THE YONEDA LEMMA AND CAYLEY'S THEOREM

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ABSTRACT. These notes are inspired by this stack exchange post. We aim to give a proof of Cayley's Theorem by viewing a group as a category with a single object, and using the Yoneda Lemma to retrieve Cayley's result. For a more explicit description of how a group can be viewed this way, see page 14 of Tom Leinster's Basic Category Theory.

For a locally small category $\mathcal{A}$, we write H_A for the image of $A \in \mathcal{A}$ under the Yoneda embedding $H_\bullet : \mathcal{A} \to [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$.

Theorem 0.1 (Yoneda). *Let $\mathcal{A}$ be a locally small category. Then*

$$[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}](H_A, X) \cong X(A)$$

naturally in $A \in \mathcal{A}$ and $X \in [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$.

Wikipedia describes the Yoneda Lemma as a 'vast generalisation' of Cayley's Theorem, which states that every group G is isomorphic to a subgroup of a symmetric group:

Theorem 0.2 (Cayley). *Let G be a group and let Y be its underlying set. Then G is isomorphic to a subgroup of* $\mathrm{Sym}(Y)$.

The proof of Cayley's theorem is relatively straightforward if one is familiar with group actions and the first isomorphism theorem for groups. Indeed, here is a purely group-theoretic proof of Theorem 0.2:

Proof. Let $\cdot$ denote the group operation of G. We define a left action of G on its underlying set Y by $(g, y) \mapsto g \cdot y$. Since G is a group, it follows directly from the group axioms that this does indeed define a left action. Now, for each $g \in G$, define a function $\overline{g} : Y \to Y$ by $\overline{g}(y) = g \cdot y$. One can check that $\overline{g}$ and $\overline{g^{-1}}$ are mutually inverse. Hence, $\overline{g} \in \mathrm{Sym}(Y)$. This gives rise to a group homomorphism

$$\begin{aligned} \Sigma : G &\to \mathrm{Sym}(Y) \\ g &\mapsto \overline{g}. \end{aligned}$$

Now, suppose $g \in \ker \Sigma$. Then we have $e = \overline{g}(e) = g \cdot e = g$. Hence, $\ker \Sigma = \{e\}$. Thus, by the first isomorphism theorem, we have

$$G \cong G/\{e\} \cong \operatorname{im} \Sigma \leq \mathrm{Sym}(Y),$$

as required. □

The categorical flavour of the above proof is strikingly apparent. Through letting the group act on itself, we have embedded it in a larger and, perhaps, nicer group. This gives us at least some idea of how the Yoneda Lemma may come into play later on.

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1. A Group as a One-Object Category

To view a group G as a one-object category, we define a corresponding category $\mathcal{G}$ with a single object $\star$. The morphisms (that is, the elements of $\mathcal{G}(\star,\star)$) are the elements of G, composition is the group operation $\cdot$ and $1_\star = e$, the identity element. Clearly, $\mathcal{G}$ is small (and hence locally small) as the class of all maps in $\mathcal{G}$ is just the underlying set of G. This is all well and good, but for the Yoneda lemma to bear any relevance, we need to go further. That is, we need to understand functors $\mathcal{G}^{\mathrm{op}} \to \mathbf{Set}$.

Proposition 1.1. *A functor $X : \mathcal{G}^{\mathrm{op}} \to \mathbf{Set}$ is a right G-set.*

Proof. Let us first consider the information that we can extract from such a functor. We have

(1) The value of X at the single object $\star$ of $\mathcal{G}^{\mathrm{op}}$;

(2) For each $g \in G$, a function $X(g) : X(\star) \to X(\star)$.

Now, since X is a set-valued functor, the above amounts to

(1) A set S

(2) For each $g \in G$, a function $\overline{g} : S \to S$.

