TWISTS OF K-THEORY AND TMF

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ABSTRACT. We explore an approach to twisted generalized cohomology from the point of view of stable homotopy theory and ∞ -category theory provided by [ABGHR]. We explain the relationship to the twisted K-theory provided by Fredholm bundles. We show how this approach allows us to twist elliptic cohomology by degree four classes, and more generally by maps to the four-stage Postnikov system $BO\langle 0\dots 4\rangle$. We also discuss Poincaré duality and umkehr maps in this setting.

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1. Introduction

In [ABGHR], we and our co-authors generalize the classical notion of Thom spectrum. Let R be an A_{∞} ring spectrum: it has a space of units GL_1R which deloops to give a classifying space BGL_1R . To a space X and a map

$$\xi \colon X \to BGL_1R$$

we associate an R-module Thom spectrum X^{ξ} . Letting S denote the sphere spectrum, one finds that BGL_1S is the classifying space for stable spherical fibrations of virtual rank 0, and X^{ξ} is equivalent to the classical Thom spectrum of the spherical fibration classified by ξ (as in, for example, [LMSM86]).

We remark in the introduction to [ABGHR] that BGL_1R classifies the twists of R-theory. More precisely, we define the ξ -twisted R-homology of X to be

$$R_k(X)_{\xi} \stackrel{\text{def}}{=} \pi_0 \text{R-mod}(\Sigma^k R, X^{\xi}) \cong \pi_k X^{\xi}$$

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and the ξ -twisted cohomology to be

$$R^k(X)_{\xi} \stackrel{\text{def}}{=} \pi_0 \text{R-mod}(X^{\xi}, \Sigma^k R),$$

where here Σ^k denotes the k-fold suspension, or equivalently smashing with S^k .

In this paper, we expand on that remark, explaining how this definition generalizes both singular cohomology with local coefficients and the twists of K-theory studied by [DK70, Ros89, AS04]. The key maneuver is to focus on the ∞ -categorical approach to Thom spectra developed in [ABGHR] (where by ∞ -categories we mean the quasicategories of [Joy02, HTT]). We show that the ∞ -category Line_R of R-modules L which admit a weak equivalence $R \simeq L$ is a model for BGL_1R : we have a weak equivalence of spaces

$$|\mathrm{Line}_R| \simeq BGL_1R.$$

One appeal of Line_R is that, by construction, it classifies what one might call "homotopy local systems" of free rank-one R-modules. This flexible notion generalizes both classical local coefficient systems and bundles of spaces (such as bundles of Fredholm operators). As one might expect, our work is closely related to the parametrized spectra of May and Sigurdsson [MS06]; we discuss the relationship further in Section 3.4.

As applications of our approach to twisted generalized cohomology, we explain how the twisting of K-theory by degree three cohomology is related to the $Spin^c$ orientation of Atiyah-Bott-Shapiro. Similarly, recall that there is a map (unique up to homotopy)

$$BSpin \xrightarrow{\lambda} K(\mathbb{Z},4)$$

whose restriction to BSU is the second Chern class. The fiber of λ is called BString, and if V is a Spin vector bundle on X, then a String structure on V is a trivialization of $\lambda(V)$; that is, a map g in the diagram

$$BString$$

$$\downarrow^{g} \qquad \downarrow^{\pi}$$

$$X \xrightarrow{V} BSpin,$$

together with a homotopy $\pi q \Rightarrow V$.

The work of Ando, Hopkins, and Rezk [AHR] constructs an E_{∞} String orientation of tmf, the spectrum of topological modular forms. (The discussion in this paper applies equally to the connective spectrum tmf and to the periodic spectrum TMF.) Associating to a vector bundle its underlying spherical fibration gives a map $BSpin \rightarrow BGL_1S$, and associated to the unit $S \rightarrow tmf$ is a map $BGL_1S \rightarrow BGL_1tmf$. Composing these, we have a map

$$k: BSpin \rightarrow BGL_1tmf.$$

We show that the E_{∞} String orientation of tmf of [AHR] implies the following.

Theorem 1.1.

(1) tmf admits twists by degree-four integral cohomology. More precisely, there is a map $h: K(\mathbb{Z},4) \to BGL_1tmf$ making the diagram

$$BSpin \xrightarrow{k} BGL_1tmf$$

$$\downarrow \downarrow =$$

$$K(\mathbb{Z}, 4) \xrightarrow{h} BGL_1tmf$$

commute up to homotopy. Thus given a map $z: X \to K(\mathbb{Z}, 4)$ (representing a class in $H^4(X, \mathbb{Z})$) we can define

$$tmf^*(X)_z \stackrel{def}{=} tmf^*(X)_{hz}.$$

(2) If V is a Spin-bundle over X, classified by

$$X \xrightarrow{V} BSpin,$$

then a homotopy $h\lambda(V) \Rightarrow k(V)$ determines an isomorphism

$$tmf^*(X^V) \cong tmf^*(X)_{\lambda(V)}.$$

of modules over $tm f^*(X)$.

Theorem 1.1 is well-known to the experts (e.g. Hopkins, Lurie, Rezk, and Strickland). As the reader will see, with the approach to twisted cohomology presented here, it is an immediate consequence of the String orientation.

We also explain how twisted generalized cohomology is related to Poincaré duality. We briefly describe some work in preparation, concerning twisted umkehr maps in generalized cohomology. As special cases, we recover the twisted K-theory umkehr map constructed by Carey and Wang, and we construct an umkehr map in twisted elliptic cohomology. As we explain, our interest in the twisted elliptic cohomology umkehr map arose from conversations with Hisham Sati.

Finally, we report two applications of twisted equivariant elliptic cohomology: we recall a result (due independently to the first author [And00] and Jacob Lurie), relating twisted equivariant elliptic cohomology to representations of loop groups, and we explain work of the first author and John Greenlees, relating twisted equivariant elliptic cohomology to the equivariant sigma orientation.

Remark 1.2. If E is a cohomology theory and X is a space, then $E^*(X)$ will refer to the unreduced cohomology. If Z is a spectrum, then we write $E^*(Z)$ for the spectrum cohomology. We write Σ_{\perp}^{∞} for the functor

$$\Sigma_+^{\infty}$$
: (spaces) $\xrightarrow{\text{disjoint basepoint}}$ (pointed spaces) $\xrightarrow{\Sigma^{\infty}}$ (spectra),

so we have by definition

$$E^*(X) \cong E^*(\Sigma^{\infty}_+ X),$$

while the reduced cohomology is

$$\widetilde{E}^*(X) \cong E^*(\Sigma^{\infty}X).$$

If V is a vector bundle over X of rank r, then we write X^V for its Thom spectrum: this is equivalent to the suspension spectrum of the Thom space, so $E^*(X^V)$ is the reduced cohomology of the Thom space. Thus the Thom isomorphism, if it exists, takes the form

$$E^*(X) \cong E^{*+r}(X^V).$$

Remark 1.3. The ∞ -category Line_R is not the largest category we could use to construct twists of R-theory. If R is an E_{∞} ring spectrum, i.e. a commutative ring spectrum, then we could consider the ∞ -category Pic(R), consisting of invertible R-modules: R-modules L for which there exists an R-module M such that $L \wedge_R M \simeq R$.

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2. Classical examples of twisted generalized cohomology

2.1. Geometric models for twisted K-theory. Let \mathcal{H} be a complex Hilbert space, and let \mathcal{F} be its space of Fredholm operators. Then \mathcal{F} is a representing space for K-theory: Atiyah showed [Ati69] that

$$K(X) \cong \pi_0 \operatorname{map}(X, \mathcal{F}) = \pi_0 \Gamma(X \times \mathcal{F} \to X).$$

Atiyah and Segal [AS04] develop the following approach to twisted K-theory. The unitary group $U = U(\mathcal{H})$ of \mathcal{H} acts on the space \mathcal{F} of Fredholm operators by conjugation. Associated to a principal PU-bundle $P \to X$, then, we can form the bundle

$$\xi = P \times_{PU} \mathcal{F} \to X$$

with fiber \mathcal{F} . They define the P-twisted K-theory of X to be

$$K(X)_P = \pi_0 \Gamma(\xi \to X).$$

Thus one twists K(X) by PU-bundles over X; isomorphism classes of these are classified by $\pi_0 \operatorname{map}(X, BPU)$; as BPU is a model for $K(\mathbb{Z}, 3)$, we have have $\pi_0 \operatorname{map}(X, BPU) \cong H^3(X; \mathbb{Z})$.

We warn the reader that this summary neglects important and delicate issues which Atiyah and Segal address with care, for example concerning the choice of topology on U and PU.

Another approach to twisted K-theory passes through algebraic K-theory; again we neglect important operator-theoretic matters, referring the reader to [Ros89, BM00, BCMMS] for details. Conjugation induces an action of PU on the algebra K of compact operators on \mathcal{H} . Thus from the PU-bundle $P \to X$ we can form the bundle $P \times_{PU} K$. Let $\mathcal{A} = \Gamma(P \times_{PU} K \to X)$. This is a (non-unital) C^* -algebra, and

$$K(X)_P \cong K(\mathcal{A}).$$

If P is the trivial bundle, then $A \cong \text{map}(X, \mathcal{K})$, and we have isomorphisms

$$K(\mathcal{A}) \cong K(\operatorname{map}(X, \mathbb{C})) \cong K(X).$$

Both of these approaches to twisted K-theory are based on the idea that from a PU-bundle we can build a bundle of copies of the representing space for K-theory, and both have had a number of successes. They demand a good deal of information about K-theory, and they exploit features of models of K-theory which may not be available in other cohomology theories.

May and Sigurdsson show how to implement the construction of Atiyah and Segal in the setting of their theory of parametrized stable homotopy theory [MS06, §22]. Specifically, they give a construction of certain twisted cohomology theories associated to parametrized spectra, and explain how the Atiyah-Segal definition fits into their framework. However, the approach of May and Sigurdsson also takes advantage of good features of known models for K-theory which may not be available in other cohomology theories.

In this paper we explain another way to locate twisted K-theory in stable homotopy theory. Our constructions continue to demand a good deal of K-theory, for example that it be an A_{∞} or E_{∞} ring spectrum, but many generalized cohomology theories E satisfy our demands, and so our approach works in those cases as well.

Our approach incorporates and generalizes the construction of Thom spectra of vector bundles, and so clarifies standard results concerning twisted E-theory, such as the relationship to the Thom isomorphism and Poincaré duality. It also generalizes the classical notion of (co)homology with local coefficients, as we now explain.

2.2. Cohomology with local coefficients. Let X be a space, and let $\Pi_{\leq 1}(X)$ be its fundamental groupoid. We recall that a local coefficient system on X is a functor¹

$$\{A\}: \Pi_{\leq 1}(X) \to (\text{Abelian groups}).$$

Given a local system $\{A\}$ on X, we can form the twisted singular homology $H_*(X; \{A\})$ and cohomology $H^*(X; \{A\})$.

Example 2.1. For example, if $\pi: E \to X$ is a Serre fibration, then associating to a point $p \in X$ the fiber $F_p = \pi^{-1}(p)$ gives rise to a representation

$$F_{\bullet} : \Pi_{\leq 1}(X) \to \text{Ho(spaces)}.$$

(Here Ho denotes the homotopy category obtained by inverting the weak equivalences.) Applying singular cohomology in degree r produces the local coefficient system $\{H^r(F_{\bullet})\}$.

Example 2.2. If V is a vector bundle over X of rank r, then taking the fiberwise one-point compactification V^+ provides a Serre fibration, and so we have the local coefficient system

$$\{\widetilde{H}^r(V_{\bullet}^+); \mathbb{Z}\}$$
 (2.3)

¹Since the fundamental groupoid is a groupoid, it is equivalent to consider covariant or contravariant functors.

whose value at $p \in X$ is the cohomology group $\widetilde{H}^r(V_p^+; \mathbb{Z})$.

From the Serre spectral sequence it follows that there is an isomorphism

$$H^*(X; {\widetilde{H}^r(V_{\bullet}^+)}) \cong H^{*+r}(X^V; \mathbb{Z})$$

between the twisted cohomology of X with coefficients in the system $\{\widetilde{H}^r(V_{\bullet}^+)\}$ and the cohomology of the Thom spectrum of V.

An orientation of V is a trivialization of the local system (2.3), that is, an isomorphism of functors

$$\{\widetilde{H}^r(V_{\bullet}^+)\} \cong \mathbb{Z}$$

(where \mathbb{Z} denotes the evident constant functor). It follows immediately that an orientation of V determines a Thom isomorphism

$$H^*(X; \mathbb{Z}) \cong H^{*+r}(X^V; \mathbb{Z}).$$

3. Bundles of module spectra

3.1. **The problem.** Let X be a space. We seek a notion of local system of spectra ξ on X, generalizing the bundles of Fredholm operators in $\S 2.1$ and the local systems of $\S 2.2$. In particular, if $E \to X$ is a Serre fibration as in Example 2.1, then the classical local system $\{H^r(F_{\bullet})\}$ should arise from a bundle of spectra $F_{\bullet} \wedge H\mathbb{Z}$ by passing to homotopy groups.

From this example, we quickly see that while it is reasonable to ask ξ to associate to each point $p \in X$ a spectrum ξ_p , it is too much to expect to associate to a path $\gamma \colon I \to X$ from p to q an isomorphism of fibers; instead, we expect a homotopy equivalence

$$\xi_{\gamma} \colon \xi_p \to \xi_q$$
.

Moreover, an (endpoint-preserving) homotopy of paths $H: \gamma \to \gamma'$ should give rise to a path

$$\xi_H \colon \xi_{\gamma} \to \xi_{\gamma'}$$

in the space of homotopy equivalences from ξ_p to ξ_q .

