Lecture 7

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Lecture 7

Discrete optimization modeling

Model fitting as an optimization problem

- Discussed in Lecture 4 about model fitting
  - gave several formulations for model fitting
  - they all are optimization problems
  - discussed in Lecture 4 a calculus-based approach to solving the least squares formulation of model fitting
- Discuss here about a discrete approach to optimization, in particular to model fitting
  - focus on linear programming
  - Chebyshev criterion (minimize the greatest deviation) leads to a linear program in the case of linear models

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Content

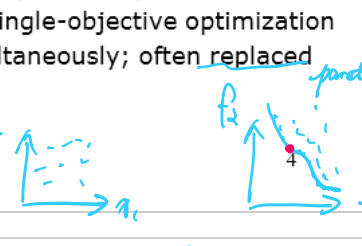
- Overview
- Linear programming
  - geometric solutions
  - algebraic solutions

Generalities

- Discrete optimization problems: the variables are assumed to take only discrete values
  - large field of research, many different branches
- Focus here on the following type of discrete optimization problems:
  - Optimize (i.e. minimize or maximize) functions $f_1(x_1,\dots,x_m)$, $f_2(x_1,\dots,x_m)$, …, $f_n(x_1,\dots,x_m)$ such that the following constraints are satisfied:
    - $g_1(x_1,\dots,x_m)\geq b_1$, …, $g_{p_1}(x_1,\dots,x_m)\geq b_{p_1}$
    - $g_{p_1+1}(x_1,\dots,x_m)=b_{p_1+1}$, …, $g_{p_1+p_2}(x_1,\dots,x_m)=b_{p_1+p_2}$
    - $g_{p_1+p_2+1}(x_1,\dots,x_m)\leq b_{p_1+p_2+1}$, …, $g_{p_1+p_2+p_3}(x_1,\dots,x_m)\leq b_{p_1+p_2+p_3}$
  - $(x_1,\dots,x_m)$ are the variables of the model
  - $f_1, f_2, \dots, f_n$ are the objective functions (to be optimized)
    - multi-objective optimization often reduced to single-objective optimization
    - multiple objectives may not be optimized simultaneously; often replaced with a single combined objective

$n=1$, single-objective opt.

$n>1$, multi-objective opt.



$\min f = w_1 f_1 + w_2 f_2$

m discrete independent variables $x_1,\dots,x_m$.

n cost/utility functions $\begin{cases} f_1(x_1,\dots,x_m) \\ \vdots \\ f_n(x_1,\dots,x_m)\end{cases}$ dependent variables

@f

Constraints: $\begin{cases} g_1(x_1,\dots,x_m)\geq b_1 \\ \vdots \\ g_{p_1}(x_1,\dots,x_m)\geq b_{p_1}\end{cases}$ Inequality! @g

$\begin{cases} h_1(x_1,\dots,x_m)=c_1 \\ \vdots \\ h_{p_2}(x_1,\dots,x_m)=c_{p_2}\end{cases}$ Equality! @h

$h_1\geq c_1$

$-h_1\geq -c_1$

Example: production schedule

- Carpenter making tables and bookcases. Question: how many of each should he make each week so that he maximizes his profit?
  - net profit for selling a table: 25\$, a bookcase: 30\$
  - amount of lumber available weekly: 600 board feet
  - amount of labor per week: 40 hours
  - lumber needed for a table: 20 board feet; for a bookcase: 30 board feet
  - work needed for a table: 5 hours; for a bookcase: 4 hours
  - he has firm contracts for 4 tables and 2 bookcases every week
- Problem formulation: maximize $25x_1+30x_2$ such that:
  - $20x_1+30x_2\leq 600$ (lumber)
  - $5x_1+4x_2\leq 40$ (labor)
  - $x_1\geq 4$ (contract)
  - $x_2\geq 2$ (contract)

(x_1) (x_2)

$\max_{x_1,x_2} f = 25x_1+30x_2$

$\begin{cases} 20x_1+30x_2\leq 600 \\ 5x_1+4x_2\leq 40 \\ x_1\geq 4 \\ x_2\geq 2\end{cases}$

Linear programs

- A discrete optimization problem is called a linear program if:
  - there is a unique objective function
  - the objective functions and the constraint functions are linear
  - the objective functions and the constraint functions are non-parametric
  - the variables take value over ~~rational~~ real numbers
- If there are no constraint functions: unconstrained problem
  - Example: model fitting through the sum of absolute deviations: minimize $\Sigma_i|y_i-y(x_i)| = r$ $\quad y(x_i)=ax_i+b \quad -r\leq ax_i+b\leq r$
- A linear program is called an integer program if at least one variable is restricted to integer values only

NP-hard!

