j\ (E_iAE_k'g,\ AE_l'g) &&= \delta_j^{\,i}(E_k'E_iAg,\ E_l'Ag)\\ &= \delta_j^{\,i}\,(E_l'E_k'E_iAg,\ Ag) &&= \delta_j^{\,i}\delta_k^{\,l}(E_k'E_iAg,\ Ag).\end{aligned}$$

These results imply that if $A\epsilon\boldsymbol{M}$, then

$$\begin{aligned}\|Ag\|^2 &= \left\|\left(\sum_{i=1}^n E_i\right)A\left(\sum_{j=1}^n E_j\right)g\right\|^2 = \left\|\sum_{i=1}^n\sum_{j=1}^n E_iAE_jg\right\|^2 = \sum_{i=1}^n\sum_{j=1}^n\|E_iAE_jg\|^2\\ &= \sum_{i=1}^n\sum_{j=1}^n\|E_iAE_jE_jg\|^2 = \sum_{i=1}^n\sum_{j=1}^n\|E_iAE_jf_j\|^2 = \sum_{i=1}^n\sum_{j=1}^n\|E_iAE_jW_j'W_jf_1\|^2\\ &= \sum_{i=1}^n\sum_{j=1}^n\|W_j'E_iAE_jW_jf_1\|^2 = \sum_{i=1}^n\sum_{j=1}^n\|W_j'E_1'E_iAE_jW_jf_1\|^2\\ &= \sum_{i=1}^n\sum_{j=1}^n\|E_1'E_iAE_jW_jf_1\|^2 = \sum_{i=1}^n\sum_{j=1}^n\|E_iAE_jW_jE_1'f_1\|^2\\ &= \sum_{i=1}^n\sum_{j=1}^n\|E_iAE_jW_jf_1\|^2 = \sum_{i=1}^n\sum_{j=1}^n\|W_i^*E_iAE_jW_jf_1\|^2\\ &= \sum_{i=1}^n\sum_{j=1}^n\|W_i^*E_iAE_jW_jE_1f\|^2 = \sum_{i=1}^n\sum_{j=1}^n\|W_i^*E_iAE_jW_jf\|^2,\end{aligned}$$

remembering that W_j' is isometric on the range of E_1', W_i^* on the range of E_i and $W_jE_1=W_j$. Substituting A^* for A and interchanging i and j we also have

$$\|A^*g\|^2 = \sum_{i=1}^n\sum_{j=1}^n\|W_j^*E_jA^*E_iW_if\|^2 = \sum_{i=1}^n\sum_{j=1}^n\|(W_i^*E_iAE_jW_j)^*f\|^2.$$

But recalling the properties of W_j and W_i^* we have

$$E_1W_i^*E_iAE_jW_j = W_i^*E_iAE_jW_j = W_i^*E_iAE_jW_jE_1.$$

Thus Lemma 1.4.3 with $A=W_i^*E_iAE_jW_j$ yields $\|(W_i^*E_iAE_jW_j)^*f\|^2 \leqq K\|W_i^*E_iAE_jW_jf\|^2$ and with this, we obtain from the above equations for $\|Ag\|^2$ and $\|A^*g\|^2$ that $\|A^*g\|^2\leqq K\|Ag\|^2$.

This result may be stated as follows.

LEMMA 1.5.1. *If K is any number >1, there exists a $g\epsilon\mathfrak{H}$, $g\neq 0$ such that for every $A\epsilon\boldsymbol{M}$, $\|A^*g\|^2\leqq K\|Ag\|^2$. Obviously we may assume that $\|g\|=1$.*

1.6. We now have

LEMMA 1.6.1. *Let K and g be as in Lemma 1.5.1. Then if E and F, $\epsilon\boldsymbol{M}$, are two projections such that $D_M(E)=D_M(F)$, then $K^{-1}(Fg, g)\leqq(Eg, g)\leqq K(Fg, g)$.*

Inasmuch as $D_M(E)=D_M(F)$, there exists a partially isometric operator W such that $W^*W=F$, $WW^*=E$, $W\epsilon\boldsymbol{M}$ (cf. R.O., Definition 8.2.1, Definition 6.1.1, and Lemma 4.3.1). Now $(Fg, g)=(W^*Wg, g)=(Wg, Wg)=\|Wg\|^2$, $(Eg, g)=(WW^*g, g)=(W^*g, W^*g)=\|W^*g\|^2$. Lemma 1.5.1 with $A=W$ implies $(Eg, g)\leqq K(Fg, g)$. With $A=W^*$, the same lemma implies $(Fg, g)\leqq K(Eg, g)$ or $K^{-1}(Fg, g)\leqq(Eg, g)$. We have now shown our lemma.

Suppose $E\epsilon\boldsymbol{M}$ is such that $D_M(E)=1/m$, where m is an integer. Then there exist $m-1$ projections $E_2, \cdots, E_m$, such that when we let $E_1=E$, we have $\sum_{j=1}^m E_j=1$, $E_iE_j=0$, for $i\neq j$, $E_j\epsilon\boldsymbol{M}$, $D_M(E_j)=1/m$. Now returning to the g of Lemmas 1.5.1 and 1.6.1, we have

$$1=\|g\|^2=(g, g)=\left(\sum_{j=1}^m E_jg, g\right)=\sum_{j=1}^m (E_jg, g).$$

Let $(E_1g, g)=\alpha$. Then Lemma 1.6.1 implies that $K\alpha\geqq(E_jg, g)\geqq K^{-1}\alpha$, for every j. Summing over j, gives $Km\alpha\geqq 1\geqq K^{-1}m\alpha$ or $K\alpha\geqq 1/m\geqq K^{-1}\alpha$ which is the same as

$$(+)\qquad K(Eg, g)\geqq D_M(E)\geqq K^{-1}(Eg, g),$$

since $D_M(E)=1/m$.

Let us study the class Σ of E's in $\boldsymbol{M}$ which satisfy the equation $(+)$. Now if $F_1, \cdots, F_q$ or $F_1, F_2, \cdots$ satisfy $(+)$ and $F_iF_j=0$, $i\neq j$, then $\sum_{j=1}^q F_j$ or $\sum_{j=1}^\infty F_j$ also satisfies $(+)$. But if $F\epsilon\boldsymbol{M}$ is such that $D_M(F)=q/p$, $F=\sum_{j=1}^q F_j$ for mutually orthogonal F_j's $\epsilon\boldsymbol{M}$ with $D_M(F_j)=1/p$ and hence satisfying $(+)$. Thus if $D_M(F)=q/p$, F satisfies $(+)$.

Let E be any projection of $\boldsymbol{M}$, $D_M(E)=\alpha$. Let $\{\alpha_i\}$, $\alpha_i\geqq 0$ be a sequence of rational numbers, $\sum_{i=1}^\infty\alpha_i=\alpha$. Then there exists a set of mutually orthogonal projections $\{E_i\}$ such that $E_i\leqq E$, $E_i\epsilon\boldsymbol{M}$, $D_M(E_i)=\alpha_i$. To show this, suppose $E_1, \cdots, E_{i-1}$ have been chosen with these properties. Then $D_M(E-\sum_{j=1}^{i-1}E_j)=\sum_{j=i}^\infty\alpha_j\geqq\alpha_i$. Hence an E_i can be chosen in such a way that $E_i\epsilon\boldsymbol{M}$ is $\leqq E-\sum_{j=1}^{i-1}E_j$ with $D_M(E_i)=\alpha_i$. Now $D_M(E-\sum_{i=1}^\infty E_i)$ $=\alpha-\sum_{i=1}^\infty\alpha_i=0$ or $E=\sum_{i=1}^\infty E_i$. Since each E_i satisfies $(+)$, E does too. Since E is arbitrary we have

LEMMA 1.6.2. *Let K and g be as in Lemma 1.5.1, then for every $E\epsilon\boldsymbol{M}$*

$$K(Eg, g)\geqq D_M(E)\geqq K^{-1}(Eg, g).$$

1.7. From Lemma 1.6.2 we can conclude, using Lemma 1.4.2, that if $A \epsilon \mathbf{M}$ is positive definite, then

$$K(Ag, g) \geqq Tr_{\mathbf{M}}(A) \geqq K^{-1}(Ag, g).$$

We can state our result as follows.

THEOREM I. *Let $\mathbf{M}$ be a factor in case* II_1. *Let $\mathbf{M}$ and $\mathbf{M}'$ be such that when $D_{\mathbf{M}}$ and $D_{\mathbf{M}'}$ are normalized in such a way that the range of $D_{\mathbf{M}}$ is the interval* $(0, 1)$ *and* $C=1$ (cf. §1.1), *then the range of $D_{\mathbf{M}'}$ is an interval* $(0, \alpha)$ *with* $0<\alpha\leqq\infty$, *then* $\alpha\geqq 1$. *Then to every* $K>1$, *there exists a* $g \epsilon \mathfrak{H}$ *such that for every positive definite* $A \epsilon \mathbf{M}$,

$$K(Ag, g) \geqq Tr_{\mathbf{M}}(A) \geqq K^{-1}(Ag, g).$$

CHAPTER II. IMMEDIATE CONSEQUENCES

2.1. We are now able to show that $Tr_{\mathbf{M}}$ has the following two properties when $\mathbf{M}$ is in a finite case (cf. R.O., Theorem VIII); that is, when $\mathbf{M}$ is in a case I_n $(n=1, 2, \cdots)$ or II_1.

PROPERTY I. *For all Hermitian A and $B \epsilon \mathbf{M}$*

$$Tr_{\mathbf{M}}(A+B) = Tr_{\mathbf{M}}(A) + Tr_{\mathbf{M}}(B).$$

PROPERTY II. *$Tr_{\mathbf{M}}(A)$ is weakly continuous if A is subject to either of these conditions*:

(i) *A is uniformly bounded (that is, $|||A|||\leqq D$ for some fixed D)*;

(ii) *A is definite.*

These properties are obviously independent of the normalization of $D_{\mathbf{M}}(E)$ and $Tr_{\mathbf{M}}(A)$.

If $\mathbf{M}$ is in cases I_n, $n=1, 2, \cdots$, these are of course well known results about $Tr_{\mathbf{M}}(A)$ (cf. R.O., p. 220). So we may assume $\mathbf{M}$ to be in case II_1.

Then we have in the normalization of R.O., Theorem VIII, three possibilities for the factorization $\mathbf{M}$, $\mathbf{M}'$: II_1, II_∞; II_1, II_1 with $C\leqq 1$; II_1, II_1 with $C\geqq 1$. The two first ones correspond to $\alpha\geqq 1$ in the normalization of §1.1, and since we shall only use Theorem I, from §1.1, we may treat these two cases together.

We therefore consider first the conditions under which Theorem I holds: the normalization of §1.1, and $\alpha \geqq 1$. We obtain an equivalent statement of Theorem I with $A \epsilon \mathbf{M}$, Hermitian and not necessarily definite.

Let K, g be as in Theorem I, then $K^{-1}(g, g)=K^{-1}\|g\|^2\leqq Tr_{\mathbf{M}}(1)=1$, or $\|g\|^2\leqq K$. Furthermore if $A \epsilon \mathbf{M}$ is Hermitian, then $A=A_1-A_2$, where A_1 and A_2 are both positive definite and $A_1A_2=0=A_2A_1$. (Take $A_1=\frac{1}{2}(A+|A|)$,

$A_2=\frac{1}{2}(|A|-A)$ (cf. (15), Chapter VI, or (19), p. 203 ff.).) Since A_1 and A_2 commute, $Tr_M(A)=Tr_M(A_1)-Tr_M(A_2)$ by R.O., Lemma 15.3.4. It is a consequence of Theorem 1 that, if $\epsilon=K-1$,

$$|Tr_M(A_1)-(A_1g,g)|\leqq\epsilon(A_1g,g);\quad |Tr_M(A_2)-(A_2g,g)|\leqq\epsilon(A_2g,g).$$

These with the above equation for $Tr_M(A)$ imply

$$|Tr_M(A)-(Ag,g)|\leqq\epsilon((A_1g,g)+(A_2g,g)).$$

But the bounds of A_1 and A_2 are each not more than that of A. Also $\|g\|^2\leqq 1+\epsilon$. So if A has the bound D, we have

$$|Tr_M(A)-(Ag,g)|\leqq 2\epsilon(1+\epsilon)D.$$

Let A, B, $A+B$ have the respective bounds D_1, D_2, D_3. Substituting in (1) each of these operators, inasmuch as $((A+B)g,g)=(Ag,g)+(Bg,g)$, we get

$$|Tr_M(A+B)-Tr_M(A)-Tr_M(B)|\leqq 2\epsilon(1+\epsilon)(D_1+D_2+D_3),$$

which since ϵ may be taken arbitrarily small, implies Property I.

We turn our attention to Property II, (i). Consider the set Σ of all Hermitian $A\epsilon M$, with a bound less than or equal to a fixed D. We will show that to every $A\epsilon\Sigma$ and to every $\eta>0$, there exists a weak neighborhood of A, $U(A;g;g;\eta/3)$ such that for all $B\epsilon\Sigma$ and $B\epsilon U(A;g;g;\eta/3)$ (i.e., $|((B-A)g,g)|<\eta/3$) we have $|Tr_M(B)-Tr_M(A)|<\eta$. Letting ϵ be such that $\epsilon(1+\epsilon)D\leqq\eta/3$, as above we can find a g such that for A and every $B\epsilon\Sigma$, $|Tr_M(A)-(Ag,g)|\leqq\epsilon(1+\epsilon)D\leqq\eta/3$, $|Tr_M(B)-(Bg,g)|\leqq\eta/3$. In particular for $B\epsilon U(A;g;g;\eta/3)$ (cf. above) which means that we also have $|((B-A)g,g)|<\eta/3$, these inequalities imply that $|Tr_M(A)-Tr_M(B)|<\eta$.

Thus we have shown the weak neighborhood continuity of $Tr_M(A)$ with of course a restriction. From this the sequential continuity follows immediately. (If $A_n\rightarrow A$, then the A_n are uniformly bounded and for any $\eta>0$, almost all the A_n must be in the above given neighborhood.) In this situation the two kinds of continuity are equivalent (cf. (18), pp. 383–384), but we will use only the sequential.

$Tr_M(A)$ is also weakly continuous when considered only for the positive definite $A\epsilon M$. For suppose A is such and an $\epsilon>0$ is given. Let the K of Theorem I be chosen in such a way that $1<K\leqq 2$, and $(K-1)Tr_M(A)<\epsilon/5$, and let g correspond to this K as there. Let η be chosen in such a way that $\eta\leqq\epsilon/5$. Then if a positive definite $B\epsilon M$ is in $U(A;g;g;\eta)$, we have $|Tr_M(A)-Tr_M(B)|<\epsilon$ (because $|((B-A)g,g)|<\eta\leqq\epsilon/5$, $|Tr_M(A)-(Ag,g)|$

$\leqq (K-1)Tr_M(A) < \epsilon/5$, and, in addition, $|Tr_M(B) - (Bg, g)| \leqq (K-1)(Bg, g)$ $\leqq (K-1)((Ag, g)+\eta) \leqq (K-1)(Tr_M(A)+\epsilon/5+\eta) \leqq 3\epsilon/5)$.

Thus we have settled the factorizations II_1, II_∞ and II_1, II_1 with $C \leqq 1$. But we still must consider the case II_1, II_1 with $C > 1$. Let m be an integer such that $C/m \leqq 1$, E_m an m-dimensional unitary space. Form $E_m \otimes \mathfrak{H}$ (cf. R.O., Theorem I). Let $\boldsymbol{N}$ be the ring of operators on E_m, $\boldsymbol{N}'$ the corresponding set of operators on $E \otimes \mathfrak{H}$ (cf. R.O., Lemma 2.3.1 and Lemma 2.3.2), $\boldsymbol{B}$ the set of all operators in $\mathfrak{H}$, $\boldsymbol{B}^{(2)}$ the corresponding set in $E \otimes \mathfrak{H}$.

Under the above correspondence of $\boldsymbol{B}$ and $\boldsymbol{B}^{(2)}$, $\boldsymbol{M} \sim \boldsymbol{M}^{(2)}$, $\boldsymbol{M}' \sim \boldsymbol{M}'^{(2)}$. The first and hence each of the others is a full ring isomorphism (R.O., Lemma 2.3.4 and (22), §4). Thus $\boldsymbol{M}^{(2)}$ and $\boldsymbol{M}'^{(2)}$ are in case II_1 (R.O., Theorem IX). Now let $\phi_1, \cdots, \phi_m$ be a complete orthonormal set in E_m and let us think of the set up of R.O., §2.4. By R.O., Lemma 2.4.5, we have $\boldsymbol{M}^{(2)\prime} = R(\boldsymbol{N}^{(1)}, \boldsymbol{M}'^{(2)})$ and by R.O., Lemma 11.5.2, $R(\boldsymbol{N}^{(1)}, \boldsymbol{M}'^{(2)})$ is in case II_1 since as we have mentioned before, $\boldsymbol{M}'^{(2)}$ is in this case.

Thus the coupled factorization, $\boldsymbol{M}^{(2)}$, $\boldsymbol{M}^{(2)\prime}$, is in case II_1, II_1 and $\boldsymbol{M}^{(2)}$ is fully ring isomorphic to $\boldsymbol{M}$. Furthermore such an isomorphism preserves D_M in the standard normalization (R.O., Theorem IX) and hence the trace. Now suppose we have shown that for $\boldsymbol{M}^{(2)}$, $\boldsymbol{M}^{(2)\prime}$ we have $C^{(2)} \leqq 1$. Then $\boldsymbol{M}^{(2)}$ has Properties I and II, by the above and $\boldsymbol{M}$ must have them to be the isomorphism. (As m is finite, even weak operator topology is conserved under the mapping $\boldsymbol{B} \sim \boldsymbol{B}^{(2)}$.)

