Lecture 9

Dimensional analysis

Content

- Focus in this lecture on modeling of some physical processes
- Dimensional analysis
  - Generalities
  - Examples
  - Buckingham's theorem
- Similitude

Background

- Explored so far mostly hypothesis-driven and data-driven modeling
  - In the former case, we know the main interactions and test how well they fit the data
    - Focus on how much the hypothesis can explain the data
    - Mostly a problem of model fitting
  - In the later case, we only have data and are willing to accept any (reasonable) model confirming them
    - Interested in describing the trends of the data
    - Often a problem of interpolation

$y = kx \quad y = e^{cx}$

$X = \frac{x_0 \cdots x_n}{x_0 \cdots x_j}$

$f(x_1, x_2 \ldots x_n) = y$

$3 \times 3 \times \cdots \times 3$

3^n

$y = f(x)$

- Discuss here a different approach to modeling, based on "dimensional analysis"
  - In this case we know the main variables important for the model
    - We know their units of measure (or dimension)
  - We do not know their interactions
  - We do not have the data at the moment when we build the model
  - We have in this case a problem of model identification

meter. second. kg.

Dimensional analysis – basics

- It is a method for determining how the selected variables are related
- Important also in reducing significantly the amount of experimental data needed later on for fitting the model
- Main ideas in dimensional analysis:
  - Physical quantities have dimensions
  - Physical laws are not altered by changing the units measuring dimensions
  - Any equation must be expressed in the same dimension on both sides of the equality (equations must be *dimensionally correct*)

Example: a simple pendulum

$T = f(r, g, m, \theta)$

- Problem: for a given pendulum, determine its period
  - the time required for the pendulum bob to swing through one complete cycle and return to its initial position
- Assumptions
  - Identify the factors influencing the period:
    - the length r,
    - the acceleration due to the gravity g
    - The mass m
    - The initial angle θ
    - Frictional forces
    - The drag on the pendulum
  - For simplicity, assume that the hinge is frictionless, the drag force is negligible, and the mass of the pendulum is concentrated at one end of the pendulum

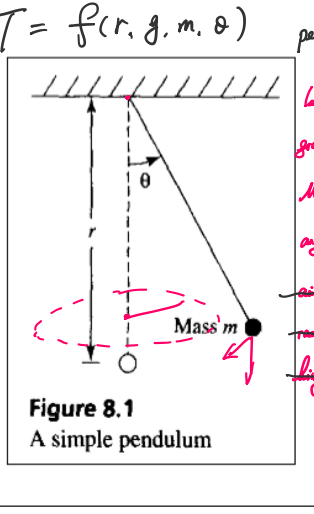

Figure 8.1
A simple pendulum

Giordano et al. A first course in mathematical modeling. (3rd edition), Page 293

period: $T = \frac{1}{2}\sqrt{\frac{l}{g}}$

length: r

gravity: g $\sin\theta \approx \theta$ $\theta \ll 1$

Mass: m.

angle: θ

~~air friction~~ no

~~radius of ball~~ mass pt.

~~drag friction~~ x

Example: a simple pendulum

- Problem formulation: determine/approximate the function t=f(r,m,θ,g)
  Const.
- Data-driven approach
  - Gravity is constant
  - t only depends on r,m,θ
  - Collect enough data on the value of t depending on various values of r,m,θ
  - Question: how many experiments are needed?
    - Assume that N points are to be considered for each variable
    - Then we need 3^N experiments
  - Clear need to reduce the number of independent variables as much as possible
    - In fact, minimize the number of independent variables in function f

