THE STACK OF LOCAL SYSTEMS WITH RESTRICTED VARIATION AND GEOMETRIC LANGLANDS THEORY WITH NILPOTENT SINGULAR SUPPORT

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Introduction

0.1. **Starting point.** Classically, Langlands conjectured a bijection between irreducible automorphic representations for a reductive group G and spectral data involving the dual group $\check{G}$.

P. Deligne (for GL_1), V. Drinfeld (for GL_2) and G. Laumon (for GL_n) realized Langlands-style phenomena in algebraic geometry. In their setting, the fundamental objects of interest are *Hecke eigensheaves*. This theory works over an arbitrary ground field k, and takes as an additional input a *sheaf theory* for varieties over that field. Specializing to $k = \overline{\mathbb{F}}_q$ and étale sheaves, one recovers special cases of Langlands's conjectures by taking the trace of Frobenius.

Inspired by these works, Beilinson and Drinfeld proposed the *categorical Geometric Langlands Conjecture*

$$\text{D-mod}(\text{Bun}_G) \simeq \text{IndCoh}_{\text{Nilp}}(\text{LocSys}_{\check{G}}(X)) \tag{0.1}$$

for X a smooth projective curve over a field k of characteristic zero. Here the left-hand side D-mod(Bun_G) is a sheaf-theoretic analogue of the space of unramified automorphic functions, and the right hand side is defined in [AG].

There are (related) discrepancies between this categorical conjecture and more classical conjectures.

(i) Hecke eigensheaves make sense in any sheaf theory, while the Beilinson-Drinfeld conjecture applies only in the setting of D-modules.

(ii) Hecke eigensheaves categorify the arithmetic Langlands correspondence through the trace of Frobenius construction, while the Beilinson-Drinfeld conjecture bears no direct relation to automorphic functions.

(iii) Langlands's conjecture parametrizes *irreducible* automorphic representations, while the Beilinson-Drinfeld conjecture provides a spectral decomposition of (a sheaf-theoretic analogue of) the whole space of (unramified) automorphic functions.

These differences provoke natural questions:

–Is there a categorical geometric Langlands conjecture that applies in any sheaf-theoretic context, in particular, in the étale setting over finite fields?

–The trace construction attaches automorphic functions to particular étale sheaves on Bun_G; is there a direct relationship between the *category* of étale sheaves on Bun_G and the *space* of automorphic functions?

–Is it possible to give a spectral description of the space of classical automorphic functions, not merely its irreducible constituents?

0.2. **Summary.**

0.2.1. In this paper, we provide positive answers to the three questions raised above.

Our Conjecture 14.2.4 provides an analogue of the categorical Geometric Langlands Conjecture that is suited to any ground field and any sheaf theory.

Our Conjecture 15.3.5 proposes a closer relationship between sheaves on Bun_G and unramified automorphic functions than was previously considered. As such, it allows one to extract new, concrete conjectures on automorphic functions from our categorical Geometric Langlands Conjecture, see right below.

Our Conjecture 16.6.11 describes the space of unramified automorphic functions over a function field in spectral terms, refining Langlands's conjectures in this setting.

In sum, the main purpose of this work is to propose a variant of the categorical Beilinson-Drinfeld conjecture that makes sense over finite fields, and in that setting, to connect it with the arithmetic Langlands program.

0.2.2. This paper contains two main ideas. The first of them is the introduction of a space

$$\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)$$

of G-*local systems with restricted variation* on X. In our Conjecture 14.2.4, $\mathrm{LocSys}_{\check{G}}^{\mathrm{restr}}(X)$ replaces $\mathrm{LocSys}_{\check{G}}(X)$ from the original conjecture of Beilinson and Drinfeld.

We discuss $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)$ in detail later in the introduction. For now, let us admit it into the discussion as a black box.

Then our Conjecture 14.2.4 asserts

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \simeq \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}}^{\mathrm{restr}}(X)), \tag{0.2}$$

where the left-hand side is the category of ind-constructible sheaves on Bun_G with nilpotent singular support; we study this category in detail in Sect. 10.

0.2.3. In addition, we make some progress toward Conjecture 14.2.4.

Our Theorem 10.5.2 provides an action of $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}^{\mathrm{restr}}(X))$ on $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ compatible with Hecke functors. We regard this result as a spectral decomposition of the category $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ over $\mathrm{LocSys}_{\check{G}}^{\mathrm{restr}}(X)$. This theorem is a counterpart of [Ga5, Corollary 4.5.5], which applied in the D-module setting and whose proof used completely different methods.

Using these methods, we settle long-standing conjectures on the structure of Hecke eigensheaves. Our Corollary 10.3.8 shows that Hecke eigensheaves have nilpotent singular support, as predicted by G. Laumon in [Lau, Conjecture 6.3.1]. In addition, our Corollary 11.6.7 shows that in the D-module setting, any Hecke eigensheaf has regular singularities, as predicted by Beilinson-Drinfeld in [BD1, Sect. 5.2.7].

0.2.4. The second main idea of this paper is that of categorical trace. It appears in our Conjecture 15.3.5, which we title the *Trace Conjecture*. This conjecture predicts a stronger link between geometric and arithmetic Langlands than was previously considered:

Suppose $k = \overline{\mathbb{F}}_q$ and that X and G are defined over $\mathbb{F}_q$, and therefore carry Frobenius endomorphisms. The Trace Conjecture asserts that the categorical trace of the functor $(\mathrm{Frob}_{\mathrm{Bun}_G})_*$ on $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ maps isomorphically to the space of (compactly supported) unramified automorphic functions

$$\mathrm{Autom} := \mathrm{Funct}_c(\mathrm{Bun}_G(\mathbb{F}_q)).$$

More evocatively: we conjecture that a trace of Frobenius construction recovers the *space* of automorphic forms from the *category* $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$, much as one classically extracts a automorphic functions from an automorphic sheaf by a trace of Frobenius construction.

Combined with our Theorem 10.5.2, the Trace Conjecture gives rise the spectral decomposition of Autom along the set of isomorphism classes of semi-simple Langlands parameters, recovering the (unramified case of) V. Lafforgue's result.

Moreover, if we combine the Trace Conjecture with our version of the categorical Geometric Langlands Conjecture (i.e., Conjecture 14.2.4), we obtain a full description of the space of (unramified) automorphic functions in terms of Langlands parameters (and not just the spectral decomposition):

$$\mathrm{Autom} \simeq \Gamma(\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X), \omega_{\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)}),$$

where $\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)$ is the algebraic stack of Frobenius-fixed points, i.e.,

$$\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X) := (\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))^{\mathrm{Frob}},$$

where Frob is the automorphism of $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$ induced by the geometric Frobenius on X. This is our Conjecture 16.6.11, as referenced in Sect. 0.2.1.

0.3. **Some antecedents.** Before discussing the contents of this paper in more detail, we highlight two points that are *not* original to our work.

0.3.1. *Work of Ben-Zvi and Nadler.* Observe that in Conjecture 14.2.4 we consider the subcategory $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \subset \mathrm{Shv}(\mathrm{Bun}_G)$ of (ind-constructible) sheaves with nilpotent singular support, a hypothesis with no counterpart in the Beilinson-Drinfeld setting of D-modules.

This idea of considering this subcategory, which is so crucial to our work, is due to D. Ben-Zvi and D. Nadler, who did so in their setting of *Betti* Geometric Langlands, see [BN].

0.3.2. Let us take a moment to clarify the relationship between our work and [BN].

For $k = \mathbb{C}$, Ben-Zvi and Nadler consider the larger category $\mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)$ of *all* (possibly not ind-constructible) sheaves on $\mathrm{Bun}_G(\mathbb{C})$, considered as a complex stack via its analytic topology. Let $\mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \subset \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)$ be the full subcategory consisting of objects with nilpotent singular support. Ben-Zvi and Nadler conjectured an equivalence

$$\mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \simeq \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}}(X)), \tag{0.3}$$

where in the right-hand side $\mathrm{LocSys}_{\check{G}}(X)$ is the Betti version of the stack of $\check{G}$-local systems on X.

Let us compare this conjectural equivalence with the Beilinson-Drinfeld version (0.1). The latter is particular to D-modules, while (0.3) is particular to topological sheaves. Our (0.2) sits in the middle between the two: when $k = \mathbb{C}$ our $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ can be thought of as a full subcategory of both $\mathrm{D\text{-}mod}(\mathrm{Bun}_G)$ and $\mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$.

Similarly, our $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$ is an algebro-geometric object that is embedded into both the de Rham and Betti versions of $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$. Now, the point is that $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$ can be defined abstractly, so that it makes sense in any sheaf-theoretic context, along with the conjectural equivalence (0.2).

Remark 0.3.3. We should point out another source of initial evidence towards the relationship between $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ and $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$:

It was a discovered by D. Nadler and Z. Yun in [NY] that when we apply Hecke functors to objects from $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$, we obtain objects in $\mathrm{Shv}(\mathrm{Bun}_G \times X)$ that *behave like local systems along* X; see Theorem 10.2.3 for a precise assertion.

Remark 0.3.4. We should also emphasize that what enabled us to even talk about $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ in the context of ℓ-adic sheaves was the work of A. Beilinson [Be2] and T. Saito [Sai], where the singular support of étale sheaves over any ground field was defined and studied.

0.3.5. *Work of V. Lafforgue.* Our Trace Conjecture is inspired by the work [VLaf1] of V. Lafforgue on the arithmetic Langlands correspondence for function fields.

A distinctive feature of Geometric Langlands is that Hecke functors are defined not merely at points $x \in X$ of a curve, but extend over all of X, and moreover, extend over X^I for any finite set I. These considerations lead to the distinguished role played by the *factorization algebras* of [BD2] and *Ran space* in geometric Langlands theory.

In his work, V. Lafforgue showed that the existence of Hecke *functors* over powers of a curve has implications for automorphic *functions*. Specifically, he used the existence of these functors to construct *excursion operators*, and used these excursion operators to define the spectral decomposition of automorphic functions (over function fields) as predicted by the Langlands conjectures.

0.3.6. In [GKRV], a subset of the authors of this paper attempted to reinterpret V. Lafforgue's constructions using categorical traces. It provided a toy model for the spectral decomposition in [VLaf1] in the following sense:

In *loc.cit.* one starts with an abstract category $\mathcal{C}$ equipped with an action of Hecke functors *in the Betti setting* and an endofunctor $\Phi : \mathcal{C} \to \mathcal{C}$ (to be thought of as a prototype of Frobenius), and obtains a spectral decomposition of the vector space $\mathrm{Tr}(\Phi, \mathcal{C})$ along a certain space, which could be thought of as a Betti analog of the coarse moduli space of arithmetic Langlands parameters.

Now, the present work allows to carry the construction of [GKRV] in the actual setting of applicable to the study of automorphic functions: we take our $\mathcal{C}$ to be the ℓ-adic version of $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ (for a curve X over $\overline{\mathbb{F}}_q$).

In Sect. 16, we revisit V. Lafforgue's work, and show how our Trace Conjecture recovers and (following ideas of V. Drinfeld) refines the main results of [VLaf1] in the unramified case.

0.4. **Contents.** This paper consists of four parts.

In Part I we define and study the properties of the stack $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$.

In Part II we establish a general spectral decomposition result that produces an action of $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$ on a category $\mathbf{C}$, equipped with what one can call a *lisse Hecke action*.

In Part III we study the properties of the category $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$. We should say right away that in this Part we prove two old-standing conjectures: that Hecke eigensheaves have nilpotent singular support, and that (in the case of D-modules) all sheaves with nilpotent singular support have regular singularities.

In Part IV we study the applications of the theory developed hereto to the Langlands theory.

Below we will review the main results of each of the Parts.

0.5. **Overview: the stack $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$.** In this subsection we let G be an arbitrary algebraic group over a field of coefficients e.

0.5.1. Let us start by recalling the definition of the (usual) algebraic stack $\mathrm{LocSys}_{\mathsf{G}}(X)$ of G-local systems on X in the context of sheaves in the classical topology (to be referred to as the *Betti* context).

On the first pass, let us take $\mathsf{G} = GL_n$.

Choose a base point $x \in X$. For an affine test scheme $S = \mathrm{Spec}(A)$ over e, an S-point of $\mathrm{LocSys}_{GL_n}(X)$ is an A-module E_S, locally free of rank n, equipped with an action of $\pi_1(X, x)$.

For an arbitrary G, the definition is obtained from the one for GL_n via Tannakian formalism.

0.5.2. We now give the definition of $\mathrm{LocSys}^{\mathrm{restr}}_{GL_n}(X)$, still in the Betti context. Namely $\mathrm{LocSys}^{\mathrm{restr}}_{GL_n}(X)$ is a subfunctor of $\mathrm{LocSys}_{GL_n}(X)$ that corresponds to the following condition:

We require that the action of $\pi_1(X,x)$ on E_S be locally finite.

For an arbitrary G, one imposes this condition for each finite-dimensional representation $\mathsf{G} \to GL_n$.

When A is Artinian, the above condition is automatic, so the formal completions of $\mathrm{LocSys}_{\mathsf{G}}(X)$ and $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ at any point are the same. The difference appears for A that have positive Krull dimension.

With that we should emphasize that $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ is *not* entirely formal, i.e., it is *not* true that any S-point of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ factors though an S'-point with S' Artinian. For example, for $\mathsf{G} = \mathbb{G}_a$, the map

$$\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X) \to \mathrm{LocSys}_{\mathsf{G}}(X)$$

is an isomorphism.

0.5.3. Let us explain the terminology "restricted variation", again in the example of $\mathsf{G} = GL_n$,

The claim is that when we move along $S = \mathrm{Spec}(A)$, the corresponding representation of $\pi_1(X,x)$ does not change too much, in the sense that the isomorphism class of its semi-simplification is constant (as long as S is connected).

Indeed, let us show that for every $\gamma \in \pi_1(X,x)$ and every $\lambda \in \mathsf{e}$, the generalized λ-eigenspace of γ on $E_s := E_S \underset{A,s}{\otimes} \mathsf{e}$ has a constant dimension as s moves along S.

Indeed, due to the locally finiteness condition, we can decompose E_S into a direct sum of generalized eigenspaces for γ

$$E_S = \underset{\lambda}{\oplus} E_S^{(\lambda)},$$

where each $E_S^{(\lambda)}$ is an A-submodule, and being a direct summand of a locally free A-module, it is itself locally free.

The same phenomenon will happen for any G: an S-point of $\mathrm{LocSys}_{\mathsf{G}}(X)$ factors through $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ if and only for all e-points of S, the resulting G-local systems on X all have the same semi-simplification.

0.5.4. We are now ready to give the general definition of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$.

Within the given sheaf theory, we consider the full subcategory

$$\mathrm{Lisse}(X) \subset \mathrm{Shv}(X)^{\mathrm{constr}}$$

of local systems (of finite rank).

Consider its ind-completion, denoted $\mathrm{IndLisse}(X)$. Finally, let $\mathrm{QLisse}(X)$ be the left completion of $\mathrm{IndLisse}(X)$ in the natural t-structure[1]. Now, for an affine test scheme $S = \mathrm{Spec}(A)$, an S-point of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ is a symmetric monoidal functor

$$\mathrm{Rep}(\mathsf{G}) \to (A\text{-mod}) \otimes \mathrm{QLisse}(X),$$

required to be right t-exact with respect to the natural t-structures.

By definition, e-points of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ are just G-local systems on X.

Two remarks are in order:

[1]The last step of left completion is unnecessary if X is a *categorical* $K(\pi,1)$, see Sect. B.2.1, which is the case of curves of genus > 0. However, left completion is *non-trivial* for $X = \mathbb{P}^1$, i.e., $\mathrm{IndLisse}(\mathbb{P}^1) \neq \mathrm{QLisse}(\mathbb{P}^1)$, see Sect. B.2.4

(i) In the definition of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ one can (and should!) allow S to be a *derived* affine scheme over e (i.e., we allow A to be a connective commutative e-algebra). Thus, $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ is inherently an object of derived algebraic geometry[2].

(ii) The definition of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ uses the *large* category $(A\text{-}\mathsf{mod}) \otimes \mathrm{QLisse}(X)$. When we evaluate our functor on *truncated* affine schemes, we can replace $\mathrm{QLisse}(X)$ by $\mathrm{IndLisse}(X) = \mathrm{Ind}(\mathrm{Lisse}(X))$ (see Proposition 1.5.6), and so we can express the definition in terms of small categories. But for an arbitrary S, it is essential to work with the entire $\mathrm{QLisse}(X)$, to ensure *convergence* (see Sect. 1.5).

0.5.5. As defined above, $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ is just a functor on (derived) affine schemes, so is just a prestack. But what kind of prestack is it? I.e., can we say something about its geometric properties?

The majority of Part I is devoted to investigating this question.

0.5.6. First, let us illustrate the shape that $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ has in the Betti context. Recall that in this case we have the usual moduli stack $\mathrm{LocSys}_{\mathsf{G}}(X)$, which is a quotient of the affine scheme $\mathrm{LocSys}^{\mathrm{rigid}_x}_{\mathsf{G}}(X)$ (that classifies local systems with a trivialization at x) by G.

Assume that G is reductive, and let

$$\mathrm{LocSys}^{\mathrm{coarse}}_{\mathsf{G}}(X) := \mathrm{LocSys}^{\mathrm{rigid}_x}_{\mathsf{G}}(X)/\!/\mathsf{G} := \mathrm{Spec}(\Gamma(\mathrm{LocSys}_{\mathsf{G}}(X), \mathcal{O}_{\mathrm{LocSys}_{\mathsf{G}}(X)}))$$

be the corresponding coarse moduli space. We have the tautological map

$$r : \mathrm{LocSys}_{\mathsf{G}}(X) \to \mathrm{LocSys}^{\mathrm{coarse}}_{\mathsf{G}}(X), \tag{0.4}$$

and recall that two e-points of $\mathrm{LocSys}_{\mathsf{G}}(X)$ lie in the same fiber of this map if and only if they have isomorphic semi-simplifications.

We can describe $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ as the disjoint union of formal completions of the fibers of r over e-points of $\mathrm{LocSys}^{\mathrm{coarse}}_{\mathsf{G}}(X)$ (see Theorem 3.6.7).

In particular, we note one thing that $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ is *not*: it is *not* an algebraic stack (or union of such), because it has all these formal directions.

Remark 0.5.7. The above explicit description of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ in the Betti case may suggest that it is in general a "silly" object. Indeed, why would we want a moduli space in which all irreducible local systems belong to different connected components?

However, as the results in Parts III and IV of this paper show, $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ is actually a natural object to consider, in that it is perfectly adapted to the study of $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$, and thereby to applications to the arithmetic theory.

For example, formula (0.11) below is the reflection on the automorphic side of the above decomposition of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ as a disjoint union. See also (0.13) for a version of the Geometric Langlands Conjecture with nilpotent singular support. Finally, see formula (0.21) for an expression for the space of automorphic functions in terms of Frobenius-fixed locus on $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$.

0.5.8. For a general sheaf theory, we prove the following theorem concerning the structure of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$. Let $\mathrm{LocSys}^{\mathrm{restr},\mathrm{rigid}_x}_{\mathsf{G}}(X)$ be the fiber product

$$\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X) \underset{\mathrm{pt}/\mathsf{G}}{\times} \mathrm{pt},$$

where

$$\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X) \to \mathrm{pt}/\mathsf{G}$$

is the map corresponding to taking the fiber at a chosen base point $x \in X$. So

$$\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X) \simeq \mathrm{LocSys}^{\mathrm{restr},\mathrm{rigid}_x}_{\mathsf{G}}(X)/\mathsf{G}.$$

[2]In fact, our definition of the usual $\mathrm{LocSys}_{\mathsf{G}}(X)$ in the Betti context was a bit of a euphemism: for the correct definition in the context of derived algebraic geometry, one has to use the entire fundamental groupoid of X, and not just π_1; the difference does not matter, however, when we evaluate on classical test affine schemes, while the distinction between $\mathrm{LocSys}_{\mathsf{G}}(X)$ and $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ happens at the classical level.

We prove (in Theorem 1.3.2) that $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X)$ is a disjoint union of ind-affine ind-schemes $\mathcal{Y}$ (locally almost of finite type), each of which is a *formal affine scheme*.

We recall that a prestack $\mathcal{Y}$ is a formal affine scheme if it an be written as a formal completion

$$\mathrm{Spec}(R)^{\wedge}_{Y},$$

where R is a connective e-algebra (but not necessarily almost of finite type over e) and $Y \simeq \mathrm{Spec}(R')$ is a Zariski closed subset in $\mathrm{Spec}(R)$, where R' is a (classical, reduced) e-algebra of finite type.

This all may sound a little too technical, but the upshot is that the $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)$ fails to be an algebraic stack precisely to the same extent as in the Betti case, and the extent of this failure is such that we can control it very well.

To illustrate the latter point, in Sect. 5 we study the category $\mathrm{QCoh}(\mathcal{Y})$ on formal affine schemes (or quotients of these by groups) and show that its behavior is very close to that of $\mathrm{QCoh}(-)$ on affine schemes (which is *not at all* the case of $\mathrm{QCoh}(-)$ on arbitrary ind-schemes).

0.5.9. As we have seen in Sect. 0.5.6, in the Betti context, the prestack $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)$ splits into a disjoint union of prestacks $\mathcal{Z}_\sigma$ parameterized by isomorphism classes of semi-simple G-local systems[3] σ on X. Moreover, the underlying reduced prestack of each $\mathcal{Z}_\sigma$ is an algebraic stack.

In Sect. 2 we prove that the same is true in any sheaf theory. Furthermore, for each σ, we construct a *uniformization map*

$$\underset{\mathsf{P}}{\sqcup}\, \mathrm{LocSys}_{\mathsf{P},\sigma_{\mathsf{M}}}^{\mathrm{restr}}(X) \to \mathcal{Z}_\sigma,$$

which is proper and surjective at dominant at the reduced level, where:

- The disjoint union runs over the set over parabolic subgroups P, such that σ can be factored via an *irreducible* local system σ_{M} for some/any Levi splitting $\mathsf{P} \hookleftarrow \mathsf{M}$ (here M is the Levi quotient of P);
- $\mathrm{LocSys}_{\mathsf{P},\sigma_{\mathsf{M}}}^{\mathrm{restr}}(X)$ is the *algebraic stack*

$$\mathrm{LocSys}_{\mathsf{P}}^{\mathrm{restr}}(X) \underset{\mathrm{LocSys}_{\mathsf{M}}^{\mathrm{restr}}(X)}{\times} \mathrm{pt}\,/\,\mathrm{Aut}(\sigma_{\mathsf{M}}).$$

0.5.10. Let G be again reductive. For a general sheaf theory, we do not have the picture involving (0.4) that we had in the Betti case. However, we do have a formal part of it.

Namely, let $\mathcal{Z}$ be a connected component of $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)$. This is an ind-algebraic stack, which can be written as

$$\underset{i}{\mathrm{colim}}\, \mathcal{Z}_i,$$

where each $\mathcal{Z}_i$ is an algebraic stack isomorphic to the quotient of a (derived) affine scheme by G.

We can consider the ind-affine ind-scheme

$$\mathcal{Z}^{\mathrm{coarse}} := \underset{i}{\mathrm{colim}}\, \mathrm{Spec}(\Gamma(\mathcal{Z}_i, \mathcal{O}_{\mathcal{Z}_i})),$$

and the map

$$r : \mathcal{Z} \to \mathcal{Z}^{\mathrm{coarse}}. \tag{0.5}$$

In Theorem 4.4.2 we prove that:

(i) $\mathcal{Z}^{\mathrm{coarse}}$ is a formal affine scheme (see Sect. 0.5.8 for what this means) whose underlying reduced scheme is pt;

(ii) The map (0.5) makes $\mathcal{Z}$ into a *relative algebraic stack* over $\mathcal{Z}^{\mathrm{coarse}}$.

[3]When G is not reductive, the parameterization is by the same set for the maximal reductive quotient of G.

0.5.11. Assume for a moment that we work in the context of D-modules (to be referred to as the *de Rham* context). In this case, we also have the algebraic stack $\mathrm{LocSys}_{\mathsf{G}}(X)$ classifying de Rham local systems (but we do not have the picture with the coarse moduli space).

We have a naturally defined map

$$\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X) \to \mathrm{LocSys}_{\mathsf{G}}(X),$$

and in Sect. 3.1 we show that, as in the Betti case, it identifies $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ with the disjoint union of formal completions along the closed substacks, indexed by isomorphism classes of semi-simple G-local systems σ, each equal to the image of the map

$$\bigsqcup_{\mathsf{P}} \mathrm{LocSys}_{\mathsf{P},\sigma_{\mathsf{M}}}(X) \to \mathrm{LocSys}_{\mathsf{G}}(X),$$

where

$$\mathrm{LocSys}_{\mathsf{P},\sigma_{\mathsf{M}}}(X) := \mathrm{LocSys}_{\mathsf{P}}(X) \underset{\mathrm{LocSys}_{\mathsf{M}}(X)}{\times} \mathrm{pt} / \mathrm{Aut}(\sigma_{\mathsf{M}}),$$

where σ_{M} is an irreducible M-local system that induces σ for G.

0.6. **Overview: spectral decomposition.** Part II contains one of the two the main results of this paper, Theorem 6.1.4.

0.6.1. We again start with a motivation in the Betti context.

Let $\mathcal{X}$ be a connected space, and let $\mathbf{C}$ be a DG category.

In this case, we have the notion of action of $\mathrm{Rep}(\mathsf{G})^{\otimes \mathcal{X}}$ on $\mathbf{C}$, see [GKRV, Sect. 1.7]. It consists of a compatible family of functors

$$\mathrm{Rep}(\mathsf{G})^{\otimes I} \to \mathrm{End}(\mathbf{C}) \otimes (\mathrm{Vect}^{\mathcal{X}}_{\mathsf{e}})^{\otimes I}, \quad I \in \mathrm{fSet},$$

where $\mathrm{Vect}^{\mathcal{X}}_{\mathsf{e}}$ is the DG category $\mathrm{Funct}(\mathcal{X}, \mathrm{Vect}_{\mathsf{e}})$ (it can be thought of as the category of local systems of vector spaces on $\mathcal{X}$, *not necessarily* of finite rank), and fSet is the category of finite sets.

Now, we have the stack of Betti local systems $\mathrm{LocSys}_{\mathsf{G}}(\mathcal{X})$ and we can consider actions of the symmetric monoidal category $\mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}(\mathcal{X}))$ on $\mathbf{C}$.

The tautological defined symmetric monoidal functor

$$\mathrm{Rep}(\mathsf{G}) \otimes \mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}(\mathcal{X})) \to \mathrm{Vect}^{\mathcal{X}}_{\mathsf{e}}$$

gives rise to a map (of ∞-groupoids)

$$\{\text{Actions of } \mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}(\mathcal{X})) \text{ on } \mathbf{C}\} \to \{\text{Actions of } \mathrm{Rep}(\mathsf{G})^{\otimes \mathcal{X}} \text{ on } \mathbf{C}\}. \tag{0.6}$$

A relatively easy result (see [GKRV, Theorem 1.5.5]) says that the map (0.6) is an equivalence (of ∞-groupoids).

0.6.2. We now transport ourselves to the context of algebraic geometry. Let X be a connected scheme over k and $\mathbf{C}$ be a e-linear DG category. By an action of $\mathrm{Rep}(\mathsf{G})^{\otimes X}$ on $\mathbf{C}$ we shall mean a compatible collection of functors

$$\mathrm{Rep}(\mathsf{G})^{\otimes I} \to \mathrm{End}(\mathbf{C}) \otimes \mathrm{QLisse}(X)^{\otimes I}, \quad I \in \mathrm{fSet}\,.$$

As before, we have the tautological symmetric monoidal functor

$$\mathrm{Rep}(\mathsf{G}) \otimes \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)) \to \mathrm{QLisse}(X),$$

and we obtain a map

$$\{\text{Actions of } \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(\mathcal{X})) \text{ on } \mathbf{C}\} \to \{\text{Actions of } \mathrm{Rep}(\mathsf{G})^{\otimes X} \text{ on } \mathbf{C}\}. \tag{0.7}$$

One can can ask whether the map (0.7) is an isomorphism as well. Our Spectral Decomposition theorem says that it is, provided that $\mathbf{C}$ is dualizable as a DG category[4]. We are sure that the proof we give is not the optimal one; there should exist a more robust argument for statements of this kind (see Sect. 0.6.3 for the general framework).

[4]We are confident that the result holds without the dualizability on assumption $\mathbf{C}$; we were just unable to prove it in this more general case.

The majority of Part II is devoted to the proof of this theorem. In the rest of this subsection we will indicate the general framework for the proof and some of the ideas involved.

0.6.3. Let $\mathbf{H}$ be a symmetric monoidal category.

We define another symmetric monoidal category, denoted $\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$ by the universal property that for a test symmetric monoidal category $\mathbf{C}$, we have

$$\mathrm{Funct}^{\mathrm{SymMon}}(\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}), \mathbf{C}) \simeq \mathrm{Funct}^{\mathrm{SymMon}}(\mathrm{Rep}(\mathsf{G}), \mathbf{C} \otimes \mathbf{H}).$$

It is not difficult to show (essentially mimicking [GKRV, Theorem 1.2.4]) that actions of $\mathrm{Rep}(\mathsf{G})^{\otimes X}$ are the same as functors from $\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \mathrm{QLisse}(X))$, considered just as a monoidal category, to $\mathrm{End}(\mathbf{C})$.

Assume now that $\mathbf{H}$ is equipped with a t-structure. We define the prestack $\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$ so that its value on a test affine scheme S consists of symmetric monoidal functors

$$\mathrm{Rep}(\mathsf{G}) \to \mathrm{QCoh}(S) \otimes \mathbf{H},$$

required to be right t-exact with respect to the tensor product t-structure on the right-hand side.

We have a tautologically defined functor

$$\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}) \to \mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})). \tag{0.8}$$

One can ask: under what conditions on $\mathbf{H}$ is the functor (0.8) an equivalence? Unfortunately, we do not know the answer. We do know, however, that it is an equivalence for $\mathbf{H} = \mathrm{Vect}_{\mathsf{e}}^{\mathcal{X}}$ for a connected space $\mathcal{X}$ (this is [GKRV, Theorem 1.5.5], quoted in the present paper as Theorem 7.1.2).

Note that if we knew that (0.8) is an equivalence for $\mathbf{H} = \mathrm{QLisse}(X)$, we would know that (0.7) is an equivalence as well.

0.6.4. What we do prove (see Theorem 6.2.11) is that the functor dual to (0.8) (i.e., one obtained by considering $\mathrm{Funct}_{\mathrm{cont}}(-, \mathrm{Vect}_{\mathsf{e}})$) is an equivalence when $\mathbf{H}$ is a particularly well-behaved Tannakian category; see Sect. 6.2.5 for the precise conditions (our $\mathrm{QLisse}(X)$ is an example of such a Tannakian category).

The catch here is that we do not know that $\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$ is dualizable as a DG category (we do know this about $\mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}))$ thanks to Theorem 1.3.2, which is applicable to $\mathrm{QLisse}(X)$ replaced by $\mathbf{H}$, and the material from Sect. 5). So the statement for the dual functor is weaker than for (0.8) itself.

In addition, we show that, under the same assumptions on $\mathbf{H}$, the original functor (0.8) is *localization*, i.e., admits a fully faithful right adjoint (but we do not know whether this right adjoint is continuous).

In any case, we show that the assertion of Theorem 6.2.11 is sufficient to deduce that (0.7) is an isomorphism for $\mathbf{C}$ dualizable.

0.6.5. Let us indicate some ideas involved in the proof of Theorem 6.2.11, i.e., that the functor

$$\mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}))^{\vee} \to \underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\vee}$$

is an equivalence.

First, we try a frontal assault to try to prove that (0.8) is an equivalence without any conditions on $\mathbf{H}$ (Sect. 7.8). And we "almost" succeed. We manage to prove that the map

$$\mathrm{Funct}^{\mathrm{SymMon}}(\mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})), \mathbf{C}) \to \mathrm{Funct}^{\mathrm{SymMon}}(\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}), \mathbf{C})$$

is an equivalence when $\mathbf{C}$ has a compact unit (or is the inverse limit of symmetric monoidal categories with this property). Note that $\mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}))$, being the category of quasi-coherent sheaves on a prestack, has this property. But we do not know whether this is the case for $\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$.

Although the attempt in Sect. 7.8 does not quite give us what we need, it provides some particular cases. First, it does allow to prove that (0.8) is an equivalence when

$$\mathbf{H} = \mathsf{h}\text{-mod},$$

where h is a connective Lie algebra over e (this is Corollary 7.5.5).

Then we can improve a little on the methods from Sect. 7.8 and show that (0.8) is an equivalence when $\mathbf{H} = \mathrm{Rep}(\mathsf{H})$, where H is a *reductive* group. This is done in Sects. 8.1-8.3.

We then combine the ideas from the above two cases and show that (0.8) is an equivalence when

$$\mathbf{H} = (\mathsf{h}, \mathsf{H}_{\mathrm{red}})\text{-mod}$$

for a Harish-Chandra pair $(\mathsf{h}, \mathsf{H}_{\mathrm{red}})$ with $\mathsf{H}_{\mathrm{red}}$ reductive. This is Theorem 8.4.3.

Finally, we show that (0.8) is an equivalence for $\mathbf{H} = \mathrm{Rep}(\mathsf{H})$ for a finite-dimensional algebraic group H. Namely, we deduce this formally from the case of Harish-Chanda pairs using the fully faithful embedding

$$\mathrm{Rep}(\mathsf{H}) \hookrightarrow (\mathsf{h}, \mathsf{H}_{\mathrm{red}})\text{-mod},$$

where $\mathsf{h} := \mathrm{Lie}(\mathsf{H})$ and $\mathsf{H}_{\mathrm{red}}$ is the reductive part of H for some choice of Levi splitting. The deduction is explained in Sects. 7.2.1-7.2.8.

0.6.6. Thus, we can now prove that (0.8) is an equivalence for $\mathbf{H} = \mathrm{Rep}(\mathsf{H})$ for an algebraic group H. At this point it seems natural to treat the case of a general Tannakian $\mathbf{H}$ by approximating it by categories of the form $\mathrm{Rep}(\mathsf{H})$. But we were not able to make it work.

Even for $\mathbf{H} = \mathrm{Rep}(\mathsf{H})$ for H being a *pro*-algebraic group

$$\mathsf{H} = \lim_{i} \mathsf{H}_i$$

with H_i finite-dimensional, the adjunctions seem to go the wrong way, and we could not get the two sides of (0.8) as limits/colimits of the two sides involving $\mathrm{Rep}(\mathsf{H}_i)$.

So, we (almost) abandon these attempts and try something else.

0.6.7. In Sect. 9 we reimpose our stringent conditions on $\mathbf{H}$, so that $\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$ is the quotient of an affine formal scheme by G. We consider the diagonal map

$$\Delta_{\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})} : \mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}) \to \mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}) \times \mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$$

and give an explicit description of the object

$$(\Delta_{\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})})_*(\mathcal{O}_{\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})}) \in \mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}) \times \mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})),$$

which is in fact the unit of the self-duality on $\mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}))$.

This gives us a tool to compute certain Hom spaces explicitly, and we use this to prove directly that the functor

$$\mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}))^\vee \to \underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^\vee \tag{0.9}$$

is fully faithful.

Thus, for the proof of Theorem 6.2.11 we only need to show that the functor (0.9) is essentially surjective.

We show that both sides carry natural t-structures and reduce the question of essential surjectivity to one at the level of the corresponding abelian categories.

Finally, for abelian categories, we are able to reduce the question of essential surjectivity to the case when $\mathbf{H}$ is of the form $\mathrm{Rep}(\mathsf{H})$, and there we already know that the functor is an equivalence.

0.7. **Overview:** $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$**.** In this subsection we take G to be a reductive group and we will study the category $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ of sheaves on Bun_G (within any of our contexts) with singular support in the nilpotent cone $\mathrm{Nilp} \subset T^*(\mathrm{Bun}_G)$.

0.7.1. The stack Bun_G is non quasi-compact, and what allows us to work efficiently with the category $\mathrm{Shv}(\mathrm{Bun}_G)$ is the fact that we can simultaneously think of it as a *limit*, taken over poset of quasi-compact open substacks $\mathcal{U} \subset \mathrm{Bun}_G$,

$$\lim_{\mathcal{U}} \mathrm{Shv}(\mathcal{U}),$$

with transition functors given by restriction, and *also as a colimit*

$$\operatorname*{colim}_{\mathcal{U}} \mathrm{Shv}(\mathcal{U}),$$

with transition functors given by !-extension.

We now take $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$. More or less by definition, we still have

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) := \lim_{\mathcal{U}} \mathrm{Shv}_{\mathrm{Nilp}}(\mathcal{U}),$$

but we run into trouble with the colimit presentation:

In order for such presentation to exist, we should be able to find a cofinal family of quasi-compact opens, such that for every pair $\mathcal{U}_1 \overset{j}{\hookrightarrow} \mathcal{U}_2$ from this family, the functor $j_!$ sends

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathcal{U}_1) \to \mathrm{Shv}_{\mathrm{Nilp}}(\mathcal{U}_2).$$

Fortunately, we can find such a family; its existence is guaranteed by Theorem 10.1.4.

0.7.2. Thus, we can access the category $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ via the corresponding categories on the quasi-compacts. But our technical troubles are not over:

We do not know whether the categories $\mathrm{Shv}_{\mathrm{Nilp}}(\mathcal{U})$ are compactly generated. Such questions (for an arbitrary algebraic stack or even scheme $\mathcal{Y}$, with a fixed $\mathcal{N} \subset T^*(\mathcal{Y})$) may be non-trivial. For example, it is *not* true in general that $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})$ is generated by objects that are compact in $\mathrm{Shv}(\mathcal{Y})$. We refer the reader to Sect. C where some general facts pertaining to these issues are summarized.

Yet, it turns that the situation with our particular pair $(\mathrm{Bun}_G, \mathrm{Nilp})$ is very favorable. Namely, we have Theorem 10.1.6, which says that $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ is generated by objects that are compact in the ambient category $\mathrm{Shv}(\mathrm{Bun}_G)$.

Two remarks are in order:

(i) We wish to emphasize that Theorem 10.1.6 is not some kind of general result, but uses the specifics of the situation. In particular, it uses the Spectral Decomposition theorem from Part II, and also our second main result, Theorem 10.3.3, described below.

(ii) Although the statement of Theorem 10.1.6 may appear as too technical (why do we care about compactness from a bird's-eye view?), it is necessary to even formulate the Trace Conjecture in Part IV, see Sect. 0.8.4.

0.7.3. We now proceed to formulating the other key results in Part III.

We consider the Hecke action on $\mathrm{Shv}(\mathrm{Bun}_G)$. Now, the subcategory

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \subset \mathrm{Shv}(\mathrm{Bun}_G)$$

has the following key feature with respect to this action:

According to [NY], combined with [GKRV, Theorem A.3.8], the Hecke functors

$$\mathrm{H}(-,-) : \mathrm{Rep}(\check{G})^{\otimes I} \otimes \mathrm{Shv}(\mathrm{Bun}_G) \to \mathrm{Shv}(\mathrm{Bun}_G \times X^I), \quad I \in \mathrm{fSet}, \tag{0.10}$$

send the subcategory

$$\mathrm{Rep}(\check{G})^{\otimes I} \otimes \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \subset \mathrm{Rep}(\check{G})^{\otimes I} \otimes \mathrm{Shv}(\mathrm{Bun}_G)$$

to

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{QLisse}(X)^{\otimes I} \subset \mathrm{Shv}(\mathrm{Bun}_G \times X^I).$$

This means that $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ carries an action of $\mathrm{Rep}(\check{G})^{\otimes X}$, i.e., we find ourselves in the setting of the Spectral Decomposition theorem. (There is a technical issue that at this point we do not yet know that $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ is dualizable, so we cannot directly apply Theorem 6.1.4 to it; but we are able to get around this issue using t-structures.)

Thus, we obtain the first main result of this paper, which we call the Spectral Decomposition theorem (it appears as Theorem 10.5.2 in the main body of the paper):

Theorem 0.7.4. *The category* $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ *carries a monoidal action of* $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$.

Theorem 0.7.4 has an obvious ideological significance. For example, it immediately implies that the category $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ splits as a direct sum

$$\underset{\sigma}{\oplus}\, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)_\sigma, \tag{0.11}$$

indexed by isomorphism classes of semi-simple $\check{G}$-local systems.

However, in addition, we use Theorem 0.7.4 extensively to prove a number of structural results about $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$. For example, we use it to prove: (i) the compact generation of $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ (this is Theorem 11.2.1); (ii) Theorem 10.1.6 mentioned above; (iii) the fact that objects from $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ have regular singularities (see below); (iv) the tensor product property of $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ (Theorem 11.5.3, see below).

0.7.5. We now come to the second main result of this paper (it appears as Theorem 10.3.3 in the main body of the paper), which is in some sense a converse to the assertion of [NY] mentioned above:

Theorem 0.7.6. *Let* $\mathcal{F}$ *be an object of* $\mathrm{Shv}(\mathrm{Bun}_G)$*, such that the Hecke functors* (0.10) *send it to*

$$\mathrm{Shv}(\mathrm{Bun}_G) \otimes \mathrm{QLisse}(X),$$

then $\mathcal{F} \in \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$.

A particular case of this assertion was conjectured by G. Laumon. Namely, [Lau, Conjecture 6.3.1] says that Hecke eigensheaves have nilpotent singular support.

0.7.7. The combination of Theorems 0.7.4 and 0.7.6 allows us to establish a whole array of results about $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$, in conjunction with another tool: Beilinson's spectral projector, whose definition we will now recall.

Let us first start with a single $\check{G}$-local system σ. We can consider the category

$$\mathrm{Hecke}_\sigma(\mathrm{Shv}(\mathrm{Bun}_G))$$

of Hecke eigensheaves on Bun_G with respect to σ.

We have a tautological forgetful functor

$$\mathbf{oblv}_{\mathrm{Hecke}_\sigma} : \mathrm{Hecke}_\sigma(\mathrm{Shv}(\mathrm{Bun}_G)) \to \mathrm{Shv}(\mathrm{Bun}_G), \tag{0.12}$$

and Beilinson's spectral projector is a functor

$$R^{\mathrm{enh}}_\sigma : \mathrm{Shv}(\mathrm{Bun}_G) \to \mathrm{Hecke}_\sigma(\mathrm{Shv}(\mathrm{Bun}_G)),$$

left adjoint to (0.12).

A feature of the functor R^{enh}_σ is that the composition

$$\mathrm{Shv}(\mathrm{Bun}_G) \overset{R^{\mathrm{enh}}_\sigma}{\longrightarrow} \mathrm{Hecke}_\sigma(\mathrm{Shv}(\mathrm{Bun}_G)) \overset{\mathbf{oblv}_{\mathrm{Hecke}_\sigma}}{\longrightarrow} \mathrm{Shv}(\mathrm{Bun}_G)$$

is given by an explicit *integral Hecke functor*[5].

However, now that we have $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$, we can consider a version of the functor R^{enh}_σ is families: Let S be an affine scheme equipped with a map

$$f : S \to \mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X).$$

[5]I.e., a colimit of functors (0.10) for explicit objects of $\mathrm{Rep}(\check{G})^{\otimes I}$, as I ranges over the category of finite sets.

Then it again makes sense to consider the category of *Hecke eigensheaves* parametrized by S:

$$\mathrm{Hecke}(S, \mathrm{Shv}(\mathrm{Bun}_G)).$$

It is endowed with a forgetful functor

$$\mathrm{Hecke}(S, \mathrm{Shv}(\mathrm{Bun}_G)) \to \mathrm{QCoh}(S) \otimes \mathrm{Shv}(\mathrm{Bun}_G)$$

(i.e., forget the eigenproperty), which we can then compose with

$$\mathrm{QCoh}(S) \otimes \mathrm{Shv}(\mathrm{Bun}_G) \overset{\Gamma(S,-)\otimes \mathrm{Id}}{\longrightarrow} \mathrm{Shv}(\mathrm{Bun}_G).$$

We have a version of Beilinson's spectral projector, which is now a functor, denoted in this paper by

$$R^{\mathrm{enh}}_S : \mathrm{Shv}(\mathrm{Bun}_G) \to \mathrm{Hecke}(S, \mathrm{Shv}(\mathrm{Bun}_G)),$$

left adjoint to the composition

$$\mathrm{Hecke}(S, \mathrm{Shv}(\mathrm{Bun}_G)) \to \mathrm{QCoh}(S) \otimes \mathrm{Shv}(\mathrm{Bun}_G) \to \mathrm{Shv}(\mathrm{Bun}_G).$$

Let us note that the definition of functor R^{enh}_S only uses the existence of $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$. We do not need to use Theorems 0.7.4 and 0.7.6 to prove its existence or to establish its properties.

0.7.8. However, let us now use the functor R^{enh}_S in conjunction with Theorems 0.7.4 and 0.7.6.

First, Theorem 0.7.6 implies that the inclusion

$$\mathrm{Hecke}(S, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)) \subset \mathrm{Hecke}(S, \mathrm{Shv}(\mathrm{Bun}_G))$$

is an equality.

And Theorem 0.7.4 implies that the category $\mathrm{Hecke}(S, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G))$ identifies with

$$\mathrm{QCoh}(S) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))}{\otimes} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G).$$

Thus, we obtain that the functor R^{enh}_S provides a left adjoint to the functor

$$\mathrm{QCoh}(S) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))}{\otimes} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \overset{f_*\otimes \mathrm{Id}}{\longrightarrow} \mathrm{QCoh}(S) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))}{\otimes} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \simeq$$
$$\simeq \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \hookrightarrow \mathrm{Shv}(\mathrm{Bun}_G).$$

This construction has a number of consequences:

(i) It allows to prove the compact generation of $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ (left adjoints can be used to construct compact generators); this is Theorem 11.2.1.

(ii) We can extend the construction of R^{enh}_S to prestacks, and taking $S = \mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$, we obtain a functor

$$R^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)} : \mathrm{Shv}(\mathrm{Bun}_G) \to \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G),$$

which we prove to be isomorphic to the *right* adjoint of the embedding $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \hookrightarrow \mathrm{Shv}(\mathrm{Bun}_G)$; this is Theorem 11.3.4.

(iii) We construct explicit generators of $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ by applying the functor $R^{\mathrm{enh}}_{\mathcal{Y}}$ (for some particularly chosen $f : \mathcal{Y} \to \mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$) to δ-function objects in $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$. This leads to the theorem that all objects in $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ have regular singularities (in the de Rham context); this is Main Corollary 11.6.6. Combined with Corollary 10.3.8, we obtain that all Hecke eigensheaves have regular singularities; this is Main Corollary 11.6.7.

(iv) We use the above generators of $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ to prove the (unexpected, but important for future applications) property that the tensor product functor

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \to \mathrm{Shv}_{\mathrm{Nilp}\times\mathrm{Nilp}}(\mathrm{Bun}_G \times \mathrm{Bun}_G)$$

is an equivalence; this is Theorem 11.5.3 (see the discussion in Sect. 11.5.1 regarding why such an equivalence is not something we should expect on general grounds).

0.8. **Overview: Langlands theory.** In this subsection we let X be a curve over k, and we will work with any of the sheaf-theoretic contexts from our list.

0.8.1. Having set up the theories of $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$ and $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$, we are now in the position to state a version of the (categorical) Geometric Langlands Conjecture, with nilpotent singular support: this is Conjecture 14.2.4. It says that we have an equivalence

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \simeq \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)), \tag{0.13}$$

as categories equipped with an action of $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$.

Here in the right-hand side, $\mathrm{IndCoh}_?(-)$ stands for the category of ind-coherent sheaves with prescribed *coherent* singular support, a theory developed in [AG]. (In *loc.cit.*, this theory was developed for quasi-smooth schemes/algebraic stacks, but in Sect. 14.1 we show that it equally applicable to objects such as our $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}$.) In our case $? = \mathrm{Nilp}$, the *global nilpotent cone* in $\mathrm{Sing}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$, see Sect. 14.2.2[6].

0.8.2. Note that Conjecture 14.2.4 may be the first instance when a categorical statement is suggested for automorphic sheaves in the context of ℓ-adic sheaves.

That said, both the de Rham and Betti contexts have their own forms of the (categorical) Geometric Langlands Conjecture. In the de Rham context, this is an equivalence

$$\text{D-mod}(\mathrm{Bun}_G) \simeq \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}\,\check{G}(X)), \tag{0.14}$$

as categories equipped with an action of $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))$.

In the Betti context, this is an equivalence

$$\mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \simeq \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}}(X)), \tag{0.15}$$

as categories equipped with an action of $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))$, where $\mathrm{Shv}^{\mathrm{all}}_?(-)$ stands for the category of all sheaves (i.e., not necessarily ind-constructible ones) with a prescribed singular support.

We show that in each of these contexts, our Conjecture 14.2.4 is a formal consequence of (0.14) (resp., (0.15)), respectively. In fact, we show that the two sides in Conjecture 14.2.4 are obtained from the two sides in (0.14) (resp., (0.15)) by

$$\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \underset{\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))}{\otimes} -.$$

That said, we show (assuming Hypothesis 14.4.2) that the restricted version of GLC (i.e., (0.13)) actually implies the full de Rham version, i.e., (0.14). Probably, a similar argument can show that (0.13) implies the full Betti version (i.e., (0.15)) as well.

0.8.3. For the rest of this subsection we will work over the ground field $k = \overline{\mathbb{F}}_q$, but assume that our geometric objects (i.e., X and G) are defined over $\mathbb{F}_q$, so that they carry the geometric Frobenius endomorphism.

We now come to the second main theme of this paper, the Trace Conjecture.

For any (quasi-compact) algebraic stack $\mathcal{Y}$ over $\overline{\mathbb{F}}_q$, but defined over $\mathbb{F}_q$. we can consider the endomorphism (in fact, automorphism) of $\mathrm{Shv}(\mathcal{Y})$ given by Frobenius pushforward, $(\mathrm{Frob}_{\mathcal{Y}})_*$. Since $\mathrm{Shv}(\mathcal{Y})$ is a compactly generated (and, hence, dualizable) category, we can consider the categorical trace of $(\mathrm{Frob}_{\mathcal{Y}})_*$ on $\mathrm{Shv}(\mathcal{Y})$:

$$\mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}(\mathcal{Y})) \in \mathrm{Vect}_{\mathsf{e}}\,.$$

The Grothdendieck passage from Weil sheaves on $\mathcal{Y}$ to functions on $\mathcal{Y}(\mathbb{F}_q)$ can be upgraded to a map

$$\mathrm{LT} : \mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}(\mathcal{Y})) \to \mathrm{Funct}(\mathcal{Y}(\mathbb{F}_q)),$$

compatible with *-pullbacks and !-pushforwards, see Theorem 15.1.8.

[6]It should not be confused with $\mathrm{Nilp} \subset T^*(\mathrm{Bun}_G)$: the two uses of Nilp have different meanings, and occur on different sides of Langlands duality.

However, the map LT is *not at all* an isomorphism (unless $\mathcal{Y}$ has finitely many isomorphism classes of $\overline{\mathbb{F}}_q$-points).

0.8.4. We apply the above discussion to $\mathcal{Y} = \mathrm{Bun}_G$. Since Bun_G is not quasi-compact, the local term map is in this case a map

$$\mathrm{LT} : \mathrm{Tr}((\mathrm{Frob}_{\mathrm{Bun}_G})_*, \mathrm{Shv}(\mathrm{Bun}_G)) \to \mathrm{Funct}_c(\mathrm{Bun}_G(\mathbb{F}_q)), \tag{0.16}$$

where $\mathrm{Funct}_c(-)$ stands for functions with finite support.

In what follows we will denote

$$\mathrm{Autom} := \mathrm{Funct}_c(\mathrm{Bun}_G(\mathbb{F}_q)).$$

This is the space of compactly supported unramified automorphic functions.

As we just mentioned, the map (0.16) is *not* an isomorphism (unless X is of genus 0).

0.8.5. We now consider the full subcategory

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \hookrightarrow \mathrm{Shv}(\mathrm{Bun}_G).$$

Since it is stable under the action of the Frobenius, and since the above embedding admits a continuous right adjoint (it is here that Theorem 10.1.6 is crucial), we obtain a map

$$\mathrm{Tr}((\mathrm{Frob}_{\mathrm{Bun}_G})_*, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)) \to \mathrm{Tr}((\mathrm{Frob}_{\mathrm{Bun}_G})_*, \mathrm{Shv}(\mathrm{Bun}_G)).$$

Our Trace Conjecture (Conjecture 15.3.5) says that the composite map

$$\begin{multline}\mathrm{Tr}((\mathrm{Frob}_{\mathrm{Bun}_G})_*, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)) \to \mathrm{Tr}((\mathrm{Frob}_{\mathrm{Bun}_G})_*, \mathrm{Shv}(\mathrm{Bun}_G)) \overset{\mathrm{LT}}{\to} \\ \to \mathrm{Funct}_c(\mathrm{Bun}_G(\mathbb{F}_q)) = \mathrm{Autom}\end{multline} \tag{0.17}$$

is an isomorphism.

Remark 0.8.6. The sheaves-functions correspondence has been part of the geometric Langlands program since its inception by V. Drinfeld: in his 1982 paper, he constructed a Hecke eigenfunction corresponding to a 2-dimensional local system on X by first constructing the corresponding sheaf and then taking the associated functions.

Constructions of this sort allow to produce particular elements in Autom that satisfy some desired properties.

Our Trace Conjecture is an improvement in that it, in principle, allows to deduce statements about the *space* Autom from statements of $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ as a *category*.

0.8.7. In fact, the Trace Conjecture is a particular case of a more general statement, Conjecture 15.5.7, which we refer to as the Shtuka Conjecture.

Namely, for a finite set I and $V \in \mathrm{Rep}(\check{G})^{\otimes I}$ consider the Hecke functor

$$\mathrm{H}(V, -) : \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \to \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{QLisse}(X^I).$$

Generalizing the categorical trace construction, we can consider the trace of this functor, precomposed with $(\mathrm{Frob}_{\mathrm{Bun}_G})_*$. The result will be an object that we denote

$$\widetilde{\mathrm{Sht}}_{I,V} \in \mathrm{QLisse}(X^I) \subset \mathrm{Shv}(X^I).$$

Our Shtuka Conjecture says that we have a canonical isomorphism

$$\widetilde{\mathrm{Sht}}_{I,V} \simeq \mathrm{Sht}_{I,V}, \tag{0.18}$$

where $\mathrm{Sht}_{I,V} \in \mathrm{Shv}(X^I)$ is the shtuka cohomology, see Sect. 15.5.1 where we recall the definition.

Note that the validity of (0.18) implies that the objects $\mathrm{Sht}_{I,V}$ belong to $\mathrm{QLisse}(X^I) \subset \mathrm{Shv}(X^I)$.

The latter fact has been unconditionally established by C. Xue in [Xue2], which provides a reality check for our Shtuka Conjecture.

0.8.8. We will now explain how the Trace Conjecture recovers V. Lafforgue's spectral decomposition of Autom along the arithmetic Langlands parameters.

The ind-stack $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$ (which is an algebro-geometric object over $\mathsf{e} = \overline{\mathbb{Q}}_\ell$!) carries an action of Frobenius, by transport of structure; we denote it by Frob. Denote

$$\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}} := \left(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)\right)^{\mathrm{Frob}}.$$

A priori, $\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)$ is also an *ind*-algebraic stack, but we prove (see Theorem 16.1.4) that $\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)$ is an actual algebraic stack (locally almost of finite type). We also prove that it is quasi-compact (i.e., even though $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$ had infinitely many connected components, only finitely many of them survive the Frobenius).

The algebra

$$\mathcal{E}xc := \Gamma(\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X), \mathcal{O}_{\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)})$$

receives a map from V. Lafforgue's algebra of excursion operators; this map is surjective at the level oh H^0, see Sect. 16.2.2.

0.8.9. The categorical meaning of $\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)$ is that the category $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X))$ identifies with the category of Hochschild chains of Frob^* acting on $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$.

We will now apply the relative version of the trace construction from [GKRV, Sect. 3.8], and attach to the pair

$$(\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G), (\mathrm{Frob}_{\mathrm{Bun}_G})_*),$$

viewed as acted on by the pair

$$(\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)), \mathrm{Frob}^*),$$

its class

$$\mathrm{cl}(\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G), (\mathrm{Frob}_{\mathrm{Bun}_G})_*) \in \mathrm{HH}(\mathrm{Frob}^*, \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))) \simeq \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)).$$

We denote the resulting object of $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X))$ by

$$\mathrm{Drinf} \in \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)).$$

By [GKRV, Theorem 3.8.5], we have

$$\Gamma(\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X), \mathrm{Drinf}) \simeq \mathrm{Tr}((\mathrm{Frob}_{\mathrm{Bun}_G})_*, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)).$$

Combining with the Trace Conjecture (see (0.17)) we thus obtain an isomorphism

$$\Gamma(\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X), \mathrm{Drinf}) \simeq \mathrm{Autom}. \tag{0.19}$$

In particular, the tautological action of $\mathcal{E}xc$ on $\Gamma(\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X), \mathrm{Drinf})$ gives rise to an action of $\mathcal{E}xc$ on Autom. This recovers V. Lafforgue's spectral decomposition.

0.8.10. The ideological significance of the isomorphism (0.19) is that it provides a *localization* picture for Autom.

Namely, it says that behind the vector space Autom stands a finer object, namely, a quasi-coherent sheaf (this is our Drinf) on the moduli *stack* of Langlands parameters (this is our $\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)$), such that Autom is recovered as its global sections.

In other words, Autom is something that lives over the *coarse moduli space*

$$\mathrm{LocSys}^{\mathrm{arithm,coarse}}_{\check{G}}(X) := \mathrm{Spec}(\mathcal{E}xc),$$

and it is obtained as direct image along the tautological map

$$r : \mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X) \to \mathrm{LocSys}^{\mathrm{arithm,coarse}}_{\check{G}}(X)$$

from a finer object, namely Drinf, on the *moduli stack*.

Remark 0.8.11. The notation Drinf has the following origin: upon learning of V. Lafforgue's work [VLaf1], V. Drinfeld suggested that the objects

$$\widetilde{\mathrm{Sht}}_{I,V}$$

mentioned above should organize themselves into an object of $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X))$. (However, at the time there was not yet a definition of $\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)$.)

Now, with our definition of $\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)$, the Shtuka Conjecture, i.e., (0.18), is precisely the statement that the object Drinf constructed above realizes Drinfeld's vision.

0.8.12. A particular incarnation of the localization phenomenon of Autom is the following.

Fix an e-point of $\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)$ corresponding to an *irreducible* Weil $\check{G}$-local system σ. In Theorem 16.1.6 we show that such a point corresponds to a connected component of $\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)$, isomorphic to $\mathrm{pt}\,/\,\mathrm{Aut}(\sigma)$.

The restriction of Drinf to this connected component is then a representation of the (finite) group $\mathrm{Aut}(\sigma)$. The corresponding direct summand on Autom is obtained by taking $\mathrm{Aut}(\sigma)$-invariants in this representation.

0.8.13. Finally, let us juxtapose the Trace Conjecture with the Geometric Langlands Conjecture (0.13). We obtain an isomorphism

$$\mathrm{Autom} \simeq \mathrm{Tr}(\mathrm{Frob}^!, \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))).$$

Now, a (plausible, and much more elementary) Conjecture 16.6.7 says that the inclusion

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \hookrightarrow \mathrm{IndCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$$

induces an isomorphism

$$\mathrm{Tr}(\mathrm{Frob}^!, \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))) \simeq \mathrm{Tr}(\mathrm{Frob}^!, \mathrm{IndCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))). \tag{0.20}$$

Now, for any quasi-smooth stack $\mathcal{Y}$ with an endomorphism ϕ, we have

$$\mathrm{Tr}(\phi^!, \mathrm{IndCoh}(\mathcal{Y})) \simeq \Gamma(\mathcal{Y}^\phi, \omega_{\mathcal{Y}^\phi}).$$

Hence, the right-hand side in (0.20) identifies with

$$\Gamma(\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X), \omega_{\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)}).$$

Summarizing, we obtain that the combination of the above three conjectures yields an isomorphism

$$\mathrm{Autom} \simeq \Gamma(\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X), \omega_{\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)}). \tag{0.21}$$

This gives a conjectural expression for the space of (unramified) automorphic functions purely in terms of the stack of arithmetic Langlands parameters.

0.9. **Notations and conventions.** The notations in this paper will largely follow those adopted in [GKRV].

0.9.1. *Algebraic geometry.* There will be two algebraic geometries present in this paper.

On the one hand, we fix a ground field k (assumed algebraically closed, but of arbitrary characteristic) and we will consider algebro-geometric objects over k. This algebraic geometry will occur on the *geometric/automorphic* side of Langlands correspondence.

Thus, X will be a scheme over k (in Parts III and IV of the paper, X will be a complete curve), G will be a reductive group over k, Bun_G will be the stack of G-bundles on X, etc.

The algebro-geometric objects over k will be *classical*, i.e., *non-derived*; this is because we will study sheaf theories on them that are insensitive to the derived structure (such as ℓ-adic sheaves, or D-modules).

All algebro-geometric objects over k will be *locally of finite type* (see [GR1, Chapter 2, Sect. 1.6.1] for what this means). We will denote by $\mathrm{Sch}_{\mathrm{ft},/k}$ the category of schemes of finite type over k.

On the other hand, we will have a field of coefficients e (assumed algebraically closed *and of characteristic zero*), and we will consider *derived* algebro-geometric objects over e, see Sect. 0.9.6 below.

The above two kinds of algebro-geometric objects do not generally mix unless we work with D-modules, in which case $k = \mathsf{e}$ is a field of characteristic zero.

0.9.2. *Higher categories.* This paper will substantially use the language of ∞-categories[7], as developed in [Lu1].

We let Spc denote the ∞-category of spaces.

Given an ∞-category $\mathbf{C}$, and a pair of objects $\mathbf{c}_1, \mathbf{c}_2 \in \mathbf{C}$, we let $\mathrm{Maps}_{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2) \in \mathrm{Spc}$ the mapping space between them.

Recall that given an ∞-category $\mathbf{C}$ that contains filtered colimits, an object $\mathbf{c} \in \mathbf{C}$ is said to be compact if the Yoneda functor $\mathrm{Maps}_{\mathbf{C}}(\mathbf{c}, -) : \mathbf{C} \to \mathrm{Spc}$ preserves filtered colimits. We let $\mathbf{C}^c \subset \mathbf{C}$ denote the full subcategory spanned by compact objects.

Given a functor $F : \mathbf{C}_1 \to \mathbf{C}_2$ between ∞-categories, we will denote by F^R (resp., F^L) its right (resp., left) adjoint, provided that it exists.

0.9.3. *Higher algebra.* Throughout this paper we will be concerned with *higher algebra* over a field of coefficients, denoted e (as was mentioned above, throughout the paper, e will be assumed algebraically closed and of characteristic zero).

We will denote by $\mathrm{Vect}_{\mathsf{e}}$ the stable ∞-category of chain complexes of e-modules, see, e.g., [GaLu, Example 2.1.4.8].

We will regard $\mathrm{Vect}_{\mathsf{e}}$ as equipped with a symmetric monoidal structure (in the sense on ∞-categories), see, e.g., [GaLu, Sect. 3.1.4]. Thus, we can talk about commutative/associative algebra objects in $\mathrm{Vect}_{\mathsf{e}}$, see, e.g., [GaLu, Sect. 3.1.3].

0.9.4. *DG categories.* We will denote by DGCat the ∞-category of (presentable) cocomplete stable ∞-categories, *equipped with a module structure over* $\mathrm{Vect}_{\mathsf{e}}$ *with respect to the symmetric monoidal structure on the ∞-category of cocomplete stable ∞-categories given by the Lurie tensor product*, see [Lu2, Sect. 4.8.1]. We will refer to objects of DGCat as "DG categories". We emphasize that 1-morphisms in DGCat are in particular colimit-preserving.

For a given DG category $\mathbf{C}$, and a pair of objects $\mathbf{c}_1, \mathbf{c}_2 \in \mathbf{C}$, we have a well-defined "inner Hom" object $\mathcal{H}om_{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2) \in \mathrm{Vect}_{\mathsf{e}}$, characterized by the requirement that

$$\mathrm{Maps}_{\mathrm{Vect}_{\mathsf{e}}}(V, \mathcal{H}om_{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2)) \simeq \mathrm{Maps}_{\mathcal{C}}(V \otimes \mathbf{c}_1, \mathbf{c}_2), \quad V \in \mathrm{Vect}_{\mathsf{e}}\,.$$

The category DGCat itself carries a symmetric monoidal structure, given by Lurie tensor product over Vect:

$$\mathbf{C}_1, \mathbf{C}_2 \rightsquigarrow \mathbf{C}_1 \otimes \mathbf{C}_2.$$

In particular, we can talk about the ∞-category of associative/commutative algebras in DGCat, which we denote by $\mathrm{DGCat}^{\mathrm{Mon}}$ (resp., $\mathrm{DGCat}^{\mathrm{SymMon}}$), and refer to as monoidal (resp., symmetric monoidal) DG categories.

Unless specified otherwise, all monoidal/symmetric monoidal DG categories will be assumed unital. Given a monoidal/symmetric monoidal DG category $\mathcal{A}$, we will denote by $\mathbf{1}_{\mathcal{A}}$ its unit object.

[7]We will often omit the adjective "infinity" and refer to ∞-categories simply as "categories".

0.9.5. *t-structures.* Given a DG category $\mathbf{C}$, we can talk about a t-structures on it. For example, the category $\mathrm{Vect}_{\mathsf{e}}$ carries a natural t-structure.

Given a t-structure on $\mathbf{C}$, we will denote by

$$\mathbf{C}^{\leq n},\ \mathbf{C}^{\geq n},\ \mathbf{C}^{\heartsuit}$$

the corresponding subcategories (according to *cohomological* indexing conventions), and also

$$\mathbf{C}^{<\infty} = \underset{n}{\cup} \mathbf{C}^{\leq n},\ \mathbf{C}^{>-\infty} = \underset{n}{\cup} \mathbf{C}^{\geq -n}.$$

We will refer to the objects of $\mathbf{C}^{\leq 0}$ (resp., $\mathbf{C}^{\geq 0}$) as *connective* (resp., *coconnective*) with respect to the given t-structure.

0.9.6. *Derived algebraic geometry over* e. Most of the work in the present paper involves algebraic geometry on the spectral side of the Langlands correspondence. This is somewhat atypical to most work in geometric Langlands.

As was mentioned above, algebraic geometry on the spectral side occurs over the field e and is derived. The starting point of derived algebraic geometry over e is the category $\mathrm{Sch}^{\mathrm{aff}}_{/\mathsf{e}}$ of *derived affine schemes* over e, which is by definition the opposite category of the category of connective commutative algebras in $\mathrm{Vect}_{\mathsf{e}}$.

All other algebro-geometric objects over e will be *prestacks*, i.e., accessible functors

$$(\mathrm{Sch}^{\mathrm{aff}}_{/\mathsf{e}})^{\mathrm{op}} \to \mathrm{Spc}\,.$$

Inside the category $\mathrm{PreStk}_{/\mathsf{e}}$ of all prestacks, one singles out various subcategories. One such subcategory is $\mathrm{PreStk}_{\mathrm{laft},/\mathsf{e}}$ that consists of prestacks *locally almost of finite type* (see [GR1, Chapter 2, Sect. 1.7]). We set

$$\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft},/\mathsf{e}} := \mathrm{PreStk}_{\mathrm{laft},/\mathsf{e}} \cap \mathrm{Sch}^{\mathrm{aff}}_{/\mathsf{e}}\,.$$

We refer the reader to [GR1, Chapter 3] for the assighment

$$\mathcal{Y} \in \mathrm{PreStk}_{/\mathsf{e}} \rightsquigarrow \mathrm{QCoh}(\mathcal{Y}) \in \mathrm{DGCat}\,.$$

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Part I: the (pre)stack $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ and its properties

Let us make a brief overview of the contents of this Part.

In Sect. 1 we define the prestack $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ and state Theorem 1.3.2, pertaining to its geometric properties. We establish the deformation theory properties of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ (and of its variant $\mathrm{LocSys}^{\mathrm{restr},\mathrm{rigid}_x}_{\mathsf{G}}(X)$), leading to the conclusion that $\mathrm{LocSys}^{\mathrm{restr},\mathrm{rigid}_x}_{\mathsf{G}}(X)$ is an ind-affine ind-scheme.

In Sect. 2 we finish the proof of Theorem 1.3.2 by combining the following two results. One is Theorem 2.1.2, which says that the underlying *reduced* prestack of $\mathrm{LocSys}^{\mathrm{restr},\mathrm{rigid}_x}_{\mathsf{G}}(X)$ is a disjoint union of affine schemes. The other is a general result due to J. Lurie (we quote it as Theorem 2.1.4), which gives a deformation theory criterion for an ind-scheme to be a formal scheme (for completeness, we supply a proof in Sect. A). We prove Theorem 2.1.2 by constructing a uniformization of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ that has to do with parabolic subgroups in G and *irreducible* local systems for their Levi subgroups. In the process, we show that the set of connected components of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ is in bijection with the set of isomorphism classes of irreducible G-local systems on X.

In Sect. 3 we compare $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ with the *usual* $\mathrm{LocSys}_{\mathsf{G}}(X)$ in the two contexts in which the latter is defined, i.e., de Rham and Betti. We show that in both cases, the resulting map $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X) \to \mathrm{LocSys}_{\mathsf{G}}(X)$ identifies $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ with the disjoint union of formal completions of closed substacks, each corresponding to G-local systems with a fixed semi-simplification. Additionally, in the Betti context, we show that these closed substacks are exactly the fibers of the map

$$r : \mathrm{LocSys}_{\mathsf{G}}(X) \to \mathrm{LocSys}^{\mathrm{coarse}}_{\mathsf{G}}(X),$$

where $\mathrm{LocSys}^{\mathrm{coarse}}_{\mathsf{G}}(X)$ is the corresponding coarse moduli space.

In Sect. 4, we assume that G is reductive. First, we establish two more geometric properties of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$: namely, that it is *mock-affine* and *mock-proper*. Then we state and prove another structural result, Theorem 4.4.2, which says that $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ admits a coarse moduli space at the formal level, see Sect. 0.5.10.

In Sect. 5 we show that the category $\mathrm{QCoh}(-)$ of a formal affine scheme has properties largely analogous to that of $\mathrm{QCoh}(-)$ of an affine scheme. The material from this section will be applied when we will study the action of $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X))$ on $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$.

1. The restricted version of the stack of local systems

We let X be a connected scheme over the ground field k.

In this section we will define $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ as a prestack and consider a few examples.

We then proceed to the study of deformation theory (=infinitesimal) properties of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$. Most of these properties follow easily from the definition, apart from some issues of convergence.

Finally, we show that $\mathrm{LocSys}^{\mathrm{restr},\mathrm{rigid}_x}_{\mathsf{G}}(X)$ is an ind-scheme by reducing to the assertion that for a pair of algebraic groups G_1 amd G_2, the prestack of maps $\mathbf{Maps}_{\mathrm{Grp}}(G_1, G_2)$ is an ind-affine ind-scheme.

1.1. **Lisse sheaves.**
In this subsection we will introduce one of our main actors–the category of lisse sheaves on X.

1.1.1. We will work in one of the following sheaf-theoretic contexts:

$$X \mapsto \mathrm{Shv}(X)^{\mathrm{constr}}, \quad X \in \mathrm{Sch}_{\mathrm{ft},/k}\,.$$

- Constructible sheaves of e-vector spaces on the topological space underlying X, when $k = \mathbb{C}$ (to be referred to as the *Betti context*);
- Holonomic D-modules X, when $\mathrm{char}(k) = 0$ (to be referred to as the *de Rham context*);
- Constructible $\overline{\mathbb{Q}}_\ell$-adic sheaves on X, when $\mathrm{char}(k) \neq \ell$ (to be referred to as the *ℓ-adic context*).

In the Betti context, we will sometimes consider also the category of all sheaves of vector spaces on X; we will denote it by $\mathrm{Shv}^{\mathrm{all}}(X)$, see Sect. 10.4.1.

In the de Rham complex we will sometimes consider also the category of all D-modules on X, denoted D-mod(X).

1.1.2. We will denote by e the field of coefficients of our sheaves. In three contexts above, this is e (an arbitrary field of characteristic 0), k and $\overline{\mathbb{Q}}_\ell$, respectively.

1.1.3. In any of the above contexts, we set

$$\mathrm{Shv}(X) := \mathrm{Ind}(\mathrm{Shv}(X)^{\mathrm{constr}}).$$

The perverse t-structure on $\mathrm{Shv}(X)^{\mathrm{constr}}$ induces t-structures on $\mathrm{Shv}(X)$. Its heart is the ind-completion $\mathrm{Ind}(\mathrm{Perv}(X))$ of the category $\mathrm{Perv}(X)$ of perverse sheaves on X.

We record the following result, proved in Sect. B.1:

Theorem 1.1.4. *The category* $\mathrm{Shv}(X)$ *is left-complete in its t-structure.*

Remark 1.1.5. In addition to the preverse t-structure on $\mathrm{Shv}(X)^{\mathrm{constr}}$, one can consider the *usual* t-structure. It is a characterized by the requirement that the functors of *-fiber at closed points of X be t-exact. By ind-extension, the usual t-structure on $\mathrm{Shv}(X)^{\mathrm{constr}}$ defines a t-structure on $\mathrm{Shv}(X)$, which we will refer to as the "usual" t-structure.

We note that the analog of Theorem 1.1.4 remains valid for the usual t-structure, due to the fact that the two t-structures are a finite distance apart (bounded by $\dim(X)$).

That said, unless explicitly stated otherwise, we will work with the perverse t-structure.

1.1.6. Assume that X is smooth. Let

$$\mathrm{Perv}_{\mathrm{lisse}}(X) \subset \mathrm{Perv}(X)$$

be the full subcategory of *lisse* objects.

In each of the above contexts, we mean the following:

- In the Betti context, $\mathrm{Perv}_{\mathrm{lisse}}(X)$ consists of local systems on X of finite rank;
- In the de Rham context, $\mathrm{Perv}_{\mathrm{lisse}}(X)$ consists of $\mathcal{O}$-coherent D-modules;
- In the ℓ-adic coontext, $\mathrm{Perv}_{\mathrm{lisse}}(X)$ consists of ℓ-adic local systems on X.

Let $\mathrm{Ind}(\mathrm{Perv}_{\mathrm{lisse}}(X))$ be the ind-completion of $\mathrm{Perv}_{\mathrm{lisse}}(X)$, viewed as a full abelian subcategory in $\mathrm{Ind}(\mathrm{Perv}(X))$.

1.1.7. The following definition will be central for this paper:

Definition 1.1.8. *We let*

$$\mathrm{QLisse}(X) \subset \mathrm{Shv}(X)$$

be the full subcategory, consisting of objects all of whose cohomologies belong to

$$\mathrm{Ind}(\mathrm{Perv}_{\mathrm{lisse}}(X)) \subset \mathrm{Ind}(\mathrm{Perv}(X)).$$

We will also sometimes use the notation

$$\mathrm{Shv}_{\{0\}}(X) := \mathrm{QLisse}(X).$$

1.1.9. By Theorem 1.1.4, the category $\mathrm{QLisse}(X)$ is left-complete in its t-structure.

Unfortunately, for a general X we will not be able to say almost anything about the general categorical properties of $\mathrm{QLisse}(X)$. For example, we do not know whether it is compactly generated or even dualizable.

That said, our main application is when X is an algebraic curve, in which case we do know that $\mathrm{QLisse}(X)$ is compactly generated, see Sect. B.2.5.

1.1.10. We endow $\mathrm{Shv}(X)^{\mathrm{constr}}$ with a symmetric monoidal structure given by the $\overset{!}{\otimes}$ operation. It induces a symmetric monoidal structure on $\mathrm{Shv}(X)$, by ind-extension.

Further, it restricts to a symmetric monoidal structure on $\mathrm{QLisse}(X)$.

Note that the unit object for this monoidal structure is ω_X, the dualizing object on X.

When we consider $\mathrm{QLisse}(X)$ as a symmetric monoidal category, we will shift its original t-structure by $[\dim(X)]$. The effect of this shift is that ω_X lies in $\mathrm{QLisse}(X)^{\heartsuit}$, and the monoidal operation is t-exact.

Remark 1.1.11. One can define the category $\mathrm{QLisse}(X)$ for X that is not necessarily smooth by appealing to the *usual* (rather than the perverse) t-structure on $\mathrm{Shv}(X)^{\mathrm{constr}}$.

With this definition, $\mathrm{QLisse}(X)$ acquires a symmetric monoidal structure from the $\overset{*}{\otimes}$ operation on $\mathrm{Shv}(X)$.

One can define a different embedding $\mathrm{QLisse}(X) \hookrightarrow \mathrm{Shv}(X)$ by tensoring objects from $\mathrm{QLisse}(X)$ by ω_X. This functor is symmetric monoidal, where we equip $\mathrm{QLisse}(X)$ with the symmetric monoidal structure just defined, and where we regard $\mathrm{Shv}(X)$ as a symmetric monoidal category via the $\overset{!}{\otimes}$ operation.

The latter realization of $\mathrm{QLisse}(X)$ as a symmetric monoidal subcategory of $(\mathrm{Shv}(X), !)$ coincides with the one in Sect. 1.1.10 in the smooth case.

1.2. **Another version of lisse sheaves.** Along with $\mathrm{QLisse}(X)$ we can consider its variant, denoted $\mathrm{IndLisse}(X)$, introduced below. Its main advantage is that $\mathrm{IndLisse}(X)$ is compactly generated, by design.

1.2.1. Denote

$$\mathrm{Lisse}(X) := \mathrm{QLisse}(X) \cap \mathrm{Shv}(X)^{\mathrm{constr}}.$$

This is the full subcategory of $\mathrm{Shv}(X)^{\mathrm{constr}}$ consisting of objects, all of whose cohomologies belong to $\mathrm{Perv}_{\mathrm{lisse}}(X)$.

Remark 1.2.2. One can characterize the subcategory

$$\mathrm{Lisse}(X) \subset \mathrm{Shv}(X)$$

as the subcategory of objects dualizable with respect to the $\overset{!}{\otimes}$ symmetric monoidal structure on $\mathrm{Shv}(X)$.

1.2.3. Set

$$\mathrm{IndLisse}(X) := \mathrm{Ind}(\mathrm{Lisse}(X)).$$

In other words, $\mathrm{IndLisse}(X)$ is the full subcategory in $\mathrm{Shv}(X) := \mathrm{Ind}(\mathrm{Shv}(X)^{\mathrm{constr}})$ generated by $\mathrm{Lisse}(X)$.

1.2.4. We have a tautologically defined fully faithful functor

$$\mathrm{IndLisse}(X) \to \mathrm{QLisse}(X). \tag{1.1}$$

The functor (1.1) is t-exact and induces an equivalence

$$(\mathrm{IndLisse}(X))^{\geq -n} \to (\mathrm{QLisse}(X))^{\geq -n}$$

for any n.

From here it follows that the functor (1.1) identifies $\mathrm{QLisse}(X)$ with the left completion of $\mathrm{IndLisse}(X)$.

1.2.5. However, the functor (1.1) is *not* always an equivalence. For example, it fails to be such for $X = \mathbb{P}^1$, see Sect. B.2.5.

Equivalently, the category $\mathrm{IndLisse}(X)$ is *not* necessarily complete in its t-structure.

That said, as we will see in Sect. B.2.5, the functor (1.1) is an equivalence for all curves X (projective or affine) different from $\mathbb{P}^1$.

Remark 1.2.6. The definitions of $\mathrm{Lisse}(X)$ and $\mathrm{IndLisse}(X)$ are applicable when X is not necessarily smooth (see Remark 1.1.11).

Note that all the assertions regarding left completions remain valid due to Remark 1.1.5.

1.2.7. One should consider $\mathrm{IndLisse}(X)$ as a "really nice" symmetric monoidal category, in that it is compactly generated and rigid (see [GR1, Chapter 1, Sect. 9.2] for what this means).

Moreover, one can pick compact generators that belong to the heart of $\mathrm{IndLisse}(X)$, and they will have a cohomological dimension bounded by $\dim(X)$.

One thing that $\mathrm{IndLisse}(X)$ is *not* is that it is *not* the derived category of its heart, see Sect. B.2.1.

1.3. **Definition of** $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ **as a functor.**

1.3.1. We define the prestack $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}$ over the field e of coefficients by sending $S \in \mathrm{Sch}^{\mathrm{aff}}_{/\mathsf{e}}$ to the space of *right t-exact symmetric monoidal functors*

$$\mathrm{Rep}(\mathsf{G}) \to \mathrm{QCoh}(S) \otimes \mathrm{QLisse}(X). \tag{1.2}$$

The main result of this Part is the following:

Main Theorem 1.3.2. *The prestack* $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ *is a disjoint union of prestacks each of which can be written as a quotient by* G *of a prestack* $\mathcal{Y}$ *with the following properties:*

(a) $\mathcal{Y}$ *is locally almost of finite type*[8]*;*

(b) ${}^{\mathrm{red}}\mathcal{Y}$ *is an affine (classical, reduced) scheme;*

(c) $\mathcal{Y}$ *is an ind-scheme;*

(d) $\mathcal{Y}$ *can be written as*

$$\underset{i\geq 0}{\mathrm{colim}}\ \mathrm{Spec}(R_i) \tag{1.3}$$

where R_i *are connective commutative* e*-algebras of the following form: there exists a connective commutative* e*-algebra* R *and elements* $f_1, ..., f_n \in R$ *so that*

$$R_i = R \underset{\mathsf{e}[t_{1,i},\dots,t_{n,i}]}{\otimes} \mathsf{e}, \quad t_{m,i} \mapsto f_m^i.$$

Remark 1.3.3. Points (a,b,c) of Theorem 1.3.2 can be combined to the following statement: $\mathcal{Y}$ can be written as *filtered* colimit

$$\mathcal{Y} \simeq \underset{i}{\mathrm{colim}}\, Y_i, \tag{1.4}$$

where all Y_i are affine schemes almost of finite type[9], and the maps $Y_i \to Y_j$ are closed nil-isomorphisms (i.e., closed embeddings that induce isomorphisms of the underlying reduced prestacks), see [GR2, Chapter 2, Corollary 1.8.6(a)].

[8]See [GR1, Chapter 1, Sect. 1.7.2] for what this means.
[9]See [GR1, Chapter 1, Sect. 1.7.1] for what this means.

Remark 1.3.4. Note, however, that points (a,b,c) of Theorem 1.3.2 do *not* include the assertion contained in (d). For example, if we take

$$\mathcal{Y} := \underset{n}{\operatorname{colim}}\, (\mathbb{A}^n)^\wedge_0,$$

then this $\mathcal{Y}$ admits a presentation as in (1.4) (with the specified properties), but it does *not* admit a presentation (1.3) (the reason is that prestacks of the latter form admit *cotangent spaces*, while the former only *pro-cotangent spaces*, see Sect. 1.6).

The property of admitting a presentation as in (1.3) insures, among other things, that the category $\mathrm{QCoh}(\mathcal{Y})$ is particularly well-behaved (has many properties similar to those of $\mathrm{QCoh}(-)$ of an affine scheme, see Sect. 5).

Prestacks $\mathcal{Y}$ satisfying (d) are called *formal affine schemes*.

Finally, we emphasize that the commutative algebra R that appears in (d) is *not* necessarily almost of finite type over e.

Remark 1.3.5. For the validity of Theorem 1.3.2 in the Betti context, we can work more generally: instead of starting with an algebraic variety X over $\mathbb{C}$, we can let X be a topological space homotopy equivalent to a compact CW complex.

1.3.6. *Example.* Let $\mathsf{G} = \mathbb{G}_m$. As we shall see in Corollary 2.3.3, in this case the *underlying reduced prestack* of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ is a disjoint union, over the set of isomorphism classes of one-dimensional local systems on X, of copies of $\mathrm{pt}/\mathbb{G}_m$.

In Sect. 1.6 we will see that for each 1-dimensional local system (i.e., an e-point of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathbb{G}_m}(X)$), the tangent space to $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ at this point identifies with

$$\mathrm{C}^\cdot(X, \mathsf{e}_X)[1] \in \mathrm{Vect}_{\mathsf{e}}, \tag{1.5}$$

i.e., it looks like the tangent space of the "usual would-be" $\mathrm{LocSys}_{\mathbb{G}_m}(X)$. (Tangent spaces are defined for prestacks that admit deformation theory and are locally almost of finite type, see [GR2, Chapter 1, Sect. 4.4].)

1.3.7. *Example.* Let $\mathsf{G} = \mathbb{G}_a$. We claim that in this case $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ is the algebraic stack associated with the object (1.5), i.e.,

$$\mathrm{Maps}(\mathrm{Spec}(R), \mathrm{LocSys}^{\mathrm{restr}}_{\mathbb{G}_a}(X)) = \tau^{\leq 0}(R \otimes \mathrm{C}^\cdot(X, \mathsf{e}_X)[1]), \tag{1.6}$$

where we view an object of $(\mathrm{Vect}_{\mathsf{e}})^{\leq 0}$ as a space by the Dold-Kan functor (see [GR1, Chapter 1, Sect. 10.2.3]).

Indeed, the space of symmetric monoidal functors from $\mathrm{Rep}(\mathbb{G}_a)$ to any symmetric monoidal category $\mathbf{A}$ identifies with

$$\tau^{\leq 0}(\mathcal{E}nd_{\mathbf{A}}(\mathbf{1}_{\mathbf{A}})[1]).$$

In our case $\mathbf{A} = R\text{-mod} \otimes \mathrm{QLisse}(X)$ so $\mathbf{1}_{\mathbf{A}} = R \otimes \mathsf{e}_X$, whence (1.6).

1.3.8. *Notation.* In what follows, for $V \in \mathrm{Vect}_{\mathsf{e}}$ we will use the notation $\mathrm{Tot}(V)$ for the corresponding prestack, i.e.,

$$\mathrm{Hom}(S, \mathrm{Tot}(V)) = \tau^{\leq 0}(V \otimes \Gamma(S, \mathcal{O}_S)), \quad S \in {}^{\mathrm{cl}}\mathrm{Sch}^{\mathrm{aff}}_{/\mathsf{e}}. \tag{1.7}$$

For example, when $V \in \mathrm{Vect}^{\heartsuit}_{\mathsf{e}} \cap \mathrm{Vect}^c_{\mathsf{e}}$, we have

$$\mathrm{Tot}(V) = \mathrm{Spec}(\mathrm{Sym}(V^\vee)).$$

Thus, (1.6) is saying that

$$\mathrm{LocSys}^{\mathrm{restr}}_{\mathbb{G}_a}(X) \simeq \mathrm{Tot}(\mathrm{C}^\cdot(X, \mathsf{e}_X)[1]).$$

1.3.9. *Example.* This is a preview of the material in Sect. 3:

Let our sheaf-theoretic context be either Betti or de Rham, so in both cases we have the usual algebraic stack $\mathrm{LocSys}_{\mathsf{G}}(X)$. In this case we will show that there exists a forgetful map

$$\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X) \to \mathrm{LocSys}_{\mathsf{G}}(X), \tag{1.8}$$

which identifies $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)$ with the disjoint union of formal completions of a collection of pairwise non-intersecting Zariski-closed reduced substacks of $\mathrm{LocSys}_{\mathsf{G}}(X)$, such that every e-point of $\mathrm{LocSys}_{\mathsf{G}}(X)$ belongs to (exactly) one of these substacks. Furthermore, we will be able to describe the corresponding reduced substacks explicitly.

1.4. **Rigidification.** Let us choose a base point $x \in X$. We will introduce a cousin of $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)$, denoted $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X)$, that has to do with choosing a trivialization of our local systems at x.

1.4.1. Given the base point $x \in X$, consider the corresponding fiber functor

$$\mathrm{QLisse}(X) \xrightarrow{\mathrm{ev}_x} \mathrm{Vect}_{\mathsf{e}}. \tag{1.9}$$

Consider the prestack $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X)$ that sends $S \in \mathrm{Sch}^{\mathrm{aff}}_{/\mathsf{e}}$ to the space of symmetric monoidal functors (1.2), equipped with an isomorphism between the composition

$$\mathrm{Rep}(\mathsf{G}) \to \mathrm{QCoh}(S) \otimes \mathrm{QLisse}(X) \xrightarrow{\mathrm{Id} \otimes \mathrm{ev}_x} \mathrm{QCoh}(S)$$

and

$$\mathrm{Rep}(\mathsf{G}) \xrightarrow{\mathbf{oblv}_{\mathsf{G}}} \mathrm{Vect}_{\mathsf{e}} \xrightarrow{\mathrm{unit}} \mathrm{QCoh}(S) \tag{1.10}$$

(as symmetric monoidal functors). Note that the latter identification implies that the functor (6.11) is right t-exact.

In other words,

$$\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X) \simeq \mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X) \underset{\mathrm{pt}/\mathsf{G}}{\times} \mathrm{pt},$$

where the map

$$\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X) \to \mathrm{pt}/\mathsf{G}$$

is given by (1.9).

1.4.2. We have a natural action of G on $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X)$, and it is easy to see that $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)$ identifies with the étale quotient of $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X)$ by this action.

Hence, in order to prove Theorem 1.3.2, it suffices to show the following:

Theorem 1.4.3. *The prestack* $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X)$ *is a disjoint union of prestacks* $\mathcal{Y}$ *with the properties* (a)-(d) listed in Theorem 1.3.2.

1.5. **Convergence.** In this subsection we begin the investigation of infinitesimal properties of $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)$ (resp., $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X)$).

We start with the most basic one–the property of being convergent[10]. It is here that it will become important that we are working with $\mathrm{QLisse}(X)$ and not $\mathrm{IndLisse}(X)$.

1.5.1. Thus, we need to show that for a (derived) affine test-scheme S, the map

$$\mathrm{Maps}(S, \mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)) \to \lim_n \mathrm{Maps}({}^{\leq n}S, \mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)) \tag{1.11}$$

is an isomorphism, where $S \mapsto {}^{\leq n}S$ denotes the n-th coconnective truncation, i.e., the operation

$$R \mapsto \tau^{\geq -n}(R)$$

at the level of rings.

[10] See [GR1, Chapter 2, Sect. 1.4] for what this means.

1.5.2. In what follows we will repeatedly use the following assertion:

Lemma 1.5.3. *Let* $\mathbf{C}$ *be a category equipped with a t-structure. Then for* S *as above we have:*

(a) *If* $\mathbf{C}$ *is left complete, then* $\mathbf{C} \otimes \mathrm{QCoh}(S)$ *is also left complete.*

(a') *More generally, if* $\mathbf{C}^\wedge$ *is the left completion of* $\mathbf{C}$*, then the functor*

$$\mathrm{QCoh}(S) \otimes \mathbf{C} \to \mathrm{QCoh}(S) \otimes \mathbf{C}^\wedge$$

identifies $\mathrm{QCoh}(S) \otimes \mathbf{C}^\wedge$ *with the left completion of* $\mathbf{C}$*.*

(b) *If* $\mathbf{C}$ *is left complete, then the functor*

$$(\mathbf{C} \otimes \mathrm{QCoh}(S))^{\leq 0} \to \lim_n (\mathbf{C} \otimes \mathrm{QCoh}({}^{\leq n}S))^{\leq 0}$$

is an equivalence.

Proof. Point (a) follows from the fact that the functor

$$\mathbf{C} \otimes \mathrm{QCoh}(S) \overset{\mathrm{Id} \otimes \Gamma(S,-)}{\longrightarrow} \mathbf{C}$$

is t-exact and commutes with limits.

Point (b) follows from point (a) and the fact that for any n, the functor

$$(\mathbf{C} \otimes \mathrm{QCoh}(S))^{\leq 0, \geq -n} \to (\mathbf{C} \otimes \mathrm{QCoh}({}^{\leq m}S))^{\leq 0, \geq -n}$$

is an equivalence, whenever $m \geq n$.

□

1.5.4. We are now ready to prove that (1.11) is an equivalence.

Proof. Since $\mathrm{Rep}(\mathsf{G})$ is the derived category of its heart, the space of right t-exact symmetric monoidal functors

$$\mathrm{Rep}(\mathsf{G}) \to \mathrm{QCoh}(S) \otimes \mathrm{QLisse}(X)$$

is isomorphic to the space of symmetric monoidal functors

$$\mathrm{Rep}(\mathsf{G})^\heartsuit \to (\mathrm{QCoh}(S) \otimes \mathrm{QLisse}(X))^{\leq 0},$$

and similarly for every ${}^{\leq n}S$.

The assertion now follows from Lemma 1.5.3.

□

1.5.5. Recall the fully faithful embedding (1.1). We will now show how to rewrite the functor of points of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ in terms of the (more manageable) category $\mathrm{IndLisse}(X)$:

Proposition 1.5.6. *Suppose that* S *is eventually coconnective*[11]*. Then the space of (right t-exact) symmetric monoidal functors*

$$\mathrm{Rep}(\mathsf{G}) \to \mathrm{QCoh}(S) \otimes \mathrm{IndLisse}(X)$$

maps isomorphically to the space of (right t-exact) symmetric monoidal

$$\mathrm{Rep}(\check{G}) \to \mathrm{QCoh}(S) \otimes \mathrm{QLisse}(X).$$

Proof. The space of (continuous) symmetric monoidal functors

$$\mathrm{Rep}(\mathsf{G}) \to \mathrm{QCoh}(S) \otimes \mathrm{IndLisse}(X)$$

maps isomorphically to the space of symmetric monoidal functors

$$\mathrm{Rep}(\mathsf{G})^c \to \mathrm{QCoh}(S) \otimes \mathrm{IndLisse}(X),$$

and similarly for $\mathrm{IndLisse}(X)$ replaced by $\mathrm{QLisse}(X)$.

[11] I.e., the cohomologies of its structure ring live in finitely many degrees.

Since every object of $\mathrm{Rep}(\mathsf{G})^c$ is dualizable, it suffices to show that the embedding

$$\mathrm{QCoh}(S) \otimes \mathrm{IndLisse}(X) \hookrightarrow \mathrm{QCoh}(S) \otimes \mathrm{QLisse}(X) \tag{1.12}$$

induces an equivalence on the subcategories of dualizable objects.

By Lemma 1.5.3(a'), the functor (1.12) identifies $\mathrm{QCoh}(S) \otimes \mathrm{QLisse}(X)$ with the left completion of $\mathrm{QCoh}(S) \otimes \mathrm{IndLisse}(X)$.

Hence, it suffices to show that (for S eventually coconnective), any dualizable object in the category $\mathrm{QCoh}(S) \otimes \mathrm{QLisse}(X)$ is bounded below (in the sense of the t-structure).

We have a conservative t-exact functor $\mathrm{ev}_x : \mathrm{QLisse}(X) \to \mathrm{Vect}_{\mathsf{e}}$, so the functor

$$\mathrm{QCoh}(S) \otimes \mathrm{QLisse}(X) \overset{\mathrm{Id} \otimes \mathrm{ev}_x}{\longrightarrow} \mathrm{QCoh}(S)$$

is also conservative.

Hence, it is enough to show that $\mathrm{Id} \otimes \mathrm{ev}_x$ sends dualizable objects to objects bounded below. However, $\mathrm{Id} \otimes \mathrm{ev}_x$ is symmetric monoidal, the assertion follows from the fact that dualizable objects in $\mathrm{QCoh}(S)$ (for S eventually coconnective) are bounded below.

□

1.6. **Deformation theory.** In this subsection we will show that the prestack $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ (and its version $\mathrm{LocSys}^{\mathrm{restr},\mathrm{rigid}_x}_{\mathsf{G}}(X)$) admit deformation theory.

1.6.1. Our local goal is to prove the following:

Proposition 1.6.2.

(a) *The prestack* $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ *admits a* (-1)*-connective corpresentable deformation theory.*

(b) *For* $S \in \mathrm{Sch}^{\mathrm{aff}}_{/\mathsf{e}}$ *and an* S*-point*

$$\mathsf{F} : \mathrm{Rep}(\mathsf{G}) \to \mathrm{QCoh}(S) \otimes \mathrm{QLisse}(X)$$

of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$*, the cotangent space* $T^*_{\mathsf{F}}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)) \in \mathrm{QCoh}(S)^{\leq 1}$ *identifies with*

$$(\mathrm{Id} \otimes \mathrm{C}^{\cdot}_c(X, -))(\mathsf{F}(\mathfrak{g}^\vee) \otimes \omega_X)[-1].$$

Remark 1.6.3. We refer the reader to [GR2, Chapter 1, Definition 7.1.5(a)], where it is explained what it means to admit a $(-n)$-connective corpresentable deformation theory. In fact, there are three conditions:

(i) The first one is that the prestack admits deformation theory (i.e. admits pro-cotangent spaces that are functorial in the test-scheme, and is infinitesimally cohesive);

(ii) The adjective "corepresentable" refers to the fact that the pro-cotangent spaces are actually objects (of $\mathrm{QCoh}(S)^{<\infty}$, where S is the test-scheme), and not only pro-objects.

(iii) The adjective "$(-n)$-connective" refers to the fact that cotangent spaces actually belongs to $\mathrm{QCoh}(S)^{\leq n}$.

As a formal corollary of Proposition 1.6.2, we obtain:

Corollary 1.6.4.

(a) *The prestack* $\mathrm{LocSys}^{\mathrm{restr},\mathrm{rigid}_x}_{\mathsf{G}}(X)$ *admits a connective corepresentable deformation theory.*

(b) *For* $S \in \mathrm{Sch}^{\mathrm{aff}}_{/\mathsf{e}}$ *and an* S*-point of* $\mathrm{LocSys}^{\mathrm{restr},\mathrm{rigid}_x}_{\mathsf{G}}(X)$*, the corresponding cotangent space belongs to*

$$\mathrm{QCoh}(S)^c \cap \mathrm{QCoh}(S)^{\leq 0} \subset \mathrm{QCoh}(S)^{\leq 0}.$$

1.6.5. The rest of this subsection is devoted to the proof of Proposition 1.6.2. First, we show that $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ admits deformation theory. Given that $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ is convergent, and applying [GR2, Chapter 1, Proposition 7.2.5], we have to show that for a push-out diagram of eventually coconnective affine schemes

$$\begin{tikzcd} S_1 \arrow[r] \arrow[d] & S_2 \arrow[d] \\ S_1' \arrow[r] & S_2', \end{tikzcd} \tag{1.13}$$

where $S_1 \to S_1'$ is a nilpotent embedding (see [GR2, Chapter 1, Sect. 5.5.1]), the diagram

$$\begin{tikzcd} \mathrm{Maps}(S_2', \mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)) \arrow[r] \arrow[d] & \mathrm{Maps}(S_1', \mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)) \arrow[d] \\ \mathrm{Maps}(S_2, \mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)) \arrow[r] & \mathrm{Maps}(S_1, \mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)) \end{tikzcd}$$

is a pullback square of spaces.

