SOME CALCULATIONS OF THE RIEMANN ZETA-FUNCTION

By A. M. TURING

Introduction

IN June 1950 the Manchester University Mark 1 Electronic Computer was used to do some calculations concerned with the distribution of the zeros of the Riemann zeta-function. It was intended in fact to determine whether there are any zeros not on the critical line in certain particular intervals. The calculations had been planned some time in advance, but had in fact to be carried out in great haste. If it had not been for the fact that the computer remained in serviceable condition for an unusually long period from 3 p.m. one afternoon to 8 a.m. the following morning it is probable that the calculations would never have been done at all. As it was, the interval $2\pi.63^2 < t < 2\pi.64^2$ was investigated during that period, and very little more was accomplished.

The calculations were done in an optimistic hope that a zero would be found off the critical line, and the calculations were directed more towards finding such zeros than proving that none existed. The procedure was such that if it had been accurately followed, and if the machine made no errors in the period, then one could be sure that there were no zeros off the critical line in the interval in question. In practice only a few of the results were checked by repeating the calculation, so that the machine might well have made an error.

If more time had been available it was intended to do some more calculations in an altogether different spirit. There is no reason in principle why computation should not be carried through with the rigour usual in mathematical analysis. If definite rules are laid down as to how the computation is to be done one can predict bounds for the errors throughout. When the computations are done by hand there are serious practical difficulties about this. The computer will probably have his own ideas as to how certain steps should be done. When certain steps may be omitted without serious loss of accuracy he will wish to do so. Furthermore he will probably not see the point of the 'rigorous' computation and will probably say 'If you want more certainty about the accuracy why not just take more figures?' an argument difficult to counter. However, if the calculations are being done by an automatic computer one can feel sure that this kind of indiscipline

does not occur. Even with the automatic computer this rigour can be rather tiresome to achieve, but in connexion with such a subject as the analytical theory of numbers, where rigour is the essence, it seems worth while. Unfortunately, although the details were all worked out, practically nothing was done on these lines. The interval $1414 < t < 1608$ was investigated and checked, but unfortunately at this point the machine broke down and no further work was done. Furthermore this interval was subsequently found to have been run with a wrong error value, and the most that can consequently be asserted with certainty is that the zeros lie on the critical line up to $t = 1540$, Titchmarsh having investigated as far as 1468 (Titchmarsh (5)).

   This paper is divided into two parts. The first part is devoted to the analysis connected with the problem. All the results obtained in this part are likely to be applicable to any further calculations to the same end, whether carried out on the Manchester Computer or by any other means. The second part is concerned with the means by which the results were achieved on the Manchester Computer.

PART I. GENERAL

1. The Θ notation

   In analysis it is customary to use the notation $O\{f(x)\}$ to indicate 'some function whose ratio to $f(x)$ is bounded'. In the theory of a computation one needs a similar notation, but one is interested in the value of the bound concerned. We therefore use the notation $\Theta(\alpha)$ to indicate 'some number not greater in modulus than α'. The symbol Θ has been chosen partly because of a typographical similarity to 0, partly because of the relation with the use of ϑ to indicate 'a number less than 1'.

2. The approximate functional equation

   We shall use throughout the notation of Ingham (1) and Titchmarsh (3) without special definition. Our problem is to investigate the distribution of zeros of $\zeta(s)$ for large t. This will presumably depend on being able to calculate $\zeta(\sigma+it)$ or some closely associated function for large t, and σ not too far from $\frac{1}{2}$. We have to consider what formula to use and what associated function. For $\sigma > 1$ it is possible to use the defining series $\zeta(s) = \sum_1^\infty n^{-s}$, but this is too far from $\sigma = \frac{1}{2}$. For $0 < \sigma < 1$ there are other formulae which also involve calculating a number of terms of this series, but it is always necessary to take at least $t/2\pi$ terms.

   Alternatively one can use the functional equation

$$\zeta(s)\Gamma(\tfrac{1}{2}s)\pi^{-\frac{1}{2}s} = \zeta(1-s)\Gamma(\tfrac{1}{2}-\tfrac{1}{2}s)\pi^{-\frac{1}{2}+\frac{1}{2}s}$$

and take $t/2\pi$ terms of the series $\zeta(1-s) = \sum_{1}^{\infty} n^{s-1}$. Another possible method which might suggest itself is to calculate at a number of points in the region $\sigma > 1$ and extrapolate, but this again involves much the same amount of work. However, if one considers an interpolation formula involving both values from the region $\sigma > 1$ and from the region $\sigma < 0$ one finds that it is possible to calculate the function by taking only about $\sqrt{(t/2\pi)}$ terms of the series $\sum n^{-s}$ and an equal number from $\sum n^{s-1}$. This result is embodied in

THEOREM 1. *Let m and ξ be respectively the integral and non-integral parts of $\tau^{\frac{1}{2}}$ and*

$$\tau \geqslant 64,$$

$$\kappa(\tau) = \frac{1}{4\pi i}\log\frac{\Gamma(\frac{1}{4}+\pi i\tau)}{\Gamma(\frac{1}{4}-\pi i\tau)} - \tfrac{1}{4}\tau\log\pi,$$

$$Z(\tau) = \zeta(\tfrac{1}{2}+2\pi i\tau)e^{-2\pi i\kappa(\tau)},$$

$$\kappa_1(\tau) = \tfrac{1}{2}(\tau\log\tau-\tau-\tfrac{1}{2}),$$

$$h(\xi) = \frac{\cos 2\pi(\xi^2-\xi-\frac{1}{16})}{\cos 2\pi\xi}.$$

Then $Z(\tau)$ is real and

$$Z(\tau) = 2\sum_{n=1}^{m} n^{-\frac{1}{2}}\cos 2\pi\{\tau\log n-\kappa(\tau)\}+(-)^{m+1}\tau^{-\frac{1}{4}}h(\xi)+\Theta(1\cdot09\tau^{-\frac{3}{4}}),$$

$$\kappa(\tau) = \kappa_1(\tau)+\Theta(0\cdot006\tau^{-1}).$$

It will be seen that $Z(\tau)$ may also be defined as being $\zeta(\frac{1}{2}+2\pi i\tau)$ for τ real, $0 < \tau < 1$, and elsewhere by analytic continuation. The theorem could be proved by the argument outlined above, but is more conveniently proved by the method given as Theorem 22 of Titchmarsh (3). The numerical details are given in Titchmarsh (4). A more elaborate remainder is given there and is valid for $\tau \geqslant \delta$. The validity of the remainder given here follows trivially from it.

This formula can only give a limited accuracy, although it is nearly always adequate. If greater accuracy is required the formula given in Turing (6) may be applied. These agree with Titchmarsh's expression in the sum of m terms, but $h(\xi)$ is replaced by another sum.