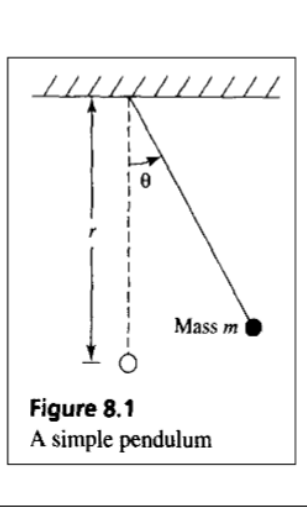

Figure 8.1
A simple pendulum

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About dimensions

- Some elementary concepts in physics:
  - mass, length, time, velocity, acceleration, force, energy, work, pressure

    M kg, L meter, T second, $L \cdot T^{-1}$, $L \cdot T^{-2}$, $M \cdot L \cdot T^{-2}$, $M \cdot L^2 \cdot T^{-2}$, $M \cdot L^2 \cdot T^{-2}$, $M \cdot L^{-1} \cdot T^{-2}$
  - For each such concept there is a unit of measurement
  - For a physical law, e.g. F=ma, to make sense, the units of measurement on both sides of the equality have to be consistent
    - If the mass is measured in kilograms and the acceleration is measured in meters per second squared, then the force must be measured in newtons: the MKS (meter-kilogram-second) mass systems
    - It would be inconsistent with the equation F=ma to measure mass in slugs, acceleration in feet per second squared, and force in newtons
      - It would become consistent if force were expressed in pounds: the American Engineering System of measurement

$F = m \cdot a$

About dimensions

- The three main physical quantities we consider in this lecture are mass, length, and time
  - Associate with these quantities the (generic) dimensions M, L, and T, resp.
    - We do not specify which unit of measurement to use for each
  - The dimension of other quantities with respect to M, L, T follow from basic laws of physics
  - Numerical (dimensionless) constants may also be involved in defining some more complex quantities; they are ignored in finding the dimension of that concept
  - There are also dimensional variables that in our application may be considered a constant, such as the gravity – for them we need to consider their dimension
- More complex entities, such as momentum or kinetic energy, may be defined as products of simpler entities, which in turn are products of M, L, T
- Each product will get a dimension associated to it, in terms of a product $M^nL^pT^q$
  - Dimensionless quantities: n=p=q=0

$g = 9.8\, L \cdot T^{-2}$ $G = \frac{m \cdot M}{r^2} \cdot g$

$h = 6.6 \times 10^{-}$

$[M^0 \cdot L^0 \cdot T^0] = r^a m^b g^c \theta^d$

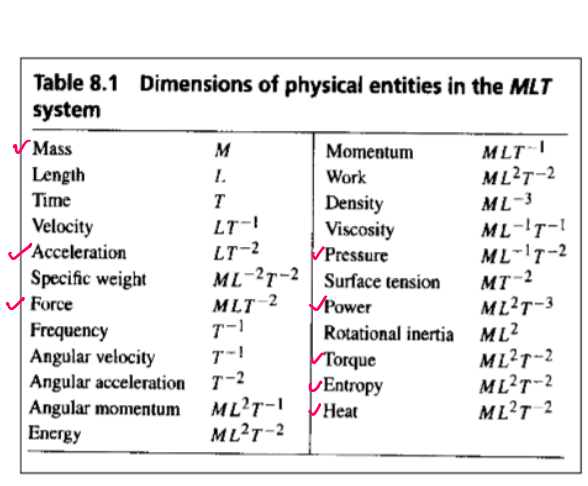

Table 8.1 Dimensions of physical entities in the *MLT* system

✓Mass	M	Momentum	MLT^{-1}
Length	L	Work	ML^2T^{-2}
Time	T	Density	ML^{-3}
Velocity	LT^{-1}	Viscosity	$ML^{-1}T^{-1}$
✓Acceleration	LT^{-2}	✓Pressure	$ML^{-1}T^{-2}$
Specific weight	$ML^{-2}T^{-2}$	Surface tension	MT^{-2}
✓Force	MLT^{-2}	✓Power	ML^2T^{-3}
Frequency	T^{-1}	Rotational inertia	ML^2
Angular velocity	T^{-1}	✓Torque	ML^2T^{-2}
Angular acceleration	T^{-2}	✓Entropy	ML^2T^{-2}
Angular momentum	ML^2T^{-1}	✓Heat	ML^2T^{-2}
Energy	ML^2T^{-2}		

