LECTURE: "THE ARITHMETIC OF THE LANGLANDS PROGRAM"

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Contents

1. Remarks on the lecture

- Aim: General introduction to the Langlands program
- Focus: More on "picture", proofs usually by reference. More "local, geometric" than "global, analytic".
- Prerequisites: Vary a lot (algebraic number theory, algebraic groups, algebraic geometry, harmonic analysis, representation theory,...). Usually detailed knowledge will not be necessary to understand the statements/picture, at least in some example.
- Original plan: Yield background material for, now cancelled, trimester progam *"The Arithmetic of the Langlands program"* of the HIM, which was originally scheduled in 2020 ($\Rightarrow$ explains title of lecture)
- Topics to be covered: Let's see...
- Hint: More analytic stuff on automorphic forms/Jacquet-Langlands in Edgar Assing's lecture "Topics in Automorphic forms"
- Disclaimer: I am by far not a person with serious knowledge/understanding of the Langlands program, thus I may oversimplify/overcomplicate things or miss subtelties. For definite/correct statements, one has to consult the references.
- Reference: Getz/Hahn: "An introduction to automorphic representations with a view towards the trace formula" [GH19], more references during the course.
- Style of lecture: I will explain this file via zoom (the layout is meant to be optimized for the use of the file in zoom). Any suggestion on improvement is welcome!
- Video/Audio: Please turn off your video/audio (except for questions/comments). Do not record me!
- Questions: via chat (in zoom), or orally (make me aware of you, e.g., raise hand via zoom, or speak), or by e-mail.
- Problems with technic: This file will be uploaded on the webpage.

Questions?

Today:

- Rough introduction to the Langlands program, and a first example of its arithmetic significance.

Setup:

- G reductive group over $\mathbb{Q}$, most important example: $G = \mathrm{GL}_n$ for some $n \geq 1$.
  - an affine, algebraic group $G/\mathbb{Q}$ is reductive if it has no closed normal subgroups isomorphic to $\mathbb{G}_a$.
- $\mathbb{A}_f := \{(x_p)_p \in \prod\limits_p \mathbb{Q}_p \mid x_p \in \mathbb{Z}_p \text{ for all but finitely many primes } p\ \}$
  "ring of finite adéles"
- $\mathbb{A} := \mathbb{A}_f \times \mathbb{R}$
  "ring of adéles"

Then:

- $\mathbb{A}$ is a locally compact topological ring
  - On $\mathbb{A}_f$: unique ring topology such that $\prod\limits_p \mathbb{Z}_p$ is open.
  - On $\mathbb{A}$: product topology.
- $\mathbb{Q} \subseteq \mathbb{A}$ discrete subring (via diagonal embedding).
- $G(\mathbb{A}) \cong G(\mathbb{A}_f) \times G(\mathbb{R})$ is a locally compact topological group
  - Choose an embedding $G \subseteq \mathrm{Spec}(\mathbb{Q}[X_1, \ldots, X_m])$ for some $m \geq 1$, and take the subspace topology of induced embedding $G(\mathbb{A}) \subseteq \mathbb{A}^m$, cf. [GH19, Theorem 2.2.1.].
- $G(\mathbb{Q}) \subseteq G(\mathbb{A})$ is a discrete subgroup.

Facts:

- On any locally compact topological group H exists a right-invariant Haar measure, cf. [GH19, Section 3.2.].
  - unique up to a scalar in $\mathbb{R}_{>0}$
- On $G(\mathbb{A})$ the right Haar measure is also left-invariant, and it descends to a $G(\mathbb{A})$-invariant measure on $G(\mathbb{Q})\backslash G(\mathbb{A})$, cf. [GH19, Lemma 3.5.4.], [GH19, Lemma 3.5.3].

Definition 2.1. *We set*

$$L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$$

as the space of measurable functions $f\colon G(\mathbb{Q})\backslash G(\mathbb{A}) \to \mathbb{C}$ *such that*

$$\int_{G(\mathbb{Q})\backslash G(\mathbb{A})} |f|^2 < \infty$$

where the integration is w.r.t. the $G(\mathbb{A})$*-invariant measure on* $G(\mathbb{Q})\backslash G(\mathbb{A})$.

As usual: Two functions in the L^2-space have to be identified if they agree outside a set of measure zero.

Facts:

- $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$ is a Hilbert space, i.e., it has an inner product.
- the $G(\mathbb{A})$-action coming from right translation on it is unitary, i.e., preserves the inner product.

Aim of the Langlands program (crude form):

- Decompose the Hilbert space $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$ as a representation of $G(\mathbb{A})$, according to arithmetic data.

Questions?

What does "decomposing" mean, roughly?

- $\widehat{G} :=$ isomorphism classes of irreducible, unitary $G(\mathbb{A})$-representations.
- Roughly, decomposing means to write (as a unitary $G(\mathbb{A})$-representation)

$$L^2(G(\mathbb{Q})\backslash G(\mathbb{A})) \text{ “}\cong\text{” } \bigoplus_{[\pi]\in\widehat{G}} \pi^{\oplus m_\pi}$$

  with $m_\pi \in \mathbb{N}\cup\{\infty\}$ multiplicity of π.
- Looking for such a decomposition is too naive.

Example: $G = \mathbb{G}_m = \mathrm{GL}_1$

- In this case,

$$G(\mathbb{Q})\backslash G(\mathbb{A}) \cong \mathbb{Q}^\times\backslash\mathbb{A}^\times$$

  is a locally compact abelian group
  $\Rightarrow$ can use abstract Fourier theory to analyze $L^2(\mathbb{Q}^\times\backslash\mathbb{A}^\times)$.

Abstract Fourier theory (cf. [DE14, Chapter 3]**):**

- A any locally compact abelian group
- $\widehat{A} =$ isom. classes of irreducible, unitary representations of A
- $\widehat{A} \cong \mathrm{Hom}_{\mathrm{cts}}(A, S^1)$ is again a locally compact abelian group.
  - via pointwise multiplication and the compact-open topology
- $A \cong \hat{\hat{A}}$ as topological groups via $a \mapsto (\chi \mapsto \chi(a))$.
- A is discrete if and only if $\widehat{A}$ is compact.
  "$\Rightarrow$" Choose $\bigoplus\limits_I \mathbb{Z} \twoheadrightarrow A$, then $\widehat{A}$ embeds into $\prod\limits_I S^1$. "$\Leftarrow$" Use $A \cong \hat{\hat{A}}$ and definition of compact-open topology.)

Theorem 2.2 (Plancherel theorem, cf. [DE14, Chapter 3.4])**.** *The Fourier transform*

$$\mathcal{F}\colon L^1(A)\cap L^2(A) \to L^2(\widehat{A})$$

defined by

$$f \mapsto (\chi \mapsto \int\limits_A f(x)\chi(x)d\mu(x))$$

extends to a unitary isomorphism

$$\mathcal{F}\colon L^2(A) \cong L^2(\widehat{A})$$

of Hilbert spaces.

Fact:

- $\mathcal{F}$ is an A-equivariant isomorphism where A acts on $L^2(\widehat{A})$ via

$$a\cdot g(\chi) := \chi(a)g(\chi)$$

  for $a\in A$, $g \in L^2(\widehat{A}), \chi\in\widehat{A}$.

Examples:

- If $A = S^1$, then
$$\mathbb{Z} \cong \widehat{A},\ n \mapsto \chi_n \text{ with } \chi_n(z) = z^n,\ z \in S^1.$$
Thus
$$L^2(S^1) \cong \widehat{\bigoplus_{n\in\mathbb{Z}}} \mathbb{C}\chi_n$$
is the Hilbert space direct sum of the S^1-equivariant subspaces
$$\mathbb{C}\chi_n \subseteq L^2(S^1).$$
Concretely: each L^2-function $f\colon S^1 \to \mathbb{C}$ can uniquely be written as
$$f = \sum_{n\in\mathbb{Z}} a_n\chi_n, \quad \text{where } a_n \in \mathbb{C},\ \sum_{n\in\mathbb{Z}} |a_n|^2 < \infty.$$
$\Rightarrow$ We found a nice decomposition of the unitary S^1-representation $L^2(S^1)$ into irreducible subspaces.
- If $A = \mathbb{R}$, then $\widehat{\mathbb{R}} \cong \mathbb{R}$ via
$$\mathbb{R} \to \widehat{\mathbb{R}},\ x \mapsto \chi_x, \quad \text{where } \chi_x\colon \mathbb{R} \to S^1,\ y \mapsto e^{2\pi i x y}.$$
The isomorphism
$$\mathcal{F}\colon L^2(\mathbb{R}) \cong L^2(\mathbb{R})$$
implies therefore that each $f \in L^2(\mathbb{R})$ can be written as
$$f(x) = \int_{\mathbb{R}} g(y)\chi_y(x)dy$$
for a unique function $g \in L^2(\mathbb{R})$. Thus,
$$L^2(\mathbb{R}) \cong \int_{\mathbb{R}} \chi_y dy$$
is a Hilbert *integral* of representation. We cannot do better: $\chi_x \notin L^2(\mathbb{R})$ for any $x \in \mathbb{R}$, and $L^2(\mathbb{R})$ contains no irreducible subrepresentation of $\mathbb{R}$, cf. [GH19, Section 3.7].
  - Also no irreducible quotient, the statement about quotients in [GH19, Lemma 3.7.1] is wrong, I think.

Back to $G = \mathbb{G}_m$:

- Factoring each $n \in \mathbb{Q}^\times$ into $n = \pm p_1 \dots p_k$ with p_i prime implies
$$\mathbb{Q}^\times \backslash \mathbb{A}^\times \cong \prod_p \mathbb{Z}_p^\times \times \mathbb{R}_{>0}.$$
- Let us ignore $\mathbb{R}_{>0}$ and look at
$$L^2(\prod_p \mathbb{Z}_p^\times) \cong \widehat{\bigoplus_{\chi}} \mathbb{C}\chi.$$
with $\chi\colon \prod_p \mathbb{Z}_p^\times \to S^1$ all continuous characters of $\prod_p \mathbb{Z}_p^\times$.
- The characters χ are "arithmetic" data, namely
$$\mathrm{Gal}(\mathbb{Q}(\mu_\infty)/\mathbb{Q}) \cong \prod_p \mathbb{Z}_p^\times,$$
where $\mathbb{Q}(\mu_\infty) = \bigcup_{n\in\mathbb{N}} \mathbb{Q}(e^{\frac{2\pi i}{n}})$ is the cyclotomic extension of $\mathbb{Q}$.

- Thus the $\chi \circ \alpha$ are (continuous) 1-dimensional Galois representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.
  - Namely: $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \xrightarrow{\alpha} \prod_{p \text{ prime}} \mathbb{Z}_p^\times \xrightarrow{\chi} S^1 \subseteq \mathbb{C}^\times$
- By Kronecker-Weber *each* 1-dimensional Galois representation

$$\sigma \colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{C}^\times$$

  is of the form $\sigma = \chi \circ \alpha$ for some χ.

Theorem 2.3 (Kronecker-Weber)**.** *The field $\mathbb{Q}(\mu_\infty)$ is the maximal abelian extension of $\mathbb{Q}$, i.e., each finite Galois extension $F/\mathbb{Q}$ with abelian Galois group $\mathrm{Gal}(F/\mathbb{Q})$ is contained in $\mathbb{Q}(\mu_\infty)$.*

The final decomposition for $G = \mathbb{G}_m$:

- $$L^2(\mathbb{R}_{>0}\mathbb{Q}^\times \backslash \mathbb{A}^\times) \cong \widehat{\bigoplus_{\sigma \colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{C}^\times}} \mathbb{C}\chi_\sigma$$
  with χ_σ characterised by $\chi_\sigma \circ \alpha = \sigma$.
- This (easy) example is prototypical for what one aims for in the Langlands program.

Questions?

Back to general G: Geometry of $G(\mathbb{Q})\backslash G(\mathbb{A})$

- There exists a central subgroup $A_G \subseteq G(\mathbb{R})$, with $A_G \cong \mathbb{R}^r_{>0}$, such that

$$[G] := A_G G(\mathbb{Q})\backslash G(\mathbb{A})$$

  has finite volume, cf. [GH19, Theorem 2.6.2.].
  - A_G is the connected component of the maximal split subtorus of the center $Z(G)$ of G, e.g., $A_{\mathrm{GL}_n} \subseteq \mathrm{GL}_n(\mathbb{R})$ is the group of scalar matrix with entries in $\mathbb{R}_{>0}$ while A_{SL_n} is trivial.
- Tamagawa measure = some canonical measure on $G(\mathbb{A})$, and

$$\tau(G) := \mathrm{vol}([G]) = \text{ Tamagawa number of } G.$$

  Knowledge of $\tau(G)$ is arithmetically interesting. Cf. [Col11, Appendix B], [Lur, Lecture 1].
  - $\tau(\mathrm{SL}_n) = 1 \Rightarrow \mathrm{vol}(\mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})) = \zeta(2)\zeta(3)\ldots\zeta(n)$, with $\zeta(s) =$ Riemann ζ-function
- $G(\mathbb{Q})\backslash G(\mathbb{A})$ and $[G]$ are profinite coverings of some real manifold, e.g.,

$$\mathrm{GL}_n(\mathbb{Z})\backslash(\prod_p \mathrm{GL}_n(\mathbb{Z}_p) \times \mathrm{GL}_n(\mathbb{R})) \text{ with diagonal action of } \mathrm{GL}_n(\mathbb{Z})$$

  which covers the real manifold

$$\mathrm{GL}_n(\mathbb{Z})\backslash \mathrm{GL}_n(\mathbb{R}).$$

  Similarly, $[G]$ covers $\mathrm{GL}_n(\mathbb{Z})\mathbb{R}_{>0}\backslash \mathrm{GL}_n(\mathbb{R})$. Cf. [GH19, Section 2.6], [Del73, (0.1.4.1)].
- $K_\infty \subseteq G(\mathbb{R})$ maximal compact connected subgroup $\Rightarrow [G]$ is a principal K_∞-bundle (or K_∞-torsor) over $[G]/K_\infty$.
  - $G = \mathrm{GL}_n \Rightarrow K_\infty \cong \mathrm{SO}_n(\mathbb{R})$

- $G = \mathrm{GL}_2$, then by Möbius transformations

$$A_G\backslash \mathrm{GL}_2(\mathbb{R})/K_\infty \xrightarrow{\sim} \mathbb{H}^{\pm},\ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto g(i) := \frac{a \cdot i + b}{c \cdot i + d}$$

  with

$$\mathbb{H}^{\pm} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) \neq 0\}$$

  the upper/lower halfplane

$$\Rightarrow [\mathrm{GL}_2]/K_\infty \cong \mathrm{GL}_2(\mathbb{Z})\backslash(\mathrm{GL}_2(\prod_p \mathbb{Z}_p) \times \mathbb{H}^{\pm})$$

- $K^\infty \subseteq G(\mathbb{A}_f)$ compact-open ("a level subgroup")

$$\Rightarrow \quad G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K^\infty = \coprod_{i=1}^{m} G(\mathbb{Q})g_i R K^\infty$$

  is finite (cf. [GH19, Theorem 2.6.1]) and

$$[G]/K_\infty K^\infty \cong \coprod_{i=1}^{m} \Gamma_i\backslash X$$

  where

$$X := A_G\backslash G(\mathbb{R})/K_\infty$$

  is a real manifold, and

$$\Gamma_i := G(\mathbb{Q}) \cap G(\mathbb{R})g_i K^\infty g_i^{-1} \subseteq G(\mathbb{R})$$

  is a *congruence subgroup*, cf. [GH19, Section 2.6].
  - One usually has to assume that K^∞ is sufficiently small, so that Γ_i acts freely on X, cf. [GH19, Definition 15.2].
- $G = \mathrm{GL}_2$, then

$$\Gamma_i\backslash X \cong \Gamma\backslash\mathbb{H}^{\pm} \qquad (\text{e.g., } \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H})$$

  with $\Gamma \subseteq \mathrm{GL}_2(\mathbb{Z})$ a discrete subgroup containing

$$\Gamma(m) := \ker(\mathrm{GL}_2(\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{Z}/m))$$

  for some $m \geq 0$. (later $\Rightarrow$ relation to modular forms).
- Usually $\dim(\Gamma_i\backslash X) > 0 \Rightarrow$ geometry more complicated than in case $G = \mathbb{G}_m$.
- The $\Gamma_i\backslash X$ are interesting real manifolds, e.g., sometimes actually complex manifolds and *canonically defined over number fields*.
- $[G]$ compact ($\Leftrightarrow$ each $\Gamma_i\backslash X$ compact) if and only if any $\mathbb{G}_m \subseteq G$ lies in center, cf. [GH19, Theorem 2.6.2].
  - e.g., G the units in a non-split quaternion algebra over $\mathbb{Q}$, i.e., units in a $\mathbb{Q}$-algebra with presentation $\langle x, y | x^2 = a, y^2 = b, xy = -yx\rangle$, for suitable $a, b \in \mathbb{Q}^\times$. Note that $G(\mathbb{R}) \cong \mathrm{GL}_2(\mathbb{R})$ if $ab < 0$. Cf. [GS17, Chapter 1].

More examples: Weil restrictions

- $F/\mathbb{Q}$ finite extension, H reductive over F
  $\Rightarrow$ reductive group $G := \mathrm{Res}_{F/\mathbb{Q}}(H)$ over $\mathbb{Q}$.
  - G represents the functor $R \mapsto H(R \otimes_{\mathbb{Q}} F)$ on the category of $\mathbb{Q}$-algebras, cf. [CGP15, Appendix A] or [GH19, Section 1.4].

- $G(\mathbb{Q})\backslash G(\mathbb{A}) = H(F)\backslash H(\mathbb{A}_F)$ with $\mathbb{A}_F$ ring of adéles for F ($\Rightarrow$ no need to discuss general F).
- $H = \mathbb{G}_{m,F} \Rightarrow$ desired decomposition of $L^2([G])$ is equivalent to class field theory for F.
- F imaginary quadratic, $H = \mathrm{GL}_{2,F} \Rightarrow G(\mathbb{R}) \cong \mathrm{GL}_2(\mathbb{C})$, $K_\infty \cong U(n)$ unitary group and
$$X = A_G\backslash G(\mathbb{R})/K_\infty$$
is the 3-dimensional hyperbolic space $\mathbb{H}^3$. The quotients $\Gamma_i\backslash X$ are arithmetic hyperbolic 3-manifolds, cf. [Thu82, Section 4 & 5].
    - $\mathbb{H}^3$ can be modelled on the quaternions $\{q = x + y \cdot i + z \cdot j \mid x, y \in \mathbb{R},\ z > 0\}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})$ acts as $q \mapsto (aq+b)(cq+d)^{-1}$, where $\mathbb{C} = \{x + y \cdot i \mid x, y \in \mathbb{R}\}$

Questions?

Definition 2.4 ([GH19, Definition 3.3])**.** *Let G be a reductive group over $\mathbb{Q}$. An automorphic representation (in the L^2-sense) is an irreducibe unitary $G(\mathbb{A})$-representation π which is isomorphic to a subquotient of $L^2([G])$.*

Facts:

- $G(\mathbb{A})$ acts on $[G] = A_G G(\mathbb{Q})\backslash G(\mathbb{A})$ from the right $\Rightarrow G(\mathbb{A})$ acts unitarily on $L^2([G])$
    - After choice of a right-invariant Haar measure on $G(\mathbb{A})$...
- Necessarily π is trivial on A_G. Usually harmless: Schur's lemma $\Rightarrow$ can find for irreducible unitary $G(\mathbb{A})$-repr. π a character $\chi\colon G(\mathbb{R}) \to \mathbb{C}^\times$ with $\chi \otimes \pi$ trivial on A_G. Cf. [GH19, Section 6.5].
    - For $G = \mathrm{GL}_n$ take χ of the form $g \mapsto |\det(g)|^s)$ for some $s \in \mathbb{R}_{>0}$.
- If $[G]$ is not compact, $L^2([G])$ will decompose into a "discrete" and a "continuous" part, cf. [GH19, Section 10.4].
- Recall $\widehat{G} =$ isom. classes of irreducible unitary $G(\mathbb{A})$-representations.
- $\widehat{G}$ has a natural topology, the Fell topology, cf. [GH19, Section 3.8].
    - It generalizes the compact-open topology on $\widehat{A}$ for A locally compact abelian.
- Let $\pi \in \widehat{G}$, $x \in \pi$, and $\varphi\colon \pi \to \mathbb{C}$ continuous $\mathbb{C}$-linear. Then
$$f_{x,\varphi}\colon G \to \mathbb{C},\ g \mapsto \varphi(gx)$$
is a matrix coefficient of π. Roughly, two $\pi, \pi' \in \widehat{G}$ are close if their matrix coefficients agree for the compact-open topology.

Theorem 2.5 ([GH19, Theorem 3.9.4])**.** *There exists a measurable multiplicity function $m\colon \widehat{G} \to \{1, 2, \dots, \infty\}$ and a measure μ on $\widehat{G}$ such that*
$$L^2([G]) \cong \int_{\widehat{G}} (\widehat{\bigoplus}^{m(\pi)} \pi) d\mu(\pi).$$
The m and μ are unique up changes on sets of measure 0.

The discrete and continuous part of $L^2([G])$:

- We don't need the exact meaning of the integral in 2.5, its occurence is due to the fact that some representations of $G(\mathbb{A})$ "contribute" to $L^2([G])$ without being subrepresentations.
  - Compare with the discussion of the Fourier transform for $\mathbb{R}$.
- Points $\pi \in \widehat{G}$ with $\mu(\pi) > 0$ appear (with multiplicity $m(\pi)$) as *sub*representations in $L^2([G])$.
- By Theorem 2.5

$$L^2([G]) \cong L^2_{\mathrm{disc}}([G]) \oplus L^2_{\mathrm{cont}}([G])$$

  where $L^2_{\mathrm{disc}}([G])$ is the Hilbert sum of all irreducible, unitary *sub*representations of $G(\mathbb{A})$, and $L^2_{\mathrm{cont}}([G])$ its orthogonal complement, cf. [GH19, Section 9.1].
- By definition, a closed, connected subgroup $P \subseteq G$ is parabolic if G/P is proper over $\mathrm{Spec}(\mathbb{Q})$. The quotient of P by its unipotent radical is its Levi quotient M. Cf. [GH19, Section 1.9].
  - In GL_n: take decomposition $n = n_1 + n_2 + \ldots + n_k$, P is subgroup of block upper triangular matrices in GL_n with blocks of size $n_1, \ldots, n_k$. The Levi quotient is $M =$ block diagonal matrices. Up to conjugation all parabolics in GL_n are of this form.
- Langlands proved that the continuous part $L^2_{\mathrm{cont}}([G])$ can be described via "inducing" the discrete parts $L^2_{\mathrm{disc}}([M])$ for all Levi quotients of parabolics $P \subseteq G$, $P \neq G$, cf. [GH19, Section 10.4]).
  - This is his theory of "Eisenstein series", and it is a starting point of the Langlands program.
- In this course, $L^2_{\mathrm{disc}}([G])$ will be more important than $L^2([G])$.
- $[G]$ is compact if and only G has no proper parabolics (defined over $\mathbb{Q}$), this confirms that $L^2([G]) = L^2_{\mathrm{disc}}([G])$ in this case.

Questions?

3. From modular forms to automorphic representations, part I

Today (and next time):

- Construct automorphic representations for GL_2 associated to modular forms
- Keep some piece of paper ready!

Last time:

- G reductive over $\mathbb{Q}$, $\mathbb{A} = \mathbb{A}_f \times \mathbb{R}$ ring of adéles
- $[G] = A_G G(\mathbb{Q})\backslash G(\mathbb{A})$ adélic quotient, has finite invariant volume
- Langlands program: Decompose
$$L^2([G])$$
according to arithmetic data.
- $G = \mathbb{G}_m \Rightarrow$
$$L^2(\mathbb{R}_{>0}\mathbb{Q}^\times\backslash\mathbb{A}^\times) \cong \widehat{\bigoplus_{\sigma\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\to\mathbb{C}^\times}} \mathbb{C}\circ\chi_\sigma.$$
- Geometry of $[G]$:
  - $K_\infty \subseteq G(\mathbb{R})$ some maximal compact, connected
  - $K = K^\infty \subseteq G(\mathbb{A}_f)$ some compact-open subgroup
  - Then
$$\begin{array}{ll} [G] & \\ \Big\downarrow{\scriptstyle K\text{-torsor (in particular: a profinite covering)}} & \\ [G]/K & \text{a real manifold, at least for sufficiently small } K \\ \Big\downarrow{\scriptstyle K_\infty\text{-torsor}} & \\ [G]/KK_\infty = \coprod_{i=1}^m \Gamma_i\backslash X & \text{disjoint union of arithmetic manifolds} \end{array}$$
    with $\Gamma_i \subseteq G(\mathbb{R})$ some (discrete) congruence subgroups determined by K, and
$$X := A_G\backslash G(\mathbb{R})/K_\infty.$$
- An irreducible unitary representation π of $G(\mathbb{A})$ is automorphic if it is isomorphic to a subquotient of $L^2([G])$.
- Examples:
  - $G = \mathbb{G}_m$, then the automorphic representations of $G(\mathbb{A})$ are the characters of $\mathbb{A}^\times$ which are trivial on $\mathbb{Q}^\times\cdot\mathbb{R}_{>0}$ (by definition of $[G]$)
  - $[G]$ has finite volume
    $\Rightarrow$ the trivial representation (=constant functions on $[G]$) is automorphic, and in $L^2_{\mathrm{disc}}([G])$.
- Have the general orthogonal decomposition
$$L^2([G]) \cong L^2_{\mathrm{disc}}([G]) \oplus L^2_{\mathrm{cont}}([G])$$
with $L^2_{\mathrm{disc}}([G])$ the closure of the sum of all irreducible *sub*representations of $G(\mathbb{A})$.

- Langlands $\Rightarrow L^2_{\text{cont}}([G])$ described via $L^2_{\text{disc}}([M])$ for M running through Levi quotients of parabolic subgroups $P \subseteq G, P \neq G$.

Questions ?

For $G = \mathrm{GL}_2$:

- $A_{\mathrm{GL}_2} \cong \mathbb{R}_{>0}$ embedded as scalar matrices into $\mathrm{GL}_2(\mathbb{R})$
- We have
$$\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}) \cong \mathrm{GL}_2(\mathbb{Z})\backslash(\mathrm{GL}_2(\widehat{\mathbb{Z}}) \times \mathrm{GL}_2(\mathbb{R})),$$
where
$$\widehat{\mathbb{Z}} := \varprojlim_m \mathbb{Z}/m \cong \prod_p \mathbb{Z}_p.$$
- Indeed:
  - $\mathrm{GL}_2(\mathbb{A}_f)$ is the restricted product of the $\mathrm{GL}_2(\mathbb{Q}_p)$ w.r.t. to the $\mathrm{GL}_2(\mathbb{Z}_p)$, i.e.,
$$\mathrm{GL}_2(\mathbb{A}_f) = \{(A_p)_p \in \prod_p \mathrm{GL}_2(\mathbb{Q}_p) \mid A_p \in \mathrm{GL}_2(\mathbb{Z}_p) \text{ for all but finitely many } p\}$$
(cf. [GH19, Proposition 2.3.1]).
  - $\mathbb{Q} \subseteq \mathbb{A}_f$ is dense (use Chinese remainder theorem).
  - $\mathbb{A}_f^\times = \mathbb{Q}^\times \cdot \widehat{\mathbb{Z}}^\times$ (use prime factorization).
  - $\mathrm{GL}_2(\mathbb{Z}_p), \mathrm{GL}_2(\mathbb{Q}_p)$ generated by elementary and diagonal matrices
$\Rightarrow \mathrm{GL}_2(\mathbb{A}_f) = \mathrm{GL}_2(\mathbb{Q})\mathrm{GL}_2(\widehat{\mathbb{Z}})$
$\Rightarrow \mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}_f) \cong \mathrm{GL}_2(\mathbb{Q}) \cap \mathrm{GL}_2(\widehat{\mathbb{Z}})\backslash \mathrm{GL}_2(\widehat{\mathbb{Z}}) \cong \mathrm{GL}_2(\mathbb{Z})\backslash\mathrm{GL}_2(\widehat{\mathbb{Z}})$.
$\Rightarrow \mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}) \cong \mathrm{GL}_2(\mathbb{Z})\backslash(\mathrm{GL}_2(\widehat{\mathbb{Z}}) \times \mathrm{GL}_2(\mathbb{R}))$
  - The proof works for all $n \geq 1$ and shows:
$$\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A}) \cong \mathrm{GL}_n(\mathbb{Z})\backslash(\mathrm{GL}_n(\widehat{\mathbb{Z}}) \times \mathrm{GL}_n(\mathbb{R})).$$
- $K_m = \ker(\mathrm{GL}_2(\widehat{\mathbb{Z}}) \to \mathrm{GL}_2(\mathbb{Z}/m))$ for $m \geq 1$.
  - these groups are cofinal within all compact-open subgroups in $\mathrm{GL}_2(\mathbb{A}_f)$.
  - we have
$$\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}_f)/K_m \cong \mathrm{GL}_2(\mathbb{Z})\backslash\mathrm{GL}_2(\widehat{\mathbb{Z}})/K_m$$
and $\mathrm{GL}_2(\widehat{\mathbb{Z}})/K_m \cong \mathrm{GL}_2(\mathbb{Z}/m)$.
  - $\mathrm{SL}_2(\mathbb{Z}/m)$ is generated by elementary matrices: use the euclidean algorithm and the magic identity
$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -a^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$
for any invertible element a in some ring R. Check!
$\Rightarrow \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/m)$ surjective
  - We obtain
$$\mathrm{GL}_2(\mathbb{Z})\backslash\mathrm{GL}_2(\widehat{\mathbb{Z}})/K_m \cong \{\pm 1\}\backslash(\mathbb{Z}/m)^\times$$
via the determinant.
- $K_\infty = \mathrm{SO}_2(\mathbb{R}) \subseteq \mathrm{GL}_2(\mathbb{R})$

- $X = A_{\mathrm{GL}_2}\backslash \mathrm{GL}_2(\mathbb{R})/K_\infty \cong \mathbb{H}^\pm$ via
$$A_{\mathrm{GL}_2}\backslash \mathrm{GL}_2(\mathbb{R})/K_\infty \xrightarrow{\sim} \mathbb{H}^\pm,\ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto g(i) := \frac{a\cdot i + b}{c \cdot i + d}$$
- In the end:
$$\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f)/K_m \times X) \cong \coprod_{\{\pm 1\}\backslash(\mathbb{Z}/m)^\times} \Gamma(m)\backslash\mathbb{H}^\pm$$
with
$$\Gamma(m) := K_m \cap \mathrm{GL}_2(\mathbb{Z}) = \ker(\mathrm{GL}_2(\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{Z}/m)).$$

Finiteness of the volume of $[G]$ for $G = \mathrm{GL}_2$: A direct argument

- $K_\infty K_m$ compact
$\Rightarrow$ It suffices to show that $[\mathrm{GL}_2]/K_\infty K_m$ has finite volume.
$\Rightarrow$ Suffices to see that $\Gamma(m)\backslash\mathbb{H}^\pm$ has finite volume.
- Up to scalar in $\mathbb{R}_{>0}$, the $\mathrm{GL}_2(\mathbb{R})$-invariant measure on $\mathbb{H}^\pm$ is given by the volume form
$$\frac{1}{y^2} dx \wedge dy.$$
- Indeed:
  - Have to show
$$g^*(\frac{1}{y^2}dx\wedge dy) = \frac{1}{y^2}dx \wedge dy$$
for $g \in \mathrm{GL}_2(\mathbb{R})$.
  - Write $dx \wedge dy = \frac{i}{2} dz \wedge d\overline{z}$, $z = x + iy$.
  - Statement true for
$$g = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \quad a \in \mathbb{R}.$$
  - Statement true for A_{GL_2}.
  - Statement true for
$$g = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \quad a \in \mathbb{R}^\times.$$
  - Statement true for
$$g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
i.e., for $z \mapsto \frac{1}{z} = \frac{\overline{z}}{|z|^2}$.
  - From the magic identity we get invariance for lower triangular matrices, and thus for all of $\mathrm{GL}_2(\mathbb{R})$. Check!
- A known fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$ is given by
$$D := \{z \in \mathbb{H} \mid |z| > 1, |\mathrm{Re}(z)| < 1/2\}$$
and finiteness of the volume of $\mathrm{GL}_2(\mathbb{Z})\backslash\mathbb{H}^\pm$ follows from
$$\int_D \frac{1}{y^2} dx \wedge dy < \infty.$$

- Finally, $\Gamma(m) \subseteq \mathrm{GL}_2(\mathbb{Z})$ is of finite index
  $\Rightarrow \Gamma(m)\backslash\mathbb{H}^{\pm}$ is of finite volume, as desired.

Questions ?

Modular forms classically (cf. [DS05]**):**

- $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ congruence subgroup, later sufficiently small.
- $k \in \mathbb{Z}$
- A function $f\colon \mathbb{H}^{\pm} \to \mathbb{C}$ is a modular form of weight k for Γ if
  - $f(\gamma \cdot z) = (cz+d)^k f(z)$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$
  - f is holomorphic
  - f is holomorphic at the cusps of Γ
- "holomorphic at cusp"
- $M_k(\Gamma) =$ the $\mathbb{C}$-vector space of modular forms of weight k for Γ.

Let us call, for the moment, a function weakly modular if it satisfies the first two conditions.

Holomorphic at cusps:

- For each cusp σ a weakly modular form f for Γ has a Fourier expansion, i.e., up to translating the cusp to ∞ there is an equality
  $$f(z) = \sum_{n \in \mathbb{Z}} a_n q^{n/h}$$
  for $q = e^{2\pi i z}$, h the smallest periodicity of f, and $a_n \in \mathbb{C}$, $n \in \mathbb{Z}$.
- f is holomorphic at the cusp, i.e., a modular form, if $a_n = 0$ for $n < 0$.
- In this case, f is a cusp form if $a_0 = 0$.
- Let $S_k(\Gamma) \subseteq M_k(\Gamma)$ be the subspace of cusp forms.