Therefore it remains to show that $C^{(2)} \leqq 1$. Consider the closed linear manifold, $(\phi_1 \otimes f; f \epsilon \mathfrak{H}) = (\text{say})\ \mathfrak{M}$, in $E_m \otimes \mathfrak{H}$. The correspondence of $\mathfrak{M}$ and $\mathfrak{H}$ given by $\phi_1 \otimes f \sim f$ is unitary. Under it, $\boldsymbol{M}^{(2)}_{\mathfrak{M}}$ and $R(\boldsymbol{N}^{(1)}, \boldsymbol{M}'^{(2)})_{\mathfrak{M}}$ (cf. R.O., Definition 11.3.1) are unitarily equivalent to $\boldsymbol{M}$ and $\boldsymbol{M}'$ (cf. R.O., §2.4). Thus the C for $\boldsymbol{M}$, $\boldsymbol{M}'$ is the same as that of $\boldsymbol{M}^{(2)}_{\mathfrak{M}}$ and $R(\boldsymbol{N}^{(1)}, \boldsymbol{M}'^{(2)})_{\mathfrak{M}}$. Then from the proof of Lemma 11.4.3, we can conclude that $C = (D_{M^{(2)}}(\mathfrak{H})/D_{M^{(2)\prime}}(\mathfrak{M}))C^{(2)} = mC^{(2)}$ or $C^{(2)} = C/m \leqq 1$.

2.2. Previously $Tr_M(A)$ has only been considered for Hermitian $A \epsilon \boldsymbol{M}$. We now define it for all $A \epsilon \boldsymbol{M}$.

Definition 2.2.1. *If* $A \epsilon \boldsymbol{M}$, *then* $A = B + iC$, *where* $B = \frac{1}{2}(A + A^*)$, $C = -\frac{1}{2}i(A - A^*)$ *are Hermitian, and we define* $Tr_M(A) = Tr_M(B) + iTr_M(C)$.

$Tr_M(A)$ has the following property.

Property III. $(*)$ *$Tr_M(A)$ agrees with the previous definition of $Tr_M(A)$ if A is Hermitian.*

$(**)$ *For uniformly bounded or for definite A's, $Tr_M(A)$ is weakly continuous.*

(i) $Tr_M(1)=1$;
(ii) $Tr_M(\rho A)=\rho Tr_M(A)$, *$\rho$ complex*;
(iii) $Tr_M(A+B)=Tr_M(A)+Tr_M(B)$;
(iv) $Tr_M(A)\geqq 0$, *if A is definite*;
(iv)′ *If A is definite and $\neq 0$, $Tr_M(A)>0$*;
(v) $Tr_M(A^*)=\overline{Tr_M(A)}$;
(vi) $Tr_M(AB)=Tr_M(BA)$;
(vi)′ $Tr_M(U^{-1}AU)=Tr_M(A)$, *if U is unitary.*

Now, $(*)$ and (v) are immediate consequences of the definition. (i), (iv), and (iv)′ follow from $(*)$ and R.O., Theorem XIII. A^* is a weakly continuous additive function of A. Thus B and C are too, and we may infer (iii) and $(**)$ from Properties I and II respectively. To show (ii) we first note that if A is Hermitian by Definition 2.2.1, $Tr_M(iA)=iTr_M(A)$ and if a is real by $(*)$ $Tr_M(aA)=aTr_M(A)$. Then (ii) can be shown by a calculation involving (iii).

To show (vi) suppose A and $B\epsilon M$ are Hermitian. Let $C=A+iB$, $C^*=A-iB$. Then $Tr_M(CC^*)=Tr_M(C^*C)$ (by $(*)$, cf. proof of Lemma 1.4.3, equation (β)). Substituting for C, expanding the products according to the distributive law and collecting terms, we get $2i(Tr_M(AB)-Tr_M(BA))=0$ or $Tr_M(AB)=Tr_M(BA)$. From this we can show (vi) in general by using Definition 2.2.1, (ii) and (iii). (vi)′ follows from (vi) by letting $A=U^{-1}A$, $B=U$ and using the associative law.

Property IV. (i)–(iv), (v), (vi)′ *of Property* III *characterize $Tr_M(A)$ uniquely.*

Let $Tr'(A)$ have the listed properties. Then if A is Hermitian, $Tr'(A)$ is real by (v). This and (i)–(iv), (vi)′ imply that for $A\epsilon M$ and Hermitian $Tr'(A)=Tr_M(A)$ (R.O., Theorem XIII). By (ii) and (iii), we then have that for an arbitrary $C\epsilon M$, $C=A+iB$, A and B Hermitian, $Tr'(C)=Tr'(A+iB)$ $=Tr'(A)+Tr'(iB)=Tr'(A)+iTr'(B)=Tr_M(A)+iTr_M(B)=Tr_M(A+iB)$ $=Tr_M(C)$ or $Tr'(C)=Tr_M(C)$.

Chapter III. The exact form of $Tr_M(A)$ (for $\alpha\geqq 1$)

3.1. We again assume M in case II_1. Some lemmas about families of projections and spectral forms in M are needed. Of these the two last ones have some interest of their own.

Lemma 3.1.1. *For any two projections E, $F\epsilon M$ with $E\leqq F$, and any $\alpha\geqq D_M(E)$, $\leqq D_M(F)$, a projection $G\epsilon M$ with $E\leqq G\leqq F$, $D_M(G)=\alpha$ exists.*

$F-E$ is a projection, and $0\leqq\alpha-D_M(E)\leqq D_M(F)-D_M(E)=D_M(F-E)$

$\leqq 1$. Thus a projection $G' \epsilon \boldsymbol{M}$ with $D_{\boldsymbol{M}}(G') = \alpha - D_{\boldsymbol{M}}(E)$ exists (because $\boldsymbol{M}$ is in case II_1). So $D_{\boldsymbol{M}}(G') \leqq D_{\boldsymbol{M}}(F-E)$ and therefore a projection $G'' \epsilon \boldsymbol{M}$ with $G'' \leqq F-E$, $D_{\boldsymbol{M}}(G'') = D_{\boldsymbol{M}}(G') = \alpha - D_{\boldsymbol{M}}(E)$ exists. G'' is thus orthogonal to E, and therefore $G''+E$ is a projection and $D_{\boldsymbol{M}}(G''+E) = D_{\boldsymbol{M}}(G'') + D_{\boldsymbol{M}}(E) = \alpha$. Besides, $G''+E \geqq E$ and $G''+E \leqq (F-E)+E = F$. So $G = G''+E$ has the desired properties.

LEMMA 3.1.2. *For any two projections $E, F \epsilon \boldsymbol{M}$ with $E \leqq F$ a family of projections $G(\alpha) \epsilon \boldsymbol{M}$ defined for all α with $D_{\boldsymbol{M}}(E) \leqq \alpha \leqq D_{\boldsymbol{M}}(F)$ exists, which possesses the following properties:*

(i) *$G(\alpha) = E$ or F if $\alpha = D_{\boldsymbol{M}}(E)$ or $D_{\boldsymbol{M}}(F)$ respectively.*

(ii) *$\alpha \leqq \beta$ implies $G(\alpha) \leqq G(\beta)$.*

(iii) *$D_{\boldsymbol{M}}(G(\alpha)) = \alpha$.*

Choose a sequence $\rho_1, \rho_2, \cdots$ which lies and is dense in the interval $D_{\boldsymbol{M}}(E) \leqq \alpha \leqq D_{\boldsymbol{M}}(F)$, with $\rho_1 = D_{\boldsymbol{M}}(E)$, $\rho_2 = D_{\boldsymbol{M}}(F)$.

We will now define a sequence of projections $G(\rho_1), G(\rho_2), \cdots, \epsilon \boldsymbol{M}$, so that (i)–(iii) hold for $\alpha = \rho_1, \rho_2, \cdots$. Put first $G(\rho_1) = E$, $G(\rho_2) = F$; then (i)–(iii) hold for $\alpha = \rho_1, \rho_2$. Assume now that for a $j = 3, 4, \cdots$ the $G(\rho_1), \cdots, G(\rho_{j-1})$ are already defined, so that (i)–(iii) holds for $\alpha = \rho_1, \cdots, \rho_{j-1}$, we will now define $G(\rho_j)$ without violating (i)–(iii).

Consider the ρ_i, $i = 1, \cdots, j-1$, with $\rho_i \leqq \rho_j$ (they exist: $\rho_1 = D_{\boldsymbol{M}}(E) \leqq \rho_j$); let $\rho_{i'}$ be the greatest one. Consider the ρ_i, $i = 1, \cdots, j-1$, with $\rho_i \geqq \rho_j$ (they exist: $\rho_2 = D_{\boldsymbol{M}}(F) \geqq \rho_j$); let $\rho_{i''}$ be the smallest one. Thus $\rho_{i'} \leqq \rho_{i''}$ and so $G(\rho_{i'}) \leqq G(\rho_{i''})$. Besides $D_{\boldsymbol{M}}(G(\rho_{i'})) = \rho_{i'} \leqq \rho_j \leqq \rho_{i''} = D_{\boldsymbol{M}}(G(\rho_{i''}))$. So Lemma 3.1.1 can be applied to $E = G(\rho_{i'})$, $F = G(\rho_{i''})$, $\alpha = \rho_j$, and we define $G(\rho_j) = G$.

Thus $G(\rho_{i'}) \leqq G(\rho_j) \leqq G(\rho_{i''})$, therefore $\rho_i \leqq \rho_j$, $i = 1, \cdots, j-1$, implies $\rho_i \leqq \rho_{i'}$, $G(\rho_i) \leqq G(\rho_{i'}) \leqq G(\rho_j)$ and $\rho_i \geqq \rho_j$, $i = 1, \cdots, j-1$, implies $\rho_i \geqq \rho_{i''}$, $G(\rho_i) \geqq G(\rho_{i''}) \geqq G(\rho_j)$. Besides $D_{\boldsymbol{M}}(G(\rho_j)) = \rho_j$. So (i)–(iii) hold for $\alpha = \rho_1, \cdots, \rho_{j-1}, \rho_j$, too.

Thus all $G(\rho_1), G(\rho_2), \cdots$ are defined. Owing to (ii), $\lim_{\rho_i \to \alpha, \rho_i \geqq \alpha} G(\rho_i)$ exists for all α with $D_{\boldsymbol{M}}(E) = \rho_1 \leqq \alpha \leqq \rho_2 = D_{\boldsymbol{M}}(F)$. If α is equal to a ρ_j, then this limit is meant to denote the value at ρ_j. So we can extend the definition of $G(\alpha)$ to all above α by defining

$$G(\alpha) = \lim_{\rho_i \to \alpha,\, \rho_i \geqq \alpha} G(\rho_i).$$

Now (i) remains true, and (ii), (iii) extend by continuity to all above α.

LEMMA 3.1.3. *Let $A \epsilon \boldsymbol{M}$ be a definite operator with bound $\leqq 1$. Then there exists a monotonous non-decreasing and right semi-continuous function $\phi(\alpha)$ for $0 \leqq \alpha \leqq 1$ with $0 \leqq \phi(\alpha) \leqq 1$ and a resolution of identity $E(\alpha) \epsilon \boldsymbol{M}$* (cf. (16), p. 92, or R.O., Definition 15.1.1) *with the following properties:*

(i) $E(0)=0$, $E(1)=1$,
(ii) $D_M(E(\alpha))=\alpha$,
(iii) $(Af, g)=\int_0^1\phi(\alpha)d(E(\alpha)f, g)$ *for any* f, g, *or symbolically,*

$$A = \int_0^1 \phi(\alpha)dE(\alpha).$$

Let $F(\lambda)$ be the resolution of identity corresponding to A. As A is definite, so $F(0)=0$; as the bound of A is $\leqq 1$, so $F(1)=1$.

Consider now the two following functions:

$$\psi(\lambda) = D_M(F(\lambda)),$$
$$\phi(\alpha) = \underset{0\leqq\lambda\leqq 1,\ \psi(\lambda)\leqq\alpha}{\text{l.u.b.}} \quad \lambda.$$

$\psi(\lambda)$ is monotonous non-decreasing and right semi-continuous in $0\leqq\lambda\leqq 1$, because $F(\lambda)$ is a resolution of identity, and besides $\psi(0)=0$, $\psi(1)=1$. Thus $\phi(\alpha)$ is monotonous non-decreasing and right semi-continuous in $0\leqq\alpha\leqq 1$, and its values are all in $0\leqq\lambda\leqq 1$. One verifies these facts, as well as those which follow, easily; besides, their discussion may be found in (19) (numbers in parentheses refer to the bibliography in R.O., pp. 125–126) on p. 193. (Our $\psi(\lambda)$, $\phi(\alpha)$ corresponds to the $\phi(a)$, $\psi(b)$ there, thus they are both "m-functions," and $\phi(\alpha)$ is the "measure function" of $\psi(\lambda)$.) The relation between $\phi(\alpha)$ and $\psi(\lambda)$ is symmetric (cf. loc. cit., $\psi(\lambda)$ is the "measure function" of $\phi(\alpha)$, we must define for the empty set in $0\leqq\lambda\leqq 1$ the l.u.b. 0); that is,

$$\psi(\lambda) = \underset{0\leqq\alpha\leqq 1}{\text{l.u.b.}} (\phi(\alpha) \leqq \lambda).$$

Finally (cf. loc. cit.)

$$(*) \qquad \psi(\phi(\psi(\lambda))) = \psi(\lambda).$$

That is, $D_M(F(\phi(\psi(\lambda)))=D_M(F(\lambda))$. But $F(\phi(\psi(\lambda)))\gtreqless F(\lambda)$, and therefore this implies

$$(**) \qquad F(\phi(\psi(\lambda))) = F(\lambda).$$

If $\phi(\alpha)<1$, then choose μ with $\phi(\alpha)<\mu\leqq 1$, and let $\mu\to\phi(\alpha)$. Under these conditions the definition of $\phi(\alpha)$ implies $D_M(F(\mu))>\alpha$, so $D_M(F(\phi(\alpha)))\geqq\alpha$. For $\phi(\alpha)=1$ this holds, too: $D_M(F(\phi(\alpha)))=D_M(F(1))=D_M(1)=1\geqq\alpha$. On the other hand, if $\phi(\alpha)>0$, then a sequence of μ with $0\leqq\mu\leqq\phi(\alpha)$, $\mu\to\phi(\alpha)$ and $D_M(F(\mu))\leqq\alpha$ exists, so $D_M(F(\phi(\alpha)-0))\leqq\alpha$. This holds for $\phi(0)=0$, too, if we identify $F(0-0)$ with $F(0)=0$: $D_M(F(0-0))=D_M(0)=0\leqq\alpha$. So we have

$$(\#) \qquad D_M(F(\phi(\alpha) - 0)) \leqq \alpha \leqq D_M(F(\phi(\alpha))).$$

Form now for every $0 \leqq \lambda \leqq 1$, $E = F(\lambda - 0)$, $F = F(\lambda)$ and apply Lemma 3.1.2. A family of projections $G(\alpha) \epsilon \boldsymbol{M}$, $D_M(F(\lambda - 0)) \leqq \alpha \leqq D_M(F(\lambda))$, results. Choose for every $0 \leqq \lambda \leqq 1$ such a family, and hold it fixed: $G(\lambda, \alpha)$. (If $F(\lambda - 0) = F(\lambda)$, then necessarily $\alpha = D_M(F(\lambda))$, and $G(\lambda, \alpha) = F(\lambda)$. So only the λ with $F(\lambda - 0) \neq F(\lambda)$, the point-proper values of A, are important. Their set is finite or enumerably infinite.)

Now define for every $0 \leqq \alpha \leqq 1$

$$(\S) \qquad E(\alpha) = G(\phi(\alpha), \alpha)$$

(this expression has a meaning owing to (#)).

We have now by definition $D_M(E(\alpha)) = \alpha$, so condition (ii) is satisfied. For $\alpha = 0, 1$, this gives $E(0) = 0$, $E(1) = 1$, so condition (i) is satisfied, too. If $\alpha \leqq \beta$, then $\phi(\alpha) \leqq \phi(\beta)$. The relation $\phi(\alpha) = \phi(\beta)$ gives

$$E(\alpha) = G(\phi(\alpha), \alpha) = G(\phi(\beta), \alpha) \leqq G(\phi(\beta), \beta) = E(\beta),$$

while $\phi(\alpha) < \phi(\beta)$ gives

$$E(\alpha) = G(\phi(\alpha), \alpha) \leqq F(\phi(\alpha)) \leqq F(\phi(\beta) - 0) \leqq G(\phi(\beta), \beta) = E(\beta).$$

So we see that

$$(\cdot) \qquad \alpha \leqq \beta \quad \text{implies} \quad E(\alpha) \leqq E(\beta).$$

Furthermore $\beta > \alpha$ gives

$$D_M(E(\beta) - E(\alpha)) = D_M(E(\beta)) - D_M(E(\alpha)) = \beta - \alpha,$$

so $\lim_{\beta \to \alpha, \beta > \alpha} D_M(E(\beta) - E(\alpha)) = 0$. But

$$\lim_{\beta \to \alpha, \beta > \alpha} D_M(E(\beta) - E(\alpha)) = D_M\left(\lim_{\beta \to \alpha, \beta > \alpha} E(\beta) - E(\alpha)\right) = D_M(E(\alpha + 0) - E(\alpha))$$

so $D_M(E(\alpha + 0) - E(\alpha)) = 0$, $E(\alpha + 0) = E(\alpha)$, or

$$(\cdot\,\cdot) \qquad \lim_{\beta \to \alpha, \beta > \alpha} E(\beta) = E(\alpha).$$

(i), $(\cdot)$, $(\cdot\,\cdot)$ state that $E(\alpha)$ is a resolution of unity.

It remains for us to prove (iii).

Consider the expression $\int_0^1 \phi(\alpha) d(\|E(\alpha)f\|^2)$. $\phi(\alpha)$ and $\|E(\alpha)f\|^2$ are both monotonous non-decreasing functions of α if α increases from 0 to 1, then these functions increase from $\phi(0)$ to $\phi(1) = 1$ and from 0 to $\|f\|^2$ respectively. Thus our integral is, by partial integration, equal to $\|f\|^2 - \int_0^1 \|E(\alpha)f\|^2 d\phi(\alpha)$. We may now introduce a new variable of integration: $\lambda = \phi(\alpha)$. Then we have (cf. loc. cit., p. 198) $\int_0^1 \|E(\alpha)f\|^2 d\phi(\alpha) = \int_0^1 \|E(\psi(\lambda))f\|^2 d\lambda$. But by (§) $E(\psi(\lambda)) = G(\phi(\psi(\lambda)), \psi(\lambda))$, and by (*) $\psi(\phi(\psi(\lambda))) = D_M(F(\phi(\psi(\lambda))))$. So the definition of $G(\mu, \alpha)$ gives $G(\phi(\psi(\lambda)), \psi(\lambda)) = F(\phi(\psi(\lambda)))$. By (* *) this is $F(\lambda)$, so we have $E(\psi(\lambda)) = F(\lambda)$. Thus our original expression is equal to $\|f\|^2 - \int_0^1 \|F(\lambda)f\|^2 d\lambda$, and this becomes by another partial integration $\int_0^1 \lambda d\|F(\lambda)f\|^2$, that is, (Af, f). Thus we have proved:

$$(Af, f) = \int_0^1 \phi(\alpha) d\|E(\alpha)f\|^2.$$

Replacing herein f by $f \pm g/2$, and subtracting, gives the real part of (iii). Now replacing f, g by if, g gives the imaginary part of (iii). Thus the proof is completed.

