AUTOMORPHIC FUNCTIONS AS THE TRACE OF FROBENIUS

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Abstract. We prove that the trace of the Frobenius endofunctor of the category of automorphic sheaves with nilpotent singular support maps isomorphically to the space of unramified automorphic functions, settling a conjecture from [AGKRRV1]. More generally, we show that traces of Frobenius-Hecke functors produce shtuka cohomologies.

Contents

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Introduction

0.1. **Overview.** This work is part of a series, following [AGKRRV1] and [AGKRRV2], attempting to understand the (unramified, function field) arithmetic Langlands conjectures via geometric and categorical techniques. We begin with an overview of the problems considered here.

0.1.1. Some of the most striking applications of geometric representation theory pass through the sheaves-functions dictionary of Grothendieck–Deligne.

Recall the setting: one has an algebraic stack $\mathcal{Y}$ over $\overline{\mathbb{F}}_q$, assumed to be defined over $\mathbb{F}_q$, and an ℓ-adic Weil sheaf $\mathcal{F}$ on $\mathcal{Y}$. For a rational point $y \in \mathcal{Y}_0(\mathbb{F}_q)$, one takes the trace of Frobenius on the stalk $y^*(\mathcal{F})$ to obtain an element of $\overline{\mathbb{Q}}_\ell$; this defines a function $\mathrm{funct}(\mathcal{F}) : \mathcal{Y}_0(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell$.

One finds that many functions of interest in harmonic analysis over finite fields arise by this procedure, and that the perspective offered by sheaf theory provides deep insights into function theory.

For example, this is the case in the theory of automorphic functions (for function fields), whose realizations via ℓ-adic sheaves exhibit explicit constructions of Langlands's conjectures.

0.1.2. In this paper, we establish a higher categorical analogue of the sheaves-functions dictionary. Instead of passing from sheaves to functions, we pass from categories (of sheaves) to vector spaces (of functions).

As in the previous setting, we decategorify using trace of Frobenius. However, whereas before we considered the trace of a Frobenius endomorphism of a vector space (and thus produced a scalar), we now consider the trace of a Frobenius endofunctor of a category (and thus produce a vector space).

Unlike the usual Grothendieck–Deligne paradigm, where one may take a general algebraic stack $\mathcal{Y}$, our results are specialized to spherical automorphic functions. The results of this paper establish conjectures from [AGKRRV1], which specify a relation between the category of automorphic sheaves and the vector space of automorphic functions via the trace of Frobenius.

0.1.3. We give a precise formulation of our main results below. However, by way of motivation, we highlight the following application, which illustrates how sheaf-theoretic considerations provide insights into the classical theory of automorphic functions.

In [AGKRRV1], we introduced a version of the geometric Langlands conjecture suitable for ℓ-adic sheaves. From this conjecture, combined with the *Trace Conjecture*, proved in this paper, we deduced that the space of compactly supported spherical automorphic functions (denoted $\mathrm{Funct}_c(\mathrm{Bun}_G(\mathbb{F}_q))$ in the body of this text) can be described as

$$\mathrm{Funct}_c(\mathrm{Bun}_G(\mathbb{F}_q)) \simeq \Gamma(\mathrm{LocSys}^{\mathrm{arthm}}_{\check{G}}(X), \omega_{\mathrm{LocSys}^{\mathrm{arthm}}_{\check{G}}(X)}). \tag{0.1}$$

In the above formula, X is the smooth projective curve corresponding to our choice of function field and $\mathrm{LocSys}^{\mathrm{arthm}}_{\check{G}}(X)$ is the algebraic stack over $\overline{\mathbb{Q}}_\ell$, defined in [AGKRRV1], parametrizing (unramified) Langlands parameters.[1]

In the everywhere unramified function field setting, this conjecture provides an interesting alternative to Langlands's original perspective: for general reductive G, the above conjecture yields a compete description of the space of automorphic functions in terms of Galois data, whereas classical Langlands conjectures only concern L-packets.

Remark 0.1.4. The work [VLaf] of V. Lafforgue and its extension [Xue1] by C. Xue provide a decomposition of the space of (not necessarily unramified!) automorphic functions in terms of Langlands parameters. It should come as no surprise that our results are closely related to their work.

First, our work is also based on considering cohomologies of shtukas.

And second, our constructions show that $\mathrm{Funct}_c(\mathrm{Bun}_G(\mathbb{F}_q))$ arises as global sections of *some* quasi-coherent sheaf $\mathrm{Drinf}^{\mathrm{arithm}}$ on $\mathrm{LocSys}^{\mathrm{arthm}}_{\check{G}}(X)$. (The conjecture expressed by formula (0.1) says that $\mathrm{Drinf}^{\mathrm{arithm}}$ is the dualizing sheaf of $\mathrm{LocSys}^{\mathrm{arthm}}_{\check{G}}(X)$.) The existence of $\mathrm{Drinf}^{\mathrm{arithm}}$ recovers the spectral decomposition of $\mathrm{Funct}_c(\mathrm{Bun}_G(\mathbb{F}_q))$ along the set of classical Langlands parameters, see Remark 3.4.2.

We refer to [AGKRRV1, Sect. 16] for further discussion of the relation to V. Lafforgue's work.

Remark 0.1.5. The principle that geometric methods enrich Langlands's conjectures is an old one, dating (at least) to [De] and [Dr]. But precise refinements of Langlands's conjectures using geometric ideas have emerged recently. The present paper provides one example. Similarly, the Fargues–Scholze geometrization program (see [Fa]) aims to produce a more robust form of the local Langlands conjectures.

0.2. **Formulation of the main result.**

We now proceed to the statement of our main result.

0.2.1. *Notation.* Throughout the paper, we use algebraic geometry over the two fields $k := \overline{\mathbb{F}}_q$ and $\mathsf{e} := \overline{\mathbb{Q}}_\ell$ (where $\ell \in \mathbb{F}_q^\times$).

When we work over $\overline{\mathbb{F}}_q$, we generally work with algebraic stacks $\mathcal{Y}$ that are assumed to be defined over $\mathbb{F}_q$; we abuse notation somewhat in letting $\mathcal{Y}(\mathbb{F}_q)$ denote the groupoid of $\mathbb{F}_q$-points of the corresponding stack. We let $\mathrm{Frob}_{\mathcal{Y}} : \mathcal{Y} \to \mathcal{Y}$ denote the geometric (q-)Frobenius morphism, whose stack of fixed-points $\mathcal{Y}^{\mathrm{Frob}}$ is a discrete, and identifies with (the étale sheafification of) the groupoid $\mathcal{Y}(\mathbb{F}_q)$.

Let X be a smooth projective curve over $\overline{\mathbb{F}}_q$, but assumed defined over $\mathbb{F}_q$. Let $G/\overline{\mathbb{F}}_q$ be a reductive group, considered over $\mathbb{F}_q$ via its split form. Let Bun_G denote the moduli stack of principal G-bundles on X.

We refer to Sect. 0.6 for further details on our conventions.

0.2.2. *Categorical trace.* Throughout this paper, all DG categories are enriched over the field e. In particular, $\mathrm{Vect} = \mathrm{Vect}_{\mathsf{e}}$ (the DG category of chain complexes of e-vector spaces), and so on.

We remind that the category DGCat of cocomplete DG categories is equipped with a canonical symmetric monoidal structure $\otimes$, the *Lurie tensor product*. On general grounds, this means one may speak about categorical duals and traces as follows.

If $\mathbf{C} \in \mathrm{DGCat}$ is dualizable, there is another DG category $\mathbf{C}^\vee$ equipped with canonical unit and counit maps

$$\mathrm{u}_{\mathbf{C}} : \mathrm{Vect} \to \mathbf{C} \otimes \mathbf{C}^\vee$$

and

$$\mathrm{ev}_{\mathbf{C}} : \mathbf{C} \otimes \mathbf{C}^\vee \to \mathrm{Vect}\,.$$

For an endofunctor $\Phi : \mathbf{C} \to \mathbf{C}$ of $\mathbf{C}$, we have $\mathrm{Tr}(\Phi, \mathbf{C}) \in \mathrm{Vect}$ defined as

$$\mathrm{Tr}(\Phi, \mathbf{C}) := \mathrm{ev}\left((\Phi \otimes \mathrm{Id})(\mathrm{u}_{\mathbf{C}})\right).$$

[1]An alternative construction of $\mathrm{LocSys}^{\mathrm{arthm}}_{\check{G}}(X)$ due to P. Scholze and X. Zhu, may be found in [Zhu]. Their construction proceeds along very different lines, but conjecturally produces an equivalent object.

We refer to [GKRV] for further discussion.

0.2.3. *Categories of sheaves.* We consider the category $\mathrm{Shv}(\mathrm{Bun}_G)$ of *automorphic sheaves*. Precisely, $\mathrm{Shv}(\mathrm{Bun}_G)$ is the category of ind-constructible $\overline{\mathbb{Q}}_\ell$-sheaves on Bun_G. As in [AGKRRV1], we also consider its full subcategory

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \subset \mathrm{Shv}(\mathrm{Bun}_G) \tag{0.2}$$

of objects with singular support in the global nilpotent cone.

The categories $\mathrm{Shv}(\mathrm{Bun}_G)$ and $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ have favorable finiteness properties: by [AGKRRV1, Theorems 10.1.4 and 10.1.6], they are compactly generated and therefore dualizable in DGCat. Moreover, the embedding $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \to \mathrm{Shv}(\mathrm{Bun}_G)$ preserves compactness.

0.2.4. Pushforward with respect to the geometric Frobenius endomorphism (see Sect. 0.6.2) defines an auto-equivalence $(\mathrm{Frob}_{\mathrm{Bun}_G})_*$ of $\mathrm{Shv}(\mathrm{Bun}_G)$, which preserves the subcategory $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$.

Hence, it makes sense to consider the categorical trace

$$\mathrm{Tr}((\mathrm{Frob}_{\mathrm{Bun}_G})_*, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)) \in \mathrm{Vect}\,.$$

As we describe below, the goal of this paper is to show that this vector space maps isomorphically onto the space automorphic functions via the sheaves-functions dictionary.

0.2.5. For any quasi-compact algebraic stack $\mathcal{Y}$ over $\overline{\mathbb{F}}_q$ and defined over $\mathbb{F}_q$, there is a canonical *local term* map (see [AGKRRV1, Sect. 15.2]),

$$\mathrm{LT}_{\mathcal{Y}} : \mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}(\mathcal{Y})) \to \mathrm{Funct}(\mathcal{Y}(\mathbb{F}_q)),$$

where $\mathrm{Funct}(-)$ stands for the (classical) vector space e-valued functions.

In addition, in *loc. cit.*, we extended this construction to non-quasi-compact stacks (such as Bun_G). In this setting, the above map takes the form

$$\mathrm{LT}_{\mathcal{Y}} : \mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}(\mathcal{Y})) \to \mathrm{Funct}_c(\mathcal{Y}(\mathbb{F}_q)),$$

where $\mathrm{Funct}_c(\mathcal{Y}(\mathbb{F}_q)) \subset \mathrm{Funct}(\mathcal{Y}(\mathbb{F}_q))$ is the subspace of compactly supported e-valued functions.

Remark 0.2.6. By functoriality of traces, any (possibly lax) Weil sheaf

$$(\mathcal{F} \in \mathrm{Shv}(\mathcal{Y})^c, \alpha : \mathcal{F} \to (\mathrm{Frob}_{\mathcal{Y}})_*(\mathcal{F}))$$

on $\mathcal{Y}$ defines an element

$$\mathrm{cl}(\mathcal{F}, \alpha) \in \mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}(\mathcal{Y})).$$

According to [AGKRRV1, Sect. 15.2], the value of the map $\mathrm{LT}_{\mathcal{Y}}$ on $\mathrm{cl}(\mathcal{F}, \alpha)$ produces the corresponding Grothendieck–Deligne function, denoted $\mathrm{funct}(\mathcal{F})$.

0.2.7. By the above, we obtain a map

$$\mathrm{Tr}((\mathrm{Frob}_{\mathrm{Bun}_G})_*, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)) \to \mathrm{Tr}((\mathrm{Frob}_{\mathrm{Bun}_G})_*, \mathrm{Shv}(\mathrm{Bun}_G)) \overset{\mathrm{LT}_{\mathrm{Bun}_G}}{\to} \mathrm{Funct}_c(\mathrm{Bun}_G(\mathbb{F}_q)), \tag{0.3}$$

where we note that the vector space $\mathrm{Funct}_c(\mathrm{Bun}_G(\mathbb{F}_q))$ is the space of compactly supported unramified automorphic functions.

Main Theorem 0.2.8. *The composition* (0.3) *is an isomorphism.*

This result was proposed as a conjecture in [AGKRRV1], where it was termed the *Trace Conjecture*.

Remark 0.2.9. Informally, one can view Theorem 0.2.8 as saying that "there are enough weak Weil sheaves on Bun_G with nilpotent singular support to recover all automorphic functions, and any relations between the automorphic functions defined by such sheaves have categorical origins."

Of course, the fact that we are considering sheaves with nilpotent singular support is crucial here. If we did not have the singular support condition, we would obviously have enough sheaves to recover all functions. However, the relations imposed by sheaves would not match the relations on functions.

Remark 0.2.10. The phrase in quotation makes in the previous remark would be a correct assertion if the phrase inside the quotation marks was understood in the derived sense.

More precisely, for a (compactly generated) category $\mathbf{C}$ with an endofunctor Φ, the trace object $\mathrm{Tr}(\Phi, \mathbf{C}) \in \mathrm{Vect}$ is computed as the geometric realization of a certain canonically defined simplicial object of Vect; let us denote it $\mathrm{Tr}(\Phi, \mathbf{C})^\bullet$.

In particular, we have a map $\mathrm{Tr}(\Phi, \mathbf{C})^0 \to \mathrm{Tr}(\Phi, \mathbf{C})$ (here the superscript 0 denotes the space of 0-simplices), and hence a map

$$H^0(\mathrm{Tr}(\Phi, \mathbf{C})^0) \to H^0(\mathrm{Tr}(\Phi, \mathbf{C})).$$

Now, the image of the latter map is the span of the classes

$$\mathrm{cl}(\mathbf{c}, \alpha), \quad \mathbf{c} \in \mathbf{C}^c, \alpha : \mathbf{c} \to \Phi(\mathbf{c}).$$

However, it is not true, in general, that $H^0(\mathrm{Tr}(\Phi, \mathbf{C}))$ is spanned by the above classes: higher cohomologies of higher simplices can also contribute.

0.3. **Insertion of Hecke functors.** So far, we have not mentioned Langlands duality. Remarkably, it plays an essential role in the proof of Theorem 0.2.8.

In fact, our proof of Theorem 0.2.8 constructs isomorphisms between two families of objects, which are, roughly speaking, indexed by Hecke functors. As we explain below, this amounts to a proof of [AGKRRV1, Conjecture 15.5.7], the so-called *Shtuka Conjecture* of *loc. cit*[2]

Moreover, our method of proof relies in an essential way on consideration of *all* Hecke functors simultaneously.

We presently explain what these objects are, and what our results about them assert.

0.3.1. Let $\check{G}$ denote the Langlands dual group of G considered as an algebraic group over the coefficient field $\mathsf{e} := \overline{\mathbb{Q}}_\ell$. We let $\mathrm{Rep}(\check{G})$ denote the (symmetric monoidal) DG category of representations of $\check{G}$.

0.3.2. First, recall the formalism of Hecke functors in geometric Langlands. The key feature is that they may be indexed by tuples of moving, possibly colliding points.

Precisely, given a finite set I and an object $V \in \mathrm{Rep}(\check{G})^{\otimes I}$, we can consider the Hecke functor

$$\mathsf{H}(V, -) : \mathrm{Shv}(\mathrm{Bun}_G) \to \mathrm{Shv}(\mathrm{Bun}_G \times X^I). \tag{0.4}$$

Restricting to $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \subset \mathrm{Shv}(\mathrm{Bun}_G)$, we obtain a functor that lands in the subcategory

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{QLisse}(X)^{\otimes I} \subset \mathrm{Shv}(\mathrm{Bun}_G \times X^I)$$

(see Corollary 1.3.5), where $\mathrm{QLisse}(X)$ denotes the category of lisse sheaves on X, see Sect. 0.6.4.

0.3.3. As we recall in Sect. 1.4.1, there is a symmetric monoidal category $\mathrm{Rep}(\check{G})_{\mathrm{Ran}}$, the *Ran version* of the category $\mathrm{Rep}(\check{G})$, that is the universal source of Hecke functors.

More precisely, there is an action of $\mathrm{Rep}(\check{G})_{\mathrm{Ran}}$ on $\mathrm{Shv}(\mathrm{Bun}_G)$ by *integral Hecke functors*, preserving the subcategory $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$. By design, this action encodes the Hecke actions for varying finite sets I. We refer to Sect. 1.5.1 for the construction.

We denote the monoidal product on $\mathrm{Rep}(\check{G})_{\mathrm{Ran}}$ by $\star$ and its monoidal unit by $\mathbf{1}_{\mathrm{Rep}(\check{G})_{\mathrm{Ran}}}$.

[2]We draw the reader's attention to our conventions regarding the cohomological shift in the Shtuka Conjecture, see Remarks 1.3.7 and 3.2.6. In particular, in [AGKRRV1, Conjecture 15.5.7], the embedding $\mathrm{QLisse}(X^I) \hookrightarrow \mathrm{Shv}(X^I)$ must be understood in the sense of (1.6) of the present paper.

0.3.4. Below, we will be considering functors $\mathcal{S} : \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \to \mathrm{Vect}$.

By the definition of $\mathrm{Rep}(\check{G})_{\mathrm{Ran}}$, such functors amount to compatible systems of functors

$$\mathcal{S}_I : \mathrm{Rep}(\check{G})^{\otimes I} \to \mathrm{Shv}(X^I)$$

defined for $I \in \mathrm{fSet}$.

In examples, the functors $\mathcal{S}_I$ tend to be more familiar avatars of the functor $\mathcal{S}$, so it is convenient to reference them.

0.3.5. We have a functor

$$\mathrm{Sht}^{\mathrm{Tr}} : \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \to \mathrm{Vect}$$

constructed as the composition

$$\mathrm{Rep}(\check{G})_{\mathrm{Ran}} \to \mathrm{End}_{\mathrm{DGCat}}(\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)) \overset{-\circ(\mathrm{Frob}_{\mathrm{Bun}_G})_*}{\to} \mathrm{End}_{\mathrm{DGCat}}(\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)) \overset{\mathrm{Tr}}{\to} \mathrm{Vect}\,.$$

In other words, we take $\mathcal{V} \in \mathrm{Rep}(\check{G})_{\mathrm{Ran}}$, form the corresponding Hecke functor, compose with Frobenius, and take the trace of the resulting endofunctor of $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$.

By construction, we have

$$\mathrm{Sht}^{\mathrm{Tr}}(\mathbf{1}_{\mathrm{Rep}(\check{G})_{\mathrm{Ran}}}) = \mathrm{Tr}((\mathrm{Frob}_{\mathrm{Bun}_G})_*, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)).$$

Remark 0.3.6. To make the above more explicit, we describe the functors $\mathrm{Sht}^{\mathrm{Tr}}_I : \mathrm{Rep}(\check{G})^{\otimes I} \to \mathrm{Shv}(X^I)$ for a finite set I.

Precomposing (0.4) with $(\mathrm{Frob}_{\mathrm{Bun}_G})_*$, we obtain a functor

$$\mathsf{H}(V,-) \circ (\mathrm{Frob}_{\mathrm{Bun}_G})_* : \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \to \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{QLisse}(X)^{\otimes I},$$

and we can consider its *parameterized* trace

$$\mathrm{Tr}(\mathsf{H}(V,-) \circ (\mathrm{Frob}_{\mathrm{Bun}_G})_*, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)) \in \mathrm{QLisse}(X)^{\otimes I}.$$

Unwinding the constructions, the resulting functor

$$\mathrm{Rep}(\check{G})^{\otimes I} \to \mathrm{QLisse}(X)^{\otimes I},$$

followed by the embedding

$$\mathrm{QLisse}(X)^{\otimes I} \hookrightarrow \mathrm{Shv}(X^I),$$

is our $\mathrm{Sht}^{\mathrm{Tr}}_I$.

0.3.7. On the other hand, following [VLaf], to the data $(I \in \mathrm{fSet}, V \in \mathrm{Rep}(\check{G})^{\otimes I})$ we can attach the "shtuka cohomology", which is an object

$$\mathrm{Sht}_I(V) \in \mathrm{Shv}(X^I),$$

see Sect. 3.1 in the main body of the paper. These functors satisfy the requisite compatibilities needed to define a functor

$$\mathrm{Sht} : \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \to \mathrm{Vect}\,.$$

Example 0.3.8. For $I = \emptyset$, the functor $\mathrm{Sht}_\emptyset$ amounts to a map Vect $\to$ Vect, i.e., a vector space, which corresponds to $\mathrm{Sht}(\mathbf{1}_{\mathrm{Rep}(\check{G})_{\mathrm{Ran}}})$. As is standard, this vector space is $\mathrm{Funct}_c(\mathrm{Bun}_G(\mathbb{F}_q))$.

0.3.9. Our main result asserts the following.

Main Theorem 0.3.10. *There is a canonical equivalence*

$$\mathrm{Sht}^{\mathrm{Tr}} \simeq \mathrm{Sht}$$

of functors $\mathrm{Rep}(\check{G})_{\mathrm{Ran}} \to \mathrm{Vect}$. *Moreover, the resulting map*

$$\mathrm{Tr}((\mathrm{Frob}_{\mathrm{Bun}_G})_*, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)) = \mathrm{Sht}^{\mathrm{Tr}}(\mathbf{1}_{\mathrm{Rep}(\check{G})_{\mathrm{Ran}}}) \overset{\sim}{\to} \mathrm{Sht}(\mathbf{1}_{\mathrm{Rep}(\check{G})_{\mathrm{Ran}}}) = \mathrm{Funct}_c(\mathrm{Bun}_G(\mathbb{F}_q))$$

is the local term map.

Clearly this result refines Theorem 0.2.8.

It follows that the functors $\mathrm{Sht}_I^{\mathrm{Tr}}$ and Sht_I canonically coincide. This is exactly the assertion of the Shtuka Conjecture [AGKRRV1, Conjecture 15.5.7].

0.4. **Outline of the proof.** We now give an overview of the proof of Theorem 0.3.10. It requires a few additional objects and some relations between them.

0.4.1. First, we recall the (non-algebraic) stack

$$\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$$

over e introduced in [AGKRRV1], the *stack of local systems with restricted variation.*

We have a naturally defined symmetric monoidal localization functor

$$\mathrm{Loc} : \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \to \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)),$$

see Sect. 2.3.

0.4.2. We use *Beilinson's spectral projector*, which is a certain object

$$\mathcal{R} \in \mathrm{Rep}(\check{G})_{\mathrm{Ran}}$$

related to $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$. It has the following properties:

(i) $\mathrm{Loc}(\mathcal{R}) \simeq \mathcal{O}_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}$.

(ii) The functor $\mathrm{Sht}^{\mathrm{Tr}}$ is canonically isomorphic to $\mathrm{Sht}(\mathcal{R} \star -)$, i.e., act by $\mathcal{R}$ and then apply Sht.

The above Property (i) is one of the key results of the paper [AGKRRV1] (it follows from what is stated in this paper as Theorem 2.4.6, which is in turn [AGKRRV1, Theorem 9.1.3]).

Property (ii) is a straightforward calculation using the main result of the paper [AGKRRV2], and appears here as Theorem 4.2.2.

0.4.3. Using the result of [Xue2] mentioned above, we show in Corollary 3.1.4 that Sht factors canonically as a composition

$$\mathrm{Rep}(\check{G})_{\mathrm{Ran}} \overset{\mathrm{Loc}}{\longrightarrow} \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \overset{\mathrm{Sht}_{\mathrm{Loc}}}{\longrightarrow} \mathrm{Vect}$$

for a certain functor $\mathrm{Sht}_{\mathrm{Loc}}$.

At this point, the result follows from Properties (i) and (ii) above: we have

$$\mathrm{Sht}^{\mathrm{Tr}} \simeq \mathrm{Sht}_{\mathrm{Loc}} \circ \mathrm{Loc}(\mathcal{R} \star -),$$

and because Loc is monoidal and sends $\mathcal{R}$ to the unit of $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$, we have

$$\mathrm{Loc}(\mathcal{R} \star -) \simeq \mathrm{Loc},$$

so combining we obtain

$$\mathrm{Sht}^{\mathrm{Tr}} = \mathrm{Sht}_{\mathrm{Loc}} \circ \mathrm{Loc}(\mathcal{R} \star -) \simeq \mathrm{Sht}_{\mathrm{Loc}} \circ \mathrm{Loc} \simeq \mathrm{Sht},$$

which is the assertion of Theorem 0.3.10. as desired.

0.4.4. It remains to show that after evaluating on the unit object of $\operatorname{Rep}(\check{G})_{\operatorname{Ran}}$, the isomorphism $\operatorname{Sht}^{\operatorname{Tr}} \simeq \operatorname{Sht}$ is given by the local term map.

This is the content of Theorem 5.1.3, which requires some additional ideas.

0.5. **Organization of the paper.** We now describe how the present paper is structured.

0.5.1. In Sect. 1 we review the formalism of Hecke functors acting on $\operatorname{Shv}(\operatorname{Bun}_G)$.

In particular, we introduce the Ran version of the category $\operatorname{Rep}(\check{G})$, denoted $\operatorname{Rep}(\check{G})_{\operatorname{Ran}}$, which is a monoidal category that acts on $\operatorname{Shv}(\operatorname{Bun}_G)$ by *integral Hecke functors*.

This section does not contain any results original to this paper.

0.5.2. In Sect. 2 we review some notions associated with the *stack of local systems with restricted variation*, introduced in [AGKRRV1], and denoted $\operatorname{LocSys}^{\operatorname{restr}}_{\check{G}}(X)$.

Apart from the definition of $\operatorname{LocSys}^{\operatorname{restr}}_{\check{G}}(X)$, the main points are:

(i) Description of the category $\operatorname{QCoh}(\operatorname{LocSys}^{\operatorname{restr}}_{\check{G}}(X))$ as the category of compatible collections of functors

$$\operatorname{Rep}(\check{G})^{\otimes I} \to \operatorname{QLisse}(X)^{\otimes I}, \quad I \in \operatorname{fSet};$$

(ii) The localization functor

$$\operatorname{Loc} : \operatorname{Rep}(\check{G})_{\operatorname{Ran}} \to \operatorname{QCoh}(\operatorname{LocSys}^{\operatorname{restr}}_{\check{G}}(X));$$

(iii) Construction of Beilinson's spectral projector, which is an explicit object $\mathcal{R} \in \operatorname{Rep}(\check{G})_{\operatorname{Ran}}$, one of whose main properties is the isomorphism

$$\operatorname{Loc}(\mathcal{R}) \simeq \mathcal{O}_{\operatorname{LocSys}^{\operatorname{restr}}_{\check{G}}(X)}.$$

(iv) Corollary 2.3.3, which asserts that a functor $\mathcal{S} : \operatorname{Rep}(\check{G})_{\operatorname{Ran}} \to \operatorname{Vect}$ factors (uniquely) through the localization functor exactly when the functors $\mathcal{S}_I$ are valued in $\operatorname{QLisse}(X)^{\otimes I} \subset \operatorname{Shv}(X^I)$.

The entirety of the material of this section is a reformulation of the results in Parts I and II of [AGKRRV1].

0.5.3. In Sect. 3 we review the shtuka construction and some of its variants.

First, we introduce the functor $\operatorname{Sht} : \operatorname{Rep}(\check{G})_{\operatorname{Ran}} \to \operatorname{Vect}$.

We then quote the main result from [Xue2], which we interpret as saying that the functors Sht_I take values in $\operatorname{QLisse}(X)^{\otimes I} \subset \operatorname{Shv}(X^I)$. Using (iv) above, this yields the existence of the functor $\operatorname{Sht}_{\operatorname{Loc}}$ from above.

By duality, the functor $\operatorname{Sht}_{\operatorname{Loc}}$ yields an object of $\operatorname{QCoh}(\operatorname{LocSys}^{\operatorname{restr}}_{\check{G}}(X))$, which we denote by Drinf.

0.5.4. In Sect. 4 we formulate and prove the main result of this paper, Theorem 4.1.2, which is the assertion of Theorem 0.3.10 modulo compatibility with local terms. As particular cases, this statement contains both the (unrefined) Trace Conjecture and the Shtuka Conjecture.

The argument is the one outlined above.

0.5.5. In Sects. 5 and 6 we show that the isomorphism

$$\operatorname{Tr}(\operatorname{Frob}_*, \operatorname{Shv}_{\operatorname{Nilp}}(\operatorname{Bun}_G)) \simeq \operatorname{Funct}_c(\operatorname{Bun}_G(\mathbb{F}_q))$$

from Theorem 4.1.2 induces the local term map (0.3), as in the statement of Theorem 0.3.10.

0.5.6. The trace calculation in this paper is based on considering shtukas, as defined in [VLaf] and further studied in [Xue1, Xue2], and on a *non-standard* self-duality of the category $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$, studied in [AGKRRV2] and reviewed in Sect. 4.3.

In the Appendix we attempt to re-run the calculation of $\mathrm{Tr}(\mathrm{Frob}_*, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G))$ by relying on the usual Verdier self-duality of $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$. In the process we encounter a new object that we name "co-shtukas".

0.6. **Notations and conventions.** The notations in this paper largely follow those of [AGKRRV1] and [AGKRRV2].

0.6.1. *Algebraic geometry.* There will be disjoint "two algebraic geometries" at play in this paper: one on the automorphic side, and another on the spectral side.

On the automorphic side, our algebraic geometry will be over the ground field k, which in this paper is $\overline{\mathbb{F}}_q$. Our algebro-geometric objects will be either schemes or algebraic stacks locally of finite type over k. In this paper we will not need more general prestacks. Moreover, the algebraic geometry that we consider over k is classical (i.e. *not* derived).

On the spectral side, our algebraic geometry will be over the field of coefficients $\mathsf{e} := \overline{\mathbb{Q}}_\ell$, see below. We will consider just one algebro-geometric object over e–the (pre)stack $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$ (see Sect. 2), but it will play quite a prominent role. Importantly, the algebraic geometry we consider over e is *derived*: by default, all schemes, stacks, etc. over e are derived.

0.6.2. *Frobenius endomorphism.* Let $\mathcal{Y}$ be an algebraic stack over $\overline{\mathbb{F}}_q$, but defined over $\mathbb{F}_q$. In this case, we can consider the geometric Frobenius endomorphism of $\mathcal{Y}$, denoted

$$\mathrm{Frob}_{\mathcal{Y}} : \mathcal{Y} \to \mathcal{Y}.$$

Thus, whenever we refer to the Frobenius endomorphism of $\mathcal{Y}$, we will assume that $\mathcal{Y}$ is defined over $\mathbb{F}_q$. This is the case of our curve X, the reductive group G, and the stack Bun_G of principal G-bundles on X.

0.6.3. *Higher algebra.* We will work with DG categories over the field of coefficients $\mathsf{e} := \overline{\mathbb{Q}}_\ell$.

All our conventions and notations regarding DG categories are imported from [AGKRRV2, Sects. 0.5.2-0.5.3].

There will be two kinds of sources that feed into higher algebra, i.e., the sources of DG categories.