Modular forms as sections of line bundles on modular curves:

- The embedding $\mathbb{H}^{\pm} \subseteq \mathbb{P}^1_{\mathbb{C}},\ z \mapsto [z:1]$ is $\mathrm{GL}_2(\mathbb{R})$-equivariant.
- The $\mathrm{GL}_2(\mathbb{C})$-action on $\mathbb{P}^1_{\mathbb{C}}$ lifts (naturally) to $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(k)$:
  - Sufficient for $k = -1$
  - But $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-1)$ is associated to the $\mathbb{G}_m$-torsor
    $$\mathbb{A}^2_{\mathbb{C}} \setminus \{0\} \to \mathbb{P}^1_{\mathbb{C}},\ (z_1, z_2) \mapsto [z_1 : z_2],$$
    which is naturally $\mathrm{GL}_2(\mathbb{C})$-equivariant.
- We obtain a Γ-equivariant morphism
  $$\begin{tikzcd} \mathcal{O}_{\mathbb{H}^{\pm}}(k) := \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(k) \times_{\mathbb{P}^1_{\mathbb{C}}} \mathbb{H}^{\pm} \arrow[d] \\ \mathbb{H}^{\pm} \end{tikzcd}$$
- Modding out Γ yields:
  $$\begin{tikzcd} \omega^{\otimes k} := \Gamma\backslash\mathcal{O}_{\mathbb{H}^{\pm}}(k) \arrow[d] \\ \Gamma\backslash\mathbb{H}^{\pm}, \end{tikzcd}$$
  with $\omega^{\otimes k}$ a holomorphic line bundle over $\Gamma\backslash\mathbb{H}^{\pm}$

- The space
$$H^0(\Gamma\backslash\mathbb{H}^\pm, \omega^{\otimes k})$$
of holomorphic sections of $\omega^{\otimes k}$ identifies with weakly modular forms for Γ.
- Indeed:
  - For $k \in \mathbb{Z}$ define the holomorphic $\mathrm{GL}_2(\mathbb{R})$-equivariant line bundle
  $$L_k := \mathbb{H}^\pm \times \mathbb{C}$$
  over $\mathbb{H}^\pm$ with $\mathrm{GL}_2(\mathbb{R})$-acting on the left by
  $$g \cdot (z, \lambda) := (\frac{az+b}{cz+d}, (cz+d)^{-k}\lambda).$$
  for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})$.
  - The map
  $$\begin{aligned} &L_{-1}^* \to \mathcal{O}_{\mathbb{H}^\pm}(-1)^* = \mathbb{H}^\pm \times_{\mathbb{P}^1_{\mathbb{C}}} \mathbb{A}^2_{\mathbb{C}} \setminus \{0\}, \\ &(z, \lambda) \mapsto (z, (\lambda z, \lambda)) \end{aligned}$$
  is a holomorphic isomorphism of $\mathrm{GL}_2(\mathbb{R})$-equivariant line bundles over $\mathbb{H}^\pm$. Here $(-)^*$ is the complement of the zero section in a line bundle. Check!
  - By taking tensor powers:
  $$L_k \cong \mathcal{O}_{\mathbb{H}^\pm}(k)$$
  for all $k \in \mathbb{Z}$, as $\mathrm{GL}_2(\mathbb{R})$-equivariant holomorphic vector bundles.
  - A holomorphic section
  $$\mathbb{H}^\pm \to L_k, \ z \mapsto (z, f(z))$$
  is Γ-equivariant if and only if $f(z)$ satisfies
  $$f(\frac{az+b}{cz+d}) = (cz+d)^k f(z)$$
  for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. Check!
  - This shows that
  $$H^0(\Gamma\backslash\mathbb{H}^\pm, \omega^{\otimes k})$$
  identifies with the space of weakly modular forms for Γ.
- In particular, we have a natural embedding
$$M_k(\Gamma) \subseteq H^0(\Gamma\backslash\mathbb{H}^\pm, \omega^{\otimes k})$$
with image defined by condition of being "holomorphic at the cusps":
  - ω extends canonically to the canonical compactification $\overline{\Gamma\backslash\mathbb{H}^\pm}$ of the algebraic curve $\Gamma\backslash\mathbb{H}^\pm$
  - $M_k(\Gamma)$ is the image of $H^0(\overline{\Gamma\backslash\mathbb{H}^\pm}, \omega^{\otimes k})$
  $\Rightarrow$ In particular, $M_k(\Gamma)$ is finite-dimensional over $\mathbb{C}$. Moreover, ω is ample on $\overline{\Gamma\backslash\mathbb{H}^\pm}$ which implies that $M_k(\Gamma) = 0$ for $k < 0$ and $M_k(\Gamma)$ gets "big" for $k \gg 0$.
  - $S_k(\Gamma) \cong H^0(\overline{\Gamma\backslash\mathbb{H}^\pm}, \omega^{\otimes k}(-D))$ where $D = \overline{\Gamma\backslash\mathbb{H}^\pm} \setminus \Gamma\backslash\mathbb{H}^\pm$ is the (reduced) divisor at infinity.

Questions ?

From sections of $\omega^{\otimes k}$ to functions on $[G] = [\mathrm{GL}_2]$:

- Let $K \subseteq \mathrm{GL}_2(\mathbb{A}_f)$ compact-open and write
$$\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f/K) \times \mathbb{H}^\pm) \cong \coprod_{i=1}^{m} \Gamma_i \backslash \mathbb{H}^\pm$$
- $\omega^{\otimes k}$ defines by pullback from $\mathbb{H}^\pm$ a (complex) line bundle $\omega^{\otimes k}$ on
$$\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f/K) \times \mathbb{H}^\pm).$$
- The spaces $M_k(\Gamma_i)$ of modular forms for Γ_i, $i = 1, \ldots, m$, embed into the space of holomorphic sections
$$H^0(\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f)/K \times \mathbb{H}^\pm), \omega^{\otimes k}).$$
- Set
$$M_k(K) := \bigoplus_{i=1}^{m} M_k(\Gamma_i).$$
- Then $M_k(K) \subseteq H^0(\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f)/K \times \mathbb{H}^\pm), \omega^{\otimes k})$ is defined by condition of being holomorphic at all cusps.
- $\omega^{\otimes k}$ defines by pullback a $G(\mathbb{A}_f)$-equivariant line bundle, again written $\omega^{\otimes k}$, on
$$\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^\pm)$$
- Define
$$\begin{aligned} & H^0(\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^\pm), \omega^{\otimes k}) \\ := \ & \varinjlim_K H^0(\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f)/K \times \mathbb{H}^\pm), \omega^{\otimes k}). \end{aligned}$$
  and
$$M_k := \varinjlim_K M_k(K).$$
- The line bundle $\omega^{\otimes k}$ is $\mathrm{GL}_2(\mathbb{R})$-equivariantly trivial after pullback along
$$A_{\mathrm{GL}_2}\backslash \mathrm{GL}_2(\mathbb{R}) \to \mathbb{H}^\pm, \ g \mapsto g(i).$$
  Namely, for $k = -1$, there is the $\mathrm{GL}_2(\mathbb{R})$-equivariant section Check!
$$\begin{aligned} & s \colon A_{\mathrm{GL}_2}\backslash \mathrm{GL}_2(\mathbb{R}) \to A_{\mathrm{GL}_2}\backslash \mathrm{GL}_2(\mathbb{R}) \times_{\mathbb{H}^\pm} \mathcal{O}_{\mathbb{H}^\pm}(-1) \cong A_{\mathrm{GL}_2}\backslash \mathrm{GL}_2(\mathbb{R}) \times_{\mathbb{P}^1_{\mathbb{C}}} \mathbb{A}^2_{\mathbb{C}} \setminus \{0\}, \\ & g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (g, |\det(g)|^{-1/2}(ai + b, ci + d)). \end{aligned}$$
- In particular, pullback of sections defines a morphism
$$\Phi \colon H^0(\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^\pm), \omega^{\otimes k}) \to C^\infty([\mathrm{GL}_2])$$
  which on the infinite component is induced by pullback along
$$A_{\mathrm{GL}_2}\backslash \mathrm{GL}_2(\mathbb{R}) \mapsto \mathbb{H}^\pm, \ g \mapsto g(i)$$
  and the isomorphism
$$A_{\mathrm{GL}_2}\backslash \mathrm{GL}_2(\mathbb{R}) \times_{\mathbb{H}^\pm} L_k \cong A_{\mathrm{GL}_2}\backslash \mathrm{GL}_2(\mathbb{R}) \times_{\mathbb{H}^\pm} \mathcal{O}_{\mathbb{H}^\pm}(k) \cong A_{\mathrm{GL}_2}\backslash \mathrm{GL}_2(\mathbb{R}) \times \mathbb{C}$$

where the last one is induced by the section s. One checks, by reducing to the case $k=-1$, that each section $f\colon \mathbb{H}^{\pm} \to L_k$ is send to the function

$$g \mapsto \varphi_f(g) := |\det(g)|^{k/2}(ci+d)^{-k} f(\frac{ai+b}{ci+d})$$

for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})$. Indeed, for $k=-1$, the pullback of

$$\mathbb{H}^{\pm} \to L_{-1},\ z \mapsto (z, f(z))$$

and the corresponding section

$$A_{\mathrm{GL}_2}\backslash \mathrm{GL}_2(\mathbb{R}) \to A_{\mathrm{GL}_2}\backslash \mathrm{GL}_2(\mathbb{R}) \times \mathbb{C},\ g \mapsto (g, \varphi_f(g))$$

are related, as sections of $A_{\mathrm{GL}_2}\backslash \mathrm{GL}_2(\mathbb{R}) \times_{\mathbb{P}^1_{\mathbb{C}}} \mathbb{A}^2_{\mathbb{C}} \setminus \{0\}$ by the formula Check!

$$f(g(i))(g(i), 1) = \varphi_f(g)|\det(g)|^{1/2}(ai+b, ci+d)$$

which forces

$$\varphi_f(g) = |\det(g)|^{-1/2}(ci+d) f(g(i))$$

as desired.

- Note that the factor $|\det(g)|^{-1/2}$ is necessary only because by definition elements of $L^2([\mathrm{GL}_2])$ or $C^\infty([\mathrm{GL}_2])$ are constant along A_{GL_2}.

We used the following general definition.

Definition 3.1 (Smooth functions on $[G]$). *Let G be a reductive group over $\mathbb{Q}$. Then a function $f\colon [G] \to \mathbb{C}$ is smooth if it is stable under some $K \subseteq G(\mathbb{A}_f)$ compact-open and the resulting function*

$$f\colon [G]/K \to \mathbb{C}$$

is smooth, i.e., C^∞. (Note that $[G]/K$ is a real manifold.)

Upshot:

- We constructed a canonical morphism

$$M_k \subseteq H^0(\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^{\pm}), \omega^{\otimes k}) \xrightarrow{\Phi} C^\infty([\mathrm{GL}_2]),$$

  where $M_k = \varinjlim_K M_k(K)$ and $M_k(K)$ is a sum of spaces of modular forms $\bigoplus_{i=1}^m M_k(\Gamma_i)$ for various congruence subgroups $\Gamma_i \subseteq \mathrm{GL}_2(\mathbb{Z})$.
- The image of M_k in $H^0(\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^{\pm}), \omega^{\otimes k})$ is defined by the condition of "holomorphicity at the cusps".

Questions ?

When do modular forms give rise to functions in $L^2([\mathrm{GL}_2])$?

- More generally, pick a section $f \in H^0(\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^{\pm}, \omega^{\otimes k})$.
- Question: When is

$$\Phi(f) \in C^\infty([\mathrm{GL}_2])$$

  in $L^2([\mathrm{GL}_2])$?
- There exists $K \subseteq \mathrm{GL}_2(\mathbb{A}_f)$ compact-open such that

$$f \in H^0(\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f)/K \times \mathbb{H}^{\pm}), \omega^{\omega k}),$$

and $\Phi(f)$ is L^2 if and only if $\Phi(f) \in C^\infty([\mathrm{GL}_2]/K) \subseteq C^\infty([\mathrm{GL}_2])$ (as K is compact).

$\Rightarrow$ Reduce to question:
For $\Gamma \subseteq \mathrm{GL}_2(\mathbb{Z})$ some congruence subgroup, $f \in H^0(\Gamma\backslash\mathbb{H}^\pm, \omega^{\otimes k})$. When is

$$\Gamma A_{\mathrm{GL}_2}\backslash \mathrm{GL}_2(\mathbb{R}) \to \mathbb{C},\ g \mapsto \varphi_f(g) = |\det(g)|^{k/2}(ci+d)^{-k} f(g(i))$$

square-integrable, i.e., in L^2?

- Claim:
$$|\varphi_f(g)|^2 = |f(g(i))|^2 |\mathrm{Im}(g(i))|^k$$
for any $g \in \mathrm{GL}_2(\mathbb{R})$.
- Equivalently,
$$|\det(g)|^{-1/2}|ci+d| \stackrel{?}{=} |\mathrm{Im}(g(i))|^{1/2}$$
for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})$.
- This is true as both sides are invariant under
  - $K_\infty = \mathrm{SO}_2(\mathbb{R})$-invariant from the right,
  - the subgroup $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \subseteq \mathrm{GL}_2(\mathbb{R})$ acting from the left,
  - the diagonal matrices in $\mathrm{GL}_2(\mathbb{R})$ acting from the left. Check these!
- Hence
$$\int\limits_{\Gamma A_{\mathrm{GL}_2}\backslash \mathrm{GL}_2(\mathbb{R})} |\varphi_f(g)|^2 dg = \int\limits_{\Gamma\backslash\mathbb{H}^\pm} |f(z)|^2 |y|^k \frac{1}{y^2} dx \wedge dy$$
(Recall that we integrate over "the" invariant measure.)
- $\Gamma\backslash\mathbb{H}^\pm$ has finitely many cusps $\Rightarrow$ Reduce to local statement at cusps
- Write f in Fourier expansion, wlog at ∞:
$$f(z) = \sum_{n\in\mathbb{Z}} a_n q^{n/h}.$$
- Then:
$$|y|^k f(z) = 2\pi |\log(|q|)|^k \sum_{n\in\mathbb{Z}} a_n q^{n/h}$$
as $y = \log(|q|)$. Check!
- Now for $0 < r \leq 1$
$$\int\limits_{0<|q|\leq r} |\log(|q|)|^{k-2} \sum_{n\in\mathbb{Z}} a_n q^{n/h} \frac{i}{2} dq \wedge d\overline{q} < \infty$$
if $a_n = 0$ for $n \leq 0$, i.e., if f is cuspidal.

Upshot:

- The map $M_k \subseteq H^0(\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^\pm), \omega^k) \xrightarrow{\Phi} C^\infty([\mathrm{GL}_2])$ induces a canonical inclusion
$$\Phi\colon S_k \to L^2([\mathrm{GL}_2]),$$
where $S_k := \varinjlim_K S_k(K)$.

Observations:

- $\mathrm{GL}_2(\mathbb{A}_f)$ acts on $H^0(\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^\pm), \omega^k)$, $C^\infty([\mathrm{GL}_2])$, $L^2([\mathrm{GL}_2])$, M_k, S_k and all inclusions, in particular Φ, are equivariant.
- E.g., for $g \in \mathrm{GL}_2(\mathbb{A}_f)$ and varying $K \subseteq \mathrm{GL}_2(\mathbb{A}_f)$ the isomorphism

$$M_k(K) \xrightarrow{g} M_k(g^{-1}Kg) \subseteq M_k$$

  defines the action $M_k \xrightarrow{g} M_k$ as $M_k = \varinjlim_K M_k(K)$.
- $\mathrm{GL}_2(\mathbb{A}_f)$ does not act on $M_k(K), S_k(K), \ldots$ for any fixed $K \subseteq \mathrm{GL}_2(\mathbb{A}_f)$.

Next tasks:

- Describe S_k as a $\mathrm{GL}_2(\mathbb{A}_f)$-representation.
- Characterize the image of S_k in $L^2([\mathrm{GL}_2])$.

Questions ?

4. From modular forms to automorphic representations, part II

Today:

- Continue construction of automorphic representations associated to cusp forms.

Questions from last time:

1) Can one classify $\mathrm{GL}_2(\mathbb{R})$-equivariant line bundles on $\mathbb{H}^{\pm}$?
2) Similarly: Can one classify $\mathrm{GL}_2(\mathbb{Z})$-equivariant line bundles on $\mathbb{H}^{\pm}$?

- For 1):
  - Assume G Lie group, H a closed subgroup. Then the category
    $$\{G\text{-equivariant complex vector bundles on } G/H\}$$
    is equivalent to
    $$\{\text{finite dimensional } \mathbb{C}\text{-representations of } H\}.$$
    "$\Rightarrow$" Look at the representation of H at the coset $1 \cdot H \in G/H$.
    "$\Leftarrow$" Pass to the quotient
    $$\mathcal{V} := (G \times V)/H$$
    for the action $h \cdot (g, v) := (gh^{-1}, hv)$.
  - Useful:
    $$\sigma \colon H \to \mathrm{GL}(V)$$
    finite-dimensional $\mathbb{C}$-representation of H. Then:
    Continuous resp. smooth sections
    $$G/H \to \mathcal{V}$$
    identify with continuous resp. smooth functions
    $$f \colon G \to V$$
    such that $f(gh) = \rho(h)^{-1} \cdot f(v)$. Check!
  - In our case, $G = \mathrm{GL}_2(\mathbb{R})$, $H = A_{\mathrm{GL}_2} K_\infty$ with $K_\infty \cong \mathrm{SO}_2(\mathbb{R})$. Then
    $$H \cong \mathbb{C}^\times$$
    and the irreducible representations of H are given by
    $$z \mapsto z^p \overline{z}^q, \qquad p, q \in \mathbb{Z}.$$
    $\Rightarrow$ There exist more $\mathrm{GL}_2(\mathbb{R})$-equivariant line bundles on $\mathbb{H}^{\pm}$.
  - Only the ones for $q = 0$ define *holomorphic* line bundles.
  - The $\mathrm{SL}_2(\mathbb{R})$-equivariant line bundles on $\mathbb{H}$ are all isomorphic to $\omega^{\otimes k}$ for some $k \in \mathbb{Z}$.
  - Potentially confusing: Exists modular forms of half-integral weight.
  - Explanation: These are not sections of an $\mathrm{SL}_2(\mathbb{R})$-equivariant line bundle on $\mathbb{H}$.
  - But sections of an $\widetilde{\mathrm{SL}_2(\mathbb{R})}$-equivariant line bundle.
  - Here: $\widetilde{\mathrm{SL}_2(\mathbb{R})} \to \mathrm{SL}_2(\mathbb{R})$, the "metaplectic group", the degree 2-covering of $\mathrm{SL}_2(\mathbb{R})$ (a non-algebraic Lie group).
  - Note: $\pi_1(\mathrm{SL}_2(\mathbb{R}), 1) \cong \pi_1(\mathrm{S}^1, 1) \cong \mathbb{Z}$.

    - More details: [Del73].
- For 2):
    - $\mathcal{M}_{1,1}$ = stack of elliptic curves over $\mathrm{Spec}(\mathbb{C})$.
    - Analytic stack associated to $\mathcal{M}_{1,1}$ is the stacky quotient $[\mathrm{GL}_2(\mathbb{Z})\backslash\mathbb{H}^{\pm}]$.
    - Mumford:
$$\mathrm{Pic}(\mathcal{M}_{1,1}) \cong \mathbb{Z}/12,$$
generated by ω, with the relation $\omega^{\otimes 12} \cong \mathcal{O}$ induced by the cusp form $\Delta \in S_{12}(\mathrm{SL}_2(\mathbb{Z}))$, cf. [Mum65].
    - Note: Does not imply $M_k(\Gamma) \cong M_{k+12}(\Gamma)$ for $k \in \mathbb{Z}$, $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ congruence subgroup.
    - Reason: The canonical extensions of $\omega^{\otimes k}$ and $\omega^{\otimes k+12}$ to the compactification of $\mathcal{M}_{1,1}$ differ.
    - I don't know if all (holomorphic) line bundles on $[\mathrm{GL}_2(\mathbb{Z})\backslash\mathbb{H}^{\pm}]$ arise from an algebraic line bundle on $\mathcal{M}_{1,1}$, i.e., are algebraic.

Last time:

- We constructed a $\mathrm{GL}_2(\mathbb{A}_f)$-equivariant embedding
$$\Phi\colon M_k \to C^{\infty}([\mathrm{GL}_2]),$$
inducing a $\mathrm{GL}_2(\mathbb{A}_f)$-equivariant embedding
$$\Phi\colon S_k \to L^2([\mathrm{GL}_2]).$$
- Here
$$M_k = \varinjlim_{K\subseteq \mathrm{GL}_2(\mathbb{A}_f)} M_k(K)$$
with
$$M_k(K) = \bigoplus_{i=1}^{m} M_k(\Gamma_i)$$
a sum of spaces of modular forms for congruence subgroups $\Gamma_i \subseteq \mathrm{GL}_2(\mathbb{Z})$.
- Similarly:
$$S_k = \varinjlim_{K} S_k(K)$$
with $S_k(K)$ a sum of spaces of cusp forms for congruence subgroups.
- Today: How can one characterize the images
$$\Phi(M_k) \subseteq C^{\infty}([\mathrm{GL}_2])$$
resp.
$$\Phi(S_k) \subseteq L^2([\mathrm{GL}_2])?$$
.
- Next time: How does S_k decompose as a $\mathrm{GL}_2(\mathbb{A}_f)$-representation?

Recall the construction of Φ:

- Consider $\mathrm{GL}_2(\mathbb{R})$-equivariant line bundle
$$L_k \to \mathbb{H}^{\pm}$$
associated to the representation $z \mapsto z^k$ of $\mathbb{C}^{\times} = A_{\mathrm{GL}_2}\mathrm{SO}_2(\mathbb{R})$.
- By pullback: Obtain line bundle $\omega^{\otimes k}$ on $\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^{\pm})$.

- Smooth, holomorphic,... sections of $\omega^{\otimes k}$ identify with smooth, holomorphic,... functions
$$f\colon \mathrm{GL}_2(\mathbb{A}_f)\times \mathbb{H}^{\pm}\to \mathbb{C},\ (g,z)\mapsto f(g,z)$$
satisfying
$$f(\gamma g,\gamma z)=(cz+d)^k f(g,z)$$
for
$$\gamma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in \mathrm{GL}_2(\mathbb{Q}).$$
- Define the function
$$j\colon \mathrm{GL}_2(\mathbb{R})\times \mathbb{H}^{\pm}\to \mathbb{C}^\times,\ (\begin{pmatrix} a & b\\ c & d\end{pmatrix},z)\mapsto cz+d.$$
Then
$$j(gh,z)=j(g,hz)j(h,z)$$
for all $g,h\in \mathrm{GL}_2(\mathbb{R}), z\in \mathbb{H}^{\pm}$. Check!
- For $f\colon \mathrm{GL}_2(\mathbb{A}_f)\times \mathbb{H}^{\pm}\to \mathbb{C}$ as above the function
$$\varphi_f\colon \mathrm{GL}_2(\mathbb{A}_f)\times \mathrm{GL}_2(\mathbb{R})\to \mathbb{C},\ (g,g_\infty)\mapsto j(g_\infty,i)^{-k}f(g,g_\infty i)$$
is $\mathrm{GL}_2(\mathbb{Q})$-equivariant, i.e., lies in $C^\infty(\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}))$.
- This is not the function we denoted by $\Phi(f)$ last time: It is not invariant by A_{GL_2}!
- This can be fixed by multiplying it with the function
$$\mathrm{GL}_2(\mathbb{R})\to \mathbb{C}^\times,\ g\mapsto |\det(g)|^{k/2}.$$
- Thus
$$\Phi(f)(g,g_\infty):=|\det(g_\infty)|^{k/2}\varphi_f(g,g_\infty)$$
is the associated function in $C^\infty([\mathrm{GL}_2])$ from last time.
- Recall
$$\mathbb{C}^\times\cong A_{\mathrm{GL}_2}\mathrm{SO}_2(\mathbb{R})=\{\begin{pmatrix} a & -b\\ b & a\end{pmatrix}\}\subseteq \mathrm{GL}_2(\mathbb{R}).$$
- Then
$$\varphi_f(g,g_\infty z)=z^{-k}\varphi_f(g,g_\infty)$$
for all $z\in\mathbb{C}^\times\subseteq \mathrm{GL}_2(\mathbb{R}), (g,g_\infty)\in \mathrm{GL}_2(\mathbb{A})$. Check!
- We obtain that $f\mapsto \varphi_f$ identifies *smooth* sections
$$\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}_f)\times \mathbb{H}^{\pm}\to \omega^{\otimes k}$$
with smooth functions
$$\varphi\colon \mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A})\to \mathbb{C}$$
such that
$$\varphi(g,g_\infty z)=z^{-k}\varphi(g,g_\infty)$$
for all $(g,g_\infty)\in \mathrm{GL}_2(\mathbb{A}_f)\times \mathrm{GL}_2(\mathbb{R})$, $z\in\mathbb{C}^\times$.

Questions?

Upshot and next aims:

- Describe the image of M_k in $C^\infty(\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}))$ under $f \mapsto \varphi_f$ by conditions soley on $\varphi \in C^\infty(\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}))$.
- An additional condition will determine the image of S_k.
  $\Rightarrow$ Obtain a description of $S_k \subseteq L^2([\mathrm{GL}_2])$.
- We already know that an element φ in the image of M_k must satisfy

$$\varphi(g, g_\infty z) = z^{-k}\varphi(g, g_\infty)$$

  for $z \in \mathbb{C}^\times$ and $(g, g_\infty) \in \mathrm{GL}_2(\mathbb{A})$ and that such φ descent to a smooth function

$$f_\varphi \colon \mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^\pm \to \mathbb{C}, (g, z) \mapsto f(g, z)$$

  satisfying modularity for $\mathrm{GL}_2(\mathbb{Q})$.
- Need to check:
  - When is f_φ holomorphic?
  - When is f_φ holomorphic at the cusps?

When is f_φ holomorphic?

- Reduces to: Given a smooth function

$$f \colon \mathbb{H}^\pm \to \mathbb{C}.$$

  Which condition on

$$\varphi_f \colon \mathrm{GL}_2(\mathbb{R}) \to \mathbb{C},\ g \mapsto j(g, i)^{-k} f(gi)$$

  guarantees that f is holomorphic?

The strategy:

- The Lie algebra

$$\mathfrak{g} := \mathfrak{gl}_2(\mathbb{R}) \cong \mathrm{Mat}_{2,2}(\mathbb{R})$$

  of $\mathrm{GL}_2(\mathbb{R})$ acts on $C^\infty(\mathrm{GL}_2(\mathbb{R}))$ by deriving the action of $\mathrm{GL}_2(\mathbb{R})$ by *right* translations.
- Make this action explicit on φ_f.
- Then construct an element $Y \in \mathfrak{g}_\mathbb{C}$ such that

$$Y * \varphi_f = 0 \quad \Leftrightarrow \quad f \text{ is holomorphic.}$$

Infinitesimal actions (cf. [GH19, Section 4.2]**):**

- H any Lie group with Lie algebra $\mathfrak{h} := T_1 H$ ("tangent space at identity")
- V any representation of H, with V a Hausdorff topological $\mathbb{C}$-vector space.
- For $X \in \mathfrak{h}$, defined by some path $\gamma \colon (-\varepsilon, \varepsilon) \to H$ with $\gamma(0) = 1$, and $v \in V$ consider

$$(-\varepsilon, \varepsilon) \setminus \{0\} \to V,\ t \mapsto \frac{\gamma(t)v - v}{t}.$$

  If the limit exists, we write

$$X * v := \varinjlim_{t \to 0} \frac{\gamma(t)v - v}{t}.$$

- An element $v \in V$ is called smooth if

$$X_1 * (\ldots * (X_n * v)) \ldots)$$

  exists for all $X_1, \ldots, X_n \in \mathfrak{h}$.

- $V_{\mathrm{sm}} \subseteq V$ the subspace of smooth vectors in V.
- V_{sm} stable under H.
- V_{sm} is a representation of $\mathfrak{h}$, equivalently a module under the enveloping algebra for $\mathfrak{h}$:

$$U(\mathfrak{h}) := \bigoplus_{n \geq 0} \mathfrak{h}^{\otimes n} / \langle X \otimes Y - Y \otimes X - [X, Y] \mid X, Y \in \mathfrak{h} \rangle.$$

- Here: $[-,-]$ the Lie bracket of $\mathfrak{h}$.

Questions?

The explicit action of $\mathfrak{g} = \mathfrak{gl}_2(\mathbb{R})$ on φ_f:

- Recall: $f \in \mathbb{H}^{\pm} \to \mathbb{C}$ a smooth function.
- Recall:

$$\varphi_f \colon \mathrm{GL}_2(\mathbb{R}) \to \mathbb{C},\ g \mapsto j(g, i)^{-k} f(gi)$$

with

$$j(g, z) := cz + d \in \mathbb{C}$$

for

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R}),\ z \in \mathbb{H}^{\pm}.$$

- Recall:

$$j(gh, z) = j(g, hz) j(h, z)$$

for $g, h \in \mathrm{GL}_2(\mathbb{R}), z \in \mathbb{H}^{\pm}$.
- We know

$$\varphi_f(gz) = z^{-k} \varphi_f(g)$$

for $z \in \mathbb{C}^{\times}$, $g \in \mathrm{GL}_2(\mathbb{R})$

$\Rightarrow \mathbb{C} \cong \mathrm{Lie}(\mathbb{C}^{\times}) \subseteq \mathfrak{g}$ acts via the linear form

$$\mathbb{C} \mapsto \mathbb{C},\ z \mapsto -kz$$

on $\varphi_f(g)$. Note that

$$\mathrm{Lie}\ \mathbb{C}^{\times} = \langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \rangle_{\mathbb{R}}$$

and the anti-diagonal matrix acts by multiplying with $-ki$. Check!
- Consider the subgroup

$$U := \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \subseteq \mathrm{GL}_2(\mathbb{R}).$$

Then

$$\mathrm{Lie}(U) = \langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \rangle_{\mathbb{R}}.$$

- Define

$$\tilde{\varphi}_f \colon \mathrm{GL}_2(\mathbb{R}) \times \mathbb{H}^{\pm} \to \mathbb{C},\ (g, z) \mapsto j(g, z)^{-k} f(gz).$$

- For a fixed $g \in \mathrm{GL}_2(\mathbb{R})$:

$$\frac{\partial \tilde{\varphi}_f}{\partial \bar{z}}(g, z) = \det(g) j(g, z)^{-k-2} \frac{\partial f}{\partial \bar{z}}(gz).$$

Check! Needs that $z \mapsto gz$ has $\frac{\partial}{\partial z}$-derivative $\frac{\det(g)}{j(g,z)^2}$.

- Thus, f is holomorphic if and only if
$$\frac{\partial \tilde{\varphi}_f}{\partial \overline{z}}(g,i) = 0$$
for all $g \in \mathrm{GL}_2(\mathbb{R})$.
- For any path $\gamma\colon (-\varepsilon,\varepsilon) \to U$ with $\gamma(0) = 1$ we get
$$\varphi_f(g\gamma(t)) = \tilde{\varphi}_f(g\gamma(t), i) = \tilde{\varphi}_f(g, \gamma(t)i)$$
because $j(\gamma(t), i) = 1$.

Check!

- Set
$$X_\gamma := \frac{\partial \gamma}{\partial t}(0) \in \mathrm{Lie}(U).$$
- Thus
$$\begin{aligned} & X_\gamma * \varphi_f \\ = \ & \tfrac{\partial}{\partial t}(\varphi_f(g\gamma(t))_{|t=0} \\ = \ & \tfrac{\partial \tilde{\varphi}_f}{\partial z}(g,i)\tfrac{\partial}{\partial t}(\gamma(t)\cdot i)_{|t=0} + \tfrac{\partial \tilde{\varphi}_f}{\partial \overline{z}}(g,i)\overline{\tfrac{\partial}{\partial t}(\gamma(t)\cdot i)}_{|t=0}. \end{aligned}$$

Check! (Use the chain rule for Wirtinger derivatives.)

- Consider
$$\gamma_1\colon (-\varepsilon,\varepsilon) \to U,\ t \mapsto \begin{pmatrix} 1+t & 0 \\ 0 & 1 \end{pmatrix}.$$
Then $\gamma_1(t)\cdot i = (1+t)i$ and thus
$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} * \varphi_f(g) = i\frac{\partial \tilde{\varphi}_f}{\partial z}(g,i) - i\frac{\partial \tilde{\varphi}_f}{\partial \overline{z}}(g,i).$$
- Consider
$$\gamma_2\colon (-\varepsilon,\varepsilon) \to U,\ t \mapsto \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},$$
then $\gamma_1(t)\cdot i = i + t$ and thus
$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} * \varphi_f(g) = \frac{\partial \tilde{\varphi}_f}{\partial z}(g,i) + \frac{\partial \tilde{\varphi}_f}{\partial \overline{z}}(g,i).$$
- In particular,
$$i\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} * \varphi_f(g) + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} * \varphi_f(g) = 2\frac{\partial \tilde{\varphi}_f}{\partial \overline{z}}(g,i)$$
- Thus
$$\begin{pmatrix} i & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{g}_{\mathbb{C}}.$$
would be a candidate for Y.
- But we can do better:
$$\begin{aligned} Y := \ & \tfrac{1}{2}\begin{pmatrix} -1 & i \\ i & 1 \end{pmatrix} \\ & = \tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \tfrac{1}{2}i\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + i\begin{pmatrix} i & 1 \\ 0 & 0 \end{pmatrix} \end{aligned}$$
satisfies
$$Y * \varphi_f(g) = \underbrace{-\frac{k}{2}\varphi_f(g) + i(-\frac{ki}{2})\varphi_f(g)}_{=0} + 2i\frac{\partial \tilde{\varphi}_f}{\partial \overline{z}}(g,i) = 2i\frac{\partial \tilde{\varphi}_f}{\partial \overline{z}}(g,i).$$

Check!

- Why better?
  - Set
$$H := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad X := \frac{1}{2}\begin{pmatrix} -1 & -i \\ -i & 1 \end{pmatrix}.$$
  - Then H, X, Y is an $\mathfrak{sl}_2$-triple in $\mathfrak{g}_{\mathbb{C}}$.
  - Namely, in the basis
$$\begin{pmatrix} -i \\ 1 \end{pmatrix}, \begin{pmatrix} i \\ 1 \end{pmatrix},$$
    it is given by
$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Check!

Questions?

When is f_φ holomorphic at the cusps?

- Consider a function in Fourier expansion:
$$f(z) = \sum_{n \in \mathbb{Z}} a_n e^{2\pi i z n},$$
  say on
$$\{z \in \mathbb{C} \mid |\mathrm{Re}(z)| < 1, \mathrm{Im}(z) > 1\}.$$
- Recall
$$|e^{2\pi i z}| = e^{-2\pi y}$$
  with $y = \mathrm{Im}(z)$.
- Then $a_n = 0$ for $n < 0$ if and only if
$$|f(x+iy)| \leq C y^N$$
  for some $C, N \in \mathbb{R}_{>0}$ and $y \to \infty$.
- By definition this means f is of *moderate growth*, or *slowly increasing*.

