Lecture 11

Modeling with systems of ODEs

Modeling with differential equations

Systems of ODEs

- Modeling strategy
  - Focus on the change in the values of all variables
  - We describe the change as a function of the current values of all variables
  - If the process is deterministic and it takes place continuously in time, the model may be formulated in terms of differential equations

x(t) y(t)

x'(t) y'(t)

Step 1. Theory of ODE system!

Step 2. Model ODE (sys) solution?

Systems of differential equations

- Very often, models describe interactive situations: several variables involved, their change depends on all the other variables
  - An animal may serve as prey for another
  - Two species may depend on each other for mutual support
  - Two or more species may compete for the same resource
- Model through a system of differential equations
  - Each equation gives the rate of change of one variable as a function of the current values of all variables
  - Solve all equations simultaneously
- Bad news: the equations are often non-linear and in general they cannot be solved analytically
- Good news: many numerical techniques exist to analyze the behavior of such (systems of) equations

x(t) y(t)

competitive model → $\begin{cases} x'(t) = a\,x(t) - c\,x(t)y(t) & \text{: predator} \\ y'(t) = b\,y(t) - d\,x(t)y(t) & \text{: predator} \end{cases}$

collaborative model → $\begin{cases} x'(t) = a\,x(t) - c\,x(t)y(t) & \text{: predator} \\ y'(t) = -b\,y(t) + d\,x(t)y(t) & \text{: prey} \end{cases}$

Difficulties: ① Nonlinearity

② No explicit solutions.

Quantitative sols ⇒ Qualitative behaviours

定量 定性

Content

- Basic notions
- Graphical solutions
- Examples
  - competitive hunter model
  - predator-prey model
  - economic aspects of an arms race

Autonomous systems of ODEs

- Definition. A system of ODEs of the form $dx_i/dt=f_i(x_1,x_2,\dots x_n)$ is called autonomous
  - In other words, no independent variable t on the right hand side
  - the system is not time-dependent; "pseudo-time"
  - it only depends on the variation of its variables

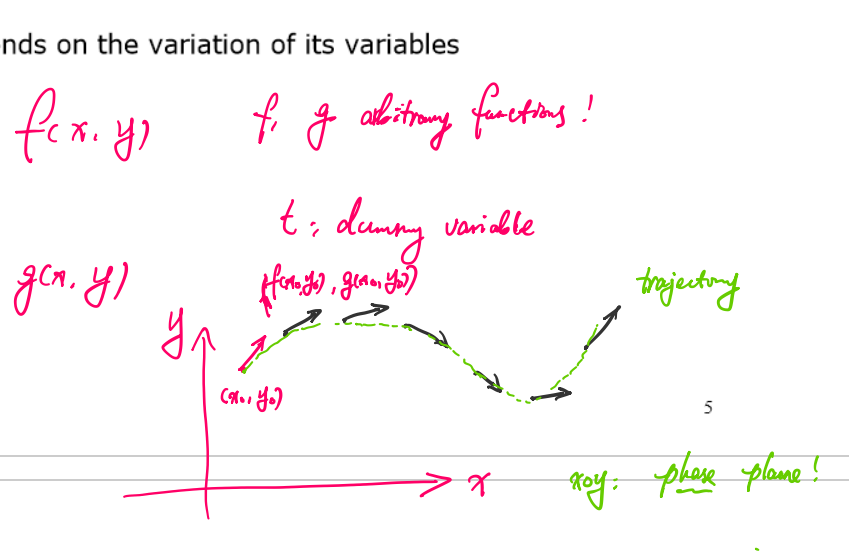
$$\begin{cases} \dfrac{d\,x(t)}{dt} = f(x,y) \\ \dfrac{dy(t)}{dt} = g(x,y) \end{cases}$$

f, g: arbitrary functions!

t: dummy variable

$x(0) = x_0$

$y(0) = y_0$



Equilibria

- Equilibrium points / steady states
  - A system of ODEs $dx_i/dt=f_i(x_1,x_2,\dots x_n)$, $i=1,2,\dots,n$
  - $a=(a_1,a_2,\dots,a_n)$ is an equilibrium point for the system if $f_i(a)=0$, for all i
  - Equivalently, if $x_i(0)=a_i$, for all i, then the system has a solution the constant functions $x_i(t)=a_i$
  - **Stable equilibrium**
    - a is **stable** if it is true that x(0) being "close" to a implies that x(t) is "close" to a, for all t>0
    - a is called **unstable** otherwise
  - **Asymptotically stable**
    - a is asymptotically stable if it is true that x(0) being "close" to a implies that $\lim_{t\to\infty} x(t)=a$

