

FORMS AND COVERS OF REDUCTIVE GROUPS, AND THE L -GROUP

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1. Classification of reductive groups over a separably closed field

For this section we assume k to be separably closed (unless otherwise noted). [For now, this section is missing a lot of results, as one needs to establish the analogs of results that we proved for semisimple Lie algebras in characteristic zero, for reductive groups.]

We saw in Theorem 1.2 a simple combinatorial description for diagonalizable groups, and we would like to have a similar description for more general reductive groups. This is not possible in the sense of getting an equivalence of categories¹, but at least we can fully describe the isomorphism classes this way.

Let G be reductive (over k), and let T be a maximal torus in G . Let $X^\bullet(T)$, $X_\bullet(T)$ be the character and cocharacter groups of T . The adjoint action of T on \mathfrak{g} is semisimple, and we have a decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha,$$

where the α 's, here, are eigencharacters $\alpha : T \rightarrow \mathbb{G}_m$.

By Proposition 4.8, \mathfrak{g}_0 is just the Lie algebra of T , however, *to accommodate the case of positive characteristic*, we will not be using \mathfrak{t} to denote this Lie algebra, but the real vector space

$$\mathfrak{t} = X_\bullet(T) \otimes \mathbb{R}$$

(and, similarly, $\mathfrak{t}^* = X^\bullet(T) \otimes \mathbb{R}$). For real algebraic groups, this \mathfrak{t}^* can be identified with the dual of the Lie algebra, by the map that assigns to any character χ its differential $d\chi$ at the identity.

¹There is a good reason for it: To get an equivalence of categories one must consider the category of all G -representations, cf. Tannaka-Krein duality. For diagonalizable groups this category is described easily in terms of combinatorial data, this is no longer the case for other groups.

Definition 1.1 (Root datum). A *root datum* is a quadruple $(X, \Phi, \check{X}, \check{\Phi})$, where X, \check{X} are two lattices (finitely generated, torsion-free abelian groups) in duality, $\Phi \subset X$ and $\check{\Phi} \subset \check{X}$ are finite subsets, such that there exists a bijection $\Phi \ni \alpha \leftrightarrow \check{\alpha} \in \check{\Phi}$, satisfying:

- (1) $\langle \alpha, \check{\alpha} \rangle = 2$;
- (2) the endomorphisms of X, \check{X} defined by $w_\alpha(x) := x - \langle x, \check{\alpha} \rangle \alpha$, $w_{\check{\alpha}}(\check{x}) = \check{x} - \langle \alpha, \check{x} \rangle \check{\alpha}$ preserve Φ and $\check{\Phi}$.

A *based root datum* is a root datum as above, together with a choice of positive roots $\Phi^+ \subset \Phi$ (in the sense of Definition 4.1).

Remarks 1.2. (1) The last axiom is equivalent to: the endomorphisms w_α preserve Φ and generate a finite group (the Weyl group W).
 (2) The condition $\langle \alpha, \check{\alpha} \rangle = 2$ characterizes the bijection $\Phi \leftrightarrow \check{\Phi}$, so it need not be part of the data.
 (3) The \mathbb{R} -span of Φ in $X^* \otimes \mathbb{R}$, together with Φ and the Weyl group of automorphisms generated by the w_α 's, forms a root system, as can be easily verified; hence the definition of based root datum in terms of based root systems.

We now define the appropriate notion of morphisms between root data.

Definition 1.3. An *isogeny* of root data $(X, \Phi, \check{X}, \check{\Phi}) \rightarrow (X', \Phi', \check{X}', \check{\Phi}')$ is a homomorphism $f : X \rightarrow X'$ with the following properties:

- (1) f is injective, and with finite cokernel — hence, so is its adjoint $f^* : \check{X}' \rightarrow \check{X}$;
- (2) f and f^* induce bijections between the subsets of roots and coroots, respectively.

Proposition 1.4. *Given a connected reductive group G and a maximal torus T with set of roots $\Phi \subset X^*(T)$, and given $\alpha \in \Phi$, consider the subtorus $T_\alpha = \ker(\alpha)^\circ \subset T$. Let L_α be the centralizer of T_α (which is connected by Proposition 3.2, and L'_α its derived subgroup. Then L'_α is isomorphic to SL_2 or PGL_2 , and therefore there is a unique cocharacter $\check{\alpha} : \mathbb{G}_m \rightarrow T \cap L'_\alpha$ with $\langle \alpha, \check{\alpha} \rangle = 2$. If w_α is the nontrivial element of the Weyl group of T inside of L'_α , then the elements w_α generate the full Weyl group of T in G , $W = \mathcal{N}_G(T)/T$.*

Proof. Omitted. □

Definition 1.5. In the notation of Proposition 1.4, the elements $\check{\alpha} \in X_*(T)$ associated to the roots $\alpha \in X^*(T)$ are the *coroots* of the torus T in G .

Definition 1.6. Let $(G, T), (G', T')$ be two pairs consisting of a connected reductive group and a maximal torus. A *central isogeny* $(G', T') \rightarrow (G, T)$ is a morphism $G' \rightarrow G$ sending $T' \rightarrow T$, surjective, whose kernel is a finite central group scheme. A central isogeny $G' \rightarrow G$ is a morphism which induces a central isogeny $(G', T') \rightarrow (G, T)$ for some maximal tori T', T (or, equivalently, for any maximal torus T' , and T the image of T').

Remark 1.7. A surjective morphism $(G', T') \rightarrow (G, T)$ with finite kernel gives rise to an injective map $X^*(T) \rightarrow X^*(T')$ with finite cokernel, and a bijection between the roots of T on G and the roots of T' on G' . The requirement that the kernel be a central group scheme can be reformulated as follows: the map

induces isomorphisms between the corresponding root spaces of T' and T in \mathfrak{g}' , resp. \mathfrak{g} . This requirement is automatically satisfied in characteristic zero, but not in positive characteristic. For example, if G is defined over a finite field with q elements, the Frobenius morphism F_q (see the proof of Theorem 4.7) restricts to the Frobenius morphism on the subgroups $\mathfrak{g}_\alpha \simeq \mathbb{G}_a$, which is not an isomorphism. The kernel of Frobenius on \mathfrak{g}_α does not have any nontrivial points over the finite field, but it is not a central group scheme.

Now we define some categories of reductive groups with extra data that will be used in the classification.

Definition 1.8. We let Red be the category whose objects are connected reductive groups G over k , and whose morphisms are central isogenies $G' \rightarrow G$. We let Red_T be the category whose objects are pairs $(G \supset T)$ of reductive groups with maximal tori, and whose morphisms are central isogenies of pairs $(G', T') \rightarrow (G, T)$. We let $Red_{B,T}$ be the category whose objects are triples $(G \supset B \supset T)$ of reductive groups with Borel subgroups and maximal tori thereof, and whose morphisms are central isogenies of pairs $(G', T') \rightarrow (G, T)$ sending B' to B .

We let RD be the category whose objects are root data $(X, \Phi, \check{X}, \check{\Phi})$ and whose morphisms are central isogenies. We let RD^+ be the category whose objects are root data, together with a choice $\Phi^+ \subset \Phi$ of positive roots.

Definition 1.9. If G is a connected reductive group over a field k which is not necessarily separably closed, it is said to be *quasisplit* if there exists a Borel subgroup over k , and *split* if there exists a maximal torus which is split over k (hence also a Borel subgroup).

Theorem 1.10 (Classification over the separable closure). *Assume k separably closed. Given a connected reductive group G and a maximal torus T , the quadruple $\Psi(G, T) = (X^*(T), \Phi, X_*(T), \check{\Phi})$, where $\Phi, \check{\Phi}$ denote, respectively, the sets of roots and coroots of T in G , is a root datum.*

The assignment $(G, T) \mapsto \Psi(G, T)$ is a functor $Red \rightarrow RD^{op}$, in the notation of Definition 1.8.

This functor is a bijection on isomorphism classes of objects, and every morphism $\Psi(G, T) \rightarrow \Psi(G', T')$ in RD lifts to a morphism (i.e., central isogeny) $(G', T') \rightarrow (G, T)$, uniquely up to precomposing with conjugation by elements of T' , or postcomposing with conjugation by elements of T .

More generally, the same statement holds if k is not separably closed, but we consider the full subcategory of such pairs (G, T) where T (and, hence, G) is split over k .

Proof. Omitted. See [Spr79] for more details and references. \square

We will now see two variants of this theorem: One, where we choose more data in order to rigidify the functor, and another, where we make no choices (i.e., we do not choose tori).

If, in the context of Theorem 1.10, we choose a Borel subgroup $B \supset T$, it gives rise to a based root datum, with Φ^+ being the set of roots appearing in the Lie algebra of B . We will generally denote by Δ the subset of simple roots.

Definition 1.11. A *pinning* of a triple (G, B, T) consisting of a connected reductive group G , a Borel subgroup, and a maximal torus T is a choice of isomorphisms $p_\alpha : \mathfrak{g}_\alpha \simeq \mathbb{G}_a$, for every simple root $\alpha \in \Delta$.