These homotopy coherence issues quickly lead one to consider representations of not merely the fundamental groupoid $\Pi_{\leq 1}(X)$, but the whole fundamental ∞ -groupoid $\Pi_{\leq \infty}(X)$, that is, the singular complex Sing X. Quasicategories make it both natural and inevitable to consider such representations.

 $3.2. \infty$ -categories from spaces and from simplicial model categories. Recall that a quasicategory is a simplicial set which has fillings for all inner horns. Thus one source of quasicategories is spaces. If X is a space, then its singular complex $\operatorname{Sing} X$ is a Kan complex: it has fillings for all horns. From the point of view of quasicategories, where 1-simplices correspond to morphisms, this means that all morphisms are invertible up to (coherent higher) homotopy. Thus Kan complexes may be identified with " ∞ -groupoids".

We also recall (from [HTT, Appendix A and 1.1.5.9]) how simplicial model categories give rise to quasicategories. This procedure is an important source of quasicategories, and it provides intuition about how quasicategories encode homotopy theory.

If \mathcal{M} is a simplicial model category, then we can define \mathcal{M}° to be the full subcategory consisting of cofibrant and fibrant objects. The simplicial nerve of \mathcal{M}° ,

$$\mathscr{C} = N\mathscr{M}^{\circ},$$

is the quasicategory associated to \mathcal{M} .

By construction, \mathscr{C} is the simplicial set in which

- (1) the vertices \mathscr{C}_0 are cofibrant-fibrant objects of \mathscr{M} ;
- (2) \mathscr{C}_1 consists of maps

$$L \to M$$

between cofibrant-fibrant objects;

(3) \mathcal{C}_2 consists of diagrams (not necessarily commutative)



together with a homotopy from gf to h in the mapping space (simplicial set) $\mathcal{M}(L, M)$; and so forth.

In particular, in \mathscr{C} the equivalences correspond to weak equivalences in \mathscr{M}° , that is, homotopy equivalences. Thus we may sometimes refer to the equivalences in a quasicategory as weak equivalences or homotopy equivalences.

A simplicial model category \mathcal{M} has an associated homotopy category ho \mathcal{M} , and an ∞ -category \mathcal{C} has a homotopy category ho \mathcal{C} . As one would expect, there is an equivalence of categories (enriched over the homotopy category of spaces)

$$ho \mathcal{M} \simeq ho N \mathcal{M}^{\circ}$$
.

By analogy to the model category situation, if $\mathscr C$ is a quasicategory and ho $\mathscr C\simeq \mathscr D$, then we shall say that $\mathscr C$ is a "model for $\mathscr D$ ".

3.3. The ∞ -category of A-modules. Let $\mathscr S$ be a symmetric monoidal ∞ -category of spectra. Lurie constructs such an ∞ -category from scratch ([DAGI] introduces an ∞ -category of spectra, which is shown to be monoidal in [DAGII], and symmetric monoidal in [DAGIII]). Lurie shows that his ∞ -category is equivalent to the symmetric monoidal ∞ -category arising from the symmetric spectra of [HSS00], and so by [MMSS01] it is equivalent to the symmetric monoidal ∞ -categories of spectra arising from various classical symmetric monoidal simplicial model categories of spectra. Let S be the sphere spectrum.

Definition 3.1. An S-algebra is a monoid (strictly speaking, an algebra, since the relevant monoidal structure is not given by the cartesian product) in \mathscr{S} . We write Alg(S) for the ∞ -category of S-algebras, and CommAlg(S) for the ∞ -category of commutative S-algebras.

Using [DAGII, DAGIII] and [MMSS01] as above, one learns that the symmetric monoidal structure on $\mathscr S$ is such that $\mathrm{Alg}(S)$ is a model for A_∞ ring spectra, and $\mathrm{CommAlg}(S)$ is a model for E_∞ ring spectra, so the reader is free to use his or her favorite method to produce ∞ -categories equivalent to $\mathrm{Alg}(S)$ and $\mathrm{CommAlg}(S)$.

Definition 3.2. If A is an S-algebra, we let Mod_A be the ∞ -category of A-modules.

Example 3.3. An S-module is just a spectrum, and so Mod_S is the ∞ -category of spectra.

3.4. Bundles of spaces and spectra. The purpose of this section is to introduce the ∞ -categorical model of parametrized spectra we work with in this paper and compare it to the May-Sigurdsson notion of parametrized spectra [MS06].

We begin by reviewing some models for the ∞ -category of spaces over a cofibrant topological space X. On the one hand, we have the topological model category \mathscr{T}/X of spaces over X, obtained from the topological model category of spaces by forming the slice category; i.e., the weak equivalences and fibrations are determined by the forgetful functor to spaces. We will refer to this model structure as the "standard" model structure on \mathscr{T}/X . This is Quillen equivalent to the corresponding simplicial model category structure on simplicial sets over Sing X, which in turn is Quillen equivalent to the simplicial model category of simplicial presheaves on the simplicial category $\mathfrak{C}[\operatorname{Sing} X]$ (with, say, the projective model structure) [HTT, §2.2.1.2]. Here \mathfrak{C} denotes the left adjoint to the simplicial nerve; it associates a simplicial category to a simplicial set [HTT, §1.1.5].

Remark 3.4. The Quillen equivalence between simplicial presheaves and parametrized spaces depends on the fact that the base is an ∞ -groupoid (Kan complex) as opposed to an ∞ -category; there is a more general

theory of "right fibrations" (and, dually, "left fibrations"), but over a Kan complex a right fibration is a left fibration (and conversely) and therefore a Kan fibration.

On the level of ∞ -categories, this yields an equivalence

$$\operatorname{St} \colon \operatorname{NSet}_{\Delta/\operatorname{Sing} X}^{\circ} \longrightarrow \operatorname{Fun}(\operatorname{Sing} X^{\operatorname{op}}, \operatorname{NSet}_{\Delta}^{\circ});$$

the map, called the *straightening* functor, rigidifies a fibration over Sing X into a presheaf of ∞ -groupoids on Sing X whose value at the point x is equivalent to the fiber over x [HTT, §3.2.1].

A distinct benefit of the presheaf approach is a particularly straightforward treatment of the base-change adjunctions. Given a map of spaces $f: Y \to X$, we may restrict a presheaf of ∞ -groupoids F on Sing X to a presheaf of ∞ -groupoids f^*F on Sing Y. This gives a functor, on the level of ∞ -categories, from spaces over X to spaces over Y, such that the fiber of f^*F over the point y of Y is equivalent to the fiber of F over f(y). Moreover, f^* admits both a left adjoint $f_!$ and a right adjoint f_* , given by left and right Kan extension along the map Sing $Y^{\text{op}} \to \text{Sing } X^{\text{op}}$, respectively. Note that this is left and right Kan extension in the ∞ -categorical sense, which amounts to homotopy left and right Kan extension on the level of simplicial categories or model categories. On the level of model categories of presheaves, there is an additional subtlety:

$$f^* : \operatorname{Fun}(\mathfrak{C}[\operatorname{Sing} X^{\operatorname{op}}], \operatorname{Set}_{\Delta}) \longrightarrow \operatorname{Fun}(\mathfrak{C}[\operatorname{Sing} Y^{\operatorname{op}}], \operatorname{Set}_{\Delta})$$

is a right Quillen functor for the projective model structure, with (derived) left adjoint $f_!$, and a left Quillen functor for the injective model structure, with (derived) right adjoint f_* , on the above categories of (simplicial) presheaves. Of course the identity adjunction gives a Quillen equivalence between these two model structures, but nevertheless one is forced to switch back and forth between projective and injective model structures if one wishes to simultaneously consider both base-change adjunctions.

Now we may stabilize either of the equivalent ∞ -categories

$$\operatorname{NSet}_{\Delta/\operatorname{Sing} X}^{\circ} \simeq \operatorname{Fun}(\operatorname{Sing} X^{\operatorname{op}}, \operatorname{NSet}_{\Delta}^{\circ})$$

by forming the ∞ -category of spectrum objects in \mathscr{C}_* ; here \mathscr{C}_* denotes the ∞ -category of *pointed* objects in \mathscr{C} . If \mathscr{C} is an ∞ -category with finite limits, then so is \mathscr{C}_* , and $\operatorname{Stab}(\mathscr{C})$ is defined as the inverse limit of the tower

$$\operatorname{Stab}(\mathscr{C}) = \lim \{ \cdots \xrightarrow{\Omega} \mathscr{C}_* \xrightarrow{\Omega} \mathscr{C}_* \}$$

associated to the loops endomorphism $\Omega \colon \mathscr{C}_* \to \mathscr{C}_*$ of \mathscr{C}_* . In other words, a spectrum object in an ∞ -category \mathscr{C} (with finite limits) is a sequence of pointed objects $A = \{A_0, A_1, \ldots\}$ together with equivalences $A_n \simeq \Omega A_{n+1}$ for each natural number n.

Thus, our category of parametrized spectra is the stabilization $\operatorname{Stab}(\operatorname{N}(\operatorname{Set}_{\Delta}/\operatorname{Sing}X)^{\circ})$. For our purposes, it turns out to be much more convenient to use the presheaf model; there is an equivalence of ∞ -categories

$$\operatorname{Stab}(\operatorname{N}(\operatorname{Set}_{\Delta}/\operatorname{Sing}X)^{\circ}) \simeq \operatorname{Fun}(\operatorname{Sing}X^{\operatorname{op}},\mathscr{S}).$$

Note that a functor $F: \operatorname{Sing} X \to \mathscr{S}$ associates to each point x of X a spectrum F_x , to each path $x_0 \to x_1$ in X a map of spectra (necessarily a homotopy equivalence) $F_{x_0} \to F_{x_1}$, and so on for higher-dimensional simplices of X.

Given a presentable ∞ -category \mathscr{C} , the stabilization $\operatorname{Stab}(\mathscr{C})$ is itself presentable, and the functor Ω^{∞} : $\operatorname{Stab}(\mathscr{C}) \to \mathscr{C}$ admits a left adjoint $\Sigma^{\infty} : \mathscr{C} \to \operatorname{Stab}(\mathscr{C})$ [DAGI, Proposition 15.4]. Just as Ω^{∞} is natural in presentable ∞ -categories and right adjoint functors, dually, Σ^{∞} is natural in presentable ∞ -categories and left adjoint functors [DAGI, Corollary 15.5]. In particular, given a map of spaces $f: Y \to X$, the adjoint pairs $(f_!, f^*)$ and (f^*, f_*) defined above extend to the stabilizations, yielding a restriction functor

$$f^* : \operatorname{Fun}(\operatorname{Sing} X^{\operatorname{op}}, \mathscr{S}) \longrightarrow \operatorname{Fun}(\operatorname{Sing} Y^{\operatorname{op}}, \mathscr{S})$$

which admits a left adjoint f_1 and a right adjoint f^* , again given by left and right Kan extension, respectively.

We can also formally stabilize suitable model categories, using Hovey's work on spectra in general model categories [Hov01]. Specifically, given a left proper cellular model category \mathscr{C} and an endofunctor of \mathscr{C} , Hovey constructs a cellular model category $\operatorname{Sp}^{\mathbb{N}}\mathscr{C}$ of spectra. When \mathscr{C} is additionally a simplicial symmetric monoidal model category, the endofunctor given by the tensor with S^1 yields a simplicial symmetric monoidal

model category of symmetric spectra $\operatorname{Sp}^{\Sigma}\mathscr{C}$ (as well as a simplicial model category $\operatorname{Sp}^{\mathbb{N}}\mathscr{C}$ of prespectra). These models of the stabilization are functorial in left Quillen functors which are suitably compatible with the respective endofunctors (see [Hov01, 5.2]).

In order to compare our model of parametrized spectra over X to the May-Sigurdsson model, we use the following consistency result.

Proposition 3.5. Let \mathscr{C} be a left proper cellular simplicial model category and write $\operatorname{Sp}^{\mathbb{N}}\mathscr{C}$ for the cellular simplicial model category of spectra generated by the tensor with S^1 . Then there is an equivalence of ∞ -categories

$$N(Sp^{\mathbb{N}}\mathscr{C})^{\circ} \simeq Stab(N\mathscr{C}^{\circ}).$$

Proof. The functors $\operatorname{Ev}_n \colon \operatorname{Sp}^{\mathbb{N}} \mathscr{C} \to \mathscr{C}$, which associate to a spectrum its n^{th} -space A_n , induce a functor (of ∞ -categories)

$$f \colon \mathcal{N}(\mathcal{Sp}^{\mathbb{N}}\mathscr{C})^{\circ} \to \lim \{ \cdots \xrightarrow{\Omega} \mathcal{N}\mathscr{C}_{*}^{\circ} \xrightarrow{\Omega} \mathcal{N}\mathscr{C}_{*}^{\circ} \} \simeq \operatorname{Stab}(\mathcal{N}\mathscr{C}^{\circ})$$

which is evidently essentially surjective. To see that it is fully faithful, it suffices to check that for cofibrant-fibrant spectrum objects A and B in $\operatorname{Sp}^{\mathbb{N}}\mathscr{C}$, there is an equivalence of mapping spaces

$$\operatorname{map}(A, B) \simeq \operatorname{holim}\{\cdots \xrightarrow{\Omega} \operatorname{map}(A_1, B_1) \xrightarrow{\Omega} \operatorname{map}(A_0, B_0)\},\$$

where Ω : map $(A_{n+1}, B_{n+1}) \to \text{map}(A_n, B_n)$ sends $A_{n+1} \to B_{n+1}$ to $A_n \simeq \Omega A_{n+1} \to \Omega B_{n+1} \simeq B_n$. Since any cofibrant A is a retract of a cellular object, inductively we can reduce to the case in which $A = F_m X$, i.e., the shifted suspension spectrum on a cofibrant object X of \mathscr{C}_* . Then $\text{map}(A, B) \simeq \text{map}(X, B_m)$ by adjunction. The latter is in turn equivalent to $\text{map}(\Sigma^{n-m}X, B_n)$, where we interpret $\Sigma^{n-m}X = *$ for m > n, in which case the homotopy limit is equivalent to that of the homotopically constant (above degree n) tower whose n^{th} term is $\text{map}(\Sigma^{n-m}X, B_n)$.