Finally we need the following lemma.

LEMMA 3.1.4. *Let $E(\alpha) \epsilon M$ be a resolution of unity in $a \leqq \alpha \leqq b$, $E(a) = 0$, $E(b) = 1$. Then a bounded Hermitian operator $B \epsilon M$ with $(Bf, g) = \int_a^b \lambda d(E(\lambda)f, g)$, symbolically $B = \int_a^b \lambda dE(\lambda)$, exists. For every bounded Baire function $\psi(\alpha)$ an operator $C = \psi(B) \epsilon M$ with $(Cf, g) = \int_a^b \psi(\lambda) d(E(\lambda)f, g)$ symbolically $C = \int_a^b \psi(\lambda) dE(\lambda)$ exists. If $0 \leqq \psi(\lambda) \leqq 1$, then C is Hermitian, definite, and of bound* 1. *For all $\psi(\lambda)$*

$$Tr_M(C) = \int_a^b \psi(\lambda) dD_M(E(\lambda)).$$

The existence of the bounded B is a well known fact about spectral forms, as all $E(\lambda) \epsilon M$ so $B \epsilon M$ (see (18), p. 389, Theorem 1). $C = \psi(B)$ exists and is bounded ((19), p. 205), and $B \epsilon M$ implies $C \epsilon M$ ((19), p. 213, Theorem 6). $0 \leqq \psi(\alpha) \leqq 1$ implies the Hermitian, definite character of C and $1 - C$ ((19), p. 205) (C Hermitian: by Property b; C and $1 - C$ definite: by Property e), with $F(x) = \psi(x)$ or $1 - \psi(x)$ and $G(x) = H(x) = (F(x))^{1/2}$. Thus the bound of C is $\leqq 1$. (Alternatively (16), p. 113, Theorem 4*, could be used.)

It remains to prove the final formula for the trace. If it holds for $\psi_1(\lambda), \psi_2(\lambda), \cdots$, and the $\psi_n(\lambda)$, $n = 1, 2, \cdots$ are uniformly bounded and everywhere convergent in $a \leqq \lambda \leqq b$, then it holds for $\psi(\lambda) = \lim_{n \to \infty} \psi_n(\lambda)$ too: $\psi(B) = \text{strong} \lim_{n \to \infty} \psi_n(B)$, the $\psi_n(B)$ being uniformly bounded, by (19), p. 205, Property h), and Tr_M is continuous for strong (even for weak) convergence by Property III, (* *). Thus our relation holds for all bounded Baire functions $\psi(\lambda)$ if it holds for all continuous functions $\psi(\lambda)$. And it holds for these, if it holds for all intervalwise constant functions $\psi(\lambda)$. But these are linear aggregates of functions $\psi_{c,d}(\lambda) = 1$ for $c < \lambda \leqq d = 0$ otherwise, where $a \leqq c \leqq d \leqq b$. So it suffices to consider these. Now clearly

$$\psi_{c,d}(B) = E(d) - E(c),$$
$$Tr_M(\psi_{c,d}(B)) = D_M(E(d) - E(c)) = D_M(E(d)) - D_M(E(c)),$$

and

$$\int_a^b \psi_{c,d}(\alpha)dD_M(E(\alpha)) = D_M(E(d)) - D_M(E(c)),$$

completing the proof.

3.2. We return now to the normalization of §1.1, and assume that an $\boldsymbol{M}$ in case II_1 is given with $\alpha \geqq 1$.

Consider a g and K which are related as in Theorem I. Our objective is to prove Theorem II below, but we begin by considering these two maximum problems.

(A) For a given λ with $0 \leqq \lambda \leqq 1$ consider all projections $F \epsilon \boldsymbol{M}$ with $D_M(F) = \lambda$, and the corresponding values of (Fg, g). Prove that these (Fg, g) possess a maximum m_a which they assume for a certain $F = F_0$ from the above class.

(B) For a given λ with $0 \leqq \lambda \leqq 1$ consider all definite operators $B \epsilon \boldsymbol{M}$ of bound $\leqq 1$ with $Tr_M(B) = \lambda$, and the corresponding values of (Bg, g). Prove that these (Bg, g) possess a maximum m_b which they assume for a certain $B = B_0$ from the above class.

LEMMA 3.2.1. *Problem* (B) *possesses a solution* $B = B_0$, *the maximum* m_b *fulfills* $\lambda K^{-1} \leqq m_b \leqq \lambda K$.

For the B of the class described in Problem (B) we have by Theorem I, $\lambda K^{-1} \leqq (Bg, g) \leqq \lambda K$. So these (Bg, g) possess a l.u.b. m_b' and $\lambda K^{-1} \leqq m_b' \leqq \lambda K$.

Select now a sequence $B_1, B_2, \cdots$ from this class, so that $\lim_{n\to\infty}(B_n g, g) = m_b'$. As the $B_1, B_2, \cdots$ are uniformly bounded, there exists a subsequence $B_{n_1}, B_{n_2}, \cdots$ $(n_1 < n_2 < \cdots)$ such that $B_0 = \text{weak lim } B_{n_i}$ exists. As all B_{n_i} are $\epsilon \boldsymbol{M}$, definite, and of bound $\leqq 1$, the same is true for B_0. As all $Tr_M(B_{n_i}) = \lambda$, so Property II or III, $(**)$ gives $Tr_M(B_0) = \lambda$. Finally $(B_0 g, g) = \lim_{i\to\infty}(B_{n_i} g, g) = \lim_{n\to\infty}(B_n g, g) = m_b'$.

Thus B_0 belongs to our class, and $(B_0 g, g) = m_b'$. Therefore the l.u.b. m_b' is a maximum: $m_b' = m_b$, and $\lambda K^{-1} \leqq m_b \leqq \lambda K$. So the proof is completed.

LEMMA 3.2.2. *The* B_0 *of Lemma* 3.2.1 *can be chosen as a projection.*

Consider the B_0 of Lemma 3.2.1. By Lemmas 3.1.3 and 3.1.4 we have $B_0 = \phi(B)$, where $B = \int_0^1 \alpha dE(\alpha)$ (symbolically), $E(\alpha)$ being a resolution of unity with $E(\alpha) \epsilon \boldsymbol{M}$, $D_M(E(\alpha)) = \alpha$, and $\phi(\alpha)$ a monotonous non-decreasing and right semi-continuous function.

Consider any Baire function $\psi(\alpha)$, $0 \leqq \alpha \leqq 1$, with $0 \leqq \psi(\alpha) \leqq 1$. Form $\psi(B) = \int_0^1 \psi(\alpha) dE(\alpha)$ (symbolically), using Lemma 3.1.4. Then $\psi(B)$ is $\epsilon \boldsymbol{M}$, definite, and of bound $\leqq 1$, and

$$Tr_M(\psi(B)) = \int_0^1 \psi(\alpha)dD_M(E(\alpha)) = \int_0^1 \psi(\alpha)d\alpha.$$

Thus $\psi(B)$ belongs to the class of Problem (B) if and only if $\int_0^1\psi(\alpha)d\alpha=\lambda$. Besides $(\psi(B)g,\, g)=\int_0^1\psi(\alpha)d(\|E(\alpha)g\|^2)$.

Apply this to $\psi(\alpha)=\phi(\alpha)$, $\psi(B)=\phi(B)=B_0$. We get $\int_0^1\phi(\alpha)d\alpha=\lambda$, $\int_0^1\phi(\alpha)d(\|E(\alpha)g\|^2)=m_b$. And the maximum property of B_0 gives the following. For every Baire function $\psi(\alpha)$, $0\leqq\alpha\leqq 1$, with $0\leqq\psi(\alpha)\leqq 1$ the equation $\int_0^1\psi(\alpha)d\alpha=\lambda$ implies $\int_0^1\psi(\alpha)d\|E(\alpha)g\|^2\leqq m_b$.

If we had $\phi(1-\lambda)=0$, then the monotony of $\phi(\alpha)$ would give $\phi(\alpha)=0$ for $0\leqq\alpha\leqq 1-\lambda$, and the right semi-continuity $\phi(\alpha)\leqq\frac{1}{2}$ for $1-\lambda<\alpha\leqq 1-\lambda+\epsilon'$ for a suitable $\epsilon'>0$. Besides $\phi(\alpha)$ is always $\leqq 1$. Thus

$$\int_0^1 \phi(\alpha)d\alpha \leqq 0(1-\lambda) + \tfrac{1}{2}\epsilon' + (\lambda-\epsilon')1 = \lambda - \tfrac{1}{2}\epsilon' < \lambda$$

contradicting $\int_0^1\phi(\alpha)d\alpha=\lambda$. Therefore $\phi(1-\lambda)>0$.

If we had $\phi(1-\lambda-0)=1$, then the monotony of $\phi(\alpha)$ would give $\phi(\alpha)=1$ for $1-\lambda\leqq\alpha\leqq 1$ and the definition of $\phi(1-\lambda-0)$, $\phi(\alpha)\geqq\frac{1}{2}$ for $1-\lambda-\epsilon'\leqq\alpha<1-\lambda$, for a suitable $\epsilon'>0$. Besides $\phi(\alpha)$ is always $\geqq 0$. Thus

$$\int_0^1 \phi(\alpha)d\alpha \geqq (1-\lambda-\epsilon')\cdot 0 + \frac{\epsilon'}{2} + \lambda\cdot 1 = \lambda + \frac{\epsilon'}{2} > \lambda$$

contradicting $\int_0^1\phi(\alpha)d\alpha=\lambda$. Therefore $\phi(1-\lambda-0)<1$.

So we can choose a δ, $0<\delta<1$, with $\phi(1-\lambda)>\delta$, $\phi(1-\lambda-0)<1-\delta$. Thus $1-\lambda\leqq\alpha\leqq 1$ implies $(\phi(\alpha)-\delta)/(1-\delta)\geqq(\phi(1-\lambda)-\delta)/(1-\delta)\geqq 0$ and $\leqq(1-\delta)/(1-\delta)=1$, and $0\leqq\alpha\leqq 1-\lambda$ implies $\phi(\alpha)/(1-\delta)\geqq 0$ and $\leqq(1-\delta)/(1-\delta)=1$. Now put $\phi_1(\alpha)=1$ for $1-\lambda\leqq\alpha\leqq 1$ and $\phi_1(\alpha)=0$ for $0\leqq\alpha\leqq 1-\lambda$. Then we have $0\leqq\phi_1(\alpha)\leqq 1$ for all $0\leqq\alpha\leqq 1$; and also, for $\phi_2(\alpha)=(\phi(\alpha)-\delta\phi_1(\alpha))/(1-\delta)$, $0\leqq\phi_2(\alpha)\leqq 1$ for all $0\leqq\alpha\leqq 1$ (we proved this above separately for $1-\lambda\leqq\alpha\leqq 1$ and for $0\leqq\alpha\leqq 1-\lambda$). Besides

$$(*)\qquad \phi(\alpha) = \delta\phi_1(\alpha) + (1-\delta)\phi_2(\alpha).$$

Now clearly $\int_0^1\phi_1(\alpha)d\alpha=\lambda$. This and $\int_0^1\phi(\alpha)d\alpha=\lambda$ and $(*)$ give $\int_0^1\phi_2(\alpha)d\alpha=\lambda$. Therefore $\int_0^1\phi_1(\alpha)d\|E(\alpha)g\|^2\leqq m_b$, $\int_0^1\phi_2(\alpha)d\|E(\alpha)g\|^2\leqq m_b$. But $(*)$ gives

$$\delta\int_0^1 \phi_1(\alpha)d\|E(\alpha)g\|^2 + (1-\delta)\int_0^1 \phi_2(\alpha)d\|E(\alpha)g\|^2 = \int_0^1 \phi(\alpha)d\|E(\alpha)g\|^2 = m_b.$$

Therefore necessarily $\int_0^1\phi_1(\alpha)d\|E(\alpha)g\|^2=m_b$. In other words, for $B_1=\phi_1(B)$ which belongs to the class of Problem (B), we have $(B_1g,\, g)=m_b$.

Thus $B_1=\phi_1(B)$ is a solution of Problem (B), too. But one verifies easily that $B_1=\phi_1(B)=1-E(\lambda)$, and so B_1 is a projection.

Therefore replacement of B_0 by B_1 completes the proof.

LEMMA 3.2.3. *Problems* (A) *and* (B) *possess a common solution* $E_0=B_0$, *and for the maxima we have* $m_a=m_b$.

The class of the E in Problem (A) is obviously a subclass of the B in Problem (B): It consists of all projections of the latter class. (This is so, because for projections E, $D_M(E)=Tr_M(E)$ (cf. R.O., Lemma 15.3.1).) Now Lemma 3.2.2 states, that a solution of Problem (B) exists, which is a projection and thus belongs to the class of Problem (A). Therefore it is a solution of Problem (A) too, and $m_a=m_b$.

Problems (A) and (B) can be modified, when a projection $G=P_{\mathfrak{N}}\epsilon\mathfrak{M}$ is given, in the following sense: Replace $\mathfrak{H}$, $\boldsymbol{M}$ by $\mathfrak{N}$, $\boldsymbol{M}_{(\mathfrak{N})}$ (cf. R.O., §11.3). Then $D_M(E)$ must be replaced by $D_M(E)/D_M(G)$, in order to conserve the standard normalization and so the normalization of §1.1 requires replacing of $D_{M'}(E')$ by $D_{M'}(E')/D_M(G)$. This α is replaced by $\alpha/D_M(G)$ and so $\alpha\geqq 1$ remains true. Lemma 3.2.3 is modified, in so far as λ is replaced by $\lambda/D_M(G)$ and $\boldsymbol{M}$ in Problems (A) and (B) is to be replaced by $\boldsymbol{M}_{(\mathfrak{N})}$. This means that projections $F\epsilon\boldsymbol{M}$ must be $\leqq G$, and operators $B\epsilon\boldsymbol{M}$ must fulfill $BG=GB=B$.

Combining this and the corresponding changes in Lemma 3.2.1 (observe that $(B_0g, g)=(E_0g, g)=\|E_0g\|^2$), we have

COROLLARY. *Let a projection* $G\epsilon\boldsymbol{M}$ *be given. Replace Problems* (A) *and* (B) *by Problems* (A_G) *and* (B_G), *which arise by imposing these further restrictions*: *In* (A_G) *the projections* $F\epsilon\boldsymbol{M}$ *are* $F\leqq G$. *In* (B_G) *the operators* $B\epsilon\boldsymbol{M}$ *fulfill* $BG=GB=B$. *Assume* $0\leqq\lambda\leqq D_M(G)$. *Then Problems* ($\mathrm{A}_G$) *and* ($\mathrm{B}_G$) *possess a common solution* $E_0=B_0$, *and we have for the maxima* $K^{-1}\|E_0g\|^2\leqq m_a=m_b\leqq K\|E_0g\|^2$.

3.3. We hold g fixed, and make the following

DEFINITION 3.3.1. *Let* E, G *be two projections* $\epsilon\boldsymbol{M}$, $E\leqq G$. *We say that* E *reduces* g *with respect to* G *if for every* $A\epsilon\boldsymbol{M}$ *with* $AG=GA=A$

$$(AEg, g) = (EAg, g).$$

If G *may be taken* $=1$, *we say that* E *reduces* g.

LEMMA 3.3.1. *Let* G *be a projection* $\epsilon\boldsymbol{M}$, E_0 *a solution of Problem* (A_G) (cf. the corollary to Lemma 3.2.3). *Then* E_0 *reduces* g *with respect to* G.

If U is unitary, $\epsilon\boldsymbol{M}$, and commutes with G, then $U^{-1}E_0U$ is a projection $\epsilon\boldsymbol{M}$, it is $\leqq U^{-1}GU=G$, and $D_M(U^{-1}E_0U)=D_M(E_0)=\lambda$. So $(U^{-1}E_0Ug, g)$

$\leqq (E_0g, g)$. Now $(U^{-1}E_0Ug, g) = (U^*E_0Ug, g) = (E_0Ug, Ug)$. So $(E_0g, g) - (E_0Ug, Ug) \geqq 0$.

Let A be Hermitian, ϵM, and commute with G. Put for some $\epsilon \gtrless 0$ $\phi_\epsilon(\alpha) = (1+i\epsilon\alpha)/(1-i\epsilon\alpha)$. As $|\phi_\epsilon(\alpha)| = 1$, so $U_\epsilon = \phi_\epsilon(A)$ is unitary, and it is ϵM and commutes with G along with A. Further $\phi_\epsilon(\alpha) = 1 + 2i\epsilon\alpha + \epsilon^2\psi_\epsilon(\alpha)$, where $\psi_\epsilon(\alpha) = -\alpha^2/(1-i\epsilon\alpha)$. So $|\psi_\epsilon(\alpha)| \leqq D^2$ if $|\alpha| \leqq D$, and therefore $|||\psi_\epsilon(A)||| \leqq |||A|||^2$ (cf., for instance, (16), p. 113, Theorem 4*). Now

$$\begin{aligned} 0 \leqq (E_0g, g) - (E_0Ug, Ug) = (E_0g, g) \\ - (E_0(1 + 2i\epsilon A + \epsilon^2\psi_\epsilon(A))g, (1 + 2i\epsilon A + \epsilon^2\psi_\epsilon(A))g) \\ = 2\epsilon(i(AE_0 - E_0A)g, g) + O(\epsilon^2). \end{aligned}$$

So we have

$$2\epsilon i((AE_0 - E_0A)g, g) + O(\epsilon^2) \geqq 0.$$

As $\epsilon \gtrless 0$, this necessitates $((AE_0 - E_0A)g, g) = 0$, that is,

$$(AE_0g, g) = (E_0Ag, g).$$

This equation extends from the Hermitian $A \epsilon M$ to all $A \epsilon M$, which commute with G. Put for such an A, $A_1 = \frac{1}{2}(A + A^*)$, $A_2 = -\frac{1}{2}i(A - A^*)$, then it holds for A_1, A_2 and so for $A = A_1 + iA_2$. Thus we have established that E_0 reduces g with respect to G.