$n\sim f_{\ell}(n)$

$f(x_1,\dots,x_m) = a_1x_1+\dots+a_mx_m+b = \vec{a}^T\vec{x}+b$

$g_i(x_1,\dots,x_m) = d_{i_1}x_1+\dots+d_{i_m}x_m+d_{i_{m+1}}$ — $|$ + .

$\geq c_i \quad i=1,\dots,p$

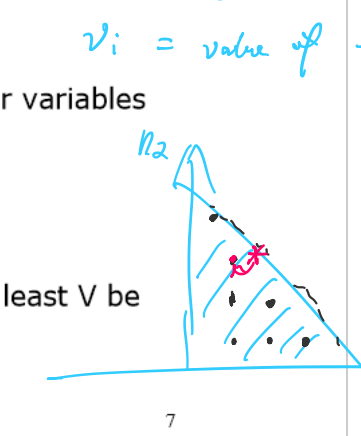
$f(\omega, x_1,\dots,x_m)$

Example of an integer problem

- The knapsack problem
  - Given a set of items, each with a weight and a value
  - Determine the number of each item to include in a collection so that the total weight is less than or equal to a given limit and the total value is as large as possible.
- Formulation:
  - we have m items with weights and values as follows: $(w_1,v_1),\dots,(w_m,v_m)$
  - we have a rucksack of capacity at most C
  - maximize $n_1v_1+\dots+n_mv_m$ over the non-negative integer variables $(n_1,\dots,n_m)$ such that $n_1w_1+\dots+n_mw_m\leq C$
- Computational complexity
  - NP-hard in the formulation above
  - NP-complete as a decision problem: can a value of at least V be achieved without exceeding total weight W?

w_i: weight of item i

v_i = value of item i



Example of a multiobjective program

- Investment problem: an investor has \$40.000 to invest in
  - savings with return 7% x
  - municipal bonds at 9% y
  - stocks that have consistently averaged at 14% z
  - various option have different degrees of risk
- Goals for the investment:
  1. yearly return of at least \$5000 $\quad 7\%x + 9\%y + 14\%z \geq 5000$
  2. invest at least \$10.000 in stocks $\quad z\geq 10000.$
  3. investment in stocks should not exceed the combined total in bonds and savings $\quad z\leq x+y$
  4. have a liquid savings account between \$5000 and \$15.000 $\quad 5000\leq x\leq 15000$
- A first attempt to a solution: invest as much as possible in the highest-return option available, as little as possible in the lowest-return option available
  - savings have the smallest return: only invest \$5000 in them (constraint 4)
  - the rest into stocks and bonds with as much as possible into stocks: \$20.000 in stocks, \$15.000 in bonds (see 3)
  - return: 0.07x5000 + 0.09x15000 + 0.14x20000=4500, violating constraint 1

$z=20000$

$y=15000$

$x=5000$

Example (continued)

- Denote by x,y,z the investment in savings, bonds, stocks, resp.
- Formulate the constraints:
  1. $0.07x+0.09y+0.14z\geq 5000$ ✓
  2. $z\geq 10000$ ✓
  3. $z\leq x+y$ ✓
  4. $5000\leq x\leq 15000$ ✓
  5. $x+y+z\leq 40000$ ✓
- Clear from our first attempt at a solution (Greedy) that not all constraints can be satisfied
- Reformulate the problem: we can fail goals 3 and 4, but we should minimize the total amount with which we fail the two goals $\quad 1:1.$
  - Minimize G_3+G_4 such that:
    1. $0.07x+0.09y+0.14z\geq 5000$
    2. $z\geq 10000$
    3. $z-G_3\leq x+y$
    4. $5000-G_4\leq x\leq 15000$
    5. $x+y+z\leq 40000$
  - where x,y,z are non-negative
  - A solution for the reformulated problem can be found, e.g., through the Simplex method

stock increases!

saving reduces!