For this, using Proposition 1.5.6, for S eventually coconnective, we interpret $\mathrm{Maps}(S, \mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X))$ as the space of right t-exact symmetric monoidal functors

$$\mathrm{Rep}(\mathsf{G})^c \to (\mathrm{QCoh}(S) \otimes \mathrm{IndLisse}(X))^{\mathrm{dualizable}}.$$

Hence, it suffices to show that in the situation of (1.13), the diagram

$$\begin{tikzcd} (\mathrm{QCoh}(S_2') \otimes \mathrm{IndLisse}(X))^{\mathrm{dualizable}} \arrow[r] \arrow[d] & (\mathrm{QCoh}(S_1') \otimes \mathrm{IndLisse}(X))^{\mathrm{dualizable}} \arrow[d] \\ (\mathrm{QCoh}(S_2) \otimes \mathrm{IndLisse}(X))^{\mathrm{dualizable}} \arrow[r] & (\mathrm{QCoh}(S_1) \otimes \mathrm{IndLisse}(X))^{\mathrm{dualizable}} \end{tikzcd} \tag{1.14}$$

is a pullback square of (small) symmetric monoidal categories. This follows by repeating the proof of [GR2, Chapter 7, Corollary A.2.2]. We include the argument for completeness.

1.6.6. First, as in [GR2, Chapter 7, Sect. A.2.3], we reduce to the case when the affine schemes involved are almost of finite type. In this case, by [GR2, Chapter 7, Theorem A.1.2], we have a pullback square of symmetric monoidal categories

$$\begin{tikzcd} \mathrm{IndCoh}(S_2') \arrow[r] \arrow[d] & \mathrm{IndCoh}(S_1') \arrow[d] \\ \mathrm{IndCoh}(S_2) \arrow[r] & \mathrm{IndCoh}(S_1), \end{tikzcd}$$

equipped with the !-tensor product. Hence,

$$\begin{tikzcd} \mathrm{IndCoh}(S_2') \otimes \mathrm{IndLisse}(X) \arrow[r] \arrow[d] & \mathrm{IndCoh}(S_1') \otimes \mathrm{IndLisse}(X) \arrow[d] \\ \mathrm{IndCoh}(S_2) \otimes \mathrm{IndLisse}(X) \arrow[r] & \mathrm{IndCoh}(S_1) \otimes \mathrm{IndLisse}(X) \end{tikzcd}$$

is also a pullback square, and hence so is

$$\begin{tikzcd} (\mathrm{IndCoh}(S_2') \otimes \mathrm{IndLisse}(X))^{\mathrm{dualizable}} \arrow[r] \arrow[d] & (\mathrm{IndCoh}(S_1') \otimes \mathrm{IndLisse}(X))^{\mathrm{dualizable}} \arrow[d] \\ (\mathrm{IndCoh}(S_2) \otimes \mathrm{IndLisse}(X))^{\mathrm{dualizable}} \arrow[r] & (\mathrm{IndCoh}(S_1) \otimes \mathrm{IndLisse}(X))^{\mathrm{dualizable}}. \end{tikzcd}$$

Finally, as in [GR1, Chapter 6, Lemma 3.3.7], we show that for any $S \in \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft},/\mathsf{e}}$, the functor of tensoring by the dualizing complex of S defines an equivalence

$$(\mathrm{QCoh}(S) \otimes \mathrm{IndLisse}(X))^{\mathrm{dualizable}} \to (\mathrm{IndCoh}(S) \otimes \mathrm{IndLisse}(X))^{\mathrm{dualizable}}.$$

1.6.7. To prove the remaining assertions of Proposition 1.6.2, it suffices to perform the calculation of point (b). Let $\mathcal{M}$ be an object of $\mathrm{QCoh}(S)^{\leq 0}$, and let $S_{\mathcal{M}} \in \mathrm{Sch}^{\mathrm{aff}}_{/\mathsf{e}}$ be the corresponding split square-zero extension of S. We need to construct an isomorphism

$$\mathrm{Maps}(S_{\mathcal{M}}, \mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)) \underset{\mathrm{Maps}(S, \mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X))}{\times} \{*\} \simeq \tau^{\leq 0}\left(\Gamma\left(S, \mathcal{M} \otimes (\mathrm{Id} \otimes \mathrm{C}^{\cdot}(X,-))(\mathsf{F}(\mathfrak{g}))\right)[1]\right).$$

Let $\mathbf{A}$ be a symmetric monoidal category and let $\mathbf{a} \in \mathbf{A}$ be an object. We regard $\mathbf{1}_{\mathbf{A}} \oplus \mathbf{a}$ as an object of $\mathrm{ComAlg}(\mathbf{A})$. Consider the category

$$(\mathbf{1}_{\mathbf{A}} \oplus \mathbf{a})\text{-mod}(\mathbf{A})$$

as a symmetric monoidal category, equipped with a symmetric monoidal functor back to $\mathbf{A}$, given by

$$- \underset{\mathbf{1}_{\mathbf{A}} \oplus \mathbf{a}}{\otimes} \mathbf{1}_{\mathbf{A}}.$$

We have the following general claim:

Proposition 1.6.8. *Given a symmetric monoidal functor*

$$\mathsf{F} : \mathrm{Rep}(\mathsf{G}) \to \mathbf{A},$$

the space of its lifts to a functor

$$\mathrm{Rep}(\mathsf{G}) \to (\mathbf{1}_{\mathbf{A}} \oplus \mathbf{a})\text{-mod}(\mathbf{A})$$

identifies canonically with

$$\mathrm{Maps}_{\mathbf{A}}(\mathbf{1}_{\mathbf{A}}, \mathbf{a} \otimes \mathsf{F}(\mathfrak{g})[1]).$$

Applying this to

$$\mathbf{A} := \mathrm{QCoh}(S) \otimes \mathrm{QLisse}(X), \quad \mathbf{a} = \mathcal{M} \otimes \mathsf{e}_X,$$

we obtain the desired result.

Proof of Proposition 1.6.8. For a symmetric monoidal category $\mathbf{A}'$, the datum of a symmetric monoidal functor $\mathsf{F}' : \mathrm{Rep}(\mathsf{G}) \to \mathbf{A}'$ is equivalent to the datum of a commutative algebra object $\mathcal{R} := \mathsf{F}'(\mathcal{O}_{\mathsf{G}}) \in \mathbf{A}'$, equipped with an action of G, i.e., a commutative algebra object in $\mathbf{A}' \otimes \mathrm{Rep}(\mathsf{G})$ with the following property:

The map

$$\mathcal{R} \otimes \mathcal{R} \to \mathrm{coInd}_{\mathsf{G}}^{\mathsf{G} \times \mathsf{G}}(\mathcal{R}),$$

obtained by adjunction from the product map

$$\mathcal{R} \otimes \mathcal{R} \to \mathcal{R},$$

is an isomorphism.

Given such object $\mathcal{R}$ corresponding to a given functor F, we wish to classify its lifts to an $(\mathbf{1}_{\mathbf{A}} \oplus \mathbf{a})$-algebra $\mathcal{R}'$, so that

$$\mathbf{1}_{\mathbf{A}} \underset{\mathbf{1}_{\mathbf{A}} \oplus \mathbf{a}}{\otimes} \mathcal{R}' \simeq \mathcal{R}.$$

The assertion of the proposition follows now from the fact that, by functoriality, the cotangent space of $\mathcal{R}$, viewed as a commutative algebra object in $\mathbf{A}' \otimes \mathrm{Rep}(\mathsf{G})$, identifies with $\mathsf{F}(\mathfrak{g}^\vee)$.

□

1.7. **A Tannakian intervention.** In order to access the classical prestack underlying $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$, we will approximate the category $\mathrm{QLisse}(X)$ by the category of representations of a pro-algebraic group. This will be done using the usual Tannakian formalism.

1.7.1. Recall the fiber functor

$$\mathrm{QLisse}(X) \xrightarrow{\mathrm{ev}_x} \mathrm{Vect}_{\mathsf{e}},$$

see (1.9). We precompose it with

$$\mathrm{IndLisse}(X) \to \mathrm{QLisse}(X),$$

and thus obtain a symmetric monoidal functor

$$\mathrm{IndLisse}(X) \to \mathrm{Vect}_{\mathsf{e}},$$

which preserves compactness and hence admits a colimit-preserving right adjoint. The composite

$$\mathrm{ev}_x \circ (\mathrm{ev}_x)^R$$

has a structure of *comonad* on $\mathrm{Vect}_{\mathsf{e}}$. Hence, it is given by tensor product with a co-algebra, to be denoted $\mathcal{H}$.

1.7.2. Since ev_x is t-exact, its right adjoint is left t-exact, and hence $\mathcal{H} \in (\mathrm{Vect}_{\mathsf{e}})^{\geq 0}$. Therefore, the category $\mathcal{H}$-comod acquires a t-structure, uniquely characterized by the requirement that that the forgetful functor

$$\mathbf{oblv}_{\mathcal{H}} : \mathcal{H}\text{-comod} \to \mathrm{Vect}_{\mathsf{e}}$$

is t-exact.

The functor ev_x lifts to a functor

$$(\mathrm{ev}_x)^{\mathrm{enh}} : \mathrm{IndLisse}(X) \to \mathcal{H}\text{-comod}; \tag{1.15}$$

this functor is t-exact by construction.

We claim:

Proposition 1.7.3. *The functor* $(\mathrm{ev}_x)^{\mathrm{enh}}$ *induces an equivalence*

$$(\mathrm{IndLisse}(X))^{>-\infty} \to (\mathcal{H}\text{-comod})^{>-\infty}.$$

1.7.4. We will prove Proposition 1.7.3(a) in the following more general framework. We will start with a DG category $\mathbf{C}$ equipped with a functor $\mathsf{F} : \mathbf{C} \to \mathrm{Vect}_{\mathsf{e}}$ that admits a continuous right adjoint. We will make the following assumptions:

- $\mathbf{C}$ is equipped with a t-structure compatible with filtered colimits, in which it is right complete;
- The functor F is t-exact and is conservative on $\mathbf{C}^{\heartsuit}$.

Let $\mathcal{H}$ denote the resulting co-associative co-algebra in $\mathrm{Vect}_{\mathsf{e}}$.

We claim that in this case, the resulting functor

$$\mathsf{F}^{\mathrm{enh}} : \mathbf{C} \to \mathcal{H}\text{-comod}$$

induces an equivalence

$$(\mathbf{C})^{>-\infty} \to (\mathcal{H}\text{-comod})^{>-\infty}.$$

Indeed, the functor

$$\mathsf{F} : (\mathbf{C})^{>-\infty} \to (\mathrm{Vect}_{\mathsf{e}})^{>-\infty}$$

is conservative, and commutes with totalizations of cosimplicial objects. Hence, it satisfies the conditions of the Barr-Beck-Lurie theorem.

1.7.5. As a formal corollary of Proposition 1.7.3, we obtain:

Corollary 1.7.6. *The functor* (1.15) *induces an equivalence between* $\mathrm{QLisse}(X)$ *and the left completion of* $\mathcal{H}$*-comod.*

1.7.7. The symmetric monoidal structure on $(\mathrm{IndLisse}(X), \mathrm{ev}_x)$ defines on $\mathcal{H}$ a structure of commutative Hopf algebra, so that $(\mathcal{H}\text{-comod}, \mathbf{oblv}_{\mathcal{H}})$ is symmetric monoidal, and so is the functor

$$\mathrm{ev}_x^{\mathrm{enh}} : \mathrm{IndLisse}(X) \to \mathcal{H}\text{-comod}.$$

We can now give yet a third interpretation for values of the functor $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)$ on eventually coconnective affine schemes:

Proposition 1.7.8. *For an eventually coconnective affine scheme S, the the functor $\mathrm{ev}_x^{\mathrm{enh}}$ induces an equivalence between the space of (right t-exact) symmetric monoidal functors*

$$\mathrm{Rep}(\mathsf{G}) \to \mathrm{QCoh}(S) \otimes \mathrm{QLisse}(X)$$

and the space of (right t-exact) symmetric monoidal functors

$$\mathrm{Rep}(\mathsf{G}) \to \mathrm{QCoh}(S) \otimes \mathcal{H}\text{-comod}.$$

The proof repeats the argument proving Proposition 1.5.6.

1.7.9. For the sake of completeness, we will now discuss a more precise version of Corollary 1.7.6. Let $\mathcal{H}$ be any coalgebra in $\mathrm{Vect}_{\mathsf{e}}^{\geq 0}$. Let us describe explicitly the category

$$(\mathcal{H}\text{-comod})^\wedge.$$

Consider the following comonad on $\mathrm{Vect}_{\mathsf{e}}$, denoted $\widehat{\mathcal{H}}$:

$$\widehat{\mathcal{H}}(V) = \lim_n (\mathcal{H} \otimes \tau^{\geq -n}(V)).$$

Let

$$\widehat{\mathcal{H}}\text{-comod}$$

be the category of $\widehat{\mathcal{H}}$-comodules acting on $\mathrm{Vect}_{\mathsf{e}}$.

Proposition 1.7.10. *There exists a canonical equivalence*

$$(\mathcal{H}\text{-comod})^\wedge \simeq \widehat{\mathcal{H}}\text{-comod}.$$

Proof. The category $\widehat{\mathcal{H}}$-comod carries a natural t-structure compatible with the forgetful functor to $\mathrm{Vect}_{\mathsf{e}}$.

It is clear that for every n, we have

$$(\widehat{\mathcal{H}}\text{-comod})^{\geq -n} \simeq (\mathcal{H}\text{-comod})^{\geq -n}.$$

Now, it is easy to see that $\widehat{\mathcal{H}}$-comod is left-complete in its t-structure. Indeed, for a an object

$$\{\mathbf{c}^n \in (\widehat{\mathcal{H}}\text{-comod})^{\geq -n}\},$$

its limit in $\widehat{\mathcal{H}}$-comod is given by he vector space

$$\lim_n \mathbf{oblv}_{\widehat{\mathcal{H}}}(\mathbf{c}^n),$$

and the action of $\widehat{\mathcal{H}}$ is given by

$$\lim_n \mathbf{oblv}_{\widehat{\mathcal{H}}}(\mathbf{c}^n) \to \lim_n (\mathcal{H} \otimes \mathbf{oblv}_{\widehat{\mathcal{H}}}(\mathbf{c}^n)) =: \widehat{\mathcal{H}} \left(\lim_n \mathbf{oblv}_{\widehat{\mathcal{H}}}(\mathbf{c}^n) \right).$$

□

1.8. **Proof of ind-representability.** In this subsection we will prove that $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X)$ is an ind-affine ind-scheme locally almost of finite type.

We will do so by reducing to the case when instead of the category $\mathrm{QLisse}(X)$ we are dealing with the category $\mathrm{Rep}(\mathsf{H})$ of representations of an algebraic group H.

1.8.1. Recall (see [GR2, Definition 1.1.2]) that prestack $\mathcal{Y}$ is an *ind-scheme* if it can be written as a *filtered* colimit

$$\mathcal{Y} \simeq \underset{i}{\operatorname{colim}}\, Y_i, \tag{1.16}$$

where Y_i are (quasi-compact) schemes and the transition maps $Y_i \to Y_j$ are closed embeddings.

An ind-scheme is *ind-affine* if all Y_i can be chosen to be affine.

An ind-scheme is locally almost of finite type as a prestack if all Y_i can be chosen to be almost of finite type see ([GR2, Corollary 1.7.5(a)]).

1.8.2. By Corollary 1.6.4(a) combined with [GR2, Chapter 2, Corollary 1.3.13], in order to show that $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X)$ is an ind-affine ind-scheme, it suffices to show that its classical truncation ${}^{\mathrm{cl}}\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X)$ is a classical ind-affine ind-scheme.

Similarly, by Corollary 1.6.4(b) combined with [GR2, Chapter 1, Theorem 9.1.2], in order to show that $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X)$ is locally almost of finite type, it suffices to show that ${}^{\mathrm{cl}}\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X)$ is locally of finite type as a classical prestack.

1.8.3. Let S be a truncated affine scheme. From Proposition 1.7.8 we obtain that the value of $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X)$ on S is the category of symmetric monoidal functors

$$\mathrm{Rep}(\mathsf{G}) \to \mathrm{QCoh}(S) \otimes \mathcal{H}\text{-comod}$$

equipped with an identification of the composition

$$\mathrm{Rep}(\mathsf{G}) \to \mathrm{QCoh}(S) \otimes \mathcal{H}\text{-comod} \overset{\mathbf{oblv}_{\mathcal{H}}}{\to} \mathrm{QCoh}(S)$$

and

$$\mathrm{Rep}(\mathsf{G}) \overset{\mathbf{oblv}_{\mathsf{G}}}{\to} \mathrm{Vect}_{\mathsf{e}} \overset{\mathrm{unit}}{\to} \mathrm{QCoh}(S).$$

The category of such functors is equivalent to the category of maps of Hopf algebras over $\mathcal{O}_S$

$$\mathcal{O}_S \otimes \mathcal{O}_{\mathsf{G}} \to \mathcal{O}_S \otimes \mathcal{H}. \tag{1.17}$$

1.8.4. Let $\mathcal{H}^0$ denote the 0-th cohomology of $\mathcal{H}$. This is a classical commutative Hopf algebra. Denote

$$\mathrm{Gal}_{X,x} := \mathrm{Spec}(\mathcal{H}^0);$$

this is the algebraic Galois (a.k.a. Tannakian) group corresponding to the category $\mathrm{QLisse}(X)^{\heartsuit}$ with fiber functor ev_x.

Let S be a classical affine scheme. We obtain that the value of ${}^{\mathrm{cl}}\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X)$ on S is the set of homomorphisms of group-schemes over S

$$S \times \mathrm{Gal}_{X,x} \to S \times \mathsf{G}. \tag{1.18}$$

We claim that this functor, to be denoted

$$\mathbf{Maps}_{\mathrm{Grp}}(\mathrm{Gal}_{X,x}, \mathsf{G}),$$

is representable by an ind-affine ind-scheme locally of finite type.

1.8.5. Indeed, write $\mathrm{Gal}_{X,x}$ as a filtered colimit

$$\mathrm{Gal}_{X,x} := \lim_{\alpha}\, \mathrm{Gal}^{\alpha}_{X,x},$$

where the transition maps are surjective.

Then,

$$\mathbf{Maps}_{\mathrm{Grp}}(\mathrm{Gal}_{X,x}, \mathsf{G}) \simeq \underset{\alpha}{\operatorname{colim}}\, \mathbf{Maps}_{\mathrm{Grp}}(\mathrm{Gal}^{\alpha}_{X,x}, \mathsf{G}),$$

and the transition maps are closed embeddings.

Hence, it suffices to show the following:

Proposition 1.8.6. *For a pair of algebraic groups* H *and* G*, the prestack*

$$\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H},\mathsf{G}), \quad S \mapsto \mathrm{Hom}_{\mathrm{Grp\text{-}Sch}_{/S}}(S\times\mathsf{H}, S\times\mathsf{G})$$

is representable by an ind-affine ind-scheme locally almost of finite type.

Proof. Below we will show that the classical prestack underlying $\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H},\mathsf{G})$ is an ind-affine ind-scheme locally of finite type. This will imply that $\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H},\mathsf{G})$ is an ind-affine ind-scheme locally almost of finite type at the derived level because it is easy to control the deformation theory of $\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H},\mathsf{G})$: for an affine test scheme S, and an S-point ϕ of $\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H},\mathsf{G})$, the cotangent space

$$T^*_\phi(\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H},\mathsf{G}))\in\mathrm{QCoh}(S)$$

identifies with

$$\mathrm{Fib}((\mathfrak{g})^\vee\otimes\mathcal{O}_S\to\mathrm{coinv}_{\mathsf{H}}((\mathfrak{g})^\vee\otimes\mathcal{O}_S)),$$

see the proof of Proposition 2.3.9 below. In the above formula the notations are as follows:

- $\mathfrak{g}$ is the Lie algebra of G and we regard it as a representation of H via ϕ;
- $\mathrm{coinv}_{\mathsf{H}}$ stands for the functor of invariants $\mathrm{Rep}(\mathsf{H})\to\mathrm{Vect}_{\mathsf{e}}$.

From now in we will consider the underlying classical prestacks and omit the superscript "cl" from the nottaion.

Consider the prestack

$$\mathbf{Maps}_{\mathrm{Sch}}(\mathsf{H},\mathsf{G}), \quad S\mapsto\mathrm{Hom}_{\mathrm{Sch}_{/S}}(S\times\mathsf{H},S\times\mathsf{G}).$$

Since a closed subfunctor of an ind-affine ind-scheme locally of finite type is again an ind-affine ind-scheme locally of finite type, it suffices to show that for a pair of affine schemes of finite type Y_1, Y_2, the prestack

$$\mathbf{Maps}_{\mathrm{Sch}}(Y_1,Y_2), \quad S\mapsto\mathrm{Hom}(S\times Y_1,Y_2)$$

is an ind-affine ind-scheme locally of finite type.

Let $Y_2\hookrightarrow Y_2'$ be a closed embedding of affine schemes of finite type. It is easy to see that the corresponding map

$$\mathbf{Maps}_{\mathrm{Sch}}(Y_1,Y_2)\to\mathbf{Maps}_{\mathrm{Sch}}(Y_1,Y_2')$$

is a closed embedding of functors.

Hence, we can assume that Y_2 is the affine space associated with a (finite-dimensional) vector space $V\in\mathrm{Vect}_{\mathsf{e}}^{\heartsuit}$, to be denoted $\mathrm{Tot}(V)$ (see (1.7)).

However, for any prestack $\mathcal{Y}$,

$$\mathbf{Maps}_{\mathrm{Sch}}(\mathcal{Y},\mathrm{Tot}(V))\simeq\mathrm{Tot}(W),$$

where

$$W:=V\otimes\Gamma(\mathcal{Y},\mathcal{O}_{\mathcal{Y}})\in\mathrm{Vect}_{\mathsf{e}}^{\heartsuit}.$$

Finally, we note that $\mathrm{Tot}(W)$ is indeed a classical ind-affine ind-scheme locally of finite type: writing

$$W\simeq\underset{i}{\mathrm{colim}}\,W_i,$$

with W_i finite dimensional, we have

$$\mathrm{Tot}(W)\simeq\underset{i}{\mathrm{colim}}\ \mathrm{Tot}(W_i),$$

while

$$\mathrm{Tot}(W_i)\simeq\mathrm{Spec}(\mathrm{Sym}(W_i^\vee)).$$

□

2. Uniformization and the end of proof of Theorem 1.3.2

In this section we will finish the proof of Theorem 1.3.2, while introducing a tool of independent interest: a uniformization of $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)$ by *algebraic stacks* associated to parabolic subgroups in G and *irreducible* local systems for their Levi quotients.

2.1. **What is there left to prove?**

2.1.1. We claim that in order to finish the proof of Theorem 1.3.2, it remains to show the following:

Theorem 2.1.2. *The underlying reduced prestack of* $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X)$ *is a disjoint union of affine schemes.*

Let us show how Theorem 2.1.2 implies Theorem 1.3.2.

2.1.3. Indeed, we have already shown that $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X)$ is an ind-affine ind-scheme locally almost of finite type. Combined with Theorem 2.1.2, this implies points (a,b,c) of Theorem 1.3.2.

To prove point (d), we quote the following result, which is a particular case of [Lu3, Theorem 18.2.3.2] (combined with [GR3, Proposition. 6.7.4]):

Theorem 2.1.4. *Let* $\mathcal{Y}$ *be an ind-scheme locally almost of finite type with the following properties:*

(i) ${}^{\mathrm{red}}\mathcal{Y}$ *is an affine scheme;*

(ii) *For any* $(S,y) \in \mathrm{Sch}^{\mathrm{aff}}_{/\mathcal{Y}}$, *the cotangent space* $T^*_y(\mathcal{Y}) \in \mathrm{Pro}(\mathrm{QCoh}(S)^{\leq 0})$ *actually belongs to* $\mathrm{QCoh}(S)^{\leq 0}$.

Then $\mathcal{Y}$ *can be written in the form* (1.3)*.*

For the sake of completeness, we will outline the proof of Theorem 2.1.4 in Sect. A.

2.2. **Uniformization.** In this subsection we will begin the proof of Theorem 2.1.2. The method is based on constructing an algebraic stack that maps dominantly onto ${}^{\mathrm{red}}\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)$. This construction will also shed some light on "what $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)$ looks like".

2.2.1. Having proved that $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X)$ is an ind-affine ind-scheme locally almost of finite type, we know that each connected component ${}^{\mathrm{red}}\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X)$ is a filtered colimit of reduced affine schemes along closed embeddings. Hence, in order to prove Theorem 2.1.2, it suffices to show that these colimits stabilize.

We will achieve this by the following construction. We will find an *algebraic stack* locally almost of finite type $\widetilde{\mathrm{LocSys}}_{\mathsf{G}}^{\mathrm{restr}}(X)$, equipped with a map

$$\pi : \widetilde{\mathrm{LocSys}}_{\mathsf{G}}^{\mathrm{restr}}(X) \to \mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)$$

with the following properties at the level of the underlying reduced prestacks:

- Each connected component of $\widetilde{\mathrm{LocSys}}_{\mathsf{G}}^{\mathrm{restr}}(X)$ is quasi-compact and irreducible;
- π is schematic and proper on every connected component of $\widetilde{\mathrm{LocSys}}_{\mathsf{G}}^{\mathrm{restr}}(X)$;
- π is dominant (at the level of the underlying reduced prestacks);
- The set of connected components of $\widetilde{\mathrm{LocSys}}_{\mathsf{G}}^{\mathrm{restr}}(X)$ splits as a union of finite clusters, and elements from different clusters have non-intersecting images in $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)$.

It is clear that an existence of such $(\widetilde{\mathrm{LocSys}}_{\mathsf{G}}^{\mathrm{restr}}(X), \pi)$ would imply the required properties of ${}^{\mathrm{red}}\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X)$.

2.2.2. We shall now proceed to the construction of $\widetilde{\mathrm{LocSys}}_{\mathsf{G}}^{\mathrm{restr}}(X)$. Let $\mathrm{Par}(\mathsf{G})$ be the (po)set of standard parabolics in G.

For every $\mathsf{P} \in \mathrm{Par}(\mathsf{G})$, let M denote its Levi quotient. The maps

$$\mathsf{G} \leftarrow \mathsf{P} \to \mathsf{M}$$

induce the maps

$$\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X) \overset{\mathsf{p}_{\mathsf{P}}}{\leftarrow} \mathrm{LocSys}_{\mathsf{P}}^{\mathrm{restr}}(X) \overset{\mathsf{q}_{\mathsf{P}}}{\to} \mathrm{LocSys}_{\mathsf{M}}^{\mathrm{restr}}(X).$$

Let σ_{M} be an *irreducible* local system for M. Choose its trivialization at x, so we obtain an e-point of $\mathrm{LocSys}_{\mathsf{M}}^{\mathrm{restr,rigid}_x}(X)$. Let $\mathrm{Stab}_{\mathsf{M}}(\sigma_{\mathsf{M}})$ denote its stabilizer in M. We obtain a closed embedding

$$\mathrm{pt}\,/\,\mathrm{Stab}_{\mathsf{M}}(\sigma_{\mathsf{M}}) \hookrightarrow \mathrm{LocSys}_{\mathsf{M}}^{\mathrm{restr}}(X).$$

Denote

$$\mathrm{LocSys}_{\mathsf{P},\sigma_{\mathsf{M}}}^{\mathrm{restr}}(X) := \mathrm{LocSys}_{\mathsf{P}}^{\mathrm{restr}}(X) \underset{\mathrm{LocSys}_{\mathsf{M}}^{\mathrm{restr}}(X)}{\times} \mathrm{pt}\,/\,\mathrm{Stab}_{\mathsf{M}}(\sigma_{\mathsf{M}}).$$

Finally, we set

$$\widetilde{\mathrm{LocSys}}_{\mathsf{G}}^{\mathrm{restr}}(X) := \underset{\mathsf{P}\in \mathrm{Par}(\mathsf{G})}{\sqcup}\ \underset{\sigma_{\mathsf{M}}}{\sqcup}\ \mathrm{LocSys}_{\mathsf{P},\sigma_{\mathsf{M}}}^{\mathrm{restr}}(X).$$

The maps p_{P} define the sought-for map

$$\pi : \widetilde{\mathrm{LocSys}}_{\mathsf{G}}^{\mathrm{restr}}(X) \to \mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X).$$

2.2.3. We first show that $\widetilde{\mathrm{LocSys}}_{\mathsf{G}}^{\mathrm{restr}}(X)$ is an algebraic stack, each of whose connected components is quasi-compact and irreducible. I.e., we have to show that each $\mathrm{LocSys}_{\mathsf{P},\sigma_{\mathsf{M}}}^{\mathrm{restr}}(X)$ has this property. For this, it is sufficient to show that the map

$$\mathsf{q}_{\mathsf{P}} : \mathrm{LocSys}_{\mathsf{P}}^{\mathrm{restr}}(X) \to \mathrm{LocSys}_{\mathsf{M}}^{\mathrm{restr}}(X)$$

is a relative algebraic stack (i.e., its base change by an derived affine scheme yields an algebraic stack) with fibers that are quasi-compact and irreducible.

The property of a map between prestacks to be a relative algebraic stack with fibers that are quasi-compact and irreducible is stable under compositions. Filtering the unipotent radical of P we reduce the assertion to the following:

Proposition 2.2.4. *Let*

$$1 \to \mathrm{Tot}(V) \to \mathsf{G}_1 \to \mathsf{G}_2 \to 1 \tag{2.1}$$

be a short exact sequence of algebraic groups, where $\mathrm{Tot}(V)$ *is the vector group associated with a* G_2*-representation* V*. Then the resulting map*

$$\mathrm{LocSys}_{\mathsf{G}_1}^{\mathrm{restr}}(X) \to \mathrm{LocSys}_{\mathsf{G}_2}^{\mathrm{restr}}(X)$$

is a relative algebraic stack whose fibers are quasi-compact and irreducible.

2.2.5. Before we prove Proposition 2.2.4, we make the following observation.

First, the datum of (2.1) is equivalent to that of an object

$$\mathrm{cl}_{\mathsf{G}_1} \in \mathrm{Maps}_{\mathrm{Rep}(\mathsf{G}_2)}(\mathrm{triv}, V[2]).$$

Let now $\mathbf{C}$ be a symmetric monoidal category, and let us be given a a symmetric monoidal functor

$$\mathsf{F} : \mathrm{Rep}(\mathsf{G}_2) \to \mathbf{C}.$$

Consider the object $\mathsf{F}(V) \in \mathbf{C}$ and

$$\mathsf{F}(\mathrm{cl}_{\mathsf{G}_1}) \in \mathrm{Maps}(\mathbf{1}_{\mathbf{C}}, \mathsf{F}(V)[2]).$$

Then the space of lifts of F to a functor

$$\mathrm{Rep}(\mathsf{G}_1) \to \mathbf{C}$$

identifies with the space of null-homotopies of $\mathsf{F}(\mathrm{cl}_{\mathsf{G}_1})$.

2.2.6. *Proof of Proposition 2.2.4.* Let us be given an affine scheme S and an S-point

$$\mathsf{F} : \mathrm{Rep}(\mathsf{G}_2) \to \mathrm{QCoh}(S) \otimes \mathrm{QLisse}(X)$$

of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}_2}(X)$. Consider the object

$$\mathsf{F}(V) \in \mathrm{QCoh}(S) \otimes \mathrm{QLisse}(X),$$

and the object

$$\mathcal{E} := (\mathrm{Id}_{\mathrm{QCoh}(S)} \otimes \mathrm{C}^{\cdot}(X, -))(\mathsf{F}(V)) \in \mathrm{QCoh}(S).$$

According to Sect. 2.2.5, we have a point

$$\mathsf{F}(\mathrm{cl}_{\mathsf{G}_1}) \in \Gamma(S, \mathcal{E}[2]),$$

and the fiber product

$$S \underset{\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}_2}(X)}{\times} \mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}_1}(X)$$

is the functor that sends $S' \to S$ to the space of null-homotopies of $\mathsf{F}(\mathrm{cl}_{\mathsf{G}_1})|_{S'}$.

Hence, it remains to show that the above functor of null-homotipies is indeed an algebraic stack over S with fibers that are quasi-compact and irreducible. For that it suffices to show that, locally on S, the object $\mathcal{E}$ is perfect of amplitude $[0, d]$ for some d, i.e., can be represented by a finite complex

$$\mathcal{E}_0 \to \mathcal{E}_1 \to ... \to \mathcal{E}_d,$$

where each $\mathcal{E}_i$ is locally free of finite rank.

This property can be checked after restriction to ${}^{\mathrm{cl}}S$. Furthermore, since the prestacks involved are locally (almost) of finite type, we can assume that S is of finite type. In this case, the required property of $\mathcal{E}$ can be checked at the level of fibers at e-points of $\mathcal{E}$. Now, the required property follows from the fact that for

$$\mathcal{V} \in \mathrm{QLisse}(X)^{\heartsuit},$$

we have

$$\mathrm{C}^{\cdot}(X, \mathcal{V}) \in \mathrm{Vect}_{\mathsf{e}}^{\geq 0, \leq 2\dim(X)}.$$

$\square$[Proposition 2.2.4]

2.2.7. Next we show that π is schematic and proper on every connected component of $\widetilde{\mathrm{LocSys}}^{\mathrm{restr}}_{\mathsf{G}}(X)$. For that it is sufficient to show that at the level of the underlying reduced prestacks, the map

$$\mathsf{p}_{\mathsf{P}} : \mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{P}}(X) \to \mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$$

is schematic, quasi-compact and proper.

However, using the fact that G/P is proper, the assertion follows from the next observation:

Proposition 2.2.8. *Let G' be a subgroup of G. Then the map*

$$\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}'}(X) \to \mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X) \underset{\mathrm{pt}/\mathsf{G}}{\times} \mathrm{pt}/\mathsf{G}',$$

given by evaluation at x, is a closed embedding.

Proof. The statement is equivalent to the assertion that

$$\mathrm{LocSys}^{\mathrm{restr},\mathrm{rigid}_x}_{\mathsf{G}'}(X) \to \mathrm{LocSys}^{\mathrm{restr},\mathrm{rigid}_x}_{\mathsf{G}}(X)$$

is a closed embedding. Given that each side admits a connective corepresentable deformation theory, it suffices to show that the above map is a closed embedding at the level of the underlying classical ind-schemes.

I.e., we have to show that given a classical affine test-scheme S and a map

$$S \to \mathrm{LocSys}^{\mathrm{restr},\mathrm{rigid}_x}_{\mathsf{G}}(X),$$

the fiber product

$$S \underset{\mathrm{LocSys}^{\mathrm{restr},\mathrm{rigid}_x}_{\mathsf{G}}(X)}{\times} \mathrm{LocSys}^{\mathrm{restr},\mathrm{rigid}_x}_{\mathsf{G}'}(X),$$

viewed as a functor on *classical* affine schemes over S, is representably by a closed subscheme of S.

Note that the value of $\mathrm{LocSys}^{\mathrm{restr},\mathrm{rigid}_x}_{\mathsf{G}}(X)$ on a classical affine scheme $\mathrm{Spec}(R)$ is the set of the algebraic Galois group $\mathrm{Gal}_{X,x}$ (see Sect. 1.8.4) on the functor

$$\mathrm{Rep}(\mathsf{G})^{\heartsuit} \overset{\mathbf{oblv}_{\mathsf{G}}}{\to} \mathrm{Vect} \overset{\otimes R}{\to} R\text{-mod},$$

and similarly for $\mathrm{LocSys}^{\mathrm{restr},\mathrm{rigid}_x}_{\mathsf{G}'}(X)$.

Choosing a faithful representation of G, we can assume that $\mathsf{G} = GL_n$, and we are dealing with a faithful representation $\mathsf{G}' \hookrightarrow GL_n$. Denote the corresponding object of $\mathrm{Rep}(\mathsf{G}')^{\heartsuit}$ by V.

For every $W \in \mathrm{Rep}(\mathsf{G}')^{\heartsuit}$ we can find integers n and m, and a one-dimensional representation ℓ of G, so that

$$\ell \hookrightarrow V^{\otimes n} \text{ and } W \otimes \ell \hookrightarrow V^{\otimes m}.$$

It is now clear that the condition that a given action of $\mathrm{Gal}_{X,x}$ on

$$V^{\otimes n} \otimes R \text{ and } V^{\otimes m} \otimes R$$

should preserve the subspaces

$$\ell \otimes R' \subset V^{\otimes n} \otimes R' \text{ and } W \otimes \ell \otimes R' \subset V^{\otimes m} \otimes R'$$

is given by a closed subfunctor of $\mathrm{Spec}(R)$. Further, the tensor compatibility of these actions is another closed condition.

$\square$

2.3. **Dominance of the uniformization morphism.** In this subsection we will begin the proof of the fact that π is dominant (at the reduced level).

2.3.1. Let S be a classical integral affine scheme of finite type, and let us be given an S-point of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$. We will argue by induction on the semi-simple rank.

If one of the maps

$$S \underset{\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)}{\times} \mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{P}}(X) \to S \tag{2.2}$$

for a P a *proper* parabolic is dominant, we are done by the induction hypothesis.

Hence, after shrinking S we can assume that the products

$$S \underset{\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)}{\times} \mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{P}}(X)$$

for all proper parabolics are *empty*. In this case, we shall say that our S-point of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ is *irreducible*.

Hence, we need to show the following:

Proposition 2.3.2. *Let G be reductive. Let S be a classical integral affine scheme of finite type, and let us be given an S-point of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ that is irreducible. Then our S-point factors through an e-point.*

Note that as a particular case, we obtain:

Corollary 2.3.3. *For $\mathsf{G} = \mathsf{T}$ being a torus, the prestack ${}^{\mathrm{red}}\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{T}}(X)$ is the disjoint union of copies of pt/T over the set of isomorphism classes of T-local systems on X with e-coefficients.*

2.3.4. The rest of this subsection is devoted to the proof of Proposition 2.3.2.

According to Sect. 1.8.4, we can think about an S-point of $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)$ as an S-family of group homomorphisms

$$\mathrm{Gal}_{X,x} \to \mathsf{G}.$$

Hence, as in Sect. 1.8.5, in order to prove Proposition 2.3.2, it suffices to prove the following:

Proposition 2.3.5. *Let G, H be algebraic groups with G reductive. Let S be a classical integral scheme of finite type, and let us be given an S-point of $\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, \mathsf{G})$, such that for no e-point $s \in S$ the resulting homomorphism $\mathsf{H} \to \mathsf{G}$ factors through a proper parabolic. Then the above S-point lands in a single orbit of the G-action on $\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, \mathsf{G})$.*

The rest of this subsection is devoted to the proof of Proposition 2.3.5

2.3.6. Consider the Levi decomposition of H

$$1 \to \mathsf{H}_u \to \mathsf{H} \to \mathsf{H}_{\mathrm{red}} \to 1.$$

We claim that the given homomorphism

$$S \times \mathsf{H} \to \mathsf{G}$$

factors via a homomorphism

$$S \times \mathsf{H}_{\mathrm{red}} \to \mathsf{G}.$$

Since S is reduced, this assertion is enough to check after restricting to e-points on S. So we can assume dealing just with a homomorphism

$$\mathsf{H} \to \mathsf{G}.$$

With no restriction of generality, we may assume that the above homomorphism is injective. We now recall the following assertion from [Se]:

Theorem 2.3.7. *For a reductive group G and a subgroup $\mathsf{H} \subset \mathsf{G}$ the following conditions are equivalent:*

(i) *H is reductive;*

(ii) *If there exists a parabolic $\mathsf{P} \subset \mathsf{G}$ that contains H, then there exists a Levi splitting $\mathsf{P} \leftrightarrows \mathsf{M}$ such that $\mathsf{H} \subset \mathsf{M}$.*

By the irreducibility assumption, our subgroup H satisfies (ii) in Theorem 2.3.7. Hence, it is reductive as claimed.

2.3.8. Thus, in order to prove Proposition 2.3.5, it suffices to establish the following:

Proposition 2.3.9. *Let H and G be a pair of algebraic groups with H reductive. Then the ind-scheme $\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, \mathsf{G})$ is the disjoint union over isomorphism classes of homomorphisms*

$$\phi : \mathsf{H} \to \mathsf{G}$$

of the (classical) schemes $\mathsf{G}/\operatorname{Stab}_{\mathsf{G}}(\phi)$, where the stabilizer is taken with respect to the action of G on $\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, \mathsf{G})$ by conjugation.

This proposition is well-known. We will supply a proof for completeness.

Proof. It is enough to show that for any e-point of $\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, \mathsf{G})$

$$\phi : \mathsf{H} \to \mathsf{G},$$

the map

$$\mathrm{coFib}(H^0(\mathrm{inv}_{\mathsf{H}}(\mathfrak{g}_\phi)) \to \mathfrak{g}) \to T_\phi(\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, \mathsf{G})) \tag{2.3}$$

is an isomorphism, where $\mathrm{inv}_{\mathsf{H}}$ stands for H-invariants, and $\mathfrak{g}_\phi$ is $\mathfrak{g}$ viewed as a H-representation via ϕ and the adjoint action.

Equivalently, this is the assertion that

$$H^0(\mathrm{inv}_{\mathsf{H}}(\mathfrak{g}_\phi))[1] \to T_\phi(\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, \mathsf{G})/\mathsf{G})$$

is an isomorphism.

We can think of $\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H},\mathsf{G})/\mathsf{G}$ as the mapping space between pt/H and pt/G, i.e.,

$$\mathbf{Maps}_{\mathrm{PreStk}}(\mathrm{pt}/\mathsf{H},\mathrm{pt}/\mathsf{G}).$$

The tangent space to the latter is

$$\Gamma(\mathrm{pt}/\mathsf{H},\mathfrak{g}_\phi[1]).$$

Now, the required result follows from the fact that

$$H^1(\mathsf{H},\mathfrak{g}_\phi)=0.$$

NB: note that validity of Proposition 2.3.9 depends on the assumption that we work over a field of characteristic 0 (in our case this is the field of coefficients e).

□

2.4. **Analysis of connected/irreducible components.** In this subsection we will prove the last remaining property of the map

$$\pi:\widetilde{\mathrm{LocSys}}^{\mathrm{restr}}_{\mathsf{G}}(X)\to \mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X),$$

namely that the set of connected components of $\widetilde{\mathrm{LocSys}}^{\mathrm{restr}}_{\mathsf{G}}(X)$ is a union of finite clusters, and elements from different clusters have non-intersecting images in $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ along π.

In addition, we will describe explicitly the set of connected components of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$.

2.4.1. Let P_1 and P_2 be a pair of standard parabolics in G, each equipped with an irreducible local system σ_{M_i} with respect to its Levi factor.

We shall say that the pairs $(\mathsf{P}_1,\sigma_{\mathsf{M}_1})$ and $(\mathsf{P}_2,\sigma_{\mathsf{M}_2})$ are *associated* if there exists a G-orbit O in $\mathsf{G}/\mathsf{P}_1\times\mathsf{G}/\mathsf{P}_2$ with the following properties:

- For some/any pair of points $(\mathsf{P}'_1,\mathsf{P}'_2)\in\mathsf{O}$, the maps
$$\mathsf{M}_1\leftarrow\mathsf{P}'_1\leftarrow\mathsf{P}'_1\cap\mathsf{P}'_2\to\mathsf{P}'_2\to\mathsf{M}_2$$
identify M_i, $i=1,2$, with the Levi factor of $\mathsf{P}'_1\cap\mathsf{P}'_2$;
- Under the resulting isomorphism $\mathsf{M}_1\simeq\mathsf{M}_2$, the local systems σ_{M_1} and σ_{M_2} are isomorphic.

2.4.2. We will prove:

Proposition 2.4.3. *The images of* $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{P}_1,\sigma_{\mathsf{M}_1}}(X)$ *and* $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{P}_2,\sigma_{\mathsf{M}_2}}(X)$ *in* $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ *have a non-empty intersection if and only if* $(\mathsf{P}_1,\sigma_{\mathsf{M}_1})$ *and* $(\mathsf{P}_2,\sigma_{\mathsf{M}_2})$ *are associated.*

Proof. The "if" direction is straightforward: find a Levi splitting of $\mathsf{P}'_1\cap\mathsf{P}'_2$. Then the resulting local system with respect to $\mathsf{P}'_1\cap\mathsf{P}'_2$ projects to the same point of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ via both $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{P}_1,\sigma_{\mathsf{M}_1}}(X)$ and $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{P}_2,\sigma_{\mathsf{M}_2}}(X)$.

Let us prove the "only if" direction. Let σ_{G} be a e-point of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ that lies in the images of both $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{P}_1,\sigma_{\mathsf{M}_1}}(X)$ and $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{P}_2,\sigma_{\mathsf{M}_2}}(X)$.

The pair of reductions of σ_{G} to P_1 and P_2 corresponds to a G-orbit O on $\mathsf{G}/\mathsf{P}_1\times\mathsf{G}/\mathsf{P}_2$. We will show that this orbit satisfies the two conditions of Sect. 2.4.1.

By assumption, we can choose parabolics P'_1 and P'_2, conjugate to P_1 and P_2, respectively, and lying on O, so that σ_{G} admits a reduction to $\mathsf{P}'_1\cap\mathsf{P}'_2$; denote this reduction by $\sigma_{1,2}$. Furthermore, σ_{M_i}, $i=1,2$, is induced from $\sigma_{1,2}$ along the map

$$\mathsf{P}'_1\cap\mathsf{P}'_2\hookrightarrow\mathsf{P}'_i\twoheadrightarrow\mathsf{M}_i. \tag{2.4}$$

We note that the image of (2.4) is a Levi subgroup in M_i, $i=1,2$. Hence, by the assumption on σ_{M_i}, the maps (2.4) are surjective, and hence identify M_i as a Levi factor of $\mathsf{P}'_1\cap\mathsf{P}'_2$.

□

2.4.4. Recall that a G-local system σ_{G} is said to be *semi-simple* if whenever we can factor it via a local system σ_{P} with $\mathsf{P} \subset \mathsf{G}$ a parabolic, then we can factor σ_{P} via a local system σ_{M} for M for *some* Levi splitting

$$\mathsf{P} \leftrightarrows \mathsf{M}.$$

Given two G-local systems σ_1 and σ_2, we shall say that σ_2 is a *semi-simplification* of σ_1 if

- σ_2 is semi-simple;
- there exists a parabolic P and a reduction $\sigma_{1,\mathsf{P}}$ of σ_1 to P, such that the *induced* local system from $\sigma_{1,\mathsf{P}}$ via
$$\mathsf{P} \to \mathsf{M} \hookrightarrow \mathsf{P} \to \mathsf{G}$$
is isomorphic to σ_2 for some/any Levi splitting $\mathsf{P} \leftrightarrows \mathsf{M}$.

The end of the proof of Proposition 2.4.3 shows:

Lemma 2.4.5.

(a) *Let $\mathsf{P} \subset \mathsf{G}$ be a parabolic and choose a Levi splitting $\mathsf{P} \leftrightarrows \mathsf{M}$. Let σ_{M} in an irreducible M-local system, and let σ_{G} denote the induced G-local system via $\mathsf{M} \to \mathsf{P} \to \mathsf{G}$. Then σ_{G} is semi-simple.*

(b) *Every G-local system admits a semi-simplification, and any two semi-simplifications of a given G-local system are isomorphic.*

(c) *For two pairs $(\mathsf{P}_1, \sigma_{\mathsf{M}_1})$ and $(\mathsf{P}_2, \sigma_{\mathsf{M}_2})$ as in point (a), the G-local systems σ_1 and σ_2 are isomorphic if and only if $(\mathsf{P}_1, \sigma_{\mathsf{M}_1})$ and $(\mathsf{P}_2, \sigma_{\mathsf{M}_2})$ are associated.*

(d) *The assignment $(\mathsf{P}, \sigma_{\mathsf{M}}) \mapsto \sigma_{\mathsf{G}}$ of point (a) establishes a bijection between classes of association of pairs $(\mathsf{P}, \sigma_{\mathsf{M}})$ and isomorphism classes of semi-simple G-local systems.*

2.4.6. We claim:

Proposition 2.4.7. *Two e-points of $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)$ belong to the same connected component of $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)$ if and only if they have isomorphic semi-simplifications.*

As a formal corollary we obtain:

Corollary 2.4.8. *Each connected component of $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)$ contains a unique isomorphism class of e-points corresponding to a semi-simple G-local system. This establishes between the set of connected components of $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)$ and the set of isomorphism classes of semi-simple G-local systems.*

Proof of Proposition 2.4.7. In one direction, we have to show that if σ_2 is a semi-simplification of σ_1, then they belong to the same connected component of $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)$. Let σ_1 factor through a parabolic P and let $\sigma_{2,\mathsf{M}}$ be corresponding semi-simple local system for the Levi factor M. With no restriction of generality, we can assume that $\sigma_{2,\mathsf{M}}$ is irreducible. In this case, both σ_1 and σ_2 belong to the image of $\mathrm{LocSys}_{\mathsf{P},\sigma_{2,\mathsf{M}}}^{\mathrm{restr}}(X)$.

In the other direction, by Proposition 2.4.3, it suffices to show that if $(\mathsf{P}_1, \sigma_{\mathsf{M}_1})$ and $(\mathsf{P}_2, \sigma_{\mathsf{M}_2})$ are associated, then the local systems in their respective images in $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)$ have isomorphic semi-simplifications. Consider the corresponding subgroup $\mathsf{P}_1' \cap \mathsf{P}_2'$, and fix its Levi splitting

$$\mathsf{P}_1' \cap \mathsf{P}_2' \leftrightarrows \mathsf{M}_{1,2}$$

and let $\sigma_{1,2}$ be the resulting irreducible local system for $\mathsf{M}_{1,2}$. Then it s clear that every local system in the image of $(\mathsf{P}_i, \sigma_{\mathsf{M}_i})$, $i = 1, 2$ has as its semi-simplification the G-local system induced by

$$\mathsf{M}_{1,2} \hookrightarrow \mathsf{P}_1' \cap \mathsf{P}_2' \hookrightarrow \mathsf{G}.$$

□

Remark 2.4.9. It is easy to see from the above argument that for given a local system σ, the map

$$\mathrm{pt} / \mathrm{Stab}_{\mathsf{G}}(\sigma) \to \mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)$$

is a closed embedding if and only if σ is semi-simple, cf. Propositions 3.3.7 and 3.5.13.

Remark 2.4.10. It is clear that the image of each $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{P},\sigma_{\mathsf{M}}}(X)$ along π is irreducible. However, it is *not* true that the images of different of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{P}_1,\sigma_{\mathsf{M}_1}}(X)$ and $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{P}_2,\sigma_{\mathsf{M}_2}}(X)$ in $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ will always produce different irreducible components:

For example, take $G = GL_2$, $\mathsf{P}_1 = \mathsf{P}_2 = \mathsf{B}$, so $\mathsf{M}_1 = \mathsf{M}_2 = \mathbb{G}_m \times \mathbb{G}_m$. Take σ_{M_1} and σ_{M_2} be given by

$$(E_1, E_2) \text{ and } (E_2, E_1),$$

where E_1 and E_2 are non-isomorphic one-dimensional local systems.

Then if X is a curve of genus ≥ 2, the images of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{P},\sigma_{\mathsf{M}_1}}(X)$ and $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{P},\sigma_{\mathsf{M}_2}}(X)$ are two distinct irreducible components of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$.

By contrast, if X is a curve of genus 1, both $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{P},\sigma_{\mathsf{M}_1}}(X)$ and $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{P},\sigma_{\mathsf{M}_2}}(X)$ are set-theoretically isomorphic to $\mathrm{pt}/\mathbb{G}_m \times \mathbb{G}_m$, and they map onto the same closed subset of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$.

3. Comparison with the Betti and de Rham versions of $\mathrm{LocSys}_{\mathsf{G}}(X)$

In this section we study the relationship between $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ with $\mathrm{LocSys}_{\mathsf{G}}(X)$ in the two contexts when the latter is defined: de Rham and Betti.

We will show that in both cases, the map

$$\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X) \to \mathrm{LocSys}_{\mathsf{G}}(X)$$

is a *formal isomorphism* with an explicit image at the reduced level.

3.1. **Relation to the Rham version.** In this subsection we will take our ground field k to be of characteristic 0. We will take $\mathsf{e} = k$ and let $\mathrm{Shv}(-)$ to be the sheaf theory of ind-holonomic D-modules.

We will study the relationship between $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ and the "usual" stack $\mathrm{LocSys}_{\mathsf{G}}(X)$ classifying de Rham local systems.

3.1.1. Note that objects of $\mathrm{Shv}(X)^{\mathrm{constr}}$ are compact as objects of $\mathrm{D\text{-}mod}(X)$. Hence, the functor

$$\mathrm{Shv}(X) \to \mathrm{D\text{-}mod}(X),$$

obtained by ind-extending the tautological embedding is fully faithful.

Therefore, so is the composite functor

$$\mathrm{QLisse}(X) \hookrightarrow \mathrm{Shv}(X) \to \mathrm{D\text{-}mod}(X).$$

We now claim:

3.1.2. Recall now (see, e.g., [AG, Sects. 10.1-2]), that the prestack $\mathrm{LocSys}_{\mathsf{G}}(X)$ is defined by sending $S \in \mathrm{Sch}^{\mathrm{aff}}_{/\mathsf{e}}$ to the space of right t-exact symmetric monoidal functors

$$\mathrm{Rep}(\mathsf{G}) \to \mathrm{QCoh}(S) \otimes \mathrm{D\text{-}mod}(X).$$

Note that the difference between $\mathrm{LocSys}_{\mathsf{G}}(X)$ and $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ is in the recipient category: in the former case this is $\mathrm{QCoh}(S) \otimes \mathrm{D\text{-}mod}(X)$ and in the latter this $\mathrm{QCoh}(S) \otimes \mathrm{QLisse}(X)$.

Note, however, that since $\mathrm{QLisse}(X) \to \mathrm{D\text{-}mod}(X)$ is fully faithful and $\mathrm{QCoh}(S)$ is dualizable, the functor

$$\mathrm{QCoh}(S) \otimes \mathrm{QLisse}(X) \subset \mathrm{QCoh}(S) \otimes \mathrm{D\text{-}mod}(X) \tag{3.1}$$

is fully faithful.

In particular, we obtain that the embedding (3.1) defines a *monomorphism*

$$\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X) \to \mathrm{LocSys}_{\mathsf{G}}(X). \tag{3.2}$$

3.1.3. *Example.* Let us explain how the difference between $\mathrm{LocSys}_{\mathbb{G}_m}(X)$ and $\mathrm{LocSys}^{\mathrm{restr}}_{\mathbb{G}_m}(X)$ plays out in the simplest case when $\mathsf{G} = \mathbb{G}_m$.

Take S to be classical. Then S-points of $\mathrm{LocSys}_{\mathbb{G}_m}(X)$ are line bundles over $S \times X$, equipped with a connection along X. Trivializing this line bundle locally, the connection corresponds to a section of

$$\mathcal{O}_S \boxtimes \Omega^{1,\mathrm{cl}}_X,$$

i.e., a function on S with values in closed 1-forms on X.

By contrast, if our S-point lands in $\mathrm{LocSys}^{\mathrm{restr}}_{\mathbb{G}_m}(X)$, and if we further assume that S is intergal, by Example 1.3.6, our line bundle with connection is necessarily pulled back from X.

Let us now take $\mathsf{G} = \mathbb{G}_a$. Then it follows from Sect. 1.3.7 that the map

$$\mathrm{LocSys}^{\mathrm{restr}}_{\mathbb{G}_a}(X) \to \mathrm{LocSys}_{\mathbb{G}_a}(X)$$

is an isomorphism.

3.1.4. We now claim:

Proposition 3.1.5. *The map* (3.2) *is a formal isomorphism, i.e., identifies* $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ *with its formal completion inside* $\mathrm{LocSys}_{\mathsf{G}}(X)$.

Proof. We need to show that the map

$$\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X) \to (\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X))_{\mathrm{dR}} \underset{(\mathrm{LocSys}_{\mathsf{G}}(X))_{\mathrm{dR}}}{\times} \mathrm{LocSys}_{\mathsf{G}}(X)$$

is an isomorphism.

I.e., we have to show that for $S \in \mathrm{Sch}^{\mathrm{aff}}_{/\mathsf{e}}$ and a map

$$S \to \mathrm{LocSys}_{\mathsf{G}}(X), \tag{3.3}$$

such that the composite map

$${}^{\mathrm{red}}S \to S \to \mathrm{LocSys}_{\mathsf{G}}(X)$$

factors through $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$, the initial map (3.3) factors though $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ as well.

Thus, we need to show that given a functor

$$\mathsf{F} : \mathrm{Rep}(\mathsf{G}) \to \mathrm{QCoh}(S) \otimes \text{D-mod}(X), \tag{3.4}$$

such that the composite functor

$$\mathrm{Rep}(\mathsf{G}) \to \mathrm{QCoh}(S) \otimes \text{D-mod}(X) \to \mathrm{QCoh}({}^{\mathrm{red}}S) \otimes \text{D-mod}(X),$$

lands in

$$\mathrm{QCoh}({}^{\mathrm{red}}S) \otimes \mathrm{QLisse}(X) \subset \mathrm{QCoh}({}^{\mathrm{red}}S) \otimes \text{D-mod}(X), \tag{3.5}$$

the functor (3.4) also lands in

$$\mathrm{QCoh}(S) \otimes \mathrm{QLisse}(X) \subset \mathrm{QCoh}(S) \otimes \text{D-mod}(X). \tag{3.6}$$

By Lemma 1.5.3(b), we can assume that S is eventually coconnective. Hence, by Proposition 1.5.6, in (3.6) we can replace $\mathrm{QLisse}(X)$ by $\mathrm{IndLisse}(X)$.

Let ι denote the embedding $\mathrm{IndLisse}(X) \hookrightarrow \text{D-mod}(X)$. It sends compacts to compacts, hence admits a continuous right adjoint, to be denoted ι^R.

We need to show that the natural transformation

$$\mathsf{F} \to (\mathrm{Id} \otimes \iota^R) \circ (\mathrm{Id} \otimes \iota) \circ \mathsf{F}$$

is an isomorphism.

Let f denote the embedding ${}^{\mathrm{red}}S \to S$. We know that

$$(f^* \otimes \mathrm{Id}) \circ \mathsf{F} \to (f^* \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \iota^R) \circ (\mathrm{Id} \otimes \iota) \circ \mathsf{F} \simeq (\mathrm{Id} \otimes \iota^R) \circ (\mathrm{Id} \otimes \iota) \circ (f^* \otimes \mathrm{Id}) \circ \mathsf{F}$$

is an isomorphism.

This implies the assertion since for S eventually coconnective, the functor

$$f^* \otimes \mathrm{Id} : \mathrm{QCoh}(S) \otimes \mathbf{C} \to \mathrm{QCoh}({}^{\mathrm{red}}S) \otimes \mathbf{C}$$

is conservative for any DG category $\mathbf{C}$ (indeed, $\mathrm{QCoh}(S)$ is generated under finite limits by the essential image of f_*).

□

3.1.6. From now on, until the end of this subsection we will assume that X is proper. In this case by [AG, Sect. 10.3], we know that $\mathrm{LocSys}_{\mathsf{G}}(X)$ is an algebraic stack locally almost of finite type.

We claim:

Theorem 3.1.7. *The map*

$$\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X) \to \mathrm{LocSys}_{\mathsf{G}}(X)$$

is a closed embedding at the reduced level *for* each connected component of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$.

This theorem will be proved in Sect. 3.3. In the course of the proof we will also describe the closed substacks of ${}^{\mathrm{red}}\mathrm{LocSys}_{\mathsf{G}}(X)$ that arise as images of connected components of ${}^{\mathrm{red}}\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$.

Combined with Proposition 3.1.5, we obtain:

Corollary 3.1.8. *The subfunctor*

$$\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X) \subset \mathrm{LocSys}_{\mathsf{G}}(X)$$

is the disjoint union of formal completions of a collection of pairwise non-intersecting closed substacks of ${}^{\mathrm{red}}\mathrm{LocSys}_{\mathsf{G}}(X)$.

Remark 3.1.9. The closed substacks of ${}^{\mathrm{red}}\mathrm{LocSys}_{\mathsf{G}}(X)$ appearing in Corollary 3.1.8 will be explicitly described in Remark 3.3.8.

Remark 3.1.10. Let $\mathsf{G}' \to \mathsf{G}$ be a closed embedding. The argument proving Proposition 2.2.8 shows that the diagram

$$\begin{CD}
\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}'}(X) @>>> \mathrm{LocSys}_{\mathsf{G}'}(X) \\
@VVV @VVV \\
\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X) @>>> \mathrm{LocSys}_{\mathsf{G}}(X)
\end{CD}$$

is a fiber square.

3.2. A digression: ind-closed embeddings.

3.2.1. Let us recall the notion of *ind-closed embedding* of prestacks (see [GR3] Sect. 2.7.2).

First, if S is an affine scheme and $\mathcal{Y}$ is a prestack mapping to it, we shall say that this map is an *ind-closed* embedding if $\mathcal{Y}$ is an ind-scheme and for some/any presentation of $\mathcal{Y}$ as (1.16), the resulting maps

$$Y_i \to S$$

are closed embeddings.

We shall say that a map of prestacks $\mathcal{Y}_1 \to \mathcal{Y}_2$ is an ind-closed embedding if its base change by an affine scheme yields an ind-closed embedding.

Remark 3.2.2. Let us emphasize the difference between "ind-closed embedding" and "closed embedding". For example, the inclusion of the disjoint union of infinitely many copies of pt onto $\mathbb{A}^1$ is an ind-closed embedding but not a closed embedding. Similarly, the map

$$\mathrm{Spf}(\mathsf{e}[\![t]\!]) \to \mathbb{A}^1$$

is an ind-closed embedding but not a closed embedding.

3.2.3. From Corollary 3.1.8 we obtain:

Corollary 3.2.4. *The map* $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X) \to \mathrm{LocSys}_{\mathsf{G}}(X)$ *is an ind-closed embedding.*

Remark 3.2.5. An ind-closed embedding that is also a formal isomorphism is necessarily of the form described in Corollary 3.1.8.

More precisely, let $\mathcal{Y}_1 \to \mathcal{Y}_2$ be a map of prestacks that is a formal isomorphism, i.e., the map

$$\mathcal{Y}_1 \to (\mathcal{Y}_1)_{\mathrm{dR}} \underset{(\mathcal{Y}_2)_{\mathrm{dR}}}{\times} \mathcal{Y}_2$$

is an isomorphism. Let $\mathcal{Y}_2$ be an algebraic stack. Then the following conditions are equivalent:

- It is a closed embedding at the reduced level;
- It is an ind-closed embedding;
- $\mathcal{Y}_1$ is obtained as the disjoint union of formal completions of a collection of pairwise non-intersecting closed subfunctors of ${}^{\mathrm{red}}\mathcal{Y}_2$.

3.3. Uniformization and the proof of Theorem 3.1.7.

3.3.1. For a standard parabolic P consider the diagram

$$\mathrm{LocSys}_{\mathsf{G}}(X) \overset{\mathsf{p_P}}{\leftarrow} \mathrm{LocSys}_{\mathsf{P}}(X) \overset{\mathsf{q_P}}{\to} \mathrm{LocSys}_{\mathsf{M}}(X).$$

Fix an irreducible local system σ_{M} for M and denote

$$\mathrm{LocSys}_{\mathsf{P},\sigma_{\mathsf{M}}}(X) := \mathrm{LocSys}\,\mathsf{P}(X) \underset{\mathrm{LocSys}_{\mathsf{M}}(X)}{\times} \mathrm{pt}\,/\,\mathrm{Stab}_{\mathsf{M}}(\sigma_{\mathsf{M}}).$$

3.3.2. We have a commutative diagram

$$\begin{tikzcd} \mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{P},\sigma_{\mathsf{M}}}(X) \arrow[r] \arrow[d] & \mathrm{LocSys}_{\mathsf{P},\sigma_{\mathsf{M}}}(X) \arrow[d] \\ \mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X) \arrow[r] & \mathrm{LocSys}_{\mathsf{G}}(X). \end{tikzcd}$$

Consider the composite morphism

$$\begin{tikzcd} \mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{P},\sigma_{\mathsf{M}}}(X) \arrow[r] & \mathrm{LocSys}_{\mathsf{P},\sigma_{\mathsf{M}}}(X) \arrow[d] \\ & \mathrm{LocSys}_{\mathsf{G}}(X) \end{tikzcd} \tag{3.7}$$

We claim that in order to prove Theorem 3.1.7, it suffices to show that the morphism (3.7) is schematic and proper. This essentially a diagram chase; here is the argument.

3.3.3. Indeed, consider the irreducible closed substack of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$, equal to the image of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{P},\sigma_{\mathsf{M}}}(X)$; denote it by $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G},\mathsf{P},\sigma_{\mathsf{M}}}(X)$.

Since the morphism

$$\underset{\mathsf{P}\in\mathrm{Par}(\mathsf{G})}{\sqcup}\ \underset{\sigma_{\mathsf{M}}}{\sqcup}\ \mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{P},\sigma_{\mathsf{M}}}(X) \to \mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$$

is surjective at the level of e-points (and given Proposition 2.4.3), we obtain that is enough to show that the composite map

$$\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G},\mathsf{P},\sigma_{\mathsf{M}}}(X) \to \mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X) \to \mathrm{LocSys}_{\mathsf{G}}(X)$$

is a closed embedding (at the reduced level) for all $(\mathsf{P},\sigma_{\mathsf{M}})$.

Consider the algebraic stack (in fact, an algebraic space)

$$\mathrm{LocSys}^{\mathrm{rigid}_x}_{\mathsf{G}}(X) := \mathrm{LocSys}_{\mathsf{G}}(X) \underset{\mathrm{pt}\,/\mathsf{G}}{\times} \mathrm{pt},$$

and the corresponding maps

$$\mathrm{LocSys}^{\mathrm{restr},\mathrm{rigid}_x}_{\mathsf{G},\mathsf{P},\sigma_{\mathsf{M}}}(X) \to \mathrm{LocSys}^{\mathrm{restr},\mathrm{rigid}_x}_{\mathsf{G}}(X) \to \mathrm{LocSys}^{\mathrm{rigid}_x}_{\mathsf{G}}(X).$$

It is enough to show that the latter composite map is a closed embedding (at the reduced level). Let us base change this map by an affine scheme S:

$$S' := S \underset{\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{rigid}_x}(X)}{\times} \mathrm{LocSys}_{\mathsf{G},\mathsf{P},\sigma_{\mathsf{M}}}^{\mathrm{restr},\mathrm{rigid}_x}(X) \to S.$$

Since ${}^{\mathrm{red}}\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X)$ is a disjoint union of affine schemes (by Theorem 1.3.2), we obtain that ${}^{\mathrm{red}}\mathrm{LocSys}_{\mathsf{G},\mathsf{P},\sigma_{\mathsf{M}}}^{\mathrm{restr},\mathrm{rigid}_x}(X)$ is an affine scheme. From here, and the fact that $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{rigid}_x}(X)$ has an affine diagonal, we obtain that ${}^{\mathrm{red}}S'$ is also also an affine scheme.

Set

$$S'' := S \underset{\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{rigid}_x}(X)}{\times} \mathrm{LocSys}_{\mathsf{P},\sigma_{\mathsf{M}}}^{\mathrm{restr},\mathrm{rigid}_x}(X).$$

Assume that we know that the morphism (3.7) is schematic and proper. Then so is the morphism

$$\mathrm{LocSys}_{\mathsf{P},\sigma_{\mathsf{M}}}^{\mathrm{restr},\mathrm{rigid}_x}(X) \to \mathrm{LocSys}_{\mathsf{G}}^{\mathrm{rigid}_x}(X).$$

We obtain that S'' is a scheme, proper over S. Further, the map $S'' \to S'$ is surjective at the level of e-points. From here we obtain that ${}^{\mathrm{red}}S'$ is proper (and, hence, finite) over S.

Since we also know that $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X) \to \mathrm{LocSys}_{\mathsf{G}}(X)$ is a monomorphism, we obtain that

$${}^{\mathrm{red}}S' \to S$$

is a closed embedding, as required.

3.3.4. Thus, it remains to show that the map (3.7) is schematic and proper. This follows from the combination of the following three assertions:

Proposition 3.3.5. *The map*

$$\mathrm{LocSys}_{\mathsf{P},\sigma_{\mathsf{M}}}^{\mathrm{restr}}(X) \to \mathrm{LocSys}_{\mathsf{P},\sigma_{\mathsf{M}}}(X)$$

is an isomorphism.

Proposition 3.3.6. *The map*

$$\mathsf{p} : \mathrm{LocSys}_{\mathsf{P}}(X) \to \mathrm{LocSys}_{\mathsf{G}}(X)$$

is schematic and proper.

Proposition 3.3.7. *For a reductive group* G *and an irreducible local system* σ*, the resulting map*

$$\mathrm{pt} / \mathrm{Stab}_{\mathsf{G}}(\sigma) \to \mathrm{LocSys}_{\mathsf{G}}(X)$$

is a closed embedding.

Remark 3.3.8. Note that the combination of the above three propositions describes the ind-closed substack

$${}^{\mathrm{red}}\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X) \subset {}^{\mathrm{red}}\mathrm{LocSys}_{\mathsf{G}}(X).$$

Namely, it equals the disjoint union over classes of association of $(\mathsf{P}, \sigma_{\mathsf{M}})$ of the unions of the images of the maps

$${}^{\mathrm{red}}\mathrm{LocSys}_{\mathsf{P},\sigma_{\mathsf{M}}}(X) \to {}^{\mathrm{red}}\mathrm{LocSys}_{\mathsf{G}}(X)$$

within a given class.

3.3.9. We now prove the above three propositions.

The assertion of Proposition 3.3.5 follows by tracing the proof of Proposition 2.2.4: namely, in the situation of *loc.cit.*, for an S-point of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}_2}$, the map

$$S \underset{\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}_2}}{\times} \mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}_1} \to S \underset{\mathrm{LocSys}_{\mathsf{G}_2}(X)}{\times} \mathrm{LocSys}_{\mathsf{G}_1}(X)$$

is an isomorphism. Indeed, in both cases, this fiber product classifies null-homotopies for the same class.

Proposition 3.3.6 is well-known: it follows from the fact that the map

$$\mathrm{LocSys}_{\mathsf{P}}(X) \to \mathrm{LocSys}_{\mathsf{G}}(X) \underset{\mathrm{pt}/\mathsf{G}}{\times} \mathrm{pt}/\mathsf{P}$$

is a closed embedding, where $\mathrm{LocSys}_{\mathsf{G}}(X) \to \mathrm{pt}/\mathsf{G}$ is obtained by taking the fiber at some point $x \in X$.

It remains to prove Proposition 3.3.7.

3.3.10. *Proof of Proposition 3.3.7.*

Algebraic proof. It is enough to show that for a smooth affine curve C over e and a point $c \in C$, given a map

$$C \to \mathrm{LocSys}_{\mathsf{G}}(X), \tag{3.8}$$

such that the composite map

$$(C - c) \to C \to \mathrm{LocSys}_{\mathsf{G}}(X)$$

factors as

$$(C - c) \to \mathrm{Spec}(\mathsf{e}) \xrightarrow{\sigma} \mathrm{LocSys}_{\mathsf{G}}(X),$$

then the initial map (3.8) also factors as

$$C \to \mathrm{Spec}(\mathsf{e}) \xrightarrow{\sigma} \mathrm{LocSys}_{\mathsf{G}}(X). \tag{3.9}$$

The maps (3.8) and (3.9) correspond to G-bundles $\mathcal{F}^1_{\mathsf{G}}$ and $\mathcal{F}^2_{\mathsf{G}}$ on $C \times X$, each equipped with a connection, and we are given an isomorphism of these data over $(C - c) \times X$. We wish to show that this isomorphism extends over all $C \times X$.

This is easy to show if G was a torus. Hence, we obtain that for the induced bundles with respect to $\mathsf{G}/[\mathsf{G},\mathsf{G}]$, the given isomorphism indeed extends over over all $C \times X$. Modifying by means of local system with respect to Z_{G}, we can thus assume that the induced local systems for $\mathsf{G}/[\mathsf{G},\mathsf{G}]$ are trivial. Hence, we can replace G by $[\mathsf{G},\mathsf{G}]$, i.e., we can assume that G is semi-simple.

Let η_X denote the generic point of X. Then the relative position of $\mathcal{F}^1_{\mathsf{G}}$ and $\mathcal{F}^2_{\mathsf{G}}$ at $c \times \eta_X$ is a cell of the affine Grassmannian of G, and hence corresponds to a coweight λ of G, which is 0 if and only if the isomorphism between $\mathcal{F}^1_{\mathsf{G}}$ and $\mathcal{F}^2_{\mathsf{G}}$ extends over all $C \times X$.

Furthermore, the restrictions of both $\mathcal{F}^1_{\mathsf{G}}$ and $\mathcal{F}^2_{\mathsf{G}}$ to $c \times \eta_X$ acquire a reduction to the corresponding standard parabolic P (it corresponds to those vertices i of the Dynkin diagram, for which $\langle \check{\alpha}_i, \lambda \rangle = 0$). These reductions to P are horizontal with respect to the connection along η_X.

Now, $\mathcal{F}^2_{\mathsf{G}}$ is the constant family corresponding to σ, so $\mathcal{F}^2_{\mathsf{G}}|_{c \times X}$ is also given by σ. By the valuative criterion, its reduction to P over $c \times \eta_X$ extends to all of $c \times X$. However, since σ was assumed irreducible, we have $\mathsf{P} = \mathsf{G}$. Hence, $\lambda = 0$, as required.

□

Analytic proof. We can assume that $k = \mathbb{C}$. Clearly,

$$\mathrm{pt}/\mathrm{Stab}_{\mathsf{G}}(\sigma) \to \mathrm{LocSys}_{\mathsf{G}}(X)$$

is a locally closed embedding. To prove that it is a closed embedding, it is enough to show that its image is closed *in the analytic topology.*

We identify the analytic stack underlying ${}^{\mathrm{cl}}\mathrm{LocSys}_{\mathsf{G}}(X)$ with its Betti version (see Sect. 3.4.7). Hence, it is enough to show that the map

$$\mathrm{pt}\,/\,\mathrm{Stab}_{\mathsf{G}}(\sigma) \to \mathrm{LocSys}_{\mathsf{G}}(X)$$

is a closed embedding for the *Betti* version.

However, in this case the assertion follows from Proposition 3.5.13 below.

□

3.4. **The Betti version of** $\mathrm{LocSys}_{\mathsf{G}}(X)$**.** From this point until the end of this section we let e be an arbitrary algebraically closed field of characteristic 0.

3.4.1. Let $\mathcal{X}$ be a connected object of Spc. We define the prestack $\mathrm{LocSys}_{\mathsf{G}}(\mathcal{X})$ to be

$$(\mathrm{pt}\,/\mathsf{G})^{\mathcal{X}} = \mathbf{Maps}(\mathcal{X}, \mathrm{pt}\,/\mathsf{G}).$$

I.e., for $S \in \mathrm{Sch}^{\mathrm{aff}}_{/\mathsf{e}}$,

$$\mathrm{Maps}(S, \mathrm{LocSys}_{\mathsf{G}}(\mathcal{X})) = \mathrm{Maps}_{\mathrm{Spc}}(\mathcal{X}, \mathrm{Maps}(S, \mathrm{pt}\,/\mathsf{G})).$$

The fact that $\mathrm{pt}\,/\mathsf{G}$ admits (-1)-connective corepresentable deformation theory formally implies that the same is true for $\mathrm{LocSys}_{\mathsf{G}}(\mathcal{X})$.

3.4.2. Assume for a moment that $\mathcal{X}$ is compact, i.e., is a retract of a space that can be obtained by a finite operation of taking push-outs from $\{*\} \in \mathrm{Spc}$.

In this case, it is clear from the definitions that $\mathrm{LocSys}_{\mathsf{G}}(\mathcal{X})$ is locally almost of finite type.

3.4.3. We claim:

Proposition 3.4.4. *The prestack* $\mathrm{LocSys}_{\mathsf{G}}(\mathcal{X})$ *is a derived algebraic stack. It can be realized as a quotient of an affine scheme locally almost by an action of* G*.*

Proof. Choose a base point $x \in \mathcal{X}$. Denote

$$\mathrm{LocSys}^{\mathrm{rigid}_x}_{\mathsf{G}}(\mathcal{X}) := \mathrm{LocSys}_{\mathsf{G}}(\mathcal{X}) \underset{\mathrm{pt}\,/\check{G}}{\times} \mathrm{pt},$$

where the map $\mathrm{LocSys}_{\mathsf{G}}(\mathcal{X}) \to \mathrm{pt}\,/\check{G}$ is given by restriction to x.

We have a natural action of G on $\mathrm{LocSys}^{\mathrm{rigid}_x}_{\mathsf{G}}(\mathcal{X})$ so that

$$\mathrm{LocSys}_{\mathsf{G}}(\mathcal{X}) \simeq \mathrm{LocSys}^{\mathrm{rigid}_x}_{\mathsf{G}}(\mathcal{X})/\mathsf{G}.$$

We will show that $\mathrm{LocSys}^{\mathrm{rigid}_x}_{\mathsf{G}}(\mathcal{X})$ is an affine scheme. The fact that $\mathrm{LocSys}_{\mathsf{G}}(\mathcal{X})$ admits (-1)-connective corepresentable deformation theory implies that $\mathrm{LocSys}^{\mathrm{rigid}_x}_{\mathsf{G}}(\mathcal{X})$ admits connective corepresentable deformation theory (here we use the assumption that $\mathcal{X}$ is connected).

Hence, by [Lu3, Theorem 18.1.0.1], in order to show that $\mathrm{LocSys}^{\mathrm{rigid}_x}_{\mathsf{G}}(\mathcal{X})$ is an affine scheme, it suffices to show that

$${}^{\mathrm{cl}}\mathrm{LocSys}^{\mathrm{rigid}_x}_{\mathsf{G}}(\mathcal{X})$$

is a classical affine scheme.

Denote

$$\Gamma := \pi_1(X, x).$$

It follows from the definitions that for $S \in {}^{\mathrm{cl}}\,\mathrm{Sch}^{\mathrm{aff}}_{/\mathsf{e}}$, the space $\mathrm{Maps}(S, \mathrm{LocSys}^{\mathrm{rigid}_x}_{\mathsf{G}}(\mathcal{X}))$ is a set of homomorphisms $\Gamma \to \mathsf{G}$, parameterized by S.

I.e., $\mathrm{LocSys}^{\mathrm{rigid}_x}_{\mathsf{G}}(\mathcal{X})$ is a subfunctor of

$$S \mapsto \mathrm{Maps}(S, \mathsf{G})^{\Gamma} \simeq \mathrm{Maps}(S, \mathsf{G}^{\Gamma}),$$

consisting of elements that obey the group law, i.e.,

$$\mathsf{G}^{\Gamma} \underset{\mathsf{G}^{\Gamma \times \Gamma}}{\times} \mathrm{pt}\,.$$

Since G^Γ and $\mathsf{G}^{\Gamma\times\Gamma}$ are affine schemes, we obtain that so is $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{rigid}_x}(\mathcal{X})$.

$\square$

3.4.5. Let us now rewrite the definition of $\mathrm{LocSys}_{\mathsf{G}}(\mathcal{X})$ slightly differently. Consider the DG category

$$\mathrm{Vect}_{\mathsf{e}}^{\mathcal{X}} \simeq \mathrm{Funct}(\mathcal{X}, \mathrm{Vect}_{\mathsf{e}}).$$

For any DG category $\mathbf{C}$, we have a tautological functor

$$\mathbf{C} \otimes \mathrm{Vect}_{\mathsf{e}}^{\mathcal{X}} \to \mathbf{C}^{\mathcal{X}}, \tag{3.10}$$

which is an equivalence if $\mathbf{C}$ is dualizable (or if $\mathcal{X}$ is finite).

Furthermore $\mathrm{Vect}_{\mathsf{e}}^{\mathcal{X}}$ has a natural symmetric monoidal structure, and if $\mathbf{C}$ is also symmetric monoidal, the functor (3.10) is symmetric monoidal.

Assume for a moment that $\mathbf{C}$ has a t-structure. Then $\mathbf{C}^{\mathcal{X}}$ also acquires a t-structure (an object is connective/coconnetive if its value for any $x \in X$ is connective/coconnective). In particular, $\mathrm{Vect}_{\mathsf{e}}^{\mathcal{X}}$ has a t-structure. With respect to these t-structures, the functor (3.10) is t-exact.

3.4.6. We obtain that the value of $\mathrm{LocSys}_{\mathsf{G}}(\mathcal{X})$ on S can be rewritten as the space of right t-exact symmetric monoidal functors

$$\mathrm{Rep}(\mathsf{G}) \to \mathrm{QCoh}(S) \otimes \mathrm{Vect}_{\mathsf{e}}^{\mathcal{X}}.$$

3.4.7. Let now X be CW complex. Let $\mathrm{Shv}_{\mathrm{loc.const.}}(X)$ be the category of sheaves of e-vector spaces *with locally constant cohomologies.*

We define the prestack $\mathrm{LocSys}_{\mathsf{G}}(X)$ as follows. It sends an affine scheme S to the space of right t-exact symmetric monoidal functors

$$\mathrm{Rep}(\mathsf{G}) \to \mathrm{QCoh}(S) \otimes \mathrm{Shv}_{\mathrm{loc.const.}}(X).$$

3.4.8. Let us write X as a geometric realization of an object $\mathcal{X}$ of Spc.

In this case we have a canonical t-exact equivalence of symmetric monoidal categories

$$\mathrm{Shv}_{\mathrm{loc.const.}}(X) \simeq \mathrm{Vect}_{\mathsf{e}}^{\mathcal{X}}.$$

Hence, we obtain that in this case we have a canonical isomorphism of prestacks

$$\mathrm{LocSys}_{\mathsf{G}}(X) \simeq \mathrm{LocSys}_{\mathsf{G}}(\mathcal{X}).$$

Thus, the results pertaining to $\mathrm{LocSys}_{\mathsf{G}}(\mathcal{X})$ that we have reviewed above carry over to $\mathrm{LocSys}_{\mathsf{G}}(X)$ as well.

3.5. **Relationship of the restricted and Betti versions.** In this subsection we let X be a smooth algebraic variety over $\mathbb{C}$.

3.5.1. Consider the functor

$$\mathrm{QLisse}(X) \to \mathrm{Shv}_{\mathrm{loc.const.}}(X) \tag{3.11}$$

We claim:

Proposition 3.5.2. *The functor* (3.11) *is fully faithful.*

Remark 3.5.3. Note that, unlike the de Rham case, in the Betti setting, the fully faithulness of (3.11) is not a priori evident (because objects from $\mathrm{Shv}(X)^{\mathrm{constr}}$ are *not* compact as objects in the category of *all* sheaves of e-vector spaces on X).

Proof. Since both categories are left-complete and (3.11) is t-exact, it is sufficient to show that it induces fully faithful functors

$$(\mathrm{QLisse}(X))^{\geq -n} \to (\mathrm{Shv}_{\mathrm{loc.const.}}(X))^{\geq -n}. \tag{3.12}$$

Now,

$$(\mathrm{IndLisse}(X))^{\geq -n} \to (\mathrm{QLisse}(X))^{\geq -n}$$

is an equivalence, and hence the functor (3.12) sends compacts to compacts.

Since $(\mathrm{IndLisse}(X))^{\geq -n}$ is compactly generated (by $(\mathrm{Lisse}(X))^{\geq -n}$) and

$$\mathrm{Lisse}(X) \to \mathrm{Shv}_{\mathrm{loc.const.}}(X)$$

is fully faithful, we obtain that (3.12) is fully faithful.

□

Remark 3.5.4. The material of this and the next subsection is equally applicable, when instead of X we take a compact connected CW complex. In this case we let $\mathrm{QLisse}(X)$ be the full subcategory of $\mathrm{Shv}_{\mathrm{loc.const.}}(X)$ consisting of objects such that each of their cohomologies (with respect to the usual t-structure) is locally finite as a representation of $\pi_1(X, x)$; see Remark 1.1.11.

3.5.5. The functor (3.11) defines a map

$$\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X) \to \mathrm{LocSys}_{\mathsf{G}}(X). \tag{3.13}$$

As in the de Rham case, from Proposition 3.5.2 we obtain that the map (3.13) is a monomorphism.

Remark 3.5.6. This remark is parallel to Remark 3.1.3. Let us explain how the difference between $\mathrm{LocSys}_{\mathsf{G}}(X)$ and $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ plays out in the simplest case when $\mathsf{G} = \mathbb{G}_m$. Take $S = \mathrm{Spec}(R)$ to be classical.

In this case, an S-point of $\mathrm{LocSys}_{\mathbb{G}_m}(X)$ is a homomorphism

$$\pi_1(X) \to R^\times.$$

By contrast, if we further assume S to be reduced, then an S-point of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathbb{G}_m}(X)$ is a homomorphism

$$\pi_1(X) \to \mathsf{e}^\times.$$

Take now $\mathsf{G} = \mathbb{G}_a$. In this case, by Example Sect. 1.3.7, the map

$$\mathrm{LocSys}^{\mathrm{restr}}_{\mathbb{G}_a}(X) \to \mathrm{LocSys}_{\mathbb{G}_a}(X)$$

is an isomorphism.

Remark 3.5.7. A remark parallel to Remark 3.1.10 holds in the Betti context as well, i.e., for a closed embedding $\mathsf{G}' \to \mathsf{G}$, the diagram

$$\begin{tikzcd} \mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}'}(X) \arrow[r] \arrow[d] & \mathrm{LocSys}_{\mathsf{G}'}(X) \arrow[d] \\ \mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X) \arrow[r] & \mathrm{LocSys}_{\mathsf{G}}(X) \end{tikzcd}$$

is a fiber square.

3.5.8. We have also the following statements that are completely parallel with the de Rham situation (with the same proofs):

Proposition 3.5.9. *The map* (3.13) *is a formal isomorphism, i.e., identifies* $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ *with its formal completion inside* $\mathrm{LocSys}_{\mathsf{G}}(X)$.

Theorem 3.5.10. *The map*

$$\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X) \to \mathrm{LocSys}_{\mathsf{G}}(X)$$

is a closed embedding at the reduced level *for* each connected component of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$.

Corollary 3.5.11. *The subfunctor*

$$\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X) \subset \mathrm{LocSys}_{\mathsf{G}}(X)$$

is the disjoint union of formal completions of a collection of pairwise non-intersecting closed substacks of ${}^{\mathrm{red}}\mathrm{LocSys}_{\mathsf{G}}(X)$.

Corollary 3.5.12. *The map* $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X) \to \mathrm{LocSys}_{\mathsf{G}}(X)$ *is an ind-closed embedding.*

Note, however, that we still have to supply a proof of the Betti version of Proposition 3.3.7:

Proposition 3.5.13. *For a reductive group* G *and an irreducible local system* σ*, the resulting map*

$$\mathrm{pt} / \mathrm{Stab}_{\mathsf{G}}(\sigma) \to \mathrm{LocSys}_{\mathsf{G}}(X)$$

is a closed embedding.

The proof is given in Sect. 3.6.5 below.

Remark 3.5.14. Note that as in Remark 3.3.8, we obtain that the image of

$$ {}^{\mathrm{red}}\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X) \to {}^{\mathrm{red}}\mathrm{LocSys}_{\mathsf{G}}(X) \tag{3.14}$$

is the ind-closed substack

$${}^{\mathrm{red}}\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X) \subset {}^{\mathrm{red}}\mathrm{LocSys}_{\mathsf{G}}(X)$$

that equals the disjoint union over classes of association of $(\mathsf{P}, \sigma_{\mathsf{M}})$ of the unions of the images of the maps

$${}^{\mathrm{red}}\mathrm{LocSys}_{\mathsf{P},\sigma_{\mathsf{M}}}(X) \to {}^{\mathrm{red}}\mathrm{LocSys}_{\mathsf{G}}(X)$$

within a given class.

In Sect. 3.6 below we will give an alternative description of the image of (3.14), which is specific to the Betti situation.

3.6. **The coarse moduli space of homomorphisms.** Let X be as in Sect. 3.5. We will give a more explicit description of ${}^{\mathrm{red}}\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ as a subfunctor of ${}^{\mathrm{red}}\mathrm{LocSys}_{\mathsf{G}}(X)$.

3.6.1. Let $\mathsf{G}_{\mathrm{red}}$ denote the reductive quotient of G. We have a fiber square

$$\begin{tikzcd} \mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X) \arrow[r] \arrow[d] & \mathrm{LocSys}_{\mathsf{G}}(X) \arrow[d] \\ \mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}_{\mathrm{red}}}(X) \arrow[r] & \mathrm{LocSys}_{\mathsf{G}_{\mathrm{red}}}(X). \end{tikzcd} \tag{3.15}$$

Hence, in order to describe $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ as a subfunctor of $\mathrm{LocSys}_{\mathsf{G}}(X)$, it is enough to do so for G replaced by $\mathsf{G}_{\mathrm{red}}$. So, from now until the end of this subsection we will assume that G is reductive.

3.6.2. Choose a base point $x \in X$. Denote $\Gamma := \pi_1(X, x)$. As was remarked in the course of the proof of Proposition 3.4.4, the classical algebraic stack ${}^{\mathrm{cl}}\mathrm{LocSys}_{\mathsf{G}}(X)$ identifies with the classical stack underlying the quotient

$$\mathbf{Maps}_{\mathrm{Grp}}(\Gamma, \mathsf{G})/\operatorname{Ad}(\mathsf{G}),$$

where $\mathbf{Maps}_{\mathrm{Grp}}(\Gamma, \mathsf{G})$ is the affine scheme of homomorphism of the finitely generated discrete group Γ to G.

Consider now the coarse moduli space

$$\mathbf{Maps}_{\mathrm{Grp}}(\Gamma, \mathsf{G})/\!/\mathrm{Ad}(\mathsf{G}) := \mathrm{Spec}(\mathcal{O}_{\mathbf{Maps}_{\mathrm{Grp}}(\Gamma,\mathsf{G})})^{\mathsf{G}}).$$

We have a natural projection

$$r : \mathbf{Maps}_{\mathrm{Grp}}(\Gamma, \mathsf{G})/\operatorname{Ad}(\mathsf{G}) \to \mathbf{Maps}_{\mathrm{Grp}}(\Gamma, \mathsf{G})/\!/\mathrm{Ad}(\mathsf{G}). \tag{3.16}$$

3.6.3. Recall the notion of semi-simple local system and semi-simplification (see Sect. 2.4.4). The same definitions apply to homomorphisms $\Gamma \to \mathsf{G}$.

We have the following fundamental result of [Ri]:

Theorem 3.6.4.

(a) *Two* e*-points of the stack* $\mathbf{Maps}_{\mathrm{Grp}}(\Gamma, \mathsf{G})/\operatorname{Ad}(\mathsf{G})$ *map to the same point in the affine scheme* $\mathbf{Maps}_{\mathrm{Grp}}(\Gamma, \mathsf{G})/\!/\mathrm{Ad}(\mathsf{G})$ *if and only if they have isomorphic semi-simplifications.*

(b) *For an* e*-point* σ *of* $\mathbf{Maps}_{\mathrm{Grp}}(\Gamma, \mathsf{G})/\operatorname{Ad}(\mathsf{G})$*, the resulting map*

$$\mathrm{pt}/\operatorname{Stab}_{\mathsf{G}}(\sigma) \to \mathbf{Maps}_{\mathrm{Grp}}(\Gamma, \mathsf{G})/\operatorname{Ad}(\mathsf{G})$$

is a closed embedding if and only if σ *is semi-simple.*

3.6.5. Note that Theorem 3.6.4(b) immediately implies the assertion of Proposition 3.5.13: indeed any irreducible homomorphism is semi-simple.

Note, however, that we can also derive Proposition 3.5.13 from point (a) of Theorem 3.6.4:

Proof. Let σ be irreducible, and consider the closed substack

$$\mathrm{pt} \underset{\mathbf{Maps}_{\mathrm{Grp}}(\Gamma,\mathsf{G})/\!/\mathrm{Ad}(\mathsf{G})}{\times} \mathbf{Maps}_{\mathrm{Grp}}(\Gamma, \mathsf{G})/\operatorname{Ad}(\mathsf{G}) \subset \mathbf{Maps}_{\mathrm{Grp}}(\Gamma, \mathsf{G})/\operatorname{Ad}(\mathsf{G}),$$

where

$$\mathrm{pt} \to \mathbf{Maps}_{\mathrm{Grp}}(\Gamma, \mathsf{G})/\!/\mathrm{Ad}(\mathsf{G})$$

is given by $r(\sigma)$.

By Theorem 3.6.4(a) and the irreducibility assumption on σ, the above stack contains a unique isomorphism class of e-points. Hence, the map

$$\mathrm{pt}/\operatorname{Stab}_{\mathsf{G}}(\sigma) \to \mathrm{pt} \underset{\mathbf{Maps}_{\mathrm{Grp}}(\Gamma,\mathsf{G})/\!/\mathrm{Ad}(\mathsf{G})}{\times} \mathbf{Maps}_{\mathrm{Grp}}(\Gamma, \mathsf{G})/\operatorname{Ad}(\mathsf{G})$$

is an isomorphism of the underlying reduced substacks. In particular, it is a closed embedding.

$\square$

3.6.6. We now claim:

Theorem 3.6.7. *The subfunctor*

$${}^{\mathrm{red}}\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X) \subset {}^{\mathrm{red}}\mathrm{LocSys}_{\mathsf{G}}(X)$$

is the disjoint union of the fibers of the map r *of* (3.16)*.*

The rest of this subsection is devoted to the proof of Theorem 3.6.7.

3.6.8. We will prove the following slightly more precise statement (which would imply Theorem 3.6.7 in view of Remark 3.5.14):

Fix a class of association of pairs $(\mathsf{P}, \sigma_\mathsf{M})$. For each element in this class pick a Levi splitting

$$\mathsf{P} \leftrightarrows \mathsf{M},$$

and consider the induced G-local system. Note, however, that by the definition of association, these G-local systems are all isomorphic (for different elements $(\mathsf{P}, \sigma_\mathsf{M})$ in the given class); denote it by σ.

We will show that the reduced substack underlying

$$\text{pt} \underset{\mathbf{Maps}_{\mathrm{Grp}}(\Gamma,\mathsf{G})/\!/\mathrm{Ad}(\mathsf{G})}{\times} \mathbf{Maps}_{\mathrm{Grp}}(\Gamma, \mathsf{G})/\operatorname{Ad}(\mathsf{G}) \subset \mathbf{Maps}_{\mathrm{Grp}}(\Gamma, \mathsf{G})/\operatorname{Ad}(\mathsf{G}) \tag{3.17}$$

(where $\text{pt} \to \mathbf{Maps}_{\mathrm{Grp}}(\Gamma, \mathsf{G})/\!/\mathrm{Ad}(\mathsf{G})$ is given by $r(\sigma)$), equals the union of the images of the maps

$$\mathrm{LocSys}_{\mathsf{P},\sigma_\mathsf{M}}(X) \to \mathrm{LocSys}_\mathsf{G}(X), \tag{3.18}$$

where the union is taken over the pairs $(\mathsf{P}, \sigma_\mathsf{M})$ in our chosen class of association.

3.6.9. We claim:

Proposition 3.6.10. *Let $\mathsf{P} \leftrightarrows \mathsf{M}$ be a parabolic with a Levi splitting. Let σ_M be a M-local system, and let σ be the induced G-local system. Then the composite*

$$^{\mathrm{red}}\mathrm{LocSys}_{\mathsf{P},\sigma_\mathsf{M}}(X) \to {}^{\mathrm{red}}\mathrm{LocSys}_\mathsf{G}(X) \simeq {}^{\mathrm{red}}\mathbf{Maps}_{\mathrm{Grp}}(\Gamma, \mathsf{G})/\operatorname{Ad}(\mathsf{G}) \to {}^{\mathrm{red}}\mathbf{Maps}_{\mathrm{Grp}}(\Gamma, \mathsf{G})/\!/\mathrm{Ad}(\mathsf{G})$$

factors through

$$\text{pt} \xrightarrow{r(\sigma)} {}^{\mathrm{red}}\mathbf{Maps}_{\mathrm{Grp}}(\Gamma, \mathsf{G})/\!/\mathrm{Ad}(\mathsf{G}).$$

Proof. Note that all e-points of

$$\mathrm{LocSys}_\mathsf{G}(X) \simeq \mathbf{Maps}_{\mathrm{Grp}}(\Gamma, \mathsf{G})/\operatorname{Ad}(\mathsf{G})$$

obtained from e-points of $\mathrm{LocSys}_{\mathsf{P},\sigma}(X)$ have σ as their semi-simplification.

Hence, the assertion of the proposition follows from Theorem 3.6.4(a). $\square$

We will now deduce from Proposition 3.6.10 the description of (3.17) as the union of the images of the maps (3.18).

3.6.11. Indeed, on the one hand, Proposition 3.6.10 implies that the images of the maps (3.18) (at the reduced level) indeed lie in the fiber (3.17).