Moderate growth for general reductive groups:

- The moderate growth condition generalizes to arbitrary G reductive over $\mathbb{Q}$.
- For each embedding $\rho\colon G \hookrightarrow \mathrm{SL}_{n,\mathbb{Q}}$ (not into $\mathrm{GL}_{n,\mathbb{Q}}$) we obtain a norm
$$|g| := \sup_v \max_{i,j=1,\ldots,n} \{|\rho(g)_{i,j}{}_v|_v\}$$
  on $G(\mathbb{A})$, where v runs through all primes and ∞.
- A function $\varphi\colon G(\mathbb{A}) \to \mathbb{C}$ is of moderate growth or slowly increasing if
$$|\varphi(g)| \leq C|g|^N$$
  for suitable $C, N \in \mathbb{R}_{>0}$, cf. [GH19, Definition 6.4.].
- This definition does not depend on the embeddding ρ.

Exercise:

- Consider a (holomorphic) section
$$f \in H^0(\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^{\pm}), \omega^{\otimes k}).$$

- Then: $\Phi(f) \in C^\infty([\mathrm{GL}_2])$ is of moderate growth
  $\Leftrightarrow f$ is holomorphic at the cusps
- In other words, if and only if f lies in M_k.

Upshot:

- We can describe the image of

$$M_k \to C^\infty(\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A})),\ f \mapsto \varphi_f.$$

- Namely, $\varphi \in C^\infty(\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}))$ lies in the image if and only if
  - $\varphi(g, g_\infty z) = z^{-k}\varphi(g, g_\infty z)$ for $(g, g_\infty) \in \mathrm{GL}_2(\mathbb{A})$, $z \in \mathbb{C}^\times$.
  - φ is of moderate growth.
  - For each $g \in \mathrm{GL}_2(\mathbb{A}_f)$

$$Y * \varphi(g, -) = 0,$$

    where $Y \in \mathfrak{g}_\mathbb{C} = \mathfrak{gl}_2(\mathbb{R})_\mathbb{C}$ is the element constructed before.
- For $k \geq 1$ we will see that there exists a unique, up to isomorphism, irreducible $(\mathfrak{g}_\mathbb{C}, \mathrm{O}_2(\mathbb{R}))$-module D_{k-1} containing a non-zero element v, such that $Y * v = 0$, and $\mathbb{C}^\times$ acts on v via the character $z \mapsto z^{-k}$, cf. [Del73, Section 2.1].
- Using D_{k-1} one can rewrite our result as saying that

$$M_k \cong \mathrm{Hom}_{(\mathfrak{g}_\mathbb{C}, \mathrm{O}_2(\mathbb{R}))}(D_{k-1}, C^\infty_{\mathrm{mg}}(\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}))),$$

  where $C^\infty_{\mathrm{mg}}(\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A})) \subseteq C^\infty(\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}))$ denotes the subset of functions with moderate growth.

A condition for cuspidality:

- Let $\varphi = \varphi_f \in C^\infty(\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}))$ for some $f \in M_k$.
- Then $f \in M_k$ if and only if

$$\varphi_B(g) := \int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})} \varphi(ng)dn = 0$$

  for all $g \in \mathrm{GL}_2(\mathbb{A})$, where

$$N = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$$

  is the unipotent radical in the standard Borel

$$B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

  and dn an $N(\mathbb{A})$-invariant measure on $N(\mathbb{Q})\backslash N(\mathbb{A})$.
- Namely:
  - We may assume that f, and thus $\varphi = \varphi_f$, is invariant under some principal congruence subgroup $K(m) \subseteq \mathrm{GL}_2(\mathbb{A}_f)$.

- Consider $g = 1$. Then using $N(\mathbb{Q})\backslash N(\mathbb{A}) \cong (m\widehat{\mathbb{Z}} \times \mathbb{R})/m\mathbb{Z}$:

$$\begin{aligned}
& \int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})} \varphi_f(n)dn \\
= & \int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})} j(n_\infty, i)^{-k} f(n, n_\infty i) d(n, n_\infty) \\
= & \int\limits_{\mathbb{R}/m\mathbb{Z}} \int\limits_{m\widehat{\mathbb{Z}}} f(a, i+x) da dx \\
= & \mu(m\widehat{\mathbb{Z}}) \int\limits_{\mathbb{R}/m\mathbb{Z}} f(1, i+x)dx \\
= & \mu(m\widehat{\mathbb{Z}}) a_0
\end{aligned}$$

  with $\mu(m\widehat{\mathbb{Z}})$ the volume of $m\widehat{\mathbb{Z}}$ and

$$f(1, z) = \sum_{n=0}^{\infty} a_n q^{n/m}$$

  the Fourier expansion of the modular form $f(1, -)$ for $\Gamma(m)$ at the cusp ∞.
- Indeed: The crucial point is that

$$\int\limits_{\mathbb{R}/m\mathbb{Z}} e^{2\pi i \frac{n}{m} x} dx = 0$$

  if $n > 0$.
- The condition $\varphi_B(g) = 0$ for *all* $g \in \mathrm{GL}_2(\mathbb{A})$ is then equivalent to the vanishing of the constant Fourier coefficients at *all* cusps.
- We leave the details as an exercise, cf. [Del73, Rappel 1.3.2.].

• Note: Because φ is $\mathrm{GL}_2(\mathbb{Q})$-invariant for left translations, we can replace B by any conjugate gBg^{-1} with $g \in \mathrm{GL}_2(\mathbb{Q})$.

Questions?

5. From modular forms to automorphic representations, part III

Last time:

- Let $\varphi \in C^\infty(\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}))$.
- Then $\varphi = \varphi_f$ for some $f \in M_k$ if and only if
  - $\varphi(g, g_\infty z) = z^{-k}\varphi(g, g_\infty)$ for $(g, g_\infty) \in \mathrm{GL}_2(\mathbb{A})$, $z \in \mathbb{C}^\times$.
  - φ is of moderate growth.
  - For each $g \in \mathrm{GL}_2(\mathbb{A}_f)$
  $$Y * \varphi(g, -) = 0,$$
  where $Y \in \mathfrak{g}_\mathbb{C} = \mathfrak{gl}_2(\mathbb{R})_\mathbb{C}$ is a suitably constructed, natural element.
- $f \in M_k$ is a cusp form if and only if for all $B \subseteq \mathrm{GL}_{2,\mathbb{Q}}$ proper parabolic, i.e., Borel, with unipotent radical $N \subseteq \mathrm{GL}_{2,\mathbb{Q}}$:
  $$\int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})} \varphi_f(ng)dn = 0$$
  for all $g \in \mathrm{GL}_2(\mathbb{A})$.

Today:

- Describe S_k as a $\mathrm{GL}_2(\mathbb{A}_f)$-representation.
- This will yield our main source of examples for automorphic representations.

Observation:

- M_k and S_k are *smooth* and *admissible* $\mathrm{GL}_2(\mathbb{A}_f)$-representations in the following sense.

Definition 5.1. *Let G be a locally profinite group (like $\mathrm{GL}_2(\mathbb{A}_f)$). A representation G on a $\mathbb{C}$-vector space V is called smooth if*
$$V = \bigcup_{K \subseteq G} V^K,$$
where K runs through the compact-open subgroups of G and
$$V^K := \{v \in V \mid kv = v \text{ for all } k \in K\}.$$
Equivalently, V is smooth if the action morphism
$$G \times V \to V$$
is continuous with V carrying the discrete topology. Denote by
$$\mathrm{Rep}^\infty_\mathbb{C} G$$
the (abelian) category of smooth representations of G on $\mathbb{C}$-vector spaces.

Definition 5.2. *Let G be a locally profinite group. A smooth representation V of G is called admissible if*
$$\dim_\mathbb{C} V^K < \infty$$
for all $K \subseteq G$ compact-open.

Examples:

- $M_k = \bigcup\limits_{K \subseteq \mathrm{GL}_2(\mathbb{A}_f)} M_k^K$ with $M_k^K = M_k(K)$ is smooth and admissible because the $M_k(K)$ are finite dimensional.

- Similarly: S_k is an admissible representation of $\mathrm{GL}_2(\mathbb{A}_f)$.
- For G reductive over $\mathbb{Q} \Rightarrow C^\infty(G(\mathbb{Q})\backslash G(\mathbb{A}))$ is a smooth $G(\mathbb{A}_f)$-representation, but not admissible.
- $L^2([G])$ is not a smooth $G(\mathbb{A}_f)$-representation.

The Hecke algebra (of a locally profinite group):

- G a locally profinite group.
  - e.g., $\mathrm{GL}_2(\mathbb{A}_f)$, $\mathrm{GL}_n(\mathbb{Q}_p)$, $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \ldots$
- For $K \subseteq G$ compact-open, exists a unique (left resp. right invariant) Haar measure μ on G with $\mu(K) = 1$.
- Concretely:
  - By translation invariance sufficient to define μ for $K' \subseteq G$ compact-open subgroup.
  - $K'' := K' \cap K$ compact-open $\Rightarrow$ of finite index in K resp. K'.
  - Set
$$\mu(K') := \frac{[K' : K'']}{[K : K'']}.$$
  - This works and is also the unique possibility.
  - Note that μ takes actually values in $\mathbb{Q}$.
- Fix a left invariant Haar measure μ on G.
- Define the "Hecke algebra"
$$\mathcal{H}(G) := C_c^\infty(G)$$
of G as the set of locally constant, compactly supported functions $G \to \mathbb{C}$.
- $\mathcal{H}(G)$ is an algebra via convolution:
$$f * g(x) := \int\limits_G f(xy)g(y^{-1})dy$$
for $f, g \in \mathcal{H}(G)$. Check associativity!
- The multiplication is a purely algebraic operation:
  - Each $f \in \mathcal{H}(G)$ is a finite sum
$$f = \sum_{i=1}^m a_i \chi_{g_i K_i}$$
for some $g_i \in G$, $K_i \subseteq G$ compact-open subgroups.
  - Wlog: $K_1 = \ldots = K_m$ (by shrinking).
  - Let $K \subseteq G$ be compact-open and $g_1, g_2 \in G$. Then
$$g_1 K g_2 K = \coprod_{j=1}^m h_j K$$
for some $h_j \in G$ and
$$\chi_{g_1 K} * \chi_{g_2 K} = \sum_{j=1}^m \mu(K) \chi_{h_j K}.$$
Check!
- $\mathcal{H}(G)$ has a lot of idempotents:
$$e_K := \frac{1}{\mu(K)} \chi_K$$

with $K \subseteq G$ compact-open subgroups, χ_K the characteristic function of K, but in general no identity. Check!

Modules for the Hecke algebra (cf. [BRa]**):**

- $\mathcal{H}(G)$ is a replacement for the group algebra $\mathbb{C}[G]$, which is better suited to study *smooth* representations of G.
- If $V \in \mathrm{Rep}^{\infty}_{\mathbb{C}} G$, then
$$f * v := \int\limits_G f(h) hv dh$$
for $f \in \mathcal{H}(G)$, $v \in V$, defines an action of $\mathcal{H}(G)$ on V.
- Concretely: If $K \subseteq G$ is compact-open and sufficiently small, such that $v \in V$ is fixed by K and $f = \sum\limits_{i=1}^{m} a_i \chi_{g_i K}$ with $a_i \in \mathbb{C}, g_i \in G$, then
$$f * v = \sum_{i=1}^{m} a_i g_i v.$$
- Assume $K \subseteq G$ compact-open. Define the Hecke algebra relative to K
$$\mathcal{H}(G, K) := e_K \mathcal{H}(G) e_K.$$
Then $\mathcal{H}(G, K)$ is an algebra with unit e_K.
- Let V be any smooth representation of G. Then
$$V^K = e_K * V$$
for any $K \subseteq G$ compact-open. Check!
- In particular, $\mathcal{H}(G, K) \cong C_c^{\infty}(K \backslash G / K)$ (K-biinvariant, compactly supported functions on G)

Important property:

- If $V \in \mathrm{Rep}^{\infty}_{\mathbb{C}} G$ is irreducible and $V^K \neq 0$, then V^K is irreducible as an $\mathcal{H}(G, K)$-module.
  - Indeed: If $M \subseteq V^K$ is a non-trivial $\mathcal{H}(G, K)$-submodule, then
  $$V = \mathcal{H}(G) * M$$
  by irreducibility and thus
  $$V^K = e_K(\mathcal{H}(G) * M)) = e_K \mathcal{H}(G) e_K M = M.$$
- Conversely, if V^K is irreducible or zero for all $K \subseteq G$ compact-open, then V is irreducible if $V \neq 0$. Check!

From irreducible, admissible $\mathrm{GL}_n(\mathbb{A}_f)$-representations to Hecke eigensystems:

- V irreducible, admissible $\mathrm{GL}_n(\mathbb{A}_f)$-representation
- Recall
$$K(m) = \ker(\mathrm{GL}_n(\widehat{\mathbb{Z}}) \to \mathrm{GL}_n(\mathbb{Z}/m)), \ m \geq 0,$$
form basis of compact-open neighborhoods of the identity.
- We know:
  - $V^{K(m)}$ is an irreducible $\mathcal{H}(\mathrm{GL}_n(\mathbb{A}_f), K(m))$-module for $m \gg 0$.
  - Admissibility + Schur's lemma $\Rightarrow \mathrm{End}_{\mathcal{H}(\mathrm{GL}_n(\mathbb{A}_f), K(m))}(V^{K(m)}) \cong \mathbb{C}$.

- Set
$$S := \bigcap_{m\geq 0,\ V^{K(m)}\neq 0} \{p \text{ prime dividing } m\},$$
$$\widehat{\mathbb{Z}}^S := \prod_{p\notin S} \mathbb{Z}_p$$
and
$$\mathbb{A}_f^S := \mathbb{Q} \otimes \widehat{\mathbb{Z}}^S$$
the adéles away from S.
- Note: For $p \notin S$ there exists $0 \neq v \in V$ fixed by $\mathrm{GL}_n(\mathbb{Z}_p) \subseteq \mathrm{GL}_n(\mathbb{A}_f)$.
- Define
$$\mathbb{T}^S := \mathcal{H}(\mathrm{GL}_n(\mathbb{A}_f^S), \mathrm{GL}_n(\widehat{\mathbb{Z}}^S)),$$
the "Hecke-algebra away from S"
- Very important:
$$\mathbb{T}^S \text{ is commutative!}$$
- Namely, consider
$$\sigma\colon \mathrm{GL}_n(\mathbb{A}_f^S) \to \mathrm{GL}_n(\mathbb{A}_f^S),\ g \mapsto g^{\mathrm{tr}}.$$
Then:
  - σ preserves $\mathrm{GL}_n(\widehat{\mathbb{Z}}^S)$.
  - $\sigma(f_1 * f_2) = \sigma(f_2) * \sigma(f_1)$ for $f_1, f_2 \in \mathbb{T}^S$.
  - The cosets $\mathrm{GL}_n(\widehat{\mathbb{Z}}^S) g \mathrm{GL}_n(\widehat{\mathbb{Z}}^S)$ with $g \in \mathrm{GL}_n(\mathbb{A}_f)$ a diagonal matrix span $\mathbb{T}^S$ (by the elementary divisor theorem).
  - Conclusion: $\sigma(f) = f$ for all $f \in \mathbb{T}^S$ and thus
$$f_1 * f_2 = \sigma(f_1 * f_2) = \sigma(f_2) * \sigma(f_1) = f_2 * f_1$$
as desired.
- This argument is called Gelfand's trick.
- In fact, $\mathbb{T}^S$ is central in $\mathcal{H}(\mathrm{GL}_n(\mathbb{A}_f), K(m))$ if all prime divisors of m lie in S.
  - If S_1, S_2 are two, possibly infinite, but disjoint sets of primes and $g_i \in \mathrm{GL}_n(\mathbb{A}_f^{S_i})$, then
$$K(m)g_1K(m)g_2K(m) = K(m)g_2K(m)g_1K(m),$$
which implies the claim.
- Claim: V defines a natural morphism
$$\sigma_V\colon \mathbb{T}^S \to \mathbb{C}$$
of $\mathbb{C}$-algebras.
- Indeed:
  - For $p \notin S$ choose $0 \neq v \in V$ fixed by $\mathrm{GL}_n(\mathbb{Z}_p)$. Then V is fixed by $K(m)$ for some $m \geq 0$ with $p \nmid m$.
  - $V^{K(m)}$ is irreducible, and thus $\mathcal{H}(\mathrm{GL}_n(\mathbb{Q}_p), \mathrm{GL}_n(\mathbb{Z}_p))$ acts via a homomorphism
$$\sigma_{V,p}\colon \mathcal{H}(\mathrm{GL}_n(\mathbb{Q}_p), \mathrm{GL}_n(\mathbb{Z}_p)) \to \mathrm{End}_{\mathcal{H}(\mathrm{GL}_n(\mathbb{A}_f),K(m))}(V^{K(m)}) \cong \mathbb{C},$$
which is independent of v.

- The $\sigma_{V,p}$ combine to σ_V.

- We have not been very careful about Haar measures, or how the $\sigma_{V,p}$ combine to σ_V.

Definition 5.3. *A Hecke eigensystem, or system of Hecke eigenvalues, is a maximal ideal in $\mathbb{T}^S$ where S is a finite set of primes and $\mathbb{T}^S := \mathcal{H}(\mathrm{GL}_n(\mathbb{A}_f^S), \mathrm{GL}_n(\widehat{\mathbb{Z}}^S))$.*

Observations:

- $\mathbb{T}^S$ is of countable dimension and thus each Hecke eigensystem has residue field $\mathbb{C}$.
- If $S \subseteq S'$ are two finite sets of primes, each Hecke eigensystem for S induces one for S'. $\Rightarrow$ We call these Hecke eigensystems equivalent.
- Each irreducible, admissible $\mathrm{GL}_n(\mathbb{A}_f)$-representation yields by the above construction a system of Hecke eigenvalues: $V \mapsto \mathfrak{m}_V := \ker(\sigma_V)$.
- A Hecke eigensystem for S is equivalently a collection of $\mathbb{C}$-algebra homomorphisms
$$\sigma_p \colon \mathcal{H}(\mathrm{GL}_n(\mathbb{Q}_p), \mathrm{GL}_n(\mathbb{Z}_p)) \to \mathbb{C}$$
for $p \notin S$.

A full description of $\mathcal{H}(\mathrm{GL}_2(\mathbb{Q}_p), \mathrm{GL}_2(\mathbb{Z}_p))$:

- Set $G = \mathrm{GL}_2(\mathbb{Q}_p)$, $K = \mathrm{GL}_2(\mathbb{Z}_p)$.
- Normalize Haar measure such that $\mu(K) = 1$.
- Elementary divisor theorem: $\mathcal{H}(G, K)$ is free on the elements
$$K \begin{pmatrix} p^i & 0 \\ 0 & p^j \end{pmatrix} K$$
with $(i, j) \in \mathbb{Z}^{2,+} := \{(i, j) \in \mathbb{Z}^2 \mid i \geq j\}$.
- Claim:
$$f \colon \mathbb{C}[X_1, X_2^{\pm 1}] \xrightarrow{\simeq} \mathcal{H}(G, K)$$
via
$$\begin{aligned} X_1 &\mapsto K \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} K, \\ X_2 &\mapsto K \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} K. \end{aligned}$$
- Here we identify a double coset with its characteristic function.
- Indeed:
  - The element
  $$K \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} K$$
  is central in $\mathcal{H}(G, K)$ as
  $$KzK \cdot KgK = KgK \cdot KzK$$
  for all $g \in G$, and $z \in G$ in the center.
  - Moreover, let us check that each
  $$K \begin{pmatrix} p^i & 0 \\ 0 & 1 \end{pmatrix} K$$
  with $i \geq 1$ lies in the image of f.

- Using induction we get that the n-fold product
$$K\begin{pmatrix}p & 0\\ 0 & 1\end{pmatrix}K\cdot\ldots\cdot K\begin{pmatrix}p & 0\\ 0 & 1\end{pmatrix}K$$
agrees with the set of matrices in $\mathrm{Mat}_{2,2}(\mathbb{Z}_p)$ of determinant of valuation n.
- Thus, the product
$$K\begin{pmatrix}p & 0\\ 0 & 1\end{pmatrix}K\cdot\ldots\cdot K\begin{pmatrix}p & 0\\ 0 & 1\end{pmatrix}K$$
is the characteristic function of this set.
- As this characterstic function has the characteristic function of
$$K\begin{pmatrix}p^n & 0\\ 0 & 1\end{pmatrix}K$$
as a summand with coefficient 1 we are done.
- In particular, we see that $\mathcal{H}(G,K)$ must be commutative without using Gelfand's trick.
- Now it is not difficult to conclude that f must be an isomorphism. Indeed,
$$\mathrm{Spec}(\mathcal{H}(G,K))\subseteq\mathrm{Spec}(\mathbb{C}[X_1,X_2^{\pm1}])$$
defines a subscheme whose projection to $\mathrm{Spec}(\mathbb{C}[X_2^{\pm1}])$ is free of infinite rank, but this is only possible for $\mathrm{Spec}(\mathbb{C}[X_1,X_2^{\pm1}])$ itself.

Upshot:

- Each irreducible, admissible $\mathrm{GL}_2(\mathbb{A}_f)$-representation V yields canonically a system of Hecke eigenvalues
$$\tilde{a}_p\in\mathbb{C},\ \tilde{b}_p\in\mathbb{C}^\times$$
for all primes p outside a finite set of primes S, which is naturally determined by V.
- Concretely: If $v\in\mathrm{GL}_2(\mathbb{A}_f)$ is fixed by $K(m)$ and $p\nmid m$, then
$$\tilde{T}_p(v):=\mathrm{GL}_2(\mathbb{Z}_p)\begin{pmatrix}p & 0\\ 0 & 1\end{pmatrix}\mathrm{GL}_2(\mathbb{Z}_p)*v=\tilde{a}_pv,$$
"the Hecke operator at p", and
$$\tilde{S}_p(v):=\mathrm{GL}_2(\mathbb{Z}_p)\begin{pmatrix}p & 0\\ 0 & p\end{pmatrix}\mathrm{GL}_2(\mathbb{Z}_p)*v=\tilde{b}_pv.$$
- Let $Z\subseteq\mathrm{GL}_2$ be the center, i.e., the scalar matrices.
- Let $\psi\colon\mathbb{A}_f^\times\cong Z(\mathbb{A}_f)\to\mathbb{C}^\times$ be the central character of V (exists by Schur's lemma). Then
$$\tilde{b}_p=\psi((1,\ldots,1,p,1,\ldots,1)$$
for p prime such that $\mathbb{Z}_p^\times\subseteq\mathbb{A}_f^\times$ acts trivially on V.

General results on the $\mathrm{GL}_2(\mathbb{A}_f)$-representation S_k:

- The $\mathrm{GL}_2(\mathbb{A}_f)$-representation S_k decomposes into a direct sum

$$S_k \cong \bigoplus_{i \in I} V_i$$

of irreducible, admissible representations V_i.

- Indeed:
  - $S_k \overset{\Phi}{\subseteq} L^2([\mathrm{GL}_2])$ and thus the $\mathbb{C}$-vector space S_k is equipped with the non-degenerate $\mathrm{GL}_2(\mathbb{A}_f)$-equivariant hermitian pairing

$$(\cdot,\cdot)\colon S_k \times S_k \to \mathbb{C},\ (f,g) \mapsto \int\limits_{[\mathrm{GL}_2]} \overline{\Phi(f)}\Phi(g)$$

  - Each irreducible $\mathrm{GL}_2(\mathbb{A}_f)$-subquotient of S_k is already a direct summand (use orthogonal complements).
  - Using Zorn's lemma each smooth non-zero $\mathrm{GL}_2(\mathbb{A}_f)$-representation V has an irreducible subquotient (Wlog: V generated by some $v \in V$. Then apply Zorn's lemma to the set of submodules not containing v.)
- Shalika/Piatetski-Shapiro: For each finite set S of primes and each systems of Hecke eigenvalues $\{\tilde{a}_p, \tilde{b}_p\}_{p \notin S}$ there exists at most one irreducible subrepresentation $V \subseteq S_k$ with this system of Hecke eigenvalues.
  - This is a special case of the "strong multiplicty one theorem" for cuspidal automorphic representations for GL_n, cf. [GH19, Theorem 11.7.2] (maybe we'll prove this later for $n = 2$).
- Consequence: Assume that $0 \neq f \in S_k$ is fixed by $K(m)$, and an eigenvector for the $\tilde{T}_p, \tilde{S}_p-$operators for $p \nmid m$. Then $V := \langle f \rangle_{\mathrm{GL}_2(\mathbb{A}_f)}$ is irreducible.
  - Reason: Each $v \in V$ yields a system of Hecke eigenvalues, which is equivalent to the one for f. Now apply strong multiplicity one.

The decomposition of S_k indexed by newforms:

- Atkin-Lehner/Casselman: For each irreducible $\mathrm{GL}_2(\mathbb{A}_f)$-subrepresentation $V \subseteq S_k$, there exists some $m \in \mathbb{N}$ and some $0 \neq v \in V$ such that v is fixed by the subgroup

$$K_1(m) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid c \equiv 0, d \equiv 1 \bmod m\} \subseteq \mathrm{GL}_2(\widehat{\mathbb{Z}})$$

- Set

$$\Gamma_1(m) := \mathrm{GL}_2(\mathbb{Z}) \cap K_1(m).$$

- Remarks:
  - Our notation is different than in many sources, where $\Gamma_1(m) \subseteq \mathrm{SL}_2(\mathbb{Z})$.
  - The minimal m for which there exists such a non-zero $v \in V$ is called the conductor of V. In this case, v is unique up to a scalar.
  - Maybe more details later, cf. [Del73].
- The morphism

$$\begin{array}{ccc} \Gamma_1(m)\backslash\mathbb{H}^{\pm} & \xrightarrow{\simeq} & \mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f)/K_1(m) \times \mathbb{H}^{\pm}) \\ [z] & \mapsto & [(1,z)] \end{array}$$

is an isomorphism, compatible with $\omega^{\otimes k}$.

- In particular,
$$M_k(\Gamma_1(m)) \cong M_k(K_1(m)).$$
- Let $\tilde{f} \in M_k(K_1(m))$.
- Let $f \in M_k(\Gamma_1(m))$ be the section corresponding to $\tilde{f}$.
- Then
$$\tilde{f}(g, z) = \tilde{f}(\gamma, z) = j(\gamma^{-1}, z)^{-k} f(\gamma^{-1} z).$$
where $g = \gamma h$ with $\gamma \in \mathrm{GL}_2(\mathbb{Q}), h \in K_1(m)$. Check!
- Fix p prime, $p \nmid m$.
- $\tilde{T}_p, \tilde{S}_p$ act on $M_k(K_1(m))$. Thus, by transport of structure on $M_k(\Gamma_1(m))$.
- Let us make $\tilde{S}_p$ on $M_k(\Gamma_1(m))$ explicit.
- We have to find an expression
$$\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} = \gamma h, \text{ with } \gamma \in \mathrm{GL}_2(\mathbb{Q}), \quad h \in K_1(m).$$
- Because p, m are prime there exists $a, b \in \mathbb{Z}$, such that
$$A := \begin{pmatrix} a & b \\ m & p \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}).$$
- Then
$$A_{\mathrm{diag}} \cdot \begin{pmatrix} p^{-1} & 0 \\ 0 & p^{-1} \end{pmatrix}_{\mathrm{diag}} \cdot \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}_p \in K_1(m),$$
where $(-)_{\mathrm{diag}}$ denotes the diagonal embedding for $\mathrm{GL}_2(\mathbb{Q})$ and $(-)_p$ the embedding at p.
- In other words, we can take
$$\gamma := \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} A^{-1}$$
- Thus $\tilde{S}_p$ acts on $M_k(\Gamma_1(m))$ via
$$f \mapsto (z \mapsto j(\gamma^{-1}, z)^{-k} f(\gamma^{-1} z)).$$
- The group
$$\Gamma_0(m) := \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid c \equiv 1 \bmod m \}$$
normalizes $\Gamma_1(m)$ and thus there is an action of $\Gamma_0(m)/\Gamma_1(m) \cong (\mathbb{Z}/m)^\times$ via
$$(B, f) \mapsto j(B, z)^{-k} f(Bz).$$
for $B \in \Gamma_0(m), f \in M_k(\Gamma_1(m))$.
- Thus, we obtain a decomposition
$$M_k(\Gamma_1(m)) = \bigoplus_{\chi \colon (\mathbb{Z}/m)^\times \to \mathbb{C}^\times} M_k(\Gamma_0(m), \chi)$$
with $M_k(\Gamma_0(m), \chi)$ the space of modular forms for $\Gamma_0(m)$ with nebentypus χ, i.e., modular forms for $\Gamma_1(m)$ satisfying
$$f(\frac{az+b}{cz+d})) = \chi(d)(cz+d)^k f(z) \quad \text{ for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(m).$$

- For each $\chi\colon (\mathbb{Z}/m)^\times \to \mathbb{C}^\times$ there exists a unique character
$$\psi_\chi\colon \mathbb{Q}^\times\backslash\mathbb{A}^\times/(1+m\widehat{\mathbb{Z}})\cap\widehat{\mathbb{Z}}^\times \to \mathbb{C}^\times$$
such that $\psi_\chi(r) = r^{-k}$ for $r \in \mathbb{R}_{>0}$.
- We get that
$$M_k(\Gamma_0(m),\chi) \cong M_k(K_1(m),\psi_\chi)$$
where the RHS denotes the ψ_χ-eigenspace for the action of $\mathbb{A}^\times$ on $M_k(K_1(m))$.
- Let us finish the description of $\tilde{S}_p$ on $M_k(\Gamma_1(M))$.
- The action of $\tilde{S}_p$ is given on $M_k(\Gamma_0(m),\chi) \subseteq M_k(\Gamma_1(m))$ by
$$\gamma^{-1} = \begin{pmatrix} p^{-1} & 0 \\ 0 & p^{-1}\end{pmatrix} A$$
where $\begin{pmatrix} p^{-1} & 0 \\ 0 & p^{-1}\end{pmatrix}$ acts by p^k and $A \in \Gamma_0(m)$ by $\chi(p)$.
- Let us describe now the action (by transport of structure) of $\tilde{T}_p$ on $M_k(\Gamma_0(m),\chi)$.
- We have
$$\mathrm{GL}_2(\mathbb{Z}_p)\begin{pmatrix} p & 0 \\ 0 & 1\end{pmatrix}\mathrm{GL}_2(\mathbb{Z}_p) = \coprod_{j=0}^{p-1}\begin{pmatrix} p & j \\ 0 & 1\end{pmatrix}\mathrm{GL}_2(\mathbb{Z}_p)\coprod\begin{pmatrix} 1 & 0 \\ 0 & p\end{pmatrix}\mathrm{GL}_2(\mathbb{Z}_p).$$
- We again have to find factorizations into
$$\gamma\cdot h,\quad \gamma\in\mathrm{GL}_2(\mathbb{Q}),\quad h\in K_1(m).$$
- We can take the factorization
$$\begin{pmatrix} p & j \\ 0 & 1\end{pmatrix}_p = \begin{pmatrix} p & j \\ 0 & 1\end{pmatrix}_{\mathrm{diag}}\cdot h$$
for $j = 0,\dots,p-1$, with $h \in K_1(m)$, i.e.,
$$\gamma^{-1} = \begin{pmatrix} p^{-1} & -p^{-1}j \\ 0 & 1\end{pmatrix}.$$
- For
$$\begin{pmatrix} 1 & 0 \\ 0 & p\end{pmatrix}$$
we find, similar to the argument above,
$$\gamma^{-1} = \begin{pmatrix} a & b \\ m & p\end{pmatrix}\cdot\begin{pmatrix} 1 & 0 \\ 0 & p^{-1}\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & p^{-1}\end{pmatrix}\cdot\begin{pmatrix} a & p^{-1}b \\ pm & p\end{pmatrix}.$$
- Putting these together we obtain
$$\tilde{T}_p(f) = p^k\chi(p)f(pz) + \sum_{j=0}^{p-1} f(\frac{z-j}{p}).$$
- If $f \in M_k(\Gamma_0(m),\chi)$ is written in Fourier expansion
$$f(z) = \sum_{n=0}^{\infty} a_n q^n,\quad q = e^{2\pi i z},$$
then
$$\tilde{T}_p(f)(z) = \sum_{n=0}^{\infty} p a_{np} q^n + \sum_{n=0}^{\infty} p^k\chi(p)a_n q^{pn}.$$

- This follows from

$$\sum_{j=0}^{p-1} e^{2\pi i \frac{-jn}{p}} = \begin{cases} p, & p|n \\ 0, & \text{otherwise.} \end{cases}$$

Check!

- In particular, if $f \in M_k(\Gamma_0(m), \chi)$ is an eigenvector for $\tilde{T}_p$ and $a_1 = 1$, then

$$T_p(f) = a_p f,$$

where

$$T_p = \frac{1}{p}\tilde{T}_p.$$

In other words, the Hecke eigenvalues are the Fourier coefficients of the normalized eigenform.

Theorem 5.4. *The* $\mathrm{GL}_2(\mathbb{A}_f)$*-representation* S_k *decomposes into irreducibles as*

$$S_k \cong \bigoplus_f \langle f \rangle_{\mathrm{GL}_2(\mathbb{A}_f)}$$

with f *running through the set of (normalized) newforms for* $\Gamma_0(N)$ *with nebentypus* χ *for varying* N *and Dirichlet characters* $\chi\colon (\mathbb{Z}/N)^\times \to \mathbb{C}^\times$.

Newforms (roughly):

- A newform is an eigenform realising the minimal N among all eigenforms with an equivalent system of Hecke eigenvalues.

Proof. After the above preparation we can now quote [DS05, Proposition 5.8.4], [DS05, Theorem 5.8.2] which imply that for each system of Hecke eigenvalues appearing in S_k there exists a unique normalized (i.e., in Fourier expansion $a_1 = 1$) newform with equivalent system of Hecke eigenvalues. □

Questions?

6. Langlands reciprocity for newforms

Last time:

- We proved the decomposition
$$S_k \cong \bigoplus_f \langle f \rangle_{\mathrm{GL}_2(\mathbb{A}_f)}$$
into irreducibles.
- Here f is running through the set of (normalized) newforms of weight k for $\Gamma_0(N)$ with nebentypus χ for varying N and varying Dirichlet characters
$$\chi\colon (\mathbb{Z}/N)^\times \to \mathbb{C}^\times.$$
- To each irreducible, admissible $\mathrm{GL}_2(\mathbb{A}_f)$-representation π we associated a system of Hecke eigenvalues
$$\{\tilde{a}_p(\pi), \tilde{b}_p(\pi)\}_{p \notin S}$$
for $S := \{p \text{ prime with } \pi^{\mathrm{GL}_n(\mathbb{Z}_p)} = 0\}$. In fact, $\tilde{a}_p(\pi)$ resp. $\tilde{b}_p(\pi)$ is the eigenvalue of the double coset
$$\mathrm{GL}_2(\mathbb{Z}_p)\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\mathrm{GL}_2(\mathbb{Z}_p)$$
resp.
$$\mathrm{GL}_2(\mathbb{Z}_p)\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}\mathrm{GL}_2(\mathbb{Z}_p)$$
when acting on a non-zero vector fixed by $\mathrm{GL}_2(\mathbb{Z}_p)$.
- If $\pi = \langle f \rangle_{\mathrm{GL}_2(\mathbb{A}_f)}$ for a normalized newform $f \in S_k(\Gamma_0(N), \chi)$, then $S = \{\text{prime divisors of } N\}$ and
$$\tilde{a}_p(\pi) = pa_p, \quad \tilde{b}_p(\pi) = p^k\chi(p)$$
if $p \notin S$, where $f(q) = \sum\limits_{n=1}^{\infty} a_n q^n$ is the Fourier expansion of f.