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discrete time

Dynamic programming

- A type of optimization where not all decision have to be taken at the same time – they can be taken in stages
- Example:
  - a rancher with an initial herd of k cattle
  - he wants to retire after N years, when he will sell all remaining cattle
  - each year he decides how many cattle to sell: profit p_i in year i
  - the number of cattle kept in year i will double in year i+1
  - characteristic of the problem: each year he takes a (local) decision of taking a larger profit vs building the future; the optimization goal is a global one, concerned with the profit after N years.

$C_0 = k$

p_0

$C_1 = 2(k-p_0)$

$\vdots$

$C_N = 2(C_{N-1}-p_{N-1})$

- Focus in the remaining of the lecture on solving linear programs
  - geometric solutions
  - algebraic solutions
  - the Simplex method

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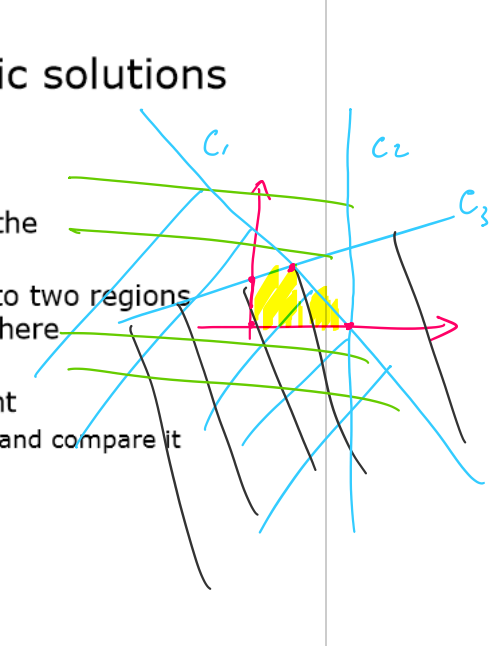
1. Linear programming: geometric solutions

Linear programs: geometric solutions

- Basic insight:
  - a linear equation $a_1x_1+a_2x_2=b$ can be seen as a line in the bidimensional plane (x_1,x_2) $\quad$ in $\mathbb{R}^2$
  - the line will separate the bidimensional plane (x_1,x_2) into two regions (half-planes), one where $a_1x_1+a_2x_2<b$, and the other where $a_1x_1+a_2x_2>b$
  - identify which region of the plane satisfies the constraint
    - choose any point (x_1,x_2) in the region, calculate $a_1x_1+a_2x_2$ and compare it to b
  - repeat the same for all constraints
  - take the intersection of all the identified regions

$f = \vec{a}^T\cdot\vec{x}+b$

  - Note: the same is true for more than 2 variables; a linear equation $a_1x_1+\dots+a_nx_n=b$ can be seen as a hyperplane (dimension n-1) in the n-dimensional plane



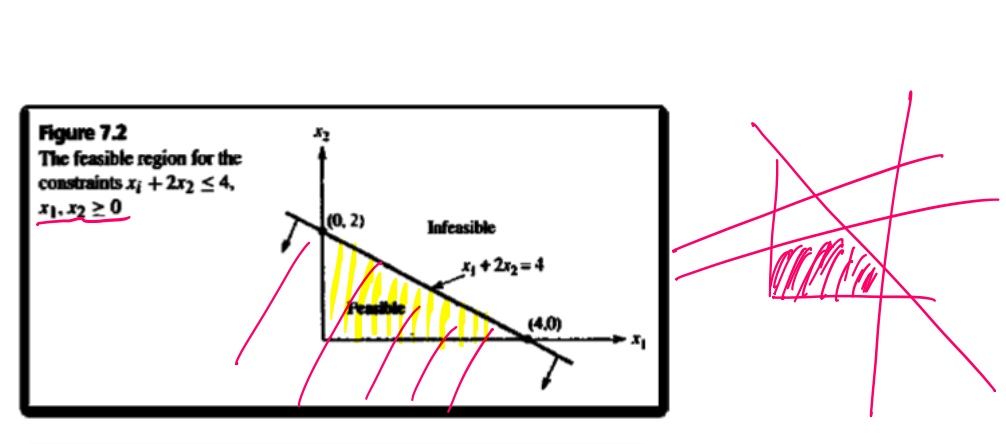


Figure 7.2 The feasible region for the constraints $x_1+2x_2\leq 4$, $x_1, x_2\geq 0$