The function $h(\xi)$ is troublesome to calculate, largely because the numerator and denominator both vanish at $\xi = \frac{1}{4}$ and $\xi = \frac{3}{4}$, so that a special method would have to be applied for the neighbourhood of these points. The alternative of using a table and interpolation suggests itself. This possibility quickly leads to the suggestion of replacing the function by some polynomial which approximates it well enough in the region concerned.

In fact the polynomial $0 \cdot 373 + 2 \cdot 160(\xi - \tfrac{1}{2})^2$ is quite adequate, for we have

THEOREM 2. *If* $|\xi - \tfrac{1}{2}| < \tfrac{1}{2}$ *we have*

$$h(\xi) = 0 \cdot 373 + 2 \cdot 160(\xi - \tfrac{1}{2})^2 + \Theta(0 \cdot 0153)$$

and if $|\xi - \tfrac{1}{2}| < 0 \cdot 53$ *we have* $h(\xi) = 0 \cdot 373 + 2 \cdot 160(\xi - \tfrac{1}{2})^2 + \Theta(0 \cdot 0243)$.

This result is rather unexpectedly troublesome to prove. Its proof will be given in slightly more detail than it deserves, treating it as an example of 'rigorous computation'.

It may be said: 'As this is a purely numerical result surely it can be proved by straight computation.' This is in effect what is done, but it is not possible to avoid theory entirely. The function was calculated for the values $0, \tfrac{1}{30}, \tfrac{2}{30}, \ldots, \tfrac{16}{30}, \tfrac{17}{30}$ of $\xi - \tfrac{1}{2}$ with an error $\Theta(10^{-4})$, and was found to satisfy the inequality with some margin. But nothing further can be deduced even if the differences are taken into account, unless something is known about the general behaviour of the function. An upper bound for the second derivative would be sufficient, but the labour of even the formal differentiation is discouraging, and the accidental singularities make the situation considerably worse. However, the function is integral, and it is therefore possible to obtain an inequality for any derivative by means of Cauchy's integral formula, taken in combination with an inequality for the function itself on a suitable contour. The method actually applied will be seen to be very similar to this. Instead of Cauchy's integral formula we use

$$f(\xi) - P(\xi)$$

$$= \frac{(\xi - \xi_1)(\xi - \xi_2)(\xi - \xi_3)(\xi - \xi_4)}{2\pi i} \int \frac{f(u)\, du}{(u - \xi)(u - \xi_1)(u - \xi_2)(u - \xi_3)(u - \xi_4)},$$

where the function $f(\xi)$ is regular inside the anti-clockwise contour of integration, and $P(\xi)$ is the cubic polynomial agreeing with $f(\xi)$ at $\xi_1, \xi_2, \xi_3, \xi_4$. This equation follows from the fact that the right-hand side vanishes at the points $\xi_1, \xi_2, \xi_3, \xi_4$ and is of the form of $f(\xi)$ added to a cubic polynomial. We actually take the contour to be the square whose vertices are $\tfrac{1}{2} \pm i, \tfrac{1}{2} \pm 1$. One can prove without difficulty that $|h(\xi)| < \cosh \pi$ on this square and that if $f(\xi) = h(\xi) - 0 \cdot 373 - 2 \cdot 160(\xi - \tfrac{1}{2})^2$ then $|f(\xi)| < 14 \cdot 3$ on the square. Taking $\xi_1, \xi_2, \xi_3, \xi_4$ to be of form $n/30$ and two of them to be on either side of ξ one easily deduces $|f(\xi) - P(\xi)| < 0 \cdot 0033$ if $|\xi - \tfrac{1}{2}| < 0 \cdot 053$, and a consideration of the values at the calculated points and the differences gives $|P(\xi)| < 0 \cdot 021$ if $|\xi - \tfrac{1}{2}| < 0 \cdot 53$ and $|P(\xi)| < 0 \cdot 012$ if $|\xi - \tfrac{1}{2}| < \tfrac{1}{2}$. It will be seen that the use of this approximation to $h(\xi)$ gives an extra error in $Z(\tau)$ of the order of $\tau^{-\frac{1}{4}}$ whereas Titchmarsh's formula has an error

of order only $\tau^{-\frac{1}{4}}$; but the errors are not equal until τ is over 2000, and both are then quite small. In the actual calculation described in Part II there were other errors of order as large as $\tau^{\frac{1}{4}}$.

Titchmarsh's formula as stated is valid only when the right value of m is used, i.e. if $\tau^{\frac{1}{2}} = m+\xi$ and $|\xi-\frac{1}{2}| \leqslant \frac{1}{2}$. This may be inconvenient as one may occasionally wish to go a little outside the range. One may justify doing so by means of

THEOREM 3. *Theorem 1 is valid with the error $\Theta(1{\cdot}09\tau^{-\frac{1}{4}})$ replaced by $\Theta(1{\cdot}15m^{-\frac{1}{4}})$ if the condition that m and ξ be the integral and non-integral parts of $\tau^{\frac{1}{2}}$ is replaced by the condition that m be an integer and*

$$\tau^{\frac{1}{2}} = m+\xi, \qquad |\xi-\tfrac{1}{2}| < 0{\cdot}53.$$

The new error introduced is

$$(-)^m \tau^{-\frac{1}{4}}\left[\frac{\cos 2\pi(\xi^2-\xi-\frac{1}{16})}{\cos 2\pi\xi} + \frac{\cos 2\pi\{(\xi-1)^2-(\xi-1)-\frac{1}{16}\}}{\cos 2\pi(\xi-1)}\right] -$$
$$-2(m+1)^{-\frac{1}{4}}\cos 2\pi[(m+\xi)^2\log(m+1)-\kappa\{(m+\xi)^2\}]]$$

in the case that $1 < \xi < 1{\cdot}03$. But we have

$$\frac{\cos 2\pi(\xi^2-\xi-\frac{1}{16})}{\cos 2\pi\xi} + \frac{\cos 2\pi\{(\xi-1)^2-(\xi-1)-\frac{1}{16}\}}{\cos 2\pi(\xi-1)} = 2\cos 2\pi(\xi^2-2\xi-\tfrac{1}{16}).$$

Also if we put

$$j(\xi) = (m+\xi)^2\log(m+1)-\kappa_1\{(m+\xi)^2\}-\tfrac{1}{2}m^2-m+\tfrac{1}{2}+\xi^2-2\xi-\tfrac{1}{16},$$

then $j(\xi)$ and its first two derivatives vanish at $\xi = 1$ and $j'''(\xi) = \dfrac{-2}{m+\xi}$.