We now recall the May-Sigurdsson setup. Given a space X, let $(\mathcal{T}/X)_*$ denote the category of spaces over and under X (ex-spaces). Although this category has a model structure induced by the standard model structure on \mathcal{T}/X , one of the key insights of May and Sigurdsson is that for the purposes of parametrized homotopy theory it is essential to work with a variant they call the qf-model structure [MS06, 6.2.6]. This model structure is Quillen equivalent to the standard model structure on ex-spaces [MS06, 6.2.7]; however, its cofiber and fiber sequences are compatible with classical notions of cofibration and fibration (described in terms of extension and lifting properties).

May and Sigurdsson then construct a stable model structure on the categories \mathscr{S}_X of orthogonal spectra in $(\mathscr{T}/X)_*$ [MS06, 12.3.10]. This model structure is based on the qf-model structure on ex-spaces, leveraging the diagrammatic viewpoint of [MMSS01, MM02]. Similarly, they construct a stable model structure on the category \mathscr{P}_X of prespectra in $(\mathscr{T}/X)_*$; the forgetful functor $\mathscr{S}_X \to \mathscr{P}_X$ is a Quillen equivalence [MS06, 12.3.10].

Using [MS06, 12.3.14], we see that after passing to ∞ -categories the category \mathscr{P}_X is in turn equivalent to the category $\operatorname{Sp}^{\mathbb{N}}(\mathscr{T}/X)_*$; the formal stabilization of the qf-model structure on $(\mathscr{T}/X)_*$ with respect to the fiberwise smash with S^1 . Using Proposition 3.5 and the fact that the qf-model structure is Quillen equivalent to the standard model structure, we obtain equivalences of ∞ -categories

$$\begin{split} \mathrm{N}(\mathscr{S}_X)^\circ &\to \mathrm{N}(\mathscr{P}_X)^\circ \to \mathrm{N}(\mathrm{Sp}^{\mathbb{N}}(\mathscr{T}/X)_*)^\circ \\ &\to \mathrm{Stab}(\mathrm{N}(\mathscr{T}/X)^\circ) \to \mathrm{Stab}(\mathrm{N}(\mathrm{Set}_\Delta/\operatorname{Sing}X)^\circ) \to \mathrm{Fun}(\mathrm{Sing}\,X^\mathrm{op},\mathscr{S}). \end{split}$$

Thus we obtain the following comparison theorem.

Theorem 3.6. There is an equivalence of ∞ -categories between the simplicial nerve of the May-Sigurdsson category of parametrized orthogonal spectra $N(\mathscr{S}_X)^{\circ}$ and the ∞ -category $Fun(\operatorname{Sing} X^{\operatorname{op}}, \mathscr{S})$ of presheaves on X with values in spectra.

Furthermore, the derived base-change functors we construct via the stabilization of the presheaves agree with the derived base-change functors constructed by May and Sigurdsson. To see this, observe that it suffices to check this for f^* ; compatibility then follows formally for the adjoints f_* and $f_!$. Moreover, since

 f^* on the categories of spectra is obtained as the suspension of f^* on spaces, we can reduce to checking that the right derived functor of $f^*: (\mathscr{T}/X)_* \to (\mathscr{T}/Y)_*$ in the qf-model structure is compatible with the right derived functor of $f^*: \operatorname{Fun}(\mathfrak{C}(\operatorname{Sing} X^{\operatorname{op}}), \operatorname{Set}_{\Delta}) \to \operatorname{Fun}(\mathfrak{C}(\operatorname{Sing} Y^{\operatorname{op}}), \operatorname{Set}_{\Delta})$ in the projective model structure. By the work of [MS06, §9.3], it suffices to check the compatibility for f^* in the q-model structure. Since both versions of f^* that arise here are Quillen right adjoints, this amounts to the verification that the diagram

$$\operatorname{Fun}(\mathfrak{C}(\operatorname{Sing} X^{\operatorname{op}}), \operatorname{Set}_{\Delta}) \xrightarrow{f^*} \operatorname{Fun}(\mathfrak{C}(\operatorname{Sing} Y^{\operatorname{op}}), \operatorname{Set}_{\Delta})$$

$$\downarrow^{\operatorname{Un}} \qquad \qquad \downarrow^{\operatorname{Un}}$$

$$\operatorname{Set}_{\Delta}/X \xrightarrow{f^*} \operatorname{Set}_{\Delta}/Y$$

commutes when applied to fibrant objects, where here Un denotes the unstraightening functor (which is the right adjoint of the Quillen equivalence). Finally, this follows from [HTT, 2.2.1.1].

3.5. Bundles of A-modules and A-lines. If X is a space, let Sing X be its singular complex. The work of the previous section justifies the following definition.

Definition 3.7. A bundle or homotopy local system of A-modules over X is a map of simplicial sets

$$f \colon \operatorname{Sing} X \to \operatorname{Mod}_A$$
.

Similarly if Y is any ∞ -groupoid, then a bundle of A-modules over Y is just a map of simplicial sets

$$f: Y \to \mathrm{Mod}_A$$
.

Thus f assigns

- (0) to each point $p \in X$ an A-module f(p);
- (1) to each path γ from p to q a map of A-modules

$$f(\gamma) \colon f(p) \to f(q);$$
 (3.8)

(2) to each 2-simplex $\sigma \colon \Delta^2 \to X$, say

$$\begin{array}{c}
p \\
\sigma_{01} \downarrow \\
q \xrightarrow{\sigma_{12}} r,
\end{array}$$

a path $f(\sigma)$ in $\operatorname{Mod}_A(f(p), f(r))$ from $f(\sigma_{12})f(\sigma_{01})$ to $f(\sigma_{02})$;

and so forth.

Recall [HTT, 1.2.7.3] that if Y is a simplicial set and \mathcal{C} is an ∞ -category, then the simplicial mapping space \mathcal{C}^Y is the ∞ -category Fun (Y,\mathcal{C}) of functors from Y to \mathcal{C} .

Definition 3.9. The ∞ -category of bundles of A-modules over X is the simplicial mapping space

$$\operatorname{Mod}_A^X \stackrel{\operatorname{def}}{=} \operatorname{Fun}(\operatorname{Sing} X, \operatorname{Mod}_A).$$

Remark 3.10. We have not set up the framework necessary to work directly with the bundle of A-modules associated to $f \colon \operatorname{Sing} X \to \operatorname{Mod}_A$ (although see Theorem 3.6). Nonetheless, the notation of bundle and pullback is compelling, and so we write \mathscr{M} for the identity map

$$Mod_A \to Mod_A$$
,

and if $f : \operatorname{Sing} X \to \operatorname{Mod}_A$ is a map of ∞ -categories, then we may write $f^* \mathcal{M}$ as a synonym for f, when we want to emphasize its bundle aspect.

Recall that Sing X is a Kan complex or ∞ -groupoid: it satisfies the extension condition for all horns. Viewing an ∞ -category as a model for a homotopy theory, an ∞ -groupoid models a homotopy theory in which all the morphisms are homotopy equivalences.

In particular, the map $f(\gamma)$ in (3.8) is necessarily an equivalence: the A-modules f(p) will vary through weak equivalences as p varies over a path component of X. We shall be particularly interested in the case that these fibers are free rank-one A-modules.

Definition 3.11. An A-line is an A-module L which admits a weak equivalence

$$L \xrightarrow{\cong} A$$
.

The ∞ -category Line_A is the maximal ∞ -groupoid in Mod_A generated by the A-lines. We write j for the inclusion

$$j: \operatorname{Line}_A \to \operatorname{Mod}_A$$

and $\mathcal{L} \stackrel{\text{def}}{=} j^* \mathcal{M}$ for the tautological bundle of A-lines over Line_A.

By construction Line_A is a Kan complex, and we regard it as the classifying space for bundles of A-lines. If X is a space, then a map

$$f \colon \operatorname{Sing} X \to \operatorname{Line}_A$$
.

assigns

- (0) to each point $p \in X$ an A-line f(p);
- (1) to each path γ from p to q an equivalence map of A-lines

$$f(\gamma)$$
: $f(p) \simeq f(q)$;

(2) to each 2-simplex $\sigma \colon \Delta^2 \to X$, say

$$\begin{array}{c}
p \\
\sigma_{01} \downarrow \\
q \xrightarrow{\sigma_{12}} r,
\end{array}$$

a path $f(\sigma)$ in Line_A(f(p), f(r)) from $f(\sigma_{12})f(\sigma_{01})$ to $f(\sigma_{02})$;

and so forth

Definition 3.12. The simplicial mapping space $\operatorname{Line}_A^X = \operatorname{Fun}(\operatorname{Sing} X, \operatorname{Line}_A)$ is an ∞ -category (in fact, a Kan complex); we call it the the ∞ -category or space of A-lines over X.

We develop twisted A-theory starting from Line_A in §5. Before doing so, we briefly discuss other aspects of the ∞ -category Line_A .

3.6. Line_A and GL_1A . By construction, Line_A is connected, and so equivalent to the maximal ∞ -groupoid $B \operatorname{Aut}(A)$ on the single A-module A. As we discuss in [ABGHR, §6], it is an important point that the space of morphisms $\operatorname{Aut}(A) = \operatorname{Line}_A(A, A)$ is not a group, or even a monoid, but instead merely a group-like A_∞ space.

Nevertheless, Line_A is not only the classifying space for bundles of A-lines, but it is a delooping of Aut(A). To see this, let Triv(A) be the ∞ -category of A-lines L, equipped with an equivalence $L \stackrel{\simeq}{\to} A$. Then [ABGHR, Prop. 7.38] Triv(A) is contractible, and the map

$$Triv(A) \to Line_A$$

is a Kan fibration, with fiber Aut(A).

Classical infinite loop space theory provides another model for homotopy type $\operatorname{Aut}(A)$. Namely, let A be an A_{∞} ring spectrum in the sense of [LMSM86]: so $\pi_0 \Omega^{\infty} A$ is a ring. Let $GL_1 A$ be the pull-back in the diagram

$$GL_1A \longrightarrow \Omega^{\infty}A$$

$$\downarrow \qquad \qquad \downarrow$$

$$(\pi_0\Omega^{\infty}A)^{\times} \longrightarrow \pi_0\Omega^{\infty}A.$$

Then GL_1A is a group-like A_{∞} space: π_0GL_1A is a group. We show that

$$GL_1A \simeq |\operatorname{Aut}(A)|.$$

Since the geometric realization of a Kan fibration is a Serre fibration [Qui68], the fibration

$$\operatorname{Aut}(A) \to \operatorname{Triv}(A) \to \operatorname{Line}_A$$

gives rise to a fibration

$$GL_1A \simeq |\operatorname{Aut}(A)| \to |\operatorname{Triv}(A)| \simeq * \to |\operatorname{Line}_A|.$$
 (3.13)

Thus $|\text{Line}_A|$ provides a model for the delooping BGL_1A . It has the virtue that we have already given a precise description of the vertices of the simplicial mapping space

$$\operatorname{Line}_A^X = \operatorname{map}(\operatorname{Sing} X, \operatorname{Line}_A) \simeq \operatorname{map}(X, |\operatorname{Line}_A|).$$

Example 3.14. If S is the sphere spectrum, then $\Omega^{\infty}S$ is the space $QS^0 = \Omega^{\infty}\Sigma^{\infty}S^0$, and

$$GL_1S = Q_{\pm 1}S^0,$$

i.e., the unit components. The space $BGL_1S \simeq |\mathrm{Line}_S|$ is the classifying space for stable spherical fibrations of virtual rank 0. It follows the space of S-lines over X is homotopy equivalent to the space of spherical fibration of virtual rank 0.

Example 3.15. The classical *J*-homomorphism is a map

$$J\colon O\to GL_1S$$
,

which deloops to give a map

$$BJ \colon BO \to BGL_1S$$
.

One sees that this is the map which takes a virtual vector bundle of rank 0 to its associated stable spherical fibration; we may regard this as associating to a vector bundle its bundle of S-lines.

Example 3.16. If A is an S-algebra, then the unit of A induces a map

$$BGL_1S \to BGL_1A$$
.

In our setting, this map arises from the map of ∞ -categories

$$Mod_S \to Mod_A$$

given by $M \mapsto M \otimes_S A = M \wedge_S A$, which restricts to give a map of ∞ -categories

$$Line_S \to Line_A$$
.

Example 3.17 ([MQRT77]). Let $H\mathbb{Z}$ be the integral Eilenberg-MacLane spectrum. Then $\Omega^{\infty}H\mathbb{Z} \simeq K(\mathbb{Z},0) \simeq \mathbb{Z}$, and so $GL_1H\mathbb{Z} \simeq \{\pm 1\} \simeq \mathbb{Z}/2$, and $BGL_1H\mathbb{Z} \simeq B\mathbb{Z}/2 \simeq K(\mathbb{Z}/2,1)$.

Remark 3.18. If A = K, the spectrum representing complex K-theory, then Aut(K) has the homotopy type of the space of K-module equivalences $K \to K$. Atiyah and Segal [AS04] build twists of K-theory from PU-bundles. They remark that one can more generally build twists of K-theory from G-bundles, where G is the group of strict K-module automorphisms of K-theory. Our space Aut(K) generalizes this idea.