One will be various categories produced out of ℓ-adic sheaves on the automorphic side. Another will be categories of quasi-coherent sheaves on the spectral side (specifically, the category of quasi-coherent sheaves on $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$).

0.6.4. *Sheaves.* For a scheme S of finite type, we let $\mathrm{Shv}(S)^{\mathrm{constr}}$ denote the category of constructible $\overline{\mathbb{Q}}_\ell$-adic sheaves on S, viewed as a (small) DG category over the field of coefficients $\mathsf{e} = \overline{\mathbb{Q}}_\ell$.

We let $\mathrm{Shv}(S)$ denote the (cocomplete) DG category $\mathrm{Ind}(\mathrm{Shv}(S)^{\mathrm{constr}})$. We extend the assignment

$$S \mapsto \mathrm{Shv}(S)$$

from schemes to algebraic stacks by the procedure of [AGKRRV2, Sect. 1.1.1].

For a given stack $\mathcal{Y}$, we will denote by

$$\underline{\mathsf{e}}_{\mathcal{Y}}, \omega_{\mathcal{Y}} \in \mathrm{Shv}(\mathcal{Y})$$

the constant and dualizing sheaves, respectively.

0.6.5. *Singular support.* Let $\mathcal{Y}$ be an algebraic stack and $\mathcal{N}$ a conical Zariski-closed subset of $T^*(\mathcal{Y})$. We will denote by

$$\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y}) \subset \mathrm{Shv}(\mathcal{Y})$$

the corresponding full subcategory, defined as in [AGKRRV1, Sects. B.4.1 and C.5.1].

If $\mathcal{Y}$ is smooth and $\mathcal{N}$ is the zero-section, usually denoted $\{0\}$, we will also use the notation $\mathrm{QLisse}(\mathcal{Y})$ for $\mathrm{Shv}_{\{0\}}(\mathcal{Y})$.

0.6.6. *Functors (co)defined by kernels.* In a few places in this paper we will make reference to functors *defined* or *codefined* by a kernel. We refer the reader to [AGKRRV2, Sects. A.1 and C.2.1], where these notions are introduced.

Following *loc. cit.*, given a functor $\mathrm{Shv}(\mathcal{Y}_1) \to \mathrm{Shv}(\mathcal{Y}_2)$, defined or codefined by a kernel F, we will denote by

$$\mathrm{Id} \otimes \mathsf{F} : \mathrm{Shv}(\mathcal{Z} \times \mathcal{Y}_1) \to \mathrm{Shv}(\mathcal{Z} \times \mathcal{Y}_2)$$

for an algebraic stack $\mathcal{Z}$ (thought of as a stack of parameters).

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1. The Hecke action

In this section we will recall the pattern of Hecke acton of the category $\mathrm{Rep}(\check{G})$ on $\mathrm{Shv}(\mathrm{Bun}_G)$, and some related formalism. The section contains no original material.

1.1. **Hecke functors.**

1.1.1. In this paper, by the phenomenon of Hecke action we will understand a system of functors, defined for every finite set I and every algebraic stack $\mathcal{Z}$ (thought of as a stack of parameters):

$$\mathsf{H}^! : \mathrm{Rep}(\check{G})^{\otimes I} \otimes \mathrm{Shv}(\mathcal{Z} \times \mathrm{Bun}_G) \to \mathrm{Shv}(\mathcal{Z} \times \mathrm{Bun}_G \times X^I). \tag{1.1}$$

For a fixed $V \in \mathrm{Rep}(\check{G})^{\otimes I}$, we will denote by $\mathsf{H}^!(V,-)$ the resulting functor

$$\mathrm{Shv}(\mathcal{Z} \times \mathrm{Bun}_G) \to \mathrm{Shv}(\mathcal{Z} \times \mathrm{Bun}_G \times X^I).$$

1.1.2. The Hecke functions (1.1) are associative in the following sense: we have a natural commutative diagram of functors

$$\begin{tikzcd}
\mathrm{Rep}(\check{G})^{\otimes I} \otimes \mathrm{Rep}(\check{G})^{\otimes I} \otimes \mathrm{Shv}(\mathcal{Z} \times \mathrm{Bun}_G) \arrow[r, "\mathrm{mult}"] \arrow[d, "\mathrm{Id} \otimes \mathsf{H}^!"'] & \mathrm{Rep}(\check{G})^{\otimes I} \otimes \mathrm{Shv}(\mathcal{Z} \times \mathrm{Bun}_G) \arrow[dd, "\mathsf{H}^!"] \\
\mathrm{Rep}(\check{G})^{\otimes I} \otimes \mathrm{Shv}(\mathcal{Z} \times \mathrm{Bun}_G \times X^I) \arrow[d, "\mathsf{H}^!"'] & \\
\mathrm{Shv}(\mathcal{Z} \times \mathrm{Bun}_G \times X^I \times X^I) \arrow[r, "(\mathrm{Id} \times \Delta)^!"] & \mathrm{Shv}(\mathcal{Z} \times \mathrm{Bun}_G \times X^I),
\end{tikzcd}$$

where

$$\mathrm{mult} : \mathrm{Rep}(\check{G})^{\otimes I} \otimes \mathrm{Rep}(\check{G})^{\otimes I} \to \mathrm{Rep}(\check{G})^{\otimes I}$$

is the tensor product functor and $\Delta_{X^I} : X^I \to X^I \times X^I$ is the diagonal embedding. The data of associativity of (1.1) comes additionally with higher coherence for higher powers of $\mathrm{Rep}(\check{G})^{\otimes I}$.

We can rephrase that as saying that the category $\mathrm{Shv}(\mathcal{Z}\times \mathrm{Bun}_G \times X^I)$ is a module category for the monoidal category $\mathrm{Rep}(\check{G})^{\otimes I}$, and the action is $\mathrm{Shv}(X^I)$-linear (i.e. it is a module category for the monoidal category $\mathrm{Rep}(\check{G})^{\otimes I}\otimes \mathrm{Shv}(X^I)$).

1.1.3. The functors (1.1) are naturally compatible with maps between finite sets. Namely, for a map $\psi : I\to J$, we have a data of commutativity for the diagram

$$\begin{tikzcd} \mathrm{Rep}(\check{G})^{\otimes I}\otimes \mathrm{Shv}(\mathcal{Z}\times \mathrm{Bun}_G) \arrow[r, "\mathsf{H}^!"] \arrow[d, "\mathrm{Id}\otimes \mathrm{mult}^\psi"'] & \mathrm{Shv}(\mathcal{Z}\times \mathrm{Bun}_G \times X^I) \arrow[d, "(\mathrm{Id}\times\Delta_\psi)^!"] \\ \mathrm{Rep}(\check{G})^{\otimes J}\otimes \mathrm{Shv}(\mathcal{Z}\times \mathrm{Bun}_G) \arrow[r, "\mathsf{H}^!"] & \mathrm{Shv}(\mathcal{Z}\times \mathrm{Bun}_G \times X^J) \end{tikzcd} \tag{1.2}$$

where

$$\mathrm{mult}^\psi : \mathrm{Rep}(\check{G})^{\otimes I}\to \mathrm{Rep}(\check{G})^{\otimes J}$$

is the functor given by the symmetric monoidal structure on $\mathrm{Rep}(\check{G})$, and

$$\Delta_\psi : X^J \to X^I$$

the diagonal map defined by ψ.

The above data of commutativity are endowed with a homotopy coherent system of compatibilities for compositions of maps of finite sets.

Moreover, this data is compatible with the associativity described in Sect. 1.1.2. Namely, the functor

$$(\mathrm{Id}\times\Delta_\psi)^! : \mathrm{Shv}(\mathcal{Z}\times \mathrm{Bun}_G \times X^I)\to \mathrm{Shv}(\mathcal{Z}\times \mathrm{Bun}_G \times X^J)$$

is a functor of $\mathrm{Shv}(X^I)\otimes \mathrm{Rep}(\check{G})^{\otimes I}$-module categories.

1.1.4. A feature of the functors $\mathsf{H}^!(V,-)$ is that they are functors that are both *defined and codefined by kernels*; see Sect. 0.6.6 for what this means. In practical terms, this implies that for a map $\mathcal{Z}_1\to\mathcal{Z}_2$ between algebraic stacks, we have a datum of commutativity for the diagrams

$$\begin{tikzcd} \mathrm{Shv}(\mathcal{Z}_1\times \mathrm{Bun}_G) \arrow[r, "\mathsf{H}^!(V,-)"] \arrow[d] & \mathrm{Shv}(\mathcal{Z}_1\times \mathrm{Bun}_G \times X^I) \arrow[d] \\ \mathrm{Shv}(\mathcal{Z}_2\times \mathrm{Bun}_G) \arrow[r, "\mathsf{H}^!(V,-)"] & \mathrm{Shv}(\mathcal{Z}_2\times \mathrm{Bun}_G \times X^I), \end{tikzcd}$$

where the vertical arrows are given by either *- or !- pushforwards, and also for the diagrams

$$\begin{tikzcd} \mathrm{Shv}(\mathcal{Z}_1\times \mathrm{Bun}_G) \arrow[r, "\mathsf{H}^!(V,-)"] & \mathrm{Shv}(\mathcal{Z}_1\times \mathrm{Bun}_G \times X^I) \\ \mathrm{Shv}(\mathcal{Z}_2\times \mathrm{Bun}_G) \arrow[r, "\mathsf{H}^!(V,-)"] \arrow[u] & \mathrm{Shv}(\mathcal{Z}_2\times \mathrm{Bun}_G \times X^I), \arrow[u] \end{tikzcd}$$

where the vertical arrows are given by either !- or *- pullbacks.

Moreover, this datum of commutativity is functorial in $V\in \mathrm{Rep}(\check{G})^{\otimes I}$, and is compatible with the datum of commutativity of the diagrams (1.2).

1.2. **The ULA property of the Hecke action.**

1.2.1. Another key feature of the functors $\mathsf{H}^!$ is that for $\mathcal{F}\in \mathrm{Shv}(\mathcal{Z}\times \mathrm{Bun}_G)^c$ and $V\in (\mathrm{Rep}(\check{G})^{\otimes I})^c$, the object

$$\mathsf{H}^!(V,\mathcal{F})\in \mathrm{Shv}(\mathcal{Z}\times \mathrm{Bun}_G \times X^I)$$

is ULA with respect to the projection

$$\mathcal{Z}\times \mathrm{Bun}_G \times X^I \to X^I.$$

1.2.2. Let us denote by H^* the functor

$$\mathrm{Rep}(\check{G})^{\otimes I} \otimes \mathrm{Shv}(\mathcal{Z} \times \mathrm{Bun}_G) \to \mathrm{Shv}(\mathcal{Z} \times \mathrm{Bun}_G \times X^I)$$

defined as

$$\mathsf{H}^*(V, \mathcal{F}) := \mathsf{H}^!(V, \mathcal{F}) \overset{!}{\otimes} p_3^!(\underline{\mathsf{e}}_{X^I}).$$

Note that for a fixed I, the difference between $\mathsf{H}^!$ and H^* amounts to a cohomological shift by $2|I|$ since $\underline{\mathsf{e}}_{X^I} \simeq \omega_{X^I}[-2|I|]$.

1.2.3. The ULA property of the objects $\mathsf{H}^!(V, \mathcal{F})$ implies that we have canonical isomorphisms

$$\mathsf{H}^!(V, \mathcal{F}) \overset{!}{\otimes} p_3^!(\mathcal{M}) \simeq \mathsf{H}^*(V, \mathcal{F}) \overset{*}{\otimes} p_3^*(\mathcal{M}), \quad \mathcal{M} \in \mathrm{Shv}(X^I). \tag{1.3}$$

Furthermore, for a map of finite sets $\psi : I \to J$, we have a data of commutativity for the diagram

$$\begin{tikzcd} \mathrm{Rep}(\check{G})^{\otimes I} \otimes \mathrm{Shv}(\mathcal{Z} \times \mathrm{Bun}_G) \arrow[r, "\mathsf{H}^*"] \arrow[d, "\mathrm{Id} \otimes \mathrm{mult}^\psi"'] & \mathrm{Shv}(\mathcal{Z} \times \mathrm{Bun}_G \times X^I) \arrow[d, "(\mathrm{Id} \times \Delta_\psi)^*"] \\ \mathrm{Rep}(\check{G})^{\otimes J} \otimes \mathrm{Shv}(\mathcal{Z} \times \mathrm{Bun}_G) \arrow[r, "\mathsf{H}^*"] & \mathrm{Shv}(\mathcal{Z} \times \mathrm{Bun}_G \times X^J), \end{tikzcd} \tag{1.4}$$

endowed with a homotopy coherent system of compatibilities for compositions of maps of finite sets.

1.2.4. Thus, we can regard the functors $\mathsf{H}^*(V, -)$ also as defined and codefined by kernels, and they have the formal properties parallel to those of the functors $\mathsf{H}^!(V, -)$.

1.3. Hecke action on $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$.

1.3.1. One of the main actor in this paper is the full subcategory

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \subset \mathrm{Shv}(\mathrm{Bun}_G).$$

The following result, essentially due to [NY], describes the behavior of this subcategory under the Hecke functors (this is stated as [AGKRRV1, Theorem 10.2.3]):

Theorem 1.3.2. *The Hecke functor $\mathsf{H}^!$ for $I = \{*\}$ sends the full subcategory*

$$\mathrm{Rep}(\check{G}) \otimes \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \subset \mathrm{Rep}(\check{G}) \otimes \mathrm{Shv}(\mathrm{Bun}_G)$$

to the full subcategory

$$\mathrm{Shv}_{\mathrm{Nilp} \times \{0\}}(\mathrm{Bun}_G \times X) \subset \mathrm{Shv}(\mathrm{Bun}_G \times X).$$

1.3.3. Recall also (see [AGKRRV1, Theorem B.5.8]) that for any algebraic stack $\mathcal{Y}$ and a conical half-dimensional $\mathcal{N} \subset T^*(\mathcal{Y})$, the (a priori fully faithful) functor

$$\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y}) \otimes \mathrm{QLisse}(X) \to \mathrm{Shv}_{\mathcal{N} \times \{0\}}(\mathcal{Y} \times X)$$

is an equivalence.

Thus, from Theorem 1.3.2 we obtain:

Corollary 1.3.4. *The Hecke functor $\mathsf{H}^!$ for $I = \{*\}$ sends the full subcategory*

$$\mathrm{Rep}(\check{G}) \otimes \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \subset \mathrm{Rep}(\check{G}) \otimes \mathrm{Shv}(\mathrm{Bun}_G)$$

to the full subcategory

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{QLisse}(X) \subset \mathrm{Shv}(\mathrm{Bun}_G \times X).$$

Iterating, from Corollary 1.3.4 we further obtain:

Corollary 1.3.5. *The Hecke functors $\mathsf{H}^!$ map the full subcategory*

$$\mathrm{Rep}(\check{G})^{\otimes I} \otimes \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \subset \mathrm{Rep}(\check{G})^{\otimes I} \otimes \mathrm{Shv}(\mathrm{Bun}_G)$$

to the full subcategory

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{QLisse}(X)^{\otimes I} \subset \mathrm{Shv}(\mathrm{Bun}_G \times X^I).$$

1.3.6. Note that for a scheme or stack $\mathcal{Y}$, we can consider $\mathrm{QLisse}(\mathcal{Y})$ as a full subcategory of $\mathrm{Shv}(\mathcal{Y})$ in two different ways.

One is the tautological embedding[3]

$$\mathrm{QLisse}(\mathcal{Y}) \hookrightarrow \mathrm{Shv}(\mathcal{Y}), \quad \mathcal{L} \mapsto \mathcal{L}; \tag{1.5}$$

it endows $\mathrm{QLisse}(\mathcal{Y})$ with a symmetric monoidal structure induced by the !-tensor product on $\mathrm{Shv}(\mathcal{Y})$.

We also have a different embedding:

$$\mathrm{QLisse}(\mathcal{Y}) \hookrightarrow \mathrm{Shv}(\mathcal{Y}), \quad \mathcal{L} \mapsto \mathcal{L} \overset{!}{\otimes} \underline{\mathsf{e}}_{\mathcal{Y}}; \tag{1.6}$$

it endows $\mathrm{QLisse}(\mathcal{Y})$ with a symmetric monoidal structure induced by the *-tensor product on $\mathrm{Shv}(\mathcal{Y})$.

However, it follows tautologically that the two symmetric monoidal structures on $\mathrm{QLisse}(\mathcal{Y})$ coincide. Moreover, the operations

$$\mathrm{QLisse}(\mathcal{Y}) \otimes \mathrm{Shv}(\mathcal{Y}) \overset{(1.5)\otimes\mathrm{Id}}{\longrightarrow} \mathrm{Shv}(\mathcal{Y}) \otimes \mathrm{Shv}(\mathcal{Y}) \overset{\overset{!}{\otimes}}{\to} \mathrm{Shv}(\mathcal{Y})$$

and

$$\mathrm{QLisse}(\mathcal{Y}) \otimes \mathrm{Shv}(\mathcal{Y}) \overset{(1.6)\otimes\mathrm{Id}}{\longrightarrow} \mathrm{Shv}(\mathcal{Y}) \otimes \mathrm{Shv}(\mathcal{Y}) \overset{\overset{*}{\otimes}}{\to} \mathrm{Shv}(\mathcal{Y})$$

define the *same* monoidal action of $\mathrm{QLisse}(\mathcal{Y})$ on $\mathrm{Shv}(\mathcal{Y})$.

We will apply this discussion to $\mathcal{Y} = X^I$.

Remark 1.3.7. Note also that when $\mathcal{Y}$ is smooth of dimension n, the embeddings (1.5) and (1.6) differ by the cohomological shift $[2n]$. Yet they should not be confused.

1.3.8. An assertion parallel to Corollary 1.3.4 holds for the H^* functors. Namely, these functors send

$$\mathrm{Rep}(\check{G})^{\otimes I} \otimes \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \subset \mathrm{Rep}(\check{G})^{\otimes I} \otimes \mathrm{Shv}(\mathrm{Bun}_G)$$

to

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{QLisse}(X)^{\otimes I} \subset \mathrm{Shv}(\mathrm{Bun}_G \times X^I).$$

where we will think of the embedding $\mathrm{QLisse}(X)^{\otimes I} \hookrightarrow \mathrm{Shv}(X^I)$ as given by (1.6).

1.3.9. Thus, we can think of the Hecke action on $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ either by means of the functors

$$\mathsf{H}^! : \mathrm{Rep}(\check{G})^{\otimes I} \otimes \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \to \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{QLisse}(X)^{\otimes I}, \tag{1.7}$$

when we think of $\mathrm{QLisse}(X^I)$ as embedded into $\mathrm{Shv}(X^I)$ via (1.5), or, equivalently, as

$$\mathsf{H}^* : \mathrm{Rep}(\check{G})^{\otimes I} \otimes \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \to \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{QLisse}(X)^{\otimes I}, \tag{1.8}$$

when we think of $\mathrm{QLisse}(X^I)$ as embedded into $\mathrm{Shv}(X^I)$ via (1.6).

The functors (1.7) and (1.8) are canonically isomorphic. Thus, in what follows we will not distinguish notationally between $\mathsf{H}^!$ and H^*, when applied to objects from $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$, and denote the corresponding functors simply by

$$\mathsf{H} : \mathrm{Rep}(\check{G})^{\otimes I} \otimes \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \to \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{QLisse}(X)^{\otimes I}. \tag{1.9}$$

For a map of finite sets $\psi : I \to J$, we have a data of commutativity for the diagram

$$\begin{CD} \mathrm{Rep}(\check{G})^{\otimes I} \otimes \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) @>{\mathsf{H}}>> \mathrm{Shv}(\mathrm{Bun}_G) \otimes \mathrm{QLisse}(X)^{\otimes I} \\ @V{\mathrm{Id}\otimes \mathrm{mult}^\psi}VV @VV{\mathrm{Id}\times \mathrm{mult}^\psi}V \\ \mathrm{Rep}(\check{G})^{\otimes J} \otimes \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) @>{\mathsf{H}}>> \mathrm{Shv}(\mathrm{Bun}_G) \otimes \mathrm{QLisse}(X)^{\otimes J}, \end{CD} \tag{1.10}$$

where in the right vertical arrow the functor

$$\mathrm{mult}^\psi : \mathrm{QLisse}(X)^{\otimes I} \to \mathrm{QLisse}(X)^{\otimes J}$$

[3]Recall that our conventions are such that the default pullback functor is !-pullback, and therefore, by definition, lisse sheaves are those that are locally !-pulled back from a point. As explained below, it will be important to also consider the usual notion of lisse sheaves, i.e. those which are locally $*$-pulled back from a point.

is given by the symmetric monoidal structure on $\mathrm{QLisse}(X)$.

These data of commutativity are endowed with a homotopy coherent system of compatibilities for compositions of maps of finite sets.

1.4. **The category** $\mathrm{Rep}(\check{G})_{\mathrm{Ran}}$**.** We will now introduce a device that allows us to express the Hecke action on $\mathrm{Shv}(\mathrm{Bun}_G)$ in terms of a single monoidal category.

1.4.1. Let $\mathcal{C}$ be a symmetric monoidal DG category. We define a new symmetric monoidal DG category $\mathcal{C}_{\mathrm{Ran}}$ by the following construction.

Let $\mathrm{TwArr}(\mathrm{fSet})$ be the category of *twisted arrows* on fSet, see [GKRV, Sect. 1.2.2].

The category $\mathcal{C}_{\mathrm{Ran}}$ is the colimit over $\mathrm{TwArr}(\mathrm{fSet})$ of the functor

$$\mathrm{TwArr}(\mathrm{fSet}) \to \mathrm{DGCat} \tag{1.11}$$

that sends

$$(I \to J) \mapsto \mathcal{C}^{\otimes I} \otimes \mathrm{Shv}(X^J).$$

Here for a map

$$\begin{tikzcd} I_1 \arrow[r] \arrow[d, "\phi_I"'] & J_1 \\ I_2 \arrow[r] & J_2, \arrow[u, "\phi_J"'] \end{tikzcd} \tag{1.12}$$

in $\mathrm{TwArr}(\mathrm{fSet})$, the corresponding functor

$$\mathcal{C}^{\otimes I_1} \otimes \mathrm{Shv}(X^{J_1}) \to \mathcal{C}^{\otimes I_2} \otimes \mathrm{Shv}(X^{J_2})$$

is given by the tensor product functor along the fibers of ϕ_I

$$\mathrm{mult}^{\phi_I} : \mathcal{C}^{\otimes I_1} \to \mathcal{C}^{\otimes I_2} \tag{1.13}$$

and the functor

$$(\Delta_{\phi_J})_! : \mathrm{Shv}(X^{J_1}) \to \mathrm{Shv}(X^{J_2}), \tag{1.14}$$

where $\Delta_{\phi_J} : X^{J_2} \to X^{J_1}$ is the diagonal map induced by ϕ_J.

1.4.2. The functor (1.11) is naturally right-lax symmetric monoidal. Therefore, the colimit $\mathcal{C}_{\mathrm{Ran}}$ carries a natural symmetric monoidal structure. Explicitly, this symmetric monoidal structure can be described as follows. For

$$V_1 \otimes \mathcal{M}_1 \in \mathcal{C}^{\otimes I_1} \otimes \mathrm{Shv}(X^{J_1}) \text{ and } V_2 \otimes \mathcal{M}_2 \in \mathcal{C}^{\otimes I_2} \otimes \mathrm{Shv}(X^{J_2}),$$

the tensor product of their images in $\mathcal{C}_{\mathrm{Ran}}$ is the image of the object

$$(V_1 \otimes V_2) \otimes (\mathcal{M}_1 \boxtimes \mathcal{M}_2) \in \mathcal{C}^{\otimes (I_1 \sqcup I_2)} \otimes \mathrm{Shv}(X^{J_1 \sqcup J_2}).$$

We will denote the resulting monoidal operation on $\mathcal{C}_{\mathrm{Ran}}$ by

$$\mathcal{V}_1, \mathcal{V}_2 \mapsto \mathcal{V}_1 \star \mathcal{V}_2.$$

We denote the unit object by $\mathbf{1}_{\mathcal{C}_{\mathrm{Ran}}}$. It arises from $(\mathrm{Id} : \emptyset \to \emptyset) \in \mathrm{TwArr}(\mathrm{fSet})$ and the corresponding map

$$\mathrm{Vect} = \mathcal{C}^{\otimes \emptyset} \otimes \mathrm{Shv}(X^{\emptyset}) \to \mathcal{C}_{\mathrm{Ran}}.$$

1.4.3. Let $(\psi : I \to J) \in \mathrm{TwArr}(\mathrm{fSet})$ be given.

We denote by

$$\mathrm{ins}_\psi : \mathcal{C}^{\otimes I} \otimes \mathrm{Shv}(X^J) \to \mathcal{C}_{\mathrm{Ran}}$$

the corresponding functor.

In the important special case $\psi = \mathrm{Id}_I : I \to I$, we use the notation ins_I in place of $\mathrm{ins}_{\mathrm{Id}_I}$.

1.4.4. We will apply the above discussion to the case $\mathcal{C} = \operatorname{Rep}(\check{G})$.

We denote the resulting (symmetric monoidal) category by $\operatorname{Rep}(\check{G})_{\operatorname{Ran}}$.

1.5. **A Ran version of the Hecke action.**

1.5.1. We now claim that the datum of the functors (1.1) together with the compatibilities (1.2) allow to define an action of $\operatorname{Rep}(\check{G})_{\operatorname{Ran}}$ on $\operatorname{Shv}(\mathcal{Z} \times \operatorname{Bun}_G)$.

Namely, for $(I \overset{\psi}{\to} J) \in \operatorname{TwArr}(\operatorname{fSet})$ and

$$V \otimes \mathcal{M} \in \operatorname{Rep}(\check{G})^{\otimes I} \otimes \operatorname{Shv}(X^J),$$

we let the corresponding endofunctor of $\operatorname{Shv}(\mathcal{Z} \times \operatorname{Bun}_G)$ be the composition

$$\operatorname{Shv}(\mathcal{Z} \times \operatorname{Bun}_G) \overset{\mathsf{H}^!(V,-)}{\longrightarrow} \operatorname{Shv}(\mathcal{Z} \times \operatorname{Bun}_G \times X^I) \overset{-\overset{!}{\otimes} p_3^!((\Delta_\psi)_*(\mathcal{M}))}{\longrightarrow} \to \operatorname{Shv}(\mathcal{Z} \times \operatorname{Bun}_G \times X^I) \overset{(p_{1,2})_*}{\longrightarrow} \operatorname{Shv}(\mathcal{Z} \times \operatorname{Bun}_G). \tag{1.15}$$

1.5.2. Note, however, that using (1.3), we can rewrite the expression in (1.15) as

$$\operatorname{Shv}(\mathcal{Z} \times \operatorname{Bun}_G) \overset{\mathsf{H}^*(V,-)}{\longrightarrow} \operatorname{Shv}(\mathcal{Z} \times \operatorname{Bun}_G \times X^I) \overset{-\overset{*}{\otimes} p_3^*((\Delta_\psi)_*(\mathcal{M}))}{\longrightarrow} \to \operatorname{Shv}(\mathcal{Z} \times \operatorname{Bun}_G \times X^I) \overset{(p_{1,2})_!}{\longrightarrow} \operatorname{Shv}(\mathcal{Z} \times \operatorname{Bun}_G). \tag{1.16}$$

1.5.3. The interpretation of the Hecke action via (1.15) implies that it commutes with !-pullbacks and *-pushforwards along maps $f : \mathcal{Z}_1 \to \mathcal{Z}_2$. And the interpretation of the Hecke action via (1.16) implies that it commutes with *-pullbacks and !-pushforwards along maps $f : \mathcal{Z}_1 \to \mathcal{Z}_2$.

This implies that for a given $\mathcal{V} \in \operatorname{Rep}(\check{G})_{\operatorname{Ran}}$ its Hecke action is a functor both *defined and codefined by a kernel*, (see Sect. 0.6.6 for what this means).

We will denote the resulting endofunctor of $\operatorname{Shv}(\operatorname{Bun}_G)$ by

$$\mathcal{F} \mapsto \mathcal{V} \star \mathcal{F}$$

and of $\operatorname{Shv}(\mathcal{Z} \times \operatorname{Bun}_G)$ by $\operatorname{Id} \otimes (\mathcal{V} \star -)$ (see Sect. 0.6.6 for the $\otimes$ notation).

We will refer to endofunctors of $\operatorname{Shv}(\operatorname{Bun}_G)$ (or, more generally, $\operatorname{Shv}(\mathcal{Z} \times \operatorname{Bun}_G)$) that arise in this way as *integral Hecke functors*.

1.5.4. For later use, we introduce the following notation. For $\mathcal{V} \in \operatorname{Rep}(\check{G})_{\operatorname{Ran}}$ we let

$$\mathcal{K}_{\mathcal{V}} \in \operatorname{Shv}(\operatorname{Bun}_G \times \operatorname{Bun}_G)$$

denote the object equal to

$$(\operatorname{Id} \otimes (\mathcal{V} \star -))((\Delta_{\operatorname{Bun}_G})_!(\underline{\mathsf{e}}_{\operatorname{Bun}_G})).$$

1.5.5. By Theorem 1.3.2, the action of $\operatorname{Rep}(\check{G})_{\operatorname{Ran}}$ on $\operatorname{Shv}(\operatorname{Bun}_G)$ preserves the full subcategory

$$\operatorname{Shv}_{\operatorname{Nilp}}(\operatorname{Bun}_G) \subset \operatorname{Shv}(\operatorname{Bun}_G).$$

1.6. **The dual category of** $\operatorname{Rep}(\check{G})_{\operatorname{Ran}}$**.** For what follows we will need to recall some constructions pertaining to duality on $\operatorname{Rep}(\check{G})_{\operatorname{Ran}}$. We will do so in the general setting of Sect. 1.4.

Thus, we take $\mathcal{C}$ to be a general symmetric monoidal DG category.

1.6.1. Assume that $\mathcal{C}$ is dualizable, and that for every $(I \overset{\psi}{\to} J) \in \operatorname{TwArr}(\operatorname{fSet})$, the functor

$$\operatorname{mult}^\psi : \mathcal{C}^{\otimes I} \to \mathcal{C}^{\otimes J}$$

is such that the dual functor $(\mathcal{C}^{\otimes J})^\vee \to (\mathcal{C}^{\otimes I})^\vee$ admits a left adjoint.

In this case, one shows that the category $\mathcal{C}_{\operatorname{Ran}}$ is dualizable (see, e.g., [GR, Chapter 1, Proposition 6.3.4]).

1.6.2. The dual category $(\mathcal{C}_{\mathrm{Ran}})^\vee$ is the category of continuous functors $\mathcal{C}_{\mathrm{Ran}} \to \mathrm{Vect}$, and hence it can be described as

$$\lim_{(I \overset{\psi}{\to} J) \in \mathrm{TwArr}(\mathrm{fSet})^{op}} (\mathcal{C}^{\otimes I} \otimes \mathrm{Shv}(X^J))^\vee, \tag{1.17}$$

where the limit is formed using the functors dual to the ones used in the formation of the colimit in Sect. 1.4.1.

