Student 02.05.2024

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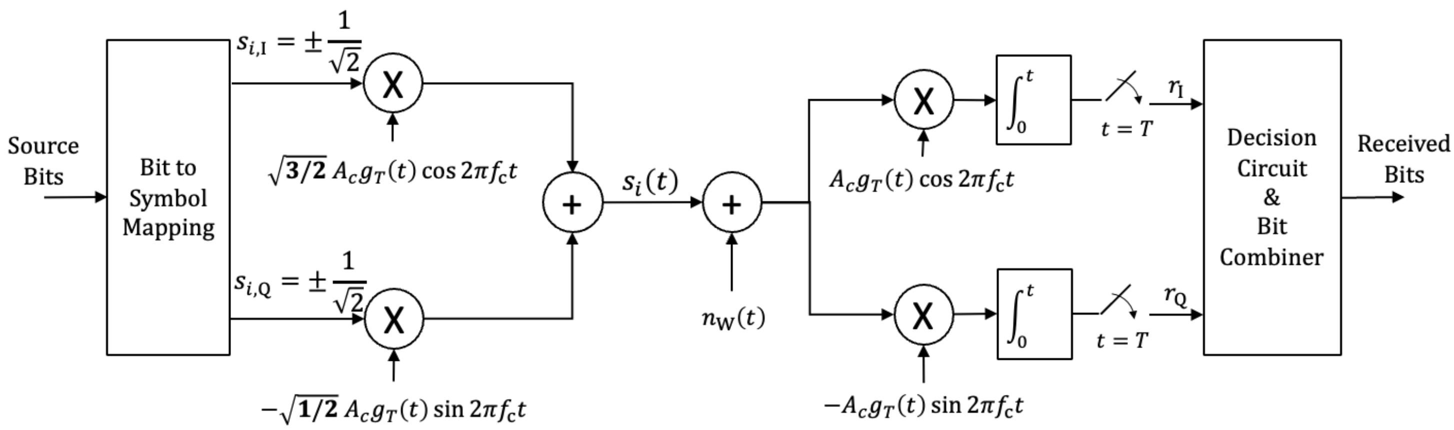
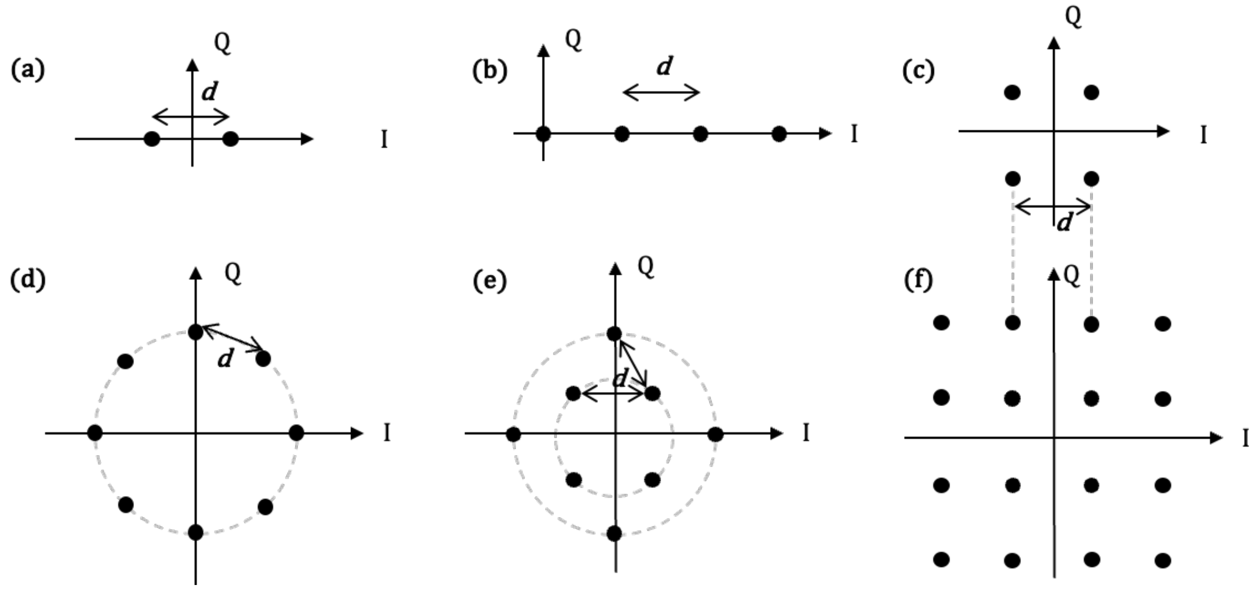
EHB 308E
COMMUNICATION II
Midterm Exam II