LEMMA 3.3.2. *If E reduces g relatively to G and G reduces g, then E reduces g.*

We must show that for every $A \epsilon M$, $(AEg, g) = (EAg, g)$. Now

$$\begin{aligned} (EAg, g) = (GEAg, g) = (G(EA)g, g) = ((EA)Gg, g) = (EGAGg, g) \\ = (GAGEg, g) = (GAEg, g) = ((AE)Gg, g) = (AEg, g). \end{aligned}$$

LEMMA 3.3.3. *If $\{E_i\}$ is a sequence of projections $E_i \epsilon M$ each of which reduce g and $E_i \cdot E_j = 0$ if $i \neq j$, then $\sum_{i=1}^{\infty} E_i$ reduces g. If E_1 and E_2 reduce g and $E_1 \geqq E_2$ then $E_1 - E_2$ reduces g.*

This is immediate for $A \epsilon M$ implies A is bounded.

LEMMA 3.3.4. *Let $\{E_i\}$ be a sequence with the same properties as in Lemma 3.3.3. Let $E = \sum_{i=1}^{\infty} E_i$. Then for $A \epsilon M$,*

$$(EAg, g) = (AEg, g) = (EAEg, g) = \sum_{i=1}^{\infty} (E_iAE_ig, g).$$

Also if E reduces g, $EF = 0$, $F \epsilon M$, then $(EAFg, g) = (FAEg, g) = 0$.

We have $(EAg, g) = (E(EA)g, g) = (EAEg, g) = (E(AE)g, g) = ((AE)Eg, g) = (AEg, g)$. Hence we have $(EAg, g) = ((\sum_{i=1}^{\infty} E_i)Ag, g) = \sum_{i=1}^{\infty} (E_iAg, g)$

$=\sum_{i=1}^{\infty}(E_iAE_ig, g)$. Also $(EAFg, g)=(E(AF)g, g)=((AF)Eg, g)=(A(FE)g, g)$ $=0=((EF)Ag, g)=(E(FA)g, g)=((FA)Eg, g)=(FAEg, g)$.

LEMMA 3.3.5. *Let $\{E_i\}$ be a sequence of mutually orthogonal projections. If $E=\sum_{i=1}^{\infty}E_i$ and A and $E_i \epsilon M$,*

$$Tr_M(EAE) = \sum_{i=1}^{\infty} Tr_M(E_iAE_i).$$

By Property III, (vi), $Tr_M(EAE)=Tr_M(AE\cdot E)=Tr_M(AE)$. Similarly $Tr_M(E_iAE_i)=Tr_M(AE_i)$. By Property III, $(**)$, and (iii) $Tr_M(AE)$ $=Tr_M(\sum_{i=1}^{\infty}AE_i)=\sum_{i=1}^{\infty}Tr_M(AE_i)$. Thus $Tr_M(EAE)=\sum_{i=1}^{\infty}Tr_M(E_iAE_i)$. Note that we have also demonstrated the convergence of $\sum_{i=1}^{\infty}Tr_M(E_iAE_i)$.

LEMMA 3.3.6. *Let $\{E_i\}$ be a sequence of projections each $E_i \geqq {}_p\lambda$ $(\leqq {}_p\lambda)$ for g. Furthermore let us suppose that each E_i reduces g and $E_i\cdot E_j=0$ if $i\neq j$. Then $E=\sum_{i=1}^{\infty}E_i \geqq {}_p\lambda$ $(\leqq {}_p\lambda)$.*

Let A be positive definite, ϵM and such that $EA=AE=A$. Then by Lemma 3.3.4

$$(Ag, g) = (EAg, g) = \sum_{i=1}^{\infty}(E_iAE_ig, g).$$

Now E_iAE_i is positive definite and is unchanged by left- or right-multiplication with E_i. Hence Lemma 1.4.2 yields $(E_iAE_ig, g)\geqq \lambda Tr_M(E_iAE_i)$. Thus by Lemma 3.2.6

$$(Ag, g) = \sum_{i=1}^{\infty}(E_iAE_ig, g) \geqq \lambda\sum_{i=1}^{\infty} Tr_M(E_iAE_i) = \lambda Tr_M(A).$$

Lemma 1.4.2 now implies $E\geqq {}_p\lambda$.

3.4. We still suppose that g is held fixed.

LEMMA 3.4.1. *Let $E\neq 0$ be a projection, ϵM, and such that $a\leqq {}_pE\leqq {}_pb$. Let a λ with $0<\lambda<D_M(E)$ be given. Then there exists a projection $E_0\epsilon M$ with the following properties*: (i) $E_0\leqq E$; (ii) $D_M(E_0)=\lambda$; (iii) E_0 *reduces g with respect to E*; *and* (iv) *there exists a ξ_0 with $a\leqq {}_pE-E_0\leqq {}_p\xi_0\leqq {}_pE_0\leqq {}_pb$.*

Consider the solution E_0 of Problem (A_G) with $G=E$, in the corollary to Lemma 3.2.3. This E_0 is a projection ϵM and fulfills (i), (ii) by definition, and it fulfills (iii) by Lemma 3.3.1. As to (iv), both E_0 and $E-E_0$ are $\geqq {}_pa$ and $\leqq {}_pb$ along with E, by Lemma 1.2.3.

So we must only find a ξ with $E-E_0\leqq {}_p\xi\leqq {}_pE_0$. Let Γ' be the set of all η's, for which there exists an $F\leqq E_0$ with $F<\eta$; and let Γ'' be the set of all η's for which there exists an $F\leqq E-E_0$ with $F>\eta$. Γ' and Γ'' are open intervals, and clearly every $\eta > b$ belongs to Γ', and every $\eta < a$ belongs to Γ''. So there exists either a ξ_0 such that every η in Γ' is $\geqq \xi_0$ and every η in Γ'' is $\leqq \xi_0$, or Γ' and Γ'' have a common element ξ_1.

If the former holds, then $F \leqq E_0$ implies $F \geqq \eta$ for all $\eta < \xi_0$, and so $F \geqq \xi_0$, that is, $E_0 \geqq {}_p\xi_0$. Similarly $E - E_0 \leqq {}_p\xi_0$. So in this case the rest of (iv) is proved, too.

If the latter holds, then we have even a $\xi_1 - \delta$ in Γ' and $\xi_1 + \delta$ in Γ'' for some suitable $\delta > 0$. So an $F_1 \leqq E_0$ with $F_1 < \xi_1 - \delta$ exists, and an $F_2 \leqq E - E_0$ with $F_2 > \xi_1 + \delta$. By Lemma 1.2 there exist two F_3 and F_4 (both $\neq 0$), with $F_3 \leqq F_1$, $F_4 \leqq F_2$ and $F_3 \leqq {}_p\xi_1 - \delta$, $F_4 \geqq {}_p\xi_1 + \delta$. We may assume that $D_M(F_3) = D_M(F_4)$, since otherwise we may replace F_3, F_4 by two $F_3' \leqq F_3$, $F_4' \leqq F_4$ with $D_M(F_3') = D_M(F_4') = \min\,(D_M(F_3), D_M(F_4))$ (remembering Lemma 1.2.3).

Now as $F_3 \leqq E_0$, $F_4 \leqq E - E_0$, so $E_0 - F_3 + F_4$, is a projection $\leqq E$, and

$$D_M(E_0 - F_3 + F_4) = D_M(E_0) - D_M(F_3) + D_M(F_4) = D_M(E_0) = \lambda,$$

while

$$\begin{aligned}((E_0 - F_3 + F_4)g, g) &= (E_0 g, g) - (F_3 g, g) + (F_4 g, g) \\ &\geqq m_a - (\xi_1 - \delta) D_M(F_3) + (\xi_1 + \delta) D_M(F_4) \\ &\geqq m_a - (\xi_1 - \delta) D_M(F_3) + (\xi_1 + \delta) D_M(F_3) \\ &= m_a + 2\delta D_M(F_3) > m_a,\end{aligned}$$

contradicting the maximum property of m_a. So this case cannot arise.

The proof is therefore completed.

Lemma 3.4.2. *We can define for all* $0 \leqq \alpha \leqq 1$ *a family of projections* $E(\alpha)$ *and a function* $\xi(\alpha)$ *with the following properties*:

(i) $E(0) = 0$, $E(1) = 1$, (ii) $\alpha \leqq \beta$ *implies* $E(\alpha) \leqq E(\beta)$,
(iii) $\xi(0) = K$, $\xi(1) = K^{-1}$, (iv) $\alpha \leqq \beta$ *implies* $\xi(\alpha) \geqq \xi(\beta)$,
(v) $D_M(E(\alpha)) = \alpha$, (vi) $E(\alpha)$ *reduces* g,
(vii) $\alpha < \beta$ *implies* $\xi(\beta - 0) \leqq {}_pE(\beta) - E(\alpha) \leqq {}_p\xi(\alpha + 0)$.

Choose a sequence $\rho_1, \rho_2, \cdots$ which lies and is everywhere dense in $0 \leqq \alpha \leqq 1$, with $\rho_1 = 0$, $\rho_2 = 1$. We will define $E(\alpha)$ and $\xi(\alpha)$ for $\alpha = \rho_1, \rho_2, \cdots$ so that they fulfill (i)–(vi) and

(vii)′ $E(\alpha) \geqq {}_p\xi(\alpha)$, $1 - E(\alpha) \leqq {}_p\xi(\alpha)$,

for all $\alpha = \rho_1, \rho_2, \cdots$.

Put first $E(0) = 0$, $E(1) = 1$, $\xi(0) = K$, $\xi(1) = K^{-1}$. Then $\alpha = \rho_1, \rho_2$ are taken care of, and (i)–(vi) as well as (vii)′ hold for $\alpha = \rho_1, \rho_2$. Assume now that for a $j = 3, 4, \cdots$ the $E(\alpha)$, $\xi(\alpha)$ for $\alpha = \rho_1, \cdots, \rho_{j-1}$ are already defined, so that (i)–(vi), (vii)′ hold for $\alpha = \rho_1, \cdots, \rho_{j-1}$, we will now define $E(\rho_j)$, $\xi(\rho_j)$ without violating (i)–(vi), (vii)′.

Consider the ρ_i, $i=1,\cdots,j-1$ with $\rho_i\leqq\rho_j$ (they exist: $\rho_1=0\leqq\rho_j$); let $\rho_{i'}$ be the greatest one. Consider the ρ_i, $i=1,\cdots,j-1$, with $\rho_i\geqq\rho_j$ (they exist: $\rho_2=1\geqq\rho_j$); let $\rho_{i''}$ be the smallest one. As $\rho_{i'}$, $\rho_{i''}\neq\rho_j$, so we have $\rho_{i'}<\rho_j<\rho_{i''}$. So (ii) gives $E(\rho_{i'})\leqq E(\rho_{i''})$, and (v) gives $D_M(E(\rho_{i''})-E(\rho_{i'}))=\rho_{i''}-\rho_i>\rho_j-\rho_{i'}>0$.

Apply now Lemma 3.4.1 to $E=E(\rho_{i''})-E(\rho_{i'})$, $\lambda=\rho_j-\rho_{i'}$ with $a=\xi(\rho_{i''})$, $b=\xi(\rho_{i'})$. Owing to (vii)$'$ and Lemma 1.2.3 we have $E(\rho_{i''})-E(\rho_{i'})\leqq E(\rho_{i''})$ and therefore $\geqq_p\xi(\rho_{i''})$, and $E(\rho_{i''})-E(\rho_{i'})\leqq 1-E(\rho_{i'})$ and therefore $\leqq_p\xi(\rho_{i'})$. That is, $a\leqq_p E\leqq_p b$. An $E_0\leqq E(\rho_{i''})-E(\rho_{i'})$ and a ξ_0 result, put $E(\rho_j)=E(\rho_{i'})+E_0$ (this is a projection ϵM) and $\xi(\rho_j)=\xi_0$. Let us now consider (i)–(vi), (vii)$'$ for $\alpha=\rho_1,\cdots,\rho_{j-1},\rho_j$.

(i), (iii) are unaffected.

In (ii), (iv) the only new possibilities are $\alpha=\rho_i\leqq\rho_j=\beta$ and $\alpha=\rho_j\leqq\rho_i=\beta$, where $i=1,\cdots,j-1$. In the first case $\rho_i\leqq\rho_{i'}$, so $E(\alpha)=E(\rho_i)\leqq E(\rho_{i'})\leqq E(\rho_{i'})+E_0=E(\rho_j)=E(\beta)$, and $\xi(\alpha)=\xi(\rho_i)\geqq\xi(\rho_{i'})=b\geqq\xi_0=\xi(\rho_j)=\xi(\beta)$. In the second case $\rho_i\geqq\rho_{i''}$, so $E(\beta)=E(\rho_i)\geqq E(\rho_{i''})=E(\rho_i)+E(\rho_{i''})-E(\rho_{i'})\geqq E(\rho_i)+E_0=E(\rho_j)=E(\alpha)$, and $\xi(\beta)=\xi(\rho_i)\leqq\xi(\rho_{i''})=a\leqq\xi_0=\xi(\rho_j)=\xi(\alpha)$. So (ii), (iv) remain true.

In (v) we need only $D_M(E(\rho_j))=\rho_j$. Now $D_M(E(\rho_j))=D_M(E(\rho_{i'})+E_0)=D_M(E(\rho_{i'}))+D_M(E_0)=\rho_{i'}+(\rho_j-\rho_{i'})=\rho_j$.

In (vi) we need only that $E(\rho_j)$ reduces g. As $E(\rho_j)=E(\rho_{i'})+E_0$, and $E(\rho_{i'})$ reduces g, we must only show that E_0 reduces g (use Lemma 3.3.3). But E_0 reduces g with respect to $E=E(\rho_{i''})-E(\rho_{i'})$ by Lemma 3.4.1, (iii), and $E(\rho_{i''})-E(\rho_{i'})$ reduces g because $E(\rho_{i''})$ and $E(\rho_{i'})$ do (again use Lemma 3.3.3), therefore E_0 reduces g by Lemma 3.3.2. So the property (vi) remains true.

In (vii)$'$ we need only $E(\rho_j)\geqq_p\xi(\rho_j)$, $1-E(\rho_j)\leqq_p\xi(\rho_j)$. Now $E(\rho_j)=E(\rho_{i'})+E_0$, and $E(\rho_{i'})\geqq_p\xi(\rho_{i'})\geqq\xi(\rho_j)$, $E_0\geqq_p\xi_0=\xi(\rho_j)$ by Lemma 3.4.1, (iv), so $E(\rho_j)\geqq_p\xi(\rho_j)$ by Lemma 3.3.6. On the other hand $1-E(\rho_j)=1-E(\rho_{i''})+((E(\rho_{i''})-E(\rho_{i'}))-E_0)$ and by Lemma 3.4.1, (iv), $1-E(\rho_{i''})\leqq_p\xi(\rho_{i''})\leqq\xi(\rho_j)$, $((E(\rho_{i''})-E(\rho_{i'}))-E_0)=E-E_0\leqq_p\xi_0=\xi(\rho_j)$ so that $1-E(\rho_j)\leqq_p\xi(\rho_j)$ by Lemma 3.3.6. Therefore (vii)$'$ remains true.

Thus we have verified (i)–(vi), (vii)$'$ for $\alpha=\rho_1,\cdots,\rho_{j-1},\rho_j$. Therefore all $\alpha=\rho_1,\rho_2,\cdots$ are taken care of, and (i)–(vi), (vii)$'$ hold for all of them. Owing to (ii) and (iv) $\lim_{\rho_i\to\alpha,\rho_i\geqq\alpha}E(\rho_i)$ and $\lim_{\rho_i\to\alpha,\rho_i>\alpha}\xi(\rho_i)$ exist for all α with $0\leqq\alpha\leqq 1$. If α is equal to a ρ_j, then this limit is meant to denote the value at ρ_j. So we can extend the definitions of $E(\alpha)$ and $\xi(\alpha)$ to all above α by defining

$$E(\alpha)=\lim_{\rho_i\to\alpha,\,\rho_i\geqq\alpha}E(\rho_i),\qquad \xi(\alpha)=\lim_{\rho_i\to\alpha,\,\rho_i\geqq\alpha}\xi(\rho_i).$$

The statements (i), (iii) are unaffected by this extension, while (ii), (iv)–(vi) extend by continuity to all $0 \leqq \alpha \leqq 1$.

Let us now consider (vii). Assume $\alpha < \beta$. Choose a sequence ρ_{i_n}, $n = 0, \pm 1, \pm 2, \cdots$, with $\alpha < \cdots < \rho_{i_{-2}} < \rho_{i_{-1}} < \rho_{i_0} < \rho_{i_1} < \rho_{i_2} < \cdots < \beta$ and $\lim_{n \to -\infty} \rho_{i_n} = \alpha$, $\lim_{n \to \infty} \rho_{i_n} = \beta$. We have $\rho_{i_{-1}} > \rho_{i_{-2}} > \cdots > \alpha$, so that $E(\rho_{i_{-1}}) \geqq E(\rho_{i_{-2}}) \geqq \cdots \geqq E(\alpha)$. Therefore $\lim_{n \to -\infty} E(\rho_{i_n})$ exists, and is $\geqq E(\alpha)$. Now we have

$$D_M\left(\lim_{n \to -\infty} E(\rho_{i_n}) - E(\alpha)\right) = \lim_{n \to -\infty} D_M(E(\rho_{i_n})) - D_M(E(\alpha))$$
$$= \lim_{n \to -\infty} \rho_{i_n} - \alpha = \alpha - \alpha = 0, \quad \lim_{n \to -\infty} E(\rho_{i_n}) = E(\alpha).$$

Similarly

$$\lim_{n \to \infty} E(\rho_{i_n}) = E(\beta).$$

Therefore

$$\sum_{m=-\infty}^{\infty} (E(\rho_{i_{m+1}}) - E(\rho_{i_m})) = \lim_{n \to \infty} \sum_{m=-n}^{n-1} (E(\rho_{i_{m+1}}) - E(\rho_{i_m}))$$
$$= \lim_{n \to \infty} (E(\rho_{i_n}) - E(\rho_{i_{-n}})) = E(\beta) - E(\alpha).$$

We have $E(\rho_{i_{m+1}}) - E(\rho_{i_m}) \leqq E(\rho_{i_{m+1}})$, therefore (vii)′ (for $\alpha = \rho_{i_{m+1}}$) and Lemma 1.2.3 give $E(\rho_{i_{m+1}}) - E(\rho_{i_m}) \geqq {}_p\xi(\rho_{i_{m+1}}) \geqq \xi(\beta - 0)$. Then Lemma 3.3.6 gives $E(\beta) - E(\alpha) \geqq {}_p\xi(\beta - 0)$. On the other hand we have $E(\rho_{i_{m+1}}) - E(\rho_{i_m}) \leqq 1 - E(\rho_{i_m})$ therefore (vii)′ (for $\alpha = \rho_{i_m}$) and Lemma 1.2.3 give $E(\rho_{i_{m+1}}) - E(\rho_{i_m}) \leqq {}_p\xi(\rho_{i_m}) \leqq \xi(\alpha + 0)$. Then Lemma 3.3.6 gives $E(\beta) - E(\alpha) \leqq {}_p\xi(\alpha + 0)$. Thus we have proved

$$\xi(\beta - 0) \leqq {}_pE(\beta) - E(\alpha) \leqq {}_p\xi(\alpha + 0),$$

that is, (vii).

