The Poisson Distribution

The Poisson distribution is used to model the number of random occurrences of some phenomenon in a specified unit of space or time. For example,

- The number of phone calls in a 30-minute period.
- The number of defective parts produced during an eight hour shift.
- The number emergency assistance calls received in an hour.

For a Poisson random variable, the probability that ***X*** is some value ***x*** is given by the formula:

$$P(X = x) = \frac{\lambda^{x} e^{-\lambda}}{x!} \qquad x = 0, 1, \ldots$$

where: **λ** : mean number of occurrences in a specified interval (time period).

x : number of occurrences of interest.

e : Base of the natural logarithmic function ln (≈ 2.71828)

Example

On an average day in January, a furnace repair person receives 4 calls for emergency furnace repair service. Find the probability that this person will receive 5 calls today. (i.e., $\lambda = 4$, $X = 5$)

$$P(X = 5) = \frac{4^{5} e^{-4}}{5!} = 0.1563$$

Find the probability that this person will receive fewer than 4 calls today. (i.e., $\lambda = 4$, $X < 4$)

$$\begin{aligned} P(X < 4) &= P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) \\ &= 0.0183 + 0.0733 + 0.1465 + 0.1954 \\ &= 0.4335 \end{aligned}$$

Problem Statement

The plant manager of a toy manufacturing plant has been given an order for 10,000 new toys with a shipping date 10 weeks from now. He currently has 8 machines that produce 100 toys per week per machine, assuming one shift of 40 hours per week. He needs to know if the machines need to run overtime to meet the order and if so how much overtime is necessary. The goal is to minimize overtime.

Calculate how many toys the 8 machines will produce in 10 weeks working one 40 hour shift. Do the machines need to run additional shifts (each shift for each machine is 40 hours per week)?

Write a formula for calculating the number of weeks with an extra shift to meet this order in 10 weeks.

Now, in actuality, the machines have been known to produce on the average 2 defective toys in a week (one shift) per machine. Also, the machines periodically require maintenance. This occurs on the average of 1 machine down for maintenance per week (one shift), with the maintenance requiring 1 hour. With these uncertainties calculate the necessary extra weeks of overtime to meet the order.

Given:

T_{total} = 10,000 toys
8 machines
10 weeks production time
100 toys/machine · shift-week

Determine R, the rate of production for all 8 machines for one shift-week

R = (8 machines * 100 toys/ machine · shift-week)
= 800 toys/shift-week

Determine Tp, the production for 10 one shift-weeks

Tp = 800 toys/shift-week * 10 shift-weeks
Tp = 8000 toys

But we need 10000, so we are 2000 short.
How many shift-weeks to add (as overtime) to production?

2000 toys * 1 shift-week/800 toys = 2.5 shift-weeks of overtime

Write a formula for calculating the number of weeks with an extra shift to meet this order in 10 weeks.

S (shift-weeks) = **T** (toys) / **R** (toys/shift-week)

Dealing with uncertainty

The number of defective toys produced are the number of toys lost each shift-week.

Average 2 defective toys in one shift-week per machine.

Use Poisson Distribution as model.