1. (40p) Transmitter and receiver diagrams of a digital communication system is given below. Sampled signals at decision circuit input is $r_\mathrm{I} = a_\mathrm{I} + n_{k,\mathrm{I}}$, and $r_\mathrm{Q} = a_\mathrm{Q} + n_{k,\mathrm{Q}}$where $n_{k,\mathrm{I}}, n_{k,\mathrm{Q}} \sim \mathcal{N}\left(\mu = 0, \sigma^2 = \frac{N_0}{2} E_s\right)$. <u>Transmitter</u> has I/Q gain imbalance problem where in-phase and quadrature branches have different gains as shown in the figure. A-priori probabilities of all symbols are equal: $P_i = 1/M,\ \ i = 0, \dots, M-1$. Gray mapping is used at transmitter. $g_\mathrm{T}(t) = \begin{cases} 1 & 0 \le t \le T \\ 0 & \text{elsewhere} \end{cases}$.



a) Write the expression for $s_\mathrm{i}(t)$. What is M? Name the modulation.
b) Find the average symbol energy and average bit energy (E_s and E_b) in terms of A_c and T.
c) Draw the constellation diagram seen at sampled received signal (without noise). Indicate the coordinate values of all symbol points & mapped bits to each symbol.
d) Derive average probability of symbol error P_e using Q-function.
e) Evaluate if gain imbalance increases or decreases error probability for the same symbol energy E_s, compared to no-imbalance case where $P_\mathrm{e} \approx 2Q\left(\sqrt{E_s/N_0}\right)$. **Hint:** Q-function decreases exponentially with distance between points (refer to notes).
f) Can <u>receiver</u> compensate for I/Q gain imbalance to get the error rate as if no imbalance occurred at <u>transmitter</u>? If yes, how? If no, show why not?

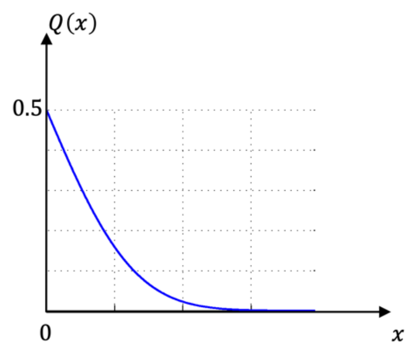
2. (30p) Constellations given below have <u>identical</u> <u>minimum distance</u> between closest neighbors to get approximately same bit error probability P_b when used with Gray mapping. Also, symbol durations are all same, T seconds. Answer the following questions briefly:
a) Determine and compare bitrate (R_b) of all modulations above.
b) Which of above modulation/s can be designed to be constant envelope?
c) Determine and compare average bit energies, E_b, in terms of *d.*
d) Order the modulations according to their approximate symbol error probability P_e
e) Which of above require both in-phase and quadrature receiver chains?
f) For which modulation/s we can use a non-coherent receiver?
g) Which modulation/s use only antipodal signaling ($s_\mathrm{i}(t) = a_i g_T(t)$)?



3. (30p) $s_i(t) = A_c \cos\left[2\pi\left(f_c + \left(i - \frac{1}{2}\right)\Delta f\right)t\right]$, $0 \le t < T$ and $i = 0{,}1$ in an BFSK system.
   a) Draw a coherent receiver for the 2-FSK system given above using correlator type receiver. Express the operations performed in each block.
   b) Show that the correlation function between 2 possible signals can be written as $\rho = \frac{\int s_0(t)s_1(t)dt}{\int s_0^2(t)dt} \approx sinc(2\Delta f T)$. What is the minimum frequency separation to guarantee an orthogonal BFSK transmission?
   c) With orthogonal BFSK case, write the signal as linear combination of its orthonormal basis functions $\left(s_i(t) = \sum_{j=1}^{2} s_{ij}\phi_j(t)\right)$. Determine $\phi_j(t)$ and of s_{ij} for every i and j.
   d) Assume we want to transmit 2 bits per **FSK** symbol (M=4) without increasing bandwidth (using the same set of basis functions, i.e., frequencies). Design that modulation by assigning s_{ij} values for all 4 possible symbols ($i = 0, \ldots, 3$) and $j = 1, 2$. Draw the signal space constellation for designed FSK scheme.