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Dimensions

$F = ma$ $M \cdot L \cdot T^{-2}$ $M \cdot L \cdot T^{-1}$ + $L^2 \cdot T^{-2}$

- Dimensionally compatible sums
  - All terms in a sum must have compatible dimensions
  - Example: $\boxed{F=mv+v^2}$ must be incorrect: mv has dimension MLT^{-1}, whereas v^2 has dimension L^2T^{-2}
- Dimensional homogeneity
  - Equations that are true regardless of the system of units for its variables are called dimensionally homogenous
  - Example: the time a body falls a distance s under gravity g, neglecting air resistance

$T \quad t = \sqrt{\frac{2s}{g}}$ is dimensionally homogenous $\quad \sqrt{\frac{L}{L/T^2}} = T$

$T \quad t = \sqrt{\frac{s}{16.1}}$ is not dimensionally homogenous $\quad \sqrt{L}$

Applying dimensional analysis

- Main assumption: the model is given by a dimensionally homogenous equation in terms of the appropriate variables
- Task: determine the form of that equation
- Approach: find an appropriate dimensionless equation and solve it for the dependent variable of interest
  - Choose which are the relevant variables to consider
  - Determine/describe all possible dimensionless products of those variables
  - Use some of these products to construct dimensionally homogenous equations

$L.H.S. = R.H.S. = M^\alpha \cdot L^\beta \cdot T^\gamma$

$0 = [M^0 L^0 T^0]$

$S - S$

Another example

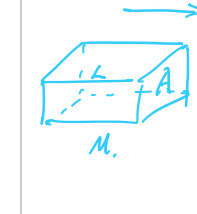
- Wind force on a van: you are driving a van on a highway with gusty winds
  - Problem: how does the speed of the van affect the wind force you are experiencing?
  - Assumptions: the force F of the wind is affected by the speed v of the van and the surface area A of the van directly exposed to the wind's direction
  - Hypothesis: the force is proportional to some power of the speed times some power of the surface area: $F=kv^aA^b$

Variable				
Dimension				

  - In terms of dimensions, the equation above is $MLT^{-2}=(M^0L^0T^0)(LT^{-1})^a(L^2)^b$
  - But this cannot be correct: M has a zero exponent on the right hand side
  - Conclusion: error in the model formulation
  - Rework the hypothesis: the strength of the wind is also affected by its density (density of air): $F=kv^aA^b\rho^c$
  - Dimensionally: $MLT^{-2}=(M^0L^0T^0)(LT^{-1})^a(L^2)^b(ML^{-3})^c$

$= M^c \cdot L^{a+2b-3c} \cdot T^{-a}$

$c = 1, \quad a = 2 \quad b = 1$



$M \cdot L \cdot T^{-2}$

$F := f(v, A, l, m)$

$L \cdot T^{-1} \quad L^2 \quad L \quad M$

$= k\, v^\alpha A^\beta m^\gamma = (L \cdot T^{-1})^\alpha \cdot (L^2)^\beta \cdot (M)^\gamma = M^\gamma \cdot L^{\alpha+2\beta} \cdot T^{-\alpha}$

$\rho^\gamma = (L \cdot T^{-1})^\alpha \cdot (L^2)^\beta \cdot (\frac{M}{L^3})^\gamma$

$\gamma = 1. \quad \alpha = 2 \quad \beta = -\frac{1}{2}$

$\alpha + 2\beta = 1$

Example (continued)