We have established (i)–(vii) for all $0 \leqq \alpha \leqq 1$, and so the proof is completed.

LEMMA 3.4.3. *Let α_i, $i = 0, 1, \cdots, n$ be a set of numbers such that $0 = \alpha_0 \leqq \alpha_1 \leqq \cdots \leqq \alpha_{n-1} \leqq \alpha_n = 1$ and let λ_i, $i = 1, \cdots, n$ be a set of positive real numbers. Put (with the $E(\alpha)$, $\xi(\alpha)$ of Lemma* 3.4.2)

$$g' = \sum_{i=1}^{n} \lambda_i (E(\alpha_i) - E(\alpha_{i-1}))g,$$

$$C_1 = \max_{i=1,\cdots,n} \lambda_i^2 \xi(\alpha_{i-1} + 0); \qquad C_2 = \min_{i=1,\cdots,n} \lambda_i^2 \xi(\alpha_i - 0).$$

Then for all definite $A \epsilon M$

$$C_1 Tr_M(A) \geqq (Ag', g') \geqq C_2 Tr_M(A).$$

We have

$$(Ag', g') = \left(A\sum_{i=1}^{n}\lambda_i(E(\alpha_i) - E(\alpha_{i-1}))g,\ \sum_{j=1}^{n}\lambda_j(E(\alpha_j) - E(\alpha_{j-1}))g\right)$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_i\lambda_j(A(E(\alpha_i) - E(\alpha_{i-1}))g, (E(\alpha_j) - E(\alpha_{j-1}))g)$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_i\lambda_j((E(\alpha_j) - E(\alpha_{j-1}))A(E(\alpha_i) - E(\alpha_{i-1}))g, g).$$

As every $E(\alpha_k)-E(\alpha_{k-1})$ reduces g (by Lemma 3.4.2, (vi), and Lemma 3.3.3) and as $(E(\alpha_j)-E(\alpha_{j-1}))\cdot(E(\alpha_i)-E(\alpha_{i-1}))=0$ for $j\neq i$, Lemma 3.3.4 permits us to drop all terms with $i\neq j$ from the above sum $\sum_{i,j=1}^{n}$. Therefore it becomes $\sum_{i=1}^{n}\lambda_i^2((E(\alpha_i)-E(\alpha_{i-1}))A(E(\alpha_i)-E(\alpha_{i-1}))g, g)$.

Now $E(\alpha_i)-E(\alpha_{i-1})\leqq_p\xi(\alpha_{i-1}+0)$ by Lemma 3.4.2, (vii). Therefore it is evident from Lemma 1.4.1 that $((E(\alpha_i)-E(\alpha_{i-1}))A(E(\alpha_i)-E(\alpha_{i-1}))g, g)$ $\leqq\xi(\alpha_{i-1}+0)Tr_M((E(\alpha_i)-E(\alpha_{i-1}))A(E(\alpha_i)-E(\alpha_{i-1})))$. So

$$(Ag', g') \leqq \sum_{i=1}^{n}\lambda_i^2\xi(\alpha_{i-1} + 0)Tr_M((E(\alpha_i) - E(\alpha_{i-1}))A(E(\alpha_i) - E(\alpha_{i-1})))$$
$$\leqq C_1\sum_{i=1}^{n} Tr_M((E(\alpha_i) - E(\alpha_{i-1}))A(E(\alpha_i) - E(\alpha_{i-1})))$$
$$\leqq C_1Tr_M(A)$$

(use Lemma 3.3.5). Similarly $E(\alpha_i)-E(\alpha_{i-1})\geqq_p\xi(\alpha_i-0)$ by Lemma 3.4.2, (vii). Therefore Lemma 1.4.1 gives $((E(\alpha_i)-E(\alpha_{i-1}))A(E(\alpha_i)-E(\alpha_{i-1}))g, g)$ $\geqq\xi(\alpha_i-0)Tr_M((E(\alpha_i)-E(\alpha_{i-1}))A(E(\alpha_i)-E(\alpha_{i-1})))$. So

$$(Ag', g') \geqq \sum_{i=1}^{n}\lambda_i^2\xi(\alpha_i - 0)Tr_M((E(\alpha_i) - E(\alpha_{i-1}))A(E(\alpha_i) - E(\alpha_{i-1})))$$
$$\geqq C_2\sum_{i=1}^{n} Tr_M((E(\alpha_i) - E(\alpha_{i-1}))A(E(\alpha_i) - E(\alpha_{i-1}))) = C_2Tr_M(A)$$

(use Lemma 3.3.5). Thus the proof is completed.

3.5. We can now take the decisive step.

Lemma 3.5.1. *Let g and K be as in Theorem* I. *Let K' be any number such that $1<K'\leqq K$. Then there is a g' which is related to K' as g is to K in Theorem* I, *and such that*

$$\|g' - g\| \leqq (K - 1)K^{1/2}.$$

Let n be the smallest positive integer with $(K')^n\geqq K$. Put $\theta=K^{1/n}$, so

$1<\theta\leqq K'$. Choose the α_i, $i=0, 1, \cdots, n$ with $0=\alpha_0\leqq\alpha_1\leqq\cdots\leqq\alpha_{n-1}\leqq\alpha_n=1$, so that $\xi(\alpha_i+0)\leqq\theta^{n-2i}\leqq\xi(\alpha_i-0)$. (For $i=0$, $\theta^{n-2i}=\theta^n=K$; for $i=n$, $\theta^{n-2i}=\theta^{-n}=K^{-1}$.) Put $\lambda_i=\theta^{i-(n+1)/2}$. Now apply Lemma 3.4.3. We have $\lambda_i^2\xi(\alpha_{i-1}+0)\leqq\theta^{2i-(n+1)}\theta^{n-2(i-1)}=\theta\leqq K'$ and $\lambda_i^2\xi(\alpha_i-0)\geqq\theta^{2i-(n+1)}\cdot\theta^{n-2i}\geqq\theta^{-1}\geqq(K')^{-1}$, so $C_1\leqq K'$ and $C_2\geqq(K')^{-1}$. Therefore the g' of that lemma gives for every definite $A\epsilon\boldsymbol{M}$: $K'Tr_{\boldsymbol{M}}(A)\geqq(Ag', g')\geqq(K')^{-1}Tr_{\boldsymbol{M}}(A)$; that is, $K'(Ag',g')\geqq Tr_{\boldsymbol{M}}(A)\geqq(K')^{-1}(Ag',g')$. So it is related to K' as g and K are in Theorem I.

Compute now $\|g'-g\|$. The projections $E(\alpha_i)-E(\alpha_{i-1})$, $i=1,\cdots,n$, are mutually orthogonal, therefore the $(E(\alpha_i)-E(\alpha_{i-1}))g$, $i=1,\cdots,n$ are mutually orthogonal too. Now we have

$$\begin{aligned}\|g'-g\|^2 &= \left\|\sum_{i=1}^n\lambda_i(E(\alpha_i)-E(\alpha_{i-1}))g-\sum_{i=1}^n(E(\alpha_i)-E(\alpha_{i-1}))g\right\|^2\\ &= \left\|\sum_{i=1}^n(\lambda_i-1)(E(\alpha_i)-E(\alpha_{i-1}))g\right\|^2\\ &= \sum_{i=1}^n(\lambda_i-1)^2\|(E(\alpha_i)-E(\alpha_{i-1}))g\|^2\\ &\leqq \max_{i=1,\cdots,n}(\lambda_i-1)^2\sum_{i=1}^n\|(E(\alpha_i)-E(\alpha_{i-1}))g\|^2\\ &= \max_{i=1,\cdots,n}(\lambda_i-1)^2\cdot\|g\|^2.\end{aligned}$$

Now clearly

$$\max_{i=1,\cdots,n}(\lambda_i-1)^2=(\lambda_n-1)^2=(\theta^{(n+1)/2}-1)^2\leqq(\theta^n-1)^2=(K-1)^2$$

so that $\|g'-g\|\leqq(K-1)\|g\|$.

But $A=1$ in Theorem I gives

$$\|g\|^2=(g, g)\leqq KTr_{\boldsymbol{M}}(1)=K, \quad \|g\|\leqq K^{1/2}.$$

Therefore $\|g'-g\|\leqq(K-1)K^{1/2}$. Thus the proof is completed.

Let $K_i=1+2^{-(i+2)}$, $i=0, 1, 2, \cdots$. We define inductively a sequence of elements g_i, $i=0, 1, 2, \cdots$. Let g_0 be related to K_0 as g is to k in Theorem I. Suppose g_i has been defined and is related to K_i as in Theorem I. Then in Lemma 3.5.1 let $g=g_i$, $K=K_i$, $K'=K_{i+1}$. We then define g_{i+1} as g'. Then g_{i+1} and K_{i+1} are related as g and K in Theorem I and we also have

$$\|g_{i+1}-g_i\|\leqq(K_i-1)K_i^{1/2}=\frac{1}{2^{i+2}}K_i^{1/2}\leqq\frac{1}{2^{i+1}}\cdot$$

Then for $p>0$

$$\|g_{n+p} - g_n\| = \left\| \sum_{i=n}^{n+p-1} (g_{i+1} - g_i) \right\| \leqq \sum_{i=n}^{n+p-1} \|g_{i+1} - g_i\| \leqq \sum_{i=n}^{n+p-1} \frac{1}{2^{i+1}}$$
$$\leqq \sum_{i=n}^{\infty} \frac{1}{2^{i+1}} = \frac{1}{2^n}.$$

This implies that the g_i's satisfy the Cauchy condition. Let g be their limit, and let A again be definite and $\epsilon \boldsymbol{M}$. Then, since A is bounded,

$$(Ag, g) = \lim_{i\to\infty} K_i(Ag_i, g_i) \geqq Tr_{\boldsymbol{M}}(A) \geqq \lim_{i\to\infty} K_i^{-1}(Ag_i, g_i) = (Ag, g).$$

Thus, if $A \epsilon \boldsymbol{M}$ is definite, $(Ag, g) = Tr_{\boldsymbol{M}}(A)$.

For $A \epsilon \boldsymbol{M}$, Hermitian, we have as in §2.1, under Property II, $A = A_1 - A_2$, A_1 and A_2, $\epsilon \boldsymbol{M}$ and positive definite. Property I of §2.1 then yields $(Ag, g) = Tr_{\boldsymbol{M}}(A)$. If A is merely $\epsilon \boldsymbol{M}$, then Definition 2.2.1 now implies that $(Ag, g) = Tr_{\boldsymbol{M}}(A)$. We have thus demonstrated

THEOREM II. *Let $\boldsymbol{M}$ be a factor in case* II, *with* $\alpha \geqq 1$. (*In the normalization of* §1.1. *In the normalization of R.O., Theorem* VIII *this means $\boldsymbol{M}$, $\boldsymbol{M}'$ are either in case* II_1, II_∞ *or in case* II_1, II_1 *with* $C \leqq 1$, *standard normalization.*) *Then there exists a $g \epsilon \mathfrak{H}$ such that*

$$(Ag, g) = Tr_{\boldsymbol{M}}(A).$$

We now drop the restriction that $\alpha \geqq 1$. Let $\boldsymbol{M}$, $\boldsymbol{M}'$ be in case II_1, II_1, $\alpha < 1$, and let m be an integer such that $m\alpha \geqq 1$ or in the standard normalization of R.O., Theorem X, $C/m \leqq 1$. Let $E_1, E_2, \cdots, E_m$ be m projections each of which is $\epsilon \boldsymbol{M}$, $D_{\boldsymbol{M}}(E_i) = 1/m$, $E_iE_j = 0$ if $i \neq j$. Let the range of E_i be $\mathfrak{M}_i$. Let us recall R.O., §11.3 using $\mathfrak{M}_i$ for $\mathfrak{M}$.

Consider the $\boldsymbol{M}_{(\mathfrak{M}_i)} = \boldsymbol{M}_i$ and $\boldsymbol{M}'_{(\mathfrak{M}_i)} = \boldsymbol{M}_i'$ of Definition 11.3.1. By Lemma 11.3.2, these constitute a factorization in $\mathfrak{M}_i$. Now we can take for $A \epsilon \boldsymbol{M}$ and such that $E_iA = AE_i = A$, $D_{\boldsymbol{M}_i}(A_{E_i}) = D_{\boldsymbol{M}}(A)$ and for $A' \epsilon \boldsymbol{M}'$, $D_{\boldsymbol{M}_i'}(A_{E_i}) = D_{\boldsymbol{M}'}(A')$, $D_{\boldsymbol{M}_i}$ and $D_{\boldsymbol{M}_i'}$ are dimension functions for $\boldsymbol{M}_i$ and $\boldsymbol{M}_i'$ respectively (Lemma 11.3.6).

The ranges of $D_{\boldsymbol{M}(\mathfrak{M})}$ and $D_{\boldsymbol{M}'(\mathfrak{M})}$ are by Lemma 11.3.7, the intervals $(0, 1/m)$, $(0, \alpha)$ respectively. So we can obtain the standard normalization of R.O., Theorem X by multiplying by m and $1/\alpha$ respectively. But in this latter normalization, by Lemma 11.4.3, $C = (D_{\boldsymbol{M}}(E_i)/D_{\boldsymbol{M}'}(1))C_0 = C_0/m \leqq 1$, where $D_{\boldsymbol{M}}$ and $D_{\boldsymbol{M}'}$ refer to the standard normalization for $\boldsymbol{M}$ and $\boldsymbol{M}'$ as in R.O., Theorem X.

By Theorem II above this implies that there is a $g_i^0 \epsilon \mathfrak{M}_i$ such that, if $A_0 \epsilon \boldsymbol{M}_i$, then $Tr_{\boldsymbol{M}'}(A_0) = (A_0 g_i^0, g_i^0)$. But now if $A \epsilon \boldsymbol{M}$ is such that $E_iA = AE_i = A$, let $A_0 = A_{E_i}$. From the above it will be seen that $Tr_{\boldsymbol{M}}(A)$

$= (1/m)Tr_{M_i}(A_0) = (1/m)(A_0 g_i^0, g_i^0) = (1/m)(Ag_i^0, g_i^0) = (Ag_i, g_i)$, if $g_i = (1/m)^{1/2} g_i^0$.

Now if $A \epsilon \boldsymbol{M}$, we have by Lemma 3.2.6

$$Tr_M(A) = \sum_{i=1}^{m} Tr_M(E_i A E_i) = \sum_{i=1}^{m} (E_i A E_i g_i, g_i)$$

$$= \sum_{i=1}^{m} (A E_i g_i, E_i g_i) = \sum_{i=1}^{m} (A g_i, g_i).$$

If $\boldsymbol{M}$, $\boldsymbol{M}'$ is in case I_m, $\mathrm{I}_{m'}$, $m < \infty$, we may proceed as follows. By R.O., Lemma 8.6.1, $\mathfrak{H} = E_m \otimes \mathfrak{H}_2$, where E_m is an m-dimensional euclidean space. Let $\phi_1, \cdots, \phi_m$ be a complete orthonormal set in E_m and $f \epsilon \mathfrak{H}_2$ with $\|f\| = 1$. Then if we let $g_i = \phi_i \otimes f$, we easily verify that the above formula for Tr_M holds. Thus we have

THEOREM III. *If $\boldsymbol{M}$ is a factor in a finite case, then there exists a finite number of elements $g_i \epsilon \mathfrak{H}$ such that*

$$Tr_M(A) = \sum_{i=1}^{m} (A g_i, g_i).$$

Suppose that $A \epsilon \boldsymbol{M}$ is held fixed. Then consider the weak neighborhood $U = U(A; g_1, \cdots, g_m; g_1, \cdots, g_m; \epsilon/m)$. $X \epsilon U$ implies $|((X - A)g_i, g_i)| < \epsilon/m$ for $i = 1, \cdots, m$. Hence if X is also $\epsilon \boldsymbol{M}$, Theorem III implies $|Tr_M(X) - Tr_M(A)| < \epsilon$. This proves

THEOREM IV. *If $\boldsymbol{M}$ is a factor in a finite case, $Tr_M(A)$ is weakly continuous.*

CHAPTER IV. THE ISOMORPHISM OF $\boldsymbol{M}$, $\boldsymbol{M}'$ AND $\mathfrak{H}$ (FOR $\alpha = 1$)

4.1. We assume that $\boldsymbol{M}$, $\boldsymbol{M}'$ is in case II_1, II_1 and $\alpha = 1$ or what is the same thing that $C = 1$ and we have the standard normalization. We know by the discussion of R.O., §§11.3 and 11.4 how all other cases II can be reduced to this case.

As $\alpha = 1$, Theorem II holds. We define

DEFINITION 4.1.1. *A g which satisfies Theorem* II *is uniformly distributed with respect to $\boldsymbol{M}$; abbreviated g u.d.r. $\boldsymbol{M}$.*

LEMMA 4.1.1. *If g is u.d.r. $\boldsymbol{M}$, then* (i) $\|g\| = 1$, (ii) $E_g^{M'} = 1$, (iii) $E_g^{M} = 1$ (cf. R.O., Definition 5.1.1).