Given a pair (G, B) of a connected reductive group and a Borel subgroup, whose unipotent radical we will denote by N , an *algebraic Whittaker datum* is a homomorphism $\ell : N \rightarrow \mathbb{G}_a$ which restricts to a nontrivial linear map on the one-parameter subgroup $G_\alpha \simeq \mathbb{G}_a$ associated to every simple root α . (In particular, its differential induces a pinning; vice versa, given a pinning as above, there is a unique such ℓ with $d\ell|_{\mathfrak{g}_\alpha} = p_\alpha \cdot$)

We let Red_{PIN} denote the category whose objects are pinned reductive groups, and whose morphisms are central isogenies preserving the Borel, the torus, and the pinning.

Pinnings allow us to rigidify the morphisms lifted from morphisms of (based) root data:

Theorem 1.12. *The assignment $(G, B, T, P) \rightarrow \Psi^+(G, B, T)$, where $\Psi^+(G, B, T)$ denotes the based root datum consisting of $\Psi(G, T)$ (notation as in Theorem 1.10) with Φ^+ the positive roots associated to B , is an equivalence of categories: $\text{Red}_{\text{PIN}} \rightarrow \text{RD}^+$.*

Proof. Omitted. See, again, [Spr79]. \square

Finally, a version of the classification for morphisms $G' \rightarrow G$, without any choices: Recall that to any reductive group G , we have associated its (universal) Cartan \mathbb{A}^G , endowed with sets of roots and positive roots $\Phi^+ \subset \Phi \subset X^*(\mathbb{A}^G)$. (Recall that in this section we are assuming the field to be separably closed.) We will denote by $\Psi^+(\mathbb{A}^G)$ this based root system.

Lemma 1.13. *The association $G \rightarrow \mathbb{A}^G$ is functorial in the category Red of reductive groups with central isogenies.*

Proof. If $f : G \rightarrow G'$ is a central isogeny, it sends a Borel subgroup $B \subset G$ onto a Borel subgroup $B' \subset G'$, inducing morphisms of their reductive quotients: $\mathbb{A}^G = B/N \rightarrow \mathbb{A}^{G'} = B'/N'$. If we choose another Borel subgroup $B_1 \subset G$, there is a $g \in G$ with $gBg^{-1} = B_1$, hence $f(g)B'f(g)^{-1} = B'_1 := f(B_1)$, and the morphism $\mathbb{A}^G \rightarrow \mathbb{A}^{G'}$ induced from $B_1 \rightarrow B'_1$ is the same as before, given how we have identified \mathbb{A}^G as the quotient of *any* Borel subgroup. \square

Theorem 1.14. *The assignment $G \mapsto \Psi^+(\mathbb{A}^G)$ is a functor $\text{Red} \rightarrow (\text{RD}^+)^{\text{op}}$, in the notation of Definition 1.8. It induces a bijection on isomorphism classes of objects, and for any morphism $\Psi^+(\mathbb{A}^G) \rightarrow \Psi^+(\mathbb{A}^{G'})$ there is a morphism $G' \rightarrow G$, unique up to inner automorphisms.*

Proof. This follows from Theorem 1.10, and the conjugacy of Cartan subgroups. \square

1.15. Simply connected and adjoint groups.

Definition 1.16. Let $(X, \Phi, \check{X}, \check{\Phi})$ be a root datum, let $R \subset X$, $\check{R} \subset \check{X}$ be the subgroups spanned by the roots, resp. coroots, and let $P = \check{R}^*$, $\check{P} = R^*$ the dual lattices.

The root datum is called *semisimple* if \mathcal{R} is of finite index in X .

Assume this to be the case, so that we have containments with finite quotients $P \supset X \supset R$ and $\check{P} \supset \check{X} \supset \check{R}$. We say that the root datum is *simply-connected* if $X = P$, equivalently $\check{X} = \check{R}$, and *adjoint* if $X = \mathcal{R}$, equivalently $\check{X} = \check{P}$.

Definition 1.17. A reductive group is *adjoint* if it has trivial center, and *simply connected* if it admits no nontrivial central isogeny from a connected smooth group over k .

The relationship between this notion of being simply connected, and the algebrogeometric one, will become clear in Proposition 1.19 and Remark 1.20.

Remark 1.18. The “center” mentioned in Definition 1.17 needs to be understood as a group scheme. For example, the center of SL_p in characteristic p is the non-trivial group scheme μ_p of p -th roots of unity (which has only one point over \mathbb{F}_p).

Proposition 1.19. *Let (G, T) be a connected reductive group with a maximal torus over k , and let $\Psi(G, T)$ be the associated root datum.*

- (1) *G is semisimple iff $\Psi(G, T)$ is semisimple.*
- (2) *G is adjoint iff $\Psi(G, T)$ is adjoint.*
- (3) *G is simply connected (in the sense of Definition 1.17) iff $\Psi(G, T)$ is simply connected.*
- (4) *In characteristic zero, G is simply connected as a scheme (or, equivalently, $G(\mathbb{C})$ is simply connected as a topological space when $k = \mathbb{C}$) iff $\Psi(G, T)$ is simply connected.*

Proof. (1) A reductive group is semisimple iff its center is finite. The center belongs to T , and coincides with the common kernel of all roots. In other words, in terms of the equivalence of diagonalizable groups and finitely generated abelian groups (Theorem 1.2), the character group of the center is the cokernel of the inclusion $R \rightarrow X$. Hence, it is finite iff R is of finite index in X .

(Notice that the center is not necessarily reduced; for example, the center of SL_p in characteristic p is the non-reduced group scheme μ_p of p -th roots of unity.)

- (2) Continuing along the same lines, the center is trivial iff $R = X$.
- (3) A central isogeny $G' \rightarrow G$ which is not an isomorphism, mapping some maximal tori $T' \rightarrow T$, induces an isogeny $\Psi(G, T) \rightarrow \Psi(G', T')$, which is completely determined by the map of cocharacter groups $X_*(T') \rightarrow X_*(T)$. This map is injective, of finite cokernel, and has to preserve coroot lattices, so if $X_*(T)$ is equal to the coroot lattice, the map is an isomorphism.
- (4) By [BS13, Theorem 1], if p is the characteristic exponent of the field, every prime-to- p étale Galois cover $G' \rightarrow G$ is a central isogeny. In particular, in characteristic zero, G is simply connected in the sense of Definition 1.17 iff it is simply connected in the sense of étale topology. Moreover, if $k = \mathbb{C}$, the étale fundamental group is the profinite completion of the topological fundamental group of $G(\mathbb{C})$.

□

Remark 1.20. In positive characteristic, the étale fundamental group is always infinite, for a smooth affine scheme $X = \mathrm{Spec}(R)$ of positive dimension. For example, we have the Artin–Schreier \mathbb{Z}/p -covers, $X' = \mathrm{Spec} R[y]/(y^p - y - f)$, if $f \in R$ is chosen appropriately.

1.21. Automorphisms. Given an automorphism $f : X \rightarrow X$ of an affine variety X over a field k , its defining homomorphism $f^* : k[X] \rightarrow k[X]$ restricts to an

automorphism $V \rightarrow V$ on a generating, finite-dimensional subspace V of $k[X]$. Therefore, the automorphism group $\text{Aut}(X)$ has a natural structure as the ind-algebraic group

$$\varinjlim_V \text{Aut}(X, V),$$

where V runs over all finite-dimensional, generating subspaces of $k[X]$, and $\text{Aut}(X, V) \subset \text{GL}(V)$ is the subgroup of those automorphisms of V (as a vector space) which induce an automorphism of X (as an algebraic variety).

Definition 1.22. The *group of inner automorphisms*, $\text{Inn}(G)$, is the image of the natural morphism $G \rightarrow \text{Aut}(G)$ given by the conjugation action of G on itself. It is also called the *adjoint group* of G , and denoted by G_{ad} .

The quotient $\text{Aut}(G)/\text{Inn}(G)$ is the group of *outer automorphisms* of G .

The kernel of the map $G \rightarrow \text{Aut}(G)$ is the center of G , so the group $G_{\text{ad}} = \text{Inn}(G)$ is the quotient of G by its center.

For reductive groups the automorphism group is of finite type. This follows from the fact that, by the classification of reductive groups in terms of root data, the outer automorphism group is the group of automorphisms of based root data.

Proposition 1.23. *The functor from reductive groups to based root data of Theorem 1.14 gives rise to a short exact sequence of (ind-)algebraic groups:*

$$(1.23.1) \quad 0 \rightarrow \text{Inn}(G) \rightarrow \text{Aut}(G) \rightarrow \text{Aut}\Psi^+(\mathbb{A}^G) \rightarrow 0.$$

In particular, $\text{Out}(G) = \text{Aut}\Psi^+(\mathbb{A}^G)$.