$\begin{cases} f(x,y) = 0 \\ g(x,y) = 0 \end{cases}$

sum

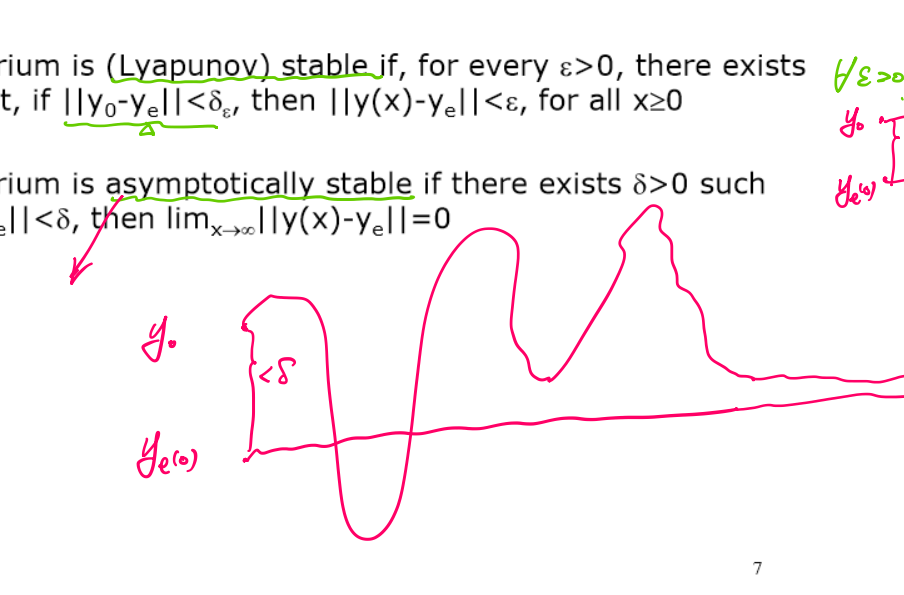
pts in phase plane $\begin{cases} \text{Equilibrium pts:} \begin{cases} \text{Stable Equilibrium pts} - \{\text{Asymptotically Stable Equilibrium pts.} \\ \text{Unstable Equilibrium pts.} \end{cases} \\ \text{Non-equilibrium pts.} \end{cases}$

Lyapunov stability (optional slide)

- Consider an autonomous dynamical system $dy/dx=f(y(x))$, $y(0)=y_0$, where $f:\Re^n \to \Re^n$ is continuous and has an equilibrium y_e
- The equilibrium is (Lyapunov) stable if, for every ε>0, there exists $\delta_\varepsilon>0$ such that, if $||y_0-y_e||<\delta_\varepsilon$, then $||y(x)-y_e||<\varepsilon$, for all x≥0
- The equilibrium is asymptotically stable if there exists δ>0 such that, if $||y_0-y_e||<\delta$, then $\lim_{x\to\infty}||y(x)-y_e||=0$

$\vec{y} = \begin{pmatrix} x \\ y \end{pmatrix}$

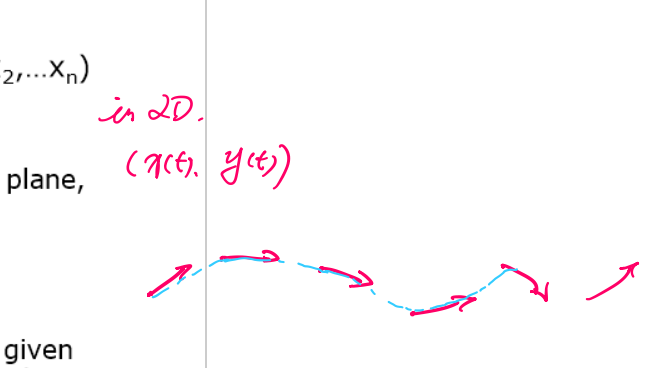
$\forall \varepsilon > 0\ \exists \delta = \delta(\varepsilon) > 0$ s.t. $\forall\ ||y_0 - y_e(t)|| < \delta_0 \Rightarrow ||y(t) - y_e(t)|| < \varepsilon$



Graphical solutions

- Consider autonomous systems of first-order ODEs $dx/dt=f_i(x_1,x_2,\dots x_n)$
  - not time dependent
  - consider its solution as describing a trajectory in the n-dimensional plane, with coordinates $(x_1(t),x_2(t),\dots x_n(t))$
    - the n-dimensional plane $(x_1,x_2,\dots x_n)$ is called a phase plane
    - convenient to think about it as the movement of a particle
  - autonomous system implies that the direction of movement from a given point on the trajectory only depends on that point, not on the time when the particle arrived in that point
    - Consequence: no trajectory can cross itself unless it is a closed curve (periodic)
    - Consequence: at most one trajectory going through any given point
    - Equivalently: two different trajectories cannot intersect

in 2D (x(t), y(t))



well-posedness: uniqueness of sols

Graphical solutions

- Consider autonomous systems of first-order ODEs $dx/dt=f_i(x_1,x_2,\dots x_n)$
  - if $(e_1,e_2,\dots e_n)$ is an equilibrium point, then the only trajectory going through that point is the constant one
    - Consequence: a trajectory that starts outside an equilibrium point can only reach the equilibrium asymptotically, not in a finite amount of time
- A non-constant motion of a particle can have one of the following 3 behaviors:
  - approaches an equilibrium point
  - moves along or approaches asymptotically a closed path
  - at least one of the trajectory components becomes arbitrarily large as t tends to infinity

① Equilibrium pts

Example

✱1 $\vec{Y} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ $\dfrac{d\vec{Y}}{dt} = A\vec{Y}$ $A = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$

if d=1 $y' = ay$ $y = Ce^{at}$

if d=2 $\boxed{\vec{Y} = C e^{\lambda_i t} \vec{\xi}_i}$ i=1,2

LHS $\dfrac{d\vec{Y}}{dt} = C\lambda_i e^{\lambda_i t}\vec{\xi}_i$

RHS $A\vec{Y} = C e^{\lambda_i t} A\vec{\xi}_i$ } LHS=RHS

$A\vec{\xi}_i = \lambda_i \vec{\xi}_i$ i=1,2

Solve $\det(A - \lambda I) = 0$ λ_1 λ_2

Null$(A - \lambda_i I)$ $\vec{\xi}_1$, $\vec{\xi}_2$

$\vec{Y} = \sum_{i=1}^{2} C_i e^{\lambda_i t} \vec{\xi}_i \Leftarrow \begin{cases} x(0) = x_0 \\ y(0) = y_0 \end{cases}$