On the other hand, take an e-point σ' in the fiber (3.17), and let $(\mathsf{P}', \sigma_{\mathsf{M}'})$ be a pair such σ' lies in the image of

$$\mathrm{LocSys}_{\mathsf{P}',\sigma_{\mathsf{M}'}}(X) \to \mathrm{LocSys}_\mathsf{G}(X).$$

We need to show that $(\mathsf{P}', \sigma_{\mathsf{M}'})$ lies in our class of association. However, by Proposition 3.6.10, the G-local system, induced from $\sigma_{\mathsf{M}'}$, is isomorphic to σ. This implies the result by Lemma 2.4.5(c).

4. The formal coarse moduli space

In this section we will assume that G is reductive. The goal of this section is to establish a version, adapted to $\mathrm{LocSys}^{\mathrm{restr}}_\mathsf{G}(X)$, of the picture

$$r : \mathrm{LocSys}_\mathsf{G}(X) \to \mathrm{LocSys}^{\mathrm{coarse}}_\mathsf{G}(X)$$

that we have in the Betti case (see Sect. 4.5.1).

Prior to doing so, we show that $\mathrm{LocSys}^{\mathrm{restr}}_\mathsf{G}(X)$ has the following two geometric properties: it is *mock-affine* and *mock-proper*.

The main result of this section is Theorem 4.4.2, which constructs the desired picture

$$r : \mathcal{Z} \to \mathcal{Z}^{\mathrm{restr}}$$

for each connected component $\mathcal{Z}$ of $\mathrm{LocSys}^{\mathrm{restr}}_\mathsf{G}(X)$.

In the course of the proof we will encounter another fundamental feature of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ (Theorem 4.8.6). Recall that at the classical level, when we can think of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ as the prestack of homomorphisms $\mathrm{Gal}(X,x) \to \mathsf{G}$. The claim is that on each component of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$, these homomorphisms factor via a particular quotient of $\mathrm{Gal}(X,x)$ which is *topologically finitely generated*.

4.1. The "mock-properness" of ${}^{\mathrm{red}}\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}$.

4.1.1. Let $\mathcal{Z}$ be a quasi-compact algebraic stack over e. Let

$$\mathrm{Coh}(\mathcal{Z}) \subset \mathrm{QCoh}(\mathcal{Z})$$

be the full subcategory consisting of objects whose pullback under a smooth cover (equivalently, any map)

$$S \to \mathcal{Z}, \quad S \in \mathrm{Sch}^{\mathrm{aff}}_{/\mathsf{e}}$$

belongs to $\mathrm{Coh}(S) \subset \mathrm{QCoh}(S)$.

We shall say that $\mathcal{Z}$ is *mock-proper* if the functor

$$\Gamma(\mathcal{Z}, -) : \mathrm{QCoh}(\mathcal{Z}) \to \mathrm{Vect}_{\mathsf{e}}$$

sends $\mathrm{Coh}(\mathcal{Z})$ to $\mathrm{Vect}^c_{\mathsf{e}}$.

Remark 4.1.2. This definition is equivalent to one in [Ga3, Sect. 6.5]. Indeed, the subcategory

$$\mathrm{D\text{-}mod}(\mathcal{Z})^c \subset \mathrm{D\text{-}mod}(\mathcal{Z})$$

is generated under finite colimits by the image of $\mathrm{Coh}(\mathcal{Z})$ along induction functor

$$\mathbf{ind}_{\mathrm{D\text{-}mod}} : \mathrm{QCoh}(\mathcal{Z}) \to \mathrm{D\text{-}mod}(\mathcal{Z}).$$

4.1.3. *Examples.*

(i) If $\mathcal{Z}$ is a scheme, then it is mock-proper as a stack if and only it is proper as a scheme.

(ii) The stack pt/H is mock-proper for any algebraic group H.

(iii) For a (finite-dimensional) vector space V, the stack $\mathrm{Tot}(V)/\mathbb{G}_m$ is mock-proper. (This is just the fact that for a finitely generated graded $\mathrm{Sym}(V^\vee)$-module, its degree 0 component is finite-dimensional as a vector space.)

4.1.4. Let $\mathcal{Z}$ be a connected component of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$. Recall that according to Theorem 1.3.2, its underlying reduced prestack ${}^{\mathrm{red}}\mathcal{Z}$ is actually an algebraic stack.

We will prove:

Theorem 4.1.5. *The algebraic stack ${}^{\mathrm{red}}\mathcal{Z}$ is mock-proper.*

The rest of the subsection is devoted to the proof of this result.

4.1.6. Recall (see Sect. 2.4) that to $\mathcal{Z}$ there corresponds a class of association of pairs (P,σ), where P is a parabolic in G and σ is an irreducible local system with respect to the Levi quotient M of P.

Moreover, the resulting morphism

$$\pi : \underset{(\mathsf{P},\sigma)}{\sqcup} {}^{\mathrm{red}}\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{P},\sigma_{\mathsf{M}}}(X) \to {}^{\mathrm{red}}\mathcal{Z}$$

(the union is taken over the given class of association) is proper and surjective at the level of geometric points.

We claim that the category $\mathrm{Coh}({}^{\mathrm{red}}\mathcal{Z})$ is generated under finite colimits by the essential image of

$$\mathrm{Coh}(\underset{(\mathsf{P},\sigma)}{\sqcup} {}^{\mathrm{red}}\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{P},\sigma_{\mathsf{M}}}(X))$$

along π_*.

Indeed, this follows from the next general assertion:

Lemma 4.1.7. *Let $\pi : \mathcal{Z}' \to \mathcal{Z}$ be a proper map between algrebraic stacks, surjective at the level of geometric points. Then* $\mathrm{Coh}(\mathcal{Z})$ *is generated under finite colimits by the essential image of* $\mathrm{Coh}(\mathcal{Z}')$ *along* π_*.

Proof. First, since π is proper, the functor π_* does indeed send $\mathrm{Coh}(\mathcal{Z}')$ to $\mathrm{Coh}(\mathcal{Z})$. Since $\mathrm{IndCoh}(\mathcal{Z}')$ is generated by $\mathrm{Coh}(\mathcal{Z}')$ (see [DrGa1, Proposition 3.5.1]), the assertion of the lemma is equivalent to the fact that the essential image of $\mathrm{IndCoh}(\mathcal{Z}')$ along

$$\pi_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(\mathcal{Z}') \to \mathrm{IndCoh}(\mathcal{Z})$$

generates $\mathrm{IndCoh}(\mathcal{Z})$. This is equivalent to the fact that the right adjoint

$$\pi^! : \mathrm{IndCoh}(\mathcal{Z}) \to \mathrm{IndCoh}(\mathcal{Z}')$$

is conservative. However, the latter is [Ga4, Proposition 8.1.2]. □

4.1.8. Thus, we obtain that it suffices to show that for a parabolic P with Levi quotient M and a M-local system σ_{M}, the algebraic stack

$$\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{P},\sigma_{\mathsf{M}}}(X)$$

is mock-proper.

Let $\mathrm{LocSys}^{\mathrm{restr},\mathrm{rigid}_x}_{\mathsf{P},\sigma_{\mathsf{M}}}(X)$ be the following (algebraic) stack: it classifies the data of

$$(\sigma_{\mathsf{P}}, \alpha, \epsilon),$$

where:

- σ_{P} is a point of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{P}}(X)$,
- α is an identification $\mathsf{M} \overset{\mathsf{P}}{\times} \sigma_{\mathsf{P}} \simeq \sigma_{\mathsf{M}}$ (so that the pair $(\sigma_{\mathsf{P}}, \alpha)$ is a point of
$$\mathrm{pt} \underset{\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{M}}(X)}{\times} \mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{P}}(X);$$
- ϵ is an identification
$$\sigma_{\mathsf{P},x} \simeq \mathsf{P} \overset{\mathsf{M}}{\times} \sigma_{\mathsf{M},x},$$
compatible with the datum of α.

The stack $\mathrm{LocSys}^{\mathrm{restr},\mathrm{rigid}_x}_{\mathsf{P},\sigma_{\mathsf{M}}}(X)$ carries an action of $\mathrm{Aut}(\sigma_{\mathsf{M}})$ (by changing the datum of α); in particular, it is acted on by $Z(\mathsf{M})^0$, the connected component of the center of M. In addition, it carries a commuting action of the (unipotent) group

$$(\mathsf{N}_{\mathsf{P}})_{\sigma_{\mathsf{M},x}}$$

(by changing the datum of ϵ), where:

- N_{P} is the unipotent radical of P;
- $(\mathsf{N}_{\mathsf{P}})_{\sigma_{\mathsf{M},x}}$ is the twist of N_{P} by the M-torsor $\sigma_{\mathsf{M},x}$, using the adjoint action of M on N_{P}.

We have:

$$\mathrm{LocSys}^{\mathrm{restr},\mathrm{rigid}_x}_{\mathsf{P},\sigma_{\mathsf{M}}}(X)/\,\mathrm{Aut}(\sigma_{\mathsf{M}}) \times (\mathsf{N}_{\mathsf{P}})_{\sigma_{\mathsf{M},x}} \simeq \mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{P},\sigma_{\mathsf{M}}}(X).$$

Choose a coweight $\mathbb{G}_m \to Z(\mathsf{M})^0$, dominant and regular with respect to P (i.e., one such that the adjoint action of $\mathbb{G}_m$ on $\mathfrak{n}_{\mathsf{P}}$ has positive eigenvalues).

It suffices to show that the algebraic stack

$$\mathrm{LocSys}^{\mathrm{restr},\mathrm{rigid}_x}_{\mathsf{P},\sigma_{\mathsf{M}}}(X)/\mathbb{G}_m$$

is mock-proper.

4.1.9. Note that the proof in Sects. 2.2.3-2.2.6 of the fact that the morphism

$$\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{P}}(X) \to \mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{M}}(X)$$

is a relative algebraic stack implies that $\mathrm{LocSys}^{\mathrm{restr},\mathrm{rigid}_x}_{\mathsf{P},\sigma_{\mathsf{M}}}(X)$ is actually an affine (derived) scheme.

Furthermore, the fact that $\mathbb{G}_m$ acts on $\mathfrak{n}_{\mathsf{P}}$ with positive eigenvalues implies that the action of $\mathbb{G}_m$ on $\mathrm{LocSys}^{\mathrm{restr},\mathrm{rigid}_x}_{\mathsf{P},\sigma_{\mathsf{M}}}(X)$ is *contracting*:

Recall (see [DrGa3, Sect. 1.4.4]) that an action of $\mathbb{G}_m$ on an affine scheme Z is said to be contracting if it can be extended to an action of the monoid $\mathbb{A}^1$, so that the action of $0 \in \mathbb{A}^1$ factors as

$$Z \to \mathrm{pt} \to Z.$$

The required result follows now from the next general assertion:

Lemma 4.1.10. *Let Z be an affine scheme, equipped with a contracting action of $\mathbb{G}_m$. Then the algebraic stack $Z/\mathbb{G}_m$ is mock-proper.*

Proof. Write $Z = \mathrm{Spec}(A)$. The $\mathbb{G}_m$-action on Z equips A with a grading. The fact that the $\mathbb{G}_m$-action is contracting is equivalent to the fact that the grading on A is non-negative and that the map $\mathsf{e} \to A^0$ is an isomorphism.

The category $\mathrm{QCoh}(Z/\mathbb{G}_m)$ consist of complexes M of graded A-modules. The subcategory $\mathrm{Coh}(Z/\mathbb{G}_m) \subset \mathrm{QCoh}(Z/\mathbb{G}_m)$ corresponds to the condition that M be cohomologically bounded and $H^0(M)$ be finitely generated over $H^0(A)$.

The functor

$$\Gamma(Z/\mathbb{G}_m, -) : \mathrm{Coh}(Z/\mathbb{G}_m) \to \mathrm{Vect}_{\mathsf{e}}$$

takes M to its degree 0 component M^0.

This implies the assertion of the lemma.

□

4.2. A digression: ind-algebraic stacks.

4.2.1. Let $\mathcal{Z}$ be a prestack.

We shall say that $\mathcal{Z}$ is an *ind-algebraic stack* it is *convergent* and for every n, the nth coconnective truncation ${}^{\leq n}\mathcal{Z}$, can be written as

$$ {}^{\leq n}\mathcal{Z} \simeq \underset{i\in I}{\mathrm{colim}}\, \mathcal{Z}_{i,n}, \tag{4.1}$$

where:

- Each $\mathcal{Z}_{i,n}$ is a quasi-compact n-coconnective algebraic stack locally of finite type;
- The category I of indices is filtered;
- The transition maps $\mathcal{Z}_{i,n} \to \mathcal{Z}_{j,n}$ are closed embedding.

We claim:

Lemma 4.2.2. *Let $\mathcal{Z}$ be an n-coconnective ind-algebraic stack. Then:*

(a) *The maps $\mathcal{Z}_{i,n} \to \mathcal{Z}$ are closed embeddings.*

(b) *The family*

$$i \mapsto (\mathcal{Z}_{i,n} \to \mathcal{Z})$$

is confinal in the category of n-coconnective algebraic quasi-compact stacks equipped with a closed embedding into $\mathcal{Z}$.

The proof of parallel to [GR3, Lemma 1.3.6][12].

[12]The n-coconnectivity condition is important here: we use it when we say that an n-coconnective quasi-compact algebraic stack can be written as a *finite* limit of affine schemes, sheafified in the étale/fppf topology.

4.2.3. We now claim:

Lemma 4.2.4. *Let a prestack $\mathcal{Z}$ be equal to the quotient $\mathcal{Y}/\mathsf{G}$, where $\mathcal{Y}$ is an ind-scheme, and G is an algebraic group. Then $\mathcal{Z}$ is an ind-algebraic stack.*

Proof. The convergence condition easily follows from the fact that both $\mathcal{Y}$ and pt/G are convergent. Thus, we may assume that $\mathcal{Y}$ is n-coconnective. We need to show that we can write $\mathcal{Y}$ as a filtered colimit

$$\mathcal{Y} \simeq \underset{i \in I}{\mathrm{colim}}\, Y_i,$$

where:

- Each Y_i is a quasi-compact scheme, stable under the G-action;
- The transition maps $Y_i \to Y_j$ are closed embeddings, compatible with the action of G.

This can be proved by repeating the argument of [GR3, Theorem 5.1.1] (in other words, we run the proof of *loc. cit.* where instead of $\mathrm{QCoh}(-)$ we use $\mathrm{QCoh}(-)^{\mathsf{G}}$).

□

4.2.5. As corollary, combining with Theorem 1.3.2, we obtain:

Corollary 4.2.6. *Every connected component of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ is an ind-algebraic stack.*

4.3. **Mock-affineness and coarse moduli spaces.**

4.3.1. Let $\mathcal{Z}$ be an algebraic stack. We shall say that $\mathcal{Z}$ is *mock-affine* if the functor of global sections

$$\Gamma(\mathcal{Z}, -) : \mathrm{QCoh}(\mathcal{Z}) \to \mathrm{Vect}_{\mathsf{e}}$$

is t-exact.

Clearly, $\mathcal{Z}$ is mock-affine if and only if its underlying classical stack ${}^{\mathrm{cl}}\mathcal{Z}$ is mock-affine.

4.3.2. *Example.* Let $\mathcal{Z}$ be of the form Y/G, where Y is affine scheme and G is a *reductive* algebraic group. Then (assuming that e has characteristic zero) the stack $\mathcal{Z}$ is mock-affine.

4.3.3. Let $\mathcal{Z}$ be an ind-algebraic stack. We shall say that $\mathcal{Z}$ is mock-affine if ${}^{\mathrm{cl}}\mathcal{Z}$ admits a presentation (4.1) whose terms are mock-affine.

By Lemma 4.2.2, this is equivalent to requiring that for every algebraic stack $\mathcal{Z}'$ equipped with a closed embedding $\mathcal{Z}' \to \mathcal{Z}$, the stack $\mathcal{Z}'$ is mock-affine.

4.3.4. From Theorem 1.3.2, we obtain that each connected component of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ is mock-affine.

4.3.5. Let $\mathcal{Z}$ be a mock-affine algebraic stack. In particular, the e-algebra

$$\Gamma(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}})$$

is connective.

Further, for every n,

$$\tau^{\geq -n}(\Gamma(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}})) \simeq \Gamma({}^{\leq n}\mathcal{Z}, \mathcal{O}_{\mathrm{red}\,\mathcal{Z}}).$$

We define the coarse moduli space $\mathcal{Z}^{\mathrm{coarse}}$ of $\mathcal{Z}$ to be the affine scheme

$$\mathrm{Spec}(\Gamma(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}})).$$

By construction, we have a canonical projection

$$r : \mathcal{Z} \to \mathcal{Z}^{\mathrm{coarse}}.$$

4.3.6. Let $\mathcal{Z}$ be a mock-affine ind-algebraic stack. For every n consider the n-coconnective ind-affine ind-scheme

$$^{\leq n}\mathcal{Z}^{\mathrm{coarse}} := \underset{i}{\mathrm{colim}}\ \mathrm{Spec}(\Gamma(\mathcal{Z}_{i,n}, \mathcal{O}_{\mathcal{Z}_{i,n}}))$$

for $^{\leq n}\mathcal{Z}$ written as in Sect. 4.1 (by Lemma 4.2.2, this definition is independent of the presentation).

We define the ind-affine ind-scheme $\mathcal{Z}^{\mathrm{coarse}}$ to be the convergent completion of

$$\underset{n}{\mathrm{colim}}\ {}^{\leq n}\mathcal{Z}^{\mathrm{coarse}}. \tag{4.2}$$

I.e., this is a convergent prestack whose value on eventually coconnective affine schemes is given by the colimit (4.2).

We have a canonical projection

$$r : \mathcal{Z} \to \mathcal{Z}^{\mathrm{coarse}}.$$

4.3.7. We claim:

Lemma 4.3.8. *Let $\mathcal{Z}$ be a mock-affine ind-algebraic stack satisfying:*

- *$\mathcal{Z}$ is locally almost of finite type;*
- *$^{\mathrm{red}}\mathcal{Z}$ a mock-proper algebraic stack.*
- *$^{\mathrm{red}}\mathcal{Z}$ is connected.*

Then $\mathcal{Z}^{\mathrm{coarse}}$ has the following properties:

- *It is locally almost of finite type;*
- *$^{\mathrm{red}}(\mathcal{Z}^{\mathrm{coarse}}) \simeq \mathrm{pt}$.*

Proof. To prove that $\mathcal{Z}^{\mathrm{coarse}}$ is locally almost of finite type, it suffices to show that for every n, and a presentation of $^{\leq n}\mathcal{Z}$ as in Sect. 4.1, the rings $\Gamma(\mathcal{Z}_{i,n}, \mathcal{O}_{\mathcal{Z}_{i,n}})$ are finite-dimensional over e. However, this follows from the mock-properness assumption.

This also implies that $^{\mathrm{red}}(\mathcal{Z}^{\mathrm{coarse}})$ is Artinian, i.e., is the union of finite many copies of pt. The connectedness assumption on $^{\mathrm{red}}\mathcal{Z}$ implies that there is only one copy.

□

4.4. Coarse moduli spaces for connected components of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$.

4.4.1. We apply the discussion from Sect. 4.3 to $\mathcal{Z}$ being a connected component of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$. We now ready to state the main result of this subsection:

Main Theorem 4.4.2. *Let $\mathcal{Z}$ being a connected component of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$, and consider the corresponding map*

$$r : \mathcal{Z} \to \mathcal{Z}^{\mathrm{coarse}}.$$

We have:

(a) *The map r makes $\mathcal{Z}$ into a relative algebraic stack over $\mathcal{Z}^{\mathrm{coarse}}$, i.e., the base change of r by an affine scheme yields an algebraic stack.*

(b) *The ind-scheme $\mathcal{Z}^{\mathrm{coarse}}$ is a formal affine scheme (see Remark 1.3.4 for what this means).*

A consequence of Theorem 4.4.2 of particular importance for the sequel is:

Corollary 4.4.3. *The fiber product*

$$\mathrm{pt} \underset{\mathcal{Z}^{\mathrm{coarse}}}{\times} \mathcal{Z}^{\mathrm{rigid}_x}$$

is an affine scheme.

The above corollary can be equivalently stated as follows:

Corollary 4.4.4. *The fiber product*

$$\mathrm{pt} \underset{\mathcal{Z}^{\mathrm{coarse}}}{\times} \mathcal{Z}$$

is an algebraic stack[13].

[13]It follows automatically that it is quasi-compact and locally almost of finite type

Remark 4.4.5. We emphasize that the assertion of Corollary 4.4.3 (resp., Corollary 4.4.4) is that the corresponding fiber products do *not* have ind-directions.

They may be non-reduced, but the point is that they are (locally) schemes, as opposed to formal schemes.

4.5. **Coarse moduli space in the Betti setting.** In this subsection we return to the context of Sect. 3.4, i.e., we let $\mathcal{X}$ be a connected object of Spc; we fix a base point x. We will illustrate explicitly what Theorem 4.4.2 in this setting.

4.5.1. In the Betti case, we have a well-defined algebraic stack $\mathrm{LocSys}_{\mathsf{G}}(\mathcal{X})$. Since G was assumed reductive, we obtain that the algebra

$$\Gamma(\mathrm{LocSys}_{\mathsf{G}}(\mathcal{X}), \mathcal{O}_{\mathrm{LocSys}_{\mathsf{G}}(\mathcal{X})})$$

is connective.

Indeed, write

$$\mathrm{LocSys}_{\mathsf{G}}(\mathcal{X}) \simeq \mathrm{LocSys}_{\mathsf{G}}^{\mathrm{rigid}_x}(\mathcal{X})/\mathsf{G},$$

so

$$\Gamma(\mathrm{LocSys}_{\mathsf{G}}(\mathcal{X}), \mathcal{O}_{\mathrm{LocSys}_{\mathsf{G}}(\mathcal{X})}) \simeq \left(\Gamma(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{rigid}_x}(\mathcal{X}), \mathcal{O}_{\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{rigid}_x}(\mathcal{X})})\right)^{\mathsf{G}}.$$

Define

$$\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{coarse}}(\mathcal{X}) := \mathrm{Spec}\left(\Gamma(\mathrm{LocSys}_{\mathsf{G}}(\mathcal{X}), \mathcal{O}_{\mathrm{LocSys}_{\mathsf{G}}(\mathcal{X})})\right).$$

Let r denote the natural projection

$$\mathrm{LocSys}_{\mathsf{G}}(\mathcal{X}) \to \mathrm{LocSys}_{\mathsf{G}}^{\mathrm{coarse}}(\mathcal{X}).$$

4.5.2. Let $\Gamma := \pi_1(\mathcal{X}, x)$. It is clear that

$${}^{\mathrm{cl}}\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{coarse}}(\mathcal{X}) \simeq {}^{\mathrm{cl}}\mathbf{Maps}_{\mathrm{Grp}}(\Gamma, \mathsf{G})/\!/\mathsf{G},$$

see Sect. 3.6.2 (the above isomorphism is true at the derived level if $\mathcal{X} \simeq B(\Gamma)$).

4.5.3. Assume now that $\mathcal{X}$ is a homotopy type of a compact CW complex, see Sect. 3.5. Adapting the above definitions, we obtain the affine scheme $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{coarse}}(X)$ and the map

$$r : \mathrm{LocSys}_{\mathsf{G}}(X) \to \mathrm{LocSys}_{\mathsf{G}}^{\mathrm{coarse}}(X). \tag{4.3}$$

Let

$$\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr,coarse}}(X)$$

be the disjoint union of formal completions of $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{coarse}}(X)$ at its e-points.

Note that Theorem 3.6.7 can be reformulated as saying that we have a Cartesian diagram

$$\begin{array}{ccc} \mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X) & \longrightarrow & \mathrm{LocSys}_{\mathsf{G}}(\mathcal{X}) \\ \downarrow & & \downarrow \\ \mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr,coarse}}(X) & \longrightarrow & \mathrm{LocSys}_{\mathsf{G}}^{\mathrm{coarse}}(X). \end{array} \tag{4.4}$$

4.5.4. For a fixed $\sigma \in \mathrm{LocSys}_{\mathsf{G}}^{\mathrm{coarse}}(X)$, let $\mathcal{Z}_\sigma \subset \mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)$ be the corresponding connected component of $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)$.

It is clear from (4.4) that

$$(\mathcal{Z}_\sigma)^{\mathrm{coarse}} \simeq (\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{coarse}}(X))^\wedge_\sigma, \tag{4.5}$$

where the right-hand side is the formal completion of $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{coarse}}(X)$ at σ.

The isomorphism (4.5) makes both assertions of Theorem 4.4.2 manifest. Indeed, point (a) follows from the fact that the projection

$$\mathcal{Z}_\sigma \xrightarrow{r} (\mathcal{Z}_\sigma)^{\mathrm{coarse}}$$

is a base change of the map (4.3), while $\mathrm{LocSys}_{\mathsf{G}}(X)$ is an algebraic stack.

4.6. **Maps of groups.** Prior to proving Theorem 4.4.2, we will consider another situation, where a statement of this nature can be established explicitly.

Logically, Theorem 4.4.2 will not rely on Theorem 4.6.6 stated in the present subsection. Yet its proof will use the same circle of ideas.

4.6.1. Let H and G be be (finite-dimensional) algebraic groups, and consider the prestacks

$$\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, \mathsf{G}) \text{ and } \mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, \mathsf{G})/\mathsf{G}.$$

In Proposition 1.8.6 we have already established that $\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, \mathsf{G})$ is an ind-affine ind-scheme. Furthermore, if H is reductive, we know by Proposition 2.3.9 that $\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, \mathsf{G})$ is a disjoint union of (classical smooth) affine schemes.

Let $\mathcal{Z}$ be a connected component of $\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, \mathsf{G})/\mathsf{G}$, and let $\mathcal{Z}^{\mathrm{rigid}}$ be its preimage in $\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, \mathsf{G})$. We are going to explicitly realize $\mathcal{Z}^{\mathrm{rigid}}$ as the completion of an affine scheme along a Zariski closed subset.

4.6.2. Choose a Levi splitting of H, i.e.,

$$\mathsf{H} := \mathsf{H}_{\mathrm{red}} \ltimes \mathsf{H}_u.$$

We have a natural projection

$$\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, \mathsf{G}) \to \mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}_{\mathrm{red}}, \mathsf{G}).$$

Fix a point $\phi \in \mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}_{\mathrm{red}}, \mathsf{G})$, and set

$$\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, \mathsf{G})_\phi := \mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, \mathsf{G}) \underset{\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}_{\mathrm{red}}, \mathsf{G})}{\times} \{\phi\}.$$

This is an ind-scheme, equipped with an action of $\mathrm{Stab}_{\mathsf{G}}(\phi)$. We will see shortly that $\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, \mathsf{G})_\phi$ is connected. So, our $\mathcal{Z}$ is of the form

$$\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, \mathsf{G})_\phi / \mathrm{Stab}_{\mathsf{G}}(\phi). \tag{4.6}$$

We are going to exhibit $\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, \mathsf{G})_\phi$ as the completion of an affine scheme along a Zariski closed subset, such that the entire situation carries an action of $\mathrm{Stab}_{\mathsf{G}}(\phi)$.

4.6.3. Note that $\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, \mathsf{G})_\phi$ identifies with

$$\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}_u, \mathsf{G})^{\mathsf{H}_{\mathrm{red}}},$$

where $\mathsf{H}_{\mathrm{red}}$ acts on G via ϕ.

Consider the affine scheme

$$\mathbf{Maps}_{\mathrm{Lie}}(\mathfrak{h}_u, \mathfrak{g})$$

(see Proposition 7.4.2 below), and its closed subscheme

$$\mathbf{Maps}_{\mathrm{Lie}}(\mathfrak{h}_u, \mathfrak{g})^{\mathsf{H}_{\mathrm{red}}}.$$

We have a naturally defined map

$$\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}_u, \mathsf{G})^{\mathsf{H}_{\mathrm{red}}} \to \mathbf{Maps}_{\mathrm{Lie}}(\mathfrak{h}_u, \mathfrak{g})^{\mathsf{H}_{\mathrm{red}}}. \tag{4.7}$$

We claim:

Proposition 4.6.4. *The map* (4.7) *realizes* $\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}_u, \mathsf{G})^{\mathsf{H}_{\mathrm{red}}}$ *as the formal completion of the affine scheme* $\mathbf{Maps}_{\mathrm{Lie}}(\mathfrak{h}_u, \mathfrak{g})^{\mathsf{H}_{\mathrm{red}}}$ *along the closed subset consisting of those maps*

$$\mathfrak{h}_u \to \mathfrak{g}$$

whose image is contained in the nilpotent cone of $\mathfrak{g}$.

As a formal consequence, we obtain that $\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, \mathsf{G})_\phi$ is connected. Indeed, the action of $\mathbb{G}_m$ by dilations contracts it to a single point.

Proof of Proposition 4.6.4. We interpret

$$\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}_u, \mathsf{G})$$

as $\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathrm{Rep}(\mathsf{H}_u))^{\mathrm{rigid}}$, see Sect. 6.4.1, and $\mathbf{Maps}_{\mathrm{Lie}}(\mathsf{h}, \mathsf{g})$ as

$$\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathsf{h}\text{-mod})^{\mathrm{rigid}},$$

see Corollary 7.5.4.

The fact that (4.7) is an ind-closed embedding and a formal isomorphism follows now from the fact that the restriction functor

$$\mathrm{Rep}(\mathsf{H}_u) \to \mathsf{h}_u\text{-mod}$$

is fully faithful, whose essential image consists of objects, all of whose cohomologies are such that the action of h_u on them is nilpotent.

This description also implies the stated description of the essential image at the reduced level. $\square$

4.6.5. Assume now that G be reductive. Fix ϕ as above, and let $\mathcal{Z}_\phi$ be the corresponding connected component of $\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, \mathsf{G})/\mathsf{G}$, i.e.,

$$\mathcal{Z}_\phi \simeq \mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, \mathsf{G})_\phi / \mathrm{Stab}_{\mathsf{G}}(\phi).$$

By the above formula, $\mathcal{Z}_\phi$ is mock-affine. We claim that $\mathcal{Z}_\phi$ is also mock-proper. This follows by the same argument as in the case of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$.

Consider the resulting ind-affine ind-scheme $\mathcal{Z}_\phi^{\mathrm{coarse}}$ and the corresponding map

$$r : \mathcal{Z}_\phi \to \mathcal{Z}_\phi^{\mathrm{coarse}}.$$

We will prove:

Theorem 4.6.6.

(a) *The map r makes $\mathcal{Z}_\phi$ into a relative algebraic stack over $\mathcal{Z}_\phi^{\mathrm{coarse}}$.*

(b) *The ind-scheme $\mathcal{Z}_\phi^{\mathrm{coarse}}$ is a formal affine scheme.*

4.7. Proof of Theorem 4.6.6.

4.7.1. *How should we think about ϕ?* First, we note that the fact that ${}^{\mathrm{red}}(\mathcal{Z}_\phi^{\mathrm{coarse}}) \simeq \mathrm{pt}$ implies that the action of G on $\mathcal{Z}_\phi^{\mathrm{rigid}}$ has a *unique* closed orbit. I.e., the stack $\mathcal{Z}_\phi$ has a unique closed point.

In terms of the identification

$$\mathcal{Z}_\phi \simeq \mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}_u, \mathsf{G})^{\mathsf{H}_{\mathrm{red}}} / \mathrm{Stab}_{\mathsf{G}}(\phi).,$$

this closed point corresponds to the trivial map $\mathsf{H}_u \to \mathsf{G}$. The corresponding homomorphism

$$\mathsf{H} \to \mathsf{G}$$

factors as

$$\mathsf{H} \to \mathsf{H}_{\mathrm{red}} \to \mathsf{G},$$

where the map $\mathsf{H}_{\mathrm{red}} \to \mathsf{G}$ is our ϕ.

4.7.2. *A reduction step.* We claim that in order to prove both points of the theorem, it suffices to show that

$$\mathcal{W}_\phi^{\mathrm{rigid}} := \mathrm{pt} \underset{\mathcal{Z}_\phi^{\mathrm{coarse}}}{\times} \mathcal{Z}_\phi^{\mathrm{rigid}} \tag{4.8}$$

is an affine scheme.

Indeed, for point (a), it suffices to show that if $\mathcal{Z}_\phi^{\mathrm{coarse}}$ is written as

$$\underset{i}{\mathrm{colim}}\ \mathrm{Spec}(A_i)$$

with A_i Artinian, then each

$$\mathrm{Spec}(A_i) \underset{\mathcal{Z}_\phi^{\mathrm{coarse}}}{\times} \mathcal{Z}_\phi^{\mathrm{rigid}}$$

is an affine scheme.

However, since A_i is Artinian, this would follow once we know that the further base change

$$\mathrm{pt} \underset{\mathrm{Spec}(A_i)}{\times} \left(\mathrm{Spec}(A_i) \underset{\mathcal{Z}_\phi^{\mathrm{coarse}}}{\times} \mathcal{Z}_\phi^{\mathrm{rigid}}\right)$$

is an affine scheme[14]. However the latter prestack is the same as the prestack $\mathcal{W}_\phi^{\mathrm{rigid}}$ of (4.8).

For point (b), by Theorem 2.1.4, it suffices to show that the tangent space to $\mathcal{Z}_\phi^{\mathrm{coarse}}$ at its unique closed point, viewed as an object of $\mathrm{Vect}_{\mathsf{e}}^{\geq 0}$, is finite-dimensional in each degree. For that it suffices to check that the !-pullback of $T_{\mathrm{pt}}(\mathcal{Z}_\phi^{\mathrm{coarse}})$ to $\mathcal{W}_\phi^{\mathrm{rigid}}$, viewed as an object of $\mathrm{IndCoh}(\mathcal{W}_\phi^{\mathrm{rigid}})$, is such that all its cohomologies are in $\mathrm{Coh}(\mathcal{W}_\phi^{\mathrm{rigid}})^\heartsuit$.

We have a fiber sequence

$$T(\mathcal{W}_\phi^{\mathrm{rigid}}) \to T(\mathcal{Z}_\phi^{\mathrm{rigid}})|_{\mathcal{W}_\phi^{\mathrm{rigid}}} \to T_{\mathrm{pt}}(\mathcal{Z}_\phi^{\mathrm{coarse}})|_{\mathcal{W}_\phi^{\mathrm{rigid}}}.$$

The cohomologies of $T(\mathcal{W}_\phi^{\mathrm{rigid}}) \in \mathrm{IndCoh}(\mathcal{W}_\phi^{\mathrm{rigid}})$ lie in $\mathrm{Coh}(\mathcal{W}_\phi^{\mathrm{rigid}})^\heartsuit$ because $\mathcal{W}_\phi^{\mathrm{rigid}}$ is an affine scheme (locally of finite type).

Now, $T(\mathcal{Z}_\phi^{\mathrm{rigid}})|_{\mathcal{W}_\phi^{\mathrm{rigid}}}$ also has cohomologies lying in $\mathrm{Coh}(\mathcal{W}_\phi^{\mathrm{rigid}})^\heartsuit$ by the description of the tangent complex to $\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, \mathsf{G})$ in the proof of Proposition 2.3.9: the tangent space to $\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, \mathsf{G})/\mathrm{Ad}(\mathsf{G})$ at a map $\mathsf{H} \to \mathsf{G}$ is given by $\mathrm{inv}_{\mathsf{H}}(\mathfrak{g}[1])$.

4.7.3. In order to prove that $\mathcal{W}_\phi^{\mathrm{rigid}}$ is an affine scheme, we will use the following construction (to be carried out in Sect. 4.7.7).

We will find an affine scheme $\mathrm{Spec}(A)$ and a map

$$r' : {}^{\mathrm{cl}}\mathcal{Z} \to \mathrm{Spec}(A)$$

with the following property:

Consider the point of $\mathrm{Spec}(A)$ equal to the image under r' of the unique closed point of $\mathcal{Z}$, i.e., one corresponding to

$$\mathsf{H} \twoheadrightarrow \mathsf{H}_{\mathrm{red}} \xrightarrow{\phi} \mathsf{G}.$$

Then we need that the classical prestack underlying the fiber product

$$\mathrm{pt} \underset{\mathrm{Spec}(A)}{\times} {}^{\mathrm{cl}}\mathcal{Z}^{\mathrm{rigid}} \tag{4.9}$$

be a (classical) affine scheme (as opposed to an ind-affine ind-scheme).

Equivalently, we need that the classical prestack underlying the fiber product

$$\mathrm{pt} \underset{\mathrm{Spec}(A)}{\times} {}^{\mathrm{cl}}\mathcal{Z}$$

[14]Indeed, given an ind-scheme $\mathcal{Y}$ over $\mathrm{Spec}(A)$ with A Artinian, if $\mathcal{Y} \underset{\mathrm{Spec}(A)}{\times} \mathrm{pt}$ is a scheme, then $\mathcal{Y}$ is a scheme.

be a (classical) algebraic stack (as opposed to an ind-algebraic stack).

Assuming the existence of such a construction, let us proceed with the proof of the fact that $\mathcal{W}_\phi^{\mathrm{rigid}_x}$ is an affine scheme.

4.7.4. Let $\mathrm{Spec}(A)$ and r' be as above. First, we claim that we can extend r' to a map at the derived level,

$$\mathcal{Z}_\phi \to \mathrm{Spec}(A)$$

which we will denote by the same symbol r'.

Indeed, with no restriction of generality, we can assume that A is a classical polynomial algebra, so the datum of r' amounts to a collection G-invariant elements in $\Gamma({}^{\mathrm{cl}}\mathcal{Z}_\phi^{\mathrm{rigid}}, \mathcal{O}_{{}^{\mathrm{cl}}\mathcal{Z}_\phi^{\mathrm{rigid}}})$ or $\Gamma(\mathcal{Z}_\phi^{\mathrm{rigid}}, \mathcal{O}_{\mathcal{Z}_\phi^{\mathrm{rigid}}})$ for the classical and derived versions of r', respectively.

Now, since $\mathcal{Z}_\phi^{\mathrm{rigid}}$ is a formal affine scheme, the map

$$\Gamma(\mathcal{Z}_\phi^{\mathrm{rigid}}, \mathcal{O}_{\mathcal{Z}_\phi^{\mathrm{rigid}}}) \to \Gamma({}^{\mathrm{cl}}\mathcal{Z}_\phi^{\mathrm{rigid}}, \mathcal{O}_{{}^{\mathrm{cl}}\mathcal{Z}_\phi^{\mathrm{rigid}}})$$

is an isomorphism on H^0. Hence, so is the map

$$\Gamma(\mathcal{Z}_\phi^{\mathrm{rigid}}, \mathcal{O}_{\mathcal{Z}_\phi^{\mathrm{rigid}}})^{\mathsf{G}} \to \Gamma({}^{\mathrm{cl}}\mathcal{Z}_\phi^{\mathrm{rigid}}, \mathcal{O}_{{}^{\mathrm{cl}}\mathcal{Z}_\phi^{\mathrm{rigid}}})^{\mathsf{G}},$$

since G is reductive. Hence every element can be lifted.

4.7.5. We now claim that the fiber product

$$\mathrm{pt} \underset{\mathrm{Spec}(A)}{\times} \mathcal{Z}_\phi^{\mathrm{rigid}}$$

itself is an affine scheme.

Indeed, its underlying classical prestack is a classical affine scheme, by assumption. Further, it has a connective co-representable deformation theory, because $\mathcal{Z}_\phi^{\mathrm{rigid}}$ has this property. Hence, it is indeed an affine scheme by [Lu3, Theorem 18.1.0.1].

4.7.6. We are now ready to prove that $\mathcal{W}_\phi^{\mathrm{rigid}}$ is an affine scheme.

Note that for $\mathrm{Spec}(A)$ as above, the map

$$r' : \mathcal{Z}_\phi \to \mathrm{Spec}(A)$$

canonically factors as

$$\mathcal{Z}_\phi \xrightarrow{r} \mathcal{Z}_\phi^{\mathrm{coarse}} \to \mathrm{Spec}(A).$$

Let us base change these maps by

$$\mathrm{pt} \to \mathrm{Spec}(A),$$

corresponding to the point $r'(\phi) \in \mathrm{Spec}(A)$.

Thus, from r, we obtain a map

$$\mathrm{pt} \underset{\mathrm{Spec}(A)}{\times} \mathcal{Z}_\phi \to \mathrm{pt} \underset{\mathrm{Spec}(A)}{\times} \mathcal{Z}_\phi^{\mathrm{coarse}}. \tag{4.10}$$

The map (4.10) realizes $\mathrm{pt} \underset{\mathrm{Spec}(A)}{\times} \mathcal{Z}_\phi^{\mathrm{coarse}}$ as

$$\left(\mathrm{pt} \underset{\mathrm{Spec}(A)}{\times} \mathcal{Z}_\phi\right)^{\mathrm{coarse}}.$$

The left-hand side in (4.10) is

$$(\mathrm{pt} \underset{\mathrm{Spec}(A)}{\times} \mathcal{Z}_\phi^{\mathrm{rigid}})/\mathsf{G},$$

and hence, by Sect. 4.7.5, is a mock-affine *algebraic stack* (as opposed to ind-algebraic stack).

From here, we obtain that the right-hand side in (4.10) is an affine *scheme* (as opposed to ind-scheme).

Therefore, the map

$$\mathrm{pt} \underset{\mathrm{Spec}(A)}{\times} \mathcal{Z}_\phi^{\mathrm{rigid}} \to \mathrm{pt} \underset{\mathrm{Spec}(A)}{\times} \mathcal{Z}_\phi^{\mathrm{coarse}}.$$

is a map between affine schemes. Hence, its further pullback with respect to

$$\mathrm{pt} \to \mathrm{pt} \underset{\mathrm{Spec}(A)}{\times} \mathcal{Z}_\phi^{\mathrm{coarse}}$$

is still an affine scheme. But the latter pullback is the prestack $\mathcal{W}_\phi^{\mathrm{rigid}}$ of (4.8).

Thus, $\mathcal{W}_\phi^{\mathrm{rigid}}$ is an affine scheme, as required.

4.7.7. We will now perform the construction stated in Sect. 4.7.3. Let

$$\mathfrak{a} := \mathsf{g}/\!/\mathsf{G}$$

be the characteristic variety of g. This is an affine scheme equipped with an action of $\mathbb{G}_m$.

We let $\mathrm{Spec}(A)$ be the affine scheme

$$\mathbf{Maps}_{\mathrm{Sch}}(\mathsf{h}_u, \mathfrak{a})^{\mathbb{G}_m}.$$

It is easy to see that this is indeed an affine scheme.

We define map r' as the composition

$$\mathcal{Z}_\phi \to \mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, \mathsf{G})/\mathsf{G} \to \mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}_u, \mathsf{G})/\mathsf{G} \to \mathbf{Maps}_{\mathrm{Lie}}(\mathsf{h}_u, \mathsf{g})/\mathsf{G} \to \\ \to \mathbf{Maps}_{\mathrm{Sch}}(\mathsf{h}_u, \mathsf{g})^{\mathbb{G}_m}/\mathsf{G} \to \mathbf{Maps}_{\mathrm{Sch}}(\mathsf{h}_u, \mathfrak{a})^{\mathbb{G}_m}$$

We will show that (4.9) is an affine scheme (actually, at the derived level).

It suffices to show that

$$\mathrm{pt} \underset{\mathbf{Maps}_{\mathrm{Sch}}(\mathsf{h}_u, \mathfrak{a})^{\mathbb{G}_m}}{\times} \mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, \mathsf{G})_\phi$$

is an affine scheme.

We rewrite the latter as

$$\mathrm{pt} \underset{\mathbf{Maps}_{\mathrm{Sch}}(\mathsf{h}_u, \mathfrak{a})^{\mathbb{G}_m}}{\times} \mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}_u, \mathsf{G})^{\mathsf{H}_{\mathrm{red}}},$$

and consider the fiber product

$$\mathrm{pt} \underset{\mathbf{Maps}_{\mathrm{Sch}}(\mathsf{h}_u, \mathfrak{a})^{\mathbb{G}_m}}{\times} \mathbf{Maps}_{\mathrm{Lie}}(\mathsf{h}_u, \mathsf{g})^{\mathsf{H}_{\mathrm{red}}},$$

which is an affine scheme, because $\mathbf{Maps}_{\mathrm{Lie}}(\mathsf{h}_u, \mathsf{g})^{\mathsf{H}_{\mathrm{red}}}$ is such.

Hence, it suffices to show that the map

$$\mathrm{pt} \underset{\mathbf{Maps}_{\mathrm{Sch}}(\mathsf{h}_u, \mathfrak{a})^{\mathbb{G}_m}}{\times} \mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}_u, \mathsf{G})^{\mathsf{H}_{\mathrm{red}}} \to \mathrm{pt} \underset{\mathbf{Maps}_{\mathrm{Sch}}(\mathsf{h}_u, \mathfrak{a})^{\mathbb{G}_m}}{\times} \mathbf{Maps}_{\mathrm{Lie}}(\mathsf{h}_u, \mathsf{g})^{\mathsf{H}_{\mathrm{red}}} \tag{4.11}$$

is an isomorphism.

By Proposition 4.6.4, the above map a priori realizes the left-hand side as the formal completion of the right-hand side corresponding to locus of maps

$$\mathsf{h}_u \to \mathsf{g}$$

whose image is contained in the nilpotent cone.

However, the condition that the composition

$$\mathsf{h}_u \to \mathsf{g} \to \mathfrak{a}$$

is zero implies that the above locus is the entire right-hand side in (4.11)

□[Theorem 4.6.6]

4.8. **Proof of Theorem 4.4.2.** This subsection is devoted to the proof of Theorem 4.4.2.

4.8.1. *A reduction step.* Let σ be a semi-simple local system, and let $\mathcal{Z}_\sigma$ be the corresponding connected component of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ corresponding to σ.

Repeating the steps in Sects. 4.7.2-4.7.4, it suffices to find a map

$$r' : {}^{\mathrm{cl}}\mathcal{Z}_\sigma \to \mathrm{Spec}(A),$$

such that the classical prestack underlying the fiber product

$$\mathrm{pt} \underset{\mathrm{Spec}(A)}{\times} {}^{\mathrm{cl}}\mathcal{Z}_\sigma$$

is a (classical) algebraic stack, where $\mathrm{pt} \to \mathrm{Spec}(A)$ is the image of the unique closed point of $\mathcal{Z}_\sigma$.

4.8.2. Recall that the prestack ${}^{\mathrm{cl}}\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ identifies with the classical prestack underlying

$$\mathbf{Maps}_{\mathrm{Grp}}(\mathrm{Gal}_{X,x}, \mathsf{G})/\operatorname{Ad}(\mathsf{G}),$$

where $\mathrm{Gal}_{X,x}$ is as in Sect. 1.8.4.

Choose a of Levi splitting

$$\mathrm{Gal}_{X,x} \simeq (\mathrm{Gal}_{X,x})_{\mathrm{red}} \ltimes (\mathrm{Gal}_{X,x})_u,$$

see [HM] Theorem 3.2.

By Corollary 2.4.8, the point σ corresponds to a homomorphism $\sigma : (\mathrm{Gal}_{X,x})_{\mathrm{red}} \to \mathsf{G}$ (see Sect. 4.7.1). Let

$$\mathbf{Maps}_{\mathrm{Grp}}(\mathrm{Gal}_{X,x}, \mathsf{G})_\sigma$$

have the same meaning as in Sect. 4.6.2, so that

$${}^{\mathrm{cl}}\mathcal{Z}_\sigma \simeq \mathbf{Maps}_{\mathrm{Grp}}(\mathrm{Gal}_{X,x}, \mathsf{G})_\sigma/\operatorname{Stab}_{\mathsf{G}}(\sigma),$$

and

$$\mathbf{Maps}_{\mathrm{Grp}}(\mathrm{Gal}_{X,x}, \mathsf{G})_\sigma \simeq \mathbf{Maps}_{\mathrm{Grp}}((\mathrm{Gal}_{X,x})_u, \mathsf{G})^{(\mathrm{Gal}_{X,x})_{\mathrm{red}}},$$

where $(\mathrm{Gal}_{X,x})_{\mathrm{red}}$ acts on G via σ.

4.8.3. Being a pro-unipotent group, we can write $(\mathrm{Gal}_{X,x})_u$ as

$$\lim_\alpha \mathsf{H}_\alpha,$$

where α runs over a filtered family of indices, the groups H_α are finite-dimensional and unipotent and the transition maps

$$\mathsf{H}_{\alpha_2} \to \mathsf{H}_{\alpha_1}$$

are surjective.

With no restriction of generality, we can assume that the $(\mathrm{Gal}_{X,x})_{\mathrm{red}}$-action on $(\mathrm{Gal}_{X,x})_u$ comes from a compatible family of actions on the H_α's.

We have:

$$\mathbf{Maps}_{\mathrm{Grp}}((\mathrm{Gal}_{X,x})_u, \mathsf{G})^{(\mathrm{Gal}_{X,x})_{\mathrm{red}}} \simeq \operatorname*{colim}_\alpha \mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}_\alpha, \mathsf{G})^{(\mathrm{Gal}_{X,x})_{\mathrm{red}}}.$$

4.8.4. For each index α, let $\mathsf{h}_{\alpha,\sigma\text{-isotyp}}$ be the maximal Lie algebra quotient of

$$\mathsf{h}_{u,\alpha} := \mathrm{Lie}(\mathsf{H}_\alpha)$$

on which the action of $(\mathrm{Gal}_{X,x})_{\mathrm{red}}$ has only the same isotypic components as those that appear in $\mathsf{g} := \mathrm{Lie}(\mathsf{G})$, where the latter is acted on by $(\mathrm{Gal}_{X,x})_{\mathrm{red}}$ via σ.

Let $\mathsf{H}_{\alpha,\sigma\text{-isotyp}}$ denote the corresponding quotient of H_α. Note that the map

$$\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}_{\alpha,\sigma\text{-isotyp}}, \mathsf{G})^{(\mathrm{Gal}_{X,x})_{\mathrm{red}}} \to \mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}_\alpha, \mathsf{G})^{(\mathrm{Gal}_{X,x})_{\mathrm{red}}}$$

is an isomorphism.

Set

$$\mathsf{H}_{\sigma\text{-isotyp}} := \lim_{\alpha} \mathsf{H}_{\alpha,\sigma\text{-isotyp}}.$$

The map

$$\mathbf{Maps}_{\mathrm{Grp}}((\mathrm{Gal}_{X,x})_u, \mathsf{G})^{(\mathrm{Gal}_{X,x})_{\mathrm{red}}} \to \mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}_{\sigma\text{-isotyp}}, \mathsf{G})^{(\mathrm{Gal}_{X,x})_{\mathrm{red}}}$$

is an isomorphism as well.

4.8.5. Consider the composition

$$\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}_{\sigma\text{-isotyp}}, \mathsf{G})^{(\mathrm{Gal}_{X,x})_{\mathrm{red}}} \to \mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}_{\sigma\text{-isotyp}}, \mathsf{G}) \to \mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}_{\sigma\text{-isotyp}}, \mathsf{G})/\operatorname{Ad}(\mathsf{G}),$$

where the first map is a closed embedding, and the second map is schematic.

We obtain that in order to find a pair $(\mathrm{Spec}(A), r')$ for the original ${}^{\mathrm{cl}}\mathcal{Z}_\sigma$ (see Sect. 4.8.1), it is sufficient to do so for the ind-algebraic stack $\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}_{\sigma\text{-isotyp}}, \mathsf{G})/\operatorname{Ad}(\mathsf{G})$.

This will result from the combination of the following two assertions, both of independent interest:

Theorem 4.8.6. *The pro-algebraic group $\mathsf{H}_{\sigma\text{-isotyp}}$ is topologically finitely generated*[15].

Theorem 4.8.7. *The assertion of Theorem 4.6.6 holds for H, which is a topologically finitely generated pro-algebraic group (with G being a reductive group).*

Remark 4.8.8. The proof of Theorem 4.8.7 will *not* reply on the proof of Theorem 4.6.6 given earlier.

Since every finite-dimensional algebraic group is (topologically) finitely generated (see below), the proof of Theorem 4.8.7 that we will give will also give an alternative proof also for Theorem 4.6.6.

4.9. **Proof of Theorem 4.8.6.**

4.9.1. Let Free_n be the free group on n letters, and let $\mathrm{Free}_n^{\mathrm{alg}}$ be its pro-algebraic envelope, i.e.,

$$\mathrm{Hom}_{\mathrm{Grp}}(\mathrm{Free}_n^{\mathrm{alg}}, \mathsf{H}) \simeq \mathsf{H}^{\times n}, \quad \mathsf{H} \in \mathrm{Alg.\,Groups}\,.$$

4.9.2. Let H be a pro-algebraic group, written as

$$\lim_{\alpha} \mathsf{H}_\alpha.$$

A map $\mathrm{Free}_n^{\mathrm{alg}} \to \mathsf{H}$ is then the same as an n-tuple $\underline{g}$ of elements in $\mathsf{H}(\mathsf{e})$.

We shall say that an n-tuple $\underline{g}$ *topologically generates* H if the corresponding map $\mathrm{Free}_n^{\mathrm{alg}} \to \mathsf{H}$ is such that all the composite maps

$$\mathrm{Free}_n^{\mathrm{alg}} \to \mathsf{H} \to \mathsf{H}_\alpha$$

are surjective.

This is equivalent to the condition that the Zariski closure of the abstract group generated by the images of the elements of $\underline{g}$ in H_α is all of H_α

4.9.3. We will say that H is *topologically finitely generated* if it admits a finite set of topological generators.

It is easy to see that any finite-dimensional algebraic group is finitely generated.

[15]See Sect. 4.9.3 below for what this means.

4.9.4. Consider $\mathsf{h} := \mathrm{Lie}(\mathsf{H})$ as a pro-finite dimensional vector space. The following is elementary:

Lemma 4.9.5. *Let V be a finite-dimensional subspace of h such that for every α, the image of V in $\mathsf{h}_\alpha := \mathrm{Lie}(\mathsf{H}_\alpha)$ generates it as a Lie algebra. Then H is topologically finitely generated.*

4.9.6. Assume now that H is pro-unipotent. We claim:

Proposition 4.9.7. *Assume that $\mathsf{h}/[\mathsf{h},\mathsf{h}]$ is finite-dimensional. Then H is topologically finitely generated.*

Proof. Let $\mathsf{V} \subset \mathsf{h}$ be a finite-dimensional vector space that projects surjectively onto $\mathsf{h}/[\mathsf{h},\mathsf{h}]$. By Lemma 4.9.5, it suffices to see that for any α, the image of V in h_α generates it as a Lie algebra.

But this follows from the next property of nilpotent Lie algebras: if a subspace V' in a nilpotent finite-dimesnional Lie algebra h' projects surjectively onto $\mathsf{h}'/[\mathsf{h}',\mathsf{h}']$, then V' generates h' as a Lie algebra.

□

4.9.8. We will prove Theorem 4.8.6 by applying Proposition 4.9.7 to $\mathsf{H}_{\sigma\text{-isotyp}}$.

Note that the

$$\mathsf{h}_{\sigma\text{-isotyp}}/[\mathsf{h}_{\sigma\text{-isotyp}},\mathsf{h}_{\sigma\text{-isotyp}}]$$

is the maximal pro-abelian quotient of $\mathrm{Lie}(\mathrm{Gal}_{X,x})_u)$ on which $(\mathrm{Gal}_{X,x})_{\mathrm{red}}$ via isotypic components that appear in its action on g via σ.

Hence, it is enough to show that the vector space

$$\mathrm{Hom}_{\mathrm{Vect}_{\mathsf{e}}^{\heartsuit}}(\mathrm{Lie}((\mathrm{Gal}_{X,x})_u)/[\mathrm{Lie}((\mathrm{Gal}_{X,x})_u),\mathrm{Lie}((\mathrm{Gal}_{X,x})_u],\mathsf{g}))^{(\mathrm{Gal}_{X,x})_{\mathrm{red}}}$$

is finite-dimensional.

Note, however, that the above vector space is the same as

$$H^1(\mathrm{Lie}((\mathrm{Gal}_{X,x})_u),\mathsf{g})^{(\mathrm{Gal}_{X,x})_{\mathrm{red}}},$$

and the latter identifies with the tangent space to

$$\mathbf{Maps}_{\mathrm{Grp}}((\mathrm{Gal}_{X,x})_u,\mathsf{G})^{(\mathrm{Gal}_{X,x})_{\mathrm{red}}}$$

at the point corresponding to the trivial map $(\mathrm{Gal}_{X,x})_u \to \mathsf{G}$.

Further, the latter vector space identifies with

$$H^0\left(T_\sigma(\mathbf{Maps}_{\mathrm{Grp}}(\mathrm{Gal}_{X,x},\mathsf{G})/\mathrm{Ad}(\mathsf{G}))\right) \simeq H^0\left(T_\sigma({}^{\mathrm{cl}}\mathcal{Z}_\sigma)\right),$$

and also with

$$H^0\left(T_\sigma(\mathcal{Z}_\sigma)\right).$$

However, the latter is finite-dimensional by Proposition 1.6.2(b).

□[Theorem 4.8.6]

4.10. **Proof of Theorem 4.8.7.**

4.10.1. Write

$$\mathsf{H} \simeq \lim_{\alpha} \mathsf{H}_\alpha.$$

Let $\mathsf{H}' \to \mathsf{H}$ be a homomorphism of pro-algebraic groups, such that for every α the composite map

$$\mathsf{H}' \to \mathsf{H} \to \mathsf{H}_\alpha$$

is surjective.

We claim that if the the assertion of Theorem 4.6.6 holds for H' then it holds for H_1 (for a given G).

4.10.2. First, we note that the map

$$\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H},\mathsf{G})/\operatorname{Ad}(\mathsf{G}) \to \mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}',\mathsf{G})/\operatorname{Ad}(\mathsf{G})$$

is a closed embedding (at the classical level).

Let $\mathcal{Z}_\phi$ be a connected component of $\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H},\mathsf{G})/\operatorname{Ad}(\mathsf{G})$ containing a given map

$$\phi : \mathsf{H}_{\mathrm{red}} \to \mathsf{G}.$$

Let $\mathcal{Z}'_\phi$ be the corresponding connected component of $\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}',\mathsf{G})/\operatorname{Ad}(\mathsf{G})$.

As in Sects. 4.7.2-4.7.4, it suffices to find a map

$${}^{\mathrm{cl}}\mathcal{Z}_\phi \to \operatorname{Spec}(A),$$

such that the classical prestack underlying the fiber product

$$\mathrm{pt} \underset{\operatorname{Spec}(A)}{\times} {}^{\mathrm{cl}}\mathcal{Z}_\phi$$

is an algebraic stack.

If Theorem 4.6.6 is true for H', we can find such a map for ${}^{\mathrm{cl}}\mathcal{Z}'_\phi$. Now, the map

$$\mathrm{pt} \underset{\operatorname{Spec}(A)}{\times} {}^{\mathrm{cl}}\mathcal{Z}_\phi \to \mathrm{pt} \underset{\operatorname{Spec}(A)}{\times} {}^{\mathrm{cl}}\mathcal{Z}'_\phi$$

is a closed embedding, and the assertion follows.

4.10.3. Thus, by the assumption on H and Sect. 4.10.1, we can replace H by $\mathrm{Free}_n^{\mathrm{alg}}$.

Note now that the prestack

$$\mathbf{Maps}_{\mathrm{Grp}}(\mathrm{Free}_n^{\mathrm{alg}},\mathsf{G})/\operatorname{Ad}(\mathsf{G})$$

is the same as (the Betti) $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(\mathcal{X})$, where $\mathcal{X}$ is the bouquet of n copies of S^1.

Hence, in this case, the assertion of Theorem 4.6.6 follows from Sect. 4.5.4.

$\square$[Theorem 4.8.7]

5. Properties of the category of quasi-coherent sheaves

In this section we will study properties of the category of quasi-coherent sheaves on a formal affine scheme, and then apply the results to $\mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X))$.

The special feature of formal schemes among general ind-schemes is the following: for an ind-scheme $\mathcal{Y}$, the category $\mathrm{QCoh}(\mathcal{Y})$ is by definition the inverse limit of the categories $\mathrm{QCoh}(Y_i)$ for closed subschemes $Y_i \hookrightarrow \mathcal{Y}$. The functors in this inverse systems are given by *-pullback and they do not generally admit left adjoints. So we do not in general know whether $\mathrm{QCoh}(\mathcal{Y})$ is compactly generated.

However, in the case of formal affine schemes, the situation will be much better.

5.1. **Quasi-coherent sheaves on a formal affine scheme.**

5.1.1. Let $\mathcal{Y}$ be an formal affine scheme. I.e., $\mathcal{Y}$ is a prestack that can be written as

$$\operatorname*{colim}_{n\geq 0} \operatorname{Spec}(R_n) \tag{5.1}$$

as in Theorem 1.3.2(d).

In this subsection we will describe some favorable properties enjoyed by $\mathrm{QCoh}(\mathcal{Y})$ for such $\mathcal{Y}$. In general, QCoh of an ind-scheme is unwieldy, but Proposition 5.1.4 below allows to get one's hand on $\mathrm{QCoh}(\mathcal{Y})$ for $\mathcal{Y}$ a formal affine scheme.

5.1.2. Fix a presentation of $\mathcal{Y}$ as in (5.1); denote by i_∞ the resulting map $\mathcal{Y} \to \operatorname{Spec}(R)$. Set

$$Y_n := \operatorname{Spec}(R_n) \overset{i_n}{\hookrightarrow} \operatorname{Spec}(R).$$

For $n_1 \leq n_2$, let i_{n_1,n_2} denote the corresponding map $Y_{n_1} \to Y_{n_2}$.

Let $U \overset{j}{\hookrightarrow} \operatorname{Spec}(R)$ be the (open) complement of $\operatorname{Spec}(R_1)$. Let

$$\operatorname{QCoh}(\operatorname{Spec}(R))_{\mathcal{Y}} \overset{(i_\infty)_!}{\hookrightarrow} \operatorname{QCoh}(\operatorname{Spec}(R))$$

be the inclusion of the full subcategory consisting of objects with *set-theoretic* support on Y_1 (i.e., these are objects whose restriction to U vanishes). This inclusion admits a right adjoint, denoted $i^!_\infty$; explicitly, for every $\mathcal{F} \in \operatorname{QCoh}(\operatorname{Spec}(R))$ we have the Cousin exact triangle

$$(i_\infty)_! \circ (i_\infty)^!(\mathcal{F}) \to \mathcal{F} \to j_* \circ j^*(\mathcal{F}).$$

Furthermore, we can explicitly write the functor $(i_\infty)_! \circ (i_\infty)^!$ as

$$\operatorname*{colim}_n (i_n)_* \circ (i_n)^!, \tag{5.2}$$

where we note that each $i^!_n$ is continuous because i_n is a regular embedding. (Note, however, that for fixed n_1, n_2, the functor $(i_{n_1,n_2})^!$, right adjoint to $(i_{n_1,n_2})_*$, is discontinuous.)

5.1.3. Consider the composite functor

$$\operatorname{QCoh}(\operatorname{Spec}(R))_{\mathcal{Y}} \overset{(i_\infty)_!}{\hookrightarrow} \operatorname{QCoh}(\operatorname{Spec}(R)) \overset{(i_\infty)^*}{\to} \operatorname{QCoh}(\mathcal{Y}). \tag{5.3}$$

The following is established in [GR3, Proposition 7.1.3]:

Proposition 5.1.4. *The functor* (5.3) *is an equivalence.*

From here we formally obtain:

Corollary 5.1.5.
(a) *There exists a (unique) equivalence* $\operatorname{QCoh}(\operatorname{Spec}(R))_{\mathcal{Y}} \simeq \operatorname{QCoh}(\mathcal{Y})$*, under which the functor*

$$(i_\infty)^! : \operatorname{QCoh}(\operatorname{Spec}(R)) \to \operatorname{QCoh}(\operatorname{Spec}(R))_{\mathcal{Y}}$$

goes over to the functor

$$(i_\infty)^* : \operatorname{QCoh}(\operatorname{Spec}(R)) \to \operatorname{QCoh}(\mathcal{Y}).$$

(b) *The functor* $(i_\infty)^*$ *realizes* $\operatorname{QCoh}(\mathcal{Y})$ *both as a colocalization and a localization of* $\operatorname{QCoh}(\operatorname{Spec}(R))$ *with respect to the essential image of* $\operatorname{QCoh}(U)$ *along* j_*.

We now claim:

Lemma 5.1.6. *Let* $\mathcal{Y}$ *and* Y_n *be as above.*

(a) *The category* $\operatorname{QCoh}(\operatorname{Spec}(R))_{\mathcal{Y}}$ *is compactly generated by the objects* $(i_n)_*(\mathcal{O}_{Y_n})$*. The subcategory of compact objects in* $\operatorname{QCoh}(\operatorname{Spec}(R))_{\mathcal{Y}}$ *is closed under the monoidal operation.*

(b) *The functor*

$$\operatorname*{colim}_n \operatorname{QCoh}(Y_n) \to \operatorname{QCoh}(\operatorname{Spec}(R))_{\mathcal{Y}}, \tag{5.4}$$

where the colimit is formed using the functors

$$(i_{n_1,n_2})_* : \operatorname{QCoh}(Y_{n_1}) \to \operatorname{QCoh}(Y_{n_2}),$$

and where the map is (5.4) *is given by* $\{(i_n)_*\}$*, is an equivalence.*

Proof. The fact that the objects $(i_n)_*(\mathcal{O}_{Y_n})$ generate $\mathrm{QCoh}(\mathrm{Spec}(R))_{\mathcal{Y}}$ follows from (5.2). The fact that they are compact follows from the fact that they are compact as objects of $\mathrm{QCoh}(\mathrm{Spec}(R))$. The fact that the subcategory of compact is closed under the monoidal operation follows from the corresponding fact for $\mathrm{QCoh}(\mathrm{Spec}(R))$.

This proves point (a).

For point (b), we first rewrite $\underset{n}{\mathrm{colim}}\, \mathrm{QCoh}(Y_n)$ as

$$\lim_n \mathrm{QCoh}(Y_n),$$

where the limit is formed using the *discontinuous* functors

$$(i_{n_1,n_2})^! : \mathrm{QCoh}(Y_{n_2}) \to \mathrm{QCoh}(Y_{n_1}),$$

see [GR1, Chapter 1, Proposition 2.5.7]. The functor right adjoint to (5.4) is given by the compatible collection of functors

$$\{(i_n)^!\} : \mathrm{QCoh}(\mathcal{Y}) \to \lim_n \mathrm{QCoh}(Y_n).$$

Given point (a), it suffices to show that this right adjoint is fully faithful. For that, we have to show that for $\mathcal{M}', \mathcal{M}'' \in \mathrm{QCoh}(\mathrm{Spec}(R))_{\mathcal{Y}}$, the map

$$\mathcal{H}om_{\mathrm{QCoh}(\mathrm{Spec}(R))_{\mathcal{Y}}}(\mathcal{M}', \mathcal{M}'') \to \lim_n \mathcal{H}om_{\mathrm{QCoh}(Y_n)}((i_n)^!(\mathcal{M}'), (i_n)^!(\mathcal{M}''))$$

is an isomorphism.

We rewrite the RHS as

$$\lim_n \mathcal{H}om_{\mathrm{QCoh}(\mathrm{Spec}(R))_{\mathcal{Y}}}((i_n)_* \circ (i_n)^!(\mathcal{M}'), \mathcal{M}'') \simeq \mathcal{H}om_{\mathrm{QCoh}(\mathrm{Spec}(R))_{\mathcal{Y}}}\left(\underset{n}{\mathrm{colim}}\, (i_n)_* \circ (i_n)^!(\mathcal{M}'), \mathcal{M}''\right),$$

and the assertion follows from (5.2).

□

5.1.7. Let $i_{n,\infty}$ denote the map $Y_n \to \mathcal{Y}$. Note that by Corollary 5.1.5, the functor

$$(i_{n,\infty})_* : \mathrm{QCoh}(Y_n) \to \mathrm{QCoh}(\mathcal{Y}),$$

right adjoint to

$$(i_{n,\infty})^* : \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{QCoh}(Y_n),$$

identifies with $(i_\infty)^* \circ (i_n)_*$; in particular, it is continuous.

Hence, from Lemma 5.1.6, we obtain:

Corollary 5.1.8.
(a) *The category* $\mathrm{QCoh}(\mathcal{Y})$ *is compactly generated by the objects* $(i_n)_*(\mathcal{O}_{Y_n})$. *The subcategory of compact objects in* $\mathrm{QCoh}(\mathcal{Y})$ *is closed under the monoidal operation.*

(b) *The functor*

$$\underset{n}{\mathrm{colim}}\, \mathrm{QCoh}(Y_n) \to \mathrm{QCoh}(\mathcal{Y}), \tag{5.5}$$

given by $\{(i_{n,\infty})_*\}$, *is an equivalence.*

5.2. Some 2-categorical properties.

5.2.1. Recall what it means for a prestack to be 1-affine, see [Ga2, Definition 1.3.7]. From [Ga2, Theorem 2.3.1], we obtain that our $\mathcal{Y}$ is 1-affine.

In particular, from [Ga2, Proposition 3.1.9], we obtain that for any prestack $\mathcal{Z} \to \mathcal{Y}$ and an affine scheme $f : S \to \mathcal{Y}$, the natural functor

$$\mathrm{QCoh}(S) \underset{\mathrm{QCoh}(\mathcal{Y})}{\otimes} \mathrm{QCoh}(\mathcal{Z}) \to \mathrm{QCoh}(S \underset{\mathcal{Y}}{\times} \mathcal{Z})$$

is an equivalence.

5.2.2. Moreover, we obtain that for any (S,f) as above, the category $\mathrm{QCoh}(S)$ is canonically self-dual *as a module* category over $\mathrm{QCoh}(\mathcal{Y})$, where the counit is given by

$$\mathrm{QCoh}(S) \underset{\mathrm{QCoh}(\mathcal{Y})}{\otimes} \mathrm{QCoh}(S) \to \mathrm{QCoh}(S) \xrightarrow{f_*} \mathrm{QCoh}(\mathcal{Y}) \tag{5.6}$$

and the unit by

$$\mathrm{QCoh}(S) \xrightarrow{(\Delta_{S/\mathcal{Y}})_*} \mathrm{QCoh}(S \underset{\mathcal{Y}}{\times} S) \simeq \mathrm{QCoh}(S) \underset{\mathrm{QCoh}(\mathcal{Y})}{\otimes} \mathrm{QCoh}(S). \tag{5.7}$$

5.2.3. Finally we observe the following: let $\mathbf{C}$ a module category over $\mathrm{QCoh}(\mathcal{Y})$. Then it is dualizable as such if and only if it is dualizable as a plain DG category.

This follows from the fact that for a pair of categories $\mathbf{C}_1$ and $\mathbf{C}_2$ as above, the functor

$$\mathbf{C}_1 \underset{\mathrm{QCoh}(\mathrm{Spec}(R))}{\otimes} \mathbf{C}_2 \to \mathbf{C}_1 \underset{\mathrm{QCoh}(\mathcal{Y})}{\otimes} \mathbf{C}_2$$

is an equivalence (since $\mathrm{QCoh}(\mathcal{Y})$ is a localization of $\mathrm{QCoh}(\mathrm{Spec}(R))$), and the corresponding property of $\mathrm{QCoh}(\mathrm{Spec}(R))$ (as the latter is *rigid*, see [GR1, Proposition 9.4.4]).

5.3. **Independence of presentation.** We continue to assume that $\mathcal{Y}$ is a formal affine scheme. Lemma 5.1.6(b) allows to write $\mathrm{QCoh}(\mathcal{Y})$ as a *colimit* of QCoh on affine schemes. In this subsection we show how to write $\mathrm{QCoh}(\mathcal{Y})$ as a *colimit* in a way independent of the presentation (5.1).

This is needed in order to perform some functorial constructions with $\mathrm{QCoh}(\mathcal{Y})$ in the next subsection.

5.3.1. First, we notice:

Lemma 5.3.2. *The diagonal map $\Delta_{\mathcal{Y}} : \mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$ is affine.*

Proof. Fix a presentation of $\mathcal{Y}$ as in (5.1). Then the map $\Delta_{\mathcal{Y}}$ can be obtained as the base change of the diagonal map $\Delta_{\mathrm{Spec}(R)} : \mathrm{Spec}(R) \to \mathrm{Spec}(R) \times \mathrm{Spec}(R)$, i.e., the square

$$\begin{tikzcd} \mathcal{Y} \arrow[r, "\Delta_{\mathcal{Y}}"] \arrow[d] & \mathcal{Y} \times \mathcal{Y} \arrow[d] \\ \mathrm{Spec}(R) \arrow[r, "\Delta_{\mathrm{Spec}(R)}"] & \mathrm{Spec}(R) \times \mathrm{Spec}(R) \end{tikzcd} \tag{5.8}$$

is Cartesian. □

5.3.3. Let S be an affine scheme, equipped with a map f to $\mathcal{Y}$. Note that f is *affine* as a map of prestacks (by Lemma 5.3.2). Hence, the functor f_*, right adjoint to f^* is continuous.

From Corollary 5.1.8(b), we obtain:

Corollary 5.3.4. *The functor*

$$\underset{(S,f)}{\mathrm{colim}}\, \mathrm{QCoh}(S) \to \mathrm{QCoh}(\mathcal{Y}), \tag{5.9}$$

is an equivalence, where:

- *The index category is either of the following:*

$$\mathrm{Sch}^{\mathrm{aff}}_{/\mathcal{Y}},\ \mathrm{Sch}^{\mathrm{aff}}_{/\mathcal{Y},\mathrm{closed}},$$

  where the subscript "closed" indicates that we consider only closed embeddings[16] $S \to \mathcal{Y}$*;*
- *The colimit is formed is the pushforward functors $(f_{1,2})_* : \mathrm{QCoh}(S_1) \to \mathrm{QCoh}(S_2)$ for*

$$f_{1,2} : S_1 \to S_2, \quad f_2 \circ f_{1,2} = f_1.$$

- *The map in* (5.9) *is given by* $\{\mathrm{QCoh}(S) \xrightarrow{f_*} \mathrm{QCoh}(\mathcal{Y})\}$.

[16] When $\mathcal{Y}$ locally almost of finite type as a prestack, we can further allow $(\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft},/\mathrm{e}})_{/\mathcal{Y}}$ and $(\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft},/\mathrm{e}})_{/\mathcal{Y},\mathrm{closed}}$ as index categories in the above colimit.

Proof. Fix a presentation of $\mathcal{Y}$ as in (1.3). The assertion follows from the fact that that the family $Y_n \overset{i_{n,\infty}}{\longrightarrow} \mathcal{Y}$ is cofinal in any of the above categories. □

5.4. **Self-duality of** QCoh **of a formal affine scheme.** Let $\mathcal{Y}$ be as above. In this subsection we will explore the self-duality property of $\mathrm{QCoh}(\mathcal{Y})$.

Recall that if Y is an affine scheme, the functors

$$\mathrm{QCoh}(Y) \otimes \mathrm{QCoh}(Y) \overset{\otimes}{\to} \mathrm{QCoh}(Y) \overset{\Gamma(Y,-)}{\to} \mathrm{Vect}_{\mathsf{e}} \tag{5.10}$$

and

$$\mathrm{Vect}_{\mathsf{e}} \overset{\mathsf{e}\mapsto \mathcal{O}_Y}{\longrightarrow} \mathrm{QCoh}(Y) \overset{(\Delta_*)_Y}{\longrightarrow} \mathrm{QCoh}(Y \times Y) \simeq \mathrm{QCoh}(Y) \otimes \mathrm{QCoh}(Y) \tag{5.11}$$

define an identification

$$\mathrm{QCoh}(Y) \simeq \mathrm{QCoh}(Y)^\vee.$$

This would fail for a general prestack, and in particular ind-scheme. For a formal affine scheme $\mathcal{Y}$, the functor $\Gamma(\mathcal{Y},-)$ is still discontinuous, so (5.10) cannot serve as a counit of a self-duality. Yet, we will see that (5.11) does form the unit of a self-duality.

5.4.1. Note that by Corollary 5.1.8(a), the category $\mathrm{QCoh}(\mathcal{Y})$ is compactly generated, and hence dualizable. In particular, for *any* prestack $\mathcal{Y}'$, the external tensor product functor

$$\mathrm{QCoh}(\mathcal{Y}) \otimes \mathrm{QCoh}(\mathcal{Y}') \to \mathrm{QCoh}(\mathcal{Y} \times \mathcal{Y}')$$

is an equivalence, see [GR1, Chapter 3, Proposition 3.1.7].

5.4.2. By Lemma 5.3.2, the diagonal map for $\mathcal{Y}$ is affine, so we can consider the object

$$(\Delta_{\mathcal{Y}})_*(\mathcal{O}_{\mathcal{Y}}) \in \mathrm{QCoh}(\mathcal{Y} \times \mathcal{Y}).$$

We claim:

Proposition 5.4.3. *The object*

$$(\Delta_{\mathcal{Y}})_*(\mathcal{O}_{\mathcal{Y}}) \in \mathrm{QCoh}(\mathcal{Y} \times \mathcal{Y}) \simeq \mathrm{QCoh}(\mathcal{Y}) \otimes \mathrm{QCoh}(\mathcal{Y})$$

is the unit of a duality.

Proof. Fix a presentation of $\mathcal{Y}$ as in (5.1). It is easy to see that the functors

$$\mathrm{QCoh}(\mathrm{Spec}(R))_{\mathcal{Y}} \otimes \mathrm{QCoh}(\mathrm{Spec}(R))_{\mathcal{Y}} \overset{(i_\infty)_! \otimes (i_\infty)_!}{\longrightarrow}$$
$$\to \mathrm{QCoh}(\mathrm{Spec}(R)) \otimes \mathrm{QCoh}(\mathrm{Spec}(R)) \overset{\otimes}{\to} \mathrm{QCoh}(\mathrm{Spec}(R)) \overset{\Gamma(\mathrm{Spec}(R),-)}{\longrightarrow} \mathrm{Vect}_{\mathsf{e}}$$

and

$$\mathrm{Vect}_{\mathsf{e}} \overset{(\Delta_{\mathrm{Spec}(R)})_*(\mathcal{O}_{\mathrm{Spec}(R)})}{\longrightarrow} \mathrm{QCoh}(\mathrm{Spec}(R) \times \mathrm{Spec}(R)) \simeq$$
$$\simeq \mathrm{QCoh}(\mathrm{Spec}(R)) \otimes \mathrm{QCoh}(\mathrm{Spec}(R)) \overset{(i_\infty)^! \otimes (i_\infty)^!}{\longrightarrow} \mathrm{QCoh}(\mathrm{Spec}(R))_{\mathcal{Y}} \otimes \mathrm{QCoh}(\mathrm{Spec}(R))_{\mathcal{Y}}$$

define a self-duality on $\mathrm{QCoh}(\mathrm{Spec}(R))_{\mathcal{Y}}$.

Now the assertion follows from the fact that with respect to the equivalence of Corollary 5.1.5(a), the object

$$((i_\infty)^! \otimes (i_\infty)^!)((\Delta_{\mathrm{Spec}(R)})_*(\mathcal{O}_{\mathrm{Spec}(R)}) \in \mathrm{QCoh}(\mathrm{Spec}(R))_{\mathcal{Y}} \otimes \mathrm{QCoh}(\mathrm{Spec}(R))_{\mathcal{Y}} \simeq$$
$$\simeq \mathrm{QCoh}(\mathrm{Spec}(R) \times \mathrm{Spec}(R))_{\mathcal{Y}\times\mathcal{Y}}$$

goes over to

$$((i_\infty)^* \otimes (i_\infty)^*)((\Delta_{\mathrm{Spec}(R)})_*(\mathcal{O}_{\mathrm{Spec}(R)}) \in \mathrm{QCoh}(\mathcal{Y}) \otimes \mathrm{QCoh}(\mathcal{Y}) \simeq \mathrm{QCoh}(\mathcal{Y} \times \mathcal{Y}),$$

and the latter identifies with $(\Delta_{\mathcal{Y}})_*(\mathcal{O}_{\mathcal{Y}})$, by base change along (5.8). □

We claim:

Lemma 5.4.4. *For an affine scheme S and a map $S \xrightarrow{f} \mathcal{Y}$, with respect to the above self-duality on* $\operatorname{QCoh}(\mathcal{Y})$ *and the canonical self-duality on* $\operatorname{QCoh}(S)$*, the functor f^* is the dual of the functor f_*.*

Proof. We need to establish an isomorphism

$$(f \times \mathrm{id}_{\mathcal{Y}})^* \circ (\Delta_{\mathcal{Y}})_*(\mathcal{O}_{\mathcal{Y}}) \simeq (\mathrm{id}_S \times f)_* \circ (\Delta_S)_*(\mathcal{O}_S). \tag{5.12}$$

By passing to right adjoints along the vertical arrows in the commutative diagram

$$\begin{tikzcd} \operatorname{QCoh}(S) & \operatorname{QCoh}(\mathcal{Y}) \arrow[l, "f^*"'] \\ \operatorname{QCoh}(S \times \mathcal{Y}) \arrow[u, "(\mathrm{Graph}_f)^*"] & \operatorname{QCoh}(\mathcal{Y} \times \mathcal{Y}) \arrow[l, "(f \times \mathrm{id}_{\mathcal{Y}})^*"'] \arrow[u, "(\Delta_{\mathcal{Y}})^*"'] \end{tikzcd}$$

we obtain a diagram that *a priori* commutes up to a natural transformation

$$\begin{tikzcd} \operatorname{QCoh}(S) \arrow[d, "(\mathrm{Graph}_f)_*"'] & \operatorname{QCoh}(\mathcal{Y}) \arrow[l, "f^*"'] \arrow[d, "(\Delta_{\mathcal{Y}})_*"] \\ \operatorname{QCoh}(S \times \mathcal{Y}) & \operatorname{QCoh}(\mathcal{Y} \times \mathcal{Y}). \arrow[l, "(f \times \mathrm{id}_{\mathcal{Y}})^*"'] \end{tikzcd} \tag{5.13}$$

However, it is easy to see from the equivalence $\operatorname{QCoh}(\mathcal{Y}) \simeq \operatorname{QCoh}(\operatorname{Spec}(R))_{\mathcal{Y}}$ that the above natural transformation in (5.13) is actually an isomorphism.

Evaluating the two circuits of (5.13) on $\mathcal{O}_{\mathcal{Y}} \in \operatorname{QCoh}(\mathcal{Y})$, we obtain the desired isomorphism in (5.12). □

5.5. **The functor of !-global sections.** As was mentioned above, for a formal affine scheme, the functor of global sections

$$\Gamma(\mathcal{Y}, -) = \mathcal{H}om_{\operatorname{QCoh}(\mathcal{Y})}(\mathcal{O}_{\mathcal{Y}}, -), \quad \operatorname{QCoh}(\mathcal{Y}) \to \mathrm{Vect}_{\mathsf{e}}$$

is discontinuous.

In this subsection we will introduce its substitute, denoted $\Gamma_!(\mathcal{Y}, -)$.

5.5.1. Let $\Gamma_!(\mathcal{Y}, -)$ denote the functor

$$\operatorname{QCoh}(\mathcal{Y}) \to \mathrm{Vect}_{\mathsf{e}},$$

dual to the functor

$$\mathrm{Vect}_{\mathsf{e}} \xrightarrow{\mathcal{O}_{\mathcal{Y}}} \operatorname{QCoh}(\mathcal{Y})$$

with respect to the self-duality

$$\operatorname{QCoh}(\mathcal{Y})^{\vee} \simeq \operatorname{QCoh}(\mathcal{Y}) \tag{5.14}$$

of Proposition 5.4.3.

We will now describe the above functor $\Gamma_!(\mathcal{Y}, -)$ more explicitly. First, we claim:

Corollary 5.5.2. *For an affine scheme S and a map $S \xrightarrow{f} \mathcal{Y}$, there is a canonical isomorphism*

$$\Gamma_!(\mathcal{Y}, -) \circ f_* \simeq \Gamma(S, -) : \operatorname{QCoh}(S) \to \mathrm{Vect}_{\mathsf{e}}\,.$$

Proof. By Lemma 5.4.4, the functors dual to both sides identify with

$$\mathrm{Vect}_{\mathsf{e}} \xrightarrow{\mathcal{O}_S} \operatorname{QCoh}(S).$$

□

Note that Corollary 5.5.2 allows to describe the functor $\Gamma_!(\mathcal{Y},-)$ as follows: in terms of the presentation (5.9), it corresponds to the compatible collection of functors

$$\Gamma(S,-):\mathrm{QCoh}(S)\to\mathrm{Vect}_{\mathsf{e}}\,.$$

This functor should *not* be confused with the *discontinuous* functor

$$\Gamma(\mathcal{Y},-):\mathrm{QCoh}(\mathcal{Y})\to\mathrm{Vect}_{\mathsf{e}},$$

corepresented by $\mathcal{O}_{\mathcal{Y}}$.

Remark 5.5.3. For a choice of the presentation of $\mathcal{Y}$ as in (5.1), in terms of the identification $\mathrm{QCoh}(\mathcal{Y})\simeq\mathrm{QCoh}(\mathrm{Spec}(R))_{\mathcal{Y}}$, the functor $\Gamma_!(\mathcal{Y},-)$ corresponds to the composition

$$\Gamma(\mathrm{Spec}(R),-)\circ(i_\infty)_!.$$

We did not give this as a definition of the functor $\Gamma_!(\mathcal{Y},-)$ in order for the definition to be manifestly independent of the presentation of $\mathcal{Y}$ as in (5.1).

5.5.4. We now claim:

Proposition 5.5.5. *The counit for the self-duality* (5.14) *is given by*

$$\mathrm{QCoh}(\mathcal{Y})\otimes\mathrm{QCoh}(\mathcal{Y})\overset{\otimes}{\to}\mathrm{QCoh}(\mathcal{Y})\overset{\Gamma_!(\mathcal{Y},-)}{\longrightarrow}\mathrm{Vect}_{\mathsf{e}}\,. \tag{5.15}$$

Proof. Choose a presentation of $\mathcal{Y}$ as in (5.1). Then with respect to the identification $\mathrm{QCoh}(\mathcal{Y})\simeq\mathrm{QCoh}(\mathrm{Spec}(R))_{\mathcal{Y}}$, the functor (5.15) corresponds to

$$\mathrm{QCoh}(\mathrm{Spec}(R))_{\mathcal{Y}}\otimes\mathrm{QCoh}(\mathrm{Spec}(R))_{\mathcal{Y}}\to\mathrm{QCoh}(\mathrm{Spec}(R))\otimes\mathrm{QCoh}(\mathrm{Spec}(R))\overset{\otimes}{\to}$$
$$\to\mathrm{QCoh}(\mathrm{Spec}(R))\overset{\Gamma(\mathrm{Spec}(R),-)}{\longrightarrow}\mathrm{Vect}_{\mathsf{e}}\,.$$

□

Finally, we claim:

Proposition 5.5.6. *The category* $\mathrm{QCoh}(\mathcal{Y})$ *carries a t-structure, uniquely characterized by the requirement that the functor* $\Gamma_!(\mathcal{Y},-)$ *is t-exact. Furthermore,* $\mathrm{QCoh}(\mathcal{Y})$ *is left-complete in this t-structure.*

Proof. Choose a presentation as in (5.1). Then the assertion of the proposition follows from Proposition 5.1.4:

The corresponding t-structure on $\mathrm{QCoh}(\mathrm{Spec}(R))_{\mathcal{Y}}$ is the unique one for which the functor $(i_\infty)_!$ is t-exact. □

5.6. **Consequences for** $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X)$**.** The prestack $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X)$ is a disjoint union of formal affine schemes. So the theory developed in the previous subsections is immediately applicable to $\mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X))$. In this subsection we will record the relevant results for future reference.

5.6.1. Consider the prestack $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X)$.

From Theorem 1.4.3 and Lemma 5.3.2, we obtain that $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}}$ an affine diagonal. In particular, for an affine scheme S and a map $f:S\to\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X)$, the functor f_*, right adjoint to f^*, is continuous.

From Corollaries 5.1.8 and 5.3.4 we obtain:

Corollary 5.6.2.
(a) *The category* $\mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X))$ *is compactly generated; the subcategory of compact objects is closed under the monoidal operation.*
(b) *The functor*

$$\underset{(S,f)}{\mathrm{colim}}\,\mathrm{QCoh}(S)\to\mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X)), \tag{5.16}$$

is an equivalence, where:

- *The index category is any of*

$$\mathrm{Sch}^{\mathrm{aff}}_{/\,\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X)},\ \ (\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft},/\mathsf{e}})_{/\,\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X)},$$

$$\mathrm{Sch}^{\mathrm{aff}}_{/\,\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X),\mathrm{closed}},\ \ (\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft},/\mathsf{e}})_{/\,\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X),\mathrm{closed}};$$

- *The colimit is formed is the pushforward functors* $(f_{1,2})_* : \mathrm{QCoh}(S_1) \to \mathrm{QCoh}(S_2)$ *for*

$$f_{1,2} : S_1 \to S_2, \quad f_2 \circ f_{1,2} = f_1.$$

- *The map in* (5.9) *is given by* $\{\mathrm{QCoh}(S) \xrightarrow{f_*} \mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X))\}$.

5.6.3. Next, we claim:

Corollary 5.6.4. *The object*

$$(\Delta_{\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X)})_*(\mathcal{O}_{\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X)}) \in \mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X) \times \mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X)) \simeq$$
$$\simeq \mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X)) \otimes \mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X))$$

defines the unit of a self-duality on $\mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X))$.

Corollary 5.6.5. *For an affine scheme S and a map* $S \xrightarrow{f} \mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X)$, *with respect to the above self-duality on* $\mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X))$ *and the canonical self-duality on* $\mathrm{QCoh}(S)$, *the functor* f^* *is the dual of the functor* f_*.

5.6.6. Let

$$\Gamma_c(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X), -)$$

be the functor dual to

$$\mathrm{Vect}_{\mathsf{e}} \xrightarrow{\mathcal{O}_{\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X)}} \mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X))$$

with respect to the above self-duality on $\mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X))$.

In terms of the presentation (5.16), it corresponds to the compatible collection of functors

$$\Gamma(S, -) : \mathrm{QCoh}(S) \to \mathrm{Vect}_{\mathsf{e}}, \quad S \in \mathrm{Sch}^{\mathrm{aff}}_{/\,\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X)}.$$

Finally, as in Proposition 5.5.5, we have:

Corollary 5.6.7. *The counit for the self-duality on* $\mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X))$ *is given by*

$$\mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X)) \otimes \mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X)) \overset{\otimes}{\to} \tag{5.17}$$
$$\to \mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X)) \xrightarrow{\Gamma_!(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X), -)} \mathrm{Vect}_{\mathsf{e}}.$$

5.6.8. From Sect. 5.2.1 we obtain:

Corollary 5.6.9. *The prestack* $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X)$ *is 1-affine. Moreover:*

(a) *For* $\mathcal{Z} \to \mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X)$ *and an affine scheme* $f : S \to \mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X)$, *the natural functor*

$$\mathrm{QCoh}(S) \underset{\mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X))}{\otimes} \mathrm{QCoh}(\mathcal{Z}) \to \mathrm{QCoh}(S \underset{\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X)}{\times} \mathcal{Z})$$

is an equivalence.

(b) *For* (S, f) *as above, the category* $\mathrm{QCoh}(S)$ *is canonically self-dual as a module category over* $\mathrm{QCoh}(\mathcal{Y})$, *where the counit and the unit are given by* (5.6) *and* (5.7), *respectively.*

(c) *A module category over* $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X)$ *is dualizable if and only if it is such as a plain DG category.*

5.7. **Consequences for** $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)$**.** In this subsection we will (try to) transport the results about $\mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr,rigid}_x}(X))$ to $\mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X))$.

Most of it will be automatic, apart from the issue of compact generation, treated in the next subsection.

5.7.1. We now consider the prestack $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)$, which is isomorphic to

$$\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr,rigid}_x}(X)/\mathsf{G}.$$

The fact that $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr,rigid}_x}(X)$ has an affine diagonal implies that the same is true for $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)$.

5.7.2. The category $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr,rigid}_x}(X)$ is equipped with an action of the group G, and the category $\mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X))$ is equipped with an action of $\mathrm{Rep}(\mathsf{G})$ so that we have

$$\mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr,rigid}_x}(X)) \simeq \mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)) \underset{\mathrm{Rep}(\mathsf{G})}{\otimes} \mathrm{Vect}_{\mathsf{e}}$$

and

$$\mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)) \simeq \Big(\mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr,rigid}_x}(X))\Big)^{\mathsf{G}}.$$

5.7.3. Recall now (see [Ga2, Theorem 2.2.2]) that the stack pt/G is 1-affine, which means that the operations

$$\mathbf{C} \mapsto \mathbf{C} \underset{\mathrm{Rep}(\mathsf{G})}{\otimes} \mathrm{Vect}_{\mathsf{e}}, \quad \mathrm{Rep}(\mathsf{G})\text{-}\mathbf{mod} \to \mathsf{G}\text{-}\mathbf{mod}$$

and

$$\mathbf{C}' \mapsto (\mathbf{C}')^{\mathsf{G}}, \quad \mathsf{G}\text{-}\mathbf{mod} \to \mathrm{Rep}(\mathsf{G})\text{-}\mathbf{mod}$$

define mutually inverse equivalences of categories.

In particular, an object $\mathbf{C} \in \mathrm{Rep}(\mathsf{G})\text{-}\mathbf{mod}$ is dualizable (this is equivalent to being dualizable as a plain DG category, since $\mathrm{Rep}(\mathsf{G})$ is rigid, see [GR1, Chapter 1, Proposition 9.4.4]) if and only if

$$\mathbf{C}' := \mathbf{C} \underset{\mathrm{Rep}(\mathsf{G})}{\otimes} \mathrm{Vect}_{\mathsf{e}}$$

is dualizable as an object of $\mathsf{G}\text{-}\mathbf{mod}$ (this is equivalent to being dualizable as a plain DG category).

Note also that as a consequence of 1-affineness, we obtain that the functor

$$\mathbf{C} \mapsto \mathbf{C} \underset{\mathrm{Rep}(\mathsf{G})}{\otimes} \mathrm{Vect}_{\mathsf{e}}, \quad \mathrm{Rep}(\mathsf{G})\text{-}\mathbf{mod} \to \mathrm{DGCat} \tag{5.18}$$

is conservative.

5.7.4. In particular, from Corollary 5.6.2(a) we obtain:

Corollary 5.7.5. *The category* $\mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X))$ *is dualizable.*

Remark 5.7.6. In fact, we will see that that $\mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X))$ is compactly generated; but we will need to work harder to show that, see Sect. 5.8.

5.7.7. From Corollary 5.6.2(b) we obtain:

Corollary 5.7.8. *The functor*

$$\underset{(S,f)}{\mathrm{colim}}\, \mathrm{QCoh}(S) \to \mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)), \tag{5.19}$$

is an equivalence, where:

- *The index category is either of*

$$\mathrm{Sch}^{\mathrm{aff}}_{/\,\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)},\ (\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft},/\mathsf{e}})_{/\,\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)};$$

- *The colimit is formed is the pushforward functors* $(f_{1,2})_* : \mathrm{QCoh}(S_1) \to \mathrm{QCoh}(S_2)$ *for*

$$f_{1,2} : S_1 \to S_2, \quad f_2 \circ f_{1,2} = f_1.$$

- *The map in* (5.9) *is given by* $\{\mathrm{QCoh}(S) \overset{f_*}{\to} \mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X))\}$.

Proof. Note the fact that the functor

$$\mathrm{Sch}^{\mathrm{aff}}_{/\,\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)} \to \mathrm{Sch}^{\mathrm{aff}}_{/\,\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X)}, \quad S \mapsto S \underset{\mathrm{pt}\,/\mathsf{G}}{\times} \mathrm{pt}$$

is cofinal. Hence, the colimit in the left-hand side of (5.16) is obtained from the colimit in the left-hand side of (5.19) by applying the functor (5.18).

Hence, the assertion follows from the fact that (5.18) is conservative.

□

5.7.9. Next, we claim

Corollary 5.7.10.
(a) *The object*

$$(\Delta_{\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)})_*(\mathcal{O}_{\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)}) \in \mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X) \times \mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)) \simeq$$
$$\simeq \mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)) \otimes \mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X))$$

defines the unit of a self-duality on $\mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X))$.

(b) *For an affine scheme* S *and a map* $S \overset{f}{\to} \mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)$, *with respect to the above self-duality on* $\mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X))$ *and the canonical self-duality on* $\mathrm{QCoh}(S)$, *the functor* f^* *is the dual of the functor* f_*.

Proof. We will prove point (a). Point (b) would follow as in the proof of Lemma 5.4.4.

Let $\mathbf{C}_1$ and $\mathbf{C}_2$ be module categories over a symmetric monoidal category $\mathbf{A}$. Note that the natural projection

$$\mathbf{C}_1 \otimes \mathbf{C}_2 \overset{\mathrm{mult}}{\longrightarrow} \mathbf{C}_1 \underset{\mathbf{A}}{\otimes} \mathbf{C}_2$$

admits a continuous right adjoint, to be denoted mult^R.

Let us be given an object $u \in \mathbf{C}_1 \underset{\mathrm{Rep}(\mathsf{G})}{\otimes} \mathbf{C}_2$ which is the unit of a duality between $\mathbf{C}_1$ and $\mathbf{C}_2$ as $\mathbf{A}$-module categories. Then

$$\mathrm{mult}^R(u) \in \mathbf{C}_1 \otimes \mathbf{C}_2$$

is the unit of a duality between $\mathbf{C}_1$ and $\mathbf{C}_2$ as plain DG categories.

We apply this to $\mathbf{A} = \mathrm{Rep}(\mathsf{G})$. Note that by 1-affineness, the condition on u is equivalent to the fact that the image of u along

$$\mathbf{C}_1 \underset{\mathrm{Rep}(\mathsf{G})}{\otimes} \mathbf{C}_2 \to \mathrm{Vect}_{\mathsf{e}} \underset{\mathrm{Rep}(\mathsf{G})}{\otimes} \left(\mathbf{C}_1 \underset{\mathrm{Rep}(\mathsf{G})}{\otimes} \mathbf{C}_2\right) \simeq \left(\mathrm{Vect}_{\mathsf{e}} \underset{\mathrm{Rep}(\mathsf{G})}{\otimes} \mathbf{C}_1\right) \otimes \left(\mathrm{Vect}_{\mathsf{e}} \underset{\mathrm{Rep}(\mathsf{G})}{\otimes} \mathbf{C}_2\right) \tag{5.20}$$

defines a duality between $\mathrm{Vect}_{\mathsf{e}} \underset{\mathrm{Rep}(\mathsf{G})}{\otimes} \mathbf{C}_1$ and $\mathrm{Vect}_{\mathsf{e}} \underset{\mathrm{Rep}(\mathsf{G})}{\otimes} \mathbf{C}_2$ as plain categories.