Moreover, any pinning on G (Definition 1.11) gives rise to a splitting $\text{Out}(G) \rightarrow \text{Aut}(G)$, characterized by the fact that its image consists of the automorphisms preserving the pinning.

Proof. Applying the functor of Theorem 1.14 to automorphisms of G , we get a homomorphism $\text{Aut}(G) \rightarrow \text{Aut}\Psi^+(\mathbb{A}^G)$. By Theorem 1.14, this homomorphism is surjective, and its kernel is precisely the group of inner automorphisms.

The existence of a unique splitting that fixes a given pinning follows from Theorem 1.12. \square

2. Forms and Galois descent

Definition 2.1. Let G be a linear algebraic group over a field k or over an extension L/k . A *form* of G over k is a linear algebraic group G' over k , such that $G_L \simeq G'_L$.

The goal of this section is to explain how isomorphism classes of forms of G are described by the Galois cohomology set $H^1(\Gamma, \text{Aut}(G_{k^s}))$, where $\Gamma = \text{Gal}(k^s/k)$, the Galois group of the separable closure of k . There is nothing special about classification of linear algebraic groups here — the same arguments apply to any category of objects that satisfy *effective descent*. We introduce these notions in some generality, but not the full generality of faithfully flat descent (see [Sta19, Tag 0238]).

Let Aff_k be the category of affine schemes over the field k . A *k -groupoid* is a category with a functor $\mathcal{F} \rightarrow \text{Aff}_k$ satisfying two axioms, which can be summarized as “base change exists”, and “fiber categories are groupoids”. To formulate them, denote by \mathcal{F}_U the full subcategory living over an object $U \in \text{Aff}_k$. The axioms are:

- (1) Given $U \in \text{Aff}_k$, $u \in \mathcal{F}_U$, and $f : V \rightarrow U$ a morphism, there exists a morphism $\tilde{f} : v \rightarrow u$ in \mathcal{F} lying over f .
- (2) For any pair of morphisms $\tilde{f} : u \rightarrow z$, $\tilde{g} : v \rightarrow z$ in \mathcal{F} , lying over $f : U \rightarrow Z$ and $g : V \rightarrow Z$ in Aff_k , and any $h : U \rightarrow V$ with $f = g \circ h$, there exists a unique lift $\tilde{h} : u \rightarrow v$ with $\tilde{f} = \tilde{g} \circ \tilde{h}$.

If v is as in the first axiom, we will feel free to write $v = u_V$, i.e., as a base change of u . Notice that, for every other choice v' , the second axiom provides a canonical isomorphism $v' \rightarrow v$, corresponding to the identity map on V . For $f : V \rightarrow U$ and $u \in \mathcal{F}_U$, one has a unique morphism $u_V \rightarrow u'_V$ over any morphism $\phi : u \rightarrow u'$; we will be denoting that by $f^*\phi$. For a composition $W \rightarrow V \rightarrow U$, we will feel free to identify $(u_V)_W$ with u_W — see [Sta19, Tag 02XN] for a clarification of these issues.

Now consider a Grothendieck topology on Aff_k , turning it into a site. For notational simplicity, we will represent every cover as a single morphism $U \rightarrow X$, instead of a family of morphisms. The groupoid $\mathcal{F} \rightarrow \text{Aff}_k$ is called a *prestack* (of groupoids) if, for any $U \in \text{Aff}_k$, and any $x, y \in \mathcal{F}_U$, the presheaf $V \mapsto \text{Mor}(x_V, y_V)$ is a sheaf on the site Aff_U , and it is called a *stack* (of groupoids) if *descent data are effective*.

These are two conditions that need to be formulated for arbitrary covers, but we just explain their meaning for the case of covers corresponding to finite Galois extensions L/k with Galois group Γ ; the generalization to arbitrary U and arbitrary covers is straightforward, see [Sta19, Tag 0268].

Denote $U = \text{Spec}(L) \rightarrow X = \text{Spec}(k)$, and notice that there is a canonical isomorphism $U \times_X U \simeq \Gamma \times U$, equivariant with respect to the action of Γ on the first copy, taking $U \times \{1\}$ to the diagonal. A *descent datum* for L/k consists of an object $u \in \mathcal{F}_U$, together with (iso)morphisms $\phi_\gamma : \gamma u \rightarrow u$ (over the *identity* of U), where γu is the base change of u via the map $\gamma^{-1} : U \rightarrow U$, and the ϕ_γ 's are required to satisfy the cocycle condition $\phi_{\gamma_1 \gamma_2} = \phi_{\gamma_1} \circ \gamma_1^* \phi_{\gamma_2}$. Equivalently, a descent datum is an isomorphism $\phi : u_1 \rightarrow u_2$, where u_i is the pullback via the i -th projection $U \times_X U \rightarrow U$, and the isomorphism is required to satisfy the natural cocycle condition with respect to the triple product $U \times_X U \times_X U$.

Let $\mathcal{F}_{L/k}$ denote the category of descent data (u, ϕ) . (The morphisms in this category are the obvious ones, morphisms of the u 's commuting with the ϕ 's.) There is a canonical functor

$$(2.1.1) \quad \mathcal{F}_k \rightarrow \mathcal{F}_{L/k}$$

taking an object $x \in \mathcal{F}_k = \mathcal{F}_X$ to its base change x_L , together with the descent data induced from the Galois automorphisms of $U = \text{Spec}(L)$ over k . (Notice that a morphism $\gamma u \rightarrow u$ over the identity in U is equivalent to a morphism $u \rightarrow u$ over the map $\gamma : U \rightarrow U$.)

Then, \mathcal{F}_k satisfies *effective descent* (together with the prestack condition) with respect to the Galois cover $L \rightarrow k$, if the functor (2.1.1) is an *equivalence*. Here are two basic examples where it happens:

Example 2.2. The category \mathcal{F}_k of vector spaces over k satisfies effective descent with respect to any separable extension L/k . (It can be extended to a k -stack \mathcal{F} by considering vector bundles over an arbitrary basis.)

Example 2.3. The category \mathcal{F}_k of algebras over k satisfies effective descent with respect to L/k . Indeed, let B be an L -algebra. The descent datum amounts to an isomorphism $\phi : B \otimes L \simeq L \otimes B$ (tensor products over k), and the cocycle condition

states that the triangle consisting of $B \otimes L \otimes L$, $L \otimes B \otimes L$ and $L \otimes L \otimes B$, applying ϕ to get isomorphisms between them, commutes. We then take our algebra A over k to consist of those sections of B whose pullbacks to $L \otimes B \xrightarrow{\phi} B \otimes L$ (geometrically, in the notation above: whose pullbacks to $U \times_X U$ under both projections) coincide, i.e., $A = \ker(B \rightarrow L \otimes B)$, where the map is given by $b \mapsto 1 \otimes b - \phi(b \otimes 1)$. One checks that A is a k -algebra, and $B \simeq A \otimes L$ compatibly with descent data, see [Sta19, Tag 0244].

Example 2.4. Galois descent is also satisfied for any quasi-projective scheme, see [BLR90, §6.2, Example B].

Example 2.3 is the basic one for our purposes. It immediately extends to groupoids of affine varieties with extra structure given by “closed” conditions, such as algebraic groups.

Finally, descent data can be parametrized by torsors for the automorphism group:

Proposition 2.5. *Given $(u, \phi) \in \mathcal{F}(L/k)$, let $A = \text{Aut}_{\mathcal{F}_L}(x)$, considered as a Γ -module (possibly non-abelian) with action $\gamma a := \phi_\gamma \circ a \circ \phi_\gamma^{-1}$. Then, we have a natural equivalence of categories*

$$\mathcal{F}(L/k)/u \xrightarrow{\sim} A - \text{Tors}^\Gamma,$$

where $\mathcal{F}(L/k)/u$ denotes the category of descent data (u', ϕ') with $u' \simeq u$, and $A - \text{Tors}^\Gamma$ denotes the category of (set-theoretic) right A -torsors T with a compatible left Γ -action, i.e., $\gamma(t \cdot a) = \gamma t \cdot \gamma a$ for all $\gamma \in \Gamma, a \in A$, and $t \in T$.

The functors are:

- $(u', \phi') \rightarrow \text{Isom}(u, u')$, and
- $T \rightarrow (u' = T \times^A u, \phi')$, where $\phi' = \{\phi'_\gamma\}$ is obtained by acting diagonally on T (via the given Galois action) and on u (via ϕ).

Proof. Left to the reader. □

If Γ is a finite group, and A is a (possibly non-abelian) group with a Γ -action, then isomorphism classes of A -torsors with a Γ -action are naturally parametrized by the 1st cohomology set $H^1(\Gamma, A)$, which is the pointed set of A -orbits on the set of 1-cocycles $Z^1(\Gamma, A) = \{c : \Gamma \rightarrow A \mid c(\gamma_1 \gamma_2) = c(\gamma_1) \cdot {}^{\gamma_1} c(\gamma_2)\}$, where A acts by twisted conjugation $(a \cdot c)(\gamma) = ac(\gamma)^\gamma a^{-1}$. The parametrization takes a cocycle c to the A -torsor T which can be identified with A as a right- A -set, but with Galois action twisted by c , i.e., if we let a' be the element of T corresponding to an element a of A , then ${}^\gamma a' = c(\gamma) \cdot \gamma a$.