Determine C_1, C_2 by substituting initial conditions

- dx/dt=-x+y, dy/dt=-x-y
- Analytic solution: $x(t)=e^{-t}\sin(t)$, $y(t)=e^{-t}\cos(t)$
- Equilibrium point: dx/dt=dy/dt=0 leads to x=y=0
- Note: $x^2(t)+y^2(t)=e^{-2t}$, i.e., the trajectory is a circular spiral with decreasing radius approaching 0

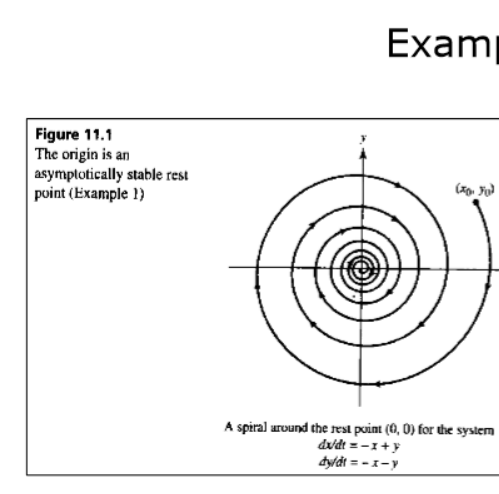


Figure 11.1 The origin is an asymptotically stable rest point (Example 1)

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Example: a competitive hunter model

- Assume we have a small pond that we desire to stock with game fish, say trout and bass. The problem we want to solve is whether it is possible for the two species to coexist
- Hypothesis
  - Unlimited amount of food available
  - The space is a limitation for the co-existance of the two species
    - The more there is of the other species, the smaller the growth rate;
    - Assume the causes to be in the competition for space – model it as proportional in the number of possible interactions between the two species
- Model formulation
  - The change in the level of trout X(t): dX/dt = aX(t) – bX(t)Y(t)
  - Similar reasoning for the level of bass Y(t): dY/dt=mY(t) – nX(t)Y(t)

Example: a competitive hunter model

- Model: dX/dt = aX(t) – bX(t)Y(t), dY/dt=mY(t) – nX(t)Y(t)
- Question: can the two populations reach an equilibrium where both are non-zero
  - Answer: ax-bxy=0, my-nxy=0
  - Solution: either x=y=0, or x=m/n, y=a/b

Let $\frac{dx}{dt} = \frac{dy}{dt} = 0$

- Difficulty: impossible to start with exactly the equilibrium values (they might not even be integers)
  - so, we cannot expect to start in an equilibrium point
  - study the property of the equilibrium, hoping it is a stable one

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Example: a competitive hunter model (continued)

dX/dt = aX(t) – bX(t)Y(t), dY/dt=mY(t) – nX(t)Y(t)

- Equilibrium points: (0,0), (m/n, a/b)
- Additional question: what is the behavior if we start close to the equilibrium point?
  - Solution: we study the tendency of X(t), Y(t) to increase/decrease around the equilibrium point. For this, we study the sign of the derivatives of X(t), Y(t)
  - dX/dt≥0 ⇔ aX-bXY≥0 ⇔ a/b≥Y
  - dY/dt≥0 ⇔ mY-nXY≥0 ⇔ m/n ≥ X

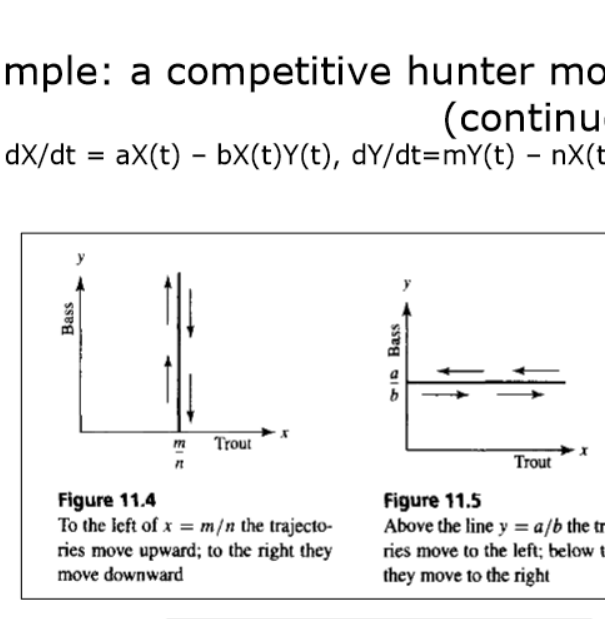
Figure 11.4 To the left of x = m/n the trajectories move upward; to the right they move downward

Figure 11.5 Above the line y = a/b the trajectories move to the left; below the line they move to the right