Useful notes:

- $\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$, $\Pi\left(\frac{t}{T}\right) \overset{FT}{\leftrightarrow} T\text{sinc}(fT)$
- $Q(x) = \frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-\frac{u^2}{2}}du$



- $\cos a \cos b = \frac{1}{2}(\cos(a+b) + \cos(a-b))$
- $\cos a \sin b = \frac{1}{2}(\sin(a+b) - \sin(a-b))$
- $\sin a \sin b = \frac{1}{2}(\cos(a-b) - \cos(a+b))$
- $\int \cos(ax)dx = \frac{1}{a}\sin(ax) + C$

Attention:

- Turned off your mobile phone completely during the exam.
- Do not leave the room in first 45 minutes.
- Write down you name in question and answer sheets and submit both once done.

All the best...

1 a) $s_i(t) = s_{i,I}\sqrt{\frac{3}{2}}A_c g_T(t)\cos 2\pi f_c t - s_{i,Q}\frac{1}{\sqrt{2}}A_c g_T(t)\sin 2\pi f_c t \qquad s_{iI}, s_{iQ} \in \{\pm\frac{1}{\sqrt{2}}\}$

6 $\boxed{M=4}$ $\qquad \boxed{QPSK}$ or $\boxed{4\text{-PSK}}$, or $\boxed{4\text{-QAM}}$

8 b) $E_s = \sum_{i=0}^{3} P_i E_i = E_i \quad$ for $\forall i \in \{0,\dots 3\}$ since $P_i = 1/4$

$$E_i = \int_{-\infty}^{\infty} s_i^2(t)\,dt = \int_{-\infty}^{\infty} s_{iI}^2 \cdot \frac{3}{2}\cdot A_c^2 g_T^2(t)\cos^2 2\pi f_c t + \int_{-\infty}^{\infty} s_{iQ}^2\,\frac{1}{2}A_c^2 g_T^2(t)\sin^2 2\pi f_c t$$

$$-\int_{-\infty}^{\infty} s_{iI}\cdot s_{iQ}\sqrt{3}A_c^2 g_T^2(t)\cos 2\pi f_c t\cdot\sin 2\pi f_c t$$

$$= \frac{3}{2}s_{iI}^2 A_c^2\int_0^T\left(\frac{1}{2}+\xrightarrow{0}\cancel{\frac{1}{2}\cos 4\pi f_c t}\right)dt + \frac{1}{2}s_{iQ}^2 A_c^2\int_0^T\left(\frac{1}{2}-\xrightarrow{0}\cancel{\frac{1}{2}\cos 4\pi f_c t}\right)dt$$

$$-A_c^2\sqrt{3}\,s_{iI}\,s_{iQ}\xrightarrow{0}\cancel{\int^T\frac{1}{2}(\sin 4\pi f_c t - \sin 0)\,dt} = \frac{3}{4}A_c^2 T\,s_{iI}^2 + \frac{1}{4}A_c^2 T\,s_{iQ}^2$$

Since $s_{iI}, s_{iQ} \in \{-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\} \Rightarrow E_i = \frac{1}{2}A_c^2 T$ for all i.

$$\boxed{E_s = E_i = \frac{1}{2}A_c^2 T} \qquad \boxed{E_b = \frac{E_s}{\log_2 M} = \frac{1}{4}A_c^2 T}$$

8 c-) Levels on in-phase branch.