Thus, we obtain:

Proposition 2.6. *The equivalence of 2.5 induces an isomorphism of pointed sets:*

$$(\mathcal{F}(L/k)/u)^\sim \xrightarrow{\sim} H^1(\Gamma, A),$$

where the left-hand side denotes isomorphism classes of descent data for u over L/k , with the chosen descent datum as the base point.

Proof. One just needs to check that isomorphism classes of Γ -equivariant A -torsors are indeed parametrized by $H^1(\Gamma, A)$. Left to the reader. □

Remark 2.7. Notice that, although Proposition 2.5 classifies a family of descent data depending only on the isomorphism class of u , the classification, and the Galois structure on the automorphism group A , depend on the chosen descent structure on u . This choice makes the isomorphism classes of descent data into a pointed set, which corresponds to the distinguished point (the class of the trivial torsor) in $H^1(\Gamma, A)$.

3. Forms of reductive groups

Lemma 3.1. *If G, G' are two reductive groups over a field k that are isomorphic over the algebraic closure \bar{k} , they are isomorphic over a finite separable extension $L \subset k^s$.*

Proof. This is a corollary of the existence of maximal \bar{k} -tori over k , Theorem 4.7, the fact that those split over a finite separable extension, Theorem 1.2, and the classification of split reductive groups in terms of root data, Theorem 1.14. \square

By “continuous descent data” over a separable closure k^s , in a k -groupoid \mathcal{F} as above, we will mean pairs (u, ϕ) consisting of $u \in \mathcal{F}_{k^s}$ and isomorphisms $\phi_\gamma : \gamma u \rightarrow u$ for $\gamma \in \Gamma = \text{Gal}(k^s/k)$, such that *these descent data are induced from descent data over a finite Galois extension L/k* , in the obvious way (i.e., u is the base change of some object $u' \in \mathcal{F}_L$, and ϕ is obtained by extending scalars from some descent datum ϕ' over L).

Proposition 3.2. *The category of reductive groups over k is equivalent to the category of continuous descent data (G, ϕ) , where G is a reductive group over a fixed separable closure k^s , and ϕ is an isomorphism $G \times_k k^s \simeq k^s \times_k G$.*

Given a reductive group G over k , the set of isomorphism classes of forms G' of G over k is in natural bijection with the Galois cohomology set $H_{\text{cont}}^1(\Gamma, \text{Aut}(G_{k^s}))$, defined as the set of $G(k^s)$ -orbits of continuous 1-cocycles $\Gamma \rightarrow \text{Aut}(G_{k^s})$ (i.e., factoring through a finite extension).

Proof. The first statement follows from Lemma 3.1, and effectiveness of descent for linear algebraic groups (an easy corollary of Example 2.3). The second follows from Proposition 2.6. \square

Thus, we are led to study the cohomology of the automorphism group of G (over the separable closure). From now on, for every linear algebraic group over k , we will be writing simply $H^1(\Gamma, G)$ for the continuous cohomology set $H_{\text{cont}}^1(\Gamma, G(k^s))$.

Proposition 3.3. *Given a reductive group G over a field k , there is an exact sequence*

$$(3.3.1) \quad H^1(\Gamma, \text{Inn}(G)) \rightarrow H^1(\Gamma, \text{Aut}(G)) \rightarrow H^1(\Gamma, \text{Out}(G)) \rightarrow 0.$$

Moreover, if we assume that G is split over k , and we fix a pinning $(G, B, T, \{p_\alpha\}_{\alpha \in \Delta})$ over k (Definition 1.11), the resulting splitting $\text{Out}(G) \rightarrow \text{Aut}(G)$ of Proposition 1.23 induces a splitting

$$H^1(\Gamma, \text{Out}(G)) \rightarrow H^1(\Gamma, \text{Aut}(G))$$

whose image corresponds to the quasisplit forms of G .

Moreover, the cohomology groups of (3.3.1), together with this splitting, classify isomorphism classes in the following sequence of categories:

$$\{G_{\text{ad}}\text{-torsors over } k\} \rightarrow \{\text{forms of } G \text{ over } k\} \rightrightarrows \{\text{quasisplit forms of } G \text{ over } k\},$$

compatibly with the inclusion functor on the right, and the functor that assigns to a right G -torsor T the G -automorphism group $\text{Aut}^G(T)$.

Proof. The sequence (3.3.1) follows immediately by applying the long exact sequence of cohomology to the short exact sequence of (1.23.1). To prove surjectivity of the map to $H^1(\Gamma, \text{Out}(G))$, it is enough to assume that G is split, since the choice of a different form only affects the base point. (Such a form always exists by Theorem 1.10.) Then, the splitting $\text{Out}(G) \rightarrow \text{Aut}(G)$ of Proposition 1.23 gives rise to a splitting of the corresponding cohomology groups, and in particular proves surjectivity. Finally, the identification of cohomology groups with isomorphism classes of objects in the stated categories follows from effective descent and Proposition 2.6. In all cases, the stated groups are the groups of isomorphisms of a given object in the stated category (for example, $G_{\text{ad}} = \text{Inn}(G)$ is the automorphism group of a G_{ad} -torsor), except for the category of quasispit forms of G over k , which has more automorphisms than the outer automorphisms of G , and therefore requires some explanation.

By Proposition 1.23, $\text{Out}(G)$ can be identified with automorphisms of the based root datum $\Psi^+(\mathbb{A}^G)$. By Theorem 1.12, this can be identified with the automorphism group of a pinned quadruple (G, B, T, p) (over k^s) or, equivalently, a quadruple (G, B, T, ℓ) , where ℓ is an algebraic Whittaker datum for B (Definition 1.11). Such quadruples make sense over k , as well (and its Galois extensions), thus, by Galois descent, the set $H^1(\Gamma, \text{Out}(G))$ classifies isomorphism classes of such quadruples over k . But the forgetful map $(G, B, T, \ell) \rightarrow (G, B)$ is a bijection on isomorphism classes, because every pair (G, B) admits a maximal torus $T \subset B$ and a Whittaker datum ℓ , and any two such are conjugate by an element of $B_{\text{ad}} \subset G_{\text{ad}}$. [More details to be added.]

□

Definition 3.4. A *pure inner form* of an algebraic group G over k is a G -torsor T ; the term is often used to refer to the G -automorphism group of T , but with the understanding that a G -torsor has been fixed. An *inner form* of G is a pure inner form R for the adjoint group $G_{\text{ad}} = \text{Inn}(G)$; the term is often used to refer to the form $R \times^{\text{Inn}(G)} G$, but with the understanding that a G_{ad} -torsor has been fixed.

4. The L -group and the C -group

Let G be a reductive group over a field k , fix a separable closure k^s of k with Galois group Γ , and let $\Psi^+(G)$ be the associated based root datum of G_{k^s} , by Theorem 1.14. Since the universal Cartan \mathbb{A}^G is defined over k (Proposition 5.3), and has a canonical set of positive roots, there is an action of the Galois group Γ on its character group which preserves the positive roots, hence an action

$$(4.0.1) \quad \Gamma \rightarrow \text{Aut}(\Psi^+(G)).$$

Remark 4.1. By Proposition 1.23, we have a canonical isomorphism $\text{Out}(G) \simeq \text{Aut}(\Psi^+(G))$. Viewed as a Galois cocycle into $\text{Aut}(\Psi^+(G))$ (with trivial Galois action, corresponding to the split form of G), the corresponding cohomology class in $H^1(\Gamma, \text{Out}(G))$ is the one attached to the outer class of G by Proposition 3.3.

Notice that the universal Cartan groups for inner forms are canonically isomorphic, in an order-preserving way, so that the action (4.0.1), hence also the L -group that we are about to define, are identical for two groups that are inner forms of each other.

Definition 4.2. Let G be a reductive group over a field k , and fix a separable closure k_s with Galois group Γ . Let $\Psi^+(G)$ be the based root datum of G , and $\check{\Psi}^+(G)$ the dual based root datum, obtained by interchanging the character and the cocharacter lattices. The *Langlands dual group* \check{G} is the pinned group (provided by the equivalence of categories of Theorem 1.12) with based root datum $\check{\Psi}^+(G)$. The L -group is the semidirect product ${}^L G = \check{G} \rtimes \Gamma$, where Γ acts through (4.0.1), by the pinned automorphisms provided by the equivalence of Theorem 1.12.

Notice that the L -group can be thought of as being defined over any field, in fact over \mathbb{Z} . The ring of definition that one uses for the L -group depends on the coefficients of representations of G that one considers.