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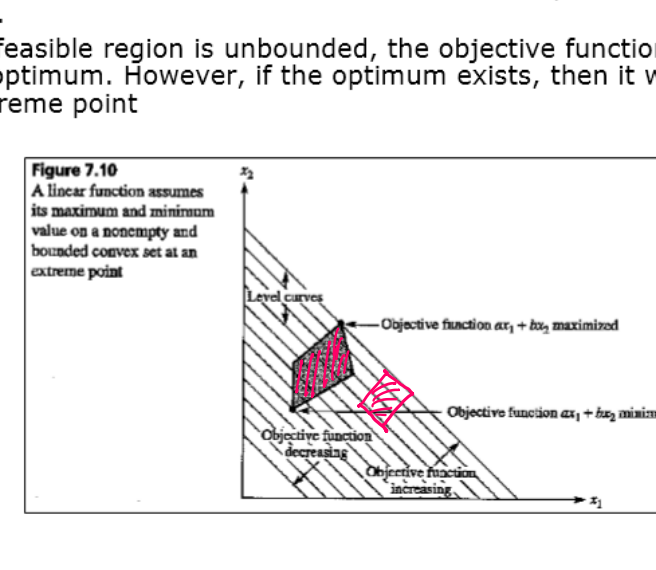
Geometric solutions to linear programs (2)

- Theorem. The points satisfying the constraint of a linear program (the feasible region) form a convex set.
  - If the feasible region is bounded, then the objective function attains both its maximum and its minimum value at extreme points of the feasible region.
  - If the feasible region is unbounded, the objective function may not have a finite optimum. However, if the optimum exists, then it will be attained at an extreme point



x_2
Level curves
Objective function ax_1+bx_2 maximized
Objective function ax_1+bx_2 minimized
Objective function decreasing
Objective function increasing
x_1

Figure 7.10 A linear function assumes its maximum and minimum value on a nonempty and bounded convex set at an extreme point

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Example

$60x_1+30x_2$ $\quad 10x_1+30x_2$

- Recall the carpenter's problem: maximize $25x_1+30x_2$ such that:
  - $20x_1+30x_2\leq 600$ (lumber)
  - $5x_1+4x_2\leq 40$ (labor)
  - $x_1\geq 0$ (contract)
  - $x_2\geq 0$ (contract)
- Find the solution at the extreme points
  - A(0,0): \$0
  - B(24,0): \$600
  - C(12,15): \$750
  - D(0,23): \$690
- Answer: make 12 tables and 15 bookshelves for a profit of \$750

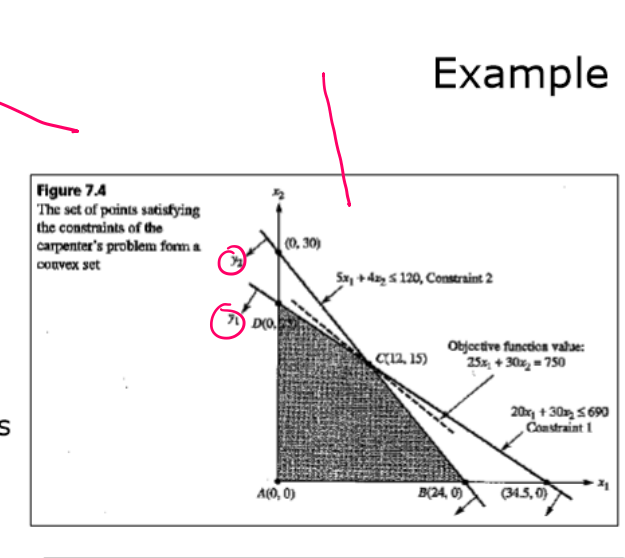

Figure 7.4 The set of points satisfying the constraints of the carpenter's problem form a convex set

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① draw the picture;

② calculate the intersection pts.