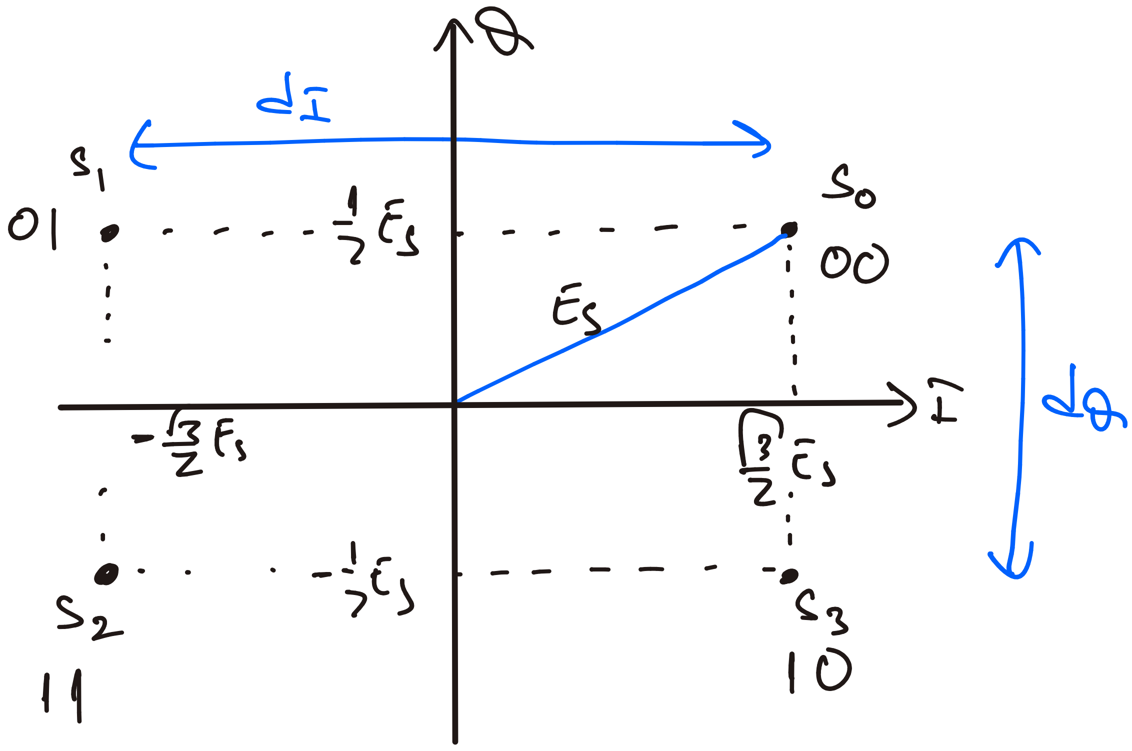
$$r_I = \int_0^t (s_i(t)+n_w(t))\cdot A_c g_T(t)\cos 2\pi f_c t\,dt\Big|_{t=T}$$

$$= \frac{\sqrt{3}}{\sqrt{2}}A_c^2 s_{iI}\int_0^t g_T^2(t)\cos^2 2\pi f_c t\,dt\Big|_{t=T} - \frac{1}{\sqrt{2}}A_c^2 s_{iQ}\int_0^t g_T^2(t)\cos 2\pi f_c t\,\sin 2\pi f_c t\,dt\Big|_{t=T}$$

$$+ A_c\int_0^t n_w(t)\cdot g_T(t)\cdot\cos 2\pi f_c t\,dt\Big|_{t=T} = \underbrace{\sqrt{\frac{3}{8}}A_c^2 T\,s_{iI}}_{V_{i,I} = \pm\frac{\sqrt{3}}{2}E_s} + n_{2,I}$$

Similarly for Q branch:

$$r_Q = \int_0^t (s_i(t)+n_w(t))(-A_c g_T(t)\sin 2\pi f_c t)\,dt\Big|_{t=T} = \underbrace{\frac{1}{\sqrt{8}}A_c^2 T\,s_{iQ}}_{V_{i,Q} = \pm\frac{1}{2}E_s} + n_{2,Q}$$



d) $P_e = 1-(1-P_{b,I})(1-P_{b,Q}) = P_{b,I} + P_{b,Q} - P_{b,I}\cdot P_{b,Q}$

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$$P_{b,I} = Q\left(\frac{d_I}{2\sqrt{N_I}}\right) = Q\left(\frac{\sqrt{3}E_s}{2\sqrt{\frac{N_0}{2}E_s}}\right) = Q\left(\sqrt{\frac{3}{2}}\sqrt{\frac{E_s}{N_0}}\right)$$

$$P_{b,Q} = Q\left(\frac{d_Q}{2\sqrt{N_Q}}\right) = Q\left(\frac{E_s}{2\sqrt{\frac{N_0}{2}E_s}}\right) = Q\left(\frac{1}{\sqrt{2}}\sqrt{\frac{E_s}{N_0}}\right)$$

$$P_e = Q\left(\sqrt{\frac{3}{2}}\sqrt{\frac{E_s}{N_0}}\right) + Q\left(\frac{1}{\sqrt{2}}\sqrt{\frac{E_s}{N_0}}\right) - Q\left(\sqrt{\frac{3E_s}{2N_0}}\right)\cdot Q\left(\sqrt{\frac{E_s}{2N_0}}\right)$$