Remark 4.3. The choice of separable closure, and the definition of the L -group as a semidirect product (hence, a distinguished splitting $\Gamma \rightarrow {}^L G$) are not completely justified. It is better to think of the L -group as a *sheaf of pinned reductive groups over the étale site of $\text{Spec} F$* . Hence, instead of talking about “the” based root datum of G (defined using its weight lattice over a fixed separable extension), we have based root data for every separable extension over which G splits, and isomorphisms between them for every morphism of such fields. In particular, the descent data give rise to the Galois action.

Although the Langlands program with complex coefficients is often formulated in terms of the L -group, this is not the most natural dual group to consider, and in particular does not work well with integer coefficients. A more natural choice is the C -group, which we introduce now, following [BG14] and ideas of Joseph Bernstein. The description of [BG14] is combinatorial, but, following Bernstein, we will describe the C -group in terms of the Langlands dual of a natural extension of G .

Lemma 4.4. *Let G be a connected reductive group, \mathcal{B} its flag variety of Borel subgroups, Ω the canonical bundle (bundle of volume forms) on \mathcal{B} . Then, there is a canonical square root $\Omega^{\frac{1}{2}}$, i.e., a line bundle over \mathcal{B} whose square is Ω , characterized by the fact that it admits a linearization for the action of the simply connected cover G_{sc} of the derived group of G .*

We recall that “linearization” means an action of the group on the bundle, i.e., on its sheaf of sections, compatible with its action on the base, i.e., an isomorphism of the two pullbacks under the projection and action maps $G_{sc} \times \mathcal{B} \rightrightarrows \mathcal{B}$ which is compatible with the group structure.

Proof. By Galois descent and the uniqueness of the square root with this property, it is enough to assume that the field is algebraically closed. If $B \in \mathcal{B}$ is a Borel subgroup, 2ρ is the sum of positive roots of G , and we use exponential notation when thinking about the corresponding character $e^{2\rho} : B \rightarrow \mathbb{G}_m$, then Ω is a G -linear bundle under the right action of G on \mathcal{B} , and B acts by the character $e^{2\rho}$ on its fiber [exercise!]. In other words, Ω is induced from the character $e^{2\rho}$ of B ; its total space is $\mathbb{G}_{a,2\rho} \times^B G$, where $\mathbb{G}_{a,2\rho}$ is the line with an action of B by this character.

If B_{sc} is the corresponding Borel of G_{sc} , then $\mathcal{B} = B \backslash G = B_{sc} \backslash G_{sc}$, and the restriction of 2ρ to B_{sc} admits a unique square root, namely ρ . Thus, there is a unique G_{sc} -linear bundle $\Omega^{\frac{1}{2}}$ over \mathcal{B} whose square is Ω . \square

Definition 4.5. Let G be a connected reductive group, \mathcal{B} its flag variety of Borel subgroups, $\Omega^{\frac{1}{2}}$ the canonical square root of the canonical bundle on \mathcal{B} (Lemma 4.4). The *canonical extension* \tilde{G} is the group of pairs (g, τ) , where $g \in G$ and τ is an isomorphism: $g^* \Omega^{\frac{1}{2}} \xrightarrow{\sim} \Omega^{\frac{1}{2}}$.

Remark 4.6. Notice that we consider G as acting on the right on \mathcal{B} , so pullback is a left action on line bundles: $(g_1 g_2)^* \Omega = g_1^* (g_2^* \Omega)$. The composition law on the extended group is $(g_1, \tau_1) \cdot (g_2, \tau_2) = (g_1 g_2, \tau_1 \circ (g_1^* \tau_2))$. The extended group in a central linear algebraic extension:

$$(4.6.1) \quad 1 \rightarrow \mathbb{G}_m \rightarrow \tilde{G} \rightarrow G \rightarrow 1,$$

where the central \mathbb{G}_m is the group of scalar automorphisms of the line bundle $\Omega^{\frac{1}{2}}$. It acts on \mathcal{B} through the quotient G , and, by definition, *the line bundle $\Omega^{\frac{1}{2}}$ is canonically \tilde{G} -linear*.

On the other hand, if G is simply connected, then, by the definition of $\Omega^{\frac{1}{2}}$, it already has a (unique) G -linearization; hence, in this case, $\tilde{G} = \mathbb{G}_m \times G$, canonically.

Definition 4.7. For a reductive group G over a field k (in a fixed separable closure with Galois group Γ), the *extended Langlands dual group* $\check{\tilde{G}}$ of G is the Langlands dual group (Definition 4.2) of the canonical extension \tilde{G} (Definition 4.5), and the *C -group* ${}^C G$ is the L -group of the canonical extension.

The C -group comes with a canonical character dual to the sequence (4.6.1):

$$(4.7.1) \quad 1 \rightarrow {}^L G \rightarrow {}^C G \rightarrow \mathbb{G}_m \rightarrow 1,$$

which will be called the *cyclotomic character* of the C -group.

Remark 4.8. In terms of based root data, if X denotes the weight lattice in $\Psi^+(G)$, and \tilde{X} the coweight lattice, the weight lattice \tilde{X} of \tilde{G} is generated inside of the vector space $X_{\mathbb{Q}} \oplus \mathbb{Q}$ by X and the element $\tilde{\rho} = (\rho, 1)$; its projection $\mathbb{Z} \subset \mathbb{Q}$ is the weight lattice of the central \mathbb{G}_m , and $\tilde{\rho}$ is the character from which the \tilde{G} -linear line bundle $\Omega^{\frac{1}{2}}$ is induced.

Notice that the element $e^{2\rho}(-1)$ is a canonical central element of the dual group $\check{\tilde{G}}$ (possibly trivial), thus defining a map $\mu_2 \rightarrow \check{\tilde{G}}$. The extended dual group $\check{\tilde{G}}$ is the group $\check{\tilde{G}} \times^{\mu_2} \mathbb{G}_m$.

5. The real case: compact Lie groups and Lie algebras

The main reference for this section is [Bou05, Ch. IX §1].

Definition 5.1. Let \mathfrak{g} be a finite-dimensional real or complex Lie algebra. The *group of inner automorphisms* $\text{Inn}(\mathfrak{g})$ is the connected immersed subgroup of $\text{GL}(\mathfrak{g})$ (see 4.9) whose Lie algebra is $\text{ad}(\mathfrak{g})$.

For compact or semisimple Lie groups, this coincides with the group $\mathcal{E}(\mathfrak{g})$ of Definition 5.1, but we will not prove that.

Proposition 5.2. *Let \mathfrak{g} be a (finite-dimensional) real Lie algebra. The following conditions are equivalent:*

- (1) \mathfrak{g} is isomorphic to the Lie algebra of a compact Lie group.
- (2) The group $\text{Inn}(\mathfrak{g})$ is compact.
- (3) \mathfrak{g} has a positive definite, invariant symmetric bilinear form.

- (4) The adjoint representation of \mathfrak{g} is semisimple, and for every $x \in \mathfrak{g}$, $\text{ad}(x)$ is semisimple with purely imaginary eigenvalues.
- (5) \mathfrak{g} is a direct sum of its center and its maximal semisimple ideal, and the Killing form is negative semi-definite.

If, moreover, $\mathfrak{g} = \text{Lie}(G)$ for some connected Lie group G , the above are equivalent to any of the following:

- (6) The group $\text{Ad}(G) \subset \text{GL}(\mathfrak{g})$ is compact.
- (7) There is a Riemannian metric on G invariant under left and right translations.

Finally, in that case the exponential map $\mathfrak{g} \rightarrow G$ is surjective.

Proof. (1) \Rightarrow (2): If $\mathfrak{g} = \text{Lie}(G)$ with G compact and connected, then $\text{Inn}(\mathfrak{g}) = G/Z$, where Z is the center, hence is compact.

(2) \Rightarrow (3): Choose any positive definite symmetric bilinear form on \mathfrak{g} , and average over the compact group $\text{Inn}(G)$ to obtain an invariant one.

(3) \Rightarrow (4): The orthogonal complement, with respect to a definite invariant form, of an invariant subspace is invariant, which proves semisimplicity. The operators $\text{ad}(x)$ are then anti-self-adjoint, which implies that they are diagonalizable with purely imaginary eigenvalues.

(4) \Rightarrow (5): If the adjoint representation is semisimple, \mathfrak{g} is the direct sum of its center and its simple ideals. If B denotes the Killing form, then $B(x, x) = \text{tr}(\text{ad}(x)^2)$, and since the eigenvalues are all imaginary, this is ≤ 0 .

(5) \Rightarrow (1): The center of the Lie algebra is isomorphic to the Lie algebra of a compact real torus, so we are reduced to the case that \mathfrak{g} is semisimple, with negative Killing form. Then, $\text{Inn}(\mathfrak{g})$ is a closed [reference] subgroup of $\text{GL}(\mathfrak{g})$, and since it preserves the negative definite form B , it belongs to the compact orthogonal subgroup $O(B)$, thus is compact (with Lie algebra \mathfrak{g}).