Using the Verdier self-duality on $\mathrm{Shv}(X^J)$, we can rewrite

$$(\mathcal{C}_{\mathrm{Ran}})^\vee \simeq \lim_{(I \overset{\psi}{\to} J) \in \mathrm{TwArr}(\mathrm{fSet})^{op}} \mathbf{Maps}_{\mathrm{DGCat}}(\mathcal{C}^{\otimes I}, \mathrm{Shv}(X^J)), \tag{1.18}$$

where $\mathbf{Maps}_{\mathrm{DGCat}}(-,-)$ stands for the DG category of continuous e-linear functor between two objects of DGCat.

Explicitly, the transition functors in (1.18) are defined as follows. For a morphism in TwArr(fSet) given by (1.12), the corresponding functor

$$\mathbf{Maps}_{\mathrm{DGCat}}(\mathcal{C}^{\otimes I_2}, \mathrm{Shv}(X^{J_2})) \to \mathbf{Maps}_{\mathrm{DGCat}}(\mathcal{C}^{\otimes I_1}, \mathrm{Shv}(X^{J_1}))$$

is given by precomposition (1.13) and postcomposition with

$$(\Delta_{\phi_J})^! : X^{J_2} \to X^{J_1},$$

which is the functor dual to (1.14), under the Verdier self-duality of $\mathrm{Shv}(X^?)$.

Remark 1.6.3. Suppose for a moment that $\mathcal{C}$ is rigid. In this case, we have a natural identification $\mathcal{C}^\vee \simeq \mathcal{C}$, and we can further rewrite the right-hand side in (1.18) as

$$\lim_{(I \overset{\psi}{\to} J) \in \mathrm{TwArr}(\mathrm{fSet})^{op}} \mathcal{C}^{\otimes I} \otimes \mathrm{Shv}(X^J),$$

where the limit is formed using the functors *right adjoint* to the ones used in the formation of the colimit in Sect. 1.4.1. Hence, the above limit is isomorphic to the colimit

$$\operatorname*{colim}_{(I \overset{\psi}{\to} J) \in \mathrm{TwArr}(\mathrm{fSet})} \mathcal{C}^{\otimes I} \otimes \mathrm{Shv}(X^J)$$

(see [GR, Chapter 1, Proposition 2.5.7]), i.e., to $\mathcal{C}_{\mathrm{Ran}}$ itself.

This implies that for $\mathcal{C}$ rigid, the category $\mathcal{C}_{\mathrm{Ran}}$ is naturally self-dual. However, we will not use this self-duality for the purposes of this paper.

1.6.4. Consider the $(\infty,2)$-category

$$\mathrm{DGCat}^{\mathrm{fSet}} := \mathrm{Funct}(\mathrm{fSet}, \mathrm{DGCat}).$$

There will be several DG categories of interest in this paper that will arise as

$$\mathbf{Maps}_{\mathrm{DGCat}^{\mathrm{fSet}}}(\mathfrak{C}_1, \mathfrak{C}_2)$$

for some particular $\mathfrak{C}_1, \mathfrak{C}_2 \in \mathrm{DGCat}^{\mathrm{fSet}}$.

Concretely, an object of $\mathbf{Maps}_{\mathrm{DGCat}^{\mathrm{fSet}}}(\mathfrak{C}_1, \mathfrak{C}_2)$ is a collection of functors between DG categories

$$\mathfrak{C}_1(I) \to \mathfrak{C}_2(I), \quad I \in \mathrm{fSet}$$

that make the diagrams

$$\begin{CD} \mathfrak{C}_1(I) @>>> \mathfrak{C}_2(I) \\ @V{\mathfrak{C}_1(\psi)}VV @VV{\mathfrak{C}_2(\psi)}V \\ \mathfrak{C}_1(J) @>>> \mathfrak{C}_2(J) \end{CD}$$

commute for $I \overset{\psi}{\to} J$, along with a homotopy coherent system of higher compatibilities.

1.6.5. Here are the first few objects of $\mathrm{DGCat}^{\mathrm{fSet}}$ that we will need.

One is the object denoted $\mathcal{C}^{\otimes\,\mathrm{fSet}}$, and defined by

$$I \in \mathrm{fSet} \rightsquigarrow \mathcal{C}^{\otimes I} \in \mathrm{DGCat},$$

where the functoriality is furnished by the symmetric monoidal structure on $\mathcal{C}$.

Another object, denoted $\mathrm{Shv}^!(X^{\mathrm{fSet}})$, is defined by

$$I \in \mathrm{fSet} \rightsquigarrow \mathrm{Shv}(X^I) \in \mathrm{DGCat},$$

where for $I \xrightarrow{\phi} J$, the corresponding functor $\mathrm{Shv}(X^I) \to \mathrm{Shv}(X^J)$ is $(\Delta_\phi)^!$.

1.6.6. Consider the category

$$\mathbf{Maps}_{\mathrm{DGCat}^{\mathrm{fSet}}}(\mathcal{C}^{\otimes\,\mathrm{fSet}}, \mathrm{Shv}^!(X^{\mathrm{fSet}})).$$

Note that this category identifies with the limit (1.18) (see e.g. [GKRV, Lemma 1.3.12]).

1.6.7. Thus, to summarize, we obtain a canonical equivalence

$$\mathbf{Maps}_{\mathrm{DGCat}^{\mathrm{fSet}}}(\mathcal{C}^{\otimes\,\mathrm{fSet}}, \mathrm{Shv}^!(X^{\mathrm{fSet}})) \simeq (\mathcal{C}_{\mathrm{Ran}})^\vee. \tag{1.19}$$

Explicitly, given

$$\mathcal{S} : \mathcal{C}_{\mathrm{Ran}} \to \mathrm{Vect},$$

the corresponding system of functors

$$\mathcal{S}_I : \mathcal{C}^{\otimes I} \to \mathrm{Shv}(X^I)$$

is recovered as follows:

We precompose $\mathcal{S}$ with ins_I to obtain a functor

$$\mathcal{C}^{\otimes I} \otimes \mathrm{Shv}(X^I) \to \mathrm{Vect}\,.$$

By Verdier duality, the datum of the latter functor is equivalent to the datum of a functor $\mathcal{S}_I$:

$$\mathcal{S} \circ \mathrm{ins}_I(c \otimes \mathcal{M}) = \mathrm{C}^\cdot(X^I, \mathcal{S}_I(c) \overset{!}{\otimes} \mathcal{M}), \quad c \in \mathcal{C}^{\otimes I},\ \mathcal{M} \in \mathrm{Shv}(X^I).$$

1.6.8. Vice versa, the pairing

$$\mathcal{C}_{\mathrm{Ran}} \otimes \mathbf{Maps}_{\mathrm{DGCat}^{\mathrm{fSet}}}(\mathcal{C}^{\otimes\,\mathrm{fSet}}, \mathrm{Shv}^!(X^{\mathrm{fSet}})) \to \mathrm{Vect}$$

is explicitly given as follows:

For an object $\{\mathcal{S}_I\} \in \mathbf{Maps}_{\mathrm{DGCat}^{\mathrm{fSet}}}(\mathcal{C}^{\otimes\,\mathrm{fSet}}, \mathrm{Shv}^!(X^{\mathrm{fSet}}))$, the corresponding functor

$$\mathcal{S} : \mathcal{C}_{\mathrm{Ran}} \to \mathrm{Vect}$$

is such that for $(I \xrightarrow{\psi} J) \in \mathrm{TwArr}(\mathrm{fSet})$, the resulting functor

$$\mathcal{C}^{\otimes I} \otimes \mathrm{Shv}(X^J) \xrightarrow{\mathrm{ins}_\psi} \mathcal{C}_{\mathrm{Ran}} \xrightarrow{\mathcal{S}} \mathrm{Vect},$$

equals

$$\mathcal{C}^{\otimes I} \otimes \mathrm{Shv}(X^J) \xrightarrow{\mathrm{mult}^\psi \otimes \mathrm{Id}} \mathcal{C}^{\otimes J} \otimes \mathrm{Shv}(X^J) \xrightarrow{\mathcal{S}_J \otimes \mathrm{Id}} \mathrm{Shv}(X^J) \otimes \mathrm{Shv}(X^J) \to \mathrm{Vect},$$

where the last arrow is the Verdier duality pairing on $\mathrm{Shv}(X^J)$, i.e.,

$$\mathrm{Shv}(X^J) \otimes \mathrm{Shv}(X^J) \xrightarrow{\Delta^!_{X^J}} \mathrm{Shv}(X^J) \xrightarrow{\mathrm{C}^\cdot(X^J, -)} \mathrm{Vect}\,.$$

2. Quasi-coherent sheaves on $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$

Although the statement of the Trace Conjecture does not involve Langlands duality, we will need some of its ingredients for the proof. Indeed, one of the key tools in the proof will be the category of quasi-coherent sheaves on the (pre)stack $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$, classifying local systems with restricted variation with respect to the Langlands dual group $\check{G}$ of G.

2.1. **The (pre)stack** $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$**.** We start by recalling the definition of the prestack

$$\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X),$$

following [AGKRRV1, Sect. 1.3].

2.1.1. For a test affine (derived) scheme S, we let $\mathrm{Maps}(S, \mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$ be the space of right t-exact symmetric monoidal functors

$$\mathrm{Rep}(\check{G}) \to \mathrm{QCoh}(S) \otimes \mathrm{QLisse}(X).$$

According to [AGKRRV1, Theorem 1.3.2], the prestack $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$ can be written as the quotient $\mathcal{Z}/\check{G}$, where $\mathcal{Z}$ is a disjoint union of *formal affine schemes* locally almost of finite type (over the field of coefficients e).

2.1.2. The main results of this paper will be based on considering the (symmetric monoidal) category

$$\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)).$$

We will now explain a certain feature that this category possesses, which is a consequence of properties of $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$ as a prestack.

(i) First, according to [AGKRRV1, Lemma 5.3.2], the diagonal map

$$\Delta_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)} : \mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X) \to \mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X) \times \mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$$

is affine, so the functor

$$(\Delta_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)})_* : \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \to \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X) \times \mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$$

is continuous.

(ii) Second, according to [AGKRRV1, Corollary 5.7.5], the category $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$ is dualizable. By [GR, Chapter 3, Proposition 3.1.7], this implies that for any prestack $\mathcal{Y}$ over e, the functor of external tensor product

$$\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \otimes \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X) \times \mathcal{Y})$$

is an equivalence.

In particular, we can view

$$(\Delta_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)})_*(\mathcal{O}_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)})$$

as an object of

$$\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \otimes \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)).$$

(iii) And third, according to [AGKRRV1, Corollary 5.7.10], the above object

$$(\Delta_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)})_*(\mathcal{O}_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}) \in \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \otimes \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$$

defines the unit of a self-duality on $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$.

2.1.3. *The functor of "sections with scheme-theoretic support".* We can view the structure sheaf $\mathcal{O}_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)} \in \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$ as defining a functor

$$\mathrm{Vect} \to \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)). \tag{2.1}$$

Note that the object $\mathcal{O}_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)} \in \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$ is not compact, so the functor of global sections

$$\Gamma(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X), -) : \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \to \mathrm{Vect},$$

the right adjoint to (2.1), is *not* continuous.

However, due to the self-duality of $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$, we can consider the functor *dual* to (2.1), which is a functor

$$\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \to \mathrm{Vect}, \tag{2.2}$$

which we will denote by $\Gamma_!(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X), -)$, and refer to as the functor of *sections with scheme-theoretic support*.

Remark 2.1.4. The terminology for $\Gamma_!(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X), -)$ is explained by the following assertion (see [AGKRRV1, Corollary 5.7.8 and Sect. 5.7.11]):

Proposition 2.1.5.
(a) *The functor*

$$\underset{S\in (\mathrm{Sch}_{\mathrm{aft}})_{/\,\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}}{\mathrm{colim}} \mathrm{QCoh}(S)$$

is an equivalence, where the transition functors in the family are

$$f: S_1 \to S_2 \rightsquigarrow f_* : \mathrm{QCoh}(S_1) \to \mathrm{QCoh}(S_2).$$

(b) *In terms of the above equivalence, the functor $\Gamma_!(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X), -)$ corresponds to the functor*

$$\underset{S\in (\mathrm{Sch}_{\mathrm{aft}})_{/\,\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}}{\mathrm{colim}} \mathrm{QCoh}(S) \to \mathrm{Vect}$$

given by the compatible family of functors

$$\Gamma(S, -) : \mathrm{QCoh}(S) \to \mathrm{Vect}\,.$$

2.1.6. *The tautological objects.* For a finite set I and an object $V \in \mathrm{Rep}(\check{G})^{\otimes I}$, let

$$\mathrm{Ev}(V) \in \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \otimes \mathrm{QLisse}(X)^{\otimes I}$$

be the corresponding tautological object:

For $S \to \mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$ corresponding to a symmetric monoidal functor

$$\Phi_S : \mathrm{Rep}(\check{G}) \to \mathrm{QCoh}(S) \otimes \mathrm{QLisse}(X),$$

the pullback of $\mathrm{Ev}(V)$ to S, viewed as an object in $\mathrm{QCoh}(S) \otimes \mathrm{QLisse}(X)^{\otimes I}$ equals the value on V of the functor

$$\mathrm{Rep}(\check{G})^{\otimes I} \overset{\Phi_S^{\otimes I}}{\to} (\mathrm{QCoh}(S) \otimes \mathrm{QLisse}(X))^{\otimes I} \to \mathrm{QCoh}(S) \otimes \mathrm{QLisse}(X)^{\otimes I},$$

where the second arrow is given by the I-fold tensor product functor

$$\mathrm{QCoh}(S)^{\otimes I} \to \mathrm{QCoh}(S).$$

2.2. **Description of the category $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$.** The prestack $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$ was defined using the symmetric monoidal categories $\mathrm{Rep}(\check{G})$ and $\mathrm{QLisse}(X)$. Therefore, it would not be very surprising to have a description of the category $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$ purely in terms of functors between the above two categories.

In this subsection, we will provide such a description, following [AGKRRV1].

2.2.1. Recall the category $\mathrm{DGCat}^{\mathrm{fSet}}$, see Sect. 1.6.4, and the object

$$\mathrm{QLisse}(X)^{\otimes\,\mathrm{fSet}} \in \mathrm{DGCat}^{\mathrm{fSet}},$$

see Sect. 1.6.5.

Note that we could also consider the object $\mathrm{QLisse}(X^{\mathrm{fSet}})$:

$$I \in \mathrm{fSet} \rightsquigarrow \mathrm{QLisse}(X^I) \in \mathrm{DGCat}\,.$$

We have a naturally defined map in $\mathrm{DGCat}^{\mathrm{fSet}}$

$$\mathrm{QLisse}(X)^{\otimes\,\mathrm{fSet}} \to \mathrm{QLisse}(X^{\mathrm{fSet}}). \tag{2.3}$$

However, from [AGKRRV1, Theorem B.5.8 and Corollary B.3.7], we obtain:

Lemma 2.2.2. *The map* (2.3) *is an isomorphism.*

2.2.3. Consider now the DG category $\mathbf{Maps}_{\mathrm{DGCat}^{\mathrm{fSet}}}(\mathrm{Rep}(\check{G})^{\otimes\,\mathrm{fSet}}, \mathrm{QLisse}(X)^{\otimes\,\mathrm{fSet}})$, and the following functor, to be denoted coLoc:

$$\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \to \mathbf{Maps}_{\mathrm{DGCat}^{\mathrm{fSet}}}(\mathrm{Rep}(\check{G})^{\otimes\,\mathrm{fSet}}, \mathrm{QLisse}(X)^{\otimes\,\mathrm{fSet}}). \tag{2.4}$$

Namely, coLoc sends $\mathcal{F} \in \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$ to the collection of functors

$$\mathcal{F}_I : \mathrm{Rep}(\check{G})^{\otimes I} \to \mathrm{QLisse}(X)^{\otimes I}$$

defined by

$$\mathcal{F}_I(V) := \Gamma_!(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X), \mathcal{F} \otimes \mathrm{Ev}(V)),$$

where $\Gamma_!$ is as in Sect. 2.1.3.

The following is one of the main results of the paper [AGKRRV1] (see Theorem 6.2.11 and Sect. 9.2.3 in *loc. cit.*):

Theorem 2.2.4. *The functor* coLoc *is an equivalence.*

Remark 2.2.5. We note that the results of this subsection and the previous one hold more generally: instead of a curve X one can take any scheme of finite type (not necessarily proper).

The only place were properness was used was Lemma 2.2.2, but Theorem 2.2.4 does not rely on it.

2.3. **Localization.** Recall the category $\mathrm{Rep}(\check{G})_{\mathrm{Ran}}$ (see Sect. 1.4.1). In this paper, we will use it as a device to relate the phenomenon of Hecke action and the category $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$.

In this subsection we will see how $\mathrm{Rep}(\check{G})_{\mathrm{Ran}}$ is related to $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$.

2.3.1. Consider the symmetric monoidal category $\mathrm{Rep}(\check{G})_{\mathrm{Ran}}$, see Sect. 1.4.1. We are going to construct a symmetric monoidal functor

$$\mathrm{Loc} : \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \to \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \tag{2.5}$$

that plays a key role in this work.

For an individual

$$(I \overset{\psi}{\to} J) \in \mathrm{TwArr}(\mathrm{fSet}),$$

the corresponding functor

$$\mathrm{Loc}^{I \overset{\psi}{\to} J} : \mathrm{Rep}(\check{G})^{\otimes I} \otimes \mathrm{Shv}(X^J) \to \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$$

sends $V \in \mathrm{Rep}(\check{G})^{\otimes I}$ to the functor $\mathrm{Shv}(X^J) \to \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$ equal to

$$\begin{aligned} \mathrm{Shv}(X^J) &\overset{\mathrm{Ev}(V)\otimes\mathrm{Id}}{\longrightarrow} \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \otimes \mathrm{QLisse}(X)^{\otimes I} \otimes \mathrm{Shv}(X^J) \overset{\mathrm{Id}\otimes\mathrm{mult}^{\psi}\otimes\mathrm{Id}}{\longrightarrow} \\ &\to \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \otimes \mathrm{QLisse}(X)^{\otimes J} \otimes \mathrm{Shv}(X^J) \longrightarrow \\ &\to \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \otimes \mathrm{Shv}(X^J) \overset{\mathrm{Id}\otimes\mathrm{C}^{\cdot}_c(X^J,-)}{\longrightarrow} \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)), \end{aligned}$$

where the third arrow using the canonical action of $\mathrm{QLisse}(X)^{\otimes J} \simeq \mathrm{QLisse}(X^J)$ on $\mathrm{Shv}(X^J)$, see Sect. 1.3.6.

It is easy to see that the functors $\mathrm{Loc}^{I \overset{\psi}{\to} J}$ indeed combine to define a functor, to be denote Loc, as in (2.5). Moreover, this functor carries a naturally defined symmetric monoidal structure.

2.3.2. Consider the dual functor

$$\mathrm{Loc}^\vee : \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \to (\mathrm{Rep}(\check{G})_{\mathrm{Ran}})^\vee.$$

Recall now that the category $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$ is self-dual (see Sect. 2.1.2), and that the category $(\mathrm{Rep}(\check{G})_{\mathrm{Ran}})^\vee$ can be described as $\mathbf{Maps}_{\mathrm{DGCat}^{\mathrm{fSet}}}(\mathcal{C}^{\otimes\, \mathrm{fSet}}, \mathrm{Shv}^!(X^{\mathrm{fSet}}))$ (see (1.19)).

Thus, we can view $\mathrm{Loc}^\vee$ as a functor

$$\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \to \mathbf{Maps}_{\mathrm{DGCat}^{\mathrm{fSet}}}(\mathcal{C}^{\otimes\, \mathrm{fSet}}, \mathrm{Shv}^!(X^{\mathrm{fSet}})).$$

Unwinding the definitions, we obtain that $\mathrm{Loc}^\vee$ identifies with the composition

$$\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \overset{\mathrm{coLoc}}{\to} \mathbf{Maps}_{\mathrm{DGCat}^{\mathrm{fSet}}}(\mathcal{C}^{\otimes\, \mathrm{fSet}}, \mathrm{QLisse}(X)^{\otimes\, \mathrm{fSet}}) \simeq$$
$$\simeq \mathbf{Maps}_{\mathrm{DGCat}^{\mathrm{fSet}}}(\mathcal{C}^{\otimes\, \mathrm{fSet}}, \mathrm{QLisse}(X^{\mathrm{fSet}})) \to \mathbf{Maps}_{\mathrm{DGCat}^{\mathrm{fSet}}}(\mathcal{C}^{\otimes\, \mathrm{fSet}}, \mathrm{Shv}^!(X^{\mathrm{fSet}})),$$

where the last arrow is given by the (fully faithful) functor (1.5).

Hence, combining with Theorem 2.2.4, we obtain:

Corollary 2.3.3.

(a) *The functor* $\mathrm{Loc}^\vee$ *is fully faithful.*

(b) *An object* $\mathcal{S} \in (\mathrm{Rep}(\check{G})_{\mathrm{Ran}})^\vee$ *lies in the essential image of* $\mathrm{Loc}^\vee$ *if and only if the corresponding family of functors* $\{\mathcal{S}_I\}$

$$\mathrm{Rep}(\check{G})^{\otimes I} \to \mathrm{Shv}(X^I)$$

takes values in $\mathrm{QLisse}(X^I) \subset \mathrm{Shv}(X^I)$.

2.3.4. Let

$$(\mathrm{Rep}(\check{G})_{\mathrm{Ran}})^\vee_{\mathrm{QLisse}} \subset (\mathrm{Rep}(\check{G})_{\mathrm{Ran}})^\vee$$

be the full subcategory that under the equivalence (1.19) corresponds to

$$\mathbf{Maps}_{\mathrm{DGCat}^{\mathrm{fSet}}}(\mathcal{C}^{\otimes\, \mathrm{fSet}}, \mathrm{QLisse}(X^{\mathrm{fSet}})) \subset \mathbf{Maps}_{\mathrm{DGCat}^{\mathrm{fSet}}}(\mathcal{C}^{\otimes\, \mathrm{fSet}}, \mathrm{Shv}^!(X^{\mathrm{fSet}})),$$

where the embedding $\mathrm{QLisse}(X^{\mathrm{fSet}}) \hookrightarrow \mathrm{Shv}^!(X^{\mathrm{fSet}})$ is (1.5).

We obtain that Corollary 2.3.3 can be reformulated as follows:

Corollary 2.3.5. *The functor* $\mathrm{Loc}^\vee$ *defines an equivalence*

$$\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \to (\mathrm{Rep}(\check{G})_{\mathrm{Ran}})^\vee_{\mathrm{QLisse}}.$$

2.4. **Beilinson's spectral projector.** In this subsection we will introduce a certain object

$$\mathcal{R} \in \mathrm{Rep}(\check{G})_{\mathrm{Ran}},$$

that will play a crucial role in the proof of the main results in this paper.

2.4.1. Let

$$R_{\check{G}} \in \mathrm{Rep}(\check{G}) \otimes \mathrm{Rep}(\check{G})$$

denote the regular representation.

For $(I \overset{\psi}{\to} J) \in \mathrm{TwArr}(\mathrm{fSet})$ let $R^{I \overset{\psi}{\to} J}_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}$ be the object of the tensor product category

$$\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \otimes \left(\mathrm{Rep}(\check{G})^{\otimes I} \otimes \mathrm{Shv}(X^J)\right),$$

equal to the image of $(R_{\check{G}})^{\otimes I} \in \mathrm{Rep}(\check{G})^{\otimes I} \otimes \mathrm{Rep}(\check{G})^{\otimes I}$ along the functor

$$\mathrm{Rep}(\check{G})^{\otimes I} \otimes \mathrm{Rep}(\check{G})^{\otimes I} \overset{\mathrm{Ev}(-)\otimes \mathrm{Id}}{\longrightarrow} \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \otimes \mathrm{QLisse}(X)^{\otimes I} \otimes \mathrm{Rep}(\check{G})^{\otimes I} \overset{\mathrm{Id}\otimes \mathrm{mult}^{\psi} \otimes \mathrm{Id}}{\longrightarrow}$$
$$\to \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \otimes \mathrm{QLisse}(X)^{\otimes J} \otimes \mathrm{Rep}(\check{G})^{\otimes I} \to$$
$$\to \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \otimes \mathrm{Shv}(X^J) \otimes \mathrm{Rep}(\check{G})^{\otimes I} = \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \otimes \mathrm{Rep}(\check{G})^{\otimes I} \otimes \mathrm{Shv}(X^J).$$

2.4.2. Mapping each $\mathrm{Rep}(\check{G})^{\otimes I} \otimes \mathrm{Shv}(X^J)$ to $\mathrm{Rep}(\check{G})_{\mathrm{Ran}}$, the assignment

$$(I \overset{\psi}{\to} J) \mapsto R^{I \overset{\psi}{\to} J}_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}$$

naturally extends to a functor

$$\mathrm{TwArr}(\mathrm{fSet}) \to \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \otimes \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \tag{2.6}$$

2.4.3. We define the object

$$R_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)} \in \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \otimes \mathrm{Rep}(\check{G})_{\mathrm{Ran}}$$

to be the colimit of the functor (2.6) over $\mathrm{TwArr}(\mathrm{fSet})$.

Remark 2.4.4. Recall that the category $\mathrm{Rep}(\check{G})_{\mathrm{Ran}}$ is canonically self-dual as a DG category, see Remark 1.6.3. Let

$$\mathrm{u}_{\mathrm{Rep}(\check{G})_{\mathrm{Ran}}} \in \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \otimes \mathrm{Rep}(\check{G})_{\mathrm{Ran}}$$

be the unit of this self-duality.

One can show that we have a canonical isomorphism

$$R_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)} \simeq (\mathrm{Loc} \otimes \mathrm{Id})(\mathrm{u}_{\mathrm{Rep}(\check{G})_{\mathrm{Ran}}}),$$

as objects of $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \otimes \mathrm{Rep}(\check{G})_{\mathrm{Ran}}$.

2.4.5. Consider the object

$$(\mathrm{Id} \otimes \mathrm{Loc})(R_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}) \in \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \otimes \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)). \tag{2.7}$$

The following is [AGKRRV1, Theorem 9.1.3]:

Theorem 2.4.6. *The image of* $(\mathrm{Id} \otimes \mathrm{Loc})(R_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)})$ *under*

$$\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \otimes \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \overset{\sim}{\to} \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X) \times \mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$$

identifies canonically with

$$(\Delta_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)})_*(\mathcal{O}_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}).$$

2.4.7. Denote

$$\mathcal{R} := (\Gamma_!(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X), -) \otimes \mathrm{Id})(R_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}) \in \mathrm{Rep}(\check{G})_{\mathrm{Ran}},$$

where $\Gamma_!(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X), -)$ is as in Sect. 2.1.3.

From Theorem 2.4.6 we obtain:

Corollary 2.4.8. *The object*

$$\mathrm{Loc}(\mathcal{R}) \in \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$$

identifies canonically with $\mathcal{O}_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}$.

3. The reciprocity law for shtuka cohomology

The main result of this section is Corollary 3.1.4, which asserts that the functor Sht introduced below factors through the localization functor Loc. We will deduce it from a theorem of C. Xue on lisseness of shtuka cohomology.

As an application, we construct an object $\mathrm{Drinf} \in \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}})$ that encodes the cohomology of shtuka moduli spaces.

3.1. **Functorial shtuka cohomology.** In this subsection we will interpret shtuka cohomology as a functor

$$\mathrm{Sht} : \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \to \mathrm{Vect},$$

i.e., an object of the category $(\mathrm{Rep}(\check{G})_{\mathrm{Ran}})^\vee$.

3.1.1. Recall the functor

$$\mathcal{V} \in \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \rightsquigarrow \mathcal{K}_{\mathcal{V}} \in \mathrm{Shv}(\mathrm{Bun}_G \times \mathrm{Bun}_G),$$

see Sect. 1.5.4.

Let $\mathrm{Graph}_{\mathrm{Frob}_{\mathrm{Bun}_G}} : \mathrm{Bun}_G \to \mathrm{Bun}_G \times \mathrm{Bun}_G$ be the graph of Frobenius, i.e., the map

$$\mathrm{Bun}_G \overset{\Delta_{\mathrm{Bun}_G}}{\longrightarrow} \mathrm{Bun}_G \times \mathrm{Bun}_G \overset{\mathrm{Frob}_{\mathrm{Bun}_G} \times \mathrm{Id}}{\longrightarrow} \mathrm{Bun}_G \times \mathrm{Bun}_G\,.$$

We define the functor $\mathrm{Sht} : \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \to \mathrm{Vect}$ as the composition

$$\mathrm{Rep}(\check{G})_{\mathrm{Ran}} \overset{\mathcal{V} \mapsto \mathcal{K}_{\mathcal{V}}}{\longrightarrow} \mathrm{Shv}(\mathrm{Bun}_G \times \mathrm{Bun}_G) \overset{(\mathrm{Graph}_{\mathrm{Frob}_{\mathrm{Bun}_G}})^*}{\longrightarrow} \mathrm{Shv}(\mathrm{Bun}_G) \overset{\mathrm{C}^\cdot_c(\mathrm{Bun}_G,-)}{\longrightarrow} \mathrm{Vect}\,.$$

3.1.2. We will prove:

Theorem 3.1.3. *The object* $\mathrm{Sht} \in (\mathrm{Rep}(\check{G})_{\mathrm{Ran}})^\vee$ *belongs to the full subcategory*

$$(\mathrm{Rep}(\check{G})_{\mathrm{Ran}})^\vee_{\mathrm{QLisse}} \subset (\mathrm{Rep}(\check{G})_{\mathrm{Ran}})^\vee.$$

Applying Corollary 2.3.5, from Theorem 3.1.3, we obtain:

Corollary 3.1.4. *The functor* $\mathrm{Sht} : \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \to \mathrm{Vect}$ *factors as* $\mathrm{Sht}_{\mathrm{Loc}} \circ \mathrm{Loc}$ *for a uniquely defined functor* $\mathrm{Sht}_{\mathrm{Loc}} : \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \to \mathrm{Vect}$.

3.2. **Proof of Theorem 3.1.3 and relation to the usual shtuka cohomology.** Once we relate the functor Sht to shtuka cohomology, the proof of Theorem 3.1.3 will be almost immediate from a recent theorem of C. Xue [Xue2].

3.2.1. For a finite set I, consider the functor

$$\mathrm{Rep}(\check{G})^{\otimes I} \to \mathrm{Shv}(X^I),$$

to be denoted Sht_I, that sends $V \in \mathrm{Rep}(\check{G})^{\otimes I}$ to

$$\mathrm{Sht}_I(V) := (p_2)_! \circ (\mathrm{Graph}_{\mathrm{Frob}_{\mathrm{Bun}_G}})^* \circ \mathsf{H}^*(V,-) \circ \Delta_!(\underline{\mathsf{e}}_{\mathrm{Bun}_G}). \tag{3.1}$$

The above functor Sht_I is the usual functor of (compactly supported) shtuka cohomology, studied by [VLaf].

3.2.2. We now quote the following crucial result of [Xue2]:

Theorem 3.2.3. *The functor* Sht_I *takes values in* $\mathrm{QLisse}(X^I) \subset \mathrm{Shv}(X^I)$.