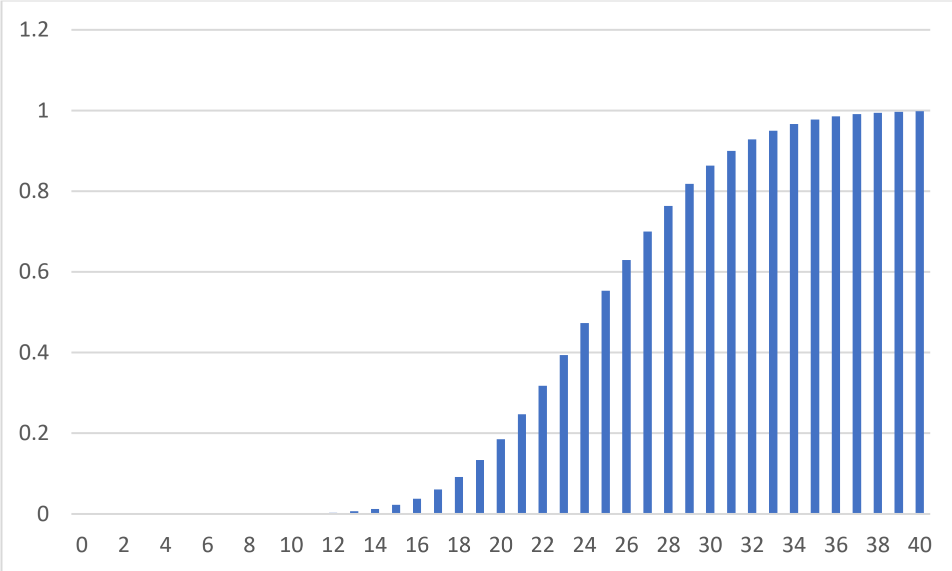
$\lambda = 2$

P(X = 0) = 0.1353
P(X = 1) = 0.2707
P(X = 2) = 0.2707
P(X = 3) = 0.1804
P(X = 4) = 0.0902
P(X = 5) = 0.0361

The above is for an interval of one shift-week, which is a subinterval. There are actually 12.5 shift-weeks to consider. So the λ for the interval we are interested in is 2 * 12.5 = 25.

Let L = toys lost for the interval per machine: Probabilites and chart on next page:

L	P(L)	Sum(P(L))
0	1.38879E-11	1.38879E-11
1	3.47199E-10	3.61087E-10
2	4.33998E-09	4.70107E-09
3	3.61665E-08	4.08676E-08
4	2.26041E-07	2.66908E-07
5	1.1302E-06	1.39711E-06
6	4.70918E-06	6.10629E-06
7	1.68185E-05	2.29248E-05
8	5.25578E-05	7.54826E-05
9	0.000145994	0.000221477
10	0.000364985	0.000586462
11	0.000829511	0.001415973
12	0.001728149	0.003144122
13	0.003323363	0.006467484
14	0.005934576	0.012402061
15	0.009890961	0.022293021
16	0.015454626	0.037747647
17	0.022727391	0.060475038
18	0.031565821	0.092040859
19	0.041533975	0.133574834
20	0.051917469	0.185492303
21	0.06180651	0.247298813
22	0.070234671	0.317533484
23	0.076342033	0.393875517
24	0.079522951	0.473398469
25	0.079522951	0.55292142
26	0.076464376	0.629385796
27	0.070800349	0.700186145
28	0.063214597	0.763400742
29	0.054495342	0.817896084
30	0.045412785	0.863308869
31	0.036623214	0.899932083
32	0.028611886	0.928543969
33	0.021675671	0.95021964
34	0.015937993	0.966157633
35	0.011384281	0.977541914
36	0.007905751	0.985447665
37	0.005341723	0.990789388
38	0.003514292	0.99430368
39	0.002252751	0.996556431
40	0.001407969	0.997964401



The above is a plot of the probabilities

Let L = 40 toys lost for the interval per machine as nearly 100% probability

T_{lost} = L * 8 machines = 40 * 8 = 320 toys lost

Substitute into formula:

Shift-weeks = 320 toys / 800 toys/shift-week = 0.4 shift-weeks

Dealing with uncertainty, Part 2

Average 1 machine down for maintenance in one shift-week with maintenance requiring 1 hour to complete.

Use Poisson Distribution as model.