Hence by the mean value theorem $|j(\xi)| < \dfrac{(\xi-1)^3}{3(m+1)}$. Using also

$$|\kappa\{(m+\xi)^2\}-\kappa_1\{(m+\xi)^2\}| < \frac{0{\cdot}006}{(m+1)^2}$$

we see that the new error is at most

$$4\pi(m+1)^{-\frac{1}{4}}\left(\frac{(\xi-1)^3}{3(m+1)}+0{\cdot}006(m+1)^{-2}\right)+2|(m+\xi)^{-\frac{1}{4}}-(m+1)^{-\frac{1}{4}}|,$$

which is less than $0{\cdot}052(m+1)^{-\frac{1}{4}}$ since $m \geqslant 7$, $|\xi-1| < 0{\cdot}03$. A similar argument applies for the case $-0{\cdot}03 < \xi < 0$.

3. Principles of the calculations

We may now consider that with the aid of Theorems 1, 2, 3 we are in a position to calculate $Z(\tau)$ for any desired τ. How can we use this to obtain results about the distribution of the zeros? So long as the zeros are on the critical line the result is clearly applicable to enable us to find their position

to an accuracy limited only by the accuracy to which we can find $Z(\tau)$. If there are zeros off the line we can find their position as follows. Suppose we have calculated $Z(\tau)$ for $\tau_1, \tau_2, ..., \tau_N$. Then we can approximate $Z(\tau)$ in the neighbourhood of these points by means of the polynomial $P(\tau)$ agreeing with $Z(\tau)$ at these points. The accuracy of the approximation may be determined as in Theorem 2. Suppose that in this way we find that $|Z(\tau)-P(\tau)| < \epsilon$ and $|P''(\tau)| < \epsilon'$ for $|\tau-\tau'| < \delta$ and that $|P(\tau')| < \epsilon''$ and $|P'(\tau')-a| < \epsilon'''$, then we see that

$$|Z(\tau)-a(\tau-\tau')| < \epsilon+\epsilon''+\tfrac{1}{2}\epsilon'\delta^2+\epsilon'''\delta$$

for $|\tau-\tau'| < \delta$, and we may conclude by Rouché's theorem that $Z(\tau)$ has a zero within this circle if $|a| > \epsilon'''+\tfrac{1}{2}\epsilon'\delta+\dfrac{\epsilon+\epsilon''}{\delta}$. This, however, is a tiresome

procedure, and should be avoided unless we have good reason to believe that such a zero is really present. If there are any such zeros we may expect that the first ones to appear will be rather close to the critical line, and they will show themselves by the curve of $Z(\tau)$ approaching the zero line and receding without crossing it: in other words by behaving like a quadratic expression with complex zeros. In the absence of such behaviour we wish to prove that there are no complex zeros without using this interpolation procedure. Let us suppose that we have been investigating the range $T_0 < \tau < T_1$ and that we have found a certain number of real zeros in the interval. If by some means we can determine the total number of zeros in the rectangle $|\mathscr{I}\tau| < 2, T_0 < \mathscr{R}\tau < T_1$ (say) and find it to equal the number of changes of sign found, then we can be sure that there were no zeros off the critical line in this rectangle. This total number of zeros can be determined by calculating the function at various points round the rectangle. This might normally be expected to involve even more work than the calculations on the critical line. Fortunately, with the function concerned, the calculations on the lines $|\mathscr{I}\tau| = 2$ are not necessary. It is well known that the change in the argument of $Z(\tau)$ on these lines can be calculated to within $\tfrac{1}{2}\pi$ in terms of the gamma function. It remains to find the change on the lines $\mathscr{R}\tau = T_0$ and $\mathscr{R}\tau = T_1$. In principle this could be done by approximating $Z(\tau)$ with a polynomial, using an interpolation formula based on values calculated on the critical line. Since this interpolation procedure is necessary only at the ends of the interval investigated this would be a considerably smaller burden than the repeated application of it throughout the interval required by the method previously suggested. It will, however, be shown later on that even this application of the interpolation procedure is unnecessary, but for the sake of argument we will suppose for the moment that it is done. We may suppose then that the

total number of zeros in the rectangle is known. If this differs from the number of changes of sign which have been found, then the deficit must be ascribed to a combination of four causes. Some may be due to pairs of complex zeros, some to pairs of changes of sign which were missed due to insufficiently many values $Z(\tau)$ being calculated, some to the accuracy of some of the values being inadequate to establish that changes of sign had occurred. Finally there may be some multiple zeros on the critical line. Each source accounts for an even number of zeros provided that the accuracy is sufficient for there to be no doubt about the signs of $Z(T_0)$ and $Z(T_1)$. By calculating further values and increasing the accuracy we can remove some of the discrepancies, but we cannot do anything about the multiple zeros by mere calculation. Assuming that there are no multiple zeros it is possible in principle to make sure that all the real zeros have been found by calculating $Z(\tau)$ at a sufficient number of real points, but the number of points would be many more than are required for finding all the real zeros. It is better to find the complex zeros in the manner already described.

   To summarize. The method recommended is first to find the total number of zeros in the rectangle by methods to be described later. Then to calculate the function at sufficient points to account for all the zeros, either by changes of sign or as complex zeros determined by the use of Rouché's theorem. We know no way of dealing with multiple zeros, and simply hope that none are present.

4. Evaluation of $N(t)$

   For reasons explained in the last section it is desirable to be able to determine the number of zeros of $Z(\tau)$ in a region $T_0 < \tau < T_1$. In practice this is best done by determining separately the numbers in the regions $0 < \mathcal{R}\tau < T_0$ and $0 < \mathcal{R}\tau < T_1$. If we write $\pi S(t)$ for the argument of $\zeta(\tfrac{1}{2}+it)$ obtained by continuation along a line parallel to the real axis from $\infty+it$, where the argument is defined to be zero, we have

$$N(T) = 2\kappa\left(\frac{T}{2\pi}\right)+1+S(T),$$

where $N(T)$ is the number of zeros of $\zeta(\sigma+it)$ in the region $0 < t < T$. The problem is thus reduced to the determination of $S(T)$. If the sign of $Z\left(\dfrac{T}{2\pi}\right)$ is known, the value of $S(T)$ is known modulo 2. It is not therefore necessary to obtain $S(T)$ to any great accuracy. The principle of the method is that if $S_1(t) = \displaystyle\int_0^t S(u)\, du$ then $S_1(t)$ is known to be $O(\log t)$. If then the positions of the zeros are known in an interval of length L, $S(t)$ will be known

modulo 2 in this interval, the additive even integer being the same through-out. Hence $S_1(t_0+L)-S_1(t_0)$ will be known modulo $2L$, and if L is suffi-ciently large this will determine it exactly and thereby determine $S(t)$ throughout the interval. In order to complete the details of this argument it is necessary to replace the O result by a Θ result. It would also be desirable to try and arrange to manage with very limited knowledge of the positions of the zeros.