Remark 3.19. As we explain in [ABGHR, §6], for many algebras A (including the sphere S), the group $\operatorname{Aut}_{strict}(A)$ of strict A-module automorphisms of A cannot provide a sufficiently rich theory of bundles of A-modules. For example, Lewis's Theorem [Lew91] implies that there is no model for the sphere spectrum S such that the classifying space B $\operatorname{Aut}_{strict}(S)$ classifies stable spherical fibrations. (See also [MS06, §22.2] for discussion of this issue.)

4. The generalized Thom spectrum

Let A be an S-algebra, let X be a space, and let f be a bundle of A-lines over X, that is, a map of simplicial sets

$$f \colon \operatorname{Sing} X \to \operatorname{Line}_A$$
.

Although the ∞ -category Line_A is not cocomplete (it doesn't even have sums), the ∞ -category Mod_A of A-modules is complete and cocomplete. This allows us in [ABGHR] to make the following definition.

Definition 4.1. The *Thom spectrum* of f is the colimit

$$X^f \stackrel{\mathrm{def}}{=} \operatorname{colim} \left(\operatorname{Sing} X \xrightarrow{f} \operatorname{Line}_A \xrightarrow{j} \operatorname{Mod}_A \right).$$

Equivalently, X^f is the left Kan extension $L_{\pi}jf$ in the diagram

$$X \xrightarrow{f} \operatorname{Line}_{A} \xrightarrow{j} \operatorname{Mod}_{A}.$$

$$\pi \downarrow \qquad \qquad \qquad \downarrow$$

$$* - - \stackrel{f}{L_{p}(fj)}$$

The colimit and left Kan extension here are ∞ -categorical colimits: they are generalizations of the notion of homotopy colimit and homotopy left Kan extension. It is an important achievement of ∞ -category theory to give a sensible definition of these colimits.

Let $A_X: X \to * \to \text{Line}_A$ be the map which picks out A, considered as the constant A-line over X. The colimit means that we have an equivalence of mapping spaces

$$\operatorname{Mod}_A(X^f, A) \simeq \operatorname{Mod}_A^X(f^*j^*\mathcal{M}, A_X).$$

Notice also that we have a natural inclusion

$$\operatorname{Line}_{A}^{X}(f^{*}\mathcal{L}, A_{X}) \to \operatorname{Mod}_{A}^{X}(f^{*}j^{*}\mathcal{M}, A_{X}):$$
 (4.2)

a map of bundles of A-modules $f^*\mathcal{L} \to A_X$ is a map of bundles of A-lines if it is an equivalence over every point of X, and one checks that the inclusion (4.2) is the inclusion of a set of path components.

Definition 4.3. The space of orientations of X^f is the pull-back in the diagram

$$\operatorname{orient}(X^f, A) \longrightarrow \operatorname{Mod}_A(X^f, A)$$

$$\simeq \downarrow \qquad \qquad \downarrow \simeq \qquad (4.4)$$

$$\operatorname{Line}_A^X(f^*\mathscr{L}, A_X) \longrightarrow \operatorname{Mod}_A^X(f^*j^*\mathscr{M}, A_X).$$

That is, the space of orientations orient (X^f, A) is the subspace of A-module maps $X^f \to A$ which correspond, under the equivalence

$$\operatorname{Mod}_A(X^f, A) \simeq \operatorname{Mod}_A^X(f^*j^*\mathcal{M}, A_X),$$

to fiberwise equivalences $f^*\mathcal{L} \to A_X$.

This appealing notion of orientation expresses orientations as fiberwise equivalences of bundles of spectra. The following results from [ABGHR] explain how our Thom spectra and orientations generalize the classical notions.

Theorem 4.5. Suppose that

$$\xi \colon \operatorname{Sing} X \to \operatorname{Line}_S$$

corresponds to a map

$$g: X \to BGL_1S$$
.

Then X^{ξ} is equivalent to the classical Thom spectrum X^g of the spherical fibration classified by g. It follows (see Example 3.16) that if f is the composition

$$f \colon \operatorname{Sing} X \xrightarrow{\xi} \operatorname{Line}_S \to \operatorname{Line}_A$$

then $X^f \simeq X^{\xi} \wedge_S A \simeq X^g \wedge_S A$ is equivalent to the classical Thom spectrum tensored with A.

We can then study the space of orientations of X^f via the equivalences

$$\operatorname{Mod}_A(X^f, A) \simeq \operatorname{Mod}_A(X^g \wedge_S A, A) \simeq \operatorname{Mod}_S(X^g, A),$$
 (4.6)

and we find that

Proposition 4.7. A map $\alpha: X^f \to A \in \operatorname{Mod}_A(X^f, A)$ is in orient (X^f, A) if and only if it corresponds to an orientation $\beta \colon X^g \to A$ of the classical Thom spectrum, that is, if and only if

$$(X_{+} \xrightarrow{z} A) \mapsto (X^{g} \xrightarrow{\Delta} X_{+} \wedge X^{g} \xrightarrow{z \wedge \beta} A \wedge A \to A)$$

induces an isomorphism

$$A^*(X_+) \cong A^*(X^g).$$

Our theory leads to an obstruction theory for orientations. Let $\operatorname{map}_f(\operatorname{Sing} X, \operatorname{Triv}(A))$ be the simplicial set which is the pull-back in the diagram

$$\begin{split} \operatorname{map}_f(\operatorname{Sing} X, \operatorname{Triv}(A)) & \longrightarrow & \operatorname{map}(\operatorname{Sing} X, \operatorname{Triv}(A)) \\ & \downarrow & & \downarrow \\ \{f\} & \longrightarrow & \operatorname{map}(\operatorname{Sing} X, \operatorname{Line}_A). \end{split}$$

That is, $\operatorname{map}_{f}(\operatorname{Sing} X, \operatorname{Triv}(A))$ is the mapping simplicial set of lifts in the diagram

$$\begin{array}{c}
\operatorname{Triv}(A) \\
\downarrow \\
\operatorname{Sing} X \xrightarrow{f} \operatorname{Line}_{A}.
\end{array} (4.8)$$

The obstruction theory for orientations of the bundle of A-modules is given by the following.

Theorem 4.9. Let $f: \operatorname{Sing} X \to \operatorname{Line}_A$ be a bundle of A-lines over X, and let X^f be the associated A-module Thom spectrum. Then there is an equivalence

$$\operatorname{map}_f(\operatorname{Sing} X,\operatorname{Triv}(A)) \simeq \operatorname{Line}_A^X(f,\iota) \simeq \operatorname{orient}(X^f,A).$$

In particular, the bundle $f^*\mathcal{L}$ admits an orientation if and only if f is null-homotopic.

Example 4.10. This theorem recovers and slightly generalizes the obstruction theory of [MQRT77] (which treats the case that A is a E_{∞} ring spectrum, that is, a commutative S-algebra). Let $g: X \to BGL_1S$ be a stable spherical fibration. Then g admits a Thom isomorphism in A-theory if and only if the composition

$$X \xrightarrow{g} BGL_1S \simeq |\text{Line}_S| \to |\text{Line}_A| \simeq BGL_1A$$

is null.

Example 4.11. This example appears in [MQRT77]. Let $H\mathbb{Z}$ be the integral Eilenberg-MacLane spectrum. From Example 3.17 we have $BGL_1H\mathbb{Z} \simeq K(\mathbb{Z}/2,1)$. The obstruction to orienting a vector bundle V/X in singular cohomology is the map

$$X \xrightarrow{V} BO \xrightarrow{BJ} BGL_1S \to BGL_1H\mathbb{Z} \simeq K(\mathbb{Z}/2,1);$$

this is just the first Stiefel-Whitney class.

5. Twisted generalized cohomology

Now we consider twisted generalized cohomology in the language of sections 3 and 4. Let A be an S-algebra, and let

$$f \colon \operatorname{Sing} X \to \operatorname{Line}_A$$

or, equivalently, $f: X \to BGL_1A$ (see §3.6) classify a bundle of A-lines over X. As in Definition 4.1, let

$$X^f = \operatorname{colim}\left(\operatorname{Sing} X \xrightarrow{f} \operatorname{Line}_A \xrightarrow{j} \operatorname{Mod}_A\right)$$
₁₃

be the indicated A-module. We think of X^f as the f-twisted cohomology object associated to the bundle f, and we make the following

Definition 5.1. The f-twisted A homology and cohomology groups of X are

$$A^n(X)_f \stackrel{\text{def}}{=} \pi_0 \text{Mod}_A(X^f, \Sigma^n A)$$

$$A_n(X)_f \stackrel{\text{def}}{=} \pi_0 \text{Mod}_A(\Sigma^n A, X^f).$$

Equivalently, we have

$$A^{n}(X)_{f} = \pi_{-n}F_{A}(X^{f}, A)$$

$$A_{n}(X)_{f} = \pi_{n}F_{A}(A, X^{f}) \cong \pi_{n}X^{f}.$$

Here if V and W are A-modules, then $F_A(V, W)$ is the function spectrum of A-module maps from V to W: it is a spectrum such that

$$\Omega^{\infty} F_A(V, W) \simeq \operatorname{Mod}_A(V, W).$$

Thus for $n \geq 0$,

$$\pi_n F_A(V, W) \cong \pi_n \operatorname{Mod}_A(V, W) \cong \operatorname{Mod}_A(\Sigma^n V, W) \cong \operatorname{Mod}_A(V, \Sigma^{-n} W).$$

Example 5.2. Suppose that V is a vector bundle over X. Then we can form the map

$$j(V): X \xrightarrow{V} BO \xrightarrow{BJ} BGL_1S \to BGL_1A.$$

and also the twisted cohomology

$$A^*(X)_{j(V)} = \pi_0 \text{Mod}_A(X^{j(V)}, \Sigma^* A).$$

Since by Theorem 4.5

$$X^{j(V)} \simeq X^V \wedge A$$
,

we have

$$A^*(X)_{j(V)} \cong \pi_0 \text{Mod}_S(X^V, \Sigma^* A) = A^*(X^V),$$
 (5.3)

so in this case the twisted cohomology is just the cohomology of the Thom spectrum.

The definition is not quite a direct generalization of that Atiyah and Segal. Let S_X : Sing $X \to \text{Mod}_S$ be the constant functor which attaches to each point of X the sphere spectrum S. Theorem 3.6 and the work of [MS06, §22] allow us to describe their construction as attaching to the bundle of A-lines $f^*\mathscr{L}$ over X the group

$$A(X)_{f,AS} = \pi_0 \Gamma(f/X) \cong \pi_0 \operatorname{Mod}_S^X(S_X, f^* \mathscr{L}).$$

To compare this definition to ours, we must assume (as is the case for K-theory, for example) that A is a commutative S-algebra. In that case, if L is an A-module, then the dual spectrum

$$L^{\vee} \stackrel{\text{def}}{=} F_A(L, A)$$

is again an A-module. As in the case of classical commutative rings, the operation $L\mapsto L^\vee$ defines an involution

$$\operatorname{Line}_A \xrightarrow{(-)^{\vee}} \operatorname{Line}_A$$

on the ∞ -category of A-lines, such that

$$L^{\vee} \wedge_A M \simeq F_A(L, M);$$

in particular

$$\operatorname{Mod}_{S}(S, L^{\vee}) \simeq \operatorname{Mod}_{A}(A, L^{\vee}) \simeq \operatorname{Mod}_{A}(L, A).$$

Remark 5.4. As we have written it, we have an evident functor $\operatorname{Line}_A \to (\operatorname{Line}_A)^{\operatorname{op}}$. As Line_A is an ∞ -groupoid, it admits an equivalence $\operatorname{Line}_A \simeq \operatorname{Line}_A^{\operatorname{op}}$.

This is an ∞ -categorical approach to the following: if A is a commutative S-algebra then GL_1A is a sort of commutative group. More precisely, it is a commutative group-like monoid in the ∞ -category of spaces, or equivalently it is a group-like E_{∞} -space. As such it has an involution

$$-1: GL_1A \to GL_1A$$

which deloops to a map

$$B(-1): BGL_1A \to BGL_1A.$$

In any case, given a map

$$f \colon \operatorname{Sing} X \to \operatorname{Line}_A$$
,

classifying the bundle $f^*\mathcal{L}$, we may form the map

$$-f = f^{\vee} \colon \operatorname{Sing} X \xrightarrow{f} \operatorname{Line}_A \xrightarrow{(-)^{\vee}} \operatorname{Line}_A$$

so that $(-f)^*\mathcal{L}$ is the fiberwise dual of $f^*\mathcal{L}$, and then one has

$$\Gamma((-f)^* \mathscr{L}) = \operatorname{Mod}_S^X(S_X, (-f)^* \mathscr{L})$$

$$\simeq \operatorname{Mod}_A^X(A_X, (-f)^* \mathscr{L})$$

$$\simeq \operatorname{Mod}_A^X(f^* \mathscr{L}, A)$$

$$\simeq \operatorname{Mod}_A(X^f, A).$$

That is, the cohomology object we associate to $f: X \to BGL_1A$ is the one which Atiyah and Segal associate to $-f: X \to BGL_1A$,

$$A^*(X)_f \cong A^*(X)_{-f,AS}$$
.

Of course Atiyah and Segal also explain how to construct a K-line from a PU-bundle over X: in our language, they construct a map

$$BPU \simeq K(\mathbb{Z},3) \to BGL_1K.$$

From our point of view the existence of this map can be phrased as a question about the $Spin^c$ orientation of complex K-theory.