Furthermore, as the domain of X is $\mathcal{G}^{\mathrm{op}}$, then the order of composition is reversed. That is,

$$\overline{(g \cdot h)} = X(g \cdot h) = X(h) \circ X(g) = \overline{h} \circ \overline{g}$$

for all $g, h \in G$. Define a function $\phi : S \times G \to S$ by $\phi(s, g) := \overline{g}(s)$. We claim that this defines a right action of G on S. Indeed, we have

$$\phi(s, e) = \overline{e}(s) = (X(1_\star))(s) = 1_{X(\star)}(s) = 1_S(s) = s,$$

and

$$\phi(\phi(s, g), h) = \overline{h}(\phi(s, g)) = \overline{h}(\overline{g}(s)) = (\overline{h} \circ \overline{g})(s) = \overline{(g \cdot h)}(s) = \phi(s, gh)$$

for any $g, h \in G$ and $s \in S$. Thus, ϕ defines a right action of G on S. Hence, the functor X is simply a set equipped with a right action by G, a right G-set. $\square$

With this in mind, we are one step closer to understanding just why the Yoneda Lemma is relevant. However, we also need to develop an understanding of the Yoneda embedding of $\mathcal{G}$, and what natural transformations $H_\star \to X$ correspond to for functors $X : \mathcal{G}^{\mathrm{op}} \to \mathbf{Set}$. Hence:

Proposition 1.2. *Let $X \in [\mathcal{G}^{\mathrm{op}}, \mathbf{Set}]$. Natural transformations $H_\star \to X$ are in a one-to-one correspondence with G-equivariant maps $H_\star(\star) \to X(\star)$.*

Proof. By Proposition 1.1, both $H_\star$ and X are right G-sets. Let $\alpha : H_\star \to X$ be a natural transformation. What information can we extract from α? We have right G-sets $S = H_\star(\star)$, $\tilde{S} = X(\star)$ and a map $\alpha_\star : S \to \tilde{S}$ such that

$$\begin{tikzcd} S \arrow[r, "H_\star(g)"] \arrow[d, "\alpha_\star"'] & S \arrow[d, "\alpha_\star"] \\ \tilde{S} \arrow[r, "X(g)"'] & \tilde{S} \end{tikzcd}$$

commutes, for any $g \in G$.

Now, let $\phi : S \times G \to S$ and $\varphi : \tilde{S} \times G \to \tilde{S}$ be the respective right actions of G on S and $\tilde{S}$ introduced in the proof of Proposition 1.1. Write $\tilde{g}$ for the function $X(g) : \tilde{S} \to \tilde{S}$, and $\overline{g}$ for the function $H_\star(g) : S \to S$. Then, reading off the naturality square, we have

$$\varphi(\alpha_\star(s), g) = \tilde{g}(\alpha_\star(s)) = (X(g) \circ \alpha_\star)(s) = (\alpha_\star \circ H_\star(g))(s) = \alpha_\star(\overline{g}(s)) = \alpha_\star(\phi(s, g)).$$

for any $g \in G$ and $s \in S$. Hence, $\alpha_\star$ is a G-equivariant map. Since $\star$ is the sole object of $\mathcal{G}$, we have $\alpha = \alpha_\star$, and a natural transformation $\alpha : H_\star \to X(\star)$ is thus a G-equivariant map $H_\star(\star) \to X(\star)$.

Now, let $f : H_\star(\star) = S \to \tilde{S} = X(\star)$ be a G-equivariant map. Let $g \in G$, and consider the square

$$\begin{tikzcd} S \arrow[r, "H_\star(g)"] \arrow[d, "f"'] & S \arrow[d, "f"] \\ \tilde{S} \arrow[r, "X(g)"'] & \tilde{S}. \end{tikzcd}$$

We have that

$$\begin{aligned}(f \circ H_\star(g))(s) = f(H_\star(g))(s) = f(\overline{g}(s)) = f(\phi(s, g)) &= \varphi(f(s), g) = \tilde{g}(f(s)) \\ &= (X(g) \circ f)(s),\end{aligned}$$

for any $s \in S$. Hence, the above square commutes, and (as $\star$ is the sole object of $\mathcal{G}$, and $g \in G$ was taken to be arbitrary) f defines a natural transformation $H_\star \to X$. $\square$

Strictly speaking, we did not construct a bijection between the set of objects of $[\mathcal{G}^{\mathrm{op}}, \mathbf{Set}]$ and the set of G-equivariant maps $H_\star(\star) \to H_\star(\star)$. However, one can easily see how this bijection could be constructed. Informally:

(1) Given a natural transformation $\alpha : H_\star \to X$, the natural transformation induced by $\alpha_\star$ is α;

(2) Given a G-equivariant map $f : H_\star(\star) \to X(\star)$, f is the component at $\star$ of the natural transformation induced by f.