THEOREM 4. *If* $t_2 > t_1 > 168\pi$, *then*

$$S_1(t_2)-S_1(t_1) = \Theta\left(2{\cdot}30+0{\cdot}128\log\frac{t_2}{2\pi}\right).$$

The proof of this follows Theorem 40 of Titchmarsh (3). The essential step is

LEMMA 1. *If* $t_2 > t_1 > 0$, *then*

$$\pi\{S_1(t_2)-S_1(t_1)\} = \mathscr{R}\int_{\frac{1}{2}+it_2}^{\infty+it_2} \log\zeta(s)\,ds - \mathscr{R}\int_{\frac{1}{2}+it_1}^{\infty+it_1}\log\zeta(s)\,ds.$$

We apply Cauchy's theorem to $\log\zeta(s)$ and the rectangle with vertices $\frac{1}{2}+it_1$, $\frac{1}{2}+it_2$, $R+it_2$, $R+it_1$ and appropriate detours round the branch lines from zeros within the rectangle. The real part of the integral is

$$-\mathscr{R}\int_{\frac{1}{2}+it_2}^{R+it_2}\log\zeta(s)\,ds + \int_{\frac{1}{2}+it_1}^{\frac{1}{2}+it_2}\arg\zeta(s)(-i\,ds)+$$

$$+\mathscr{R}\int_{\frac{1}{2}+it_1}^{R+it_1}\log\zeta(s)\,ds - \int_{R+it_1}^{R+it_2}\arg\zeta(s)(-i\,ds),$$

no contribution arising from the detours. The last of these integrals tends to 0 as $R\to\infty$ and the second is $\pi\{S_1(t_2)-S_1(t_1)\}$.

LEMMA 2. *If* $\tau \geqslant 64$, *we have*

$$|\zeta(\tfrac{1}{2}+2\pi i\tau)| < 4\tau^{\frac{1}{4}}.$$

Since $|h(\zeta)| < 0{\cdot}95$ we have, by Theorem 1,

$$|\tau(\tfrac{1}{2}+2\pi i\tau)| = |Z(\tau)| < 2\sum_{1\leqslant r\leqslant\tau^{\frac{1}{2}}} r^{-\frac{1}{2}}+1{\cdot}2\tau^{-\frac{1}{4}}+0{\cdot}95\tau^{-\frac{1}{4}}$$

and

$$\sum_{1\leqslant r\leqslant\tau^{\frac{1}{2}}} r^{-\frac{1}{2}} < 1+\int_1^{\tau^{\frac{1}{2}}} x^{-\frac{1}{2}}\,dx = 2\tau^{\frac{1}{4}}-1.$$

LEMMA 3.

$$|\zeta(1\cdot25+it)| < \zeta(1\cdot25) < 4\cdot6,$$

$$\left|\int_{1\cdot25+it}^{\infty} \log\zeta(s)\,ds\right| < \int_{1\cdot25}^{\infty} \log\zeta(\sigma)\,d\sigma < 1\cdot17,$$

$$\left|\frac{\zeta'}{\zeta}(1\cdot5+it)\right| < \frac{\zeta'}{\zeta}(1\cdot5) < 2\cdot62,$$

$$\left|\int_{1\cdot5+it}^{2\cdot5+it} \log\zeta(s)\,ds\right| < \int_{1\cdot5}^{2\cdot5} \log\zeta(\sigma)\,d\sigma < 0\cdot548,$$

$$\left|\int_{1\cdot5+it}^{\infty} \log\zeta(s)\,ds\right| < \int_{1\cdot5}^{\infty} \log\zeta(\sigma)\,d\sigma < 0\cdot997,$$

$$\tfrac{1}{2}\log\pi > 0\cdot572.$$

These results are all based on the tables in Jahnke–Emde (2), p. 323. An error of two units in the last place is assumed. To the extent that we do not know how these tables were obtained we depart from the principles of the 'rigorous computation'.

LEMMA 4. *If* $\tfrac{1}{2} < \sigma < \tfrac{5}{4}$ *and* $t > 168\pi$, *then*

$$|\zeta(s)| < 4\cdot5t^{3/8-\frac{1}{2}\sigma}.$$

Consider $f(s)$:

$$f(s) = \zeta(s)\left(\frac{s-\frac{1}{2}}{i}\right)^{-3/8+\frac{1}{2}\sigma}\exp\left[\frac{-4\pi i}{s-\frac{1}{2}-127\cdot5\pi i}\right].$$

Now $\left|\exp\left[\dfrac{-4\pi i}{s-\frac{1}{2}-127\cdot5\pi i}\right]\right| = \exp\left[\dfrac{-4\pi(t-127\cdot5\pi)}{(t-127\cdot5\pi)^2+(\sigma-\frac{1}{2})^2}\right].$

Hence, by Lemma 3, $|f(s)| < 4$ on the line $\sigma = \frac{1}{2}$. Elsewhere, if $\tfrac{1}{2} < \sigma < \tfrac{5}{4}$, $t > 128\pi$, we have

$$\log\left|\left(\frac{s-\frac{1}{2}}{i}\right)^{-3/8+\frac{1}{2}s}\right| = \tfrac{1}{2}(-\tfrac{3}{8}+\tfrac{1}{4}\sigma)\log\{t^2+(\sigma-\tfrac{1}{2})^2\}+\tfrac{1}{4}t\tan^{-1}\frac{\sigma-\frac{1}{2}}{t}$$

$$\leqslant -\tfrac{1}{32}\log t^2+\tfrac{3}{16}.$$

Hence on the line $\sigma = \frac{5}{4}$, $t \geqslant 128\pi$ we have

$$|f(s)| < \zeta(\tfrac{5}{4})e^{\frac{3}{16}}(128\pi)^{-\frac{1}{16}} < 4.$$

Finally on the line $t = 128\pi$, $\tfrac{1}{2} \leqslant \sigma \leqslant \tfrac{5}{4}$ we have

$$|f(s)| < |\zeta(s)|\exp\left[\frac{-2\pi^2}{\frac{1}{4}\pi^2+\frac{9}{16}}\right] \quad\text{and}\quad |\zeta(s)| < (128\pi)^{\frac{1}{4}}$$

by equation (8) on p. 27 of Ingham (1). Hence $|f(s)| < 4$ on the whole boundary of the strip $t \geqslant 128\pi$, $\tfrac{1}{2} < \sigma < \tfrac{5}{4}$, and, since certainly $f(s) = O(t)$,

we have $|f(s)| < 4$ throughout the strip by the Phragmén–Lindelöf theorem. From this it follows that

$$|\zeta(s)| < 4e^{0\cdot 1} \left| \left(\frac{s-\frac{1}{2}}{i}\right)^{\frac{3}{8}-\frac{1}{4}s} \right|$$

$$< 4\cdot 5 t^{\frac{3}{8}-\frac{1}{4}\sigma} \quad \text{for} \quad t > 168\pi.$$

The purpose of the factor $\exp\left[\dfrac{-4\pi i}{s-\frac{1}{2}-127\cdot 5\pi i}\right]$ is merely to enable us to do without accurate knowledge of $\zeta(s)$ over the end of the strip.