3.2.4. Let us show how Theorem 3.2.3 implies the assertion of Theorem 3.1.3.

By base change (and using the fact that X is proper), we obtain that for $I \in \mathrm{fSet}$, the functor

$$\mathrm{Rep}(\check{G})^{\otimes I} \otimes \mathrm{Shv}(X^I) \overset{\mathrm{ins}_I}{\to} \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \overset{\mathrm{Sht}}{\to} \mathrm{Vect}$$

is given by sending

$$V \in \mathrm{Rep}(\check{G})^{\otimes I},\, \mathcal{M} \in \mathrm{Shv}(X^I) \;\mapsto \mathrm{C}^{\cdot}\left(X^I, (\mathrm{Sht}_I(V) \overset{*}{\otimes} \omega_{X^I}) \overset{!}{\otimes} \mathcal{M}\right).$$

Thus, the object of $\mathbf{Maps}_{\mathrm{DGCat}^{\mathrm{fSet}}}(\mathrm{Rep}(\check{G})^{\otimes\,\mathrm{fSet}}, \mathrm{Shv}^!(X^{\mathrm{fSet}}))$ corresponding to Sht is given by

$$V \in \mathrm{Rep}(\check{G})^{\otimes I} \;\mapsto \mathrm{Sht}_I(V) \overset{*}{\otimes} \omega_{X^I} \simeq \mathrm{Sht}_I(V)[2|I|],$$

see Sect. 1.6.7.

This object belongs to $\mathrm{QLisse}(X^I)$ by Theorem 3.2.3, as required.

Remark 3.2.5. Note that by Theorem 3.2.3, the expression $\mathrm{C}^{\cdot}\left(X^I, (\mathrm{Sht}_I(V) \overset{*}{\otimes} \omega_{X^I}) \overset{!}{\otimes} \mathcal{M}\right)$, which appears above, can be also rewritten as

$$\mathrm{C}^{\cdot}\left(X^I, \mathrm{Sht}_I(V) \overset{*}{\otimes} \mathcal{M}\right),$$

see Sect. 1.2.3.

Remark 3.2.6. Along with the object $\mathrm{Shv}^!(X^{\mathrm{fSet}}) \in \mathrm{DGCat}^{\mathrm{fSet}}$, we can consider the object, denoted $\mathrm{Shv}^*(X^{\mathrm{fSet}})$, whose value on I is again the category $\mathrm{Shv}(X^I)$, but now for $I \overset{\psi}{\to} J$ we use the functor

$$(\Delta_\psi)^* : \mathrm{Shv}(X^I) \to \mathrm{Shv}(X^J).$$

As in Sect. 1.3.6, we have the fully faithful embeddings

$$\mathrm{Shv}^*(X^{\mathrm{fSet}}) \hookleftarrow \mathrm{QLisse}(X^{\mathrm{fSet}}) \hookrightarrow \mathrm{Shv}^!(X^{\mathrm{fSet}}),$$

given by (1.5) and (1.6).

By its construction, the system of functors $\{\mathrm{Sht}_I\}$ is naturally an object of the category $\mathbf{Maps}_{\mathrm{DGCat}^{\mathrm{fSet}}}(\mathrm{Rep}(\check{G})^{\otimes\,\mathrm{fSet}}, \mathrm{Shv}^*(X^{\mathrm{fSet}}))$, and we can view Theorem 3.2.3 as saying that it actually belongs to the essential image of the functor

$$\mathbf{Maps}_{\mathrm{DGCat}^{\mathrm{fSet}}}(\mathrm{Rep}(\check{G})^{\otimes\,\mathrm{fSet}}, \mathrm{QLisse}(X^{\mathrm{fSet}})) \hookrightarrow \mathbf{Maps}_{\mathrm{DGCat}^{\mathrm{fSet}}}(\mathrm{Rep}(\check{G})^{\otimes\,\mathrm{fSet}}, \mathrm{Shv}^*(X^{\mathrm{fSet}})).$$

Now, the object of $\mathbf{Maps}_{\mathrm{DGCat}^{\mathrm{fSet}}}(\mathrm{Rep}(\check{G})^{\otimes\,\mathrm{fSet}}, \mathrm{Shv}^!(X)^{\mathrm{fSet}})$ corresponding to Sht equals the image of the resulting object of $\mathbf{Maps}_{\mathrm{DGCat}^{\mathrm{fSet}}}(\mathrm{Rep}(\check{G})^{\otimes\,\mathrm{fSet}}, \mathrm{QLisse}(X^{\mathrm{fSet}}))$ under

$$\mathbf{Maps}_{\mathrm{DGCat}^{\mathrm{fSet}}}(\mathrm{Rep}(\check{G})^{\otimes\,\mathrm{fSet}}, \mathrm{QLisse}(X^{\mathrm{fSet}})) \hookrightarrow \mathbf{Maps}_{\mathrm{DGCat}^{\mathrm{fSet}}}(\mathrm{Rep}(\check{G})^{\otimes\,\mathrm{fSet}}, \mathrm{Shv}^!(X^{\mathrm{fSet}})).$$

Thus, we denote by the same symbol $\{\mathrm{Sht}_I\}$ the above objects of the categories

$$\mathbf{Maps}_{\mathrm{DGCat}^{\mathrm{fSet}}}(\mathrm{Rep}(\check{G})^{\otimes\,\mathrm{fSet}}, \mathrm{Shv}^*(X^{\mathrm{fSet}})) \hookleftarrow \mathbf{Maps}_{\mathrm{DGCat}^{\mathrm{fSet}}}(\mathrm{Rep}(\check{G})^{\otimes\,\mathrm{fSet}}, \mathrm{QLisse}(X^{\mathrm{fSet}})) \hookrightarrow$$
$$\hookrightarrow \mathbf{Maps}_{\mathrm{DGCat}^{\mathrm{fSet}}}(\mathrm{Rep}(\check{G})^{\otimes\,\mathrm{fSet}}, \mathrm{Shv}^!(X^{\mathrm{fSet}})).$$

Example 3.2.7. We have a canonical isomorphism

$$\mathrm{Sht}(\mathbf{1}_{\mathrm{Rep}(\check{G})_{\mathrm{Ran}}}) \simeq \mathrm{Sht}_\emptyset(\mathsf{e}) = \mathrm{C}_c\left(\mathrm{Bun}_G, (\mathrm{Graph}_{\mathrm{Frob}_{\mathrm{Bun}_G}})^* \circ (\Delta_{\mathrm{Bun}_G})_!(\underline{\mathsf{e}}_{\mathrm{Bun}_G})\right) \simeq \mathrm{Funct}_c(\mathrm{Bun}_G(\mathbb{F}_q))$$

where the last isomorphism is by base-change.

3.3. **Drinfeld's sheaf.**

3.3.1. By Corollary 3.1.4, there is a canonical functor $\mathrm{Sht}_{\mathrm{Loc}}: \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \to \mathrm{Vect}$, i.e.,

$$\mathrm{Sht}_{\mathrm{Loc}} \in \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))^{\vee}.$$

As $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$ is canonically self-dual (cf. Sect. 2.1.2), there is a corresponding object[4]

$$\mathrm{Drinf} \in \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)).$$

3.3.2. Let us observe some basic properties of Drinf.

First, we have

$$\Gamma_c(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X), \mathrm{Drinf}) \simeq \mathrm{Sht}_{\mathrm{Loc}}(\mathcal{O}_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}).$$

by duality.

As

$$\mathcal{O}_{\mathrm{LocSys}^{\mathrm{restr}}} = \mathrm{Loc}(\mathbf{1}_{\mathrm{Rep}(\check{G})_{\mathrm{Ran}}}),$$

we deduce from Example 3.2.7 that there is a canonical isomorphism

$$\Gamma_c(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X), \mathrm{Drinf}) = \mathrm{Sht}(\mathbf{1}_{\mathrm{Rep}(\check{G})_{\mathrm{Ran}}}) = \mathrm{Funct}_c(\mathrm{Bun}_G(\mathbb{F}_q)). \tag{3.2}$$

3.3.3. More generally, recall from Sect. 2.2.3 that for any $\mathcal{F} \in \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$ and a finite set I, one defines a functor

$$\mathcal{F}_I : \mathrm{Rep}(\check{G})^{\otimes I} \to \mathrm{QLisse}(X^I).$$

By construction, for $\mathcal{F} = \mathrm{Drinf}$, the functor Drinf_I coincides with Sht_I. In this manner, we see that Drinf encodes cohomology of shtuka moduli spaces.

3.4. **Some remarks.** We now provide some additional remarks on the object Drinf.

Remark 3.4.1. Recall (see [AGKRRV1, Sect. 16.1]) that the stack $\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)$ is defined as

$$(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))^{\mathrm{Frob}}.$$

Let ι denote the forgetful map

$$\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X) \to \mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X).$$

One can show that the object Drinf can be *a priori* obtained as $\iota_*(\mathrm{Drinf}^{\mathrm{arithm}})$ for a canonically defined object

$$\mathrm{Drinf}^{\mathrm{arithm}} \in \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)).$$

This additional structure on Drinf encodes the equivariance of the objects

$$\mathrm{Sht}_I(V) \in \mathrm{Shv}(X^I), \quad V \in \mathrm{Rep}(\check{G})^{\otimes I}$$

with respect to the *partial Frobenius* maps acting on X^I. See also Sect. 4.5.6.

Remark 3.4.2. The object $\mathrm{Drinf}^{\mathrm{arithm}}$ allows to recover the spectral decomposition of the space of automorphic functions along classical Langlands parameters, established in [VLaf] for the cuspidal subspace and extended in [Xue1] to the entire space.

Namely, by (3.2), we have:

$$\Gamma(\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X), \mathrm{Drinf}^{\mathrm{arithm}}) \simeq \mathrm{Funct}_c(\mathrm{Bun}_G(\mathbb{F}_q)).$$

Set

$$\mathcal{A} := \Gamma(\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X), \mathcal{O}_{\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)});$$

this is is a commutative DG algebra over e that lives in non-positive cohomological degrees. Set

$$\mathrm{LocSys}^{\mathrm{arithm,coarse}}_{\check{G}}(X) := \mathrm{Spec}(\mathcal{A});$$

this is an affine (derived) scheme over e.

[4]This object is named after V. Drinfeld, since it was his idea, upon learning about V. Lafforgue's work, that shtuka cohomology should be encoded by a quasi-coherent sheaf on the stack of Langlands parameters.

Let $\mathcal{A}^0$ denote the 0-th cohomology of $\mathcal{A}$, so that $\mathrm{Spec}(\mathcal{A}^0)$ is the classical affine scheme ${}^{\mathrm{cl}}\mathrm{LocSys}^{\mathrm{arithm,coarse}}_{\check{G}}(X)$ underlying $\mathrm{LocSys}^{\mathrm{arithm,coarse}}_{\check{G}}(X)$

Now, one can show that the set of e-points of $\mathrm{Spec}(\mathcal{A}^0)$ is in bijection with isomorphism classes of semi-simple Frobenius-equivariant $\check{G}$-local systems on X, see [AGKRRV1, Corollary 2.4.8][5]. I.e., we can view ${}^{\mathrm{cl}}\mathrm{LocSys}^{\mathrm{arithm,coarse}}_{\check{G}}(X)$ as the scheme of classical Langlands parameters.

By construction, $\mathcal{A}$ acts on the space of global actions of any object of $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X))$. In particular, we obtain an action of $\mathcal{A}$ on $\mathrm{Funct}_c(\mathrm{Bun}_G(\mathbb{F}_q))$. However, since $\mathrm{Funct}_c(\mathrm{Bun}_G(\mathbb{F}_q))$ sits in cohomologcal degree 0, this action factors through an action of $\mathcal{A}^0$ on $\mathrm{Funct}_c(\mathrm{Bun}_G(\mathbb{F}_q))$.

Thus, we can view $\mathrm{Funct}_c(\mathrm{Bun}_G(\mathbb{F}_q))$ as global sections of a canonically defined quasi-coherent sheaf on ${}^{\mathrm{cl}}\mathrm{LocSys}^{\mathrm{arithm,coarse}}_{\check{G}}(X)$. This indeed may be viewed as a spectral decomposition of $\mathrm{Funct}_c(\mathrm{Bun}_G(\mathbb{F}_q))$ over classical Langlands parameters.

Furthermore, one can show that $\mathcal{A}^0$ is a quotient of V. Lafforgue's algebra of excursion operators. So the above action of $\mathcal{A}^0$ recovers the action of the excursion algebra on $\mathrm{Funct}_c(\mathrm{Bun}_G(\mathbb{F}_q))$, established in [Xue1].

Remark 3.4.3. The above construction of the object Drinf (resp., $\mathrm{Drinf}^{\mathrm{arithm}}$) was specific to the *everywhere unramified* situation. In a subsequent publication, we will show this construction can be generalized to allow for level structure.

I.e., given a divisor $D\subset X$ define over $\mathbb{F}_q$, one can construct objects

$$\mathrm{Drinf}_D\in \mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X-D) \text{ and } \mathrm{Drinf}^{\mathrm{arithm}}_D\in \mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X-D)$$

that encode shtuka cohomology with level structure.

What is for now a far-fetched goal is to interpret $\mathrm{Drinf}^{\mathrm{arithm}}_D$ also as categorical trace, see Sect. 4.5.7 for what we mean by that.

4. Calculating the trace

In this section we will prove the main result of this paper, Corollary 4.1.4, which asserts that the space of (compactly supported) automorphic functions can be obtained as the (categorical) trace of Frobenius on the category of automorphic sheaves with nilpotent singular support.

4.1. **Traces of Frobenius-Hecke operators.**

4.1.1. Recall that the $\mathrm{Rep}(\check{G})_{\mathrm{Ran}}$ action on $\mathrm{Shv}(\mathrm{Bun}_G)$ preserves its subcategory $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$. Therefore, we obtain a functor

$$\mathrm{Rep}(\check{G})_{\mathrm{Ran}}\to \mathbf{Maps}_{\mathrm{DGCat}}(\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G),\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G))$$

sending $\mathcal{V}$ to the functor $\mathcal{V}\star -$.

Note also that the subcategory $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)\subset \mathrm{Shv}(\mathrm{Bun}_G)$ is preserved by the endofunctor $(\mathrm{Frob}_{\mathrm{Bun}_G})_*$, see [AGKRRV1, Sect. 15.3.3].

We define a functor

$$\mathrm{Sht}^{\mathrm{Tr}}:\mathrm{Rep}(\check{G})_{\mathrm{Ran}}\to \mathrm{Vect}$$

as the functor

$$\mathcal{V}\mapsto \mathrm{Tr}\left((\mathcal{V}\star -)\circ (\mathrm{Frob}_{\mathrm{Bun}_G})_*,\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G),\right).$$

In other words, we compose $\mathcal{V}\star -$ with pushforward along the geometric Frobenius $\mathrm{Frob}_{\mathrm{Bun}_G}$ endomorphism of Bun_G and form the trace (as an endofunctor of the (dualizable) DG category $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$).

Our main theorem asserts:

Theorem 4.1.2. *There is a canonical isomorphism of functors* $\mathrm{Sht}\simeq \mathrm{Sht}^{\mathrm{Tr}}:\mathrm{Rep}(\check{G})_{\mathrm{Ran}}\to \mathrm{Vect}$.

We will prove this result in Sect. 4.2.

[5]The quoted result of [AGKRRV1] applies to $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$, but the case of $\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)$ is similar.

4.1.3. For the moment, we assume Theorem 4.1.2 and deduce further results from it.

First, observe that by definition, we have

$$\mathrm{Sht}^{\mathrm{Tr}}(\mathbf{1}_{\mathrm{Rep}(\check{G})_{\mathrm{Ran}}}) = \mathrm{Tr}((\mathrm{Frob}_{\mathrm{Bun}_G})_*, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)).$$

On the other hand, by Example 3.2.7 we have

$$\mathrm{Sht}(\mathbf{1}_{\mathrm{Rep}(\check{G})_{\mathrm{Ran}}}) = \mathrm{Funct}_c(\mathrm{Bun}_G(\mathbb{F}_q)).$$

Hence, we obtain:

Corollary 4.1.4. *There exists a canonical isomorphism in* Vect

$$\mathrm{Tr}((\mathrm{Frob}_{\mathrm{Bun}_G})_*, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)) \simeq \mathrm{Funct}_c(\mathrm{Bun}_G(\mathbb{F}_q)).$$

4.1.5. By (1.19), the functor $\mathrm{Sht}^{\mathrm{Tr}}$ corresponds to a system of functors $\{\mathrm{Sht}^{\mathrm{Tr}}_I\}$

$$\mathrm{Sht}^{\mathrm{Tr}}_I : \mathrm{Rep}(\check{G})^{\otimes I} \to \mathrm{Shv}(X^I). \tag{4.1}$$

From Theorem 4.1.2 we immediately obtain:

Corollary 4.1.6. *For an individual finite set I, the functors* Sht_I *and* $\mathrm{Sht}^{\mathrm{Tr}}_I$

$$\mathrm{Rep}(\check{G})^{\otimes I} \to \mathrm{QLisse}(X)^{\otimes I}$$

are canonically isomorphic.

Let us describe the functors $\mathrm{Sht}^{\mathrm{Tr}}_I$ explicitly.

4.1.7. We will use the following construction.

Recall that if $\mathbf{C}$ is a dualizable DG category and $T : \mathbf{C} \to \mathbf{C} \otimes \mathbf{D}$ is a functor, we can consider the *relative trace* object

$$\mathrm{Tr}(T, \mathbf{C}) \in \mathbf{D}$$

defined as the composition

$$\mathrm{Vect} \overset{\mathrm{u}_{\mathbf{C}}}{\to} \mathbf{C}^\vee \otimes \mathbf{C} \overset{\mathrm{Id} \otimes T}{\to} \mathbf{C}^\vee \otimes \mathbf{C} \otimes \mathbf{D} \overset{\mathrm{ev}_{\mathbf{C}} \otimes \mathrm{Id}}{\to} \mathbf{D}.$$

4.1.8. Let I be a finite set. Recall that we have a canonically defined functor

$$\mathsf{H} : \mathrm{Rep}(\check{G})^{\otimes I} \otimes \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \to \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{QLisse}(X^I),$$

see (1.9).

In particular, for $V \in \mathrm{Rep}(\check{G})^{\otimes I}$ we can consider the object

$$\mathrm{Tr}(\mathsf{H}(V, -) \circ (\mathrm{Frob}_{\mathrm{Bun}_G})_*, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)) \in \mathrm{QLisse}(X^I),$$

and this operation defines a functor

$$\mathrm{Rep}(\check{G})^{\otimes I} \to \mathrm{QLisse}(X^I). \tag{4.2}$$

Unwinding the definitions, it is easy to see that the functor $\mathrm{Sht}^{\mathrm{Tr}}_I$ of (4.1) identifies with the composition of (4.2) and the embedding $\mathrm{QLisse}(X^I) \to \mathrm{Shv}(X^I)$ of (1.5).

4.1.9. Hence, from Corollary 4.1.6 we obtain:

Corollary 4.1.10. *For an individual finite set I, the functor*

$$\mathrm{Sht}_I : \mathrm{Rep}(\check{G})^{\otimes I} \to \mathrm{QLisse}(X^I)$$

of (3.1) *identifies canonically with the functor*

$$V \mapsto \mathrm{Tr}(\mathsf{H}(V, -) \circ (\mathrm{Frob}_{\mathrm{Bun}_G})_*, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)).$$

Corollary 4.1.10 is the *Shtuka Conjecture* from [AGKRRV1] (Conjecture 15.5.7 in *loc.cit.*).

4.1.11. Finally, we note that Theorem 4.1.2 may be tautologically be reformulated in terms of Drinfeld sheaves.

First, as was noted above, the functors $\mathrm{Sht}_I^{\mathrm{Tr}}$ take values in $\mathrm{QLisse}(X^I)$. Therefore, by Corollary 2.3.5, $\mathrm{Sht}^{\mathrm{Tr}}$ factors uniquely as $\mathrm{Sht}^{\mathrm{Tr}}_{\mathrm{Loc}} \circ \mathrm{Loc}$ for a uniquely defined functor

$$\mathrm{Sht}^{\mathrm{Tr}}_{\mathrm{Loc}} : \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \to \mathrm{Vect}\,.$$

As in Sect. 3.3, by self-duality of $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$, to $\mathrm{Sht}^{\mathrm{Tr}}_{\mathrm{Loc}}$ there corresponds an object $\mathrm{Drinf}^{\mathrm{Tr}} \in \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$.

From Theorem 4.1.2, we obtain:

Corollary 4.1.12. *The objects* Drinf *and* $\mathrm{Drinf}^{\mathrm{Tr}}$ *of* $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}})$ *are canonically isomorphic.*

4.2. **Proof of Theorem 4.1.2.**

4.2.1. Recall the object $\mathcal{R} \in \mathrm{Rep}(\check{G})_{\mathrm{Ran}}$ of Sect. 2.4. The proof of Theorem 4.1.2 will be based on the following result.

Theorem 4.2.2. *There is a canonical isomorphism*

$$\mathrm{Sht}^{\mathrm{Tr}} \simeq \mathrm{Sht}(\mathcal{R} \star -)$$

as functors $\mathrm{Rep}(\check{G})_{\mathrm{Ran}} \to \mathrm{Vect}$.

Remark 4.2.3. As the proof of Theorem 4.2.2 will show, we will rather construct an isomorphism

$$\mathrm{Sht}^{\mathrm{Tr}} \simeq \mathrm{Sht}(- \star \mathcal{R}),$$

and we will swap $\mathrm{Sht}(- \star \mathcal{R})$ for $\mathrm{Sht}(\mathcal{R} \star -)$ using the fact that the category $\mathrm{Rep}(\check{G})_{\mathrm{Ran}}$ is *symmetric monoidal*.

Our preference of one over the other is purely notational.

4.2.4. We postpone the proof of Theorem 4.2.2 to Sect. 4.4. It is essentially a calculation using the results of [AGKRRV2], which we recall in Sect. 4.3.

For the present, we assume Theorem 4.2.2 and deduce Theorem 4.1.2 from it.

4.2.5. The argument is straightforward at this point using Theorem 3.1.3. Recall that *loc. cit.* provides a factorization $\mathrm{Sht} = \mathrm{Sht}_{\mathrm{Loc}} \circ \mathrm{Loc}$.

Now recall that Loc is monoidal and sends $\mathcal{R}$ to the structure sheaf by Corollary 2.4.8. Therefore,

$$\mathrm{Loc}(\mathcal{R} \star -) \simeq \mathrm{Loc}(-)$$

as functors $\mathrm{Rep}(\check{G})_{\mathrm{Ran}} \to \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$.

By Theorem 4.2.2, we obtain

$$\mathrm{Sht}^{\mathrm{Tr}} \simeq \mathrm{Sht}(\mathcal{R} \star -) \simeq \mathrm{Sht}_{\mathrm{Loc}} \circ \mathrm{Loc}(\mathcal{R} \star -) \simeq \mathrm{Sht}_{\mathrm{Loc}} \circ \mathrm{Loc}(-) \simeq \mathrm{Sht}$$

as desired.

4.3. **Self-duality for** $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$**.** To prove Theorem 4.2.2, we use the explicit description of the dual of $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ from [AGKRRV2]. We review these results below.

4.3.1. Consider the object

$$\mathcal{K}_{\mathcal{R}} \in \mathrm{Shv}(\mathrm{Bun}_G \times \mathrm{Bun}_G),$$

where $\mathcal{R} \in \mathrm{Rep}(\check{G})_{\mathrm{Ran}}$ was defined in Sect. 2.4, and where the notation $\mathcal{K}_{-}$ is an in Sect. 1.5.4.

A priori, it is defined as an object of the category $\mathrm{Shv}(\mathrm{Bun}_G \times \mathrm{Bun}_G)$. However, we have the following key result, which follows from [AGKRRV2, Sect. 2.1.3 and Proposition 2.1.10]:

Theorem 4.3.2. *The object* $\mathcal{K}_{\mathcal{R}}$ *lies in the essential image of the fully faithful functor*

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \to \mathrm{Shv}(\mathrm{Bun}_G \times \mathrm{Bun}_G).$$

4.3.3. For a stack $\mathcal{Y}$ let
$$\mathrm{ev}^*_{\mathcal{Y}} : \mathrm{Shv}(\mathcal{Y}) \otimes \mathrm{Shv}(\mathcal{Y}) \to \mathrm{Vect}$$
be the functor given by
$$\mathcal{F}_1, \mathcal{F}_2 \mapsto \mathrm{C}^\cdot_c(\mathcal{Y}, \mathcal{F}_1 \overset{*}{\otimes} \mathcal{F}_2).$$
Warning: in general, the pairing $\mathrm{ev}^*_{\mathcal{Y}}$ is *not* perfect.

4.3.4. Take $\mathcal{Y} = \mathrm{Bun}_G$, and we restrict the pairing $\mathrm{ev}^*_{\mathrm{Bun}_G}$ to
$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \subset \mathrm{Shv}(\mathrm{Bun}_G) \otimes \mathrm{Shv}(\mathrm{Bun}_G).$$

We now quote the following result (see [AGKRRV2, Theorem 2.1.5]):

Theorem 4.3.5. *The object*
$$\mathcal{K}_{\mathcal{R}} \in \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$$
together with the pairing
$$\mathrm{ev}^*_{\mathrm{Bun}_G} : \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$$
define an identification
$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)^\vee \simeq \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G).$$

4.4. **Proof of Theorem 4.2.2.**

4.4.1. Let us be given a dualizable DG category $\mathbf{C}$ and functors $T, S : \mathbf{C} \to \mathbf{C}$.

In this case, recall that we can tautologically compute $\mathrm{Tr}(S \circ T, \mathbf{C}) \in \mathrm{Vect}$ as the composition
$$\mathrm{Vect} \xrightarrow{\mathrm{u}_{\mathbf{C}}} \mathbf{C}^\vee \otimes \mathbf{C} \xrightarrow{T^\vee \otimes S} \mathbf{C}^\vee \otimes \mathbf{C} \xrightarrow{\mathrm{ev}_{\mathbf{C}} \otimes \mathrm{Id}} \mathrm{Vect}\,.$$

4.4.2. For $\mathcal{V} \in \mathrm{Rep}(\check{G})_{\mathrm{Ran}}$, we deduce from Theorem 4.3.5 and the above that we can compute $\mathrm{Sht}^{\mathrm{Tr}}(\mathcal{V})$ as the composition
$$\mathrm{Vect} \xrightarrow{\mathcal{K}_{\mathcal{R}}} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \xrightarrow{((\mathrm{Frob}_{\mathrm{Bun}_G})_*)^\vee \otimes (\mathcal{V} \star -)}$$
$$\to \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \xrightarrow{- \overset{*}{\otimes} -} \mathrm{Shv}(\mathrm{Bun}_G) \xrightarrow{\mathrm{C}_c(\mathrm{Bun}_G, -)} \mathrm{Vect}\,.$$
Here $((\mathrm{Frob}_{\mathrm{Bun}_G})_*)^\vee$ is the dual functor to $(\mathrm{Frob}_{\mathrm{Bun}_G})_*$ with respect to the duality of Theorem 4.3.5.

4.4.3. To proceed further, we need to compute $((\mathrm{Frob}_{\mathrm{Bun}_G})_*)^\vee$.

Lemma 4.4.4. *In the above notation, the functor*
$$((\mathrm{Frob}_{\mathrm{Bun}_G})_*)^\vee : \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \to \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$$
is canonically isomorphic to $(\mathrm{Frob}_{\mathrm{Bun}_G})^*$.

Proof. For an algebraic stack $\mathcal{Y}$, the geometric Frobenius morphism $\mathrm{Frob}_{\mathcal{Y}}$ is finite, so we have a canonical isomorphism
$$(\mathrm{Frob}_{\mathcal{Y}})_! \simeq (\mathrm{Frob}_{\mathcal{Y}})_*. \tag{4.3}$$

Therefore, we need to compute the dual to $(\mathrm{Frob}_{\mathrm{Bun}_G})_! : \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \to \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$.

Note that $(\mathrm{Frob}_{\mathrm{Bun}_G})_!$ is an auto-equivalence of $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$. Moreover, the unit and counit of the duality are equivariant with respect to this autoequivalence. It follows formally under such circumstances that the dual to $(\mathrm{Frob}_{\mathrm{Bun}_G})_!$ is its inverse.

Observe that the inverse to $(\mathrm{Frob}_{\mathrm{Bun}_G})_! \simeq (\mathrm{Frob}_{\mathrm{Bun}_G})_*$ is its left adjoint $(\mathrm{Frob}_{\mathrm{Bun}_G})^*$, completing the argument. (This argument shows we could also take $(\mathrm{Frob}_{\mathrm{Bun}_G})^!$ in the assertion of the lemma.)

□

4.4.5. For $\mathcal{V} \in \mathrm{Rep}(\check{G})_{\mathrm{Ran}}$, we deduce that we can compute $\mathrm{Sht}^{\mathrm{Tr}}(\mathcal{V})$ as the composition:

$$\mathrm{Vect} \xrightarrow{\mathcal{K}_{\mathcal{R}}} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \overset{(\mathrm{Frob}_{\mathrm{Bun}_G})^* \otimes (\mathcal{V} \star -)}{\longrightarrow}$$
$$\to \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \xrightarrow{-\overset{*}{\otimes}-} \mathrm{Shv}(\mathrm{Bun}_G) \overset{\mathrm{C}_c(\mathrm{Bun}_G,-)}{\longrightarrow} \mathrm{Vect}\,.$$

Note that

$$(\mathrm{Id} \otimes (\mathcal{V} \star -))(\mathcal{K}_{\mathcal{R}}) \simeq \mathcal{K}_{\mathcal{V}\star\mathcal{R}} \simeq \mathcal{K}_{\mathcal{R}\star\mathcal{V}}.$$

Therefore, we can compute $\mathrm{Sht}^{\mathrm{Tr}}(\mathcal{V})$ as the composition

$$\mathrm{Vect} \xrightarrow{\mathcal{K}_{\mathcal{R}\star\mathcal{V}}} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \overset{(\mathrm{Frob}_{\mathrm{Bun}_G})^* \otimes \mathrm{Id}}{\longrightarrow}$$
$$\to \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \overset{\Delta^*_{\mathrm{Bun}_G}(-\boxtimes-)}{\longrightarrow} \mathrm{Shv}(\mathrm{Bun}_G) \overset{\mathrm{C}_c(\mathrm{Bun}_G,-)}{\longrightarrow} \mathrm{Vect}\,.$$

This coincides with

$$\mathrm{Vect} \xrightarrow{\mathcal{K}_{\mathcal{R}\star\mathcal{V}}} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \overset{(\mathrm{Graph}_{\mathrm{Frob}_{\mathrm{Bun}_G}})^*}{\longrightarrow} \mathrm{Shv}(\mathrm{Bun}_G) \overset{\mathrm{C}_c(\mathrm{Bun}_G,-)}{\longrightarrow} \mathrm{Vect}\,.$$

By definition, this composition is $\mathrm{Sht}(\mathcal{R} \star \mathcal{V})$, concluding the argument.

4.5. **Interpretation as enhanced trace.** The contents of this section are an extended remark and are not necessary for the rest of the paper.

4.5.1. Recall (see [AGKRRV1, Theorem 10.5.2]) that the category $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ carries a monoidal action of $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$.

Moreover, the action of $(\mathrm{Frob}_{\mathrm{Bun}_G})_*$ is compatible with the action of the monoidal automorphism of Frob^* of $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$, where Frob is the automorphism of $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$, induced by Frobenius endomorphism Frob_X of X.