Lemma 1.3. *Let $\phi : H_\star(\star) \times G \to H_\star(\star)$ be the right action of G on $H_\star(\star)$ introduced in Proposition* 1.1*. Then*

$$\phi(g, h) = g \cdot h$$

for all $g, h \in G$.

Proof. Recall that $(H_\star(g))(h) = h \cdot g$ for $g, h \in G$, as $\cdot$ is composition in $\mathcal{G}$. Hence, we have that

$$\phi(g, h) = (H_\star(h))(g) = g \cdot h$$

for any $g, h \in G$. $\square$

Lemma 1.4. *Write Y for the underlying set of G. Then each G-equivariant map $f : H_\star(\star) = \mathcal{G}(\star, \star) = Y \to Y = H_\star(\star) = \mathcal{G}(\star, \star) = Y$ is an element of* $\mathrm{Sym}(Y)$*, and furthermore*

$$[\mathcal{G}^{\mathrm{op}}, \mathbf{Set}](H_\star, H_\star) \leq \mathrm{Sym}(Y).$$

Proof. For a G-equivariant map $f : Y \to Y$, we have

$$f(e) \cdot g = \phi(f(e), g) = f(\phi(e, g)) = f(e \cdot g) = f(g).$$

Hence, $f(g) = f(e) \cdot g$ for all $g \in G$. Thus, f is wholly determined by its value at e, which is just an element of G. In other words, f maps $g \in G$ to $ag \in G$, where a is some fixed element of G. By the cancellation property and the fact that a has an inverse $a^{-1} \in G$, it follows that f is a bijection. That is, $f \in \mathrm{Sym}(Y)$.

Let f_1, f_2 be G-equivariant maps $Y \to Y$. We have $f_1(g) = ag$, $f_2(g) = bg$ for some $a, b \in G$. Since inverses are unique, we must have $f_2^{-1}(g) = b^{-1}g$, and thus

$$\begin{aligned}(f_1 \circ f_2^{-1})(\phi(g,h)) = (f_1 \circ f_2^{-1})(g \cdot h) = f_1(b^{-1} \cdot (g \cdot h)) &= ab^{-1} \cdot (gh) \\ &= ab^{-1}g \cdot h \\ &= \phi(f_1 f_2^{-1}(g), h).\end{aligned}$$

Thus, $f_1 f_2^{-1}$ is a G-equivariant map. By the subgroup test and Proposition 1.2, the result follows. □

2. Recovery of Cayley's Theorem

The Yoneda Lemma establishes a bijection $[\mathcal{G}^{\mathrm{op}}, \mathbf{Set}](H_\star, H_\star) \to H_\star(\star)$; $\alpha \mapsto \alpha_\star(1_\star)$. Recall that $H_\star(\star) = Y$, the underlying set of G. Equip Y with the operation $\cdot$ and note that $G = (Y, \cdot)$ by definition. Let $\alpha, \beta \in [\mathcal{G}^{\mathrm{op}}, \mathbf{Set}](H_\star, H_\star)$. By the proof of Proposition 1.4, we know that $\alpha_\star(g) = ag$, $\beta_\star(g) = bg$ for some $a, b \in G$. Hence,

$$(\alpha_\star \circ \beta_\star)(1_\star) = \alpha_\star(\beta_\star(1_\star)) = \alpha_\star(\beta_\star(e)) = \alpha_\star(b) = ab = \alpha_\star(1_\star) \cdot \beta_\star(1_\star).$$

It follows that the bijection established by the Yoneda Lemma is an isomorphism $[\mathcal{G}^{\mathrm{op}}, \mathbf{Set}](H_\star, H_\star) \to G$. Hence, Lemma 1.4 allows us to recover Cayley's Theorem.

3. Conclusion and Further Discussion

It is clear that it is much more straightforward to prove Cayley's theorem using purely group-theoretic arguments. However, we have gathered some insight into how the Yoneda lemma 'generalises' Cayley's result. For reassurance that the Yoneda lemma really does give us more than an elongated proof of a relatively rudimentary result in group theory, see <u>here</u>.