LEMMA 5. *If* $t > 168\pi$, *then*

$$\mathscr{R} \int_{\frac{1}{2}+it}^{\infty+it} \log \zeta(s)\, ds \leqslant 2\cdot 30 + 0\cdot 12 \log t.$$

For $\quad \mathscr{R} \displaystyle\int_{\frac{1}{2}+it}^{\infty+it} \log \zeta(s)\, ds \leqslant 1\cdot 17 + \int_{0\cdot 5}^{1\cdot 25} \log|\zeta(s)|\, ds,$ by Lemma 3,

$$\leqslant 1\cdot 17 + \int_{0\cdot 5}^{1\cdot 25} \{\log 4\cdot 5 + (\tfrac{3}{8} - \tfrac{1}{4}\sigma)\log t\}\, d\sigma,$$

$$\text{by Lemma 4,}$$

$$= 1\cdot 17 + 0\cdot 75 \log 4\cdot 5 + \tfrac{15}{128} \log t$$

$$< 2\cdot 30 + 0\cdot 12 \log t.$$

It is certainly possible to improve the coefficient of $\log t$ in this result at the expense of the constant. The coefficient of $\log t$ could be reduced at any rate to $0\cdot 052$ using results stated on pp. 25, 26 of Titchmarsh (3).

LEMMA 6.

$$\frac{\zeta(s)\zeta(s+2)}{\{\zeta(s+1)\}^2} = \frac{s^2}{s^2-1} \frac{\{\Gamma(\frac{1}{2}s+\frac{3}{2})\}^2}{\Gamma(\frac{1}{2}s+1)\Gamma(\frac{1}{2}s+2)} \prod_{\rho} \frac{(s-\rho)(s-\rho+2)}{(s-\rho+1)^2},$$

where the product is over the non-trivial zeros of the zeta-function.

This is an immediate consequence of the Weierstrass product for the zeta-function.

LEMMA 7. *If* $k = 1\cdot 49$, $\mathscr{R}a \geqslant 0$, *then*

$$\mathscr{R}\left(\psi(a)+\frac{k}{a}\right) = \mathscr{R}\left[\int_{a-1}^{a} \log\frac{z}{z+1}\, dz + \frac{k}{a}\right] \geqslant 0.$$

It is easily seen that if $\mathscr{R}a = 0$ then $\mathscr{R}\psi(a) = \mathscr{R}(k/a) = 0$. Also that $\mathscr{R}(k/a) \geqslant 0$ for $\mathscr{R}a \geqslant 0$ and that $\psi(a)$ is continuous at 0. $\psi(a)+(k/a) \to 0$ as $a \to \infty$. Hence applying the maximum modulus principle (or rather, the minimum real part principle) to $\psi(a)+(k/a)$ and various regions

$$\mathscr{R}a \geqslant 0, \qquad 0 < \epsilon < |a| < R,$$

we see by allowing $\epsilon \to 0$, $R \to \infty$ that the minimum real part must be achieved either on the boundary $\mathscr{R}a = 0$ or on the real axis (which may be a singularity). It only remains therefore to establish our inequality for the real axis. At any stationary point we must have

$$0 = \mathscr{R}\left(\psi'(a) - \frac{k}{a^2}\right) = -\log\left|1 - \frac{1}{a^2}\right| - \frac{k}{a^2}.$$

This equation only has solutions near to $0\cdot91$ and $1\cdot2$ both of which correspond to minima of $\psi(a) + (k/a)$. There is no ordinary maximum separating them, but there is a singularity at $a = 1$. By computations near to these minima, and knowledge of an upper bound for the second derivative of the function in intervals enclosing them, one can show that the values at the minima are positive. The value at the lesser minimum (near $0\cdot91$) is about $0\cdot0087$. Hence $\psi(a) + (k/a) > 0$ on the real axis as required.

LEMMA 8. If $\mathscr{R}z > 0$, then

$$\frac{\Gamma'(z)}{\Gamma(z)} = \log z - \frac{1}{2z} + \Theta\left(\frac{2}{\pi^2|(\mathscr{I}z)^2 - (\mathscr{R}z)^2|}\right).$$

We use the formula

$$\frac{\Gamma'(z)}{\Gamma(z)} = \log z - \frac{1}{2z} + 2\int_0^\infty \frac{u\,du}{(u^2+z^2)(e^{2\pi u} - 1)}$$

and take the line of integration to be $\mathscr{R}u = \mathscr{I}u = v > 0$. Then

$$\left|\frac{u}{e^{2\pi u} - 1}\right| < \frac{e^{-\pi v}}{\pi\sqrt{2}}, \qquad |u^2 + z^2| > |(\mathscr{I}z)^2 - (\mathscr{R}z)^2|.$$

No poles are encountered in the change of line of integration since $\mathscr{R}z > 0$.

LEMMA 9. If $t > 50$, then

$$-\mathscr{R}\int_{\frac{1}{2}+it}^\infty \log\zeta(s)\,ds < 4\cdot9 + 0\cdot245\log\frac{t}{2\pi}.$$

We have

$$\mathscr{R}\int_{\frac{1}{2}+it}^\infty \log\zeta(s)\,ds = \mathscr{R}\int_{\frac{1}{2}+it}^\infty \log\frac{\zeta(s)\zeta(s+2)}{\{\zeta(s+1)\}^2}\,ds +$$

$$+ \mathscr{R}\int_{\frac{1}{2}+it}^\infty \log\zeta(s)\,ds + \mathscr{R}\int_{\frac{3}{2}+it}^{\frac{5}{2}+it} \log\zeta(s)\,ds$$

$$\geqslant \mathscr{R}\int_{\frac{1}{2}+it}^\infty \log\frac{\zeta(s)\zeta(s+2)}{\{\zeta(s+1)\}^2}\,ds - 1\cdot545,$$

by Lemma 3.

Also, by Lemma 6,

$$\int_{\frac{1}{2}+it}^{\infty} \log \frac{\zeta(s)\zeta(s+2)}{\{\zeta(s+1)\}^2}\, ds = \sum_{\rho} \int_{\frac{1}{2}+it}^{\frac{3}{2}+it} \log \frac{s-\rho}{s-\rho+1}\, ds -$$

$$- \int_{\frac{1}{2}+it}^{\frac{3}{2}+it} \log \frac{\Gamma(\frac{1}{2}s+1)}{\Gamma(\frac{1}{2}s+\frac{3}{2})}\, ds + \int_{\frac{1}{2}+it}^{\frac{3}{2}+it} \log \frac{s}{s-1}\, ds.$$

Now if $\frac{1}{2} < \sigma < \frac{3}{2}$, then $|s-1| < |s|$, and therefore

$$\mathcal{R} \int_{\frac{1}{2}+it}^{\frac{3}{2}+it} \log \frac{s}{s-1}\, ds \geqslant 0.$$