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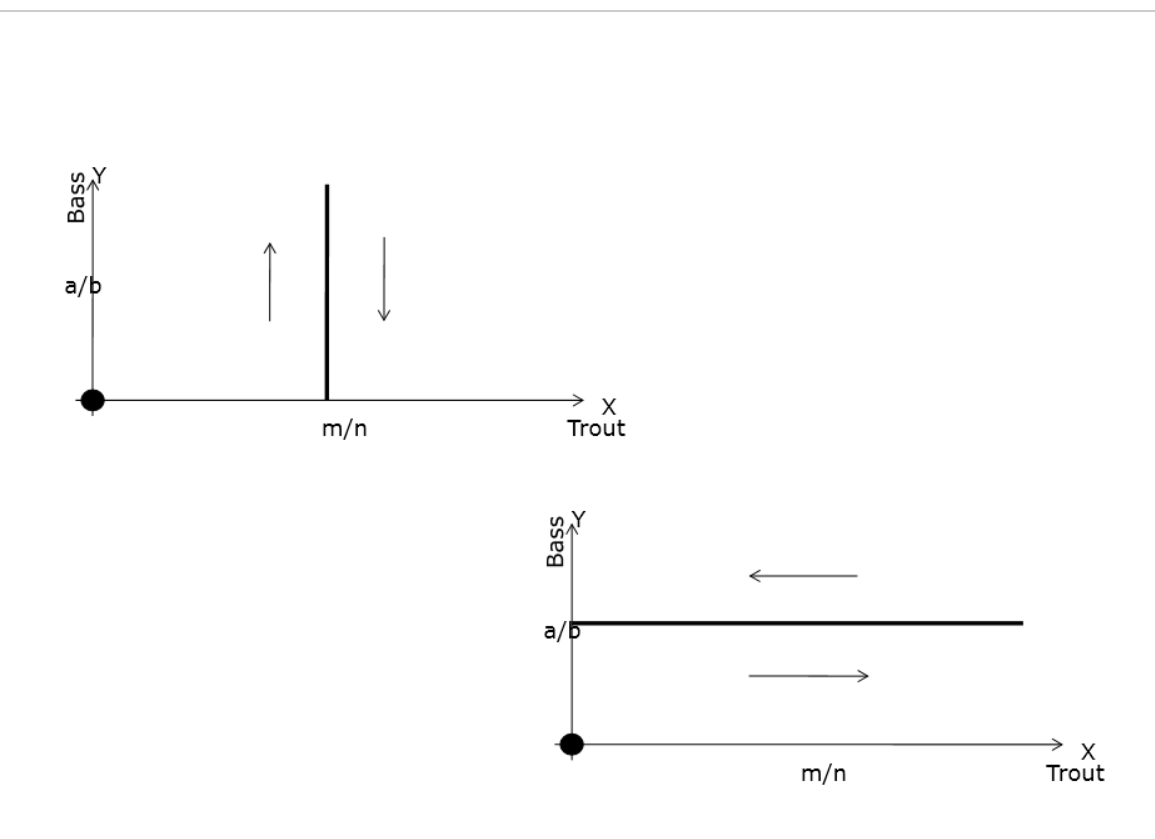
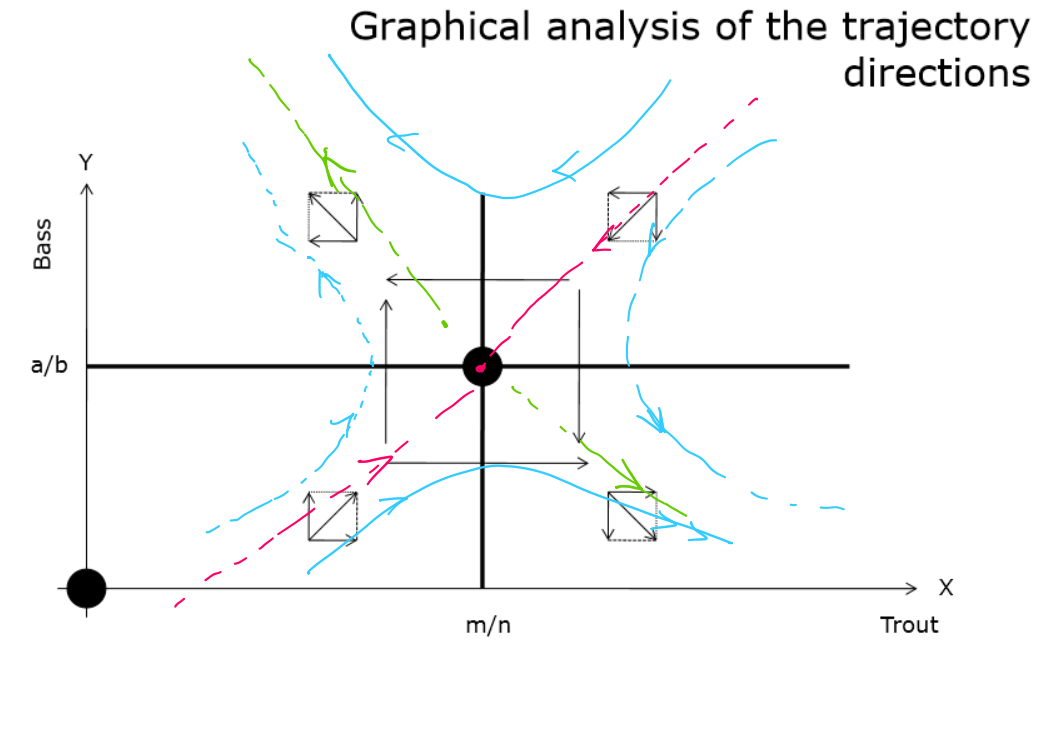
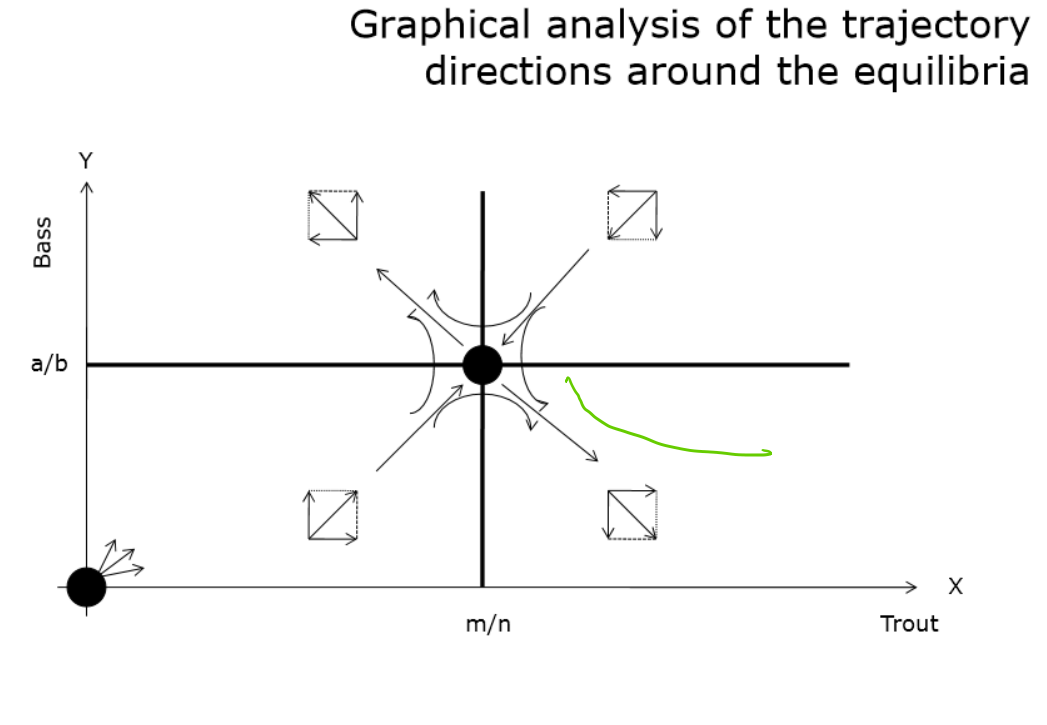


