1 Question 1

Prove that a discrete state space stochastic process which satisfies Markovian property has

$$P^{(n+m)} = P^{(n)}P^{(m)}$$

and thus

$$P^{(n)} = P^{(n-1)}P = P \times \cdots \times P = P^n,$$

where $P^{(n)}$ denotes the matrix of n-step transition probabilities and P^n denotes the nth power of the matrix P.

1.1 The probability :

Let the n-step transition probabilities be denoted by $p_{ij}^{(n)}$: the probability that a process in state i will be in state j after n additional transitions. That is,

$$p_{ij}^{(n)} = P\{X_{n+m} = j | X_m = i\} \; n \geq 0, \; i, \; j \geq 0,$$

This probability does not depend on m, either!

$$p_{ij}^{(1)} = p_{ij}$$

1.2 Proof :

$$
\begin{aligned}
p_{ij}^{(n+m)} &= P\{X_{n+m} = j | X_0 = i\} \\
&= \sum_{k=0}^{\infty} P\{X_{n+m} = j, X_n = k | X_0 = i\} \\
&= \sum_{k=0}^{\infty} P\{X_{n+m} = j | X_n = k, X_0 = i\} P\{X_n = k | X_0 = i\} \\
&= \sum_{k=0}^{\infty} P\{X_{n+m} = j | X_n = k\} P\{X_n = k | X_0 = i\} \\
&= \sum_{k=0}^{\infty} p_{ik}^{(n)} p_{kj}^{(m)}.
\end{aligned}
$$

In matrix form, we have

$$P^{(n+m)} = P^{(n)} P^{(m)}$$

and thus

$$P^{(n)} = P^{(n-1)} P = P \times \cdots \times P = P^n,$$

where $P^{(n)}$ denotes the matrix of n-step transition probabilities and $P^{(n)}$ denote the nth power of the matrix P.

$$p_{ij}^{(n+m)} = \sum_{k=0}^{\infty} p_{ik}^{(n)} p_{kj}^{(m)} \quad \text{for all } n, m \geq 0, \text{ all } i, j \geq 0.$$

In that, we have assumed that, without loss any generality, the state space E is {0, 1, 2, $\cdots$}. If we do not assume this, and let the state space be E, then the equation should be written as

$$p_{ij}^{(n+m)} = \sum_{k \in E} p_{ik}^{(n)} p_{kj}^{(m)} \quad \text{for all } n, m \geq 0, \text{ all } i, j \in E.$$

2 Question 2

Consider adding a pizza delivery service as an alternative to the dining halls. Table 1 gives the transition percentages based on a student survey. Determine the long-term percentages eating at each place. Try several different starting values. Is equilibrium achieved in each case? If so, what is the final distribution of students in each case?

		Next state		
		Grease Dining Hall	Sweet Dining Hall	Pizza delivery
Present state	Grease Dining Hall	0.25	0.25	0.50
	Sweet Dining Hall	0.10	0.30	0.60
	Pizza delivery	0.05	0.15	0.80

Table 1: Survey of dining at College USA

$$p = \begin{bmatrix} 0.25 & 0.25 & 0.50 \\ 0.10 & 0.30 & 0.60 \\ 0.05 & 0.15 & 0.80 \end{bmatrix}$$

from $qp = p$, we have :

$$q = \begin{bmatrix} 0.0741 & 0.1852 & 0.7407 \end{bmatrix}$$

So that we can see for every case, assume that the total number of customers are n, then in the equilibrium, Grease Dining Hall has $0.0741n$ customers, Sweet Dining Hall has $0.1852n$ customers, Pizza delivery has $0.7407n$ customers.