Today:

- Traces of Frobenii for ℓ-adic Galois representations.
- Statement of Langlands program for S_k.

Traces of Frobenii for ℓ-adic representations:

- ℓ some prime.
- $\overline{\mathbb{Q}}_\ell = \varinjlim\limits_{E/\mathbb{Q}_\ell} E$ is a topological field via colimit topology.
- Let W be a finite dimensional $\overline{\mathbb{Q}}_\ell$-vector space.
- Let $\sigma\colon G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}(W)$ be a continuous representation (an "ℓ-adic Galois representation")
- Recall: For each prime p there exists an embedding
$$G_{\mathbb{Q}_p} := \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to G_{\mathbb{Q}},$$
well-defined up to conjugacy in $G_{\mathbb{Q}}$.
- $G_{\mathbb{Q}_p}$ sits in the exact sequence:
$$1 \to I_p \to G_{\mathbb{Q}_p} \to \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) = (\mathrm{Frob}_p^{\mathrm{geom}})^{\widehat{\mathbb{Z}}} \to 1$$

with I_p the inertia subgroup, and

$$\mathrm{Frob}_p^{\mathrm{geom}}\colon \overline{\mathbb{F}}_p \to \overline{\mathbb{F}}_p,\ x \mapsto x^{1/p}$$

the *geometric* Frobenius at p (= inverse of usual Frobenius).

- Let S be a finite set of primes. We say σ is unramified outside S if $\sigma(I_p) = 1$ for all $p \notin S$.
- If σ unramified outside S and $n := \dim_{\overline{\mathbb{Q}}_\ell} W$, we can associate to σ a system

$$\{c_{1,p}, \ldots, c_{n,p}\}_{p \notin S}$$

of elements $c_{1,p}, \ldots, c_{n,p} \in \overline{\mathbb{Q}}_\ell$. Namely,

$$c_{i,p} := \mathrm{Tr}(\sigma(\mathrm{Frob}_p^{\mathrm{geom}})|\Lambda^i W)$$

- Note: The $c_{i,p}$ are well-defined because $\sigma(I_p) = 1$.
- Note: The $c_{i,p} = \mathrm{Tr}(\sigma(\mathrm{Frob}_p^{\mathrm{geom}})|\Lambda^i W)$ depend only on p as the trace is invariant under conjugation.
- If W is semisimple, then σ is determined by the

$$\{c_{1,p}, \ldots, c_{n,p}\}_{p \notin S}$$

for each finite set of primes S such that W is unramified outside S, cf. [Ser97, I-10].
- In fact, the collection $\{c_{1,p}\}_{p \notin S}$ is sufficient (+ W semisimple and unramified outside S).
- Indeed:
  - Consider $\Lambda := \overline{\mathbb{Q}}_\ell[G_{\mathbb{Q},S}]$, where $G_{\mathbb{Q},S}$ is the quotient of $G_{\mathbb{Q}}$ by the closure of the subgroup generated by the $I_p, p \notin S$.
  - Assume W, W' are two semisimple, continuous $G_{\mathbb{Q},S}$-representations with

$$\mathrm{Tr}(\mathrm{Frob}_p^{\mathrm{geom}}|W) = \mathrm{Tr}(\mathrm{Frob}_p^{\mathrm{geom}}|W')$$

    for all $p \notin S$.
  - By Chebotarev density and continuity this implies

$$\mathrm{Tr}(\lambda|W) = \mathrm{Tr}(\lambda|W'),$$

    for all $\lambda \in \Lambda$.
  - For irreducible, pairwise non-isomorphic Λ-modules $W_1, \ldots, W_m$ and each $i = 1, \ldots, m$, there exists $\mu_i \in \Lambda$, such that

$$\mu_i = 1$$

    on W_i, but

$$\mu_i = 0$$

    on $W_j, j \neq i$ (this is a version of the Chinese remainder theorem).
  - Write

$$W = \bigoplus_i^m W_i^{\oplus n_i}, \quad W' = \bigoplus_{i=1}^m W_i^{\oplus n_i'}$$

    in isotypic components with $W_1, \ldots, W_m$ irreducible, pairwise non-isomorphic, and $n_i, n_i' \in \mathbb{N}$, possibly zero.

- Then

$$n_i \dim_{\overline{\mathbb{Q}}_\ell} W_i = \mathrm{Tr}(\mu_i|W) = \mathrm{Tr}(\mu_i|W') = n_i' \dim_{\overline{\mathbb{Q}}_\ell} W_i$$

for all $i = 1, \ldots, m$, i.e., $n_i = n_i', i = 1, \ldots, m$ and $W \cong W'$ as desired.

- This statement can be seen as an analog of "strong multiplicity one" for cuspidal automorphic representations for GL_n mentioned last time.

Arithmetic significance of the traces of Frobenii:

- Let $\sigma \colon G_{\mathbb{Q}} \to \mathrm{GL}(W)$ be an ℓ-adic representation, unramified outside S.
- The traces

$$c_{1,p} := \mathrm{Tr}(\sigma(\mathrm{Frob}_p^{\mathrm{geom}})|W)$$

encode significant arithmetic information.

- For example, assume $F/\mathbb{Q}$ is Galois with $\mathrm{Gal}(F/\mathbb{Q}) \cong S_3$ and

$$\sigma \colon G_{\mathbb{Q}} \twoheadrightarrow \mathrm{Gal}(F/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$$

is inflated from the unique irreducible, 2-dimensional representation of S_3. Set

$$S := \{p \text{ ramified in } F\}$$

and for $p \notin S$

$$\mathrm{Frob}_{F,p}^{\mathrm{geom}}$$

as the image of $\mathrm{Frob}_p^{\mathrm{geom}} \in G_{\mathbb{Q},S}$ in $\mathrm{Gal}(F/\mathbb{Q}) \cong S_3$.

- Then:
  * $\mathrm{Tr}(\mathrm{Frob}_p^{\mathrm{geom}}|W) = 2 \Leftrightarrow \mathrm{Frob}_{F,p}^{\mathrm{geom}} \in \{1\} \Leftrightarrow p$ splits completely in $\mathcal{O}_F$,
  * $\mathrm{Tr}(\mathrm{Frob}_p^{\mathrm{geom}}|W) = 0 \Leftrightarrow \mathrm{Frob}_{F,p}^{\mathrm{geom}} \in \{(1,2),(1,3),(2,3)\} \Leftrightarrow p$ splits into three distinct primes in $\mathcal{O}_F$,
  * $\mathrm{Tr}(\mathrm{Frob}_p^{\mathrm{geom}}|W) = -1 \Leftrightarrow \mathrm{Frob}_{F,p}^{\mathrm{geom}} \in \{(1,2,3),(1,3,2)\} \Leftrightarrow p$ splits into two distinct primes in $\mathcal{O}_F$.
- Thus: Knowledge of traces of Frobenii for all ℓ-adic Galois representations (with finite image) implies knowledge of the decomposition of unramified primes in *all* finite extensions of $\mathbb{Q}$.
- Other examples of ℓ-adic representations:

$$H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell)$$

for proper, smooth schemes X over $\mathbb{Q}$, and $i \geq 0$.

- The $G_{\mathbb{Q}}$-action is induced by functoriality of $H^i_{\text{ét}}(-, \overline{\mathbb{Q}}_\ell)$ from the (right) action of $G_{\mathbb{Q}}$ on

$$X_{\overline{\mathbb{Q}}} = X \times_{\mathrm{Spec}(\mathbb{Q})} \mathrm{Spec}(\overline{\mathbb{Q}}).$$

- Proper, smooth base change + Grothendieck-Lefschetz trace formula $\Rightarrow$ If X has good reduction at p and $p \neq \ell$, then

$$\sum_{i=0}^{2\dim(X)} (-1)^i \mathrm{Tr}(\mathrm{Frob}_p^{\mathrm{geom}}|H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell)) = \sharp \mathcal{X}(\mathbb{F}_p),$$

where $\mathcal{X} \to \mathrm{Spec}(\mathbb{Z}_{(p)})$ is a proper, smooth model of X at p.

- The last example motivates why we took the geometric Frobenius and not the arithmetic Frobenius $\mathrm{Frob}_p^{\mathrm{arith}} = (\mathrm{Frob}_p^{\mathrm{geom}})^{-1}$.

Questions?

The whole point of Langlands reciprocity:

- $\boxed{\text{Traces of Frobenii are supposed to match Hecke eigenvalues!}}$
- Thus, arithmetic information is expected to be encoded in automorphic information, which might be more concrete.
- Example (Hecke):
  - Let F be the splitting field of $x^3 - x - 1$.
  - Then F is the Hilbert class field of $\mathbb{Q}(\sqrt{-23})$, and Galois over $\mathbb{Q}$ with Galois group S_3.
  - Consider as before the associated 2-dimensional, irreducible Galois representation $\sigma\colon G_{\mathbb{Q}} \to \mathrm{Gal}(F/\mathbb{Q}) \hookrightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$.
  - Hecke proved that there exists a normalized newform $f(q) = \sum\limits_{i=1}^{\infty} a_n q^n$ such that
    $$a_p = \mathrm{Tr}(\sigma(\mathrm{Frob}_p^{\mathrm{geom}}))$$
    for all primes p with $p \nmid 23$.
  - One can make f more explicit: Considering ramification, f must lie in $S_1(\Gamma_0(23), \chi)$ with $\chi\colon (\mathbb{Z}/23)^\times \to \mathbb{C}^\times$ the quadratic character determining
    $$\mathbb{Q}(\sqrt{-23}) \subseteq \mathbb{Q}(\zeta_{23}),$$
    and people knowing modular forms will tell us that we must have
    $$f(q) = q \prod_{i=1}^{\infty} (1 - q^i)(1 - q^{23i}).$$
  - We can now calculate the developement of f (or look it up in some database like LMFDB):
    $$f(q) = q - q^2 - q^3 + \ldots + q^{58} + 2q^{59} + \ldots$$
    and deduce how primes decompose in F, e.g., 59 splits completely!
  - This example was posted by Matthew Emerton on mathoverflow. More insightful posts by him can be found via his webpage: `http://www.math.uchicago.edu/~emerton/` .

Find and read them!

- Fermat's theorem was proved by Wiles/Taylor using the following strategy (initiated by Frey, and complemented by Serre, Ribet):
  - Assume $u^p + v^p + w^p = 0$ with $u, v, w \in \mathbb{Q}$, $uvw \neq 0$ and $p \geq 3$.
  - Consider the elliptic curve E over $\mathbb{Q}$ with (affine) Weierstraß equation
    $$y^2 = x(x + u^p)(x - v^p)$$
    (after possibly manipulating u, v, w a bit).
  - Consider the Galois representation
    $$\sigma\colon G_{\mathbb{Q}} \to \mathrm{GL}(H^1_{\text{ét}}(E_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell)) \cong \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell).$$
  - Show that σ is modular, i.e., there exists a normalized newform $f_1(q) = \sum\limits_{n=1}^{\infty} a_n q^n$ of weight 2 such that
    $$\mathrm{Tr}(\sigma(\mathrm{Frob}^{\mathrm{geom}})) = a_p$$
    for almost all primes p.

- Using a theorem of Ribet, one concludes that f_1 must be congruent to a normalized newform f_2 in

$$S_2(\Gamma_0(2)),$$

  i.e., the Fourier coefficients of f_1, f_2 are algebraic integers and congruent modulo some prime.
- But $S_2(\Gamma_0(2)) = 0$, and thus f_2 cannot exist.
- This yields the desired contradiction.
- For more details, cf. [Wil95], [Rib90].

Langlands reciprocity for newforms:

- Fix a prime ℓ and an isomorphism

$$\iota\colon \overline{\mathbb{Q}}_\ell \cong \mathbb{C}.$$

- For newforms the Langlands program combined with the Fontaine-Mazur conjecture predicts a bijection

$$\mathrm{LL}\colon \mathcal{A}_{\mathrm{mod}} \xrightarrow{\simeq} \mathcal{G}_{\mathrm{mod}}$$

  from the set

$$\mathcal{A}_{\mathrm{mod}} := \{\text{irreducible } \mathrm{GL}_2(\mathbb{A}_f)\text{-subrepresentations } \pi \subseteq \bigoplus_{k\geq 1} S_k\},$$

  to a certain set

$$\mathcal{G}_{\mathrm{mod}}$$

  consisting of irreducible, 2-dimensional Galois representations $\sigma\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$
- LL should satisfy that for all $\pi \in \mathcal{A}_{\mathrm{mod}}$ with $\sigma := \mathrm{LL}(\pi)$ we have

$$\iota(\mathrm{Tr}(\sigma(\mathrm{Frob}_p^{\mathrm{arith}}))) = a_p(\pi), \quad \iota(\det(\sigma(\mathrm{Frob}_p^{\mathrm{arith}}))) = b_p(\pi)$$

  for p outside some specified finite set S of primes.

  In a previous version I considered the geometric Frobenius, which was wrong.
- Here, the eigenvalues $a_p(\pi)$ resp. $b_p(\pi)$ are used and not the $\tilde{a}_p(\pi), \tilde{b}_p(\pi)$.
- Note: Such a bijection LL is uniquely determined (by the respective multiplicity one theorems), if it exists.
- How is S specified? Let us call a prime p unramified for σ resp. π if

$$\sigma(I_p) = 1$$

  resp.

$$\pi^{\mathrm{GL}_2(\mathbb{Z}_p)} \neq 0.$$

- Then a prime $p \neq \ell$ is conjecturally unramified for π if and only if it is unramified for $\sigma := \mathrm{LL}(\pi)$.
- The matching of Hecke eigenvalues states then more precisely that $\sigma(\mathrm{Frob}_p^{\mathrm{arith}})$ should have characteristic polynomial

$$X^2 - \iota^{-1}(a_p(\pi))X + \iota^{-1}(b_p(\pi))$$

  for each unramified prime $p \neq \ell$ for π.
- It is the next aim of the course to indicate how the map LL can be constructed, i.e., how to associate Galois representations to normalized newforms. For this we follow results of Deligne, cf. [Del71b], and Deligne/Serre, cf. [DS74].

The set $\mathcal{G}_{\mathrm{mod}}$:

- The set $\mathcal{G}_{\mathrm{mod}}$ was defined by Fontaine/Mazur in relation with their remarkable conjecture on geometric Galois representations, cf. [FM95].
- Let $\sigma\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ be an irreducible, 2-dimensional ℓ-adic representation.
- Then by definition $\sigma \in \mathcal{G}_{\mathrm{mod}}$ if and only if
  - σ unramified at almost all p, i.e., $\sigma(I_p) = 1$ for p outside some finite set of primes S.
  - σ is odd, i.e., for each complex conjugation $c \in \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \subseteq G_{\mathbb{Q}}$ we have
$$\det(\sigma(c)) = -1.$$
  - The restriction
$$\sigma_\ell := \sigma_{|G_{\mathbb{Q}_\ell}}\colon G_{\mathbb{Q}_\ell} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$$
is *de Rham* with Hodge-Tate weights $0, \omega$ for some $\omega \in \mathbb{N}$ (note that $\omega = 0$ is allowed).
- If $\sigma = \mathrm{LL}(\pi)$ with $\pi \subseteq S_k$, then conjecturally $\omega = k - 1$.
- Using the work of many people (Wiles, Kisin, Emerton, Skinner-Wiles, Pan,...) any $\sigma \in \mathcal{G}_{\mathrm{mod}}$ with distinct Hodge-Tate weights is modular if $\ell \geq 5$, cf. [Pan19, Theorem 1.0.4.].
- If $\sigma \in \mathcal{G}_{\mathrm{mod}}$ has finite image, then σ is modular, cf. [PS16], [GH19, Section 13.4.] (and the references therein).
- Conjecturally: HT-weights 0 $\Leftrightarrow$ image of σ finite.

de Rham representations:

- p prime (the previous ℓ).
- $K/\mathbb{Q}_p$ a discretely valued, non-archimedean extension (e.g., $K/\mathbb{Q}_p$ finite).
- $\sigma\colon G_K := \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_m(\overline{\mathbb{Q}}_p)$ a continuous representation.
- Then σ has coefficients in some finite extension $E/\mathbb{Q}_p$ (by compactness of G_K as $\overline{\mathbb{Q}}_p$ carries the colimit topology here).
- We obtain a $\mathbb{Q}_p$-linear representation, a "local p-adic Galois representation",
$$\rho\colon G_K \to \mathrm{GL}_m(E) \subseteq \mathrm{GL}_n(\mathbb{Q}_p),$$
where $n := m \cdot \dim_{\mathbb{Q}_p} E$.
- Upshot: Let V be a finite dimensional $\mathbb{Q}_p$-vector space and $\rho\colon G_K \to \mathrm{GL}(V)$ a continuous representation.
- Then V is called de Rham if
$$\dim_K (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K} = \dim_{\mathbb{Q}_p} V$$
for a certain field extension B_{dR} of K with G_K-action, cf. [BC09, Section 6].
- Namely: B_{dR} is Fontaine's field of p-adic periods, cf. [Fon94], [BC09, Definition 4.4.7]. Abstractly,
$$B_{\mathrm{dR}} \cong \mathbb{C}_K((t)),$$
where $\mathbb{C}_K$ is the completion of $\overline{K}$ for its p-adic valuation (this topology is coarser than the colimit topology on $\overline{K}$).

- B_{dR} is a discretely valued field, with residue field $\mathbb{C}_K$, the deduced decreasing filtration $\mathrm{Fil}^\bullet B_{\mathrm{dR}}$ is G_K-stable and as $\mathbb{C}_K$-semilinear G_K-representations

$$\mathrm{gr}^\bullet B_{\mathrm{dR}} \cong \bigoplus_{j\in\mathbb{Z}} \mathbb{C}_K(j),$$

where $\mathbb{C}_K(j) := \mathbb{C}_K \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(j)$, with

$$\mathbb{Z}_p(1) := T_p\mu_{p^\infty}(\mathbb{C}_K)$$

the p-adic Galois representation associated to the cyclotomic character

$$\chi_{\mathrm{cyc}}\colon G_K \to \mathbb{Z}_p^\times \cong \mathrm{Aut}(\mu_{p^\infty}(\mathbb{C}_K)).$$

- Important Theorem of Tate, cf. [BC09, Theorem 2.2.7]: Let $\chi\colon G_K \to \mathbb{Q}_p^\times$ be a character. Then

$$H^0(G_K, \mathbb{C}_K(\chi)) = \begin{cases} K, & \text{if } \chi_{|I_K} \text{ has finite image} \\ 0, & \text{otherwise} \end{cases},$$

here $I_K \subseteq G_K$ is the inertia subgroup.
- In particular:

$$H^0(G_K, \mathbb{C}_K(j)) = 0, \quad j \neq 0$$

as $\chi^j_{\mathrm{cyc}|I_K}$ has infinite image if $j \neq 0$.
- Conclusion: If $\dim_{\mathbb{C}} V = 1$, then V is de Rham if and only if V is isomorphic to $\chi \cdot \chi^j_{\mathrm{cyc}}$ for some character $\chi\colon G_K \to \mathbb{Q}_p^\times$ with I_K of finite order, and $j \in \mathbb{Z}$

$\Rightarrow$ There exists many representations, which are not de Rham: $(\chi^{p-1}_{\mathrm{cyc}})^a$ with $a \in \mathbb{Z}_p \setminus \mathbb{Z}$.
- Facts:
  - $\rho\colon G_K \to \mathrm{GL}(V)$ is de Rham if $\rho_{|G_{K'}}$ is de Rham for some finite extension K' of K, cf. [BC09, Proposition 6.3.8].
  - de Rham representations are stable under subquotients, duals and tensor products, cf. [BC09, Section 6.1].
- Let V be a de Rham representation of dimension n. Then

$$V \otimes_{\mathbb{Q}_p} \mathbb{C}_K \cong \mathbb{C}_K(j_1) \oplus \ldots \oplus \mathbb{C}_K(j_n)$$

as $\mathbb{C}_K$-semilinear G_K-representations, where $j_1, \ldots, j_n \in \mathbb{Z}$. The unordered collection $(j_1, \ldots, j_n)$ is called the collection of Hodge-Tate weights for V.

Strategy for constructing LL (roughly):

- Let $\pi \in \mathcal{A}_{\mathrm{mod}}$, with associated system of Hecke eigenvalues

$$\{a_p, b_p\}_{p\notin S}.$$

- Assume for simplicity that $\pi \subseteq S_2$, i.e., associated to a weight 2 newform.
- Then we will construct a $\mathrm{GL}_2(\mathbb{A}_f)$-equivariant embedding

$$S_2 \hookrightarrow H^1(\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^\pm), \mathbb{C}) := \varinjlim_K H^1(\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f)/K \times \mathbb{H}^\pm), \mathbb{C})$$

of S_2 into cohomology.
- The given isomorphism $\iota\colon \overline{\mathbb{Q}}_\ell \cong \mathbb{C}$ yields an isomorphism

$$H^1(\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^\pm), \mathbb{C}) \cong H^1(\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^\pm), \overline{\mathbb{Q}}_\ell)$$

- For $K \subseteq \mathrm{GL}_2(\mathbb{A}_f)$ compact-open, the complex manifold
$$\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f)/K \times \mathbb{H}^{\pm})$$
is actually *algebraic*, i.e., given by $\tilde{X}_K(\mathbb{C})$ for some quasi-projective scheme $\tilde{X}_K \to \mathrm{Spec}(\mathbb{C})$.
- The étale comparison theorem implies
$$H^1(\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^{\pm}), \overline{\mathbb{Q}}_\ell) \cong H^1_{\text{ét}}(\tilde{X}_K, \overline{\mathbb{Q}}_\ell).$$
- Now the miracle happens: For $K \subseteq \mathrm{GL}_2(\mathbb{A}_f)$
$\boxed{\text{the quasi-projective scheme } \tilde{X}_K \to \mathrm{Spec}(\mathbb{C}) \text{ is canonically defined over } \mathrm{Spec}(\mathbb{Q})!}$
In other words, $\tilde{X}_K \cong X_K \times_{\mathrm{Spec}(\mathbb{Q})} \mathrm{Spec}(\mathbb{C})$ for (compatible) quasi-projective schemes $X_K \to \mathrm{Spec}(\mathbb{Q})$.
- In particular,
$$H^1_{\text{ét}}(\tilde{X}_K, \overline{\mathbb{Q}}_\ell) \cong H^1_{\text{ét}}(X_{K,\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell)$$
by invariance of étale cohomology under change of algebraically closed base fields. But the RHS carries an action of $G_{\mathbb{Q}}$!
- Look now at the $G_{\mathbb{Q}} \times \mathrm{GL}_2(\mathbb{A}_f)$-module
$$H^1_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell) := \varinjlim_K H^1_{\text{ét}}(X_{K,\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell).$$
- Then we will check that the $G_{\mathbb{Q}}$-module
$$LL(\pi) := \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{A}_f)}(\pi, H^1_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell))$$
is 2-dimensional and that we can express the traces of Frobenii by Hecke eigenvalues.
- Replacing $\overline{\mathbb{Q}}_\ell$ by some local system a similar strategy works for weight $k \geq 2$.
- However, this does not help for $k = 1$. Here one has to use that a weight 1 modular form can be congruent to some modular form of weight $k \geq 2$, and use the previous case.

Questions?

7. Modular curves as moduli of elliptic curves (by Ben Heuer)

Last time:

- First rough sketch of construction of

$$\mathrm{LL}\colon \mathcal{A}_{\mathrm{mod}} \xrightarrow{\simeq} \mathcal{G}_{\mathrm{mod}}$$

$$\mathcal{A}_{\mathrm{mod}} := \{\text{irreducible } \mathrm{GL}_2(\mathbb{A}_f)\text{-subrepresentations } \pi \subseteq \bigoplus_{k\geq 1} S_k\},$$
$$\mathcal{G}_{\mathrm{mod}} := \{\sigma : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell),\ \text{unramified outside finite set } S\text{, odd, de Rham}\}$$

- Crucial point for construction: The complex manifold

$$\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f)/K \times \mathbb{H}^{\pm})$$

  (1) is algebraic over $\mathrm{Spec}(\mathbb{C})$,
  (2) admits canonical model X_K over $\mathrm{Spec}(\mathbb{Q})$.
  This gives rise to the Galois action on l-adic étale cohomology $H^1_{\text{ét}}(X_{K,\bar{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell)$.

Today:

- Explain why the algebraic model X_K exists.
- Reinterpret notions defined before in algebro-geometric terms: modular forms, compactification, q-expansions, adelic Hecke action, ...

Reminder on elliptic curves Reference: [Silverman: The Arithmetic...]

- Let K be any field. We have the following equivalent definitions:

  Definition 7.1. *An **elliptic curve** over K is a*
  – *connected smooth projective curve $E|K$ of genus 1 with point $O \in E(K)$.*
  – *connected smooth projective algebraic group $E|K$ of dimension 1.*
  – *non-singular plane cubic curve, defined by a Weierstraß equation*

$$E : y^2 = x^3 + ax + b, \quad a, b \in K$$

  (if $\mathrm{char} K \notin \{2, 3\}$, otherwise more terms are required).

- Fact: The group scheme is automatically commutative (!).
- Fact: The non-singularity can be expressed as a condition on the discriminant:

$$E \text{ non-singular} \quad \Leftrightarrow \quad \Delta(a, b) = -16(4a^3 + 27b^2) \neq 0.$$

- Fact: Given E, the Weierstraß equation is not unique. But the **j-invariant**

$$j(E) := j(a, b) = 1728\frac{4a^3}{4a^3 + 27b^2}$$

  is independent of choice of Weierstraß equation.
- Fact: If K is algebraically closed, we have

$$E \cong E' \Leftrightarrow j(E) = j(E').$$

- More generally, we can replace $\mathrm{Spec}(K)$ by any base scheme S.

  Definition 7.2. *An elliptic curve over S is a proper smooth curve $E \to S$ with geometrically connected fibres of genus 1, together with a point $0 : S \to E$.*

- This can be thought of as family of elliptic curves parametrised by S.

- Again, this automatically has a group structure:

Theorem (Abel)**.** *There is a unique way to endow an elliptic curve $E \to S$ with the structure of a commutative S-group scheme s.t. $0 =$ identity section.*

- Zariski-locally on S, one can describe E in terms of a Weierstraß equation.

Definition 7.3. *A morphism of elliptic curves $E \to E'$ over S is a homomorphism of S-group schemes. An* ***isogeny*** *is an fppf-surjective homomorphism with finite flat kernel.*

Example 7.4. For any $N \in \mathbb{Z}$, the group scheme structure gives the morphism "multiplication by N", which is denoted by $[N] : E \to E$.

- Fact: $[N]$ is finite flat of rank N^2. It is finite étale if N is invertible on S.
- In particular, $[N]$ is an isogeny.

Definition 7.5. *$E[N] := \ker[N] \subseteq E$, the N-**torsion subgroup**. This is a finite flat closed subgroup scheme. It is finite étale if N is invertible on S.*

- More generally, for any finite flat subgroup scheme $D \subseteq E[N]$ of exponent N, $\exists$ unique elliptic curve E/D with isogeny $\varphi : E \to E/D$ with kernel D.
- $[N]$ induces a **dual isogeny** $\varphi^\vee : E/D \to E$ with kernel $E[N]/D$.

Elliptic curves over $K = \mathbb{C}$ Reference: [Diamond–Shurman: A first course...]

- Let $E|\mathbb{C}$ be an elliptic curve. Then $E(\mathbb{C})$ has the structure of a compact complex Lie group. As such, it can be canonically uniformised:
- Let $\mathrm{Lie}E$ be the tangent space at $0 \in E(\mathbb{C})$. Then $\mathrm{Lie}E \cong \mathbb{C}$ and there is an isomorphism of complex Lie groups

$$E(\mathbb{C}) = \mathrm{Lie}E/H_1(E, \mathbb{Z}) \cong \mathbb{C}/\Lambda, \quad \text{where } \Lambda \cong \mathbb{Z}^2.$$

- For any $N \in \mathbb{Z}$, we have

$$E(\mathbb{C})[N] = (\tfrac{1}{N}\Lambda)/\Lambda \subseteq \mathbb{C}/\Lambda.$$

- Note: $E(\mathbb{C})[N] = (\frac{1}{N}\mathbb{Z}/\mathbb{Z})^2 = (\mathbb{Z}/N)^2$ is of rank N^2, as claimed.
- Upshot: $E(\mathbb{C})$ is a **1-dimensional complex torus**: A quotient of $\mathbb{C}$ by a lattice $\mathbb{Z}^2 \cong \Lambda \subseteq \mathbb{C}$ such that $\Lambda \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{C}$.
- morphisms of complex tori := morphisms of complex Lie groups

Theorem. *There is an equivalence of categories*

$$\{\textit{elliptic curves over } \mathbb{C}\} \to \{1\textit{-dim complex tori}\}$$

- Essential surjectivity: Let $\Lambda \subseteq \mathbb{C}$ as above. Define the Weierstraß $\wp$-function

$$\wp(z, \Lambda) = \frac{1}{z^2} + \sum_{\substack{w \in \Lambda \\ w \neq 0}} \left(\frac{1}{z^2} - \frac{1}{(z-w)^2} \right)$$

- Then $\wp(z) = \wp(z, \Lambda)$ is holomorphic on $\mathbb{C}\backslash\Lambda$ with complex derivative

$$\wp'(z) = -2 \sum_{w \in \Lambda} \frac{1}{(z-w)^3}.$$

- This is clearly Λ-periodic $\rightsquigarrow$ $\wp$ and $\wp'$ are meromorphic functions on $\mathbb{C}/\Lambda$.

Theorem. *The functions $\wp$ and $\wp'$ are related by a Weierstraß equation*

$$E_\Lambda : \wp'^2 = 4(\wp^3 + g_2(\Lambda)\wp + g_3(\Lambda))$$

for explicit $g_2(\Lambda), g_3(\Lambda) \in \mathbb{C}$. We thus get an isomorphism of Lie groups

$$(\wp, \wp') : \mathbb{C}/\Lambda \xrightarrow{\sim} E_\Lambda(\mathbb{C}).$$

- Sending $\mathbb{C}/\Lambda \mapsto E_\Lambda$ defines a quasi-inverse

$$\{\text{1-dim complex tori}\} \to \{\text{elliptic curves over } \mathbb{C}\}.$$

Complex moduli spaces of elliptic curves

- We can now parametrise complex elliptic curves up to isomorphism
- For this, we just have to determine when two lattices Λ_1, Λ_2 satisfy

$$\mathbb{C}/\Lambda_1 \cong \mathbb{C}/\Lambda_2.$$

- Fact: Any homomorphism of complex tori is multiplication by $a \in \mathbb{C}$.
- $\Rightarrow$ this happens iff $\Lambda_1 = a\Lambda_2$ for some $a \in \mathbb{C}^\times$.
- Can now uniformise as follows: Write $\Lambda = w_1\mathbb{Z} \oplus w_2\mathbb{Z}$.
- The condition $\Lambda \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{C}$ implies $w_1/w_2 \in \mathbb{C} \setminus \mathbb{R} = \mathbb{H}^\pm$.
- $\Rightarrow$ For every complex torus E, there is $\tau \in \mathbb{H}^\pm$ such that

$$E \cong E_\tau := \mathbb{C}/\Lambda_\tau, \quad \Lambda_\tau = \mathbb{Z} \oplus \tau\mathbb{Z}$$

- Finally, for $\tau, \tau' \in \mathbb{H}^\pm$, we have

$$\mathbb{Z} + \tau\mathbb{Z} = \mathbb{Z} + \tau'\mathbb{Z} \Leftrightarrow \exists \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z}) : \gamma\tau = \frac{a\tau + b}{c\tau + d} = \tau'$$

- This shows: $\tau \mapsto E_\tau$ defines

$$\mathrm{GL}_2(\mathbb{Z})\backslash\mathbb{H}^\pm \cong \{\text{complex tori}\}/\sim \cong \{\text{elliptic curves over } \mathbb{C}\}/\sim$$

This gives our space from earlier,

$$\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H} = \mathrm{GL}_2(\mathbb{Z})\backslash\mathbb{H}^\pm = \mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f)/K \times \mathbb{H}^\pm) \text{ for } K = \mathrm{GL}_2(\hat{\mathbb{Z}})$$

an interpretation as a "moduli space" (so far, in a weak sense):

$$\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H} = \{\text{elliptic curves over } \mathbb{C}\}/\sim$$

- $\mathbb{C}$ algebraically closed $\Rightarrow$ j-invariant defines bijection (in fact biholomorphism)

$$j : \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H} \xrightarrow{\sim} \mathbb{C}.$$

- Via the covering map $\mathbb{H}^\pm \to \mathrm{GL}_2(\mathbb{Z})\backslash\mathbb{H}^\pm$, we get from the above:

$$\begin{aligned} \mathbb{H}^\pm =& \{\text{complex tori with ordered basis } \alpha : \mathbb{Z}^2 \xrightarrow{\sim} \Lambda\}/\sim \\ =& \{\text{elliptic curves } E \text{ over } \mathbb{C} \text{ with } \alpha : \mathbb{Z}^2 \xrightarrow{\sim} H_1(E, \mathbb{Z})\}/\sim \end{aligned}$$

- What about moduli interpretations of other levels? Recall for $N \in \mathbb{N}$, we had the level $K = K(N)$,

$$X_K := \mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f)/K \times \mathbb{H}^\pm) = \coprod \Gamma(N)\backslash\mathbb{H}^\pm$$

where

$$\Gamma(N) = \{\gamma \in \mathrm{GL}_2(\mathbb{Z}) | \gamma \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \bmod N\}.$$

- These are precisely the automorphisms of Λ that preserve the subgroup

$$E_\Lambda[N] = \tfrac{1}{N}\Lambda/\Lambda \subseteq \mathbb{C}/\Lambda.$$

Consequently, we have the more general statement

$$\begin{aligned}\Gamma(N)\backslash\mathbb{H}^\pm =& \{\text{complex tori with ordered basis } \alpha : (\mathbb{Z}/N)^2 \xrightarrow{\sim} \tfrac{1}{N}\Lambda/\Lambda\}/\sim \\ =& \{\text{elliptic curves } E \text{ over } \mathbb{C} \text{ with } \alpha : (\mathbb{Z}/N)^2 \xrightarrow{\sim} E(\mathbb{C})[N]\}/\sim .\end{aligned}$$

- In between $\mathrm{GL}_2(\mathbb{Z})$ and $\Gamma(N)$, we had the groups

$$\Gamma_1(N) = \{\gamma \in \mathrm{GL}_2(\mathbb{Z}) | \gamma \equiv \left(\begin{smallmatrix} * & * \\ 0 & 1\end{smallmatrix}\right) \bmod N\},$$

$$\Gamma_0(N) = \{\gamma \in \mathrm{GL}_2(\mathbb{Z}) | \gamma \equiv \left(\begin{smallmatrix} * & * \\ 0 & *\end{smallmatrix}\right) \bmod N\}.$$

- The modular curves associated to these have moduli interpretations

$$\begin{aligned}\Gamma_1(N)\backslash\mathbb{H}^\pm =& \{\text{elliptic curves } E|\mathbb{C} \text{ with point } Q \in E(\mathbb{C})[N] \text{ of exact order } N\}/\sim \\ \Gamma_0(N)\backslash\mathbb{H}^\pm =& \{\text{elliptic curves } E|\mathbb{C} \text{ with cyclic subgroup } C \subseteq E(\mathbb{C})[N] \text{ of rank } N\}/\sim .\end{aligned}$$

Motivating Example: The Legendre family

- Consider the complex manifold $U = \mathbb{P}^1\backslash\{0, 1, \infty\}$ in the variable λ.
- Over U, we have a complex family of elliptic curves:
- The **Legendre family** is cut out of $\mathbb{P}^2_U \to U$ by the Weierstraß equation

$$\mathbb{E}_\lambda : Y^2 = X(X-1)(X-\lambda).$$

Proposition 7.6. (1) *We have* $\mathbb{E}_\lambda[2] = \{O, (0,0), (1,0), (\lambda, 0)\}$*. In particular, there is a natural isomorphism* $\mathbb{E}_\lambda[2] = (\underline{\mathbb{Z}/2\mathbb{Z}})^2$*.*
(2) *For any elliptic curve E over $\mathbb{C}$ together with a trivialisation*

$$\alpha : (\underline{\mathbb{Z}/2\mathbb{Z}})^2 \xrightarrow{\sim} E[2],$$

there is a unique point $x \in U$ such that $E = (\mathbb{E}_\lambda)_x$.