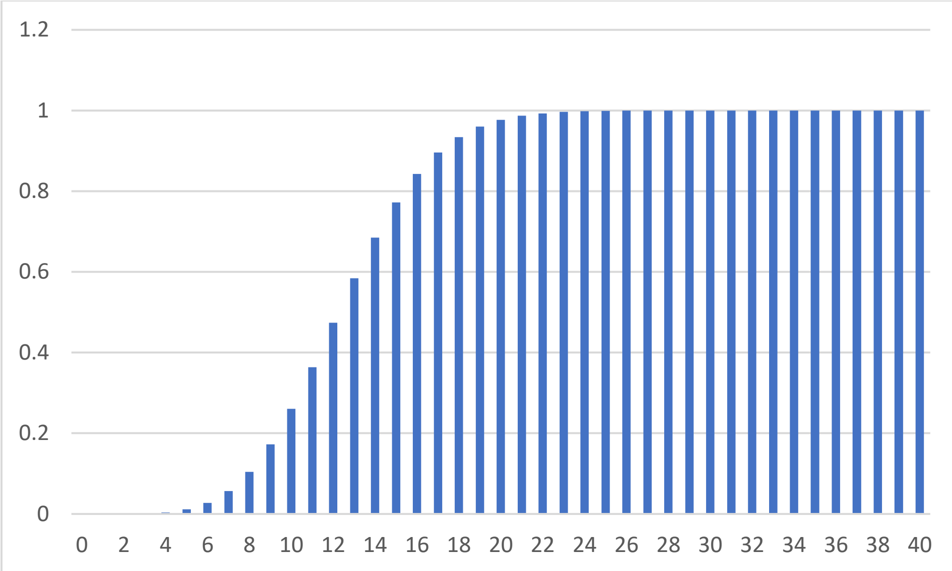
$\lambda = 1$

P(X = 0) = 0.3679
P(X = 1) = 0.3679
P(X = 2) = 0.1839
P(X = 3) = 0.0613
P(X = 4) = 0.0153
P(X = 5) = 0.0031

The above is for one shift-week, which is a subinterval, we need to consider for 12.9 shift-weeks as our interval. So the $\lambda = 12.9 * 1 = 12.9$

Let H = hours lost to maintenance: Probabilites and chart on next page:

H	P(H)	Sum of P(H)
0	2.49805E-06	2.49805E-06
1	3.22248E-05	3.47229E-05
2	0.00020785	0.000242573
3	0.000893756	0.001136329
4	0.002882364	0.004018693
5	0.007436498	0.011455191
6	0.015988472	0.027443663
7	0.029464469	0.056908132
8	0.047511456	0.104419588
9	0.068099754	0.172519342
10	0.087848683	0.260368025
11	0.103022546	0.363390571
12	0.110749237	0.474139808
13	0.10989732	0.584037128
14	0.10126253	0.685299658
15	0.087085776	0.772385435
16	0.070212907	0.842598342
17	0.053279206	0.895877547
18	0.038183431	0.934060978
19	0.02592454	0.959985518
20	0.016721328	0.976706847
21	0.010271673	0.98697852
22	0.006022936	0.993001455
23	0.003378081	0.996379536
24	0.001815719	0.998195255
25	0.000936911	0.999132166
26	0.000464852	0.999597018
27	0.000222096	0.999819114
28	0.000102323	0.999921437
29	4.5516E-05	0.999966953
30	1.95719E-05	0.999986524
31	8.14443E-06	0.999994669
32	3.28322E-06	0.999997952
33	1.28344E-06	0.999999236
34	4.86953E-07	0.999999722
35	1.79477E-07	0.999999902
36	6.43125E-08	0.999999966
37	2.24225E-08	0.999999989
38	7.61184E-09	0.999999996



Let H = hours lost to maintenance, with a probability of 100% that 22 hours lost over the interval of interest:

22 hours/40 hours per shift week = 0.55 shift-weeks

Putting it all together:

Total production time =

12.5 shift-weeks + 0.4 shift-weeks + 0.55 shift-weeks

Total production time = 13.45 shift-weeks

Any time over 10 shift-weeks is overtime. Therefore the plant manager will have to schedule approximately 3.5 shift-weeks of overtime during the 10 week run of production on this new toy.

Using Excel (formulas) to generate the first probability table for L:

	A	B	C
1	L	P(L)	Sum(P(L))
2	0	=((2*12.5)^A2*EXP(-2*12.5))/FACT(A2)	=B2
3	1	=((2*12.5)^A3*EXP(-2*12.5))/FACT(A3)	=B2+B3
4	2	=((2*12.5)^A4*EXP(-2*12.5))/FACT(A4)	=B4+C3
5	3	=((2*12.5)^A5*EXP(-2*12.5))/FACT(A5)	=B5+C4
6	4	=((2*12.5)^A6*EXP(-2*12.5))/FACT(A6)	=B6+C5

Using Excel (formulas) to generate the second probability table for H:

	A	B	C
1	H	P(H)	Sum of P(H)
2	0	=((1*12.9)^A2*EXP(-1*12.9))/FACT(A2)	=B2
3	1	=((1*12.9)^A3*EXP(-1*12.9))/FACT(A3)	=B2+B3
4	2	=((1*12.9)^A4*EXP(-1*12.9))/FACT(A4)	=B4+C3
5	3	=((1*12.9)^A5*EXP(-1*12.9))/FACT(A5)	=B5+C4
6	4	=((1*12.9)^A6*EXP(-1*12.9))/FACT(A6)	=B6+C5