To prove (i), in Theorem II take A as 1. Then $1 = Tr_M(1) = (g, g) = \|g\|^2$.

To prove (ii), in Theorem II take A as $E_g^{M'}$. Then

$$1 = \|g\|^2 = (E_g^{M'} g, g) = Tr_M(E_g^{M'}) = D_M(E_g^{M'})$$

or $D_M(E_g^{M'}) = 1$ which in our normalization implies $E_g^{M'} = 1$.

Consider (iii). Since we have the standard normalization and $C=1$, $D_{M'}(E_g^M)=D_M(E_g^{M'})=1$. This implies $E_g^M=1$.

DEFINITION 4.1.2. *Let g be $\epsilon\mathfrak{H}$. Let $Q_g(\boldsymbol{M})$ consist of all operators $Z\epsilon U(\boldsymbol{M})$, i.e., Z is linear closed $\eta\boldsymbol{M}$ and with a dense domain* (cf. R.O., Definition 4.2.1), *for which Zg exists.*

Now consider any $f\epsilon\mathfrak{H}$ such that $E_f^{M'}=E_f^M=1$. By R.O., Lemma 9.2.1, if $h\epsilon\mathfrak{H}$, then $h=XYf$, where X and Y are $\epsilon U(\boldsymbol{M})$. But if $\boldsymbol{M}$ is in case II_1 by R.O., Theorem XV, $Z=[XY]$ is $\epsilon U(\boldsymbol{M})$, and, since $Z\supseteq XY$, Zf exists, and we also have $h=XYf=Zf$; i.e., every $h\epsilon\mathfrak{H}$ is in the form Zf, where Z is $\epsilon U(\boldsymbol{M})$.

Furthermore this Z is uniquely determined by h. For suppose Z_1 and $Z_2\epsilon U(\boldsymbol{M})$ are such that $Z_1f=Z_2f$ or $(Z_1-Z_2)f=0$. Then $[Z_1-Z_2]$ is by R.O., Theorem XV, $\epsilon U(\boldsymbol{M})$ and since $[Z_1-Z_2]\supseteq Z_1-Z_2[Z_1-Z_2]f$ exists and is zero. Let A' be $\epsilon\boldsymbol{M}'$. Then since $A'[Z_1-Z_2]\subseteq[Z_1-Z_2]A'$, $[Z_1-Z_2]A'f=A'[Z_1-Z_2]f=0$. Thus $[Z_1-Z_2]$ is zero on the set of $A'f$, $A'\epsilon\boldsymbol{M}$. But since $E_f^{M'}=1$, this latter set is everywhere dense in $\mathfrak{H}$. Thus $[Z_1-Z_2]$ is zero on a dense set and since it is closed it must be zero. R.O., Theorem XV, then yields

$$Z_1=[Z_1-0]=[Z_1-[Z_1-Z_2]]=[[Z_1-Z_1]+Z_2]=[0+Z_2]=Z_2.$$

So we have proved

LEMMA 4.1.2. *If f is such that $E_f^{M'}=E_f^M=1$ (in particular, if f is u.d.r. $\boldsymbol{M}$), then $h=Zf$ defines a one-to-one correspondence of $\mathfrak{H}$ and the set of all $Z\epsilon U(\boldsymbol{M})$ for which Zf exists (i.e., between $\mathfrak{H}$ and $Q_f(\boldsymbol{M})$).*

Since our assumptions on f are symmetric in $\boldsymbol{M}$ and $\boldsymbol{M}'$, we also have

LEMMA 4.1.3. *Let f be as in Lemma 4.1.2. Then $h=Z'f$ defines a one-to-one correspondence of $\mathfrak{H}$ and the set of all $Z'\epsilon U(\boldsymbol{M}')$ for which $Z'f$ exists (i.e., between $\mathfrak{H}$ and $Q_f(\boldsymbol{M}')$).*

Thus for $h\epsilon\mathfrak{H}$ we have three correspondences defined by the equations $Zf=h=Z'f$, if f is u.d.r. $\boldsymbol{M}$: (1) the correspondence $\mathfrak{J}_M$ of $\mathfrak{H}$ and $Q_f(\boldsymbol{M})$, (2) the correspondence $\mathfrak{J}_{M'}$ of $\mathfrak{H}$ and $Q(\boldsymbol{M}')$, and (3) the correspondence $\mathfrak{J}_{M,M'}$ of $Q_f(\boldsymbol{M})$ and $Q_f(\boldsymbol{M}')$.

4.2. We now investigate these three correspondences. We know that they are one-to-one. They are obviously linear and of course along with $\mathfrak{H}$ the sets $Q_f(\boldsymbol{M})$ and $Q_f(\boldsymbol{M}')$ are linear. But inasmuch as $\mathfrak{J}_{M,M'}$ is a correspondence of operators, we would like to know the algebraic properties of this correspondence as well as certain properties of $Q_f(\boldsymbol{M})$ and $Q_f(\boldsymbol{M}')$. In particular we would like to know:

(i) To what extent can the operation $[XY]$ be performed in $Q_f(\boldsymbol{M})$ (or $Q_f(\boldsymbol{M}')$)?

(ii) To what extent can the operation X^* be performed in $Q_f(\boldsymbol{M})$ (or $Q_f(\boldsymbol{M}')$)?

(iii) Is the property of being bounded invariant under $\mathfrak{J}_{\boldsymbol{M},\boldsymbol{M}'}$; i.e., since $\boldsymbol{M}\subset Q_f(\boldsymbol{M})$, $\boldsymbol{M}'\subset Q_f(\boldsymbol{M}')$, is $\boldsymbol{M}$ mapped on $\boldsymbol{M}'$ by $\mathfrak{J}_{\boldsymbol{M},\boldsymbol{M}'}$?

(iv) Are the properties of being Hermitian, definite, or unitary invariant under $\mathfrak{J}_{\boldsymbol{M},\boldsymbol{M}'}$?

LEMMA 4.2.1. *If f is u.d.r. $\boldsymbol{M}$, then every Uf, $U\epsilon\boldsymbol{M}$ and unitary, is u.d.r. $\boldsymbol{M}$ also.*

If $A\epsilon\boldsymbol{M}$, then

$$(AUf,\, Uf) = (U^{-1}AUf,\, f) = Tr_{\boldsymbol{M}}(U^{-1}AU) = Tr_{\boldsymbol{M}}(A).$$

LEMMA 4.2.2. *If f and g are each u.d.r. $\boldsymbol{M}$, then there exists a $U'\epsilon\boldsymbol{M}'$ and unitary with $g=U'f$.*

If $h=Af$, $A\epsilon\boldsymbol{M}$, define h^* as Ag. The correspondence $h\rightleftarrows h^*$ is linear, and, since

$$\begin{aligned}\|h^*\|^2 &= \|Ag\|^2 = (Ag,\, Ag) = (A^*Ag,\, g) \doteq Tr_{\boldsymbol{M}}(A^*A) = (A^*Af,\, f)\\ &= (Af,\, Af) = \|Af\|^2 = \|h\|^2,\end{aligned}$$

it is isometric. Thus the equation $Wh=h^*$ defines a linear isometric and hence one-valued operator W. Since $E_f^{\boldsymbol{M}}=E_g^{\boldsymbol{M}}=1$, both domain and range of W are everywhere dense and hence its closure $\tilde{W}$ is unitary.

Now if h is in the domain of W, $h=Af$, $A\epsilon\boldsymbol{M}$. Then, for every $B\epsilon\boldsymbol{M}$, $BWh=Bh^*=BAg=WBAf=WBh$ or $BW\subseteq WB$. This implies that $[BW]\subseteq[WB]$. But $B\tilde{W}=\tilde{B}\tilde{W}\subseteq[BW]$, while since B is bounded $[WB]=\tilde{W}B$. Hence $B\tilde{W}\subseteq\tilde{W}B$. If in particular B is unitary, R.O., Lemma 4.2.2 now yields that $\tilde{W}$ is $\eta\boldsymbol{M}'$ and R.O., Lemma 4.2.1 implies that $\tilde{W}$ is $\epsilon\boldsymbol{M}'$.

Letting $A=1$ in the definition of W, we get $g=Wf=\tilde{W}f$.

LEMMA 4.2.3. *If f and g are u.d.r. $\boldsymbol{M}$, then there exists a $U\epsilon\boldsymbol{M}$ and unitary such that $g=Uf$.*

By Lemma 4.1.2, $g=Zf$ where Z is $\epsilon U(\boldsymbol{M})$. Using the canonical decomposition for Z, $Z=BW$, where B is self-adjoint and definite and W is partially isometric and as indicated in the proof of Lemma 1.4.3 may be taken as unitary; that is, $g=BWf$, where W is unitary and B definite and self-adjoint. We must show that $B=1$.

By Lemma 4.2.1, $f_0=Wf$ is u.d.r. $\boldsymbol{M}$. Hence we must show that if f_0 and g are u.d.r. $\boldsymbol{M}$ and $g=Bf_0$, where B is definite and self-adjoint, then $B=1$.

Let $E(\lambda)$ be the resolution of the identity corresponding to B. If $E(\lambda)=0$ for $\lambda<1$ and $E(\lambda)=1$ for $\lambda>1$, then $B=1$. Thus if B is not 1, either there exists a $\lambda_1<1$ for which $E(\lambda_1)\neq 0$ or a $\lambda_2>1$ for which $E(\lambda_2)\neq 1$. In the first case we have $\|E(\lambda_1)f_0\|^2=(E(\lambda_1)f_0,\ E(\lambda_1)f_0)=(E(\lambda_1)f_0,\ f_0)=Tr_M(E(\lambda_1))$ $=(E(\lambda_1)g,\ g)=\|E(\lambda_1)g\|^2=\|E(\lambda_1)Bf_0\|^2=\|BE(\lambda_1)f_0\|^2\leqq \lambda_1^2\|E(\lambda_1)f_0\|^2$ or $(1-\lambda_1^2)\|E(\lambda_1)f_0\|^2\leqq 0$. This implies $0=\|E(\lambda_1)f_0\|^2=(E(\lambda_1)f_0,f_0)=Tr_M(E(\lambda_1))$ $=D_M(E(\lambda_1))$ or $E(\lambda_1)=0$, a contradiction. In the second case a similar type of calculation yields that $\|(1-E(\lambda_2))f_0\|^2=\|B(1-E(\lambda_2))f_0\|^2\geqq\lambda_2^2\|(1-E(\lambda_2))f_0\|^2$ which again implies that $1-E(\lambda_2)=0$, a contradiction. Thus B must be 1 and our lemma is proved.

LEMMA 4.2.4. *If f is u.d.r. $\boldsymbol{M}$, and $U'\epsilon\boldsymbol{M}'$ and unitary, then $U'f$ is u.d.r. $\boldsymbol{M}$.*

If A is $\epsilon\boldsymbol{M}$,

$$(AU'f,\ U'f)=(U'^{-1}AU'f,f)=(AU'^{-1}U'f,f)=(Af,f)=Tr_M(A).$$

LEMMA 4.2.5. *Unitary operators correspond to unitary operators under $\mathfrak{J}_{M,M'}$; i.e., if f is u.d.r. $\boldsymbol{M}$, $U\epsilon\boldsymbol{M}$, $U'\epsilon\boldsymbol{M}'$ and $Uf=U'f$, then if either U or U' is unitary the other is too.*

Let $U\epsilon\boldsymbol{M}$ be unitary, then by Lemma 4.2.1 Uf is u.d.r. $\boldsymbol{M}$. Then Lemma 4.2.2 yields that there is a unitary $U'\epsilon\boldsymbol{M}$, such that $U'f=Uf$. Thus $U\sim U'$ under $\mathfrak{J}_{M,M'}$. Similarly Lemmas 4.2.4 and 4.2.3 yield that to every unitary $U'\epsilon\boldsymbol{M}$, there exists a unitary $U\epsilon\boldsymbol{M}$ such that $U'f=Uf$.

LEMMA 4.2.6. *f u.d.r. $\boldsymbol{M}$ is equivalent to f u.d.r. $\boldsymbol{M}'$.*

Let f be u.d.r. $\boldsymbol{M}$ and $A'\epsilon\boldsymbol{M}$. Then consider $T_0(A')=(A'f,f)$. It is linear in A'. By Lemma 4.1.1, (i) $T_0(1)=1$. If A' is definite, then $T_0(A')=(A'f_0,f_0)$ $\geqq 0$. Furthermore we have $T_0(A'^*)=(A'^*f,\ f)=(f,\ A'f)=\overline{(A'f,\ f)}=\overline{T_0(A)}$. Therefore $T_0(A')$ satisfies (i)–(iv) and (v) of Property III with $\boldsymbol{M}'$ instead of $\boldsymbol{M}$.

We would like to verify next (vi)$'$ of the same property for $T_0(A')$. By Lemma 4.2.5 we have for any unitary $U'\epsilon\boldsymbol{M}$

$$\begin{aligned}T_0(U'^{-1}A'U')&=(U'^{-1}A'U'f,f)=(A'U'f,\ U'f)=(A'Uf,\ Uf)\\&=(U^{-1}A'Uf,f)=(A'(U^{-1}U)f,f)=(A'f,f)=T_0(A').\end{aligned}$$

Thus (vi)$'$ holds. Property IV of §2.2 now implies that $T_0(A')=Tr_{M'}(A')$ and hence f is u.d.r. $\boldsymbol{M}'$.

Replacing $\boldsymbol{M}$ by $\boldsymbol{M}'$ in the above argument yields the converse.

We note that the above argument also proves

LEMMA 4.2.7. *If $f\epsilon\mathfrak{H}$ is such that* (i) $\|f\|=1$ *and* (ii) *if to every unitary $U\epsilon\boldsymbol{M}$ there exists a unitary $U'\epsilon\boldsymbol{M}'$ such that $Uf=U'f$, then f is u.d.r. $\boldsymbol{M}$.*

THEOREM V. *Let $f \epsilon \mathfrak{H}$ be such that $\|f\| = 1$ and let $\boldsymbol{M}$, $\boldsymbol{M}'$ be in case* II_1, II_1 *with $C = 1$. Then the following statements are equivalent*:

(α) *f is u.d.r. $\boldsymbol{M}$.*

(β) *f is u.d.r. $\boldsymbol{M}'$.*

(γ) *If U is unitary and $\epsilon \boldsymbol{M}$, then there exists a unitary $U' \epsilon \boldsymbol{M}'$ such that $Uf = U'f$.*

(δ) *If U' is unitary and $\epsilon \boldsymbol{M}'$, then there exists a unitary $U \epsilon \boldsymbol{M}$ such that $U'f = Uf$.*

Furthermore either (α), (β), (γ), *or* (δ) *implies $E_f^{\boldsymbol{M}} = E_f^{\boldsymbol{M}'} = 1$.*

(α) is equivalent to (β) by Lemma 4.2.6. (α) and (γ) are equivalent by Lemmas 4.2.5 and 4.2.7. Replacing $\boldsymbol{M}$ by $\boldsymbol{M}'$, the last mentioned lemmas also show (β) and (δ) to be equivalent. This and Lemma 4.1.1 prove the last statement of the theorem.

In view of Theorem V, we will say that f is uniformly distributed (abbreviated u.d.) if f is u.d.r. $\boldsymbol{M}$.

COROLLARY. *If f is u.d., $\mathfrak{J}_{\boldsymbol{M},\boldsymbol{M}'}$ maps $\boldsymbol{M}(\subseteq Q_f(\boldsymbol{M}))$ on $\boldsymbol{M}'(\subseteq Q_f(\boldsymbol{M}'))$; that is, the property of being bounded is invariant under $\mathfrak{J}_{\boldsymbol{M},\boldsymbol{M}'}$.*

By Theorem V, the property of being unitary is invariant under $\mathfrak{J}_{\boldsymbol{M},\boldsymbol{M}'}$. This and the linearity of $\mathfrak{J}_{\boldsymbol{M},\boldsymbol{M}'}$ imply together that the property of having the form $A = \sum_{\nu=1}^{n} a_\nu U_\nu$ ($n = 1, 2, \cdots$; a_ν complex, U_ν unitary and in $\boldsymbol{M}$ or $\boldsymbol{M}'$ along with A) is invariant under $\mathfrak{J}_{\boldsymbol{M},\boldsymbol{M}'}$.

We will show that $A \epsilon Q_f(\boldsymbol{M})$ (or $Q_f(\boldsymbol{M}')$) is bounded if and only if it is in this form. Now if A is in this form, it is obviously bounded so we must only show that every bounded A is in this form. Since $A = A_1 + iA_2$, A_1 and A_2 Hermitian, it will be sufficient if this is shown for Hermitian A's. Since if A is in this form, cA is also, we may assume that the bound of A is $\leqq 1$. Then $1 - A^2$ is definite and $(1-A^2)^{1/2}$ exists. Letting $U = A + i(1-A^2)^{1/2}$, then $U^* = A - i(1-A^2)^{1/2}$ and U is unitary. Furthermore $A = \frac{1}{2}(U + U^*)$.

Now since the form $\sum_{\nu=1}^{n} a_\nu U_\nu$ is invariant under $\mathfrak{J}_{\boldsymbol{M},\boldsymbol{M}'}$, boundedness must be also.

THEOREM VI. *Assume that f_0 is u.d. Then $\mathfrak{J}_{\boldsymbol{M},\boldsymbol{M}'}$ is an anti-isomorphism of $\boldsymbol{M}$ and $\boldsymbol{M}'$. That is, if $A \sim A'$, $B \sim B'$, then* (i) $\alpha A \sim \alpha A'$; (ii) $A^* \sim A'^*$; (iii) $A + B \sim A' + B'$; (iv) $AB \sim B'A'$; (v) *if A (or A') is Hermitian, then A' (or A) is also.*

(i) and (iii) are obvious.

Consider (iv). We have $ABf_0 = AB'f_0 = B'Af_0 = B'A'f_0$, so that $AB \sim B'A'$. To prove (v), let U be unitary, $U \sim U'$. Then U' is unitary also by Theorem V. Similarly inasmuch as U^{-1} is unitary, if $U^{-1} \sim V'$, V' is unitary. As

$UU^{-1}=U^{-1}U=1$ by (iv) we have $U'V'=V'U'=1$, and hence $V'=(U')^{-1}$. So $U^{-1}\sim(U')^{-1}$ and $\alpha(U+U^{-1})\sim\alpha(U'+(U')^{-1})$ by (iii). If α is >0, the right side is Hermitian and the left side is any Hermitian element of $\boldsymbol{M}$ (cf. the proof of the corollary of Theorem V). This proves (v).