We apply this to $\mathbf{C}_1 = \mathbf{C}_2 = \mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X))$ and u being the image of the structure sheaf under the map

$$\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X) \to \mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X) \underset{\mathrm{pt}\,/\mathsf{G}}{\times} \mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X),$$

where we note that

$$\mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X) \underset{\mathrm{pt}\,/\mathsf{G}}{\times} \mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)) \simeq \mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)) \underset{\mathrm{Rep}(\mathsf{G})}{\otimes} \mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)).$$

We note that the image of our u under the map (5.20) identifies with

$$(\Delta_{\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X)})_*(\mathcal{O}_{\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X)}) \in \mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X) \times \mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X)) \simeq$$
$$\simeq \mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X)) \otimes \mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X))$$

Now, the assertion of the corollary follows from Corollary 5.6.4.

□

5.7.11. Let

$$\Gamma_c(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X), -)$$

be the functor dual to

$$\mathrm{Vect}_{\mathsf{e}} \overset{\mathcal{O}_{\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)}}{\longrightarrow} \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X))$$

with respect to the above self-duality on $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X))$.

In terms of the presentation (5.19), it corresponds to the compatible collection of functors

$$\Gamma(S, -) : \mathrm{QCoh}(S) \to \mathrm{Vect}_{\mathsf{e}}, \quad S \in \mathrm{Sch}^{\mathrm{aff}}_{/\,\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)}.$$

Finally, we have:

Corollary 5.7.12. *The counit for the self-duality on* $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X))$ *is given by*

(5.21)
$$\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)) \otimes \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)) \overset{\otimes}{\to} \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)) \overset{\Gamma_!(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X), -)}{\longrightarrow} \mathrm{Vect}_{\mathsf{e}}.$$

Proof. Follows from Corollary 5.6.7 in the same way as Corollary 5.7.10 follows from Corollary 5.6.4. □

From Sect. 5.2.1, and combining again with the fact that pt/G is 1-affine, we obtain:

Corollary 5.7.13. *The prestack* $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ *is 1-affine. Moreover:*

(a) *For* $\mathcal{Z} \to \mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ *and an affine scheme* $f : S \to \mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$*, the natural functor*

$$\mathrm{QCoh}(S) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X))}{\otimes} \mathrm{QCoh}(\mathcal{Z}) \to \mathrm{QCoh}(S \underset{\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)}{\times} \mathcal{Z})$$

is an equivalence.

(b) *For* (S, f) *as above, the category* $\mathrm{QCoh}(S)$ *is canonically self-dual as a module category over* $\mathrm{QCoh}(\mathcal{Y})$*, where the counit and the unit are given by* (5.6) *and* (5.7)*, respectively.*

(c) *A module category over* $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ *is dualizable if and only if it is such as a plain DG category.*

5.8. **Compact generation of** $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X))$**.** In this subsection we will prove that the category $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X))$ is compactly generated. In fact, we will explicitly describe particular a set of compact generators.

5.8.1. Let $\mathcal{Z}$ be a connected component of $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X))$ and set

$$\mathcal{Z}^{\mathrm{rigid}_x} := \mathcal{Z} \underset{\mathrm{pt}/\mathsf{G}}{\times} \mathrm{pt},$$

where the map

$$\mathrm{ev}_x : \mathcal{Z} \to \mathrm{pt}/\mathsf{G}$$

corresponds to some particular point $x \in X$.

First, we note that $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Z})$ is compact if and only if its pullback to $\mathcal{Z}^{\mathrm{rigid}_x}$ is compact. In particular, the subcategory of $\mathrm{QCoh}(\mathcal{Z})$ consisting of compact objects is closed under the monoidal operation (by Corollary 5.6.2(a)).

5.8.2. We will prove:

Theorem 5.8.3. *The category* $\mathrm{QCoh}(\mathcal{Z})$ *is compactly generated by a family of objects of the form* $\mathcal{F} \otimes \mathrm{ev}_x(V)$*, where:*

- $V \in \mathrm{Rep}(\mathsf{G})^c$*;*
- $\mathcal{F}$ *can be expressed as a finite colimit in terms of* $\mathcal{O}_{\mathcal{Z}}$*.*

The rest of this subsection is devoted to the proof of this theorem.

Remark 5.8.4. We wish to emphasize that the object $\mathcal{O}_{\mathcal{Z}} \in \mathrm{QCoh}(\mathcal{Z})$ itself is *not* compact.

5.8.5. Consider the coarse moduli space $\mathcal{Z}^{\mathrm{coarse}}$ corresponding to $\mathcal{Z}$ and the map

$$r : \mathcal{Z} \to \mathcal{Z}^{\mathrm{coarse}}.$$

Recall that according to Theorem 4.4.2(b), the ind-scheme $\mathcal{Z}^{\mathrm{coarse}}$ is actually a formal affine scheme. Write

$$\mathcal{Z}^{\mathrm{coarse}} \simeq \underset{n}{\mathrm{colim}}\ \mathrm{Spec}(R_n)$$

as in (5.1).

By Lemma 5.1.6(a), we can find compact generators $\mathcal{F}^0 \in \mathrm{QCoh}(\mathcal{Z}^{\mathrm{coarse}})$ that can be expressed as finite colimits in terms of $\mathcal{O}_{\mathcal{Z}^{\mathrm{coarse}}}$.

Set $\mathcal{F} := r^*(\mathcal{F}_0)$. We will show that these objects satisfy the requirements of the theorem.

By construction, $\mathcal{F}$ can be expressed as a finite colimit in terms of $\mathcal{O}_{\mathcal{Z}}$.

5.8.6. Let $\mathcal{F}'$ denote the further pullback of $\mathcal{F}$ to $\mathcal{Z}^{\mathrm{rigid}_x}$. Let us show that $\mathcal{F}'$ is compact.

Let r' denote the composite map

$$\mathcal{Z}^{\mathrm{rigid}_x} \to \mathcal{Z} \xrightarrow{r} \mathcal{Z}^{\mathrm{coarse}}.$$

For $\mathcal{F}'_1 \in \mathrm{QCoh}(\mathcal{Z}^{\mathrm{rigid}_x})$, we have

$$\mathcal{H}om_{\mathrm{QCoh}(\mathcal{Z}^{\mathrm{rigid}_x})}(\mathcal{F}', \mathcal{F}'_1) \simeq \mathcal{H}om_{\mathrm{QCoh}(\mathcal{Z}^{\mathrm{coarse}})}(\mathcal{F}_0, r'_*(\mathcal{F}'_1)). \tag{5.22}$$

Now, by Theorem 4.4.2(a), the map r' is *schematic*, so the functor r'_* is continuous. Hence, the right-hand side in (5.22) is continuous as a functor of $\mathcal{F}'_1$. Hence, so is the left-hand side, as required.

5.8.7. Finally, let us show that the objects $\mathcal{F} \otimes \mathrm{ev}_x(V)$ generate $\mathrm{QCoh}(\mathcal{Z})$.

First, since the above morphism r' is affine, we obtain that the objects $\mathcal{F}'$ generate $\mathrm{QCoh}(\mathcal{Z}^{\mathrm{rigid}_x})$.

For $\mathcal{F}_1 \in \mathrm{QCoh}(\mathcal{Z})$, we have

$$\mathcal{H}om_{\mathrm{QCoh}(\mathcal{Z}^{\mathrm{rigid}_x})}(\mathcal{F}', \mathcal{F}'_1) \simeq \mathcal{H}om_{\mathrm{QCoh}(\mathcal{Z})}(\mathcal{F}, \mathcal{F}_1 \otimes \mathrm{ev}_x^*(R_{\mathsf{G}})),$$

where $\mathcal{F}'_1$ denotes the pullback of $\mathcal{F}_1$ to $\mathcal{Z}^{\mathrm{rigid}_x}$, and R_{G} is the regular representation of G.

If $\mathcal{F}_1 \neq 0$ we can find $\mathcal{F}'$ so that the above $\mathcal{H}om$ is non-zero.

Further, since we already know that $\mathcal{F}$ is compact, we rewrite the latter expression as

$$\underset{V}{\mathrm{colim}}\ \mathcal{H}om_{\mathrm{QCoh}(\mathcal{Z})}(\mathcal{F}, \mathcal{F}_1 \otimes \mathrm{ev}_x^*(V)),$$

where the colimit is taken over $\mathrm{Rep}(\mathsf{G})^c_{/R_{\mathsf{G}}}$.

Hence, for some $V \in \mathrm{Rep}(\mathsf{G})$,

$$\mathcal{H}om_{\mathrm{QCoh}(\mathcal{Z})}(\mathcal{F}, \mathcal{F}_1 \otimes \mathrm{ev}_x^*(V)) \simeq \mathcal{H}om_{\mathrm{QCoh}(\mathcal{Z})}(\mathcal{F} \otimes \mathrm{ev}_x^*(V^\vee), \mathcal{F}_1)$$

is non-zero, as required.

□

Part II: Lisse Hecke actions and spectral decomposition over $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$

Let us make a brief overview of the contents of this Part.

In Sect. 6 we describe the set-up for the following question: what does it take to have an action of the monoidal category $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X))$ on a DG category $\mathbf{C}$? It turns that the appropriate input datum is what one can call *an action of* $\mathrm{Rep}(\mathsf{G})^{\otimes X}$ *on* $\mathbf{C}$. In Theorem 6.1.4 we state that these two pieces of data are indeed in bijection. We introduce an abstract framework for this result, where instead of $\mathrm{QLisse}(X)$ we are dealing with a general symmetric monoidal category $\mathbf{H}$, equipped with a t-structure. The object of study becomes the functor between symmetric monoidal categories

$$\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}) \to \mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}))$$

(see (6.6)), and we wish to show that it is an equivalence under certain conditions on $\mathbf{H}$. We state Theorem 6.2.11, which says that the dual functor

$$\mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\vee} \to \underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\vee}$$

is an equivalence when H is a Tannakian category with strong finiteness properties. We show how Theorem 6.2.11 implies Theorem 6.1.4.

In Sect. 7 we consider some particular cases of the above paradigm. Namely, we show that the functor (6.6) considered above is an equivalence in some cases of interest. First, we quote a result from [GKRV], which says that it is an equivalence for $\mathbf{H} = \mathrm{Vect}^{\mathcal{X}}_{\mathsf{e}}$ for a connected space $\mathcal{X}$. From here we deduce that its also an equiavalence for $\mathbf{H} = \mathrm{QLisse}(\mathcal{X})$ (i.e., the case of arbitrary local systems implies the case of local systems of ind-finite rank). We also show that (6.6) is an equivalence for $\mathbf{H} = \mathsf{h}$-mod for a connective Lie algebra h. Finally, in Sect. 7.8, we make an attempt to prove that (6.6) is an equivalence in general; we do not succeed, but we show that induces an equivalence on the spaces of functors with the target being a symmetric monoidal category with a compact unit (or a limit of such).

In Sect. 8 we prove that the functor (6.6) is an equivalence for $\mathbf{H} = \mathrm{Rep}(\mathsf{H})$, where H is a finite-dimensional algebraic group. We do this by bootstrapping the (not fully unsuccessful) attempt in Sect. 7.8. We first treat the case when H is semi-simple, then then the case when it is a torus, and then combine to the case when H is reductive. We then combine this with the case of Lie algebras considered in the previous section and prove that (6.6) is an equivalence for $\mathbf{H} = (\mathsf{h}, \mathsf{H}_{\mathrm{red}})$, for a Harish-Chandra pair $(\mathsf{h}, \mathsf{H}_{\mathrm{red}})$ with $\mathsf{H}_{\mathrm{red}}$ reductive. Finally, we use the methods in the implication "validity for $\mathrm{Vect}^{\mathcal{X}}_{\mathsf{e}}$ $\Rightarrow$ validity for $\mathrm{QLisse}(\mathcal{X})$" to deduce the case of $\mathbf{H} = \mathrm{Rep}(\mathsf{H})$, using the fully-faithful embedding

$$\mathrm{Rep}(\mathsf{H}) \hookrightarrow (\mathsf{h}, \mathsf{H}_{\mathrm{red}}),$$

where h is the Lie algebra of H and $\mathsf{H}_{\mathrm{red}}$ is the reductive part of H corresponding to a choice of Levi splitting.

In Sect. 9 we finish the proof of Theorem 6.2.11 by using another set of ideas. Namely, we give an explicit description of the object

$$(\Delta_{\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})})_*(\mathcal{O}_{\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})}) \in \mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}) \times \mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})),$$

and use to compute Homs in the category $\mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\vee}$. This allows us to prove explicitly that the functor

$$\mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\vee} \to \underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\vee}$$

appearing in Theorem 6.2.11 is fully faithful. To prove that it is essentially surjective, we reduce the assertion to one at the level of abelian categories, and deduce the latter from the (already established) fact that equivalence holds for $\mathbf{H} = \mathrm{Rep}(\mathsf{H})$, where H is an algebraic group.

6. The spectral decomposition theorem

In this section we introduce the state the main theorem of Part II, Theorem 6.1.4, along with a closely related abstract version, Theorem 6.2.11, and show how the latter implies the former.

We also discuss a statement equivalent to Theorem 6.2.11, where we introduce rigidification.

6.1. **Definition of a lisse action.** In this subsection we define what it means to have an action $\mathrm{Rep}(\mathsf{G})^{\otimes X}$ on $\mathbf{C}$, and state the main theorem of this part, Theorem 6.1.4, that the datum of such an action on a category $\mathbf{C}$ is equivalent to the datum of an action on $\mathbf{C}$ of the category $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X))$, provided that $\mathbf{C}$ is dualizable.

6.1.1. Let $\mathbf{C}$ be a DG category. We define the notion of *action of* $\mathrm{Rep}(\mathsf{G})^{\otimes X}$ *on* $\mathbf{C}$ by imitating [GKRV, Sects. C.1.2 and C.2.2]. Namely, this is a natural transformation between the following two functors $\mathrm{fSet} \to \mathrm{DGCat}^{\mathrm{Mon}}$:

From the functor
$$I \mapsto \mathrm{Rep}(\mathsf{G})^{\otimes I}$$
to the functor
$$I \mapsto \mathrm{End}(\mathbf{C}) \otimes \mathrm{QLisse}(X)^{\otimes I}.$$

6.1.2. Consider the symmetric monoidal category $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X))$. We claim that there is a canonically defined natural transformation between the following two functors $\mathrm{fSet} \to \mathrm{DGCat}^{\mathrm{SymMon}}$:

From the functor
$$I \mapsto \mathrm{Rep}(\mathsf{G})^{\otimes I} \tag{6.1}$$
to the functor
$$I \mapsto \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)) \otimes \mathrm{QLisse}(X)^{\otimes I}. \tag{6.2}$$

Indeed, by definition, a datum of such a natural transformation is equivalent to a compatible system of natural transformations from (6.1) to
$$I \mapsto \mathrm{QCoh}(S) \otimes \mathrm{QLisse}(X)^{\otimes I} \text{ for } S \in \mathrm{Sch}^{\mathrm{aff}}_{/\,\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)}.$$

By definition, the datum of a map $S \to \mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ is a (right t-exact) symmetric monoidal functor
$$\mathsf{F} : \mathrm{Rep}(\mathsf{G}) \to \mathrm{QCoh}(S) \otimes \mathrm{QLisse}(X).$$

The required functor
$$\mathrm{Rep}(\mathsf{G})^{\otimes I} \to \mathrm{QCoh}(S) \otimes \mathrm{QLisse}(X)^{\otimes I}$$
is then the composition
$$\mathrm{Rep}(\mathsf{G})^{\otimes I} \xrightarrow{\mathsf{F}^{\otimes I}} \mathrm{QCoh}(S)^{\otimes I} \otimes \mathrm{QLisse}(X)^{\otimes I} \to \mathrm{QCoh}(S) \otimes \mathrm{QLisse}(X)^{\otimes I},$$
where
$$\mathrm{QCoh}(S)^{\otimes I} \to \mathrm{QCoh}(S)$$
is the tensor product map.

6.1.3. From Sect. 6.1.2 we obtain that for any DG category $\mathbf{C}$, equipped with an action of $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X))$, we obtain an action of $\mathrm{Rep}(\mathsf{G})^{\otimes X}$ on $\mathbf{C}$. I.e., we obtain a map of spaces
$$\{\text{Actions of } \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)) \text{ on } \mathbf{C}\} \to \{\text{Actions of } \mathrm{Rep}(\mathsf{G})^{\otimes X} \text{ on } \mathbf{C}\}. \tag{6.3}$$

The main result of Part II of this paper is the following:

Main Theorem 6.1.4. *Assume that both* $\mathbf{C}$ *and* $\mathrm{QLisse}(X)$ *are dualizable as DG categories. Then the map* (6.3) *is an isomorphism.*

We can regard this theorem as saying that a (dualizable) category $\mathbf{C}$ equipped with an action of $\mathrm{Rep}(\mathsf{G})^{\otimes X}$, admits a spectral decomposition with respect to $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$.

6.1.5. The rest of Part II is devoted to the proof of this theorem.

Remark 6.1.6. We expect that the map (6.3) is an isomorphism for any $\mathbf{C}$, i.e., without the dualizability assumption.

6.2. **Abstract setting for Theorem 6.1.4.** In this subsection we introduce an abstract setting for Theorem 6.1.4: we replace the category $\mathrm{QLisse}(X)$ by a general symmetric monoidal category $\mathbf{H}$.

6.2.1. Let us recall the following general construction. Let $\mathbf{O}$ be a symmetric monoidal category (in practice we will take $\mathbf{O} = \mathrm{DGCat}$). Let A (resp., B) be a commutative algebra (resp., cocommutative coalgebra) object in $\mathbf{O}$. In this case one can form a commutative algebra object

$$\mathrm{coEnd}(A, B) \in \mathbf{O},$$

with the following universal property: for a commutative algebra object $A' \in \mathbf{O}$, the space of maps of commutative algebras $\mathrm{coEnd}(A, B) \to A'$ is the space of maps

$$\phi : A \otimes B \to A',$$

equipped with a datum of commutativity for the diagram

$$\begin{tikzcd}
A \otimes A \otimes B \arrow[r, "\mathrm{mult}_A \otimes \mathrm{id}_B"] \arrow[d, "\mathrm{id}_{A\otimes A} \otimes \mathrm{comult}_B"'] & A \otimes B \arrow[r, "\phi"] & A' \\
A \otimes A \otimes B \otimes B \arrow[rr, "\phi\otimes\phi"] & & A' \otimes A', \arrow[u]
\end{tikzcd}$$

along with a homotopy-coherent system of higher compatibilities.

Note that if B is dualizable as an object of $\mathbf{O}$, and so $B^\vee$ is a commutative algebra in $\mathbf{O}$, the above space of maps is the same as the space of maps of commutative algebras

$$A \to A' \otimes B^\vee.$$

In this case we will also use the notation

$$\underline{\mathrm{coHom}}(A, B^\vee) := \mathrm{coEnd}(A, B).$$

One can explicitly write down $\mathrm{coEnd}(A, B)$ as an object of $\mathbf{O}$. Namely, it is the colimit over the category $\mathrm{TwArr}(\mathrm{fSet})$ ([GKRV, Sect. 1.2.2]) of the functor that sends

$$(I \to J) \in \mathrm{TwArr}(\mathrm{fSet})$$

to

$$A^{\otimes I} \otimes B^{\otimes J}.$$

For a 1-morphism

$$\begin{tikzcd}
I_0 \arrow[r] \arrow[d] & J_0 \\
I_1 \arrow[r] & J_1 \arrow[u]
\end{tikzcd}$$

in $\mathrm{TwArr}(\mathrm{fSet})$, the corresponding map

$$A^{\otimes I_0} \otimes B^{\otimes J_0} \to A^{\otimes I_1} \otimes B^{\otimes J_1}$$

is given by the maps $A^{\otimes I_0} \to A^{\otimes I_1}$ (resp., $B^{\otimes J_0} \to B^{\otimes J_1}$), given by the commutative algebra structure on A (resp., cocommutative coalgebra structure on B).

The symmetric monoidal structure on the above colimit is induced by the operation of disjoint union on fSet.

From the fact that the category fSet is sifted, as in [GKRV, Theorem 1.2.4], we obtain:

Lemma 6.2.2. *Assume that B is dualizable as an object of $\mathbf{O}$.*
(a) *For an* associative *algebra object $C \in \mathbf{O}$, the space of maps of associative algebras*

$$\underline{\mathrm{coHom}}(A, B^\vee) \to C$$

identifies with the space of compatible collections of maps of associative *algebras*

$$A^{\otimes I} \to C \otimes (B^\vee)^{\otimes I}, \quad I \in \mathrm{fSet}\,.$$

(b) *For a* plain *object $C \in \mathbf{O}$, the space* $\mathrm{Maps}_{\mathbf{O}}(\underline{\mathrm{coHom}}(A, B^\vee), C)$ *identifies with the space of compatible collections of maps in* $\mathbf{O}$

$$A^{\otimes I} \to C \otimes (B^\vee)^{\otimes I}, \quad I \in \mathrm{fSet}\,.$$

6.2.3. We take $\mathbf{O} := \mathrm{DGCat}$, and $A := \mathrm{Rep}(\mathsf{G})$. We take $B^\vee$ to be a symmetric monoidal category $\mathbf{H}$, equipped with a t-structure and a conservative t-exact symmetric monoidal functor $\mathbf{oblv_H}$ to $\mathrm{Vect}_{\mathsf{e}}$. We will assume that $\mathbf{H}$ is dualizable as a DG category.

On the one hand, we consider the (symmetric monoidal) category

$$\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}).$$

6.2.4. On the other hand, we can consider the prestack $\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$ that sends an affine scheme S to the space of right t-exact symmetric monoidal functors

$$\mathrm{Rep}(\mathsf{G}) \to \mathrm{QCoh}(S) \otimes \mathbf{H}. \tag{6.4}$$

Since $\mathbf{H}$ is dualizable as a DG category (and hence the operation $- \otimes \mathbf{H}$ commutes with limits), the functors (6.4) assemble to a symmetric monoidal functor

$$\mathrm{Rep}(\mathsf{G}) \to \mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})) \otimes \mathbf{H}. \tag{6.5}$$

By adjunction, we obtain a symmetric monoidal functor

$$\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}) \to \mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})). \tag{6.6}$$

6.2.5. We will make the following assumptions, which can be summarized that $\mathbf{H}$ is a particularly well-behaved Tannakian category:

- The following classes of objects in $\mathbf{H}^\heartsuit$ coincide:
  (i) Compact objects;
  (ii) Objects that are sent to compact objects in $\mathrm{Vect}_{\mathsf{e}}$ by $\mathbf{oblv_H}$;
  (iii) Dualizable objects.
- The object $\mathbf{1_H}$ has the following properties:
  (i) It has bounded cohomological dimension;
  (ii) For any $\mathbf{h} \in \mathbf{H}^c \cap \mathbf{H}^\heartsuit$ and all i, the cohomologies of $\mathcal{H}om_{\mathbf{H}}(\mathbf{1_H}, \mathbf{h}) \in \mathrm{Vect}_{\mathsf{e}}$ are finite-dimensional.
- $\mathbf{H}$ is left-complete in its t-structure and the functor
  $$\mathrm{Ind}(\mathbf{H}^c \cap \mathbf{H}^\heartsuit) \to \mathbf{H}^\heartsuit$$
  is an equivalence.

Remark 6.2.6. We note that the last condition can be reformulated as saying that the functor

$$\mathbf{H}^{\mathrm{ren}} \to \mathbf{H}$$

identifies $\mathbf{H}$ with the left completion of $\mathbf{H}^{\mathrm{ren}}$, where the latter is the full subcategory of $\mathbf{H}$ generated by $\mathbf{H}^c \cap \mathbf{H}^\heartsuit$.

6.2.7. Having future needs in mind we record the following:

Lemma 6.2.8. *The unit functor* $\mathrm{Vect}_{\mathsf{e}} \to \mathbf{H}$ *admits a left adjoint (to be denoted* $\mathbf{coinv_H}$*).*

Proof. First, the functor $\mathbf{coinv_H}$ is defined on objects from $\mathbf{H}^c \cap \mathbf{H}^\heartsuit$. Indeed,

$$\mathbf{coinv_H}(h) = (\mathcal{H}om_{\mathbf{H}}(h, \mathbf{1_H}))^\vee.$$

Hence, it is defined on all of $\mathbf{H}^{\mathrm{ren}}$. Hence, it is defined on $\mathbf{H}^{\geq -n}$ for any n. We now claim that for an arbitrary $h \in \mathbf{H}$, the value of $\mathbf{coinv_H}$ on it is given by

$$\lim_n \mathbf{coinv_H}(\tau^{\geq -n}(h)).$$

Indeed, for every m, the m-the cohomology of the system

$$n \mapsto \mathbf{coinv_H}(\tau^{\geq -n}(h))$$

stabilizes (since $\mathbf{coinv_H}$ is right t-exact), and the above object has the required adjunction property by the left-completeness of $\mathbf{H}$.

□

6.2.9. An example of a category $\mathbf{H}$ satisfying the above conditions is $\mathrm{Rep}(\mathsf{H})$, where H is a (finite-dimensional) algebraic group.

Another example, central to this paper, is $\mathbf{H} = \mathrm{QLisse}(X)$. Note that the functor $\mathbf{coinv}_{\mathbf{H}}$ (from the above lemma) is given in the case by $\mathrm{C}^{\cdot}_c(X, -)$.

6.2.10. We will deduce Theorem 6.1.4 from the following more abstract result:

Theorem 6.2.11. *Under the above conditions on* $\mathbf{H}$*, the functor* (6.6) *induces an equivalence*

$$\mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})))^{\vee} \to \underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\vee}. \tag{6.7}$$

Here, for a (not necessarily dualizable) DG category $\mathbf{C}$ we denote

$$\mathbf{C}^{\vee} := \mathrm{Funct}_{\mathrm{cont}}(\mathbf{C}, \mathrm{Vect}_{\mathsf{e}}).$$

Remark 6.2.12. We expect that the functor (6.6) is itself an equivalence (cf. Remark 6.1.6). In fact, in the course of the proof of Theorem 6.2.11 we will show that this is indeed the case for *many* choices of $\mathbf{H}$. However, unfortunately, we were not able to prove this in general (the problem is that we do not a priori know that $\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$ is dualizable).

Further, we do not know whether the conditions in Sect. 6.2.5 can be relaxed for Theorem 6.2.11 to remain valid.

6.3. **The implication Theorem 6.2.11 $\Rightarrow$ Theorem 6.1.4.** Let us show how Theorem 6.2.11 implies Theorem 6.1.4.

6.3.1. Take $\mathbf{H} = \mathrm{QLisse}(X)$. By definition,

$$\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}) \simeq \mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X).$$

Define the symmetric monoidal category $\mathrm{Rep}(\mathsf{G})^{\otimes X}$ to be $\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \mathrm{QLisse}(X))$, so what we defined as an action of $\mathrm{Rep}(\mathsf{G})^{\otimes X}$ on $\mathbf{C}$ really corresponds to an action of the symmetric monoidal category $\mathrm{Rep}(\mathsf{G})^{\otimes X}$ on $\mathbf{C}$.

Thus, the map (6.6) in this case can be interpreted as a map

$$\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \mathrm{QLisse}(X)) \to \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)). \tag{6.8}$$

Under the above identifications, the map (6.3) corresponds to restriction along (6.8).

6.3.2. Thus, if we knew that (6.6) was an equivalence, we would immediately obtain the assertion of Theorem 6.1.4 (without the assumption that $\mathbf{C}$ should be dualizable). Without knowing that that (6.6) is an equivalence, but only that (6.7) is an equivalence, we argue as follows:

It is enough to show that (for $\mathbf{C}$ dualizable) and any finite set I, the functor (6.6) induces an equivalence

$$\mathrm{Maps}(\mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}))^{\otimes I}, \mathrm{End}(\mathbf{C})) \to \mathrm{Maps}(\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\otimes I}, \mathrm{End}(\mathbf{C})), \tag{6.9}$$

where $\mathrm{Maps} = \mathrm{Maps}_{\mathrm{DGCat}}$.

6.3.3. Note that in the setting of Sect. 6.2.1, for a finite set I, we have a tautological identification

$$\underline{\mathrm{coHom}}(A, B^{\vee})^{\otimes I} \simeq \underline{\mathrm{coHom}}(A^{\otimes I}, B^{\vee}).$$

In particular,

$$\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\otimes I} \simeq \underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}^I), \mathbf{H}).$$

Similarly,

$$\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^I \simeq \mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}^I), \mathbf{H}). \tag{6.10}$$

As we have seen (in Sect. 5.7), the category $\mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}))$ is dualizable, hence the functor

$$\mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}))^{\otimes I} \to \mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^I)$$

is an equivalence. Thus, (6.10) implies

$$\mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H}))^{\otimes I} \simeq \mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}^I),\mathbf{H})).$$

We have a commutative diagram

$$\begin{tikzcd}
\mathrm{Funct}_{\mathrm{cont}}(\mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H}))^{\otimes I},\mathrm{Vect}) \arrow[r] & \mathrm{Funct}_{\mathrm{cont}}(\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})^{\otimes I},\mathrm{Vect}) \\
\mathrm{Funct}_{\mathrm{cont}}(\mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}^I),\mathbf{H})),\mathrm{Vect}) \arrow[r] \arrow[u, "\sim"] & \mathrm{Funct}_{\mathrm{cont}}(\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}^I),\mathbf{H}),\mathrm{Vect}). \arrow[u, "\sim"']
\end{tikzcd}$$

By Theorem 6.2.11, applied to G^I, we obtain that the functor

$$\mathrm{Funct}_{\mathrm{cont}}(\mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H}))^{\otimes I},\mathrm{Vect}) \to \mathrm{Funct}_{\mathrm{cont}}(\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})^{\otimes I},\mathrm{Vect})$$

is an equivalence. Hence, the functor

$$\mathrm{Funct}_{\mathrm{cont}}(\mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H}))^{\otimes I},\mathbf{D}) \to \mathrm{Funct}_{\mathrm{cont}}(\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})^{\otimes I},\mathbf{D})$$

is an equivalence for any *dualizable* DG category $\mathbf{D}$.

This implies that the functor (6.9) is an equivalence, since for $\mathbf{C}$ dualizable, $\mathbf{D} := \mathrm{End}(\mathbf{C})$ is also dualizable as a DG category.

6.4. **A rigidified version.** Recall that along with $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$, it was convenient to consider its variant $\mathrm{LocSys}^{\mathrm{restr},\mathrm{rigid}_x}_{\check{G}}(X)$, where we added the data of rigidification at a point. The advantage of $\mathrm{LocSys}^{\mathrm{restr},\mathrm{rigid}_x}_{\check{G}}(X)$ was that it was "almost" an affine scheme.

In this subsection we will introduce the corresponding rigidified objects on both sides of (6.6). The reason for doing so is that for some manipulations they will be more convenient to work with than the original ones.

6.4.1. Let $\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})^{\mathrm{rigid}}$ be the prestack that sends an affine scheme S to the space of symmetric monoidal functors

$$\mathrm{Rep}(\mathsf{G}) \to \mathrm{QCoh}(S)\otimes\mathbf{H}, \tag{6.11}$$

equipped with the identification of the composition

$$\mathrm{Rep}(\mathsf{G}) \to \mathrm{QCoh}(S)\otimes\mathbf{H} \overset{\mathrm{Id}_{\mathrm{QCoh}(S)}\otimes\mathbf{oblv}_{\mathbf{H}}}{\longrightarrow} \mathrm{QCoh}(S)$$

with the forgetful functor

$$\mathrm{Rep}(\mathsf{G}) \overset{\mathbf{oblv}_{\mathsf{G}}}{\to} \mathrm{Vect}_{\mathsf{e}} \overset{\mathcal{O}_S}{\longrightarrow} \mathrm{QCoh}(S),$$

as symmetric monoidal functors. (Note that the latter identification implies that the functor (6.11) is right t-exact.)

In other words,

$$\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})^{\mathrm{rigid}} = \mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H}) \underset{\mathrm{pt}/\mathsf{G}}{\times} \mathrm{pt},$$

where the map

$$\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H}) \to \mathrm{pt}/\mathsf{G} \simeq \mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}),\mathrm{Vect}_{\mathsf{e}})$$

is given by $\mathbf{oblv}_{\mathbf{H}}$.

The group G acts naturally on $\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})^{\mathrm{rigid}}$, and we have

$$\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H}) \simeq \mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})^{\mathrm{rigid}}/\mathsf{G}.$$

So,

$$\mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})^{\mathrm{rigid}}) \simeq \mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})) \underset{\mathrm{Rep}(\mathsf{G})}{\otimes} \mathrm{Vect}_{\mathsf{e}}.$$

6.4.2. Composition with the fiber functor $\mathbf{oblv_H}$ defines a symmetric monoidal functor

$$\mathrm{Rep}(\mathsf{G}) \simeq \underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \mathrm{Vect}_{\mathsf{e}}) \to \underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}).$$

Denote

$$\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\mathrm{rigid}} := \underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}) \underset{\mathrm{Rep}(\mathsf{G})}{\otimes} \mathrm{Vect}_{\mathsf{e}}.$$

By construction, for a symmetric monoidal category $\mathbf{A}$, the datum of a symmetric monoidal functor

$$\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\mathrm{rigid}} \to \mathbf{A}$$

is equivalent to that of a symmetric monoidal functor

$$\mathrm{Rep}(\mathsf{G}) \to \mathbf{A} \otimes \mathbf{H},$$

equipped with an identification of the composition

$$\mathrm{Rep}(\mathsf{G}) \to \mathbf{A} \otimes \mathbf{H} \overset{\mathrm{Id}_{\mathbf{A}} \otimes \mathbf{oblv_H}}{\longrightarrow} \mathbf{A},$$

with the forgetful functor

$$\mathrm{Rep}(\mathsf{G}) \overset{\mathbf{oblv}_{\mathsf{G}}}{\to} \mathrm{Vect}_{\mathsf{e}} \overset{\mathbf{1_A}}{\longrightarrow} \mathbf{A}.$$

6.4.3. As in Sect. 6.2.3, we have a symmetric monoidal functor

$$\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\mathrm{rigid}} \to \mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\mathrm{rigid}}). \tag{6.12}$$

Since pt/G is 1-affine (see Sect. 5.7.3), the functor

$$- \underset{\mathrm{Rep}(\mathsf{G})}{\otimes} \mathrm{Vect}_{\mathsf{e}} : \mathrm{Rep}(\mathsf{G})\text{-}\mathbf{mod} \to \mathrm{DGCat}$$

is conservative.

Hence, the statement of Theorem 6.2.11 is equivalent to the following one:

Theorem 6.4.4. *Under the assumptions on* $\mathbf{H}$ *specified in Sect. 6.2.5, the functor* (6.12) *induces an equivalence*

$$(\mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\mathrm{rigid}}))^{\vee} \to ((\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\mathrm{rigid}})^{\vee}. \tag{6.13}$$

7. Some particular cases

In this section we will establish some variants and particular cases of Theorem 6.2.11. These ideas will be used later in Sect. 8 for the proof of the case of Theorem 6.2.11 when $\mathbf{H} = \mathrm{Rep}(\mathsf{H})$, where H is a (finite-dimensional) algebraic group.

7.1. **Proof in the Betti context.** In subsection we let $\mathcal{X}$ be an object of Spc. We will assume that $\mathcal{X}$ is connected, and we will choose a base point $x \in X$.

We will quote the result from [GKRV] that says that (6.6) is an equivalence for $\mathbf{H} = \mathrm{Vect}_{\mathsf{e}}^{\mathcal{X}}$ (and we will later reprove it, see Sect. 7.7). Using the validity of the equivalence (6.6) for $\mathbf{H} = \mathrm{Vect}_{\mathsf{e}}^{\mathcal{X}}$ we will formally deduce the validity of the equivalence (6.6) for $\mathbf{H} = \mathrm{QLisse}(X)$, where X is the geometric realization of $\mathcal{X}$ (thereby proving Theorem 6.1.4 in the Betti context). The method of deduction will be used again in Sect. 8.

7.1.1. Note that the (symmetric monoidal) category

$$\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \mathrm{Vect}_{\mathsf{e}}^{\mathcal{X}})$$

is the same as what in [GKRV, Sect. 1.2.1] was denoted $\mathrm{Rep}(\mathsf{G})^{\otimes \mathcal{X}}$.

Note also that for $\mathbf{H} := \mathrm{Vect}_{\mathsf{e}}^{\mathcal{X}}$, the prestack $\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$ identifies with $\mathrm{LocSys}_{\mathsf{G}}(\mathcal{X})$.

Under these identifications, the map (6.6) is the map [GKRV, Equation (1.19)]. Hence, from [GKRV, Theorem 1.5.5] we obtain:

Theorem 7.1.2. *The functor* (6.6) *is an equivalence for* $\mathbf{H} = \mathrm{Vect}_{\mathsf{e}}^{\mathcal{X}}$.

Remark 7.1.3. Note that the category $\mathbf{H} = \mathrm{Vect}_{\mathsf{e}}^{\mathcal{X}}$ is *not* Tannakian in that it is very far from satisfying the assumptions of Theorem 6.2.11.

So the assertion of Theorem 7.1.2 is *not* a particular case of Theorem 6.2.11.

7.1.4. From now on we will assume that $\mathcal{X}$ is compact. Let X be the geometric realization of $\mathcal{X}$.

We will now show how Theorem 7.1.2 implies Theorem 6.1.4 for $\mathbf{H} = \mathrm{QLisse}(X)$ in the Betti context.

Remark 7.1.5. Note that the validity of Theorem 6.1.4 in the Betti context implies its validity in the de Rham context for X proper[17] (by Lefschetz principle, as Riemann-Hilbert identifies the symmetric monoidal categories $\mathrm{QLisse}(X)$ in the two contexts once $k = \mathsf{e} = \mathbb{C}$). However, in order to deduce Theorem 6.1.4 in the ℓ-adic context we will have to work much harder.

7.2. **The implication Theorem 7.1.2 $\Rightarrow$ Theorem 6.1.4.** Let $\mathbf{C}$ be a dualizable DG category.

We wish to show that the map

$$\{\text{Actions of } \mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)) \text{ on } \mathbf{C}\} \to \{\text{Actions of } \mathrm{Rep}(\mathsf{G})^{X} \text{ on } \mathbf{C}\}. \tag{7.1}$$

is an equivalence.

We know (by Theorem 7.1.2) that the map

$$\{\text{Actions of } \mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}(\mathcal{X})) \text{ on } \mathbf{C}\} \to \{\text{Actions of } \mathrm{Rep}(\mathsf{G})^{\mathcal{X}} \text{ on } \mathbf{C}\}. \tag{7.2}$$

is an equivalence.

7.2.1. Consider the commutative square

$$\begin{array}{ccc} \{\text{Actions of } \mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}(X)) \text{ on } \mathbf{C}\} & \xrightarrow[\sim]{(7.2)} & \{\text{Actions of } \mathrm{Rep}(\mathsf{G})^{\mathcal{X}} \text{ on } \mathbf{C}\} \\ \uparrow & & \uparrow \\ \{\text{Actions of } \mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)) \text{ on } \mathbf{C}\} & \xrightarrow[(7.1)]{} & \{\text{Actions of } \mathrm{Rep}(\mathsf{G})^{X} \text{ on } \mathbf{C}\}. \end{array} \tag{7.3}$$

We need to show that the bottom horizontal arrow is an equivalence. We will show that both vertical arrows are fully faithful, and that their essential images match up under the equivalence given by the top horizontal arrow.

[17] And for any X, if we take $\mathrm{Shv}(X)^{\mathrm{constr}}$ to mean the *regular* holonomic category.

7.2.2. Recall that an action of $\mathrm{Rep}(\mathsf{G})^{\otimes \mathcal{X}}$ on $\mathbf{C}$ consists of a compatible family of functors

$$\mathrm{Rep}(\mathsf{G}) \to \mathrm{End}(\mathbf{C}) \otimes \mathrm{Vect}_{\mathsf{e}}^{\mathcal{X}^I}, \quad I \in \mathrm{fSet},$$

see [GKRV, Proposition 1.7.2].

Recall also that for any finite set I the functors

$$\mathrm{QLisse}(X)^{\otimes I} \to \mathrm{QLisse}(X^I)$$

and

$$\mathrm{QLisse}(X^I) \to \mathrm{Shv}_{\mathrm{loc.const.}}(X^I) \simeq \mathrm{Vect}_{\mathsf{e}}^{\mathcal{X}^I}$$

are fully faithful (the latter, by Proposition 3.5.2). Hence, so is the composition

$$\mathrm{QLisse}(X)^{\otimes I} \to \mathrm{Vect}_{\mathsf{e}}^{\mathcal{X}^I}.$$

Therefore, since $\mathbf{C}$ is dualiazable, the functor

$$\mathrm{End}(\mathbf{C}) \otimes \mathrm{QLisse}(X)^{\otimes I} \to \mathrm{End}(\mathbf{C}) \otimes \mathrm{Vect}_{\mathsf{e}}^{\mathcal{X}^I}$$

is also fully faithful.

This implies that the right vertical arrow in (7.3) is fully faithful.

7.2.3. For the proof that the left vertical arrow in (7.3) is fully faithful we will consider the following general situation.

Let $\mathcal{Y}_1 \to \mathcal{Y}_2$ be as in Remark 3.2.5, i.e., $\mathcal{Y}_2$ is an algebraic stack and $\mathcal{Y}_1$ is the disjoint union of formal completions of pairwise non-intersecting closed subfunctors $\mathcal{Z}_\alpha$ of $\mathcal{Y}_2$. Consider the (symmetric monoidal) restriction functor

$$\mathrm{QCoh}(\mathcal{Y}_2) \to \mathrm{QCoh}(\mathcal{Y}_1). \tag{7.4}$$

Assume that $\mathcal{Y}_2$ is locally almost of finite type (this assumption is needed in order to apply [GR3, Proposition 7.1.3]).

Then, by *loc. cit.*, the functor (7.4), viewed as a functor between plain DG categories, admits a fully faithful left adjoint. This implies that for any (symmetric monoidal) DG category $\mathbf{D}$, restriction along (7.4) defines a fully faithful embedding

$$\mathrm{Funct}_{\mathrm{cont}}(\mathrm{QCoh}(\mathcal{Y}_1), \mathbf{D}) \to \mathrm{Funct}_{\mathrm{cont}}(\mathrm{QCoh}(\mathcal{Y}_2), \mathbf{D}), \tag{7.5}$$

whose essential image consists of those functors that vanish on the full subcategory of $\mathrm{QCoh}(\mathcal{Y}_1)$ consisting of objects $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y}_1)$ for which their *-restrictions to all $\mathcal{Z}_\alpha$ vanish.

This formally implies that for any monoidal category $\mathbf{A}$ the map

$$\mathrm{Maps}_{\mathrm{DGCat}^{\mathrm{Mon}}}(\mathrm{QCoh}(\mathcal{Y}_1), \mathbf{A}) \to \mathrm{Maps}_{\mathrm{DGCat}^{\mathrm{Mon}}}(\mathrm{QCoh}(\mathcal{Y}_2), \mathbf{A})$$

is fully faithful, with the essential image described as above.

7.2.4. We apply the discussion in Sect. 7.2.5 to the embedding

$$\mathrm{LocSys}_{\mathsf{G}}(X)^{\mathrm{restr}} \hookrightarrow \mathrm{LocSys}_{\mathsf{G}}(X), \tag{7.6}$$

see Corollary 3.5.11. We obtain that the left vertical arrow in (7.3) is fully faithful.

Hence, it remains to show that the essential images of the vertical arrows in (7.3) match under the equivalence given by the top horizontal arrow.

Thus, let us be given an action of $\mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}(X))$ on $\mathbf{C}$, such that the resulting action of $\mathrm{Rep}(\mathsf{G})^{\otimes \mathcal{X}}$ factors through an action of $\mathrm{Rep}(\mathsf{G})^{\otimes X}$. We wish to show that the action of $\mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}(X))$ factors through $\mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X))$.

7.2.5. Let us return to the setting of Sect. 7.2.3.

Assume for a moment that $\mathcal{Y}_2$ is an affine scheme S. Then the subcategory of $\mathrm{QCoh}(S)$ consisting of objects whose *-restrictions to all $\mathcal{Z}_\alpha$ vanish is generated by sky-scrapers of the generic points of irreducible closed subschemes of S that are *not* contained in one of the $\mathcal{Z}_\alpha$.

Hence, in this case, the essential image of (7.5) consists of those functors

$$\mathsf{F} : \mathrm{QCoh}(S) \to \mathbf{D},$$

such that

$$\mathsf{F}(\iota_*(\mathsf{e}')) = 0$$

for all field-valued points

$$\mathrm{Spec}(\mathsf{e}') \xrightarrow{\iota} S$$

that do *not* factor through one of the $\mathcal{Z}_\alpha$'s.

Let now $\mathrm{QCoh}(S)$ act on a DG category $\mathbf{C}$. We obtain that this action factors through $\mathrm{QCoh}(\mathcal{Y}_1)$ if and only if for every (e', ι) as above, the tensor product

$$\mathrm{Vect}_{\mathsf{e}'} \underset{\mathrm{QCoh}(S)}{\otimes} \mathbf{C}$$

vanishes.

Let now $\mathcal{Y}_2$ be an algebraic stack of the form S/H, where S is an affine scheme and H is algebraic group. For a $\mathrm{QCoh}(\mathcal{Y}_2)$-module category $\mathbf{C}$, consider

$$\mathrm{Vect}_{\mathsf{e}} \underset{\mathrm{Rep}(\mathsf{G})}{\otimes} \mathbf{C}$$

as acted on by $\mathrm{QCoh}(S)$.

We obtain that the above criterion for when the given $\mathrm{QCoh}(\mathcal{Y}_2)$-action factors through $\mathrm{QCoh}(\mathcal{Y}_1)$ remains valid: it suffices to show that the products

$$\mathrm{Vect}_{\mathsf{e}'} \underset{\mathrm{QCoh}(\mathcal{Y}_2)}{\otimes} \mathbf{C}$$

vanish for all field-valued points

$$\mathrm{Spec}(\mathsf{e}') \xrightarrow{\iota} \mathcal{Y}_2$$

that do *not* factor through one of the $\mathcal{Z}_\alpha$'s.

7.2.6. We apply the above discussion to (7.6).

Thus, we have a monoidal functor

$$\mathsf{F} : \mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}(X)) \to \mathrm{End}(\mathbf{C}),$$

and we are assuming that the corresponding functor

$$\widetilde{\mathsf{F}} : \mathrm{Rep}(\mathsf{G}) \to \mathrm{End}(\mathbf{C}) \otimes \mathrm{Shv}_{\mathrm{loc.const.}}(X)$$

takes values in

$$\mathrm{End}(\mathbf{C}) \otimes \mathrm{QLisse}(X) \subset \mathrm{End}(\mathbf{C}) \otimes \mathrm{Shv}_{\mathrm{loc.const.}}(X).$$

We wish to show that F factors via

$$\mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}(X)) \twoheadrightarrow \mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}(X)^{\mathrm{restr}}).$$

By Sect. 7.2.5 we need to show that given a map

$$\iota : \mathrm{Spec}(\mathsf{e}') \to \mathrm{LocSys}_{\mathsf{G}}(X),$$

if the category

$$\mathbf{C}' := \mathrm{Vect}_{\mathsf{e}'} \underset{\mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}(X))}{\otimes} \mathbf{C}$$

is non-zero, then ι factors through $\mathrm{LocSys}_{\mathsf{G}}(X)^{\mathrm{restr}}$.

7.2.7. Let F' denote the composition of F with

$$\mathrm{End}(\mathbf{C}) \to \mathrm{End}(\mathbf{C}'),$$

and consider the corresponding functor

$$\widetilde{\mathsf{F}}' : \mathrm{Rep}(\mathsf{G}) \to \mathrm{End}(\mathbf{C}') \otimes \mathrm{Shv}_{\mathrm{loc.const.}}(X).$$

By construction, F' factors as

$$\mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}(X)) \xrightarrow{\iota^*} \mathrm{Vect}_{\mathsf{e}'} \to \mathrm{End}(\mathbf{C}')$$

and hence $\widetilde{\mathsf{F}}'$ factors as

$$\mathrm{Rep}(\mathsf{G}) \to \mathrm{Vect}_{\mathsf{e}'} \otimes \mathrm{Shv}_{\mathrm{loc.const.}}(X) \to \mathrm{End}(\mathbf{C}') \otimes \mathrm{Shv}_{\mathrm{loc.const.}}(X). \tag{7.7}$$

Now, since $\widetilde{\mathsf{F}}$ takes values in $\mathrm{End}(\mathbf{C}) \otimes \mathrm{QLisse}(X)$, we obtain that $\widetilde{\mathsf{F}}'$ takes values in

$$\mathrm{End}(\mathbf{C}') \otimes \mathrm{QLisse}(X) \subset \mathrm{End}(\mathbf{C}') \otimes \mathrm{Shv}_{\mathrm{loc.const.}}(X).$$

Let us denote the first arrow in (7.7) by Φ. We need to show that it takes values in

$$\mathrm{Vect}_{\mathsf{e}'} \otimes \mathrm{QLisse}(X) \subset \mathrm{Vect}_{\mathsf{e}'} \otimes \mathrm{Shv}_{\mathrm{loc.const.}}(X),$$

assuming that $\mathbf{C}'$ is non-zero.

7.2.8. Note that since $\mathbf{C}$ is dualizable, then so is $\mathbf{C}'$ as a plain DG category, and hence also as a DG category over e'. Hence, if $\mathbf{C}' \neq 0$, then there exists a non-zero continuous $\mathrm{Vect}_{\mathsf{e}'}$-linear functor $\mathsf{T} : \mathbf{C}' \to \mathrm{Vect}_{\mathsf{e}'}$. Pick some $\mathbf{c}' \in \mathbf{C}'$ such that $\mathsf{T}(\mathbf{c}') \neq 0$.

For $V \in \mathrm{Rep}(\mathsf{G})$ consider the object

$$\widetilde{\mathsf{F}}'(V)(\mathbf{c}') \in \mathbf{C}' \otimes \mathrm{Shv}_{\mathrm{loc.const.}}(X).$$

Note that we have

$$\mathsf{T}(\mathbf{c}') \underset{\mathsf{e}'}{\otimes} \Phi(V) \simeq (\mathsf{T} \otimes \mathrm{Id})(\widetilde{\mathsf{F}}'(V)(\mathbf{c}'))$$

as objects in

$$\mathrm{Vect}_{\mathsf{e}'} \otimes \mathrm{Shv}_{\mathrm{loc.const.}}(X).$$

Since $\mathsf{T}(\mathbf{c}') \neq 0$, it suffices to show that

$$(\mathsf{T} \otimes \mathrm{Id})(\widetilde{\mathsf{F}}'(V)(\mathbf{c}')) \in \mathrm{Vect}_{\mathsf{e}'} \otimes \mathrm{QLisse}(X). \tag{7.8}$$

Now, since $\widetilde{\mathsf{F}}'$ takes values in $\mathrm{End}(\mathbf{C}') \otimes \mathrm{QLisse}(X)$, we have

$$\widetilde{\mathsf{F}}'(V)(\mathbf{c}') \in \mathbf{C}' \otimes \mathrm{QLisse}(X),$$

and hence (7.8).

7.3. **Complements: Rham and Betti spectral actions.** In this subsection we will make a brief digression, and consider the de Rham or Betti contexts, in which the "usual" $\mathrm{LocSys}_{\mathsf{G}}(X)$ is defined. Let us be given a category $\mathbf{C}$ equipped with an action of $\mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}(X))$.

We will explicitly describe the full subcategory

$$\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)) \underset{\mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}(X))}{\otimes} \mathbf{C} \subset \mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}(X)) \underset{\mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}(X))}{\otimes} \mathbf{C} = \mathbf{C},$$

where we view

$$\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \simeq \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}$$

as a colocalization of $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))$.

7.3.1. Let us return to the context of Sect. 3. I.e., we will assume that our sheaf theory is either de Rham or Betti, so we have a well-defined algebraic stack $\mathrm{LocSys}_{\mathsf{G}}(X)$.

For a given $V \in \mathrm{Rep}(\mathsf{G})$ we have the tautological object

$$\mathcal{E}_V \in \mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}(X)) \otimes \mathrm{Shv}(X).$$

Let $\mathbf{C}$ be a DG category, equipped with an action of $\mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}(X))$. In particular, for $V \in \mathrm{Rep}(\mathsf{G})$, we have the Hecke functor

$$\mathrm{H}(V, -) : \mathbf{C} \to \mathbf{C} \otimes \mathrm{Shv}(X),$$

corresponding to the action of the object $\mathcal{E}_V$ above.

7.3.2. Let is first specialize to the Betti context.

Let

$$\mathbf{C}^{\mathrm{fin.mon.}} \subset \mathbf{C}$$

be the full subcategory consisting of objects $\mathbf{c} \in \mathbf{C}$, for which

$$\mathrm{H}(V, \mathbf{c}) \subset \mathbf{C} \otimes \mathrm{QLisse}(X) \subset \mathbf{C} \otimes \mathrm{Shv}_{\mathrm{loc.const.}}(X).$$

As in [GKRV, Proposition C.2.5], one shows that the category $\mathbf{C}^{\mathrm{fin.mon.}}$ is stable under the action of $\mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}(X))$ and it carries an action of $\mathrm{Rep}(\mathsf{G})^{\otimes X}$.

7.3.3. The argument in Sects. 7.2.6-7.2.8 proves the following:

Proposition 7.3.4. *The full subcategory* $\mathbf{C}^{\mathrm{fin.mon.}} \subset \mathbf{C}$ *equals*

$$\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)) \underset{\mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}(X))}{\otimes} \mathbf{C} \subset \mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}(X)) \underset{\mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}(X))}{\otimes} \mathbf{C} = \mathbf{C},$$

where we view

$$\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \simeq \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}$$

as a colocalization of $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))$.

7.3.5. Let us now specialize to the de Rham context.

Let

$$\mathbf{C}^{\mathrm{lisse}} \subset \mathbf{C}$$

be the full subcategory consisting of objects $\mathbf{c} \in \mathbf{C}$, for which

$$\mathrm{H}(V, \mathbf{c}) \subset \mathbf{C} \otimes \mathrm{QLisse}(X) \subset \mathbf{C} \otimes \mathrm{D\text{-}mod}(X).$$

As in [GKRV, Proposition C.2.5], one shows that the category $\mathbf{C}^{\mathrm{lisse}}$ is stable under the action of $\mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}(X))$ and it carries an action of $\mathrm{Rep}(\mathsf{G})^{\otimes X}$.

7.3.6. The argument in Sects. 7.2.6-7.2.8 is still applicable to the ind-closed embedding

$$\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X) \to \mathrm{LocSys}_{\mathsf{G}}(X)$$

and we obtain:

Proposition 7.3.7. *The full subcategory* $\mathbf{C}^{\mathrm{lisse}} \subset \mathbf{C}$ *equals*

$$\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)) \underset{\mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}(X))}{\otimes} \mathbf{C} \subset \mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}(X)) \underset{\mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}(X))}{\otimes} \mathbf{C} = \mathbf{C},$$

where we view

$$\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \simeq \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}$$

as a colocalization of $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))$.

A statement somewhat weaker than Proposition 7.3.7 appeared in [GKRV] as Conjecture C.5.5 of *loc. cit.*

7.4. **The case of Lie algebras.** With are now going to establish a variant of Theorem 7.1.2, where instead of an object $\mathcal{X} \in \mathrm{Spc}$ we have a Lie algebra $\mathsf{h} \in \mathrm{LieAlg}(\mathrm{Vect}^{\leq 0}_{\mathsf{e}})$.

7.4.1. Let $\mathfrak{g}$ denote the Lie algebra of G. Consider the prestack, denoted $\mathbf{Maps}_{\mathrm{Lie}}(\mathsf{h}, \mathsf{g})$, that sends an affine scheme S to the space of maps of Lie algebras

$$\mathsf{h} \to \mathsf{g} \otimes \mathcal{O}_S.$$

We claim:

Proposition 7.4.2.

(a) *The prestack* $\mathbf{Maps}_{\mathrm{Lie}}(\mathsf{h}, \mathsf{g})$ *is an affine scheme.*

(b) *The scheme* $\mathbf{Maps}_{\mathrm{Lie}}(\mathsf{h}, \mathsf{g})$ *is almost of finite type if* h *is finite-dimensional in each degree.*

For the proof of Proposition 7.4.2 (as well as that of Theorem 7.4.5 below) we recall that any object in $\mathrm{LieAlg}(\mathrm{Vect}_{\mathsf{e}}^{\leq 0})$ can be written as a sifted colimit of the objects of the form

$$\mathbf{free}_{\mathrm{Lie}}(V), \quad V \in \mathrm{Vect}_{\mathsf{e}}^{\leq 0}. \tag{7.9}$$

Proof of Proposition 7.4.2. Since a limit of affine schemes is an affine scheme, for the proof of point (a) we can assume that h is of the form (7.9). Then

$$\mathbf{Maps}_{\mathrm{Lie}}(\mathbf{free}_{\mathrm{Lie}}(V), \mathsf{g})$$

is the affine scheme

$$\mathrm{Spec}(\mathrm{Sym}(V \otimes \mathsf{g}^*)).$$

For the proof of point (b) we argue as follows. Assume that h is finite-dimensional in each degree. Then ${}^{\mathrm{cl}}\mathbf{Maps}_{\mathrm{Lie}}(\mathsf{h}, \mathsf{g})$ is the classical scheme that classifies maps of (classical) Lie algebras $H^0(\mathsf{h}) \to \mathsf{g}$; in particular it is a closed subscheme in the affine space of the vector space Hom from $H^0(\mathsf{h})$ to $\mathfrak{g}$, i.e.,

$$\mathrm{Tot}(\mathrm{Hom}_{\mathrm{Vect}_{\mathsf{e}}}(\mathbf{oblv}_{\mathrm{Lie}}(H^0(\mathsf{h})), \mathbf{oblv}_{\mathrm{Lie}}(\mathsf{g}))),$$

and so is of finite type. By [GR2, Chapter 1, Theorem 9.1.2], it remains to show that for a classical scheme S of finite type and an S-point F of $\mathbf{Maps}_{\mathrm{Lie}}(\mathsf{h}, \mathsf{g})$, the cotangent space

$$T^*_{\mathsf{F}}(\mathbf{Maps}_{\mathrm{Lie}}(\mathsf{h}, \mathsf{g})) \in \mathrm{QCoh}(S)^{\leq 0}$$

has coherent cohomologies. However, we have:

$$T^*_{\mathsf{F}}(\mathbf{Maps}_{\mathrm{Lie}}(\mathsf{h}, \mathsf{g})) \simeq \mathrm{coFib}\left(\mathsf{g} \otimes \mathcal{O}_S \to \mathrm{C}_\cdot(\mathsf{h}, \mathsf{g} \otimes \mathcal{O}_S)\right)[-1],$$

where $\mathsf{g} \otimes \mathcal{O}_S$ is viewed as a h-module via F. This implies the required assertion as $\mathrm{C}_\cdot(\mathsf{h}, -)$ can be computed by the standard Chevalley complex.

$\square$

7.4.3. Let us place ourselves again in the context of Sect. 6.2.1 with $\mathbf{O} = \mathrm{DGCat}$, $A := \mathrm{Rep}(\mathsf{G})$ and $B^\vee = \mathsf{h}\text{-mod}$, viewed as a symmetric monoidal category with respect to the (usual) operation of tensor product of h-modules.

Note that the functor

$$M_1, M_2 \mapsto \mathrm{C}_\cdot(\mathsf{h}, M_1 \otimes M_2), \quad \mathsf{h}\text{-mod} \otimes \mathsf{h}\text{-mod} \to \mathrm{Vect}_{\mathsf{e}}$$

defines an identification $\mathsf{h}\text{-mod} \simeq \mathsf{h}\text{-mod}^\vee$ as DG categories.

Consider the (symmetric monoidal) category

$$\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \mathsf{h}\text{-mod}).$$

7.4.4. We have a naturally defined adjoint action of G on $\mathbf{Maps}_{\mathrm{Lie}}(\mathfrak{h},\mathfrak{g})$ and consider the quotient

$$\mathbf{Maps}_{\mathrm{Lie}}(\mathfrak{h},\mathfrak{g})/\operatorname{Ad}(\mathsf{G}).$$

The (symmetric monoidal) restriction functor

$$\operatorname{Rep}(\mathsf{G}) \to \mathfrak{g}\text{-mod} \tag{7.10}$$

gives rise to a symmetric monoidal functor

$$\operatorname{Rep}(\mathsf{G}) \to \operatorname{QCoh}(S) \otimes \mathfrak{h}\text{-mod}$$

for every affine scheme S and an S-point of $\mathbf{Maps}_{\mathrm{Lie}}(\mathfrak{h},\mathfrak{g})/\operatorname{Ad}(\mathsf{G})$.

Since $\mathfrak{h}$-mod is dualizable as a category, the above collection of functors give rise to a functor

$$\operatorname{Rep}(\mathsf{G}) \to \operatorname{QCoh}(\mathbf{Maps}_{\mathrm{Lie}}(\mathfrak{h},\mathfrak{g})/\operatorname{Ad}(\mathsf{G})) \otimes \mathfrak{h}\text{-mod}. \tag{7.11}$$

I.e., we obtain a (symmetric monoidal) functor

$$\underline{\operatorname{coHom}}(\operatorname{Rep}(\mathsf{G}), \mathfrak{h}\text{-mod}) \to \operatorname{QCoh}(\mathbf{Maps}_{\mathrm{Lie}}(\mathfrak{h},\mathfrak{g})/\operatorname{Ad}(\mathsf{G})) \tag{7.12}$$

We will prove:

Theorem 7.4.5. *The functor* (7.12) *is an equivalence.*

7.5. **Consequences and interpretations of Theorem 7.4.5.** Before we prove Theorem 7.4.5 we will discuss some of its consequences.

7.5.1. First, the 1-affineness of $\operatorname{pt}/\mathsf{G}$ implies that the statement of Theorem 7.4.5 is equivalent to the following one:

Theorem 7.5.2. *For a symmetric monoidal category* $\mathbf{A}$*, restriction along* (7.11) *defines an equivalence from the space of symmetric monoidal functors*

$$\operatorname{QCoh}(\mathbf{Maps}_{\mathrm{Lie}}(\mathfrak{h},\mathfrak{g})) \to \mathbf{A} \tag{7.13}$$

to the space of symmetric monoidal functors

$$\operatorname{Rep}(\mathsf{G}) \to \mathbf{A} \otimes \mathfrak{h}\text{-mod}, \tag{7.14}$$

equipped with an identification of the composition

$$\operatorname{Rep}(\mathsf{G}) \to \mathbf{A} \otimes \mathfrak{h}\text{-mod} \overset{\operatorname{Id}_{\mathbf{A}} \otimes \mathbf{oblv}_{\mathfrak{h}}}{\longrightarrow} \mathbf{A}$$

with

$$\operatorname{Rep}(\mathsf{G}) \overset{\mathbf{oblv}_{\mathsf{G}}}{\longrightarrow} \operatorname{Vect}_{\mathsf{e}} \overset{\mathbf{1}_{\mathbf{A}}}{\to} \mathbf{A}.$$

Taking $\mathbf{A}$ to be $\operatorname{QCoh}(S)$, for an affine scheme S, from Theorems 7.4.5 and 7.5.2, we obtain:

Corollary 7.5.3. *Restriction along* (7.10) *identifies:*

(i) *The space of maps* $S \to \mathbf{Maps}_{\mathrm{Lie}}(\mathfrak{h},\mathfrak{g})$ *with the space of symmetric monoidal functors*

$$\operatorname{Rep}(\mathsf{G}) \to \operatorname{QCoh}(S) \otimes \mathfrak{h}\text{-mod},$$

equipped with an identification of the composition

$$\operatorname{Rep}(\mathsf{G}) \to \operatorname{QCoh}(S) \otimes \mathfrak{h}\text{-mod} \overset{\mathbf{oblv}_{\mathfrak{h}}}{\to} \operatorname{QCoh}(S)$$

with

$$\operatorname{Rep}(\mathsf{G}) \overset{\mathbf{oblv}_{\mathsf{G}}}{\longrightarrow} \operatorname{Vect}_{\mathsf{e}} \overset{\mathcal{O}_S}{\to} \mathcal{O}_S.$$

(ii) *The space of maps* $S \to \mathbf{Maps}_{\mathrm{Lie}}(\mathfrak{h},\mathfrak{g})/\mathsf{G}$ *with the space of right t-exact symmetric monoidal functors*

$$\operatorname{Rep}(\mathsf{G}) \to \operatorname{QCoh}(S) \otimes \mathfrak{h}\text{-mod}.$$

Note that the last corollary can be reformulated as follows:

Corollary 7.5.4. *The naturally defined maps*

$$\mathbf{Maps}_{\mathrm{Lie}}(\mathsf{h}, \mathsf{g}) \to \mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathsf{h}\text{-mod})^{\mathrm{rigid}}$$

and

$$\mathbf{Maps}_{\mathrm{Lie}}(\mathsf{h}, \mathsf{g})/\mathsf{G} \to \mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathsf{h}\text{-mod})$$

are isomorphisms.

Combining Theorem 7.4.5 and Corollary 7.5.4 we obtain:

Corollary 7.5.5. *The map* (6.6) *is an equivalence for* $\mathbf{H} = \mathsf{h}\text{-mod}$.

Remark 7.5.6. Note that in a way parallel to Remark 7.1.3, the symmetric monoidal category category $\mathbf{H} = \mathsf{h}\text{-mod}$ is *not* Tannakian (being very far from satisfying the assumptions of Theorem 6.2.11).

So the assertion of Corollary 7.5.5 is *not* a particular case of Theorem 6.2.11.

That said, when $H^0(\mathsf{h})$ is nilpotent, Corollary 7.5.5 is a particular case of Theorem 7.1.2: indeed, the theory of rational homotopy type implies that there exists a pointed space $\mathcal{X}$ such that the pair $(\mathrm{Vect}_{\mathsf{e}}^{\mathcal{X}}, \mathrm{ev}_x)$ is equivalent to $(\mathsf{h}\text{-mod}, \mathbf{oblv}_{\mathsf{h}})$.

7.5.7. Finally, from Lemma 6.2.2, we obtain:

Corollary 7.5.8. *The functor* (7.12) *defines an isomorphism between:*

(a) *For a* monoidal $\mathbf{C}$*, from the space of* monoidal *functors*

$$\mathrm{QCoh}(\mathbf{Maps}_{\mathrm{Lie}}(\mathsf{h}, \mathsf{g})/\mathsf{G}) \to \mathbf{C}$$

to the space of compatible collections of monoidal functors

$$\mathrm{Rep}(\mathsf{G})^{\otimes I} \to \mathbf{C} \otimes (\mathsf{h}\text{-mod})^{\otimes I}, \quad I \in \mathrm{fSet}\,.$$

(b) *For a plain category* $\mathbf{C}$*, from the category of functors*

$$\mathrm{QCoh}(\mathbf{Maps}_{\mathrm{Lie}}(\mathsf{h}, \mathsf{g})/\mathsf{G}) \to \mathbf{C}$$

to the space of compatible collections of functors

$$\mathrm{Rep}(\mathsf{G})^{\otimes I} \to \mathbf{C} \otimes (\mathsf{h}\text{-mod})^{\otimes I}, \quad I \in \mathrm{fSet}\,.$$

7.5.9. Note now that due to Proposition 7.4.2(a), the category $\mathrm{QCoh}(\mathbf{Maps}_{\mathrm{Lie}}(\mathsf{h}, \mathsf{g})/\mathsf{G})$ is dualizable and self-dual. Hence, Corollary 7.5.8(b) gives a description of the category $\mathrm{QCoh}(\mathbf{Maps}_{\mathrm{Lie}}(\mathsf{h}, \mathsf{g})/\mathsf{G})$. Let us spell out what this description says concretely:

For every $V \in \mathrm{Rep}(\mathsf{G})$, we consider the $\mathrm{Ad}(\mathsf{G})$-equivariant quasi-coherent sheaf on $\mathbf{Maps}_{\mathrm{Lie}}(\mathsf{h}, \mathsf{g})$

$$\mathcal{V} := V \otimes \mathcal{O}_{\mathbf{Maps}_{\mathrm{Lie}}(\mathsf{h}, \mathsf{g})},$$

which is equipped with a tautological action of h.

With these notations, for a given $\mathcal{M} \in \mathrm{QCoh}(\mathbf{Maps}_{\mathrm{Lie}}(\mathsf{h}, \mathsf{g})/\operatorname{Ad}(\mathsf{G}))$, the corresponding functors

$$\mathrm{Rep}(\mathsf{G})^{\otimes I} \to \mathbf{C} \otimes (\mathsf{h}\text{-mod})^{\otimes I}$$

send an I-tuple V_i of G-representations to

$$\Gamma\left(\mathsf{g}/\operatorname{Ad}(\mathsf{G}), \mathcal{M} \otimes \bigotimes_i \mathcal{V}_i\right),$$

equipped with the I-tuple of pairwise commuting actions of h.

7.6. **Proof of Theorem 7.4.5.** In this subsection we will prove Theorem 7.4.5 in the form of Theorem 7.5.2. The proof will be, in some sense, tautological. We will later use the same idea to reprove Theorem 7.1.2, and even to attempt to prove that (6.6) is an equivalence in general, see Sect. 7.8.

7.6.1. *Step 1.* We will first show that we can assume that $\mathbf{A}$ is of the form A-mod for some $A \in \mathrm{ComAlg}(\mathrm{Vect}_{\mathsf{e}})$. Namely, we will show that both (7.13) and (7.14) factor canonically via A-mod, where

$$A := \mathcal{E}nd_{\mathbf{A}}(\mathbf{1}_{\mathbf{A}}).$$

For (7.13) this is obvious: for any affine scheme $Y = \mathrm{Spec}(R)$, symmetric monoidal functors

$$\mathrm{QCoh}(Y) = R\text{-mod} \to \mathbf{A}$$

are in bijection with maps of commutative algebras $R \to \mathcal{E}nd_{\mathbf{A}}(\mathbf{1}_{\mathbf{A}}) =: A$, and the latter are the same as symmetric monoidal functors

$$R\text{-mod} \to A\text{-mod}.$$

For (7.14) we argue as follows. We have a tautological (symmetric monoidal) functor

$$A\text{-mod} =: \mathbf{A}' \to \mathbf{A}, \tag{7.15}$$

which, however, is not necessarily fully faithful (because $\mathbf{1}_{\mathbf{A}} \in \mathbf{A}$ is not necessarily compact).

The datum of (7.14) for $\mathbf{A}$ and $\mathbf{A}'$, respectively, amounts to defining an action of h on

$$\underline{V} \otimes \mathbf{1}_{\mathbf{A}} \tag{7.16}$$

(viewed as an object of either $\mathbf{A}$ or $\mathbf{A}'$) for every $V \in \mathrm{Rep}(\mathsf{G})^c$, in a way compatible with the tensor structure (here $V \to \underline{V}$ denotes the forgetful functor $\mathrm{Rep}(\mathsf{G}) \to \mathrm{Vect}_{\mathsf{e}}$).

The datum of such action consists of a compatible family of diagrams

$$U(\mathsf{h})^{\otimes I} \otimes \underline{V} \otimes \mathbf{1}_{\mathbf{A}} \to \underline{V} \otimes \mathbf{1}_{\mathbf{A}}, \quad V \in \mathrm{Rep}(\mathsf{G}^J)^c,\ I, J \in \mathrm{fSet}\,. \tag{7.17}$$

The assertion follows now from the fact that the functor (7.15) *does* induce an isomorphisms on the mapping space from objects of the form

$$W_1 \otimes \mathbf{1}_{\mathbf{A}}, \quad W_1 \in \mathrm{Vect}_{\mathsf{e}}$$

to objects of the form

$$W_2 \otimes \mathbf{1}_{\mathbf{A}}, \quad W_2 \in \mathrm{Vect}_{\mathsf{e}}^c\,.$$

7.6.2. *Step 2.* Thus, we can assume that $\mathbf{A} = A$-mod for $A \in \mathrm{ComAlg}(\mathrm{Vect}_{\mathsf{e}})$. Next we claim that we can assume that A is connective. More precisely, we claim that (7.13) and (7.14) factor canonically via A'-mod, where $A' := \tau^{\leq 0}(A)$.

This is again obvious for (7.13): for $R \in \mathrm{ComAlg}(\mathrm{Vect}_{\mathsf{e}}^{\leq 0})$, a map $R \to A$ factors canonically through a map $R \to A'$.

For (7.14) we argue as follows: since $\mathrm{Rep}(\mathsf{G})$ is the derived category of its heart and the tensor product operation is t-exact, in (7.16) we can assume that $V \in \mathrm{Rep}(\mathsf{G})^{\heartsuit} \cap \mathrm{Rep}(\mathsf{G})^c$. Hence, in (7.17) we can also assume that

$$V \in \mathrm{Rep}(\mathsf{G}^J)^{\heartsuit} \cap \mathrm{Rep}(\mathsf{G}^J)^c.$$

Now, in this case, maps in (7.17), which correspond to points in

$$\mathrm{Maps}_{\mathrm{Vect}_{\mathsf{e}}}(U(\mathsf{h})^{\otimes I} \otimes \underline{V}, A \otimes \underline{V})$$

factor canonically via

$$\mathrm{Maps}_{\mathrm{Vect}_{\mathsf{e}}}(U(\mathsf{h})^{\otimes I} \otimes \underline{V}, A' \otimes \underline{V}).$$

7.6.3. *Step 3.* Thus, we can assume that $\mathbf{A} = A\text{-mod}$ for $A \in \mathrm{ComAlg}(\mathrm{Vect}_{\mathsf{e}}^{\leq 0})$. Denote $S := \mathrm{Spec}(A)$. Then (7.13) is the same as maps of affine schemes

$$S \to \mathbf{Maps}_{\mathrm{Lie}}(\mathsf{h}, \mathsf{g}). \tag{7.18}$$

Similarly, we claim that (7.14) is also the same as (7.18). Indeed, both sides take sifted colimits in h to limits in spaces, so we can assume that h is of the form (7.9). Moreover, we can assume that

$$V \in \mathrm{Vect}_{\mathsf{e}}^{\leq 0} \cap \mathrm{Vect}_{\mathsf{e}}^{c}.$$

Note that in this case, we have

$$\mathbf{Maps}_{\mathrm{Lie}}(\mathsf{h}, \mathsf{g}) = \mathrm{Tot}(\mathcal{H}om_{\mathrm{Vect}_{\mathsf{e}}}(V, \mathbf{oblv}_{\mathrm{Lie}}(\mathsf{g}))),$$

so (7.18) is the space

$$\mathrm{Maps}_{\mathrm{Vect}_{\mathsf{e}}}(V, A \otimes \mathbf{oblv}_{\mathrm{Lie}}(\mathsf{g})).$$

Let A' be the split square-zero extension of A equal to

$$A' = A \otimes (\mathsf{e} \oplus \epsilon \cdot V^*), \quad \epsilon^2 = 0$$

The description of the data of (7.14) in Step 1 above shows that this space identifies with the space of automorphisms of the symmetric monoidal functor

$$\mathrm{Rep}(\mathsf{G}) \xrightarrow{\mathbf{oblv}_{\mathsf{G}}} \mathrm{Vect}_{\mathsf{e}} \xrightarrow{A'} A'\text{-mod},$$

equipped with the trivialization of the automorphism of the composite functor

$$\mathrm{Rep}(\mathsf{G}) \xrightarrow{\mathbf{oblv}_{\mathsf{G}}} \mathrm{Vect}_{\mathsf{e}} \xrightarrow{A'} A'\text{-mod} \to A\text{-mod}.$$

By Tannaka duality, the latter is the same as the space of maps

$$\mathrm{Spec}(A') \to \mathsf{G},$$

equipped with the trivialization of the composite

$$S = \mathrm{Spec}(A) \to \mathrm{Spec}(A') \to \mathsf{G}.$$

By deformation theory, we rewrite the latter as

$$\mathrm{Maps}_{\mathrm{Vect}_{\mathsf{e}}}(\mathsf{g}^*, A \otimes V^*) \simeq \mathrm{Maps}_{\mathrm{Vect}_{\mathsf{e}}}(V, A \otimes \mathbf{oblv}_{\mathrm{Lie}}(\mathsf{g})),$$

as required.

□[Theorem 7.4.5]

7.7. **Back to the Betti case.** Note that one can prove Theorem 7.1.2, along the lines of the proof of Theorem 7.4.5. Let is sketch the argument.

7.7.1. Let

$$\Omega(\mathcal{X}, x) \in \mathrm{Grp}(\mathrm{Spc})$$

be the loop space of $\mathcal{X}$ based at x. We have

$$\mathrm{LocSys}_{\mathsf{G}}(\mathcal{X})^{\mathrm{rigid}_x} \simeq \mathbf{Maps}_{\mathrm{Grp}}(\Omega(\mathcal{X}, x), \mathsf{G})$$

and

$$\mathrm{Vect}_{\mathsf{e}}^{\mathcal{X}} \simeq \Omega(\mathcal{X}, x)\text{-mod}.$$

Thus, we wish to prove that for a symmetric monoidal category, the space of symmetric monoidal functors

$$\mathrm{QCoh}(\mathbf{Maps}_{\mathrm{Grp}}(\Omega(\mathcal{X}, x), \mathsf{G})) \to \mathbf{A} \tag{7.19}$$

maps isomorphically to the space of symmetric monoidal functors

$$\mathrm{Rep}(\mathsf{G}) \to \mathbf{A} \otimes \Omega(\mathcal{X}, x)\text{-mod},$$

equipped with an identification of the composite

$$\mathrm{Rep}(\mathsf{G}) \to \mathbf{A} \otimes \Omega(\mathcal{X}, x)\text{-mod} \xrightarrow{\mathrm{Id}_{\mathbf{A}} \otimes \mathbf{oblv}_{\Omega(\mathcal{X}, x)}} \mathbf{A} \tag{7.20}$$

with

$$\mathrm{Rep}(\mathsf{G}) \overset{\mathbf{oblv}_{\mathsf{G}}}{\to} \mathrm{Vect}_{\mathsf{e}} \overset{\mathbf{1_A}}{\to} \mathbf{A}.$$

7.7.2. Repeating the argument in Sects. 7.6.1-7.6.2, we can assume that $\mathbf{A} = A\text{-mod}$ for some $A \in \mathrm{ComAlg}(\mathrm{Vect}_{\mathsf{e}}^{\leq 0})$. In this case, we rewrite (7.19) as the space of commutative algebra homomorphisms

$$\mathcal{O}_{\mathbf{Maps}_{\mathrm{Grp}}(\Omega(\mathcal{X},x),\mathsf{G})} \to A,$$

i.e., as the space of maps

$$S := \mathrm{Spec}(A) \to \mathbf{Maps}_{\mathrm{Grp}}(\Omega(\mathcal{X},x),\mathsf{G}),$$

and further as the space of maps in Grp(Spc)

$$\Omega(X,x) \to \mathrm{Maps}_{\mathrm{Sch}^{\mathrm{aff}}}(S,\mathsf{G}) \tag{7.21}$$

We rewrite (7.20) as the space of actions of $\Omega(X,x)$ on the functor

$$\mathrm{Rep}(\mathsf{G}) \overset{\mathbf{oblv}_{\mathsf{G}}}{\to} \mathrm{Vect}_{\mathsf{e}} \overset{A}{\to} A\text{-mod} = \mathrm{QCoh}(S). \tag{7.22}$$

Now, the fact that the map from (7.19) to (7.20) is an isomorphism is an expression of Tannaka duality. Namely, the group $\mathrm{Maps}_{\mathrm{ComAlg}}(S,\mathsf{G})$ maps isomorphically to the group of automorphisms of the functor (7.22).

Remark 7.7.3. Note that Corollary 7.5.5 implies Theorem 6.2.11 in the particular case when $\mathbf{H} = \mathrm{Rep}(\mathsf{H})$, where H is a *unipotent* algebraic group. In fact we claim that in this case, the functor (6.6) is an equivalence.

Indeed, the proof is a word-by-word repetition of the argument in Sects. 7.2.1- 7.2.8 using the fact that the restriction functor

$$\mathrm{Rep}(\mathsf{H}) \to \mathsf{h}\text{-mod}, \quad \mathsf{h} = \mathrm{Lie}(\mathsf{H})$$

is fully faithful (it is here that we use the assumption that H is unipotent).

7.8. **A (failed?) attempt to prove that** (6.6) **is an equivalence.** We would like to be able to show that (6.6) is an equivalence. This amounts to saying that for a symmetric monoidal category $\mathbf{A}$, the functor

$$\mathrm{Maps}_{\mathrm{DGCat}^{\mathrm{SymMon}}}(\mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})),\mathbf{A}) \to \mathrm{Maps}_{\mathrm{DGCat}^{\mathrm{SymMon}}}(\mathrm{Rep}(\mathsf{G}),\mathbf{A}\otimes\mathbf{H}), \tag{7.23}$$

arising from (6.5), is an isomorphism.

We will be able to do so under the additional assumption that the unit object $\mathbf{1_A} \in \mathbf{A}$ is compact and $\mathbf{A}$ is dualizable as a DG category.

Remark 7.8.1. Note that the fact that (7.23) is an isomorphism for target categories $\mathbf{A}$ in which $\mathbf{1_A} \in \mathbf{A}$ is compact implies that this assertion remains valid for those $\mathbf{A}$ that can be written as *limits* of symmetric monoidal categories with this property (indeed, both sides in (7.23) take limits to limits).

In particular, if we knew that both

$$\mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})) \text{ and } \underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})$$

can be written as limits of symmetric monoidal categories with a compact unit, we would know that (6.6) is an equivalence in general.

Being the category of quasi-coherent sheaves on a prestack, the category $\mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H}))$ can indeed be written as a limit of symmetric monoidal categories with a compact unit.

So, the missing piece is that we do not know how to prove the corresponding fact that for $\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})$.

7.8.2. We will prove an equivalent fact that if $\mathbf{A}$ is a symmetric monoidal category with a compact unit, then the space of symmetric monoidal functors

$$\mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\mathrm{rigid}}) \to \mathbf{A} \tag{7.24}$$

maps isomorphically to the space of symmetric monoidal functors

$$\mathsf{F} : \mathrm{Rep}(\mathsf{G}) \to \mathbf{A} \otimes \mathbf{H}, \tag{7.25}$$

equipped with an identification of the composite

$$\mathrm{Rep}(\mathsf{G}) \to \mathbf{A} \otimes \mathbf{H} \overset{\mathrm{Id}_{\mathbf{A}} \otimes \mathbf{oblv}_{\mathbf{H}}}{\longrightarrow} \mathbf{A}$$

with

$$\mathrm{Rep}(\mathsf{G}) \overset{\mathbf{oblv}_{\mathsf{G}}}{\to} \mathrm{Vect}_{\mathsf{e}} \overset{\mathbf{1}_{\mathbf{A}}}{\longrightarrow} \mathbf{A}.$$

7.8.3. Set $A := \mathcal{E}nd_{\mathbf{A}}(\mathbf{1}_{\mathbf{A}})$. We claim that in both (7.24) and (7.25) we can replace $\mathbf{A}$ by A-mod.

Indeed, set $\mathbf{A}' := A$-mod; denote by Φ the corresponding symmetric monoidal functor $\mathbf{A}' \to \mathbf{A}$. The assumption that $\mathbf{1}_{\mathbf{A}}$ is compact implies that Φ is fully faithful.

7.8.4. We claim that any functor as in (7.24) takes values in $\mathbf{A}' \subset \mathbf{A}$. Indeed, by Theorem 1.4.3(b) (which is applicable to $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}$ replaced by $\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\mathrm{rigid}}$, see Sect. 9.1.1 below), the prestack $\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\mathrm{rigid}}$ is isomorphic to the disjoint union of prestacks of the form (1.3).

We claim that any symmetric monoidal functor

$$\mathrm{QCoh}(\underset{i \geq 0}{\mathrm{colim}}\ \mathrm{Spec}(R_i)) \to \mathbf{A}$$

(we use the notations of Theorem 1.3.2(d)) takes values in $\mathbf{A}'$. Indeed, by Sect. 7.2.5, it is enough to establish the corresponding fact for functor

$$\mathrm{QCoh}(\mathrm{Spec}(R)) \to \mathbf{A},$$

and the latter is obvious as the generator R gets sent to $\mathbf{1}_A$.

7.8.5. We will now show that any functor F in (7.25) takes values in

$$\mathbf{A}' \otimes \mathbf{H} \subset \mathbf{A} \otimes \mathbf{H}.$$

Recall that Φ denotes the embedding $\mathbf{A}' \to \mathbf{A}$; let Φ^R denote its right adjoint. We need to show that the map

$$((\Phi \circ \Phi^R) \otimes \mathrm{Id}_{\mathbf{H}})(\mathsf{F}(V)) \simeq (\Phi \otimes \mathrm{Id}_{\mathbf{H}}) \circ (\Phi^R \otimes \mathrm{Id}_{\mathbf{H}})(\mathsf{F}(V)) \to \mathsf{F}(V) \tag{7.26}$$

is an isomorphism.

Since

$$(\mathrm{Id}_{\mathbf{A}} \otimes \mathbf{oblv}_{\mathbf{H}})(\mathsf{F}(V)) \simeq \underline{V} \otimes \mathbf{1}_{\mathbf{A}} \in \mathbf{A}',$$

we know that (7.26) becomes an isomorphism after applying $\mathrm{Id}_{\mathbf{A}} \otimes \mathbf{oblv}_{\mathbf{H}}$.

Now the required assertion follows from the fact that $\mathrm{Id}_{\mathbf{A}} \otimes \mathbf{oblv}_{\mathbf{H}}$ is conservative (the latter conservativity is since $\mathbf{oblv}_{\mathbf{H}}$ is conservative and $\mathbf{A}$ is dualizable).

7.8.6. Thus, we can assume that $\mathbf{A} = A\text{-mod}$ for $A \in \text{ComAlg}(\text{Vect}_{\mathsf{e}})$. Furthermore, as in Sect. 7.6.2, we can assume that A is connective.

Then by construction, the datum of (7.25) amounts to a map

$$S := \text{Spec}(A) \to \mathbf{coMaps}(\text{Rep}(\mathsf{G}), \mathbf{H})^{\text{rigid}}.$$

We will show that the datum in (7.24) also amounts to such a map. Indeed, let us write $\mathbf{coMaps}(\text{Rep}(\mathsf{G}), \mathbf{H})^{\text{rigid}}$ as a union of prestacks of the form (1.3). We claim that in the notations of *loc. cit.*, the datum of a symmetric monoidal functor

$$\text{QCoh}(\underset{i\geq 0}{\text{colim}}\ \text{Spec}(R_i)) \to A\text{-mod} \tag{7.27}$$

is equivalent to the datum of a map

$$S := \text{Spec}(A) \to \underset{i\geq 0}{\text{colim}}\ \text{Spec}(R_i). \tag{7.28}$$

Indeed, by Sect. 7.2.5, the datum of a symmetric monoidal functor (7.27) amounts to that of a symmetric monoidal functor

$$R\text{-mod} \to A\text{-mod},$$

for which

$$A\text{-mod} \underset{R\text{-mod}}{\otimes} \text{Vect}_{\mathsf{e}'} = 0$$

for any e'-point of $\text{Spec}(R)$ that does not land in $\text{Spec}(R_0)$.

The latter is the same as a S-point of $\text{Spec}(R)$, such that

$$S \underset{\text{Spec}(R)}{\otimes} \text{Spec}(\mathsf{e}') = \emptyset$$

for any e'-point of $\text{Spec}(R)$ that does not land in $\text{Spec}(R_0)$. I.e., this is the same as a datum in (7.28).

8. Proof of Theorem 6.2.11 for $\mathbf{H} = \text{Rep}(\mathsf{H})$ for an algebraic group H

In this section we will show that Theorem 6.2.11 holds for $\mathbf{H}$ of the form $\text{Rep}(\mathsf{H})$ for H an algebraic group. In fact, we will show that in this case, the map (6.6) is an equivalence.

We will do so by directly treating the case of a reductive group, essentially by tweaking the method in Sect. 7.8. We will then combine it with Theorem 7.4.5 to prove that (6.6) is an equivalence for $\mathbf{H} = (\mathfrak{h}, \mathsf{H}_{\text{red}})\text{-mod}$ for a Harish-Chandra pair $(\mathfrak{h}, \mathsf{H}_{\text{red}})$ with H_{red} reductive.

Finally, we will show that the result for $\mathbf{H} = (\mathfrak{h}, \mathsf{H}_{\text{red}})\text{-mod}$ implies the result for $\mathbf{H} = \text{Rep}(\mathsf{H})$ by the method of Sects. 7.2.1-7.2.8.

8.1. **The case when H is semi-simple.** In this subsection we take H to be semi-simple. We will prove:

Theorem 8.1.1. *The functor* (6.6) *is an equivalence for* $\mathbf{H} = \text{Rep}(\mathsf{H})$.

The proof will be obtained by adapting the method from Sect. 7.8.

8.1.2. We will follow the argument of Sect. 7.8, i.e., we will show that the space of maps (7.24) maps isomorphically to the space of maps (7.25).

Let A and $\mathbf{A}'$ be as in Sect. 7.8.3. We will first show that, as in Sect. 7.8, the space of maps (7.24) (resp., (7.25)) is isomorphic to the corresponding space with $\mathbf{A}$ replaced by $\mathbf{A}' = A\text{-mod}$.

8.1.3. Let us show that (7.24) remains unchanged if we replace $\mathbf{A}$ by $\mathbf{A}' = A$-mod. For that end, we will show that $\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathrm{Rep}(\mathsf{H}))^{\mathrm{rigid}}$ is a an affine scheme, so that the argument in Sect. 7.8.3 is applicable.

We have:

$$\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathrm{Rep}(\mathsf{H}))^{\mathrm{rigid}} = \mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, \mathsf{G}).$$

A priori, by Proposition 1.8.6, this is an ind-affine ind-scheme. However, by Proposition 2.3.9, it is also a disjoint union of schemes. So combining, we obtain that $\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathrm{Rep}(\mathsf{H}))^{\mathrm{rigid}}$ is a disjoint union of affine schemes.

It remains to show that it has finitely many connected components.

Choose a faithful representation $\mathsf{G} \hookrightarrow GL_n$ for some n. The morphism

$$\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, \mathsf{G}) \to \mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, GL_n)$$

is a closed embedding, so it suffices to show that $\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, GL_n)$ has finitely many connected components. Equivalently, we have to show that $\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, GL_n)/GL_n$ has finitely many isomorphism classes of e-points. However, the latter is just the fact that H has finitely many isomorphism classes of representations of a given dimension (it is here that we use the fact that H is semi-simple).

Remark 8.1.4. We have shown that the unit object in $\mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathrm{Rep}(\mathsf{H}))^{\mathrm{rigid}})$ is compact. So, a posteriori, once we prove that (6.6) is an equivalence, we will know that the unit object in $\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \mathrm{Rep}(\mathsf{H}))^{\mathrm{rigid}}$ is also compact. However, we cannot apply Sect. 7.8 right away, since we did not a priori know the latter compactness statement.

8.1.5. We will now show the space in (7.25) remains unchanged if we replace $\mathbf{A}$ by $\mathbf{A}' := A$-mod.

The datum of (7.25) amounts to a compatible family of lifts of the objects

$$\mathbf{1}_{\mathbf{A}} \otimes \underline{V}, \quad V \in \mathrm{Rep}(\mathsf{G}^I)^c \cap \mathrm{Rep}(\mathsf{G}^I)^{\heartsuit}, \quad I \in \mathrm{fSet} \tag{8.1}$$

to objects of

$$\mathbf{A} \otimes \mathrm{Rep}(\mathsf{H}).$$

We now use the fact that $\mathrm{Rep}(\mathsf{H})$ is equivalent, as a DG category to a direct sum of copies of $\mathrm{Vect}_{\mathsf{e}}$ (indexed by the set of isomorphism classes of irreducible representations of H). Hence, the above family of lifts amounts to a tensor-compatible family of decompositions of objects of the form (8.1), as direct sums indexed by $\mathrm{Irrep}(\mathsf{H})$.

However, we claim:

Lemma 8.1.6. *Let $a \otimes W' \in \mathbf{A}$ be a direct summand of $\mathbf{1}_{\mathbf{A}} \otimes W$ for $W', W \in \mathrm{Vect}_{\mathsf{e}}^{\heartsuit}$ and W finite-dimensional. Then $\dim(W') \leq \dim(W)$.*

The proof of the lemma is given in Sect. 8.1.8.

From the lemma, we obtain that for a given $V \in \mathrm{Rep}(\mathsf{G}^I)^c \cap \mathrm{Rep}(\mathsf{G}^I)^{\heartsuit}$, only the directs summands corresponding to irreducible representations of H of dimension $\leq \dim(V)$ can appear; in particular, for a given V we have a *finite* direct sum decomposition (again, it is here that we use the assumption that H is semi-simple).

In other words, the above lifts are in bijection with a tensor-compatible families of idempotents on the objects (8.1). However, such spaces of idempotents are isomorphic to the spaces of A-linear idempotents on the objects

$$A \otimes \underline{V} \in A\text{-mod}, \tag{8.2}$$

as desired.

8.1.7. Thus, we have reduced the verification of the assertion that the map from the space of maps (7.24) to the space of maps (7.25) is an isomorphism to the case when $\mathbf{A}$ is of the form A-mod for $A \in \mathrm{ComAlg}(\mathrm{Vect}_{\mathsf{e}})$.

In this case, the argument in Sect. 7.8.6 goes through "as-is".

8.1.8. *Proof of Lemma 8.1.6.* Consider the idempotent on $\mathbf{1}_{\mathbf{A}} \otimes W$ corresponding to $a \otimes W'$. This is a point of $\mathcal{E}nd_{\mathbf{A}}(\mathbf{1}_{\mathbf{A}}) \otimes \mathrm{End}_{\mathsf{e}}(W)$. Hence, we can replace $\mathbf{A}$ by A-mod, where, as before, $A := \mathcal{E}nd_{\mathbf{A}}(\mathbf{1}_{\mathbf{A}})$. Further, we can replace A by $\tau^{\leq 0}(A)$.

So we can assume that $\mathbf{A} = A$-mod for a connective commutative algebra. Choose a field-valued point

$$\mathrm{Spec}(\mathsf{e}') \to \mathrm{Spec}(A),$$

at which the fiber of $a \otimes W'$ is non-zero. We obtain that

$$(a \underset{A}{\otimes} \mathsf{e}') \otimes W'$$

is a non-zero direct summand of $\mathsf{e}' \otimes W'$. Hence $\dim(W') \leq \dim(W)$.

□

8.2. **The case when H is a torus.** To simplify the notation, we will assume that H is a 1-dimensional torus, i.e., $\mathbb{G}_m$.

Thus, we are going to prove:

Theorem 8.2.1. *The functor* (6.6) *is an equivalence for* $\mathbf{H} = \mathrm{Rep}(\mathbb{G}_m) = \mathrm{Vect}_{\mathsf{e}}^{\mathbb{Z}}$.

The proof will be again obtained by tweaking the method in Sect. 7.8, but in a way different from that of the case when H was semi-simple.

8.2.2. We will again follow the argument from Sect. 7.8. However, it will *no longer be true* that, for a general $\mathbf{A}$, the space of maps (7.24) (resp., (7.25)) is isomorphic to the corresponding space with $\mathbf{A}$ replaced by $\mathbf{A}' = A$-mod. Rather, we will modify the initial $\mathbf{A}$ so that it becomes true.

Remark 8.2.3. Unlike the case when H was reductive, it is *not* true that the unit object in $\mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathrm{Rep}(\mathbb{G}_m))^{\mathrm{rigid}})$ is compact. Indeed,

$$\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathrm{Rep}(\mathbb{G}_m))^{\mathrm{rigid}} = \mathbf{Maps}_{\mathrm{Grp}}(\mathbb{G}_m, \mathsf{G}) \tag{8.3}$$

is an *infinite* disjoint union of affine schemes.

8.2.4. First, by Sect. 8.1.3, the prestack (8.3) is the disjoint union over the set of conjugacy classes of cocharacters $\lambda : \mathbb{G}_m \to \mathsf{G}$ of $\mathsf{G}/\mathrm{Stab}_{\mathsf{G}}(\lambda)$.

Pick a faithful representation

$$\mathsf{G} \hookrightarrow GL_n, \tag{8.4}$$

and consider the corresponding closed embedding

$$\mathbf{Maps}_{\mathrm{Grp}}(\mathbb{G}_m, \mathsf{G}) \hookrightarrow \mathbf{Maps}_{\mathrm{Grp}}(\mathbb{G}_m, GL_n).$$

We will regard $\mathbf{Maps}_{\mathrm{Grp}}(\mathbb{G}_m, \mathsf{G})$ as the disjoint union of preimages of connected components of $\mathbf{Maps}_{\mathrm{Grp}}(\mathbb{G}_m, GL_n)$; the latter are indexed by partitions of n:

$$n = \sum_{d \in \mathbb{Z}} n_d, \quad n_d \geq 0. \tag{8.5}$$

For a given partition $\underline{d}$, let

$$\mathbf{Maps}_{\mathrm{Grp}}(\mathbb{G}_m, \mathsf{G})_{\underline{d}} \tag{8.6}$$

denote the corresponding subscheme of $\mathbf{Maps}_{\mathrm{Grp}}(\mathbb{G}_m, \mathsf{G})$.