To see this, recall that Atiyah, Bott, and Shapiro [ABS64] produce a Thom isomorphism in complex K-theory for $Spin^c$ -bundles. According to Theorem 4.9, this corresponds to the arrow labeled ABS in the diagram

$$K(\mathbb{Z},2) \xrightarrow{} GL_1K$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$BSpin^c \xrightarrow{ABS} \operatorname{Triv}(K) \simeq *$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$BSO \xrightarrow{BJ} \operatorname{Line}_S \xrightarrow{(-) \land K} \operatorname{Line}_K \simeq BGL_1K$$

$$\beta w_2 \downarrow \qquad \qquad \parallel$$

$$K(\mathbb{Z},3) - - - \xrightarrow{BABS} - - - \rightarrow BGL_1K$$

The map ABS induces a map

$$ABS: K(\mathbb{Z}, 2) \to GL_1K,$$
 (5.5)

and the diagram suggests that we ask whether this map deloops to give a map

$$B(ABS): K(\mathbb{Z},3) \to BGL_1K$$
,

as indicated. Not surprisingly, this question is related to the multiplicative properties of the orientation.

The Thom spectrum associated $MSpin^c$ associated to $BSpin^c$ is a commutative S-algebra, as is K-theory. The construction of Atiyah-Bott-Shapiro produces a map of spectra

$$t: MSpin^c \to K$$

and Michael Joachim [Joa04] shows that t can be refined to a map of commutative S-algebras. As we shall see in $\S 6$, it follows that $K(\mathbb{Z},2) \to GL_1K$ is a map of infinite loop spaces.

We use these ideas to twist K-theory by maps $X \to K(\mathbb{Z},3)$ in §7.

6. Multiplicative orientations and comparison of Thom spectra

The most familiar orientations are exponential: the Thom class of a Whitney sum is the product of the Thom classes. For example, consider the case of Spin bundles and real K-theory, KO. Atiyah, Bott, and Shapiro show that the Dirac operator associates to a spin vector bundle $V \to X$ a Thom class $t(V) \in KO(X^V)$. If MSpin is the Thom spectrum of the universal Spin bundle over BSpin, then we can view their construction as corresponding to a map of spectra

$$t: MSpin \rightarrow KO.$$

If $W \to Y$ is another spin vector bundle, then

$$(X \times Y)^{V \oplus W} \simeq X^V \wedge Y^W,$$

and it turns out that with respect to the resulting isomorphism

$$KO((X \times Y)^{V \oplus W}) \cong KO(X^V \wedge Y^W),$$

one has

$$t(V \oplus W) = t(V) \wedge t(W). \tag{6.1}$$

Now the sum of vector bundles gives MSpin the structure of a ring spectrum, and so the multiplicative property (6.1) (together with a unit condition, which says that $t(\underline{0}) = 1$) corresponds to the fact that

$$t: MSpin \rightarrow KO$$

is a map of monoids in the homotopy category of spectra.

It is important that t is in fact a map of commutative monoids in the ∞ -category of spectra. More precisely, MSpin and KO are both commutative S-algebras, and it turns out [Joa04, AHR] that t is a map of commutative S-algebras.

The construction of classical Thom spectra such as MSpin, MSO, MU as commutative S-algebras (equivalently, E_{∞} ring spectra) is due to [MQRT77, LMSM86]. In this section, we discuss the theory from the ∞ -categorical point of view. We'll see (Remark 6.23) that this gives a way to think about the comparison of our Thom spectrum to classical constructions. It also provides some tools we use to build twists of K-theory from PU-bundles.

We begin with a question. Suppose that A is a commutative S-algebra. Under what conditions on a map f should we expect that the Thom

$$X^f = \operatorname{colim}(\operatorname{Sing} X \xrightarrow{f} \operatorname{Line}_A \xrightarrow{j} \operatorname{Mod}_A)$$

is a commutative A-algebra? And in that situation, how do we understand A-algebra maps out of X^f ?

In the context of ∞ -categories, spaces play the role which sets play in the context of classical categories, and so we begin by studying the situation of a discrete commutative ring R and a set X. In that case, an R-line is just a free rank-one R-module, and a bundle of R-lines ξ over X is just a collection of R-lines, indexed by the points $x \in X$. We can think of this as a functor

$$\xi \colon X \to \mathrm{Line}_R$$

from X, considered as a discrete category, to the category of R-lines: free rank-one R-modules and isomorphisms. The "Thom spectrum"

$$X^{\xi} = \operatorname{colim}\left(X \xrightarrow{\xi} \operatorname{Line}_R \to \operatorname{Mod}_R\right)$$

is easily seen to be the sum

$$X^{\xi} \cong \bigoplus_{x \in X} \xi_x.$$

Now suppose that R is a commutative ring, so that Mod_R is a symmetric monoidal category, and Line_R is the maximal sub-groupoid of Mod_R generated by R. If X is a discrete abelian group, then we may consider X as a symmetric monoidal category with objects the elements of X. It is then not difficult to check the following.

Proposition 6.2. If $\xi: X \to \text{Line}_R$ is a map of symmetric monoidal categories, then X^{ξ} has structure of a commutative R-algebra.

The analogue of this result holds in the ∞ -categorical setting; see Theorem 6.21 below. It is possible to give a direct proof; instead we sketch the circuitous proof given in [ABGHR], as some of the results which arise along the way will be useful in sections 7 and 8.

We begin with another construction of the R-module X^{ξ} in the discrete associative case. Let GL_1R be the group of units of R. Note that the free abelian group functor

$$\mathbb{Z}$$
: (sets) \to (abelian goups) (6.3)

induces a functor

$$\mathbb{Z}$$
: (groups) \longrightarrow (rings): GL_1 (6.4)

whose right adjoint is GL_1 . In particular, we have a natural map of rings

$$\mathbb{Z}[GL_1R] \to R$$

and so the colimit-preserving functor

$$(GL_1R\text{-sets}) \xrightarrow{\mathbb{Z}[-] \otimes_{\mathbb{Z}[GL_1R]}R} \operatorname{Mod}_R.$$

This functor restricts to an equivalence of categories

$$\mathbb{Z}[-] \otimes_{\mathbb{Z}[GL_1R]} R \colon \operatorname{Tors}(GL_1R) \xrightarrow{\longleftarrow} \operatorname{Line}_R \colon T. \tag{6.5}$$

Here $Tors(GL_1R)$ is the category of GL_1R -torsors, and the inverse equivalence is the functor T which associates to an R-line L the GL_1R -torsor

$$T(L) \stackrel{\text{def}}{=} \operatorname{Line}_R(R, L) \cong \{u \in L | Ru \cong L\}.$$

That is, we have the following diagram of categories which commutes up to natural isomorphism

$$\begin{array}{c} \operatorname{Line}_R & \longrightarrow \operatorname{Mod}_R \\ \mathbb{Z}[-] \otimes_{\mathbb{Z}[GL_1R]} R & & & & \\ \mathbb{Z}[-] \otimes_{\mathbb{Z}[GL_1R]} R & & & & \\ \operatorname{Tors}(GL_1R) & \longrightarrow & (GL_1R\text{-sets}). \end{array}$$

Moreover, the vertical arrows preserve colimits, and the left vertical arrows comprise an equivalence.

If ξ is a bundle of R-lines over X, we write $P(\xi)$ for the GL_1R -set

$$P(\xi) = \operatorname{colim}\left(X \xrightarrow{\xi} \operatorname{Line}_R \xrightarrow{T} \operatorname{Tors}(GL_1R) \to (GL_1R\text{-sets})\right).$$

That is, $P(\xi)$ is the GL_1R -torsor over X whose fiber at $x \in X$ is $P(\xi)_x = T(\xi_x)$.

Proposition 6.6. There is a natural isomorphism of R-modules

$$X^{\xi} \cong \mathbb{Z}[P(\xi)] \otimes_{\mathbb{Z}[GL_1R]} R.$$

Proof. We have

$$X^{\xi} = \operatorname{colim}\left(X \xrightarrow{\xi} \operatorname{Line}_{R} \to \operatorname{Mod}_{R}\right)$$

$$\cong \operatorname{colim}\left(X \xrightarrow{T\xi} \operatorname{Tors}(GL_{1}R) \to (GL_{1}R\text{-sets}) \xrightarrow{\mathbb{Z}[-] \otimes_{\mathbb{Z}[GL_{1}R]}R} \operatorname{Mod}_{R}\right)$$

$$\cong \mathbb{Z}[P(\xi)] \otimes_{\mathbb{Z}[GL_{1}R]} R.$$

| Figure 1. | Selected | instances | of t | he anal | ogv | Sets:Spaces: | ·Cateo | $\text{cories} \infty$ -c | ategories |
|-----------|----------|-----------|------|----------|-----|----------------|---------|---------------------------|-----------|
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| Categorical notion | ∞ -categorical notion | Alternate name/description |
|--------------------------|---------------------------------|---|
| Category | ∞-category | |
| Set | ∞-groupoid | Space/Kan complex |
| Monoid | monoidal ∞-groupoid | A_{∞} space; *-algebra |
| Group | group-like monoidal | group-like A_{∞} space |
| | ∞ -groupoid | |
| Abelian group | group-like symmetric | group-like E_{∞} space; |
| | monoidal ∞ -groupoid | (-1)-connected spectrum |
| Abelian group | spectrum | |
| The ring \mathbb{Z} | The sphere spectrum S | |
| Ring | Monoid in spectra | S -algebra or A_{∞} ring spectrum |
| Commutative ring | Commutative monoid in spectra | Commutative S-algebra or E_{∞} ring spectrum |
| The functor \mathbb{Z} | The functor Σ_+^{∞} | |
| Mod_R | Mod_A | |
| Line_R | Line_A | BGL_1A |
| GL_1R | $Line_A(A, A) = Aut_A(A)$ | GL_1A |
| GL_1R -set | GL_1A -space | $A_{\infty} GL_1A$ -space |
| GL_1R -torsors | $Tors(GL_1A)$ | |
| \otimes | \wedge | |

Now suppose that R is a commutative ring, and X is an abelian group. If $\xi \colon X \to \mathrm{Line}_R$ is a symmetric monoidal functor, then

$$GL_1R \to P(\xi) \to X$$

is an extension of abelian groups. The adjunction (6.4) restricts further to an adjunction

$$\mathbb{Z}$$
: (abelian groups) \longrightarrow (commutative rings): GL_1 , (6.7)

and so we have maps of commutative rings

$$\mathbb{Z}[P(\xi)] \leftarrow \mathbb{Z}[GL_1R] \rightarrow R.$$

The isomorphism of Proposition 6.6 has the following consequence.

Proposition 6.8. If R is a commutative ring and $\xi \colon X \to \operatorname{Line}_R$ is a symmetric monoidal functor, then X^{ξ} is a commutative R-algebra; indeed, it is the pushout in the category of commutative rings

$$\mathbb{Z}[GL_1R] \longrightarrow R$$

$$\downarrow \qquad \qquad \downarrow$$

$$\mathbb{Z}[P(\xi)] \longrightarrow X^{\xi}.$$

The preceding discussion generalizes elegantly and directly to spaces and spectra. The generalization illustrates how spaces play the role in ∞ -categories that sets play in categories. The reader may find it useful to consult the table in Figure 6.

Let \mathcal{T} be an ∞ -category of spaces. The analogue of the adjunction (6.3) is

$$\Sigma_{+}^{\infty} \colon \mathscr{T} \xrightarrow{\longleftarrow} \mathscr{S} \colon \Omega^{\infty}. \tag{6.9}$$

Definition 6.10. Let Alg(*) be the ∞ -category of monoids in the ∞ -category of \mathscr{T} .

We introduce the awkward name *-algebra to emphasize that these are more general than monoids with respect to the classical product of topological spaces. The symmetric monoidal structure on \mathscr{T} is such that Alg(*) is a model for the ∞ -category of A_{∞} spaces. A *-algebra X is group-like if $\pi_0 X$ is a group.

Definition 6.11. We write $Alg(*)^{\times}$ for the ∞ -category of group-like monoids.

The adjunction (6.9) restricts to an adjunction

$$\Sigma_{+}^{\infty} : \operatorname{Alg}(*)^{\times} \xrightarrow{\longrightarrow} \operatorname{Alg}(S) : GL_{1}.$$
 (6.12)

Remark 6.13. If X is a group-like monoid in \mathscr{T} , then Sing X is a group-like monoidal ∞ -groupoid. One way to see that a group-like A_{∞} space is the appropriate generalization of a group is to observe that if Z is an object in a *category*, then $\operatorname{End}(Z)$ is a monoid, and $\operatorname{Aut}(Z)$ is a group. If Z is an object in an ∞ -category, then $\operatorname{End}(Z)$ is a monoidal ∞ -groupoid, and $\operatorname{Aut}(Z)$ is group-like.

In particular, as we have already discussed in §3.6, one construction of the right adjoint GL_1 is the following. If A is an S-algebra, then it has an ∞ -category of modules Mod_A . In Mod_A we have the maximal $\operatorname{sub-}\infty$ -groupoid Line_A whose objects are weakly equivalent to A. We can define

$$GL_1A = Aut(A) = Line_A(A, A)$$

to be the subspace of (the geometric realization of) $\operatorname{Mod}_A(A, A)$ consisting of homotopy equivalences: it is a group-like *-algebra. We also write BGL_1A for the full ∞ -subcategory of Line_A on the single object A: this is the ∞ -groupoid with $\operatorname{Aut} A$ as its simplicial set of of morphisms.

Since GL_1A is a (group-like) monoid in the symmetric monoidal ∞ -category of spaces, we can form the ∞ -category of GL_1A -spaces, and then define $Tors(GL_1A)$ to be the maximal subgroupoid whose objects are GL_1A -spaces P which admit an equivalence of GL_1A -spaces

$$GL_1A \simeq P$$
.