- **Upshot:** $\mathbb{E}_\lambda \to U$ is **universal elliptic curve** with level $\Gamma(2)$, in some sense.
- This is all algebraic! Can make sense of $\mathbb{E}_\lambda \to U$ as morphism of schemes $\mathbb{C}$.
- Even better: These schemes are already defined over $\mathbb{Q}$.
- Can elaborate this argument to show that

$$\Gamma(2)\backslash\mathbb{H}^\pm =: X_{\Gamma(2)}(\mathbb{C}) = \mathbb{P}^1\backslash\{0, 1, \infty\}.$$

- NB: In particular, the compactification is $X^*_{\Gamma(2)} = \mathbb{P}^1$.
- This shows that $S_2(\Gamma_0(2)) = 0$, which we used for Fermat's Last Theorem.

Idea:

> Get algebraic model of $\Gamma(N)\backslash\mathbb{H}^\pm$ over $\mathbb{Q}$ by passing from $\mathbb{C}$ to moduli of elliptic curves over general $\mathbb{Q}$-schemes S.

- **Question:** Is there a scheme representing the functor

$$\mathcal{P}_{\mathrm{SL}_2(\mathbb{Z})} : S \mapsto \{\text{elliptic curves over } S\}/\sim$$

on schemes over $\mathbb{Q}$? This would be a scheme $\mathcal{X} \to \mathbb{Q}$ with $\mathbb{C}$-points $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$.
- **Answer:** No. An easy way to see this is the phenomenon of twists:

- Can have two non-isomorphic $E \not\cong E'$ over $\mathbb{Q}$ that become isomorphic over a quadratic extension K. In particular, $j(E) = j'(E)$.
- Clearly, two $\mathbb{Q}$-points of $\mathcal{X}$ agree iff they do over K. So $\mathcal{X}$ cannot exist!
- Same problem for the Legendre family above: This does not represent

$$S \mapsto \{\text{elliptic curves } E \text{ over } S \text{ with } \alpha : \underline{(\mathbb{Z}/2\mathbb{Z})}^2 \xrightarrow{\sim} E[2]\}/\sim$$

  because this fails to account for quadratic twists.
- Fact: Twists of elliptic curves E over a field K correspond to 1-cocycles

$$G_{\overline{K}|K} \to \mathrm{Aut}(E).$$

  Quadratic twists come from the involution $([-1] : E \to E) \in \mathrm{Aut}(E)$.
- **Slogan: Nontrivial automorphisms are bad for representability**
- **Upshot:** In order to get a representable functor, need additional data.

Moduli problems of elliptic curves [Katz–Mazur: Arithmetic moduli...]

- Consider the moduli functors on schemes over $\mathbb{Z}[\frac{1}{N}]$:

$$\mathcal{P}_{\Gamma_0(N)} \colon S \mapsto \{(E|S, C \subseteq E[N] \text{ cyclic subgroup scheme of rank } N)\}/\sim,$$
$$\mathcal{P}_{\Gamma_1(N)} \colon S \mapsto \{(E|S, Q \in E[N](S) \text{ point of exact order } N)/\sim,$$
$$\mathcal{P}_{\Gamma(N)} \colon S \mapsto \{(E|S, \alpha \colon \underline{(\mathbb{Z}/N\mathbb{Z})}^2 \xrightarrow{\sim} E[N] \text{ isom of group schemes})/\sim.$$

- We can also form "mixed moduli problems" like

$$\mathcal{P}_{\Gamma_1(N)\cap\Gamma_0(p)} := \mathcal{P}_{\Gamma_1(N)} \times_{\mathcal{P}_{\mathrm{SL}_2(\mathbb{Z})}} \mathcal{P}_{\Gamma_0(p)} \colon S \mapsto \{(E, Q, C)\}/\sim$$

- Crucial point: Let $(E, C) \in \mathcal{P}_{\Gamma_0(N)}(S)$. The automorphism $[-1] : E \to E$ sends $C \mapsto C$. In particular, $[-1] \in \mathrm{Aut}(E, C)$.
- In contrast, let $(E, Q) \in \mathcal{P}_{\Gamma_1(N)}(S)$, then $[-1] : E \to E$ sends $Q \in E[N]$ to $-Q \in E[N]$, which is **different if $\mathbf{N \geq 3}$**. In particular, $[-1] \notin \mathrm{Aut}(E, Q)$.
- Similarly for $\mathcal{P}_{\Gamma(N)}(S)$. This explains the problem with the Legendre family: $N = 2$ is too small to deal with the automorphism $[-1]$! We need $N \geq 3$.
- **Upshot:** The moduli problems $\mathcal{P}_{\Gamma(N)}$ and $\mathcal{P}_{\Gamma_1(N)}$ are "**rigid**" (:= have no non-trivial automorphisms) for $N \geq 3$, in contrast to $\mathcal{P}_{\Gamma_0(N)}$ and $\mathcal{P}_{\Gamma(2)}$.

Theorem. *Let $N \geq 3$ and p any prime.*

(1) *The moduli problems $\mathcal{P}_{\Gamma(N)}$ and $\mathcal{P}_{\Gamma_1(N)}$ are each representable by smooth affine curves $X_{\Gamma(N)}$ and $X_{\Gamma_1(N)}$ over $\mathbb{Z}[\frac{1}{N}]$.*

(2) *For any $n \in \mathbb{N}$, the moduli problem $\mathcal{P}_{\Gamma(N)\cap\Gamma_0(p^n)}$ is representable by a flat affine curve $X_{\Gamma_1(N)\cap\Gamma_0(p^n)}$ over $\mathbb{Z}[\frac{1}{N}]$ that is smooth over $\mathbb{Z}[\frac{1}{pN}]$.*

- Remark: In order to represent $\mathcal{P}_{\mathrm{SL}_2(\mathbb{Z})}$, one can pass from schemes to the bigger category of *stacks*. Get "moduli stack of elliptic curves" $\mathcal{M}_{1,1}$.
- Between varying level structures $\Gamma \subseteq \Gamma' \subseteq \Gamma(N)$, have forgetful morphisms

$$X_{\Gamma'} \to X_{\Gamma}.$$

  These are all finite étale over $\mathbb{Z}[\frac{1}{N}]$.

- $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ acts from the right on $\mathcal{P}_{\Gamma(N)}$ by precomposition
$$\alpha \mapsto \alpha \circ \gamma.$$
Alternatively, this is a left action via precomposition with $\gamma^\vee = \gamma^{-1}\det\gamma$.
- This induces a $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$-action on $X_{\Gamma(N)}$.
- Similarly, there is a natural $(\mathbb{Z}/N\mathbb{Z})^\times$-action on $X_{\Gamma_1(N)}$.
- We can then *define* a modular curve
$$X_{\Gamma_0(N)} := X_{\Gamma_1(N)}/(\mathbb{Z}/N\mathbb{Z})^\times.$$
- This does not represent $\mathcal{P}(\Gamma_0(N))$, but it is "as close as possible".
- In particular, $X_{\Gamma_0(N)}(K) = \mathcal{P}(\Gamma_0(N))(K)$ for algebraically closed K.

Compactification

- The j-invariant associated to Weierstraß equations defines a finite flat function
$$j : X \to \mathbb{A}^1.$$
- By normalisation in $\mathbb{A}^1 \to \mathbb{P}^1$, can define
$$j : X^* \to \mathbb{P}^1,$$
still a finite flat morphism. Think of X^* as X plus a finite divisor of points.
- Fact: X^* is smooth and proper.
- Reason: This can be seen using the Tate curve over $\mathbb{Z}[\frac{1}{N}][[q]]$.
- $X^*(\mathbb{C})$ is the smooth compactification $X^*_{\mathbb{C}}$ of $\Gamma\backslash\mathbb{H}^\pm$ mentioned earlier.
- Remark: X^* is a moduli space of "generalised elliptic curves".

Geometric Modular forms References: [Katz], [Loeffler: lectures notes]

- Fix $N \geq 3$. Let $X = X_{\Gamma_1(N)}$.
- Goal: Geometric reinterpretation of modular forms as sections of sheaves on X:

Definition 7.7. *Let $\omega := e^*\Omega^1_{E|X}$ where $e : X \to E$ is the identity section.*

- Since $E \to X$ is smooth of dimension 1, this is a line bundle.
- Fact: This extends uniquely to a line bundle ω on X^*. Reason: The universal elliptic curve extends to a group scheme $E^* \to X^*$, take $e^*\Omega^1_{E^*|X^*}$.

Proposition 7.8. *The analytification of $\omega^{\otimes k}$ is naturally isomorphic to the sheaf ω^k of modular forms of weight k on $X^*(\mathbb{C})$.*

Sketch of proof. Recall: $\mathbb{H}^\pm$ is moduli space of $E|\mathbb{C}$ with $\alpha : \mathbb{Z}^2 \xrightarrow{\sim} H_1(E,\mathbb{Z})$.
Integration $\int_{\alpha(1,0)}$ defines an isomorphism $e^*\Omega^1_{E|\mathbb{C}} \xrightarrow{\sim} \mathbb{C}$.
Get canonical trivialisation of ω over $\mathbb{H}^\pm$. Check $\mathrm{GL}_2(\mathbb{Z})$-action coincides. □

- In particular, the sheaf ω already has an algebraic model over $\mathbb{Z}[\frac{1}{N}]$.

Definition 7.9. *For any $\mathbb{Z}[\frac{1}{N}]$-algebra R, can define*
$$M_k(\Gamma_1(N), R) = \Gamma(X^*_{\Gamma_1(N)} \times_{\mathrm{Spec}(\mathbb{Z}[1/N])} \mathrm{Spec}(R), \omega^{\otimes k}),$$
*the R-module of **modular forms of weight k with coefficients in R**.*

- Fact: This is a finite free $\mathbb{Z}[\frac{1}{N}]$-module.

- Fact: Using Tate curves, get q-expansions in $R[[q]]$. $\rightsquigarrow$ Can define cusp forms.
- Fact: If R, S are flat $\mathbb{Z}[\frac{1}{N}]$-algebras (e.g. any $\mathbb{Q}$-algebras), then

$$M_k(\Gamma_1(N), R) \otimes_R S = M_k(\Gamma_1(N), S).$$

- **Upshot:** Modular forms are manifestly objects of algebraic geometry!

8. The Eichler–Shimura relation (by Ben Heuer)

Last time:

- There is a $\mathrm{GL}_2(\mathbb{Z})$-equivariant bijection.

$$\begin{array}{ccc} \mathbb{H}^{\pm} & \to & \{\text{elliptic curves } E \text{ over } \mathbb{C} \text{ with } \alpha\colon \mathbb{Z}^2 \xrightarrow{\cong} H_1(E,\mathbb{Z})\}/\sim \\ \tau & \mapsto & (E_\tau := \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau), \alpha\colon \mathbb{Z}^2 \to \mathbb{Z}+\mathbb{Z}\tau,\ (1,0)\mapsto 1, (0,1)\mapsto \tau) \end{array}$$

- Corollary:

$$\mathrm{GL}_2(\mathbb{Z})\backslash(\mathrm{GL}_2(\widehat{\mathbb{Z}}) \times \mathbb{H}^{\pm}) \cong \{\text{elliptic curves } E \text{ over } \mathbb{C} \text{ with } \alpha\colon \widehat{\mathbb{Z}}^2 \xrightarrow{\sim} TE\}/\sim,$$

  where $TE := \varprojlim_{N\in\mathbb{N}} E[N]$ is the full adèlic Tate module of E.
- This can be defined over $\mathbb{Q}$!
- Set $\widehat{X} \to \mathrm{Spec}(\mathbb{Q})$ as the moduli scheme representing the functor

$$\mathbb{Q}\text{−}\mathbf{Sch} \to \mathbf{Sets}, \quad S \mapsto \{\text{elliptic curves } E \text{ over } S \text{ with } \alpha\colon \widehat{\mathbb{Z}}^2 \xrightarrow{\sim} TE\}/\sim.$$

  Here, $TE = \varprojlim_N E[N]$ is an inverse limit of finite, étale group schemes over S
- $-1 \in \mathrm{GL}_2(\widehat{\mathbb{Z}})$ acts trivially on $\widehat{X}$.
- For $K \subseteq \mathrm{GL}_2(\widehat{\mathbb{Z}})$ compact-open with $K \subseteq K_1(3)$ (note in particular, $-1 \notin K$) have smooth quasi-projective curve

$$X_K = \widehat{X}/K.$$

- E.g.: If $K = K(N)$, $N \geq 3$, then

$$X_{K(N)}$$

  parametrizes elliptic curves E together with trivialization $(\mathbb{Z}/N)^2 \xrightarrow{\sim} E[N]$.
- Other examples, $K = K_1(N) \rightsquigarrow \mathbb{Z}/N \hookrightarrow E[N]$, $K = K_0(N) \rightsquigarrow$ subgroups.

Next goal:

- Use modular curves to reinterpret Hecke operators geometrically.
- From geometry of modular curves, deduce fundamental relation between Galois action and Hecke action: The Eichler–Shimura relation.

The adélic action:

- Clearly, $\mathrm{GL}_2(\widehat{\mathbb{Z}})$ acts on $\widehat{X}$, by precomposition on $\alpha\colon \widehat{\mathbb{Z}}^2 \cong TE$.
- But: Want $\mathrm{GL}_2(\mathbb{A}_f)$-action to get Hecke operators

$\Rightarrow$ work with elliptic curves up to quasi-isogeny.
- Recall: $\mathbb{A}_f = \widehat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ and

$$\mathrm{GL}_2(\mathbb{Z})\backslash(\mathrm{GL}_2(\widehat{\mathbb{Z}}) \times \mathbb{H}^{\pm}) \cong \mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^{\pm}).$$

- On elliptic curves can interpret this as follows.
- Recall that the LHS geometrizes to $\widehat{X}$, which represents the functors

$$P : \mathbb{Q}\text{−}\mathbf{Sch} \to \mathbf{Sets}, \quad S \mapsto \{(E|S, \alpha : \widehat{\mathbb{Z}}^2 \xrightarrow{\sim} TE)\}/\text{isomorphism}$$

- Let $VE = TE \otimes_{\mathbb{Z}} \mathbb{Q}$ be the rational adèlic Tate module, and define

$$P' : \mathbb{Q}\text{−}\mathbf{Sch} \to \mathbf{Sets}, \quad S \mapsto \{(E|S, \alpha : \mathbb{A}_f^2 \xrightarrow{\sim} VE)\}/\text{quasi-isogeny}$$

Definition 8.1. *Recall: An isogeny is a surjective homomorphism $E \to E'$ with finite flat kernel. A **quasi-isogeny** is an element of*

$$\{E \to E' \text{ isogeny}\} \otimes_{\mathbb{Z}} \mathbb{Q}.$$

*A p-**quasi-isogeny** is an element of*

$$\{E \to E' \text{ isogeny of degree a power of } p\} \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{p}].$$

- Then $P \cong P'$. Indeed:
  - Clearly have natural transformation $P \to P'$. Sketch for inverse:
  - Take $(E, \alpha : \mathbb{A}_f^2 \xrightarrow{\sim} VE) \in P'(S)$. Multiply by isogeny $N : E \to E$ until α^{-1} restricts to
$$TE \to \widehat{\mathbb{Z}}^2.$$
    Reduce both sides mod $M \in \mathbb{N}$. Let D be the kernel of the induced map
$$E[M] \to (\mathbb{Z}/M)^2,$$
    this stabilises for $M \gg 0$. The dual isogeny $E/D \to E$ induces
$$\alpha : \widehat{\mathbb{Z}}^2 \xrightarrow{\sim} T(E/D).$$
  - This defines unique representative in $P(S)$. Thus $P(S) \twoheadrightarrow P'(S)$.
- This shows that the natural $\mathrm{GL}_2(\hat{\mathbb{Z}})$-action extends to a $\mathrm{GL}_2(\mathbb{A}_f)$-action.
- BThis does not fix the projection to any X_K because it changes E.
- Example: $\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$ acts by sending $E \mapsto E/D$ where $D \subseteq E[p]$ is the subgroup generated by the image of $\alpha(1,0)$ under $T_pE \twoheadrightarrow E[p]$.
- Fact: All of this still works after compactification.

Geometric interpretation of Hecke operators

- Let $V = C^\infty(\mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^\pm)$ be the smooth $\mathrm{GL}_2(\mathbb{A}_f)$-representation of smooth functions
$$f : \mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^\pm \to \mathbb{C},\ (g, z) \mapsto f(g, z),$$
  i.e., f is fixed under some compact-open subgroup in $\mathrm{GL}_2(\mathbb{A}_f)$ and $z \mapsto f(g, z)$ is smooth for any $g \in \mathrm{GL}_2(\mathbb{A}_f)$.
- Fix a prime p.
- Let $\varphi \in \mathcal{H}(\mathrm{GL}_2(\mathbb{Q}_p))$ be any element of the Hecke algebra at p, i.e., a locally constant function $\varphi : \mathrm{GL}_2(\mathbb{Q}_p) \to \mathbb{C}$ with compact support.
- Recall that for $f \in V$ we have
$$\varphi * f(g, z) = \int\limits_{\mathrm{GL}_2(\mathbb{Q}_p)} \varphi(h) f(gh, z) dh,$$
  where we chose the Haar measure on $\mathrm{GL}_2(\mathbb{Q}_p)$ with volume 1 on K_p.
- The integral above can be rewritten as follows:
- Consider the following diagram:
$$\begin{tikzcd} & \mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^\pm \times \mathrm{GL}_2(\mathbb{Q}_p) \arrow[dl, "q_2"'] \arrow[dr, "q_1"] & \\ \mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^\pm & & \mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^\pm, \end{tikzcd}$$

$$q_1(g,z,h) = (g,z),$$
$$q_2(g,z,h) = (gh,z).$$

- Such a diagram is called a **correspondence**.
- We also have the projection to the third factor:

$$\pi\colon \mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^\pm \times \mathrm{GL}_2(\mathbb{Q}_p) \to \mathrm{GL}_2(\mathbb{Q}_p)$$

Lemma 8.2. *For $f : \mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^\pm \to \mathbb{C}$ in V we have*

$$\varphi * f = q_{1,!}(q_2^*(f) \cdot \pi^*(\varphi)),$$

where $(-)^*$ means the pullback of a function and $(-)_!$ integrating along the fiber (which makes sense because φ has compact support).

- *Proof.* The fiber of q_1 over $(g,z) \in \mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^\pm$ is

$$q_1^{-1}(g,z) = \{(g,z,h) |\ h \in \mathrm{GL}_2(\mathbb{Q}_p)\} \cong \mathrm{GL}_2(\mathbb{Q}_p)$$

and $q_2^*(f) \cdot \pi^*(\varphi)$ is the function

$$(g,z,h) \mapsto f(gh,z)\varphi(h).$$

Applying $q_{1,!}$, the integral over this fibre, gives

$$\int\limits_{\mathrm{GL}_2(\mathbb{Q}_p)} \varphi(h) f(gh,z) dh = \varphi * f(g,z).$$

Note that this is well-defined because φ has compact support. □

Passage to finite level

- Let N be coprime to p and let $K := K_1(N) \subseteq \mathrm{GL}_2(\mathbb{A}_f)$. Then $K_p = \mathrm{GL}_2(\mathbb{Z}_p)$.
- Write K^p for the prime-to-p-part of K, i.e. $K = K_p K^p$ and $K^p \cap K_p = 1$.
- Assume that f is fixed under K, i.e., arises by pullback from a function

$$\tilde{f}\colon \mathrm{GL}_2(\mathbb{A}_f)/K \times \mathbb{H}^\pm \to \mathbb{C}.$$

- Assume $\varphi \in \mathcal{H}(\mathrm{GL}_2(\mathbb{Q}_p), K_p)$, i.e., that φ is K_p-biinvariant.
- φ is the pullback of a function with finite support

$$\tilde{\varphi}\colon K_p \backslash \mathrm{GL}_2(\mathbb{Q}_p)/K_p \to \mathbb{C}.$$

- In this case, $\varphi * f$ is again K-invariant, i.e., the pullback of a function

$$\widetilde{\varphi * f}\colon \mathrm{GL}_2(\mathbb{A}_f)/K \times \mathbb{H}^\pm \to \mathbb{C}.$$

- We can change the above diagram (while retaining notation) to:

$$\begin{tikzcd}
& (\mathrm{GL}_2(\mathbb{A}_f)/K^p \times \mathbb{H}^\pm) \times^{K_p} \mathrm{GL}_2(\mathbb{Q}_p)/K \arrow[dl, "q_2"'] \arrow[dr, "q_1"] & \\
\mathrm{GL}_2(\mathbb{A}_f)/K \times \mathbb{H}^\pm & & \mathrm{GL}_2(\mathbb{A}_f)/K \times \mathbb{H}^\pm.
\end{tikzcd}$$

Here $- \times^{K_p} -$ is the contracted product (i.e., the quotient for the action $k \cdot (g,z,h) := (gk^{-1}, z, kh)$) and

$$q_2([g,z,h]) = [gh,z], \quad q_1([g,z,h]) = [g,z],$$

where square brackets indicate equivalence classes.

- We again have a projection
$$\begin{aligned}\pi\colon (\mathrm{GL}_2(\mathbb{A}_f)/K^p \times \mathbb{H}^\pm) \times^{K_p} \mathrm{GL}_2(\mathbb{Q}_p)/K_p \to& K_p\backslash \mathrm{GL}_2(\mathbb{Q}_p)/K_p,\\ [g,z,h] \mapsto& [h].\end{aligned}$$
- Next, mod out by $\mathrm{GL}_2(\mathbb{Q})$ on the left: Recall this gives a complex manifold
$$X_{\Gamma_1(N)}(\mathbb{C}) = \mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f)/K \times \mathbb{H}^\pm)$$
- We then need to replace functions f by sections in $\omega^k,\ k \in \mathbb{Z}$ on this.
- More precisely, we obtain

$$\begin{tikzcd} & \mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f)/K^p \times \mathbb{H}^\pm) \times^{K_p} \mathrm{GL}_2(\mathbb{Q}_p)/K_p \arrow[dl, "q_2"'] \arrow[dr, "q_1"] & \\ X_{\Gamma_1(N)}(\mathbb{C}) & & X_{\Gamma_1(N)}(\mathbb{C}). \end{tikzcd} \tag{1}$$

  and for $f \in H^0(X_{\Gamma_1(N)}(\mathbb{C}), \omega^k)$, i.e., a weakly modular form of level $K_1(N)$,
$$\varphi * f = q_{1,!}(q_2^*(f) \cdot \pi^*(\varphi)).$$
- To make this precise, use: We have a canonical isomorphism
$$q_2^*(\omega^k) \cong q_1^*(\omega^k)$$
  since ω^k is defined via pullback of a $\mathrm{GL}_2(\mathbb{Q})$-equivariant line bundle on $\mathbb{H}^\pm$.
- Thus can regard $q_2^*(f) \cdot \pi^*(\varphi) \in q_2^*(\omega^k)$ as section of $q_1^*(\omega^k)$.
- Sum up along the fibers to obtain the section
$$q_{1,!}(q_2^*(f) \cdot \pi^*(\varphi)) \in \Gamma(X_{\Gamma_1(N)}(\mathbb{C}), \omega^k).$$

Interpretation in terms of moduli

- On elliptic curves, (1) corresponds to a **Hecke correspondence**:
$$\begin{tikzcd} & \{(\iota : E_1 \dashrightarrow E_2, \alpha)\} \arrow[dl, "q_2"'] \arrow[dr, "q_1"] & \\ \{(E_2, \alpha')\} & & \{(E_1, \alpha)\} \end{tikzcd}$$
  where
  - (E_1, α) is an elliptic curve over $\mathbb{C}$ with a $\Gamma_1(N)$-level structure
  $$\alpha : \mathbb{Z}/N\mathbb{Z} \hookrightarrow E_1[N]$$
  - Equivalently, this is the datum of a point $Q := \alpha(1) \in E_1[N](\mathbb{C})$.
  - $(\iota : E_1 \dashrightarrow E_2, \alpha)$ is the data of elliptic curves E_1, E_2 over $\mathbb{C}$ with α a $\Gamma_1(N)$-level structure on E_1, and ι a p-quasi-isogeny:
  - $q_1 : (\iota : E_1 \dashrightarrow E_2, \alpha) \mapsto (E_1, \alpha)$.
  - $q_2 : (\iota : E_1 \dashrightarrow E_2, \alpha) \mapsto (E_2, \alpha')$ where α' is the composition
  $$\mathbb{Z}/N\mathbb{Z} \xhookrightarrow{\alpha} E_1[N] \xrightarrow[\sim]{\iota} E_2[N]$$
  (use: ι is an isomorphism on N-torsion as $(p, N) = 1$). Equivalently, this sends Q to the image Q' in $E_2[N]$.
  - The map π sends $(\iota : E_1 \dashrightarrow E_2, \alpha)$ to the class of the isomorphism
  $$T_p(\iota)\colon T_pE_1 \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\sim} T_pE_2 \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

More concretely, choose any bases of T_pE_1 resp. T_pE_2. Then $T_p(\iota)$ is represented by a matrix $A \in \mathrm{GL}_2(\mathbb{Q}_p)$. The class of $A \in K_p\backslash\mathrm{GL}_2(\mathbb{Q}_p)/K_p$ is independent of the chosen basis.

- Problem: Functor of p-quasi-isogenies not representable (only by ind-scheme).
- However: Mainly interested in φ given by the characteristic functions for

$$K_p \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} K_p \quad \leftrightarrow \quad S_p,$$
$$K_p \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} K_p \quad \leftrightarrow \quad T_p.$$

- Then term $\varphi(h)$ in defining integral is supported on this double coset.
- $\rightsquigarrow$ Hecke correspondences are supported on subspaces represented by schemes!
- For S_p, consider tuples $(\iota : E_1 \dashrightarrow E_2, \alpha)$ of the form

$$([p] : E \to E, \alpha)$$

i.e. $E_1 = E_2$ and $\iota : E_1 \to E_2$ is multiplication $[p]$.

- For T_p, instead need to consider

$$(\iota \colon E_1 \to E_2, \alpha)$$

where $\iota \colon E_1 \to E_2$ is an isogeny of degree p.

- Equivalently, this is the datum of a cyclic subgroup $\ker \iota \subseteq E_1$ of rank p.
- Last time: These are now representable by schemes, even over $\mathrm{Spec}(\mathbb{Q})$!

Hecke action on modular curves – perspective of algebraic geometry

- We now pass to the finite level $K = K_1(N)$ for some $p \nmid N$.
- Corresponding modular curve $X_{\Gamma_1(N)}$ over $\mathbb{Z}[\frac{1}{N}]$ represents pairs (E, Q) over $\mathbb{Z}[\frac{1}{N}]$-schemes S of an elliptic curve E over S with a point

$$Q \in E[N](S).$$

- We start with $\langle p \rangle := S_p$. By the above, just need to multiply Q by p.
- For this, use action of $(\mathbb{Z}/N\mathbb{Z})^\times$ on $X_{\Gamma_1(N)}$.
- By the above, the correspondence for S_p now becomes

$$\begin{tikzcd} & X_{\Gamma_1(N)} \arrow[dl, "p\cdot"', "\sim"] \arrow[dr, equal, "\mathrm{id}"] & \\ X_{\Gamma_1(N)} & & X_{\Gamma_1(N)}. \end{tikzcd}$$

- Fact: For any $a \in (\mathbb{Z}/N\mathbb{Z})^\times$, have canonical isomorphism $a^*\omega = \omega$.
- Since the right map is an isomorphism, the integration over fibers is trivial.
- Everything extends to compactifications.
- Putting everything together, we have proved:

Proposition 8.3. *Consider the operator defined as the composition*

$$M_k(\Gamma_1(N), \mathbb{Z}[\tfrac{1}{N}]) = H^0(X^*_{\Gamma_1(N)}, \omega^k) \xrightarrow{p^*} H^0(X^*_{\Gamma_1(N)}, p^*\omega^k) = M_k(\Gamma_1(N), \mathbb{Z}[\tfrac{1}{N}]).$$

Then its base-change to $\mathbb{C}$ is the Hecke operator S_p.

- For T_p, instead of isogenies $(\iota : E \to E', \alpha)$, parametrise triples (E, D, Q), where $D = \ker \iota$ finite cyclic subgroup of rank p. Get maps of moduli functors

$$\pi_1 : (E, D, Q) \mapsto (E, Q),$$
$$\pi_2 : (E, D, Q) \mapsto (E/D, Q'), \quad Q' := Q + D = \text{image of } Q \text{ under } E[N] \to (E/D)[N]$$

- These induce a Hecke correspondence

$$\begin{tikzcd} & X_{\Gamma_0(p)\cap\Gamma_1(N)} \arrow[dl, "\pi_2"'] \arrow[dr, "\pi_1"] & \\ X_{\Gamma_1(N)} & & X_{\Gamma_1(N)}. \end{tikzcd}$$

- Fact: π_1 and π_2 extend to compactifications $X^*_{\Gamma_0(p)\cap\Gamma_1(N)} \to X^*_{\Gamma_1(N)}$.
- Fact: π_1 and π_2 are finite flat of degree $p+1$, even on compactifications.
- Fact: There are canonical isomorphisms $\pi_1^*\omega = \pi_2^*\omega$
- Since π_1 has finite fibers, the integral occurring in the definition of the Hecke operator becomes a discrete sum, which we can interpret as a trace map

$$\mathrm{Tr}_\pi : \pi_{1,!}\pi_1^*\omega \to \omega$$

- Combining all this, we have proved:

Proposition 8.4. *The base-change to $\mathbb{C}$ of the operator*

$$H^0(X^*_{\Gamma_1(N)}, \omega^{\otimes k}) \xrightarrow{\pi_2^*} H^0(X^*_{\Gamma_1(N)\cap\Gamma_0(p)}, \omega^{\otimes k}) \xrightarrow{\mathrm{Tr}_{\pi_1}} H^0(X^*_{\Gamma_1(N)}, \omega^{\otimes k}).$$

is the Hecke operator T_p.

- Note: This is all defined already over $\mathbb{Q}$ (even over $\mathbb{Z}[\frac{1}{N}]$)! Consequence:

Corollary 8.5. *The eigenvalues of S_p, T_p on $M_k(\Gamma_1(N), \mathbb{C})$ are algebraic.*

- To give a more explicit description of T_p, reinterpret in terms of divisors:
- Let $\mathrm{Div}(X_{\Gamma_1(N)})$ = set of divisors on X ($\approx$ formal sums of points).
- We can then reinterpret $\langle p\rangle := S_p$ and T_p as operators on $\mathrm{Div}(X_{\Gamma_1(N)})$:

$$S_p \colon \mathrm{Div}(X_{\Gamma_1(N)}) \to \mathrm{Div}(X_{\Gamma_1(N)}), \quad [E, Q] \mapsto [E, p \cdot Q],$$
$$T_p \colon \mathrm{Div}(X_{\Gamma_1(N)}) \to \mathrm{Div}(X_{\Gamma_1(N)}), \quad [E, Q] \mapsto \sum_{D \subseteq E[p]} [E/D, Q + D],$$

where E is an elliptic curve, and $Q \in E[N]$ a point of order N.

Modular curves in characteristic p

- Let E be an elliptic curve over a scheme S of characteristic p.
- Then $E[N]$ is étale for all $p \nmid N$.
- The morphism $[p] : E \to E$ factors into

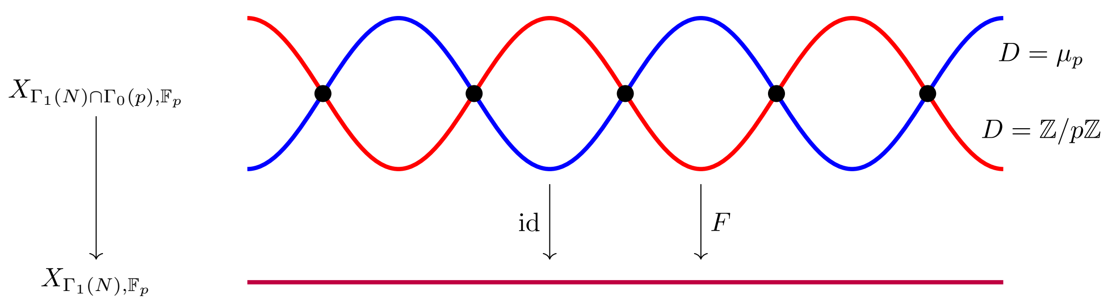
$$E \xrightarrow{F} E^{(p)} \xrightarrow{V} E,$$

the Frobenius and Verschiebung isogeny. Here $E^{(p)} = E \times_{S,F} S$.
- $\Rightarrow$ always have $\ker F \subseteq E[p]$. This is a connected, i.e. $\ker F(S) = 0$.