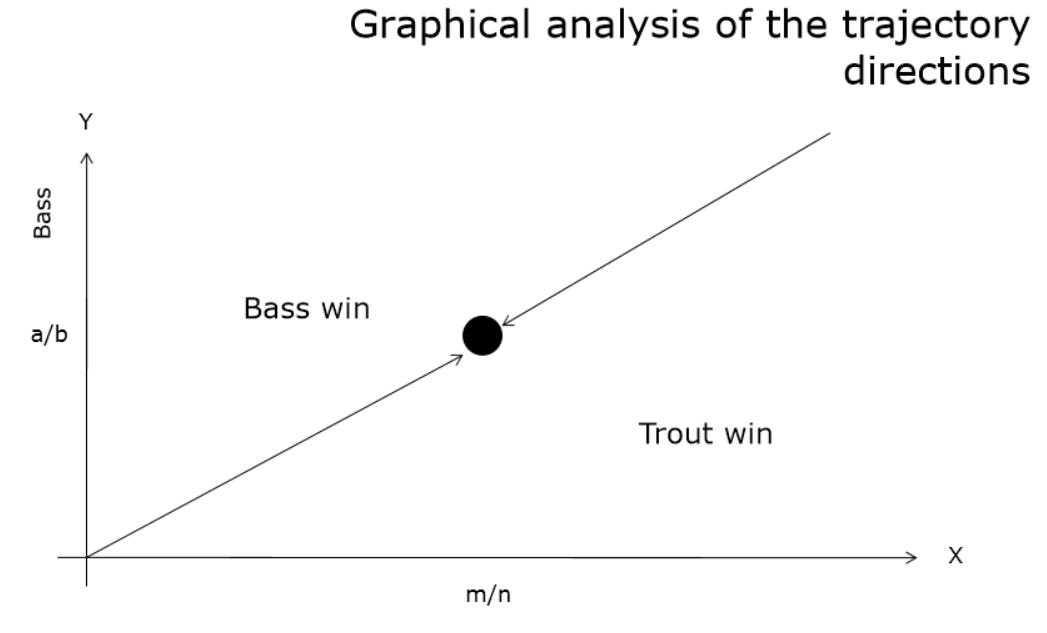
Graphical analysis of the trajectory directions



Graphical analysis of the trajectory directions around the equilibria



Graphical analysis of the trajectory directions



Conclusion: the co-existance of the two species is highly improbable

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Limits of graphical analysis

- Not always possible to determine the nature of the motion near an equilibrium based on graphical analysis
  - Example: the behavior in Fig 11.9 through graphical analysis is satisfied by all 3 trajectories in Fig 11.10
  - Example: The trajectory in Fig 11.10c could be either growing unboundedly or approach a closed curve