The adjunction (6.12) provides a map of S-algebras

$$\Sigma^{\infty}_{+}GL_{1}A \to A$$
,

and so a (∞ -category) colimit-preserving functor

$$\Sigma_{+}^{\infty}(-) \wedge_{\Sigma_{-}^{\infty}GL_{1}A} A : (GL_{1}A\text{-spaces}) \to \operatorname{Mod}_{A},$$

which restricts to an equivalence of ∞ -categories

$$\Sigma^{\infty}_{+}(-) \wedge_{\Sigma^{\infty}_{+}GL_{1}A} A : \operatorname{Tors}(GL_{1}A) \to \operatorname{Line}_{A}.$$

The inverse equivalence is the functor

$$T: \operatorname{Line}_A \to \operatorname{Tors}(GL_1A)$$

which to an A-line L associates the GL_1A -torsor

$$T(L) \stackrel{\text{def}}{=} \operatorname{Line}_A(A, L).$$

Putting all these together, we have the homotopy commutative diagram of ∞-categories

$$\begin{array}{ccc}
\operatorname{Line}_{A} & \longrightarrow & \operatorname{Mod}_{A} \\
\Sigma_{+}^{\infty}(-) \wedge_{\Sigma_{+}^{\infty}GL_{1}A} & & \uparrow \\
\operatorname{Tors}(GL_{1}A) & \longrightarrow & (GL_{1}A\text{-spaces}),
\end{array}$$

in which the vertical arrows preserve ∞ -categorical colimits, and the left vertical arrows comprise an equivalence.

Now let X be a space, and let

$$\xi \colon X \to \mathrm{Line}_A$$

be a bundle of A-lines over X. Recall that

$$X^{\xi} = \operatorname{colim}\left(\operatorname{Sing} X \xrightarrow{\xi} \operatorname{Line}_A \to \operatorname{Mod}_A\right).$$

On the other hand, let

$$P(\xi) = \operatorname{colim}\left(\operatorname{Sing} X \xrightarrow{\xi} \operatorname{Line}_A \xrightarrow{T} \operatorname{Tors}(GL_1A) \to (GL_1A\operatorname{-spaces})\right).$$

We have the following analogue of Proposition 6.6.

Proposition 6.14. There is a natural equivalence of A-modules

$$X^{\xi} \simeq \Sigma_{+}^{\infty} P(\xi) \wedge_{\Sigma_{+}^{\infty} GL_{1}A} A. \tag{6.15}$$

Now we turn to the commutative case.

Definition 6.16. We write CommAlg(*) for the ∞ -category of commutative monoids in \mathscr{T} . It is equivalent to the nerve of the simplicial category of (cofibrant and fibrant) E_{∞} spaces. We write CommAlg(*) $^{\times}$ for the ∞ -category of group-like commutative monoids, which models group-like E_{∞} spaces.

The adjunction (6.12) restricts to the analogue of the adjunction (6.7), namely

$$\Sigma_{+}^{\infty} : \operatorname{CommAlg}(*)^{\times} \xrightarrow{\longleftarrow} \operatorname{CommAlg}(S) : GL_{1}$$
 (6.17)

The reader will notice that in the table in Figure 6, we mention two models for "abelian groups", namely, group-like E_{∞} spaces and spectra. It is a classical theorem of May [May72, May74], reviewed for example in [ABGHR, §3], that the functor Ω^{∞} induces an equivalence of ∞ -categories

$$\Omega^{\infty}$$
: $((-1)$ -connected spectra) \simeq CommAlg(*) $^{\times}$,

and so we may rewrite the adjunction (6.17) as

$$\Sigma_{+}^{\infty}\Omega^{\infty} : ((-1)\text{-connected spectra}) \xrightarrow{\simeq} \text{CommAlg}(*)^{\times} \xrightarrow{\Sigma_{+}^{\infty}} \text{CommAlg}(S) : gl_{1}$$
 (6.18)

(The left adjoints are written on top, but the pair of adjoints on the left is an equivalence of ∞ -categories). Note that we have introduced the functor gl_1 , with the property that if A is a commutative S-algebra, then

$$GL_1A \simeq \Omega^{\infty} gl_1A$$
.

We also define $bgl_1A = \Sigma gl_1A$: then $\Omega^{\infty}bgl_1A \simeq BGL_1A$.

Thus a map of spectra

$$f: b \to bql_1A$$

may be viewed equivalently as a map of group-like commutative *-algebras

$$\Omega^{\infty} f \colon B = \Omega^{\infty} b \to BGL_1 A$$

or as a map of symmetric monoidal ∞-groupoids

$$\xi \colon \operatorname{Sing} B \to \operatorname{Line}_A.$$
 (6.19)

Form the pull-back diagram

$$gl_{1}A = gl_{1}A$$

$$\downarrow \qquad \qquad \downarrow$$

$$p \longrightarrow egl_{1}A \simeq *$$

$$\downarrow \qquad \qquad \downarrow$$

$$b \stackrel{f}{\longrightarrow} bgl_{1}A.$$

$$(6.20)$$

One checks [ABGHR, Lemma 8.23] that

$$P(\xi) \simeq \Omega^{\infty} p$$
,

and so we have the following.

Proposition 6.21. The Thom spectrum of the map ξ of symmetric monoidal ∞ -groupoids (6.19) is a commutative A-algebra; indeed, we have

$$X^{\xi} \simeq \Sigma^{\infty}_{+} P(\xi) \wedge_{\Sigma^{\infty}_{+} GL_{1}A} A \simeq \Sigma^{\infty}_{+} \Omega^{\infty} p \wedge_{\Sigma^{\infty}_{+} \Omega^{\infty} gl_{1}A} A.$$

Example 6.22. Taking A to be the sphere spectrum in Proposition 6.21, we recover the result of [LMSM86] that the Thom spectrum of an ∞ -loop map

$$B \to BGL_1S$$

is a commutative S-algebra.

Remark 6.23. The formula (6.15) provides one way to see that our Thom spectrum coincides with the classical Thom spectrum of [MQRT77, LMSM86]. One way to compute the smash product in (6.15) is to realize it as a two-sided bar construction [EKMM96, Proposition 7.5]. In particular if $\xi: X \to BGL_1S$ classifies a spherical fibration, then one expects

$$X^{\xi} \simeq \Sigma_{+}^{\infty} P(\xi) \wedge_{\Sigma_{+}^{\infty} GL_{1}S} S \simeq B(\Sigma_{+}^{\infty} P(\xi), \Sigma_{+}^{\infty} GL_{1}S, S). \tag{6.24}$$

It is not difficult to see that the constructions of [MQRT77, LMSM86] provide careful models for the two-sided bar construction on the right-hand side of (6.24). Note that some care is required to make this proposal precise; see §8.6 of [ABGHR] for details.

7. Application: $K(\mathbb{Z},3)$, twisted K-theory, and the $Spin^c$ orientation

7.1. Recall that $BSpin^c$ participates in a fibration of infinite loop spaces

$$K(\mathbb{Z},2) \to BSpin^c \to BSO \xrightarrow{bw_2} K(\mathbb{Z},3),$$
 (7.1)

where bw_2 is the composite of the usual w_2 with the \mathbb{Z} -Bockstein

$$BSO \xrightarrow{w_2} K(\mathbb{Z}/2,2) \xrightarrow{b} K(\mathbb{Z},3).$$

Passing to Thom spectra in (7.1) we have a map of commutative S-algebras

$$\Sigma^{\infty}_{\perp}K(\mathbb{Z},2) \to MSpin^c$$
.

It's a theorem of Joachim [Joa04] that the orientation

$$MSpin^c \to K$$

of Atiyah-Bott-Shapiro is map of commutative S-algebras, and so we have a sequence of maps of commutative S-algebras

$$\Sigma^{\infty}_{+}K(\mathbb{Z},2) \to MSpin^{c} \to K.$$
 (7.2)

The $(\Sigma_{+}^{\infty}\Omega^{\infty}, gl_1)$ adjunction (6.18) produces from (7.2) a map of infinite loop spaces

$$K(\mathbb{Z},2) \to GL_1K$$

which we deloop once to view as a map

$$T: K(\mathbb{Z},3) \to BGL_1K$$
.

That is, the fact that the Atiyah-Bott-Shapiro orientation is a map of (commutative) S-algebras implies that we have a homotopy-commutative diagram

$$K(\mathbb{Z},2) \longrightarrow GL_1K$$

$$\downarrow \qquad \qquad \downarrow$$

$$BSpin^c \longrightarrow \qquad *$$

$$\downarrow \qquad \qquad \downarrow$$

$$BSO \stackrel{j}{\longrightarrow} BGL_1K$$

$$\beta w_2 \downarrow \qquad \qquad \downarrow =$$

$$K(\mathbb{Z},3) \stackrel{T}{\longrightarrow} BGL_1K.$$

$$(7.3)$$

Now suppose given a map $\alpha \colon X \to K(\mathbb{Z},3)$. Then we may form the Thom spectrum $X^{T\alpha}$, and define

$$K^n(X)_{\alpha} \stackrel{\text{def}}{=} \pi_0 \text{Mod}_K(X^{T\alpha}, \Sigma^n K).$$

We then have the following.

Proposition 7.4. A map $\alpha: X \to K(\mathbb{Z},3)$ gives rise to a twist $K^*(X)_{\alpha}$ of the K-theory of X. A choice of homotopy

$$T\beta w_2 \Rightarrow i$$

in the diagram (7.3) above determines, for every oriented vector bundle V over X, an isomorphism

$$K^*(X)_{\beta w_2(V)} \cong K^*(X^V).$$
 (7.5)

In this way, the characteristic class $\beta w_2(V)$ determines the K-theory of the Thom spectrum of V.

Proof. We prove the isomorphism (7.5) to show how simple it is from this point of view. Consider the homotopy commutative diagram

$$X \xrightarrow{V} BSO \xrightarrow{j} BGL_1K$$

$$\beta w_2 \downarrow \qquad \qquad \downarrow =$$

$$K(\mathbb{Z}, 3) \xrightarrow{T} BGL_1K.$$

Omitting the gradings, we have

$$K^{0}(X)_{\beta w_{2}(V)} = \pi_{0} \operatorname{Mod}_{K}(X^{T\beta w_{2}(V)}, K)$$

$$\cong \pi_{0} \operatorname{Mod}_{K}(X^{j(V)}, K)$$

$$\cong \pi_{0} \operatorname{Mod}_{K}(X^{V} \wedge K, K)$$

$$\cong \pi_{0} \mathscr{S}(X^{V}, K)$$

$$= K^{0}(X^{V}).$$

The first isomorphism uses the construction of X^{ξ} together with the fact that $T\beta w_2$ and j are homotopic as maps $BSO \to BGL_1K$. The second isomorphism is Theorem 4.5.

Remark 7.6. We needn't have started with an oriented bundle. For example, let F be the fiber in the sequence

$$F \to BSpin^c \to BO$$
.

This is a fibration of infinite loop spaces, and so it deloops to give

$$F \to BSpin^c \to BO \xrightarrow{\gamma} BF$$
.

The same argument produces an E_{∞} map

$$\Sigma^{\infty}_{+}F \to MSpin^{c} \to K$$

whose adjoint

$$F \to GL_1K$$

deloops to

$$\zeta \colon BF \to BGL_1K$$
,

and if V is any vector bundle then

$$K(X^V) \cong K(X)_{\zeta\gamma(V)}.$$

7.2. Khorami's theorem. At this point we are in a position to state a remarkable result of M. Khorami [Kho10]. $K(\mathbb{Z},2)$ is a group-like commutative *-algebra, and so we have a bundle

$$K(\mathbb{Z},2) \to EK(\mathbb{Z},2) \simeq * \to BK(\mathbb{Z},2) \simeq K(\mathbb{Z},3).$$

(One can build this bundle a number of ways: by modeling $K(\mathbb{Z},2) \simeq PU$ as mentioned in §2.1, or using infinite loop space theory, or using the ∞ -category technology described above.)

Given a map $\zeta: X \to K(\mathbb{Z},3)$, we can form the pull-back $K(\mathbb{Z},2)$ -bundle

$$Q \longrightarrow EK(\mathbb{Z}, 2)$$

$$\downarrow \qquad \qquad \downarrow$$

$$X \stackrel{\zeta}{\longrightarrow} K(\mathbb{Z}, 3).$$

On the other hand, let P be the pull-back

$$P \longrightarrow EGL_1K$$

$$\downarrow \qquad \qquad \downarrow$$

$$X \stackrel{T\zeta}{\longrightarrow} BGL_1K$$

as in Proposition 6.21. One can check that

$$\Sigma_+^{\infty} P \simeq \Sigma_+^{\infty} Q \wedge_{\Sigma_+^{\infty} K(\mathbb{Z},2)} \Sigma_+^{\infty} GL_1 K,$$

and so the Thom spectrum whose homotopy calculates the ζ -twisted K-theory is

$$X^{T\zeta} \simeq \Sigma_{+}^{\infty} P \wedge_{\Sigma_{+}^{\infty} GL_{1}K} K \simeq \Sigma_{+}^{\infty} Q \wedge_{\Sigma_{+}^{\infty} K(\mathbb{Z},2)} \Sigma_{+}^{\infty} GL_{1}K \wedge_{\Sigma_{+}^{\infty} GL_{1}K} K \simeq \Sigma_{+}^{\infty} Q \wedge_{\Sigma_{+}^{\infty} K(\mathbb{Z},2)} K, \tag{7.7}$$

where the map of commutative S-algebras $\Sigma^{\infty}_{\perp}K(\mathbb{Z},2) \to K$ is (7.2).

The formula (7.7) implies that there is a spectral sequence (see for example [EKMM96, Theorem 4.1])

$$\operatorname{Tor}_{*}^{K_{*}K(\mathbb{Z},2)}(K_{*}Q,K_{*}) \Rightarrow \pi_{*}X^{T\zeta} \cong K_{*}(X)_{\zeta}. \tag{7.8}$$

Note that $K_*K(\mathbb{Z},2) \cong K_*\{\beta_1,\beta_2,\ldots\}$, so K_* is not a flat $K_*K(\mathbb{Z},2)$ -module. Nevertheless Khorami proves the following.