Using the isomorphism

$$\mathbf{Maps}_{\mathrm{Grp}}(\mathbb{G}_m, \mathsf{G}) \simeq \mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathrm{Rep}(\mathbb{G}_m))^{\mathrm{rigid}},$$

we will use (8.6) to define a decomposition

$$\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathrm{Rep}(\mathbb{G}_m))^{\mathrm{rigid}} = \underset{\underline{d}}{\sqcup} \mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathrm{Rep}(\mathbb{G}_m))^{\mathrm{rigid}}_{\underline{d}}.$$

8.2.5. Let $\underline{d}$ be a partition as in (8.5). We will say a symmetric monoidal functor

$$\mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}),\mathrm{Rep}(\mathbb{G}_m))^{\mathrm{rigid}}) \to \mathbf{A} \tag{8.7}$$

has *weight* $\underline{d}$ if it factors as

$$\mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}),\mathrm{Rep}(\mathbb{G}_m))^{\mathrm{rigid}}) \to \mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}),\mathrm{Rep}(\mathbb{G}_m))^{\mathrm{rigid}}_{\underline{d}}) \to \mathbf{A}.$$

As in Sect. 8.1.3, the datum in (8.7) of *specified weight* $\underline{d}$ is equivalent to a similar datum for $\mathbf{A}$ replaced by A-mod, where $A = \mathcal{E}nd_{\mathbf{A}}(\mathbf{1}_{\mathbf{A}})$. This is due to the fact that each individual $\mathbf{Maps}_{\mathrm{Grp}}(\mathbb{G}_m,\mathsf{G})_{\underline{d}}$ *is* an affine scheme.

8.2.6. In general, given a datum (8.7), the decomposition of

$$\mathbf{1}_{\mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}),\mathrm{Rep}(\mathbb{G}_m))^{\mathrm{rigid}})} = \mathcal{O}_{\mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}),\mathrm{Rep}(\mathbb{G}_m))^{\mathrm{rigid}}}$$

as

$$\underset{\underline{d}}{\oplus}\, \mathcal{O}_{\mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}),\mathrm{Rep}(\mathbb{G}_m))^{\mathrm{rigid}}_{\underline{d}}}$$

defines an idempotent decomposition

$$\mathbf{1}_{\mathbf{A}} \simeq \underset{\underline{d}}{\oplus}\, \mathbf{1}_{\mathbf{A},\underline{d}}, \tag{8.8}$$

and hence a decomposition

$$\mathbf{A} = \underset{\underline{d}}{\Pi}\, \mathbf{A}_{\underline{d}}, \tag{8.9}$$

so that each of the resulting functors

$$\mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}),\mathrm{Rep}(\mathbb{G}_m))^{\mathrm{rigid}}) \to \mathbf{A}_{\underline{d}}$$

has weight $\underline{d}$.

8.2.7. Let us now be given a symmetric monoidal functor

$$\mathsf{F} : \mathrm{Rep}(\mathsf{G}) \to \mathbf{A} \otimes \mathrm{Rep}(\mathbb{G}_m), \tag{8.10}$$

so that the composite

$$\mathrm{Rep}(\mathsf{G}) \overset{\mathsf{F}}{\to} \mathbf{A} \otimes \mathrm{Rep}(\mathbb{G}_m) \overset{\mathrm{Id}_{\mathbf{A}} \otimes \mathbf{oblv}_{\mathbb{G}_m}}{\longrightarrow} \mathbf{A}$$

identifies with

$$\mathrm{Rep}(\mathsf{G}) \overset{\mathbf{oblv}_{\mathsf{G}}}{\to} \mathrm{Vect}_{\mathsf{e}} \overset{\mathbf{1}_{\mathbf{A}}}{\to} \mathbf{A}.$$

The datum of F defines a grading on every $\mathbf{1}_{\mathbf{A}} \otimes \underline{V}$ for every $V \in \mathrm{Rep}(\mathsf{G})$:

$$\mathbf{1}_{\mathbf{A}} \otimes \underline{V} \simeq \underset{d\in\mathbb{Z}}{\oplus}\, (\mathbf{1}_{\mathbf{A}} \otimes V)^d.$$

8.2.8. Let V be the representation of G corresponding to (8.4). We will say that F has weight $\underline{d}$ if for every $d \in \mathbb{Z}$, the direct summand of

$$A \otimes \underline{V} \simeq \mathcal{H}om_{\mathbf{A}}(\mathbf{1}_{\mathbf{A}}, \mathbf{1}_{\mathbf{A}} \otimes \underline{V})$$

given by

$$(A \otimes V)^d := \mathcal{H}om_{\mathbf{A}}(\mathbf{1}_{\mathbf{A}}, (\mathbf{1}_{\mathbf{A}} \otimes V)^d)$$

is a projective A-module of rank n_d, where n_d is as in (8.5).

8.2.9. It is clear that if F has a *specified weight* $\underline{d}$, then for any $V' \in \mathrm{Rep}(\mathsf{G})^c \cap \mathrm{Rep}(\mathsf{G})^{\heartsuit}$, the object $\mathbf{1}_{\mathbf{A}} \otimes \underline{V}'$ has only finitely many non-zero graded components. As in Sect. 8.1.5, we obtain that the datum of (8.10) of *specified degree* $\underline{d}$ is equivalent to a similar datum for $\mathbf{A}$ replaced by A-mod.

8.2.10. It is clear that the map (6.6) sends the data of (8.7) of degree $\underline{d}$ to the data (8.10) of degree $\underline{d}$. Further, the argument in Sect. 7.8.6 shows that this map is an isomorphism.

8.2.11. Finally, we claim that for any data (8.10), we have a decomposition (8.9), such that each of the resulting functors

$$\mathrm{Rep}(\mathsf{G}) \to \mathbf{A} \otimes \mathrm{Rep}(\mathbb{G}_m) \to \mathbf{A}_{\underline{d}} \otimes \mathrm{Rep}(\mathbb{G}_m)$$

has weight $\underline{d}$.

We find the corresponding decomposition (8.8) as follows: for $V \in \mathrm{Rep}(\mathsf{G})^c \cap \mathrm{Rep}(\mathsf{G})^\heartsuit$ with $\dim(V) = n$, we have

$$\mathbf{1}_{\mathbf{A}} \otimes \Lambda^n_{\mathsf{e}}(\underline{V}) \simeq \Lambda^n_{\mathbf{A}}(\mathbf{1}_{\mathbf{A}} \otimes \underline{V}) \simeq \underset{\underline{d}}{\oplus} \left(\bigotimes_d \Lambda^{n_d}_{\mathbf{A}}((\mathbf{1}_{\mathbf{A}} \otimes V)^d) \right). \tag{8.11}$$

The left-hand side in (8.11) is an invertible object in the symmetric monoidal category $\mathbf{A}$. From here, we obtain that the direct sum decomposition of the right-hand side defines a decomposition (8.8) so that the direct summand

$$\bigotimes_d \Lambda^{n_d}_{\mathbf{A}}((\mathbf{1}_{\mathbf{A}} \otimes V)^d)$$

is of the form

$$\mathbf{1}_{\mathbf{A},\underline{d}} \otimes \mathcal{H}om_{\mathbf{A}} \left(\mathbf{1}_{\mathbf{A}}, \Lambda^{n_d}_{\mathbf{A}}((\mathbf{1}_{\mathbf{A}} \otimes V)^d) \right),$$

where $\mathcal{H}om_{\mathbf{A}} \left(\mathbf{1}_{\mathbf{A}}, \Lambda^{n_d}_{\mathbf{A}}((\mathbf{1}_{\mathbf{A}} \otimes V)^d) \right)$ is an invertible module over

$$A_{\underline{d}} := \mathcal{E}nd_{\mathbf{A}}(\mathbf{1}_{\mathbf{A},\underline{d}}).$$

8.3. **The case of a general reductive group.** In this subsection we will prove:

Theorem 8.3.1. *The functor* (6.6) *is an equivalence for* $\mathbf{H} = \mathrm{Rep}(\mathsf{H})$*, where* H *is a reductive group.*

The proof will be essentially a combination of the ideas used in the case when H was semi-simple and $\mathbb{G}_m$, respectively.

8.3.2. We need to show that the map from the space of maps of symmetric monoidal categories

$$\mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathrm{Rep}(\mathsf{H}))^{\mathrm{rigid}}) \to \mathbf{A} \tag{8.12}$$

to the space of maps of symmetric monoidal categories

$$\mathrm{Rep}(\mathsf{G}) \to \mathbf{A} \otimes \mathrm{Rep}(\mathsf{H}), \tag{8.13}$$

equipped with an identification of the composition

$$\mathrm{Rep}(\mathsf{G}) \to \mathbf{A} \otimes \mathrm{Rep}(\mathsf{H}) \overset{\mathrm{Id}_{\mathbf{A}} \otimes \mathbf{oblv}_{\mathsf{H}}}{\longrightarrow} \mathbf{A}$$

with

$$\mathrm{Rep}(\mathsf{G}) \overset{\mathbf{oblv}_{\mathsf{G}}}{\to} \mathrm{Vect}_{\mathsf{e}} \overset{\mathbf{1}_{\mathbf{A}}}{\to} \mathbf{A}$$

is an isomorphism.

8.3.3. Denote $\mathsf{T}_0 := Z(\mathsf{H})^0$, the neutral connected component of the center of H.

Restriction defines a map from the spaces of maps in (8.12) and (8.13) for H to the corresponding space for T_0. Note that we already know by Sect. 8.2 that the map between the latter spaces is an isomorphism.

In particular, we can assume that we are dealing with the connected components of the spaces (8.12) and (8.13), respectively, for which the corresponding functor

$$\mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathrm{Rep}(\mathsf{T}_0))^{\mathrm{rigid}}) \to \mathbf{A}$$

factors through a single connected component of

$$\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathrm{Rep}(\mathsf{T}_0))^{\mathrm{rigid}} \simeq \mathbf{Maps}_{\mathrm{Grp}}(\mathsf{T}_0, \mathsf{G}),$$

corresponding to a conjugacy class of maps $\mathsf{T}_0 \to \mathsf{G}$.

8.3.4. In this case, the argument in Sect. 8.1 applies: we replace the fact that a semi-simple group has only a finite number of isomorphism classes of representations of a given dimension by the fact that for a reductive group there is a finite number of isomorphism classes of representations of a given dimension *and* with a given finite set of central characters.

8.4. **The case of a general algebraic group (via Harish-Chandra pairs).** In this subsection we will finally prove:

Theorem 8.4.1. *The functor* (6.6) *is an equivalence for* $\mathbf{H} = \mathrm{Rep}(\mathsf{H})$*, where* H *is a finite-dimensional algebraic group.*

We have already established the case when H was reductive. We will approach a general H by first replacing it by a Harish-Chandra pair $(\mathfrak{h}, \mathsf{H}_{\mathrm{red}})$, and then bootstrapping the fact that (6.6) is an equivalence for $\mathbf{H} = \mathrm{Rep}(\mathsf{H})$ from the fact that it is an equivalence for $\mathbf{H} = (\mathfrak{h}, \mathsf{H}_{\mathrm{red}})$-mod.

8.4.2. Choose a Levi decomposition

$$1 \to \mathsf{H}_u \to \mathsf{H} \leftrightarrows \mathsf{H}_{\mathrm{red}} \to 1.$$

In particular, we can think of $\mathsf{H}_{\mathrm{red}}$ as a subgroup of H. Consider the Harish-Chandra pair $(\mathfrak{h}, \mathsf{H}_{\mathrm{red}})$ and the corresponding symmetric monoidal category $(\mathfrak{h}, \mathsf{H}_{\mathrm{red}})$-mod, which is by definition the same as representations of the formal completion $\mathsf{H}^\wedge$ of H along $\mathsf{H}_{\mathrm{red}}$.

On the one hand, we can consider the prestack

$$\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}^\wedge, \mathsf{G}),$$

which, as in Proposition 7.4.2(a) and Sect. 8.1.3, we show is isomorphic to a disjoint union of affine schemes.

On the other hand, we can consider the symmetric monoidal category

$$\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), (\mathfrak{h}, \mathsf{H}_{\mathrm{red}})\text{-mod})^{\mathrm{rigid}}.$$

As in Sect. 7.4, we have a canonically defined map

$$\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), (\mathfrak{h}, \mathsf{H}_{\mathrm{red}})\text{-mod})^{\mathrm{rigid}} \to \mathrm{QCoh}(\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}^\wedge, \mathsf{G})). \tag{8.14}$$

We will prove:

Theorem 8.4.3. *The functor* (8.14) *is an equivalence.*

As in the case of Theorem 7.4.5, the assertion of Theorem 8.4.3 implies:

Corollary 8.4.4. *The functor* (6.6) *is an equivalence for* $\mathbf{H} = (\mathfrak{h}, \mathsf{H}_{\mathrm{red}})$-mod.

8.4.5. Before we proceed to the proof of Theorem 8.4.3 , let us show how it implies Theorem 8.4.1.

Given a symmetric monoidal category $\mathbf{A}$, we need to show that the map from the space of maps

$$\mathrm{QCoh}(\mathbf{Maps}(\mathsf{H}, \mathsf{G})) \to \mathbf{A}$$

to the space of symmetric monoidal functors

$$\mathsf{F} : \mathrm{Rep}(\mathsf{G}) \to \mathbf{A} \otimes \mathrm{Rep}(\mathsf{H}),$$

equipped with an identification of the composite

$$\mathrm{Rep}(\mathsf{G}) \to \mathbf{A} \otimes \mathrm{Rep}(\mathsf{H}) \overset{\mathrm{Id}_{\mathbf{A}} \otimes \mathbf{oblv}_{\mathsf{H}}}{\longrightarrow} \mathbf{A}$$

with

$$\mathrm{Rep}(\mathsf{G}) \overset{\mathbf{oblv}_{\mathsf{G}}}{\to} \mathrm{Vect}_{\mathsf{e}} \overset{\mathbf{1}_{\mathbf{A}}}{\longrightarrow} \mathbf{A},$$

is an isomorphism.

The assertion of Theorem 8.4.3 is that the map from the space of symmetric monoidal functors

$$\mathrm{QCoh}(\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}^\wedge, \mathsf{G})) \to \mathbf{A} \tag{8.15}$$

to the space of symmetric monoidal functors

$$\operatorname{Rep}(\mathsf{G}) \to \mathbf{A} \otimes (\mathfrak{h}, \mathsf{H}_{\mathrm{red}})\text{-mod}, \tag{8.16}$$

equipped with an identification of the composition

$$\operatorname{Rep}(\mathsf{G}) \to \mathbf{A} \otimes (\mathfrak{h}, \mathsf{H}_{\mathrm{red}})\text{-mod} \overset{\mathrm{Id}_{\mathbf{A}} \otimes \mathbf{oblv}_{\mathsf{H}^\wedge}}{\longrightarrow} \mathbf{A}$$

with

$$\operatorname{Rep}(\mathsf{G}) \overset{\mathbf{oblv}_{\mathsf{G}}}{\to} \mathrm{Vect}_{\mathsf{e}} \overset{\mathbf{1}_{\mathbf{A}}}{\to} \mathbf{A},$$

is an isomorphism.

We claim that the latter isomorphism implies the former. Indeed, this follows by the argument in Sects. 7.2.1-7.2.8 using the fact that the restriction functor

$$\operatorname{Rep}(\mathsf{H}) \to (\mathfrak{h}, \mathsf{H}_{\mathrm{red}})\text{-mod}$$

is fully faithful.

8.4.6. The rest of this subsection is devoted to the proof of Theorem 8.4.3. Note that $\mathsf{H}^\wedge$ can be written as a semi-direct product

$$\mathsf{H}_{\mathrm{red}} \ltimes \mathsf{H}_u^\wedge,$$

where $\mathsf{H}_u^\wedge$ is the formal completion of H_u at the origin. We will prove Theorem 8.4.3 more generally for semi-direct products

$$\mathsf{H}_{\mathrm{red}} \ltimes \mathsf{H}_1^\wedge,$$

where:

- $\mathsf{H}_{\mathrm{red}}$ is a reductive group;
- $\mathsf{H}_1^\wedge$ is formal derived group attached via [GR2, Chapter 6, Theorem 3.1.4] to a connective Lie algebra $\mathfrak{h}_1$ equipped with an action of $\mathsf{H}_{\mathrm{red}}$, i.e., an object of the category $\operatorname{LieAlg}(\operatorname{Rep}(\mathsf{H}_{\mathrm{red}}))$, whose underlying object of $\operatorname{Rep}(\mathsf{H}_{\mathrm{red}})$ is connective.

8.4.7. By Sect. 8.3, we know that the datum of a symmetric monoidal functor

$$\operatorname{QCoh}(\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}_{\mathrm{red}}, \mathsf{G})) \to \mathbf{A}, \tag{8.17}$$

is equivalent to that of a symmetric monoidal functor

$$\operatorname{Rep}(\mathsf{G}) \to \mathbf{A} \otimes \operatorname{Rep}(\mathsf{H}_{\mathrm{red}}) \tag{8.18}$$

Consider the functor

$$\operatorname{Rep}(\mathsf{H}^\wedge) \to \operatorname{Rep}(\mathsf{H}_{\mathrm{red}}),$$

given by restriction, and the corresponding map

$$\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}^\wedge, \mathsf{G}) \to \mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}_{\mathrm{red}}, \mathsf{G}).$$

Thus, we need to show that the datum of extension of (8.17) to a functor

$$\operatorname{QCoh}(\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}^\wedge, \mathsf{G})) \to \mathbf{A} \tag{8.19}$$

is equivalent to the datum of lifting (8.18) to a functor

$$\operatorname{Rep}(\mathsf{G}) \to \mathbf{A} \otimes \operatorname{Rep}(\mathsf{H}^\wedge). \tag{8.20}$$

8.4.8. First off, the argument in Sects. 7.6.1-7.6.2 and Sect. 8.3 reduces the assertion to the case when $\mathbf{A} = A\text{-mod}$ for $A \in \operatorname{ComAlg}(\mathrm{Vect}_{\mathsf{e}}^{\leq 0})$.

Next, as in Sect. 7.6.3, we can assume that $\mathfrak{h}_1$ is of the form

$$\mathbf{free}_{\mathrm{Lie}}(V), \quad V \in \operatorname{Rep}(\mathsf{H}_{\mathrm{red}})^c \cap \operatorname{Rep}(\mathsf{H}_{\mathrm{red}})^{\leq 0}.$$

8.4.9. By Sect. 8.3, the datum of (8.17) (equivalently, (8.18)) is equivalent to that of a map

$$\mathrm{Spec}(A) \to \mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}_{\mathrm{red}}, \mathsf{G}),$$

or equivalently a map of group prestacks over $\mathrm{Spec}(A)$

$$\phi : \mathrm{Spec}(A) \times \mathsf{H}_{\mathrm{red}} \to \mathrm{Spec}(A) \times \mathsf{G}. \tag{8.21}$$

As in Sect. 7.6.3, the datum of extension of (8.17) to (8.19) is equivalent to that of a map

$$A \otimes V \to A \otimes \mathbf{oblv}_{\mathrm{Lie}}(\mathfrak{g}) \tag{8.22}$$

in the category of A-modules equipped with an action of $\mathsf{H}_{\mathrm{red}}$, where $A \otimes \mathbf{oblv}_{\mathrm{Lie}}(\mathfrak{g})$ acquires an action of $\mathsf{H}_{\mathrm{red}}$ via the map ϕ of (8.21).

8.4.10. Let

$$A' := A \otimes \mathbf{1}_{\mathrm{Rep}(\mathsf{H}_{\mathrm{red}})} \oplus \epsilon \cdot V^* \in \mathrm{ComAlg}(\mathrm{Rep}(\mathsf{H}_{\mathrm{red}})) \simeq \mathrm{ComAlg}(\mathrm{QCoh}(\mathrm{pt}/\mathsf{H}_{\mathrm{red}})),$$

which we regard as a split square-zero of

$$A_{\mathrm{Rep}(\mathsf{H}_{\mathrm{red}})} := A \otimes \mathbf{1}_{\mathrm{Rep}(\mathsf{H}_{\mathrm{red}})}.$$

Then the datum of extension of (8.18) to (8.20) is equivalent to that of an automorphism of the symmetric monoidal functor

$$\mathrm{Rep}(\mathsf{G}) \xrightarrow{\phi} A_{\mathrm{Rep}(\mathsf{H}_{\mathrm{red}})}\text{-mod}(\mathrm{Rep}(\mathsf{H}_{\mathrm{red}})) \to A'\text{-mod}(\mathrm{Rep}(\mathsf{H}_{\mathrm{red}})) \tag{8.23}$$

with the trivialization of the induced automorphism of the symmetric monoidal functor

$$\mathrm{Rep}(\mathsf{G}) \xrightarrow{\phi} A_{\mathrm{Rep}(\mathsf{H}_{\mathrm{red}})}\text{-mod}(\mathrm{Rep}(\mathsf{H}_{\mathrm{red}})) \to A'\text{-mod}(\mathrm{Rep}(\mathsf{H}_{\mathrm{red}})) \to A_{\mathrm{Rep}(\mathsf{H}_{\mathrm{red}})}\text{-mod}(\mathrm{Rep}(\mathsf{H}_{\mathrm{red}})), \tag{8.24}$$

the latter being the initial functor

$$\mathrm{Rep}(\mathsf{G}) \xrightarrow{\phi} A_{\mathrm{Rep}(\mathsf{H}_{\mathrm{red}})}\text{-mod}(\mathrm{Rep}(\mathsf{H}_{\mathrm{red}})).$$

The map ϕ allows to form the quotient $\mathrm{Spec}(A) \times \mathsf{G}/\mathrm{Ad}(\mathsf{H}_{\mathrm{red}})$ as a group-scheme over

$$\mathrm{Spec}(A) \times \mathrm{pt}/\mathsf{H}_{\mathrm{red}} = \mathrm{Spec}_{\mathrm{pt}/\mathsf{H}_{\mathrm{red}}}(A_{\mathrm{Rep}(\mathsf{H}_{\mathrm{red}})}).$$

By Tannaka duality, the datum of (8.23) with a trivialization of (8.24) is equivalent to that of a map

$$\mathrm{Spec}_{\mathrm{pt}/\mathsf{H}_{\mathrm{red}}}(A') \to \mathrm{Spec}(A) \times \mathsf{G}/\mathrm{Ad}(\mathsf{H}_{\mathrm{red}}),$$

as schemes over $\mathrm{Spec}(A) \times \mathrm{pt}/\mathsf{H}_{\mathrm{red}}$, equipped with the identification of the composite map

$$\mathrm{Spec}(A) \times \mathrm{pt}/\mathsf{H}_{\mathrm{red}} \to \mathrm{Spec}_{\mathrm{pt}/\mathsf{H}_{\mathrm{red}}}(A') \to \mathrm{Spec}(A) \times \mathsf{G}/\mathrm{Ad}(\mathsf{H}_{\mathrm{red}})$$

with the unit section.

By deformation theory, the latter is the same as a datum of a map (8.22), as required.

9. Proof of Theorem 6.2.11

In this section we will prove Theorem 6.2.11 in general.

We have made extensive preparations for the proof in the general case, namely, we have shown that (6.6) is an equivalence for $\mathbf{H} = \mathrm{Rep}(\mathsf{H})$ for a finite-dimensional algebraic group H. However, unfortunately, these preparations do not lead us to the proof: we do not know how to approximate a given $\mathbf{H}$ by categories of the form $\mathrm{Rep}(\mathsf{H})$ so that we can deduce the validity of Theorem 6.2.11 for the former from the case of the latter.

So we will try a different route. We will prove directly that the functor (6.7), induced by (6.6), is *fully faithful*.

Once we do this, it will remain to establish the essential surjectivity. And here we will be able to use the already established case of $\mathbf{H} = \mathrm{Rep}(\mathsf{H})$.

9.1. **The diagonal of $\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})$.** In order to prove that the functor (6.7) is fully faithful, we should acquire an ability to make computations in the category $\mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})$.

As a preparation, in this subsection, we will give an explicit description of the object

$$(\Delta_{\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})})_*(\mathcal{O}_{\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})}) \in \mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H}) \times \mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})),$$

which is in fact the unit of a self-duality on $\mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})$.

9.1.1. First, we note that Theorem 1.3.2 (resp., Theorem 1.4.3) applies verbatim to $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)$ (resp., $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X)$) replaced by $\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})$ (resp., $\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})^{\mathrm{rigid}}$). Indeed, the proof only used the properties of the category $\mathrm{Rep}(\mathsf{G})$ that are listed as conditions for Theorem 6.2.11.

In particular, the material in Sects. 5.6-5.7 carries over to the present situation.

9.1.2. We construct a commutative algebra object

$$R_{\mathbf{coMaps}} \in \mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H}) \times \mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})) \simeq \\ \simeq \mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})) \otimes \mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})) \tag{9.1}$$

as follows.

Let us be given a pair of affine schemes S_1 and S_2 and maps

$$\sigma_i : S_i :\to \mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H}),$$

corresponding to symmetric monoidal functors

$$\mathsf{F}_i : \mathrm{Rep}(\mathsf{G}) \to \mathrm{QCoh}(S_i) \otimes \mathbf{H}, \quad i = 1, 2.$$

We will describe the pullback of $R_{\mathbf{coMaps}}$ to $S_1 \times S_2$. Namely, $R_{\mathbf{coMaps}}|_{S_1\times S_2}$ is the colimit over $\mathrm{TwArr}(\mathrm{fSet})$ of the functor that sends

$$(I \xrightarrow{\phi} J) \in \mathrm{TwArr}(\mathrm{fSet})$$

to the image of

$$R_{\mathsf{G}}^{\otimes I} \in (\mathrm{Rep}(\mathsf{G}) \otimes \mathrm{Rep}(\mathsf{G}))^{\otimes I} \simeq \mathrm{Rep}(\mathsf{G})^{\otimes I} \otimes \mathrm{Rep}(\mathsf{G})^{\otimes I}$$

along

$$\mathrm{Rep}(\mathsf{G})^{\otimes I} \otimes \mathrm{Rep}(\mathsf{G})^{\otimes I} \overset{\mathsf{F}_1\otimes\mathsf{F}_2}{\to} \mathrm{QCoh}(S_1) \otimes \mathrm{QCoh}(S_2) \otimes \mathbf{H}^{\otimes I} \otimes \mathbf{H}^{\otimes I} \overset{\mathrm{Id}_{\mathrm{QCoh}(S_1)} \otimes \mathrm{Id}_{\mathrm{QCoh}(S_2)} \otimes \mathrm{mult}_{\mathbf{H}^{\otimes I}}}{\longrightarrow}$$

$$\to \mathrm{QCoh}(S_1) \otimes \mathrm{QCoh}(S_2) \otimes \mathbf{H}^{\otimes I} \overset{\mathrm{Id}_{\mathrm{QCoh}(S_1)} \otimes \mathrm{Id}_{\mathrm{QCoh}(S_2)} \otimes \mathrm{mult}^{\phi}_{\mathbf{H}}}{\longrightarrow}$$

$$\to \mathrm{QCoh}(S_1) \otimes \mathrm{QCoh}(S_2) \otimes \mathbf{H}^{\otimes J} \overset{\mathrm{Id}_{\mathrm{QCoh}(S_1)} \otimes \mathrm{Id}_{\mathrm{QCoh}(S_2)} \otimes (\mathbf{coinv}_{\mathbf{H}})^{\otimes J}}{\longrightarrow}$$

$$\to \mathrm{QCoh}(S_1) \otimes \mathrm{QCoh}(S_2),$$

where:

- $\mathrm{mult}_{\mathbf{H}^{\otimes I}}$ denotes the tensor product functor $\mathbf{H}^{\otimes I} \otimes \mathbf{H}^{\otimes I} \to \mathbf{H}^{\otimes I}$;
- R_{G} denotes the regular representation of G, regarded as an object of $\mathrm{Rep}(\mathsf{G}) \otimes \mathrm{Rep}(\mathsf{G})$;
- $\mathrm{mult}^{\phi}_{\mathbf{H}}$ denotes the tensor product map $\mathbf{H}^{\otimes I} \to \mathbf{H}^{\otimes J}$;
- $\mathbf{coinv}_{\mathbf{H}}$ denotes the left adjoint to the unit functor $\mathrm{Vect}_{\mathsf{e}} \to \mathbf{H}$, see Lemma 6.2.8.

In the formation of the colimit, the transition maps are given by:

- The maps
$$\mathrm{mult}^{I_0\to I_1}_{\mathbf{H}\otimes\mathbf{H}}(R_{\mathsf{G}}^{\otimes I_0}) \to R_{\mathsf{G}}^{\otimes I_1} \text{ for } I_0 \to I_1,$$
given by the commutative algebra structure on R_{G} where $\mathrm{mult}^{I_0\to I_1}_{\mathbf{H}\otimes\mathbf{H}}$ denotes the resulting functor $(\mathbf{H} \otimes \mathbf{H})^{\otimes I_0} \to (\mathbf{H} \otimes \mathbf{H})^{\otimes I_1}$.

- The natural transformations
$$\mathbf{coinv}_{\mathbf{H}}^{\otimes J_0} \circ \mathrm{mult}_{\mathbf{H}}^{J_1 \to J_0} \to \mathbf{coinv}_{\mathbf{H}}^{\otimes J_1} \text{ for } J_1 \to J_0,$$
obtained by adjunction.

The assignment
$$(\sigma_i : S_i \to \mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}),\ i = 1, 2) \rightsquigarrow R_{\mathbf{coMaps}}|_{S_1 \times S_2}$$
is manifestly compatible with pullbacks, so it gives rise to a well-defined object of
$$\mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}) \times \mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})).$$
The commutative algebra structure on $R_{\mathbf{coMaps}}$ is induced by the operation of disjoint union on fSet.

We claim:

Theorem 9.1.3. *There exists a canonical isomorphism of commutative algebra objects in the category* $\mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})) \otimes \mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}))$*:*
$$R_{\mathbf{coMaps}} \simeq (\Delta_{\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})})_*(\mathcal{O}_{\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})}).$$

The rest of this subsection is devoted to the proof of this theorem.

9.1.4. Let (σ_i, F_i), $i = 1, 2$ be as above. Let σ_i' denote the compisite
$$S_1 \times S_2 \to S_i \xrightarrow{\sigma_i} \mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}),$$
and let F_i' denote the resulting symmetric monoidal functor
$$\mathrm{Rep}(\mathsf{G}) \xrightarrow{\mathsf{F}_i} \mathrm{QCoh}(S_i) \otimes \mathbf{H} \to \mathrm{QCoh}(S_1 \times S_2) \otimes \mathbf{H}.$$

The assertion of Theorem 9.1.3 is that the space of isomorphisms of symmetric monoidal functors $\mathsf{F}_1' \to \mathsf{F}_2'$ maps isomorphically to the space of maps of commutative algebras in $\mathrm{QCoh}(S_1 \times S_2)$
$$R_{\mathbf{coMaps}}|_{S_1 \times S_2} \to \mathcal{O}_{S_1 \times S_2}.$$

9.1.5. Consider the following general situation. Let $\mathbf{A}$ be a symmetric monoidal category, and let us be given a pair of symmetric monoidal functors
$$\mathsf{F}_1, \mathsf{F}_2 : \mathrm{Rep}(\mathsf{G}) \to \mathbf{A}.$$

Consider the commutative algebra object $R_{\mathsf{F}_1,\mathsf{F}_2} \in \mathbf{A}$, obtained by applying to $R_{\mathsf{G}} \in \mathrm{Rep}(\mathsf{G}) \otimes \mathrm{Rep}(\mathsf{G})$ the symmetric monoidal functor
$$\mathrm{Rep}(\mathsf{G}) \otimes \mathrm{Rep}(\mathsf{G}) \xrightarrow{\mathsf{F}_1 \otimes \mathsf{F}_2} \mathbf{A} \otimes \mathbf{A} \xrightarrow{\mathrm{mult}} \mathbf{A}.$$

Then it is easy to see that the space of isomorphisms between F_1 and F_2 identifies canonically with the space of maps of commutative algebras
$$R_{\mathsf{F}_1,\mathsf{F}_2} \to \mathbf{1}_{\mathbf{A}}.$$

9.1.6. Hence, in order to prove Theorem 9.1.3, it remains to show that the space of maps of commutative algebras in $\mathbf{H} \otimes \mathrm{QCoh}(S_1 \times S_2)$
$$(\mathrm{Id}_{\mathrm{QCoh}(S_1 \times S_2)} \otimes \mathrm{mult}_{\mathbf{H}})(\mathsf{F}_1' \otimes \mathsf{F}_2')(R_{\mathsf{G}}) \to \mathcal{O}_{S_1 \times S_2} \otimes \mathbf{1}_{\mathbf{H}}$$
is canonically isomorphic to the space of maps of commutative algebras in $\mathrm{QCoh}(S_1 \times S_2)$
$$R_{\mathbf{coMaps}}|_{S_1 \times S_2} \to \mathcal{O}_{S_1 \times S_2}.$$

This follows from the next general assertion.

9.1.7. Let $\mathbf{A}$ and $\mathbf{A}'$ be a pair of symmetric monoidal categories, and let $\Phi : \mathbf{A}' \to \mathbf{A}$ be a symmetric monoidal functor that admits a left adjoint, denoted Φ^L, as a functor of plain DG categories.

Then the induced functor
$$\Phi : \mathrm{ComAlg}(\mathbf{A}') \to \mathrm{ComAlg}(\mathbf{A})$$
admits a left adjoint, to be denoted $\Phi^{L,\mathrm{ComAlg}}$, which is described as follows.

Define the functor
$$\widetilde{\Phi}^{L,\mathrm{ComAlg}} : \mathrm{ComAlg}(\mathbf{A}) \to \mathrm{ComAlg}(\mathbf{A}')$$
as follows:

Its value on $R \in \mathrm{ComAlg}(\mathbf{A})$ is given by the colimit over $\mathrm{TwArr}(\mathrm{fSet})$ of the functor that sends
$$(I \overset{\phi}{\to} J) \in \mathrm{TwArr}(\mathrm{fSet}) \tag{9.2}$$
to
$$\mathrm{mult}^J_{\mathbf{A}'} \circ (\Phi^L)^{\otimes J} \circ \mathrm{mult}^\phi_{\mathbf{A}}(R^{\otimes I}),$$
where $\mathrm{mult}^J_{\mathbf{A}'}$ is the J-fold tensor product functor
$$(\mathbf{A}')^{\otimes J} \to \mathbf{A}'.$$

In the formation of the colimit, transition maps are defined as in Sect. 9.1.2, and the commutative algebra structure on $\widetilde{\Phi}^{L,\mathrm{ComAlg}}$ is induced by the operation of disjoint union on fSet.

Proposition 9.1.8. *The functor $\widetilde{\Phi}^{L,\mathrm{ComAlg}}$ is canonically isomorphic to the left adjoint, denoted $\Phi^{L,\mathrm{ComAlg}}$, of*
$$\Phi : \mathrm{ComAlg}(\mathbf{A}') \to \mathrm{ComAlg}(\mathbf{A}).$$

This proposition is apparently well-known. We sketch a proof for competeness:

Proof. For $R' \in \mathrm{ComAlg}(\mathbf{A}')$, the space of maps of commutative algebras $\widetilde{\Phi}^{L,\mathrm{ComAlg}}(R) \to R'$ is the limit over $\mathrm{TwArr}(\mathrm{fSet})$ of the functor that sends (9.2) to the space of maps
$$(\Phi^L)^{\otimes J} \circ \mathrm{mult}^\phi_{\mathbf{A}}(R^{\otimes I}) \to (R')^{\otimes J},$$
which is the same as the space of maps
$$\mathrm{mult}^\phi_{\mathbf{A}}(R^{\otimes I}) \to \Phi^{\otimes J}((R')^{\otimes J}).$$

However, the latter limit is the same as the space of compatible maps
$$R^{\otimes I} \to \Phi^{\otimes I}((R')^{\otimes I}), \quad I \in \mathrm{fSet},$$
i.e., the space of maps $R \to \Phi(R')$ in $\mathrm{ComAlg}(\mathbf{A}')$, as required.

$\square$

Remark 9.1.9. The above description of the left adjoint to $\Phi : \mathrm{ComAlg}(\mathbf{A}') \to \mathrm{ComAlg}(\mathbf{A})$ is most familiar in the context of *factorization homology*. Namely, take
$$\mathbf{A} = (\mathrm{Shv}(X), \overset{!}{\otimes}),\ \mathbf{A}' = \mathrm{Vect}_{\mathsf{e}},\ \Phi(\mathsf{e}) = \omega_X.$$
Then the functor $\Phi^{L,\mathrm{ComAlg}}$ is the functor of factorization homology
$$\mathrm{ComAlg}^!(\mathrm{Shv}(X)) \to \mathrm{Vect}_{\mathsf{e}}.$$
It is naturally a composition of two functors:
$$R \mapsto \mathrm{Fact}(R), \quad \mathrm{ComAlg}^!(\mathrm{Shv}(X)) \to \mathrm{Shv}(\mathrm{Ran}(X)),$$
where
$$\mathrm{Fact}(R) \simeq \underset{(I \overset{\phi}{\to} J) \in \mathrm{TwArr}(\mathrm{fSet})}{\mathrm{colim}} (\Delta_J)_!(\mathrm{mult}^\phi_{\mathbf{A}}(R^{\otimes I}))$$
(here Δ_J is the diagonal map $X^J \to \mathrm{Ran}(X)$), followed by
$$\mathrm{C}^\cdot_c : \mathrm{Shv}(\mathrm{Ran}(X)) \to \mathrm{Vect}_{\mathsf{e}}.$$

The composite functor recovers our formula for $\widetilde{\Phi}^{L,\mathrm{ComAlg}}$.

9.1.10. We apply Proposition 9.1.8 to

$$\mathbf{A} := \mathrm{QCoh}(S_1 \times S_2) \otimes \mathbf{H},\ \mathbf{A}' = \mathrm{QCoh}(S_1 \times S_2),\ \Phi = \mathrm{Id}_{\mathrm{QCoh}(S_1 \times S_2)} \otimes \mathbf{1}_{\mathbf{H}}$$

with

$$R := (\mathrm{Id}_{\mathrm{QCoh}(S_1 \times S_2)} \otimes \mathrm{mult}_{\mathbf{H}})(\mathsf{F}'_1 \otimes \mathsf{F}'_2)(R_{\mathsf{G}})$$

and

$$R' := \mathcal{O}_{S_1 \times S_2},$$

whence the isomorphism stated in Sect. 9.1.5.

9.2. **Proof of fully-faithfulness.**

9.2.1. In this subsection, we will show that the functor

$$\mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}))^{\vee} \to \underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\vee} \tag{9.3}$$

induced by (6.6), is *fully faithful*.

Remark 9.2.2. The same proof will show that the functor

$$\mathrm{Funct}_{\mathrm{cont}}(\mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})), \mathbf{C}) \to \mathrm{Funct}_{\mathrm{cont}}(\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}), \mathbf{C})$$

is fully faithful for any target DG category $\mathbf{C}$. The latter fact implies that the initial functor (6.6) is actually a *localization*.

9.2.3. As a first step, we will describe the functor (9.3) more explicitly. Namely, recall from Lemma 6.2.2(b), that an object F of the category $\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\vee}$ is a compatible collection of functors

$$\mathsf{F}_I : \mathrm{Rep}(\mathsf{G})^{\otimes I} \to \mathbf{H}^{\otimes I}, \quad I \in \mathrm{fSet}\,. \tag{9.4}$$

Recall also that by Corollary 5.7.10, we have a canonical identification

$$\mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}))^{\vee} \simeq \mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})).$$

Let $\mathcal{M}$ be an object of $\mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$. Then the description of the counit of the self-duality on $\mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}))$ given by Corollary 5.7.12 implies that the collection of functors (9.4), corresponding to $\mathcal{M}$ is given by

$$\mathrm{Rep}(\mathsf{G})^{\otimes I} \to \mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}) \otimes \mathbf{H}^{\otimes I} \overset{\mathcal{M}\otimes -}{\longrightarrow} \tag{9.5}$$
$$\to \mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}) \otimes \mathbf{H}^{\otimes I} \overset{\Gamma_!(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}), -)\otimes \mathrm{Id}_{\mathbf{H}^{\otimes I}}}{\longrightarrow} \mathbf{H}^{\otimes I},$$

where the first arrow comes from the functor (6.6).

9.2.4. Taking into account Corollary 5.7.8, in order to prove the fully faithfulness of (9.3), we need to show the following:

Let us be given a pair of affine schemes S_1 and S_2 and maps $\sigma_i : S_i \to \mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$. Let $\mathcal{M}_i$ be objects of $\mathrm{QCoh}(S_i)$. Let F_i denote the objects of the category $\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\vee}$ attached to

$$(\sigma_i)_*(\mathcal{M}_i) \in \mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})) \simeq \mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}))^{\vee}$$

by the functor (9.3).

We need to show that the map

$$\mathcal{H}om_{\mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}))}((\sigma_1)_*(\mathcal{M}_1), (\sigma_2)_*(\mathcal{M}_2)) \to \mathcal{H}om_{\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\vee}}(\mathsf{F}_1, \mathsf{F}_2), \tag{9.6}$$

induced by the functor (9.3), is an isomorphism.

9.2.5. By base change, we rewrite the LHS in (9.6) as

$$\mathcal{H}om_{\mathrm{QCoh}(S_2)}\left((p_2)_*\left((p_1)^*(\mathcal{M}_1)\otimes(\mathcal{O}_{S_1}\underset{\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})}{\times}S_2)\right),\mathcal{M}_2\right),$$

where p_i, $i=1,2$, are the two projections $S_1\times S_2\to S_i$. By Theorem 9.1.3, we can rewrite this further as

$$\mathcal{H}om_{\mathrm{QCoh}(S_2)}\left((p_2)_*\left((p_1)^*(\mathcal{M}_1)\otimes R_{\mathbf{coMaps}}|_{S_1\times S_2}\right),\mathcal{M}_2\right).$$

Hence, we need to establish the isomorphism

$$\mathcal{H}om_{\mathrm{QCoh}(S_2)}\left((p_2)_*((p_1)^*(\mathcal{M}_1)\otimes R_{\mathbf{coMaps}}|_{S_1\times S_2}),\mathcal{M}_2\right)\simeq\mathcal{H}om_{\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})^\vee}(\mathsf{F}_1,\mathsf{F}_2). \tag{9.7}$$

9.2.6. In order to unburden the notation, we will assume that $S_1=S_2=\mathrm{pt}$ and $\mathcal{M}_1=\mathcal{M}_2=\mathsf{e}$. The proof in the general case is the same. Thus, the LHS in (9.7) becomes

$$\mathcal{H}om_{\mathrm{Vect}_{\mathsf{e}}}((\sigma_1\times\sigma_2)^*(R_{\mathbf{coMaps}}),\mathsf{e}), \tag{9.8}$$

while F_i correspond, in terms of (9.4), to the functors

$$\mathsf{F}_i^{\otimes I}:\mathrm{Rep}(\mathsf{G})^{\otimes I}\to\mathbf{H}^{\otimes I}.$$

We rewrite (9.8) as the limit over TwArr(fSet) of the functor that sends $(I\overset{\phi}{\to}J)$ to

$$\begin{aligned}\mathcal{H}om_{\mathrm{Vect}_{\mathsf{e}}}&\left(\mathbf{coinv}_{\mathbf{H}^{\otimes J}}\circ\mathrm{mult}^\phi_{\mathbf{H}}\circ\mathrm{mult}_{\mathbf{H}^{\otimes I}}\circ(\mathsf{F}_1^{\otimes I}\otimes\mathsf{F}_2^{\otimes I})(R_{\mathsf{G}}^{\otimes I}),\mathsf{e}\right)\simeq\\ &\simeq\mathcal{H}om_{\mathbf{H}^{\otimes J}}\left(\mathrm{mult}^\phi_{\mathbf{H}}\circ\mathrm{mult}_{\mathbf{H}^{\otimes I}}\circ(\mathsf{F}_1^{\otimes I}\otimes\mathsf{F}_2^{\otimes I})(R_{\mathsf{G}}^{\otimes I}),\mathbf{1}_{\mathbf{H}}^{\otimes J}\right)\simeq\\ &\simeq\mathcal{H}om_{\mathbf{H}^{\otimes J}}\left(\mathrm{mult}_{\mathbf{H}^{\otimes J}}\circ(\mathrm{mult}^\phi_{\mathbf{H}}\otimes\mathrm{mult}^\phi_{\mathbf{H}})\circ(\mathsf{F}_1^{\otimes I}\otimes\mathsf{F}_2^{\otimes I})(R_{\mathsf{G}}^{\otimes I}),\mathbf{1}_{\mathbf{H}}^{\otimes J}\right)\simeq\\ &\simeq\mathcal{H}om_{\mathbf{H}^{\otimes J}}\left(\mathrm{mult}_{\mathbf{H}^{\otimes J}}\circ(\mathrm{mult}^\phi_{\mathbf{H}}\otimes\mathrm{mult}^\phi_{\mathbf{H}})\circ(\mathsf{F}_2^{\otimes I}\otimes\mathsf{F}_1^{\otimes I})(R_{\mathsf{G}}^{\otimes I}),\mathbf{1}_{\mathbf{H}}^{\otimes J}\right)\end{aligned} \tag{9.9}$$

.

We rewrite the RHS in (9.7) also as the limit over TwArr(fSet) of the functor that sends $(I\overset{\phi}{\to}J)$ to

$$\begin{aligned}&\mathcal{H}om_{\mathrm{Funct}_{\mathrm{cont}}(\mathrm{Rep}(\mathsf{G})^{\otimes I},\mathbf{H}^{\otimes J})}(\mathrm{mult}^\phi_{\mathbf{H}}\circ\mathsf{F}_1^{\otimes I},\mathrm{mult}^\phi_{\mathbf{H}}\circ\mathsf{F}_2^{\otimes I})\simeq\\ &\simeq\mathcal{H}om_{\mathrm{Rep}(\mathsf{G})^{\otimes I}\otimes\mathbf{H}^{\otimes J}}\left((\mathrm{Id}_{\mathrm{Rep}(\mathsf{G})^{\otimes I}}\otimes(\mathrm{mult}^\phi_{\mathbf{H}}\circ\mathsf{F}_1^{\otimes I}))(R_{\mathsf{G}}^{\otimes I}),(\mathrm{Id}_{\mathrm{Rep}(\mathsf{G})^{\otimes I}}\otimes(\mathrm{mult}^\phi_{\mathbf{H}}\circ\mathsf{F}_2^{\otimes I}))(R_{\mathsf{G}}^{\otimes I})\right).\end{aligned} \tag{9.10}$$

9.2.7. We now claim that the expressions in (9.9) and (9.10) match term-wise. Namely, we claim that the value of the right adjoint of the functor

$$\mathrm{Rep}(\mathsf{G})^{\otimes I}\otimes\mathbf{H}^{\otimes J}\overset{\mathsf{F}_2^{\otimes I}\otimes\mathrm{Id}}{\longrightarrow}\mathbf{H}^{\otimes I}\otimes\mathbf{H}^{\otimes J}\overset{\mathrm{mult}^\phi_{\mathbf{H}}\otimes\mathrm{Id}}{\longrightarrow}\mathbf{H}^{\otimes J}\otimes\mathbf{H}^{\otimes J}\overset{\mathrm{mult}_{\mathbf{H}^{\otimes J}}}{\longrightarrow}\mathbf{H}^{\otimes J}$$

on $\mathbf{1}_{\mathbf{H}}^{\otimes J}\in\mathbf{H}^{\otimes J}$ identifies canonically with

$$(\mathrm{Id}_{\mathrm{Rep}(\mathsf{G})^{\otimes I}}\otimes(\mathrm{mult}^\phi_{\mathbf{H}}\circ\mathsf{F}_2^{\otimes I}))(R_{\mathsf{G}}^{\otimes I}).$$

Indeed, since we are dealing with symmetric monoidal functors between rigid categories, we can replace "right adjoint" by "dual". Now the assertion follows from the fact that R_{G} is the unit for the self-duality on $\mathrm{Rep}(\mathsf{G})$.

9.2.8. The fact that the isomorphism (9.7) established above is indeed the one induced by the map (9.6) is a straightforward verification.

9.3. **End of proof of Theorem 6.2.11.** In this subsection we will finish the proof of the fact that the functor (9.3) is an equivalence.

Given that we already know that (9.3) is fully faithful, it remains to establish the essential surjectivity. We will reduce this question to one at the level of abelian categories, and then deduce the required essential surjectivity from the case of $\mathbf{H}=\mathrm{Rep}(\mathsf{H})$.

9.3.1. By the 1-affineness of pt/G, the fully faithfulness assertion proved in the previous section implies that the functor

$$\mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})^{\mathrm{rigid}})^\vee \to (\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})^{\mathrm{rigid}})^\vee \tag{9.11}$$

is fully faithful. The assertion that (9.3) is an equivalence is equivalent to the fact that (9.11) is an equivalence.

9.3.2. We will think of objects $\mathsf{F} \in (\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})^{\mathrm{rigid}})^\vee$ as compatible collections of functors

$$\mathsf{F}_I : \mathrm{Rep}(\mathsf{G})^{\otimes I} \to \mathbf{H}^{\otimes I},$$

equipped with the following additional data: for every decomposition $I' \sqcup I'' = I$, we are given the datum of commutativity for

$$\begin{tikzcd} \mathrm{Rep}(\mathsf{G})^{\otimes I} \arrow[r, "\mathsf{F}_I"] \arrow[d, "\mathrm{Id}_{\mathrm{Rep}(\mathsf{G})^{\otimes I'}} \otimes \mathbf{oblv}_{\mathsf{G}}^{\otimes I''}"'] & \mathbf{H}^{\otimes I} \arrow[d, "\mathrm{Id}_{\mathbf{H}^{\otimes I'}} \otimes \mathbf{oblv}_{\mathbf{H}}^{\otimes I''}"] \\ \mathrm{Rep}(\mathsf{G})^{\otimes I'} \arrow[r, "\mathsf{F}_{I'}"] & \mathbf{H}^{\otimes I'}. \end{tikzcd} \tag{9.12}$$

In particular, we have a commutative square

$$\begin{tikzcd} \mathrm{Rep}(\mathsf{G})^{\otimes I} \arrow[r, "\mathsf{F}_I"] \arrow[d, "\mathbf{oblv}_{\mathsf{G}^I}"'] & \mathbf{H}^{\otimes I} \arrow[d, "(\mathbf{oblv}_{\mathbf{H}})^{\otimes I}"] \\ \mathrm{Vect}_{\mathsf{e}} \arrow[r, "\mathsf{F}_\emptyset"] & \mathrm{Vect}_{\mathsf{e}}. \end{tikzcd} \tag{9.13}$$

Since the functors $(\mathbf{oblv}_{\mathbf{H}})^{\otimes I}$ are conservative, we obtain that the assignment

$$\mathsf{F} \mapsto \mathsf{F}_\emptyset, \tag{9.14}$$

viewed as a functor

$$(\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})^{\mathrm{rigid}})^\vee \to \mathrm{Vect},$$

is conservative.

9.3.3. From the above description of $(\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})^{\mathrm{rigid}})^\vee$, we obtain that it carries a t-structure, uniquely characterized by the property that the functor (9.14) is t-exact.

Since $\mathbf{H}$ is left-complete in its t-structure, we obtain that $(\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})^{\mathrm{rigid}})^\vee$ is also left-complete in its t-structure.

9.3.4. We can think of objects of $((\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})^{\mathrm{rigid}})^\vee)^\heartsuit$ as follows. We can identify the abelian category $\mathbf{H}^\heartsuit$ with the category of representations of a pro-algebraic group, to be denoted H.

Then the datum of an object of $((\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})^{\mathrm{rigid}})^\vee)^\heartsuit$ is a vector space $\mathsf{F}_\emptyset \in \mathrm{Vect}_{\mathsf{e}}^\heartsuit$, equipped with the following additional structure:

(*) For every finite set I and $W \in \mathrm{Rep}(\mathsf{G}^I)$, we are given an action of H^I on $\mathsf{F}_\emptyset \otimes \mathbf{oblv}_{\mathsf{G}^I}(W)$.

We have the following two compatibility conditions:

- (i) For a map $I \to J$ and $W \in \mathrm{Rep}(\mathsf{G}^I)$, the action of H^J on $\mathsf{F}_\emptyset \otimes \mathbf{oblv}_{\mathsf{G}^I}(W)$, obtained from the diagonal embedding $\mathsf{H}^J \to \mathsf{H}^I$, equals the action obtained from identifying

  $$\mathsf{F}_\emptyset \otimes \mathbf{oblv}_{\mathsf{G}^I}(W) \simeq \mathsf{F}_\emptyset \otimes \mathbf{oblv}_{\mathsf{G}^J}(\mathrm{Res}^{\mathsf{G}^I}_{\mathsf{G}^J}(W))$$

  and the H^J-action on

  $$\mathsf{F}_\emptyset \otimes \mathbf{oblv}_{\mathsf{G}^J}(\mathrm{Res}^{\mathsf{G}^I}_{\mathsf{G}^J}(W)),$$

  coming from the data (*) for J and $\mathrm{Res}^{\mathsf{G}^I}_{\mathsf{G}^J}(W) \in \mathrm{Rep}(\mathsf{G}^J)$.

- (ii) For every element $i \in I$, the action of the i-th factor $\mathsf{H} \hookrightarrow \mathsf{H}^I$ on $\mathsf{F}_\emptyset \otimes \mathbf{oblv}_{\mathsf{H}^I}(W)$ equals the action obtained by identifying
$$\mathsf{F}_\emptyset \otimes \mathbf{oblv}_{\mathsf{G}^I}(W) \simeq \mathsf{F}_\emptyset \otimes \mathbf{oblv}_{\mathsf{G}}(\mathrm{Res}^{\mathsf{G}^I}_{\mathsf{G}}(W))$$
(here $\mathsf{G} \to \mathsf{G}^I$ is also the embedding of the i-th factor) and the H-action on
$$\mathsf{F}_\emptyset \otimes \mathbf{oblv}_{\mathsf{G}}(\mathrm{Res}^{\mathsf{G}^I}_{\mathsf{G}}(W)),$$
coming from the data (*) for the one-element set $\{*\}$ and $\mathrm{Res}^{\mathsf{G}^I}_{\mathsf{G}}(W) \in \mathrm{Rep}(\mathsf{G})$.

9.3.5. The functor (9.11) makes the following diagram commute:
$$\begin{CD}
\mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\mathrm{rigid}})^\vee @>{(9.11)}>> (\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\mathrm{rigid}})^\vee \\
@V{\text{Corollary 5.6.7}}V{\sim}V @VV{(9.14)}V \\
\mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\mathrm{rigid}}) @>{\Gamma_!(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\mathrm{rigid}}, -)}>> \mathrm{Vect}_{\mathsf{e}}\,.
\end{CD}$$

We equip the category $\mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\mathrm{rigid}})^\vee$ with a t-structure via its identification with $\mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\mathrm{rigid}})$. The above commutative diagram, combined with Proposition 5.5.6 implies that the functor (9.11) is t-exact.

Since both categories
$$\mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\mathrm{rigid}})^\vee \text{ and } (\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\mathrm{rigid}})^\vee$$
are left-complete in their respective t-structures, and given that we already know that (9.11) is fully faithful, in order to prove that it is an equivalence, it is enough to show that every object in the heart of the t-structure on $(\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\mathrm{rigid}})^\vee$ admits a non-zero map from an object that lies in the essential image of the functor (9.11).

9.3.6. Let F be an object in the heart of the t-structure on $(\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\mathrm{rigid}})^\vee$. We will produce a particular object in the heart of $\mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\mathrm{rigid}})^\vee$, whose image under (9.11) maps injectively to F.

Pick a faithful representation V of G. By Sect. 9.3.4 we can regard $\mathsf{F}_{\{*\}}(V)$ as a representation of H, occurring on the vector space $\mathsf{F}_\emptyset \otimes \mathbf{oblv}_{\mathsf{G}}(V)$.

For a subgroup $\mathsf{H}' \subset \mathsf{H}$ *of finite codimension* let

$$\mathsf{F}'_\emptyset \subset \mathsf{F}_\emptyset \tag{9.15}$$

be the maximal subspace, such that the action of H' on $\mathsf{F}'_\emptyset \otimes \mathbf{oblv}_{\mathsf{G}}(V)$ is trivial. It is clear that for H' small enough, the subspace $\mathsf{F}'_\emptyset$ is non-zero. Moreover, we can assume that H' is a normal subgroup.

We claim:

Proposition 9.3.7. *The subspace*

$$\mathsf{F}'_\emptyset \otimes \mathbf{oblv}_{\mathsf{G}}(V) \subset \mathsf{F}_\emptyset \otimes \mathbf{oblv}_{\mathsf{G}}(V) \tag{9.16}$$

is stable under the action of H.

Proof. Consider the action of H on $\mathsf{F}_\emptyset \otimes \mathbf{oblv}_{\mathsf{G}}(V^\vee)$. We claim that the action of H' on the subspace
$$\mathsf{F}'_\emptyset \otimes \mathbf{oblv}_{\mathsf{G}}(V^\vee) \subset \mathsf{F}_\emptyset \otimes \mathbf{oblv}_{\mathsf{G}}(V^\vee)$$
is trivial.

We consider the vector space
$$\mathsf{F}_\emptyset \otimes \mathbf{oblv}_{\mathsf{G}}(V^\vee) \otimes \mathbf{oblv}_{\mathsf{G}}(V) \otimes \mathbf{oblv}_{\mathsf{G}}(V^\vee),$$
as equipped with an action of $\mathsf{H} \times \mathsf{H} \times \mathsf{H}$, along with its subgroups
$$\mathsf{H} \times 1 \times 1,\ 1 \times \mathsf{H} \times 1,\ 1 \times 1 \times \mathsf{H}.$$

Let

$$u_V \subset \mathbf{oblv}_{\mathsf{G}}(V) \otimes \mathbf{oblv}_{\mathsf{G}}(V^\vee) \text{ and } \mathrm{ev}_V : \mathbf{oblv}_{\mathsf{G}}(V^\vee) \otimes \mathbf{oblv}_{\mathsf{G}}(V) \to \mathsf{e}$$

be the unit and the counit of the duality.

We need to show that for any $h' \in \mathsf{H}'$, $v^\vee \in \mathbf{oblv}_{\mathsf{G}}(V^\vee)$ and $f' \in \mathsf{F}'_\emptyset$, we have

$$h' \cdot \big((\mathrm{id}_{\mathsf{F}_\emptyset} \otimes \mathrm{ev}_V \otimes \mathrm{id}_{\mathbf{oblv}_{\mathsf{G}}(V^\vee)})(f' \otimes v^\vee \otimes u_V)\big) = (\mathrm{id}_{\mathsf{F}_\emptyset} \otimes \mathrm{ev}_V \otimes \mathrm{id}_{\mathbf{oblv}_{\mathsf{G}}(V^\vee)})(f' \otimes v^\vee \otimes u_V). \tag{9.17}$$

Using condition (ii), we rewrite the LHS in (9.17) as

$$(\mathrm{id}_{\mathsf{F}_\emptyset} \otimes \mathrm{ev}_V \otimes \mathrm{id}_{\mathbf{oblv}_{\mathsf{G}}(V^\vee)})\left((1,1,h') \cdot (f' \otimes v^\vee \otimes u_V)\right).$$

However, by conditions (i) and (ii), for any $h \in \mathsf{H}$, $v^\vee \in \mathbf{oblv}_{\mathsf{G}}(V^\vee)$ and $f \in \mathsf{F}_\emptyset$

$$(1,h,h) \cdot (f \otimes v^\vee \otimes u_V) = (f \otimes v^\vee \otimes u_V).$$

Hence,

$$(1,1,h) \cdot (f' \otimes v^\vee \otimes u_V) = (1,h^{-1},1) \cdot (f' \otimes v^\vee \otimes u_V).$$

By condition (i), for any $h' \in \mathsf{H}'$, $v_1^\vee, v_2^\vee \in \mathbf{oblv}_{\mathsf{G}}(V^\vee)$, $v \in \mathbf{oblv}_{\mathsf{G}}(V)$ and $f' \in \mathsf{F}'_\emptyset$ we have

$$(1,h',1) \cdot (f' \otimes v_1^\vee \otimes v \otimes v_2^\vee) = f' \otimes v_1^\vee \otimes v \otimes v_2^\vee.$$

Combining, we obtain that the desired equality in (9.17).

To prove the proposition we consider the action of $\mathsf{H} \times \mathsf{H} \times \mathsf{H}$ on

$$\mathsf{F}_\emptyset \otimes \mathbf{oblv}_{\mathsf{G}}(V) \otimes \mathbf{oblv}_{\mathsf{G}}(V^\vee) \otimes \mathbf{oblv}_{\mathsf{G}}(V).$$

We need to show that for all $h \in \mathsf{H}$, $h' \in \mathsf{H}'$, $v_1, v_2 \in \mathbf{oblv}_{\mathsf{G}}(V)$, $v^\vee \in \mathbf{oblv}_{\mathsf{G}}(V^\vee)$, $f' \in \mathsf{F}'_\emptyset$, we have

$$\begin{aligned} h' \cdot \big((\mathrm{id}_{\mathsf{F}_\emptyset} \otimes \mathrm{ev} \otimes \mathrm{id}_{\mathbf{oblv}_{\mathsf{G}}(V)})((h(f' \otimes v_1)) \otimes v^\vee \otimes v_2)\big) &= \\ &= (\mathrm{id}_{\mathsf{F}_\emptyset} \otimes \mathrm{ev} \otimes \mathrm{id}_{\mathbf{oblv}_{\mathsf{G}}(V)})((h(f' \otimes v_1)) \otimes v^\vee \otimes v_2). \end{aligned} \tag{9.18}$$

Using conditions (i) and (ii), we rewrite the LHS in (9.18) as

$$(\mathrm{id}_{\mathsf{F}_\emptyset} \otimes \mathrm{ev} \otimes \mathrm{id}_{\mathbf{oblv}_{\mathsf{G}}(V)})\left((h',h',h') \cdot (h,1,1) \cdot (f' \otimes v_1 \otimes v^\vee \otimes v_2)\right).$$

We rewrite

$$\begin{aligned} (h',h',h') \cdot (h,1,1) \cdot (f' \otimes v_1 \otimes v^\vee \otimes v_2) &= \\ &= (h,1,1) \cdot (h^{-1} \cdot h' \cdot h,1,1) \cdot (1,h',1) \cdot (1,1,h') \cdot (f' \otimes v_1 \otimes v^\vee \otimes v_2). \end{aligned}$$

Now, by the assumption on f' and condition (ii), we have

$$(1,1,h') \cdot (f' \otimes v_1 \otimes v^\vee \otimes v_2) = (f' \otimes v_1 \otimes v^\vee \otimes v_2).$$

Further,

$$(1,h',1) \cdot (f' \otimes v_1 \otimes v^\vee \otimes v_2) = (f' \otimes v_1 \otimes v^\vee \otimes v_2)$$

by what we proved above. Finally, since H' is normal, we have

$$(h^{-1} \cdot h' \cdot h,1,1) \cdot (f' \otimes v_1 \otimes v^\vee \otimes v_2) = (f' \otimes v_1 \otimes v^\vee \otimes v_2).$$

Combining, we obtain that the LHS in (9.18) equals

$$(\mathrm{id}_{\mathsf{F}_\emptyset} \otimes \mathrm{ev} \otimes \mathrm{id}_{\mathbf{oblv}_{\mathsf{G}}(V)})\left((h,1,1) \cdot (f' \otimes v_1 \otimes v^\vee \otimes v_2)\right),$$

which is the same as the RHS.

□

9.3.8. We now claim that there exists a unique object F' in the heart of the t-structure on $(\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})^{\mathrm{rigid}})^\vee$, equipped with an injective map to F, whose value on $\emptyset\in\mathrm{fSet}$ is the subspace (9.15). (In proving this we will only use the fact that V is faithful and (9.16) is stable under the action of H.)

The uniqueness is clear from the description of $((\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})^{\mathrm{rigid}})^\vee)^\heartsuit$ in Sect. 9.3.4.

To prove the existence, we need to show that for every I and $W\in\mathrm{Rep}(\mathsf{G}^I)^\heartsuit$, the subspace

$$\mathsf{F}'_\emptyset\otimes\mathbf{oblv}_{\mathsf{G}^I}(W)\subset\mathsf{F}_\emptyset\otimes\mathbf{oblv}_{\mathsf{G}^I}(W)$$

is a stable under the action of H^I.

By condition (ii) in Sect. 9.3.4, we reduce the verification to the case when I is the one-element set $\{*\}$. Furthermore, we can assume that W is finite-dimensional.

By the same logic, if we take $V^{\otimes n}\in\mathrm{Rep}(\mathsf{G}^n)$ (for V chosen in Sect. 9.3.6), then the subspace

$$\mathsf{F}'_\emptyset\otimes\mathbf{oblv}_{\mathsf{G}^n}(V^{\otimes n})\subset\mathsf{F}_\emptyset\otimes\mathbf{oblv}_{\mathsf{G}^n}(V^{\otimes n})$$

is stable under the action of H^n,

By condition (i) applied to $\{1,....,n\}\to\{*\}$, we obtain that if we take $V^{\otimes n}$, viewed as a representation of (the diagonal copy of) G, the subspace

$$\mathsf{F}'_\emptyset\otimes\mathbf{oblv}_{\mathsf{G}}(V^{\otimes n})\subset\mathsf{F}_\emptyset\otimes\mathbf{oblv}_{\mathsf{G}}(V^{\otimes n})$$

is stable under the action H.

If W is a G-subrepresentation of $V^{\otimes n}$, we have

$$\mathsf{F}'_\emptyset\otimes\mathbf{oblv}_{\mathsf{G}}(W)=\mathsf{F}'_\emptyset\otimes\mathbf{oblv}_{\mathsf{G}}(V^{\otimes n})\cap\mathsf{F}_\emptyset\otimes\mathbf{oblv}_{\mathsf{G}}(W)\subset\mathsf{F}_\emptyset\otimes\mathbf{oblv}_{\mathsf{G}}(V^{\otimes n}).$$

Hence, for any such W, the subspace

$$\mathsf{F}'_\emptyset\otimes\mathbf{oblv}_{\mathsf{G}}(W)\subset\mathsf{F}_\emptyset\otimes\mathbf{oblv}_{\mathsf{G}}(W) \tag{9.19}$$

is a stable under the action of H.

This implies that (9.19) is stable under the action of H for any finite-dimensional $W\in\mathrm{Rep}(\mathsf{G})^\heartsuit$. Indeed, since V was assumed faithful, for any W we can find a one-dimensional representation ℓ such that

$$\ell\subset V^{\otimes l}$$

for some l and

$$W\otimes\ell\subset\underset{n}{\oplus}V^{\otimes n}.$$

9.3.9. Let F' be the object in the heart of the t-structure on $(\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})^{\mathrm{rigid}})^\vee$ constructed above. We will show that this F' belongs to the essential image of an object in the heart of the t-structure on $\mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})^{\mathrm{rigid}})^\vee$ along (9.11).

By construction, for any $I\in\mathrm{fSet}$, the corresponding functor

$$\mathsf{F}'_I:\mathrm{Rep}(\mathsf{G})^{\otimes I}\to\mathbf{H}^{\otimes I},$$

when evaluated on

$$(\mathrm{Rep}(\mathsf{G})^{\otimes I})^\heartsuit\simeq\mathrm{Rep}(\mathsf{G}^I)^\heartsuit$$

takes values in

$$\mathrm{Rep}((\mathsf{H}/\mathsf{H}')^I)^\heartsuit\subset\mathrm{Rep}(\mathsf{H}^I)^\heartsuit\simeq(\mathbf{H}^{\otimes I})^\heartsuit.$$

Since $\mathrm{Rep}(\mathsf{G}^I)$ is the derived category of its heart, we can factor the collection $\{\mathsf{F}'_I\}$ through a compatible collection of functors with values in $\mathrm{Rep}(\mathsf{H}/\mathsf{H}')^{\otimes I}$.

I.e., F' belongs to the essential image of the functor

$$\Big((\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}),\mathbf{H}')^{\mathrm{rigid}})^\vee\Big)^\heartsuit\to\Big((\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})^{\mathrm{rigid}})^\vee\Big)^\heartsuit,$$

where $\mathbf{H}':=\mathrm{Rep}(\mathsf{H}/\mathsf{H}')$.

The required assertion follows now from the commutative diagram

$$\begin{tikzcd}
\left((\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}')^{\mathrm{rigid}})^{\vee}\right)^{\heartsuit} \arrow[r] & \left((\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\mathrm{rigid}})^{\vee}\right)^{\heartsuit} \\
\left(\mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}')^{\mathrm{rigid}})^{\vee}\right)^{\heartsuit} \arrow[r] \arrow[u] & \left(\mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\mathrm{rigid}})^{\vee}\right)^{\heartsuit} \arrow[u]
\end{tikzcd},$$

since the functor

$$\mathrm{QCoh}(\mathbf{coMaps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}')^{\mathrm{rigid}})^{\vee} \to (\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}')^{\mathrm{rigid}})^{\vee}$$

is a t-exact equivalence, by Theorem 8.4.1.

Part III: The category of automorphic sheaves with nilpotent singular support

Let us make a brief overview of the contents of this Part.

In Sect. 10 we introduce and study the category $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$. First, we state two technical results that allow us to work conveniently with it. One is Theorem 10.1.4, which says that Bun_G can be covered by quasi-compact open substacks, such that the functor of !-extension from each of them preserves the nilpotence of singular support. The other is Theorem 10.1.6, which says that $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ is generated by objects that are compact in the ambient category $\mathrm{Shv}(\mathrm{Bun}_G)$. We notice that (up to some issues of left completeness) $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ carries an action $\mathrm{Rep}(\check{G})^{\otimes X}$, and applying our Spectral Decomposition theorem, we obtain $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ carries a monoidal action of $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}^{\mathrm{restr}}(X))$. We also state the second main result of this paper, Theorem 10.3.3, which says that if an object $\mathcal{F} \in \mathrm{Shv}(\mathrm{Bun}_G)$ is such that the Hecke action on it is lisse, then $\mathcal{F}$ belongs to $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$. This implies, in particular, that Hecke eigensheaves have nilpotent singular support.

In Sect. 11 we introduce yet another tool in the study of $\mathrm{Shv}(\mathrm{Bun}_G)$–Beilinson's spectral projector, denoted R_S^{enh}, which is defined for an affine scheme S equipped with a map $f; S \to \mathrm{LocSys}_{\check{G}}^{\mathrm{restr}}(X)$. This is a functor, given by an explicit Hecke operator that provides a left adjoint to the functor

$$\mathrm{Hecke}(S, \mathrm{Shv}(\mathrm{Bun}_G)) \to \mathrm{QCoh}(S) \otimes \mathrm{Shv}(\mathrm{Bun}_G) \to \mathrm{Shv}(\mathrm{Bun}_G).$$

Using our Spectral Decomposition theorem and Theorem 10.3.3, we interpret R_S^{enh} as a left adjoint to the functor

$$\mathrm{QCoh}(S) \underset{\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}^{\mathrm{restr}}(X))}{\otimes} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \xrightarrow{f_* \otimes \mathrm{Id}}$$
$$\to \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}^{\mathrm{restr}}(X)) \underset{\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}^{\mathrm{restr}}(X))}{\otimes} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \simeq \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \hookrightarrow \mathrm{Shv}(\mathrm{Bun}_G).$$

Using this functor we prove an array of structural results about $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$, including an explicit description of the right adjoint to the embedding

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \hookrightarrow \mathrm{Shv}(\mathrm{Bun}_G),$$

and the fact that all objects in $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ have regular singularities (in the de Rham setting).

In Sect. 12 we prove Theorem 10.1.4 about the preservation of nilpotence of singular support for certain open embeddings $\mathcal{U} \overset{j}{\hookrightarrow} \mathrm{Bun}_G$. The proof follows closely the strategy of [DrGa2]: by the same method as in *loc.cit.*, it turns out that we can control the singular support of the extension in a *contractive* situation.

In Sect. 13 we prove Theorem 10.3.3.

10. Automorphic sheaves with nilpotent singular support

In this section we will introduce and study the category $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$.

The central results of this section are:

–Theorem 10.5.2, which says that $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ carries a monoidal action of $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}^{\mathrm{restr}}(X))$;

–Theorem 10.3.3 that any object of $\mathrm{Shv}(\mathrm{Bun}_G)$ on which the Hecke action is lisse, belongs to $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$.

10.1. **Definition and basic properties.** In this subsection we define the category $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ and formulate two results (Theorems 10.1.4 and 10.1.6) that ensure that it is well-behaved.

10.1.1. From now on we let X be a smooth and complete curve and G a reductive group, over a ground field k (assumed algebraically closed).

We fix a reductive group G and consider Bun_G, the moduli space of principal G-bundles on X. We will be studying the category

$$\mathrm{Shv}(\mathrm{Bun}_G)$$

of sheaves on Bun_G.

When $\mathrm{Shv}(-) = \text{D-mod}(-)$, the foundations of the theory of sheaves on algebraic stacks are described in [DrGa1]. When $\mathrm{Shv}(-)$ is one of the constructible theories, the basic of the theory are developed in Sect. C.

10.1.2. Let $\mathrm{Nilp} \subset T^*(\mathrm{Bun}_G)$ be the nilpotent cone. Consider the full subcategory

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \subset \mathrm{Shv}(\mathrm{Bun}_G),$$

see Sect. C.5.

Furthermore, for an open $\mathcal{U} \subset \mathrm{Shv}(\mathrm{Bun}_G)$, we can consider the full subcategory

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathcal{U}) \subset \mathrm{Shv}(\mathcal{U}).$$

10.1.3. We have the following result, which insures that the category $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ can be obtained as a colimit of the corresponding categories on quasi-compact open substacks of Bun_G, see Sect. C.5.12:

Theorem 10.1.4. *The stack* $\mathrm{Shv}(\mathrm{Bun}_G)$ *can be written as a union of quasi-compact open substacks*

$$\mathcal{U}_i \overset{j_i}{\hookrightarrow} \mathrm{Bun}_G$$

such that the extension functors

$$(j_i)_!, (j_i)_* : \mathrm{Shv}(\mathcal{U}_i) \to \mathrm{Shv}(\mathrm{Bun}_G)$$

send $\mathrm{Shv}_{\mathrm{Nilp}}(\mathcal{U}_i)^{\mathrm{constr}} \to \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)^{\mathrm{constr}}$.

The proof will be given in Sect. 12.

Note that by the definition of the categories $\mathrm{Shv}_{\mathrm{Nilp}}(\mathcal{U}_i)$, $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$, the conclusion of the theorem is equivalent to the statement that for $(\mathcal{U}_i, j_i)$ as above, the functors $(j_i)_!$ and $(j_i)_*$ send

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathcal{U}_i) \to \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G).$$

10.1.5. In Sect. 11.4.2 we will prove:

Theorem 10.1.6. *The category* $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ *is generated by objects that are compact in the ambient category* $\mathrm{Shv}(\mathrm{Bun}_G)$.

Remark 10.1.7. In the terminology of Sect. C.5.12, the above theorem says that the pair $(\mathrm{Bun}_G, \mathrm{Nilp})$ is *renormalization-adapted* and *constraccessible*.

The first of these properties follows from the fact that Bun_G is locally a quotient, see Proposition C.5.9 (and is in fact expected to hold for any pair $(\mathcal{Y}, \mathcal{N})$ of an algebraic stack and a subset of its cotangent bundle, see Conjecture C.5.10).

The property of being constraccessible is far from being tautological; it reflects a particular feature of the pair $(\mathrm{Bun}_G, \mathrm{Nilp})$ (for example, it fails for $(\mathbb{P}^1, \{0\})$, see Remark B.4.4).

Remark 10.1.8. The proof of Theorem 10.1.6 relies on Theorem 10.3.3. In particular, its validity uses the assumption on the characteristic of the ground field k in Sect. 10.3.

10.2. **Hecke action on the category with nilpotent singular support.** In this subsection we recall the pattern of Hecke action on $\mathrm{Shv}(\mathrm{Bun}_G)$, and the particular feature that the subcategory

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \subset \mathrm{Shv}(\mathrm{Bun}_G)$$

with respect to this action.

10.2.1. The following result encodes the phenomenon of Hecke action of $\mathrm{Rep}(\check{G})$ on $\mathrm{Shv}(\mathrm{Bun}_G)$ (see [GKRV, Proposition B.2.3]):

Theorem 10.2.2. *There exists a compatible family of actions of*

$$\mathrm{Rep}(\check{G})^{\otimes I} \otimes \mathrm{Shv}(X^I) \text{ on } \mathrm{Shv}(\mathrm{Bun}_G \times X^I), \quad I \in \mathrm{fSet}, \tag{10.1}$$

extending the tautological action of

$$\mathrm{Shv}(X^I) \text{ on } \mathrm{Shv}(\mathrm{Bun}_G \times X^I), \quad I \in \mathrm{fSet}\,.$$

The following is was established in [NY, Theorem 5.2.1] (see also [GKRV, Theorem B.5.2]) :

Theorem 10.2.3. *The Hecke functor*

$$\mathrm{H}(-,-) : \mathrm{Rep}(\check{G}) \otimes \mathrm{Shv}(\mathrm{Bun}_G) \to \mathrm{Shv}(\mathrm{Bun}_G \times X) \tag{10.2}$$

sends

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \subset \mathrm{Shv}(\mathrm{Bun}_G)$$

to the full subcategory

$$\mathrm{Shv}_{\mathrm{Nilp}\times\{0\}}(\mathrm{Bun}_G \times X) \subset \mathrm{Shv}(\mathrm{Bun}_G \times X).$$

10.2.4. Note that the Hecke functors

$$\mathrm{H}(-,-) : \mathrm{Rep}(\check{G})^{\otimes I} \otimes \mathrm{Shv}(\mathrm{Bun}_G) \to \mathrm{Shv}(\mathrm{Bun}_G \times X^I) \tag{10.3}$$

send

$$((\mathrm{Rep}(\check{G})^{\otimes I})^c \otimes \mathrm{Shv}(\mathrm{Bun}_G)^c \subset \mathrm{Rep}(\check{G})^{\otimes I} \otimes \mathrm{Shv}(\mathrm{Bun}_G)$$

to

$$\mathrm{Shv}(\mathrm{Bun}_G \times X^I)^c \subset \mathrm{Shv}(\mathrm{Bun}_G \times X^I).$$

Recall that

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)^{\mathrm{access}} \subset \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \tag{10.4}$$

denotes the full subcategory generated by the essential image of

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \cap \mathrm{Shv}(\mathrm{Bun}_G)^c \subset \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G),$$

and similarly for

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G \times X^I)^{\mathrm{access}} \subset \mathrm{Shv}_{\mathrm{Nilp}\times\{0\}}(\mathrm{Bun}_G \times X^I)^{\mathrm{access}}.$$

(We note that Theorem 10.1.6 implies that the inclusion (10.4) is actually an equality, we cannot use this yet, as it will create a vicious circle.)

Remark 10.2.5. The above definition of $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)^{\mathrm{access}}$ coincides with the one given in Sect. C.5.11. This is due to the combination of Theorem 10.1.4 and the fact that Bun_G is locally a quotient, see Proposition C.5.9.

10.2.6. From Theorem 10.2.3 we obtain that the Hecke functor (10.2) sends

$$\mathrm{Rep}(\check{G}) \otimes \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)^{\mathrm{access}} \subset \mathrm{Rep}(\check{G}) \otimes \mathrm{Shv}(\mathrm{Bun}_G)$$

to

$$\mathrm{Shv}_{\mathrm{Nilp}\times\{0\}}(\mathrm{Bun}_G \times X)^{\mathrm{access}} \subset \mathrm{Shv}(\mathrm{Bun}_G \times X).$$

Recall now that Theorem C.6.5 implies that the (a priori fully faithful) functor

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)^{\mathrm{access}} \otimes \mathrm{IndLisse}(X) \to \mathrm{Shv}_{\mathrm{Nilp}\times\{0\}}(\mathrm{Bun}_G \times X)^{\mathrm{access}}$$

is an equivalence[18].

[18]Note that according to Theorem C.6.7, the functor

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{QLisse}(X) \to \mathrm{Shv}_{\mathrm{Nilp}\times\{0\}}(\mathrm{Bun}_G \times X)$$

is also an equivalence.

10.2.7. Combining with the fully faithful embedding

$$\mathrm{IndLisse}(X) \hookrightarrow \mathrm{QLisse}(X)$$

we obtain:

Corollary 10.2.8. *The Hecke functor* (10.2) *sends*

$$\mathrm{Rep}(\check{G}) \otimes \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)^{\mathrm{access}} \subset \mathrm{Rep}(\check{G}) \otimes \mathrm{Shv}(\mathrm{Bun}_G)$$

to the full subcategory

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)^{\mathrm{access}} \otimes \mathrm{QLisse}(X) \subset \mathrm{Shv}(\mathrm{Bun}_G \times X).$$

Remark 10.2.9. Note that once we prove Theorem 10.1.6, from the above, we would know that the Hecke functors $\mathrm{H}(V,-)$ of (10.2) send $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ to

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{IndLisse}(X) \subset \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{QLisse}(X).$$

See Remark 10.6.6 for why this was a priori supposed to be the case.

10.2.10. Iterating, we obtain that for any $I \in \mathrm{fSet}$, the Hecke functor (10.3) sends

$$\mathrm{Rep}(\check{G})^{\otimes I} \otimes \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)^{\mathrm{access}} \subset \mathrm{Rep}(\check{G}) \otimes \mathrm{Shv}(\mathrm{Bun}_G)$$

to the full subcategory

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)^{\mathrm{access}} \otimes \mathrm{QLisse}(X)^{\otimes I} \subset \mathrm{Shv}(\mathrm{Bun}_G \times X^I).$$

Thus, combining with Theorem 10.2.2, we obtain:

Corollary 10.2.11. *There exists a compatible family of monoidal functors*

$$\mathrm{Rep}(\check{G})^{\otimes I} \to \mathrm{End}(\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)^{\mathrm{access}}) \otimes \mathrm{QLisse}(X)^{\otimes I}, \quad I \in \mathrm{fSet}\,.$$

Thus, in the terminology of Sect. 6.1.1, we obtain that the Hecke action gives rise to an action of $\mathrm{Rep}(\check{G})^{\otimes X}$ on $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)^{\mathrm{access}}$.

10.3. **A converse to Theorem 10.2.3.** In this subsection we state the second main result of this paper, Theorem 10.3.3.

It implies, among the rest, that Hecke eigensheaves have nilpotent singular support.

For the validity of Theorem 10.3.3 we will have to make the following assumption on $\mathrm{char}(k)$:

Assumption: *There exists a non-degenerate G-equivariant pairing bilinear form on $\mathfrak{g}$, whose restriction to the center of any Levi subalgebra remains non-degenerate.*

From now on, we will assume that the above assumption on $\mathrm{char}(k)$ is satisfied.

10.3.1. Let

$$\mathrm{Shv}(\mathrm{Bun}_G)^{\text{Hecke-lisse}} \subset \mathrm{Shv}(\mathrm{Bun}_G)$$

be the full subcategory consisting of objects $\mathcal{F}$ such that for all $V \in \mathrm{Rep}(\check{G})$, for the Hecke functor (10.2) we have

$$\mathrm{H}(V, \mathcal{F}) \in \mathrm{Shv}(\mathrm{Bun}_G) \otimes \mathrm{QLisse}(X) \subset \mathrm{Shv}(\mathrm{Bun}_G \times X).$$

We can phrase Corollary 10.2.8 as saying that

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \subset \mathrm{Shv}(\mathrm{Bun}_G)^{\text{Hecke-lisse}}. \tag{10.5}$$

The following was proposed as a conjecture in [GKRV] (it appears as Conjecture C.2.8 in *loc.cit.*):

Main Theorem 10.3.2. *The inclusion* (10.5) *is an equality.*

In fact, we will prove a result that a priori looks stronger, but is in fact equivalent (due to Theorem C.6.7):

Main Theorem 10.3.3. *Let $\mathcal{F} \in \mathrm{Shv}(\mathrm{Bun}_G)$ be such that for for all $V \in \mathrm{Rep}(\check{G})$, the singular support of the object*

$$\mathrm{H}(V, \mathcal{F}) \in \mathrm{Shv}(\mathrm{Bun}_G \times X)$$

is contained in $T^*(\mathrm{Bun}_G) \times \{0\} \subset T^*(\mathrm{Bun}_G \times X)$. Then $\mathcal{F} \in \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$.

10.3.4. *Example.* The assertion of Theorem 10.3.3 is easy for $G = \mathbb{G}_m$. Note that in this case $\mathrm{Bun}_G = \mathrm{Pic}$, and

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) = \mathrm{QLisse}(\mathrm{Pic}).$$

Proof. The Hecke functor for the standard character of $\check{G} = \mathbb{G}_m$ is the pullback functor with respect to the addition map

$$\mathrm{add} : \mathrm{Pic} \times X \to \mathrm{Pic}\,.$$

Let us be given an object $\mathcal{F} \in \mathrm{Shv}(\mathrm{Pic})$ such that

$$\mathrm{add}^!(\mathcal{F}) \in \mathrm{Shv}(\mathrm{Pic} \times X)$$

belongs to

$$\mathrm{Shv}(\mathrm{Pic}) \otimes \mathrm{QLisse}(X) \subset \mathrm{Shv}(\mathrm{Pic} \times X)$$

.

We wish to show that $\mathcal{F}$ belongs to QLisse(Pic). It is easy to see that it is enough to prove that $\mathcal{F}|_{\mathrm{Pic}^d}$ belongs to $\mathrm{QLisse}(\mathrm{Pic}^d)$ for some/any d.

By [GKRV, Proposition C.2.5] quoted above, for any integer d we have

$$\mathrm{add}_d^!(\mathcal{F}) \in \mathrm{Shv}(\mathrm{Pic}) \otimes \mathrm{QLisse}(X^d),$$

where add_d is the d-fold addition map

$$\mathrm{add}_d : \mathrm{Pic} \times X^d \to \mathrm{Pic}\,.$$

In particular, the !-pullback of $\mathcal{F}$ along

$$X^d \simeq \mathbf{1}_{\mathrm{Pic}} \times X^d \to \mathrm{Pic} \times X^d \to \mathrm{Pic}^d \tag{10.6}$$

belongs to $\mathrm{QLisse}(X^d)$.

Note, however, that for $d > 2g-2$, the map (10.6) is flat and surjective: indeed, it factors as

$$X^d \overset{\mathrm{sym}^d}{\longrightarrow} X^{(d)} \overset{\mathrm{AJ}_d}{\to} \mathrm{Pic}^d,$$

where AJ_d is the Abel-Jacobi map, which is smooth and surjective for $d > 2g-2$, and sym^d is finite and flat.

Then the required assertion follows from the next observation:

Lemma 10.3.5. *Let $f : Y_1 \to Y_2$ be a flat and surjective map between smooth schemes. Then if for $\mathcal{F} \in \mathrm{Shv}(X_2)$, the pullback $f^!(\mathcal{F})$ belongs to $\mathrm{QLisse}(X_1)$, then $\mathcal{F} \in \mathrm{QLisse}(X_2)$.*

Proof. The pullback functor with respect to a flat morphism is t-exact (up to a shift by the relative dimension). Hence, we can assume that $\mathcal{F}$ is an irreducible perverse sheaf. We wish to show that $\mathcal{F}$ is lisse. Since $\mathcal{F}$ is perverse and irreducible, it is lisse if and only if all of its !-fibers have the same dimension. However, this condition is enough to check after applying $f^!$. □

□

10.3.6. From Theorem 10.3.3 we obtain[19]:

Main Corollary 10.3.7. *Let $\mathcal{F} \in \mathrm{Shv}(\mathrm{Bun}_G)$ be a* loose *Hecke eigensheaf, i.e., for every $V \in \mathrm{Rep}(\check{G})^{\heartsuit}$, the object*

$$\mathrm{H}(V, \mathcal{F}) \in \mathrm{Shv}(\mathrm{Bun}_G \times X)$$

is of the form $\mathcal{F} \boxtimes E_V$ for some $E_V \in \mathrm{QLisse}(X)$. Then $\mathcal{F} \in \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$.

As a particular case, we obtain the following statement that was proposed as a conjecture by G. Laumon ([Lau, Conjecture 6.3.1]):

Main Corollary 10.3.8. *Hecke eigensheaves in* $\mathrm{Shv}(\mathrm{Bun}_G)$ *have nilpotent singular support.*

10.4. **The Betti context.** Let us temporarily place ourselves in the Betti context. I.e., we will be working over the ground field $\mathbb{C}$.

In this subsection, we will formulate Theorem 10.4.11 which explains how to single out *constructible* sheaves with nilpotent singular support among *all* sheaves with nilpotent singular support in terms of the Hecke action.

10.4.1. Recall that according to our conventions (see Sect. 1.1.1), for a prestack $\mathcal{Y}$, we denote by $\mathrm{Shv}(\mathcal{Y})$ the resulting category of ind-constructible sheaves, defined as

$$\mathrm{Shv}(\mathcal{Y}) := \lim_{S} \mathrm{Shv}(S),$$

where the limit is taken over the category of affine schemes almost of finite type mapping to $\mathcal{Y}$, with the transition functors given by !-pullback. This limit can be rewritten as a colimit

$$\mathrm{Shv}(\mathcal{Y}) := \operatorname{colim}_{S} \mathrm{Shv}(S),$$

where the transition functors are give by !-puhsforward.

Now, following [GKRV, Sect. A.1.4-A.1.5], we can also consider a different category, to be denoted $\mathrm{Shv}^{\mathrm{all}}(\mathcal{Y})$,

$$\mathrm{Shv}^{\mathrm{all}}(\mathcal{Y}) := \operatorname{colim}_{S} \mathrm{Shv}^{\mathrm{all}}(S),$$

where $\mathrm{Shv}^{\mathrm{all}}(S)$ is the category of *all*, i.e., not necessarily ind-constructible sheaves.

This colimit can be rewritten as a limit

$$\mathrm{Shv}(\mathcal{Y}) := \lim_{S} \mathrm{Shv}(S)$$

taken with respect to the !-pullback functors, but this limit is taken in the category of DG categories and *not necessarily* continuous functors (the latter because the functor of !-pullback for an arbitrary map between manifolds is no longer continuous).

That said, if $\mathcal{Y}$ is an algebraic stack, we can write

$$\mathrm{Shv}(\mathcal{Y}) := \lim_{S} \mathrm{Shv}(S),$$

where now the index category is that of affine schemes with a *smooth* map to $\mathcal{Y}$ (and smooth maps between such), and now the limit does take place in DGCat.

Remark 10.4.2. Note that the forgetful functor

$$\mathrm{Shv}(\mathcal{Y}) \to \mathrm{Shv}^{\mathrm{all}}(\mathcal{Y})$$

is not fully faithful. This is due to the fact that the objects from

$$\mathrm{Shv}(\mathcal{Y})^c := \mathrm{Shv}(\mathcal{Y})^{\mathrm{constr}}$$

are not compact in $\mathrm{Shv}^{\mathrm{all}}(\mathcal{Y})$.

[19]The conclusion of Corollary 10.3.7 appears in [GKRV] as Conjecture C.2.10.

10.4.3. Let now $\mathcal{N} \subset T^*(\mathcal{Y})$ be a coninical Lagrangian subspace. Then one can consider the full subcategory

$$\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(\mathcal{Y}) \subset \mathrm{Shv}^{\mathrm{all}}(\mathcal{Y}).$$

We have the following assertion, proved in Sect. C.7:

Lemma 10.4.4. *The functor*

$$\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y}) \to \mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(\mathcal{Y})$$

sends objects that are compact in $\mathrm{Shv}(\mathcal{Y})$ *to objects that are compact in* $\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(\mathcal{Y})$*.*

10.4.5. Thus, we can consider $\mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)$ and its full subcategory $\mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$. We claim:

Proposition 10.4.6. *The functor*

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \to \mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \tag{10.7}$$

preserves compactness and is fully faithful.

Proof. By Theorem 10.1.6, the category $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ is generated by objects that are compact in $\mathrm{Shv}(\mathrm{Bun}_G)$. By applying Lemma 10.4.4, we obtain that the functor (10.7) preserves compactness.

Given this, in order to prove that (10.7) is fully faithful, it suffices to show that it is fully faithful when restricted to $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)^c$. But this follows from the fact that (for any $\mathcal{Y}$) the functor

$$\mathrm{Shv}(\mathcal{Y})^{\mathrm{constr}} \to \mathrm{Shv}^{\mathrm{all}}(\mathcal{Y})$$

is fully faithful.

□

10.4.7. Recall, following [NY], that we the Hecke action on $\mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ defines a compatible family of functors

$$\mathrm{H}(-,-) : \mathrm{Rep}(\check{G})^{\otimes I} \otimes \mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \subset \mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{Shv}_{\mathrm{loc.const.}}(X^I), \quad I \in \mathrm{fSet}\,.$$

10.4.8. Let

$$(\mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G))^{\mathrm{fin.mon.}} \subset \mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$$

be the full subcategory consisting of objects $\mathcal{F}$ such that for all $V \in \mathrm{Rep}(\check{G})^{\heartsuit}$ we have

$$\mathrm{H}(V, \mathcal{F}) \in \mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{QLisse}(X) \subset \mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{Shv}_{\mathrm{loc.const.}}(X),$$

see Sect. 7.3.2.

As in [GKRV, Proposition C.2.5], one easily shows that $(\mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G))^{\mathrm{fin.mon.}}$ is stable under the Hecke action.

10.4.9. The essential image of the functor

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \to \mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$$

is contained in $(\mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G))^{\mathrm{fin.mon.}}$. One way to see this is that the intersection

$$\mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{Shv}_{\mathrm{loc.const.}}(X) \cap \mathrm{Shv}(\mathrm{Bun}_G \times X) \subset \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G \times X)$$

equals

$$\mathrm{Shv}_{\mathrm{Nilp}\times\{0\}}(\mathrm{Bun}_G \times X),$$

which by Theorem C.6.7 identifies with $\mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{QLisse}(X)$. Another way to see the stated containment is by combining Theorem 10.1.6 and Corollary 10.2.8.

10.4.10. In Sect. 11.7.3 we will prove the following result:

Theorem 10.4.11. *The inclusion*

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \hookrightarrow (\mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G))^{\mathrm{fin.mon.}}$$

is an equality.

10.5. **Spectral decomposition of the category with nilpotent singular support.** We now come to the first main point of this paper: the spectral decomposition of $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ over $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$.

10.5.1. We would like to combine Theorems 10.2.2, 10.2.3 and Theorem 6.1.4 in order to obtain an action of $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$ on $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$.

Unfortunately, we cannot do this directly for an (annoying) technical reason: we only proved Theorem 6.1.4 under the assumption that the category we are acting on is dualizable. The category $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ is dualizable (and even compactly generated) by Theorem 10.1.6; however, we cannot use this yet because the proof of Theorem 10.1.6 uses the existence of such a spectral action.

Nonetheless, we claim:

Main Theorem 10.5.2. *The action of* $\mathrm{Rep}(\check{G})^{\otimes X}$ *on* $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ *(arising from the Hecke action) comes from a uniquely defined action of* $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$.

The proof of Theorem 10.5.2 will be given in Sect. 10.6.

10.5.3. As a first corollary of Theorem 10.5.2, we obtain:

Corollary 10.5.4. *The category* $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ *splits canonically as a direct sum*

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \simeq \underset{\sigma}{\oplus} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)_\sigma,$$

where σ runs over the set of isomorphism classes of semi-simple $\check{G}$-local systems on X.

10.5.5. *Example.* Let us explain in concrete terms what Corollary 10.5.4 says in concrete terms for $G = \mathbb{G}_m$.

Recall that the geometric class field theory attaches to a 1-dimensional local system σ on X a local system $\mathcal{F}_\sigma$ on Pic. Then Corollary 10.5.4 is the assertion that QLisse(Pic) splits as a direct sum

$$\mathrm{QLisse}(\mathrm{Pic}) \simeq \underset{\sigma}{\oplus} \mathrm{QLisse}(\mathrm{Pic})_\sigma,$$

where each $\mathrm{QLisse}(\mathrm{Pic})_\sigma$ is generated by $\mathcal{F}_\sigma$.

In the particular case of Pic, such a decomposition is not difficult to establish directly: it follows from the fact that every lisse irreducible object in Shv(Pic) is isomorphic to one of the $\mathcal{F}_\sigma$ (this is the assertion that the étale fundamental group of Pic is the abelianization of the étale fundamental group of X) and the different $\mathcal{F}_\sigma$ are mutually orthogonal.

10.5.6. From Corollary 10.5.4 we obtain the following result:

Corollary 10.5.7. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be Hecke eigensheaves corresponding to G-local systems σ_1 and σ_2 with nono-isomorphic semi-simplifications. Then $\mathcal{F}_1$ and $\mathcal{F}_2$ are mutually orthogonal, i.e.,*

$$\mathcal{M}aps(\mathcal{F}_1, \mathcal{F}_2) = 0.$$

10.5.8. Assume for a moment that our ground field k has characteristic 0, and our sheaf theory Shv is that of D-modules.

Recall that in this case, it was shown in [Ga5, Corollary 4.5.5] that the functors (10.1) come from a (uniquely defined) action of the category $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))$ on $\mathrm{D\text{-}mod}(\mathrm{Bun}_G)$.

We claim:

Proposition 10.5.9. *The full subcategory*

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \subset \mathrm{D\text{-}mod}(\mathrm{Bun}_G)$$

equals

$$\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \underset{\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))}{\otimes} \mathrm{D\text{-}mod}(\mathrm{Bun}_G) \subset$$
$$\subset \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X)) \underset{\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))}{\otimes} \mathrm{D\text{-}mod}(\mathrm{Bun}_G) = \mathrm{D\text{-}mod}(\mathrm{Bun}_G),$$

where we view

$$\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \simeq \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}$$

as a colocalization of $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))$.

Proof. This is obtained by combining Proposition 7.3.7 with Theorem 10.3.3. □

10.5.10. Let us now specialize to the Betti context. Recall the setting of Sect. 10.4.7. By Theorem 7.1.2, the Hecke functors

$$\mathrm{H}(-,-) : \mathrm{Rep}(\check{G})^{\otimes I} \otimes \mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \to \mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{Shv}_{\mathrm{loc.const.}}(X^I), \quad I \in \mathrm{fSet}$$

combine to an action of $\mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}(X))$ on $\mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$.