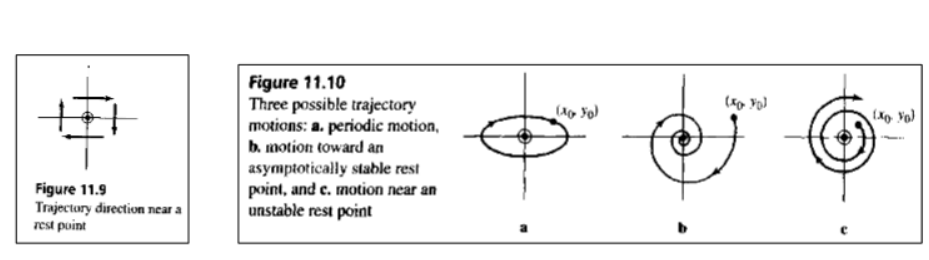
Figure 11.9 Trajectory direction near a rest point

Figure 11.10 Three possible trajectory motions: (a) periodic motion, (b) motion toward an asymptotically stable rest point, and (c) motion near an unstable rest point

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A predator-prey model

- Also known as the Lotka-Volterra model, (1910-1926)
  - A model where we have two species, one being the primary food source for the other: predator-prey model
  - Historical references: used in chemistry, ecology, biomathematics, economy
- Hypothesis
  - The prey population finds unlimited amount of food at all times
  - The food supply of the predator population depends entirely on the size of the prey population
  - Growth rate of each population taken in isolation is proportional to its size
  - Predators have limitless appetite

A predator-prey model: whales vs. krill

- Assumptions and model formulation (the Lotka-Volterra model, 1910-1926)
  - The krill population x
    - Assume that the ocean supports an indefinite growth of the krill (we do not model explicitly the plankton)
    - The population declines proportionally to the number of interactions between krill and whales
    - dx/dt=ax-bxy
  - The whale population y
    - Whale die of natural causes; constant mortality rate
    - Whales reproduce at a rate proportional to the number of interactions between krill and whales
    - dy/dt=-my+nxy
- Equilibrium points
  - dx/dt=dy/dt=0
  - (x,y)=(m/n,a/b) or (x,y)=(0,0)

A predator-prey model: graphical analysis

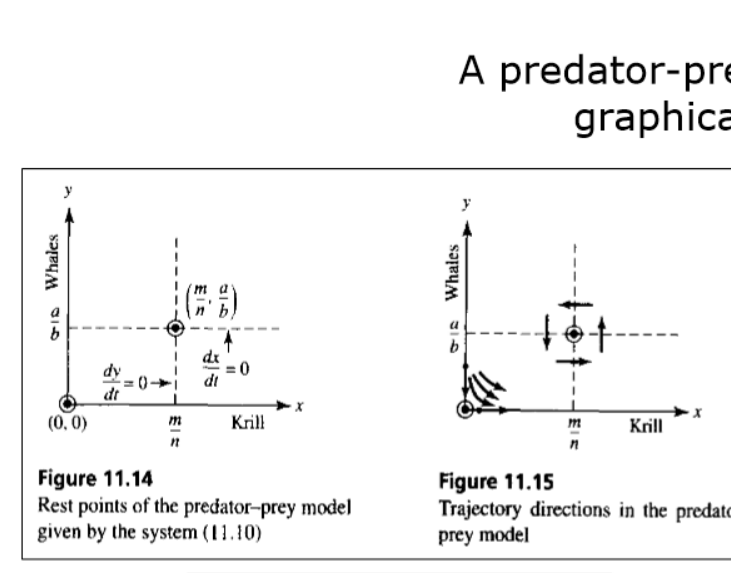
Figure 11.14 Rest points of the predator-prey model given by the system (11.10)

Figure 11.15 Trajectory directions in the predator-prey model

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Conclusion: for the analysis of the behavior around the non-zero point we need further analysis: is it periodic, (asymptotically) stable, or unstable?

Predator-prey model: analytic solution

- Note: analytic solution possible in this case
  - it reveals the periodic behavior of the model
  - skip it here; only look at the results of the numerical integration