Definition 8.6. *Over $S = \overline{\mathbb{F}}_p$, there are two cases:*

(1) *$E(\overline{\mathbb{F}}_p) = \mathbb{Z}/p\mathbb{Z}$. Then E is called ordinary, and $E[p] = \mu_p \times \mathbb{Z}/p\mathbb{Z}$.*
(2) *$E(\overline{\mathbb{F}}_p) = 0$. Then E is called supersingular, and $E[p]$ is connected.*

- B "supersingular" is *not* about smoothness. It just means "really special":
- Fact: There are only finitely many isomorphism classes of supersingular elliptic curves over $\overline{\mathbb{F}}_p$. All others are ordinary.
- Recall: $X_{\Gamma_1(N)}$ is defined over $\mathbb{Z}[\frac{1}{N}]$.
- Can therefore reduce to $\mathbb{F}_p$ to get modular curve $X_{\Gamma_1(N),\mathbb{F}_p} \to \mathrm{Spec}(\mathbb{F}_p)$.
- Similarly for $X_{\Gamma_1(N)\cap\Gamma_0(p)}$ representing (E, D, Q) with $D \subseteq E[p]$ of rank p.
- Deligne–Rapoport: The fibre $X_{\Gamma_1(N)\cap\Gamma_0(p),\mathbb{F}_p} \to \mathrm{Spec}(\mathbb{F}_p)$ is of the form



- Two copies of $X_{\Gamma_1(N),\mathbb{F}_p}$, corresponding to $D = \mu_p$ or $D = \mathbb{Z}/p\mathbb{Z}$ as subgroup of $E[p]$, with transversal intersections at supersingular points (black).
- Projection is identity (deg=1) on one copy, and Frobenius (deg=p) on other.

Eichler–Shimura relation [Diamond–Shurman], [Conrad: Appendix to Serre's...]

- Eichler–Shimura relation expresses reduction T_p mod p in terms of Frobenius:
- Base change to algebraic closure $\overline{\mathbb{F}}_p$.
- Consider morphism $\mathrm{Div}^0(X^*_{\Gamma_1(N),\overline{\mathbb{F}}_p}) \to \mathrm{Pic}^0(X^*_{\Gamma_1(N),\overline{\mathbb{F}}_p})$
- Fact: $\mathrm{Pic}^0(X^*_{\Gamma_1(N),\overline{\mathbb{F}}_p})$ is points of a group scheme over $\overline{\mathbb{F}}_p$. In particular, multiplication by p factors through Frobenius F: there is $V = F^\vee$ s.t.

$$p = V \circ F = F \circ V$$

Theorem. *(Eichler–Shimura relation) In* $\mathrm{Pic}^0(X^*_{\Gamma_1(N),\overline{\mathbb{F}}_p})$, *we have*

$$T_p = F + \langle p \rangle V.$$

- Note: There is a version for $X^*_{\Gamma_0(N),\overline{\mathbb{F}}_p}$, which famously reads

$$\boxed{T_p = F + V}$$

We get a second important variant by multiplying by F and using $FF^\vee = p$.

$$F^2 - T_pF + \langle p \rangle p = 0.$$

Proof. (Sketch) Let E be an ordinary elliptic curve over $\breve{\mathbb{Z}}_p = W(\overline{\mathbb{F}}_p)$. Let $C \subseteq E[p]$ be the "canonical subgroup":= generated by kernel of

$$E[p](\breve{\mathbb{Z}}_p) \to E[p](\overline{\mathbb{F}}_p).$$

- Key fact: For $D \subseteq E[p]$ cyclic subgroup scheme of rank p,
  (1) the isogeny $E \to E/D$ reduces to $F : \overline{E} \to \overline{E}^{(p)} \Leftrightarrow D = C$.
  (2) the isogeny $E \to E/D$ reduces to $V : \overline{E} \to \overline{E}^{(p^{-1})} \Leftrightarrow D \neq C$.
- Here $\overline{E}^{(p^{-1})} := E \times_{\overline{\mathbb{F}}_p, F^{-1}} \overline{\mathbb{F}}_p$ = base-change along inverse of Frobenius.

- F sends $Q \in E[N]$ to base-change $Q^{(p)} \in E^{(p)}[N] = E[N]^{(p)}$.
- Since $V \circ F = p$ on $E[N]$, this implies: V sends Q to $pQ^{(p^{-1})} \in E^{(p^{-1})}[N]$.
- Recall: There are $p+1$ different subgroups D of $E[p]$ over $W(\overline{\mathbb{F}}_p)$.
- Thus $T_p([E,Q]) = \sum_{D \subseteq E[p]} [E/D, Q+D]$ in $\mathrm{Div}(X_{\Gamma_1(N)})$ reduces to
$$[E^{(p)}, Q^{(p)}] + p[E^{(p^{-1})}, pQ^{(p^{-1})}] \quad \text{in } \mathrm{Div}(X_{\Gamma_1(N), \overline{\mathbb{F}}_p}).$$
- The first summand is $F[E,Q]$.
- By definition, $\langle p \rangle [E^{(p^{-1})}, Q^{(p^{-1})}] = [E^{(p^{-1})}, pQ^{(p^{-1})}]$.
- Now pass to $\mathrm{Pic}^0(X^*_{\Gamma_1(N), \overline{\mathbb{F}}_p})$. Here: $p = VF$. We therefore have
$$p[E^{(p^{-1})}, Q^{(p^{-1})}] = pF^{-1}[E,Q] = V[E,Q].$$
(technically, we need to work with $[E,Q] - [E', Q']$ to be in Div^0). $\square$

9. Galois representations associated to newforms

Exams:

- Write me to arrange an oral exam on 31.7. (second week after end of lectures).

Next aim:

- Realize newforms in cohomology.
- Finish construction of Galois representations associated to newforms of weight $k = 2$.
- Give hints on how to proceed in the case that $k \geq 2$.

De Rham cohomology:

- X a real manifold
- $C^\infty(X) =$ space of smooth functions $f\colon X \to \mathbb{C}$.
- $\mathcal{A}^i(X) =$ space of smooth i-forms on X, $i \geq 0$.
- de Rham complex of X:

$$\mathcal{A}^\bullet(X)\colon \quad \underbrace{\mathcal{A}^0(X)}_{=C^\infty(X)} \xrightarrow{d} \mathcal{A}^1(X) \xrightarrow{d} \mathcal{A}^2(X) \to \ldots$$

  with d the exterior derivative.
- de Rham cohomology of X:

$$H^*_{\mathrm{dR}}(X) := H^*(\mathcal{A}^\bullet(X))$$

- de Rham comparison isomorphism:

$$H^*_{\mathrm{dR}}(X) \cong H^*(X, \underline{\mathbb{C}})$$

  with RHS the sheaf cohomology of the constant sheaf $\underline{\mathbb{C}}$ on X.
- Moreover,

$$H^*(X, \underline{\mathbb{C}}) \cong H^*_{\mathrm{sing}}(X, \mathbb{C}).$$

  with RHS = singular cohomology of X with values in $\mathbb{C}$.
- If Γ is a discrete group acting properly discontinuously and freely on X, then

$$\mathcal{A}^\bullet(\Gamma\backslash X) \cong \mathcal{A}^\bullet(X)^\Gamma = (\underbrace{\mathcal{A}^0(X)^\Gamma}_{=C^\infty(X)^\Gamma} \xrightarrow{d} \mathcal{A}^1(X)^\Gamma \xrightarrow{d} \mathcal{A}^2(X)^\Gamma \to \ldots).$$

- Finally, we set

$$H^*_{\mathrm{dR}}(\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^\pm)) := \varinjlim_K H^*_{\mathrm{dR}}(\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f)/K \times \mathbb{H}^\pm))$$

  and

$$H^*(\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^\pm), \underline{\mathbb{C}}) := \varinjlim_K H^*(\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f)/K \times \mathbb{H}^\pm), \underline{\mathbb{C}}),$$

  where the colimit is over all compact-open subgroups $K \subseteq \mathrm{GL}_2(\mathbb{A}_f)$, and the transition maps are induced by pullback.
- Then, for $i \geq 0$, the ($\mathrm{GL}_2(\mathbb{A}_f)$-equivariant) de Rham comparison

$$H^i_{\mathrm{dR}}(\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^\pm)) \cong H^i(\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^\pm), \underline{\mathbb{C}})$$

  holds by passing to the colimit.

Upshot:

- Can construct classes in $H^*(X, \underline{\mathbb{C}})$ using (closed) differential forms.

Cohomology classes associated to modular forms of weight 2:

- Note that

$$\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f)\times\mathbb{H}^{\pm})\cong\mathrm{GL}_2(\mathbb{Q})^+\backslash(\mathrm{GL}_2(\mathbb{A}_f)\times\mathbb{H}),$$

  where $\mathrm{GL}_2(\mathbb{Q})^+\subseteq\mathrm{GL}_2(\mathbb{Q})$ is the subgroup of elements of positive determinant.
- Pick $f\in H^0(\mathrm{GL}_2(\mathbb{Q})^+\backslash(\mathrm{GL}_2(\mathbb{A}_f)\times\mathbb{H}),\omega^{\otimes k}), k\in\mathbb{Z}$.
- We view f as a function

$$f\colon \mathrm{GL}_2(\mathbb{A}_f)\times\mathbb{H}\to\mathbb{C},\ (g,z)\mapsto f(g,z)$$

  satisfying

$$f(\gamma g,\gamma z)=(cz+d)^k f(g,z)$$

  for $\gamma\in\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\mathrm{GL}_2(\mathbb{Q})^+$.
- Assume that f is of weight $k=2$.
- Then the differential form $f(g,z)dz$ on $\mathrm{GL}_2(\mathbb{A}_f)\times\mathbb{H}$ is closed and satisfies

$$\gamma^*(f(g,z)dz)=f(\gamma g,\gamma z)\gamma^*dz=\det(\gamma)f(g,z)dz$$

  for $\gamma\in\mathrm{GL}_2(\mathbb{Q})^+$.
- Indeed:
  - Closedness follows from holomorphicity as

$$d(f(g,z)dz)=\frac{\partial}{\partial z}f(g,z)dz\wedge dz-\frac{\partial}{\partial\overline{z}}f(z)dz\wedge d\overline{z}=0.$$

  - If $\gamma=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\mathrm{GL}_2(\mathbb{Q})$, then

$$\gamma^*(dz)=\frac{\det(\gamma)}{(cz+d)^2}dz.$$

- Let

$$|-|\colon\mathbb{Q}^\times\backslash\mathbb{A}^\times\to\mathbb{R}_{>0}$$

  be the adélic norm, i.e.,

$$|(x_2,x_3,\ldots,x_\infty)|:=\prod_p|x_p|_p\cdot|x_\infty|_\infty$$

  with $|-|_p\colon\mathbb{Q}_p\to\mathbb{R}_{\geq 0},\ x\mapsto p^{-v_p(x)}$ the p-adic norm, and $|-|_\infty\colon\mathbb{R}\to\mathbb{R}_{\geq 0}$ the real norm.
- $|-|$ defines the character

$$\chi_{\mathrm{no}}:=|-|\circ\det$$

  of $\mathrm{GL}_2(\mathbb{A})$, which is trivial on $\mathrm{GL}_2(\mathbb{Q})$.
- Thus, for

$$\tilde{f}(g,z):=\chi_{\mathrm{no}}((g,1))f(g,z)$$

  (with $(g,1)\in\mathrm{GL}_2(\mathbb{A})\cong\mathrm{GL}_2(\mathbb{A}_f)\times\mathrm{GL}_2(\mathbb{R})$), the form

$$\eta_f:=\tilde{f}(g,z)dz$$

  on $\mathrm{GL}_2(\mathbb{A}_f)\times\mathbb{H}$ is $\mathrm{GL}_2(\mathbb{Q})^+$-equivariant.

- Get a map, *which is not* $\mathrm{GL}_2(\mathbb{A}_f)$*-equivariant*,

$$\alpha\colon M_2 \to H^1_{\mathrm{dR}}(\mathrm{GL}_2(\mathbb{Q})^+\backslash(\mathrm{GL}_2(\mathbb{A}_f)\times\mathbb{H})) \cong H^1(\mathrm{GL}_2(\mathbb{Q})^+\backslash(\mathrm{GL}_2(\mathbb{A}_f)\times\mathbb{H}),\underline{\mathbb{C}})$$

  by sending f to $[\eta_f]$.
- For a smooth $\mathrm{GL}_2(\mathbb{A}_f)$-representation V and a smooth character

$$\chi\colon \mathrm{GL}_2(\mathbb{A}_f)\to\mathbb{C}^\times$$

  define $V(\chi) := V\otimes_{\mathbb{C}}\chi$.
- Then

$$\alpha\colon M_2(\chi_{\mathrm{no}}) \to H^1(\mathrm{GL}_2(\mathbb{Q})^+\backslash(\mathrm{GL}_2(\mathbb{A}_f)\times\mathbb{H}),\underline{\mathbb{C}})$$

  is $\mathrm{GL}_2(\mathbb{A}_f)$-equivariant.
- Let $K\subseteq\mathrm{GL}_2(\mathbb{A}_f)$ be compact-open, and set

$$X_K := \mathrm{GL}_2(\mathbb{Q})^+\backslash(\mathrm{GL}_2(\mathbb{A}_f)/K\times\mathbb{H}).$$

  with canonical compactification

$$X_K^* := \mathrm{GL}_2(\mathbb{Q})^+\backslash(\mathrm{GL}_2(\mathbb{A}_f)\times\mathbb{H}^*)$$

  where $\mathbb{H}^* := \mathbb{H}\cup\mathbb{P}^1(\mathbb{Q})$ (equipped with Satake topology).
- If $f\in S_2(K)$, then η_f extends to a holomorphic differential form on X_K^*.
  - Indeed: If $q=e^{2\pi i z}$, then

$$dq = 2\pi i q dz,$$

    i.e.,

$$dz = \frac{1}{2\pi i q}dq.$$

    Now express $f(g,z)$ at each cusp in Fourier expansion, i.e., as a function of q.
- The diagram

$$\begin{tikzcd} H^1_{\mathrm{dR}}(X_K^*) \arrow[r] \arrow[d, "\simeq"] & H^1_{\mathrm{dR}}(X_K) \arrow[d, "\simeq"] \\ H^1(X_K^*,\underline{\mathbb{C}}) \arrow[r] & H^1(X_K,\underline{\mathbb{C}}) \end{tikzcd}$$

  commutes.
- Moreover, the morphism

$$H^1_c(X_K,\underline{\mathbb{C}}) \to H^1(X_K^*,\underline{\mathbb{C}}) = H^1_c(X_K^*,\underline{\mathbb{C}})$$

  is surjective as its cokernel embeds into $H^1(\{\text{cusps}\},\underline{\mathbb{C}})=0$.
- Thus, $\alpha(S_2(K))$ lies in the *interior cohomology* $\widetilde{H}^1(X_K,\underline{\mathbb{C}})\cong H^1(X_K^*,\underline{\mathbb{C}})$ of X_K, which by definition is the image of $H^1_c(X_K,\underline{\mathbb{C}})\to H^1(X_K,\underline{\mathbb{C}})$.
- Note that the above discussion applies similarly to *anti*holomorphic modular forms, i.e., complex conjugates of modular forms.
- This yields the map

$$\overline{\alpha}\colon \overline{S_2(K)}\to\widetilde{H}^1(X_K,\underline{\mathbb{C}}),\ \overline{f}\mapsto[\chi_{\mathrm{no}}\overline{f}d\overline{z}],$$

  where $\overline{S_2(K)}$ denotes the $\mathbb{C}$-linear space of antiholomorphic modular forms.

- We can pass to infinite level and obtain the morphism

$$\alpha \oplus \overline{\alpha}\colon S_2 \oplus \overline{S_2} \to \widetilde{H}^1(\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^{\pm}), \underline{\mathbb{C}})$$

with (hopefully) self-explaining notation.

Theorem 9.1 (Eichler–Shimura)**.** *The map*

$$\alpha \oplus \overline{\alpha}\colon S_2(\chi_{no}) \oplus \overline{S_2}(\chi_{no}) \to \widetilde{H}^1(\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^{\pm}), \underline{\mathbb{C}})$$

is a $\mathrm{GL}_2(\mathbb{A}_f)$*-equivariant isomorphism.*

- The statement is equivalent to the analogous statement for all (sufficiently small) compact-open subgroup $K \subseteq \mathrm{GL}_2(\mathbb{A}_f)$.
- Thus, fix some $K \subseteq \mathrm{GL}_2(\mathbb{A}_f)$ compact-open.
- The proof will exploit the $\cup$-product pairing

$$H^1_c(X_K, \underline{\mathbb{C}}) \times H^1(X_K, \underline{\mathbb{C}}) \to H^2_c(X_K, \underline{\mathbb{C}}) \xrightarrow{\text{integrate}} \mathbb{C},$$

which under the de Rham comparison is induced from the pairing

$$(\eta_1, \eta_2) \mapsto \int\limits_{X_K} \eta_1 \wedge \eta_2$$

of differential forms, where η_1 has compact support.
- We will use the following observation:
  - $H^*_c(X_K, \underline{\mathbb{C}}) \cong H^*(\mathcal{A}^\bullet_c(X_K))$, where $\mathcal{A}^\bullet_c$ denotes differential forms with compact supports.
  - The canonical map

$$\mathcal{A}^\bullet_c(X_K) \to \mathcal{A}^\bullet_{\mathrm{rd}}(X_K) \tag{2}$$

    is a quasi-isomorphism, where the RHS denotes differentials forms of *rapid decay*, cf. [Bor80, Theorem 2].
- For $f \in S_2$ the form η_f is rapidly decreasing.
  - Use that $|e^{2\pi i n z}| = e^{-2\pi n \mathrm{Im}(z)}$ decreases rapidly if $\mathrm{Im}(z) \to \infty$ and $n \geq 1$.
- Thus α lifts naturally to a map

$$\widetilde{\alpha}\colon S_2(K) \to H^1_c(X_K, \underline{\mathbb{C}}).$$

- Consider $f \in S_2(K), g \in \overline{S_2(K)}$. Then:
  * $\widetilde{\alpha}(f) \cup \alpha(f) = 0$ as $dz \wedge dz = 0$.
  * Similarly for g.
  * Using Stokes' theorem, one proves

$$\mathrm{tr}(\widetilde{\alpha}(f) \cup \overline{\alpha}(g)) = \int_{X_K} \eta_f \wedge \eta_g.$$

  * This implies that the Poincaré pairing induces (up to a scalar) the Petterson scalar product aka L^2-pairing $\langle -, - \rangle$ on S_2.
- We can now prove that $\beta := \alpha \oplus \overline{\alpha}$ is injective.
- Indeed:
  - If $\beta(f + \overline{g}) = 0$, then

$$\beta(f + \overline{g}) \cup \overline{\alpha}(\overline{f}) = \langle f, f \rangle = 0,$$

    i.e., $f = 0$. Similarly, $\overline{g} = 0$.

- Now we check $\widetilde{H}^1(X_K, \underline{\mathbb{C}}) = 2\dim(S_2(K))$.

$\Rightarrow$ Proof of the Eichler-Shimura isomorphism is finished.

- Namely (we use lower case latters to denote dimensions):
  * $\widetilde{h}^1(X_K, \underline{\mathbb{C}})) = h^1(X_K^*, \underline{\mathbb{C}}) = 2 \cdot h^0(X_K^*, \underline{\mathbb{C}}) - \chi_{\mathrm{top}}(X_K^*)$
  * Here: χ_{top} is the *topological* Euler characteristic.
  * On the other hand:

$$\begin{array}{cc} & \dim(S_2(K)) \\ = & h^0(X_K^*, \Omega^1_{X_K^*}) \\ \overset{\mathrm{RR}}{=} & \deg(\Omega^1_{X_K^*}) + \chi_{\mathrm{hol}}(X_K^*) + h^1(X_K^*, \Omega^1_{X_K^*}) \\ = & -\chi_{\mathrm{hol}}(X_K^*) + h^0(X_K^*, \underline{\mathbb{C}}) \end{array}$$

  * Here: χ_{hol} is the *holomorphic* Euler characteristic.
  * $\chi_{\mathrm{top}}(X_K^*) = 2\chi_{\mathrm{hol}}(X_K^*)$

Galois representations associated to newforms:

- Recall that in the last two lectures we introduced a scheme

$$\widehat{X} \to \mathrm{Spec}(\mathbb{Q})$$

with $\mathrm{GL}_2(\mathbb{A}_f)$-action such that naturally

$$\widehat{X}(\mathbb{C}) \cong \mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^{\pm}).$$

- For $K \subseteq \mathrm{GL}_2(\mathbb{A}_f)$ compact-open (plus sufficiently small), get a quasi-projective, smooth curve

$$X_K \to \mathrm{Spec}(\mathbb{Q})$$

with

$$X_K(\mathbb{C}) \cong \mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f)/K \times \mathbb{H}^{\pm}).$$

(note clash in notation with previous section, where X_K was equal to RHS).

- Fix a prime ℓ.
- For $K \subseteq \mathrm{GL}_2(\mathbb{A}_f)$ can consider the interior étale cohomology

$$\widetilde{H}^1_{\text{ét}}(X_{K,\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell) := \mathrm{Im}(H^1_{c,\text{ét}}(X_{K,\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell) \to H^1_{\text{ét}}(X_{K,\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell)) \cong H^1_{\text{ét}}(X^*_{K,\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell).$$

- In the limit, we get

$$\widetilde{H}^1_{\text{ét}}(\widehat{X}_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell) := \varinjlim_K \widetilde{H}^1_{\text{ét}}(X_{K,\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell)$$

- By naturality of $H^1_{c,\text{ét}}, H^1_{\text{ét}}$ this space has a natural action of

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \times \mathrm{GL}_2(\mathbb{A}_f),$$

i.e., an action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and an action of $\mathrm{GL}_2(\mathbb{A}_f)$, and these two actions commute.

- Fix an isomorphism $\iota\colon \overline{\mathbb{Q}}_\ell \cong \mathbb{C}$.
- Using étale comparison theorems (and ι) we get the isomorphisms

$$\widetilde{H}^1_{\text{ét}}(\widehat{X}_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell) \cong H^1_{\text{ét}}(\widehat{X}_{\overline{\mathbb{C}}}, \overline{\mathbb{Q}}_\ell) \cong \widetilde{H}^1(\widehat{X}(\mathbb{C}), \overline{\mathbb{Q}}_\ell) \overset{\iota}{\cong} \widetilde{H}^1(\widehat{X}(\mathbb{C}), \underline{\mathbb{C}}).$$

- These isomorphisms are $\mathrm{GL}_2(\mathbb{A}_f)$-equivariant.
- Let $\pi \subseteq S_2$ be an irreducible $\mathrm{GL}_2(\mathbb{A}_f)$-representation.

- Using ι, we will view each smooth $\mathrm{GL}_2(\mathbb{A}_f)$-representation over $\mathbb{C}$ as a smooth representation over $\overline{\mathbb{Q}}_\ell$.
- The space

$$\widetilde{\rho}_\pi := \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{A}_f)}(\pi(\chi_{\mathrm{no}}), \widetilde{H}^1_{\text{ét}}(\widehat{X}_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell))$$

  is two-dimensional by the Eichler-Shimura isomorphism

$$\widetilde{H}^1(\widehat{X}(\mathbb{C}), \underline{\mathbb{C}}) \cong S_2(\chi_{\mathrm{no}}) \oplus \overline{S_2(\chi_{\mathrm{no}})}$$

  and because S_2 is multiplicity free as a $\mathrm{GL}_2(\mathbb{A}_f)$-representation.
- In summary, as a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \times \mathrm{GL}_2(\mathbb{A}_f)$-representation

$$\widetilde{H}^1_{\text{ét}}(\widehat{X}_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell) \cong \bigoplus_\pi \widetilde{\rho}_\pi \otimes_{\overline{\mathbb{Q}}_\ell} \pi(\chi_{\mathrm{no}}), \tag{3}$$

  where the sum is running over all irreducible $\mathrm{GL}_2(\mathbb{A}_f)$-representations $\pi \subseteq S_2$ (note that there is a natural morphism from the RHS to the LHS).

The Eichler–Shimura relation in étale cohomology:

B We need to show that Hecke eigenvalues match with traces of Frobenii. B

- Recall: To $\pi \subseteq S_k$ with system of Hecke eigenvalues

$$\{a_p(\pi), b_p(\pi) = \chi(p)p^{k-1}\}_{p \notin S},$$

  we want to attach a 2-dimensional ℓ-adic representation

$$\rho_\pi \colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell),$$

  such that for $p \notin S$ each *arithmetic* Frobenius $\mathrm{Frob}_p^{\mathrm{arith}}$ at p has characteristic polynomial

$$X^2 - a_p(\pi)X + \chi(p)p^{k-1}$$

  Confession: I stated the Langlands reciprocity for newforms wrongly!

  (we omit ι, ρ_π in the following), i.e.,

$$(\mathrm{Frob}_p^{\mathrm{arith}})^2 - a_p(\pi)\mathrm{Frob}_p^{\mathrm{arith}} + \chi(p)p^{k-1} = 0.$$

- By passing to $K_1(N)$-invariants in (3), we have (for $N \geq 3$)

$$H^1_{\text{ét}}(X^*_{\Gamma_1(N),\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell) \cong \bigoplus_\pi \widetilde{\rho}_\pi \otimes_{\overline{\mathbb{Q}}_\ell} \pi(\chi_{\mathrm{no}})^{K_1(N)}$$

  (note $X_{K_1(N)} \cong X_{\Gamma_1(N)}$).
- Let p be a prime, $p \nmid \ell N$.
- Fix a place of $\overline{\mathbb{Q}}$ over p. This determines an algebraic closure $\overline{\mathbb{F}}_p$ of $\mathbb{F}_p$.
- Recall: For an abelian variety A over a field L, and ℓ a prime, we write

$$T_\ell A = \varprojlim A[\ell^n](\overline{L}), \quad V_\ell A = T_\ell A \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

  for the ℓ-adic Tate module resp. the rationalized ℓ-adic Tate module.

Lemma 9.2. *Let p be prime, $p \nmid \ell N$. Then we have*

$$H^1_{\text{ét}}(X^*_{\Gamma_1(N),\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) = V_\ell \mathrm{Pic}^0(X^*_{\Gamma_1(N),\overline{\mathbb{F}}_p})^\vee,$$

where $(-)^\vee$ denotes the $\mathbb{Q}_\ell$-dual.

Proof. The Kummer sequence $1 \to \mu_{\ell^n} \to \mathbb{G}_m \xrightarrow{\ell^n} \mathbb{G}_m \to 1$ implies

$$H^1_{\text{ét}}(X^*_{\Gamma_1(N),\overline{\mathbb{Q}}}, \mathbb{Q}_\ell(1)) \cong V_\ell \mathrm{Pic}^0(X^*_{\Gamma_1(N),\overline{\mathbb{Q}}}).$$

The Weil pairing yields a canonical isomorphism

$$V_\ell \mathrm{Pic}^0(X^*_{\Gamma_1(N),\overline{\mathbb{Q}}})^\vee \cong V_\ell \mathrm{Pic}^0(X^*_{\Gamma_1(N),\overline{\mathbb{Q}}})(-1).$$

Finally, because $\ell \neq p$

$$V_\ell \mathrm{Pic}^0(X^*_{\Gamma_1(N),\overline{\mathbb{Q}}}) \cong V_\ell \mathrm{Pic}^0(X^*_{\Gamma_1(N),\overline{\mathbb{Q}}_p}) \cong V_\ell \mathrm{Pic}^0(X^*_{\Gamma_1(N),\overline{\mathbb{F}}_p})$$

by lifting torsion points. □

- In particular, the Galois representation

  $$H^1_{\text{ét}}(X^*_{\Gamma_1(N),\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell)$$

  is unramified at p.
- On $V_\ell \mathrm{Pic}^0(X^*_{\Gamma_1(N),\overline{\mathbb{F}}_p})$, we have the Eichler–Shimura relation, which we recall reads

  $$F^2 - \widetilde{T}_p F + p\widetilde{S}_p = 0$$

  with F the arithmetic Frobenius.
- Last time it was written T_p, but the action of the double coset

  $$\mathrm{GL}_2(\mathbb{Z}_p) \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \mathrm{GL}_2(\mathbb{Z}_p)$$

  was considered, which yields $\widetilde{T}_p$.
- Moreover, instead of $\widetilde{S}_p$, which corresponds to

  $$\mathrm{GL}_2(\mathbb{Z}_p) \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} \mathrm{GL}_2(\mathbb{Z}_p),$$

  it was written $\langle p \rangle$.

Proposition 9.3. *Let p be a prime with $p \nmid \ell N$. Then on $H^1_{\text{ét}}(X^*_{\Gamma_1(N),\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell)$, we have*

$$(\mathrm{Frob}_p^{\mathrm{geom}})^2 - \widetilde{T}_p \mathrm{Frob}_p^{\mathrm{geom}} + p\widetilde{S}_p = 0.$$

- For a normalized newform $f \in S_2(\Gamma_0(N), \chi)$ we define finally

  $$\rho_f := (\widetilde{\rho}_\pi)^\vee$$

  with π the irreducible $\mathrm{GL}_2(\mathbb{A}_f)$-representation generated by f, and

  $$\widetilde{\rho}_\pi := \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{A}_f)}(\pi(\chi_{\mathrm{no}}), \widetilde{H}^1_{\text{ét}}(\widehat{X}_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell)).$$

Some analysis of ρ_f, cf. [Rib77]**:**

- Write the (normalized) newform $f \in S_2(\Gamma_0(N), \chi)$ in Fourier expansion

  $$f(q) = \sum_{n \geq 1} a_n q^n.$$

  Then for $p \nmid \ell N$, $\mathrm{Frob}_p^{\mathrm{arith}}$ has characteristic polynomial

  $$X^2 - a_p X + \chi(p)p$$

  on ρ_f.
- Indeed:
  - By the (dual of the) Eichler–Shimura relation

    $$(\mathrm{Frob}_p^{\mathrm{arith}})^2 - \widetilde{T}_p \mathrm{Frob}_p^{\mathrm{arith}} + p\widetilde{S}_p$$

on ρ_f.

  - Recall that $\widetilde{T}_p$ has eigenvalue pa_p on $\pi^{K_1(N)}$, while $\widetilde{S}_p$ has eigenvalue $\chi(p)p^2$.
  - We had to twist the $\mathrm{GL}_2(\mathbb{A}_f)$-action on S_2 by $\chi_{\mathrm{no}} = |\det(-)|$. Thus $\widetilde{T}_p$ has eigenvalue a_p on

$$\widetilde{\rho}_\pi \otimes_{\overline{\mathbb{Q}}_\ell} \pi(\chi_{\mathrm{no}})^{K_1(N)} \subseteq H^1_{\text{ét}}(X^*_{\Gamma_1(N),\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell).$$

while $\widetilde{S}_p$ has eigenvalue $\frac{1}{p^2}\chi(p)p^2 = \chi(p)$.

- The determinant of ρ_f is

$$\psi \cdot \chi_{\mathrm{cyc}}$$

where

$$\chi_{\mathrm{cyc}} \colon G_{\mathbb{Q}} \to \overline{\mathbb{Q}}_\ell^\times$$

denotes the cyclotomic character, and

$$\psi \colon \mathbb{Q}^\times \mathbb{R}_{>0} \backslash \mathbb{A}^\times \to \overline{\mathbb{Q}}_\ell^\times \overset{\iota}{\cong} \mathbb{C}^\times$$

the (finite order) adélic character determined by χ.

- Indeed:
  - Use Chebotarev and check that both sides agree on arithmetic Frobenius elements for $p \nmid \ell N$.
- In particular, ρ_f is *odd*, i.e., the determinant of each complex conjugation is -1.
  - This uses that for all $k \geq 1$ non-triviality of $S_k(\Gamma_0(N), \chi)$ implies $\chi(-1) = (-1)^k$.
- ρ_f is irreducible, cf. [Rib77, Theorem (2.3)].
- Because, ρ_f is realized in the étale cohomology of a proper, smooth scheme over $\mathbb{Q}$, the de Rham comparison theorem in ℓ-adic Hodge theory implies that ${\rho_f}_{|G_{\mathbb{Q}_\ell}}$ is de Rham.

Outline of the construction in weights ≥ 2:

- Instead of

$$\widetilde{H}^1_{\text{ét}}(\widehat{X}_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell)$$

use

$$\widetilde{H}^1_{\text{ét}}(\widehat{X}_{\overline{\mathbb{Q}}}, \mathbb{L}^k),$$

with the local system

$$\mathbb{L}^k := \mathrm{Sym}^{k-1} R^1 f_*(\overline{\mathbb{Q}}_\ell)$$

for $f \colon E \to \widehat{X}$ the universal elliptic curve.

- Problem: $\mathbb{L}^k$ does not extend to a local system on the compactification. Thus one needs an additional argument to justify base change to $\overline{\mathbb{F}}_p$.
- Use similar strategy to obtain analogs of the Eichler–Shimura relation/isomorphism.
- Also replace the de Rham comparison/étale comparsion theorems by their versions with coefficients.

10. Galois representations for weight 1 forms (by Ben Heuer)

Last time: Constructed Galois representations attached to newforms in weight ≥ 2:

Theorem 10.1 (Deligne '71)**.** *Let $N \geq 3$ and $k \geq 2$. Let $\varepsilon : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ be a Dirichlet character. Let $f \in M_k(\Gamma_0(N), \varepsilon)$ be a newform of weight k. Let K be the number field generated by the $a_p(f)$ and ε. Then for any place λ of K over a prime $\ell \nmid N$, there is a continuous, irreducible Galois representation*

$$\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(K_\lambda),$$

unramified outside N, such that the characteristic polynomial of Frob_p is

$$X^2 - a_p X + \varepsilon(p)p^{k-1} \quad \text{for all } p \nmid N\ell.$$

Galois reps associated to weight 1 modular forms

- Goal of today: prove the following Theorem

Theorem 10.2 (Deligne–Serre '73)**.** *Let $N \geq 3$. Let $\varepsilon : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ be a Dirichlet character. Let $f \in M_1(\Gamma_0(N), \varepsilon)$ be a newform of weight 1. Then there is a continuous, odd, irreducible Galois representation*

$$\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{C}),$$

unramified outside N, such that the characteristic polynomial of Frob_p is

$$X^2 - a_p X + \varepsilon(p) \quad \text{for all } p \nmid N.$$

- This finishes the construction of newforms $\rightsquigarrow$ Galois reps.
- Note that ρ has image in $\mathrm{GL}_2(\mathbb{C})$, i.e. is an Artin representation, in contrast to the ones for weight ≥ 2, which were ℓ-adic. Why is that?
  - Easy fact: Any Artin representation has finite image.
  - In fact, ρ is defined already over a number field, so could embed into p-adic field K_λ as before. But using $\mathbb{C}$ is a way of signaling "finite image".
  - On the Galois side, see that ρ can only be finite for $k = 1$ since the determinant is required to be $\det \rho = \varepsilon\chi_{\mathrm{cycl}}^{k-1}$, and χ_{cycl} has infinite image.
- If ε is even, then $M_1(\Gamma_0(N), \varepsilon) = 0$. Hence ε is odd, which implies ρ odd.