Combining Proposition 7.3.4 with Theorem 10.4.11, we obtain:

Corollary 10.5.11. *The full subcategory*

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \subset \mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$$

equals

$$\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \underset{\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))}{\otimes} \mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \subset$$
$$\subset \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X)) \underset{\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))}{\otimes} \mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G) = \mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G),$$

where we view

$$\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \simeq \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}$$

as a colocalization of $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))$.

10.6. **Proof of Theorem 10.5.2.** Although we do not yet know that $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ is dualizable, and thus we cannot apply Theorem 6.1.4 to it, we can apply it to the category $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)^{\mathrm{access}}$, see below.

We will then bootstrap the action of $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$ on $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ by t-structure considerations.

10.6.1. Consider the full category

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)^{\mathrm{access}} \subset \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G).$$

see Sect. 10.2.4.

By Corollary 10.2.8, we have an action of $\mathrm{Rep}(\check{G})^{\otimes X}$ on $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)^{\mathrm{access}}$.

Since $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)^{\mathrm{access}}$ is compactly generated by assumption, and since $\mathrm{QLisse}(X)$ is dualiable (see Sect. B.2.5), we can apply Theorem 6.1.4.

Hence, we obtain that the above action of $\mathrm{Rep}(\check{G})^{\otimes X}$ on $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)^{\mathrm{access}}$ comes from a uniquely defined action of $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$ on $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)^{\mathrm{access}}$.

10.6.2. We now wish to extend the above action of $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$ from $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)^{\mathrm{access}}$ to all of $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$. The key fact is that $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ is the left completion of $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)^{\mathrm{access}}$ with respect to its t-structure, see Sect. C.5.4.

Recall (see Sect. 5.8.1 and Theorem 5.8.3) that the category $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$ is compactly generated and its subcategory of compact objects is closed under the monoidal operation.

Hence, in order to define a (so far, not necessarily unital) action of $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$ on $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$, it suffices to define an action on it of the monoidal subcategory of compact objects.

By the above left completion property, it suffices to show that the action of compact objects in $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$ on $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)^{\mathrm{access}}$ is given by functors whose cohomological amplitude is bounded on the left.

To establish the latter property it is enough to work at one connected component $\mathcal{Z}$ of $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$ at a time. It suffices to show that the compact generators of $\mathrm{QCoh}(\mathcal{Z})$ act by functors whose cohomological amplitude is bounded on the left.

10.6.3. For $V \in \mathrm{Rep}(\check{G})$, let $\mathcal{E}_V$ denote the corresponding tautological object in

$$\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \otimes \mathrm{QLisse}(X),$$

so that the Hecke functor

$$\mathrm{H}(V, -) : \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)^{\mathrm{access}} \to \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)^{\mathrm{access}} \otimes \mathrm{QLisse}(X)$$

is given in terms of the action of $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$ on $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)^{\mathrm{access}}$ by tensor product with $\mathcal{E}_V$.

Denote

$$\mathcal{E}_{V,x} := (\mathrm{Id} \otimes \mathrm{ev}_x)(\mathcal{E}_V) \in \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)).$$

The action of this object on $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)^{\mathrm{access}}$ is the Hecke functor $\mathrm{H}(V, -)_x$.

By Theorem 5.8.3, the category $\mathrm{QCoh}(\mathcal{Z})$ is compactly generated by objects of the form

$$\mathcal{F} \otimes \mathcal{E}_{V,x},$$

where $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Z})$ can be written as a *finite* colimit in terms of $\mathcal{O}_{\mathcal{Z}}$ and $V \in \mathrm{Rep}(\check{G})^c$. (Note, however, that $\mathcal{O}_{\mathcal{Z}}$ itself is *not* compact as an object of $\mathrm{QCoh}(\mathcal{Z})$.)

Since $\mathcal{O}_{\mathcal{Z}}$ is a direct summand of $\mathcal{O}_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}$, its action on $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)^{\mathrm{access}}$ is given by a t-exact functor. Hence, the action of the above objects $\mathcal{F}$ is given by functors of bounded cohomological amplitude. Finally, the action of $\mathcal{E}_{V,x}$, being isomorphic to $\mathrm{H}(V, -)_x$ also has a bounded amplitude for $V \in \mathrm{Rep}(\check{G})^c$.

10.6.4. It remains to show that the resulting action of $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$ on $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ is unital.

A priori, the action of $\mathcal{O}_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}$ on $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ is an idempotent. If it was not the identity idempotent, we would obtain a direct sum decomposition

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \simeq \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)_{\mathrm{good}} \oplus \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)_{\mathrm{bad}},$$

where

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)^{\mathrm{access}} \subset \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)_{\mathrm{good}},$$

since $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$ does act unitally on $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)^{\mathrm{access}}$.

However, we claim that $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)_{\mathrm{bad}} = 0$. This follows from the fact that $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)^{\mathrm{access}}$ cogenerates $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$, as the latter is the left completion of the former.

$\square$[Theorem 10.5.2]

Remark 10.6.5. Note that by the above construction, the action of the object $\mathcal{E}_V$ on $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$, viewed as a functor

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \to \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{QLisse}(X),$$

is given by the Hecke functor $\mathrm{H}(V, -)$.

In particular, we obtain that the latter lands in

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{QLisse}(X) \subset \mathrm{Shv}_{\mathrm{Nilp}\times\{0\}}(\mathrm{Bun}_G \times X) \subset \mathrm{Shv}(\mathrm{Bun}_G \times X).$$

The fact that the essential image belongs to $\mathrm{Shv}_{\mathrm{Nilp}\times\{0\}}(\mathrm{Bun}_G \times X)$ was the content of Theorem 10.2.3. So, we are strengthening this by saying that it actually belongs to the subcategory $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{QLisse}(X)$.

That said, we know that the inclusion

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{QLisse}(X) \subset \mathrm{Shv}_{\mathrm{Nilp}\times\{0\}}(\mathrm{Bun}_G \times X)$$

is an equality, but this is due to (the non-tautological) Theorem C.6.7.

Remark 10.6.6. In Remark 10.2.9 we have observed that the Hecke functors send $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ to

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)\otimes \mathrm{IndLisse}(X)\subset \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)\otimes \mathrm{QLisse}(X).$$

Let us show why this a priori follows from Theorem 10.5.2.

In fact, we claim that the objects $\mathcal{E}_V$ above in fact belong to

$$\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))\otimes \mathrm{IndLisse}(X)\subset \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))\otimes \mathrm{QLisse}(X).$$

To prove this, it is enough to show that for *cofinal* family of maps $S\to \mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$, the objects

$$\mathcal{E}_V|_S\in \mathrm{QCoh}(S)\otimes \mathrm{QLisse}(X)$$

belong to

$$\mathrm{QCoh}(S)\otimes \mathrm{IndLisse}(X)\subset \mathrm{QCoh}(S)\otimes \mathrm{QLisse}(X).$$

We will now use the fact that for X a curve, the prestack $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$ is eventually coconnective. This follows from the fact that the connected components of $\mathrm{LocSys}^{\mathrm{restr},\mathrm{rigid}_x}_{\check{G}}(X)$ are *quasi-smooth formal affine schemes*, see Sect. 14.1.1.

Hence, inside the category $\mathrm{Sch}^{\mathrm{aff}}_{/\,\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}$, a cofinal family is formed by those S that are eventually coconnective. Now, for S eventually coconnective. the fact that $\mathcal{E}_V|_S$ belongs to the subcategory $\mathrm{QCoh}(S)\otimes \mathrm{IndLisse}(X)$ is a reformulation of Proposition 1.5.6.

11. Applications of Beilinson's spectral projector

In this section we will combine our Theorems 10.5.2 and 10.3.3 with another tool – Beilinson's spectral projector – to prove some key structural theorems about $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$.

11.1. **Construction of the projector.** Beilinson's spectral projector is an explicit Hecke functor, designed to produce Hecke eigensheaves corresponding to a given local system (or a family of such).

Using Theorems 10.5.2 and 10.3.3, we will reinterpret this construction in terms of the action of $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$ on $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$, which will turn it into a mechanism of producing objects in $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ from arbitrary objects in $\mathrm{Shv}(\mathrm{Bun}_G)$.

11.1.1. Let S be an affine scheme (over e), equipped with a map

$$f:S\to \mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X).$$

To this datum we will associate a certain functor, to be denoted

$$R_S:\mathrm{Shv}(\mathrm{Bun}_G)\to \mathrm{QCoh}(S)\otimes \mathrm{Shv}(\mathrm{Bun}_G).$$

The construction will be a variant of the one used in Sect. 9.1.2. Namely, the functor R_S is the colimit over TwArr(fSet) of a family of functors

$$R_S^{I\overset{\phi}{\to}J}:\mathrm{Shv}(\mathrm{Bun}_G)\to \mathrm{QCoh}(S)\otimes \mathrm{Shv}(\mathrm{Bun}_G),\quad (I\overset{\phi}{\to}J)\in \mathrm{TwArr}(\mathrm{fSet}), \tag{11.1}$$

constructed as follows.

The functor $R_S^{I\overset{\phi}{\to}J}$ equals the composition

$$\mathrm{Shv}(\mathrm{Bun}_G)\overset{R_{\check{G}}^{\otimes I}}{\longrightarrow}(\mathrm{Rep}(\check{G})\otimes\mathrm{Rep}(\check{G}))^{\otimes I}\otimes \mathrm{Shv}(\mathrm{Bun}_G)\overset{\mathrm{mult}^\phi\otimes \mathrm{Id}}{\longrightarrow}(\mathrm{Rep}(\check{G})\otimes\mathrm{Rep}(\check{G}))^{\otimes J}\otimes \mathrm{Shv}(\mathrm{Bun}_G)\simeq$$

$$\simeq \mathrm{Rep}(\check{G})^{\otimes J}\otimes \mathrm{Rep}(\check{G})^{\otimes J}\otimes \mathrm{Shv}(\mathrm{Bun}_G)\overset{\mathsf{F}_S^J\otimes \mathrm{H}(-,-)}{\longrightarrow}\mathrm{QCoh}(S)\otimes \mathrm{QLisse}(X^J)\otimes \mathrm{Shv}(\mathrm{Bun}_G\times X^J)\to$$

$$\to \mathrm{QCoh}(S)\otimes \mathrm{Shv}(\mathrm{Bun}_G\times X^J\times X^J)\to \mathrm{QCoh}(S)\otimes \mathrm{Shv}(\mathrm{Bun}_G\times X^J)\to \mathrm{QCoh}(S)\otimes \mathrm{Shv}(\mathrm{Bun}_G),$$

where:

- $R_{\check{G}}$ denotes the regular representation of $\check{G}$, regarded as an object of $\mathrm{Rep}(\check{G})\otimes\mathrm{Rep}(\check{G})$;

- mult^ϕ denotes the tensor product functor
$$(\mathrm{Rep}(\check{G}) \otimes \mathrm{Rep}(\check{G}))^{\otimes I} \to (\mathrm{Rep}(\check{G}) \otimes \mathrm{Rep}(\check{G}))^{\otimes J}$$
along the fibers of the map ϕ;
- F^J_S denotes the functor
$$\mathrm{Rep}(\check{G})^{\otimes J} \to \mathrm{QCoh}(S) \otimes \mathrm{QLisse}(X^J)$$
corresponding to the given map $f : S \to \mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$;
- The next-to-last arrow is given by restriction along the diagonal along $X^J \to X^J \times X^J$;
- The last arrow is direct image along the projection $\mathrm{Bun}_G \times X^J \to X^J$.

The transition maps are defined in the way parallel to Sect. 9.1.2.

11.1.2. Note that given a map $f : S \to \mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$, we can consider the corresponding category of Hecke eigensheaves in $\mathrm{Shv}(\mathrm{Bun}_G)$, to be denoted $\mathrm{Hecke}(S, \mathrm{Shv}(\mathrm{Bun}_G))$.

By definition, it consists of objects $\mathcal{F} \in \mathrm{QCoh}(S) \otimes \mathrm{Shv}(\mathrm{Bun}_G)$, equipped with a compatible system of identifications
$$\mathrm{H}(V, \mathcal{F}) \simeq \mathcal{F} \underset{\mathcal{O}_S}{\boxtimes} \mathsf{F}^I_S(V), \quad I \in \mathrm{fSet},\ V \in \mathrm{Rep}(\check{G})^{\otimes I}$$
where F^I_S is as above.

In a way parallel to Theorem 9.1.3, one shows:

Theorem 11.1.3 (Beilinson)**.** *The functor R_S naturally upgrades to a functor*
$$R^{\mathrm{enh}}_S : \mathrm{Shv}(\mathrm{Bun}_G) \to \mathrm{Hecke}(S, \mathrm{Shv}(\mathrm{Bun}_G)),$$
which provides a left adjoint to the forgetful functor
$$\mathrm{Hecke}(S, \mathrm{Shv}(\mathrm{Bun}_G)) \to \mathrm{QCoh}(S) \otimes \mathrm{Shv}(\mathrm{Bun}_G) \to \mathrm{Shv}(\mathrm{Bun}_G).$$

11.1.4. Note that the image of the forgetful functor
$$\mathrm{Hecke}(S, \mathrm{Shv}(\mathrm{Bun}_G)) \to \mathrm{QCoh}(S) \otimes \mathrm{Shv}(\mathrm{Bun}_G) \to \mathrm{Shv}(\mathrm{Bun}_G)$$
by definition lands in the subcategory
$$\mathrm{Shv}(\mathrm{Bun}_G)^{\text{Hecke-lisse}} \subset \mathrm{Shv}(\mathrm{Bun}_G).$$

Applying Theorem 10.3.3, we obtain that the image of the above forgetful functor lands in
$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \subset \mathrm{Shv}(\mathrm{Bun}_G).$$

In other words, we obtain that the inclusion
$$\mathrm{Hecke}(S, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)) \subset \mathrm{Hecke}(S, \mathrm{Shv}(\mathrm{Bun}_G)) \tag{11.2}$$
is an equality.

11.1.5. Note now that we can define the category
$$\mathrm{Hecke}(S, \mathbf{C})$$
for $f : S \to \mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$ and any category $\mathbf{C}$ equipped with an action of $\mathrm{Rep}(\check{G})^{\otimes X}$; namely
$$\mathrm{Hecke}(S, \mathbf{C}) = \mathrm{Funct}_{\mathrm{Rep}(\check{G})^{\otimes X}}(\mathrm{QCoh}(S), \mathbf{C}),$$
where $\mathrm{QCoh}(S)$ is acted on by $\mathrm{Rep}(\check{G})^{\otimes X}$ via the functor
$$\mathrm{Rep}(\check{G})^{\otimes X} \to \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$$
of (6.8).

Assuming that $\mathbf{C}$ is dualizable, and applying Theorem 6.1.4, we obtain that we can regard $\mathbf{C}$ as equipped with an action of $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$, and
$$\mathrm{Hecke}(S, \mathbf{C}) \simeq \mathrm{Funct}_{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))}(\mathrm{QCoh}(S), \mathbf{C}).$$

Recall now that according to Corollary 5.7.13(b), the category $\mathrm{QCoh}(S)$ is canonically self-dual as a module over $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$, so we have a canonical equivalence

$$\mathrm{Funct}_{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))}(\mathrm{QCoh}(S), \mathbf{C}) \simeq \mathrm{QCoh}(S) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))}{\otimes} \mathbf{C}.$$

Summarizing, we obtain a canonical identification

$$\mathrm{Hecke}(S, \mathbf{C}) \simeq \mathrm{QCoh}(S) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))}{\otimes} \mathbf{C}. \tag{11.3}$$

11.1.6. Thus, combining Theorem 11.1.3, the equality (11.2), the equivalence (11.3) and the spectral decomposition assertion for $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ (i.e., Theorem 10.5.2), we obtain:

Corollary 11.1.7. *For $f : S \to \mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$, the functor*

$$R_S : \mathrm{Shv}(\mathrm{Bun}_G) \to \mathrm{QCoh}(S) \otimes \mathrm{Shv}(\mathrm{Bun}_G)$$

takes values in $\mathrm{QCoh}(S) \otimes \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$, and naturally upgrades to a functor

$$R^{\mathrm{enh}}_S : \mathrm{Shv}(\mathrm{Bun}_G) \to \mathrm{QCoh}(S) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))}{\otimes} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G).$$

The latter functor is the left adjoint of

$$\begin{aligned} \mathrm{QCoh}(S) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))}{\otimes} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \overset{f_* \otimes \mathrm{Id}}{\longrightarrow} \\ \to \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))}{\otimes} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \simeq \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \hookrightarrow \mathrm{Shv}(\mathrm{Bun}_G). \end{aligned} \tag{11.4}$$

We will use Corollary 11.1.7 to prove a number of structural results about $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$.

11.2. **Compact generation of $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$.** In this subsection we are going to prove the following assertion:

Theorem 11.2.1. *The category $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ is compactly generated.*

11.2.2. Let $\mathcal{Z}$ be an algebraic stack of the form S/H, where S is an affine scheme, equipped with a map to $f : \mathcal{Z} \to \mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$. Denote by π_S the resulting map $S \to \mathcal{Z}$.

Using the 1-affineness of pt/H, we extend the construction of the functor R_S to a functor

$$R_{\mathcal{Z}} : \mathrm{Shv}(\mathrm{Bun}_G) \to \mathrm{QCoh}(\mathcal{Z}) \otimes \mathrm{Shv}(\mathrm{Bun}_G),$$

which takes values in $\mathrm{QCoh}(\mathcal{Z}) \otimes \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$, and factors via a functor

$$R^{\mathrm{enh}}_{\mathcal{Z}} : \mathrm{Shv}(\mathrm{Bun}_G) \to \mathrm{QCoh}(\mathcal{Z}) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))}{\otimes} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G).$$

Furthermore, the functor $R^{\mathrm{enh}}_{\mathcal{Z}}$ is the left adjoint of

$$\begin{aligned} \mathrm{QCoh}(\mathcal{Z}) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))}{\otimes} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \overset{f_* \otimes \mathrm{Id}}{\longrightarrow} \\ \to \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))}{\otimes} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \simeq \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \to \mathrm{Shv}(\mathrm{Bun}_G). \end{aligned}$$

11.2.3. We are now ready to prove Theorem 11.2.1. Recall that by Theorem 4.4.2, combined by Theorem 2.1.4, we can find a family of algebraic stacks mapping to $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$

$$f_n : \mathcal{Z}_n \to \mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$$

with the following properties:

- Each $\mathcal{Z}_n$ is of the form S/H with S an affine scheme;
- Each map $f_n : \mathcal{Z}_n \to \mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$ is an affine closed embedding;
- Each of the the functors $(f_n)_* : \mathrm{QCoh}(\mathcal{Z}) \to \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$ preserve compactness;
- The essential images of the functors $(f_n)_*$ generate $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$.

We obtain that it is enough to show that each of the categories

$$\mathrm{QCoh}(\mathcal{Z}_n) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))}{\otimes} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$$

is compactly generated.

By the above, the functor

$$\mathrm{QCoh}(\mathcal{Z}_n) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))}{\otimes} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \overset{(f_n)_*\otimes \mathrm{Id}}{\longrightarrow} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \to \mathrm{Shv}(\mathrm{Bun}_G) \tag{11.5}$$

admits a left adjoint, given by $R^{\mathrm{enh}}_{\mathcal{Z}_n}$. Since $\mathrm{Shv}(\mathrm{Bun}_G)$ is compactly generated, it suffices to show that the essential image of $R^{\mathrm{enh}}_{\mathcal{Z}_n}$ generates the target category. I.e., we have to show that (11.5) is conservative.

The latter is, in turn, equivalent to the functor

$$\mathrm{QCoh}(\mathcal{Z}_n) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))}{\otimes} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \overset{(f_n)_*\otimes \mathrm{Id}}{\longrightarrow} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$$

being conservative, i.e., that the essential image of the functor

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \simeq \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))}{\otimes} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \overset{(f_n)^*\otimes \mathrm{Id}}{\longrightarrow}$$
$$\to \mathrm{QCoh}(\mathcal{Z}_n) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))}{\otimes} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$$

generates the target category.

To show that latter, it is sufficient to show that the essential image of the functor

$$f_n^* : \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \to \mathrm{QCoh}(\mathcal{Z}_n)$$

generates the target category. This is is equivalent to the functor $(f_n)_*$ being conservative. But this is indeed the case since f_n is affine.

$\square$[Theorem 11.2.1]

11.2.4. Note that Theorem 11.2.1 admits the following corollary:

Corollary 11.2.5. *Let* $\mathcal{U} \overset{j}{\hookrightarrow} \mathrm{Bun}_G$ *be an open substack such that the functor* $j_!$ *(equivalently,* j_**) sends* $\mathrm{Shv}_{\mathrm{Nilp}}(\mathcal{U})^{\mathrm{constr}}$ *to* $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)^{\mathrm{constr}}$*. Then* $\mathrm{Shv}_{\mathrm{Nilp}}(\mathcal{U})$ *is compactly generated.*

Proof. Follows from the fact that the functor

$$j^* : \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \to \mathrm{Shv}_{\mathrm{Nilp}}(\mathcal{U})$$

admits a conservative right adjoint (given by j_*).

$\square$

11.3. **The right adjoint.** In this subsection we will describe explicitly the right adjoint of the embedding

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \hookrightarrow \mathrm{Shv}(\mathrm{Bun}_G). \tag{11.6}$$

11.3.1. Let $\mathcal{Z}$ be an arbitrary prestack equipped with a map $f : \mathcal{Z} \to \mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$. Passing to the limit over the category of affine schemes over $\mathcal{Z}$, we obtain the functor $R_{\mathcal{Z}}$:

$$\mathrm{Shv}(\mathrm{Bun}_G) \to \lim_{S\to\mathcal{Z}} (\mathrm{QCoh}(S) \otimes \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)) \to \lim_{S\to\mathcal{Z}} (\mathrm{QCoh}(S) \otimes \mathrm{Shv}(\mathrm{Bun}_G)) \simeq$$
$$\simeq \Big(\lim_{S\to\mathcal{Z}} \mathrm{QCoh}(S)\Big) \otimes \mathrm{Shv}(\mathrm{Bun}_G) \simeq \mathrm{QCoh}(\mathcal{Z}) \otimes \mathrm{Shv}(\mathrm{Bun}_G),$$

where the next-to-last isomorphism is due to the fact that $\mathrm{Shv}(\mathrm{Bun}_G)$ is dualizable, and hence tensoring by it commutes with limits.

By Theorem 11.2.1, the category $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ is compactly generated, and in particular, dualizable. In particular, the operation of tensor product $- \otimes \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ commutes with *limits*, hence the functor

$$\mathrm{QCoh}(\mathcal{Z}) \otimes \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \to \lim_{S\to\mathcal{Z}} \mathrm{QCoh}(S) \otimes \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$$

is an equivalence.

In particular, we obtain that the functor $R_{\mathcal{Z}}$ takes values in the full subcategory

$$\mathrm{QCoh}(\mathcal{Z}) \otimes \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \subset \mathrm{QCoh}(\mathcal{Z}) \otimes \mathrm{Shv}(\mathrm{Bun}_G).$$

Recall now that according to Corollary 5.7.13(c), the category $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ is dualizable *also as a module* over $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$. Hence, by the same logic as above, we obtain that the above functor $R_{\mathcal{Z}}$ canonically factors via a functor

$$R^{\mathrm{enh}}_{\mathcal{Z}} : \mathrm{Shv}(\mathrm{Bun}_G) \to \mathrm{QCoh}(\mathcal{Z}) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))}{\otimes} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G),$$

Remark 11.3.2. Note, however, that for an arbitrary $\mathcal{Z}$, the functor $R^{\mathrm{enh}}_{\mathcal{Z}}$ is not a left adjoint to anything.

11.3.3. Let us apply the construction of Sect. 11.3.1 to $\mathcal{Z} = \mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$ and the identity map. We obtain a functor

$$R^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)} : \mathrm{Shv}(\mathrm{Bun}_G) \to \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G).$$

We are going to prove:

Theorem 11.3.4. *The functor $R^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}$ provides a right adjoint to the embedding* (11.6).

11.3.5. Before we proceed to the proof, let us rewrite the functor $R^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}$ somewhat differently.

Note that the functor

$$\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X) \overset{(\Delta_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)})_*}{\longrightarrow} \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X) \times \mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \simeq$$
$$\simeq \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \otimes \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \overset{\Gamma_c(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X),-)\otimes\mathrm{Id}}{\longrightarrow} \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$$

is isomorphic to the identity functor, where $\Gamma_c(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X), -)$ is as in Sect. 5.7.11.

Hence, we obtain that the functor $R^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}$ can be rewritten as

$$\mathrm{Shv}(\mathrm{Bun}_G) \overset{R_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}}{\longrightarrow}$$
$$\to \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \otimes \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \overset{\Gamma_c(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X),-)\otimes\mathrm{Id}}{\longrightarrow} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G). \tag{11.7}$$

Further, using Corollary 5.7.8, the latter functor can be rewritten as

$$\mathcal{F} \mapsto \underset{(S,f)\in\mathrm{Sch}^{\mathrm{aff}}_{/\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}}{\mathrm{colim}} (\Gamma(S,-)\otimes\mathrm{Id}) \circ R_S(\mathcal{F}). \tag{11.8}$$

11.4. **Proof of Theorem 11.3.4.**

11.4.1. As a first step, we observe that for $\mathcal{F} \in \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$, we have a canonical isomorphism

$$R^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}(\mathcal{F}) \simeq \mathcal{F}. \tag{11.9}$$

Thus, the functor $R^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}$ is a priori a left inverse of (11.6).

11.4.2. Let $\mathcal{U} \overset{j}{\hookrightarrow} \mathrm{Bun}_G$ be a quasi-compact open substack such that the functor $j_!$ (equivalently, j_*) sends $\mathrm{Shv}_{\mathrm{Nilp}}(\mathcal{U})^{\mathrm{constr}}$ to $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)^{\mathrm{constr}}$.

We will first show that the functor

$$j^* \circ R^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)} \circ j_* : \mathrm{Shv}(\mathcal{U}) \to \mathrm{Shv}_{\mathrm{Nilp}}(\mathcal{U}) \tag{11.10}$$

provides a right adjoint to the embedding

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathcal{U}) \to \mathrm{Shv}(\mathcal{U}). \tag{11.11}$$

Note that once we prove that, we will know that the functor (11.11) sends compacts to compacts (since its right adjoint is continuous). Combined with Corollary 11.2.5, this will show that the pair $(\mathcal{U}, \mathrm{Nilp})$ is constraccessible, see Sect. C.5.3. This would imply the assertion of Theorem 10.1.6.

11.4.3. Consider the following general paradigm: let us be given a triple of dualizable categories $\mathbf{C}_1, \mathbf{C}_2, \mathbf{C}$ and functors

$$\iota_1 : \mathbf{C}_1 \to \mathbf{C},\ \iota_2 : \mathbf{C}_2 \to \mathbf{C}^\vee,\ R_1 : \mathbf{C} \to \mathbf{C}_1,\ R_2 : \mathbf{C}^\vee \to \mathbf{C}_2$$

and isomorphisms of functors

$$R_1 \circ \iota_1 \simeq \mathrm{Id}_{\mathbf{C}_1},\ R_2 \circ \iota_2 \simeq \mathrm{Id}_{\mathbf{C}_2} \text{ and } (\iota_1 \circ R_1)^\vee \simeq \iota_2 \circ R_2.$$

Assume also that ι_1 and ι_2 are fully faithful.

Lemma 11.4.4. *In the above situation we have a canonical identification*

$$\mathbf{C}_1^\vee \simeq \mathbf{C}_2,$$

with respect to which we have

$$R_1^\vee \simeq \iota_2 \text{ and } R_2^\vee \simeq \iota_1.$$

Proof. Indeed, consider the functor

$$(\iota_1 \circ R_1)^\vee : \mathbf{C}^\vee \to \mathbf{C}^\vee$$

.

We claim that it factors as

$$\mathbf{C}^\vee \overset{R_1^\vee}{\to} \mathbf{C}_1^\vee \overset{\alpha}{\to} \mathbf{C}_2 \overset{\iota_2}{\to} \mathbf{C}^\vee.$$

Indeed, since ι_2 is fully faithful, we only need to show that the essential image of $R_1^\vee$ is contained in the essential image of ι_2. This is the case for the composition $R_1^\vee \circ \iota_1^\vee$ since it is isomorphic to $\iota_2 \circ R_2$. The assertion for $R_1^\vee$ follows from the fact that the functor $\iota_1^\vee$ ie essentially surjective; the latter because it admits a *right* inverse given by $R_1^\vee$.

We claim that the resulting functor α is an equivalence. Indeed, it admits a left and a right inverses, given by

$$\iota_1^\vee \circ \iota_2 \text{ and } \iota_1^\vee \circ \iota_2,$$

respectively.

□

Corollary 11.4.5. *In the above situation the functor R_1 (resp., R_2) identifies with the right adjoint of ι_1 (resp., ι_2).*

11.4.6. We apply the paradigm of Sect. 11.4.3 to $\mathbf{C} := \mathrm{Shv}(\mathcal{U})$ with its canonical self-duality (see Theorem C.2.6) and $\mathbf{C}_1 = \mathbf{C}_2 = \mathrm{Shv}_{\mathrm{Nilp}}(\mathcal{U})$, so that $\iota_1 = \iota_2$ is the natural embedding.

We let $R_1 = R_2$ be the functor (11.10). Given (11.9), we only have to show that the functor

$$\mathrm{Shv}(\mathcal{U}) \overset{j^*\circ R^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}\circ j_*}{\longrightarrow} \mathrm{Shv}_{\mathrm{Nilp}}(\mathcal{U}) \hookrightarrow \mathrm{Shv}(\mathcal{U})$$

is self-dual.

We will use the description of $R^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}$ as (11.8). Unwinding the construction, it suffices to show that for any triple

$$(I, V, \mathcal{M}), \quad I \in \mathrm{fSet},\ V \in \mathrm{Rep}(\check{G})^{\otimes I},\ \mathcal{M} \in \mathrm{Shv}(X^I),$$

the functor

$$\mathrm{Shv}(\mathcal{U}) \overset{j_*}{\to} \mathrm{Shv}(\mathrm{Bun}_G) \overset{\mathrm{H}(V,-)}{\longrightarrow} \mathrm{Shv}(\mathrm{Bun}_G \times X^I) \overset{-\overset{!}{\otimes}\mathcal{M}}{\longrightarrow} \mathrm{Shv}(\mathrm{Bun}_G \times X^I) \to \mathrm{Shv}(\mathrm{Bun}_G) \overset{j^*}{\to} \mathrm{Shv}(\mathcal{U})$$

is canonically self-dual in a way functorial in $(I, V, \mathcal{M})$.

This is obtained by diagram chase using Proposition C.3.5(vii).

11.4.7. Thus, we have established the assertion from Sect. 11.4.2. We will now prove that $R^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}$ provides a right adjoint to (11.6) on all of Bun_G.

For that it sufficient to show that the functor $R^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}$ annihilates the *right orthogonal*

$$(\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G))^{\perp} \subset \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G).$$

Since we have already established the assertion of Theorem 10.1.6, it suffices to show that for

$$\mathcal{F}_0 \in \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \cap \mathrm{Shv}(\mathrm{Bun}_G)^c \text{ and any } \mathcal{F} \in (\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G))^{\perp},$$

we have

$$\mathcal{H}om_{\mathrm{Shv}(\mathrm{Bun}_G)}(\mathcal{F}_0, R^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}(\mathcal{F})) = 0.$$

We rewrite the functor $R^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}$ as (11.8). Since $\mathcal{F}_0$ was assumed compact, it suffices to show that for any

$$(I, V, \mathcal{M}), \quad I \in \mathrm{fSet},\ V \in (\mathrm{Rep}(\check{G})^{\otimes I})^c,\ \mathcal{M} \in \mathrm{QLisse}(X^I),$$

the Hecke functor

$$\mathrm{Shv}(\mathrm{Bun}_G) \overset{\mathrm{H}(V,-)}{\longrightarrow} \mathrm{Shv}(\mathrm{Bun}_G \times X^I) \overset{-\overset{!}{\otimes}\mathcal{M}}{\longrightarrow} \mathrm{Shv}(\mathrm{Bun}_G \times X^I) \to \mathrm{Shv}(\mathrm{Bun}_G)$$

sends $(\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G))^{\perp}$ to itself.

However, the above Hecke functor admits a left adjoint, which is also a Hecke functor corresponding to $(I, V^{\vee}, \mathbb{D}(\mathcal{M}))$.

The assertion follows now from the fact that all Hecke functors preserve $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$.
$\square$[Theorem 11.3.4]

11.5. **The tensor product property.**

11.5.1. Let Y_1 and Y_2 be a pair of quasi-compact schemes (or algebraic stacks) over the ground field k. Let $\mathrm{Shv}(-)$ be a constructible sheaf theory (i.e., in the case of D-modules, we will consider the subcategory of holonomic D-modules or regular holonomic D-modules).

Consider the external tensor product functor

$$\mathrm{Shv}(Y_1) \otimes \mathrm{Shv}(Y_2) \to \mathrm{Shv}(Y_1 \times Y_2). \tag{11.12}$$

It is fully faithful (see [GKRV, Lemma A.2.6]), but very rarely an equivalence. It is an equivalence, for example, if either Y_1 or Y_2 is an algebraic stack with a *finite number of isomorphism classes of k-points*[20].

In fact, there is a clear obstruction for an object of $\mathrm{Shv}(Y_1 \times Y_2)$ to belong to the essential image of (11.12). Namely all objects in the essential image have their singular support contained in a subset of $T^*(Y_1 \times Y_2) \simeq T^*(Y_1) \times T^*(Y_2)$ of the form

$$\mathcal{N}_1 \times \mathcal{N}_2, \quad \mathcal{N}_i \subset T^*(Y_i).$$

Thus, one can wonder whether, given $\mathcal{N}_1$ and $\mathcal{N}_2$ as above, the functor

$$\mathrm{Shv}_{\mathcal{N}_1}(Y_1) \otimes \mathrm{Shv}_{\mathcal{N}_2}(Y_2) \to \mathrm{Shv}_{\mathcal{N}_1 \times \mathcal{N}_2}(Y_1 \times Y_2). \tag{11.13}$$

Now, this happens to always be the case for the Betti sheaf theory, but not for ℓ-adic sheaves over a field of positive characteristic, and not for holonomic (but irregular) D-modules.

For example, taking $Y_1 = Y_2 = \mathbb{A}^1$ and $\mathcal{N}_1 = \mathcal{N}_2 = \{0\}$, the map

$$\mathrm{QLisse}(\mathbb{A}^1) \otimes \mathrm{QLisse}(\mathbb{A}^1) \to \mathrm{QLisse}(\mathbb{A}^1 \times \mathbb{A}^1)$$

is *not* an equivalence. Indeed, the object

$$\mathrm{mult}^*(\text{A-Sch}) \in \mathrm{QLisse}(\mathbb{A}^1 \times \mathbb{A}^1)$$

does not lie in the essential image, where

$$\mathrm{mult} : \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$$

is the product map and $\text{A-Sch} \in \mathrm{Shv}(\mathbb{A}^1)$ is the Artin-Schreier local system.

That said, Theorem C.6.5 says that the functor (11.13) is an equivalence in the case when Y_1 is a proper scheme and $\mathcal{N}_1 = \{0\}$, at least up to left completions. However, an assertion of this sort would still fail even when Y_1 is proper for a more general $\mathcal{N}_1$.

11.5.2. The main result of the present subsection is the following:

Theorem 11.5.3. *The functor*

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \to \mathrm{Shv}_{\mathrm{Nilp} \times \mathrm{Nilp}}(\mathrm{Bun}_G \times \mathrm{Bun}_G) \tag{11.14}$$

is an equivalence.

The rest of this subsection is devoted to the proof of this theorem.

Using the fact that $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ is dualizable, the standard limit argument shows that the fact that (11.13) is fully faithful for quasi-compact stacks implies that the functor (11.14) is also fully faithful. Thus, the essence of the theorem is to show that its essential image generates the target category.

[20] Such as, for example, $B\backslash G/N$ or Bun_G for X of genus 0.

11.5.4. As a first step, we are going to exhibit a particular set of generators of the category $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$.

Recall that $\mathrm{Nilp} \subset T^*(\mathrm{Bun}_G)$ is Largangian, and hence is the closure of the union of conormal bundles to a collection of locally-closed connected substacks $\mathcal{Y}_i \subset \mathrm{Bun}_G$. For every i, pick a k-point y_i

$$\mathrm{Spec}(k) \overset{\iota_{y_i}}{\to} \mathrm{Bun}_G$$

belonging to $\mathcal{Y}_i$. Let $\delta_{y_i} \in \mathrm{Shv}(\mathrm{Bun}_G)$ be the corresponding !-delta function object, i.e.,

$$\delta_{y_i} = (\iota_{y_i})_!(\mathsf{e}).$$

Let

$$f_n : \mathcal{Z}_n \to \mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$$

be one of the substacks as in Sect. 11.2.3.

Consider the objects

$$R^{\mathrm{enh}}_{\mathcal{Z}_n}(\delta_{y_i}) \in \mathrm{QCoh}(\mathcal{Z}_n) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))}{\otimes} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G),$$

and

$$((f_n)_* \otimes \mathrm{Id})(R^{\mathrm{enh}}_{\mathcal{Z}_n}(\delta_{y_i})) \in \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G). \tag{11.15}$$

We claim that the objects (11.15) provide a set of compact generators for $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$.

Compactness is clear from the construction. To prove that they generate $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$, we note that

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \simeq \underset{n}{\mathrm{colim}}\, \mathrm{QCoh}(\mathcal{Z}_n) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))}{\otimes} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G).$$

Hence, it suffices to show that for a fixed n, the objects $R^{\mathrm{enh}}_{\mathcal{Z}_n}(\delta_{y_i})$ generate

$$\mathrm{QCoh}(\mathcal{Z}_n) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))}{\otimes} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G).$$

Since the functor

$$(f_n)_* \otimes \mathrm{Id} : \mathrm{QCoh}(\mathcal{Z}_n) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))}{\otimes} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \to \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$$

is conservative, the latter assertion is equivalent to the statement that if $\mathcal{F} \in \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ is *non-zero*, then *not all*

$$\mathcal{H}om_{\mathrm{Shv}(\mathrm{Bun}_G)}(\delta_{y_i}, \mathcal{F}) \simeq (\iota_{y_i})^!(\mathcal{F})$$

are zero.

However, the latter is clear: take the index i such that the corresponding stratum $\mathcal{Y}_i$ is open in the support of $\mathcal{F}$.

11.5.5. We are now ready to prove Theorem 11.5.3. It suffices to show that for all pairs i_1, i_2, n_1, n_2, the object

$$((f_{n_1} \times f_{n_2})_* \otimes \mathrm{Id})(R^{\mathrm{enh}}_{\mathcal{Z}_{n_1} \times \mathcal{Z}_{n_2}}(\delta_{y_{i_1} \times y_{i_2}})) \in \mathrm{Shv}_{\mathrm{Nilp} \times \mathrm{Nilp}}(\mathrm{Bun}_G \times \mathrm{Bun}_G)$$

lies in the essential image of the functor (11.14).

However, this object equals the tensor product

$$((f_{n_1})_* \otimes \mathrm{Id})(R^{\mathrm{enh}}_{\mathcal{Z}_{n_1}}(\delta_{y_{i_1}})) \boxtimes ((f_{n_2})_* \otimes \mathrm{Id})(R^{\mathrm{enh}}_{\mathcal{Z}_{n_2}}(\delta_{y_{i_2}})).$$

$\square$[Theorem 11.5.3]

11.6. **The de Rham context.** A construction parallel to the one in Sect. 11.1 is applicable in the de Rham context, where we can now use maps

$$S \to \mathrm{LocSys}_{\check{G}}(X)$$

instead of $S \to \mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$.

An assertion parallel to Theorem 11.1.3 holds in this situation as well, and it was in fact in this situation that Beilinson had initially carried out his construction.

However, in this setting, the assertions of both Theorems 10.1.6 and 11.3.4 can be obtained more easily; we will explain this in the present subsection.

11.6.1. Recall that in the de Rham context, the category $\mathrm{D\text{-}mod}(\mathrm{Bun}_G)$ carries an action of $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))$; we will denote it by

$$\mathcal{E} \in \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X)),\ \mathcal{M} \in \mathrm{D\text{-}mod}(\mathrm{Bun}_G) \mapsto \mathcal{E} \star \mathcal{M}.$$

Unwinding the construction, we obtain that given

$$f : S \to \mathrm{LocSys}_{\check{G}}(X),$$

the functor

$$R_S : \mathrm{D\text{-}mod}(\mathrm{Bun}_G) \to \mathrm{QCoh}(S) \otimes \mathrm{D\text{-}mod}(\mathrm{Bun}_G)$$

is given by

$$\mathcal{F} \mapsto f_*(\mathcal{O}_S) \star \mathcal{F}.$$

Furthermore, the functor R_S^{enh}, viewed as a functor

$$\mathrm{D\text{-}mod}(\mathrm{Bun}_G) \to \mathrm{QCoh}(S) \underset{\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))}{\otimes} \mathrm{D\text{-}mod}(\mathrm{Bun}_G)$$

identifies with the pullback functor

$$\mathrm{D\text{-}mod}(\mathrm{Bun}_G) \simeq \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X)) \underset{\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))}{\otimes} \mathrm{D\text{-}mod}(\mathrm{Bun}_G) \overset{f^*\otimes \mathrm{Id}}{\longrightarrow}$$
$$\to \mathrm{QCoh}(S) \underset{\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))}{\otimes} \mathrm{D\text{-}mod}(\mathrm{Bun}_G).$$

This makes the assertion of Corollary 11.1.7 manifest.

11.6.2. The assertion of Theorem 10.1.6 follows from Proposition 10.5.9 and Theorem 5.8.3: indeed, the objects of the form $\mathcal{E} \star \mathcal{M}$, where

$$\mathcal{E} \in \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))^c \simeq \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))^c_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)} \subset \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))^c$$

and $\mathcal{M} \in \mathrm{D\text{-}mod}(\mathrm{Bun}_G)^c$, are compact in $\mathrm{D\text{-}mod}(\mathrm{Bun}_G)$ and generate

$$\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \underset{\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))}{\otimes} \mathrm{D\text{-}mod}(\mathrm{Bun}_G).$$

11.6.3. Finally, the interpretation of the functor $R^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}$ as (11.7) shows that it is identifies with the functor

$$\mathrm{D\text{-}mod}(\mathrm{Bun}_G) \simeq \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X)) \underset{\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))}{\otimes} \mathrm{D\text{-}mod}(\mathrm{Bun}_G) \to$$
$$\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)} \underset{\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))}{\otimes} \mathrm{D\text{-}mod}(\mathrm{Bun}_G),$$

where the second arrow is obtained by tensor product with the functor

$$\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X)) \to \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)},$$

right adjoint to the embedding

$$\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)} \hookrightarrow \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X)).$$

Hence, it is the right adjoint to the functor

$$\text{D-mod}_{\text{Nilp}}(\text{Bun}_G) \simeq \text{QCoh}(\text{LocSys}_{\check{G}}(X))_{\text{LocSys}^{\text{restr}}_{\check{G}}(X)} \underset{\text{QCoh}(\text{LocSys}_{\check{G}}(X))}{\otimes} \text{D-mod}(\text{Bun}_G) \to \text{QCoh}(\text{LocSys}_{\check{G}}(X)) \underset{\text{QCoh}(\text{LocSys}_{\check{G}}(X))}{\otimes} \text{D-mod}(\text{Bun}_G) = \text{D-mod}(\text{Bun}_G).$$

11.6.4. Finally, we claim that the assertion of Theorem 11.5.3 can also be easily obtained from this perspective. Indeed, since $\text{D-mod}(\text{Bun}_G)$ is compactly generated, and hence dualizable, the functor

$$\text{D-mod}(\text{Bun}_G) \otimes \text{D-mod}(\text{Bun}_G) \to \text{D-mod}(\text{Bun}_G \times \text{Bun}_G)$$

is an equivalence.

Now, the equivalence in Theorem 11.5.3 can be obtained by tensoring both sides over

$$\text{QCoh}(\text{LocSys}_{\check{G}}(X)) \otimes \text{QCoh}(\text{LocSys}_{\check{G}}(X)) \simeq \text{QCoh}(\text{LocSys}_{\check{G}}(X) \times \text{LocSys}_{\check{G}}(X))$$

with

$$\text{QCoh}(\text{LocSys}^{\text{restr}}_{\check{G}}(X)) \otimes \text{QCoh}(\text{LocSys}^{\text{restr}}_{\check{G}}(X)) \simeq \text{QCoh}(\text{LocSys}^{\text{restr}}_{\check{G}}(X) \times \text{LocSys}^{\text{restr}}_{\check{G}}(X)).$$

11.6.5. *The regular singularity property.* There is, however, one new property of the category $\text{D-mod}_{\text{Nilp}}(\text{Bun}_G)$ that one obtains by the methods of the spectral projector:

Main Corollary 11.6.6. *All compact objects of* $\text{D-mod}_{\text{Nilp}}(\text{Bun}_G)$ *have regular singularities.*

Proof. It suffices to show that all objects of $\text{D-mod}_{\text{Nilp}}(\text{Bun}_G)$ lie in the ind-completion of the regular holonomic subcategory. For that it suffices to show that the compact generators of $\text{D-mod}_{\text{Nilp}}(\text{Bun}_G)$ have this property.

However, this follows from the description of the generators of $\text{D-mod}_{\text{Nilp}}(\text{Bun}_G)$ given in Sect. 11.5.4. □

Combining with Corollary 10.3.8, we obtain:

Main Corollary 11.6.7. *All Hecke eigensheaves have regular singularities.*

The above corollary was suggested as a conjecture in [BD1, Sect. 5.2.7].

11.7. **The Betti context.** We consider the category $\text{Shv}^{\text{all}}(\text{Bun}_G)$ and its full subcategory

$$\text{Shv}^{\text{all}}_{\text{Nilp}}(\text{Bun}_G) \hookrightarrow \text{Shv}^{\text{all}}(\text{Bun}_G). \tag{11.16}$$

Since $\text{Shv}^{\text{all}}(\text{Bun}_G)$ is cocomplete and the essential image of $\text{Shv}^{\text{all}}_{\text{Nilp}}(\text{Bun}_G)$ is closed under limits, the embedding (11.16) admits a left adjoint[21].

In this subsection we will describe this left adjoint in terms of the Hecke action.

11.7.1. The construction from Sect. 11.1 is applicable in the Betti context as well, and we can use maps from affines schemes to all of $\text{LocSys}_{\check{G}}(X)$ (and not just $\text{LocSys}^{\text{restr}}_{\check{G}}(X)$).

Thus, for $f : S \to \text{LocSys}_{\check{G}}(X)$ we obtain a functor

$$R_S : \text{Shv}^{\text{all}}(\text{Bun}_G) \to \text{QCoh}(S) \otimes \text{Shv}^{\text{all}}_{\text{Nilp}}(\text{Bun}_G),$$

which upgrades to a functor

$$R^{\text{enh}}_S : \text{Shv}^{\text{all}}(\text{Bun}_G) \to \text{QCoh}(S) \underset{\text{QCoh}(\text{LocSys}_{\check{G}}(X))}{\otimes} \text{Shv}^{\text{all}}_{\text{Nilp}}(\text{Bun}_G),$$

which is the left adjoint to

$$\text{QCoh}(S) \underset{\text{QCoh}(\text{LocSys}_{\check{G}}(X))}{\otimes} \text{Shv}^{\text{all}}_{\text{Nilp}}(\text{Bun}_G) \overset{f_*\otimes \text{Id}}{\longrightarrow} \to \text{QCoh}(\text{LocSys}_{\check{G}}(X)) \underset{\text{QCoh}(\text{LocSys}_{\check{G}}(X))}{\otimes} \text{Shv}^{\text{all}}_{\text{Nilp}}(\text{Bun}_G) \simeq \text{Shv}^{\text{all}}_{\text{Nilp}}(\text{Bun}_G) \hookrightarrow \text{Shv}^{\text{all}}(\text{Bun}_G).$$

[21]We are grateful to D. Nadler for explaining this to us.

This construction can be extended from affine schemes to more general prestacks (see Sect. 11.2.2). In particular, we can apply it to the identity map on $\mathrm{LocSys}_{\check{G}}(X)$.

Thus, we obtain the following consequence of (the Betti analog of) Theorem 11.1.3:

Corollary 11.7.2. *The functor $R^{\mathrm{enh}}_{\mathrm{LocSys}_{\check{G}}(X)}$ provides a left adjoint to the embedding* (11.16).

11.7.3. We will now use Corollary 11.7.2 to prove Theorem 10.4.11:

By Proposition 7.3.4, we have to show that the essential image of

$$\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)} \underset{\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))}{\otimes} \mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \hookrightarrow$$
$$\to \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X)) \underset{\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))}{\otimes} \mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \simeq \mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$$

equals

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \subset \mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G).$$

We will do so by exhibiting a set of (compact) generators of

$$\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)} \underset{\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))}{\otimes} \mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$$

and show that they belong to $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$.

Namely, let y_i be as in Sect. 11.5.4. By the same logic as in *loc. cit.*, for any $0 \neq \mathcal{F} \in \mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$, not all

$$\mathcal{H}om_{\mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)}(\delta_{y_i}, \mathcal{F})$$

are zero.

Hence, by Corollary 11.7.2, the objects

$$R^{\mathrm{enh}}_{\mathrm{LocSys}_{\check{G}}(X)}(\delta_{y_i}) \in \mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$$

generate $\mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$.

Let

$$f_n : \mathcal{Z}_n \to \mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$$

be also as in Sect. 11.5.4.

As in *loc.cit.*, we obtain that the objects

$$((f_n)_* \otimes \mathrm{Id})(R^{\mathrm{enh}}_{\mathcal{Z}_n}(\delta_{y_i}))$$

generate

$$\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)} \underset{\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))}{\otimes} \mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G).$$

However, these objects belong to $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$, by construction.

$\square$[Theorem 10.4.11]

12. Preservation of nilpotence of singular support

In this section we will prove Theorem 10.1.4. Let us indicate the main idea.

Let us ask the general question: how can we control the singular support of $f_*(\mathcal{F})$ for a morphism $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ and $\mathcal{F} \in \mathrm{Shv}(\mathcal{Y}_1)$ in terms of the singular support of $\mathcal{F}$? One situation in which we can do it is when f is proper. Namely, in this case, $\mathrm{SingSupp}(f_*(\mathcal{F}))$ is contained in the pull-push of $\mathrm{SingSupp}(\mathcal{F})$ along the diagram

$$T^*(\mathcal{Y}_2) \leftarrow \mathcal{Y}_1 \underset{\mathcal{Y}_2}{\times} T^*(\mathcal{Y}_2) \to T^*(\mathcal{Y}_1).$$

However, there is one more situation when this is possible: when f is the open embedding of stacks of the form

$$\mathbb{P}(E) \to E/\mathbb{G}_m,$$

where E is a vector bundle (over some base) and $\mathbb{P}(E)$ is its projectivization. In fact, this situation can be essentially reduced to one of a proper map, see Sect. 12.4. We call an open embedding of this form *contractive*.

The idea of the proof of Theorem 10.1.4, borrowed from [DrGa2], is to find open substacks $\mathcal{U}_i$ so that we can can calculate the singular supports of *- (or !-) extensions by reducing to the contractive situation.

12.1. **Statement of the result.** In this subsection we will give a more precise version of Theorem 10.1.4, in which we will specify what the open substacks $\mathcal{U}_i$ are.

12.1.1. Recall that the stack Bun_G admits a canonical stratification (known as the Harder-Narasimhan stratification)

$$\mathrm{Bun}_G = \underset{\theta}{\cup}\, \mathrm{Bun}_G^\theta, \quad \theta \in \Lambda^+.$$

The substacks Bun_G^θ are locally closed, and for a subset $S \subset \Lambda^+$ the following conditions are equivalent:

(i) $\underset{\theta\in S}{\cup}\, \mathrm{Bun}_G^\theta$ is open in Bun_G;

(ii) $\theta \in S$ and $\theta - \theta' \in \Lambda^{\mathrm{pos}}$ implies $\theta' \in S$.

12.1.2. For a fixed θ, let

$$\mathrm{Bun}_G^{\leq\theta} \overset{j^\theta}{\hookrightarrow} \mathrm{Bun}_G$$

denote the embedding of the open substack corresponding to $\underset{\theta'\leq\theta}{\cup}\, \mathrm{Bun}_G^{\theta'}$.

The goal of this section is to prove the following theorem:

Theorem 12.1.3. *There exists an integer c (depending on G, $\mathrm{char}(k)$)*[22]*, such that for θ satisfying*

$$\langle\theta, \check{\alpha}_i\rangle \geq (2g-2) + c, \quad \forall\, i \in I, \tag{12.1}$$

the functor

$$j^\theta_* : \mathrm{Shv}(\mathrm{Bun}_G^{\leq\theta}) \to \mathrm{Shv}(\mathrm{Bun}_G)$$

preserves the condition of having nilpotent singular support.

12.1.4. *Example.* Let X have genus 1 and $\mathrm{char}(k) = 0$, so $\theta = 0$ satisfies (12.1). Note that

$$\mathrm{Bun}_G^{\leq 0} := \mathrm{Bun}_G^{\mathrm{ss}}$$

is the semi-stable locus.

Objects of $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G^{\mathrm{ss}})$ are known as character sheaves. So, in this case, Theorem 12.1.3 says that the functor of $*$-extension from the semi-stable locus sends character sheaves to sheaves with nilpotent singular support.

12.1.5. Following [DrGa2], for θ as in Theorem 12.1.3, the functor

$$j^\theta_* : \mathrm{Shv}(\mathrm{Bun}_G^{\leq\theta}) \to \mathrm{Shv}(\mathrm{Bun}_G)$$

preserves compactness. This formally implies that the functor

$$j^\theta_! : \mathrm{Shv}(\mathrm{Bun}_G^{\leq\theta}) \to \mathrm{Shv}(\mathrm{Bun}_G),$$

left adjoint to $(j^\theta)^*$ is defined, and is related to j^θ_* by Verdier duality.

Hence, the assertion of Theorem 12.1.3 automatically applies to the functor $j^\theta_!$ as well (Verdier duality preserves singular support). In particular, it also applies to the functor

$$j^\theta_{!*} : \mathrm{Shv}(\mathrm{Bun}_G^{\leq\theta})^\heartsuit \to \mathrm{Shv}(\mathrm{Bun}_G)^\heartsuit.$$

12.2. **Set-up for the proof.** In this subsection we will explain how the calculation of extensions from the open substacks specified in Theorem 12.1.3 can be reduced to a contractive situation.

[22] For $\mathrm{char} k = 0$ one can take $c = 0$.

12.2.1. Let P be a parabolic in G with Levi quotient M and unipotent radical N. Let is call an open substack $U \subset \mathrm{Bun}_M$ *good* if for $\mathcal{P}_M \in U$, we have

$$H^1(X, V^1_{\mathcal{P}_M}) = 0 \text{ and } H^0(X, V^2_{\mathcal{P}_M}) = 0$$

for any irreducible M-representation V^1 that appears as a subquotient of $\mathfrak{g}/\mathfrak{p}$ and an irreducible representation V^2 that appears as a subquotient of $\mathfrak{n}$.

Note that the above conditions guarantee that the map

$$\mathcal{U} \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_P \hookrightarrow \mathrm{Bun}_P \overset{\mathfrak{p}}{\to} \mathrm{Bun}_G \tag{12.2}$$

is smooth and

$$\mathcal{U} \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_P \hookrightarrow \mathrm{Bun}_P \overset{\mathfrak{q}}{\to} \mathrm{Bun}_M$$

is schematic. The latter condition implies that the canonical map

$$\mathrm{Bun}_M \to \mathrm{Bun}_P$$

induces a closed embedding

$$\mathcal{U} \to \mathcal{U} \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_P\,. \tag{12.3}$$

12.2.2. We will use the following fact established in [DrGa2, Proposition 9.2.2.]:

Theorem 12.2.3. *There exists an integer c such that for θ satisfying* (12.1)*, the closed substack $\mathrm{Bun}_G - \mathrm{Bun}_G^\theta$ can be represented as a union of locally closed substacks $\mathcal{Y}$ of the following form:*

There exists a parabolic P with Levi quotient M and a good *open substack $U \subset \mathrm{Bun}_M$ such that the image $\mathcal{V}$ of the map* (12.2) *contains $\mathcal{Y}$, and the (locally closed) substack*

$$(\mathcal{U} \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_P) \underset{\mathrm{Bun}_G}{\times} \mathcal{Y} \subset \mathcal{U} \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_P$$

equals the (closed) substack

$$\mathcal{U} \subset \mathcal{U} \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_P$$

of (12.3)*.*

12.2.4. By the argument of [DrGa2, Proposition 3.7.2], in order to prove Theorem 12.1.3, it suffices to prove the following:

Let $\mathcal{Y} \subset \mathcal{V}$ be as in Theorem 12.2.3; in particular $\mathcal{Y}$ is closed in $\mathcal{V}$. Let j' denote the open embedding

$$\mathcal{V} - \mathcal{Y} \overset{j}{\hookrightarrow} \mathcal{V}.$$

Let $\mathcal{F}$ be an object of $\mathrm{Shv}(\mathcal{V} - \mathcal{Y})$ with nilpotent singular support. Then $j_*(\mathcal{F}) \in \mathrm{Shv}(\mathcal{U}')$ also has nilpotent singular support.

12.3. **What do we need to show?** Let us put ourselves in the situation of Sect. 12.2.4.

In this subsection we will formulate a general statement that estimates from the above the singular support of objects $j_*(\mathcal{F})$ in terms of the singular support of $\mathcal{F}$, see Sect. 12.3.2. We will show how this estimate implies the preservation of nilpotence of singular support.

The statement from Sect. 12.3.2 will be proved in Sect. 12.4.

12.3.1. Let $\widetilde{j}$ denote the embedding

$$\mathcal{U} \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_P - \mathcal{U} \hookrightarrow \mathcal{U} \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_P,$$

and let $\widetilde{\mathcal{F}}$ denote the pullback of $\mathcal{F}$ along the (smooth) projection

$$\mathcal{U} \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_P - \mathcal{U} \to \mathcal{V} - \mathcal{Y}.$$

Let $\mathcal{P}_M$ be a point of $\mathcal{U} \subset \mathrm{Bun}_M \subset \mathrm{Bun}_P$. Note that we have a canonical identification

$$T^*_{\mathcal{P}_M}(\mathrm{Bun}_P) \simeq \Gamma(X, \mathfrak{p}^*_{\mathcal{P}_M} \otimes \omega) \simeq \Gamma(X, \mathfrak{m}^*_{\mathcal{P}_M} \otimes \omega) \oplus \Gamma(X, \mathfrak{n}^*_{\mathcal{P}_M} \otimes \omega).$$

For $\widetilde{A} \in T^*_{\mathcal{P}_M}(\mathrm{Bun}_P)$, let A^0 and A^- denote its components in $\Gamma(X, \mathfrak{m}^*_{\mathcal{P}_M} \otimes \omega)$ and $\Gamma(X, \mathfrak{n}^*_{\mathcal{P}_M} \otimes \omega)$, respectively.

12.3.2. We claim that it is enough to show the following: let $\widetilde{\mathcal{F}}$ be *an arbitrary* object of the category $\mathrm{Shv}(\mathcal{U} \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_P - \mathcal{U})$, and let $\widetilde{A} \in T^*_{\mathcal{P}_M}(\mathrm{Bun}_P)$ belong to $\mathrm{SingSupp}(\widetilde{j}_*(\widetilde{\mathcal{F}}))$. Then there exists a point

$$\mathcal{P}'_P \in \{\mathcal{P}_M\} \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_P - \{\mathcal{P}_M\}$$

such that the image, denoted $\widetilde{A}'$, of A^0 along

$$\Gamma(X, \mathfrak{m}^*_{\mathcal{P}_M} \otimes \omega) \simeq \Gamma(X, \mathfrak{m}^*_{\mathcal{P}'_M} \otimes \omega) \hookrightarrow \Gamma(X, \mathfrak{p}^*_{\mathcal{P}'_M} \otimes \omega) \simeq T^*_{\mathcal{P}'_P}(\mathrm{Bun}_P)$$

belongs to $\mathrm{SingSupp}(\widetilde{\mathcal{F}})$.

12.3.3. Let us show how the claim in Sect. 12.3.1 implies the needed property in Sect. 12.2.4. Let $\mathcal{P}_M$ be a point of $\mathcal{U}$, and let

$$A \in \Gamma(X, \mathfrak{g}^*_{\mathcal{P}_M} \otimes \omega) \simeq T^*_{\mathcal{P}_M}(\mathrm{Bun}_G)$$

be an element contained in $\mathrm{SingSupp}(j_*(\mathcal{F}))$. We wish to show that A is nilpotent.

Consider the map

$$\Gamma(X, \mathfrak{g}^*_{\mathcal{P}_M} \otimes \omega) \to \Gamma(X, \mathfrak{p}^*_{\mathcal{P}_M} \otimes \omega). \tag{12.4}$$

Let $\widetilde{A} \in \Gamma(X, \mathfrak{p}^*_{\mathcal{P}_M} \otimes \omega)$ denote the image of A. We claim that it suffices to show that the component

$$A^0 \in \Gamma(X, \mathfrak{m}^*_{\mathcal{P}_M} \otimes \omega)$$

of $\widetilde{A}$ is nilpotent.

Indeed, identify $\mathfrak{g}$ with $\mathfrak{g}^*$ using an invariant bilinear form. Write

$$\Gamma(X, \mathfrak{g}^*_{\mathcal{P}_M} \otimes \omega) \simeq \Gamma(X, \mathfrak{g}_{\mathcal{P}_M} \otimes \omega) \simeq \Gamma(X, \mathfrak{n}_{\mathcal{P}_M} \otimes \omega) \oplus \Gamma(X, \mathfrak{m}_{\mathcal{P}_M} \otimes \omega) \oplus \Gamma(X, \mathfrak{n}^-_{\mathcal{P}_M} \otimes \omega).$$

The projection (12.4) corresponds to the projection on the last two factors. At the same time, the assumption on $\mathcal{U}$ implies that the first factor vanishes. So, we can think of A as an element of

$$\Gamma(X, \mathfrak{m}_{\mathcal{P}_M} \otimes \omega) \oplus \Gamma(X, \mathfrak{n}^-_{\mathcal{P}_M} \otimes \omega) \simeq \Gamma(X, \mathfrak{p}^-_{\mathcal{P}_M} \otimes \omega),$$

and it is nilpotent if and only if its Levi component is such.

Let $\mathcal{P}'_P$ be as in Sect. 12.3.1. Then there exists

$$A' \in T^*_{\mathcal{P}'_P}(\mathrm{Bun}_G) \simeq \Gamma(X, \mathfrak{g}^*_{\mathcal{P}'_P} \otimes \omega),$$

which belongs to $\mathrm{SingSupp}(\mathcal{F})$, and whose image along

$$\Gamma(X, \mathfrak{g}^*_{\mathcal{P}'_P} \otimes \omega) \to \Gamma(X, \mathfrak{p}^*_{\mathcal{P}'_P} \otimes \omega)$$

is contained in

$$\Gamma(X, \mathfrak{m}^*_{\mathcal{P}'_P} \otimes \omega) \subset \Gamma(X, \mathfrak{p}^*_{\mathcal{P}'_P} \otimes \omega)$$

and equals the image of A^0 under the identification

$$\Gamma(X, \mathfrak{m}^*_{\mathcal{P}'_P} \otimes \omega) \simeq \Gamma(X, \mathfrak{m}^*_{\mathcal{P}'_M} \otimes \omega) \simeq \Gamma(X, \mathfrak{m}^*_{\mathcal{P}_M} \otimes \omega).$$

By assumption, A' is nilpotent, and is contained in

$$\Gamma(X, (\mathfrak{g}/\mathfrak{n})^*_{\mathcal{P}'_P} \otimes \omega) \subset \Gamma(X, \mathfrak{g}^*_{\mathcal{P}'_P} \otimes \omega).$$

Hence, its projection along

$$\Gamma(X, (\mathfrak{g}/\mathfrak{n})^*_{\mathcal{P}'_P} \otimes \omega) \to \Gamma(X, \mathfrak{m}^*_{\mathcal{P}'_P} \otimes \omega)$$

is nilpotent as well, while the latter identifies with A^0.

12.4. **Singular support in a contractive situation.** In this subsection we will provide a general context for the proof of the claim in Sect. 12.3.2.

12.4.1. Let us be given a schematic affine map of stacks $\pi : \mathcal{W} \to \mathcal{U}$, equipped with a section $s : \mathcal{U} \to \mathcal{W}$. Assume that $\mathcal{W}$, viewed as a stack over $\mathcal{U}$, is equipped with an action of the monoid $\mathbb{A}^1$, such that the action of $0 \in \mathbb{A}^1$ on $\mathcal{W}$ equals

$$\mathcal{W} \xrightarrow{f} \mathcal{U} \xrightarrow{s} \mathcal{W}.$$

Denote by j the open embedding $\mathcal{W} - \mathcal{U} \hookrightarrow \mathcal{W}$. Let $\mathcal{F}$ be an object of $\mathrm{Shv}(\mathcal{W} - \mathcal{U})$. Assume that $\mathcal{F}$ is equivariant with respect to $\mathbb{G}_m \subset \mathbb{A}^1$, which acts on $\mathcal{W} - \mathcal{U}$.

Let u be a point of $\mathcal{U}$ and let ξ be an element of

$$T^*_u(\mathcal{U}) \oplus T^*_u(\{u\} \underset{\mathcal{U}}{\times} \mathcal{W}) \simeq T^*_u(\mathcal{W}).$$

Write ξ^0 and ξ^- for its $T^*_u(\mathcal{U})$ and $T^*_u(\{u\} \underset{\mathcal{U}}{\times} \mathcal{W})$ components, respectively.

By [DrGa2, Sect. 11.2], the claim in Sect. 12.3.1 is a particular case of the following general assertion:

Proposition 12.4.2. *Suppose that ξ belongs to* $\mathrm{SingSupp}(j_*(\mathcal{F}))$. *Then there exists a point*

$$w \in \{u\} \underset{\mathcal{U}}{\times} \mathcal{W} - \{u\}$$

and an element $\xi' \in T^*_w(\mathcal{W})$ that belongs to $\mathrm{SingSupp}(\mathcal{F})$ *such that ξ' equals the image of ξ^0 under the codifferential map*

$$T^*_u(\mathcal{U}) \to T^*_w(\mathcal{W}).$$

The rest of this subsection is devoted to the proof of Proposition 12.4.2.

12.4.3. First off, by performing a smooth base change along $\mathcal{U}$, we can assume that $\mathcal{U}$ is an affine scheme. Further, the argument in [DrGa2, Sect. 5.1.3-5.1.5] reduces the assertion to the case when $\mathcal{W} = \mathcal{U} \times \mathrm{Tot}(E)$, for a vector space E, equipped with a standard action of $\mathbb{A}^1$.

Let $\widetilde{\mathcal{W}}$ denote the product $\mathcal{U} \times \widetilde{\mathrm{Tot}}(E)$, where $\widetilde{\mathrm{Tot}}(E)$ is the blow-up at $\mathrm{Tot}(E)$ at the origin. Let p denote the projection $\widetilde{\mathcal{W}} \to \mathcal{W}$. Denote by $\widetilde{j}$ the embedding

$$\mathcal{W} - \mathcal{U} \simeq \mathcal{U} \times (\mathrm{Tot}(E) - \{0\}) \hookrightarrow \mathcal{U} \times \widetilde{\mathrm{Tot}}(E) = \widetilde{\mathcal{W}},$$

so that $j = p \circ \widetilde{j}$.

Let q denote the projection

$$\mathcal{U} \times \widetilde{\mathrm{Tot}}(E) \to \mathcal{U} \times \mathbb{P}(E),$$

and let $\widetilde{i}$ denote the inclusion

$$\mathcal{U} \times \mathbb{P}(E) \hookrightarrow \mathcal{U} \times \widetilde{\mathrm{Tot}}(E).$$

Let $\overline{\pi}$ denote the projection $\mathcal{U} \times \mathbb{P}(E) \to \mathcal{U}$. Let i denote the inclusion of the zero section

$$\mathcal{U} \to \mathcal{U} \times \mathrm{Tot}(E),$$

and $\widetilde{\pi} := \pi \circ p$, so that

$$\widetilde{\pi} \circ \widetilde{i} = i \circ \overline{\pi}.$$

12.4.4. By the assumption on $\mathcal{F}$, we have

$$\mathcal{F} = q^*(\mathcal{F}'), \quad \mathcal{F}' \in \operatorname{Shv}(\mathcal{U} \times \mathbb{P}(E)).$$

We have an exact triangle

$$q^*(\mathcal{F}') \to \widetilde{j}_*(\mathcal{F}) \to \widetilde{i}_*(\mathcal{F}')[-1],$$

so we obtain an exact triangle

$$p_* \circ q^*(\mathcal{F}') \to j_*(\mathcal{F}) \to i_* \circ \overline{\pi}_*(\mathcal{F}').$$

We will estimate the singular support of $j_*(\mathcal{F})$ at $u \in \mathcal{U} \subset \mathcal{U} \times \operatorname{Tot}(E)$ by estimating the singular supports of $p_* \circ q^*(\mathcal{F}')$ and $i_* \circ \overline{\pi}_*(\mathcal{F}')$, respectively.

12.4.5. Recall the following general assertion:

Lemma 12.4.6. *Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a proper map; let $\mathcal{F}_1 \in \operatorname{Shv}(\mathcal{Y}_1)$ and denote $\mathcal{F}_2 := f_*(\mathcal{F}_1)$. Let $y_2 \in \mathcal{Y}_2$ be a point and $\xi_2 \in T^*_{y_2}(\mathcal{Y}_2)$ an element contained in* $\operatorname{SingSupp}(\mathcal{F}_2)$*. Then there exists $y_1 \in f^{-1}(y_2)$ such that*

$$(df)^*(\xi_2) =: \xi_1 \in T^*_{y_1}(Y_1)$$

belongs to $\operatorname{SingSupp}(\mathcal{F}_1)$.

This lemma immediately implies the estimate of Proposition 12.4.2 for $i_* \circ \overline{\pi}_*(\mathcal{F}')$. We now procced to proving the corresponding estimate for $p_* \circ q^*(\mathcal{F}')$.

12.4.7. Let

$$(\xi^0 \in T^*_u(\mathcal{U}), \xi^- \in E^*) = \xi \in T_{u,0}(\mathcal{W})$$

be an element in $\operatorname{SingSupp}(p_* \circ q^*(\mathcal{F}'))$. Then by Lemma 12.4.6 there exists a line $\ell \in \mathbb{P}(E)$ such that

$$(dp)^*(\xi) \in T^*_{(u,\widetilde{i}(\ell))}(\mathcal{U} \times \widetilde{\operatorname{Tot}}(E))$$

equals

$$(dq)^*(\xi') \text{ with } \xi' \in T^*_{(u,\ell)}(\mathcal{U} \times \mathbb{P}(E)),$$

where $\xi' \in \operatorname{SingSupp}(\mathcal{F}')$.

Note that the the map

$$T^*_u(\mathcal{U}) \oplus E^* = T_{u,0}(\mathcal{U} \times \operatorname{Tot}(E)) \overset{(dp)^*}{\to} T^*_{(u,\widetilde{i}(\ell))}(\mathcal{U} \times \widetilde{\operatorname{Tot}}(E)) \simeq T^*_{(u,\ell)}(\mathcal{U} \times \mathbb{P}(E)) \oplus \ell^*$$

has as its components the maps

$$T^*_u(\mathcal{U}) \overset{(d\overline{\pi})^*}{\to} T^*_{(u,\ell)}(\mathcal{U} \times \mathbb{P}(E))$$

and

$$E^* \twoheadrightarrow \ell^*,$$

respectively.

So, the conditions on (ξ^0, ξ^-) read as follows: there exists a line $\ell \in \mathbb{P}(E)$ such that

$$(d\overline{\pi})^*(\xi^0) \in \operatorname{SingSupp}(\mathcal{F}')$$

and $\xi^- \in \ell^\perp$.

However, the first of these conditions is equivalent to the condition in Proposition 12.4.2.

13. Proof of Theorem 10.3.3

In this section we will prove Theorem 10.3.3.

We first consider the case of $G = GL_2$, which explains the main idea of the argument. We then implement this idea in a slightly more involved case of $G = GL_n$ (where it is sufficient consider the minuscule Hecke functors).

Finally, we treat the case of an arbitrary G; the proof reduces to the analysis of the local Hitchin map and affine Springer fibers.

13.1. **Estimating singular support from below.** In this subsection we will state a general result that allows to guarantee that a certain cotangent vector does belong to the singular support of a sheaf obtained as a direct image.

13.1.1. Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a schematic morphism between algebraic stacks with $\mathcal{Y}_2$ smooth. Let $\mathcal{F}_1$ be an object of $\mathrm{Shv}(\mathcal{Y}_1)$ and let $\xi_2 \neq 0$ be an element of $T^*_{y_2}(\mathcal{Y}_2)$ for some $y_2 \in \mathcal{Y}_2$.

Theorem 13.1.2. *Assume there exists a point $y_1 \in f^{-1}(y_2) \subset \mathcal{Y}_1$ such that the following conditions hold:*

(i) *The point*

$$(\xi_2, y_1) \in T^*(\mathcal{Y}_2) \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1$$

belongs to and is isolated *in the intersection*

$$(df^*)^{-1}(\mathrm{SingSupp}(\mathcal{F}_1)) \cap (\{\xi_2\} \times f^{-1}(y_2)) \subset T^*(\mathcal{Y}_2) \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1. \tag{13.1}$$

(ii) *For every cohomological degree m and for every constructible sub-object $\mathcal{F}'_1$ of $H^m(\mathcal{F}_1)$, the images along*

$$T^*(\mathcal{Y}_2) \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1 \to T^*(\mathcal{Y}_2)$$

of the irreducible components of $(df^*)^{-1}(\mathrm{SingSupp}(\mathcal{F}'_1))$ that contain the point (ξ_2, y_1) have dimension $\dim(\mathcal{Y}_2)$ at $\xi_2 \in T^*_{y_2}(\mathcal{Y}_2) \subset T^*(\mathcal{Y}_2)$.

Then ξ_2 belongs to $\mathrm{SingSupp}(f_*(\mathcal{F}_1))$.

The proof will be given in Sect. D. Several remarks are in order:

Remark 13.1.3. Note that condition (i) implies that the map

$$(df^*)^{-1}(\mathrm{SingSupp}(\mathcal{F}_1)) \to T^*(\mathcal{Y}_2)$$

is quasi-finite on a neighborhood of the point (ξ_2, y_1).

Hence, condition (ii) is equivalent to the following:

(ii') *The subvariety $(df^*)^{-1}(\mathrm{SingSupp}(\mathcal{F}'_1))$ has dimension* $\dim(\mathcal{Y}_2)$ *at the point (ξ_2, y_1).*

Remark 13.1.4. When $\mathrm{char}(k) = 0$ and we work either with holonomic D-modules, or when $k = \mathbb{C}$ and we work with constructible sheaves in the classical topology, it is known that $\mathrm{SingSupp}(\mathcal{F}_1)$ is Lagrangian, and hence $f((df^*)^{-1}(\mathrm{SingSupp}(\mathcal{F}_1)))$ is Lagrangian.

This implies that condition in point (ii) is automatic in this case.

Remark 13.1.5. When $\mathrm{char}(k) = 0$ and one works with the entire category of D-modules, we expect that an analog of Theorem 13.1.2 is true (with condition (ii) omitted). In fact, we can prove it under the assumption that $\mathcal{F}_1$ is compact. However, we were not able to prove it in general.

Instead, we proceed as follows: we prove Theorem 13.1.2 for ℓ-adic sheaves, which formally implies the assertion of Theorem 13.1.2 for constructible sheaves in the classical topology over $\mathbb{C}$, and hence by Riemann-Hilbert for *regular holonomic* D-modules.

We then formally deduce the assertion of Theorem 10.3.3 the entire category of D-modules from the regular holonomic case, see Sect. 13.7.

13.2. **The case of $G = GL_2$.** In this section we will assume that $\mathrm{char}(k) > 2$.

13.2.1. Take $G = GL_2$. To shorten the notation, we will write Bun_2 instead of Bun_{GL_2}. Let $\mathcal{F}$ be an object in $\mathrm{Shv}(\mathrm{Bun}_2)^{\text{Hecke-lisse}}$. We will show that the singular support of $\mathcal{F}$ is contained in the nilpotent cone.

Let

$$\mathrm{H} : \mathrm{Shv}(\mathrm{Bun}_2) \to \mathrm{Shv}(\mathrm{Bun}_2 \times X)$$

be the basic Hecke functor, i.e., pull-push along the diagram

$$\mathrm{Bun}_2 \xleftarrow{\overleftarrow{h}} \mathcal{H}_2 \xrightarrow{\overrightarrow{h} \times s} \mathrm{Bun}_2 \times X,$$

where $\mathcal{H}_2$ is the moduli space of triples

$$\mathcal{M} \overset{\alpha}{\hookrightarrow} \mathcal{M}', \tag{13.2}$$

where $\mathcal{M}$ and $\mathcal{M}'$ are vector bundles on X and $\mathcal{M}'/\mathcal{M}$ is a torsion sheaf of length 1 on X. The maps $\overleftarrow{h}$ and $\overrightarrow{h}$ send the triple $\mathcal{M} \overset{\alpha}{\hookrightarrow} \mathcal{M}'$ to $\mathcal{M}$ and $\mathcal{M}'$, respectively, and s sends it to the support of $\mathrm{coker}(\alpha)$.

13.2.2. We will argue by contradiction, so assume that $\mathrm{SingSupp}(\mathcal{F})$ is not contained in the nilpotent cone.

Let

$$\xi \in T^*_{\mathcal{M}}(\mathrm{Bun}_2), \quad \mathcal{M} \in \mathrm{Bun}_2$$

be an element contained in $\mathrm{SingSupp}(\mathcal{F})$. Recall that the cotangent space $T^*_{\mathcal{M}}(\mathrm{Bun}_2)$ identifies with the space of

$$A \in \mathrm{Hom}(\mathcal{M}, \mathcal{M} \otimes \omega).$$

First, we claim that $\mathrm{Tr}(A) = 0$ as an element of $\Gamma(X, \omega)$. Indeed, consider the action

$$\mathrm{Pic} \times \mathrm{Bun}_2 \to \mathrm{Bun}_2, \quad \mathcal{L}, \mathcal{M} \mapsto \mathcal{L} \otimes \mathcal{M}. \tag{13.3}$$

As in Sect. 10.3.4, it is easy to see that the pullback of $\mathcal{F}$ along (13.3) belongs to

$$\mathrm{Shv}_{\{0\} \times T^*(\mathrm{Bun}_2)}(\mathrm{Pic} \times \mathrm{Bun}_2) \subset \mathrm{Shv}(\mathrm{Pic} \times \mathrm{Bun}_2).$$

Hence, A lies in the subspace of $T^*_{\mathcal{M}}(\mathrm{Bun}_2)$ perpendicular to

$$\mathrm{Im}(T_{\mathbf{1}}(\mathrm{Pic}) \to T_{\mathcal{M}}(\mathrm{Bun}_2)) \subset T_{\mathcal{M}}(\mathrm{Bun}_2),$$

and this subspace exactly consists of those A that have trace 0.

13.2.3. Assume now that A is non-nilpotent. This means that $\det(A) \neq 0$ as an element of $\Gamma(X, \omega^{\otimes 2})$. The conditions

$$\mathrm{Tr}(A) = 0 \ \text{ and } \ \det(A) \neq 0 \tag{13.4}$$

(plus the assumption that $\mathrm{char}(k) > 2$) imply that at the generic point of X, the operator A is regular semi-simple.

Let

$$\widetilde{X} \subset T^*(X)$$

be the spectral curve corresponding to A. The fact that A is generically regular semi-simple implies that over the generic point of X, the projection

$$\widetilde{X} \to X$$

is étale.

Let $x \in X$ be a point which has two distinct preimages in $\widetilde{X}$. Let $\widetilde{x}$ be one of them. We can think of $\widetilde{x}$ as an element $T^*_x(X)$, which we will denote by ξ_x.

We will construct a point $\mathcal{M}' \in \mathrm{Bun}_2$ and $A' \in T^*_{\mathcal{M}'}(\mathrm{Bun}_2)$, such that the element

$$(A', \xi_x) \in T^*_{\mathcal{M}', x}(\mathrm{Bun}_2 \times X)$$

belongs to $\mathrm{SingSupp}(\mathrm{H}(\mathcal{F}))$.

13.2.4. For a point (13.2) of $\mathcal{H}_2$, the intersection of

$$(d\overleftarrow{h}^*)(T^*_{\mathcal{M}}(\mathrm{Bun}_2)) \cap (d(\overrightarrow{h} \times s)^*)(T^*_{\mathcal{M}',x}(\mathrm{Bun}_2)) \subset T^*_{(\mathcal{M}\overset{\alpha}{\hookrightarrow}\mathcal{M}')}(\mathcal{H}_2)$$

consists of commutative diagrams

$$\begin{tikzcd} \mathcal{M}' \arrow[r, "A'"] & \mathcal{M}' \otimes \omega \\ \mathcal{M} \arrow[u] \arrow[r, "A"] & \mathcal{M} \otimes \omega, \arrow[u, "\alpha\otimes\mathrm{id}"'] \end{tikzcd} \tag{13.5}$$

where the corresponding element of $T^*_x(X)$ is given by the induced map

$$\mathcal{M}'/\mathcal{M} \to (\mathcal{M}'/\mathcal{M}) \otimes \omega.$$

13.2.5. We can think of $\mathcal{M}$ as a torsion-free sheaf $\mathcal{L}$ on $\widetilde{X}$, which is generically a line bundle. The possible diagrams (13.5) correspond to upper modifications of

$$\mathcal{L} \hookrightarrow \mathcal{L}', \quad \mathrm{supp}_X(\mathcal{L}'/\mathcal{L}) \subset \{x\} \underset{X}{\times} \widetilde{X}$$

as coherent sheaves on $\widetilde{X}$.

By the assumption on x, there are exactly two such modifications, corresponding to the two preimages of x in $\widetilde{X}$. We let $(\mathcal{M}', A')$ be the modification corresponding to the chosen point $\widetilde{x}$, so $A' \in T^*_{\mathcal{M}'}(\mathrm{Bun}_2)$.

13.2.6. We claim that $(A', \xi_x) \in T^*_{\mathcal{M}',x}(\mathrm{Bun}_2 \times X)$ indeed belongs to $\mathrm{SingSupp}(\mathrm{H}(X))$. We will do so by applying Theorem 13.1.2 to

$$\mathcal{Y}_1 = \mathcal{H}_2,\ \mathcal{Y}_2 = \mathrm{Bun}_2 \times X,\ f = (\overrightarrow{h} \times s),\ \mathcal{F}_1 = \overleftarrow{h}^*(\mathcal{F}),$$
$$y_1 = (x, \mathcal{M} \overset{\alpha}{\hookrightarrow} \mathcal{M}'),\ y_2 = (\mathcal{M}', x),\ \xi_2 = (A', \xi_x).$$

Note that since $\overleftarrow{h}$ is smooth,

$$\mathrm{SingSupp}(\overleftarrow{h}^*(\mathcal{F})) \subset T^*(\mathcal{H}_2)$$

equals the image of

$$\mathrm{SingSupp}(\mathcal{F}) \underset{\mathrm{Bun}_2, \overleftarrow{h}}{\times} \mathcal{H}_2$$

along the codifferential of $\overleftarrow{h}$

$$\mathrm{SingSupp}(\mathcal{F}) \underset{\mathrm{Bun}_2, \overleftarrow{h}}{\times} \mathcal{H}_2 \subset T^*(\mathrm{Bun}_2) \underset{\mathrm{Bun}_2, \overleftarrow{h}}{\times} \mathcal{H}_2 \to T^*(\mathcal{H}_2).$$

13.2.7. We first verify condition (i) of Theorem 13.1.2. The fact that the point $((A', \xi_x), (s, \mathcal{M} \overset{\alpha}{\hookrightarrow} \mathcal{M}'))$ belongs to

$$\mathrm{SingSupp}(\overleftarrow{h}^*(\mathcal{F})) \underset{T^*(\mathcal{H}_2)}{\times} (T^*(\mathrm{Bun}_2 \times X) \underset{\mathrm{Bun}_2 \times X, (\overrightarrow{h} \times s)}{\times} \mathcal{H}_2) \tag{13.6}$$

follows from the construction.

In order to show that $((A', \xi_x), (\mathcal{M} \overset{\alpha}{\hookrightarrow} \mathcal{M}'))$ is isolated in (13.6), it suffices to show that the intersection

$$\left((A', \xi_x) \times (\overrightarrow{h} \times s)^{-1}(\mathcal{M}', x)\right) \cap \left(T^*(\mathrm{Bun}_2) \underset{\mathrm{Bun}_2, \overleftarrow{h}}{\times} \mathcal{H}_2\right) \subset T^*(\mathcal{H}_2)$$

is finite.

Indeed, this is the assertion that for our fixed $(\mathcal{M}', A', x)$, there are finitely many possibilities for the diagrams (13.5). However, as in Sect. 13.2.5, such diagrams are in bijection with lower modifications of $\mathcal{L}'$ as a coherent sheaf on $\widetilde{X}$ supported at x, and there are exactly two of those.

Remark 13.2.8. Note that most of the above argument would apply to Bun_n for $n \geq 2$, except for the last finiteness assertion. The latter used the fact that A is generically semi-simple, which in the case $n = 2$ is guaranteed by the conditions (13.4).

13.2.9. We now verify condition (ii) of Theorem 13.1.2. By Remark 13.1.3, it suffices to verify condition (ii') from *loc.cit.*

Note that because we are working in the constructible context, for every cohomological degree m and every constructible sub-object $\mathcal{F}'$ of $H^m(\mathcal{F})$, all irreducible components of $\mathrm{SingSupp}(\mathcal{F}')$ have dimension equal to $\dim(\mathrm{Bun}_G)$, by [Be2].

Hence, it suffices to show that the fibers of the composite map

$$\mathrm{SingSupp}(\overleftarrow{h}^*(\mathcal{F}')) \underset{T^*(\mathcal{H}_2)}{\times} (T^*(\mathrm{Bun}_2 \times X) \underset{\mathrm{Bun}_2 \times X, (\overrightarrow{h} \times s)}{\times} \mathcal{H}_2) \to \mathrm{SingSupp}(\overleftarrow{h}^*(\mathcal{F}')) \overset{\overleftarrow{h}}{\to} \mathrm{SingSupp}(\mathcal{F}')$$

have dimension ≤ 1 near $((A', \xi_x), (x, \mathcal{M} \overset{\alpha}{\hookrightarrow} \mathcal{M}'))$.

In fact, we will show that the fibers of the map

$$(T^*(\mathrm{Bun}_2) \underset{\mathrm{Bun}_2, \overleftarrow{h}}{\times} \mathcal{H}_2) \underset{T^*(\mathcal{H}_2)}{\times} (T^*(\mathrm{Bun}_2 \times X) \underset{\mathrm{Bun}_2 \times X, (\overrightarrow{h} \times s)}{\times} \mathcal{H}_2) \to (T^*(\mathrm{Bun}_2) \underset{\mathrm{Bun}_2, \overleftarrow{h}}{\times} \mathcal{H}_2) \to T^*(\mathrm{Bun}_2)$$

have dimension ≤ 1 near $((A', \xi_x), (x, \mathcal{M} \overset{\alpha}{\hookrightarrow} \mathcal{M}'))$.

Equivalently, fixing the point x, we will show that the map

$$(T^*(\mathrm{Bun}_2) \underset{\mathrm{Bun}_2, \overleftarrow{h}_x}{\times} \mathcal{H}_{2,x}) \underset{T^*(\mathcal{H}_{2,x})}{\times} (T^*(\mathrm{Bun}_2) \underset{\mathrm{Bun}_2, \overrightarrow{h}_x}{\times} \mathcal{H}_{2,x}) \to \\ \to (T^*(\mathrm{Bun}_2) \underset{\mathrm{Bun}_2, \overleftarrow{h}_x}{\times} \mathcal{H}_{2,x}) \to T^*(\mathrm{Bun}_2) \tag{13.7}$$

is finite near $(A', (\mathcal{M} \hookrightarrow \mathcal{M}'))$, where

$$\mathrm{Bun}_2 \overset{\overleftarrow{h}_x}{\longleftarrow} \mathcal{H}_{2,x} \overset{\overrightarrow{h}_x}{\longrightarrow} \mathrm{Bun}_2$$

is the fiber of

$$\mathrm{Bun}_2 \overset{\overleftarrow{h}}{\longleftarrow} \mathcal{H}_2 \overset{\overrightarrow{h}}{\longrightarrow} \mathrm{Bun}_2$$

over $x \in X$.

Since the map (13.7) is proper, it suffices to show that the point $(A', (\mathcal{M} \overset{\alpha}{\hookrightarrow} \mathcal{M}'))$ is isolated in its fiber with respect to (13.7).

However, this is a similar finiteness assertion to what we proved in Sect. 13.2.7.

13.3. **The case of $G = GL_n$.** In this section we will assume that $\mathrm{char}(k) > n$.

We will essentially follow the same argument as in the case of $n = 2$, with the difference that we will have to use all minuscule Hecke functors, and not just the basic one.

13.3.1. Let $G = GL_n$, and we will write Bun_n instead of Bun_{GL_n}. Let $\mathcal{F}$ be an object in $\mathrm{Shv}(\mathrm{Bun}_n)^{\text{Hecke-lisse}}$. We will show that the singular support of $\mathcal{F}$ is contained in the nilpotent cone.

For an integer $1 \leq i \leq n$, let

$$\mathrm{H}^n : \mathrm{Shv}(\mathrm{Bun}_n) \to \mathrm{Shv}(\mathrm{Bun}_n \times X)$$

denote the i-th Hecke functor, i.e., pull-push along the diagram

$$\mathrm{Bun}_n \overset{\overleftarrow{h}}{\longleftarrow} \mathcal{H}^i_n \overset{\overrightarrow{h} \times s}{\longrightarrow} \mathrm{Bun}_n \times X,$$

where $\mathcal{H}^i_n$ is the moduli space of quadruples $(x, \mathcal{M} \overset{\alpha}{\hookrightarrow} \mathcal{M}')$, where:

- x is a point of X;
- $\mathcal{M}$ and $\mathcal{M}'$ are rank n bundles on X;

- α is an injection of coherent sheaves

$$\mathcal{M} \overset{\alpha}{\hookrightarrow} \mathcal{M}', \tag{13.8}$$

  such that $\operatorname{coker}(\alpha)$ has length i and is *scheme-theoretically* supported at x.

For future use, let

$$\mathrm{Bun}_n \xleftarrow{\overleftarrow{h}_x} \mathcal{H}^i_{n,x} \xrightarrow{\overrightarrow{h}_x} \mathrm{Bun}_n$$

denote the fiber of the above picture over a given $x \in X$.

13.3.2. We will argue by contradiction, so assume that $\mathrm{SingSupp}(\mathcal{F})$ is not contained in the nilpotent cone.

Let

$$\xi_1 \in T^*_{\mathcal{M}}(\mathrm{Bun}_n), \quad \mathcal{M} \in \mathrm{Bun}_n$$

be an element contained in $\mathrm{SingSupp}(\mathcal{F})$. Thus ξ_1 corresponds to an element

$$A \in \mathrm{Hom}(\mathcal{M}, \mathcal{M} \otimes \omega),$$

and assume that A is non-nilpotent. Let $x \in X$ be a point such that

$$A_x \in \mathrm{Hom}(\mathcal{M}_x, \mathcal{M}_x \otimes T^*_x(X))$$

has a non-zero eigenvalue, to be denoted $\xi_x \in T^*_x(X)$. Let i denote its multiplicity (as a generalized eigenvalue). We will construct a point $\mathcal{M}' \in \mathrm{Bun}_n$ and $\xi_2 \in T^*_{\mathcal{M}}(\mathrm{Bun}_n)$, such that the element

$$(\xi_2, i \cdot \xi_x) \in T^*_{\mathcal{M}',x}(\mathrm{Bun}_n \times X)$$

belongs to $\mathrm{SingSupp}(\mathrm{H}^i(X))$ (it is here that we use the assumption that $\mathrm{char}(k) > n$).

13.3.3. For a point $(x, \mathcal{M} \overset{\alpha}{\hookrightarrow} \mathcal{M}')$ of H^i_n, the intersection

$$(d\overleftarrow{h}^*)(T^*_{\mathcal{M}}(\mathrm{Bun}_n)) \cap (d(\overrightarrow{h} \times s)^*)(T^*_{\mathcal{M}',x}(\mathrm{Bun}_n)) \subset T^*_{(x,\mathcal{M} \overset{\alpha}{\hookrightarrow} \mathcal{M}')}(\mathcal{H}^i_n)$$

consists of diagrams

$$\begin{tikzcd} \mathcal{M}' \arrow[r, "A'"] & \mathcal{M}' \otimes \omega \\ \mathcal{M} \arrow[u, "\alpha"] \arrow[r, "A"] & \mathcal{M} \otimes \omega, \arrow[u, "\alpha\otimes\mathrm{id}"'] \end{tikzcd} \tag{13.9}$$

where the corresponding element of $T^*_x(X)$ is given by the *trace* of the induced map

$$\mathcal{M}'/\mathcal{M} \to (\mathcal{M}'/\mathcal{M}) \otimes \omega.$$

13.3.4. Let $\widetilde{X} \subset T^*(X)$ be the spectral curve corresponding to A. We can think of $\mathcal{M}$ as a torsion-free sheaf $\mathcal{L}$ on $\widetilde{X}$. Its modifications

$$\mathcal{M} \overset{\alpha}{\hookrightarrow} \mathcal{M}'$$

that fit into (13.9) are in bijection with modifications

$$\mathcal{L} \overset{\widetilde{\alpha}}{\hookrightarrow} \mathcal{L}' \tag{13.10}$$

as torsion-free coherent sheaves on $\widetilde{X}$.

13.3.5. Let $\mathcal{D}_x$ be the formal disc around x, and set

$$\widetilde{\mathcal{D}}_x := \mathcal{D} \underset{X}{\times} \widetilde{X}.$$

Modifications as in (13.10) are in bijection with similar modifications of $\mathcal{L}|_{\widetilde{\mathcal{D}}_x}$.

The multi-disc $\widetilde{\mathcal{D}}_x$ can be written as

$$\widetilde{\mathcal{D}}_x := \widetilde{\mathcal{D}}^1_x \sqcup \widetilde{\mathcal{D}}^2_x,$$

where $\widetilde{\mathcal{D}}^1_x$ is the connected component containing the element $\xi_x \in T^*_x(X) \subset T^*(X)$. By assumption,

$$\widetilde{\mathcal{D}}^1_x \to \mathcal{D}_x \tag{13.11}$$

is a flat ramified cover, such that the preimage of $x \in \mathcal{D}_x$ is a "fat point" of length i. Hence, the rank of (13.11) is i.

In particular, we obtain that $\mathcal{L}|_{\widetilde{\mathcal{D}}^1_x}$, viewed as a coherent sheaf on $\mathcal{D}_x$ via pushforward along (13.11), is a vector bundle of rank equal to i.

We let the sought-for modification of $\mathcal{L}_{\mathcal{D}_x}$ be given by

$$\mathcal{L}'_{\mathcal{D}_x}|_{\widetilde{\mathcal{D}}^1_x} = \mathcal{L}'_{\mathcal{D}_x}(x)|_{\widetilde{\mathcal{D}}^1_x} \text{ and } \mathcal{L}'_{\mathcal{D}_x}|_{\widetilde{\mathcal{D}}^2_x} = \mathcal{L}'_{\mathcal{D}_x}|_{\widetilde{\mathcal{D}}^2_x},$$

i.e., we leave $\mathcal{L}$ intact on $\widetilde{\mathcal{D}}^2_x$, and twist by the divisor equal to the preimage of x on $\widetilde{\mathcal{D}}^1_x$.

13.3.6. In order to show that the pair $(\xi_2, i \cdot \xi_x)$ indeed belongs to $\mathrm{SingSupp}(\mathrm{H}^i(X))$, we will apply Theorem 13.1.2 to

$$\mathcal{Y}_1 = \mathcal{H}^i_n,\ \mathcal{Y}_2 = \mathrm{Bun}_n \times X,\ f = (\overrightarrow{h} \times s),\ \mathcal{F}_1 = \overleftarrow{h}^*(\mathcal{F}),$$

$$y_1 = (x, \mathcal{M} \overset{\alpha}{\hookrightarrow} \mathcal{M}'),\ y_2 = (\mathcal{M}', x),\ \xi_2 = (A', i \cdot \xi_x).$$

Let us verify conditions (i) and (ii) of Theorem 13.1.2. We start with condition (i).

The point

$$((A', i \cdot \xi_x), (x, \mathcal{M} \overset{\alpha}{\hookrightarrow} \mathcal{M}')) \in T^*_{(x, \mathcal{M} \overset{\alpha}{\hookrightarrow} \mathcal{M}')}(\mathcal{H}^i_n)$$

belongs to $\mathrm{SingSupp}(\overleftarrow{h}^*(\mathcal{F}))$ by assumption.

Let us show that it is isolated in the intersection

$$\left(T^*(\mathrm{Bun}_n) \underset{\mathrm{Bun}_n, \overleftarrow{h}}{\times} \mathcal{H}^i_n\right) \cap \left((A', \xi_x) \times (\overrightarrow{h} \times s)^{-1}(\mathcal{M}', x)\right) \subset T^*(\mathcal{H}^i_n). \tag{13.12}$$

13.3.7. We interpret the pair $(\mathcal{M}', A')$ as a torsion-free sheaf $\mathcal{L}'$ on $\widetilde{X}$, and the intersection (13.12) consists of its lower modifications (13.10), such that the quotient $\mathcal{L}'/\mathcal{L}$, viewed as a coherent sheaf on X, is scheme-theoretically supported at x and has length i.