In this case, following [GKRV, Sect. 3.8.2], to the pair $(\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G), (\mathrm{Frob}_{\mathrm{Bun}_G})_*)$ we can attached its *enhanced trace*,

$$\mathrm{Tr}^{\mathrm{enh}}_{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))}((\mathrm{Frob}_{\mathrm{Bun}_G})_*, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)) \in \mathrm{HH}_\bullet(\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)), \mathrm{Frob}^*),$$

where $\mathrm{HH}_\bullet(\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)), \mathrm{Frob}^*)$ is the (symmetric monoidal) category of Hochschild chains on the (symmetric) monoidal category $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$ with respect to the (symmetric) monoidal endofunctor Frob^*.

Furthermore, we have

$$\mathrm{HH}_\bullet(\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)), \mathrm{Frob}^*) \simeq \mathrm{QCoh}((\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))^{\mathrm{Frob}}),$$

see [GKRV, Sect. 3.7.3].

4.5.2. Recall that we denote

$$\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X) := (\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))^{\mathrm{Frob}},$$

and by ι the forgetful map

$$\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X) \to \mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)),$$

see [AGKRRV1, Sect. 16.1].

Thus, we can interpret the above object $\mathrm{Tr}^{\mathrm{enh}}_{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))}((\mathrm{Frob}_{\mathrm{Bun}_G})_*, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G))$ as an object, denoted

$$\mathrm{Drinf}^{\mathrm{Tr,arithm}} \in \mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X).$$

We have

$$\mathrm{Drinf}^{\mathrm{Tr}} \simeq \iota_*(\mathrm{Drinf}^{\mathrm{Tr,arithm}}); \tag{4.4}$$

this assertion is essentially [GKRV, Theorem 4.4.4].

4.5.3. Recall the objects

$$\mathrm{Ev}(V) \in \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \otimes \mathrm{QLisse}(X)^{\otimes I}.$$

By construction, the corresponding objects

$$(\iota^* \otimes \mathrm{Id})(\mathrm{Ev}(V)) \in \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)) \otimes \mathrm{QLisse}(X)^{\otimes I}$$

carry a structure of equivariance with respect to the partial Frobenius endmorphisms acting along X^I.

By (4.4) and the projection formula, we have

$$\mathrm{Sht}^{\mathrm{Tr}}_I(V) \simeq (\Gamma(\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X), -) \otimes \mathrm{Id}) \left((\iota^* \otimes \mathrm{Id})(\mathrm{Ev}(V)) \otimes \mathrm{Drinf}^{\mathrm{Tr,arithm}})\right).$$

This interpretation shows that the the objects

$$\mathrm{Sht}^{\mathrm{Tr}}_I(V) \in \mathrm{QLisse}(X)^{\otimes I} \subset \mathrm{Shv}(X^I)$$

carry a natural structure of equivariance with respect to the partial Frobenius endomorphisms on X^I.

4.5.4. Having proved Theorem 4.1.2, and hence Corollary 4.1.6, we obtain that the objects

$$\mathrm{Sht}_I(V) \in \mathrm{Shv}(X^I)$$

also carry a natural structure of equivariance with respect to the partial Frobenius endomorphisms on X^I.

However, it is not difficult to show that this structure identifies with the of partial Frobenius equivariance on shtukas (see in [VLaf]). The matching between the two is is essentially [GKRV, Lemma 4.5.4].

4.5.5. Now, given an object $\mathcal{F} \in \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$, the datum required to exhibit it as

$$\iota_*(\mathcal{F}^{\mathrm{arithm}})$$

for some $\mathcal{F}^{\mathrm{arithm}} \in \mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)$ is equivalent to a *compatible* collection of structure of partial Frobenius equivariance on the associated functors

$$\mathcal{F}_I : \mathrm{Rep}(\check{G})^{\otimes I} \to \mathrm{QLisse}(X)^{\otimes I}, \quad V \mapsto (\Gamma(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X), -) \otimes \mathrm{Id})(\mathrm{Ev}(V) \otimes \mathcal{F}).$$

4.5.6. As was mentioned in Remark 3.4.1, the corresponding structure of partial Frobenius equivariance can be directly constructed on the functors Sht_I, thereby producing an object

$$\mathrm{Drinf}^{\mathrm{arithm}} \in \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)).$$

Thus, the assertion in Sect. 4.5.4 can interpreted as an isomorphism

$$\mathrm{Drinf}^{\mathrm{arithm}} \simeq \mathrm{Drinf}^{\mathrm{Tr,arithm}},$$

which induces the isomorphism

$$\mathrm{Drinf} \simeq \mathrm{Drinf}^{\mathrm{Tr}}$$

of Theorem 4.1.2 by applying ι_*.

4.5.7. Furthermore, as was mentioned in Remark 3.4.3, the corresponding structure of partial Frobenius equivariance can be constructed also on the functors

$$\mathrm{Sht}_{I,D} : \mathrm{Rep}(\check{G})^{\otimes I} \to \mathrm{QLisse}(X - D)^{\otimes I}$$

that encode the cohomology of shtukas with level structure, thereby producing an object

$$\mathrm{Drinf}^{\mathrm{arithm}}_D \in \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X - D)).$$

What we do not have at the moment is the interpretation of this object $\mathrm{Drinf}^{\mathrm{arithm}}_D$ as enhanced categorical trace (nor of the functors $\mathrm{Sht}_{I,D}$ as just categorical traces).

5. Local terms

5.1. **Formulation of the problem.**

5.1.1. In this section we will construct four maps

$$\mathrm{LT}^{\mathrm{naive}}, \mathrm{LT}^{\mathrm{true}}, \mathrm{LT}^{\mathrm{Serre}}, \mathrm{LT}^{\mathrm{Sht}} : \mathrm{Tr}((\mathrm{Frob}_{\mathrm{Bun}_G})_*, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)) \to \mathrm{Funct}_c(\mathrm{Bun}_G(\mathbb{F}_q))$$

that we refer to as *local term* morphisms.

At this point, we have only encountered the last of these four morphisms: by definition, $\mathrm{LT}^{\mathrm{Sht}}$ is the isomorphism of Corollary 4.1.4.

5.1.2. Assuming the construction of the other three morphisms, we can state the main result of this section and the next:

Theorem 5.1.3. *The four morphisms* $\mathrm{LT}^{\mathrm{naive}}$, $\mathrm{LT}^{\mathrm{true}}$, $\mathrm{LT}^{\mathrm{Serre}}$, *and* $\mathrm{LT}^{\mathrm{Sht}}$ *are equal.*

Remark 5.1.4. By Corollary 4.1.4, the source and target of each local term map is in $\mathrm{Vect}^{\heartsuit}$, so equality (as opposed to homotopy) is the relevant notion.

5.1.5. As $\mathrm{LT}^{\mathrm{Sht}}$ is an isomorphism, from Theorem 5.1.3 we deduce:

Corollary 5.1.6. *Each of the four local term morphisms in an isomorphism.*

In the case of $\mathrm{LT}^{\mathrm{naive}}$ (see below), this corollary recovers the Trace Conjecture as formulated in [AGKRRV1, Conjecture 15.3.5].

We now proceed to the construction of the local term morphisms.

5.2. **Naive and true local terms.** In what follows, we fix an algebraic stack $\mathcal{Y}$, which plays the role of Bun_G. We will add certain additional assumptions to $\mathcal{Y}$ as we proceed.

The material in this subsection closely follows [AGKRRV1, Sect. 15.1-15.2], to which we refer the reader for more detail.

5.2.1. *Naive local term.* Any algebraic stack $\mathcal{Y}$ has the property that $\mathrm{Shv}(\mathcal{Y})$ is compactly generated and every compact object is !-extended from some quasi-compact open in $\mathcal{Y}$ (see [AGKRRV1, Sect. C.1.1]).

Suppose $y \in \mathcal{Y}(\mathbb{F}_q)$ is given. In this case, the functor $y^* : \mathrm{Shv}(\mathcal{Y}) \to \mathrm{Vect}$ preserves compact objects and intertwines $(\mathrm{Frob}_{\mathcal{Y}})_*$ with the identity functor (by (4.3)). Therefore, functoriality of traces yields an induced map

$$\mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}(\mathcal{Y})) \to \mathrm{Tr}(\mathrm{Id}, \mathrm{Vect}) = \mathsf{e}.$$

In the quasi-compact case, this yields a map

$$\mathrm{LT}^{\mathrm{naive}}_{\mathcal{Y}} : \mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}(\mathcal{Y})) \to \mathrm{Funct}(\mathcal{Y}(\mathbb{F}_q)).$$

Remark 5.2.2. By definition, this construction satisfies the compatibility referenced in Remark 0.2.6.

5.2.3. *True local term.* Suppose first that $\mathcal{Y}$ is quasi-compact. Suppose in addition that $\mathcal{Y}$ is locally of the form Z/H, where Z is a scheme of finite type and H is an affine algebraic group.

As in [AGKRRV2, Sects. 1.1.3], these assumptions imply that $\mathrm{Shv}(\mathcal{Y})$ is canonically self-dual (via Verdier duality), with pairing, denoted $\mathrm{ev}^!_{\mathcal{Y}}$:

$$\mathrm{Shv}(\mathcal{Y}) \otimes \mathrm{Shv}(\mathcal{Y}) \subset \mathrm{Shv}(\mathcal{Y} \times \mathcal{Y}) \xrightarrow{\Delta^!_{\mathcal{Y}}} \mathrm{Shv}(\mathcal{Y}) \xrightarrow{\mathrm{C}^{\cdot}_{\blacktriangle}(\mathcal{Y}, -)} \mathrm{Vect}\,.$$

The unit for this duality, denoted

$$\mathrm{u}_{\mathrm{Shv}(\mathcal{Y})} \in \mathrm{Shv}(\mathcal{Y}) \otimes \mathrm{Shv}(\mathcal{Y})$$

is obtained by applying the right adjoint to the embedding

$$\mathrm{Shv}(\mathcal{Y}) \otimes \mathrm{Shv}(\mathcal{Y}) \hookrightarrow \mathrm{Shv}(\mathcal{Y} \times \mathcal{Y}) \tag{5.1}$$

to $(\Delta_{\mathcal{Y}})_*(\omega_{\mathcal{Y}})$.

In what follows we will not distinguish notationally between $\mathrm{u}_{\mathrm{Shv}(\mathcal{Y})}$ and its image under the fully faithful functor (5.1). Thus, by adjunction we obtain a map

$$\mathrm{u}_{\mathrm{Shv}(\mathcal{Y})} \to (\Delta_{\mathcal{Y}})_*(\omega_{\mathcal{Y}}).$$

From here, we obtain a map

$$\mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}(\mathcal{Y})) \simeq \mathrm{C}^\cdot_\blacktriangle\left(\mathcal{Y}, \Delta^!_{\mathcal{Y}} \circ ((\mathrm{Frob}_{\mathcal{Y}})_* \otimes \mathrm{Id})(\mathrm{u}_{\mathrm{Shv}(\mathcal{Y})})\right) \simeq \mathrm{C}^\cdot_\blacktriangle\left(\mathcal{Y}, \Delta^!_{\mathcal{Y}} \circ (\mathrm{Frob}_{\mathcal{Y}} \times \mathrm{Id})_*(\mathrm{u}_{\mathrm{Shv}(\mathcal{Y})})\right) \to$$
$$\to \mathrm{C}^\cdot_\blacktriangle\left(\mathcal{Y}, \Delta^!_{\mathcal{Y}} \circ (\mathrm{Frob}_{\mathcal{Y}} \times \mathrm{Id})_* \circ (\Delta_{\mathcal{Y}})_*(\omega_{\mathcal{Y}})\right) \simeq \mathrm{C}^\cdot_\blacktriangle(\mathcal{Y}^{\mathrm{Frob}}, \omega_{\mathcal{Y}^{\mathrm{Frob}}}) \simeq \mathrm{Funct}(\mathcal{Y}(\mathbb{F}_q))$$

whose composition we denote by

$$\mathrm{LT}^{\mathrm{true}}_{\mathcal{Y}} : \mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}(\mathcal{Y})) \to \mathrm{Funct}(\mathcal{Y}(\mathbb{F}_q)).$$

5.2.4. We have (see [GV, Theorem 0.4]):

Theorem 5.2.5.

(a) *The maps* $\mathrm{LT}^{\mathrm{true}}_{\mathcal{Y}}$ *and* $\mathrm{LT}^{\mathrm{naive}}_{\mathcal{Y}}$ *are canonically homotopic.*

(b) *For a schematic map* $f : \mathcal{Y}_1 \to \mathcal{Y}_2$*, the diagram*

$$\begin{tikzcd}
\mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}_1})_*, \mathrm{Shv}(\mathcal{Y}_1)) \arrow[r, "\mathrm{LT}^{\mathrm{true}}_{\mathcal{Y}_1}"] \arrow[d] & \mathrm{Funct}(\mathcal{Y}_1(\mathbb{F}_q)) \arrow[d] \\
\mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}_2})_*, \mathrm{Shv}(\mathcal{Y}_2)) \arrow[r, "\mathrm{LT}^{\mathrm{true}}_{\mathcal{Y}_2}"] & \mathrm{Funct}(\mathcal{Y}_2(\mathbb{F}_q))
\end{tikzcd}$$

is commutative, where the left vertical arrow is induced by the functor $f_! : \mathrm{Shv}(\mathcal{Y}_1) \to \mathrm{Shv}(\mathcal{Y}_2)$*, and the right vertical arrow is given by pushforward.*

(c) *The commutative diagram in point (b) is compatible with the identification of point (a) with the (tautologically) commutative diagram*

$$\begin{tikzcd}
\mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}_1})_*, \mathrm{Shv}(\mathcal{Y}_1)) \arrow[r, "\mathrm{LT}^{\mathrm{naive}}_{\mathcal{Y}_1}"] \arrow[d] & \mathrm{Funct}(\mathcal{Y}_1(\mathbb{F}_q)) \arrow[d] \\
\mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}_2})_*, \mathrm{Shv}(\mathcal{Y}_2)) \arrow[r, "\mathrm{LT}^{\mathrm{naive}}_{\mathcal{Y}_2}"] & \mathrm{Funct}(\mathcal{Y}_2(\mathbb{F}_q)).
\end{tikzcd}$$

5.2.6. Let now $\mathcal{Y}$ be not necessarily quasi-compact. We will consider the poset of quasi-compact open substacks

$$\mathcal{U} \overset{j}{\hookrightarrow} \mathcal{Y},$$

and the corresponding functors $j_! : \mathrm{Shv}(\mathcal{U}) \to \mathrm{Shv}(\mathcal{Y})$.

The category $\mathrm{Shv}(\mathcal{Y})$ is compactly generated by the essential images of $j_!|_{\mathrm{Shv}(\mathcal{U})^c}$. Furthermore, the induced map

$$\underset{\mathcal{U}}{\mathrm{colim}}\ \mathrm{Tr}((\mathrm{Frob}_{\mathcal{U}})_*, \mathrm{Shv}(\mathcal{U})) \to \mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}(\mathcal{Y}))$$

is an isomorphism (see [AGKRRV1, Sect. 15.1.10]).

Using the commutative diagrams in Theorem 5.2.5(b),(c), this allows to define the maps

$$\mathrm{LT}^{\mathrm{naive}}_{\mathcal{Y}} \text{ and } \mathrm{LT}^{\mathrm{true}}_{\mathcal{Y}}$$

from $\mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}(\mathcal{Y}))$ to

$$\underset{\mathcal{U}}{\mathrm{colim}}\ \mathrm{Funct}(\mathcal{U}(\mathbb{F}_q)) \simeq \mathrm{Funct}_c(\mathcal{Y}(\mathbb{F}_q)).$$

Moreover, by Theorem 5.2.5(a), these two maps are canonically homotopic.

5.2.7. *The case of* Bun_G. We now specialize to the case of Bun_G. We consider the full subcategory

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \hookrightarrow \mathrm{Shv}(\mathrm{Bun}_G). \tag{5.2}$$

As was mentioned earlier, it is preserved by the endofunctor $(\mathrm{Frob}_{\mathrm{Bun}_G})_*$. Furthermore, it is generated by objects that are compact in the ambient category $\mathrm{Shv}(\mathrm{Bun}_G)$, see [AGKRRV1, Theorem 10.1.6].

In particular, the category $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ is itself compactly generated, and the embedding (5.2) preserves compactness. Thus, we have a well-defined map

$$\mathrm{Tr}((\mathrm{Frob}_{\mathrm{Bun}_G})_*, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)) \to \mathrm{Tr}((\mathrm{Frob}_{\mathrm{Bun}_G})_*, \mathrm{Shv}(\mathrm{Bun}_G)).$$

Composing with the maps $\mathrm{LT}^{\mathrm{naive}}_{\mathrm{Bun}_G}$ and $\mathrm{LT}^{\mathrm{true}}_{\mathrm{Bun}_G}$, we obtain two maps

$$\mathrm{Tr}((\mathrm{Frob}_{\mathrm{Bun}_G})_*, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)) \rightrightarrows \mathrm{Funct}_c(\mathrm{Bun}_G(\mathbb{F}_q)),$$

that we will denote $\mathrm{LT}^{\mathrm{naive}}$ and $\mathrm{LT}^{\mathrm{true}}$, respectively.

However, the homotopy between $\mathrm{LT}^{\mathrm{naive}}_{\mathrm{Bun}_G}$ and $\mathrm{LT}^{\mathrm{true}}_{\mathrm{Bun}_G}$ (see Sect. 5.2.6) gives rise to a homotopy between $\mathrm{LT}^{\mathrm{naive}}$ and $\mathrm{LT}^{\mathrm{true}}$.

Remark 5.2.8. Let us explain the practical implication of the equality $\mathrm{LT}^{\mathrm{naive}} = \mathrm{LT}^{\mathrm{Sht}}$, stated in Theorem 5.1.3.

Let $\mathcal{F}$ be an an object of $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)^c$ equipped with a weak Weil structure, i.e., a map

$$\alpha : \mathcal{F} \to (\mathrm{Frob}_{\mathrm{Bun}_G})_*(\mathcal{F}). \tag{5.3}$$

To such a pair $(\mathcal{F}, \alpha)$, we can attach its class

$$\mathrm{cl}(\mathcal{F}, \alpha) \in \mathrm{Tr}((\mathrm{Frob}_{\mathrm{Bun}_G})_*, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G))$$

(see [GKRV, Sect. 3.4.3]). Thus, using Corollary 4.1.4, to $(\mathcal{F}, \alpha)$ we can attach an element of

$$\mathrm{Funct}_c(\mathrm{Bun}_G(\mathbb{F}_q)),$$

i.e., a compactly supported automorphic function.

Now, the content of Theorem 5.1.3 is that the above element of $\mathrm{Funct}_c(\mathrm{Bun}_G(\mathbb{F}_q))$ equals the function attached to $\mathcal{F}$, viewed as a weak Weil sheaf via α, by the usual sheaf-function correspondence, i.e., by taking pointwise traces of the Frobenius.

5.3. **Serre local term.** In this subsection we will define the last remaining map in Sect. 5.1.3, denoted $\mathrm{LT}^{\mathrm{Serre}}$.

5.3.1. Let $\mathcal{Y}$ be an algebraic stack, and let $\mathcal{N} \subset T^*(\mathcal{Y})$ be a conical Zariski-closed subset. Consider the (fully faithful) embedding

$$\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y}) \otimes \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y}) \to \mathrm{Shv}(\mathcal{Y} \times \mathcal{Y}). \tag{5.4}$$

Denote

$$\text{ps-u}_{\mathcal{Y}} := \Delta_!(\underline{\mathsf{e}}_{\mathcal{Y}}) \in \mathrm{Shv}(\mathcal{Y} \times \mathcal{Y}),$$

and let

$$\text{ps-u}_{\mathcal{Y},\mathcal{N}} \in \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y}) \otimes \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})$$

be obtained by applying to $\text{ps-u}_{\mathcal{Y}}$ the right adjoint to the functor (5.4). We will not distinguish notationally between $\text{ps-u}_{\mathcal{Y},\mathcal{N}}$ and its image along (5.4).

The counit of the adjunction defines a map

$$\text{ps-u}_{\mathcal{Y},\mathcal{N}} \to \text{ps-u}_{\mathcal{Y}}\,. \tag{5.5}$$

5.3.2. The map (5.5) gives rise to a natural transformation

$$(\mathrm{ev}^*_{\mathcal{Y}}\otimes \mathrm{Id})\circ(\mathrm{Id}\otimes \text{ps-u}_{\mathcal{Y},\mathcal{N}}) = ((\mathrm{C}^\cdot_c(\mathcal{Y},-)\circ \Delta^*_{\mathcal{Y}})\otimes \mathrm{Id})\circ(\mathrm{Id}\otimes \text{ps-u}_{\mathcal{Y},\mathcal{N}})\to \\ \to ((\mathrm{C}^\cdot_c(\mathcal{Y},-)\circ \Delta^*_{\mathcal{Y}})\otimes \mathrm{Id})\circ(\mathrm{Id}\otimes \text{ps-u}_{\mathcal{Y}})\simeq \mathrm{Id} \tag{5.6}$$

as endofunctors of $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})$ (see Sect. 4.3.3 for the $\mathrm{ev}^*_{\mathcal{Y}}$ notation).

Recall, following [AGKRRV2, Sect. 3.3], that the pair $(\mathcal{Y},\mathcal{N})$ is said to be *Serre* if the natural transformation (5.6) is an isomorphism. If this is the case, then the data of

$$\text{ps-u}_{\mathcal{Y},\mathcal{N}}\in \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})\otimes \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y}) \text{ and } \mathrm{ev}^*_{\mathcal{Y}}:\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})\otimes \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})\to \mathrm{Vect}$$

define the unit and counit of a self-duality on $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})$.

5.3.3. Suppose that $(\mathcal{Y},\mathcal{N})$ is Serre. As with the true local term morphism, we define

$$\mathrm{LT}^{\mathrm{Serre}}_{\mathcal{Y},\mathcal{N}}:\mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*,\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y}))\to \mathrm{Funct}_c(\mathcal{Y}(\mathbb{F}_q))$$

as the composition

$$\mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*,\mathrm{Shv}(\mathcal{Y}))\simeq \mathrm{C}^\cdot_c\left(\mathcal{Y},\Delta^*_{\mathcal{Y}}\circ((\mathrm{Frob}_{\mathcal{Y}})_*\otimes\mathrm{Id})(\text{ps-u}_{\mathcal{Y},\mathcal{N}})\right)\simeq \\ \simeq \mathrm{C}^\cdot_c\left(\mathcal{Y},\Delta^*_{\mathcal{Y}}\circ((\mathrm{Frob}_{\mathcal{Y}})_!\otimes\mathrm{Id})(\text{ps-u}_{\mathcal{Y},\mathcal{N}})\right)\overset{(5.5)}{\longrightarrow}\mathrm{C}^\cdot_c\left(\mathcal{Y},\Delta^*_{\mathcal{Y}}\circ((\mathrm{Frob}_{\mathcal{Y}})_!\otimes\mathrm{Id})(\text{ps-u}_{\mathcal{Y}})\right)= \\ = \mathrm{C}^\cdot_c\left(\mathcal{Y},\Delta^*_{\mathcal{Y}}\circ(\mathrm{Graph}_{\mathrm{Frob}_{\mathcal{Y}}})_!(\mathsf{e}_{\mathcal{Y}})\right)\simeq \mathrm{C}^\cdot_c(\mathcal{Y}^{\mathrm{Frob}},\mathsf{e}_{\mathcal{Y}^{\mathrm{Frob}}})\simeq \mathrm{Funct}_c(\mathcal{Y}(\mathbb{F}_q)).$$

5.3.4. We take $\mathcal{Y}=\mathrm{Bun}_G$ and $\mathcal{N}=\mathrm{Nilp}$. We will use the notation $\mathsf{P}_{\mathrm{Nilp}}$ as a short-hand for the endofunctor $\mathcal{R}\star-$ of $\mathrm{Shv}(\mathrm{Bun}_G)$. Similarly, for a stack $\mathcal{Z}$, we will write $(\mathrm{Id}\otimes\mathsf{P}_{\mathrm{Nilp}})$ instead of $\mathrm{Id}\otimes(\mathcal{R}\star-)$.

Using this notation, we have

$$\mathcal{K}_{\mathcal{R}}:=(\mathrm{Id}\otimes\mathsf{P}_{\mathrm{Nilp}})(\text{ps-u}_{\mathrm{Bun}_G})\in \mathrm{Shv}(\mathrm{Bun}_G\times\mathrm{Bun}_G).$$

The following is [AGKRRV2, Theorem 1.3.6]:

Theorem 5.3.5.
(a) *For a stack $\mathcal{Z}$, the endofunctor of* $\mathrm{Shv}(\mathcal{Z}\times\mathrm{Bun}_G)$ *given by* $\mathrm{Id}\otimes\mathsf{P}_{\mathrm{Nilp}}$ *is the idempotent corresponding to the embedding of the full subcategory*

$$\mathrm{Shv}_{T^*(\mathcal{Z})\times\mathrm{Nilp}}(\mathcal{Z}\times\mathrm{Bun}_G)\hookrightarrow \mathrm{Shv}(\mathcal{Z}\times\mathrm{Bun}_G), \tag{5.7}$$

preceded by its right adjoint.
(b) *The fully faithful functor*

$$\mathrm{Shv}(\mathcal{Z})\otimes\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)\to \mathrm{Shv}_{T^*(\mathcal{Z})\times\mathrm{Nilp}}(\mathcal{Z}\times\mathrm{Bun}_G)$$

is an equivalence.

5.3.6. It follows formally from Theorems 4.3.2 and 5.3.5(a) that the object $\mathcal{K}_{\mathcal{R}}$ equals the value of the right adjoint to

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)\otimes\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)\hookrightarrow \mathrm{Shv}(\mathrm{Bun}_G\times\mathrm{Bun}_G) \tag{5.8}$$

on $\text{ps-u}_{\mathrm{Bun}_G}$.

Hence, in the notations of Sect. 5.3.1 above, we can write

$$\text{ps-u}_{\mathrm{Bun}_G,\mathrm{Nilp}}\simeq \mathcal{K}_{\mathcal{R}}.$$

Finally, we note that Theorem 4.3.5 exactly says that the pair $(\mathrm{Bun}_G,\mathrm{Nilp})$ is Serre.

5.3.7. Thus, we obtain a well-defined map

$$\mathrm{LT}^{\mathrm{Serre}}:=\mathrm{LT}^{\mathrm{Serre}}_{\mathrm{Bun}_G,\mathrm{Nilp}}.$$

5.3.8. At this point, all the terms in Theorem 5.1.3 have been defined. We have already seen that $\mathrm{LT}^{\mathrm{naive}} = \mathrm{LT}^{\mathrm{true}}$.

The rest of this section is devoted to the proof of the equality $\mathrm{LT}^{\mathrm{Serre}} = \mathrm{LT}^{\mathrm{Sht}}$ in the rest of this Sect. 5.4.

Finally, we show $\mathrm{LT}^{\mathrm{Serre}} = \mathrm{LT}^{\mathrm{true}}$ in Sect. 6.

5.4. Comparison of $\mathrm{LT}^{\mathrm{Serre}}$ and $\mathrm{LT}^{\mathrm{Sht}}$.

5.4.1. We begin by constructing upgraded versions of these two local term morphisms.

More precisely, we will show that there are natural transformations

$$\widetilde{\mathrm{LT}}^{\mathrm{Serre}}, \widetilde{\mathrm{LT}}^{\mathrm{Sht}} : \mathrm{Sht}^{\mathrm{Tr}} \to \mathrm{Sht}$$

of functors

$$\mathrm{Rep}(\check{G})_{\mathrm{Ran}} \to \mathrm{Vect}$$

with the property that when we evaluate either of these natural transformations on $\mathbf{1}_{\mathrm{Rep}(\check{G})_{\mathrm{Ran}}}$, we obtain the relevant local term map

$$\mathrm{LT}^{?} : \mathrm{Tr}((\mathrm{Frob}_{\mathrm{Bun}_G})_*, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)) = \mathrm{Sht}^{\mathrm{Tr}}(\mathbf{1}_{\mathrm{Rep}(\check{G})_{\mathrm{Ran}}}) \to \mathrm{Sht}(\mathbf{1}_{\mathrm{Rep}(\check{G})_{\mathrm{Ran}}}) \simeq \mathrm{Funct}_c(\mathrm{Bun}_G(\mathbb{F}_q)).$$

5.4.2. The natural transformation $\widetilde{\mathrm{LT}}^{\mathrm{Sht}}$ (in fact, an isomorphism) has been already defined: this is the isomorphism coming from Theorem 4.1.2.

5.4.3. We will now construct $\widetilde{\mathrm{LT}}^{\mathrm{Serre}}$.

Recall the isomorphism

$$\mathrm{Sht}^{\mathrm{Tr}} \simeq \mathrm{Sht}(\mathcal{R} \star -)$$

of Theorem 4.2.2. Thus, we can interpret the sought-for map $\widetilde{\mathrm{LT}}^{\mathrm{Serre}}$ as a natural transformation

$$\mathrm{Sht}(\mathcal{R} \star -) \to \mathrm{Sht}\,. \tag{5.9}$$

5.4.4. By construction, the functors Sht and $\mathrm{Sht}(\mathcal{R} \star -)$ send $\mathcal{V} \in \mathrm{Rep}(\check{G})_{\mathrm{Ran}}$ to the vector space obtained by applying

$$\mathrm{C}^{\cdot}_c(\mathrm{Bun}_G, (\mathrm{Graph}_{\mathrm{Frob}_{\mathrm{Bun}_G}})^*(-)) : \mathrm{Shv}(\mathrm{Bun}_G \times \mathrm{Bun}_G) \to \mathrm{Vect}$$

to the objects

$$(\mathrm{Id} \otimes (\mathcal{V} \star -))(\mathrm{ps\text{-}u}_{\mathrm{Bun}_G}) \text{ and } (\mathrm{Id} \otimes (\mathcal{R} \star \mathcal{V} \star -))(\mathrm{ps\text{-}u}_{\mathrm{Bun}_G}),$$

respectively.

Recall the notation $\mathsf{P}_{\mathrm{Nilp}}$ (resp., $(\mathrm{id} \otimes \mathsf{P}_{\mathrm{Nilp}})$), see Sect. 5.3.4. Let ε denote the counit of the adjunction

$$\mathsf{P}_{\mathrm{Nilp}} \to \mathrm{id}, \tag{5.10}$$

and similarly for $(\mathrm{id} \otimes \mathsf{P}_{\mathrm{Nilp}})$.

Now, the sought-for natural transformation (5.9) is induced by the map

$$(\mathrm{Id} \otimes (\mathcal{R} \star \mathcal{V} \star -))(\mathrm{ps\text{-}u}_{\mathrm{Bun}_G}) \simeq (\mathrm{id} \otimes \mathsf{P}_{\mathrm{Nilp}}) \circ (\mathrm{Id} \otimes (\mathcal{V} \star -))(\mathrm{ps\text{-}u}_{\mathrm{Bun}_G}) \xrightarrow{\varepsilon} (\mathrm{Id} \otimes (\mathcal{V} \star -))(\mathrm{ps\text{-}u}_{\mathrm{Bun}_G}).$$

5.4.5. We will prove:

Theorem 5.4.6. *There is a canonical isomorphism*

$$\widetilde{\mathrm{LT}}^{\mathrm{Serre}} \simeq \widetilde{\mathrm{LT}}^{\mathrm{Sht}}$$

of natural transformations

$$\mathrm{Sht}^{\mathrm{Tr}} \to \mathrm{Sht}\,.$$

Clearly, Theorem 5.4.6 implies the isomorphism $\mathrm{LT}^{\mathrm{Serre}} \simeq \mathrm{LT}^{\mathrm{Sht}}$. The proof of Theorem 5.4.6 will be given in Sect. 5.6.