Theorem 7.9. In (7.8) one has $Tor_q = 0$ for q > 0, and so

$$K_*(X)_\zeta \cong K_*Q \otimes_{K_*K(\mathbb{Z},2)} K_*.$$

8. Application: Degree-four cohomology and twisted elliptic cohomology

The arguments of $\S7.1$ apply equally well to String structures and the spectrum of topological modular forms.

Recall that spin vector bundles admit a degree four characteristic class λ , which we may view as a map of infinite loop spaces

$$BSpin \xrightarrow{\lambda} K(\mathbb{Z},4).$$

Indeed this map detects the generator of $H^4BSpin \cong \mathbb{Z}$. The fiber of λ is called BString, and so we have maps of infinite loop spaces

$$K(\mathbb{Z},3) \to BString \to BSpin \xrightarrow{\lambda} K(\mathbb{Z},4).$$

Passing to Thom spectra, we get maps of commutative S-algebras

$$\Sigma^{\infty}_{+}K(\mathbb{Z},3) \to MString \to MSpin.$$

Let tmf be the spectrum of topological modular forms [Hop02]. Ando, Hopkins, and Rezk [AHR] have produced a map of commutative S-algebras

$$MString \xrightarrow{\sigma} tmf,$$

and so we have a map of commutative S-algebras

$$\Sigma^{\infty}_{\perp}K(\mathbb{Z},3) \to tmf,$$

whose adjoint (see (6.12, 6.17, 6.18))

$$K(\mathbb{Z},3) \to GL_1tmf$$

deloops to

$$T: K(\mathbb{Z},4) \to BGL_1tmf.$$

By construction, the map T makes the diagram

$$K(\mathbb{Z},3) \longrightarrow GL_1tmf$$

$$\downarrow \qquad \qquad \downarrow$$

$$BString \longrightarrow \qquad *$$

$$\downarrow \qquad \qquad \downarrow$$

$$BSpin \stackrel{j}{\longrightarrow} BGL_1tmf$$

$$\downarrow \downarrow \qquad \qquad \downarrow =$$

$$K(\mathbb{Z},4) \stackrel{T}{\longrightarrow} BGL_1tmf$$

commute up to homotopy. If $\zeta \colon X \to K(\mathbb{Z},4)$ is a map, then we may define

$$tmf(X)^k_{\zeta} \stackrel{\text{def}}{=} \pi_0 \text{Mod}_{tmf}(X^{T\zeta}, \Sigma^k tmf),$$

and so we have the following.

Proposition 8.2. A map $\zeta: X \to K(\mathbb{Z}, 4)$ gives rise to a twist $tmf^*(X)_{\zeta}$ of the tmf-theory of X. A choice of homotopy

$$T\lambda \Rightarrow j$$

in the diagram (8.1) above determines, for every map

$$V: X \to BSpin$$
,

an isomorphism of $tmf^*(X)$ -modules

$$tmf^*(X)_{\lambda(V)} \cong tmf^*(X^V). \tag{8.3}$$

In this way, the characteristic class $\lambda(V)$ determines the tmf-cohomology of the Thom spectrum of V.

Remark 8.4. As in Remark 7.6, we could have started with the fiber F in the fibration sequence of infinite loop spaces

$$F \to BString \to BO \xrightarrow{\gamma} BF$$
.

We have a map of commutative S-algebras

$$\Sigma^{\infty}_{+}F \to MString \to tmf$$

whose adjoint

$$F \rightarrow GL_1 tm f$$

deloops to

$$\zeta \colon BF \to BGL_1tmf$$
,

and if $V \colon X \to BO$ classifies a virtual vector bundle, then

$$tmf^*(X^V) \cong tmf^*(X)_{\zeta\gamma(V)}$$

9. Application: Poincare duality and twisted umkehr maps

Let M be a compact smooth manifold with tangent bundle T of rank d. Embed M in \mathbb{R}^N , and then perform the Pontrjagin-Thom construction: collapse to a point the complement of a tubular neighborhood of M. If ν is the normal bundle of the embedding, this gives a map

$$S^N \to M^{\nu}$$
.

Desuspending N times then yields a map

$$\mu \colon S^0 \to M^{-T}.$$
 (9.1)

As usual, the Thom spectrum admits a relative diagonal map

$$M^{-T} \xrightarrow{\Delta} \Sigma^{\infty}_{+} M \wedge M^{-T}$$

(this is the map which gives the cohomology of the Thom spectrum the structure of a module over the cohomology of the base).

If E is a spectrum, then to a map

$$f \colon M^{-T} \to E$$

we can associate the composition

$$S^0 \xrightarrow{\mu} M^{-T} \xrightarrow{\Delta} \Sigma^{\infty}_{+} M \wedge M^{-T} \xrightarrow{1 \wedge f} \to \Sigma^{\infty}_{+} M \wedge E.$$

Milnor-Spanier-Atiyah duality says that this procedure yields an isomorphism

$$E_*(M) \cong E^{-*}(M^{-T}).$$

In the presence of a Thom isomorphism

$$E^{-*}(M^{-T}) \cong E^{d-*}(M)$$

we have Poincaré duality

$$E_*(M) \cong E^{d-*}(M)$$
.

Without a Thom isomorphism, we choose a map α

$$M \xrightarrow{\alpha} BO$$

classifying $\underline{d} - T$, and then we define $\tau(-T)$ to be the composition

$$\tau(-T): M \xrightarrow{\alpha} BO \xrightarrow{j} BGL_1S \to BGL_1E.$$

Then, following Example 5.2, we have

$$E^{d-*}(M)_{\tau(-T)} \cong E^{d-*}(M^{\underline{d}-T}) \cong E^{-*}(M^{-T}) \cong E_*(M).$$

Combining this with the results of sections 7.1 and 8, we have the following.

Proposition 9.2. Suppose that M is an oriented compact manifold of dimension d. Then

$$K_*(M) \cong K^{d-*}(M)_{-\beta w_2(M)}.$$

If M is a spin manifold, then

$$tmf_*(M) \cong tmf^{d-*}(M)_{-\lambda(M)}.$$

9.1. Twisted umkehr maps. In this section we sketch the construction of some umkehr maps in twisted generalized cohomology. Note that similar constructions are studied in [CW, MS06, Wal06]. Also, since this paper was written, we have learned that Bunke, Schneider, and Spitzweck have independently developed a similar approach to twisted umkehr maps.

Suppose that we have a family of compact spaces over X, that is, a map of ∞ -categories

$$\zeta \colon \operatorname{Sing} X \to (\operatorname{compact spaces}).$$

We also have the trivial map

$$*_X : \operatorname{Sing} X \xrightarrow{*} (\operatorname{compact spaces}).$$

If M is a compact space, let

$$DM = F(\Sigma^{\infty}_{+}M, S^{0})$$

be the Spanier-Whitehead dual of M_+ : this is a functor of ∞ -categories

$$D: (\text{compact spaces})^{\text{op}} \to \mathscr{S}.$$

The projection

$$M \to *$$

gives rise to a map of spectra

$$S^0 \cong D* \to DM. \tag{9.3}$$

Indeed if M is a compact manifold with tangent bundle T, then Milnor-Spanier-Atiyah duality says that $DM \simeq M^{-T}$, in such a way that the Pontrjagin-Thom map (9.1) identifies with (9.3).

In any case, let $S_X^0 = D*_X$. We have a natural map

$$u \colon S_X^0 \to D\zeta$$

of bundles of spectra over X. Essentially, we are applying the map (9.3) fiberwise.

It follows from Proposition 7.7 of [ABGHR] that

$$X^{S_X^0} \simeq \Sigma_{\perp}^{\infty} X.$$

As for $X^{D\zeta}$, in a forthcoming paper we prove the following.

Proposition 9.4. Suppose that ζ arises from a bundle

$$Y \xrightarrow{f} X$$

of compact manifolds, and let Tf be its bundle of tangents along the fiber. Then

$$X^{D\zeta} \simeq Y^{-Tf}$$
.

In particular, passing to Thom spectra on u gives a map of spectra

$$t: \Sigma_{+}^{\infty} X \to Y^{-Tf}. \tag{9.5}$$

This map is equivalent to the classical stable transfer map associated to f.

The map t, and indeed the idea that it arises from applying the map (9.1) fiberwise, is classical; see for example [BG75]. Casting it in our setting enables us to construct twisted versions.

More precisely, suppose that R is a commutative S-algebra, and suppose given a bundle of R-lines over X

$$\xi \colon \operatorname{Sing} X \to \operatorname{Line}_R$$
.

We then have a map of bundles of R-lines over X

$$u \wedge \xi \colon \xi \simeq S_X^0 \wedge \xi \to D\zeta \wedge \xi.$$

Thus we have constructed a twisted umkehr map

$$R^*(X^{D\zeta})_{\xi} \to R^*(X)_{\xi}.$$

In the situation of Proposition 9.4, we have a twisted transfer map

$$R^*(Y^{-Tf})_{\xi} \to R^*(X)_{\xi}.$$
 (9.6)

About this we show the following.

Proposition 9.7. Suppose that

$$Y \xrightarrow{Tf} BO \to BGL_1S \to BGL_1R$$
,

regarded as a map $\operatorname{Sing} Y \to \operatorname{Line}_R$, is homotopic to $\xi f \colon \operatorname{Sing} Y \to \operatorname{Sing} X \to \operatorname{Line}_R$. A choice of homotopy determines an isomorphism

$$R^*(Y) \cong R^*(Y^{-Tf})_{\xi},$$

and composing with the twisted transfer (9.6) we have a twisted umkehr map $R^*(Y) \to R^*(X)_{\mathcal{E}}$.

10. MOTIVATION: D-BRANE CHARGES IN K-THEORY

10.1. The Freed-Witten anomaly. Let $j: D \to X$ be an embedded submanifold, let ν be the normal bundle of j, and suppose that D carries a complex vector bundle ξ .

Suppose moreover that ν carries a $Spin^c$ -structure. Then we can form the K-theory push-forward

$$j_!: K(D) \to K(X)$$
.

In that situation Minasian and Moore and Witten discovered that it is sensible to think of the K-theory class

$$j_!(\xi) \in K(X)$$

as the "charge" of the *D*-brane *D* with Chan-Paton bundle ξ .

If ν does not carry a $Spin^c$ -structure, then we still have the Pontrjagin-Thom construction

$$X \to D^{\nu}$$
.

Suppose we have a map $H: X \to K(\mathbb{Z},3)$ making the diagram

$$D \xrightarrow{\nu} BSO$$

$$j \downarrow \qquad \qquad \downarrow bw_2$$

$$X \xrightarrow{H} K(\mathbb{Z}, 3)$$

commute up to homotopy. According to Proposition 9.7, a homotopy

$$c: bw_2 \simeq Hj$$

determines an isomorphism

$$K^*(D) \cong K^*(D^{\nu})_{-H}$$

(since $\nu = -Tj$), and then we have a twisted umkehr map

$$j_! \colon K^*(D) \to K^*(X)_{-H}.$$
 (10.1)

The class $j_!(\xi) \in K_{-H}^*(X)$ is evidently an analogue of the charge in this situation. The discovery of the condition that there exists a class H on X such that $H|_D = W_3(\nu)$ is due to Freed and Witten [FW99].

Although we discovered this push-forward in an attempt to understand Freed and Witten's condition, we were not the first: it appeared, formulated this way, in a paper of Carey and Wang [CW]. An important contribution of their work is the construction, using the twisted K-theory of [AS04], of the umkehr map (10.1).

11. An elliptic cohomology analogue

Now suppose that we are given an embedding of manifolds

$$j: M \to Y$$
,

and that $\nu = \nu(j)$ is equipped with a Spin structure. Suppose we have a map H making the diagram

$$M \xrightarrow{\nu} BSpin$$

$$j \downarrow \qquad \qquad \downarrow \lambda$$

$$Y \xrightarrow{H} K(\mathbb{Z}, 4)$$

$$(11.1)$$

commute up to homotopy. By Proposition 9.7, a homotopy $c: \lambda \nu \simeq Hj$ determines an isomorphism

$$tmf^*(M) \simeq tmf^*(D^{\nu})_{-H},$$

and then we have a homomorphism of $tmf^*(Y)$ -modules

$$tmf^*(M) \simeq tmf^*(D^{\nu})_{-H} \to tmf^*(Y)_{-H}.$$
 (11.2)

Remark 11.3. The data of a configuration like (12.3) together with the homotopy c was studied by Wang [Wan08], who calls it a *twisted String structure*. In fact, it was predicted by Kriz and Sati [KS04, Sat] that tmf should be the natural receptacle for M-brane charges.

Remark 11.4. The authors are grateful to Hisham Sati for suggesting that we think about diagrams like (11.1). The on-going investigation of the resulting twisted umkehr maps is joint work with him.

12. Twists of equivariant elliptic cohomology

At present we do not know how to twist equivariant cohomology theories in general; for that matter, equivariant Thom spectra are poorly understood. However, twists by degree four Borel cohomology play an important role in equivariant elliptic cohomology. We review two instances to give the reader a taste of the subject.

Let G be a connected and compact Lie group. In 1994, Grojnowski sketched the construction of a G-equivariant elliptic cohomology E_G , based on a complex elliptic curve of the form $C_q = \mathbb{C}/\Lambda \cong \mathbb{C}^{\times}/q^{\mathbb{Z}}$; more generally, the construction can be used to give a theory for the universal curve over the complex upper half-plane (Grojnowski's paper is now available [Gro07]). In the case of the circle, Greenlees [Gre05] has given a complete construction of a rational S^1 -equivariant elliptic spectrum.

Note also that Jacob Lurie has obtained analogous and sharper results about equivariant elliptic cohomology, in the context of his derived elliptic curves.