If $\mathfrak{g} = \text{Lie}(G)$, with G connected, then $\text{Ad}(G) = \text{Inn}(\mathfrak{g})$, as one can confirm by comparing Lie algebras. The last claim follows from the rest by constructing an $\text{Ad}(G)$ -invariant positive definite form on \mathfrak{g} (by the same averaging argument as above), and translating it by the left (or right) G -action. Vice versa, any invariant Riemannian metric, restricted to the tangent space at the identity, is such a form.

Finally, one can check [Exercise!] that, for a left- and right-invariant Riemannian metric, the group-theoretic exponential map coincides with the Riemannian exponential map, hence G is geodesically complete, and any two points can be joined by a length-minimizing geodesic (the *Hopf–Rinow theorem*). \square

Definition 5.3. A real Lie algebra \mathfrak{g} satisfying the equivalent conditions of Proposition 5.2 is called a *compact Lie algebra*.

Theorem 5.4. The Lie algebra of a connected Lie group G is compact if and only if G is the surjective image of a morphism $V \times T \times K \rightarrow G$ with finite kernel, where V is a vector space, T is a compact real torus, and K is a compact semisimple Lie group.

Proof. The direction \Leftarrow follows from Proposition 5.2.

As seen in Proposition 5.2, a compact Lie algebra is the direct sum of its center and a semisimple compact Lie algebra, and the connected component of the center has to be equal to a group of the form $V \times T$ as above, therefore we are reduced to the case when \mathfrak{g} is compact and semisimple.

In that case, the center Z is discrete, and $G/Z = \text{Ad}(G)$ is compact by Proposition 5.2, hence it suffices to prove that the group $G_{\text{ad}} = G/Z$ cannot be the image of a central isogeny with infinite kernel from a connected Lie group. Let $H \rightarrow G_{\text{ad}}$ be a central isogeny of finite degree. By Theorem 7.5, H and G_{ad} coincide with the real points of algebraic groups over \mathbb{R} , and the morphism between them is algebraic. Thus, we can apply the analysis of central isogenies of Theorem 1.10, and deduce that, since \mathfrak{g} is semisimple, the degree of any central isogeny is bounded by the index of the coroot lattice in the cocharacter lattice. Thus, G is compact. \square

Proposition 5.5. *Any semisimple Lie algebra \mathfrak{g} over \mathbb{C} has a unique split real form up to $(\text{Inn}(\mathfrak{g})$)-conjugacy, and a unique compact form up to conjugacy.*

Proof. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra, and let \mathfrak{h}_0 be the real subspace spanned by the images of \mathbb{R} under the cocharacters $\check{\Phi} \ni \check{\alpha} : \mathbf{G}_a \rightarrow \mathfrak{h}$. The Chevalley construction (omitted) shows that it extends to a real form \mathfrak{g}_0 of \mathfrak{g} , such that, for every root α , $\mathfrak{g}_{\alpha,0} := \mathfrak{g}_0 \cap \mathfrak{g}_\alpha$ is a real form of the root space \mathfrak{g}_α . Those real forms, for α in a basis Δ of the root system, determine the form, and in turn are determined by a pinning (Definition 1.11) relative to Δ . All such pinnings are conjugate under the image of $\exp(\mathfrak{h})$ in $\text{Inn}(\mathfrak{g})$, hence by the conjugacy of Cartan subalgebras we conclude that all split forms are conjugate.

Starting now with such a split form, and a pinning $\mathfrak{g}_\alpha \simeq \mathbb{G}_a$ for $\alpha \in \Delta$, let $X_\alpha \in \mathfrak{g}_\alpha$ be the element corresponding to $1 \in \mathbb{G}_a$, and let $X_{-\alpha} \in \mathfrak{g}_{-\alpha}$ be the element forming an \mathfrak{sl}_2 -triple $(h_\alpha, X_\alpha, X_{-\alpha})$, with $h_\alpha \in \mathfrak{h}$ the coroot. We can similarly choose elements $X_\alpha \in \mathfrak{g}_\alpha$ for all roots, such that $[X_\alpha, X_\beta] = N_{\alpha,\beta} X_{\alpha+\beta}$ whenever $\alpha + \beta$ is also a root, with constants $N_{\alpha,\beta}$ satisfying $N_{\alpha,\beta} = N_{\beta,\alpha}$. (This is part of the Chevalley construction, see [Bou05, Ch. VIII §2].) Then, we define another real form of \mathfrak{g} by

$$\mathfrak{g}_c = i\mathfrak{h}_0 \oplus \bigoplus_{\{\pm\alpha\} \in \Phi/\{\pm 1\}} (\mathbb{R}(X_\alpha + X_{-\alpha}) \oplus i\mathbb{R}(X_\alpha - X_{-\alpha})).$$

One checks that this is a compact form, [Bou05, Ch. IX §3.2]. Obviously, this depends only on the pinning, not on the choices of X_α for α not simple, and is generated by the summands corresponding to simple roots α .

For any other compact form \mathfrak{g}'_c , we will show that it is of this form; by the conjugacy of pinnings, we will then conclude that \mathfrak{g}'_c is G -conjugate to \mathfrak{g}_c .

To lighten notation we may already use the conjugacy of Cartan subalgebras and assume that \mathfrak{g}'_c contains a form \mathfrak{h}'_c of \mathfrak{h} . By one of the equivalent conditions of Proposition 5.2, the eigenvalues for the adjoint action of \mathfrak{h}'_c on \mathfrak{g} have to be purely imaginary, so we must have $\mathfrak{h}'_c = i\mathfrak{h}_0$. Consider the semilinear automorphism $\tau : \mathfrak{g} \rightarrow \mathfrak{g}$ corresponding to complex conjugation with respect to \mathfrak{g}'_c . It acts by -1 on \mathfrak{h}_0 , hence interchanges the spaces \mathfrak{g}_α and $\mathfrak{g}_{-\alpha}$. If, for a simple $\alpha \in \Delta$ and $X_\alpha, X_{-\alpha}$ as above, we have $\tau(X_\alpha) = cX_{-\alpha}$, then we claim that $c > 0$.

Indeed, if B is the Killing form, $0 > B(X_\alpha, X_{-\alpha}) = c^{-1}B(X_\alpha, \tau(X_{-\alpha}))$, and the quadratic form $X \mapsto B(X, \tau(X))$ is negative definite on $\text{Res}_{\mathbb{C}/\mathbb{R}} \mathfrak{g}$, since B is a \mathbb{C} -linear quadratic form on \mathfrak{g} that is negative on the fixed space \mathfrak{g}'_c of the involution τ .

If we set $X'_\alpha = c^{-\frac{1}{2}}X_\alpha$, $X'_{-\alpha} = c^{\frac{1}{2}}X_{-\alpha}$, we obtain another \mathfrak{sl}_2 -triple $(h_\alpha, X'_\alpha, X'_{-\alpha})$ with $\tau(X'_\alpha) = X'_{-\alpha}$. Then, the form \mathfrak{g}'_c is generated by the elements $X'_\alpha + X'_{-\alpha}$ and $i(X'_\alpha - X'_{-\alpha})$ with $\alpha \in \Delta$, hence is of the stated form. \square

We deduce that any complex, connected semisimple group has a unique compact form up to conjugacy, and, in fact, arrive at the following strengthening of Theorem 7.5, whose formulation is taken from class notes of Brian Conrad:

Theorem 5.6. *The functor $G \mapsto G(\mathbb{R})$ is an equivalence between: the category of \mathbb{R} -anisotropic reductive \mathbb{R} -groups whose connected components have \mathbb{R} -points, and the category of compact Lie groups. If G is such an \mathbb{R} -group then $G^0(\mathbb{R}) = G(\mathbb{R})^0$. The \mathbb{R} -group G is semisimple if and only if $G(\mathbb{R})$ has finite center, and in such cases G^0 is simply connected in the sense of algebraic groups if and only if $G(\mathbb{R})^0$ is simply connected in the sense of topology.*

Proof. [To be added] □

6. Classification of real forms in terms of Cartan involutions

Let G be a connected, complex reductive group. Proposition 5.5 implies that it has a unique, up to conjugacy, split real form. [Details omitted for now — need to explain lift from Lie algebra to group.] Fixing that form, by Proposition 3.3 we obtain a bijection between isomorphism classes of real forms of G , and $H^1(\Gamma, \text{Aut}(G))$, where $\Gamma \simeq \mathbb{Z}/2$. Equivalently, forms correspond bijectively to sections

$$\mathbb{Z}/2 \rightarrow \text{Aut}(G) \rtimes \mathbb{Z}/2$$

up to $\text{Aut}(G)$ -conjugacy.

Here, we will discuss a more classical (and useful) description of real forms, in terms of actual homomorphisms

$$\mathbb{Z}/2 \rightarrow \text{Aut}(G)$$

up to $\text{Aut}(G)$ -conjugacy. This gives rise to an extremely important correspondence between real forms of G and *symmetric spaces*, that is, spaces of the form G/G^θ , where θ is an involution (automorphism of order two) of G .