- Dimensionally: $MLT^{-2}=(M^0L^0T^0)(LT^{-1})^a(L^2)^b(ML^{-3})^c$
  - Equivalent with: c=1; a+2b-3c=1; -a=-2
  - In other words: a=2, b=1, c=1
- Final model: $F=kv^2A\rho$
- Note: for a particular situation, the density ρ is a constant
  - One might reformulate the model to say that F is proportional with v^2A
  - The reason this model seemed to fail in the beginning of our reasoning is because we assumed the proportionality constant to be dimensionless

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Example: simple pendulum revisited

- The main variables of interest:

Variable					
Dimension					

- Find all of their dimensionless products: $m^ag^bt^cr^d\theta^e$
  - Its dimension is $(M)^a(LT^{-2})^b(T)^c(L)^d(M^0L^0T^0)^e$ 1 rad 2π rad
  - In other words: $M^aL^{b+d}T^{-2b+c}$
  - Dimensionless condition: a=0; b+d=0; -2b+c=0
  - In other words: a=0, c=2b, d=-b, where b, e are arbitrary
  - Infinitely many dimensionless products
- Consider here only two of them, for b=0, e=1 and for b=1, e=0

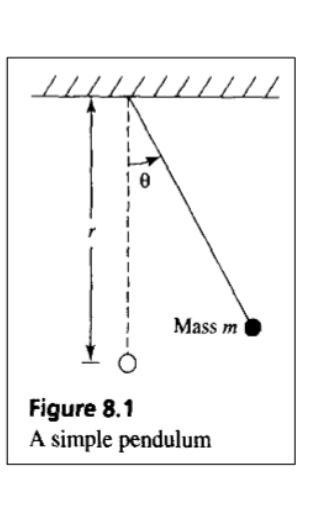

Figure 8.1
A simple pendulum

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$M^a \cdot (L \cdot T^{-2})^b \cdot T^c \cdot L^d \cdot (M^0 L^0 T^0)^e$

$= M^a \cdot L^{b+d} \cdot T^{c-2b} = [M^0 L^0 T^0]$

$a = 0 \quad b + d = 0 \quad c - 2b = 0$

$\begin{pmatrix} b \\ e \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$(a, b, c, d, e) = \begin{cases} (0, 1, 2, -1, 0) \\ (0, 0, 0, 0, 1) \end{cases}$

$\Pi_1 := g \cdot t^2 / r$

$\Pi_2 := \theta$

$f(\Pi_1, \Pi_2) = 0$

$g(\Pi_1) + h(\Pi_2) = 0$

$\Pi_1 = g^{-1} \circ h(\Pi_2)$

$\frac{g t^2}{r} = \varphi(\theta)$

$\boxed{t = \sqrt{\frac{r}{g} \cdot \varphi(\theta)}}$

$\begin{cases} \text{fix } \theta. \quad t \propto \sqrt{r} \\ \text{fix } r. \quad t \propto \varphi(\theta) \end{cases}$

Example (continued)

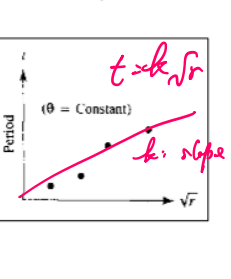
- Two dimensionless products:
  - b=0, e=1: $\Pi_1=m^0g^0t^0r^0\theta^1=\theta$
  - b=1, e=0: $\Pi_2=m^0g^1t^2r^{-1}\theta^1=gt^2/r$
- Any other solution of the linear system on the previous slide is a linear combination of the solution obtained for (b,e)=(0,1) and for (b,e)=(1,0)
- In other words, any other dimensionless product is of the form $\Pi_1^a\Pi_2^b$
- Recall our main assumption: the solution is given by a dimensionally homogenous equation in terms of the appropriate variables
  - by rewriting that equation, we get a dimensionless equation
  - but a dimensionless equation can only involve an expression in terms of $\Pi_1^a\Pi_2^b$
- In other words, a solution to the problem must be of the form $f(\Pi_1,\Pi_2)=0$, for some (as yet unknown) function f