5.5. **An algebra structure on $\mathcal{R}$.** For the proof of Theorem 5.4.6 we need to digress and discuss some structures related to the object $\mathcal{R} \in \mathrm{Rep}(\check{G})_{\mathrm{Ran}}$.

5.5.1. First, it follows from the construction that the object

$$R_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)} \in \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \otimes \mathrm{Rep}(\check{G})_{\mathrm{Ran}}$$

(see Sect. 2.4.3) carries a naturally defined structure of (non-unital) commutative algebra.

Since the functor Loc is symmetric monoidal, it carries commutative algebras to commutative algebras. The following results from the construction of the isomorphism of Theorem 2.4.6:

Proposition 5.5.2. *The isomorphism*

$$(\mathrm{Id} \otimes \mathrm{Loc})(R_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}) \simeq \Delta_*(\mathcal{O}_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)})$$

of Theorem 2.4.6 is compatible with commutative algebra structures.

5.5.3. Observe that the functor $\Gamma_!(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X), -) : \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \to \mathrm{Vect}$ is canonically right-lax symmetric monoidal.

I.e., for $\mathcal{F}_1, \mathcal{F}_2 \in \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}(X))$, there are canonical, homotopy coherent maps

$$\Gamma_!(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X), \mathcal{F}_1) \otimes \Gamma_!(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X), \mathcal{F}_2) \to \Gamma_!(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X), \mathcal{F}_1 \underset{\mathcal{O}_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}}{\otimes} \mathcal{F}_2).$$

In particular, $\Gamma_!(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X), -)$ carries (non-unital) commutative algebras in the category $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$ to (non-unital) commutative algebras in Vect.

5.5.4. Thus, we obtain that the object

$$\mathcal{R} := (\Gamma_!(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X), -) \otimes \mathrm{Id})(R_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}) \in \mathrm{Rep}(\check{G})_{\mathrm{Ran}}$$

carries a (non-unital) commutative algebra structure.

From Proposition 5.5.2 we obtain:

Corollary 5.5.5. *The isomorphism*

$$\mathrm{Loc}(\mathcal{R}) \simeq \mathcal{O}_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}$$

of Corollary 2.4.8 is compatible with commutative algebra structures.

5.5.6. In what follows we will need one more compatibility property of the commutative algebra structure on $\mathcal{R}$.

Let $\mathcal{Z}$ be an algebraic stack. Consider the endofunctor $(\mathrm{Id} \otimes \mathsf{P}_{\mathrm{Nilp}})$ of $\mathrm{Shv}(\mathcal{Z} \times \mathrm{Bun}_G)$, see Sect. 5.3.4.

On the one hand, the algebra structure on $\mathcal{R}$ yields a map

$$m : (\mathrm{Id} \otimes \mathsf{P}_{\mathrm{Nilp}}) \circ (\mathrm{Id} \otimes \mathsf{P}_{\mathrm{Nilp}}) \to (\mathrm{Id} \otimes \mathsf{P}_{\mathrm{Nilp}}). \tag{5.11}$$

On the other hand, we have the map

$$(\mathrm{Id} \otimes \mathsf{P}_{\mathrm{Nilp}}) \circ (\mathrm{Id} \otimes \mathsf{P}_{\mathrm{Nilp}}) \overset{\varepsilon \circ (\mathrm{Id} \otimes \mathsf{P}_{\mathrm{Nilp}})}{\longrightarrow} (\mathrm{Id} \otimes \mathsf{P}_{\mathrm{Nilp}}), \tag{5.12}$$

where ε is as in (5.10). (The map (5.12) is in fact an isomorphism and equals the structure on $(\mathrm{Id} \otimes \mathsf{P}_{\mathrm{Nilp}})$ of idempotent endofunctor.)

Proposition 5.5.7. *The maps* (5.11) *and* (5.12) *are canonically homotopic.*

5.5.8. Before we prove Proposition 5.5.7 let us quote its corollary that we will use in the proof of Theorem 5.4.6.

Note that for $\mathcal{V} \in \mathrm{Rep}(\check{G})_{\mathrm{Ran}}$ we have a tautological identification

$$(\mathrm{Id} \otimes \mathsf{P}_{\mathrm{Nilp}})(\mathcal{K}_{\mathcal{V}}) \simeq \mathcal{K}_{\mathcal{R}\star\mathcal{V}} \tag{5.13}$$

as objects of $\mathrm{Shv}(\mathrm{Bun}_G \times \mathrm{Bun}_G)$.

For $\mathcal{W} \in \mathrm{Rep}(\check{G})_{\mathrm{Ran}}$, let

$$\varepsilon_{\mathcal{W}} : (\mathrm{Id} \otimes \mathsf{P}_{\mathrm{Nilp}})(\mathcal{K}_{\mathcal{W}}) \to \mathcal{K}_{\mathcal{W}}$$

denote the morphism, given by the counit of the adjunction corresponding to the embedding (5.4).

Corollary 5.5.9. *For $\mathcal{V} \in \mathrm{Rep}(\check{G})_{\mathrm{Ran}}$, the diagram*

$$\begin{tikzcd}
(\mathrm{Id} \otimes \mathsf{P}_{\mathrm{Nilp}}) \circ (\mathrm{Id} \otimes \mathsf{P}_{\mathrm{Nilp}})(\mathcal{K}_{\mathcal{V}}) \arrow[r, "m"] \arrow[d, "(5.13)"', "\sim"] & (\mathrm{Id} \otimes \mathsf{P}_{\mathrm{Nilp}})(\mathcal{K}_{\mathcal{V}}) \arrow[d, "(5.13)"', "\sim"] \\
(\mathrm{Id} \otimes \mathsf{P}_{\mathrm{Nilp}})(\mathcal{K}_{\mathcal{R}\star\mathcal{V}}) \arrow[r, "\varepsilon_{\mathcal{R}\star\mathcal{V}}"] & \mathcal{K}_{\mathcal{R}\star\mathcal{V}}
\end{tikzcd}$$

commutes.

5.5.10. *Proof of Proposition 5.5.7.* Both the source and the target functor vanish on $\ker(\mathrm{Id} \otimes \mathsf{P}_{\mathrm{Nilp}})$. Hence, it is sufficient to establish the commutativity when evaluated on the full subcategory

$$\mathrm{Shv}(\mathrm{Bun}_G) \otimes \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \subset \mathrm{Shv}(\mathrm{Bun}_G \times \mathrm{Bun}_G).$$

Since all the functors involved act only on the second factor, it is sufficient to show that the map

$$\mathrm{Id} \simeq (\mathsf{P}_{\mathrm{Nilp}} \circ \mathsf{P}_{\mathrm{Nilp}})|_{\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)} \overset{m}{\to} \mathsf{P}_{\mathrm{Nilp}}|_{\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)} \simeq \mathrm{Id}$$

is the identity map.

This follows from [AGKRRV1, Theorem 10.5.2]. This theorem asserts that the action of $\mathrm{Rep}(\check{G})_{\mathrm{Ran}}$ on $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ factors uniquely through an action of $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$ via the localization functor.

Now the result follows from Corollary 5.5.5, as $\mathrm{Loc}(\mathcal{R})$, considered as a non-unital (commutative) algebra, is canonically identified with the unit object in $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$.

$\square$[Proposition 5.5.7]

5.6. **Proof of Theorem 5.4.6.**

5.6.1. *Proof of Theorem 5.4.6, Step 0.* We begin by introducing some notation.

Throughout the argument, we will replace $\mathrm{Sht}^{\mathrm{Tr}}$ with $\mathrm{Sht}(\mathcal{R} \star -)$. In particular, we consider our natural transformations as mapping

$$\widetilde{\mathrm{LT}}^{\mathrm{Serre}}, \widetilde{\mathrm{LT}}^{\mathrm{Sht}} : \mathrm{Sht}(\mathcal{R} \star -) \to \mathrm{Sht}\,.$$

For a fixed object $\mathcal{V} \in \mathrm{Rep}(\check{G})_{\mathrm{Ran}}$, we denote the corresponding maps by

$$\widetilde{\mathrm{LT}}^{\mathrm{Serre}}_{\mathcal{V}}, \widetilde{\mathrm{LT}}^{\mathrm{Sht}}_{\mathcal{V}} : \mathrm{Sht}(\mathcal{R} \star \mathcal{V}) \to \mathrm{Sht}(\mathcal{V}).$$

To simplify notation, we will use $- \otimes -$ to denote $- \underset{\mathcal{O}_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}}{\otimes} -$ and $\mathcal{O}$ to denote $\mathcal{O}_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}$.

5.6.2. *Proof of Theorem 5.4.6, Step 1.* Observe that we have a commutative diagram

$$\begin{tikzcd} \mathbf{Maps}_{\mathrm{DGCat}}(\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X), \mathrm{Vect}) \arrow[r, "F\mapsto F(\mathcal{O}\otimes -)"] \arrow[d, "\mathrm{Loc}^\vee"'] & \mathbf{Maps}_{\mathrm{DGCat}}(\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X), \mathrm{Vect}) \arrow[d, "\mathrm{Loc}^\vee"] \\ \mathbf{Maps}_{\mathrm{DGCat}}(\mathrm{Rep}(\check{G})_{\mathrm{Ran}}, \mathrm{Vect}) \arrow[r, "F\mapsto F(\mathcal{R}\star -)"] & \mathbf{Maps}_{\mathrm{DGCat}}(\mathrm{Rep}(\check{G})_{\mathrm{Ran}}, \mathrm{Vect}). \end{tikzcd}$$

Here we have used the isomorphism $\mathrm{Loc}(\mathcal{R}) \simeq \mathcal{O}$. The vertical arrows are fully faithful by Corollary 2.3.3(a), and the top horizontal arrow is tautologically isomorphic to the identity. Therefore, the bottom arrow is fully faithful when restricted to the essential image of $\mathrm{Loc}^\vee$.

The (isomorphic!) functors Sht and $\mathrm{Sht}(\mathcal{R}\star -)$ lie in the essential image of $\mathrm{Loc}^\vee$ by Theorem 3.1.3. Therefore, it suffices to identify the natural transformations $\widetilde{\mathrm{LT}}^{\mathrm{Serre}}$ and $\widetilde{\mathrm{LT}}^{\mathrm{Sht}}$ after precomposing with $\mathcal{R}\star -$.

That is, it suffices to identify the two induced natural transformations

$$\mathrm{Sht}(\mathcal{R}\star(\mathcal{R}\star -)) \to \mathrm{Sht}(\mathcal{R}\star -),$$

i.e., the two maps

$$\widetilde{\mathrm{LT}}^{\mathrm{Serre}}_{\mathcal{R}\star\mathcal{V}}, \widetilde{\mathrm{LT}}^{\mathrm{Sht}}_{\mathcal{R}\star\mathcal{V}} : \mathrm{Sht}(\mathcal{R}\star(\mathcal{R}\star\mathcal{V})) \to \mathrm{Sht}(\mathcal{R}\star\mathcal{V}), \quad \mathcal{V}\in \mathrm{Rep}(\check{G})_{\mathrm{Ran}}.$$

5.6.3. *Proof of Theorem 5.4.6, Step 2.* Let $m : \mathcal{R}\star\mathcal{R} \to \mathcal{R}$ denote the multiplication for the algebra structure on $\mathcal{R}$.

We obtain a map

$$m_{\mathcal{V}} : \mathrm{Sht}(\mathcal{R}\star\mathcal{R}\star\mathcal{V}) \to \mathrm{Sht}(\mathcal{R}\star\mathcal{V}).$$

We claim that there is a natural identification

$$m_{\mathcal{V}} \simeq \widetilde{\mathrm{LT}}^{\mathrm{Sht}}_{\mathcal{R}\star\mathcal{V}} \tag{5.14}$$

of morphisms $\mathrm{Sht}(\mathcal{R}\star\mathcal{R}\star\mathcal{V}) \to \mathrm{Sht}(\mathcal{R}\star\mathcal{V})$.

Indeed, by Corollary 5.5.5, we have a commutative diagram

$$\begin{tikzcd} \mathrm{Loc}(\mathcal{R})\otimes\mathrm{Loc}(\mathcal{R}\star\mathcal{V}) \arrow[r, "\sim"] \arrow[d, "\sim"'] & \mathrm{Loc}(\mathcal{R}\star\mathcal{R}\star\mathcal{V}) \arrow[d, "\mathrm{Loc}(m\star\mathrm{Id}_{\mathcal{V}})"] \\ \mathrm{Loc}(\mathcal{R}\star\mathcal{V}) \arrow[r, "\mathrm{Id}"] & \mathrm{Loc}(\mathcal{R}\star\mathcal{V}) \end{tikzcd}$$

where the left vertical arrow is the canonical isomorphism obtained by identifying $\mathrm{Loc}(\mathcal{V}) \simeq \mathcal{O}$. Applying $\mathrm{Sht}_{\mathrm{Loc}}$ and the definition yields the claim.

5.6.4. *Proof of Theorem 5.4.6, Step 3.* By the above, it suffices to show that there are natural identifications

$$m_{\mathcal{V}} \simeq \widetilde{\mathrm{LT}}^{\mathrm{Serre}}_{\mathcal{R}\star\mathcal{V}} \tag{5.15}$$

of morphisms $\mathrm{Sht}(\mathcal{R}\star\mathcal{R}\star\mathcal{V}) \to \mathrm{Sht}(\mathcal{R}\star\mathcal{V})$.

Now, (5.15) is obtained by applying $\mathrm{C}^\cdot_c(\mathrm{Bun}_G, (\mathrm{Graph}_{\mathrm{Frob}_{\mathrm{Bun}_G}})^*(-))$ to the commutative diagram of Corollary 5.5.9.

6. Comparison of $\mathrm{LT}^{\mathrm{true}}$ and $\mathrm{LT}^{\mathrm{Serre}}$

6.1. **Statement of the result.**

6.1.1. Let $\mathcal{Y}$ be a quasi-compact algebraic stack, and let $\mathcal{N}$ be a conical Zariski-closed subset of $T^*(\mathcal{Y})$. We will assume that the subcategory

$$\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y}) \hookrightarrow \mathrm{Shv}(\mathcal{Y}) \tag{6.1}$$

is generated by objects that are compact in $\mathrm{Shv}(\mathcal{Y})$ (in [AGKRRV1, Sect. C.5] this property of $(\mathcal{Y}, \mathcal{N})$ was termed "renormalization-adapted and constraccessible").

Assume that $\mathcal{N}$ is Frobenius-invariant, so the endofunctor $(\mathrm{Frob}_{\mathcal{Y}})_*$ of $\mathrm{Shv}(\mathcal{Y})$ preserves the subcategory $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y}) \subset \mathrm{Shv}(\mathcal{Y})$, see [AGKRRV1, Sect. 15.3.1 and Lemma 15.3.2].

In this case, the embedding (6.1) induces a map

$$\mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})) \to \mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}(\mathcal{Y})). \tag{6.2}$$

Let us denote by

$$\mathrm{LT}^{\mathrm{true}}_{\mathcal{Y},\mathcal{N}} : \mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})) \to \mathrm{Funct}(\mathcal{Y}(\mathbb{F}_q))$$

the composition of (6.2) with the map

$$\mathrm{LT}^{\mathrm{true}}_{\mathcal{Y}} : \mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}(\mathcal{Y})) \to \mathrm{Funct}(\mathcal{Y}(\mathbb{F}_q)),$$

defined in Sect. 5.2.3.

6.1.2. Assume now that the pair $(\mathcal{Y}, \mathcal{N})$ is Serre (see Sect. 5.3.2 for what this means).

Recall that in this case, we also have the map

$$\mathrm{LT}^{\mathrm{Serre}}_{\mathcal{Y},\mathcal{N}} : \mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})) \to \mathrm{Funct}(\mathcal{Y}(\mathbb{F}_q)).$$

6.1.3. Recall now the miraculous endofunctor $\mathrm{Mir}_{\mathcal{U}}$, see [AGKRRV2, Sect. 3.4]. Following *loc. cit.*, we will say that the pair $\mathcal{N}$ is *miraculous-compatible* if the endofunctor $\mathrm{Mir}_{\mathcal{U}}$ preserves the subcategory (6.1).

The main result of this section reads:

Theorem 6.1.4. *Assume that $\mathcal{N}$ is miraculous-compatible. Then the maps $\mathrm{LT}^{\mathrm{true}}_{\mathcal{Y},\mathcal{N}}$ and $\mathrm{LT}^{\mathrm{Serre}}_{\mathcal{Y},\mathcal{N}}$*

$$\mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})) \rightrightarrows \mathrm{Funct}(\mathcal{Y}(\mathbb{F}_q))$$

are canonically homotopic.

Remark 6.1.5. The assertion of Theorem 6.1.4 is far from tautological: it says that two ways to identify $\mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y}))$ with $\mathrm{Funct}(\mathcal{Y}(\mathbb{F}_q))$, corresponding to two different self-dualities on $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})$ coincide.

A somewhat analogous problem arises when we calculate the trace of the identity endofunctor on the category $\mathrm{QCoh}(Z)$, where Z is a smooth proper scheme. There are two ways to calculate the trace that correspond to two choices of self-duality data on $\mathrm{QCoh}(Z)$: the naive self-duality and Serre self-duality. Each calculation yields Hodge cohomology of Z, i.e.,

$$\underset{i}{\oplus} \Gamma(Z, \Omega^i(Z))[i].$$

However, the resulting two identifications are different, and the difference is given by the Todd class of Z. This observation lies at the core of a proof of the Grothendieck-Riemann-Roch theorem via categorical traces, see [KP].

6.1.6. Let us explain how Theorem 6.1.4 implies the equality $\mathrm{LT}^{\mathrm{true}} = \mathrm{LT}^{\mathrm{Serre}}$ (the issue here is the fact that Bun_G is not quasi-compact).

Let $\mathcal{Y}$ be a not necessarily quasi-compact algebraic stack, and let $\mathcal{N} \subset T^*(\mathcal{Y})$ be a conical Zariski-closed subset. We will recall some definitions from [AGKRRV2, Sect. A.5].

An open substack $\mathcal{U} \overset{j}{\hookrightarrow} \mathcal{Y}$ is said to be *cotruncative* if for every quasi-compact open $\mathcal{U}' \subset \mathcal{Y}$, the open embedding

$$\mathcal{U} \cap \mathcal{U}' \overset{j}{\hookrightarrow} \mathcal{U}',$$

is such that the functor

$$j_* : \mathrm{Shv}(\mathcal{U} \cap \mathcal{U}') \to \mathrm{Shv}(\mathcal{U}')$$

admits a right adjoint *as a functor defined by a kernel*.

An open substack $\mathcal{U} \overset{j}{\hookrightarrow} \mathcal{Y}$ is said to be $\mathcal{N}$-*cotruncative* if it is cotruncative, and for every stack $\mathcal{Z}$, the functor

$$(\mathrm{id} \times j)_! : \mathrm{Shv}(\mathcal{Z} \times \mathcal{U}) \to \mathrm{Shv}(\mathcal{Z} \times \mathcal{Y})$$

sends $\mathrm{Shv}_{T^*(\mathcal{Z}) \times \mathcal{N}}(\mathcal{Z} \times \mathcal{U}) \subset \mathrm{Shv}(\mathcal{Z} \times \mathcal{U})$ to $\mathrm{Shv}_{T^*(\mathcal{Z}) \times \mathcal{N}}(\mathcal{Z} \times \mathcal{Y}) \subset \mathrm{Shv}(\mathcal{Z} \times \mathcal{Y})$.

Recall that $\mathcal{Y}$ is said to be *truncatable* if we can write $\mathcal{Y}$ as a union of quasi-compact cotruncative open substacks. Finally, recall that $\mathcal{Y}$ is is said to be $\mathcal{N}$-*truncatable* if we can write $\mathcal{Y}$ as a union of quasi-compact $\mathcal{N}$-cotruncative open substacks.

6.1.7. Assume that $\mathcal{N}$ is Frobenius-invariant. By Sect. 5.2.6 we have a well-defined map

$$\mathrm{LT}^{\mathrm{true}}_{\mathcal{Y},\mathcal{N}} : \mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})) \rightrightarrows \mathrm{Funct}_c(\mathcal{Y}(\mathbb{F}_q)).$$

Let us make the following assumptions:

- $\mathcal{Y}$ is $\mathcal{N}$-truncatable;
- For every quasi-compact $\mathcal{N}$-cotruncative open substack $\mathcal{U} \subset \mathcal{Y}$,we have:
  - The pair $(\mathcal{U}, \mathcal{N}|_{\mathcal{U}})$ is renormalization-adapted and constraccessible;
  - The pair $(\mathcal{U}, \mathcal{N}|_{\mathcal{U}})$ is Serre;
  - $\mathcal{N}|_{\mathcal{U}}$ is miraculous-compatible.

It follows that in this case, the subcategory $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})$ is generated by objects that are compact in $\mathrm{Shv}(\mathcal{Y})$, and that the pair $(\mathcal{Y}, \mathcal{N})$ is Serre. So, the map

$$\mathrm{LT}^{\mathrm{Serre}}_{\mathcal{Y},\mathcal{N}} : \mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})) \to \mathrm{Funct}_c(\mathcal{Y}(\mathbb{F}_q))$$

is also well-defined by Sect. 5.3.3. Moreover, we have a commutative diagram

$$\begin{tikzcd}
\underset{\mathcal{U}}{\mathrm{colim}}\, \mathrm{Tr}((\mathrm{Frob}_{\mathcal{U}})_*, \mathrm{Shv}_{\mathcal{N}}(\mathcal{U})) \arrow[r, "\underset{\mathcal{U}}{\mathrm{colim}}\, \mathrm{LT}^{\mathrm{Serre}}_{\mathcal{U},\mathcal{N}}"] \arrow[d] & \underset{\mathcal{U}}{\mathrm{colim}}\, \mathrm{Funct}(\mathcal{U}(\mathbb{F}_q)) \arrow[d, "\sim"] \\
\mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})) \arrow[r, "\mathrm{LT}^{\mathrm{Serre}}_{\mathcal{Y},\mathcal{N}}"] & \mathrm{Funct}_c(\mathcal{Y}(\mathbb{F}_q)).
\end{tikzcd}$$

It now follows formally from Theorems 6.1.4 and 5.2.5(b) that the maps $\mathrm{LT}^{\mathrm{true}}_{\mathcal{Y},\mathcal{N}}$ and $\mathrm{LT}^{\mathrm{Serre}}_{\mathcal{Y},\mathcal{N}}$ are canonically homotopic.

6.1.8. We apply the above discussion to $\mathcal{Y} = \mathrm{Bun}_G$ and $\mathrm{Nilp} = \mathcal{N}$. The conditions in Sect. 6.1.7 are satisfied by [AGKRRV1, Theorems 10.1.4 and 10.1.6] and [AGKRRV2, Corollaries 3.6.6 and 1.6.9], respectively. This implies the desired equality

$$\mathrm{LT}^{\mathrm{true}} = \mathrm{LT}^{\mathrm{Serre}}.$$

6.2. **A geometric local term theorem.**

6.2.1. Let $\mathcal{Y}$ be a quasi-compact algebraic stack. We start by constructing a natural transformation

$$\mathrm{C}^\cdot_c(\mathcal{Y}, \Delta^*_{\mathcal{Y}} \circ (\mathrm{Id} \otimes \mathrm{Mir}_{\mathcal{Y}})(-)) \to \mathrm{C}^\cdot_\blacktriangle(\mathcal{Y}, \Delta^!_{\mathcal{Y}}(-)), \tag{6.3}$$

as functors

$$\mathrm{Shv}(\mathcal{Y} \times \mathcal{Y}) \rightrightarrows \mathrm{Vect}.$$

6.2.2. It suffices to specify the value of the natural transformation (6.3) on compact objects. Thus, we fix a compact object $\mathcal{Q} \in \mathrm{Shv}(\mathcal{Y} \times \mathcal{Y})$.

We note that the map

$$\mathrm{C}^\cdot_\blacktriangle(\mathcal{Y}, \Delta^!_{\mathcal{Y}}(\mathcal{Q})) \to \mathrm{C}^\cdot(\mathcal{Y}, \Delta^!_{\mathcal{Y}}(\mathcal{Q}))$$

is an isomorphism for $\mathcal{Q}$. So we need to construct a map

$$\mathrm{C}^\cdot_c(\mathcal{Y}, \Delta^*_{\mathcal{Y}} \circ (\mathrm{Id} \otimes \mathrm{Mir}_{\mathcal{Y}})(\mathcal{Q})) \to \mathrm{C}^\cdot(\mathcal{Y}, \Delta^!_{\mathcal{Y}}(\mathcal{Q})), \tag{6.4}$$

functorial in $\mathcal{Q} \in \mathrm{Shv}(\mathcal{Y} \times \mathcal{Y})^c$.

In what follows we will use the notation

$$\mathrm{u}_{\mathcal{Y}} := (\Delta_{\mathcal{Y}})_*(\omega_{\mathcal{Y}}) \in \mathrm{Shv}(\mathcal{Y} \times \mathcal{Y}).$$

We start with the map

$$\mathcal{Q} \boxtimes \mathbb{D}^{\mathrm{Verdier}}(\mathcal{Q}) \to \mathrm{u}_{\mathcal{Y} \times \mathcal{Y}},$$

given by Verdier duality. Applying the transposition $\sigma_{2,3}$, we interpret it as a map

$$(\mathcal{Q} \boxtimes \mathbb{D}^{\mathrm{Verdier}}(\mathcal{Q}))^{\sigma_{2,3}} \to \mathrm{u}_{\mathcal{Y}} \boxtimes \mathrm{u}_{\mathcal{Y}}\,. \tag{6.5}$$

By definition

$$(\mathrm{Id} \otimes \mathrm{Mir}_{\mathcal{Y}})(\mathrm{u}_{\mathcal{Y}}) \simeq \text{ps-u}_{\mathcal{Y}} \tag{6.6}$$

Applying the functor $\mathrm{Id} \otimes \mathrm{Id} \otimes \mathrm{Mir}_{\mathcal{Y}} \otimes \mathrm{Id}$ to the map (6.5), we obtain a map

$$((\mathrm{Id} \otimes \mathrm{Mir}_{\mathcal{Y}})(\mathcal{Q}) \boxtimes \mathbb{D}^{\mathrm{Verdier}}(\mathcal{Q}))^{\sigma_{2,3}} \to \mathrm{u}_{\mathcal{Y}} \boxtimes \text{ps-u}_{\mathcal{Y}}\,. \tag{6.7}$$

Applying the functor

$$(p_{1,3})_! \circ (\Delta_{\mathcal{Y}} \times \mathrm{id} \times \mathrm{id})^* \circ \sigma_{2,3} : \mathrm{Shv}(\mathcal{Y} \times \mathcal{Y} \times \mathcal{Y} \times \mathcal{Y}) \to \mathrm{Shv}(\mathcal{Y} \times \mathcal{Y}),$$

from (6.7) we obtain a map

$$\mathrm{C}^\cdot_c(\mathcal{Y}, \Delta^*_{\mathcal{Y}} \circ (\mathrm{Id} \otimes \mathrm{Mir}_{\mathcal{Y}})(\mathcal{Q})) \otimes \mathbb{D}^{\mathrm{Verdier}}(\mathcal{Q}) \to \mathrm{u}_{\mathcal{Y}}\,.$$

Applying Verdier duality again, we obtain the desired map (6.4).

6.2.3. Let us take $\mathcal{Q} := (\mathrm{Graph}_{\mathrm{Frob}_{\mathcal{Y}}})_*(\omega_{\mathcal{Y}}) \in \mathrm{Shv}(\mathcal{Y} \times \mathcal{Y})$.

Then the right-hand side in (6.3) identifies with

$$\mathrm{C}^\cdot_\blacktriangle(\mathcal{Y}^{\mathrm{Frob}}, \omega_{\mathcal{Y}^{\mathrm{Frob}}}) \simeq \mathrm{C}^\cdot(\mathcal{Y}^{\mathrm{Frob}}, \omega_{\mathcal{Y}^{\mathrm{Frob}}}).$$

Since $\mathrm{Frob}_{\mathcal{Y}}$ is a finite map,

$$(\mathrm{Id} \otimes \mathrm{Mir}_{\mathcal{Y}})((\mathrm{Graph}_{\mathrm{Frob}_{\mathcal{Y}}})_*(\omega_{\mathcal{Y}})) \simeq (\mathrm{Graph}_{\mathrm{Frob}_{\mathcal{Y}}})_!(\underline{\mathsf{e}}_{\mathcal{Y}}).$$

Hence, the left-hand side in (6.3) identifies with

$$\mathrm{C}^\cdot_c(\mathcal{Y}^{\mathrm{Frob}}, \underline{\mathsf{e}}_{\mathcal{Y}^{\mathrm{Frob}}}).$$

6.2.4. We will prove:

Theorem 6.2.5. *The diagram*

$$
\begin{tikzcd}
\mathrm{C}^\cdot_c(\mathcal{Y}, \Delta^*_{\mathcal{Y}} \circ (\mathrm{Id} \otimes \mathrm{Mir}_{\mathcal{Y}}) \circ (\mathrm{Graph}_{\mathrm{Frob}_{\mathcal{Y}}})_*(\omega_{\mathcal{Y}})) \arrow[r, "(6.3)"] \arrow[d, "\simeq"'] & \mathrm{C}^\cdot_\blacktriangle(\mathcal{Y}, \Delta^!_{\mathcal{Y}} \circ (\mathrm{Graph}_{\mathrm{Frob}_{\mathcal{Y}}})_*(\omega_{\mathcal{Y}})) \arrow[d, "\simeq"] \\
\mathrm{C}^\cdot_c(\mathcal{Y}^{\mathrm{Frob}}, \underline{\mathsf{e}}_{\mathcal{Y}^{\mathrm{Frob}}}) \arrow[d, "\simeq"'] & \mathrm{C}^\cdot(\mathcal{Y}^{\mathrm{Frob}}, \omega_{\mathcal{Y}^{\mathrm{Frob}}}) \arrow[d, "\simeq"] \\
\mathrm{Funct}(\mathcal{Y}(\mathbb{F}_q)) \arrow[r, "\mathrm{id}"] & \mathrm{Funct}(\mathcal{Y}(\mathbb{F}_q))
\end{tikzcd}
\tag{6.8}
$$

commutes.

The proof will be given in Sect. 6.4. We will presently show how Theorem 6.2.5 implies Theorem 6.1.4.