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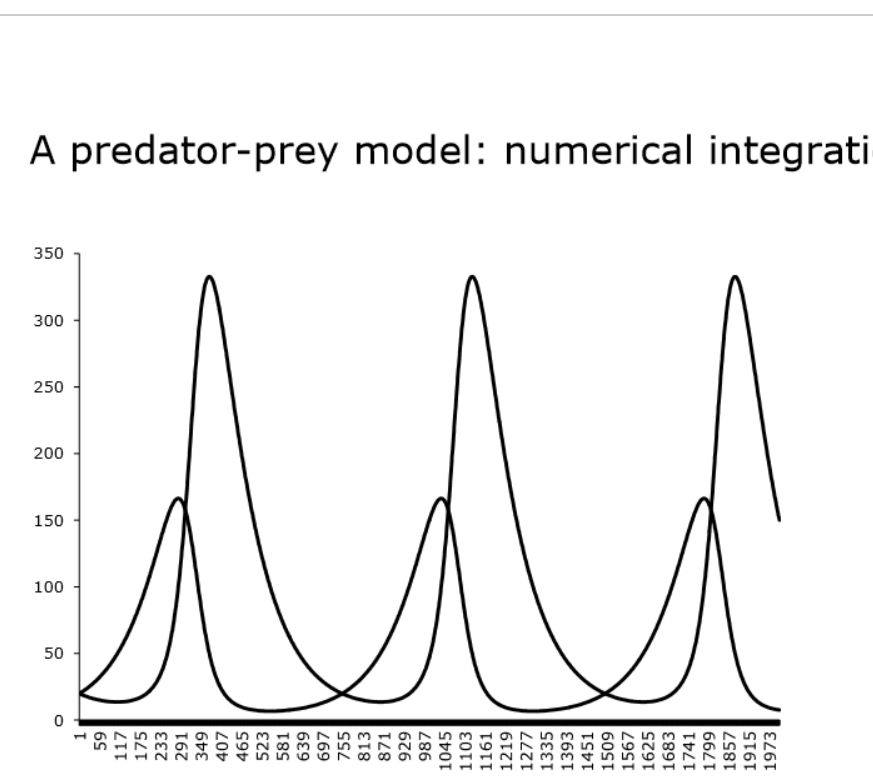
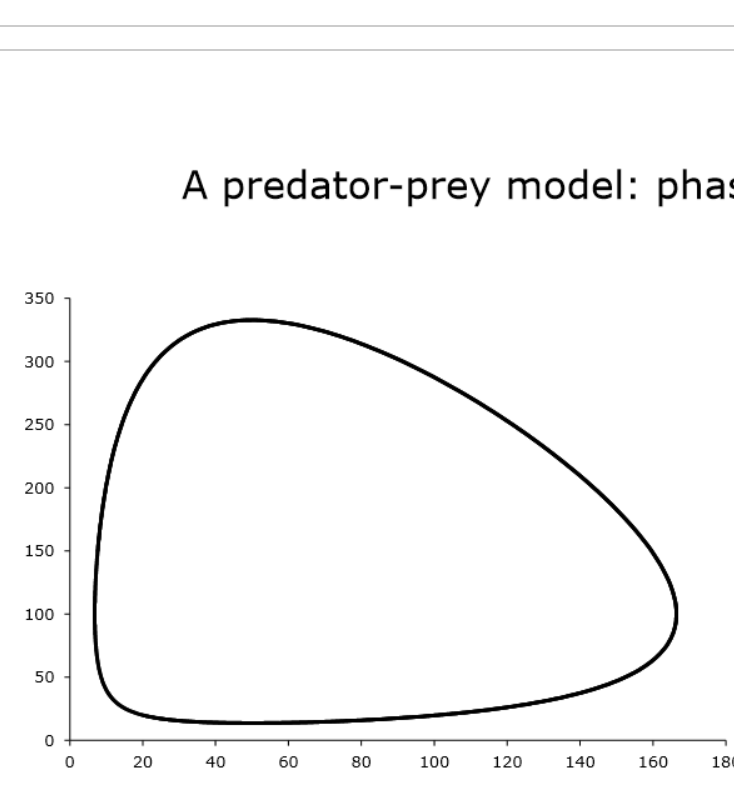
A predator-prey model: numerical integration



A predator-prey model: phase portrait



A predator-prey model: numerical integration

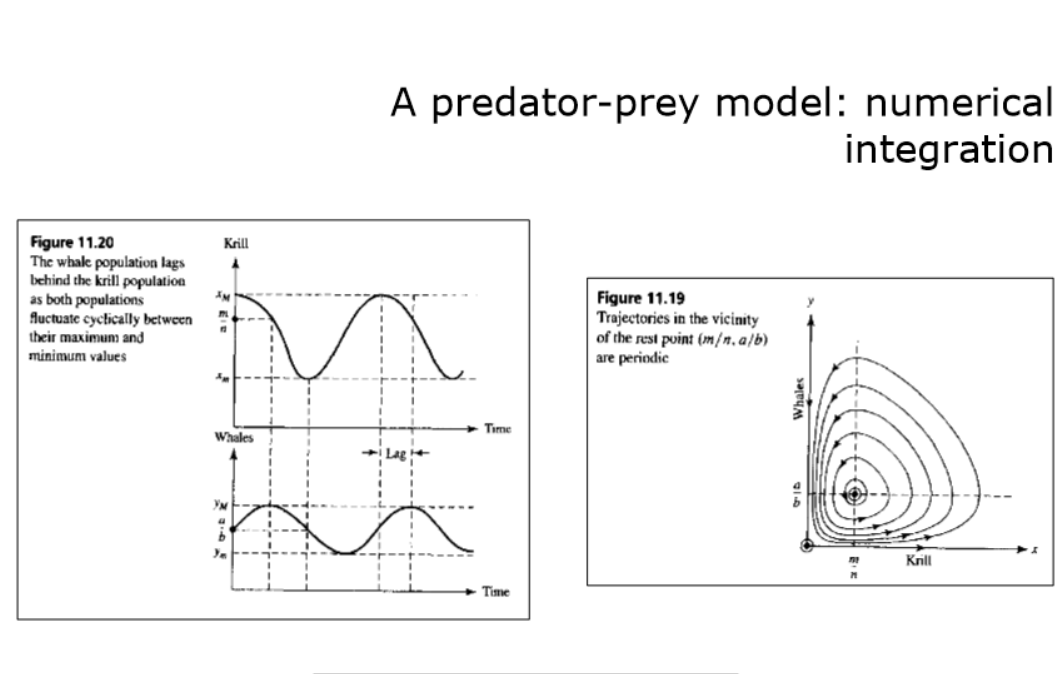


Figure 11.20 The whale population lags behind the krill population in both populations fluctuate cyclically between their maximum and minimum values