$g(\Pi_1) + h(\Pi_2) = 0$

Example (continued)

- $f(\Pi_1,\Pi_2)=0$, i.e., $f(\theta,gt^2/r)=0$
  - If we managed to solve this equation in terms of t (invert function f), then the expression we got would be $gt^2/r=\varphi(\theta)$ for some function φ
- Final solution: $t=(r/g)^{1/2}h(\theta)$, for some function $h(\theta)=(\varphi(\theta))^{1/2}$
  - Test the solution: keep g and θ constant, vary r; check the dependency between t and $r^{1/2}$
  - Determine the unknown function h: collect data on t for fixed r and various values of θ, do a data-driven modeling approach

Figure 8.4
Testing the assumptions of the simple pendulum model by plotting the period t versus the square root of the length r for constant displacement θ



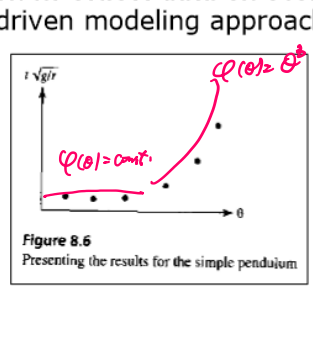


Figure 8.6
Presenting the results for the simple pendulum

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$\theta \ll 1. \quad \varphi(\theta) = \text{const.}$

The process of dimensional analysis

> Recall:
> - *Main assumption*: the solution is given by a dimensionally homogenous equation in terms of the appropriate variables
> - *Task*: determine the form of that equation
> - *Approach*: find an appropriate dimensionless equation and solve it for the dependent variable of interest

- Focus on determining the form of all dimensionless products
  - similarly as in our previous example, this leads to a linear system of equations, where each equation stands for the exponent of one of the basic dimensions you consider (in our examples, M, L, and T)
  - Elementary result in linear algebra: the solution of any linear system of equations is a linear space;
    - Idea: determine its linear basis: a complete, independent set of solutions
    - each of these "basic" solutions gives a "basic" dimensionless product
  - Any solution of the system of equations is a linear combination of the basic solutions
  - Any dimensionless product is a product of some powers of the basic dimensionless products

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The process of dimensional analysis (2)

- Main tool: Buckingham's theorem
- Intuition:
  - we have a physically meaningful (dimensionally homogenous) equation involving *n* physical variables, and these variables are expressible in terms of *k* independent fundamental physical quantities
  - then the original equation is equivalent to an equation involving a set of *p = n − k* dimensionless parameters constructed from the original variables
  - This provides a method for computing sets of dimensionless parameters from the given variables, even if the form of the equation is still unknown
  - Note: the choice of dimensionless parameters is not unique; up to the modeler to choose the most `physically meaningful' such parameters

$n = 5. \quad k = 3$

$m. g. t. r. \theta \quad g. t. r$

$\Pi_1 = \quad \Pi_2 :=$

$f(\Pi_1, \Pi_2) = 0 \quad \Pi_2 := \varphi(\Pi_1)$

The process of dimensional analysis (3)

- Buckingham's theorem / the pi-theorem (1914):

An equation

$$\mathbf{f(q_1,\ldots,q_n)=0}$$

is *dimensionally homogenous*, where q_i are physical variables expressed in terms of k independent physical units if and only if the equation can be restated as

$$\mathbf{F(\pi_1,\ldots,\pi_p)=0}$$

$p = n - k$

where π_j are *dimensionless parameters* constructed from the q_i by n-k equations of the form

$$\mathbf{\pi_j=q_1^{a_1}\ldots q_n^{a_n}}$$

where the exponents a_i are integers

Example

- Problem: determine the terminal velocity v of a raindrop falling from a motionless cloud
- Variables to consider:
  - size of the raindrop (say its radius), L
  - density ρ of the air, $M \cdot L^{-3}$
  - viscosity μ of the air (resistance to motion; sort of internal molecular friction)
  - gravity g $L \cdot T^{-2}$