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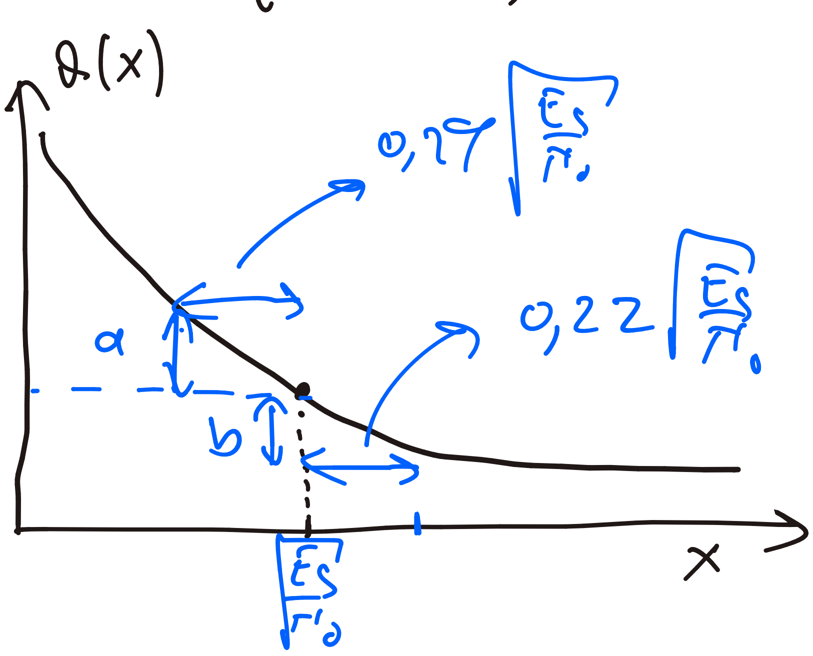
e) No-impairment case $P_e = 2Q\left(\sqrt{\frac{E_s}{N_0}}\right)$. Let's take approximate P_e as $Q\left(\sqrt{\frac{3}{2}}\sqrt{\frac{E_s}{N_0}}\right)+Q\left(\frac{1}{\sqrt{2}}\sqrt{\frac{E_s}{N_0}}\right) = Q\left(1{,}22\sqrt{\frac{E_s}{N_0}}\right)+Q\left(0{,}707\sqrt{\frac{E_s}{N_0}}\right)$



From the curve, it is certain that $|a| > |b|$, that is; decrease in P_e from first term is less than the increase of P_e due to second term.

$\Rightarrow$ $\boxed{P_e \text{ will increase!}}$

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f) No. Even if the receiver increase the gain of the quadrature branch to compensate, it will amplify the noise as well. So, receiver cannot get the same SNR (ratio) back to achieve same P_e.

2 a) $R_b = \log_2 M \cdot \frac{1}{T}$

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$$R_b^{(f)} = \frac{4}{T} > R_b^{(e)} = R_b^{(d)} = \frac{3}{T} > R_b^{(c)} = R_b^{(b)} = \frac{2}{T} > R_b^{(a)} = \frac{1}{T}$$

b) (a), (c) and (d) can be constant envelope if proper pulse shaping is selected. Others cannot because there are symbols with different amplitudes.

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c) $E_b = \frac{E_s}{\log_2 M}$

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$$E_b^{(a)} = \left(\frac{1}{2}\frac{d^2}{4}\cdot T + \frac{1}{2}\frac{d^2}{4}\cdot T\right)\cdot\frac{1}{\log_2 2} = 0{,}25\, d^2 T$$

$$E_b^{(b)} = \left(\frac{0 + d^2 + 4d^2 + 9d^2}{4}\right)\cdot\frac{1}{\log_2 4}\cdot T = 1{,}75\, d^2 T$$

$$E_b^{(c)} = \frac{d^2}{2}\cdot 4 \cdot \frac{1}{4}\cdot\frac{1}{\log_2 4}\cdot T = 0{,}25\, d^2 T$$

$$E_b^{(d)} = \left(\frac{d/2}{\sin\frac{\pi}{8}}\right)^2 \cdot 8 \cdot \frac{1}{8}\,\frac{1}{\log_2 8}\, T = \frac{d^2 T}{12 \sin^2\frac{\pi}{8}} = 0{,}56\, d^2 T$$