③ calculate the cost function.

④ compare & take the max/min

Another example

- A model fitting problem
  - The model is y=cx
  - The data is (1,2); (2,5), (3,8)
  - Fit the model to minimize the largest absolute deviation
  - Model formulation: let the largest absolute deviation be r

    |2-c|≤r; |5-2c|≤r; |8-3c|≤r
  - Equivalently:
    - r-(2-c)≥0; (constraint 1)
    - r+(2-c)≥0 (constraint 2)
    - r-(5-2c)≥0; (constraint 3)
    - r+(5-2c)≥0 (constraint 4)
    - r-(8-3c)≥0; (constraint 5)
    - r+(8-3c)≥0 (constraint 6)
  - Geometrical approach: plot the graphs of the linear functions obtained by transforming each inequality above into an equation

$\min_c \max_i |y_i - cx_i|$

indep. var. $\underline{c}$

dep. var. $\underline{r}$:= residual

min r

s.t. $|y_i - cx_i|\leq r$

$\begin{cases} y_i - cx_i\leq r \\ y_i - cx_i\geq -r\end{cases} \iff \begin{matrix} -x_i\underline{c} - \underline{r}\leq -y_i \\ -x_i\underline{c} + \underline{r}\geq -y_i\end{matrix}$

Example (continued)

- The intersection forms a convex set in the (c,r) plane with corners B-C
  - check the value of the objective function f(r)=r in points B and C
  - minimum reached in B
  - Solution: c=5/2; r=1/2
  - Check the solution: plot y=5/2 x against the data

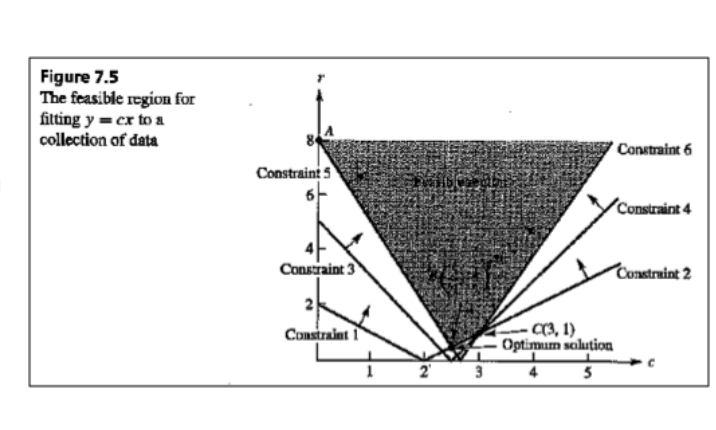

Figure 7.5 The feasible region for fitting $y=cx$ to a collection of data

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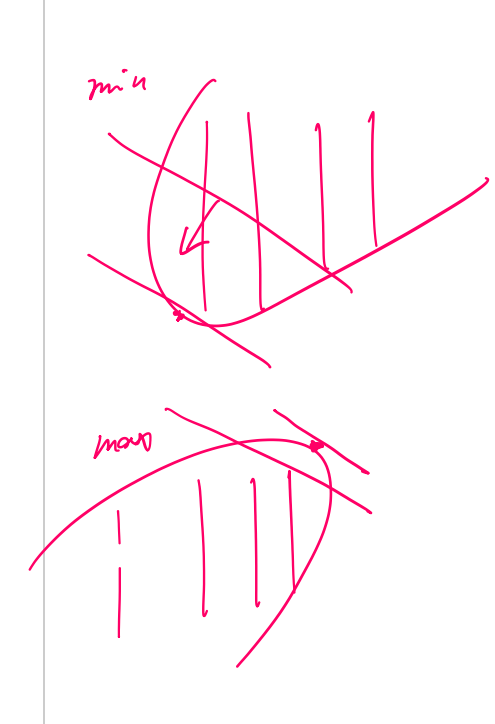