We now prove (ii). If A is $\epsilon\boldsymbol{M}$, then $A=B+iC$, $A^*=B-iC$, B, C are Hermitian and $\epsilon\boldsymbol{M}$. If $A\sim A'$, $B\sim B'$, $C\sim C'$, then B', C' are Hermitian and $A\sim B'+iC'=A'$, $A^*\sim B'-iC'=A'^*$ proving (ii).

Corollary. *The properties of being Hermitian, being definite, being a projection, as well as the numerical quantities* $|||A|||$ (*the bound of* A) *and* $Tr_{\boldsymbol{M}}(A)$ (*or* $Tr_{\boldsymbol{M}'}(A')$), *all within* $\boldsymbol{M}$ (*or* $\boldsymbol{M}'$), *are invariant under* $\mathfrak{J}_{\boldsymbol{M},\boldsymbol{M}'}$.

A Hermitian means $A=A^*$; A a projection means $A=A^*=A^2$; A definite means that there exists an Hermitian B with $A=B^2$. So all these properties are invariant under $\mathfrak{J}_{\boldsymbol{M},\boldsymbol{M}'}$. 1 is invariant ($1\epsilon\boldsymbol{M}$ and $1\epsilon\boldsymbol{M}'$, $1f_0=1f_0$). $|||A|||$ is the smallest $\alpha\geqq 0$ such that $\alpha^2\cdot 1-A^*A$ is definite, so $|||A|||$ is invariant. If $A\sim A'$, then $Af_0=A'f_0$ and $Tr_{\boldsymbol{M}}(A)=(Af_0, f_0)=(A'f_0, f_0)=Tr_{\boldsymbol{M}}(A')$ by Theorem V.

4.3. In what follows f_0 will always be assumed to be u.d. The discussion which follows could be based on an extension of the notion of $Tr_{\boldsymbol{M}}(A)$ (and of $Tr_{\boldsymbol{M}'}(A')$) to unbounded $A\eta\boldsymbol{M}$ (or $A'\eta\boldsymbol{M}'$) but we prefer an approach which avoids this.

Theorem VII. *The sets* $Q_{f_0}(\boldsymbol{M})$ *and* $Q_{f_0}(\boldsymbol{M}')$ *are independent of the choice of of u.d.* f_0. *We will therefore denote them by* $Q(\boldsymbol{M})$ *and* $Q(\boldsymbol{M}')$, *respectively. Furthermore the values of* $\|Zf_0\|$, (Xf_0, Yf_0), *where* X, Y, $Z\epsilon Q(\boldsymbol{M})$, *or* $Q(\boldsymbol{M}')$ *are independent of the choice of the u.d.* f_0.

Owing to the symmetry between $\boldsymbol{M}$ and $\boldsymbol{M}'$ it suffices to prove the statements concerning $\boldsymbol{M}$; i.e., let f_0 and g_0 be u.d., then for $A\epsilon U(\boldsymbol{M})$, Af_0 is defined if and only if Ag_0 is defined and $\|Af_0\|=\|Ag_0\|$. Owing to the symmetry in f_0 and g_0 it will be sufficient to show that Ag_0 is defined if Af_0 is and $\|Af_0\|=\|Ag_0\|$.

By Lemma 4.2.2, there exists a unitary $U'\epsilon\boldsymbol{M}'$ such that $g_0=U'f_0$. Since A is $\epsilon U(\boldsymbol{M})$, $A=U'^{-1}AU'$. Hence since Af_0 exists $Af_0=U'^{-1}AU'f_0=U'^{-1}Ag_0$ exists. This implies that Ag_0 exists and since $Ag_0=U'Af_0$, $\|Ag_0\|=\|Af_0\|$. Also if $B\epsilon Q_{f_0}(\boldsymbol{M})$, then $Bg_0=U'Bf_0$ and $(Ag_0, Bg_0)=(U'Af_0, U'Bf_0)=(Af_0, Bf_0)$.

So we must have these mappings:

$$\text{(I)}\qquad \mathfrak{J}_{\boldsymbol{M}}: \mathfrak{H}\sim Q(\boldsymbol{M});\ \mathfrak{J}_{\boldsymbol{M}'}: \mathfrak{H}\sim Q(\boldsymbol{M}');\ \mathfrak{J}_{\boldsymbol{M},\boldsymbol{M}'}: Q(\boldsymbol{M})\sim Q(\boldsymbol{M}').$$

By the corollary to Theorem V, $\mathfrak{J}_{\boldsymbol{M},\boldsymbol{M}'}$ maps $\boldsymbol{M}$ on $\boldsymbol{M}'$. So $\mathfrak{J}_{\boldsymbol{M}}$ maps $\boldsymbol{M}$ and $\mathfrak{J}_{\boldsymbol{M}'}$ maps $\boldsymbol{M}'$ on the same subset $\mathfrak{A}$ of $\mathfrak{H}$ and we have the further mappings

(II) $\qquad \mathfrak{J}_{M}: \mathfrak{A} \sim \boldsymbol{M}; \mathfrak{J}_{M'}: \mathfrak{A} \sim \boldsymbol{M}'; \mathfrak{J}_{M,M'}: \boldsymbol{M} \sim \boldsymbol{M}'.$

(II) is part of (I).

We now prove

LEMMA 4.3.1. *$\mathfrak{A}$ is a linear set, dense in $\mathfrak{H}$.*

Inasmuch as $\boldsymbol{M}$ is linear, $\mathfrak{A}$ is too.

Consider an $f \epsilon \mathfrak{H}$, $f = Zf_0$, $Z \epsilon Q(\boldsymbol{M})$. Now $Z = BU$, B self-adjoint and definite, $B \epsilon U(\boldsymbol{M})$, U unitary and $\epsilon \boldsymbol{M}$. Write B in its spectral form

$$B = \int_0^\infty \lambda dE(\lambda), \quad E(\lambda) \epsilon \boldsymbol{M}.$$

Then we have

$$[E(\mu)B] = \int_0^\mu \lambda dE(\lambda);$$

and therefore (i) $[E(\mu)B] \epsilon \boldsymbol{M}$ and (ii) if Bg is defined then we have strong $\lim_{\mu\to\infty}[E(\mu)B]g = Bg$. Thus $[E(\mu)B]U \epsilon \boldsymbol{M}$ and strong $\lim_{\mu\to\infty}[E(\mu)B]Uf_0 = BUf_0 = Zf_0 = f$. Since $[E(\mu)B]Uf_0 \epsilon \mathfrak{A}$, f is a condensation point of $\mathfrak{A}$. As this is true for any $f \epsilon \mathfrak{H}$, $\mathfrak{A}$ is dense in $\mathfrak{H}$.

DEFINITION 4.3.1. *If $f = Xf_0 = X'f_0$, $g = Yf_0 = Y'f_0$, X and $Y \epsilon Q(\boldsymbol{M})$, X' and $Y' \epsilon Q(\boldsymbol{M}')$, let $[[X]] = [[X']] = \|f\|$, $\langle\langle X, Y \rangle\rangle = \langle\langle X', Y' \rangle\rangle = (f, g)$.*

Theorem VII shows that the values of $[[X]]$, $[[X']]$, $\langle\langle X', Y' \rangle\rangle$, $\langle\langle X, Y \rangle\rangle$ are independent of the choice of the u.d. f_0.

LEMMA 4.3.2. *If X and Y are $\epsilon \boldsymbol{M}$, then $[[X]] = (Tr_{\boldsymbol{M}}(X^*X))^{1/2} = (Tr_{\boldsymbol{M}}(XX^*))^{1/2}$, $\langle\langle X, Y \rangle\rangle = Tr_{\boldsymbol{M}}(Y^*X) = Tr_{\boldsymbol{M}}(XY^*)$.*

As $[[X]] = (\langle\langle X, X \rangle\rangle)^{1/2}$, the second statement implies the first. Now clearly

$$\langle\langle X, Y \rangle\rangle = (Xf_0, Yf_0) = (Y^*Xf_0, f_0) = Tr_{\boldsymbol{M}}(Y^*X)$$

since f is u.d. We also have $Tr_{\boldsymbol{M}}(Y^*X) = Tr_{\boldsymbol{M}}(XY^*)$ by §2.2, Property III, (vi). So $\langle\langle X, Y \rangle\rangle = Tr_{\boldsymbol{M}}(Y^*X) = Tr_{\boldsymbol{M}}(XY^*)$.

LEMMA 4.3.3. *If X', $Y' \epsilon \boldsymbol{M}'$, then $[[X']] = (Tr_{\boldsymbol{M}'}(X'^*X'))^{1/2} = (Tr_{\boldsymbol{M}'}(X'X'^*))^{1/2}$, $\langle\langle X', Y' \rangle\rangle = Tr_{\boldsymbol{M}'}(Y'^*X') = Tr_{\boldsymbol{M}}(X'Y'^*)$.*

Replace $\boldsymbol{M}$ by $\boldsymbol{M}'$ in the proof of Lemma 4.3.2.

THEOREM VIII. *If we use the definitions*

$$\langle\langle X, Y \rangle\rangle = Tr_{\boldsymbol{M}}(Y^*X) = Tr_{\boldsymbol{M}}(XY^*),$$
$$[[X]] = (\langle\langle X, X \rangle\rangle)^{1/2} = (Tr_{\boldsymbol{M}}(X^*X))^{1/2} = (Tr_{\boldsymbol{M}}(XX^*))^{1/2},$$

then $\boldsymbol{M}$ is an incomplete Hilbert space. Its completion in the usual (Cantor) way gives a Hilbert space $\tilde{\boldsymbol{M}}$ which may be identified with $Q(\boldsymbol{M})$.

This is clear by the isomorphisms (I) and (II) and Lemmas 4.3.1 and 4.3.2. $\boldsymbol{M}$ and $\tilde{\boldsymbol{M}}$, that is, $Q(\boldsymbol{M})$, are isomorphic to $\mathfrak{A}$ and $\mathfrak{H}$ respectively, using the isomorphism $\mathfrak{J}_{\boldsymbol{M}}$.

The corresponding facts hold for $\boldsymbol{M}'$; their formulation is obvious.

THEOREM IX. *$X\epsilon Q(\boldsymbol{M})$ implies $X^*\epsilon Q(\boldsymbol{M})$ and* $[[X]]=[[X^*]]$.

Assume $X\epsilon Q(\boldsymbol{M})$, i.e., $X\epsilon U(\boldsymbol{M})$; Xf_0 is defined. Write $X=BU$, B self-adjoint and $\eta\boldsymbol{M}$, U unitary and $\epsilon\boldsymbol{M}$. Thus BUf_0 is defined. Now Uf_0 is u.d. (Lemma 4.2.1), hence the existence of BUf_0 implies $B\epsilon Q(\boldsymbol{M})$ (Theorem VII). This in turn implies the existence of Bf_0 and with it $X^*f_0=U^{-1}Bf_0$. So $X^*\epsilon Q(\boldsymbol{M})$.

As f_0 and Uf_0 are both u.d. (Lemma 4.2.1) we have by Definition 4.3.1, $[[X]]=\|Xf_0\|=\|BUf_0\|=[[B]]=\|Bf_0\|=\|U^{-1}Bf_0\|=\|X^*f_0\|=[[X^*]]$.

COROLLARY. *$X\sim X^*$ is an involutory conjugate anti-isomorphism of $Q(\boldsymbol{M})$ and isometric.*

$X\sim X^*$ maps $Q(\boldsymbol{M})$ on part of itself by Theorem IX. This and $X^{**}=X$ show that $X\sim X^*$ is a one-to-one and involutory mapping of $Q(M)$ on itself. It is isometric by Theorem IX; $[[X]]=[[X^*]]$ and it is obviously a conjugate anti-isomorphism.

The corresponding facts hold for $\boldsymbol{M}'$; the formulation is obvious.

4.4. We next discuss the algebra of $Q(\boldsymbol{M})$.

PROPERTY I⁰. *A, $B\epsilon Q(\boldsymbol{M})$ imply αA, A^*, $[A+B]\epsilon Q(\boldsymbol{M})$. If also either A or B is $\epsilon\boldsymbol{M}$, then $[AB]$ is $\epsilon Q(\boldsymbol{M})$.*

αA, $[A+B]$ are $\epsilon Q(\boldsymbol{M})$ since $Q(\boldsymbol{M})$ is linear (along with $\mathfrak{H}$). A^* is $\epsilon Q(\boldsymbol{M})$ by Theorem IX.

Assume now that $A\epsilon\boldsymbol{M}$, $B\epsilon Q(\boldsymbol{M})$. Then Bf_0 is defined and so is $ABf_0=[AB]f_0$. Thus $[AB]\epsilon Q(\boldsymbol{M})$. If $A\epsilon Q(\boldsymbol{M})$, $B\epsilon\boldsymbol{M}$, then $B^*\epsilon\boldsymbol{M}$, $A^*\epsilon Q(\boldsymbol{M})$, so $[B^*A^*]\epsilon Q(\boldsymbol{M})$. Since $[B^*A^*]^*=[AB]$ (R.O., Theorem XV), $[AB]$ is $\epsilon Q(\boldsymbol{M})$.

PROPERTY II⁰. (i) $[[\alpha A]]=|\alpha|\cdot[[A]]$, (ii) $[[A^*]]=[[A]]$, (iii) $[[A+B]]\leq[[A]]+[[B]]$, (iv) $[[[AB]]]\leq|||A|||\cdot[[B]]$ *and* $[[A]]\cdot|||B|||$.

(i) and (iii) hold because $Q(\boldsymbol{M})$ is isomorphic to $\mathfrak{H}$. (ii) holds by Theorem IX.

Consider (iv). We have

$$[[[AB]]]=\|[AB]f_0\|=\|ABf_0\|\leq|||A|||\cdot\|Bf_0\|=|||A|||\cdot[[B]],$$

proving the first inequality. The second follows from this, by using (ii)

$$[[[AB]]] = [[[AB]^*]] = [[[B^*A^*]]] \leqq |||\, B^* \,||| \cdot [[A^*]] = |||\, B \,||| \cdot [[A]].$$

We have $\mathfrak{H} \sim Q(\boldsymbol{M})$. Neither $\mathfrak{H}$ nor $Q(\boldsymbol{M})$ depends on the choice of the u.d. f_0 but $\mathfrak{J}_M$ does. This dependence is as follows.

Property III^0. *Let us replace the u.d. f_0 by a u.d. g_0 and let $\mathfrak{F}_M$ denote the resulting correspondence. Let U be unitary and $\epsilon \boldsymbol{M}$ and such that $Uf_0 = g_0$* (cf. Lemma 4.2.1). *Then if $X = YU$, X and $Y \epsilon Q(\boldsymbol{M})$, we have*

$$\mathfrak{J}_M : f \sim X; \qquad \mathfrak{F}_M : f \sim Y$$

if $f = Xf_0$.

This is clear since $f = Xf_0 = YUf_0 = Yg_0$.

We can now determine another notion which does not depend on the choice of the u.d. f_0.

Property IV^0. *The linear set $\mathfrak{A}$* (cf. §4.3, the correspondences (II)) *does not depend on the choice of the u.d. f_0.*

This follows from the definition of $\mathfrak{A}$ and the fact that if $X = YU$, U unitary and either X or Y is bounded, then both X and Y are bounded.

Now $Q(\boldsymbol{M}) \sim Q(\boldsymbol{M}')$ under $\mathfrak{J}_{M,M'}$ and while neither $Q(\boldsymbol{M})$ nor $Q(\boldsymbol{M}')$ depends on the choice of the u.d. f_0 yet $\mathfrak{J}_{M,M'}$ does. We now obtain this dependence.

Property V^0. *Let the u.d. f_0 be replaced by the u.d. g_0 and let the resulting correspondence between $Q(\boldsymbol{M})$ and $Q(\boldsymbol{M}')$ be denoted by $\mathfrak{F}_{M,M'}$. Then if $U \epsilon \boldsymbol{M}$ is unitary and such that $g_0 = Uf_0$, and $X = U^{-1}YU$, X and $Y \epsilon Q(\boldsymbol{M})$, then*

$$\mathfrak{J}_{M,M'} : X' \sim X; \qquad \mathfrak{F}_{M,M'} : X' \sim Y$$

if $X'f_0 = Xf_0$, $X' \epsilon Q(\boldsymbol{M}')$.

Let $X'f_0 = Xf_0$, then $X'g_0 = X'Uf_0 = UX'f_0 = UXf_0 = UXU^{-1}g_0$.

Definition 4.4.1. *The isomorphism $\mathfrak{H} \sim Q(\boldsymbol{M})$ makes correspond to every operator P in $\mathfrak{H}$ an operator P° in $Q(\boldsymbol{M})$.*

Observe that inasmuch as the elements of $Q(\boldsymbol{M})$ are operators in $\mathfrak{H}$, P° is an operator on the operators of $\mathfrak{H}$.

Theorem X. (i) *If $A \epsilon \boldsymbol{M}$, then*

$$A^\circ Z = [AZ];$$

(ii) *if $A' \epsilon \boldsymbol{M}'$ and $A' \sim A \epsilon \boldsymbol{M}$ under $\mathfrak{J}_{M,M'}$, then*

$$A'^\circ = [ZA].$$

P° is defined as follows. If $g = Xf_0$, $h = Pg$, $h = Yf_0$, X and $Y \epsilon Q(\boldsymbol{M})$, then $P^\circ X = Y$. Thus to prove (i), we have $(A^\circ Z)f_0 = A(Zf_0) = [AZ]f_0$ and to show (ii) $(A'^\circ Z)f_0 = A'Zf_0 = ZA'f_0 = ZAf_0 = [ZA]f_0$.

Corollary 1. *Let $\boldsymbol{M}^0$ be the set of all operators L_A in $Q(\boldsymbol{M})$, $A \epsilon \boldsymbol{M}$, where $L_A Z = [AZ]$, and $\boldsymbol{M}_0$ the set of all operators R_A in $Q(\boldsymbol{M})$, $A \epsilon \boldsymbol{M}$, where $R_A Z = [ZA]$. Then $\mathfrak{J}_{\boldsymbol{M}}$ carries $\boldsymbol{M}$ into $\boldsymbol{M}^0$ and $\boldsymbol{M}'$ into $\boldsymbol{M}_0$.*

This is obvious by Theorem X.