$$E_d^{(e)} = \left[\frac{d^2}{2}\cdot 4 + \Big(\frac{d}{2} + d\cdot\underbrace{\sin\frac{\pi}{3}}_{\frac{\sqrt{3}}{2}}\Big)^2\cdot 4\right]\cdot\frac{1}{8}\cdot\frac{1}{3}\, T = d^2 T\left(\frac{1}{4} + \frac{(1+\sqrt{3})^2}{8}\right)\cdot\frac{1}{3}$$

$$= 0{,}39\, d^2 T$$

$$E_d^{(f)} = \left(\frac{d^2}{2} + \frac{5d^2}{2}\cdot 2 + \frac{9d^2}{2}\right)\cdot\frac{1}{4}\cdot\frac{1}{\log_2 16}\, T = 0{,}625\, d^2 T$$

$$E_b^{(b)} > E_d^{(f)} > E_b^{(d)} > E_b^{(e)} > E_b^{(c)} = E_b^{(a)}$$

4-ASK 16QAM 8PSK 2-level 8PSK QPSK BPSK

d) $P_e \simeq P_b \cdot \log_2 M$ $\quad P_e^{(f)} > P_e^{(e)} = P_e^{(d)} > P_e^{(c)} = P_e^{(b)} > P_e^{(a)}$

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e) If phase-offset between Tx & Rx is known ⇒ (c), (d), (e) and (f).

3

" " " " " is not known ⇒ ALL

f) (b) 3 g) (a) and (b)

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3

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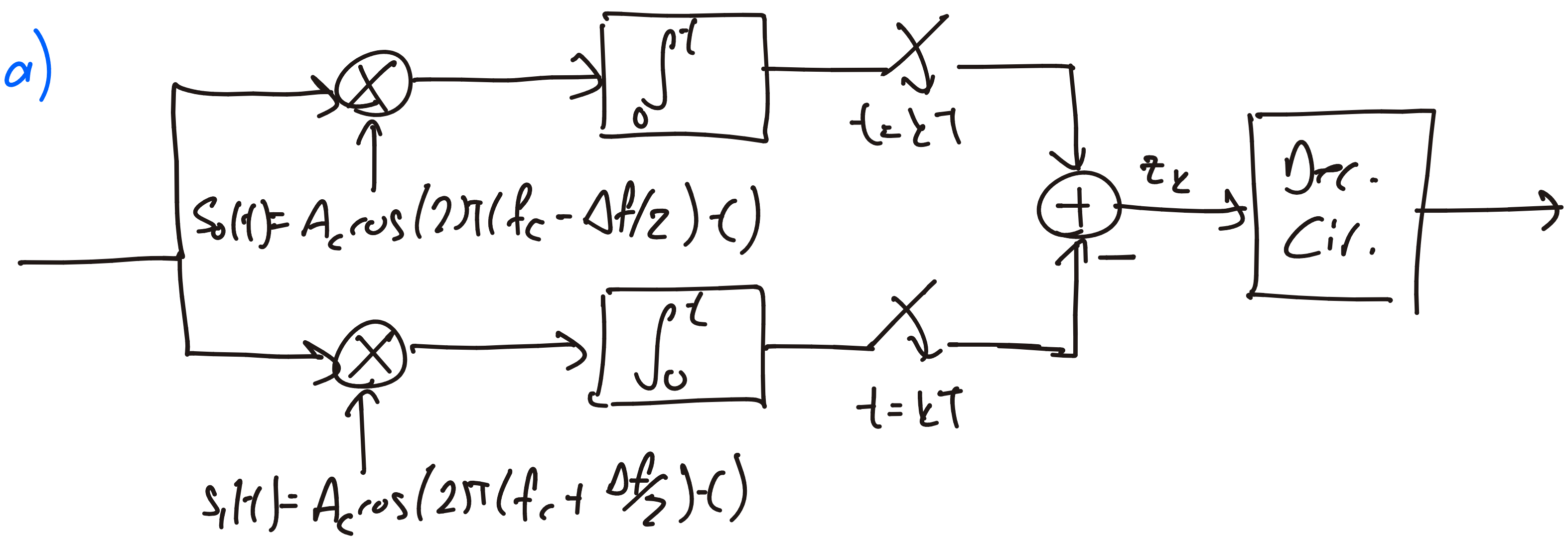
a)



The received signal is correlated with 2 possible signal, $s_0(t)$ and $s_1(t)$. Since FSK is an orthogonal scheme one of the correlator output's will have high and all other's will be 0 (in noiseless case). So, we can subtract the lower correlator output from upper to construct an "anti-podal" signal to use it as input to decision circuit. The decision rule is