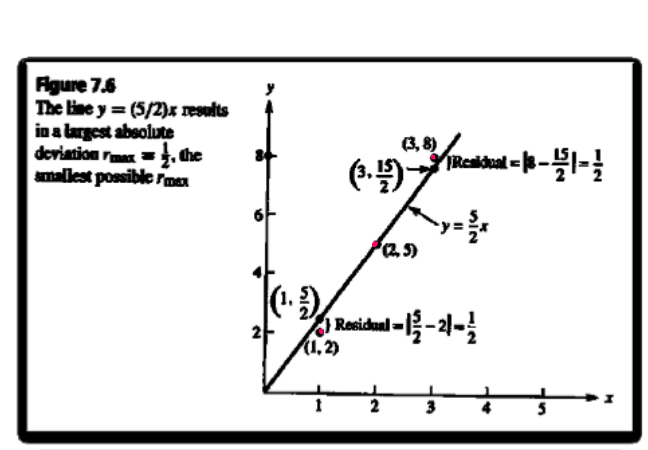

Figure 7.6 The line $y=(5/2)x$ results in a largest absolute deviation $r_{\max}=\frac{1}{2}$, the smallest possible $r_{\max}$

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Empty and unbounded feasible regions

- Empty feasible regions
  - might get empty feasible regions because of inconsistent constraints
  - Example: x<2 and x>4.
  - in this case there is obviously no solution to the optimization problem
- Unbounded feasible regions
  - we had such a case in the last example
  - depending on the optimization problem to solve, there might not be a solution
  - example: what if in the previous example we had to maximize function f(r)=2r+3

Level curves of the objective function

- Consider again the carpenter example
  - The objective function to maximize was $f(x_1,x_2)=25x_1+30x_2$
  - Note the plots below
    - The level curves are parallel to each other
    - The maximum level curve touching the feasible region will do so at an extreme point
    - It may also touch it at two extreme points simultaneously (along one of its borders)

$f = \vec{a}\cdot\vec{x}$ $\quad f=\frac{1}{2}x^TAx$ $\quad$ linprog $\quad$ quadprog

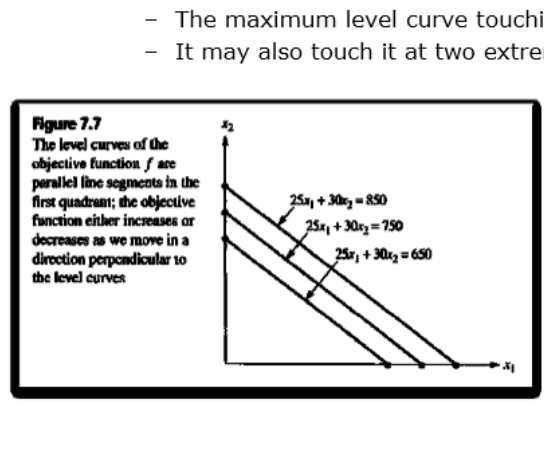

Figure 7.7 The level curves of the objective function f are parallel line segments in the first quadrant; the objective function either increases or decreases as we move in a direction perpendicular to the level curves

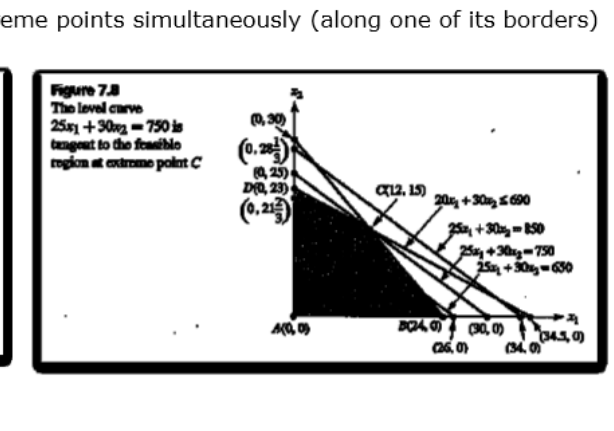

Figure 7.8 The level curve $25x_1+30x_2=750$ is tangent to the feasible region at extreme point C

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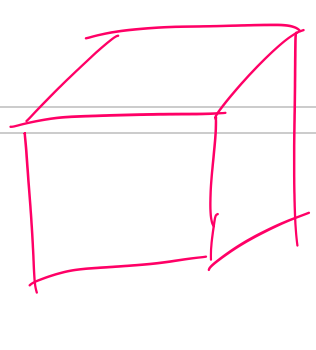
2. Linear programming: algebraic solutions

in $\mathbb{R}^n$

n-hyperplanes / intersection pts.