First sketch of construction

Compared to last time, the proof will be less geometry, more Galois representations:

- Reduce f mod p. This gives a "mod p modular form"
- Use congruences of modular forms to show there is a mod p Hecke eigenform f' with same eigenvalues but higher weight ≥ 2.
- Lift f' back to characteristic 0 using "Deligne–Serre lifting"
- Attach Galois representation $\rho_{f'}$ using weight ≥ 2 construction
- Reduce Galois representation mod ℓ
- Lift Galois representation from $\mathbb{F}_\ell$ to $\mathbb{C}$.

Introduction to mod p modular forms

- Let $N \geq 3$. Let S be any $\mathbb{Z}[\frac{1}{N}]$-scheme S.
- Recall: We had defined modular forms over S of weight k by

$$M_k(\Gamma_1(N); S) := \Gamma(X^*_{\Gamma_1(N),S}, \omega^k).$$

- ω is the ample line bundle of relative differentials on universal elliptic curve
- Can define q-expansions: Let $S = \mathrm{Spec}(A)$, then have the Tate curve $T(q)$ over $\mathbb{Z}((q)) \otimes_{\mathbb{Z}} A$. Over $\mathrm{Spec}(\mathbb{Z}((q)) \otimes_{\mathbb{Z}} A)$, have canonical isomorphism

$$\omega_{\mathbb{G}_m} = \omega_{T(q)}$$

- bundle $\omega_{\mathbb{G}_m}$ is trivial: the canonical differential $\frac{dx}{x}$ on $\mathbb{G}_m$ is invertible section
- Via this trivialisation, get for every cusp of $X^*_{\Gamma_1(N),A}$ an associated q-expansion

$$M_k(\Gamma_1(N); S) \to A[[q]].$$

Theorem 10.3 (Katz' q-expansion principle)**.** *This is injective for all k at each cusp.*

- Now apply this all to $S = \mathrm{Spec}(\mathbb{F})$ where $\mathbb{F}$ is a finite field of characteristic p: The resulting modular forms are called **mod p modular forms**.
- The following fact perhaps helps you get a feeling for mod p modular forms:

  Lemma 10.4. *For weight $k \geq 2$, any mod p modular form is the reduction of a modular form over a number field modulo p. In particular, in this case*

$$M_k(\Gamma_1(N); \mathbb{F}) = M_k(\Gamma_1(N); \mathbb{Z}[\tfrac{1}{N}]) \otimes_{\mathbb{Z}} \mathbb{F}.$$

  Proof. Using Riemann–Roch, one can show that $H^1(X^*_{\Gamma_1(N)}, \omega^{\otimes k}) = 0$. □

- However, it is often better to have a more intrinsic definition. For example:

Example of mod p modular form: The Hasse invariant

- Recall: For any elliptic curve in characteristic p, have Verschiebung isogeny

$$V : E^{(p)} \to E.$$

- In particular, have this for the universal elliptic curve.
- On relative differentials, this defines a pullback map in the other direction

$$V^* : \omega \to \omega^{\otimes p}.$$

- Tensoring with ω^{-1} yields

$$V^* : \mathcal{O} \to \omega^{\otimes(p-1)} \rightsquigarrow V^* \in \Gamma(\omega^{\otimes(p-1)}) = M_{p-1}(\Gamma_1(N); \mathbb{F}).$$

- This is the **Hasse invariant**, denoted by Ha.
- It is a mod p modular form of weight $p-1$. In fact: an eigenform of level 1.

Proposition 10.5. *The Hasse invariant has constant q-expansion $= 1 \in \mathbb{F}_p[[q]]$.*

Proof. Sketch: Use $\omega_{T(q)} = \omega_{\mathbb{G}_m}$: On $\mathbb{G}_m$ have $F = [p]$ and thus $V = 1$. □

- Note: $1 \in M_0(\Gamma_1(N); \mathbb{F}_p)$ is also a mod p modular form with q-expansion 1.
- This does not contradict the q-expansion principle as the weights are different.

Upshot: In characteristic p, we have a new phenomenon:

- Modular forms of different weight can have same q-expansion!

- E.g. for any modular form f of weight k, any $n \in \mathbb{N}$, have modular form $\mathrm{Ha}^n \cdot f$ of weight $k + n(p-1)$ with same q-expansion.
- In particular, this preserves eigenforms $\rightsquigarrow$ a system of eigenvalues for Hecke operators can correspond to eigenforms of different weights.
- BThe multiplication map

$$M_k(\Gamma_1(N); S) \xrightarrow{\cdot \mathrm{Ha}} M_{k+p-1}(\Gamma_1(N); S)$$

is injective (q-expansion principle) but not surjective!
- Reason: Ha has zeros, precisely on supersingular points of $X^*_{\Gamma_1(N)}(\overline{\mathbb{F}}_p)$
- This is one motivation for removing those points $\rightsquigarrow$ p-adic modular forms.

Eisenstein series

- There is a more classical perspective on this that we want to at least mention:

Definition 10.6.
For any even $k \geq 4$, let

$$G_k := \sum_{\substack{(n,m)\in\mathbb{Z}^2 \\ (n,m)\neq(0,0)}} \frac{1}{(m+n\tau)^k}.$$

This is a modular eigenform of weight k and level 1 with q-expansion

$$2\zeta(k)(1 - \sum_{n=0}^{\infty} \frac{4}{B_k} \sigma_{k-1}(n) q^n)$$

where $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$ and B_k is the k-th Bernoulli number.
Define the normalised Eisenstein series as

$$E_k := \frac{1}{2\zeta(k)} G_k = 1 - \frac{4}{B_k} \sum_{n=0}^{\infty} \sigma_{k-1}(n) q^n \in \mathbb{Z}_{(p)}[[q]]$$

Lemma 10.7. *Assume $p \geq 5$, then in terms of q-expansions, we have*

$$E_{p-1} \equiv 1 \bmod p.$$

Proof. This follows from Kummer's congruences for Bernoulli numbers. □

Upshot: For $p \geq 5$, the reduction mod p of the Eisenstein series E_{p-1} is = Ha:

$$E_{p-1} \equiv \mathrm{Ha} \bmod p \quad \text{“Deligne's congruence”}$$

Strategy (Deligne–Serre):

- Given eigenform of weight 1, reduce mod p.
- Multiply with Ha so it becomes eigenform of weight ≥ 2.
- Use construction from last lecture in this case.

Galois reps attached to mod p modular forms

Theorem 10.8. *Let $k \geq 1$ and let $f \in M_k(\Gamma_1(N), \varepsilon; \overline{\mathbb{F}}_p)$ be a mod p cuspidal eigenform. Let $\mathbb{F}_f$ be the (finite) field generated over $\mathbb{F}_p$ by the $a_p(f)$ and ε. Then there is a semi-simple Galois representation*

$$\rho_f : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F}_f),$$

unramified outside $N\ell$ such that for all $\ell \nmid Np$,

$$\mathrm{charpoly}(\mathrm{Frob}_p | \rho_{\overline{f}}) = X^2 - a_p X + \varepsilon(p) p^{k-1}.$$

Step 1: Reduce to case of $k \geq 2$: If $k = 1$, multiply with Ha. This does not change the q-expansion, hence is still an eigenform. But its weight is ≥ 2.

Step 2: Lift the Hecke eigensystem to characteristic 0. This is always possible in weight ≥ 2, as we shall discuss next.

Deligne–Serre lifting lemma

- Let R be a $\mathbb{Z}[\frac{1}{N}]$-algebra without zero-divisors. Let
$$\mathbb{T}_R \subseteq \mathrm{End}(M_k(\Gamma_1(N); R))$$
be the Hecke algebra: R-subalgebra generated by the Hecke operators T_l, S_l.
- A Hecke eigensystem is a ring homomorphism
$$\psi : \mathbb{T}_R \to R.$$

The archetypal example of a Hecke eigensystem is:

- Let g be an eigenform in $M_k(\Gamma_1(N); R)$, this gives rise to a Hecke eigensystem
$$\psi_g : \mathbb{T}_R \to R, \quad T \mapsto a_T \quad \text{ s.t. } T(g) = a_T g.$$

Lemma 10.9. *Assume that $R = K$ is a field. Then any Hecke eigensystem*
$$\psi : \mathbb{T}_K \to K$$
comes from a unique Hecke eigenform.

Proof.
- $\mathbb{T}_K$ is an Artinian K-algebra.
- After passing to extension, it is the product of Artinian local rings $\mathbb{T}_i$.
- Since $\mathbb{T}_K$ acts faithfully on $M := M_k(\Gamma_1(N); K)$, the submodule $M_i := \mathbb{T}_i \cdot M$ is non-zero (commutative algebra fact).
- Take any eigenvector in M_i. Then this has eigensystem ψ.
- Uniqueness: The Hecke eigenvalues determine the q-expansion. $\square$

- Let K be a number field, $\mathfrak{p}$ a place over $p \nmid N$. Let $\mathbb{F}_\mathfrak{p}$ be the residue field.
- Write $\mathcal{O}_{K,(\mathfrak{p})}$ for the valuation subring of K of $\mathfrak{p}$-integral elements.

Lemma 10.10 (Deligne–Serre Lifting Lemma)**.** *Let $k \geq 2$ and let $g \in M_k(\Gamma_1(N); \mathbb{F}_\mathfrak{p})$ be an eigenform. Then there is a finite extension $K'|K$, an extension $\mathfrak{p}'|\mathfrak{p}$ and an eigenform $\tilde{g} \in M_k(\Gamma_1(N); \mathcal{O}_{K',(\mathfrak{p}')})$ such that $\tilde{g}$ reduces to g mod $\mathfrak{p}'$.*

Proof.
- Consider the ring homomorphism
$$\mathbb{T}_{\mathcal{O}_K} \to \mathbb{T}_{\mathbb{F}_\mathfrak{p}} \xrightarrow{\psi_g} \overline{\mathbb{F}}_p.$$
- This corresponds to a maximal ideal $\mathfrak{m}$. Since
$$\mathrm{Spec}(\mathbb{T}_{\mathcal{O}_K}) \to \mathrm{Spec}(\mathcal{O}_K)$$
is locally free, can find prime ideal $\mathfrak{q} \subseteq \mathfrak{m}$ such that $\mathfrak{q} \cap \mathcal{O} = 0$.
- This defines a non-zero prime ideal of $\mathbb{T}_K$, corresponding to Hecke eigensystem
$$\mathbb{T}_K \to K'.$$
- By Lemma 10.9, this corresponds to an eigenform.
- Use uniqueness in Lemma 10.9 to see that this reduces to g. $\square$

Proof of Theorem 10.8

- Let f be mod p eigenform of weight ≥ 2, defined over extension $\mathbb{F}_f$ of $\mathbb{F}_p$.
- Lift to eigenform $\tilde{f}$ over some number field K. Let $\mathfrak{p}$ be a place over p.
- Associated to this, have Galois representation
$$\rho_{\tilde{f}} : G_{\mathbb{Q}} \to \mathrm{GL}_2(K_{\mathfrak{p}}).$$
- Since image is compact, can always find isomorphic representation of the form
$$\rho_{\tilde{f}} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{O}_{K,\mathfrak{p}}).$$
- Reduce this mod $\mathfrak{p}$ to get representation
$$\overline{\rho}_{\tilde{f}} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F}_{\mathfrak{p}}).$$
- This might not yet be semi-simple $\rightsquigarrow$ Define ρ_f as semi-simplification
$$\rho_f := \overline{\rho}_{\tilde{f}}^{ss}$$
(:=direct sum of Jordan–Hölder factors). Same trace and determinant as $\overline{\rho}_{\tilde{f}}$.
- Since $\ker \rho_f \subseteq \ker \rho_{\tilde{f}}$, this is unramified outside of pN.
- Remains to show: ρ_f already defined over $\mathbb{F}_f$: For this, STP:
$$\rho_f = \sigma \circ \rho_f \quad \text{ for all } \sigma \in \mathrm{Gal}(\mathbb{F}_{\mathfrak{p}}|\mathbb{F}_f).$$
- Certainly, all characteristic polynomials of Frob_ℓ for $\ell \nmid pN$,
$$X^2 - a_\ell X + \varepsilon(\ell),$$
are preserved by σ, since $a_\ell, \varepsilon(\ell) \in \mathbb{F}_f$ by definition. Now use:

  Theorem 10.11 (Brauer–Nesbitt)**.** *Let F be a perfect field. Let G be a finite group. Let V, W be two semi-simple finite dimensional representations of G over F. Then $V \cong W$ if and only if*
$$\mathrm{charpoly}(g|V) = \mathrm{charpoly}(g|W) \quad \textit{for all } g \in G.$$
- This finishes the construction of $f \rightsquigarrow \rho_f$. $\square$

Serre's conjecture

- A quick digression before we continue with weight 1 forms:
- In 73', Serre gave a conjecture that described exactly which mod p Galois representations arise in this way. It became known as

Conjecture 10.12 (Serre's Modularity Conjecture, or Serre's Conjecture)**.** *Let $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be a continuous Galois representation that is odd and irreducible. Then there is a mod p eigenform g such that*
$$\rho = \rho_g.$$

In 1987, Serre published an article refining this conjecture:

- He predicted the optimal level of g (in terms of the Artin conductor of ρ)
- He predicted the optimal weight (recipe in terms of ramification of ρ at p)
- He showed that his conjecture would imply Fermat's Last Theorem.
- Serre's Conjecture is now a theorem by Khare–Wintenberger (2008).

Lifting ρ_f to $\mathbb{C}$

- Back to our earlier setup: $f \in M_1(\Gamma_0(N), \varepsilon)$ eigenform
- K number field generated by eigenvalues and ε.
- Let $L :=$ set of primes of $\mathbb{Q}$ which decompose completely in K. This is an infinite set by Chebotarev.
- Let $\ell \in L$ and λ a place over ℓ in K. Let $\mathcal{O}_{K,(\lambda)} \subseteq K$ be λ-integral elements.
- BNow apply previous section with role of p played by varying $\ell \in L$.
- For any such λ, consider reduction $\overline{f}$ of f mod λ. This is a mod ℓ eigenform.
- Theorem 10.8 associates a semi-simple Galois representation

$$\rho_{\overline{f},\ell} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F}_\ell)$$

  (over $\mathbb{F}_\ell$ as λ completely split) with charpoly of Frob_p for any $p \nmid \ell N$ given by

$$\mathrm{charpoly}(\mathrm{Frob}_p | \rho_{\overline{f},\ell}) = X^2 - a_p X + \varepsilon(p).$$

- Zeros of this are n-th units roots, which we may assume are all in $\mathcal{O}_K$ (by the next lemma) and thus in $\mathbb{F}_l$. Thus

$$= (X - \overline{a})(X - \overline{b}) \quad \text{for some } \overline{a}, \overline{b} \in \mathbb{F}_\ell^\times.$$

Definition 10.13. *Let $G_\ell := \mathrm{im}(\rho_{\overline{f},\ell}) \subseteq \mathrm{GL}_2(\mathbb{F}_\ell)$. This is a finite group.*

Lemma 10.14. *If f is of weight 1, there is $A \in \mathbb{N}$ such that $|G_\ell| \leq A$ for all $\ell \in L$.*

Proof. After using an estimate on the coefficients of weight 1 modular forms this becomes purely group theoretical: use classification of subgroups of $\mathrm{GL}_2(\mathbb{F}_\ell)$. We shall not discuss it, but if you want to see it: [DS74, Lemma 8.4]. $\square$

- Enlarge K so that it contains all n-th unit roots for $n \leq A$. This shrinks L.
- Also exclude the primes $\leq A$ from L. Then L is still infinite.

Lemma 10.15. *Let G be a finite group of order prime to ℓ. Then any representation $\rho : G \to \mathrm{GL}_2(\mathbb{F}_\ell)$ lifts to a representation*

$$\tilde{\rho} : G \to \mathrm{GL}_2(\mathcal{O}_{K,(\lambda)}).$$

Proof. **Step 1: Lift to $\mathrm{GL}_2(\mathbb{Z}_\ell)$:**

- STP: any $G \to \mathrm{GL}_2(\mathbb{Z}/\ell^n\mathbb{Z})$ lifts to $G \to \mathrm{GL}_2(\mathbb{Z}/\ell^{n+1}\mathbb{Z})$.
- For this, consider the short exact sequence

$$1 \to 1 + \ell^n M_2(\mathbb{F}_\ell) \to \mathrm{GL}_2(\mathbb{Z}/\ell^{n+1}\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{Z}/\ell^n\mathbb{Z}) \to 1$$

- The first group is isomorphic to $M_2(\mathbb{F}_\ell)$ via $1 + \ell^n x \mapsto x$.
- Now use non-abelian group cohomology: Consider the left-action

$$G \text{ acts on } \mathrm{Map}(G, \mathrm{GL}_2(\mathbb{Z}/\ell^n\mathbb{Z}) \text{ via } \quad g \cdot \varphi := \varphi(g)\varphi(g)^{-1}.$$

- Then

$$\mathrm{Hom}(G, \mathrm{GL}_2(\mathbb{Z}/\ell^n\mathbb{Z})) = \mathrm{Map}(G, \mathrm{GL}_2(\mathbb{Z}/\ell^n\mathbb{Z}))^G.$$

- Apply Map to the above short exact sequence, then G gives a long exact sequence of non-abelian group cohomology

$$\cdots \to \mathrm{Hom}(G, \mathrm{GL}_2(\mathbb{Z}/\ell^{n+1})) \to \mathrm{Hom}(G, \mathrm{GL}_2(\mathbb{Z}/\ell^n)) \to H^1(G, \mathrm{Map}(G, M_2(\mathbb{F}_\ell)))$$

- Fact in group cohomology: $H^1(G, A) = 0$ if $|G|$, $|A|$ finite and coprime.

Step 2: Already defined over number field

- Since G is finite, ρ is semi-simple.
- Traces must be sums of roots of unity of order $\leq |G| \leq A \rightsquigarrow$ contained in K.
- Brauer–Nesbitt ensures that ρ is already defined over K.
- Finally, $K \cap \mathbb{Z}_\ell = \mathcal{O}_{K,(\lambda)}$ $\square$

Back to prove of Theorem 10.2 (Galois representation for weight 1 forms)

- Apply this to $\rho_g : G_\ell \hookrightarrow \mathrm{GL}_2(\mathbb{F}_\ell)$, get $G_\ell \to \mathrm{GL}_2(\mathcal{O}_{K,(\lambda)})$
- Compose with $G_\mathbb{Q} \to G_\ell$ to get Galois representation

$$\tilde{\rho}_{g,\ell} : G_\mathbb{Q} \to \mathrm{GL}_2(\mathcal{O}_{K,(\lambda)}).$$

Proposition 10.16. *The lift $\tilde{\rho}_{g,\ell} : G_\mathbb{Q} \to \mathrm{GL}_2(\mathcal{O}_{K,(\lambda)})$ is such that for all $p \nmid \ell N$,*

$$\mathrm{charpoly}(\mathrm{Frob}_p | \tilde{\rho}_{g,\ell}) = X^2 - a_p X + \varepsilon(p).$$

The key is to vary ℓ: Recall that we get $\tilde{\rho}_{g,\ell}$ for all $\ell \in L$.

Definition 10.17. *Let Y be the set of polynomials of the form $(X - \alpha)(X - \beta)$ for $\alpha, \beta \in \mathcal{O}_{K,(\lambda)}$ roots of unity of order $\leq |A|$. This is a finite set.*

- Note: Given $n \leq A$, $\ell \in L$, and n-th root of unity $x \in \mathbb{F}_\ell^\times$, there is exactly one lift to root of unity in K (existence: $n \leq A$, uniqueness: $\ell \geq A$).
- Consequence: If $F, G \in Y$, then $F \equiv G \bmod \lambda$ implies $F = G$.
- Idea: Apply this to $F = X^2 - a_p X + \varepsilon(p)$ and $G = \mathrm{charpoly}(\mathrm{Frob}_p | \tilde{\rho}_{g,\ell})$

Lemma 10.18. *$X^2 - a_p X + \varepsilon(p) \in Y$.*

Proof. For all λ over a prime in L, there is $F \in Y$ for which

$$\mathrm{charpoly}(\mathrm{Frob}_p | \rho_{g,\ell}) \equiv F \bmod \lambda$$

since $\rho_{g,\ell}$ has finite image. Thus also

$$X^2 - a_p X + \varepsilon(p) \equiv F \bmod \lambda.$$

Since Y is finite, there is F for which this holds for infinitely many λ. Thus

$$X^2 - a_p X + \varepsilon(p) = F \in Y$$

$\square$

Proof of Proposition.

- As G_ℓ finite of order $\leq A$, have

$$\mathrm{charpoly}(\mathrm{Frob}_p | \tilde{\rho}_{g,\ell}) \in Y$$

- $\mathrm{charpoly}(\mathrm{Frob}_p | \tilde{\rho}_{g,\ell}) \equiv \mathrm{charpoly}(\mathrm{Frob}_p | \rho_{g,\ell}) \equiv X^2 - a_p X + \varepsilon(p) \bmod \lambda$
- Thus both sides are in Y, which implies

$$\mathrm{charpoly}(\mathrm{Frob}_p | \rho_{g,\ell}) = X^2 - a_p X + \varepsilon(p).$$

$\square$

Remains to prove:

- ρ is odd: This is simply because ε is odd and c has order 2.
- ρ is irreducible: Use complex analytic estimate due to Rankin $\square$

This finishes the construction of eigenforms $\rightsquigarrow$ Galois representations!

Summary of the weight 1 construction

$$
\begin{tikzcd}
\left\{\begin{matrix}\text{eigenforms} \\ \text{of weight 1}\end{matrix}\right\} \arrow[r, dotted, "\text{congruence}"] \arrow[d, "\text{reduce}"] & \left\{\begin{matrix}\text{eigenforms of} \\ \text{weight} \geq 2\end{matrix}\right\} \arrow[r, "\text{Deligne's}", "\text{construction}"'] & \left\{\begin{matrix}\ell\text{-adic Galois} \\ \text{representations}\end{matrix}\right\} \arrow[dr, "\text{reduce}"] & & \left\{\begin{matrix}\text{complex Galois} \\ \text{representations}\end{matrix}\right\} \\
\left\{\begin{matrix}\text{mod } \ell \text{ eigenforms} \\ \text{of weight 1}\end{matrix}\right\} \arrow[r, "\cdot\mathrm{Ha}"] & \left\{\begin{matrix}\text{mod } \ell \text{ eigenforms} \\ \text{of weight} \geq 2\end{matrix}\right\} \arrow[u, "\text{DS-lifting}"'] \arrow[rr, dotted, "\text{mod } \ell \text{ construction}", "\text{Theorem 10.8}"'] & & \left\{\begin{matrix}\text{mod } \ell \text{ Galois} \\ \text{representations}\end{matrix}\right\} \arrow[ur, "\text{lift}"] &
\end{tikzcd}
$$

11. The Langlands program for general groups, part I

Next aims:

- More or less precise statements (for parts of) Langlands program for GL_n, or even general G reductive over $\mathbb{Q}$.

Langlands(-Clozel-Fontaine-Mazur) reciprocity for $\mathrm{GL}_{n,F}$:

- F arbitrary number field
- ℓ some prime. Fix an isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}}_\ell$.
- Then for any $n \geq 1$, there exists a (unique) bijection between
  i) the set of L-algebraic cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$,
  ii) the set of (isomorphism classes) of irreducible continuous representations $\mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ which are almost everywhere unramified, and de Rham at places dividing ℓ,

  such that the bijection matches Satake parameters with eigenvalues of Frobenius elements.

Comments:

- This is only a part of the Langlands program for $G = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_{n,F}$.
- This does not describe all automorphic representations for $\mathrm{GL}_{n,F}$, nor all Galois representations.
- Not known for $F = \mathbb{Q}, n \geq 2$, e.g., there exists L-algebraic cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A})$ (associated to Maassforms), which should correspond to *even* irreducible, 2-dimensional ℓ-adic representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ (with finite image).
- Langlands-Tunnell: If the image is solvable, automorphicity of the Galois representation is known, cf. [GH19, Theorem 13.4.5.]. This does not cover almost everywhere unramified 2-dimensional, irreducible Galois representations with image $\mathrm{SL}_2(\mathbb{F}_5)$.
- Unicity follows from the (strong) multiplicity one theorems on both sides.
- Today: Explain "L-algebraic".

Automorphic forms for general groups:

- G an arbitrary reductive group over $\mathbb{Q}$.
- Will introduce a convenient (=more algebraic) replacement for $L^2([G])$.
- Recall for $G = \mathrm{GL}_2 = \mathrm{GL}_{2,\mathbb{Q}}$, $k \in \mathbb{Z}$, the embedding

$$\Phi\colon M_k \to C^\infty(\mathrm{GL}_2(\mathbb{A})),\ f \mapsto \varphi_f.$$

- Recall the description of the image of $\Phi(M_k)$ resp. $\Phi(S_k)$, namely:
  - For $k \in \mathbb{Z}$ an element $\varphi \in C^\infty(\mathrm{GL}_2(\mathbb{A}))$ lies in the image of

$$\Phi\colon M_k \to C^\infty(\mathrm{GL}_2(\mathbb{A}))$$

    if and only if
    * $\varphi(g, g_\infty z) = z^{-k}\varphi(g, g_\infty)$ for $(g, g_\infty) \in \mathrm{GL}_2(\mathbb{A})$, $z \in \mathbb{C}^\times \subseteq \mathrm{GL}_2(\mathbb{R})$.
    * φ is of moderate growth (implies the vanishing of negative Fourier coefficients).
    * For each $g \in \mathrm{GL}_2(\mathbb{A}_f)$

$$Y * \varphi(g, -) = 0,$$

where $Y \in \mathfrak{g}_{\mathbb{C}} = \mathfrak{gl}_2(\mathbb{R})_{\mathbb{C}}$ is a suitably constructed, natural element (implies holomorphicity).

* $\varphi(\gamma g) = \varphi(g)$ for all $\gamma \in \mathrm{GL}_2(\mathbb{Q})$.

– f is moreover cuspidal if and only if

$$\int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})} \varphi_f(ng)dn = 0$$

for all unipotent radicals N in proper parabolics $P \subseteq G$ (defined over $\mathbb{Q}$).

- Fix a maximal compact (usually not connected) subgroup K_∞ of $G(\mathbb{R})$, e.g., $\mathrm{O}_2(\mathbb{R}) = \mathrm{SO}_2(\mathbb{R}) \cup s\mathrm{SO}_2(\mathbb{R}) \subseteq \mathrm{GL}_2(\mathbb{R})$ with $s = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

Definition 11.1 (cf. [GH19, Definition 6.5])**.** *Let G be a reductive group over $\mathbb{Q}$. An (adélic) automorphic form for G is a smooth function $\varphi\colon G(\mathbb{A}) \to \mathbb{C}$ such that*

1) *φ is of moderate growth (was defined in Section 4, not important for this lecture).*
2) *φ is right K_∞-finite, i.e., the functions $g \mapsto \varphi(gk)$ for $k \in K_\infty \subseteq G(\mathbb{A})$ span a finite dimensional vector space.*
3) *φ is killed by an ideal in $Z(\mathfrak{g}_{\mathbb{C}})$ of finite codimension, where $\mathfrak{g}_{\mathbb{C}}$ denotes the (complexified) Lie algebra of $G(\mathbb{R})$, and $Z(\mathfrak{g}_{\mathbb{C}})$ the center of the enveloping algebra $U(\mathfrak{g}_{\mathbb{C}})$ of $\mathfrak{g}_{\mathbb{C}}$.*
4) *φ is left $G(\mathbb{Q})$-invariant, i.e., $\varphi(\gamma g) = \varphi(g)$ for all $\gamma \in G(\mathbb{Q})$.*

The Casimir element for GL_2:

- Let us check that 3) is indeed verified for $\varphi_f\colon C^\infty(\mathrm{GL}_2(\mathbb{A})) \to \mathbb{C}$ with $f \in M_k$.
- Namely, the element Y was not in $Z(\mathfrak{gl}_{2,\mathbb{C}})$.
- Set

$$H := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad X := \frac{1}{2}\begin{pmatrix} -1 & -i \\ -i & 1 \end{pmatrix}, \quad Y := \frac{1}{2}\begin{pmatrix} -1 & i \\ i & 1 \end{pmatrix}, \quad Z := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

- Then
  *
  $$H * \varphi_f = -k\varphi_f$$
  as $z \cdot \varphi_f = z^{-k}\varphi_f$ for $z \in \mathbb{C}^\times$.
  * $Y * \varphi_f = 0$.
- $[X, Y] = XY - YX = H$ (because H, X, Y form an $\mathfrak{sl}_2$-triple)
- Define the Casimir operator

$$\Delta := \frac{1}{4}(H^2 + 2XY + 2YX).$$

Then $\Delta \in Z(\mathfrak{gl}_{2,\mathbb{C}})$.

- In fact, $Z(\mathfrak{gl}_{2,\mathbb{C}})$ is generated, as a $\mathbb{C}$-algebra, by Δ and $Z \in \mathfrak{gl}_{2,\mathbb{C}}$ (this follows from the Harish-Chandra isomorphism, cf. [GH19, Theorem 4.6.1]).
- One calculates (using $Y * \varphi_f = 0$)

$$\Delta * \varphi_f = \frac{1}{4}(H^2 - 2XY + 2YX) * \varphi_f = \frac{1}{4}(H^2 - 2H) * \varphi_f = \frac{1}{4}(k^2 - 2k)\varphi_f,$$

thus φ_f is indeed killed by the ideal $\langle \Delta - \frac{1}{4}(k^2-2k), Z+k\rangle \subseteq Z(\mathfrak{gl}_{2,\mathbb{C}})$ of finite codimension.

More on automorphic forms:

- G arbitrary reductive over $\mathbb{Q}$.
- Set $\mathcal{A}(G)$ as the space of automorphic forms, and $\mathcal{A}([G])$ as the subspace of automorphic forms, which are invariant under A_G (acting via left or right translations on $G(\mathbb{A})$).
- Unfortunately, the spaces $\mathcal{A}([G])$ and $\mathcal{A}(G)$ are not stable under $G(\mathbb{R})$, because K_∞-finiteness need not be preserved.

More on automorphic forms (continued):

- Clearly, $G(\mathbb{A}_f)$ acts on $\mathcal{A}(G)$ via $\varphi \mapsto (g \mapsto \varphi(gh))$ for $h \in G(\mathbb{A}_f)$.
- Similarly, K_∞, even $A_G K_\infty$, acts on $\mathcal{A}(G)$
- Less clear, but true (cf. [GH19, Proposition 4.4.2.]): $\mathfrak{g}_\mathbb{C}$ acts on $\mathcal{A}(G)$.

$\Rightarrow$ $U(\mathfrak{g}_\mathbb{C})$, in particular $Z(\mathfrak{g}_\mathbb{C}) \subseteq U(\mathfrak{g}_\mathbb{C})$, acts on $\mathcal{A}(G)$ by differential operators.

- The $\mathfrak{g}_\mathbb{C}$ and K_∞-action make $\mathcal{A}(G)$ into a $(\mathfrak{g}_\mathbb{C}, K_\infty)$-module.

Definition 11.2 (cf. [GH19, Definition 4.5.]). *A $(\mathfrak{g}_\mathbb{C}, K_\infty)$-module is a $\mathbb{C}$-vector space V, which is simultaneously a $\mathfrak{g}_\mathbb{C}$- and K_∞-module, such that*

1) *V is a union of finite dimensional K_∞-stable subspaces.*
2) *For $X \in \mathfrak{k} := \mathrm{Lie}(K_\infty)_\mathbb{C}$ and $\varphi \in V$ we have*

$$X * \varphi = \frac{\partial}{\partial t}(\exp(tX)\varphi)_{t=0}.$$

3) *For $h \in K_\infty$, $X \in \mathfrak{g}_\mathbb{C}$ and $\varphi \in V$ we have*

$$h(X * (h^{-1}\varphi) = (\mathrm{Ad}(h)(X)) * \varphi.$$

Comments:

- We required no topology on V, thus $(\mathfrak{g}_\mathbb{C}, K_\infty)$-modules are much more algebraic than $G(\mathbb{R})$-representations, and thus easier to handle.
- In 2) the limit is taken for the natural topology of a finite dimensional K_∞-stable subspace of V containing φ.
- For each representation of $G(\mathbb{R})$ on some Hilbert space $\mathbb{C}$-vector space, the space of smooth and K_∞-finite vectors is naturally a $(\mathfrak{g}_\mathbb{C}, K_\infty)$-module, cf. [GH19, Proposition 4.4.2.], and on *unitary* representations one does not loose any informations, cf. [GH19, Theorem 4.4.4.], namely: isomorphism classes of irreducible *unitary* representations of $G(\mathbb{R})$ inject into isomorphism classes of irreducible $(\mathfrak{g}_\mathbb{C}, K_\infty)$-modules.
- Moreover, the $(\mathfrak{g}_\mathbb{C}, K_\infty)$-modules obtained in this way are *admissible*, i.e., for each (finite dimensional) irreducible representation σ of K_∞, the σ-isotypic component is finite dimensional, cf. [GH19, Theorem 4.4.1.].
- Version of Schur's lemma $\Rightarrow$ On each irreducible, admissible $(\mathfrak{g}_\mathbb{C}, K_\infty)$-module π the center $Z(\mathfrak{g}_\mathbb{C})$ of $U(\mathfrak{g}_\mathbb{C})$ acts via some morphism

$$\omega_\pi \colon Z(\mathfrak{g}_\mathbb{C}) \to \mathbb{C}$$

of $\mathbb{C}$-algebras, called the "infinitesimal character of π".