If $z_k \geq 0$ → decide as $s_0(t)$ is transmitted.

else if $z_k < 0$ → " " $s_1(t)$ " "

b) 8

$$\int_0^T s_0(t)\cdot s_1(t)\,dt = \int_0^T A_c\cos 2\pi\left(f_c-\frac{\Delta f}{2}\right)t\cdot A_c\cos 2\pi\left(f_c+\frac{\Delta f}{2}\right)t\,dt$$

$$= \frac{A_c^2}{2}\int_0^T\left(\cos(2\pi 2f_c t)+\cos(2\pi\Delta f t)\right)dt$$

$$= \frac{A_c^2}{2}\frac{1}{2\pi 2f_c}\underbrace{\cancel{\sin(2\pi 2f_c t)}}_{0\ \ f_c\gg\frac{1}{T}}\Bigg|_0^T + \frac{A_c^2}{2}\frac{1}{2\pi\Delta f}\sin(2\pi\Delta f t)\Bigg|_0^T = \frac{A_c^2}{4\pi\Delta f}\sin(2\pi\Delta f T)$$

$$\int_0^T s_0^2(t)\,dt = \int_0^T A_c^2\cos^2 2\pi\left(f_c-\frac{\Delta f}{2}\right)t\,dt = \frac{A_c^2}{2}\int_0^T \underbrace{\cancel{\cos(4\pi f_c t)}}_{0\ \ f_c\gg\frac{1}{T}}+\underbrace{\cos(0)}_{1}\,dt = \frac{A_c^2}{2}\cdot T$$

$$\Rightarrow \frac{\int_0^T s_0(t)\cdot s_1(t)\,dt}{\int_0^T s_0^2(t)\,dt} = \frac{\sin(2\pi\Delta f T)}{2\pi\Delta f T} = \mathrm{sinc}(2\Delta f T)$$

For orthogonal BFSK the following should be satisfied;

$$\int_0^T s_0(t)\cdot s_1(t)\,dt = 0 \Rightarrow \mathrm{sinc}(2\Delta f T) = 0 \Rightarrow$$ Minimum frequency separation with coherent receiver $\boxed{\Delta f = \frac{1}{2T}}$

c) We already know that $s_0(t)$ and $s_1(t)$ are orthogonal. Let's define orthonormal basis functions as $\phi_1(t) = K \cdot s_0(t)$ and $\phi_2(t) = L \cdot s_1(t)$

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$$\int_0^T \phi_1^2(t)\,dt = 1 \Rightarrow \int_0^T K^2 A_c^2 \cos^2\left(2\pi\left(f_c - \frac{\Delta f}{2}\right)t\right) dt = 1$$

$$\Rightarrow \frac{K^2 A_c^2}{2} \int_0^T \left(1 + \cos(4\pi f_c t)\right) dt \Rightarrow \frac{K^2 A_c^2}{2} T = 1$$

$$K = \frac{1}{A_c}\sqrt{\frac{2}{T}}$$

Similarly, $L = \frac{1}{A_c}\sqrt{\frac{2}{T}}$

$$\Rightarrow \boxed{\phi_1(t) = \sqrt{\frac{2}{T}} \cos\left(2\pi\left(f_c - \frac{\Delta f}{2}\right)t\right), \quad 0 \le t \le T}$$

$$\boxed{\phi_2(t) = \sqrt{\frac{2}{T}} \cos\left(2\pi\left(f_c + \frac{\Delta f}{2}\right)t\right), \quad 0 \le t \le T}$$

$$s_i(t) = \sum_{j=1}^{2} s_{ij}\, \phi_j(t)$$

$$s_0(t) = A_c \cos\left(2\pi\left(f_c - \frac{\Delta f}{2}\right)t\right) = s_{01}\,\phi_1(t) + s_{02}\,\phi_2(t)$$

$$A_c \cos\left(2\pi\left(f_c - \frac{\Delta f}{2}\right)t\right) = s_{01}\sqrt{\frac{2}{T}} \cos\left(2\pi\left(f_c - \frac{\Delta f}{2}\right)t\right) + s_{02}\sqrt{\frac{2}{T}} \cos\left(2\pi\left(f_c + \frac{\Delta f}{2}\right)t\right)$$