6.3. Proof of Theorem 6.1.4.

6.3.1. Assumption that $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})$ is generated by objects compact in the ambient category $\mathrm{Shv}(\mathcal{Y})$ implies that the pairing $\mathrm{ev}^!_{\mathcal{Y}}$ induces a perfect pairing

$$\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y}) \otimes \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y}) \to \mathrm{Vect}\,.$$

The unit of this duality, to be denoted $\mathrm{u}_{\mathcal{Y},\mathcal{N}}$ is obtained by applying to $\mathrm{u}_{\mathcal{Y}}$ the right adjoint to the fully faithful embedding

$$\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y}) \otimes \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y}) \hookrightarrow \mathrm{Shv}(\mathcal{Y} \times \mathcal{Y}). \tag{6.9}$$

In what follows, we will not distinguish notationally between $\mathrm{u}_{\mathcal{Y},\mathcal{N}}$ and its image along (6.9). The counit of the adjunction gives rise to a map

$$\mathrm{u}_{\mathcal{Y},\mathcal{N}} \to \mathrm{u}_{\mathcal{Y}}\,. \tag{6.10}$$

6.3.2. The assumption that $\mathcal{N}$ is miraculous-compatible tautologically implies that the endofunctor $\mathrm{Id} \otimes \mathrm{Mir}_{\mathcal{Y}}$ of $\mathrm{Shv}(\mathcal{Y} \times \mathcal{Y})$ preserves the subcategory

$$\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y}) \otimes \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y}) \hookrightarrow \mathrm{Shv}(\mathcal{Y} \times \mathcal{Y}). \tag{6.11}$$

By [AGKRRV2, Corollary 3.4.7], the assumption to the pair $(\mathcal{Y},\mathcal{N})$ is Serre implies that the above functor $\mathrm{Id} \otimes \mathrm{Mir}_{\mathcal{Y}}$ intertwines the duality data given by the pair $(\mathrm{u}_{\mathcal{Y},\mathcal{N}}, \mathrm{ev}^!_{\mathcal{Y}})$ with one given by $(\text{ps-u}_{\mathcal{Y},\mathcal{N}}, \mathrm{ev}^*_{\mathcal{Y}})$.

In particular, we have a canonical isomorphism

$$\text{ps-u}_{\mathcal{Y},\mathcal{N}} \simeq (\mathrm{Id} \otimes \mathrm{Mir}_{\mathcal{Y}})(\mathrm{u}_{\mathcal{Y},\mathcal{N}}) \tag{6.12}$$

and a datum of commutativity for the diagram

$$
\begin{tikzcd}
\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y}) \otimes \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y}) \arrow[r, "\mathrm{Id} \otimes \mathrm{Mir}_{\mathcal{Y}}"] \arrow[d, "\mathrm{ev}^!_{\mathcal{Y}}"'] & \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y}) \otimes \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y}) \arrow[d, "\mathrm{ev}^*_{\mathcal{Y}}"] \\
\mathrm{Vect} \arrow[r, "\mathrm{Id}"] & \mathrm{Vect},
\end{tikzcd}
$$

i.e., an isomorphism of functors

$$\mathrm{ev}^*_{\mathcal{Y}} \circ (\mathrm{Id} \otimes \mathrm{Mir}_{\mathcal{Y}}) \simeq \mathrm{ev}^!_{\mathcal{Y}}, \quad \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y}) \otimes \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y}) \rightrightarrows \mathrm{Vect}\,. \tag{6.13}$$

6.3.3. In particular, for an endofunctor F of $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})$, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Tr}(F, \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})) \arrow[rr, "\mathrm{id}"] \arrow[d, "\sim"'] & & \mathrm{Tr}(F, \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})) \arrow[d, "\sim"] \\
\mathrm{ev}^*_{\mathcal{Y}} \circ (F \otimes \mathrm{id})(\text{ps-u}_{\mathcal{Y},\mathcal{N}}) \arrow[r, "\sim", "(6.12)"'] & \mathrm{ev}^*_{\mathcal{Y}} \circ (F \otimes \mathrm{Mir}_{\mathcal{Y}})(\mathrm{u}_{\mathcal{Y},\mathcal{N}}) \arrow[r, "\sim", "(6.13)"'] & \mathrm{ev}^!_{\mathcal{Y}} \circ (F \otimes \mathrm{id})(\mathrm{u}_{\mathcal{Y},\mathcal{N}})
\end{tikzcd} \tag{6.14}$$

6.3.4. We now use the following two observations:

(i) The diagram

$$\begin{tikzcd}
\text{ps-u}_{\mathcal{Y},\mathcal{N}} \arrow[r, "(6.12)", "\sim"'] \arrow[d, "(5.5)"'] & (\mathrm{Id} \otimes \mathrm{Mir}_{\mathcal{Y}})(\mathrm{u}_{\mathcal{Y},\mathcal{N}}) \arrow[d, "(6.10)"] \\
\text{ps-u}_{\mathcal{Y}} \arrow[r, "(6.6)"'] & (\mathrm{Id} \otimes \mathrm{Mir}_{\mathcal{Y}})(\mathrm{u}_{\mathcal{Y}})
\end{tikzcd}$$

commutes. This follows tautologically from the constructions.

(ii) The isomorphism (6.13) is canonically homotopic to the restriction of the natural transformation (6.3) along the embedding (6.11). This follows from [AGKRRV2, Proposition 4.1.4].

Concatenating, we obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{ev}^*_{\mathcal{Y}} \circ ((\mathrm{Frob}_{\mathcal{Y}})_* \otimes \mathrm{id})(\text{ps-u}_{\mathcal{Y},\mathcal{N}})) \arrow[r, "(6.13)\circ(6.12)"] \arrow[d, "(5.5)"] & \mathrm{ev}^!_{\mathcal{Y}} \circ ((\mathrm{Frob}_{\mathcal{Y}})_* \otimes \mathrm{id})(\mathrm{u}_{\mathcal{Y},\mathcal{N}})) \arrow[d, "(6.10)"] \\
\mathrm{C}^\cdot_c(\mathcal{Y}, -) \circ (\Delta_{\mathcal{Y}})^* \circ (\mathrm{Frob}_{\mathcal{Y}} \times \mathrm{id})_*(\text{ps-u}_{\mathcal{Y}}) \arrow[r, "(6.3)\circ(6.6)"] & \mathrm{C}^\cdot_\blacktriangle(\mathcal{Y}, -) \circ (\Delta_{\mathcal{Y}})^! \circ (\mathrm{Frob}_{\mathcal{Y}} \times \mathrm{id})_*(\mathrm{u}_{\mathcal{Y}}).
\end{tikzcd}$$

6.3.5. Hence, concatenating with (6.14) for $F = (\mathrm{Frob}_{\mathcal{Y}})_*$, we obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})) \arrow[r, "\mathrm{id}"] \arrow[d, "\sim"'] & \mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})) \arrow[d, "\sim"] \\
\mathrm{ev}^*_{\mathcal{Y}} \circ ((\mathrm{Frob}_{\mathcal{Y}})_* \otimes \mathrm{id})(\text{ps-u}_{\mathcal{Y},\mathcal{N}})) \arrow[r, "(6.13)\circ(6.12)"] \arrow[d, "(5.5)"] & \mathrm{ev}^!_{\mathcal{Y}} \circ ((\mathrm{Frob}_{\mathcal{Y}})_* \otimes \mathrm{id})(\mathrm{u}_{\mathcal{Y},\mathcal{N}})) \arrow[d, "(6.10)"] \\
\mathrm{C}^\cdot_c(\mathcal{Y}, -) \circ (\Delta_{\mathcal{Y}})^* \circ (\mathrm{Frob}_{\mathcal{Y}} \times \mathrm{id})_*(\text{ps-u}_{\mathcal{Y}}) \arrow[r, "(6.3)\circ(6.6)"] & \mathrm{C}^\cdot_\blacktriangle(\mathcal{Y}, -) \circ (\Delta_{\mathcal{Y}})^! \circ (\mathrm{Frob}_{\mathcal{Y}} \times \mathrm{id})_*(\mathrm{u}_{\mathcal{Y}})
\end{tikzcd}$$

6.3.6. Finally, we note that the commutative diagram (6.8) can be rephrased as

$$\begin{tikzcd}
\mathrm{C}^\cdot_c(\mathcal{Y}, -) \circ (\Delta_{\mathcal{Y}})^* \circ (\mathrm{Frob}_{\mathcal{Y}} \times \mathrm{id})_*(\text{ps-u}_{\mathcal{Y}}) \arrow[r, "(6.3)\circ(6.6)"] \arrow[d, "\sim"'] & \mathrm{C}^\cdot_\blacktriangle(\mathcal{Y}, -) \circ (\Delta_{\mathcal{Y}})^! \circ (\mathrm{Frob}_{\mathcal{Y}} \times \mathrm{id})_*(\mathrm{u}_{\mathcal{Y}}) \arrow[d, "\sim"] \\
\mathrm{C}^\cdot_c(\mathcal{Y}^{\mathrm{Frob}}, \underline{\mathsf{e}}_{\mathcal{Y}^{\mathrm{Frob}}}) \arrow[d, "\simeq"'] & \mathrm{C}^\cdot(\mathcal{Y}^{\mathrm{Frob}}, \omega_{\mathcal{Y}^{\mathrm{Frob}}}) \arrow[d, "\simeq"] \\
\mathrm{Funct}(\mathcal{Y}(\mathbb{F}_q)) \arrow[r, "\mathrm{id}"] & \mathrm{Funct}(\mathcal{Y}(\mathbb{F}_q)),
\end{tikzcd}$$

Thus, concatenating with the commutative diagram in Sect. 6.3.5 above, we obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})) \arrow[r, "\mathrm{id}"] \arrow[d, "\sim"'] & \mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})) \arrow[d, "\sim"] \\
\mathrm{ev}^*_{\mathcal{Y}} \circ ((\mathrm{Frob}_{\mathcal{Y}})_* \otimes \mathrm{id})(\text{ps-u}_{\mathcal{Y},\mathcal{N}}) \arrow[r, "(6.13)\circ(6.12)"] \arrow[d, "(5.5)"] & \mathrm{ev}^!_{\mathcal{Y}} \circ ((\mathrm{Frob}_{\mathcal{Y}})_* \otimes \mathrm{id})(\mathrm{u}_{\mathcal{Y},\mathcal{N}}) \arrow[d, "(6.10)"] \\
\mathrm{C}^\cdot_c(\mathcal{Y}, -) \circ (\Delta_{\mathcal{Y}})^* \circ (\mathrm{Frob}_{\mathcal{Y}} \times \mathrm{id})_*(\text{ps-u}_{\mathcal{Y}}) \arrow[r, "(6.3)\circ(6.6)"] \arrow[d, "\sim"'] & \mathrm{C}^\cdot_{\blacktriangle}(\mathcal{Y}, -) \circ (\Delta_{\mathcal{Y}})^! \circ (\mathrm{Frob}_{\mathcal{Y}} \times \mathrm{id})_*(\mathrm{u}_{\mathcal{Y}}) \arrow[d, "\sim"] \\
\mathrm{C}^\cdot_c(\mathcal{Y}^{\mathrm{Frob}}, \underline{\mathsf{e}}_{\mathcal{Y}^{\mathrm{Frob}}}) \arrow[d, "\simeq"'] & \mathrm{C}^\cdot(\mathcal{Y}^{\mathrm{Frob}}, \omega_{\mathcal{Y}^{\mathrm{Frob}}}) \arrow[d, "\simeq"] \\
\mathrm{Funct}(\mathcal{Y}(\mathbb{F}_q)) \arrow[r, "\mathrm{id}"] & \mathrm{Funct}(\mathcal{Y}(\mathbb{F}_q)),
\end{tikzcd}$$

in which the left composite vertical arrow is $\mathrm{LT}^{\mathrm{Serre}}_{\mathcal{Y},\mathcal{N}}$, and the right composite vertical arrow is $\mathrm{LT}^{\mathrm{true}}_{\mathcal{Y},\mathcal{N}}$.

This provides the sought-for homotopy between $\mathrm{LT}^{\mathrm{Serre}}_{\mathcal{Y},\mathcal{N}}$ and $\mathrm{LT}^{\mathrm{true}}_{\mathcal{Y},\mathcal{N}}$.

6.4. **Proof of Theorem 6.2.5.**

6.4.1. First, we recall the local version of the Grothendiek-Lefschetz trace formula, following [GV].

Let $\mathcal{Y}$ be a quasi-compact algebraic stack, and let $\mathcal{F} \in \mathrm{Shv}(\mathcal{Y})$ be a *constructible* sheaf, equipped with a weak Weil structure, i.e., a morphism

$$\alpha : \mathcal{F} \to (\mathrm{Frob}_{\mathcal{Y}})_*(\mathcal{F}),$$

or equivalently, a morphism

$$\alpha^L : \mathrm{Frob}^*_{\mathcal{Y}}(\mathcal{F}) \to \mathcal{F}.$$

On the one hand, we attach to the pair $(\mathcal{F}, \alpha^L)$ a function $\mathrm{funct}(\mathcal{F})^{\mathrm{naive}} \in \mathrm{Funct}(\mathcal{Y}(\mathbb{F}_q))$ by the standard procedure of taking the trace of Frobenius on *-fibers of $\mathcal{F}$ at $\mathbb{F}_q$-points of $\mathcal{F}$. I.e., for a Frobenius-invariant point

$$\mathrm{pt} \overset{i_y}{\to} \mathcal{Y},$$

we consider the endomorphism

$$i^*_y(\mathcal{F}) \simeq (\mathrm{Frob}_{\mathcal{Y}} \circ i_y)^*(\mathcal{F}) \simeq i^*_y \circ \mathrm{Frob}^*_{\mathcal{Y}}(\mathcal{F}) \overset{\alpha^L}{\to} i^*_y(\mathcal{F}),$$

and we set the value of $\mathrm{funct}(\mathcal{F})^{\mathrm{naive}}$ at $y \in \mathcal{Y}(\mathbb{F}_q)$ to be the trace of the above endomorphism.

On the other hand, we can attach to $(\mathcal{F}, \alpha)$ a function $\mathrm{funct}(\mathcal{F})^{\mathrm{true}} \in \mathrm{Funct}(\mathcal{Y}(\mathbb{F}_q))$, defined as follows.

Consider the canonical maps

$$\mathcal{F} \boxtimes \mathbb{D}^{\mathrm{Verdier}}(\mathcal{F}) \to (\Delta_{\mathcal{Y}})_*(\omega_{\mathcal{Y}}) \text{ and } \underline{\mathsf{e}}_{\mathcal{Y}} \to \mathcal{F} \overset{!}{\otimes} \mathbb{D}^{\mathrm{Verdier}}(\mathcal{F}).$$

From the first of these maps we produce the map

$$\mathcal{F} \boxtimes \mathbb{D}^{\mathrm{Verdier}}(\mathcal{F}) \overset{\alpha \boxtimes \mathrm{id}}{\longrightarrow} (\mathrm{Frob}_{\mathcal{Y}} \times \mathrm{id})_*(\mathcal{F} \boxtimes \mathbb{D}^{\mathrm{Verdier}}(\mathcal{F})) \to (\mathrm{Frob}_{\mathcal{Y}} \times \mathrm{id})_*(\omega_{\mathcal{Y}}) \simeq (\mathrm{Graph}_{\mathrm{Frob}_{\mathcal{Y}}})_*(\omega_{\mathcal{Y}}). \tag{6.15}$$

The function $\mathrm{funct}(\mathcal{F})^{\mathrm{true}}$, viewed as an element of

$$\mathrm{C}^\cdot(\mathcal{Y}^{\mathrm{Frob}}, \omega_{\mathcal{Y}^{\mathrm{Frob}}}) \simeq \mathrm{C}^\cdot(\mathcal{Y}, \Delta^!_{\mathcal{Y}} \circ (\mathrm{Graph}_{\mathrm{Frob}_{\mathcal{Y}}})_*(\omega_{\mathcal{Y}})),$$

corresponds to the map

$$\underline{\mathsf{e}}_{\mathcal{Y}} \to \mathcal{F} \overset{!}{\otimes} \mathbb{D}^{\mathrm{Verdier}}(\mathcal{F}) = \Delta^!_{\mathcal{Y}}(\mathcal{F} \boxtimes \mathbb{D}^{\mathrm{Verdier}}(\mathcal{F})) \overset{(6.15)}{\longrightarrow} \Delta^!_{\mathcal{Y}} \circ (\mathrm{Graph}_{\mathrm{Frob}_{\mathcal{Y}}})_*(\omega_{\mathcal{Y}}).$$

The local Grothendiek-Lefschetz trace formula says:

Theorem 6.4.2. *The functions* $\mathrm{funct}(\mathcal{F})^{\mathrm{naive}}$ *and* $\mathrm{funct}(\mathcal{F})^{\mathrm{true}}$ *are equal.*

Remark 6.4.3. When $\mathcal{F}$ is *compact*, the assertion of Theorem 6.4.2 is a particular case of that of Theorem 5.2.5. Namely, the functions $\mathrm{funct}(\mathcal{F})^{\mathrm{naive}}$ and $\mathrm{funct}(\mathcal{F})^{\mathrm{true}}$ are the values of the maps $\mathrm{LT}^{\mathrm{naive}}_{\mathcal{Y}}$ and $\mathrm{LT}^{\mathrm{true}}_{\mathcal{Y}}$ on the element

$$\mathrm{cl}(\mathcal{F},\alpha)\in \mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*,\mathrm{Shv}(\mathcal{Y})),$$

respectively.

For $\mathcal{F}$ which is constructible but not compact, the assertion of Theorem 6.4.2 can be obtained by proving a version of Theorem 5.2.5 for the *renormalized* version of the category $\mathrm{Shv}(\mathcal{Y})$, namely, one obtained as the ind-completion of the constructible subcategory of $\mathrm{Shv}(\mathcal{Y})$.

6.4.4. We precede the proof of Theorem 6.2.5 by the following observation.

Let $\mathcal{Q}$ be a *constructible* object of $\mathrm{Shv}(\mathcal{Y}\times\mathcal{Y})$. Note that the procedure in Sect. 6.2.2 defines a map

$$\psi:\mathrm{C}^{\cdot}_c(\mathcal{Y},\Delta^*_{\mathcal{Y}}\circ(\mathrm{Id}\otimes\mathrm{Mir}_{\mathcal{Y}})(\mathcal{Q}))\to \mathrm{C}^{\cdot}(\mathcal{Y},\Delta^!_{\mathcal{Y}}(\mathcal{Q})). \tag{6.16}$$

It is easy to see that the map (6.16) equals the composition of the value of the natural transformation (6.3), followed by the canonical map

$$\mathrm{C}^{\cdot}_{\blacktriangle}(\mathcal{Y},\Delta^!_{\mathcal{Y}}(\mathcal{Q}))\to \mathrm{C}^{\cdot}(\mathcal{Y},\Delta^!_{\mathcal{Y}}(\mathcal{Q})). \tag{6.17}$$

6.4.5. We are ready to launch the proof of Theorem 6.2.5. We apply the observation in Sect. 6.4.4 to

$$\mathcal{Q}:=(\mathrm{Graph}_{\mathrm{Frob}_{\mathcal{Y}}})_*(\omega_{\mathcal{Y}}).$$

Note that in this case, the map (6.17) is an isomorphism, as the corresponding map identifies with

$$\mathrm{C}^{\cdot}_{\blacktriangle}(\mathcal{Y}^{\mathrm{Frob}},\omega_{\mathcal{Y}^{\mathrm{Frob}}})\to \mathrm{C}^{\cdot}(\mathcal{Y}^{\mathrm{Frob}},\omega_{\mathcal{Y}^{\mathrm{Frob}}}).$$

We need to show that a certain map

$$\mathrm{Funct}(\mathcal{Y}(\mathbb{F}_q))=\mathrm{C}^{\cdot}_c(\mathcal{Y}^{\mathrm{Frob}},\underline{\mathsf{e}}_{\mathcal{Y}^{\mathrm{Frob}}})\to \mathrm{C}^{\cdot}(\mathcal{Y}^{\mathrm{Frob}},\omega_{\mathcal{Y}^{\mathrm{Frob}}})\simeq \mathrm{Funct}(\mathcal{Y}(\mathbb{F}_q))$$

equals the identity, where the middle arrow is the result of the construction in Sect. 6.2.2 applied to the above choice of $\mathcal{Q}$.

We interpret the above map as a functional

$$\mathrm{Funct}(\mathcal{Y}(\mathbb{F}_q))\otimes \mathrm{Funct}(\mathcal{Y}(\mathbb{F}_q))\simeq \mathrm{C}^{\cdot}_c(\mathcal{Y}^{\mathrm{Frob}},\underline{\mathsf{e}}_{\mathcal{Y}^{\mathrm{Frob}}})\otimes \mathrm{C}^{\cdot}_c(\mathcal{Y}^{\mathrm{Frob}},\underline{\mathsf{e}}_{\mathcal{Y}^{\mathrm{Frob}}})\to \mathsf{e}, \tag{6.18}$$

and we wish to show that this functional is given by

$$f_1,f_2\mapsto \underset{y\in\mathcal{Y}(\mathbb{F}_q)}{\Sigma} f_1(y)\cdot f_2(y).$$

6.4.6. Let us unwind the construction of the functional (6.18). We start with the map (6.7) for the above choice of $\mathcal{Q}$. This is a map

$$\left((\mathrm{Graph}_{\mathrm{Frob}_{\mathcal{Y}}})_!(\underline{\mathsf{e}}_{\mathcal{Y}})\boxtimes(\mathrm{Graph}_{\mathrm{Frob}_{\mathcal{Y}}})_!(\underline{\mathsf{e}}_{\mathcal{Y}})\right)^{\sigma_{2,3}}\to(\Delta_{\mathcal{Y}})_*(\omega_{\mathcal{Y}})\boxtimes(\Delta_{\mathcal{Y}})_!(\underline{\mathsf{e}}_{\mathcal{Y}}). \tag{6.19}$$

We apply to this map the functor

$$(p_{1,3})_!\circ(\Delta_{\mathcal{Y}}\times\mathrm{id}\times\mathrm{id})^*\circ\sigma_{2,3}:\mathrm{Shv}(\mathcal{Y}\times\mathcal{Y}\times\mathcal{Y}\times\mathcal{Y})\to\mathrm{Shv}(\mathcal{Y}\times\mathcal{Y}),$$

and we obtain a map

$$\mathrm{C}^{\cdot}_c(\mathcal{Y}^{\mathrm{Frob}},\underline{\mathsf{e}}_{\mathcal{Y}^{\mathrm{Frob}}})\otimes(\mathrm{Graph}_{\mathrm{Frob}_{\mathcal{Y}}})_!(\underline{\mathsf{e}}_{\mathcal{Y}})\to(\Delta_{\mathcal{Y}})_*(\omega_{\mathcal{Y}}).$$

We apply to the latter map the adjunction

$$\mathrm{C}^{\cdot}_c(\mathcal{Y},-)\circ\Delta^*_{\mathcal{Y}}:\mathrm{Shv}(\mathcal{Y}\times\mathcal{Y})\rightleftarrows\mathrm{Vect}:(\Delta_{\mathcal{Y}})_*(\omega_{\mathcal{Y}}),$$

and we obtain the desired pairing

$$\mathrm{C}^{\cdot}_c(\mathcal{Y}^{\mathrm{Frob}},\underline{\mathsf{e}}_{\mathcal{Y}^{\mathrm{Frob}}})\otimes \mathrm{C}^{\cdot}_c(\mathcal{Y}^{\mathrm{Frob}},\underline{\mathsf{e}}_{\mathcal{Y}^{\mathrm{Frob}}})\to\mathsf{e}.$$

6.4.7. Let us apply Verdier duality to the above morphisms. The dual of (6.19) is a morphism

$$(\Delta_{\mathcal{Y}})_!(\underline{\mathsf{e}}_{\mathcal{Y}}) \boxtimes (\Delta_{\mathcal{Y}})_*(\omega_{\mathcal{Y}}) \to \left((\mathrm{Graph}_{\mathrm{Frob}_{\mathcal{Y}}})_*(\omega_{\mathcal{Y}}) \boxtimes (\mathrm{Graph}_{\mathrm{Frob}_{\mathcal{Y}}})_*(\omega_{\mathcal{Y}})\right)^{\sigma_{2,3}}. \tag{6.20}$$

From this morphism we obtain an element of

$$\mathrm{C}^{\cdot}(\mathcal{Y}^{\mathrm{Frob}}, \omega_{\mathcal{Y}^{\mathrm{Frob}}}) \otimes \mathrm{C}^{\cdot}(\mathcal{Y}^{\mathrm{Frob}}, \omega_{\mathcal{Y}^{\mathrm{Frob}}})$$

by the following procedure.

We apply to (6.20) the functor

$$(p_{1,3})_* \circ (\Delta_{\mathcal{Y}} \times \mathrm{id} \times \mathrm{id})^! \circ \sigma_{2,3} : \mathrm{Shv}(\mathcal{Y} \times \mathcal{Y} \times \mathcal{Y} \times \mathcal{Y}) \to \mathrm{Shv}(\mathcal{Y} \times \mathcal{Y}),$$

and we obtain a map

$$(\Delta_{\mathcal{Y}})_!(\underline{\mathsf{e}}_{\mathcal{Y}}) \to \mathrm{C}^{\cdot}(\mathcal{Y}^{\mathrm{Frob}}, \omega_{\mathcal{Y}^{\mathrm{Frob}}}) \otimes (\mathrm{Graph}_{\mathrm{Frob}_{\mathcal{Y}}})_*(\omega_{\mathcal{Y}}).$$

Applying the adjunction

$$(\Delta_{\mathcal{Y}})_!(\underline{\mathsf{e}}_{\mathcal{Y}}) : \mathrm{Vect} \rightleftarrows \mathrm{Shv}(\mathcal{Y} \times \mathcal{Y}) : \mathrm{C}^{\cdot}(\mathcal{Y}, -) \circ \Delta_{\mathcal{Y}}^!,$$

we obtain the desired element of

$$\mathrm{C}^{\cdot}(\mathcal{Y}^{\mathrm{Frob}}, \omega_{\mathcal{Y}^{\mathrm{Frob}}}) \otimes \mathrm{C}^{\cdot}(\mathcal{Y}^{\mathrm{Frob}}, \omega_{\mathcal{Y}^{\mathrm{Frob}}}) \simeq \mathrm{Funct}(\mathcal{Y}(\mathbb{F}_q)) \otimes \mathrm{Funct}(\mathcal{Y}(\mathbb{F}_q)). \tag{6.21}$$

We wish to show that the resulting function is the characteristic function of the diagonal, i.e., its value on a $(y_1, y_2) \in \mathcal{Y}(\mathbb{F}_q) \times \mathcal{Y}(\mathbb{F}_q)$ equals the cardinality of the set of isomorphisms between the corresponding two points of the groupoid $\mathcal{Y}(\mathbb{F}_q)$.

6.4.8. However, unwinding the definitions, we obtain that the map (6.20) identifies with the map (6.15) for $\mathcal{F} = (\Delta_{\mathcal{Y}})_!(\underline{\mathsf{e}}_{\mathcal{Y}})$ and α being the tautological map α_{taut}

$$(\Delta_{\mathcal{Y}})_!(\underline{\mathsf{e}}_{\mathcal{Y}}) \simeq (\mathrm{Frob}_{\mathcal{Y}\times\mathcal{Y}})_! \circ (\Delta_{\mathcal{Y}})_!(\underline{\mathsf{e}}_{\mathcal{Y}}) \simeq (\mathrm{Frob}_{\mathcal{Y}\times\mathcal{Y}})_* \circ (\Delta_{\mathcal{Y}})_!(\underline{\mathsf{e}}_{\mathcal{Y}}).$$

From here, we obtain that the element in (6.21) constructed above equals

$$\mathrm{funct}^{\mathrm{true}}((\Delta_{\mathcal{Y}})_!(\underline{\mathsf{e}}_{\mathcal{Y}}), \alpha_{\mathrm{taut}}).$$

Applying Theorem 6.4.2, we obtain that the above element equals

$$\mathrm{funct}^{\mathrm{naive}}((\Delta_{\mathcal{Y}})_!(\underline{\mathsf{e}}_{\mathcal{Y}}), \alpha_{\mathrm{taut}}).$$

Now, the classical Grothendiek-Lefschetz trace formula about the compatibility of the assignment

$$(\mathcal{F}, \alpha) \rightsquigarrow \mathrm{funct}^{\mathrm{naive}}(\mathcal{F}, \alpha)$$

with the !-pushforward functor implies that the above function equals the direct image (=sum along the fibers) of the constant function along the map

$$\mathcal{Y}(\mathbb{F}_q) \to (\mathcal{Y} \times \mathcal{Y})(\mathbb{F}_q) \simeq \mathcal{Y}(\mathbb{F}_q) \times \mathcal{Y}(\mathbb{F}_q),$$

as required.

Appendix A. Co-shtukas

Our calculation of $\mathrm{Tr}(\mathrm{Frob}_*, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G))$ used as an ingredient the non-standard self-duality on $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$, for which the counit was $\mathrm{ev}^*_{\mathrm{Bun}_G}$.

In this Appendix we will try a different path: we will rely on the usual Verdier self-duality of $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$, and try to re-run the arguments in this paper. However, as we shall see, instead of shtukas, we will have to deal with another object, we call them "co-shtukas".

We do not have at our disposal the analogs of the results from [Xue2] that pertain to co-shtukas, so we cannot really complete the trace calculation following this path. Instead, we state a number of (mutually equivalent) conjectures, which we rather give the status of "questions", that would have allowed us to do so.

A.1. **Kernels on truncatable stacks.**

A.1.1. Let $\mathcal{Y}$ be a (not necessarily) quasi-compact algebraic stack. Following [AGKRRV2, Sect. A.5.6], we introduce the category $\operatorname{Shv}(\mathcal{Y})_{\rm co}$, as the ind-completion of the full subcategory of $\operatorname{Shv}(\mathcal{Y})$ formed by objects of the form $j_*(\mathcal{F}_\mathcal{U})$, where

$$\mathcal{U} \overset{j}{\hookrightarrow} \mathcal{Y}$$

is a quasi-compact open substack, and $\mathcal{F}_\mathcal{U}$ is a compact object in $\operatorname{Shv}(\mathcal{U})$.

Alternatively, we can write

$$\operatorname{Shv}(\mathcal{Y})_{\rm co} \simeq \underset{\mathcal{U}}{\operatorname{colim}}\ \operatorname{Shv}(\mathcal{U}), \tag{A.1}$$

where $\mathcal{U}$ runs over the poset of quasi-compact open substacks of $\mathcal{Y}$, and for $j_{1,2} : \mathcal{U}_1 \hookrightarrow \mathcal{U}_2$, the transition functor $\operatorname{Shv}(\mathcal{U}_1) \to \operatorname{Shv}(\mathcal{U}_2)$ is $(j_{1,2})_*$.

By [GR, Chapter 1, Proposition 2.5.7], we can also write

$$\operatorname{Shv}(\mathcal{Y})_{\rm co} \simeq \underset{\mathcal{U}}{\lim}\ \operatorname{Shv}(\mathcal{U}),$$

where for $j_{1,2} : \mathcal{U}_1 \hookrightarrow \mathcal{U}_2$ as above, the transition functor

$$\operatorname{Shv}(\mathcal{U}_2) \to \operatorname{Shv}(\mathcal{U}_1)$$

is $(j_{1,2})^?$, the *right* adjoint of $(j_{1,2})_*$.

The operation of !-tensor product naturally defines an action of $\operatorname{Shv}(\mathcal{Y})$ on $\operatorname{Shv}(\mathcal{Y})_{\rm co}$.

Remark A.1.2. Recall for comparison that the usual category $\operatorname{Shv}(\mathcal{Y})$ can be described as

$$\operatorname{Shv}(\mathcal{Y})_{\rm co} \simeq \underset{\mathcal{U}}{\operatorname{colim}}\ \operatorname{Shv}(\mathcal{U}),$$

with transition functors $(j_{1,2})_!$ and also as

$$\operatorname{Shv}(\mathcal{Y})_{\rm co} \simeq \underset{\mathcal{U}}{\lim}\ \operatorname{Shv}(\mathcal{U}),$$

with transition functors $(j_{1,2})^*$.