Definition 6.1. If \mathfrak{g}_0 is a semisimple Lie algebra over \mathbb{R} , with complexification \mathfrak{g} and associated antiholomorphic involution σ , a *Cartan involution* for \mathfrak{g}_0 is a holomorphic involution θ , which commutes with σ , such that $\mathfrak{g}^{\theta\sigma}$ is a compact form of \mathfrak{g} (Definition 5.3).

Given such a Cartan involution, the associated *Cartan decomposition* is the pair $(\mathfrak{k}, \mathfrak{p})$ of complementary vector subspaces of \mathfrak{g}_0 , where $\mathfrak{k} = \mathfrak{g}_0^\theta$ (in particular, \mathfrak{k} is a compact Lie subalgebra) and $\mathfrak{p} = \mathfrak{g}_0^{-\theta}$. Since the pair $(\mathfrak{k}, \mathfrak{p})$ obviously determines the involution θ , we will say that $(\mathfrak{k}, \mathfrak{p})$ is a Cartan decomposition, without reference to θ .

A *Cartan decomposition* for a real Lie group G is a pair (K, \mathfrak{p}) , where $K \subset G$ is a compact subgroup, and $\mathfrak{p} \subset \mathfrak{g}$ is an $\text{Ad}(K)$ -stable subspace such that the map $K \times \mathfrak{p} \ni (k, X) \mapsto k \exp(X)$ is a diffeomorphism.

Given such a Cartan decomposition, the map $\theta(k \exp(X)) = k \exp(-X)$ (where $k \in K$, $X \in \mathfrak{p}$) is an involution of G , called the *Cartan involution*. Since θ determines $K = G^\theta$ and $\mathfrak{p} = \mathfrak{g}_0^{-\theta}$ uniquely, we will refer to such a θ as a Cartan involution, without reference to (K, \mathfrak{p}) .

Example 6.2. Let $G = \text{GL}_n$, $K = O(n)$, the compact orthogonal group of the standard inner product on \mathbb{R}^n , $\mathfrak{k} = \text{Lie}(K)$ and \mathfrak{p} = the subspace of symmetric matrices in $\mathfrak{g}_0 = \text{Lie}(G) = \mathfrak{gl}(n, \mathbb{R})$.

Then, the pair $(\mathfrak{k}, \mathfrak{p})$ is a Cartan decomposition of \mathfrak{g}_0 , and the pair (K, \mathfrak{p}) is a Cartan decomposition of G . Indeed, the quotient space G/K can be identified with the space of positive definite quadratic forms on \mathbb{R}^n , each represented by a symmetric matrix A with positive eigenvalues, and each such matrix A has a unique square root B of the same type, which in turn is the exponential of a unique symmetric matrix X , so $A = B^2 = \exp(X)^2 = \exp(X)\exp(X)^t$, uniquely.

The Cartan involution on G is $g \mapsto g^{-t}$, on \mathfrak{g} it is $X \mapsto -X^t$, and the associated compact form of $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ is the Lie algebra of the unitary group for the standard inner product on \mathbb{C}^n .

Proposition 6.3. *If G is a connected semisimple Lie group (i.e., with semisimple Lie algebra) with Lie algebra \mathfrak{g}_0 , $(\mathfrak{k}, \mathfrak{p})$ is a Cartan decomposition of \mathfrak{g}_0 , and $K = \exp(\mathfrak{k})$. If G has finite center, then K is a compact subgroup, and (K, \mathfrak{p}) is a Cartan decomposition of G .*

Proof. The proof reduces to the analogous statement for $G = \mathrm{GL}_n(\mathbb{R})$, $K = O(n)$, as follows:

First of all, let K' be the immersed Lie subgroup of G with Lie algebra \mathfrak{k} , see Proposition 4.9. Since \mathfrak{k} is compact, by Proposition 5.2, the exponential map $\mathfrak{k} \rightarrow K'$ is surjective, hence $K' = K$.

Let θ be the associated Cartan involution, let B be the Killing form, and consider the quadratic form $q : X \mapsto B(X, \theta X)$, which is negative definite on \mathfrak{g}_0 . One easily checks that, with respect to this form, $\mathrm{ad}(X)$ is

- symmetric, if $X \in \mathfrak{p}$;
- skew-symmetric, if $X \in \mathfrak{k}$.

Thus, $\mathrm{Ad}_g(K) \subset O(q) \subset \mathrm{GL}(\mathfrak{g})$, and by Example 6.2 the map $\mathrm{Ad}_g(K) \times \mathfrak{p} \ni (g, X) \mapsto g \exp(\mathrm{ad}(X)) \in \mathrm{GL}(\mathfrak{g})$ is a closed embedding. (Notice that ad has no kernel, by semisimplicity.) Its image is both open and closed in $\mathrm{Ad}(G)$, and by connectedness it is equal to $\mathrm{Ad}(G)$. Therefore, the map $K \times \mathfrak{p} \rightarrow G$ is an open and closed embedding, and again by connectedness it is an isomorphism. Note that K is compact, because by assumption the kernel of $K \rightarrow \mathrm{Ad}_g(K)$ is finite. \square

We will eventually see that any reductive Lie group admits a Cartan decomposition, but for now we restrict our attention to complex groups (considered, by restriction of scalars, as real Lie groups).

Proposition 6.4. *Let G be a complex, connected semisimple algebraic group, and $\mathfrak{k} \subset \mathfrak{g}$ a compact form of \mathfrak{g} , which exists, by Proposition 5.5, uniquely up to conjugacy. Let $\mathfrak{p} = i\mathfrak{k} \subset \mathfrak{g}$, and set $K = \exp(\mathfrak{k})$. Then, (K, \mathfrak{p}) is a Cartan decomposition of $G(\mathbb{C})$ (viewed as a real Lie group), in the sense of Definition 6.1. Moreover, the normalizer of K is $K \exp(iZ(\mathfrak{k}))$, where $Z(\mathfrak{k})$ denotes the center of \mathfrak{k} .*

Proof. Indeed, one immediately checks that, since the Killing form $B_{\mathfrak{g}}$ is negative definite on the real form \mathfrak{k} of \mathfrak{g} , the Killing form $B_{\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}}$ is negative definite on the real form $\mathfrak{k} \oplus i\mathfrak{p}$ of $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$, hence this is a compact form. The first claim now follows from Proposition 6.3, taking into account that complex semisimple groups have finite fundamental groups, and therefore finite center (see the argument in the proof of Theorem 5.4).

For the normalizer $\mathcal{N}_G(K)$ of K , it suffices to show that $\mathcal{N}_G(K) \cap \exp(\mathfrak{p}) = \exp(i\mathfrak{Z}(\mathfrak{k}))$. If $X \in \mathfrak{p}$ is such that $\exp(X)$ normalizes K , then $[X, \mathfrak{k}] \subset \mathfrak{k}$, but on the other hand $[\mathfrak{p}, \mathfrak{k}] \subset \mathfrak{p}$, so $[X, \mathfrak{k}] = 0$, i.e., $X \in \mathfrak{p}^\mathfrak{k} = (i\mathfrak{k})^\mathfrak{k} = i\mathfrak{Z}(\mathfrak{k})$. \square

Now we have the following, see also [AT18]:

Proposition 6.5. *Let G be a connected complex semisimple algebraic group. Then:*

- (1) *Any holomorphic, resp. antiholomorphic, involution of $G(\mathbb{C})$ is algebraic, i.e., induced by an algebraic involution of G , resp. a conjugate-linear involution of $\text{Res}_{\mathbb{C}/\mathbb{R}}G$.*
For the rest of the statements, write G for $G(\mathbb{C})$.
- (2) *If G is abelian, it admits a unique compact antiholomorphic involution τ (i.e., such that G^τ is compact). It commutes with all holomorphic or antiholomorphic involutions of G .*
- (3) *For any antiholomorphic involution σ of G there exists a holomorphic involution θ , commuting with σ , such that $G^{\theta\sigma}$ is compact, and vice versa: for every such σ there exists such a θ .*

Moreover, θ is unique up to $(G^\sigma)^0$ -conjugacy, and σ is unique up to $(G^\theta)^0$ -conjugacy.

Proof. For the first statement, see [AT18, Lemma 3.1].

If G is abelian, it is a torus, and it is easy to see that the only compact antiholomorphic involution τ is the one which, on \mathfrak{g} , acts by -1 on the real subspace E generated by the differentials of cocharacters, and by $+1$ on iE . Any other holomorphic or antiholomorphic involution has to preserve the cocharacter lattice, hence the eigenspaces E and iE for τ , hence commutes with τ .

Any involution of G induces involutions on its derived group G_{der} and its center $\mathcal{Z}(G)$, and since $G = \mathcal{Z}(G)G_{\text{der}}$, the rest of the statements are now reduced to G_{der} . Thus, we may assume that G is semisimple.