Corollary 2. *$\boldsymbol{M}^0$, $\boldsymbol{M}_0$ are rings and $\boldsymbol{M}^{0\prime} = \boldsymbol{M}_0$ (all in $Q(\boldsymbol{M})$). They are factors of class* (II_1, II_1) *with* $C = 1$.

Since $\mathfrak{J}_{\boldsymbol{M}}$ is a spatial isomorphism of $\mathfrak{H}$ and $Q(\boldsymbol{M})$ which takes $\boldsymbol{M}$, $\boldsymbol{M}'$ into $\boldsymbol{M}^0$, $\boldsymbol{M}_0$ respectively, these properties hold.

We have now characterized $\mathfrak{H}$, $\boldsymbol{M}$, $\boldsymbol{M}'$ by the situation in $Q(\boldsymbol{M})$, $\boldsymbol{M}^0$, $\boldsymbol{M}_0$ to which they are spatially isomorphic. The spatial isomorphism is $\mathfrak{J}_{\boldsymbol{M}}$ which depends on an arbitrary u.d. f_0, while $Q(\boldsymbol{M})$, $\boldsymbol{M}^0$, $\boldsymbol{M}_0$ themselves do not. All the influence of the choice of f_0 is however a further transformation R_U, $X = YU$ ($U \epsilon \boldsymbol{M}$, U unitary so $R_U \epsilon \boldsymbol{M}_0$). $\mathfrak{J}_{\boldsymbol{M}}$ always carries the totality of u.d. elements g_0 of $\mathfrak{H}$ into the totality of unitary elements $U \epsilon Q(\boldsymbol{M})$, f_0 being that element which corresponds to 1.

4.5. The following important isomorphism theorem can now be proved.

Theorem XI. *Let $\mathfrak{H}_1$ and $\mathfrak{H}_2$ be two Hilbert spaces and $\boldsymbol{M}_1$, $\boldsymbol{M}_1'$ and $\boldsymbol{M}_2$, $\boldsymbol{M}_2'$ respectively be factor pairs in them, both of class* (II_1, II_1) *and with $C = 1$ in the standard normalization. Then an algebraic ring isomorphism of $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ is a necessary and sufficient condition for the existence of a spatial isomorphism of $\mathfrak{H}_1$ and $\mathfrak{H}_2$ which takes $\boldsymbol{M}_1$, $\boldsymbol{M}_1'$ into $\boldsymbol{M}_2$, $\boldsymbol{M}_2'$.*

The necessity is obvious and so we prove the sufficiency. Let $\mathfrak{F}$ be the algebraic ring isomorphism of $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$. By §2.2, Property IV, $Tr_{\boldsymbol{M}_1}(A_1) = Tr_{\boldsymbol{M}_2}(A_2)$ if $A_1 \sim A_2$ under $\mathfrak{F}$. So $[[A_1]] = [[A_2]]$, $\langle\langle A_1, B_1\rangle\rangle = \langle\langle A_2, B_2\rangle\rangle$, if $A_1 \sim A_2$, $B_2 \sim B_2$ under $\mathfrak{F}$. Thus $\mathfrak{F}$ is an isometric mapping of $\boldsymbol{M}_1$ on $\boldsymbol{M}_2$ and therefore extends by continuity in a unique way to a linear and isometric mapping of $Q(\boldsymbol{M})$ on $Q(\boldsymbol{M}')$ which we call $\mathfrak{F}$ again. $A_1 \sim A_2$, $B_1 \sim B_2$ under $\mathfrak{F}$ imply $A_1B_1 \sim A_2B_2$ under $\mathfrak{F}$ if A_1, $B_1 \epsilon \boldsymbol{M}_1$ (and thus A_2, $B_2 \epsilon \boldsymbol{M}_2$). By continuity this will even hold, if one of A_1, B_1 is in $\boldsymbol{M}_1$ and the other merely in $Q(\boldsymbol{M}_1)$ (and one of A_2, B_2 in $\boldsymbol{M}_2$, and the other merely in $Q(\boldsymbol{M}_2)$). So $\mathfrak{F}$ is a spatial isomorphism of $Q(\boldsymbol{M}_1)$ and $Q(\boldsymbol{M}_2)$ which carries $\boldsymbol{M}_1^0$, $\boldsymbol{M}_{1,0}$ into $\boldsymbol{M}_2^0$, $\boldsymbol{M}_{2,0}$ (by Corollary 1 to Theorem XI). Now (by the same corollary) $\mathfrak{J}_{\boldsymbol{M}_2}\mathfrak{F}\mathfrak{J}_{\boldsymbol{M}_1}^{-1}$ is a spatial isomorphism of $\mathfrak{H}_1$ and $\mathfrak{H}_2$ which carries $\boldsymbol{M}_1$, $\boldsymbol{M}_1'$ into $\boldsymbol{M}_2$, $\boldsymbol{M}_2'$.

Theorem XI could be extended to cover cases (II_1, II_1), where $C \gtrless 1$ in the standard normalization as well as other combinations of II_1 and II_∞. In all cases a reduction of the spatial-isomorphism questions of $\mathfrak{H}_1$ and $\mathfrak{H}_2$ to algebraic-isomorphism questions of $\mathfrak{H}_1$ and $\mathfrak{H}_2$ (plus the behavior of C) results. We will discuss these questions in detail in later publications.

Appendix

1. Consider a ring $\boldsymbol{M}$ of class II_1 in $\mathfrak{H}$. Then $\boldsymbol{M}'$ is of class II_1 or II_∞. Normalize $D_{\boldsymbol{M}}$ and $D_{\boldsymbol{M}'}$ so as to have $D_{\boldsymbol{M}}(\mathfrak{H})=1$ and $C=1$, so $D_{\boldsymbol{M}'}(\mathfrak{H})=\alpha$ (cf. §1.1). By choosing an $n=1, 2, \cdots$ with $1/n\leqq\alpha$ and then an $\mathfrak{M}'\eta\boldsymbol{M}'$ with $D_{\boldsymbol{M}'}(\mathfrak{M}')=1/n$, apply R.O., §11.3, to form $\boldsymbol{M}_{(\mathfrak{M}')}$, $\boldsymbol{M}'_{(\mathfrak{M}')}$ in $\mathfrak{M}'$. Then $\boldsymbol{M}_{(\mathfrak{M}')}$ is algebraically-ring-isomorphic to $\boldsymbol{M}$, and $D_{\boldsymbol{M}(\mathfrak{M}')}(\mathfrak{M}')=D_{\boldsymbol{M}}(\mathfrak{H})=1$ (cf. R.O., Lemmas 11.3.3 and 11.3.6). So if we are interested in the algebraical properties of $\boldsymbol{M}$ only, we may assume without any loss in generality that $\boldsymbol{M}$, $\boldsymbol{M}'$ are in case II_1, II_1 and that $\alpha=1/n$, $n=1, 2, \cdots$. Or, in standard normalization $C=n$, $n=1, 2, \cdots$ (cf. R.O., Theorem X).

Now form the direct product of $\mathfrak{H}$ with an n-dimensional Euclidean space $E_n\oplus\mathfrak{H}$ as in §2.1, for the case $C>1$, and consider $\boldsymbol{M}^{(2)}$ in $E_n\oplus\mathfrak{H}$. The argument used in §2.1 shows that $\boldsymbol{M}^{(2)\prime}=R(\boldsymbol{N}^{(1)}, \boldsymbol{M}^{(2)\prime})$ and $C^{(2)}=C/n=1$ and $\boldsymbol{M}^{(2)}$ is ring isomorphic to $\boldsymbol{M}$.

So we have for $\boldsymbol{M}^{(2)}$, $\boldsymbol{M}^{(2)\prime}$ in the standard normalization $C=1$. If we are therefore interested in the algebraical properties of $\boldsymbol{M}$ only, we may even assume without any loss in generality that $C=1$ in standard normalization; that is, $\alpha=1$ in the normalization of §1.1. We will assume this in what follows.

It may be noticed concerning subrings that the metric of $Q(\boldsymbol{M})$ (cf. Definition 4.3.1) is determined by the algebra of $\boldsymbol{M}$ (cf. §2.2). A subring demands closure in the weak operator topology, but it will be shown elsewhere that for subrings of $\boldsymbol{M}$ weak, relative closure in $\boldsymbol{M}$ for the $[[X-Y]]$ metric is equivalent to closure in the weak topology.

2. Under these conditions there is an analogy between $\boldsymbol{M}$ and the matrices of a Euclidean space E_n, described by an interesting Lebesgue-Stieltjes-Radon measure in the plane.

We can use the results of §§4.2–4.4, and thus we may form the Hilbert-space $Q(\boldsymbol{M})$ which is isomorphic to $\mathfrak{H}$, and in which $\boldsymbol{M}$, $\boldsymbol{M}'$ are located by Theorem X.

Let $E(\lambda)$, $a<\lambda<b$, be a resolution of unity, all $E(\lambda)\epsilon\boldsymbol{M}$. Put $A=\int_{-\infty}^{\infty}\lambda dE(\lambda)$ (in the well known symbolic sense). For any Borel-set of real numbers S form

$$e_S(\lambda) = \begin{cases} 1 & \text{for } \lambda\epsilon S \\ 0 & \text{for } \lambda \notin S, \end{cases}$$

and

$$E(S) = e_S(A) = \int_{-\infty}^{\infty} e_S(\lambda)dE(\lambda) = \int_S dE(\lambda)$$

(symbolically). As $\overline{e_S(\lambda)}=e_S(\lambda)=(e_S(\lambda))^2$, so $e_S(A)=e_S(A)^*=(e_S(A))^2$; that is, $E(S)=e_S(A)\epsilon \boldsymbol{M}$ is a projection. (Cf. also Maeda, Journal of Science, Hiroshima University, Ser. A, vol. 4 (1934), pp. 57–91.)

Consider now point sets in the λ, μ-plane P and in particular sets of the form $S_1 \otimes S_2$:

$$(\lambda, \mu)\epsilon S_1 \otimes S_2 \quad \text{means} \quad \lambda \epsilon S_1, \ \mu \epsilon S_2,$$

S_1, S_2 being two Borel-sets of real numbers. Define for any $X\epsilon\boldsymbol{M}$ (or even $X\epsilon Q(\boldsymbol{M})$),

$$\begin{aligned} X_{S_1\times S_2} &= E(S_1)XE(S_2) \\ \omega(X;\, S_1 \otimes S_2) &= [[X_{S_1\times S_2}]]^2 = Tr((X_{S_1\times S_2})^* X_{S_1\times S_2}) \\ &= Tr(E(S_2)X^*E(S_1)\cdot E(S_1)XE(S_2)) \\ &= Tr(E(S_2)X^*\cdot E(S_1)XE(S_2)) \\ &= Tr(E(S_1)XE(S_2)\cdot E(S_2)X^*) \\ &= Tr(E(S_1)XE(S_2)X^*). \end{aligned}$$

The first expression for $\omega(X, S_1\otimes S_2)$ shows, that it is always $\geqq 0$, the last one, that it is a totally additive set-function of S_1 and of S_2.

Denote the set of all λ by Λ, then the above facts imply:

$$\begin{aligned} \omega(X; S_1 \otimes S_2) &\leqq \omega(X; S_1 \otimes S_2) + \omega(X; (\Lambda - S_1) \otimes S_2) \\ &= \omega(X; \Lambda \otimes S_2) \\ &\leqq \omega(X; \Lambda \otimes S_2) + \omega(X; \Lambda \otimes (\Lambda - S_2)) \\ &= \omega(X; \Lambda \otimes \Lambda) = [[X_{\Lambda\times\Lambda}]]^2 \\ &= [[1\cdot X\cdot 1]]^2 = [[X]]^2. \end{aligned}$$

So we have

LEMMA A. (i) *$\omega(X; S_1\otimes S_2)$ is defined for all linear Borel-sets S_1, S_2.* (ii) *It is totally additive in S_1 as well as in S_2.* (iii) *We have always*

$$0 \leqq \omega(X; S_1 \otimes S_2) \leqq [[X]]^2 = Tr(X^*X).$$

3. We can use Lemma A to define a Lebesgue-Stieltjes-Radon measure $\mu(X; T)$ for all plane Borel-sets $T(\subset P)$ with the help of $\omega(X; S_1\otimes S_2)$.

DEFINITION A. *If T is a plane Borel-set $(T\subset P)$, then consider all sequences of linear Borel-set $S_1^{(i)}$, $S_2^{(i)}$, $i=1, 2, \cdots$ (all $S_1^{(i)}$, $S_2^{(i)} \subset\Lambda$) for which*

$$(*) \qquad T \subset \sum_{i=1}^{\infty} S_1^{(i)} \otimes S_2^{(i)}.$$

For every such sequence form the (numerical) sum

$$(**) \qquad \sum_{i=1}^{\infty} \omega(X; S_1^{(i)} \otimes S_2^{(i)}).$$

Denote the g.l.b. of all numbers $(**)$ *by* $\mu(X; T)$.

We prove now

LEMMA B. (i) $\mu(X; T)$ *is defined for all plane Borel-sets* $T \subset P$.

(ii) *It is totally additive in* T.

(iii) *We have always*

$$0 \leqq \mu(X; T) \leqq [[X]]^2 = Tr(X^*X).$$

(iv) *In particular*

$$\mu(X; S_1 \otimes S_2) = \omega(X; S_1 \otimes S_2)$$

and especially

$$\mu(X, P) = [[X]]^2 = Tr(X^*X).$$

(i) is obvious. To prove (ii), observe first that our Definition A makes

$$\mu(X; T_1 + T_2 + \cdots) \leqq \mu(X; T_1) + \mu(X; T_2) + \cdots$$

obvious, so we need to prove

$$\mu(X; T_1 + T_2 + \cdots) \geqq \mu(X; T_1) + \mu(X; T_2) + \cdots$$

only when $T_i \cdot T_j = 0$ for $i \neq j$.

Even $\mu(X; T_1 + T_2) \geqq \mu(X; T_1) + \mu(X; T_2)$ suffices. Then finite induction gives $\mu(X; T_1 + T_2 + \cdots + T_n) \geqq \mu(X; T_1) + \mu(X; T_2) + \cdots + \mu(X; T_n)$, and so $\mu(X; T_1 + T_2 + \cdots) \geqq \mu(X; T_1 + \cdots + T_n) \geqq \mu(X; T_1) + \cdots + \mu(X; T_n)$ and as this holds for all $n = 1, 2, \cdots$, it implies $\mu(X; T_1 + T_2 + \cdots) \geqq \mu(X; T_1) + \mu(X; T_2) + \cdots$.

We will prove

$$(\S) \qquad \mu(X; T) = \mu(X; TU) + \mu(X; T - TU)$$

for all T, U this gives our above inequality if $T = T_1 + T_2$, $U = T_1$. Call a U, for which (§) holds for all plane Borel-sets T, following Carathéodory (cf. (4), pp. 246–252), measurable. We must show that all U are measurable.

If $U = S_1 \otimes S_2$, then $T \subset \sum_{i=1}^{\infty} S_1^{(i)} \otimes S_2^{(i)}$ implies $TU \subset \sum_{i=1}^{\infty} S_1^{(i)} S_1 \otimes S_2^{(i)} S_2$, $T - TU \subseteq \sum_{i=1}^{\infty} S_1^{(i)} S_1 \otimes (S_2^{(i)} - S_2^{(i)} S_2) + \sum_{i=1}^{\infty} (S_1^{(i)} - S_1^{(i)} S_1) \otimes S_2^{(i)}$. Thus our Definition A gives immediately (§) with $\geqq$ in it, and as $\leqq$ is obvious (cf. above) it proves (§). So all $U = S_1 \otimes S_2$ are measurable, and therefore in particular all plane intervals (=rectangles).

Now the measurable sets U form a Borel-ring (cf., for instance, (4), loc. cit.). These considerations apply literally to the present case. Thus every plane Borel-set U is measurable. This completes the proof.

We now prove (iii). $\mu(X; T) \geqq 0$ because all expressions (* *) in Definition A are $\geqq 0$. The relation $\mu(X; T) \leqq [[X]]^2 = Tr(X^*X)$ results by putting $S_1^{(1)} = S_2^{(1)} = \Lambda$ and all other $S_1^{(i)} = S_2^{(i)} = 0$.

To prove (iv), put $S_1^{(1)} = S_1$, $S_2^{(1)} = S_2$ and all other $S_1^{(i)} = S_2^{(i)} = 0$. This gives $\mu(X; S_1 \otimes S_2) \leqq \omega(X; S_1 \otimes S_2)$. So we must prove $\geqq$ only; that is,

$$S_1 \otimes S_2 \subset \sum_{i=1}^{\infty} S_1^{(i)} \otimes S_2^{(i)} \quad \text{implies} \quad \omega(X; S_1 \otimes S_2) \leqq \sum_{i=1}^{\infty} \omega(X; S_1^{(i)} \otimes S_2^{(i)}).$$

Considering the properties of $\omega(X; S_1 \otimes S_2)$ given in Lemma A, this implication follows literally as in the paper of Łomnicki and Ulam (Fundamenta Mathematicae, vol. 23 (1934), pp. 237–278; cf. also the lecture notes of the second-named author for the year 1934–1935).

So we have proved $\mu(X; S_1 \otimes S_2) = \omega(X; S_1 \otimes S_2)$. Put now $S_1 = S_2 = \Lambda$; then

$$\mu(X; P) = \mu(X; \Lambda \otimes \Lambda) = \omega(X; \Lambda \otimes \Lambda) = [[X]]^2 = Tr(X^*X).$$

results.

4. The plane measure $\mu(X; T)$ may be used to "locate" the "position" of X in the λ, μ-plane P. It gives the entire plane a total "measure" or "weight" $[[X]]^2$ (which is >0 if $X \neq 0$), and any part T of it correspondingly a $\mu(X; T)$ ($\geqq 0$). It plays the same role, as the sum of the absolute-value-squares of all matrix-elements in a certain area T in the matrix-scheme (which may be looked at as a region in the plane) for the matrices of a finite-dimensional Euclidean space E_n (that is, $\boldsymbol{M}$ in a case (I_n), $n = 1, 2, \cdots$).

We will analyse it more thoroughly in subsequent publications.

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