- The Harish-Chandra isomorphism, cf. [GH19, Theorem 4.6.1.], furnishes

$$Z(\mathfrak{g}_{\mathbb{C}}) \cong U(\mathfrak{t}_{\mathbb{C}})^W$$

for each Cartan subalgebra $\mathfrak{t} \subseteq \mathfrak{g}$ and W the corresponding Weyl group. Note $U(\mathfrak{t}_{\mathbb{C}}) \cong \mathrm{Sym}^\bullet \mathfrak{t}_{\mathbb{C}}$ because $\mathfrak{t}_{\mathbb{C}}$ is an abelian Lie algebra.
- As W is finite, $\mathbb{C}$-algebra morphisms

$$Z(\mathfrak{g}_{\mathbb{C}}) \cong U(\mathfrak{t}_{\mathbb{C}})^W \to \mathbb{C}$$

correpond bijectively to W-orbits of $\mathbb{C}$-algebra homomorphisms

$$U(\mathfrak{t}_{\mathbb{C}}) \cong \mathrm{Sym}^\bullet \mathfrak{t}_{\mathbb{C}} \to \mathbb{C}.$$

Moreover,

$$\mathrm{Hom}_{\mathbb{C}-\mathrm{alg}}(U(\mathfrak{t}_{\mathbb{C}}), \mathbb{C}) \cong \mathrm{Hom}_{\mathbb{C}-\mathrm{lin}}(\mathfrak{t}_{\mathbb{C}}, \mathbb{C}) =: \mathfrak{t}_{\mathbb{C}}^\vee.$$

$U(\mathfrak{t}_{\mathbb{C}})^W \to U(\mathfrak{t}_{\mathbb{C}})$ is finite.
- Thus, the infinitesimal character defines a canonical W-orbit of elements in the "weight space" $\mathfrak{t}_{\mathbb{C}}^\vee$.
- Now assume that $\mathfrak{t} = \mathrm{Lie}(T)$ for some maximal torus $T \subseteq G_{\mathbb{R}}$. Then W-equivariantly

$$X^*(T_{\mathbb{C}}) \subseteq \mathfrak{t}_{\mathbb{C}}^\vee.$$

- An irreducible, admissible $(\mathfrak{g}_{\mathbb{C}}, K_\infty)$-module π is called L-algebraic if the W-orbit in $\mathfrak{t}_{\mathbb{C}}^\vee$ corresponding to the infinitesimal character ω_π lies in the finite free $\mathbb{Z}$-module $X^*(T_{\mathbb{C}}) \subseteq \mathfrak{t}_{\mathbb{C}}^\vee$, cf. [BG10, Definition 2.3.1.], [Tho, Definition 93].

$(\mathfrak{gl}_{2,\mathbb{C}}, \mathrm{O}_2(\mathbb{R}))$**-modules generated by modular forms:**

- Let $f \in M_k$. We will analyze the $(\mathfrak{gl}_{2,\mathbb{C}}, \mathrm{O}_2(\mathbb{R}))$-module $V \subseteq \mathcal{A}(\mathrm{GL}_2)$ generated by $\varphi := \varphi_f$.
- Recall

$$Y * \varphi = 0, \quad z \cdot \varphi = z^{-k}\varphi \text{ for } z \in \mathbb{C}^\times.$$

- It is not difficult to see that there exists a unique, up to isomorphism, $(\mathfrak{gl}_{2,\mathbb{C}}, \mathrm{O}_2(\mathbb{R}))$-module D'_{k-1} (the dual of D_{k-1} in [Del73, Section 2.1]) generated by such an element φ, and that $D'_{k-1} \cong V$ is irreducible.
- In particular, it only depends on k, not f.
- We obtain, as was mentioned some time ago,

$$M_k \cong \mathrm{Hom}_{(\mathfrak{gl}_{2,\mathbb{C}}, \mathrm{O}_2(\mathbb{R}))}(D'_{k-1}, \mathcal{A}(\mathrm{GL}_2)),$$

(note: the space $\mathcal{H}(G_{\mathbb{A}})$ in [Del73, Scholie 2.1.3.] agrees with our $\mathcal{A}(\mathrm{GL}_2)$ only up to inversion on $\mathrm{GL}_2(\mathbb{A})$).
- We calulated

$$\Delta * \varphi = \frac{1}{4}(k^2 - 2k)\varphi,\ Z * \varphi = -k\varphi,$$

and this determines the infinitesimal character $\omega_{D'_{k-1}}$.
- The W-orbit of $\mathbb{C}$-algebra homomorphisms $U(\mathfrak{t}_{\mathbb{C}}) \cong \mathbb{C}[Z, H] \to \mathbb{C}$ corresponding to $\omega_{D'_{k-1}}$ under the Harish-Chandra isomorphism is given by the homomorphisms

$$Z \mapsto -k,\ H \mapsto \pm(k-1).$$

- The lattice $X^*(T) \subseteq \langle Z, H\rangle_{\mathbb{C}}^{\vee}$ is given by $\mathbb{C}$-linear maps

$$Z \mapsto a, \;\; H \mapsto b$$

with $a, b \in \mathbb{Z}$ *and* $a \equiv b \bmod 2$.
- Thus, D'_{k-1} is *not* L-algebraic (it is C-algebraic in the sense of [BG10]).
- However,

$$D'_{k-1} \otimes_{\mathbb{C}} |\det|^{1/2+a}$$

is L-algebraic for any $a \in \mathbb{Z}$ because (the $(\mathfrak{gl}_{2,\mathbb{C}}, \mathrm{O}_2(\mathbb{R}))$-module associated with) $|\det|^{1/2}$ has infinitesimal character corresponding to

$$Z \mapsto 1, \;\; H \mapsto 0.$$

Upshot:

- For any G there exists the space $\mathcal{A}(G)$ of (adélic) automorphic forms, which for GL_2 naturally contains the φ_f with $f \in M_k$.
- The space $\mathcal{A}(G)$ carries a $(\mathfrak{g}_{\mathbb{C}}, K_\infty) \times G(\mathbb{A}_f)$-action, and is a more algebraic replacement for $L^2([G])$ with its $G(\mathbb{A})$-action.

A new notion of automorphic representations:

- We redefine the notion of an automorphic representation using $\mathcal{A}(G)$ instead of $L^2([G])$.
- Namely, an automorphic representation for G is an irreducible $G(\mathbb{A}_f) \times (\mathfrak{g}_{\mathbb{C}}, K_\infty)$-subquotient of $\mathcal{A}(G)$, cf. [GH19, Definition 6.8.].
- The former automorphic representations will from now on be called "L^2-automorphic representations".
- For informations how both notions relate, cf. [GH19, Section 6.5.].

Flath's theorem (cf. [GH19, Theorem 5.7.1.]**):**

- For almost all primes p the reductive group $G_{\mathbb{Q}_p} := G \times_{\mathrm{Spec}(\mathbb{Q})} \mathrm{Spec}(\mathbb{Q}_p)$ is "unramified", i.e., extends to a reductive group scheme

$$\mathcal{G}_p \to \mathrm{Spec}(\mathbb{Z}_p).$$

- Equivalently, $G_{\mathbb{Q}_p}$ is quasi-split (=contains a Borel subgroup defined over $\mathbb{Q}_p$) and split(=contains a maximal and split torus) over an unramified extension.
- Then

$$G(\mathbb{A}) = \prod_p{}' (G(\mathbb{Q}_p), \mathcal{G}_p(\mathbb{Z}_p)) \times G(\mathbb{R}),$$

where $\mathcal{G}_p(\mathbb{Z}_p) \subseteq G(\mathbb{Q}_p)$ is a compact-open subgroup (cf. [GH19, Proposition 2.3.1.]), called "hyperspecial".
- Flath's theorem (cf. [GH19, Section 5.7.] and [Fla79]) states that irreducibe, admissible $G(\mathbb{A}_f) \times (\mathfrak{g}_{\mathbb{C}}, K_\infty)$-modules π decompose (uniquely) into a "restricted tensor product"

$$\pi \cong \bigotimes_{p \text{ prime}}{}' \pi_p \otimes \pi_\infty$$

of irreducible, smooth (even admissible) $G_{\mathbb{Q}_p}$-representations π_p and an irreducible, admissible $(\mathfrak{g}_{\mathbb{C}}, K_\infty)$-module π_∞.

- Moreover, for almost all primes p the representation π_p is unramified, i.e., the group $G_{\mathbb{Q}_p}$ is unramified and $\pi_p^{\mathcal{G}_p(\mathbb{Z}_p)} \neq 0$, where $\mathcal{G}_p$ is a reductive model of $G_{\mathbb{Q}_p}$.
- Important, cf. [GH19, Theorem 7.5.1.]: $\mathcal{H}(G(\mathbb{Q}_p), \mathcal{G}_p(\mathbb{Z}_p))$ is again commutative! ($\Rightarrow$ If $\pi_p^{\mathcal{G}_p(\mathbb{Z}_p)} \neq 0$, then its dimension is one.)

Upshot:

- Thus irreducible, admissible $G(\mathbb{A}_f) \times (\mathfrak{g}_{\mathbb{C}}, K_\infty)$-modules are a collection of "local data".
- Being automorphic puts strong relations among these local data.
- For almost all primes p get homomorphism

$$\chi_p \colon \mathcal{H}(G(\mathbb{Q}_p), \mathcal{G}_p(\mathbb{Z}_p)) \to \mathbb{C},$$

  similarly to case for $\mathrm{GL}_2 \Rightarrow$ this will yield analog of a "system of Hecke eigenvalues".
- For π set

$$\pi_f := \bigotimes_p{}' \pi_p,$$

  thus

$$\pi \cong \pi_f \otimes \pi_\infty.$$

Definition 11.3 (L-algebraic automorphic representation)**.** *An automorphic representation*

$$\pi \cong \pi_f \otimes \pi_\infty$$

is L-algebraic if the irreducible, admissible $(\mathfrak{g}_{\mathbb{C}}, K_\infty)$*-module* π_∞ *is L-algebraic, cf.* [BG10, Definition 3.1.1.].

Examples:

- An automorphic representation $\pi \subseteq \mathcal{A}(\mathrm{GL}_2)$ which is generated by a modular form is not L-algebraic, but the twist

$$\pi \otimes_{\mathbb{C}} |\det|_{\text{adélic}}^{1/2+a}$$

  is for any $a \in \mathbb{Z}$.
- A continuous character $\psi \colon \mathbb{Q}^\times \backslash \mathbb{A}^\times \to \mathbb{C}$ is L-algebraic if and only if

$$\psi = \chi |-|_{\text{adélic}}^k$$

  for some $k \in \mathbb{Z}$, $|-|_{\text{adélic}}$ the adélic norm and $\chi \colon \mathbb{Q}^\times \backslash \mathbb{A}^\times \to \mathbb{C}$ a character of finite order.

12. The Langlands program for general groups, part II

Last time:

- F arbitrary number field
- The Langlands(-Clozel-Fontaine-Mazur) reciprocity conjecture for $\mathrm{GL}_{n,F}$ says:
  - ℓ some prime. Fix an isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}}_\ell$.
  - Then for any $n \geq 1$, there exists a (unique) bijection between
    - i) the set of L-algebraic cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$,
    - ii) the set of (isomorphism classes) of irreducible continuous representations $\mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ which are almost everywhere unramified, and de Rham at places dividing ℓ,

    such that the bijection matches Satake parameters with eigenvalues of Frobenius elements.
- Introduced space of (adélic) automorphic forms $\mathcal{A}(G)$ for any G.
- Flath's theorem $\Rightarrow$ automorphic representations π factorize: $\pi \cong \bigotimes'_p \pi_p \otimes \pi_\infty$.
- Explained "L-algebraic" (=infinitesimal character of π_∞ is "integral").

Today:

- Introduce "cuspidal" automorphic representations.
- Explain Satake parameters.
- Discuss some expectations of the Langlands program for general G.

Definition 12.1 ([GH19, Definition 9.2])**.** *Let $G/\mathbb{Q}$ be reductive. We call $\varphi \in L^2([G])$ resp. $\varphi \in \mathcal{A}(G)$ cuspidal if*

$$\int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})} \varphi(ng)dn = 0$$

for all unipotent radicals N of proper parabolic subgroups $P \subseteq G$ (defined over $\mathbb{Q}$), and almost all $g \in G(\mathbb{A})$.

Cuspidal automorphic representations:

- Set $L^2_{\mathrm{cusp}}([G])$ resp. $\mathcal{A}_{\mathrm{cusp}}(G)$ resp. $\mathcal{A}_{\mathrm{cusp}}([G])$ as the subspaces of cuspidal elements.
- There is an embedding with dense image

$$\mathcal{A}_{\mathrm{cusp}}([G]) \subseteq L^2_{\mathrm{cusp}}([G]),$$

  i.e., cuspidal automorphic forms satisfy a growth condition strong enough to make them L^2 (they are "rapidly decreasing"), cf. [GH19, Section 6.5.].
- An L^2-automorphic representation of $G(\mathbb{A})$ is cuspidal if it is isomorphic to a subquotient (actually subrepresentation) of $L^2_{\mathrm{cusp}}([G])$.
- Similarly, an automorphic representation is cuspidal if it occurs as a subquotient of $\mathcal{A}_{\mathrm{cusp}}(G)$.
- Gelfand, Piatetski-Shapiro: As a unitary $G(\mathbb{A})$-representation

$$L^2_{\mathrm{cusp}}([G]) \cong \widehat{\bigoplus_{\pi \in \widehat{G(\mathbb{A})}}} m_\pi \pi$$

  with each m_π finite, i.e., $m_\pi \in \mathbb{N} \cup \{0\}$, cf. [GH19, Corollary 9.1.2].
- In particular, $L^2_{\mathrm{cusp}}([G]) \subset L^2_{\mathrm{disc}}([G])$ (note: the trivial representation occurs in $L^2_{\mathrm{disc}}([G]) \setminus L^2_{\mathrm{cusp}}([G])$).

- From
$$L^2_{\mathrm{cusp}}([G]) \cong \widehat{\bigoplus_{\pi\in\widehat{G(\mathbb{A})}}} m_\pi \pi$$
on deduces the decomposition (into irreducible, admissible $G(\mathbb{A}_f)\times(\mathfrak{g}_\mathbb{C}, K_\infty)$-modules)
$$\mathcal{A}_{\mathrm{cusp}}([G]) \cong \bigoplus_{\pi\in\widehat{G(\mathbb{A})}} m_\pi \pi^{K_\infty-\mathrm{finite}},$$
where $\pi^{K_\infty-\mathrm{finite}} \subseteq \pi$ denotes the (dense) subspace of K_∞-finite vectors, cf. [GH19, Theorem 4.4.4.].
- Thus, the decomposition of $L^2_{\mathrm{cusp}}([G])$ is "the same as" the decomposition of $\mathcal{A}_{\mathrm{cusp}}([G])$.
- Piatetski-Shapiro, Shalika, cf. [GH19, Theorem 11.3.4]: F number field, $G = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_{n,F}$. Then the $G(\mathbb{A})$-representation on $L^2_{\mathrm{cusp}}([G])$ is multiplicity free ("multiplicity one").
- "Strong multiplicity one": Assume π, π' are cuspidal automorphic representations for $\mathrm{GL}_{n,F}$. If $\pi_p \cong \pi'_p$ for almost all primes p, then $\pi \cong \pi'$, cf. [GH19, 11.7.2].
- Not true without cuspidality, cf. [BG10, page 38].
- Moeglin, Waldspurger, cf. [GH19, Theorem 10.7.1.], [MW89]: $G = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_{n,F}$. Then $L^2_{\mathrm{disc}}([G])$ is known, up to parametrizing cuspidal automorphic representations of $\mathrm{GL}_{d,F}$ for $d|n$, and turns out to be multiplicity free.

Satake parameters:

- Describe π_p if π is unramified at p, i.e.,
$$\pi_p^{\mathcal{G}_p(\mathbb{Z}_p)} \neq 0$$
with $\mathcal{G}_p \to \mathrm{Spec}(\mathbb{Z}_p)$ a reductive model of $G_{\mathbb{Q}_p}$ (which exists for almost all primes p).
- Thus, describe $\mathbb{C}$-algebra homomorphisms
$$\mathcal{H}(G(\mathbb{Q}_p), K) \to \mathbb{C}$$
for $K := \mathcal{G}_p(\mathbb{Z}_p)$.
- Langlands/Satake, cf. [GH19, Theorem 7.5.1., Corollary 7.5.2.], [BG10, Section 2.1.]: There exists
  - a "natural" algebraic group $\hat{G}$ over $\overline{\mathbb{Q}}$,
  - an automorphism $\mathrm{Fr}_p\colon \hat{G}(\overline{\mathbb{Q}}) \cong \hat{G}(\overline{\mathbb{Q}})$,
  - and a bijection
$$\mathrm{Hom}_{\mathbb{C}-\mathrm{alg}}(\mathcal{H}(G(\mathbb{Q}_p), K), \mathbb{C})$$
$$\overset{1:1}{\leftrightarrow}$$
$$\{\text{Frobenius semisimple } \hat{G}(\mathbb{C}) - \text{conjugacy classes in } \hat{G}(\mathbb{C}) \rtimes \mathrm{Fr}_p^{\mathbb{Z}}\}.$$
- The group $\hat{G}_\mathbb{C}$ has the dual root datum as G. Thus for example, cf. [Bor79]:
  - if $G = \mathrm{GL}_n$, then $\hat{G} = \mathrm{GL}_{n,\mathbb{C}}$,
  - if $G = \mathrm{SL}_n$, then $\hat{G} = \mathrm{PGL}_{n,\mathbb{C}}$,
  - if $G = \mathrm{SO}_{2n}$, then $\hat{G} = \mathrm{SO}_{2n,\mathbb{C}}$,
  - if $G = \mathrm{SO}_{2n+1}$, then $\hat{G} = \mathrm{Sp}_{2n,\mathbb{C}}$,

- if $G = \mathrm{GSp}_{2n}$, then $\hat{G} = \mathrm{GSpin}_{2n+1,\mathbb{C}}$.

- If $G_{\mathbb{Q}_p}$ is split, then the automorphism Fr_p is trivial, and Frobenius semisimple conjugacy classes are just conjugacy classes of semisimple elements in $\hat{G}(\mathbb{C})$.
- If $G = \mathrm{GL}_n$, then semisimple conjugacy classes are uniquely determined by their characteristic polynomials.
- Let $\pi = \bigotimes'_p \pi_p \otimes \pi_\infty$ be an automorphic representation of G. Then we obtain the following analog of a system of Hecke eigenvalues:
  - a finite set S of primes, such that π_p is unramified for $p \notin S$,
  - for each $p \notin S$ a Frobenius semisimple conjugacy class $c_p(\pi)$ in $\hat{G}(\mathbb{C}) \rtimes \mathrm{Fr}_p^{\mathbb{Z}}$.
- For GL_n the eigenvalues of (each element in) $c_p(\pi)$ are the Satake parameters of π.
- Concretely, if the coset

$$\mathrm{GL}_n(\mathbb{Z}_p)\mathrm{Diag}(\underbrace{p,\ldots,p}_{i-\text{times}},\ \underbrace{1,\ldots,1}_{(n-i)-\text{times}})\mathrm{GL}_n(\mathbb{Z}_p),$$

  has eigenvalue $\widetilde{a}_{p,i}$ on $\pi_p^{\mathrm{GL}_n(\mathbb{Z}_p)}$, then the elements in $c_p(\pi)$ have characteristic polynomial

$$X^n - p^{\frac{(1-n)}{2}}\widetilde{a}_{p,1}X^{n-1} + \ldots + (-1)^i p^{\frac{i(i-n)}{2}}\widetilde{a}_{p,i}X^i + \ldots + (-1)^n\widetilde{a}_{p,n},$$

  cf. [GH19, Section 7.2.].
- Even more concrete, for π generated by a newform $f = \sum_{i=1}^{\infty} a_n q^n \in S_k(\Gamma_0(N),\chi)$ the $c_p(\pi), p \nmid N$, have characteristic polynomial

$$X^2 - p^{-1/2}pa_pX + \chi(p)p^k.$$

  Recall that $\pi \otimes_{\mathbb{C}} |\det|^{1/2}_{\text{adélic}}$ is L-algebraic. For $\pi \otimes_{\mathbb{C}} |\det|^{1/2}_{\text{adélic}}$ we obtain the polynomial

$$X^2 - a_pX + \chi(p)p^{k-1},$$

  which (after choosing an isomorphism $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$) was the characteristic polynomial of an arithmetic Frobenius at p.

The L-group:

- G reductive over $\mathbb{Q}$
- ℓ a prime.
- With a little work (cf. [GH19, Section 7.3.]) the group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on the reductive group $\hat{G}$ over $\overline{\mathbb{Q}}$ (the action is trivial if G is split).
- Set

$${}^LG := G(\overline{\mathbb{Q}}_\ell) \rtimes \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}).$$

- An L-parameter is by definition a $\hat{G}(\overline{\mathbb{Q}}_\ell)$-conjugacy class of continuous homomorphisms

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to {}^LG,$$

  whose projection to $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is the identity.
- If $F/\mathbb{Q}$ is finite and $G = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_{n,F}$, then an L-parameter identifies with an isomorphism class of an n-dimensional ℓ-adic Galois representation of $\mathrm{Gal}(\overline{F}/F)$.

The Buzzard–Gee conjecture for L-algebraic automorphic representations:

- Let ℓ be a prime.
- Fix an isomorphism $\iota\colon \mathbb{C} \cong \overline{\mathbb{Q}}_\ell$.
- Let π be an L-algebraic automorphic representation of G.
- Then Buzzard–Gee (cf. [BG10, Conjecture 3.2.2.]) conjecture that there exists an L-parameter
$$\rho_\pi\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to {}^LG$$
such that (in particular)
  - if $p \neq \ell$ is unramified for π, then each arithmetic Frobenius $\mathrm{Frob}_p \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ maps to the conjugacy class of
$$\iota(c_p(\pi)) \in \hat{G}(\overline{\mathbb{Q}}_\ell) \rtimes \mathrm{Frob}_p^{\mathbb{Z}}$$
under
$$\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \hat{G}(\overline{\mathbb{Q}}_\ell) \rtimes \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \twoheadrightarrow \hat{G}(\overline{\mathbb{Q}}_\ell) \rtimes \mathrm{Frob}_p^{\mathbb{Z}},$$
  - for each continuous representation ${}^LG \to \mathrm{GL}_N(\overline{\mathbb{Q}}_\ell)$, which is algebraic on $\hat{G}(\overline{\mathbb{Q}}_\ell)$, the composition
$$\mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell) \xrightarrow{\rho} {}^LG \to \mathrm{GL}_N(\overline{\mathbb{Q}}_\ell)$$
is de Rham.
- This generalizes the association of Galois representations to (a twist of) the automorphic representation associated to some newform.
- Contrary to the case of GL_n the L-parameter ρ_π is *not* conjectured to be unique, cf. [BG10, Remark 3.2.4.].
- Moreover, we may chose the same L-parameter ρ for different L-algebraic automorphic representations π.
- For GL_n the Buzzard–Gee conjecture predicts in addition to the previous Langlands-Clozel-Fontaine-Mazur conjecture the existence of Galois representations attached to possibly non-cuspidal (L-algebraic) automorphic representations. These Galois representations are then no longer conjectured to be irreducible.

The Langlands group $\mathcal{L}$:

- Conjecturally, there should exist a (very big) locally compact group $\mathcal{L}$, the (global) Langlands group, with (at least) the following properties, cf. [GH19, Section 12.6.], [Art02], [LR87]:
  - There exists a canonical surjection
$$\mathcal{L} \twoheadrightarrow \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$$
(in fact, $\mathcal{L} \twoheadrightarrow W_{\mathbb{Q}}$ with $W_{\mathbb{Q}}$ the global Weil group).
  - For each $n \geq 0$ the isomorphism classes $\mathrm{Irr}_n(\mathcal{L})$ of continuous irreducible n-dimensional $\mathbb{C}$-representations of $\mathcal{L}$ are naturally in bijection with the set Csp_n of cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A})$.

- Fix an isomorphism $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$. Then there is an injection of the set $\mathcal{G}_{\text{geom}}$ of isomorphism classes of irreducible, almost everywhere unramified ℓ-adic Galois representations $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ which are de Rham at ℓ to the set Irr of isomorphism classes of irreducible representations of $\mathcal{L}$, with image corresponding to L-algebraic cuspidal automorphic representations.
- For each place v of $\mathbb{Q}$ there is an injection

$$\mathcal{L}_v \to \mathcal{L}$$

of the local Langlands groups, which is well-defined up to conjugation in $\mathcal{L}$. Here:
  - $\mathcal{L}_v \cong W_{\mathbb{R}}$, the non-split extension of $\mathbb{C}^\times$ by $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$, if $v = \infty$.
  - $\mathcal{L}_v \cong W_{\mathbb{Q}_p} \times \mathrm{SU}_2(\mathbb{R})$ if $v = p$ prime and

$$W_{\mathbb{Q}_p} \cong I_{\mathbb{Q}_p} \rtimes \mathbb{Z} \subseteq \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \cong I_{\mathbb{Q}_p} \rtimes \hat{\mathbb{Z}}$$

is the subgroup of elements mapping to integral powers of Frobenius.
- If $\pi \cong \bigotimes'_v \pi_v \in \mathrm{Csp}_n$ corresponds to a continuous representation $\rho_\pi \colon \mathcal{L} \to \mathrm{GL}_n(\mathbb{C})$, then π_v should correspond to $\rho_{\pi|\mathcal{L}_v}$ under the (conjectural) local Langlands correspondence, cf. [GH19, Section 12.5.].

Next semester Peter is giving a lecture on this $\Rightarrow$ Follow it!

- If this conditions hold, then necessarily $\mathcal{L}^{\mathrm{ab}} \cong \mathbb{Q}^\times\backslash\mathbb{A}^\times$. Note that by global class field theory there exists a surjection $\mathbb{Q}^\times\backslash\mathbb{A}^\times \to \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.
- The existence of $\mathcal{L}$ is currently completely out of reach, even more conjecturally $\mathcal{L}$ should surject onto the "motivic Galois group" G_{mot} over $\mathbb{C}$, cf. [Art02], [LR87].

The global Langlands correspondence for general G (very, very rough):

- No precision is claimed, and all statements are close to being empty!
- For precise statements one should consult [GH19, Chapter 12] or [Art94].
- In order to parametrize non L-algebraic automorphic representations (at least those relevant for the decomposition of $L^2_{\mathrm{disc}}([G])$) one should replace the previous Galois version of L-parameters by (certain) L-parameters

$$\mathcal{L} \to {}^LG$$

(actually in the Weil form $W_{\mathbb{Q}} \ltimes \hat{G}(\mathbb{C})$ of the L-group).
- Then one hopes to construct a surjective map (cf. [GH19, Conjecture 12.6.2.])

$$\{(\text{certain) automorphic representations}\} \xrightarrow{\mathrm{LL}} \{(\text{certain) } L-\text{parameters}\}.$$

- The fibers of LL are called L-packets .
- A precise construction of a local Langlands correspondence (cf. [GH19, Conjecture 12.5.1.]) should yield a parametrization of the L-packets, cf. [GH19, (12.23)].
- For each $\pi \subseteq L^2_{\mathrm{disc}}([G])$ its multiplicity should be computable via the L-parameter $\mathrm{LL}(\pi)$ and the precise parametrizations of the L-packets, cf. [GH19, Conjecture 12.6.3.].

- However, for non-tempered representations the above should not be reasonable and one should consider Arthur parameters
$$\mathcal{L} \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G$$
instead of L-parameters, cf. [Art94].

Important stuff, which was not mentioned in the lecture:

- L-functions, [Bor79], [GH19, Chapter 11, Section 12.7.],
- Functoriality, [GH19, Section 12.6.],
- The trace formula, [Art05], [GH19, Section 18],
- The Langlands program for function fields,
- ...

A glimpse on Shimura varieties:

- Let $K_\infty \subseteq G(\mathbb{R})$ be a maximal (connected) compact subgroup.
- Let $K \subseteq G(\mathbb{A}_f)$ be a (sufficiently small) compact-open subgroup.
- Recall that
  - $$[G] \to [G]/K = G(\mathbb{Q})A_G \backslash G(\mathbb{A})/K$$
    is a profinite covering of a real manifold
  - $$[G]/K \to X_K := [G]/KK_\infty = G(\mathbb{Q})A_G \backslash G(\mathbb{A})/KK_\infty$$
    is a K_∞-bundle over X_K, which is a disjoint union of arithmetic manifolds (=quotients of a symmetric spaces by arithmetic subgroups)
- Note that implicitly X_K depends on K_∞.
- Let us set (just to simplify some notations later)
$$\widehat{X} := \varprojlim_K X_K.$$

Stuff we encountered for $G = \mathrm{GL}_2$:

(1) The upper/lower halfplane $\mathbb{H}^\pm \cong G(\mathbb{R})/K_\infty$, which is naturally a *complex* manifold
(2) The holomorphic embedding
$$\mathbb{H}^\pm \to \mathbb{P}^1_{\mathbb{C}}$$
(3) The $G(\mathbb{R})$-equivariant vector bundles
$$\omega^k$$
on $\mathbb{H}^\pm$, or by pullback on X_K for $K \subseteq G(\mathbb{A}_f)$ compact-open.
(4) For $k \in \mathbb{Z}$ the space of modular forms
$$M_k \subseteq H^0(\widehat{X}, \omega^k),$$
where the RHS denotes *holomorphic* sections.
(5) The canonical compactification X_K^* of X_K, with the extension of ω^k on it.
(6) A scheme $\widehat{\mathcal{X}} \to \mathrm{Spec}(\mathbb{Q})$, whose $\mathbb{C}$-valued points are naturally isomorphic to $\widehat{X}$.

(7) The $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \times G(\mathbb{A}_f)$-representation

$$\widetilde{H}^1_{\text{ét}}(\widetilde{\mathcal{X}}_{\overline{\mathbb{Q}}}, \mathbb{L})$$

associated to certain $G(\mathbb{A}_f)$-equivariant $\overline{\mathbb{Q}}_\ell$-local systems $\mathbb{L}$ on $\widetilde{\mathcal{X}}$, e.g., $\mathbb{L} \cong \overline{\mathbb{Q}}_\ell$.

(8) The Eichler–Shimura isomorphism relating S_k for $k \geq 2$, to certain

$$\widetilde{H}^1_{\text{ét}}(\widetilde{\mathcal{X}}_{\overline{\mathbb{Q}}}, \mathbb{L}).$$

(9) The Eichler–Shimura relation expressing a relation of the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action and the $G(\mathbb{A}_f)$-action on

$$\widetilde{H}^1_{\text{ét}}(\widetilde{\mathcal{X}}_{\overline{\mathbb{Q}}}, \mathbb{L}),$$

which was a consequence of the mod p geometry of some modular curve.

(10) A map

$$\mathrm{LL}\colon \mathcal{A}_{\mathrm{mod}} \to \mathcal{G}_{\mathrm{mod}}$$

with (conjecturally) describable image.

For general G there are unfortunately no analogs - but wait: there exists G giving rise to Shimura varieties!

- Already (1) fails for general G: The space

$$G(\mathbb{R})/K_\infty$$

  need not be *complex* manifold! (E.g., if $G = \mathrm{GL}_3$, then the real dimension $\dim(\mathrm{GL}_3(\mathbb{R})/A_{\mathrm{GL}_3(\mathbb{R})}\mathrm{SO}_3(\mathbb{R})) = \frac{3(3+1)}{2} - 1 = 5$ is odd)
- In particular, everything related to holomorphicity has no analog for such G, e.g., 2, 3, 4, 6, 7,...
- That is bad news!
- Good news: For groups G giving rise to Shimura data (like $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_{2,F}$ for $F/\mathbb{Q}$ totally real, or GSp_{2n},... , cf. [Lan17], [Mil05], [Del71a], [Del79]), the real manifold

$$G(\mathbb{R})/K_\infty A_G$$

  is a *complex* manifold.
- A Shimura datum is a pair (G, X) with G a reductive group G over $\mathbb{Q}$, and X a $G(\mathbb{R})$-conjugacy classes of morphisms $h\colon \mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to G_{\mathbb{R}}$ such that
  1) For each $h \in X$, the characters of $\mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$ acting on $\mathrm{Lie}(G_{\mathbb{R}})_{\mathbb{C}}$ (via $\mathrm{Ad} \circ h$) are either $z \mapsto \frac{z}{\overline{z}}$, $z \mapsto 1$, or $z \mapsto \frac{\overline{z}}{z}$.
  2) For each $h \in X$, the group $\{g \in G(\mathbb{C}) \mid h(i)gh(i)^{-1} = \overline{g}\}$ is compact modulo its center.
  3) The adjoint group G^{ad} has no factor, defined over $\mathbb{Q}$, for which the projection of $h \in X$ is trivial.
- E.g., take $G = \mathrm{GL}_2$, and X as the conjugacy class of the usual embedding $\mathbb{C}^\times \to \mathrm{GL}_2(\mathbb{R})$. Note that $X \cong \mathbb{H}^\pm$.
- In general, for $h \in X$ the stabilizer K_h of h in $G(\mathbb{R})$ is compact modulo the center of $G(\mathbb{R})$ (this follows from condition 2)) and $X \cong G(\mathbb{R})/K_h (\cong G(\mathbb{R})/A_G K_\infty)$.

- For $K \subseteq G(\mathbb{A}_f)$ recall

$$X_K := G(\mathbb{Q})\backslash(G(\mathbb{A}_f)/K \times X).$$

  Other notation: $\mathrm{Sh}_K(G, X)$ the Shimura variety of level K attached to (G, X).
- We get:
  (1) X is a disjoint union of hermitian symmetric domains, cf. [Mil05], [Del79].
  (2) To each $h \in X$ is naturally attached a parabolic subgroup $P_h \subseteq G(\mathbb{C})$ (cf. [CS17, Section 2.1]), and sending $h \to P_h$ defines an open embedding

$$\pi \colon X \cong G(\mathbb{R})/K_h \to \mathcal{F}l \cong G(\mathbb{C})/P_h$$

  of X into a flag variety (this generalizes the embedding $\mathbb{H}^{\pm} \to \mathbb{P}^1(\mathbb{C})$). In particular, X carries a natural complex structure. More is true (Baily-Borel): X_K is naturally a quasi-projective variety, cf. [Lan17, Theorem 2.4.1.].
  (3) The pullback of $G(\mathbb{C})$-equivariant vector bundles on $\mathcal{F}l$ gives rise to $G(\mathbb{Q})$-equivariant vector bundles on $G(\mathbb{A}_f) \times X$. By descent one one obtains automorphic vector bundles $\mathcal{E}$ on X_K for any $K \subseteq G(\mathbb{A}_f)$ compact-open, i.e., analogs of the $\omega^k, k \in \mathbb{Z}$, [Har88]. Note that $G(\mathbb{C})$-equivariant vector bundles on $\mathcal{F}l$ are equivalent to (algebraic) representations of P_h.
  (4), (5) If $\dim_{\mathbb{C}}(X_K) > 1$, there do exist compactifications X_K, but these are no longer canonical. However, it is possible to define subspaces in the cohomology of $H^*(X_K, \mathcal{E})$), which generalize M_k for $k \in \mathbb{Z}$, cf. [Har88].
  (6) The space $\widehat{X}$ (=inverse limit of complex manifolds) arises again as the $\mathbb{C}$-points of a scheme $\widehat{X} \to \mathrm{Spec}(E)$, the canonical model, defined over some number field (the "reflex field of the Shimura data"), cf. [Lan17, Theorem 2.4.3.].
  (8),(9),(10) Having the canonical model, one can attempt (sometimes sucessfully) to obtain analogs of points (18),(19),(20) and finally relate (certain) automorphic representations to (variants of) Galois representations, cf. [Har88], [BRb]. Needless to say that everything (compactifications, Eichler–Shimura isomorphism/relation, integral models, mod p geometry, interior vs intersection cohomology,...) is much more complicated!

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