$\mathbb{R}^2$

2-line / intersection pts.



Algebraic approach

- Approach:
  - Find all intersection points of the constraints $\quad A\vec{x}=b \quad \square_{n\times n}$
    - Consider all possible pairs of constraints
  - Determine which intersection points are feasible; this gives the extreme points
    - Check the intersection points against all constraints
  - Evaluate the objective function at the extreme points
  - Choose the extreme point giving the optimal value for the objective function

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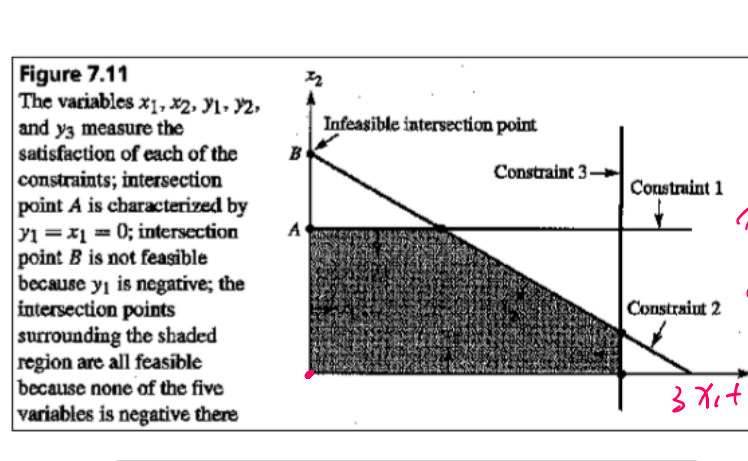

Figure 7.11 The variables x_1, x_2, y_1, y_2, and y_3 measure the satisfaction of each of the constraints; intersection point A is characterized by $y_1=x_1=0$; intersection point B is not feasible because y_1 is negative; the intersection points surrounding the shaded region are all feasible because none of the five variables is negative there

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$x_1\leq 4$ $\quad x_2\leq 2$ $\quad 3x_1+2x_2\leq 5$

$\begin{cases} x_1+y_1=4 \\ x_2+y_2=2 \\ 3x_1+2x_2+y_3=5\end{cases}$

$x_1\geq 0$ $\quad x_2\geq 0$ $\quad y_1\geq 0$ $\quad y_2\geq 0$ $\quad y_3\geq 0$

$\begin{pmatrix} x_1 \\ x_2\end{pmatrix} \rightarrow \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \\ y_3\end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 5\end{pmatrix}$ $\quad C_5^3$

$f(x_1,x_2) = \vec{a}^T\cdot\vec{x}$

$= x_1+x_2$ $\rightarrow$ $\underbrace{0 \quad \text{''} \quad \dots}$

Algebraic approach (2)

- Enumerating all intersection points for 2-dimensional problems
  - Without loss of generality assume that the constraints are linear inequalities: $a_1x_1+a_2x_2\leq b$, where $x_i\geq 0$
  - Add a new variable y_1 and transform the inequality above into a linear equation $a_1x_1+a_2x_2+y_1=b$, where $x_i\geq 0$, $y_1\geq 0$
    - $y_1=0$ is equivalent to a point on the border of the constraint
  - Repeat for all other constraints (new variable for each)
    - We will have now 2+n variables and n linear equations
    - Solve them over non-negative integers
  - Finding the intersection point comes to solving a system of two linear equations in non-negative variables
    - Some intersection points will be between two border lines, i.e., with the corresponding y variables set to 0
    - Other intersection points will be between a border line and an axis – an y variable and an x variable set to 0
    - Another intersection point will be the origin: $x_1=x_2=0$
    - In other words: for any pair of variables, set them to 0 and solve the remaining system of n equations with n variables

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Example: the carpenter's problem