$$\Rightarrow \boxed{s_{01} = A_c\sqrt{\frac{T}{2}}}, \quad \boxed{s_{02} = 0}$$

Similarly,

$$\Rightarrow \boxed{s_{11} = 0}, \quad \boxed{s_{12} = A_c\sqrt{\frac{T}{2}}}$$

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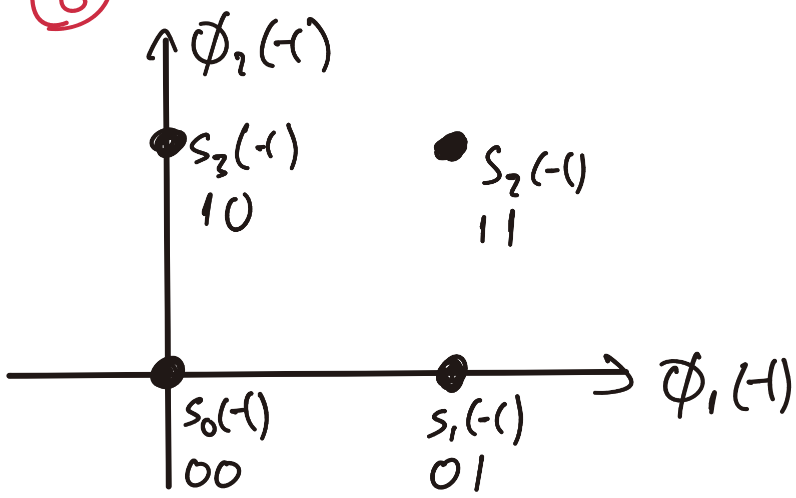
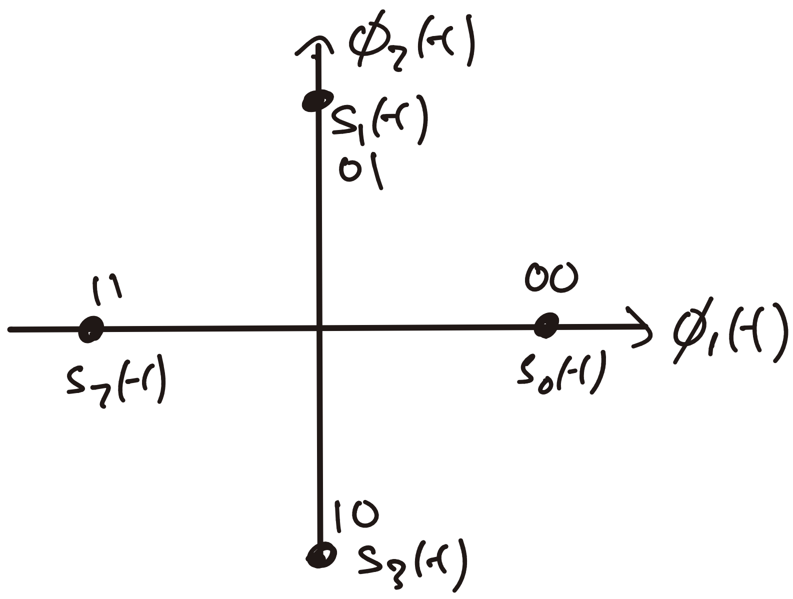
d)



$$s_0(t) = s_{01}\,\phi_1(t) + s_{02}\,\phi_2(t) \Rightarrow$$
$$s_1(t) = s_{11}\,\phi_1(t) + s_{12}\,\phi_2(t) \Rightarrow$$
$$s_2(t) = s_{21}\,\phi_1(t) + s_{22}\,\phi_2(t) \Rightarrow$$
$$s_3(t) = s_{31}\,\phi_1(t) + s_{32}\,\phi_2(t) \Rightarrow$$

$$\boxed{\begin{array}{l} s_{01} = 0,\ s_{02} = 0 \\ s_{11} = \sqrt{E_s},\ s_{12} = 0 \\ s_{21} = \sqrt{E_s},\ s_{22} = \sqrt{E_s} \\ s_{31} = 0,\ s_{32} = \sqrt{E_s} \end{array}}$$

OR



$$\boxed{\begin{array}{l} s_{01} = \sqrt{E_s},\ s_{02} = 0 \\ s_{11} = 0,\ s_{12} = \sqrt{E_s} \\ s_{21} = -\sqrt{E_s},\ s_{22} = 0 \\ s_{31} = 0,\ s_{32} = -\sqrt{E_s} \end{array}}$$