Variable						
Dimension						

- By Buckingham's theorem, we are interested in all dimensionless products of powers of these variables: $v^ar^bg^c\rho^d\mu^e$.
  - Dimensionally: $(LT^{-1})^a(L)^b(LT^{-2})^c(ML^{-3})^d(ML^{-1}T^{-1})^e$ is dimensionless
  - Equivalently: d+e=0; a+b+c-3d-e=0; -a-2c-e=0
  - Solution: b=1.5d-0.5a; c=0.5d-0.5a; e=-d; a,d arbitrary
  - Consider the independent solutions obtained for (a,d)=(1,0), (a,d)=(0,1)

$\mu = M L T^{-2} \cdot L^{-2} T$

$/L / L T^{-1}$

v

μ viscosity!

ρ :

$\mu = 1/Re$

Example (continued)

- Basic solutions: $\Pi_1=vr^{-1/2}g^{-1/2}$, $\Pi_2=r^{3/2}g^{1/2}\rho\mu^{-1}$
- Buckingham's theorem: there is a function f such that $f(vr^{-1/2}g^{-1/2}, r^{3/2}g^{1/2}\rho\mu^{-1})=0$
- Assuming we could invert this function, we obtain $vr^{-1/2}g^{-1/2}$ as a function of $r^{3/2}g^{1/2}\rho\mu^{-1}$.
  - Equivalently: $v=\sqrt{rg}\cdot h\left(\frac{r^{3/2}g^{1/2}\rho}{\mu}\right)$

    where h is some (unknown) function

$\Pi_1 = \varphi(\Pi_2)$

$\boxed{v = \sqrt{gr} \cdot \varphi(\Pi_2)}$

$\begin{cases} v \propto \sqrt{r} \\ v \propto \Pi_2 \end{cases}$

$\boxed{\mu = \frac{\Phi(\Pi_1)}{\rho \cdot g^{1/2} \cdot r^{3/2}}}$

Notes

- The n-k dimensionless products stated in Buckingham's theorem are not unique
  - for the purpose of modeling in the way we showed in the example, it is necessary to choose them in such a way that the dependent variable of interest only occurs in one of the products
  - in that way we can continue our reasoning with the inversion of the (unknown) function in Buckingham's theorem
- We did not discuss here when the function in Buckingham's theorem is invertible
  - deeper discussion related to the implicit function theorem in advanced calculus

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- Usefulness of Buckingham's theorem for modeling
  - get information about how the variables are related to each other; in some cases, lead to complete model identification
  - identify incompleteness of a model
  - help in experiment design: if $\pi_1=h(\pi_2, \pi_3,\ldots,\pi_p)$, then design the experiments so that $\pi_3,\ldots,\pi_p$ are constant to isolate the effect of π_2 on π_1
  - reduce the number of experiments needed to fully identify the model: no need to sample n variables $x_1,\ldots,x_n$, only sample the products $\pi_1, \ldots, \pi_p$, where p<n
  - helps in presenting the model in a standard form: choose the dimensionless products to be widely used coefficients in that particular field of study (Reynolds number, Froude number, Mach number, etc)

h: not our goal

$\Pi_1 = f(\Pi_2)$

$\Pi_1 = f_2(\Pi_3)$

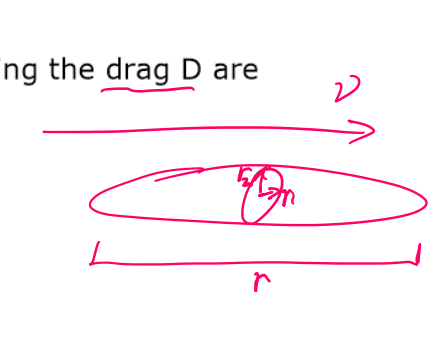
Using similitude and prototyping – an example