- Recall the carpenter's problem: maximize $25x_1+30x_2$ such that:
  - $20x_1+30x_2\leq 690$; $5x_1+4x_2\leq 120$;
  - $x_1\geq 0$; $x_2\geq 0$ (assume here for simplicity no pre-existing contract)
- Convert each of the inequalities into equations:
  - $20x_1+30x_2+y_1=690$; $5x_1+4x_2+y_2=120$;
  - $x_1\geq 0$; $x_2\geq 0$; $y_1\geq 0$; $y_2\geq 0$
- Consider the set of variables $\{x_1; x_2; y_1; y_2\}$
  - For all pairs of variables, set the chosen pair to 0 and solve the resulting system of 2 equations with 2 unknowns: 6 possibilities
  1. $x_1=x_2=0$; in this case we obtain $y_1=690$; $y_2=120$;
  2. $x_1=y_1=0$; in this case we obtain $x_2=23$; $y_2=28$;
  3. $x_1=y_2=0$; in this case we obtain $x_2=30$; $y_1=-210$; unfeasible point
  4. $x_2=y_1=0$; in this case we obtain $x_1=34.5$; $y_2=-52.5$; unfeasible point
  5. $x_2=y_2=0$; in this case we obtain $x_1=24$; $y_1=210$;
  6. $y_1=y_2=0$; in this case we obtain $x_1=12$; $x_2=15$;

$\begin{pmatrix} x_1 \\ x_2\end{pmatrix} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2\end{pmatrix}$ $\quad C_4^2 = 6$

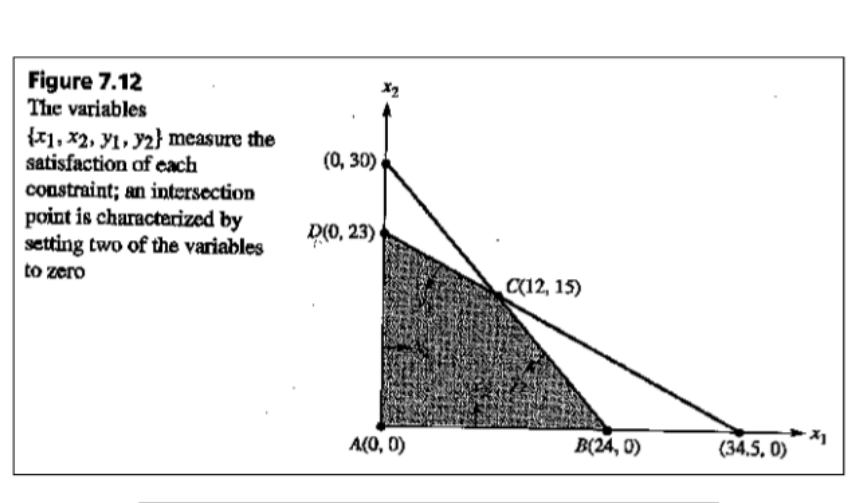

Figure 7.12 The variables $\{x_1, x_2, y_1, y_2\}$ measure the satisfaction of each constraint; an intersection point is characterized by setting two of the variables to zero

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Algebraic approach (3)

- Enumerating all intersection points for m>2 variables
  - Without loss of generality assume that the constraints are linear inequalities: $a_1x_1+\dots+a_mx_m\leq b$, where $x_i\geq 0$
  - Add a new variable y_1 and transform the inequality above into a linear equation $a_1x_1+\dots+a_mx_m+y_1=b$, where $x_i\geq 0$, $y_1\geq 0$
    - $y_1=0$ is equivalent to a point on the border of the constraint
  - Repeat for all other constraints (new variable for each)
    - We will have now m+n variables and n linear equations
    - Solve them over non-negative integers
  - For all possible sets S of m variables, set the variables in S to 0 and solve the resulting set of n equations with n unknowns to yield all intersection points
- Computational complexity
  - Identify (m+n)!/(m!n!) intersection points
  - Solve an linear system of n equations, n unknowns for each
- Note: some (potentially many/most) intersection points might not be feasible

$\begin{pmatrix} x_1 \\ \vdots \\ x_n \\ y_1 \\ \vdots \\ y_m\end{pmatrix}$

In theory, we will have $\underline{C_{m+n}^m}$ intersection pts

$\begin{cases} \text{discard all infeasible pts} \\ \text{calculate the cost functions for all feasible pts}\end{cases}$

compare & take the extreme pts