By Proposition 5.5, the set M of compact forms of \mathfrak{g} is a homogeneous space under G , and if we fix a compact form G_c , it can be identified with the quotient $G/G_c \exp(i\mathcal{Z}(\mathfrak{g}_c))$. The tangent space at the point $x = G_c$ can be identified with the quotient $\mathfrak{p}_{\text{ad}} = \mathfrak{p}/i\mathcal{Z}(\mathfrak{g}_c)$ of $\mathfrak{p} = i\mathfrak{g}_c$, and the Cartan decomposition, Proposition 6.3, shows that there is a well-defined exponential map

$$\exp_x : \mathfrak{p}_{\text{ad}} = T_x M \rightarrow M,$$

descending from the exponential map on \mathfrak{p} , which is an isomorphism.

Given σ , it induces an automorphism of order 2 of M , and our goal is to find a fixed point. Start with any point $x = G_c$, then σ induces an isomorphism σ_* between $T_x M$ and $T_{\sigma(x)} M$. We claim:

σ commutes with the exponential map, i.e., $\exp_{\sigma(x)}(\sigma_* X) = \sigma(\exp_x(X))$,
for every $X \in T_x M$.

Indeed, since σ is antiholomorphic, it maps $\mathfrak{p} = i\mathfrak{g}_c$ to $i\sigma(\mathfrak{g}_c)$, which is the space analogous to \mathfrak{p} for $\sigma(x)$.

Now, let $X \in T_x M$ be the unique element with $\sigma(x) = \exp_x(X)$, choose a preimage \tilde{X} of X in \mathfrak{p} , and set $g = \exp(\tilde{X})$ (the exponential in the group, here). Hence, $\sigma(x) = gx$, and translation by g identifies the tangent space of x with that of $\sigma(x)$, by a map which we will denote by g_* (and its inverse by g^*).

We claim that $g^* \sigma_*(X) = -X$. Indeed, $x = \sigma(\sigma(x)) = \sigma(\exp_x(X)) = \exp_{\sigma(x)}(\sigma_* X)$ by the claim above, and on the other hand $x = g^{-1}(\sigma(x)) = \exp(-\tilde{X})(\sigma(x)) =$

$\exp_{\sigma(x)}(g_*(-X))$, and by the fact that the map $\exp_{\sigma(x)}$ is an isomorphism, we deduce that $\sigma_*X = g_*(-X)$, or equivalently $g^*\sigma_*X = -X$.

Hence, the point $\exp_x(\frac{X}{2}) = \exp(\frac{\tilde{X}}{2})x$ is fixed under σ , because

$$\begin{aligned} \sigma(\exp_x(\frac{X}{2})) &= \exp_{\sigma(x)}(\sigma_*\frac{X}{2}) = \exp_{\sigma(x)}(-g_*\frac{X}{2}) = \\ &= \exp(-\frac{\tilde{X}}{2})\sigma(x) = \exp(-\frac{\tilde{X}}{2})\exp(\tilde{X})x = \exp(\frac{\tilde{X}}{2})x. \end{aligned}$$

The proof for θ in place of σ is identical.

Now we show that all points of M^σ lie in the same $(G^\sigma)^0$ -orbit. Given two such points x, x' , we can write $x' = \exp_x(X)$, for a unique element $X \in T_xM$. It is enough to show that $X \in (T_xM)^\sigma$, because then (by semisimplicity) it can be lifted to $\tilde{X} \in \mathfrak{p}^\sigma$, and then $\exp(\tilde{X})$ will belong to $(G^\sigma)^0$. Again, by the σ -equivariance of the exponential map, we get that $\exp_x(X) = x' = \sigma(x') = \sigma(\exp_x(X)) = \exp_x(\sigma_*X)$, and by the fact that \exp_x is an isomorphism we deduce that $\sigma_*X = X$.

The proof for θ is, again, identical. \square

This leads to the following two theorems which are the main results of this subsection:

Theorem 6.6. *(The group of \mathbb{R} -points of) any connected reductive algebraic group G over \mathbb{R} admits a Cartan decomposition (K, \mathfrak{p}) , see Definition 6.1. The group K is a maximal compact subgroup of $G = G(\mathbb{R})$, and all maximal compact subgroups (and all Cartan decompositions) are $G(\mathbb{R})^0$ -conjugate.*

Proof. If σ denotes the antiholomorphic involution of $G(\mathbb{C})$ which fixes $G(\mathbb{R})$, by Proposition 6.5 there exists a commuting holomorphic involution θ such that $\sigma\theta$ is compact, unique up to $G(\mathbb{R})^0$ -conjugacy. The restriction of θ to $G(\mathbb{R})$ induces a Cartan decomposition (K, \mathfrak{p}) with $K = G(\mathbb{R})^\theta$ and $\mathfrak{p} = \text{Lie}(G(\mathbb{R}))^{-\theta}$. The group K is maximal compact, because if $K' \supset K$ were compact, and $g \in K'$ did not belong to K , then by the Cartan decomposition $g = k \exp(X)$ for some $X \neq 0$ in \mathfrak{p} , but then $\exp(X)$ belongs to K' , but the powers of $\exp(X)$ have no accumulation point, a contradiction. \square

Remark 6.7. Not every non-algebraic semisimple connected Lie group (i.e., with semisimple Lie algebra) admits a Cartan decomposition. For example, by the Cartan decomposition, $K = S^1$ is a deformation retract of $G = \text{SL}_2(\mathbb{R})$, hence $\pi_1(\text{SL}_2) \simeq \mathbb{Z}$, and its fundamental cover $\widetilde{\text{SL}}_2$ is a semisimple Lie group with infinite center, no nontrivial compact subgroups (because the preimage of K is isomorphic to \mathbb{R}), and since the exponential map is not bijective, it does not admit a Cartan decomposition.

Theorem 6.8. *Let G be a connected complex reductive group. Let Ant denote the space of antiholomorphic involutions on G , and Hol the space of holomorphic involutions. There is a canonical bijection*

$$\text{Ant}/G \leftrightarrow \text{Hol}/G$$

induced by the distinguished G -orbit on $\text{Ant} \times \text{Hol}$ of those pairs (σ, θ) such that σ commutes with θ , and $\mathfrak{g}^{\sigma\theta}$ is compact.

If $\Gamma = \text{Gal}(\mathbb{C}/\mathbb{R})$, and we fix a compact real form (an antiholomorphic involution) τ on G , this bijection gives rise to bijections of pointed sets

$$H^1(\Gamma, \text{Aut}(G)) \xrightarrow{\sim} H^1(\mathbb{Z}/2, \text{Aut}(G)),$$

where $\mathbb{Z}/2$ acts trivially on $\text{Aut}(G)$, i.e.,

$$H^1(\mathbb{Z}/2, \text{Aut}(G)) = \text{Hom}(\mathbb{Z}/2, \text{Aut}(G))/\text{Aut}(G) - \text{conj}, \text{ as well as}$$

$$Z^1(\Gamma, \text{Aut}(G))/\text{Inn}(G) \xrightarrow{\sim} \text{Hom}(\mathbb{Z}/2, \text{Aut}(G))/\text{Inn}(G),$$

where Z^1 denotes the set of 1-cocycles.

Proof. Indeed, Proposition 6.5 states that the set of those pairs (σ, θ) such that σ commutes with θ , and $\mathfrak{g}^{\sigma\theta}$ is compact, forms a unique G -orbit which surjects onto both Ant and Hol .

If τ is the antiholomorphic involution corresponding to a compact form, used to define the Galois action on $\text{Aut}(G)$ by pre- and post-composition, for any $\kappa \in Z^1(\Gamma, \text{Aut}(G))$, identified with the image in $\text{Aut}(G)$ of complex conjugation, the composition $\sigma = \tau\kappa$ is also an antiholomorphic involution. This identifies $Z^1(\Gamma, \text{Aut}(G)) \simeq \text{Ant}$, equivariantly under $\text{Aut}(G)$, while $\text{Hom}(\mathbb{Z}/2, \text{Aut}(G)) = \text{Hol}$. The rest of the assertions now follow by descending the distinguished G -orbit (which is also an $\text{Aut}(G)$ -orbit) on $\text{Ant} \times \text{Hol}$. Note that $\tau \in \text{Ant}$ corresponds to the trivial involution in Hol , hence these bijections are indeed pointed. \square

Remark 6.9. Two 1-cocycles of Γ into $\text{Aut}(G)$ are conjugate by $\text{Inn}(G)$ if and only if they correspond to the same inner class, see 3.4. Thus, the quotient $Z^1(\Gamma, \text{Aut}(G))/\text{Inn}(G)$ is the union of all isomorphism classes of inner forms of all outer forms of G .

7. Other chapters

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| (2) Basic Representation Theory | (10) Representations of reductive groups over local fields |
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| (7) Linear algebraic groups | (15) GNU Free Documentation License |
| (8) Forms and covers of reductive groups, and the L -group | (16) Auto Generated Index |

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