Also $$\mathcal{R} \int_{\frac{1}{2}+it}^{\frac{3}{2}+it} \log \frac{\Gamma(\frac{1}{2}s+1)}{\Gamma(\frac{1}{2}s+\frac{3}{2})}\, ds = -\tfrac{1}{2}\mathcal{R}\, \frac{\Gamma'}{\Gamma}(\tfrac{1}{2}it+\sigma)$$

for some σ, $\frac{5}{4} < \sigma < \frac{9}{4}$, by the mean value theorems,

$$\leqslant -\tfrac{1}{4}\log(\tfrac{1}{4}t^2+\tfrac{25}{16}) - \frac{\frac{5}{2}}{t^2+\frac{25}{4}} + \frac{2}{\pi^2(\frac{1}{4}t^2-\frac{81}{16})}$$

by Lemma 8,

$$< -\tfrac{1}{2}\log \tfrac{1}{2}t \quad \text{(since } t > 50\text{)}$$

$$< -\tfrac{1}{2}\log \frac{t}{2\pi} - 0{\cdot}572, \text{ by Lemma 3.}$$

Finally

$$\mathcal{R} \sum_{\rho} \int_{\frac{1}{2}+it}^{\frac{3}{2}+it} \log \frac{s-\rho}{s-\rho+1}\, ds \geqslant -1{\cdot}49 \mathcal{R} \sum_{\rho} \frac{1}{it-\rho+\frac{3}{2}},$$

by Lemma 7,

$$=\, -1{\cdot}49\mathcal{R}\left[\frac{\zeta'}{\zeta}(it+\tfrac{3}{2}) - \tfrac{1}{2}\log \pi + \tfrac{1}{2}\frac{\Gamma'}{\Gamma}(\tfrac{1}{2}it+\tfrac{7}{4})\right],$$

by the Mittag–Leffler series for $\frac{\zeta'}{\zeta}(s)$,

$$\geqslant -1{\cdot}49\left[\mathcal{R}\, \frac{\zeta'}{\zeta}(it+\tfrac{3}{2}) - \tfrac{1}{2}\log \pi + \tfrac{1}{4}\log(\tfrac{1}{4}t^2+\tfrac{49}{16}) - \frac{7}{4t^2+49}\right],$$

by Lemma 8,

$$\geqslant -1{\cdot}49\left[\tfrac{1}{2}\log \frac{t}{2\pi} + 2{\cdot}63\right],$$

using Lemma 3 and $t > 50$.

Combining these results gives the asserted inequality.

A variant of this method enables us to reduce the coefficient of τ to $\frac{1}{2}\log 2 - \frac{1}{4} + \epsilon$, e.g. to 0·097, at the expense of the constant term.

Theorem 4 follows at once from Lemmas 1, 5, 9.

It is convenient to replace Theorem 4 by a similar result with $\kappa(\tau)$ or $\kappa_1(\tau)$ as the independent variable. This is because $\kappa(\tau)$ describes the 'expected' position of the zeros, and is therefore more informative than τ.

LEMMA 10. *If* $\tau_1 > 84$, *then*

$$\int_{\tau_1}^{\tau_2} S(2\pi\tau)\, d\kappa_1(\tau) = \Theta\{0\cdot184\log\tau_2 + 0\cdot0103(\log\tau_2)^2\}.$$

For

$$\int_{\tau_1}^{\tau_2} S(2\pi\tau)\, d\kappa_1(\tau) = \frac{1}{2\pi}\kappa_1'(\tau_1)\{S_1(2\pi\tau_2) - S_1(2\pi\tau_1)\} -$$

$$- \frac{1}{2\pi}\int_{\tau_1}^{\tau_2}\{S_1(2\pi\tau_2) - S_1(2\pi\tau)\}\kappa_1''(\tau)\, d\tau$$

$$= \Theta\left[\frac{2\cdot30 + 0\cdot128\log\tau_2}{2\pi}\left\{\kappa_1'(\tau_1) + \int_{\tau_1}^{\tau_2} |\kappa_1''(\tau)|\, d\tau\right\}\right]$$

$$= \Theta\left(\frac{2\cdot30 + 0\cdot128\log\tau_2}{4\pi}\log\tau_2\right).$$

THEOREM 5. *Let*

$$64 < \tau_{-R_1} < \tau_{1-R_1} < \cdots < \tau_0 < \cdots < \tau_{R_1-1} < \tau_{R_1}$$

and $\kappa(\tau_r) = c_r$, $\delta_r = c_r - c_0 - \frac{1}{2}r$, $\delta_{R_1} = \delta_{-R_1} = 0$, *and* $Z(\tau_r)Z(\tau_{r+1}) < 0$ *if* $1 - R_1 \leqslant r \leqslant R_2 - 2$, $\tau_{-R_1} > 84$. *Then*

$$-\frac{1}{2} + \frac{2}{R_1}\sum_{r=1-R_1}^{-1}\delta_r - \frac{0\cdot006}{\tau_{-R_1}} - \frac{2}{R_1}\{0\cdot184\log\tau_0 + 0\cdot0103(\log\tau_0)^2\}$$

$$\leqslant N(2\pi\tau_0) - 2c_0 - 1$$

$$\leqslant \frac{1}{2} + \frac{2}{R_2}\sum_{r=1}^{R_2-1}\delta_r + \frac{0\cdot006}{\tau_0} + \frac{2}{R_1}\{0\cdot184\log\tau_{R_2} + 0\cdot0103(\log\tau_{R_2})^2\}.$$

In the interval (τ_r, τ_{r+1}) we have $N(2\pi\tau) \geqslant N(2\pi\tau_0) + r$ if $0 \leqslant r \leqslant R_2 - 1$ and therefore

$$\int_{\tau_0}^{\tau_R} N(2\pi\tau)\, d\kappa_1(\tau) \geqslant \sum_{r=0}^{R-1}(c_{r+1} - c_r)\{N(2\pi\tau_0) + r\}$$

$$= \frac{1}{2}R[N(2\pi\tau_0) + \frac{1}{2}(R-1)] - \sum_{r=1}^{R}\delta_r.$$

Also

$$\int_{\tau_0}^{\tau_R} \{2\kappa(\tau)+1\}\, d\kappa_1(\tau) = c_R - c_0 + (c_R^2 - c_0^2) + \Theta\left(\frac{0\cdot006(c_R - c_0)}{\tau_0}\right)$$

$$= \tfrac{1}{2}R(1 + 2c_0 + \tfrac{1}{2}R) + \Theta\left(\frac{0\cdot003R}{\tau_0}\right).$$

The second inequality now follows since $S(2\pi\tau) = N(2\pi\tau) - 1 - 2\kappa(\tau)$ and the first may be proved similarly.

Example. It is known by computation that within distance $0\cdot05$ of each of the half-integers $547\tfrac{1}{2}$ to $554\tfrac{1}{2}$ there lie values of κ such that the corresponding value of Z has the same sign as $\cos 2\pi\kappa$. It is required to show that if τ_0 is that one of the points concerned which is within $0\cdot05$ of 551 then $N(2\pi\tau_0) = 1103$.