Lower modifications of $\mathcal{L}'$ on $\widetilde{X}$ over $x \in X$ are in bijection with lower modifications of $\mathcal{L}'|_{\widetilde{\mathcal{D}}_x}$. Those split into connected components enumerated by the length of the quotient $\mathcal{L}'/\mathcal{L}$ on *each* connected component of $\widetilde{\mathcal{D}}_x$.

Take the connected component, where the length of the modification is i over $\widetilde{\mathcal{D}}^1_x$, and 0 on all other components. We claim that this connected component consists of a single point, which corresponds to our $(x, \mathcal{M} \overset{\alpha}{\hookrightarrow} \mathcal{M}')$.

Indeed, the condition on the scheme-theoretic support of $\mathcal{L}'/\mathcal{L}$ implies that

$$\mathcal{L}'(-x) \subset \mathcal{L},$$

while the requirement on the length implies that the above inclusion is an equality.

13.3.8. Let us verify condition (ii) in Theorem 13.1.2.

Let $\mathcal{F}'$ be a constructible sub-object of $H^m(\mathcal{F})$ for some m. We know that all irreducible components of $\mathrm{SingSupp}(\mathcal{F}')$ have dimensions equal to $\dim(\mathrm{Bun}_G)$, by [Be2].

Hence, it suffices to show that the map

$$\left(T^*(\mathrm{Bun}_n) \underset{\mathrm{Bun}_n,\overleftarrow{h}}{\times} \mathcal{H}^i_n\right) \underset{T^*(\mathcal{H}^i_n)}{\times} \left(T^*(\mathrm{Bun}_n \times X) \underset{\mathrm{Bun}_n \times X, \overrightarrow{h} \times s}{\times} \mathcal{H}^i_n\right) \to \\ \to \left(T^*(\mathrm{Bun}_n) \underset{\mathrm{Bun}_n,\overleftarrow{h}}{\times} \mathcal{H}^i_n\right) \to T^*(\mathrm{Bun}_n)$$

has fibers of dimension ≤ 1 near $((A', i\cdot\xi_x),(x,\mathcal{M} \overset{\alpha}{\hookrightarrow} \mathcal{M}'))$.

Equivalently, it suffices to show that the map

$$\left(T^*(\mathrm{Bun}_n) \underset{\mathrm{Bun}_n,\overleftarrow{h}_x}{\times} \mathcal{H}^i_{n,x}\right) \underset{T^*(\mathcal{H}^i_{n,x})}{\times} \left(T^*(\mathrm{Bun}_n) \underset{\mathrm{Bun}_n, \overrightarrow{h}_x}{\times} \mathcal{H}^i_{n,x}\right) \to \\ \to \left(T^*(\mathrm{Bun}_n) \underset{\mathrm{Bun}_n,\overleftarrow{h}_x}{\times} \mathcal{H}^i_{n_x}\right) \to T^*(\mathrm{Bun}_n) \tag{13.13}$$

is finite near $(A',(\mathcal{M} \overset{\alpha}{\hookrightarrow} \mathcal{M}',\alpha))$.

Since the map (13.13) is proper, it suffices to show that the point $(A',(\mathcal{M} \overset{\alpha}{\hookrightarrow} \mathcal{M}')$ is isolated in its fiber with respect to (13.13). The latter is proved by the same consideration as in Sect. 13.3.7.

13.4. **The case of an arbitrary reductive group G.** The proof in the case of an arbitrary G will follow the same idea as in the case of GL_n. What will be different is the local analysis:

In the case of GL_n, to a cotangent vector to Bun_G (a.k.a. Higgs field), we attached its spectral curve $\widetilde{X}$, and proved the theorem by analyzing the behavior of modifications of sheaves on it.

For an arbitrary G, there is no spectral curve. Instead, our local analysis will amount to studying the fibers of the affine (parabolic) Springer map.

13.4.1. Recall the assumption on the characteristic of k: we are assuming that there exists a non-degenerate G-equivariant pairing

$$\mathfrak{g} \otimes \mathfrak{g} \to k, \tag{13.14}$$

whose restriction to the center of any Levi subalgebra remains non-degenerate. Thus assumption allows to identify $\mathfrak{g}^*$ with $\mathfrak{g}$ as G-modules, and also $\mathfrak{m}^*$ with $\mathfrak{m}$ for any Levi subgroup $M \subset G$.

13.4.2. Let $\mathcal{F} \in \mathrm{Shv}(\mathrm{Bun}_G)$ be an object with non-nilpotent singular support. We will find an irreducible representation $V^\lambda \in \mathrm{Rep}(\check{G})$, such that the corresponding Hecke functor

$$\mathrm{H}(V^\lambda,-) : \mathrm{Shv}(\mathrm{Bun}_G) \to \mathrm{Shv}(\mathrm{Bun}_G \times X),$$

sends $\mathcal{F}$ to an object of $\mathrm{Shv}(\mathrm{Bun}_G \times X)$ whose singular support is *not* contained in

$$T^*(\mathrm{Bun}_G) \times \{\text{zero-section}\} \subset T^*(\mathrm{Bun}_G) \times T^*(X) = T^*(\mathrm{Bun}_G \times X).$$

13.4.3. By our assumption on $\mathrm{char}(k)$, we can think of points of $T^*(\mathrm{Bun}_G)$ as pairs $(\mathcal{P}_G, A)$, where $\mathcal{P}_G$ is a G-bundle on X and A is an element of

$$\Gamma(X, \mathfrak{g}_{\mathcal{P}_G} \otimes \omega).$$

The Chevaley map attaches to A above a global section $\mathrm{ch}(A)$ of $\mathfrak{a}_\omega$, where the latter is the ω-twist of

$$\mathfrak{a} := \mathfrak{g}/\!/G \simeq \mathfrak{t}/\!/W,$$

where $\mathfrak{t}$ is the Cartan subalgebra and W is the Weyl group.

By assumption, $\mathrm{SingSupp}(\mathcal{F})$ contains a point $(\mathcal{P}_G, A)$ for which A is non-nilpotent, i.e., $\mathrm{ch}(A) \neq 0$. Let $x \in X$ be a point for which the value $\mathrm{ch}(A)_x$ of $\mathrm{ch}(A)$ at x is non-zero.

Choose a preimage $t \in \mathfrak{t}$ of $\mathrm{ch}(A)_x$ along the projection $\mathfrak{t} \to \mathfrak{a}$. Let M be the Levi subgroup of G equal centralizer of t. (Thus, if $\mathrm{ch}(A)_x$ were zero, we would get $M = G$, and if t was regular, we would get $M = T$, the Cartan subgroup.)

Let λ be a coweight of $Z(M)$ that is G-dominant and G-regular (the latter means that the centralizer of λ in G is contained in M). By the non-degeneracy assumption on k, we can choose λ so that the value of the pairing (13.14) on the pair (A_x, λ) in non-zero.

We claim that with this choice of λ, the singular support of the object

$$\mathrm{H}(V^\lambda, \mathcal{F}) \in \mathrm{Shv}(\mathrm{Bun}_G \times X)$$

at the point $(\mathcal{P}'_G, x) \in \mathrm{Bun}_G \times X$ will contain an element (A', ξ_x), where $\mathcal{P}'_G$ is the Hecke modification of $\mathcal{P}_G$ at x of type λ specified in Sect. 13.4.4 below, and

$$0 \neq \xi_x \in T^*_x(X).$$

The element A' will also be specified in Sect. 13.4.4 below.

13.4.4. By the choice of M, the fiber $(\mathcal{P}_{G,x}, A_x)$ of $(\mathcal{P}_G, A)$ at x admits a reduction $(\mathcal{P}_{M,x}, A_x)$ to M, so that

$$A_x \in \mathfrak{m}_{\mathcal{P}_{M,x}} \otimes T^*_x(X)$$

is such that its semi-simple part lies in

$$Z(\mathfrak{m}_{\mathcal{P}_{M,x}}) \otimes T^*_x(X) \subset \mathfrak{m}_{\mathcal{P}_{M,x}} \otimes T^*_x(X)$$

and is G-regular.

Note now that the map

$$\mathfrak{m}/\operatorname{Ad}(M) \to \mathfrak{g}/\operatorname{Ad}(G)$$

is étale at points of $\mathfrak{m}$, whose centralizer in G is contained in M. This implies that the restriction $(\mathcal{P}_G, A)|_{\mathcal{D}_x}$ admits a *unique* reduction $(\mathcal{P}_M, A)$ to M, whose value at x is the above reduction $(\mathcal{P}_{M,x}, A_x)$ of $(\mathcal{P}_{G,x}, A_x)$.

Being a cocharacter of $Z(M)$, the element λ defines a distinguished modification $\mathcal{P}'_M$ over $\mathcal{D}_x$. We let $\mathcal{P}'_G|_{\mathcal{D}_x}$ be the induced modification of $\mathcal{P}_G|_{\mathcal{D}_x}$, and we let $\mathcal{P}'_G$ denote the resulting modification of $\mathcal{P}_G$.

The centrality of λ implies that A is still regular (i.e., has no poles) as a section of $\mathfrak{m}_{\mathcal{P}'_M} \otimes \omega$. When viewed as such, we will denote it by A'. By a slight abuse of notation we will denote by the same symbol A' the resulting section of $\mathfrak{g}_{\mathcal{P}'_G} \otimes \omega$.

13.4.5. Consider the Hecke stack

$$\mathrm{Bun}_G \xleftarrow{\overleftarrow{h}} \mathcal{H}_G \xrightarrow{\overrightarrow{h}\times s} \mathrm{Bun}_G \times X.$$

For future use, denote by

$$\mathrm{Bun}_G \xleftarrow{\overleftarrow{h}_x} \mathcal{H}_{G,x} \xrightarrow{\overrightarrow{h}_x} \mathrm{Bun}_G$$

the fiber of this picture over a given $x \in X$.

We will apply Theorem 13.1.2 to

$$\mathcal{Y}_1 = \mathcal{H}_G,\ \mathcal{Y}_2 = \mathrm{Bun}_G \times X,\ f = \overrightarrow{h} \times s,\ \mathcal{F}_1 = \overleftarrow{h}^*(\mathcal{F}) \otimes \tau^*(\mathcal{V}^\lambda),$$

where:

- $\tau : \mathcal{H}_G \to \mathcal{H}_G^{\mathrm{loc}}$ is the projection on the local Hecke stack (see [GKRV, Sect.B.3.2]);
- $\mathcal{V}^\lambda \in \mathrm{Shv}(\mathcal{H}_G^{\mathrm{loc}})$ corresponds to $V^\lambda \in \mathrm{Rep}(\check{G})$ by geometric Satake.

We take $y_2 = (\mathcal{P}'_G, x)$ and y_1 corresponding to the modification $\mathcal{P}_G \overset{\alpha}{\rightsquigarrow} \mathcal{P}'_G$.

We will show the following:

(a) There exists *some* $\xi_x \in T^*_x(X)$ such that

$$((A', \xi_x), (x, \mathcal{P}_G \rightsquigarrow \mathcal{P}'_G)) \in \mathrm{SingSupp}(\overleftarrow{h}^*(\mathcal{F}) \otimes \tau^*(\mathcal{V}^\lambda));$$

(b) ξ_x is the value of the pairing (13.14) on the pair (A_x, λ), and hence, is non-zero by the choice of λ;

(c) The point $((A', \xi_x), (x, \mathcal{P}_G \rightsquigarrow \mathcal{P}'_G))$ is isolated in the intersection

$$\left(T^*(\mathrm{Bun}_G \times \mathcal{H}_G^{\mathrm{loc}}) \underset{\mathrm{Bun}_G \times \mathcal{H}_G^{\mathrm{loc}}, \overleftarrow{h} \times \tau}{\times} \overline{\mathcal{H}}^\lambda_G \right) \cap \left((A', \xi_x) \times (\overrightarrow{h} \times s)^{-1}(\mathcal{P}'_G, x) \right) \subset T^*(\mathcal{H}_G),$$

where $\overline{\mathcal{H}}^\lambda_G$ is the closure of $\mathcal{H}^\lambda_G \subset \mathcal{H}_G$, the latter being the locus of modifications of type λ.

(d) The point $(A', (\mathcal{P}_G \rightsquigarrow \mathcal{P}'_G))$ is isolated in its fiber along the map

$$\left(T^*(\mathrm{Bun}_G \times \mathcal{H}_{G,x}^{\mathrm{loc}}) \underset{\mathrm{Bun}_G \times \mathcal{H}_{G,x}^{\mathrm{loc}}, \overleftarrow{h}_x \times \tau}{\times} \overline{\mathcal{H}}^\lambda_{G,x} \right) \underset{T^*(\overline{\mathcal{H}}^\lambda_{G,x})}{\times} \left(T^*(\mathrm{Bun}_G) \underset{\mathrm{Bun}_G, \overrightarrow{h}}{\times} \overline{\mathcal{H}}^\lambda_{G,x} \right) \to$$
$$\to T^*(\mathrm{Bun}_G \times \mathcal{H}_{G,x}^{\mathrm{loc}}) \underset{\mathrm{Bun}_G \times \mathcal{H}_{G,x}^{\mathrm{loc}}, \overleftarrow{h}_x \times \tau}{\times} \overline{\mathcal{H}}^\lambda_{G,x} \to T^*(\mathrm{Bun}_G).$$

Once we establish properties (a)-(d), the assertion of Theorem 10.3.3 will follow by applying Theorem 13.1.2.

13.4.6. To prove point (a) it suffices (in fact, equivalent) to show:

(a')

$$(A', (\mathcal{P}_G \overset{\alpha}{\rightsquigarrow} \mathcal{P}'_G)) \in \mathrm{SingSupp}(\overleftarrow{h}^*_x(\mathcal{F}) \otimes \tau^*_x(\mathcal{V}^\lambda_x));$$

where

$$\tau_x : \mathcal{H}_{G,x} \to \mathcal{H}^{\mathrm{loc}}_{G,x}, \quad \mathcal{V}^\lambda_x \in \mathrm{Shv}(\mathcal{H}^{\mathrm{loc}}_{G,x})$$

are the counterparts of $(\tau, \mathcal{V}^\lambda)$ at x.

13.4.7. Recall (see, for example, [GKRV, Formula (B.22)]) that

$$T^*_{\mathcal{P}_G \overset{\alpha}{\rightsquigarrow} \mathcal{P}'_G}(\mathcal{H}^{\mathrm{loc}}_{G,x})$$

identifies with the set of pairs

$$A^{\mathrm{loc}} \in \Gamma(\mathcal{D}_x, \mathfrak{g}_{\mathcal{P}_G} \otimes \omega),\ A'^{\mathrm{loc}} \in \Gamma(\mathcal{D}_x, \mathfrak{g}_{\mathcal{P}'_G} \otimes \omega), \tag{13.15}$$

such that

$$\alpha(A^{\mathrm{loc}}) = A'^{\mathrm{loc}}$$

as elements of $\Gamma(\overset{\circ}{\mathcal{D}}_x, \mathfrak{g}_{\mathcal{P}'} \otimes \omega_X)$.

Furthermore, given

$$A \in T^*(\mathrm{Bun}_G) \simeq \Gamma(X, \mathfrak{g}_{\mathcal{P}_G} \otimes \omega),\ A' \in T^*(\mathrm{Bun}_G) \simeq \Gamma(X, \mathfrak{g}_{\mathcal{P}'_G} \otimes \omega)$$

their images in $T^*_{\mathcal{P}_G \overset{\alpha}{\rightsquigarrow} \mathcal{P}'_G}(\mathcal{H}_{G,x})$ differ by the image of an element in $T^*_{\mathcal{P}_G \overset{\alpha}{\rightsquigarrow} \mathcal{P}'_G}(\mathcal{H}^{\mathrm{loc}}_{G,x})$ if and only if

$$\alpha(A|_{X-x}) = A'|_{X-x},$$

and in this case the corresponding element of $T^*_{\mathcal{P}_G \overset{\alpha}{\rightsquigarrow} \mathcal{P}'_G}(\mathcal{H}^{\mathrm{loc}}_{G,x})$ is given in terms of (13.15) by

$$A^{\mathrm{loc}} := A|_{\mathcal{D}_x},\ A'^{\mathrm{loc}} = A'|_{\mathcal{D}_x}.$$

13.4.8. Hence, in order to prove (a'), it suffices to show that for a point

$$(\mathcal{P}_G \overset{\alpha}{\rightsquigarrow} \mathcal{P}'_G) \in \mathcal{H}^{\mathrm{loc}}_{G,x}$$

induced by a point

$$(\mathcal{P}_M \overset{\alpha}{\rightsquigarrow} \mathcal{P}'_M) \in \mathcal{H}^{\mathrm{loc}}_{M,x},$$

corresponding to λ, *any* pair

$$(A^{\mathrm{loc}}, A'^{\mathrm{loc}}) \in T^*_{\mathcal{P}_G \overset{\alpha}{\rightsquigarrow} \mathcal{P}'_G}(\mathcal{H}^{\mathrm{loc}}_{G,x})$$

belongs to $\mathrm{SingSupp}(\mathcal{V}^\lambda_x)$.

We identify

$$\mathcal{H}^{\mathrm{loc}}_{G,x} = G[\![t]\!]\backslash G(\!(t)\!)/G[\![t]\!]$$

so that the point $\mathcal{P}_G \overset{\alpha}{\rightsquigarrow} \mathcal{P}'_G$ corresponds to t^λ.

Recall that $\mathcal{V}^\lambda_x$ is the IC-sheaf on closure of the double coset of

$$t^\lambda \in G[\![t]\!]\backslash G(\!(t)\!)/G[\![t]\!].$$

Hence, the fiber of $\mathrm{SingSupp}(\mathcal{V}^\lambda_x)$ at t^λ is the conormal to this double coset, and hence equals the entire cotangent space at this point.

13.4.9. To prove point (b), we mimic the argument of [GKRV, Sect. B.6.7]. We consider $\mathcal{H}^{\mathrm{loc}}_G$, equipped with its natural crystal structure along X, and the corresponding splitting of the short exact sequence

$$0 \to T^*_x(X) \to T^*_{\mathcal{P}_G \overset{\alpha}{\rightsquigarrow} \mathcal{P}'_G}(\mathcal{H}^{\mathrm{loc}}_G) \to T^*_{\mathcal{P}_G \overset{\alpha}{\rightsquigarrow} \mathcal{P}'_G}(\mathcal{H}^{\mathrm{loc}}_{G,x}) \to 0,$$

i.e.,

$$T^*_{\mathcal{P}_G \overset{\alpha}{\rightsquigarrow} \mathcal{P}'_G}(\mathcal{H}^{\mathrm{loc}}_G) \simeq T^*_x(X) \oplus T^*_{\mathcal{P}_G \overset{\alpha}{\rightsquigarrow} \mathcal{P}'_G}(\mathcal{H}^{\mathrm{loc}}_{G,x}).$$

It suffices to show that, in terms of this identification, for an element

$$(\xi_x, (A^{\mathrm{loc}}, A'^{\mathrm{loc}})) \in T^*_{\mathcal{P}_G \overset{\alpha}{\rightsquigarrow} \mathcal{P}'_G}(\mathcal{H}^{\mathrm{loc}}_G)$$

that belongs to $\mathrm{SingSupp}(\mathcal{V}^\lambda)$, we have

$$\xi_x := \langle A^{\mathrm{loc}}_x, \lambda\rangle, \tag{13.16}$$

where A^{loc}_x is the value of A^{loc} at x.

The assertion is local, so we can assume that X is $\mathbb{A}^1$, with coordinate t. This allows to trivialize the line $T^*_x(X)$. Further, we can assume that $\mathcal{P}_G$ is trivial. Then we can think of

$$A^{\mathrm{loc}} \in \Gamma(\mathcal{D}_x, \mathfrak{g}_{\mathcal{P}_G} \otimes \omega)$$

as an element of $\mathfrak{g}[\![t]\!]$.

By [GKRV, Formula (B.33)], the element ξ_x equals

$$\mathrm{Res}(A^{\mathrm{loc}}, \lambda \cdot \frac{dt}{t}),$$

whence (13.16).

13.5. **Proof of point (c) and affine Springer fibers.** To prove point (c), it suffices to show:

(c') The point $(A', (\mathcal{P}_G \rightsquigarrow \mathcal{P}'_G))$ is isolated in the intersection

$$\left(T^*(\mathrm{Bun}_G \times \mathcal{H}^{\mathrm{loc}}_{G,x}) \underset{\mathrm{Bun}_G \times \mathcal{H}^{\mathrm{loc}}_{G,x}, \overleftarrow{h}_x \times \tau_x}{\times} \overline{\mathcal{H}}^{\lambda}_{G,x}\right) \cap \left(A' \times (\overrightarrow{h}_x)^{-1}(\mathcal{P}'_G)\right) \subset T^*(\mathcal{H}_{G,x}). \tag{13.17}$$

Point (d) in Sect. 13.4.5 is proved similarly.

13.5.1. Consider first the bigger intersection

$$\left(T^*(\mathrm{Bun}_G \times \mathcal{H}^{\mathrm{loc}}_{G,x}) \underset{\mathrm{Bun}_G \times \mathcal{H}^{\mathrm{loc}}_{G,x}, \overleftarrow{h}_x \times \tau_x}{\times} \mathcal{H}_{G_x}\right) \cap \left(A' \times (\overrightarrow{h}_x)^{-1}(\mathcal{P}'_G)\right) \subset T^*(\mathcal{H}_{G,x}). \tag{13.18}$$

By Sect. 13.4.7, the scheme in (13.18) is the space of modifications of $\mathcal{P}'_G|_{\mathcal{D}_x} \rightsquigarrow \mathcal{P}^{\mathrm{loc}}_G$, for which the element

$$A' \in \Gamma(\mathcal{D}_x, \mathfrak{g}_{\mathcal{P}'_G} \otimes \omega) \subset \Gamma(\overset{\circ}{\mathcal{D}}_x, \mathfrak{g}_{\mathcal{P}'_G} \otimes \omega) \simeq \Gamma(\overset{\circ}{\mathcal{D}}_x, \mathfrak{g}_{\mathcal{P}^{\mathrm{loc}}_G} \otimes \omega)$$

belongs to

$$\Gamma(\mathcal{D}_x, \mathfrak{g}_{\mathcal{P}^{\mathrm{loc}}_G} \otimes \omega) \subset \Gamma(\overset{\circ}{\mathcal{D}}_x, \mathfrak{g}_{\mathcal{P}^{\mathrm{loc}}_G} \otimes \omega).$$

Denote this intersection by $\mathrm{Spr}_{G,A'}$: it is isomorphic to a parahoric affine Springer fiber over the element A'. Denote the intersection (13.17) by $\mathrm{Spr}^{\leq\lambda}_{G,A'}$.

13.5.2. If we trivialize $\mathcal{P}'_G$, we can think of $\mathrm{Spr}_{G,A'}$ as a (closed) subscheme in Gr_G, and we have

$$\mathrm{Spr}^{\leq\lambda}_{G,A'} = \mathrm{Spr}_{G,A'} \cap \overline{\mathrm{Gr}}^{\lambda}_G.$$

We need to show that our particular point

$$\mathcal{P}'_G|_{\mathcal{D}_x} \rightsquigarrow \mathcal{P}_G|_{\mathcal{D}_x} \tag{13.19}$$

is isolated in $\mathrm{Spr}^{\leq\lambda}_{G,A'}$.

13.5.3. By Sect. 13.4.4, the G-bundle $\mathcal{P}'_G$ on $\mathcal{D}_x$ is equipped with a reduction to M. So along with $\mathrm{Spr}_{G,A'}$, we can consider its variant for M, to be denoted $\mathrm{Spr}_{M,A'}$. Since

$$\mathrm{Gr}_M \to \mathrm{Gr}_G$$

is a closed embedding, so is the embedding $\mathrm{Spr}_{M,A'} \hookrightarrow \mathrm{Spr}_{G,A'}$.

We claim:

Proposition 13.5.4. *The inclusion* $\mathrm{Spr}_{M,A'} \hookrightarrow \mathrm{Spr}_{G,A'}$ *is an equality.*

13.5.5. Let us show how Proposition 13.5.4 implies that (13.19) is isolated in $\mathrm{Spr}^{\leq\lambda}_{G,A'}$.

By Proposition 13.5.4, it suffices to show that the point t^λ is isolated in

$$\overline{\mathrm{Gr}}^\lambda_G \cap \mathrm{Gr}_M. \tag{13.20}$$

It suffices to show that t^λ is *open* in (13.20). Hence, it suffices to show that it is isolated in

$$\mathrm{Gr}^\lambda_G \cap \mathrm{Gr}_M. \tag{13.21}$$

Note, however, that the intersection $\mathrm{Gr}^\lambda_G \cap \mathrm{Gr}_M$ is the union of $M[\![t]\!]$-orbits Gr^μ_M over M-dominant coweights μ for which there exists $w \in W$ such that

$$\mu = w(\lambda).$$

Note that the point t^λ equals Gr^λ_M, because λ is a coweight of $Z(M)$. The assertion follows now from the regularity assumption on λ: the orbit Gr^λ_M belongs to a different connected component of Gr_M than the other Gr^μ_M with $\mu = w(\lambda)$.

13.6. **Proof of Proposition 13.5.4.**

13.6.1. Fix a Cartan subgroup $T \subset G$, and a Levi subgroup $T \subset M \subset G$. We consider the affine schemes

$$\mathfrak{a} := \mathfrak{g}/\!/\operatorname{Ad}(G) \simeq \mathfrak{t}/\!/W \text{ and } \mathfrak{a}_M := \mathfrak{m}/\!/\operatorname{Ad}(M) \simeq \mathfrak{t}/\!/W_M,$$

and a natural map between them.

Let $\overset{\circ}{\mathfrak{t}} \subset \mathfrak{t}$ be the open subset consisting of elements for which $\check{\alpha}(\mathfrak{t}) \neq 0$ for all roots $\check{\alpha}$ that are *not* roots of M. Since this subset is W_M-invariant, it corresponds to an open subset

$$\overset{\circ}{\mathfrak{a}}_M \subset \mathfrak{a}_M,$$

so that we have a Cartesian diagram

$$\begin{tikzcd} \overset{\circ}{\mathfrak{t}} \arrow[r] \arrow[d] & \mathfrak{t} \arrow[d] \\ \overset{\circ}{\mathfrak{a}}_M \arrow[r] & \mathfrak{a}_M. \end{tikzcd} \tag{13.22}$$

We will refer to $\overset{\circ}{\mathfrak{a}}_M$ as the *regular* (for the pair $M \subset G$) locus of $\mathfrak{a}_M$.

We now make the following observation:

Lemma 13.6.2. *For an element $A \in \mathfrak{m}$ the following conditions on are equivalent:*

(i) $Z_{\mathfrak{g}}(A) \subset \mathfrak{m}$*;*

(i') *The adjoint action of A on $\mathfrak{g}/\mathfrak{m}$ is invertible;*

(ii) $Z_{\mathfrak{g}}(A^{\mathrm{ss}}) \subset \mathfrak{m}$*, where A^{ss} is the semi-simple part of A;*

(ii') *The adjoint action of A^{ss} on $\mathfrak{g}/\mathfrak{m}$ is invertible;*

(iii) *The image of A in $\mathfrak{a}_M$ belongs to $\overset{\circ}{\mathfrak{a}}_M$.*

Proof. Clearly (i) $\Leftrightarrow$ (i') and (ii) $\Leftrightarrow$ (ii'). However, it is also clear that (ii) $\Leftrightarrow$ (ii'). The equivalence (iii) $\Leftrightarrow$ (ii') is the fact that the diagram (13.22) is Cartesian.

□

Let us call an element $A \in \mathfrak{m}$ *regular* (for the pair $M \subset G$) if it satisfies the equivalent conditions of Lemma 13.6.2. Regular elements of $\mathfrak{m}$ form a Zariski-open subset to be denoted $\overset{\circ}{\mathfrak{m}}$. We have a Cartesian diagram

$$\begin{tikzcd} \overset{\circ}{\mathfrak{m}} \arrow[r] \arrow[d] & \mathfrak{m} \arrow[d] \\ \overset{\circ}{\mathfrak{a}}_M \arrow[r] & \mathfrak{a}_M. \end{tikzcd}$$

We now claim:

Lemma 13.6.3. (a) *The open subset* $\overset{\circ}{\mathfrak{a}}_M \subset \mathfrak{a}_M$ *is the locus of etaleness of the map*

$$\mathfrak{a}_M \to \mathfrak{a}.$$

(b) *The open subset* $\overset{\circ}{\mathfrak{m}} \subset \mathfrak{m}$ *is the locus of etaleness of the map*

$$\mathfrak{m}/\operatorname{Ad}(M) \to \mathfrak{g}/\operatorname{Ad}(G).$$

(c) *The diagram*

$$\begin{CD} \overset{\circ}{\mathfrak{m}}/\operatorname{Ad}(M) @>>> \mathfrak{g}/\operatorname{Ad}(G) \\ @VVV @VVV \\ \mathfrak{a}_M @>>> \mathfrak{a} \end{CD}$$

is Cartesian.

Proof. Point (a) follows from the fact that an element $\mathfrak{t} \in \mathfrak{t}$ belongs to $\overset{\circ}{\mathfrak{t}}$ if and only if its stabilizer in W is contained in W_M.

Point (b) is a straightforward tangent space calculation.

For point (c), we note that by pints (a) and (b), the map

$$\overset{\circ}{\mathfrak{m}}/\operatorname{Ad}(M) \to \mathfrak{a}_M \underset{\mathfrak{a}}{\times} \mathfrak{g}/\operatorname{Ad}(G)$$

is étale. So, it is sufficient to check that it is bijective at the level of field-valued points, which is an easy exercise.

□

13.6.4. In order to proceed, we record the following lemma. For a prestack $\mathcal{Y}$ denote by $\mathcal{Y}[\![t]\!]$ the corresponding prestack of arcs:

$$\operatorname{Maps}(\operatorname{Spec}(R), \mathcal{Y}[\![t]\!]) = \operatorname{Maps}(\operatorname{Spec}(R[\![t]\!]), \mathcal{Y}).$$

Evaluation modulo t defines a map $\mathcal{Y}[\![t]\!] \to \mathcal{Y}$. We have:

Lemma 13.6.5. *Let* $\mathcal{Y}_1 \to \mathcal{Y}_2$ *be an étale map of algebraic stacks. Then the diagram of prestacks*

$$\begin{CD} \mathcal{Y}_1[\![t]\!] @>>> \mathcal{Y}_2[\![t]\!] \\ @VVV @VVV \\ \mathcal{Y}_1 @>>> \mathcal{Y}_2, \end{CD}$$

where the vertical arrows are given by evaluation mod t*, is Cartesian.*

Proof. For an algebraic stack $\mathcal{Y}$, the groupoid $\operatorname{Maps}(\operatorname{Spec}(R[\![t]\!]), \mathcal{Y})$ is the inverse limit of

$$\operatorname{Maps}(\operatorname{Spec}(R[t]/t^n), \mathcal{Y}).$$

Now the assertion follows from the definition of formal etaleness.

□

13.6.6. Let us be given a map

$$(\mathcal{P}_G, A) : \mathcal{D}_x \to \mathfrak{g}/\operatorname{Ad}(G). \tag{13.23}$$

Let us be given a regular reduction $(\mathcal{P}_{M,x}, A_x)$ of $(\mathcal{P}_{G,x}, A_x)$ to M. Since the map

$$\mathfrak{m}/\operatorname{Ad}(M) \to \mathfrak{g}/\operatorname{Ad}(G)$$

is étale at A_x, from Lemma 13.6.5, we obtain that we can uniquely lift the map (13.23) to a map

$$(\mathcal{P}_M, A) : \mathcal{D}_x \to \mathfrak{m}/\operatorname{Ad}(M),$$

so that its value at x is the lift $(\mathcal{P}_{M,x}, A_x)$ of $(\mathcal{P}_{G,x}, A_x)$.

Note, that by Lemma 13.6.3(c), the choice of $(\mathcal{P}_{M,x}, A_x)$ corresponds to a choice of lift of the point $\operatorname{ch}(A_x) \in \mathfrak{a}$ to a point of $\overset{\circ}{\mathfrak{a}}_M$, to be denoted $\operatorname{ch}_M(A_x)$.

Similarly, the datum of $(\mathcal{P}_M, A)$ corresponds to a lift of

$$\operatorname{ch}(A) : \mathcal{D}_x \to \mathfrak{a}$$

to a map $\mathcal{D}_x \to \overset{\circ}{\mathfrak{a}}_M$, to be denoted $\operatorname{ch}_M(A)$.

Note that by Lemma 13.6.5 applied to $\overset{\circ}{\mathfrak{a}}_M \to \mathfrak{a}$, the map $\operatorname{ch}_M(A)$ is the unique lift of $\operatorname{ch}(A)$, whose value at x is the lift $\operatorname{ch}_M(A_x)$ of $\operatorname{ch}(A_x)$.

13.6.7. Let us return to the setting of Proposition 13.5.4. We can assume starting with a map (13.23), and let $\mathcal{P}_M$ be as in Sect. 13.6.6 above.

We are studying the ind-scheme $\operatorname{Spr}_{G,A}$ of those modifications

$$\mathcal{P}_G \rightsquigarrow \mathcal{P}'_G,$$

for which the element

$$A \in \Gamma(\mathcal{D}_x, \mathfrak{g}_{\mathcal{P}_G}) \subset \Gamma(\overset{\circ}{\mathcal{D}}_x, \mathfrak{g}_{\mathcal{P}_G}) \simeq \Gamma(\overset{\circ}{\mathcal{D}}_x, \mathfrak{g}_{\mathcal{P}'_G})$$

belongs to

$$\Gamma(\mathcal{D}_x, \mathfrak{g}_{\mathcal{P}'_G}) \subset \Gamma(\overset{\circ}{\mathcal{D}}_x, \mathfrak{g}_{\mathcal{P}'_G}).$$

We need to show that any $(\mathcal{P}_G \rightsquigarrow \mathcal{P}'_G) \in \operatorname{Spr}_{G,A}$ is induced by a modification of M-bundles

$$\mathcal{P}_M \rightsquigarrow \mathcal{P}'_M.$$

13.6.8. By construction, the maps

$$\overset{\circ}{\mathcal{D}}_x \to \mathfrak{g}/\operatorname{Ad}(G) \to \mathfrak{a}$$

corresponding to A and A', respectively, are canonically identified. Since $\mathfrak{a}$ is an affine (hence, separated, scheme), the maps $\operatorname{ch}(A)$ and $\operatorname{ch}(A')$

$$\mathcal{D}_x \rightrightarrows \mathfrak{a},$$

corresponding to A and A', respectively, are also equal.

In particular, by Lemma 13.6.3(c), the datum of $\operatorname{ch}_M(A) =: \operatorname{ch}_M(A')$ defines a lift of

$$(\mathcal{P}'_G, A') : \mathcal{D}_x \to \mathfrak{g}/\operatorname{Ad}(G)$$

to a map

$$(\mathcal{P}'_M, A') : \mathcal{D}_x \to \mathfrak{m}/\operatorname{Ad}(M).$$

In particular, we obtain a reduction $\mathcal{P}'_M$ of $\mathcal{P}'_G$ to M. By construction, the G-bundles

$$\mathcal{P}_G = G \overset{M}{\times} \mathcal{P}_M \text{ and } \mathcal{P}'_G = G \overset{M}{\times} \mathcal{P}'_M$$

are identified over $\overset{\circ}{\mathcal{D}}_x$. To prove Proposition 13.5.4, it remains to show that the this identification is induced by an identification between $\mathcal{P}_M|_{\overset{\circ}{\mathcal{D}}_x}$ and $\mathcal{P}'_M|_{\overset{\circ}{\mathcal{D}}_x}$ as M-bundles.

13.6.9. We claim that in fact the identification of the maps

$$(\mathcal{P}_G, A)|_{\overset{\circ}{\mathcal{D}}_x} : \overset{\circ}{\mathcal{D}}_x \to \mathfrak{g}/\operatorname{Ad}(G) \text{ and } (\mathcal{P}'_G, A')|_{\overset{\circ}{\mathcal{D}}_x} : \overset{\circ}{\mathcal{D}}_x \to \mathfrak{g}/\operatorname{Ad}(G)$$

is induced by an identification of the maps

$$(\mathcal{P}_M, A)|_{\overset{\circ}{\mathcal{D}}_x} : \overset{\circ}{\mathcal{D}}_x \to \mathfrak{m}/\operatorname{Ad}(G) \text{ and } (\mathcal{P}'_M, A')|_{\overset{\circ}{\mathcal{D}}_x} : \overset{\circ}{\mathcal{D}}_x \to \mathfrak{m}/\operatorname{Ad}(G).$$

Indeed, this follows from Lemma 13.6.3(c), since the corresponding maps

$$\operatorname{ch}_M(A)|_{\overset{\circ}{\mathcal{D}}_x} : \overset{\circ}{\mathcal{D}}_x \to \overset{\circ}{\mathfrak{a}}_M \text{ and } \operatorname{ch}_M(A')|_{\overset{\circ}{\mathcal{D}}_x} : \overset{\circ}{\mathcal{D}}_x \to \overset{\circ}{\mathfrak{a}}_M$$

coincide by construction.

13.7. **Proof of Theorem 10.3.3 for non-holonomic D-modules.** We will deduce the assertion of Theorem 10.3.3 for D-mod$(-)$ from its validity for the subcategory Shv$(-)$ consisting of objects with regular holonomic cohomologies.

The proof is based on considering field extensions of the initial ground field k (cf. the proof of Observation 14.4.4).

13.7.1. By Proposition 7.3.7, it suffices to show that the inclusion

$$\begin{aligned}\operatorname{Shv}_{\operatorname{Nilp}}(\operatorname{Bun}_G) \simeq \operatorname{QCoh}(\operatorname{LocSys}^{\operatorname{restr}}_{\check{G}}(X)) \underset{\operatorname{QCoh}(\operatorname{LocSys}^{\operatorname{restr}}_{\check{G}}(X))}{\otimes} \operatorname{Shv}_{\operatorname{Nilp}}(\operatorname{Bun}_G) \simeq \\ \simeq \operatorname{QCoh}(\operatorname{LocSys}^{\operatorname{restr}}_{\check{G}}(X)) \underset{\operatorname{QCoh}(\operatorname{LocSys}_{\check{G}}(X))}{\otimes} \operatorname{Shv}_{\operatorname{Nilp}}(\operatorname{Bun}_G) \hookrightarrow \\ \hookrightarrow \operatorname{QCoh}(\operatorname{LocSys}^{\operatorname{restr}}_{\check{G}}(X)) \underset{\operatorname{QCoh}(\operatorname{LocSys}_{\check{G}}(X))}{\otimes} \operatorname{D-mod}(\operatorname{Bun}_G)\end{aligned}$$

is an equality.

Let S be an affine scheme mapping to $\operatorname{LocSys}^{\operatorname{restr}}_{\check{G}}(X)$. It suffices to show that the inclusion

$$\operatorname{QCoh}(S) \underset{\operatorname{QCoh}(\operatorname{LocSys}_{\check{G}}(X))}{\otimes} \operatorname{Shv}_{\operatorname{Nilp}}(\operatorname{Bun}_G) \hookrightarrow \operatorname{QCoh}(S) \underset{\operatorname{QCoh}(\operatorname{LocSys}_{\check{G}}(X))}{\otimes} \operatorname{D-mod}(\operatorname{Bun}_G) \tag{13.24}$$

is an equality.

13.7.2. Let $k \subset k'$ be a field extension. Set X' (resp., Bun'_G) be the base change of X (resp., Bun_G) from k to k'. Note that for any prestack $\mathcal{Y}$ over k and its base change $\mathcal{Y}'$ to k', we have

$$\operatorname{D-mod}(\mathcal{Y}') \simeq \operatorname{Vect}_{k'} \underset{\operatorname{Vect}_k}{\otimes} \operatorname{D-mod}(\mathcal{Y}). \tag{13.25}$$

For a fixed $\mathcal{N} \subset T^*(\mathcal{Y})$, we have a fully faithful embedding

$$\operatorname{Vect}_{k'} \underset{\operatorname{Vect}_k}{\otimes} \operatorname{D-mod}_{\mathcal{N}}(\mathcal{Y}) \hookrightarrow \operatorname{D-mod}_{\mathcal{N}'}(\mathcal{Y}'), \tag{13.26}$$

but which is no longer an isomorphism. (Indeed, for example, for $\mathcal{N} = \{0\}$, there are many more local systems over k' than over k.)

From (13.25) we obtain an equivalence

$$\left(\operatorname{QCoh}(S) \underset{\operatorname{QCoh}(\operatorname{LocSys}_{\check{G}}(X))}{\otimes} \operatorname{D-mod}(\operatorname{Bun}_G)\right) \underset{\operatorname{Vect}_k}{\otimes} \operatorname{Vect}_{k'} \simeq \operatorname{QCoh}(S) \underset{\operatorname{QCoh}(\operatorname{LocSys}_{\check{G}}(X'))}{\otimes} \operatorname{D-mod}(\operatorname{Bun}'_G)$$

and from (13.26) a fully faithful embedding

$$\begin{aligned}\left(\operatorname{QCoh}(S) \underset{\operatorname{QCoh}(\operatorname{LocSys}_{\check{G}}(X))}{\otimes} \operatorname{Shv}_{\operatorname{Nilp}}(\operatorname{Bun}_G)\right) \underset{\operatorname{Vect}_k}{\otimes} \operatorname{Vect}_{k'} \hookrightarrow \\ \to \operatorname{QCoh}(S) \underset{\operatorname{QCoh}(\operatorname{LocSys}_{\check{G}}(X'))}{\otimes} \operatorname{Shv}_{\operatorname{Nilp}}(\operatorname{Bun}'_G).\end{aligned} \tag{13.27}$$

However, we claim:

Lemma 13.7.3. *The inclusion* (13.27) *is an equality.*

Proof. We will show that the image of the functor (13.27) contains the generators of the target category.

Indeed, let $y_i \in \mathrm{Bun}_G$ be as Sect. 11.5.4. Let y_i' be the corresponding k'-points of Bun'_G. Then the generators of $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ (resp., $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}'_G)$) are given by $R_S^{\mathrm{enh}}(\delta_{y_i})$ (resp., $R_S^{\mathrm{enh}}(\delta_{y_i'})$), and the assertion follows.

$\square$

13.7.4. We are now ready to prove that (13.24) is an equality. Let $\mathcal{F}$ be an object in the right-hand side, which is right-orthogonal to the left-hand side. By Lemma 13.7.3 for any $k \subset k'$, the pullback $\mathcal{F}'$ to Bun'_G will have the same property.

We now clam that for any $\mathcal{F}$ as above, its image in $\mathrm{D\text{-}mod}(\mathrm{Bun}_G)$ is right-orthogonal to $\mathrm{Shv}(\mathrm{Bun}_G)$. Indeed, for any $\mathcal{F}_1 \in \mathrm{Shv}(\mathrm{Bun}_G)$, we have

$$\mathcal{H}om_{\mathrm{D\text{-}mod}(\mathrm{Bun}_G)}(\mathcal{F}_1, \mathcal{F}) \simeq \mathcal{H}om_{\mathrm{QCoh}(S) \underset{\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))}{\otimes} \mathrm{D\text{-}mod}(\mathrm{Bun}_G)}(R_S^{\mathrm{enh}}(\mathcal{F}_1), \mathcal{F}),$$

while

$$R_S^{\mathrm{enh}}(\mathcal{F}_1) \in \mathrm{QCoh}(S) \underset{\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))}{\otimes} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G).$$

Hence, we obtain that for $\mathcal{F}$ as above and any $k \subset k'$, the corresponding object $\mathcal{F}' \in \mathrm{D\text{-}mod}(\mathrm{Bun}'_G)$ is right-orthogonal to $\mathrm{Shv}(\mathrm{Bun}'_G)$.

We wish to show that $\mathcal{F} = 0$. It suffices to show that the image of $\mathcal{F}$ in $\mathrm{D\text{-}mod}(\mathrm{Bun}_G)$ is zero. This follows from the next assertion:

Lemma 13.7.5. *Let* $\mathcal{F} \in \mathrm{D\text{-}mod}(\mathcal{Y})$ *be such that for any* $k \subset k'$, *the corresponding object* $\mathcal{F}' \in \mathrm{D\text{-}mod}(\mathcal{Y}')$ *is right-orthogonal to* $\mathrm{Shv}(\mathcal{Y}')$. *Then* $\mathcal{F} = 0$.

Proof. Let $\mathcal{F} \neq 0$. Consider the underlying object $\mathbf{oblv}_{\mathrm{D\text{-}mod}}(\mathcal{F}) \in \mathrm{QCoh}(\mathcal{Y})$. Then we can find a geometric point

$$\mathrm{Spec}(k') \xrightarrow{\iota_y} \mathcal{Y},$$

so that $\iota_y^*(\mathbf{oblv}_{\mathrm{D\text{-}mod}}(\mathcal{F})) \neq 0$.

Let $\iota_{y'}$ denote the resulting geometric point of $\mathcal{Y}'$. Then $\iota_{y'}^*(\mathbf{oblv}_{\mathrm{D\text{-}mod}}(\mathcal{F}')) \neq 0$. However, the latter means that

$$\mathcal{H}om_{\mathrm{D\text{-}mod}(\mathcal{Y}')}(\delta_{y'}, \mathcal{F}') \neq 0.$$

$\square$

Part IV: Langlands theory with nilpotent singular support

Let us make a brief overview of the contents of this Part.

In Sect. 14 we state the (categorical) Geometric Langlands conjecture with restricted variation:

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \simeq \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)),$$

and compare it to the de Rham and Betti versions of the GLC. A priori the restricted version follows from these other two. However, we show that the restricted version is actually equivalent to the full de Rham version (assuming Hypothesis 14.4.2).

In Sect. 15 we formulate one of the key points of this paper, the Trace Conjecture. We start by reviewing the *local term* map

$$\mathrm{LT} : \mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}(\mathcal{Y})) \to \mathrm{Funct}_c(\mathcal{Y}(\mathbb{F}_q)),$$

where $\mathcal{Y}$ is an algebraic stack defined over $\mathbb{F}_q$, but considered over $\overline{\mathbb{F}}_q$. The Trace Conjecture says that the composition

$$\mathrm{Tr}((\mathrm{Frob}_{\mathrm{Bun}_G})_*, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)) \to \mathrm{Tr}((\mathrm{Frob}_{\mathrm{Bun}_G})_*, \mathrm{Shv}(\mathrm{Bun}_G)) \overset{\mathrm{LT}}{\to} \mathrm{Funct}_c(\mathrm{Bun}_G(\mathbb{F}_q)) =: \mathrm{Autom}$$

is an isomorphism. We then discuss a generalization of the Trace Conjecture that recovers cohomology of shtukas also as traces of functors acting on $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$.

In Sect. 16 we explain how the Trace Conjecture allows to recover V. Lafforgue's spectral decomposition of Autom with respect to (the coarse moduli space of) Langlands parameters.

We start by defining the (prestack) $\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)$ as Frobenius-fixed points on $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$); in Theorem 16.1.4 we prove that $\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)$ is actually an algebraic stack.

We view $(\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)), \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G))$ as a pair of a monoidal category with its module category, equipped with endofunctors (both given by Frobenius). In this case we can consider

$$\mathrm{cl}(\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G), (\mathrm{Frob}_{\mathrm{Bun}_G})_*) \in \mathrm{HH}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)), \mathrm{Frob}^*)$$

attached to this data (see [GKRV, Sect. 3.8.1]). We identify

$$\mathrm{HH}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)), \mathrm{Frob}^*) \simeq \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X))$$

(see [GKRV, Example 3.7.3]), and denote the resulting object

$$\mathrm{Drinf} \in \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)).$$

By design (see [GKRV, Theorem 3.8.5]), we have

$$\Gamma(\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X), \mathrm{Drinf}) \simeq \mathrm{Tr}((\mathrm{Frob}_{\mathrm{Bun}_G})_*, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)),$$

where the right-hand side is naturally acted on by

$$\mathcal{E}xc := \Gamma(\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X), \mathcal{O}_{\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)}).$$

Combining with the Trace Conjecture, we obtain an action of $\mathcal{E}xc$ on Autom, i.e., a spectral decomposition of Autom with respect to the coarse moduli space of $\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)$.

Finally, assuming the Geometric Langlands Conjecture (plus a more elementary Conjecture 16.6.7), we deduce an equivalence

$$\mathrm{Drinf} \simeq \omega_{\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)},$$

as objects of $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X))$. Combining with the Trace Conjecture, we obtain a conjectural identification

$$\mathrm{Autom} \simeq \Gamma(\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X), \omega_{\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)}),$$

i.e., a description of the space of (unramified) automorphic functions purely in terms of the stack of Langlands parameters.

In Sect. 17 we prove Theorem 16.1.4. The key tools for the proof are the properties of the map r from Theorem 4.4.2, combined with results from [De] and [LLaf].

14. Geometric Langlands Conjecture with nilpotent singular support

In this section we formulate a version of the Geometric Langlands Conjecture that involves $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$. Its main feature is that it makes sense for any sheaf theory from our list.

We then explain the relationship between this version of the Geometric Langlands Conjecture and the de Rham and Betti versions. We will show that both these versions imply the one with nilpotent singular support.

Vice versa, we show (under a certain plausible assumption, see Hypthesis 14.4.2) that the restricted version actually implies the full de Rham version.

14.1. **Digression: coherent singular support.** In this subsection we will show how to adapt the theory of singular support, developed in [AG] for quasi-smooth *schemes*, to the case of quasi-smooth *formal schemes*. We will assume that the reader is familiar with the main tenets of the paper [AG].

14.1.1. Let $\mathcal{Y}$ be a formal affine scheme (see Sect. 1.3.4), locally almost of finite type as a prestack.

We shall say that $\mathcal{Y}$ is *quasi-smooth* if for every e-point y of $\mathcal{F}$, the cotangent space $T^*_y(\mathcal{Y})$ is acyclic off degrees 0 and -1.

Equivalently, $\mathcal{Y}$ is quasi-smooth if

$$T^*(\mathcal{Y})|_{^{\mathrm{red}}\mathcal{Y}} \in \mathrm{Coh}(^{\mathrm{red}}\mathcal{Y})$$

can be locally written as a 2-step complex of vector bundles $\mathcal{E}_{-1} \to \mathcal{E}_0$.

14.1.2. We will denote by

$$T(\mathcal{Y})|_{^{\mathrm{red}}\mathcal{Y}} \in \mathrm{Coh}(^{\mathrm{red}}\mathcal{Y})^{\leq 1}$$

the *naive* dual of $T^*(\mathcal{Y})|_{^{\mathrm{red}}\mathcal{Y}}$, i.e.,

$$T(\mathcal{Y})|_{^{\mathrm{red}}\mathcal{Y}} = \underline{\mathrm{Hom}}(T^*(\mathcal{Y})|_{^{\mathrm{red}}\mathcal{Y}}, \mathcal{O}_{^{\mathrm{red}}\mathcal{Y}}).$$

We define the (reduced) scheme $\mathrm{Sing}(\mathcal{Y})$ to be the reduced scheme underlying

$$\mathrm{Spec}_{^{\mathrm{red}}\mathcal{Y}}(\mathrm{Sym}_{\mathcal{O}_{^{\mathrm{red}}\mathcal{Y}}}(T(\mathcal{Y})|_{^{\mathrm{red}}\mathcal{Y}}[1])).$$

14.1.3. Let $\mathcal{Y}$ be quasi-smooth. It follows from Theorem 2.1.4 that we can write $\mathcal{Y}$ as a filtered colimit

$$\mathcal{Y} = \underset{i}{\mathrm{colim}}\, Y_i, \tag{14.1}$$

where:

- Y_i are quasi-smooth affine schemes;
- The maps $Y_i \to Y_j$ are closed embeddings that induce isomorphisms $^{\mathrm{red}}Y_i \to {}^{\mathrm{red}}Y_j$;
- For every i, the map $Y_i \to \mathcal{Y}$ is a closed embedding such that the induced map

$$\mathrm{Sing}(\mathcal{Y}) \underset{\mathcal{Y}}{\times} Y_i \to \mathrm{Sing}(Y_i) \tag{14.2}$$

  is a closed embedding.

14.1.4. Recall that for a prestack $\mathcal{Y}$ locally almost of finite type it makes sense to talk about the category $\mathrm{IndCoh}(\mathcal{Y})$.

If $\mathcal{Y}$ is an ind-scheme, we have a well-defined (small) subcategory

$$\mathrm{Coh}(\mathcal{Y}) \subset \mathrm{IndCoh}(\mathcal{Y})^c,$$

so that $\mathrm{IndCoh}(\mathcal{Y})$ identifies with the ind-completion of $\mathrm{Coh}(\mathcal{Y})$, see [GR3, Sect. 2.4.3].

For $\mathcal{Y}$ written as (14.1), we have

$$\mathrm{IndCoh}(\mathcal{Y}) \simeq \underset{i}{\lim}\, \mathrm{IndCoh}(Y_i), \tag{14.3}$$

where the limit is formed using the !-pullback functors, and also

$$\mathrm{IndCoh}(\mathcal{Y}) \simeq \underset{i}{\mathrm{colim}}\, \mathrm{IndCoh}(Y_i),$$

where the colimit is formed inside DGCat using the *-pushforward functors, see [GR3, Sect. 2.4.2].

In terms of this identification, we have

$$\mathrm{Coh}(\mathcal{Y}) \simeq \underset{i}{\mathrm{colim}}\, \mathrm{Coh}(Y_i), \tag{14.4}$$

where the colimit is formed using the *-pullback functors, but inside the ∞-category of *not-necessarily cocomplete* DG categories.

14.1.5. The theory of singular support for quasi-smooth *schemes* developed in [AG] applies "as-is" in the case of formal affine schemes that are quasi-smooth.

In particular, to an object

$$\mathcal{M} \in \mathrm{Coh}(\mathcal{Y}),$$

one can attach its singular support $\mathrm{SingSupp}(\mathcal{M})$, which is a conical Zariski-closed subset in $\mathrm{Sing}(\mathcal{Y})$.

Explicitly, for a given $\mathcal{M} \in \mathrm{Coh}(\mathcal{Y})$, the fiber of $\mathrm{SingSupp}(\mathcal{M})$ over a given $\mathrm{Spec}(\mathsf{e}) \xrightarrow{\iota_y} \mathcal{Y}$ is the support of

$$\underset{n}{\oplus}\, H^n(\iota_y^!(\mathcal{M})),$$

viewed as a module over the algebra

$$\mathrm{Sym}^n(H^1(T_y(\mathcal{Y}))),$$

where the action is defined as in [AG, Sect. 6.1.1].

14.1.6. For a given conical Zariski-closed subset $\mathcal{N} \subset \mathrm{Sing}(\mathcal{Y})$, we can talk about a full subcategory

$$\mathrm{Coh}_{\mathcal{N}}(\mathcal{Y}) \subset \mathrm{Coh}(\mathcal{Y}),$$

consisting of objects whose singular support is contained in $\mathcal{N}$. We denote by $\mathrm{IndCoh}_{\mathcal{N}}(\mathcal{Y})$ its ind-completion, which is a full subcategory in $\mathrm{IndCoh}(\mathcal{Y})$.

One can describe the category $\mathrm{IndCoh}_{\mathcal{N}}(\mathcal{Y})$ in terms of (14.3) as follows: Given $\mathcal{N} \subset \mathrm{Sing}(\mathcal{Y})$, let $\mathcal{N}_i \subset \mathrm{Sing}(Y_i)$ be the image of

$$\mathcal{N}_i \underset{\mathcal{Y}}{\times} Y_i \subset \mathrm{Sing}(\mathcal{Y}) \underset{\mathcal{Y}}{\times} Y_i$$

under the map (14.2). Then for $Y_i \to Y_j$, the pullback functor

$$\mathrm{IndCoh}(Y_j) \to \mathrm{IndCoh}(Y_i)$$

sends

$$\mathrm{IndCoh}_{\mathcal{N}_j}(Y_j) \to \mathrm{IndCoh}_{\mathcal{N}_i}(Y_i)$$

(see [AG, Proposition 7.1.3.(a)]) and we have

$$\mathrm{IndCoh}_{\mathcal{N}}(\mathcal{Y}) \simeq \underset{i}{\lim}\, \mathrm{IndCoh}_{\mathcal{N}_i}(Y_i),$$

as subcategories in the two sides of (14.3).

14.1.7. Finally, one checks, using [AG, Proposition 7.1.3.(b)] and base change, that for a pair of indices i, j the composite functor

$$\mathrm{IndCoh}(Y_i) \overset{\text{*-pushforward}}{\longrightarrow} \mathrm{IndCoh}(\mathcal{Y}) \overset{\text{!-pullback}}{\longrightarrow} \mathrm{IndCoh}(Y_j)$$

sends

$$\mathrm{IndCoh}_{\mathcal{N}_i}(Y_i) \to \mathrm{IndCoh}_{\mathcal{N}_j}(Y_j).$$

This implies that the *-pushforward functors $\mathrm{IndCoh}(Y_i) \to \mathrm{IndCoh}(\mathcal{Y})$ send

$$\mathrm{IndCoh}_{\mathcal{N}_i}(Y_i) \to \mathrm{IndCoh}_{\mathcal{N}}(\mathcal{Y}). \tag{14.5}$$

This shows that the category $\mathrm{IndCoh}_{\mathcal{N}}(\mathcal{Y})$ is compactly generated, namely, by the essential images of the functors (14.5).

14.2. **Geometric Langlands Conjecture for $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$.**

14.2.1. Recall that we can realize $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$ as a quotient

$$\mathrm{LocSys}^{\mathrm{restr},\mathrm{rigid}_x}_{\check{G}}(X)/\check{G}.$$

According to Theorem 1.4.3, $\mathrm{LocSys}^{\mathrm{restr},\mathrm{rigid}_x}_{\check{G}}(X)$ is a disjoint union of formal affine schemes. Moreover, Proposition 1.6.2(b) implies that the cotangent spaces of $\mathrm{LocSys}^{\mathrm{restr},\mathrm{rigid}_x}_{\check{G}}(X)$ live in cohomological degrees 0 and -1 (here we use the fact that X is a curve). Hence, $\mathrm{LocSys}^{\mathrm{restr},\mathrm{rigid}_x}_{\check{G}}(X)$ is quasi-smooth.

This allows us to talk about

$$\mathrm{Sing}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)),$$

which is a (reduced) algebraic stack over ${}^{\mathrm{red}}\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$.

Furthermore, to every conical Zariski-closed $\mathcal{N} \subset \mathrm{Sing}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$ we can attach a full subcategory

$$\mathrm{IndCoh}_{\mathcal{N}}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \subset \mathrm{IndCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)).$$

14.2.2. The calculation in Sect. 1.6 allows to identify e-points of

$$\mathrm{Arth}_{\check{G}}(X) := \mathrm{Sing}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$$

with pairs

$$(\sigma, A),$$

where:

- σ is a $\check{G}$-point of $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$, i.e., a right t-exact symmetric monoidal functor
$$\mathsf{F} : \mathrm{Rep}(\check{G}) \to \mathrm{QLisse}(X);$$
- A is an element in $H^0(X, \mathsf{F}(\check{\mathfrak{g}}^*))$.

Let

$$\mathrm{Nilp} \subset \mathrm{Arth}_{\check{G}}(X)$$

be the closed subset whose e-points consist of pairs (σ, A) for which A is nilpotent. Thus, we can consider the fullcategory

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \subset \mathrm{IndCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)).$$

14.2.3. We propose the following "restricted" version of the Geometric Langlands Conjecture:

Main Conjecture 14.2.4. *There exists a canonical equivalence*

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \simeq \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)),$$

compatible with the action of $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$ *on both sides.*

14.2.5. *Example.* Let X have genus zero. Then the inclusion

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \subset \mathrm{Shv}(\mathrm{Bun}_G),$$

is an equality, and $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$ is actually an algebraic stack isomorphic to

$$(\mathrm{pt} \underset{\check{G}}{\times} \mathrm{pt})/\check{G}.$$

In this case, the assertion of Conjecture 14.2.4 is known: it follows from the (derived) geometric Satake.

14.2.6. *Example.* Let us see what Conjecture 14.2.4 says for $G = \mathbb{G}_m$. Note that in this case

$$\mathrm{Nilp} \subset \mathrm{Arth}_{\check{G}}(X)$$

is the 0-section, so

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}}^{\mathrm{restr}}(X)) = \mathrm{QCoh}(\mathrm{LocSys}_{\mathbb{G}_m}^{\mathrm{restr}}(X)).$$

Thus, Conjecture 14.2.4 says that for every isomorphism class of 1-dimensional local systems σ on X, we have an equivalence

$$\mathrm{QLisse}(\mathrm{Pic})_\sigma \simeq \mathrm{QCoh}(\mathrm{LocSys}_{\mathbb{G}_m}^{\mathrm{restr}}(X)_\sigma). \tag{14.6}$$

Up to translation by σ on $\mathrm{LocSys}_{\mathbb{G}_m}^{\mathrm{restr}}(X)_\sigma$ (resp., tensor product by $\mathcal{F}_\sigma$ on Pic), we can assume that σ is trivial, so $\mathcal{F}_\sigma$ is the constant sheaf $\underline{\mathsf{e}}_{\mathrm{Pic}}$. The corresponding equivalence (14.6) is then the following statement:

Pick a point $x \in X$. Write

$$\mathrm{LocSys}_{\mathbb{G}_m}^{\mathrm{restr}}(X)_\sigma \simeq \mathrm{pt}/\mathbb{G}_m \times \mathrm{Tot}(H^1(X, \mathsf{e}_X))^\wedge \times \mathrm{Tot}(\mathsf{e}[-1]),$$

see Sect. 1.3.6, and

$$\mathrm{Pic} \simeq \mathbb{Z} \times \mathrm{Jac}(X) \times \mathrm{pt}/\mathbb{G}_m.$$

So

$$\mathrm{QCoh}(\mathrm{LocSys}_{\mathbb{G}_m}^{\mathrm{restr}}(X)_\sigma) \simeq \mathrm{Vect}_{\mathsf{e}}^{\mathbb{Z}} \otimes \big(\mathrm{Sym}(H^1(X, \mathsf{e}_X)[-1])\text{-mod}\big) \otimes (\mathsf{e}[\xi]\text{-mod}), \quad \deg(\xi) = -1,$$

and

$$\mathrm{QLisse}(\mathrm{Pic})_\sigma \simeq \mathrm{Vect}_{\mathsf{e}}^{\mathbb{Z}} \otimes (\mathrm{Shv}(\mathrm{Jac}(X))_0) \otimes (\mathrm{Shv}(\mathrm{pt}/\mathbb{G}_m)),$$

where $\mathrm{Jac}(X)$ is the Jacobian *variety* of X, and $\mathrm{Shv}(\mathrm{Jac}(X))_0 \subset \mathrm{Shv}(\mathrm{Jac}(X))$ is the full subcategory generated by the constant sheaf.

Now the result follows from the canonical identifications

$$\mathrm{Shv}(\mathrm{pt}/\mathbb{G}_m) \simeq \mathrm{C}_\cdot(\mathbb{G}_m)\text{-mod} \simeq \mathsf{e}[\xi]\text{-mod}, \quad \deg(\xi) = -1,$$

and

$$\mathrm{Shv}(\mathrm{Jac}(X))_0 \simeq \mathcal{E}nd(\underline{\mathsf{e}}_{\mathrm{Jac}(X)})\text{-mod} \simeq \mathrm{Sym}(H^1(X, \mathsf{e}_X)[-1])\text{-mod}.$$

14.3. **Comparison to other forms of the Geometric Langlands Conjecture.**

14.3.1. Let us specialize to the de Rham setting. Recall that in this case, we have a version of the geometric Langlands conjecture from [AG, Conjecture 11.2.2], which predicts the existence of a canonical equivalence

$$\text{D-mod}(\mathrm{Bun}_G) \simeq \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}}(X)), \tag{14.7}$$

compatible with the actions of $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))$ on both sides.

We note that Conjecture 14.2.4 is a formal corollary of this statement. Namely, tensoring both sides of (14.7) with $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}^{\mathrm{restr}}(X))$ over $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))$, we obtain:

$$\begin{tikzcd}
\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}^{\mathrm{restr}}(X)) \underset{\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))}{\otimes} \text{D-mod}(\mathrm{Bun}_G) \arrow[r, "\sim"] \arrow[d, "\sim"'] & \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \arrow[d, "\sim"] \\
\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}^{\mathrm{restr}}(X)) \underset{\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))}{\otimes} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}}(X)) \arrow[r, "\sim"] & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}}^{\mathrm{restr}}(X))
\end{tikzcd}$$

where the top horizontal arrow comes from Proposition 10.5.9.

14.3.2. Let us now specialize to the Betti setting. In this case, we have a version of the geometric Langlands conjecture, proposed in [BN, Conjecture 1.5], which says that there is an equivalence

$$\mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \simeq \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}}(X)), \tag{14.8}$$

compatible with the actions of $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))$ on both sides.

We note that (14.2.4) is a formal corollary of this statement, combined with Corollary 10.5.11. Namely, tensoring both sides of (14.8) with $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$ over $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))$, we obtain:

$$\begin{tikzcd}
\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \underset{\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))}{\otimes} \mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \arrow[r, "\sim"] \arrow[d, "\sim"'] & \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \arrow[d, "\sim"] \\
\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \underset{\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))}{\otimes} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}}(X)) \arrow[r, "\sim"] & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)),
\end{tikzcd}$$

where the top horizontal arrow comes from Corollary 10.5.11.

14.4. **An inverse implication.** Above we have seen that the full de Rham version of the Geometric Langlands Conjecture implies the restricted version. Here we will show that the converse implication also takes place, under a plausible hypothesis about the de Rham version.

14.4.1. We place ourselves into the de Rham context of the Geometric Langlands Conjecture. Let us assume the following:

Hypothesis 14.4.2. *There exists a functor*

$$\mathbb{L} : \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}}(X)) \to \mathrm{D\text{-}mod}(\mathrm{Bun}_G)$$

that preserves compactness and is compatible with the actions of $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))$ *on both sides.*

This hypothesis would be a theorem if one accepted Quasi-Theorems 6.7.2 and 9.5.3 from [Ga7].

14.4.3. We now claim:

Observation 14.4.4. *Assume that the functor* $\mathbb{L}$ *from Hypothesis 14.4.2 induces an equivalence*

(14.9)

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \simeq \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \underset{\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))}{\otimes} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}}(X)) \overset{\mathrm{Id}\otimes\mathbb{L}}{\to}$$

$$\to \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \underset{\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))}{\otimes} \mathrm{D\text{-}mod}(\mathrm{Bun}_G) \simeq \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G).$$

Then the functor $\mathbb{L}$ *itself is an equivalence.*

The rest of this subsection is devoted to the proof this Observation.

14.4.5. Since the functor $\mathbb{L}$ preserves compactness, it admits a continuous right adjoint. Since $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))$ is rigid, this right adjoint is compatible with the action of $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))$.

Consider the adjunction maps

$$\mathrm{Id} \to \mathbb{L}^R \circ \mathbb{L} \text{ and } \mathbb{L} \circ \mathbb{L}^R \to \mathrm{Id} \tag{14.10}$$

. We want to show that they are isomorphisms.

We have the following general assertion:

Lemma 14.4.6. *Let* $\mathbf{C}$ *be a category acted on by* $\mathrm{QCoh}(\mathcal{Y})$*, where* $\mathcal{Y}$ *is a quasi-compact eventually coconnective algebraic stack almost of finite type with affine diagonal. Then an object* $\mathbf{c} \in \mathbf{C}$ *is zero if and only if for every field-valued point*

$$\iota : \mathrm{Spec}(\mathsf{e}') \to \mathcal{Y},$$

the image of $\mathbf{c}$ *under*

$$\iota^* : \mathbf{C} \to \mathrm{Vect}_{\mathsf{e}'} \underset{\mathrm{QCoh}(\mathcal{Y})}{\otimes} \mathbf{C}$$

vanishes.

The proof of the lemma is given below.

14.4.7. Hence, in order to prove that (14.10) are isomorphism, it suffices to show that the functor

$$\mathrm{Vect}_{\mathsf{e}'} \underset{\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))}{\otimes} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \to \mathrm{Vect}_{\mathsf{e}'} \underset{\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))}{\otimes} \text{D-mod}(\mathrm{Bun}_G) \tag{14.11}$$

is an isomorphism for all

$$\iota : \mathrm{Spec}(\mathsf{e}') \to \mathrm{LocSys}_{\check{G}}(X).$$

With no restriction of generality we can assume that e' is algebraically closed.

Let X', G' denote the base change of X, G along $\mathsf{e} \rightsquigarrow \mathsf{e}'$. Let Bun'_G denote the corresponding algebraic stack over e'. We have

$$\mathrm{LocSys}_{\check{G}'}(X') \simeq \mathrm{Spec}(\mathsf{e}') \underset{\mathrm{Spec}(\mathsf{e})}{\times} \mathrm{LocSys}_{\check{G}}(X) \text{ and } \mathrm{Bun}'_G \simeq \mathrm{Spec}(\mathsf{e}') \underset{\mathrm{Spec}(\mathsf{e})}{\times} \mathrm{Bun}_G,$$

and hence

$$\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}'}(X')) \simeq \mathrm{Vect}_{\mathsf{e}'} \underset{\mathrm{Vect}_{\mathsf{e}}}{\otimes} \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X)),$$

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}'}(X')) \simeq \mathrm{Vect}_{\mathsf{e}'} \underset{\mathrm{Vect}_{\mathsf{e}}}{\otimes} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}}(X))$$

and

$$\text{D-mod}(\mathrm{Bun}'_G) \simeq \mathrm{Vect}_{\mathsf{e}'} \underset{\mathrm{Vect}_{\mathsf{e}}}{\otimes} \text{D-mod}(\mathrm{Bun}_G).$$

Hence, we can rewrite the map in (14.11) as

$$\mathrm{Vect}_{\mathsf{e}'} \underset{\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}'}(X'))}{\otimes} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}'}(X')) \to \mathrm{Vect}_{\mathsf{e}'} \underset{\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}'}(X'))}{\otimes} \text{D-mod}(\mathrm{Bun}'_G).$$

Thus, we have reduced the verification of the isomorphism (14.11) to the case when $\mathsf{e}' = \mathsf{e}$.

14.4.8. Note now that (for $\mathsf{e}' = \mathsf{e}$), the map $\iota : \mathrm{Spec}(\mathsf{e}) \to \mathrm{LocSys}_{\check{G}}(X)$ factors as

$$\mathrm{Spec}(\mathsf{e}) \to \mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X) \to \mathrm{LocSys}_{\check{G}}(X).$$

Hence, the map (14.11) is obtained by

$$\mathrm{Vect}_{\mathsf{e}} \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))}{\otimes} -$$

from (14.9), and hence is an equivalence.

$\square$[Observation 14.4.4]

Remark 14.4.9. Note that whereas

$$\mathrm{LocSys}_{\check{G}'}(X') \simeq \mathrm{Spec}(\mathsf{e}') \underset{\mathrm{Spec}(\mathsf{e})}{\times} \mathrm{LocSys}_{\check{G}}(X),$$

the same *no longer* holds for $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$. Similarly, while

$$\text{D-mod}(\mathrm{Bun}'_G) \simeq \mathrm{Vect}_{\mathsf{e}'} \underset{\mathrm{Vect}_{\mathsf{e}}}{\otimes} \text{D-mod}(\mathrm{Bun}_G),$$

the same is no longer true for $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$.

Remark 14.4.10. A counterpart of Observation 14.4.4 would apply also in the Betti version. We are just less confident of the status of the analog of Hypothesis 14.4.2 in this case.

Note that in the Betti setting, Bun_G does not change with the change $\mathsf{e} \rightsquigarrow \mathsf{e}'$ (the geometry on the automorphic side is always over $\mathbb{C}$), however, we do have

$$\mathrm{Shv}^{\mathsf{e}',\mathrm{all}}(\mathcal{Y}) \simeq \mathrm{Vect}_{\mathsf{e}'} \underset{\mathrm{Vect}_{\mathsf{e}}}{\otimes} \mathrm{Shv}^{\mathsf{e},\mathrm{all}}(\mathcal{Y})$$

for any $\mathcal{Y}$; in particular

$$\mathrm{Shv}^{\mathsf{e}',\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \simeq \mathrm{Vect}_{\mathsf{e}'} \underset{\mathrm{Vect}_{\mathsf{e}}}{\otimes} \mathrm{Shv}^{\mathsf{e},\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$$

and

$$\mathrm{LocSys}^{\mathsf{e}'}_{\check{G}}(X) \simeq \mathrm{Spec}(\mathsf{e}') \underset{\mathrm{Spec}(\mathsf{e})}{\times} \mathrm{LocSys}_{\check{G}}(X),$$

where the superscript e' indicates that we are considering sheaves with e'-coefficients.

14.4.11. *Proof of Lemma 14.4.6.* First, by [Ga2, Theorem 2.2.6], $\mathcal{Y}$ is 1-affine[23], hence we can replace $\mathcal{Y}$ be an affine scheme $S = \mathrm{Spec}(A)$.

Let $\mathbf{c}$ be a non-zero object of $\mathbf{C}$. By Noetherian induction, we can find an irreducible subvariety $S' \subset S$ (denote the localization of A at the generic point of S by A') such that $\mathbf{c}|_{\mathrm{Spec}(A')} \neq 0$ and $\mathbf{c}|_{\mathrm{Spec}(A'')} = 0$ for any further localization A'' of A'. Let e' be the residue field of A'. We claim that $\mathbf{c}|_{\mathrm{Spec}(\mathsf{e}')} \neq 0$.

Since S was assumed eventually coconnective, we have $\mathbf{c}|_{\mathrm{Spec}(H^0(A'))} \neq 0$. Indeed, let ϕ denote the map

$$\mathrm{Spec}(H^0(A')) \to \mathrm{Spec}(A),$$

the object $\mathbf{c}$ is a (finite) extension of objects

$$\phi_*(H^i(A) \underset{H^0(A)}{\otimes} \iota^*(\mathbf{c})).$$

Thus, we have reduced to the case when A is a classical local ring, and every element of the maximal ideal $\mathfrak{m} \subset A$ acts nilpotently on $\mathbf{c}$.

Choose a regular sequence $f_1, ..., f_n \in \mathfrak{m}$ so that $A/(f_1, ..., f_n)$ is Artinian. Set $A_0 = A$ and $\mathbf{c}_0 = \mathbf{c}$ and define A_i and $\mathbf{c}_i$ inductively by

$$A_i = \mathrm{coFib}(A_{i-1} \stackrel{f_i}{\to} A_{i-1}) \text{ and } \mathbf{c}_i = \mathbf{c}|_{\mathrm{Spec}(A_i)} \simeq \mathrm{coFib}(\mathbf{c}_{i-1} \stackrel{f_i}{\to} \mathbf{c}_{i-1}).$$

By induction (given that all f_i act nilpotently on $\mathbf{c}$), we obtain that $\mathbf{c}_n \neq 0$. So, we can replace A by A_n. Note that A_n is eventually coconnective, so $\mathbf{c}|_{\mathrm{Spec}(H^0(A_n))} \neq 0$.

So, we have reduced to the case when A is a classical *Artinian* ring. Now,

$$\iota_* \circ \iota^*(\mathbf{c}) \simeq \mathsf{e}' \underset{A}{\otimes} \mathbf{c},$$

which is non-zero, because A is expressible as a finite colimit in terms of e'.

$\square$[Lemma 14.4.6]

15. The trace conjecture

Throughout this section we will be working with schemes/algebraic stacks of finite type over $\overline{\mathbb{F}}_q$, that are defined over $\mathbb{F}_q$, so that they carry the geometric Frobenius endomorphism.

Our sheaf-theoretic context will (by necessity) be that of ℓ-adic sheaves, so $\mathsf{e} = \overline{\mathbb{Q}}_\ell$.

This section contains what is the main point of this paper. We propose a conjecture that expresses the space of automorphic functions as the categorical trace of Frobenius acting on the category $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$.

15.1. **The categorical trace of Frobenius.**

[23] In most applications, we are interested in the case when $\mathcal{Y}$ is of the form S/H, where S is an affine scheme and H is an algebraic group, in which case the assertion of [Ga2, Theorem 2.2.6] easily follows from the case of pt/H.

15.1.1. Let $\mathcal{Y}$ be an algebraic stack. We consider the endofunctor (in fact, an auto-equivalence) of $\mathrm{Shv}(\mathcal{Y})$, given by $(\mathrm{Frob}_{\mathcal{Y}})_*$.

Assume first that $\mathcal{Y}$ is quasi-compact. In this case the category $\mathrm{Shv}(\mathcal{Y})$ is compactly generated, and hence dualizable. Hence, we can consider the categorical trace of $(\mathrm{Frob}_{\mathcal{Y}})_*$:

$$\mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}(\mathcal{Y})) \in \mathrm{Vect}_{\mathsf{e}}\,.$$

To $\mathcal{F} \in \mathrm{Shv}(\mathcal{Y})^c$ equipped with a map

$$\mathcal{F} \xrightarrow{\alpha} (\mathrm{Frob}_{\mathcal{Y}})_*(\mathcal{F}), \tag{15.1}$$

we can attach its class

$$\mathrm{cl}(\mathcal{F}, \alpha) \in \mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}(\mathcal{Y})),$$

see [GKRV, Sect. 3.4.3].

We will refer to the data of α as a *lax Weil structure* on $\mathcal{F}$, and to the pair $(\mathcal{F}, \alpha)$ as a *lax Weil sheaf* on $\mathcal{Y}$.

15.1.2. We claim that there is a canonically defined map, called the *Local Term*,

$$\mathrm{LT} : \mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}(\mathcal{Y})) \to \mathrm{Funct}(\mathcal{Y}(\mathbb{F}_q)), \tag{15.2}$$

where $\mathrm{Funct}(-)$ stands for the (classical) vector space of e-valued functions on the set of isomorphism classes of a given groupoid.

In fact, there are two such maps, denoted $\mathrm{LT}^{\mathrm{naive}}$ and $\mathrm{LT}^{\mathrm{true}}$.

15.1.3. The map $\mathrm{LT}^{\mathrm{naive}}$ is designed so for a lax Weil sheaf $(\mathcal{F}, \alpha)$, we have

$$\mathrm{LT}^{\mathrm{naive}}(\mathrm{cl}(\mathcal{F}, \alpha)) = \mathrm{funct}(\mathcal{F}, \alpha),$$

where $\mathrm{funct}(\mathcal{F}, \alpha)$ is the usual function on $\mathcal{Y}(\mathbb{F}_q)$ attached to $(\mathcal{F}, \alpha)$ obtained by taking traces of the Frobenius on $\mathbb{F}_q$-points:

By adjunction, the datum of α is equivalent to the datum of a map

$$\alpha^L : (\mathrm{Frob}_{\mathcal{Y}})^*(\mathcal{F}) \to \mathcal{F}.$$

Now the value of $\mathrm{funct}(\mathcal{F}, \alpha)$ on a given $y \in \mathcal{Y}(\mathbb{F}_q)$,

$$\mathrm{pt} \xrightarrow{\iota_y} \mathcal{Y}$$

equals the trace of

$$\iota_y^*(\mathcal{F}) \overset{y \text{ is Frob-invariant}}{\longrightarrow} (\mathrm{Frob}_{\mathcal{Y}} \circ \iota_y)^*(\mathcal{F}) \simeq \iota_y^* \circ \mathrm{Frob}_{\mathcal{Y}}^*(\mathcal{F}) \xrightarrow{\alpha^L} \iota_y^*(\mathcal{F}).$$

15.1.4. The actual definition of $\mathrm{LT}^{\mathrm{naive}}$ proceeds as follows. Every y as above defines a functor

$$\iota_y^* : \mathrm{Shv}(\mathcal{Y}) \to \mathrm{Vect}_{\mathsf{e}},$$

which admits a right adjoint, and is equipped with a morphism (in fact, an isomorphism)

$$\iota_y^* \circ (\mathrm{Frob}_{\mathcal{Y}})_* \to \iota_y^*. \tag{15.3}$$

Hence, by [GKRV, Sect. 3.4.1], it defines a map

$$\mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}(\mathcal{Y})) \to \mathrm{Tr}(\mathrm{Id}, \mathrm{Vect}_{\mathsf{e}}) \simeq \mathsf{e}.$$

This map is, by definition, the composition of $\mathrm{LT}^{\mathrm{naive}}$ with the evaluation map

$$\mathrm{Funct}(\mathcal{Y}(\mathbb{F}_q)) \xrightarrow{\mathrm{ev}_y} \mathsf{e}.$$

The map $\mathrm{LT}^{\mathrm{naive}}$ has the following features.

15.1.5. For a lax Weil sheaf $(\mathcal{F}_0, \alpha_0)$ on $\mathrm{Shv}(\mathcal{Y})$, consider the functor

$$\mathrm{Shv}(\mathcal{Y}) \to \mathrm{Shv}(\mathcal{Y}), \quad \mathcal{F} \mapsto \mathcal{F}_0 \overset{*}{\otimes} \mathcal{F}.$$

This functor is endowed with a natural transformation

$$\mathcal{F}_0 \overset{*}{\otimes} (\mathrm{Frob}_{\mathcal{Y}})_*(\mathcal{F}) \to (\mathrm{Frob}_{\mathcal{Y}})_*(\mathcal{F}_0 \overset{*}{\otimes} \mathcal{F}), \tag{15.4}$$

and it admits a right adjoint, given by

$$\mathcal{F} \mapsto \mathbb{D}(\mathcal{F}_0) \overset{!}{\otimes} \mathcal{F}.$$

Hence, by [GKRV, Sect. 3.4.1], it defines a map

$$\mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}(\mathcal{Y})) \to \mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}(\mathcal{Y})). \tag{15.5}$$

We claim that there is a commutative diagram

$$\begin{CD} \mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}(\mathcal{Y})) @>{\mathrm{LT}^{\mathrm{naive}}}>> \mathrm{Funct}(\mathcal{Y}(\mathbb{F}_q)) \\ @V{(15.5)}VV @VV{\mathrm{funct}(\mathcal{F}_0,\alpha_0)\cdot -}V \\ \mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}(\mathcal{Y})) @>{\mathrm{LT}^{\mathrm{naive}}}>> \mathrm{Funct}(\mathcal{Y}(\mathbb{F}_q)). \end{CD}$$

Indeed, this follows from the fact that for a given $y \in \mathcal{Y}(\mathbb{F}_q)$, we have a commutative diagram of functors

$$\begin{CD} \mathrm{Shv}(\mathcal{Y}) @>{\iota_y^*}>> \mathrm{Vect}_{\mathsf{e}} \\ @V{\mathcal{F}_0 \overset{*}{\otimes}}VV @VV{\iota_y^*(\mathcal{F}_0)\otimes -}V \\ \mathrm{Shv}(\mathcal{Y}) @>{\iota_y^*}>> \mathrm{Vect}_{\mathsf{e}} \end{CD}$$

compatible with the natural transformations (15.3) and (15.4) via the endomorphism on $\iota_y^*(\mathcal{F}_0)$ given by α_0^L. This implies that the resulting map

$$\mathsf{e} \simeq \mathrm{Tr}(\mathrm{Id}, \mathrm{Vect}_{\mathsf{e}}) \overset{(\iota_y^*(\mathcal{F}_0), \alpha_0^L)}{\longrightarrow} \mathrm{Tr}(\mathrm{Id}, \mathrm{Vect}_{\mathsf{e}}) \simeq \mathsf{e}$$

is given by

$$\mathrm{Tr}(\alpha_0^L, \iota_y^*(\mathcal{F}_0)) = \mathrm{funct}(\mathcal{F}_0, \alpha_0)(y),$$

as desired.

15.1.6. Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a map. Consider the functor

$$f^* : \mathrm{Shv}(\mathcal{Y}_2) \to \mathrm{Shv}(\mathcal{Y}_1).$$

This functor is endowed with a natural transformation (in fact, an isomorphism)

$$f^* \circ (\mathrm{Frob}_{\mathcal{Y}_2})_* \to (\mathrm{Frob}_{\mathcal{Y}_1})_* \circ f^*, \tag{15.6}$$

and it admits a right adjoint, given by f_*.

Hence, by [GKRV, Sect. 3.4.1], it defines a map

$$\mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}_2})_*, \mathrm{Shv}(\mathcal{Y}_2)) \to \mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}_1})_*, \mathrm{Shv}(\mathcal{Y}_1)). \tag{15.7}$$

We claim that there is a commutative diagram

$$\begin{CD} \mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}_2})_*, \mathrm{Shv}(\mathcal{Y}_2)) @>{\mathrm{LT}^{\mathrm{naive}}}>> \mathrm{Funct}(\mathcal{Y}_2(\mathbb{F}_q)) \\ @V{(15.7)}VV @VV{\text{pull back}}V \\ \mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}_1})_*, \mathrm{Shv}(\mathcal{Y}_1)) @>{\mathrm{LT}^{\mathrm{naive}}}>> \mathrm{Funct}(\mathcal{Y}_1(\mathbb{F}_q)), \end{CD}$$

where the right vertical arrows is given by pullback of functions along the induced map

$$\mathcal{Y}_1(\mathbb{F}_q) \to \mathcal{Y}_2(\mathbb{F}_q).$$

This is proved by the same argument as in Sect. 15.1.5 above.

15.1.7. Finally, let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be as above. Consider the functor

$$f_! : \mathrm{Shv}(\mathcal{Y}_1) \to \mathrm{Shv}(\mathcal{Y}_2).$$

This functor is endowed with a natural transformation (in fact, an isomorphism)

$$f_! \circ (\mathrm{Frob}_{\mathcal{Y}_1})_* \to (\mathrm{Frob}_{\mathcal{Y}_2})_* \circ f_!, \tag{15.8}$$

(coming from the fact that $\mathrm{Frob}_{\mathcal{Y}}$ is a finite map), and it admits a right adjoint, given by $f^!$.

Hence, by [GKRV, Sect. 3.4.1], it defines a map

$$\mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}_1})_*, \mathrm{Shv}(\mathcal{Y}_1)) \to \mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}_2})_*, \mathrm{Shv}(\mathcal{Y}_2)). \tag{15.9}$$

Theorem 15.1.8. *We have a commutative diagram*

$$\begin{tikzcd}
\mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}_1})_*, \mathrm{Shv}(\mathcal{Y}_1)) \arrow[r, "\mathrm{LT}^{\mathrm{naive}}"] \arrow[d, "(15.9)"'] & \mathrm{Funct}(\mathcal{Y}_1(\mathbb{F}_q)) \arrow[d, "\text{push forward}"] \\
\mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}_2})_*, \mathrm{Shv}(\mathcal{Y}_2)) \arrow[r, "\mathrm{LT}^{\mathrm{naive}}"] & \mathrm{Funct}(\mathcal{Y}_2(\mathbb{F}_q)),
\end{tikzcd}$$

where the right vertical arrow is given by (weighted)[24] *summation along the fiber of the induced map*

$$\mathcal{Y}_1(\mathbb{F}_q) \to \mathcal{Y}_2(\mathbb{F}_q).$$

This theorem is a version of the Grothendieck-Lefschetz trace formula. The proof will be supplied in [GaVa].

Remark 15.1.9. It is easy to prove Theorem 15.1.8 when f is a locally closed embedding. And this is the only case we will need in order to formulate Conjecture 15.3.5.

15.1.10. Let now $\mathcal{Y}$ be an algebraic stack that is not necessarily quasi-compact. We write

$$\mathcal{Y} := \underset{\mathcal{U}}{\cup} \mathcal{U},$$

where $\mathcal{U} \overset{j}{\hookrightarrow} \mathcal{Y}$ are quasi-compact open prestacks, so that

$$\mathrm{Shv}(\mathcal{Y}) \simeq \lim_{\mathcal{U}} \mathrm{Shv}(\mathcal{U}),$$

with respect to the restriction maps and also

$$\mathrm{Shv}(\mathcal{Y}) \simeq \underset{\mathcal{U}}{\mathrm{colim}}\, \mathrm{Shv}(\mathcal{U}),$$

with respect to !-pushforwards, see [DrGa2, Proposition 1.7.5].

We claim that the functors $j_! : \mathrm{Shv}(\mathcal{U}) \to \mathrm{Shv}(\mathcal{Y})$ induce an isomorphism

$$\underset{\mathcal{U}}{\mathrm{colim}}\, \mathrm{Tr}((\mathrm{Frob}_{\mathcal{U}})_*, \mathrm{Shv}(\mathcal{U})) \to \mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}(\mathcal{Y})).$$

Indeed, we have

$$\mathrm{Id}_{\mathrm{Shv}(\mathcal{Y})} \simeq \underset{\mathcal{U}}{\mathrm{colim}}\, j_! \circ j^*,$$

and hence

$$\begin{aligned}
&\mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}(\mathcal{Y})) \simeq \underset{\mathcal{U}}{\mathrm{colim}}\, \mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_* \circ j_! \circ j^*, \mathrm{Shv}(\mathcal{Y})) \simeq \\
&\simeq \underset{\mathcal{U}}{\mathrm{colim}}\, \mathrm{Tr}(j_! \circ (\mathrm{Frob}_{\mathcal{U}})_* \circ j^*, \mathrm{Shv}(\mathcal{Y})) \overset{\text{cyclicity of trace}}{\simeq} \underset{\mathcal{U}}{\mathrm{colim}}\, \mathrm{Tr}((\mathrm{Frob}_{\mathcal{U}})_* \circ j^* \circ j_!, \mathrm{Shv}(\mathcal{U})) \simeq \\
&\simeq \underset{\mathcal{U}}{\mathrm{colim}}\, \mathrm{Tr}((\mathrm{Frob}_{\mathcal{U}})_*, \mathrm{Shv}(\mathcal{U})),
\end{aligned}$$

as desired.

From here, using Theorem 15.1.8 for open embeddings, we obtain that the maps $\mathrm{LT}^{\mathrm{naive}}$ for $\mathcal{U}$ give rise to a map

$$\mathrm{LT}^{\mathrm{naive}} : \mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}(\mathcal{Y})) \to \mathrm{Funct}_c(\mathcal{Y}(\mathbb{F}_q)), \tag{15.10}$$

[24] We weigh each point by $\frac{1}{|\text{order of its group of automorphisms}|}$.

where $\mathrm{Funct}_c(-)$ stands for "functions with finite support", so

$$\mathrm{Funct}_c(\mathcal{Y}(\mathbb{F}_q)) \simeq \underset{\mathcal{U}}{\mathrm{colim}}\, \mathrm{Funct}_c(\mathcal{U}(\mathbb{F}_q)).$$

15.2. **The true local term.** We now proceed to the definition of the map

$$\mathrm{LT}^{\mathrm{true}} : \mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}(\mathcal{Y})) \to \mathrm{Funct}(\mathcal{Y}(\mathbb{F}_q)). \tag{15.11}$$

The assumption that $\mathcal{Y}$ be quasi-compact is also essential here, because the self-duality of $\mathrm{Shv}(\mathcal{Y})$ (see below) works as-is only in the quasi-compact case.

We will also assume that $\mathcal{Y}$ is duality-adapted, see Sect. C.2.4 for what this means. (According to Conjecture C.2.5, all quasi-compact algebraic stacks with an affine diagonal have this property; in Theorem C.2.6 it is shown that algebraic stacks that can locally be written as quotients are such.)

15.2.1. We recall that the algebraic stack $\mathcal{Y}^{\mathrm{Frob}}$ is *discrete*, i.e., has the form

$$\sqcup\, (\mathrm{pt}\,/\Gamma), \quad \Gamma \in \text{Finite Groups},$$

so we can identify

$$\mathrm{Funct}(\mathcal{Y}(\mathbb{F}_q)) \simeq \mathrm{C}^\cdot(\mathcal{Y}^{\mathrm{Frob}}, \omega_{\mathcal{Y}^{\mathrm{Frob}}}).$$

Let $\iota_{\mathcal{Y}}$ denote the forgetful map

$$\mathcal{Y}^{\mathrm{Frob}} \to \mathcal{Y}.$$

Let us rewrite

$$\mathrm{C}^\cdot(\mathcal{Y}^{\mathrm{Frob}}, \omega_{\mathcal{Y}^{\mathrm{Frob}}}) \simeq \mathrm{C}^\cdot(\mathcal{Y}, (\iota_{\mathcal{Y}})_*(\omega_{\mathcal{Y}^{\mathrm{Frob}}})).$$

Using base change along

$$\begin{tikzcd} \mathcal{Y}^{\mathrm{Frob}} \arrow[r, "\iota_{\mathcal{Y}}"] \arrow[d, "\iota_{\mathcal{Y}}"'] & \mathcal{Y} \arrow[d, "(\mathrm{Frob}_{\mathcal{Y}}, \mathrm{id}_{\mathcal{Y}})"] \\ \mathcal{Y} \arrow[r, "\Delta_{\mathcal{Y}}"] & \mathcal{Y} \times \mathcal{Y}, \end{tikzcd}$$

we can rewrite

$$(\iota_{\mathcal{Y}})_*(\omega_{\mathcal{Y}^{\mathrm{Frob}}}) \simeq \Delta^!_{\mathcal{Y}} \circ (\mathrm{Frob}_{\mathcal{Y}} \times \mathrm{id}_{\mathcal{Y}})_* \circ (\Delta_{\mathcal{Y}})_*(\omega_{\mathcal{Y}}).$$

To summarize, we have

$$\mathrm{Funct}(\mathcal{Y}(\mathbb{F}_q)) \simeq \mathrm{C}^\cdot(\mathcal{Y}, \Delta^!_{\mathcal{Y}} \circ (\mathrm{Frob}_{\mathcal{Y}} \times \mathrm{id}_{\mathcal{Y}})_* \circ (\Delta_{\mathcal{Y}})_*(\omega_{\mathcal{Y}})).$$

15.2.2. In order to compute $\mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}(\mathcal{Y}))$, we identify $\mathrm{Shv}(\mathcal{Y})$ with its own dual, see Sect. C.3.1. We recall that the corresponding pairing

$$\mathrm{Shv}(\mathcal{Y}) \otimes \mathrm{Shv}(\mathcal{Y}) \to \mathrm{Vect}_{\mathsf{e}}$$

is given by

$$\mathcal{F}_1, \mathcal{F}_2 \mapsto \mathrm{C}^\cdot_\blacktriangle(\mathcal{Y}, \mathcal{F}_1 \overset{!}{\otimes} \mathcal{F}_2) \simeq \mathrm{C}^\cdot_\blacktriangle(\mathcal{Y}, \Delta^!_{\mathcal{Y}}(\mathcal{F}_1 \boxtimes \mathcal{F}_2)),$$

where the notation $\mathrm{C}^\cdot_\blacktriangle$ is as in Sect. C.3.2.

Let $u_{\mathrm{Shv}(\mathcal{Y})} \in \mathrm{Shv}(\mathcal{Y}) \otimes \mathrm{Shv}(\mathcal{Y})$ be the unit of the self-duality on $\mathrm{Shv}(\mathcal{Y})$. We obtain that $\mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}(\mathcal{Y}))$ is given by

$$\mathrm{C}^\cdot_\blacktriangle(\mathcal{Y}, \Delta^!_{\mathcal{Y}} \circ \boxtimes \circ ((\mathrm{Frob}_{\mathcal{Y}})_* \otimes \mathrm{Id}_{\mathrm{Shv}(\mathcal{Y})})(u_{\mathrm{Shv}(\mathcal{Y})})),$$

where $\boxtimes$ denotes the external tensor product functor

$$\mathrm{Shv}(\mathcal{Y}) \otimes \mathrm{Shv}(\mathcal{Y}) \to \mathrm{Shv}(\mathcal{Y} \times \mathcal{Y}).$$

15.2.3. We note that

$$(\mathrm{Frob}_{\mathcal{Y}} \times \mathrm{id}_{\mathcal{Y}})_* \circ \boxtimes \simeq \boxtimes \circ ((\mathrm{Frob}_{\mathcal{Y}})_* \otimes \mathrm{Id}_{\mathrm{Shv}(\mathcal{Y})}).$$

Hence, in order to construct the map (15.11), it suffices to construct a map

$$\boxtimes(u_{\mathrm{Shv}(\mathcal{Y})}) \to (\Delta_{\mathcal{Y}})_*(\omega_{\mathcal{Y}}), \tag{15.12}$$

and a map

$$\mathrm{C}^{\cdot}_{\blacktriangle}(\mathcal{Y}, \Delta^!_{\mathcal{Y}} \circ (\mathrm{Frob}_{\mathcal{Y}} \times \mathrm{id}_{\mathcal{Y}})_* \circ (\Delta_{\mathcal{Y}})_*(\omega_{\mathcal{Y}})) \to \mathrm{C}^{\cdot}(\mathcal{Y}, \Delta^!_{\mathcal{Y}} \circ (\mathrm{Frob}_{\mathcal{Y}} \times \mathrm{id}_{\mathcal{Y}})_* \circ (\Delta_{\mathcal{Y}})_*(\omega_{\mathcal{Y}})) \simeq \\ \simeq \mathrm{Funct}(\mathcal{Y}(\mathbb{F}_q)). \tag{15.13}$$

The map (15.13) is the map (C.3). In our case, it is in fact an isomorphism, because the object

$$\Delta^!_{\mathcal{Y}} \circ (\mathrm{Frob}_{\mathcal{Y}} \times \mathrm{id}_{\mathcal{Y}})_* \circ (\Delta_{\mathcal{Y}})_*(\omega_{\mathcal{Y}}) \simeq (\iota_{\mathcal{Y}})_*(\omega_{\mathcal{Y}^{\mathrm{Frob}}}) \in \mathrm{Shv}(\mathcal{Y})$$

is compact, see Sect. C.3.2.

We proceed to the construction of the map (15.12).

15.2.4. Let $\boxtimes^R$ denote the right adjoint of the functor

$$\boxtimes : \mathrm{Shv}(\mathcal{Y}) \otimes \mathrm{Shv}(\mathcal{Y}) \to \mathrm{Shv}(\mathcal{Y} \times \mathcal{Y}).$$

We claim that we have a canonical isomorphism

$$u_{\mathrm{Shv}(\mathcal{Y})} \simeq \boxtimes^R((\Delta_{\mathcal{Y}})_*(\omega_{\mathcal{Y}})), \tag{15.14}$$

which would then give rise to the desired map (15.12) by adjunction.