Similitude

- Using prototypes
  - Sometimes it is impossible to study the actual object of interest
    - Example: effects of wave action on a large ship at sea, heat loss of a submarine, wind effects on an aircraft wing
  - Solution: construct a scaled-down model (prototype) and simulate the environment
  - Problem: how do we scale the experiments to ensure that the effects observed on the model are the same as those that will take place on the final product?
  - Approach here: use dimensional analysis

Example: drag force on a submarine

- Assumptions: the variables affecting the drag D are
  - Fluid velocity v
  - Length of the submarine: r
  - Fluid density ρ
  - Fluid viscosity μ $M \cdot L^{-1} \cdot T^{-1}$
  - Velocity of sound in the fluid c



- Describing the shape factors related to the submarine
  - Assumption: assume the submarine having an ellipsoidal shape; describe it through two ellipses – two cross-sections, one length-wise, the other width-wise
  - Shape factors
    - Length of the submarine (an axis in the length-wise cross-section): r
    - Ratio of the lengths of the two axis in the first cross-section: r_1
    - Ratio of the lengths of the two axis in the second cross-section: r_2 $A = r_1 r_2$
- In general: collect enough shape factors to be able to describe in enough details the 3D shape of the object of interest

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Example (continued)

- Dimensional analysis
  - We are highly interested in the drag D – the pressure (force per unit area) times the area
    - For geometrically similar objects, the area is proportional to r^2
    - So D is proportional to pr^2 $D^a v^b r^c \rho^d \mu^e c^f$
  - We have six fluid mechanics variables: D,v,r,ρ,μ,c
  - We have two other dimensionless (shape) factors, r_1, r_2
  - The MLT-based dimensional analysis will give an equation in 6-3=3 dimensional variables and in the 2 dimensionless variables
    - We can choose three independent variables that are widely used in physics
    - For example: Reynolds number R=vrρ/μ, Mach number M=v/c, pressure coefficient $P=p/(\rho v^2)$ $\mu = R / vr\rho$
  - Buckingham's theorem gives an equation $h(P,M,R,r_1,r_2)=0$
  - Assuming we could solve it for P, we get $P=H(M,R,r_1,r_2)$
  - Substitute $P=p/(\rho v^2)$ and we get $p= \rho v^2H(M,R,r_1,r_2)$
  - Drag force is proportional to pr^2, i.e. $D=k\rho v^2r^2H(M,R,r_1,r_2)$

Example (continued)

  - Drag force is proportional to pr^2, i.e. $D=k\rho v^2r^2H(M,R,r_1,r_2)$
  - A condition of the same form holds both for the model/prototype and for the final object: $D_m=k\rho_m v_m^2r_m^2H(M_m,R_m,r_{1m},r_{2m})$
- Conclusion: design conditions for the model
  a. $R_m=R$, i.e. $v_mr_m\rho_m/\mu_m=vr\rho/\mu$
  b. $M_m=M$, i.e. $v_m/c_m=v/c$
  c. $r_{1m}=r_1$
  d. $r_{2m}=r_2$
  - In these conditions, $D_m/D=\rho_mv_m^2r_m^2/\rho v^2r^2$
    - Compute D_m and then calculate D
- Discuss the design conditions
  - Conditions c and d imply geometric similarity
  - If the velocities are small compared to the speed of sound in a fluid, then v/c can be considered constant, as required by condition b
  - If the same fluid is used both for the model and the final object, then condition a is satisfied iff $v_mr_m=vr$, i.e. $v_m/v=r/r_m$
    - Setting the velocity of the model in this way yields $D=D_m$

Learning objectives

- Understand the principles of dimensional analysis: physical quantities have dimensions, models should be dimensionally coherent
- Be able to perform dimensional analysis for simple models
- Be able to apply Buckimgham's theorem to simple models
- Understand the application of dimensional analysis in the form of similitude analysis