A.1.3. We denote by

$$\mathrm{C}^\cdot_\blacktriangle(\mathcal{Y}, -) : \operatorname{Shv}(\mathcal{Y})_{\rm co} \to \operatorname{Vect}$$

the functor that corresponds in terms of (A.1) to the compatible family of functors

$$\mathrm{C}^\cdot_\blacktriangle(\mathcal{U}, -) : \operatorname{Shv}(\mathcal{U}) \to \operatorname{Vect}.$$

Explicitly, for an object $\mathcal{F} \in \operatorname{Shv}(\mathcal{Y})_{\rm co}$, we have

$$\mathrm{C}^\cdot_\blacktriangle(\mathcal{Y}, \mathcal{F}) \simeq \underset{\mathcal{U}}{\operatorname{colim}}\ \mathrm{C}^\cdot_\blacktriangle(\mathcal{U}, j^?(\mathcal{F})),$$

where

$$j^? : \operatorname{Shv}(\mathcal{Y})_{\rm co} \to \operatorname{Shv}(\mathcal{U})$$

is the right adjoint to the tautological functor $j_{*,\rm co} : \operatorname{Shv}(\mathcal{U}) \to \operatorname{Shv}(\mathcal{Y})_{\rm co}$.

Remark A.1.4. We have a tautologically defined functor $\operatorname{Shv}(\mathcal{Y})_{\rm co} \to \operatorname{Shv}(\mathcal{Y})$, denoted $\mathrm{Id}^{\rm naive}$, and determined by the requirement

$$\mathrm{Id}^{\rm naive}(j_{*,\rm co})(\mathcal{F}_\mathcal{U}) = j_*(\mathcal{F}_\mathcal{U})$$

for a quasi-compact $\mathcal{U} \overset{j}{\to} \mathcal{Y}$ and $\mathcal{F}_\mathcal{U} \in \operatorname{Shv}(\mathcal{U})$.

We also have the functors

$$\mathrm{C}^{\cdot,\rm bad}(\mathcal{Y}, -) : \operatorname{Shv}(\mathcal{Y}) \to \operatorname{Vect} \text{ and } \mathrm{C}^{\cdot,\rm bad}_\blacktriangle(\mathcal{Y}, -) : \operatorname{Shv}(\mathcal{Y}) \to \operatorname{Vect}.$$

Namely, for $\mathcal{F} \in \operatorname{Shv}(\mathcal{Y})$, we set

$$\mathrm{C}^{\cdot,\rm bad}(\mathcal{Y}, -) := \underset{\mathcal{U}}{\lim}\ \mathrm{C}^\cdot(\mathcal{U}, j^*(\mathcal{F})) \text{ and } \mathrm{C}^{\cdot,\rm bad}_\blacktriangle(\mathcal{Y}, -) := \underset{\mathcal{U}}{\lim}\ \mathrm{C}^\cdot_\blacktriangle(\mathcal{U}, j^*(\mathcal{F})),$$

respectively.

We have the natural transformations

$$\mathrm{C}^\cdot_\blacktriangle(\mathcal{Y},-)\to \mathrm{C}^{\cdot,\mathrm{bad}}_\blacktriangle(\mathcal{Y},-)\circ \mathrm{Id}^{\mathrm{naive}}\to \mathrm{C}^{\cdot,\mathrm{bad}}(\mathcal{Y},-)\circ \mathrm{Id}^{\mathrm{naive}}$$

as functors $\mathrm{Shv}(\mathcal{Y})_{\mathrm{co}}\to \mathrm{Vect}$, and they become isomorphisms when evaluated on compact objects of $\mathrm{Shv}(\mathcal{Y})_{\mathrm{co}}$, but they are *not at all* isomorphisms on all of $\mathrm{Shv}(\mathcal{Y})_{\mathrm{co}}$. In other words, the latter two functors are in general discontinuous.

A.1.5. Assume now that $\mathcal{Y}$ is *truncatable* (see Sect. 6.1.6). Then in the the above limits and colimits we can replace the index set of quasi-compact open substacks $\mathcal{U}$ by a cofinal subset consisting of those $\mathcal{U}$ that are *cotruncative*.

By definition, for a pair $j_{1,2}:\mathcal{U}_1\hookrightarrow \mathcal{U}_2$ of cotruncative open substacks, the corresponding functor $(j_{1,2})^?$ is defined by a kernel.

A.1.6. We continue to assume that $\mathcal{Y}$ is truncatable, and let $\mathcal{Z}$ be another algebraic stack. As was explained in [AGKRRV2, Sect. A.6.1], in this case, we can consider the category

$$\mathrm{Shv}(\mathcal{Y}\times\mathcal{Z})_{\mathrm{co}_\mathcal{Y}},$$

which can be defined as

$$\underset{\mathcal{U}_\mathcal{Y},\mathcal{U}_\mathcal{Z}}{\mathrm{colim}}\ \mathrm{Shv}(\mathcal{U}_\mathcal{Y}\times\mathcal{U}_\mathcal{Z}),$$

where the index category is that of pairs of a contruncative quasi-compact open $\mathcal{U}_\mathcal{Y}\subset\mathcal{Y}$ and a quasi-compact open $\mathcal{U}_\mathcal{Z}\subset\mathcal{Z}$, with transition functors

$$(j_{\mathcal{Y},1,2})_*\otimes(j_{\mathcal{Z},1,2})_!:\mathrm{Shv}(\mathcal{U}_{1,\mathcal{Y}}\times\mathcal{U}_{1,\mathcal{Z}})\to \mathrm{Shv}(\mathcal{U}_{2,\mathcal{Y}}\times\mathcal{U}_{2,\mathcal{Z}}),$$

which are well-defined because the embeddings

$$j_{\mathcal{Y},1,2}:\mathcal{U}_{1,\mathcal{Y}}\to\mathcal{U}_{2,\mathcal{U}=Y}$$

are cotruncative, and so the corresponding functors $(j_{\mathcal{Y},1,2})_*$ are *codefined* by kernels.

The operation of !-tensor product naturally defines an action of $\mathrm{Shv}(\mathcal{Y}\times\mathcal{Z})$ on $\mathrm{Shv}(\mathcal{Y}\times\mathcal{Z})_{\mathrm{co}_\mathcal{Y}}$.

The construction of the functor $\mathrm{C}^\cdot_\blacktriangle(\mathcal{Y},-)$ extends to a functor

$$(p_2)_\blacktriangle:\mathrm{Shv}(\mathcal{Y}\times\mathcal{Z})_{\mathrm{co}_\mathcal{Y}}\to\mathrm{Shv}(\mathcal{Z}).$$

An object

$$\mathcal{Q}\in\mathrm{Shv}(\mathcal{Y}\times\mathcal{Z})_{\mathrm{co}_Y}$$

gives rise to a functor

$$\mathsf{F}_\mathcal{Q}:\mathrm{Shv}(\mathcal{Y})\to\mathrm{Shv}(\mathcal{Z}),$$

$$\mathsf{F}_\mathcal{Q}(\mathcal{F}):=(p_2)_\blacktriangle\circ(p_1^!(\mathcal{F})\overset{!}{\otimes}\mathcal{Q}).$$

A.1.7. Let us take $\mathcal{Z}=\mathcal{Y}$. In this case we will denote the corresponding category $\mathrm{Shv}(\mathcal{Y}\times\mathcal{Z})_{\mathrm{co}_\mathcal{Y}}$ by $\mathrm{Shv}(\mathcal{Y}\times\mathcal{Y})_{\mathrm{co}_1}$.

Note that $\mathrm{u}_\mathcal{Y}:=(\Delta_\mathcal{Y})_*(\omega_\mathcal{Y})$ can be naturally regarded as an object of $\mathrm{Shv}(\mathcal{Y}\times\mathcal{Y})_{\mathrm{co}_1}$. The corresponding functor $\mathsf{F}_{\mathrm{u}_\mathcal{Y}}$ of $\mathrm{Shv}(\mathcal{Y})$ is the identity.

In addition, we note that $\Delta^!_\mathcal{Y}$ naturally defines a functor

$$\mathrm{Shv}(\mathcal{Y}\times\mathcal{Y})_{\mathrm{co}_1}\to\mathrm{Shv}(\mathcal{Y})_{\mathrm{co}}.$$

A.1.8. For $\mathcal{Q}\in\mathrm{Shv}(\mathcal{Y}\times\mathcal{Y})_{\mathrm{co}_1}$ we define

$$\mathrm{Tr}^{\mathrm{geom}}(\mathsf{F}_\mathcal{Q},\mathrm{Shv}(\mathcal{Y})):=\mathrm{C}^\cdot_\blacktriangle(\mathcal{Y},\Delta^!_\mathcal{Y}(\mathcal{Q})).$$

As in Sect. 5.2.3, we have a naturally defined map

$$\mathrm{Tr}(\mathsf{F}_\mathcal{Q},\mathrm{Shv}(\mathcal{Y}))\to\mathrm{Tr}^{\mathrm{geom}}(\mathsf{F}_\mathcal{Q},\mathrm{Shv}(\mathcal{Y})).$$

A.1.9. *Example.* Let

$$\mathcal{Q} := (\mathrm{Graph}_{\mathrm{Frob}_{\mathcal{Y}}})_*(\omega_{\mathcal{Y}}) := (\mathrm{Frob}_{\mathcal{Y}} \times \mathrm{id})_* \circ (\Delta_{\mathcal{Y}})_*(\omega_{\mathcal{Y}}) \in \mathrm{Shv}(\mathcal{Y} \times \mathcal{Y})_{\mathrm{co}_1},$$

so that the corresponding endofunctor $\mathsf{F}_{\mathcal{Q}}$ of $\mathrm{Shv}(\mathcal{Y})$ is $(\mathrm{Frob}_{\mathcal{Y}})_*$.

We claim that

$$\mathrm{Tr}^{\mathrm{geom}}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}(\mathcal{Y})) \simeq \mathrm{Funct}_c(\mathcal{Y}(\mathbb{F}_q)).$$

Indeed, we can write

$$\omega_{\mathcal{Y}} \simeq \underset{\mathcal{U}}{\mathrm{colim}}\, j_!(\omega_{\mathcal{U}}),$$

and hence

$$(\Delta_{\mathcal{Y}})_*(\omega_{\mathcal{Y}}) \simeq \underset{\mathcal{U}}{\mathrm{colim}}\, (j_* \otimes j_!) \circ (\Delta_{\mathcal{U}})_*(\omega_{\mathcal{U}}),$$

where we let $\mathcal{U}$ run over the poset of cotruncative quasi-compact open substacks of $\mathcal{Y}$.

This implies that

$$\Delta_{\mathcal{Y}}^! \circ (\mathrm{Frob}_{\mathcal{Y}} \times \mathrm{id})_* \circ (\Delta_{\mathcal{Y}})_*(\omega_{\mathcal{Y}}) \simeq \underset{\mathcal{U}}{\mathrm{colim}}\, j_* \circ \Delta_{\mathcal{U}}^! \circ (\mathrm{Graph}_{\mathrm{Frob}_{\mathcal{U}}})_*(\omega_{\mathcal{U}}),$$

so

$$\mathrm{Tr}^{\mathrm{geom}}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}(\mathcal{Y})) \simeq \underset{\mathcal{U}}{\mathrm{colim}}\, \mathrm{Funct}(\mathcal{U}(\mathbb{F}_q)),$$

and it follows from Theorem 5.2.5(b) that the transition maps in this colimit

$$\mathrm{Funct}(\mathcal{U}_1(\mathbb{F}_q)) \to \mathrm{Funct}(\mathcal{U}_2(\mathbb{F}_q))$$

are given by push-forward, so the colimit indeed identifies with $\mathrm{Funct}_c(\mathcal{Y}(\mathbb{F}_q))$.

Note that the composite map

$$\mathrm{Tr}(\mathsf{F}_{\mathcal{Q}}, \mathrm{Shv}(\mathcal{Y})) \to \mathrm{Tr}^{\mathrm{geom}}(\mathsf{F}_{\mathcal{Q}}, \mathrm{Shv}(\mathcal{Y})) \simeq \mathrm{Funct}_c(\mathcal{Y}(\mathbb{F}_q))$$

is the map $\mathrm{LT}^{\mathrm{true}}_{\mathcal{Y}}$ from Sect. 5.2.6.

A.2. **The definition of co-shtukas.**

A.2.1. For $\mathcal{V} \in \mathrm{Rep}(\check{G})_{\mathrm{Ran}}$, the endofunctor $(\mathcal{V} \star -)$ of $\mathrm{Shv}(\mathrm{Bun}_G)$ is defined by a kernel, namely by

$$\text{co-}\mathcal{K}_{\mathcal{V}} := (\mathrm{Id} \otimes (\mathcal{V} \star -))((\Delta_{\mathrm{Bun}_G})_*(\omega_{\mathrm{Bun}_G})) \in \mathrm{Shv}(\mathrm{Bun}_G \times \mathrm{Bun}_G)_{\mathrm{co}_1}. \tag{A.2}$$

Consider now the endofunctor

$$(\mathcal{V} \star -) \circ (\mathrm{Frob}_{\mathrm{Bun}_G})_*,$$

which is also given by a kernel, namely

$$(\mathrm{Id} \times \mathrm{Frob}_{\mathrm{Bun}_G})^!(\text{co-}\mathcal{K}_{\mathcal{V}}) \in \mathrm{Shv}(\mathrm{Bun}_G \times \mathrm{Bun}_G)_{\mathrm{co}_1}.$$

A.2.2. We define the functor

$$\text{co-Sht} : \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \to \mathrm{Vect}$$

to send $\mathcal{V}$ to

$$\mathrm{Tr}^{\mathrm{geom}}((\mathcal{V} \star -) \circ (\mathrm{Frob}_{\mathrm{Bun}_G})_*, \mathrm{Shv}(\mathrm{Bun}_G)),$$

which is by definition

$$\mathrm{C}^\cdot_\blacktriangle(\mathrm{Bun}_G, (\Delta_{\mathrm{Bun}_G})^! \circ (\mathrm{Frob}_{\mathrm{Bun}_G} \times \mathrm{id})^!(\text{co-}\mathcal{K}_{\mathcal{V}})) \simeq \mathrm{C}^\cdot_\blacktriangle(\mathrm{Bun}_G, (\mathrm{Graph}_{\mathrm{Frob}_{\mathrm{Bun}_G}})^!(\text{co-}\mathcal{K}_{\mathcal{V}})).$$

A.2.3. By Sect. 1.6.7, we can interpret the functor co-Sht as a family of functors

$$\text{co-Sht}_I : \operatorname{Rep}(\check{G})^{\otimes I} \to \operatorname{Shv}(X^I),$$

compatible under the functors

$$(\Delta_\phi)^! : \operatorname{Shv}(X^I) \to \operatorname{Shv}(X^J), \quad I \overset{\phi}{\to} J.$$

Explicitly, the functors co-Sht_I can be described as follows. For $V \in \operatorname{Rep}(\check{G})^{\otimes I}$, consider the object

$$(\mathrm{id} \times \mathrm{Frob}_{\mathrm{Bun}_G} \times \mathrm{id})^! \circ (\mathrm{Id} \otimes \mathsf{H}^!(V,-))((\Delta_{\mathrm{Bun}_G})_*(\omega_{\mathrm{Bun}_G})) \in \operatorname{Shv}(\mathrm{Bun}_G \times (\mathrm{Bun}_G \times X^I))_{\mathrm{co}_{\mathrm{Bun}_G}}.$$

We apply to it the functor

$$(\Delta_{\mathrm{Bun}_G} \times \mathrm{id})^! : \operatorname{Shv}(\mathrm{Bun}_G \times (\mathrm{Bun}_G \times X^I))_{\mathrm{co}_{\mathrm{Bun}_G}} \to \operatorname{Shv}(\mathrm{Bun}_G \times X^I)_{\mathrm{co}_{\mathrm{Bun}_G}},$$

and follow it by

$$(p_2)_\blacktriangle : \operatorname{Shv}(\mathrm{Bun}_G \times X^I)_{\mathrm{co}_{\mathrm{Bun}_G}} \to \operatorname{Shv}(X^I).$$

A.2.4. Thus, we regard co-Sht as an object of $(\operatorname{Rep}(\check{G})_{\mathrm{Ran}})^\vee$. We propose:

Question A.2.5. *Does the object* co-Sht *belong to the subcategory* $(\operatorname{Rep}(\check{G})_{\mathrm{Ran}})^\vee_{\mathrm{QLisse}}$ *of* $(\operatorname{Rep}(\check{G})_{\mathrm{Ran}})^\vee$*?*

Explicitly, the above question is asking whether the functors

$$\text{co-Sht}_I : \operatorname{Rep}(\check{G})^{\otimes I} \to \operatorname{Shv}(X^I)$$

take values in

$$\operatorname{QLisse}(X^I) \subset \operatorname{Shv}(X^I).$$

Remark A.2.6. In addition to co-Sht, one could consider two more functors

$$\operatorname{Rep}(\check{G})_{\mathrm{Ran}} \to \mathrm{Vect}.$$

Namely, we can send $\mathcal{V} \in \operatorname{Rep}(\check{G})_{\mathrm{Ran}}$ to

$$\mathrm{C}^{\cdot,\mathrm{bad}}(\mathrm{Bun}_G, \mathrm{Id}^{\mathrm{naive}} \circ (\mathrm{Graph}_{\mathrm{Frob}_{\mathrm{Bun}_G}})^!(\text{co-}\mathcal{K}_{\mathcal{V}})) \text{ and } \mathrm{C}^{\cdot,\mathrm{bad}}_\blacktriangle(\mathrm{Bun}_G, \mathrm{Id}^{\mathrm{naive}} \circ (\mathrm{Graph}_{\mathrm{Frob}_{\mathrm{Bun}_G}})^!(\text{co-}\mathcal{K}_{\mathcal{V}})),$$

see Remark A.1.4. Denote these functors by

$$\text{co-Sht}^{\mathrm{bad}} \text{ and } \text{co-Sht}^{\mathrm{bad}}_\blacktriangle,$$

respectively.

We have the natural transformations

$$\text{co-Sht} \to \text{co-Sht}^{\mathrm{bad}}_\blacktriangle \to \text{co-Sht}^{\mathrm{bad}},$$

but they are *not at all* isomorphisms.

For example, the values of both $\text{co-Sht}^{\mathrm{bad}}_\blacktriangle$ and $\text{co-Sht}^{\mathrm{bad}}$ on the unit object of $\operatorname{Rep}(\check{G})_{\mathrm{Ran}}$ yields $\mathrm{Funct}(\mathrm{Bun}_G(\mathbb{F}_q))$, the space of *not necessarily compactly supported* functions on $\mathrm{Bun}_G(\mathbb{F}_q)$.

A.3. **Relation to trace.**

A.3.1. Let us be again in the general setting of Sect. A.1. Let $\mathcal{N} \subset T^*(\mathcal{Y})$ be a conical Zariski-closed subset. We will assume that $\mathcal{Y}$ is $\mathcal{N}$-truncatable and that $\mathcal{N}$ is Frobenius-invariant.

We consider the full subcategory

$$\operatorname{Shv}_{\mathcal{N}}(\mathcal{Y}) \subset \operatorname{Shv}(\mathcal{Y}),$$

and let us define the full subcategory

$$\operatorname{Shv}_{\mathcal{N}}(\mathcal{Y})_{\mathrm{co}} \subset \operatorname{Shv}(\mathcal{Y})_{\mathrm{co}}$$

as

$$\underset{\mathcal{U}}{\operatorname{colim}} \operatorname{Shv}_{\mathcal{N}}(\mathcal{U}),$$

where the index category consists of $\mathcal{N}$-cotruncative quasi-compact open substacks of $\mathcal{Y}$ and the colimit is taken with respect to the *-pushforward functors.

A.3.2. Assume that for every $\mathcal{N}$-cotruncative quasi-compact open substack $\mathcal{U} \subset \mathcal{Y}$, the pair $(\mathcal{U}, \mathcal{N}|_{\mathcal{U}})$ is renormalization-adapted and constraccessible (see Sect. 6.1.1).

It follows formally that the restriction of

$$\mathrm{ev}^!_{\mathcal{Y}} := \mathrm{C}^{\cdot}_{\blacktriangle} \circ \overset{!}{\otimes} : \mathrm{Shv}(\mathcal{Y})_{\mathrm{co}} \otimes \mathrm{Shv}(\mathcal{Y}) \to \mathrm{Vect}$$

to

$$\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})_{\mathrm{co}} \otimes \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y}) \subset \mathrm{Shv}(\mathcal{Y})_{\mathrm{co}} \otimes \mathrm{Shv}(\mathcal{Y})$$

defines perfect pairing.

By construction, the corresponding contravariant equivalence

$$((\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})_{\mathrm{co}})^c)^{\mathrm{op}} \to (\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y}))^c$$

sends

$$j_{*,\mathrm{co}}(\mathcal{F}_U) \mapsto j_!(\mathbb{D}^{\mathrm{Verdier}}(\mathcal{F}_U)), \quad \mathcal{F}_U \in \mathrm{Shv}_{\mathcal{N}}(\mathcal{U})^c.$$

Denote by

$$\mathrm{u}_{\mathcal{Y},\mathcal{N}} \in \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})_{\mathrm{co}} \otimes \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})$$

the unit object for this duality.

As in Sect. 6.3.1, it is easy to see that $\mathrm{u}_{\mathcal{Y},\mathcal{N}}$ equals the image of $\mathrm{u}_{\mathcal{U}} := (\Delta_{\mathcal{Y}})_*(\omega_{\mathcal{Y}})$ under the right adjoint to the fully faithful embedding

$$\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})_{\mathrm{co}} \otimes \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y}) \hookrightarrow \mathrm{Shv}(\mathcal{Y} \times \mathcal{Y})_{\mathrm{co}_1}.$$

In what follows we will not notationally distinguish between $\mathrm{u}_{\mathcal{Y},\mathcal{N}}$ and its image under the above embedding. When viewed as such, it is equipped with a tautologically defined map

$$\mathrm{u}_{\mathcal{Y},\mathcal{N}} \to \mathrm{u}_{\mathcal{Y}}\,.$$

A.3.3. We take $\mathcal{Y} = \mathrm{Bun}_G$ and $\mathcal{N} = \mathrm{Nilp}$. It follows from Theorem 5.3.5(a) that the endofunctor $(\mathrm{Id} \otimes \mathsf{P}_{\mathrm{Nilp}})$ of $\mathrm{Shv}(\mathrm{Bun}_G \times \mathrm{Bun}_G)_{\mathrm{co}_1}$ identifies with the composition of the fully faithful embedding

$$\mathrm{Shv}_{T^*(\mathrm{Bun}_G)\times \mathrm{Nilp}}(\mathrm{Bun}_G \times \mathrm{Bun}_G)_{\mathrm{co}_1} \hookrightarrow \mathrm{Shv}(\mathrm{Bun}_G \times \mathrm{Bun}_G)_{\mathrm{co}_1},$$

preceded by its right adjoint.

We now quote the following result from [AGKRRV2, Corollary 1.5.5], which is analog of Theorem 4.3.2.

Theorem A.3.4. *The object*

$$(\mathrm{Id} \otimes \mathsf{P}_{\mathrm{Nilp}})(\mathrm{u}_{\mathrm{Bun}_G}) \in \mathrm{Shv}_{T^*(\mathrm{Bun}_G)\times \mathrm{Nilp}}(\mathrm{Bun}_G \times \mathrm{Bun}_G)_{\mathrm{co}_1}$$

belongs to the full subcategory

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)_{\mathrm{co}} \otimes \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \subset \mathrm{Shv}_{T^*(\mathrm{Bun}_G)\times \mathrm{Nilp}}(\mathrm{Bun}_G \times \mathrm{Bun}_G)_{\mathrm{co}_1}.$$

As a consequence, we obtain that the object $(\mathrm{Id} \otimes \mathsf{P}_{\mathrm{Nilp}})(\mathrm{u}_{\mathrm{Bun}_G})$ identifies canonically with $\mathrm{u}_{\mathrm{Bun}_G,\mathrm{Nilp}}$. Moreover, this isomorphism is compatible with the natural maps of both objects to $\mathrm{u}_{\mathrm{Bun}_G}$.

A.3.5. Hence, arguing as in Sect. 4.4, we obtain:

Proposition A.3.6. *There exists a canonical isomorphism of functors* $\mathrm{Rep}(\check{G})_{\mathrm{Ran}} \to \mathrm{Vect}$

$$\text{co-Sht}(\mathcal{R} \star -) \simeq \mathrm{Sht}^{\mathrm{Tr}}\,.$$

A.3.7. Recall now (see Sect. A.1.8 above) that we have a canonical natural transformation

$$\mathrm{Sht}^{\mathrm{Tr}} \to \text{co-Sht}\,. \tag{A.3}$$

In addition, as in Sect. 5.4.4, the natural transformation

$$\varepsilon : (\mathrm{Id} \otimes \mathsf{P}_{\mathrm{Nilp}}) \to \mathrm{Id}$$

defines a natural transformation

$$\text{co-Sht}(\mathcal{R} \star -) \to \text{co-Sht}\,. \tag{A.4}$$

It follows as in Sect. 5.6 that the isomorphism of Proposition A.3.6 is compatible with these maps.

A.3.8. We propose:

Question A.3.9. *Are the maps* (A.3) *and* (A.4) *isomorphisms?*

A.3.10. We claim, however, that Questions A.2.5 and A.3.9 are equivalent.

Indeed, an affirmative answer to Question A.3.9 immediately implies that

$$\text{co-Sht} \in (\mathrm{Rep}(\check{G})_{\mathrm{Ran}})^{\vee}_{\mathrm{QLisse}}$$

because $\mathrm{Sht}^{\mathrm{Tr}}$ has this property.

The inverse implication is obtained by following the logic of the proofs of Theorems 3.1.3 and 4.1.2.

A.4. **Shtukas vs coshtukas.**

A.4.1. Let us recall, following [AGKRRV2, Sect. A.6.4], that for a not necessarily quasi-compact stack $\mathcal{Y}$, we have a well-defined miraculous functor

$$\mathrm{Mir}_{\mathcal{Y}} : \mathrm{Shv}(\mathcal{Y})_{\mathrm{co}} \to \mathrm{Shv}(\mathcal{Y}).$$

Assume that $\mathcal{Y}$ is truncatable. Then more generally, for a stack $\mathcal{Z}$, we have a functor

$$(\mathrm{Mir}_{\mathcal{Y}} \otimes \mathrm{Id}) : \mathrm{Shv}(\mathcal{Y} \times \mathcal{Z})_{\mathrm{co}_{\mathcal{Y}}} \to \mathrm{Shv}(\mathcal{Y} \times \mathcal{Z}).$$

A.4.2. *Example.* Take $\mathcal{Z} = \mathcal{Y}$. Then

$$(\mathrm{Mir}_{\mathcal{Y}} \otimes \mathrm{Id})(\mathrm{u}_{\mathcal{Y}}) \simeq \text{ps-u}_{\mathcal{Y}}\,.$$

A.4.3. Let us return to the setting of Sect. A.1.7. Given an object

$$\mathcal{Q} \in \mathrm{Shv}(\mathcal{Y} \times \mathcal{Y})_{\mathrm{co}_1},$$

as in Sect. 6.2, we construct a map

$$\mathrm{C}^\cdot_c(\mathcal{Y}, \Delta^*_{\mathcal{Y}} \circ (\mathrm{Mir}_{\mathcal{Y}} \otimes \mathrm{Id})(\mathcal{Q})) \to \mathrm{C}^\cdot_\blacktriangle(\mathcal{Y}, \Delta^!_{\mathcal{Y}}(\mathcal{Q})). \tag{A.5}$$

A.4.4. Take $\mathcal{Y} = \mathrm{Bun}_G$. For $\mathcal{V} \in \mathrm{Rep}(\check{G})_{\mathrm{Ran}}$, recall the notation

$$\text{co-}\mathcal{K}_{\mathcal{V}} \in \mathrm{Shv}(\mathrm{Bun}_G \times \mathrm{Bun}_G)_{\mathrm{co}_1},$$

see Sect. A.2.

By Sect. A.4.2 and [AGKRRV2, Lemma 1.6.4], we have

$$(\mathrm{Mir}_{\mathrm{Bun}_G} \otimes \mathrm{Id})(\text{co-}\mathcal{K}_{\mathcal{V}}) \simeq \mathcal{K}_{\mathcal{V}}.$$

Hence, the map (A.5) gives rise to a natural transformation

$$\mathrm{Sht} \to \text{co-Sht}, \tag{A.6}$$

as functors $\mathrm{Rep}(\check{G})_{\mathrm{Ran}} \to \mathrm{Vect}$.

A.4.5. We propose:

Question A.4.6. *Is the natural transformation* (A.6) *an isomorphism?*

A.4.7. We claim, however, that Question A.4.6 is equivalent to Question A.3.9. Indeed, we have a (tautologically) commutative diagram

$$\begin{tikzcd} \mathrm{Sht}(\mathcal{R}\star -) \arrow[r, "(A.6)"] \arrow[d, "\sim"'] & \text{co-Sht}(\mathcal{R}\star -) \arrow[d] \\ \mathrm{Sht} \arrow[r, "(A.6)"] & \text{co-Sht}, \end{tikzcd}$$

and we also have a diagram

$$\begin{tikzcd} \mathrm{Sht}^{\mathrm{Tr}} \arrow[r, "\mathrm{id}"] \arrow[d, "\sim"'] & \mathrm{Sht}^{\mathrm{Tr}} \arrow[d, "\sim"] \\ \mathrm{Sht}(\mathcal{R}\star -) \arrow[r, "(A.6)"] & \text{co-Sht}(\mathcal{R}\star -), \end{tikzcd}$$

whose commutativity follows by the argument in Sect. 6.3.

A.4.8. Finally, we claim that the natural transformation (A.6) becomes an isomorphisms when evaluated on $\mathbf{1}_{\mathrm{Rep}(\check{G})_{\mathrm{Ran}}}$.

Indeed, this follows from Theorem 6.2.5, which can be interpreted as saying that the diagram

$$\begin{tikzcd} \mathrm{C}^{\cdot}_c(\mathcal{Y}, \Delta^*_{\mathcal{Y}} \circ (\mathrm{Graph}_{\mathcal{Y}})_!(\underline{\mathsf{e}}_{\mathcal{Y}})) \arrow[r] \arrow[d, "\sim"'] & \mathrm{C}^{\cdot}_{\blacktriangle}(\mathcal{Y}, \Delta^!_{\mathcal{Y}} \circ (\mathrm{Graph}_{\mathcal{Y}})_*(\omega_{\mathcal{Y}})) \arrow[d, "\sim"] \\ \mathrm{Funct}_c(\mathcal{Y}(\mathbb{F}_q)) \arrow[r, "\mathrm{id}"] & \mathrm{Funct}_c(\mathcal{Y}(\mathbb{F}_q)) \end{tikzcd}$$

commutes, where the top horizontal arrow is obtained by applying the natural transformation (A.5) to the object

$$\mathcal{Q} := (\mathrm{Graph}_{\mathcal{Y}})_*(\omega_{\mathcal{Y}}) \in \mathrm{Shv}(\mathcal{Y}\times\mathcal{Y})_{\mathrm{co}_1}.$$

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