Figure 11.19 Trajectories in the vicinity of the rest point (m/n, a/b) are periodic

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Example: Economic aspects of an arms race

- Problem. Two countries engaged in an arms race. Interested in whether the arms race will lead to uncontrolled spending and eventual win by the country with better economic assets
- Assumptions
  - x is the annual defense expenditures of country 1, y of country 2
  - Driving factors for rate of increase of the defense budget x
    - the bigger it is, the less it should grow: negative influence assumed to be proportional to its current budget x (economic constraint)
    - the bigger the budget of the adversary, the more it should increase: positive influence assumed to be proportional the current budget of the adversary y (military constraint)
    - some constant level of growth is needed anyway (precautionary constraint)
- Model
  - dx/dt=-ax+by+c
  - dy/dt=mx-ny+p

Example: Economic aspects of an arms race (continued)

- Model
  - dx/dt=-ax+by+c
  - dy/dt=mx-ny+p
- Equilibrium point:
  - -ax+by+c=0; mx-ny+p=0
  - x=(bp+cn)/(an-bm), y=(ap+cm)/(an-bm)
- Just to make a choice in drawing the plots, assume that an-bm>0
- Plot the two lines and consider the intersection of the two half planes
  - Region A: dx/dt>0, dy/dt<0
  - Region B: dx/dt<0, dy/dt<0
  - Region C: dx/dt>0, dy/dt>0
  - Region D: dx/dt<0, dy/dt>0

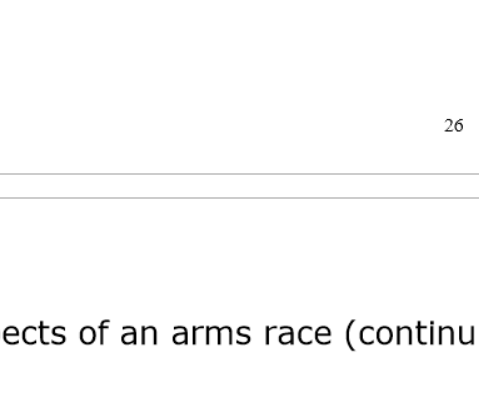
Figure 11.25 If an − bm > 0, the Model (11.19) has a unique rest point (E, F) in the first quadrant. Along the line

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Example: Economic aspects of an arms race (continued)

- dx/dt=-ax+by+c
- dy/dt=mx-ny+p
- Region A: dx/dt>0, dy/dt<0
- Region B: dx/dt<0, dy/dt<0
- Region C: dx/dt>0, dy/dt>0
- Region D: dx/dt<0, dy/dt>0

Conclusion: asymptotically stable point

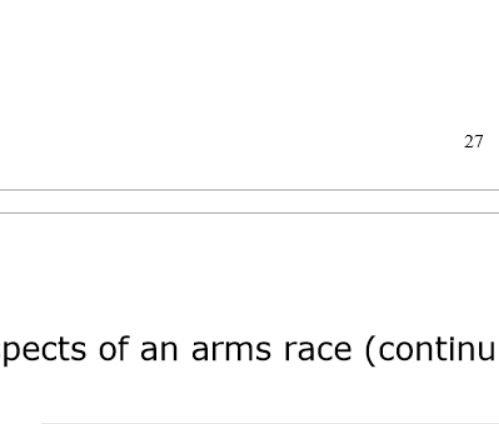
Figure 11.25 If an − bm > 0, the Model (11.19) has a unique rest point (E, F) in the first quadrant. Along the line

Figure 11.26 Composite graphical analysis of the trajectory directions in the four regions determined by the intersecting lines (Equations 11.19)

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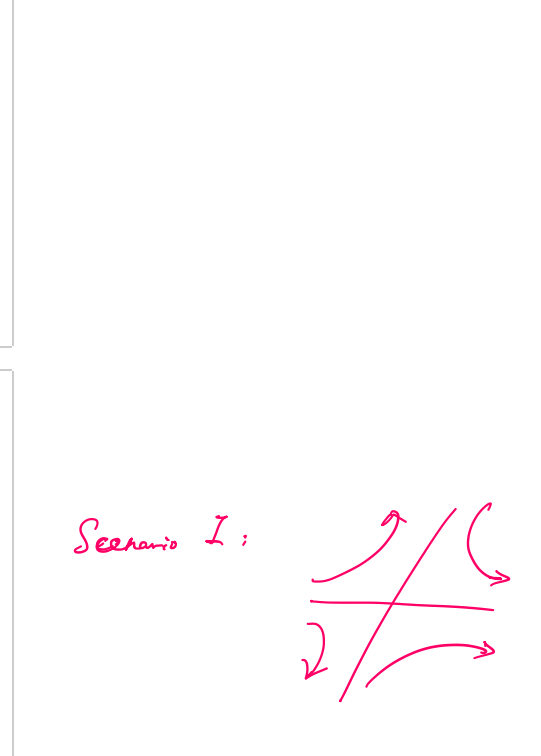

Learning objectives

- Be able to formulate the evolution of change in terms of an ODE-based model
- Be able to identify the equilibrium points of simple ODE-base models
- Understand the concepts of stable, asymptotically stable equilibrium points