We take the values concerned to be $\tau_{-7}, \tau_{-6},\ldots, \tau_7$ in Theorem 5, and define τ_{-8}, τ_8 to satisfy $\delta_{-8} = \delta_8 = 0$. Then $|\delta_r| < 0\cdot1$ for each r, $-7 \leqslant r \leqslant 7$. The conditions of Theorem 5 are satisfied and it gives

$$-1\cdot0 \leqslant N(2\pi\tau_0) - 2c_0 - 1 \leqslant 1\cdot0.$$

$N(2\pi\tau_0)$ is odd since $Z(\tau_0)\cos 2\pi\kappa(\tau_0) > 0$ and we also have

$$|c_0 - 551| < 0\cdot05.$$

The required conclusion now follows.

PART II. THE COMPUTATIONS

Essentials of the Manchester Computer

It is not intended to give any detailed account of the Manchester Computer here, but a few facts must be mentioned if the strategy of the computation is to be understood. The storage of the machine is of two kinds, known as 'electronic' and 'magnetic' storage. The electronic storage consisted of four 'pages' each of thirty-two lines of forty binary digits. The magnetic storage consisted of a certain number of tracks each of two pages of similar capacity. Only about eight of these tracks were available for the zeta-function calculations. It was possible at any time to transfer one or both pages of a track to the electronic storage by an appropriate instruction. This operation takes about 60 ms. (milliseconds). Transfers to the magnetic store from the electronic were also possible, but were in fact only used for preparatory loading of the magnetic store. The course of the calculations is controlled by instructions each of twenty binary digits. These are normally magnetically stored, but must be transferred to the electronic

store before they can be obeyed. In the initial state of the machine (with the magnetic store loaded) the electronic store is filled with zeros. A zero instruction, however, has a definite meaning, and in fact results in a transfer of instructions to the electronic store, thus initiating the calculation. Most instructions, such as transfer of 'lines' of forty digits, take 1·8 ms., but transfers to or from the magnetic store take longer, as has been mentioned, and multiplications take a time depending on the number of digits 1 in the multiplier, ranging from 3·6 ms. for a power of two to 39 ms. for $2^{40}-1$.

The results of the calculations are punched out on teleprint tape. This is a slow process in comparison with the calculations, taking about 150 ms. per character. The content of a tape may afterwards automatically be printed out with a typewriter if desired. The significance of what is printed out is determined by the 'programmer'. In the present case the output consisted mainly of numbers in the scale of 32 using the code

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
/	E	@	A	:	S	I	U	¼	D	R	J	N	F	C	K	T	Z	L	W

20	21	22	23	24	25	26	27	28	29	30	31
H	Y	P	Q	O	B	G	‖	M	X	V	£

and writing the most significant digit on the right. More conventionally the scale of 10 can be used, but this would require the storage of a conversion routine, and the writer was entirely content to see the results in the scale of 32, with which he is sufficiently familiar.

Outline of calculation method

The calculations had of course to be planned so that the total storage capacity used was within the capacity of the machine. So long as this was fulfilled it was desirable to make the time of calculation as short as possible without excessive trouble in programming. The most time-consuming part of the calculations is of course the computation of the terms

$$n^{-\frac{1}{2}} \cos 2\pi(\tau \log n - \kappa)$$

from given κ and τ. By storing tables of $\log n$ and $n^{-\frac{1}{2}}$ within the machine this was reduced essentially to two multiplications and the calculation of a cosine, together with arrangements for 'looking up' the logarithm and reciprocal square root. The cosines were obtained from a table giving $\cos(r\pi/128)$ for $0 \leqslant r \leqslant 64$ by linear interpolation and reducing to the first quadrant. This gives an error of less than 10^{-4}, which is quite sufficient accuracy for the purpose. Very much greater accuracy was of course required in the logarithms, for an error ϵ in $\log n$ gives rise to an error approaching $2\pi\tau\epsilon$ in the cosine, and $2\pi\tau$ may be very large, e.g. 25,000. These logarithms were calculated by the machine in a previous computa-

tion, and were given with an error not exceeding 2.10^{-10}. The reciprocal square roots were given with error not exceeding 10^{-5}. Both the logarithms and the reciprocal square roots were checked after loading into the magnetic store by automatic addition, the results obtained being compared with values based respectively on Stirling's formula and on the known value of $\zeta(\frac{1}{2})$. The table only went as far as $n = 63$. The tabular cosines were built up automatically from the values of $\cos(\pi/128)$ and $\sin(\pi/128)$ by using the addition formula. The values of $\cos(\pi/128)$ and $\sin(\pi/128)$ were calculated both automatically and manually. A hand-copying process was used in connexion with this table, but the final results when loaded were automatically thrice differenced and the results inspected.

The routine as a whole was checked (amongst other methods) by comparing the result given for a value of τ about 20,000 with an entirely different, slower, and simpler routine. This routine had itself been checked against a hand-computed value for $\tau = 16$ and against a value given by Titchmarsh (5) for $\tau = 201{\cdot}596$.

Since it was only necessary to calculate $\kappa(\tau)$ once for each value of τ this calculation did not have to be particularly quickly performed. It was considered sufficient to obtain the logarithm by means of a slow but simple routine taking about $1{\cdot}2$ sec. The time for each term $n^{-\frac{1}{2}} \cos 2\pi(\tau \log n - \kappa)$ was about $0{\cdot}2$ sec. With $m = 63$, and allowing for the calculation of $\kappa_1(\tau)$ this means about 14 secs. for each value of τ. The routine could be used for recording the results for given values of τ, a typical entry obtained in this way being:

$$\text{ZETAFASTG/F@Q}\tfrac{1}{4}\text{B£YNK@:ZSZ"XVMX///SA/////}\tfrac{1}{4}\text{OTNR@O//.}$$

This entry has to be divided into sequences of eight characters. In this case they are:

1. ZETAFAST. This occurs at the beginning of each entry. Its purpose is mainly to identify the document as referring to this zeta-function routine.
2. G/F@Q$\frac{1}{4}$B£. This is a number useful in checking results and called the 'cumulant'. It appears in the scale of 32, with the most significant digit on the right. This is the standard method of representing numbers on documents connected with the Manchester Computer (a decimal method can also be used if desired).
3. YNK@:ZSZ. This is also in the scale of 32 and gives the residue of $2^{40}\kappa_1(\tau)$ modulo 2^{40}. Since Z is the symbol for 17 it will be seen that $\kappa_1(\tau)$ is near to $\frac{1}{2} \bmod 1$.
4. "XVMX///. This gives the value of $2^{17}\tau$; in this case τ is about $239{\cdot}24$.
5. SA/////$\frac{1}{4}$. This was always included in the record due to a minor

difficulty in the programming. It did not seem worth while to take the trouble to eliminate it.