The functor E_G takes its values in sheaves of \mathcal{O}_{M_G} -modules, where M_G is the complex abelian variety

$$M_G = (\check{T} \otimes_{\mathbb{Z}} C_q)/W.$$

Note that the completion of M_G at the origin is spf E(BG); in general one has $E_G(X)_0^{\wedge} \cong E(EG \times_G X)$, where E is the non-equivariant elliptic cohomology associated to C_q .

Grojnowski points out that a construction of Looijenga [Loo76] (see also [And03, §5]) associates to a class $c \in H^4(BG)$ a line bundle $\mathcal{A}(c)$ over M_G . Thus if X is any G-space, then we can form the \mathcal{O}_{M_G} -module

$$E_G(X)_c \stackrel{\text{def}}{=} E_G(X) \otimes \mathcal{A}(c)$$

This $E_G(X)$ -module is a twisted form of $E_G(X)$.

12.1. Representations of loop groups. Already the case of a point is interesting: one learns that twisted equivariant elliptic cohomology carries the characters of representations of loop groups. Suppose that G is a simple and simply connected Lie group, such as SU(d) or Spin(2d). Then

$$H^4(BG) \cong \mathbb{Z}.$$

We then have the following result, due independently to the first author [And00] (who learned it from Grojnowski) and, in a much more precise form involving derived equivariant elliptic cohomology, Jacob Lurie.

Proposition 12.1. Let G be a simple and simply connected compact Lie group, and let $\phi \in H^4(BG; \mathbb{Z}) \cong \mathbb{Z}$. The character of a representation of the loop group LG of level ϕ is a section of $\mathcal{A}(\phi)$, and the Kac character formula shows that we have an isomorphism

$$R_{\phi}(LG) \cong \Gamma(E_G(*)_{\phi})$$

after tensoring with $\mathbb{Z}((q))$.

It is fun to compare this result to the work of Freed, Hopkins, and Teleman (for example [FHT]), who show that $R_{\phi}(LG)$ is the twisted G-equivariant K-theory of G. Thus we have a map

$$\Gamma E_G(*)_{\phi} \to K_G(G)_{\phi}$$
.

This map is an instance of the relationship between elliptic cohomology and the orbifold K-theory of the free loop space. We hope to provide a more extensive discussion in the future.

12.2. The equivariant sigma orientation. Let \mathbb{T} be the circle group, and suppose that V/X is a \mathbb{T} -equivariant vector bundle with structure group G (in this section we suppose that G = Spin(2d) or G = SU(d)). Let P/X be the associated principal bundle. Then

$$E_{G \times \mathbb{T}}(P) \cong E_{\mathbb{T}}(X)$$

is a sheaf of $E_G(*) = \mathcal{O}_{M_G}$ -algebras, and so we can twist $E_{\mathbb{T}}(X)$ by $\mathcal{A}(c)$ for $c \in H^4(BG) \cong \mathbb{Z}$. Let c be the generator corresponding to c_2 if G = SU(d) or the "half Pontrjagin class" λ if G = Spin(2d). Note that c determines a Borel equivariant class $c_{\mathbb{T}}$.

In [And03, AG], the authors show first of all that the twist $E_{\mathbb{T}}(X) \otimes \mathcal{A}(c)$ depends only on the equivariant degree-four class $c_{\mathbb{T}}(V) \in H^4_{\mathbb{T}}(X;\mathbb{Z})$, and so we may define

$$E_{\mathbb{T}}(X)_{c_{\mathbb{T}}(V)} = E_{\mathbb{T}}(X) \otimes \mathcal{A}(c).$$

Second, they show that the Weierstrass sigma function leads to an isomorphism

$$E_{\mathbb{T}}(X)_{c_{\mathbb{T}}(V)} \cong E_{\mathbb{T}}(X^{V}); \tag{12.2}$$

this is an analytic and equivariant form of the isomorphism (8.3).

In the case that $c_{\mathbb{T}}(V) = 0$ we conclude that

$$E_{\mathbb{T}}(X) \cong E_{\mathbb{T}}(X^V); \tag{12.3}$$

this is the T-equivariant sigma orientation in this context. More precisely, we have the following.

Proposition 12.4. Let V/X be an S^1 -equivariant SU vector bundle. Let $c_2^{\mathbb{T}}(V) \in H^4_{\mathbb{T}}(X)$ be the equivariant second Chern class of V. Let $E_{\mathbb{T}}$ denote Grojnowski's or Greenlees's \mathbb{T} -equivariant elliptic cohomology, associated to the the complex analytic elliptic curve C. Then there is a canonical isomorphism

$$E_{\mathbb{T}}(X)_{c_2^{\mathbb{T}}(V)} \cong E_{\mathbb{T}}(X^V),$$

natural in V/X. In particular if V_0 and V_1 are two such bundles with $c_2^{\mathbb{T}}(V_0) = c_2^{\mathbb{T}}(V_1)$, and

$$W = V_0 - V_1,$$

then there is a canonical isomorphism

$$E_{\mathbb{T}}(X) \cong E_{\mathbb{T}}(X^W).$$

Remark 12.5. In [And03] the author constructs the \mathbb{T} -equivariant sigma orientation in Grojnowski's equivariant elliptic cohomology, for Spin and SU bundles. The construction was motivated by the Proposition stated above, which however was given as Conjecture 1.14. In [AG] the authors construct the \mathbb{T} -equivariant sigma orientation for Greenlees's equivariant elliptic cohomology, for \mathbb{T} -equivariant SU-bundles. Proposition 12.4 appears there as Theorem 11.17. It should not be difficult to adapt the methods of these two papers to the case of Spin bundles.

Remark 12.6. The careful reader will note that in [AG] we show how to twist $E_{\mathbb{T}}^*(X)$ by $c_2^{\mathbb{T}}(Y) \in H_{\mathbb{T}}^4(X;\mathbb{Z})$: we do not there discuss twisting by general elements of $H_{\mathbb{T}}^4(X;\mathbb{Z})$. The construction of such general twists of $E_{\mathbb{T}}(X)$ is the subject of on-going work of the first author and Bert Guillou.

Remark 12.7. Lurie has obtained similar and sharper results for the elliptic cohomology associated to a derived elliptic curve. In particular he can construct the sigma orientation and twists by $H^4_T(X;\mathbb{Z})$.

References

- [ABGHR] Matthew Ando, Andrew J. Blumberg, David Gepner, Michael Hopkins, and Charles Rezk. Units of ring spectra and Thom spectra, arxiv:0810.4535v3.
- [ABS64] Michael F. Atiyah, Raoul Bott, and Arnold Shapiro. Clifford modules. Topology, 3 suppl. 1:3–38, 1964.
- [AG] Matthew Ando and J. P. C. Greenlees. Circle-equivariant classifying spaces and the rational equivariant sigma genus, http://arxiv.org/abs/0705.2687v2. Submitted.
- [AHR] Matthew Ando, Michael J. Hopkins, and Charles Rezk. Multiplicative orientations of KO and of the spectrum of topological modular forms, http://www.math.uiuc.edu/~mando/papers/koandtmf.pdf. Preprint.
- [And00] Matthew Ando. Power operations in elliptic cohomology and representations of loop groups. *Trans. Amer. Math. Soc.*, 352(12):5619–5666, 2000.
- [And03] Matthew Ando. The sigma orientation for analytic circle-equivariant elliptic cohomology. *Geometry and Topology*, 7:91–153, 2003, arXiv:math.AT/0201092.
- [AS04] Michael Atiyah and Graeme Segal. Twisted K-theory. Ukr. Mat. Visn., 1(3):287–330, 2004, arXiv:math/0407054.
- [Ati69] M. F. Atiyah. Algebraic topology and operators in Hilbert space. In Lectures in Modern Analysis and Applications. I, pages 101–121. Springer, Berlin, 1969.
- [BG75] J. C. Becker and D. H. Gottlieb. The transfer map and fiber bundles. Topology, 14:1–12, 1975.
- [BCMMS] P. Bouwknegt and A. L. Carey and V. Mathai and M. K. Murray and D. Stevenson. Twisted K-theory and K-theory of bundle gerbes. Commun. Math. Phys., 228:17–49, 2002, http://arxiv.org/abs/hep-th/0106194.
- [BM00] Peter Bouwknegt and Varghese Mathai. D-branes, B-fields and twisted K-theory. J. High Energy Phys., (3):Paper 7, 11, 2000, http://arxiv.org/abs/hep-th/0002023v3.
- [CW] Alan L. Carey and Bai-Ling Wang. Thom isomorphism and push-forward map in twisted K-theory, arxiv:math/0507414.
- [DK70] P. Donovan and M. Karoubi. Graded Brauer groups and K-theory with local coefficients. Inst. Hautes Études Sci. Publ. Math., (38):5–25, 1970.
- [EKMM96] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May. Rings, modules, and algebras in stable homotopy theory, volume 47 of Mathematical surveys and monographs. American Math. Society, 1996.
- [FHT] Daniel S. Freed, Michael J. Hopkins, and Constantin Teleman. Twisted K-theory and Loop Group Representations. arXiv:math.AT/0312155.
- [FW99] Daniel S. Freed and Edward Witten. Anomalies in string theory with D-branes. Asian J. Math., 3(4):819–851, 1999.
- [Gre05] J. P. C. Greenlees. Rational S^1 -equivariant elliptic cohomology. Topology, 44(6):1213–1279, 2005.
- [Gro07] Ian Grojnowski. Delocalized equivariant elliptic cohomology. In Elliptic cohomology: geometry, applications, and higher chromatic analogues, volume 342 of London Mathematical Society Lecture Notes. Cambridge University Press, 2007.
- [Hop02] M. J. Hopkins. Algebraic topology and modular forms. In Proceedings of the International Congress of Mathematicians, Vol. I (Beijing, 2002), pages 291–317, Beijing, 2002. Higher Ed. Press, arXiv:math.AT/0212397.
- [Hov01] , M. Hovey. Spectra and symmetric spectra in general model categories. J. Pure Appl. Algebra, 165(1):63–127, 2001.
- [HSS00] Mark Hovey, Brooke Shipley, and Jeff Smith. Symmetric spectra. J. Amer. Math. Soc., 13(1):149–208, 2000.
- [Joa04] Michael Joachim. Higher coherences for equivariant K-theory. In Structured ring spectra, volume 315 of London Math. Soc. Lecture Note Ser., pages 87–114. Cambridge Univ. Press, Cambridge, 2004.
- [Joy02] A. Joyal. Quasi-categories and Kan complexes. J. Pure Appl. Algebra, 175(1-3):207–222, 2002. Special volume celebrating the 70th birthday of Professor Max Kelly.
- [Kho10] Mehdi Khorami. A universal coefficient theorem for twisted K-theory. arXiv:math.AT/10014790
- [KS04] Igor Kriz and Hisham Sati. M-theory, type IIA superstrings, and elliptic cohomology. Adv. Theor. Math. Phys., 8(2):345-394, 2004, http://arxiv.org/abs/hep-th/0404013v3.
- [Lew91] L. G. Lewis, Jr. Is there a convenient category of spectra? J. Pure Appl. Algebra, 73:233–246, 1991.

[LMSM86] L. G. Lewis, Jr., J. P. May, M. Steinberger, and J. E. McClure. Equivariant stable homotopy theory, volume 1213 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1986. With contributions by J. E. McClure.

[Loo76] Eduard Looijenga. Root systems and elliptic curves. Inventiones Math., 38, 1976.

[DAGI] Jacob Lurie. Derived algebraic geometry I: stable ∞-categories, arXiv:math.CT/0608040.

[DAGII] Jacob Lurie. Derived algebraic geometry II: noncommutative algebra, arXiv:math.CT/0702229.

[DAGIII] Jacob Lurie. Derived algebraic geometry III: commutative algebra, arXiv:math.CT/0703204.

[HTT] Jacob Lurie. Higher Topos Theory. AIM 2006 -20, arXiv:math.CT/0608040.

[May72] J. P. May. The geometry of iterated loop spaces. Springer-Verlag, Berlin, 1972. Lectures Notes in Mathematics, Vol. 271.

[May74] J. P. May. E_{∞} spaces, group completions, and permutative categories. In *New developments in topology (Proc. Sympos. Algebraic Topology, Oxford, 1972)*, pages 61–93. London Math. Soc. Lecture Note Ser., No. 11. Cambridge Univ. Press, London, 1974.

[MQRT77] J. P. May. E_{∞} ring spaces and E_{∞} ring spectra. Springer-Verlag, Berlin, 1977. With contributions by Frank Quinn, Nigel Ray, and Jørgen Tornehave, Lecture Notes in Mathematics, Vol. 577.

[MM02] M. A. Mandell and J. P. May. Equivariant orthogonal spectra and S-modules. Mem. Amer. Math. Soc., 159(755), 2002.

[MMSS01] M. A. Mandell, J. P. May, S. Schwede, and B. Shipley. Model categories of diagram spectra. *Proc. London Math. Soc.* (3), 82(2):441–512, 2001.

[MS06] J. P. May and J. Sigurdsson. Parametrized homotopy theory, volume 132 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2006.

[Qui68] Daniel G. Quillen. The geometric realization of a Kan fibration is a Serre fibration. Proc. Amer. Math. Soc., 19:1499–1500, 1968.

[Ros89] Jonathan Rosenberg. Continuous-trace algebras from the bundle theoretic point of view. J. Austral. Math. Soc. Ser. A, 47(3):368–381, 1989.

[Sat] Hisham Sati. Geometric and topological structures related to M-branes. These proceedings.

[Wal06] Robert Waldmüller. Products and push-forwards in parametrised cohomology theories. PhD thesis.

[Wan08] Bai-Ling Wang. Geometric cycles, index theory and twisted K-homology. J. Noncommut. Geom., 2(4):497–552, 2008.

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