6. OTNR@O//. This is the value of $2^{30}Z(\tau)$ modulo 2^{40}. In this case $Z(\tau)$ is about 0·75.

The routine was not, however, used mostly for the calculation of values of $Z(\tau)$ with individually given τ. It was made to determine for itself appropriate values of τ, such as to give values of $2\kappa(\tau)$ near to successive integers. This was done by making each τ depend on the immediately previous one and on the previous κ by the formula $\tau' = \tau+(1-\delta)\alpha$, where τ', τ are the new and old values of τ respectively, δ is the difference of $2\kappa_1(\tau)$ from the nearest integer, and $(\alpha\log\tau)^{-1} = 1+\Theta(0\cdot1)$. This procedure ensured that if the initial value of $\kappa(\tau)$ differs from an integer by less than 0·125, then the succeeding values will do likewise. It was decided not to record all the values of $Z(\tau)$, partly because the inspection and filing of the teleprint tape output would have a great burden to the experimenters. Values were only recorded when the unexpected sign occurred, i.e. when $Z(\tau)\cos 2\pi\kappa(\tau) < 0$. This reduced the amount of output data by about 90 per cent.

In order that there should be no doubt about the validity of the results it is necessary that one should also record all cases where the sign of $Z(\tau)$ is doubtful because of the limited accuracy of the computation. The criterion actually used was $Z(\tau)H(\kappa) > 0\cdot31E$, where

$$H(\kappa) \equiv \kappa-\tfrac{1}{4}(\text{mod }1), \qquad |H(\kappa)| < 0\cdot31.$$

The quantity $H(\kappa)$ arises very naturally with the computer. The condition $(\alpha\log\tau)^{-1} = 1+\Theta(0\cdot1)$ ensures that (except for one or two values at the beginning of a run), $|H(\kappa)| < 0\cdot31$. The actual errors involved in the calculation were:

Error arising from using Titchmarsh's formula (Theorem 3) . .	$1\cdot15m^{-\frac{5}{4}}$
Error due to replacing $\tau^{-\frac{1}{4}}h(\xi)$ by $m^{-\frac{1}{4}}h(\xi)$	$0\cdot47m^{-\frac{5}{4}}$
Error due to replacing ξ by $\tfrac{1}{2}\tau m^{-1}-\tfrac{1}{2}m$	$1\cdot08m^{-\frac{5}{4}}$
Error from using tabulated logarithms	$5\cdot1\times10^{-10}m^{\frac{5}{4}}$
Error in replacing $\kappa(\tau)$ by $\kappa_1(\tau)$	$0\cdot15m^{-\frac{5}{4}}$
Error in calculating $\kappa_1(\tau)$	$2\times10^{-10}m^{\frac{9}{4}}$
Error from using tabulated reciprocal square roots	$1\cdot3\times10^{-4}m$
Error from using tabulated cosines and linear interpolations .	$3\cdot2\times10^{-4}m^{\frac{1}{4}}$
Error of Theorem 2	$0\cdot0243m^{-\frac{1}{4}}$

There are also numerous rounding off errors which are very small. These and all the 'cross terms' have been absorbed into the above errors so that we may put the whole error as not more than

$$E = 2\cdot85m^{-\frac{1}{4}}+0\cdot0243m^{-\frac{1}{4}}+3\cdot2\times10^{-4}m^{\frac{1}{4}}+1\cdot3\times10^{-4}m+7\cdot1\times10^{-10}m^{\frac{5}{4}},$$

e.g. for $m = 15$

$$E < 0\cdot057,$$

and for $m = 65$

$$E < 0.02.$$

The storage available was distributed as follows:

Magnetic store

Logarithms routine (for κ)	1 page
Table of logarithms and reciprocal square roots	4 pages
Routine for calculating the terms $n^{-\frac{1}{2}}\cos 2\pi(\tau \log n - \kappa)$ and table of cosines	2 pages
Remainder of routine for calculating the function $Z(\tau)$	2 pages
Input routine	2 pages
Output routine	2 pages

Electronic store, as occupied during the greater part of the time

Instructions and cosines	2 pages
Logarithms and reciprocal square roots	1 page
Miscellaneous data and working space	1 page

The principal investigation concerned the range $63^2 \leqslant \tau \leqslant 64^2$, i.e. the interval in which $m = 63$. Working at full efficiency it should have taken about 4 hours to calculate these values, the number of zeros concerned being about 1070. Full efficiency was not, however, achieved, and the calculation took about 9 hours. Only a small amount of this additional time was accounted for by duplicating the work. The special investigations in the neighbourhood of points where the unexpected sign occurred took a further 8 hours. The general reliability of the machine was checked from time to time by repeating small sections. The recorded cumulants were useful in this connexion. These cumulants were the totals of the values of $Z(\tau)$ computed since the last recorded value. If a calculation is repeated and there is agreement in cumulant value then there is a strong presumption that there is also agreement in all the individual values contributing to it. The result of this investigation, so far as it can be relied on, was that there are no complex zeros or multiple real zeros of $Z(\tau)$ in the region

$$63^2 \leqslant \tau \leqslant 64^2,$$

i.e. all zeros of $\zeta(s)$ in the region $2\pi.63^2 \leqslant t \leqslant 2\pi.64^2$ are simple zeros on the critical line.

Another investigation was also started with a view to extending the range of relatively small values of t for which the Riemann hypothesis holds. Titchmarsh has already proved that it holds up to $t = 1468$, i.e. to about $\tau = 231$. The new investigation started somewhat before $\tau = 225$ to allow a margin for the application of Theorem 5. It was intended to continue the work up to about $\tau = 500$, but an early breakdown resulted in its abandonment at $\tau = 256$. After applying Theorem 5 it would only be possible to assert the validity of the Riemann hypothesis up to about $\tau = 250$. All

this part of the calculations was done twice, the unrecorded values being confirmed by means of the 'cumulants'.

Unfortunately $0.31E$ was given the inappropriate value of $\frac{1}{128}$ and consequently we are only able to assert the validity of the Riemann hypothesis as far as $t = 1540$, a negligible advance.

REFERENCES

1. A. E. INGHAM, *The distribution of prime numbers*, Cambridge Mathematical Tracts, No. 30 (1931).
2. JAHNKE U. EMDE (1), *Tafeln höherer Funktionen*, Leipzig, 1948.
3. E. C. TITCHMARSH, *The zeta-function of Riemann*, Cambridge Mathematical Tracts, No. 26 (1930).
4. —— 'The zeros of the Riemann zeta-function', *Proc. Roy. Soc.* A, 151 (1935), 234–55.
5. —— 'The zeros of the Riemann zeta-function', ibid. 157 (1936), 261–3.
6. A. M. TURING, 'A method for the calculation of the zeta-function', *Proc. London Math. Soc.* (2) 48 (1943), 180–97.

The University,
  Manchester.