To establish (15.14) we note that for $\mathcal{F}_1, \mathcal{F}_2 \in \mathrm{Shv}(\mathcal{Y})^c$, we have

$$\mathcal{H}om_{\mathrm{Shv}(\mathcal{Y})\otimes\mathrm{Shv}(\mathcal{Y})}(\mathcal{F}_1 \otimes \mathcal{F}_2, u_{\mathrm{Shv}(\mathcal{Y})}) \simeq \mathcal{H}om_{\mathrm{Shv}(\mathcal{Y})}(\mathcal{F}_1, \mathbb{D}(\mathcal{F}_2)),$$

by the definition of the self-duality on $\mathrm{Shv}(\mathcal{Y})$ (here $\mathbb{D}$ is Verdier duality), while

$$\mathcal{H}om_{\mathrm{Shv}(\mathcal{Y})\otimes\mathrm{Shv}(\mathcal{Y})}(\mathcal{F}_1 \otimes \mathcal{F}_2, \boxtimes^R((\Delta_{\mathcal{Y}})_*(\omega_{\mathcal{Y}}))) \simeq \mathcal{H}om_{\mathrm{Shv}(\mathcal{Y}\times\mathcal{Y})}(\mathcal{F}_1 \boxtimes \mathcal{F}_2, (\Delta_{\mathcal{Y}})_*(\omega_{\mathcal{Y}})) \simeq \\ \simeq \mathcal{H}om_{\mathrm{Shv}(\mathcal{Y})}(\mathcal{F}_1, \mathbb{D}(\mathcal{F}_2))$$

as well.

15.2.5. *Example.* Let $(\mathcal{F}, \alpha)$ be a lax Weil sheaf on $\mathcal{Y}$. Unwinding the construction, we obtain that the image of

$$\mathrm{cl}(\mathcal{F}, \alpha) \in \mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}(\mathcal{Y}))$$

along the map $\mathrm{LT}^{\mathrm{true}}$, thought of as an element of

$$\mathrm{Funct}(\mathcal{Y}(\mathbb{F}_q)) \simeq \mathrm{C}^{\cdot}(\mathcal{Y}, \Delta^!_{\mathcal{Y}} \circ (\mathrm{Frob}_{\mathcal{Y}} \times \mathrm{id}_{\mathcal{Y}})_* \circ (\Delta_{\mathcal{Y}})_*(\omega_{\mathcal{Y}})) \simeq \\ \simeq \mathcal{H}om_{\mathrm{Shv}(\mathcal{Y})}(\underline{\mathsf{e}}_{\mathcal{Y}}, \Delta^!_{\mathcal{Y}} \circ (\mathrm{Frob}_{\mathcal{Y}} \times \mathrm{id}_{\mathcal{Y}})_* \circ (\Delta_{\mathcal{Y}})_*(\omega_{\mathcal{Y}})).$$

equals

$$\underline{\mathsf{e}}_{\mathcal{Y}} \to \mathcal{F} \overset{!}{\otimes} \mathbb{D}(\mathcal{F}) \simeq \Delta^!_{\mathcal{Y}}(\mathcal{F} \boxtimes \mathbb{D}(\mathcal{F})) \xrightarrow{\alpha\boxtimes\mathrm{id}} \Delta^!_{\mathcal{Y}}((\mathrm{Frob}_{\mathcal{Y}})_*(\mathcal{F}) \boxtimes \mathbb{D}(\mathcal{F})) \simeq \Delta^!_{\mathcal{Y}} \circ (\mathrm{Frob}_{\mathcal{Y}} \times \mathrm{id}_{\mathcal{Y}})_*(\mathcal{F} \boxtimes \mathbb{D}(\mathcal{F})) \to \\ \to \Delta^!_{\mathcal{Y}} \circ (\mathrm{Frob}_{\mathcal{Y}} \times \mathrm{id}_{\mathcal{Y}})_* \circ (\Delta_{\mathcal{Y}})_* \circ \Delta^*_{\mathcal{Y}}(\mathcal{F} \boxtimes \mathbb{D}(\mathcal{F})) \simeq \Delta^!_{\mathcal{Y}} \circ (\mathrm{Frob}_{\mathcal{Y}} \times \mathrm{id}_{\mathcal{Y}})_* \circ (\Delta_{\mathcal{Y}})_*(\mathcal{F} \overset{*}{\otimes} \mathbb{D}(\mathcal{F})) \to \\ \to \Delta^!_{\mathcal{Y}} \circ (\mathrm{Frob}_{\mathcal{Y}} \times \mathrm{id}_{\mathcal{Y}})_* \circ (\Delta_{\mathcal{Y}})_*(\omega_{\mathcal{Y}}).$$

Remark 15.2.6. The map (15.12) constructed above is, in general, *not* an isomorphism. In fact it is an isomorphism *if and only if* the functor

$$\mathrm{Shv}(\mathcal{Y}) \otimes \mathrm{Shv}(\mathcal{Y}) \overset{\boxtimes}{\to} \mathrm{Shv}(\mathcal{Y} \times \mathcal{Y})$$

is an equivalence, see Sect. 11.5.1.

The fact that the map (15.12) is not in general an isomorphism prevents the map (15.11) from being an isomorphism.

However (as was remarked in Sect. 11.5.1), we obtain that (15.12) is an isomorphism for algebraic stacks that have finitely many isomorphism classes of $\overline{\mathbb{F}}_q$-points, e.g., for $N\backslash G/B$, or a quasi-compact substack of Bun_G for a curve X of genus 0. Hence, (15.11) is an isomorphism in these cases as well.

15.2.7. We claim:

Theorem 15.2.8. *The maps*

$$\mathrm{LT}^{\mathrm{naive}} : \mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}(\mathcal{Y})) \to \mathrm{Funct}(\mathcal{Y}(\mathbb{F}_q))$$

and

$$\mathrm{LT}^{\mathrm{true}} : \mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}(\mathcal{Y})) \to \mathrm{Funct}(\mathcal{Y}(\mathbb{F}_q))$$

are canonically homotopic.

The proof will be supplied in [GaVa]. From now on, we will just use the symbol

$$\mathrm{LT} : \mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}(\mathcal{Y})) \to \mathrm{Funct}(\mathcal{Y}(\mathbb{F}_q))$$

for the local term map.

Remark 15.2.9. Theorem 15.2.8 implies that for a lax Weil sheaf $(\mathcal{F}, \alpha)$, the images of $\mathrm{cl}(\mathcal{F}, \alpha) \in \mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}(\mathcal{Y}))$ in $\mathrm{Funct}(\mathcal{Y}(\mathbb{F}_q))$ under the above two maps coincide.

Interpreting the image of $\mathrm{cl}(\mathcal{F}, \alpha)$ along $\mathrm{LT}^{\mathrm{true}}$ as in Sect. 15.2.5, the latter assertion becomes equivalent to one in [Var2, add ref]. The proof of Theorem 15.2.8 is an elaboration of the ideas from *loc. cit.*

15.3. **From geometric to classical: the Trace Conjecture.**

15.3.1. We start with the following observation. Let $\mathcal{Y}$ be an algebraic stack over $\overline{\mathbb{F}}_q$, but defined over $\mathbb{F}_q$. Consider the diagram

$$\begin{tikzcd}
\mathcal{Y} \arrow[r, "\mathrm{Frob}^{\mathrm{arithm}}"] \arrow[d] & \mathcal{Y} \arrow[r, "\mathrm{Frob}_{\mathcal{Y}}"] \arrow[d] & \mathcal{Y} \\
\mathrm{Spec}(\overline{\mathbb{F}}_q) \arrow[r, "\mathrm{Frob}^{\mathrm{arithm}}"] & \mathrm{Spec}(\overline{\mathbb{F}}_q),
\end{tikzcd}$$

where:

- The bottom horizontal square is the Frobenius automorphism of $\mathrm{Spec}(\mathbb{F}_q)$;
- The square is Cartesian;
- The composite top horizontal arrow is the absolute Frobenius on $\mathcal{Y}$.

For $\mathcal{N} \subset T^*(\mathcal{Y})$, let $\mathcal{N}' \subset T^*(\mathcal{Y})$ denote the base-change of $\mathcal{N}$ along $\mathrm{Frob}^{\mathrm{arithm}}$.

We claim:

Lemma 15.3.2. *The functor* $(\mathrm{Frob}_{\mathcal{Y}})_* : \mathrm{Shv}(\mathcal{Y}) \to \mathrm{Shv}(\mathcal{Y})$ *sends* $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y}) \to \mathrm{Shv}_{\mathcal{N}'}(\mathcal{Y})$.

Proof. The functor $(\mathrm{Frob}_{\mathcal{Y}})_*$ is the inverse of the pullback functor along $\mathrm{Frob}^{\mathrm{arithm}}$. Hence, it is sufficient to show that $(\mathrm{Frob}_{\mathcal{Y}})^*$ sends $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y}) \to \mathrm{Shv}_{\mathcal{N}'}(\mathcal{Y})$

Now, for any map of fields $k \to k'$, the pullback functor along

$$\mathcal{Y}' := \mathrm{Spec}(k') \underset{\mathrm{Spec}(k)}{\times} \mathcal{Y} \to \mathcal{Y}$$

sends $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y}) \subset \mathrm{Shv}(\mathcal{Y})$ to $\mathrm{Shv}_{\mathcal{N}'}(\mathcal{Y}') \subset \mathrm{Shv}(\mathcal{Y}')$.

$\square$

15.3.3. We now come to one of the central ideas of this paper.

Recall that the category $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ is compactly generated, and in particular, dualizable. By Theorem 10.1.6, the inclusion (15.15) admits a continuous right adjoint.

By Lemma 15.3.2, the action of $(\mathrm{Frob}_{\mathrm{Bun}_G})_*$ preserves the subcategory

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \subset \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G). \tag{15.15}$$

By Theorem 10.1.6, the inclusion (15.15) admits a continuous right adjoint.

Hence, by [GKRV, Sect. 3.4.1], we obtain a map

$$\mathrm{Tr}((\mathrm{Frob}_{\mathrm{Bun}_G})_*, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)) \to \mathrm{Tr}((\mathrm{Frob}_{\mathrm{Bun}_G})_*, \mathrm{Shv}(\mathrm{Bun}_G)).$$

15.3.4. We propose the following:

Main Conjecture 15.3.5. *The composite map*

$$\mathrm{Tr}((\mathrm{Frob}_{\mathrm{Bun}_G})_*, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)) \to \mathrm{Tr}((\mathrm{Frob}_{\mathrm{Bun}_G})_*, \mathrm{Shv}(\mathrm{Bun}_G)) \overset{\mathrm{LT}}{\to} \mathrm{Funct}_c(\mathrm{Bun}_G(\mathbb{F}_q))$$

is an isomorphism.

In what follows we will use the notation

$$\mathrm{Autom} := \mathrm{Funct}_c(\mathrm{Bun}_G(\mathbb{F}_q))$$

and

$$\widetilde{\mathrm{Autom}} := \mathrm{Tr}((\mathrm{Frob}_{\mathrm{Bun}_G})_*, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)).$$

So, the statement of Conjecture 15.3.5 is that the map

$$\widetilde{\mathrm{Autom}} \to \mathrm{Autom}$$

defined above, is an isomorphism.

Remark 15.3.6. Note that Conjecture 15.3.5 defines a direct bridge from the geometric Langlands theory to the classical one, since it implies that the space of automorphic functions with compact support, can be expressed as the categorical trace of the Frobenius endofunctor acting on the category of sheaves on Bun_G with nilpotent singular support.

Such a bridge allows to transport structural assertions about $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ as a category, to assertions about Autom as a vector space. We will see some examples of this in Sect. 16.

15.3.7. There are several pieces of evidence towards the validity of Conjecture 15.3.5.

(I) It is true when G is a torus. We will analyze this case in the next subsection.

(II) It is true when X is of genus 0. Indeed, in this case the inclusion

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \subset \mathrm{Shv}(\mathrm{Bun}_G)$$

is an equality, and the map LT is an isomorphism because

$$\mathrm{Shv}(\mathrm{Bun}_G) \otimes \mathrm{Shv}(\mathrm{Bun}_G) \to \mathrm{Shv}(\mathrm{Bun}_G \times \mathrm{Bun}_G) \tag{15.16}$$

is an equivalence, see Remark 15.2.6.

(III) We have seen in Remark 15.2.6 that the failure of the map LT originates in the failure of the functor (15.16) to be equivalence. Now, this obstruction goes away for $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ thanks to Theorem 11.5.3.

(IV) The lisseness property of the cohomology of shtukas, recently established by C. Xue in [Xue2], see Remark 15.5.2.

15.4. **The case of $G = \mathbb{G}_m$.** In this subsection we will verify by hand the assertion of Conjecture 15.3.5 for $G = \mathbb{G}_m$.

To simplify the notation, we will work not with the entire $\mathrm{Bun}_{\mathbb{G}_m} \simeq \mathrm{Pic}$, but with its neutral connected component Pic_0.

15.4.1. Recall that according to Sect. 14.2.6, the category

$$\mathrm{Shv}_{\{0\}}(\mathrm{Pic}_0)=\mathrm{QLisse}(\mathrm{Pic}_0)$$

is the direct sum over isomorphism classes of $\mathbb{G}_m$-local systems σ of copies of

$$\big(\mathrm{Sym}(H^1(X,\mathsf{e}_X)[-1])\text{-mod}\big)\otimes(\mathrm{C}_\cdot(\mathbb{G}_m)\text{-mod}),\tag{15.17}$$

where for every σ, we send the module

$$\big(\mathrm{Sym}(H^1(X,\mathsf{e}_X)[-1])\text{-mod}\big)\otimes(\mathsf{e})\in\big(\mathrm{Sym}(H^1(X,\mathsf{e}_X)[-1])\text{-mod}\big)\otimes(\mathrm{C}_\cdot(\mathbb{G}_m))$$

(where e denotes the augmentation module over $\mathrm{C}_\cdot(\mathbb{G}_m)$) to the Hecke eigensheaf $\mathcal{F}_\sigma\in\mathrm{QLisse}(\mathrm{Pic}_0)$ corresponding to σ.

When we compute $\mathrm{Tr}((\mathrm{Frob}_{\mathrm{Pic}_0})_*,-)$ on this category, only the direct summands, for which

$$\mathrm{Frob}_X^*(\sigma)\simeq\sigma\tag{15.18}$$

can contribute.

For each such σ choose an isomorphism in (15.18). This choice defines a Weil sheaf structure on the corresponding $\mathcal{F}_\sigma$. Further, this choice identifies the action of $\mathrm{Tr}((\mathrm{Frob}_{\mathrm{Pic}_0})_*,-)$ on the direct summand (15.17) with the action induced by the Frobenius automorphism of the algebra

$$A:=\mathrm{Sym}(H^1(X,\mathsf{e}_X)[-1])\otimes\mathrm{C}_\cdot(\mathbb{G}_m).$$

We will show that

$$\mathrm{Tr}(\mathrm{Frob},A\text{-mod})\simeq\mathsf{e},\tag{15.19}$$

and that the induced map

$$\begin{multline}\mathsf{e}\mapsto\mathrm{Tr}(\mathrm{Frob},A\text{-mod})\to\mathrm{Tr}((\mathrm{Frob}_{\mathrm{Pic}_0})_*,\mathrm{QLisse}(\mathrm{Pic}_0))\to\\ \to\mathrm{Tr}((\mathrm{Frob}_{\mathrm{Pic}_0})_*,\mathrm{Shv}(\mathrm{Pic}_0))\overset{\mathrm{LT}}{\to}\mathrm{Funct}(\mathrm{Pic}_0(\mathbb{F}_q))\end{multline}\tag{15.20}$$

sends $1\in\mathsf{e}$ to $\mathrm{funct}(\mathcal{F}_\sigma)\cdot(1-q)$.

This will prove the required assertion since the functions $\mathrm{funct}(\mathcal{F}_\sigma)$ form a basis of $\mathrm{Funct}(\mathrm{Pic}_0(\mathbb{F}_q))$, by Class Field Theory.

15.4.2. We have

$$A=A_1\otimes A_2,\quad A_1=\mathrm{Sym}(H^1(X,\mathsf{e}_X)[-1]),\ A_2=\mathrm{C}_\cdot(\mathbb{G}_m).$$

This corresponds to writing

$$\mathrm{Pic}_0\simeq\mathrm{Jac}(X)\times\mathrm{pt}/\mathbb{G}_m.$$

We will perform the calculation for each factor separately.

15.4.3. Note that if A' is a polynomial algebra

$$A'\simeq\mathrm{Sym}(V),$$

where V is equipped with an endomorphism F with no eigenvalue 1, then the functor

$$\mathrm{Vect}_{\mathsf{e}}\to A'\text{-mod},\quad\mathsf{e}\mapsto A'$$

defines an isomorphism

$$\mathsf{e}\simeq\mathrm{Tr}(\mathrm{Id},\mathrm{Vect}_{\mathsf{e}})\to\mathrm{Tr}(F,A'\text{-mod}).$$

Applying this to $A'=A_1$ and $A'=A_2$, we obtain the desired identifications

$$\mathrm{Tr}(\mathrm{Frob},A_1\text{-mod})\simeq\mathsf{e}\text{ and }\ \mathrm{Tr}(\mathrm{Frob},A_2\text{-mod})\simeq\mathsf{e},$$

as required in (15.19).

15.4.4. To prove (15.20) for A_1, we consider the composite functor

$$\mathrm{Vect}_{\mathsf{e}} \overset{\mathsf{e}\mapsto A_1}{\longrightarrow} A_1\text{-mod} \to \mathrm{QLisse}(\mathrm{Pic}_0) \to \mathrm{Shv}(\mathrm{Pic}_0),$$

equipped with its datum of compatibility with the Frobenius.

It sends

$$\mathsf{e} \mapsto \mathcal{F}_\sigma,$$

equipped with its Weil structure, to be denoted α.

Hence, the corresponding map

$$\mathsf{e} \simeq \mathrm{Tr}(\mathrm{Id}, \mathrm{Vect}_{\mathsf{e}}) \to \mathrm{Tr}((\mathrm{Frob}_{\mathrm{Pic}_0})_*, \mathrm{Shv}(\mathrm{Pic}_0))$$

sends

$$1 \in \mathsf{e} \ \mapsto \ \mathrm{cl}(\mathcal{F}_\sigma, \alpha) \in \mathrm{Tr}((\mathrm{Frob}_{\mathrm{Pic}_0})_*, \mathrm{Shv}(\mathrm{Pic}_0)).$$

Hence, its image under LT is $\mathrm{funct}(\mathcal{F}_\sigma)$.

15.4.5. We now consider A_2. The composite functor

$$\mathrm{Vect}_{\mathsf{e}} \overset{\mathsf{e}\mapsto A_2}{\longrightarrow} A_2\text{-mod} \to \mathrm{Shv}(\mathrm{pt}/\mathbb{G}_m) \overset{\text{pullback}}{\to} \mathrm{Shv}(\mathrm{pt}) = \mathrm{Vect}_{\mathsf{e}}$$

sends

$$\mathsf{e} \mapsto \mathrm{C}_\cdot(\mathbb{G}_m),$$

equipped with the natural datum of compatibility with the Frobenius.

Hence the resulting map

$$\mathsf{e} \simeq \mathrm{Tr}(\mathrm{Id}, \mathrm{Vect}_{\mathsf{e}}) \to \mathrm{Tr}((\mathrm{Frob}_{\mathrm{pt}/\mathbb{G}_m})_*, \mathrm{Shv}(\mathrm{pt}/\mathbb{G}_m)) \to \mathrm{Tr}((\mathrm{Frob}_{\mathrm{pt}})_*, \mathrm{Shv}(\mathrm{pt})) = \mathrm{Tr}(\mathrm{Id}, \mathrm{Vect}_{\mathsf{e}}) \simeq \mathsf{e}$$

sends

$$1 \in \mathsf{e} \ \mapsto \mathrm{Tr}(\mathrm{Frob}, \mathrm{C}_\cdot(\mathbb{G}_m)) = 1 - q \in \mathsf{e}.$$

15.5. **A generalization: cohomologies of shtukas.** In this subsection we will formulate a generalization of the Trace Conjecture, which gives a trace interpretation to cohomologies of shtukas.

15.5.1. Let us recall the construction of cohomologies of shtukas, following [VLaf1] and [Var1].

Let I be a finite set and V an object of $\mathrm{Rep}(\check{G})^{\otimes I}$. To this data we attach an object

$$\mathrm{Sht}_{I,V} \in \mathrm{Shv}(X^I)$$

as follows.

We consider the I-legged Hecke stack

$$\begin{tikzcd} \mathrm{Bun}_G & \mathrm{Hecke}_{X^I} \arrow[l, "\overleftarrow{h}"'] \arrow[r, "\overrightarrow{h}"] \arrow[d, "\pi"'] & \mathrm{Bun}_G \\ & X^I. & \end{tikzcd}$$

The I-legged shtuka space is defined as the fiber product

$$\begin{tikzcd} \mathrm{Sht}_I \arrow[r] \arrow[d] & \mathrm{Hecke}_{X^I} \arrow[d, "(\overleftarrow{h}, \overrightarrow{h})"] \\ \mathrm{Bun}_G \arrow[r, "(\mathrm{Frob}_{\mathrm{Bun}_G}, \mathrm{Id})"] & \mathrm{Bun}_G \times \mathrm{Bun}_G . \end{tikzcd}$$

Let π' denote the composite map

$$\mathrm{Sht}_I \to \mathrm{Hecke}_{X^I} \overset{\pi}{\to} X^I.$$

Recall that (naive) geometric Satake attaches to $V \in \mathrm{Rep}(\check{G})^{\otimes I}$ an object

$$\mathcal{S}_V \in \mathrm{Shv}(\mathrm{Hecke}_{X^I}).$$

Let $\mathcal{S}'_V \in \mathrm{Shv}(\mathrm{Sht}_I)$ denote its *-restriction to $\mathrm{Shv}(\mathrm{Sht}_I)$. Finally, we set

$$\mathrm{Sht}_{I,V} := \pi'_!(\mathcal{S}'_V) \in \mathrm{Shv}(X^I).$$

Remark 15.5.2. A recent result of [Xue2] says that the objects $\mathrm{Sht}_{I,V}$ actually belong to

$$\mathrm{QLisse}(X^I) \subset \mathrm{Shv}(X^I).$$

15.5.3. *Example.* Take $I = \emptyset$ and V to be e. Then

$$\mathrm{Sht}_\emptyset \simeq (\mathrm{Bun}_G)^{\mathrm{Frob}} \simeq \mathrm{Bun}_G(\mathbb{F}_q).$$

We obtain that

$$\mathrm{Sht}_{\emptyset,\mathsf{e}} = \mathrm{C}^\cdot_c(\mathrm{Bun}_G(\mathbb{F}_q), \underline{\mathsf{e}}_{\mathrm{Bun}_G(\mathbb{F}_q)}) \simeq \mathrm{Funct}_c(\mathrm{Bun}_G(\mathbb{F}_q)) = \mathrm{Autom}\,. \tag{15.21}$$

15.5.4. We will now construct a different system of objects

$$\widetilde{\mathrm{Sht}}_{I,V} \in \mathrm{QLisse}(X^I).$$

15.5.5. Note that the categorical trace construction has the following variant. Let $\mathbf{C}$ be a dualizable DG category and let

$$F : \mathbf{C} \to \mathbf{C} \otimes \mathbf{D},$$

where $\mathbf{D}$ is some other DG category.

Then we can consider an object

$$\mathrm{Tr}(F, \mathbf{C}) \in \mathbf{D}.$$

(The usual trace construction is when $\mathbf{D} = \mathrm{Vect}_{\mathsf{e}}$, so $\mathrm{Tr}(F, \mathbf{C}) \in \mathrm{Vect}_{\mathsf{e}}$.)

15.5.6. We apply this to $\mathbf{C} := \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$, $\mathbf{D} = \mathrm{QLisse}(X^I)$ and F being the functor

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \overset{(\mathrm{Frob}_{\mathrm{Bun}_G})_*}{\longrightarrow} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \overset{\mathrm{H}(V,-)}{\longrightarrow} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{QLisse}(X^I).$$

We set

$$\widetilde{\mathrm{Sht}}_{I,V} := \mathrm{Tr}(\mathrm{H}(V,-) \circ (\mathrm{Frob}_{\mathrm{Bun}_G})_*, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)) \in \mathrm{QLisse}(X^I).$$

We propose:

Main Conjecture 15.5.7. *The objects* $\mathrm{Sht}_{I,V}$ *and* $\widetilde{\mathrm{Sht}}_{I,V}$ *are canonically isomorphic.*

15.5.8. Consider the case of $I = \emptyset$. As we have seen in Sect. 15.5.3,

$$\mathrm{Sht}_{\emptyset,\mathsf{e}} = \mathrm{C}^\cdot_c(\mathrm{Bun}_G(\mathbb{F}_q), \underline{\mathsf{e}}_{\mathrm{Bun}_G(\mathbb{F}_q)}) \simeq \mathrm{Funct}_c(\mathrm{Bun}_G(\mathbb{F}_q)) = \mathrm{Autom}\,.$$

This is while, according to Conjecture 15.3.5,

$$\widetilde{\mathrm{Sht}}_{\emptyset,\mathsf{e}} = \mathrm{Tr}((\mathrm{Frob}_{\mathrm{Bun}_G})_*, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)) = \widetilde{\mathrm{Autom}},$$

which, according to Conjecture 15.3.5, is isomorphic to Autom.

So, Conjecture 15.3.5 is a special case of Conjecture 15.5.7. Yet the fact that the two statements amount to the same hides something non-trivial:

Recall that the map in Conjecture 15.3.5 was constructed from the identifications

$$\mathrm{C}^\cdot(\mathcal{U}^{\mathrm{Frob}}, \omega_{\mathcal{U}^{\mathrm{Frob}}}) \simeq \mathrm{Funct}(\mathcal{U}(\mathbb{F}_q))$$

for quasi-compact open substacks $\mathcal{U} \subset \mathrm{Bun}_G$.

This is while the identification (15.21) was constructed from the identifications

$$\mathrm{C}^\cdot_c(\mathcal{U}^{\mathrm{Frob}}, \underline{\mathsf{e}}_{\mathcal{U}^{\mathrm{Frob}}}) \simeq \mathrm{Funct}(\mathcal{U}(\mathbb{F}_q)).$$

So hidden in the statement is the fact that for a discrete *quasi-compact* stack, such as $\mathcal{U}^{\mathrm{Frob}}$, we have a canonical identification

$$\mathrm{C}^\cdot_c(\mathcal{U}^{\mathrm{Frob}}, \underline{\mathsf{e}}_{\mathcal{U}^{\mathrm{Frob}}}) \simeq \mathrm{C}^\cdot(\mathcal{U}^{\mathrm{Frob}}, \omega_{\mathcal{U}^{\mathrm{Frob}}}).$$

Thus, for a general (I, V), the statement of Conjecture 15.5.7 must involve a comparison between cohomology and cohomology with compact supports. Without it, we cannot even construct a map in one direction, as we did in the case of Conjecture 15.3.5.

Remark 15.5.9. A crucial piece of evidence for the validity of Conjecture 15.5.7 is provided by the result of [Xue2] mentioned in Remark 15.5.2.

15.5.10. *Partial Frobeniuses.* Recall (see [VLaf1, Sect. 3]) that the objects $\mathrm{Sht}_{I,V}$ carry an additional structure, namely, equivariance with respect to the *partial Frobenius maps*.

The construction from [GKRV, Sect. 5.3] endows the objects $\widetilde{\mathrm{Sht}}_{I,V}$ with a similar structure. (See also Remark 16.4.6 for a conceptual explanation of this structure.)

The statement of Conjecture 15.5.7 should be strengthened as follows: the isomorphism

$$\mathrm{Sht}_{I,V} \simeq \widetilde{\mathrm{Sht}}_{I,V}$$

is compatible with the structure of equivariance with respect to the partial Frobenius maps.

16. Localization of the space of automorphic functions

In this section we will introduce the space (in fact, a quasi-compact algebraic stack) of *arithmetic* Langlands parameters, denoted $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{arithm}}(X)$.

We will see how our Trace Conjecture leads to a *localization* of the space of automorphic forms onto $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{arithm}}(X)$.

16.1. The arithmetic $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}$.

16.1.1. Consider the automorphism of the symmetric monoidal category $\mathrm{QLisse}(X)$, given by pullback with respect to the Frobenius endomorphism of X:

$$\mathrm{Frob}_X^* : \mathrm{QLisse}(X) \to \mathrm{QLisse}(X).$$

By transport of structure, the prestack $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)$ acquires an automorphism, which we will denote simply by Frob.

16.1.2. Consider the prestack

$$(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X))^{\mathrm{Frob}}$$

of Frob-fixed points of $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)$.

Note that e-points of $(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X))^{\mathrm{Frob}}$ are G-local systems on X equipped with a Weil structure.

16.1.3. In Sect. 17 we will prove:

Theorem 16.1.4. *The fixed-point locus* $(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X))^{\mathrm{Frob}}$ *is a quasi-compact, mock-affine*[25] *algebraic stack, locally almost of finite type.*

16.1.5. In the same Sect. 17 we will also prove:

Theorem 16.1.6. *Assume that* G *is semi-simple. Let* σ *be an* e*-point of* $(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X))^{\mathrm{Frob}}$*, which is an irreducible as a* G*-local system with a Weil structure. Then the group of its automorphisms is finite, and the resulting map*

$$\mathrm{pt} / \mathrm{Aut}(\sigma) \to (\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X))^{\mathrm{Frob}}$$

is the embedding of a connected component.

Combining with the quasi-compactness assertion from Theorem 16.1.4, we obtain:

Corollary 16.1.7. *Let* G *be semi-simple. Then there is only a finite number of irreducible Weil* G*-local systems on* X.

[25]See Sect. 4.3.1 for what this means.

16.1.8. We will think of $(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X))^{\mathrm{Frob}}$ as the stack parameterizing G-local systems on X equipped with a Weil structure, and henceforth denote it by

$$\mathrm{LocSys}^{\mathrm{arithm}}_{\mathsf{G}}(X).$$

Remark 16.1.9. We propose $\mathrm{LocSys}^{\mathrm{arithm}}_{\mathsf{G}}(X)$ as a candidate for the stack $\mathcal{S}$, alluded to in [VLaf2, Remark 8.5].

Recently, P. Scholze (unpublished) and X. Zhu (in [Zhu]) proposed two more definitions of the stack of Weil G-local systems on X. Their definitions are different from each other, and are of completely different flavor from ours. It is likely, however, that the three definitions are actually equivalent.

Remark 16.1.10. As we shall see in Sect. 16.5.3, the stack $\mathrm{LocSys}^{\mathrm{arithm}}_{\mathsf{G}}(X)$ is *non-classical*, i.e., its structure sheaf has non-trivial negative cohomology.

16.2. **The excursion algebra.**

16.2.1. Denote

$$\mathcal{E}xc := \Gamma(\mathrm{LocSys}^{\mathrm{arithm}}_{\mathsf{G}}(X), \mathcal{O}_{\mathrm{LocSys}^{\mathrm{arithm}}_{\mathsf{G}}(X)}).$$

This is a commutative algebra object in $\mathrm{Vect}_{\mathsf{e}}$. Since $\mathrm{LocSys}^{\mathrm{arithm}}_{\mathsf{G}}(X)$ is mock-affine, the algebra $\mathcal{E}xc$ is connective.

Set

$$\mathrm{LocSys}^{\mathrm{arithm,coarse}}_{\mathsf{G}}(X) := \mathrm{Spec}(\mathcal{E}xc). \tag{16.1}$$

This is the coarse moduli space of arithmetic Langlands parameters. By construction, it is a *derived* affine scheme.

16.2.2. The algebra $\mathcal{E}xc$ is related to V. Lafforgue's algebra of excursion operators as follows.

Let $\mathrm{Weil}(X,x)^{\mathrm{discr}}$ be the Weil group of X (for some choice of a base point $x \in X$), considered as a discrete group. Set

$$\mathfrak{X}^{\mathrm{discr}} := B(\mathrm{Weil}(X,x)^{\mathrm{discr}}) \in \mathrm{Spc}\,.$$

Consider the (mock-affine) algebraic stack

$$\mathrm{LocSys}_{\check{G}}(\mathfrak{X}^{\mathrm{discr}}) \simeq \mathrm{LocSys}^{\mathrm{rigid}_x}_{\check{G}}(X^{\mathrm{discr}})/\check{G},$$

and set

$$\mathcal{E}xc^{\mathrm{discr}} := \Gamma(\mathrm{LocSys}_{\check{G}}(\mathfrak{X}^{\mathrm{discr}}), \mathcal{O}_{\mathrm{LocSys}_{\check{G}}(\mathfrak{X}^{\mathrm{discr}})}).$$

We have a naturally defined closed embedding

$$\mathrm{LocSys}^{\mathrm{arithm}}_{\mathsf{G}}(X) \hookrightarrow \mathrm{LocSys}_{\check{G}}(\mathfrak{X}^{\mathrm{discr}}),$$

which induces a map

$$\mathcal{E}xc^{\mathrm{discr}} \to \mathcal{E}xc,$$

surjective on H^0.

The algebra $\mathcal{E}xc^{\mathrm{discr}}$ is the algebra of excursion operators attached to $\mathrm{Weil}(X,x)^{\mathrm{discr}}$, see [GKRV, Sect. 2].

The algebra of excursion operators in [VLaf1] is $H^0(\mathcal{E}xc^{\mathrm{discr}})$.

16.3. **Enhanced trace and the universal shtuka.** In this subsection we will explain how the procedure of *2-categorical trace* produces from $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$, equipped with the Frobenius endofunctor, an object of $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X))$, to be denoted Drinf.

16.3.1. Recall the set-up of [GKRV, Sects. 3.6-3.8]. We start with a symmetric monoidal category $\mathcal{R}$ (assumed dualizable as a DG category), equipped with a symmetric monoidal endofunctor $F_{\mathcal{R}}$. Let $\mathcal{M}$ be an $\mathcal{R}$-module category (assumed dualizable as such), equipped with an endofunctor $F_{\mathcal{M}}$, compatible with $F_{\mathcal{R}}$.

Consider the category

$$\mathrm{HH}_\bullet(F_{\mathcal{R}}, \mathcal{R}),$$

see [GKRV, Sects. 3.7.2]. The symmetric monoidal structure on $(\mathcal{R}, F_{\mathcal{R}})$ induces a symmetric monoidal structure on $\mathrm{HH}_\bullet(F_{\mathcal{R}}, \mathcal{R})$.

Further, to $(\mathcal{M}, F_{\mathcal{M}})$ we can attach an object

$$\mathrm{Tr}^{\mathrm{enh}}(F_{\mathcal{M}}, \mathcal{M}) \in \mathrm{HH}_\bullet(F_{\mathcal{R}}, \mathcal{R}),$$

see [GKRV, Sects. 3.8.2].

Under the assumption that $\mathcal{R}$ is rigid, we have the following assertion ([GKRV, Theorem 3.8.5]):

There exists a canonical isomorphism in $\mathrm{Vect}_{\mathsf{e}}$:

$$\mathrm{Tr}(F_{\mathcal{M}}, \mathcal{M}) \simeq \mathcal{H}om_{\mathrm{HH}_\bullet(F_{\mathcal{R}}, \mathcal{R})}\left(\mathbf{1}_{\mathrm{HH}_\bullet(F_{\mathcal{R}}, \mathcal{R})}, \mathrm{Tr}^{\mathrm{enh}}(F_{\mathcal{M}}, \mathcal{M})\right), \tag{16.2}$$

where $\mathbf{1}_{\mathrm{HH}_\bullet(F_{\mathcal{R}}, \mathcal{R})}$ is the monoidal unit in $\mathrm{HH}_\bullet(F_{\mathcal{R}}, \mathcal{R})$.

Remark 16.3.2. We are going to apply (16.2) to $\mathcal{R} := \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}^{\mathrm{restr}}(X))$. This category is *not* rigid, so a justification is needed for why [GKRV, Theorem 3.8.5] is applicable:

Due to Theorem 1.3.2, $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}^{\mathrm{restr}}(X))$ is a direct sum of categories that are co-localizations of rigid categories.

First, it is easy to see that if $\mathcal{R}$ is a direct sum of (symmetric monoidal) categories $\mathcal{R}_\alpha$, the validity of (16.2) for all $\mathcal{R}_\alpha$ implies the validity of (16.2) for $\mathcal{R}$.

Next, suppose that $\mathcal{R}$ is a co-localization of a rigid symmetric monoidal category $\mathcal{R}'$. I.e., we have a symmetric monoidal functor $\phi : \mathcal{R}' \to \mathcal{R}$, which admits a left adjoint. Assume that $\mathcal{R}'$ is equipped with a symmetric monoidal endofunctor $F_{\mathcal{R}'}$ compatible with $F_{\mathcal{R}}$ via ϕ. We claim that the validity of (16.2) for $\mathcal{R}'$ (when we view $\mathcal{M}$ as a $\mathcal{R}'$-module) implies the validity of (16.2) for $\mathcal{R}$.

Indeed, this is easy to deduce from the functoriality of the assignment

$$\mathcal{R} \rightsquigarrow \mathrm{Tr}^{\mathrm{enh}}(F_{\mathcal{M}}, \mathcal{M})$$

given by [GKRV, Theorem 3.10.6].

16.3.3. Thus, we take $\mathcal{R}$ to be the (symmetric) monoidal category

$$\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}^{\mathrm{restr}}(X)),$$

and we take $F_{\mathcal{R}}$ to be given by Frob^*, where Frob is as in Sect. 16.1.1.

By [GKRV, Example 3.7.3], we have a canonical identification

$$\mathrm{HH}_\bullet(\mathrm{Frob}^*, \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}^{\mathrm{restr}}(X))) \simeq \mathrm{QCoh}((\mathrm{LocSys}_{\check{G}}^{\mathrm{restr}}(X))^{\mathrm{Frob}}) =: \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}^{\mathrm{arithm}}(X))$$

as (symmetric) monoidal categories.

The unit object

$$\mathbf{1}_{\mathrm{HH}_\bullet(\mathrm{Frob}^*, \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}^{\mathrm{restr}}(X)))} \in \mathrm{HH}_\bullet(\mathrm{Frob}^*, \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}^{\mathrm{restr}}(X)))$$

is then the structure sheaf

$$\mathcal{O}_{\mathrm{LocSys}_{\check{G}}^{\mathrm{arithm}}(X)} \in \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}^{\mathrm{arithm}}(X)).$$

16.3.4. We take the module category $\mathcal{M}$ to be $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$, and $F_{\mathcal{M}}$ to be $(\mathrm{Frob}_{\mathrm{Bun}_G})_*$, which is equipped with a natural structure of compatibility with Frob^*.

Thus, we can consider the object

$$\mathrm{Tr}^{\mathrm{enh}}((\mathrm{Frob}_{\mathrm{Bun}_G})_*, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)) \in \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)).$$

In what follows, we will denote this object by

$$\mathrm{Drinf} \in \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)).$$

In the next subsection we will explain that Drinf can be regarded as a "universal shtuka", see Proposition 16.4.5.

16.3.5. According to (16.2), we have a canonical identification

$$\widetilde{\mathrm{Autom}} := \mathrm{Tr}((\mathrm{Frob}_{\mathrm{Bun}_G})_*, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)) \simeq \Gamma(\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X), \mathrm{Drinf}). \tag{16.3}$$

In particular, we obtain an action of the algebra

$$\mathcal{E}xc := \Gamma(\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X), \mathcal{O}_{\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)})$$

on $\widetilde{\mathrm{Autom}}$.

16.3.6. Let us now combine this with Conjecture 15.3.5. We obtain:

Corollary-of-Conjecture 16.3.7. *There exists a canonical isomorphism of vector spaces*

$$\mathrm{Autom} \simeq \Gamma(\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X), \mathrm{Drinf}). \tag{16.4}$$

As a consequence, we obtain:

Corollary-of-Conjecture 16.3.8. *There exists a canonically defined action of the algebra $\mathcal{E}xc$ on* Autom.

Combining with Sect. 16.2.2, we obtain an action of the algebra $\mathcal{E}xc^{\mathrm{discr}}$ on Autom. Thus, we obtain a spectral decomposition of Autom over the affine scheme $\mathrm{LocSys}^{\mathrm{restr,coarse,arithm}}_{\check{G}}(X)$, see (16.1).

Remark 16.3.9. As we will see in Remark 16.4.6, if we furthermore input Conjecture 15.5.7 together with the compatibility from Sect. 15.5.10, we will see that the resulting action of $\mathcal{E}xc^{\mathrm{discr}}$ on Autom equals the action defined in [VLaf1] by excursion operators.

16.3.10. We can view the conclusion of Corollary 16.3.7 as "localization" of the space of automorphic functions onto the stack $\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)$ of arithmetic Langlands parameters, in the sense that we realize Autom as the space of sections of a quasi-coherent sheaf on this stack.

16.4. **Relation to shtukas.** In this subsection we will explain that the Shtuka Conjecture (i.e., Conjecture 15.5.7) implies that the object Drinf constructed above, encodes all the shtuka cohomology.

16.4.1. Recall the objects

$$\widetilde{\mathrm{Sht}}_{I,V} \in \mathrm{QLisse}(X),$$

see Sect. 15.5.6.

We will now show, following [GKRV, Sect. 5.2], how the object

$$\mathrm{Drinf} \in \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X))$$

recovers these objects, and endows them with a structure of equivariance with respect to the partial Frobenius maps.

16.4.2. For $V \in \mathrm{Rep}(\check{G})^{\otimes I}$, let $\mathcal{E}_V$ be the corresponding tautological object of

$$\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \otimes \mathrm{QLisse}(X)^{\otimes I}.$$

Namely, for $S \to \mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$, the value of $\mathcal{E}_V$ on S, viewed as an object of

$$\mathrm{QCoh}(S) \otimes \mathrm{QLisse}(X)^{\otimes I}$$

is the value on V of the symmetric monoidal functor

$$\mathrm{Rep}(\check{G})^{\otimes I} \to \mathrm{QCoh}(S)^{\otimes I} \otimes \mathrm{QLisse}(X)^{\otimes I} \to \mathrm{QCoh}(S) \otimes \mathrm{QLisse}(X)^{\otimes I},$$

where:

- The first arrow is the I tensor power of the functor $\mathrm{Rep}(\check{G}) \to \mathrm{QCoh}(S) \otimes \mathrm{QLisse}(X)$ defining the map $S \to \mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$;
- The second arrow uses the I-fold tensor product functor $\mathrm{QCoh}(S)^{\otimes I} \to \mathrm{QCoh}(S)$.

In what follows, by a slight abuse of notation, we will denote by the same character $\mathcal{E}_V$ the image of $\mathcal{E}_V$ under the fully faithful functor[26]

$$\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \otimes \mathrm{QLisse}(X)^{\otimes I} \to \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \otimes \mathrm{QLisse}(X^I).$$

16.4.3. Let

$$\mathcal{E}^{\mathrm{arithm}}_V \in \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)) \otimes \mathrm{QLisse}(X^I)$$

denote the restriction of $\mathcal{E}_V$ along

$$\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X) \to \mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X).$$

For each $i \in I$, let Frob_{i,X^I} denote the Frobenius map along the i-th factor in X^I. By construction, the object $\mathcal{E}^{\mathrm{arithm}}_V$ carries a natural structure of equivariance with respect to these endomorphisms:

$$((\mathrm{Frob}_{X^I,i})^* \otimes \mathrm{Id})(\mathcal{E}^{\mathrm{arithm}}_V) \simeq \mathcal{E}^{\mathrm{arithm}}_V.$$

16.4.4. We claim:

Proposition 16.4.5. *There exists a canonical isomorphism in* $\mathrm{QLisse}(X^I)$

$$\widetilde{\mathrm{Sht}}_{I,V} \simeq \left(\Gamma(\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X), -) \otimes \mathrm{Id}\right) (\mathrm{Drinf} \otimes \mathcal{E}^{\mathrm{arithm}}_V). \tag{16.5}$$

Proof. This is a variant of [GKRV, Theorem 4.4.6]. □

Remark 16.4.6. Note that structure of equivariance with respect to the partial Frobenius maps on $\mathcal{E}^{\mathrm{arithm}}_V$ endows the left-hand side in (16.5) with a similar structure. Thus, we obtain a structure of equivariance with respect to the partial Frobenius maps on the objects $\widetilde{\mathrm{Sht}}_{I,V}$.

As in [GKRV, Proposition 5.4.3] one shows that one can use this structure to describe the action of $\mathcal{E}xc^{\mathrm{discr}}$ on $\widetilde{\mathrm{Autom}}$ by explicit excursion operators. Thus, if one assumes Conjecture 15.5.7, this allows to match the action of $\mathcal{E}xc^{\mathrm{discr}}$ on $\widetilde{\mathrm{Autom}}$ arising from

$$\mathcal{E}xc^{\mathrm{discr}} \to \mathcal{E}xc$$

with the one defined in V. Lafforgue's work (with the extension by C. Xue in [Xue1], from the cuspidal to the general case).

[26] This functor is fact an equivalence, by Theorem C.6.7.

Remark 16.4.7. As has been remarked above, we propose our $\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)$ as a candidate for the stack sought-for in [VLaf2, Remark 8.5] and [LafZh, Sect. 6] (it was denoted $\mathcal{S}/\check{G}$ in both these papers).

The space $\mathcal{S}$ is supposed to be the affine scheme parameterizing homomorphisms from the Weil group of X (based at x) to $\check{G}$. So, our proposal for $\mathcal{S}$ itself is

$$\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X) \underset{\mathrm{pt}/\mathcal{G}}{\times} \mathrm{pt}\,.$$

Although in *loc.cit.* the space $\mathcal{S}/\check{G}$ is only defined heuristically, it is designed so that it carries a collection of quasi-coherent sheaves $\mathcal{E}^{\mathrm{arithm},\mathcal{S}}_V$ for (I, V) as above.

The goal of *loc.cit.* was to define an object

$$\mathrm{Drinf}^{\mathcal{S}} \in \mathrm{QCoh}(\mathcal{S}/\check{G}),$$

so that

$$\big(\Gamma(\mathcal{S}/\check{G}, -) \otimes \mathrm{Id}\big)\,(\mathrm{Drinf}^{\mathcal{S}} \otimes \mathcal{E}^{\mathrm{arithm},\mathcal{S}}_V) \simeq \mathrm{Sht}_{I,V}$$

Thus, assuming Conjecture 15.5.7, our $\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)$ with the object

$$\mathrm{Drinf} \in \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X))$$

achieves this goal.

16.5. **Arithmetic Arthur parameters.**

16.5.1. Fix an e-point σ of $\mathrm{LocSys}^{\mathrm{arithm}}_{\mathsf{G}}(X)$. The tangent space $T_\sigma(\mathrm{LocSys}^{\mathrm{arithm}}_{\mathsf{G}}(X))$ identifies with

$$\mathrm{Fib}\left(T_\sigma(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)) \overset{\mathrm{Frob}-\mathrm{id}}{\longrightarrow} \mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X))\right) \simeq \mathrm{Fib}\left(\mathrm{C}^\cdot(X, \check{\mathfrak{g}}_\sigma) \overset{\mathrm{Frob}-\mathrm{id}}{\longrightarrow} \mathrm{C}^\cdot(X, \check{\mathfrak{g}}_\sigma)\right)[1],$$

and thus is concentrated in the cohomological degrees $[-1, 2]$.

This implies that $\mathrm{LocSys}^{\mathrm{arithm}}_{\mathsf{G}}(X)$ has a perfect cotangent complex and is *quasi-quasi-smooth*. The latter by definition means that it can be smoothly covered by an derived affine scheme, whose cotangent spaces are are concentrated in the cohomological degrees $[-2, 0]$.

16.5.2. We have

$$H^2(T_\sigma(\mathrm{LocSys}^{\mathrm{arithm}}_{\mathsf{G}}(X))) \simeq \mathrm{coker}\left(H^2(X, \check{\mathfrak{g}}_\sigma) \overset{\mathrm{Frob}-\mathrm{id}}{\longrightarrow} H^2(X, \check{\mathfrak{g}}_\sigma)\right).$$

Hence, by duality

$$H^{-2}(T^*_\sigma(\mathrm{LocSys}^{\mathrm{arithm}}_{\mathsf{G}}(X))) \simeq \big(H^0(X, \check{\mathfrak{g}}_\sigma(1))\big)^{\mathrm{Frob}},$$

where (1) means Tate twist, and where we have identifies $\check{\mathfrak{g}}$ with its dual using an invariant form.

In other words, we can think of elements of $H^{-2}(T^*_\sigma(\mathrm{LocSys}^{\mathrm{arithm}}_{\mathsf{G}}(X)))$ as elements

$$A \in H^0(X, \check{\mathfrak{g}}_\sigma)$$

such that

$$\mathrm{Frob}(A) = q \cdot A.$$

We note that such elements A is necessarily nilpotent.

16.5.3. Note that $\mathrm{LocSys}^{\mathrm{arithm}}_{\mathsf{G}}(X)$ does contain points σ which admit a non-zero A as above. Indeed, take σ to be geometrically trivial, fix an arbitrary non-zero nilpotent element

$$A \in \check{\mathfrak{g}} \simeq H^0(X, \check{\mathfrak{g}}_\sigma),$$

and let the Weil structure be given by the image of $q \in \mathbb{G}_m$ under

$$\mathbb{G}_m \to SL_2 \to \check{G},$$

where $SL_2 \to \check{G}$ is a Jacobson-Morozov map corresponding to A.

This implies that $\mathrm{LocSys}^{\mathrm{arithm}}_{\mathsf{G}}(X)$ is *non-classical*: a classical scheme with a perfect cotangent complex is an l.c.i and hence cannot have a non-trivial H^{-2} of its cotangent spaces.

16.5.4. Let $\mathcal{Z}$ be a quasi-quasi-smooth algebraic stack. Following a suggestion of D. Beraldo, one can mimic the construction of [AG, Sect. 2.3.3] and produce a classical algebraic stack, denoted $\mathrm{Sing}_2(\mathcal{Z})$, whose e-points are pairs

$$(z,\xi), \quad z \in \mathcal{Z},\, \xi \in H^{-2}(T^*_z(\mathcal{Z})).$$

16.5.5. We will denote:

$$\mathrm{Arth}^{\mathrm{arithm}}(X) := \mathrm{Sing}_2(\mathrm{LocSys}^{\mathrm{arithm}}_{\mathsf{G}}(X)),$$

and refer to it as the stack of *arithmetic Arthur parameters*.

Thus, the stack $\mathrm{Arth}^{\mathrm{arithm}}(X)$ projects to $\mathrm{LocSys}^{\mathrm{arithm}}_{\mathsf{G}}(X)$, and the fiber over a given σ is the vector space

$$A \in H^0(X, \check{\mathfrak{g}}_\sigma),\ \mathrm{Frob}(A) = q\cdot A.$$

Remark 16.5.6. The terminology "Arthur parameters" is justified as follows:

If σ is semi-simple (as a Weil local system), then using a Jacobson-Morozov argument, we can identify the *set*

$$\{A,\ \mathrm{Frob}(A) = q\cdot A\}/\,\mathrm{Ad}(\mathrm{Aut}(\sigma))$$

with the *set*

$$\{SL_2 \to \mathrm{Aut}(\sigma)\}/\,\mathrm{Ad}(\mathrm{Aut}(\sigma)).$$

(Note, however, nilpotent elements have more automorphisms than SL_2-triples.)

16.6. **Towards an explicit spectral description of the space of automorphic functions.** In this subsection we will assume two of our Main Conjectures, 15.3.5 and 14.2.4 and (try to) deduce consequences for Autom.

16.6.1. First, putting the above two conjectures together, we obtain:

Main Conjecture 16.6.2. *We have a canonical isomorphism*

$$\mathrm{Autom} \simeq \mathrm{Tr}(\mathrm{Frob}^*, \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X))).$$

Since Frob is an automorphism of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$, in the above conjecture we could replace the functor Frob^* by $\mathrm{Frob}^!$.

Thus, assuming the above conjecture, in order to describe Autom, we wish to have an explicit description of the object

$$\mathrm{Tr}(\mathrm{Frob}^!, \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X))) \in \mathrm{Vect}_{\mathsf{e}}\,.$$

The material below was obtained as a result of communications with D. Beraldo.

16.6.3. Let $\mathcal{Y}$ be a stack equipped with an endomorphism ϕ. Then according to [GKRV, Sect. 3.5.3], we have

$$\mathrm{Tr}(\phi^*, \mathrm{QCoh}(\mathcal{Y})) \simeq \Gamma(\mathcal{Y}^\phi, \mathcal{O}_{\mathcal{Y}^\phi}).$$

A parallel computation shows that

$$\mathrm{Tr}(\phi^!, \mathrm{IndCoh}(\mathcal{Y})) \simeq \Gamma(\mathcal{Y}^\phi, \omega_{\mathcal{Y}^\phi}).$$

Furthermore, we can place ourselves in the paradigm of Sect. 16.3.1, and consider $\mathrm{QCoh}(\mathcal{Y})$ and $\mathrm{IndCoh}(\mathcal{Y})$ as module categories over $\mathrm{QCoh}(\mathcal{Y})$, equipped with compatible endofunctors.

Thus, we can consider the objects

$$\mathrm{Tr}^{\mathrm{enh}}(\phi^*, \mathrm{QCoh}(\mathcal{Y})) \text{ and } \mathrm{Tr}^{\mathrm{enh}}(\phi^!, \mathrm{IndCoh}(\mathcal{Y}))$$

in $\mathrm{QCoh}(Y^\phi)$.

A computation similar to [GKRV, Sect. 3.5.3] shows that

$$\mathrm{Tr}^{\mathrm{enh}}(\phi^*, \mathrm{QCoh}(\mathcal{Y})) \simeq \mathcal{O}_{\mathcal{Y}^\phi} \text{ and } \mathrm{Tr}^{\mathrm{enh}}(\phi^!, \mathrm{IndCoh}(\mathcal{Y})) \simeq \omega_{\mathcal{Y}^\phi}, \tag{16.6}$$

as objects of $\mathrm{QCoh}(Y^\phi)$.

16.6.4. Assume now that $\mathcal{Y}$ is quasi-smooth. Let $\mathcal{N}$ be a conical Zariski-closed subset in $\mathrm{Sing}(\mathcal{Y})$. Assume that the codifferential map

$$\mathrm{Sing}(\phi) : \mathcal{Y} \underset{\phi,\mathcal{Y}}{\times} \mathrm{Sing}(\mathcal{Y}) \to \mathrm{Sing}(\mathcal{Y}),$$

sends $\mathcal{Y} \underset{\phi,\mathcal{Y}}{\times} \mathcal{N} \subset \mathcal{Y} \underset{\phi,\mathcal{Y}}{\times} \mathrm{Sing}(\mathcal{Y})$ to $\mathcal{N} \subset \mathrm{Sing}(\mathcal{Y})$, so that the functor $\phi^!$ sends

$$\mathrm{IndCoh}_{\mathcal{N}}(\mathcal{Y}) \to \mathrm{IndCoh}_{\mathcal{N}}(\mathcal{Y}),$$

see [AG, Proposition 7.1.3(a)].

Then it makes sense to consider

$$\mathrm{Tr}(\phi^!, \mathrm{IndCoh}_{\mathcal{N}}(\mathcal{Y})) \in \mathrm{Vect}_{\mathsf{e}}\,. \tag{16.7}$$

Furthermore, we can regard $\mathrm{IndCoh}_{\mathcal{N}}(\mathcal{Y})$ as a module category over $\mathrm{QCoh}(\mathcal{Y})$ and consider the object

$$\mathrm{Tr}^{\mathrm{enh}}(\phi^!, \mathrm{IndCoh}_{\mathcal{N}}(\mathcal{Y})) \in \mathrm{QCoh}(\mathcal{Y}^{\phi}),$$

so that by (16.2) we have

$$\mathrm{Tr}(\phi^!, \mathrm{IndCoh}_{\mathcal{N}}(\mathcal{Y})) \simeq \Gamma\left(\mathcal{Y}^{\phi}, \mathrm{Tr}^{\mathrm{enh}}(\phi^!, \mathrm{IndCoh}_{\mathcal{N}}(\mathcal{Y}))\right).$$

Remark 16.6.5. Unfortunately, we do not have an explicit answer for what the above object $\mathrm{Tr}^{\mathrm{enh}}(\phi^!, \mathrm{IndCoh}_{\mathcal{N}}(\mathcal{Y}))$ is in general. We expect, however, that one can give such an answer in terms of the subset

$$\mathcal{N}^{\phi} \subset \mathrm{Sing}_2(\mathcal{Y}^{\phi}),$$

defined in (16.11) below.

Yet, we know some particular cases: by (16.6), we have

$$\mathrm{Tr}^{\mathrm{enh}}(\phi^!, \mathrm{IndCoh}_{\{0\}}(\mathcal{Y})) \simeq \mathcal{O}_{\mathcal{Y}^{\phi}} \tag{16.8}$$

and

$$\mathrm{Tr}^{\mathrm{enh}}(\phi^!, \mathrm{IndCoh}(\mathcal{Y})) \simeq \omega_{\mathcal{Y}^{\phi}}, \tag{16.9}$$

16.6.6. Note now that for $\mathcal{Y}$ quasi-smooth, the stack $\mathcal{Y}^{\phi}$ is quasi-quasi-smooth and

$$\mathrm{Sing}_2(\mathcal{Y}^{\phi}) := \{y \in \mathcal{Y},\ \phi(y) \sim y,\ \xi \in H^{-1}(T^*_y(\mathcal{Y})),\ \mathrm{Sing}(\phi)(\xi) = \xi\}. \tag{16.10}$$

Let $\mathcal{N} \subset \mathrm{Sing}(\mathcal{Y})$ be as Sect. 16.6.4. Set

$$\mathcal{N}^{\phi} \subset \mathrm{Sing}_2(\mathcal{Y}^{\phi}) \tag{16.11}$$

be the subset that in terms of (16.10) corresponds to the condition that $\xi \in \mathcal{N} \underset{\mathcal{Y}}{\times} \{y\}$.

We propose:

Conjecture 16.6.7. *Suppose that for a pair of conical subsets $\mathcal{N}_1 \subset \mathcal{N}_2$ as above, the inclusion*

$$\mathcal{N}_1^{\phi} \subset \mathcal{N}_2^{\phi}$$

is an equality. Then the inclusion functor

$$\mathrm{IndCoh}_{\mathcal{N}_1}(\mathcal{Y}) \hookrightarrow \mathrm{IndCoh}_{\mathcal{N}_2}((\mathcal{Y})$$

defines an isomorphism

$$\mathrm{Tr}^{\mathrm{enh}}(\phi^!, \mathrm{IndCoh}_{\mathcal{N}_1}(\mathcal{Y})) \simeq \mathrm{Tr}^{\mathrm{enh}}(\phi^!, \mathrm{IndCoh}_{\mathcal{N}_2}(\mathcal{Y}))$$

in $\mathrm{QCoh}(\mathcal{Y}^{\phi})$.

This conjecture is not far-fetched, and might have been already established in the works of D. Beraldo.

As a particular case, and combining with (16.9) we obtain:

Corollary-of-Conjecture 16.6.8. *Suppose that for* $\mathcal{N}$ *as above, the inclusion*

$$\mathcal{N}^\phi \subset \operatorname{Sing}_2(\mathcal{Y}^\phi)$$

is an equality. Then the inclusion functor

$$\operatorname{IndCoh}_{\mathcal{N}}(\mathcal{Y}) \hookrightarrow \operatorname{IndCoh}(\mathcal{Y})$$

defines an isomorphism

$$\operatorname{Tr}^{\mathrm{enh}}(\phi^!, \operatorname{IndCoh}_{\mathcal{N}}(\mathcal{Y})) \simeq \operatorname{Tr}^{\mathrm{enh}}(\phi^!, \operatorname{IndCoh}(\mathcal{Y})) \simeq \omega_{\mathcal{Y}^\phi}$$

in $\operatorname{QCoh}(\mathcal{Y}^\phi)$.

16.6.9. We apply this to $\mathcal{Y} = \operatorname{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$ with $\phi = \operatorname{Frob}$. We note that the inclusion

$$\operatorname{Nilp}^{\operatorname{Frob}} \hookrightarrow \operatorname{Sing}_2(\operatorname{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)) = \operatorname{Arth}^{\mathrm{arithm}}(X)$$

is indeed an equality.

Hence, combining Conjecture 14.2.4 with Corollary 16.6.8, we obtain:

Main Conjecture 16.6.10. *We have a canonical isomorphism in* $\operatorname{QCoh}(\operatorname{LocSys}^{\mathrm{arithm}}_{\check{G}}(X))$*:*

$$\operatorname{Drinf} \simeq \omega_{\operatorname{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)}.$$

Taking global sections over $\operatorname{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)$, and taking into account Conjecture 15.3.5, we obtain:

Main Conjecture 16.6.11. *We have a canonical isomorphism*

$$\operatorname{Autom} \simeq \Gamma(\operatorname{LocSys}^{\mathrm{arithm}}_{\check{G}}(X), \omega_{\operatorname{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)}).$$

17. Proofs of Theorems 16.1.4 and 16.1.6

17.1. **Proof of Theorem 16.1.4.** In this subsection we will prove Theorem 16.1.4.

The key ingredient will be provided by Theorem 4.4.2, which gives us a handle on "how far is $\operatorname{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ from being an algebraic stack", combined with some fundamental facts from algebraic geometry pertaining to Weil sheaves on curves (specifically, Weil-II and L. Lafforgue's theorem, which says that every irreducible Weil local system is pure).

17.1.1. First off, the assertion that $(\operatorname{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X))^{\operatorname{Frob}}$ is mock-affine and locally almost of finite type as a prestack follows from the corresponding property of $\operatorname{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$.

To prove the remaining assertions of the theorem, by Proposition 2.2.8, it suffices to consider the case of $\mathsf{G} = GL_n$.

The assertion of the theorem can be broken into two parts:

(a) There are only finitely many connected components of $\operatorname{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ that are invariant under Frob.

(b) For each connected component $\mathcal{Z}$ of $\operatorname{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$, the fiber product

$$(\operatorname{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X))^{\operatorname{Frob}} \underset{\operatorname{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)}{\times} \mathcal{Z}$$

is an algebraic stack (as opposed to an ind-algebraic stack, see Sect. 4.2).

17.1.2. We start by proving (a). Recall that connected components of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ are in bijection with the set of association of pairs $(\mathsf{P}, \sigma_{\mathsf{M}})$, where $\mathsf{P} \subset \mathsf{G}$ is a parabolic, and σ_{M} is an irreducible local system with respect to the Levi quotient M of P.

Let our parabolic be given by the partition

$$n = (n_1 + ... + n_1 + n_2 + ... + n_2 + ... + n_k + ... + n_k), \quad n_i \neq n_j$$

with n_i appearing m_i times.

An irreducible local system for the corresponding Levi is given by a collection of

$$(\sigma^1_{n_1}, ..., \sigma^{m_1}_{n_1}, \sigma^1_{n_2}, ..., \sigma^{m_2}_{n_2}, \sigma^1_{n_k}, ..., \sigma^{m_k}_{n_k}), \tag{17.1}$$

where each $\sigma^?_{n_i}$ is an irreducible local system of rank n_i.

We claim that there is only a finite number of possibilities for such a string (17.1), provided that its class of association is invariant under the Frobenius.

Indeed, the class of association of an M-local system as above is invariant under the Frobenius if for every $j = 1, ..., k$ there exists an element $g_j \in \Sigma_{m_j}$ (the symmetric group on m_j letters) such that

$$(\mathrm{Frob}_X(\sigma^1_{n_j}), ..., \mathrm{Frob}_X(\sigma^{m_j}_{n_j})) = (\sigma^{g_j(1)}_{n_j}, ..., \sigma^{g_j(m_j)}_{n_j}).$$

For every j let $d_j := \mathrm{ord}(g_j)$. We obtain that all local systems $\sigma^?_{n_j}$ are invariant under $(\mathrm{Frob}_X)^{d_j}$. I.e., each such local system is a geometrically irreducible local system that can be equipped with a Weil structure (with respect to $\mathbb{F}_{q^d}$).

We claim that the number of isomorphism classes of such local systems is finite.

To prove this, it suffices to show that the number of irreducible Weil local systems of a given rank r and a fixed determinant is finite. The latter follows from L. Lafforgue's theorem ([LLaf]), which says that such Weil local systems are in bijection with unramified cuspidal automorphic representations of GL_r with a fixed central character. Now, the number of such automorphic representations (for a given function field) is finite.

17.1.3. We now start tackling point (b) from Sect. 17.1.1.

Let $\mathcal{Z}$ be a connected component of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ invariant under the Frobenius. Denote

$$\mathcal{Z}^{\mathrm{rigid}_x} := \mathcal{Z} \underset{\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)}{\times} \mathrm{LocSys}^{\mathrm{restr},\mathrm{rigid}_x}_{\mathsf{G}}(X).$$

It is enough to show that

$$((\mathcal{Z})^{\mathrm{Frob}})^{\mathrm{rigid}_x} := (\mathcal{Z})^{\mathrm{Frob}} \underset{\mathcal{Z}}{\times} \mathcal{Z}^{\mathrm{rigid}_x} \tag{17.2}$$

is an affine scheme; a priori we know that it is an ind-affine ind-scheme.

It follows from Corollary 1.6.4 that $((\mathcal{Z})^{\mathrm{Frob}})^{\mathrm{rigid}_x}$ has a connective corepresentable deformation theory. Therefore, by [Lu3, Theorem 18.1.0.1], it suffices to show that ${}^{\mathrm{cl}}(((\mathcal{Z})^{\mathrm{Frob}})^{\mathrm{rigid}_x})$ is a classical affine scheme. Equivalently, it suffices to show that the underlying classical prestack of $\mathcal{Z}^{\mathrm{Frob}}$ itself is a classical algebraic stack (as opposed to an ind-algebraic stack).

17.1.4. Set

$$\mathcal{W}^{\mathrm{rigid}_x} := \mathrm{pt} \underset{\mathcal{Z}^{\mathrm{coarse}}}{\times} \mathcal{Z}^{\mathrm{rigid}_x},$$

and set

$$\mathcal{W} := \mathcal{W}^{\mathrm{rigid}_x}/\mathsf{G} \simeq \mathrm{pt} \underset{\mathcal{Z}^{\mathrm{coarse}}}{\times} \mathcal{Z}.$$

We will prove:

Proposition 17.1.5. *The map $(\mathcal{W})^{\mathrm{Frob}} \to (\mathcal{Z})^{\mathrm{Frob}}$ induces an isomorphism of the underlying classical prestacks.*

This proposition immediately implies that ${}^{\mathrm{cl}}((\mathcal{Z})^{\mathrm{Frob}})$ is an algebraic stack, since we know that $\mathcal{W}$ (and hence $(\mathcal{W})^{\mathrm{Frob}}$) is an algebraic stack, by Corollary 4.4.4.

17.1.6. Note that on the level of the underlying classical prestacks, the map

$$\mathrm{pt} \to \mathcal{Z}^{\mathrm{coarse}}$$

is fully faithful (since $\mathcal{Z}^{\mathrm{coarse}}$ is a derived scheme).

Hence, the assertion of Proposition 17.1.5 is equivalent to the following one:

Proposition 17.1.7. *The composition*

$$(\mathcal{Z})^{\mathrm{Frob}} \to \mathcal{Z} \overset{r}{\to} \mathcal{Z}^{\mathrm{coarse}} \tag{17.3}$$

factors as

$$(\mathcal{Z})^{\mathrm{Frob}} \to \mathrm{pt} \to \mathcal{Z}^{\mathrm{coarse}} \tag{17.4}$$

at the level of the underlying classical prestacks.

17.2. **Proof of Proposition 17.1.7: the pure case.** Proposition 17.1.7 says that all global functions on $\mathcal{Z}$ are constant, when restricted to $(\mathcal{Z})^{\mathrm{Frob}}$.

In this subsection we will prove this assertion on a neighborhood of a point of $(\mathcal{Z})^{\mathrm{Frob}}$ that corresponds to a *pure* local system. The proof will use Weil-II.

17.2.1. Let

$$S \to (\mathcal{Z})^{\mathrm{Frob}} \tag{17.5}$$

be a map, where $S = \mathrm{Spec}(A)$ with A classical Artinian.

It suffices to show that for any such map, the composition

$$S \to (\mathcal{Z})^{\mathrm{Frob}} \to \mathcal{Z} \overset{r}{\to} \mathcal{Z}^{\mathrm{coarse}} \tag{17.6}$$

factors as

$$S \to \mathrm{pt} \to \mathcal{Z}^{\mathrm{coarse}}. \tag{17.7}$$

We can think of (17.5) as a local system E_A on X, endowed with a Weil structure, and equipped with an action of A, whose fiber at $x \in X$ is a (locally) free A-module of rank n.

17.2.2. Let E be the Weil local system corresponding to the composition

$$\mathrm{pt} \to S \to (\mathcal{Z})^{\mathrm{Frob}}.$$

Let us first consider the case when E is *pure of weight* 0 (with respect to some identification $\overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$).

Let $\overline{E}$ denote the underlying local system, when we forget the Weil structure. Let $\mathrm{Aut}(\overline{E})$ denote the *classical* algebraic group of automorphisms of $\overline{E}$.

Varying the Weil structure on $\overline{E}$ defines a map

$$\mathrm{Aut}(\overline{E})/\,\mathrm{Ad}_{\mathrm{Frob}}(\mathrm{Aut}(\overline{E})) \to (\mathcal{Z})^{\mathrm{Frob}}, \tag{17.8}$$

where $\mathrm{Ad}_{\mathrm{Frob}}(\mathrm{Aut}(\overline{E}))$ stands for the action of $\mathrm{Aut}(\overline{E})$ on itself given by

$$g(g_1) = \mathrm{Frob}(g) \cdot g_1 \cdot g^{-1},$$

and where Frob is the automorphism of $\mathrm{Aut}(\overline{E})$ induced by

$$\mathrm{Aut}(\overline{E}) \overset{\text{functoriality}}{\longrightarrow} \mathrm{Aut}(\mathrm{Frob}(\overline{E})) \overset{\text{Weil structure}}{\longrightarrow} \mathrm{Aut}(\overline{E}).$$

17.2.3. We claim that the map (17.8) defines a formal isomorphism at E. In order to prove this, it suffices to show that the map (17.8) induces an isomorphism at the level of tangent spaces.

We have:

$$T_1(\mathrm{Aut}(\overline{E})) \simeq H^0(X, \mathrm{End}(\overline{E})),$$

and

$$T_1(\mathrm{Aut}(\overline{E})/\,\mathrm{Ad}_{\mathrm{Frob}}(\mathrm{Aut}(\overline{E}))) \simeq \mathrm{coFib}\left(H^0(X, \mathrm{End}(\overline{E})) \overset{\mathrm{Frob}-\mathrm{Id}}{\longrightarrow} H^0(X, \mathrm{End}(\overline{E}))\right).$$

We also have

$$T_E((\mathcal{Z})^{\mathrm{Frob}}) \simeq \mathrm{Fib}\left(T_{\overline{E}}(\mathcal{Z}) \overset{\mathrm{Frob}-\mathrm{Id}}{\longrightarrow} T_{\overline{E}}(\mathcal{Z})\right),$$

where

$$T_{\overline{E}}(\mathcal{Z}) \simeq \mathrm{C}^\cdot(X, \mathrm{End}(\overline{E}))[1].$$

The map that (17.8) induces at the level of tangent spaces corresponds to canonical map

$$H^0(X, \mathrm{End}(\overline{E})) \to \mathrm{C}^\cdot(X, \mathrm{End}(\overline{E})).$$

Hence, in order to show that

$$T_1(\mathrm{Aut}(\overline{E})/\,\mathrm{Ad}_{\mathrm{Frob}}(\mathrm{Aut}(\overline{E}))) \to T_E((\mathcal{Z})^{\mathrm{Frob}})$$

is an isomorphism, it suffices to show that $\mathrm{Frob}-\mathrm{Id}$ induces an isomorphism on $H^1(X, \mathrm{End}(\overline{E}))$ and $H^2(X, \mathrm{End}(\overline{E}))$. In other words, we have to show that Frob does not have eigenvalue 1 on either $H^1(X, \mathrm{End}(\overline{E}))$ or $H^2(X, \mathrm{End}(\overline{E}))$.

17.2.4. We will now use the assumption that E is pure of weight 0.

This assumption implies that the induced Weil structure on $\mathrm{End}(\overline{E})$ is also pure of weight 0. Hence, the eigenvalues of Frob on $H^1(X, \mathrm{End}(\overline{E}))$ (resp., $H^2(X, \mathrm{End}(\overline{E}))$) are algebraic numbers that under any complex embedding have Archimedean absolute values $q^{\frac{1}{2}}$ (resp., q).

In particular, these eigenvalues are different from 1.

17.2.5. Thus, we have established that the map (17.8) is a formal isomorphism at E. Hence, by deformation theory, the initial map

$$S \to (\mathcal{Z})^{\mathrm{Frob}}$$

of (17.5) can be lifted to a map

$$S \to \mathrm{Aut}(\overline{E})/\,\mathrm{Ad}_{\mathrm{Frob}}(\mathrm{Aut}(\overline{E})).$$

However, the composite map

$$\mathrm{Aut}(\overline{E})/\,\mathrm{Ad}_{\mathrm{Frob}}(\mathrm{Aut}(\overline{E})) \to (\mathcal{Z})^{\mathrm{Frob}} \to (\mathcal{Z})^{\mathrm{Frob}} \to \mathcal{Z}$$

by definition factors as

$$\mathrm{Aut}(\overline{E})/\,\mathrm{Ad}_{\mathrm{Frob}}(\mathrm{Aut}(\overline{E})) \to \mathrm{pt}/\,\mathrm{Aut}(\overline{E}) \to \mathcal{Z},$$

while the composition

$$\mathrm{pt}/\,\mathrm{Aut}(\overline{E}) \to \mathcal{Z} \to \mathcal{Z}^{\mathrm{coarse}}$$

factors as

$$\mathrm{pt}/\,\mathrm{Aut}(\overline{E}) \to \mathrm{pt} \to \mathcal{Z}^{\mathrm{coarse}}.$$

This proves the required factorization of (17.6) as (17.7) (in the case when E was pure of weight 0).

17.3. **Reduction to the pure case.** Above we have established the factorization of (17.6) as (17.7) when the Weil local system E was pure of weight 0.

In this subsection we will perform the reduction to this case. The source of pure local systems will be provided by the theorem of L. Lafforgue, which says that every irreducible Weil local system is pure.

17.3.1. Let us choose an isomorphism

$$\overline{\mathbb{Q}}_\ell \simeq \mathbb{C}, \tag{17.9}$$

so we can assign the Archimedean absolute value to elements of $\overline{\mathbb{Q}}_\ell$. With this choice, we claim that every Weil local system E' on X acquires a *canonical* (weight) filtration, indexed by real numbers

$$0 \subset ... \subset E'_{r_1} \subset E'_{r_2} \subset ... \subset E',$$

such that each subquotient

$$\mathrm{gr}_r(E')$$

is "pure of weight r" in the sense that it is of the form

$$E_0 \otimes \ell_r, \tag{17.10}$$

where:

- E_0 is pure of weight 0 (with respect to (17.9));
- ℓ_r is a character of $\mathbb{Z} = \mathrm{Weil}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$, on which the generator $1 \in \mathbb{Z}$ acts by a scalar with Archimedean absolute value $q^{\frac{r}{2}}$.

Moreover, this filtration is functorial in E' and is compatible with tensor products.

The existence and properties of such a filtration follow from the combination of the following three facts:

- For two local systems $E_0^1 \otimes \ell_{r^1}$ and $E_0^2 \otimes \ell_{r^2}$ of the form (17.10),
$$r_1 \neq r_2 \Rightarrow \mathrm{Hom}(E_0^1 \otimes \ell_{r^1}, E_0^2 \otimes \ell_{r^2}) = 0.$$
- For two local systems $E_0^1 \otimes \ell_{r^1}$ and $E_0^2 \otimes \ell_{r^2}$ of the form (17.10),
$$r_1 < r_2 \Rightarrow \mathrm{Ext}^1(E_0^1 \otimes \ell_{r^1}, E_0^2 \otimes \ell_{r^2}) = 0.$$
This follows from [De].
- Every irreducible Weil local system on X is of the form (17.10). This is a theorem of L. Lafforgue, [LLaf].

17.3.2. Applying this construction to $E' = E_A$, we obtain a filtration

$$0 = E_{A,0} \subset E_{A,1} \subset ... \subset E_{A,k} = E_A$$

by Weil local systems, stable under the action of A.

We claim that the fibers of $\mathrm{gr}_i(E_A)$ at x are (locally) free over A. For that end, it suffices to show that the induced filtration on $\mathrm{ev}_x(E_A)$ canonically splits.

Indeed, let d be such that x is defined over $\overline{\mathbb{F}}_{q^d}$. Then Frob^d acts on $\mathrm{ev}_x(E_A)$, and its action on the different subquotients

$$\mathrm{gr}_i(E_A)$$

has distinct generalized eigenvalues.

17.3.3. Thus, we obtain that we can lift our initial S-point of $(\mathcal{Z})^{\mathrm{Frob}}$ to a point of

$$(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{P}}(X))^{\mathrm{Frob}},$$

where P is the parabolic corresponding to the ranks of $\mathrm{gr}_i(E_A)$.

Let $\mathcal{Z}_{\mathsf{P}}$ denote the corresponding connected component of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{P}}(X)$, i.e., we now have a map

$$S \to (\mathcal{Z}_{\mathsf{P}})^{\mathrm{Frob}}. \tag{17.11}$$

It suffices to show that the composition

$$S \overset{(17.11)}{\longrightarrow} (\mathcal{Z}_{\mathsf{P}})^{\mathrm{Frob}} \to \mathcal{Z}_{\mathsf{P}} \to \mathcal{Z} \to \mathcal{Z}^{\mathrm{coarse}} \tag{17.12}$$

factors as

$$S \to \mathrm{pt} \to \mathcal{Z}^{\mathrm{coarse}}. \tag{17.13}$$

17.3.4. In what follows we will wish to consider the coarse moduli space corresponding to $\mathcal{Z}_{\mathsf{P}}$. The slight inconvenience is that $\mathcal{Z}_{\mathsf{P}}$ is *not* ind mock-affine (because P is not reductive). We will overcome this as follows.

Set

$$\mathcal{Z}_{\mathsf{P}}^{\text{unip-rigid}_x} \simeq \mathcal{Z}_{\mathsf{P}} \underset{\mathrm{pt}/\mathsf{P}}{\times} \mathrm{pt}/\mathsf{M}.$$

The map

$$\mathrm{pt}/\mathsf{M} \to \mathrm{pt}/\mathsf{P}$$

is smooth, so the map

$$S \overset{(17.11)}{\longrightarrow} (\mathcal{Z}_{\mathsf{P}})^{\mathrm{Frob}}$$

can be lifted to a map

$$S \to (\mathcal{Z}_{\mathsf{P}})^{\mathrm{Frob}} \underset{\mathcal{Z}_{\mathsf{P}}}{\times} \mathcal{Z}_{\mathsf{P}}^{\text{unip-rigid}_x}.$$

It suffices to show that the composition

$$S \to (\mathcal{Z}_{\mathsf{P}})^{\mathrm{Frob}} \underset{\mathcal{Z}_{\mathsf{P}}}{\times} \mathcal{Z}_{\mathsf{P}}^{\text{unip-rigid}_x} \to \mathcal{Z}_{\mathsf{P}}^{\text{unip-rigid}_x} \to \mathcal{Z}_{\mathsf{P}} \to \mathcal{Z} \to \mathcal{Z}^{\text{coarse}} \tag{17.14}$$

factors as

$$S \to \mathrm{pt} \to \mathcal{Z}^{\text{coarse}}. \tag{17.15}$$

17.3.5. Since

$$\mathcal{Z}_{\mathsf{P}}^{\text{rigid}_x} := \mathcal{Z}_{\mathsf{P}} \underset{\mathrm{pt}/\mathsf{P}}{\times} \mathrm{pt}$$

is ind-affine ind-scheme, and M is reductive, the ind-algebraic stack $\mathcal{Z}_{\mathsf{P}}^{\text{unip-rigid}_x}$ is ind mock-affine. Hence, we have the well-defined ind-affine ind-scheme

$$\mathcal{Z}_{\mathsf{P}}^{\text{unip-rigid}_x,\text{coarse}},$$

and by construction, any map

$$\mathcal{Z}_{\mathsf{P}}^{\text{unip-rigid}_x} \to \mathcal{U},$$

where $\mathcal{U}$ is an ind-affine ind-scheme, factors as

$$\mathcal{Z}_{\mathsf{P}}^{\text{unip-rigid}_x} \to \mathcal{Z}_{\mathsf{P}}^{\text{unip-rigid}_x,\text{coarse}} \to \mathcal{U}.$$

17.3.6. In particular, the map

$$\mathcal{Z}_{\mathsf{P}}^{\text{unip-rigid}_x} \to \mathcal{Z}_{\mathsf{P}} \to \mathcal{Z} \to \mathcal{Z}^{\text{coarse}}$$

that appears in (17.14) factors as

$$\mathcal{Z}_{\mathsf{P}}^{\text{unip-rigid}_x} \to \mathcal{Z}_{\mathsf{P}}^{\text{unip-rigid}_x,\text{coarse}} \to \mathcal{Z}^{\text{coarse}}.$$

Hence, it suffices to show that the composition

$$S \to (\mathcal{Z}_{\mathsf{P}})^{\mathrm{Frob}} \underset{\mathcal{Z}_{\mathsf{P}}}{\times} \mathcal{Z}_{\mathsf{P}}^{\text{unip-rigid}_x} \to \mathcal{Z}_{\mathsf{P}}^{\text{unip-rigid}_x} \to \mathcal{Z}_{\mathsf{P}}^{\text{unip-rigid}_x,\text{coarse}} \tag{17.16}$$

factors as

$$S \to \mathrm{pt} \to \mathcal{Z}_{\mathsf{P}}^{\text{unip-rigid}_x,\text{coarse}}, \tag{17.17}$$

where

$$\mathrm{pt} \to \mathcal{Z}_{\mathsf{P}}^{\text{unip-rigid}_x,\text{coarse}}$$

is the unique e-point of $\mathcal{Z}_{\mathsf{P}}^{\text{unip-rigid}_x,\text{coarse}}$, see isomorphism (17.18) below.

17.3.7. Let $\mathcal{Z}_{\mathsf{M}}$ be the connected component of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{M}}(X)$, corresponding to $\mathcal{Z}_{\mathsf{P}}$. By the argument in Sect. 4.1.9, the projection

$$\mathcal{Z}_{\mathsf{P}}^{\text{unip-rigid}_x} \to \mathcal{Z}_{\mathsf{M}}$$

induces an isomorphism

$$\mathcal{Z}_{\mathsf{P}}^{\text{unip-rigid}_x,\mathrm{coarse}} \simeq \mathcal{Z}_{\mathsf{M}}^{\mathrm{coarse}}. \tag{17.18}$$

Hence, it suffices to show that the composition

$$S \overset{(17.11)}{\longrightarrow} (\mathcal{Z}_{\mathsf{P}})^{\mathrm{Frob}} \to (\mathcal{Z}_{\mathsf{M}})^{\mathrm{Frob}} \to \mathcal{Z}_{\mathsf{M}} \to \mathcal{Z}_{\mathsf{M}}^{\mathrm{coarse}}$$

factors as

$$S \to \mathrm{pt} \to \mathcal{Z}_{\mathsf{M}}^{\mathrm{coarse}}.$$

17.3.8. Write

$$\mathsf{M} = \prod_i GL_{n_i},$$

so it is enough to prove the corresponding factorization assertion for each of the GL_{n_i} factors separately.

However, by the assumption on $\mathrm{gr}_i(E_A)$, this reduces us to the pure of weight 0 case considered in Sect. 17.2. Indeed, the resulting local systems $\mathrm{gr}_i(E_A)$ are pure of weight 0 (up to a twist by a line).

□[Theorem 16.1.4]

17.4. **Proof of Theorem 16.1.6.**

17.4.1. To prove the theorem, it suffices to show that for an irreducible Weil local system σ, the tangent space

$$T_\sigma\left((\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X))^{\mathrm{Frob}}\right) = 0.$$

We have:

$$T_\sigma\left((\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X))^{\mathrm{Frob}}\right) \simeq \mathrm{Fib}(T_\sigma\left(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)\right) \overset{\mathrm{Frob}-\mathrm{Id}}{\longrightarrow} T_\sigma\left(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)\right),$$

while

$$T_\sigma\left(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)\right) \simeq \mathrm{C}^{\cdot}(X, \mathfrak{g}_\sigma)[1].$$

So, it is enough to show that Frob does not have fixed vectors when acting on $H^i(X, \mathfrak{g}_\sigma)$, $i = 0, 1, 2$.

17.4.2. We first consider the case of $i = 0$.

Note that

$$(H^0(X, \mathfrak{g}_\sigma))^{\mathrm{Frob}}$$

is the Lie algebra of the *classical* group of automorphisms of σ as a Weil local system.

If this group has a non-trivial connected component, a standard argument implies that σ can be reduced to a proper parabolic.

17.4.3. To treat the cases $i = 1, 2$, it suffices to prove the following:

Proposition 17.4.4. *Let* G *be a semi-simple group and let* σ *be an irreducible Weil* G*-local system. Then for any* $V \in \mathrm{Rep}(\mathsf{G})^{c,\heartsuit}$*, the associated Weil local system* V_σ *is pure of weight* 0.

This proposition is likely well-known. We will provide a proof for completeness.

17.4.5. First, we recall the following general construction:

Let

$$\mathrm{Fil}_{\mathbb{R}}(\mathrm{Vect}_{\mathsf{e}}^{c,\heartsuit})$$

be the abelian symmetric monoidal category consisting of finite-dimensional vector spaces, endowed with a filtration indexed by the real numbers.

Note that the datum of a lift of a symmetric monoidal functor

$$\mathsf{F}_\sigma : \mathrm{Rep}(\mathsf{G})^{c,\heartsuit} \to \mathrm{Vect}_{\mathsf{e}}^{c,\heartsuit},$$

to a symmetric monoidal functor

$$\mathsf{F}_\sigma^{\mathrm{Fil}_{\mathbb{R}}} : \mathrm{Rep}(\mathsf{G})^{c,\heartsuit} \to \mathrm{Fil}_{\mathbb{R}}(\mathrm{Vect}_{\mathsf{e}}^{c,\heartsuit})$$

is equivalent to the datum of a reduction of σ to a parabolic P (denote its Levi quotient by M) and an element $\lambda \in \pi_{1,\mathrm{alg}}(Z^0_{\mathsf{M}}) \otimes \mathbb{R}$, such that for any $V \in \mathrm{Rep}(\mathsf{G})^{c,\heartsuit}$, the filtration on $\mathsf{F}_\sigma(V)$ is recovered as follows:

The reduction of σ to a parabolic P defines a filtration on each $\mathsf{F}_\sigma(V)$, indexed by the poset of characters of Z^0_{M},

$$(\mathsf{F}_\sigma(V))_\chi \subset \mathsf{F}_\sigma(V), \quad \chi \in \mathrm{Hom}(Z^0_{\mathsf{M}}, \mathbb{G}_m),$$

such that for a given character χ, the action of P on the subquotient

$$\mathrm{gr}_\chi(\mathsf{F}_\sigma(V))$$

factors through M with Z^0_{M} acting by χ.

Now for $r \in \mathbb{R}$, the subspace

$$(\mathsf{F}_\sigma(V))_r \subset \mathsf{F}_\sigma(V)$$

is the sum of the subspaces

$$(\mathsf{F}_\sigma(V))_\chi, \quad \langle \chi, \lambda \rangle \leq r.$$

In particular, if G is semi-simple, then $\mathsf{P} = \mathsf{G}$ if and only the lift $\mathsf{F}_\sigma^{\mathrm{Fil}_{\mathbb{R}}}$ is trivial, i.e., for every $V \in \mathrm{Rep}(\mathsf{G})^{c,\heartsuit}$

$$\mathrm{gr}_r(\mathsf{F}_\sigma(V)) = 0 \text{ for } r \neq 0. \tag{17.19}$$

This construction is functorial. In particular, if a group acts on F_σ in a way preserving its lift to $\mathsf{F}_\sigma^{\mathrm{Fil}_{\mathbb{R}}}$, then the action of this group on the initial G-torsor is induced by its action on the resulting P-torsor.

Proof of Proposition 17.4.4. Let $\mathrm{Gal}(X,x)^W$ be the Tannakian pro-algebraic group corresponding to the (abelian) symmetric monoidal category of Weil local systems on X, equipped with the fiber functor given by ev_x.

The datum of σ can viewed as a datum of a symmetric monoidal functor

$$\mathsf{F}_\sigma : \mathrm{Rep}(\mathsf{G})^{c,\heartsuit} \to \mathrm{Vect}_{\mathsf{e}}^{c,\heartsuit},$$

acted on by $\mathrm{Gal}(X,x)^W$.

Recall the setting of Sect. 17.3.1. We obtain that the canonical weight filtration on the Weil local systems V_σ defines a reduction of σ to a parabolic P. Since σ was assumed irreducible, we obtain that $\mathsf{P} = \mathsf{G}$. By (17.19), this implies

$$\mathrm{gr}_r(V_\sigma) = 0 \text{ for } r \neq 0.$$

I.e., all V_σ are pure of weight 0 as required.

$\square$

$\square$[Theorem 16.1.6]

Appendix A. Formal affine schemes vs ind-schemes

In this section we will outline the proof of Theorem 2.1.4.

A.1. **Creating the ring.** In this subsection we will state (a particular case of) [Lu3, Theorem 18.2.3.2] and deduce from it our Theorem 2.1.4.

A.1.1. Ley $\mathcal{Y}$ be an ind-affine ind-scheme, and write it as

$$\mathcal{Y} \simeq \underset{i \in I}{\operatorname{colim}} \, Y_i,$$

where:

- I is a filtered index category;
- $Y_i = \operatorname{Spec}(R_i)$'s are derived affine schemes almost of finite type;
- The transition maps $Y_i \to Y_j$ are closed embeddings, i.e., the corresponding maps $R_j \to R_i$ induce surjective maps $H^0(R_j) \twoheadrightarrow H^0(R_i)$.

We can form a commutative ring

$$R := \lim_{i \in I} R_j.$$

However, in general, we would not be able to say much about this R; in particular, we do not know that it is connective.

A.1.2. Assume now that $\mathcal{Y}$ is as in Theorem 2.1.4, i.e.,

- ${}^{\mathrm{red}}\mathcal{Y}$ is an affine scheme (to be denoted $Y_{\mathrm{red}} = \operatorname{Spec}(R_{\mathrm{red}})$);
- $\mathcal{Y}$ admits a corepresentable deformation theory, i.e., for any $(S, y) \in \mathrm{Sch}^{\mathrm{aff}}_{/\mathcal{Y}}$, the cotangent space $T^*_y(\mathcal{Y}) \in \operatorname{Pro}(\operatorname{QCoh}(S)^{\leq 0})$ actually belongs to $\operatorname{QCoh}(S)^{\leq 0}$.

In this case we claim:

Theorem A.1.3.

(a) *The ring R is connective. Furthermore, for every n, the natural map*

$$\tau^{\geq -n}(R) \to \lim_{i \in I} \tau^{\geq n}(R_j)$$

is an isomorphism.

(b) *The ideal $I := \ker(H^0(R) \to R_{\mathrm{red}})$ is finitely generated.*

(c) *The map*

$$\mathcal{Y} \to \operatorname{Spec}(R)^{\wedge}_{\operatorname{Spec}(R_{\mathrm{red}})}$$

is an isomorphism.

In the above formula, for a prestack $\mathcal{W}$ and a classical reduced prestack $\mathcal{W}^0 \to \mathcal{W}$, we denote by $\mathcal{W}^{\wedge}_{\mathcal{W}^0}$ the completion of $\mathcal{W}$ along $\mathcal{W}_0$, i.e.,

$$\operatorname{Maps}(S, \mathcal{W}^{\wedge}_{\mathcal{W}^0}) = \operatorname{Maps}(S, \mathcal{W}) \underset{\operatorname{Maps}({}^{\mathrm{red}}S, \mathcal{W})}{\times} \operatorname{Maps}({}^{\mathrm{red}}S, \mathcal{W}^0).$$

A.1.4. The assertion of Theorem A.1.3 implies that of Theorem 2.1.4. Indeed, the possibility to write $\mathcal{Y}$ as a colimit (1.3) is the content of [GR3, Proposition 6.7.4].

The rest of this section is devoted to the proof of Theorem A.1.3.

A.2. **Analysis of the classical truncation.**

A.2.1. For every index i, let I_i denote the ideal

$$\ker(H^0(R_i) \to R_{\mathrm{red}}).$$

For an integer n, we can consider its n-th power $I_i^n \subset H^0(R_i)$. We claim:

Proposition A.2.2. *For every n, the I-family*

$$i \mapsto H^0(R_i)/I_i^n$$

stabilizes.

Proof. The proof proceeds by induction on m for $n = 2^m$. We first consider the base case $m = 1$, so $n = 2$.

Thus, we wish to follow that the family

$$i \mapsto I_i/I_i^2$$

stabilizes.

For every i, consider

$$\mathrm{Fib}(T^*(Y_i)|_{Y_{\mathrm{red}}} \to T^*(Y_{\mathrm{red}})) \in \mathrm{QCoh}(Y_{\mathrm{red}})^{\leq 0}.$$

By the assumption on $\mathcal{Y}$, the inverse system

$$i \mapsto \mathrm{Fib}(T^*(Y_i)|_{Y_{\mathrm{red}}} \to T^*(Y_{\mathrm{red}}))$$

is equivalent to a constant object of $\mathrm{QCoh}(Y_{\mathrm{red}})^{\leq 0}$.

Hence, the inverse system

$$i \mapsto H^0\left(\mathrm{Fib}(T^*(Y_i)|_{Y_{\mathrm{red}}} \to T^*(Y_{\mathrm{red}}))\right)$$

is equivalent to a constant object of $\mathrm{QCoh}(Y_{\mathrm{red}})^{\heartsuit}$.

However

$$H^0\left(\mathrm{Fib}(T^*(Y_i)|_{Y_{\mathrm{red}}} \to T^*(Y_{\mathrm{red}}))\right) \simeq I_i/I_i^2$$

(see, e.g., [GR2, Chapter 1, Lemma 5.4.3(a)]), and the transition maps

$$I_j/I_j^2 \to I_i/I_i^2$$

are surjective.

This implies the stabilization assertion for $n = 2$. The induction step is carried out by the same argument:

Assume that the assertion hold for $n \leq 2^m$. Let $R_{n,\mathrm{cl}}$ denote the resulting ring (the eventual value of $H^0(R_i)/I_i^n$). Since I is filtered, we can assume that $\mathrm{Spec}(R_{n,\mathrm{cl}})$ maps to all the Y_i. Then we run the same argument as above with Y_{red} replaced by $\mathrm{Spec}(R_{n,\mathrm{cl}})$ for $n = 2^m$.

□

A.2.3. Let $R_{n,\mathrm{cl}}$ as above. Let $I_n := \ker(R_{n,\mathrm{cl}} \to Y_{\mathrm{red}})$. By construction, for $m \leq n$, we have

$$R_{n,\mathrm{cl}}/I_n^m \simeq R_{n-m,\mathrm{cl}}.$$

Set

$$R_{\mathrm{cl}} := \lim_n R_{n,\mathrm{cl}}$$

Let I denote the ideal $\ker(R_{\mathrm{cl}} \to R_{\mathrm{red}})$.

By the locally almost of finite type assumption, the ideal

$$I_2 \subset R_{2,\mathrm{cl}}$$

is finitely generated; choose generators $\overline{f}_1, ...\overline{f}_m$. Let $f_1, ..., f_m$ be their lifts to elements of ${}^{\mathrm{cl}}I_\infty$.

The following is a standard convergence argument:

Lemma A.2.4.

(a) *The elements* $f_1, ..., f_m$ *generate* I.

(b) *For any* n*, the ideal* $I^n \subset R_{\mathrm{cl}}$ *is closed in the* I*-adic topology on* R_{cl}*, and the inclusion*

$$I^n \subset \ker(R_{\mathrm{cl}} \to R_{n,\mathrm{cl}})$$

is an equality.

A.2.5. By construction, we obtain that the map

$${}^{\mathrm{cl}}\mathcal{Y} \to \mathrm{Spec}(R_{\mathrm{cl}})_{Y_{\mathrm{red}}}$$

is an isomorphism.

A.3. **Derived structure.**

A.3.1. For $k \geq 0$ consider the k-th coconnective truncation of $\mathcal{Y}$, denoted ${}^{\leq k}\mathcal{Y}$. Write

$${}^{\leq k}\mathcal{Y} = \underset{i \in I}{\mathrm{colim}}\, Y_{i,k},$$

where $Y_{i,k} = \mathrm{Spec}(R_{i,k}) \in {}^{\leq k}\mathrm{Sch}^{\mathrm{aff}}_{/\mathsf{e}}$ and I is a filtered index category.

Set

$$R_k := \lim_{i \in I} R_{i,k}.$$

Using induction on k, we will prove the following statements:

- The ring R_k connective and for any $k' \leq k$, the map $\tau^{\geq -k'}(R_k) \to R_{k'}$ is an isomorphism. In particular, the map
$$H^0(R_k) \to R_{\mathrm{cl}}$$
is an isomorphism.
- The map ${}^{\leq k}\mathcal{Y} \to \mathrm{Spec}(R_k)_{Y_{\mathrm{red}}}$ is an isomorphism.

Once we prove this, the assertion of Theorem A.1.3 will follow by taking the limit over k.

A.3.2. The base of the induction is the case $k = 0$, which has been considered in Sect. A.2. We will now carry out the induction step. Thus, we will assume that the statement is true for k and prove it for $k + 1$.

Thus, we write

$${}^{\leq k+1}\mathcal{Y} = \underset{i \in I}{\mathrm{colim}}\, Y_{i,k+1}.$$

Let $Y_{i,k} = \tau^{\geq k}(Y_{i,k})$ and

$$0 \to I_{i,k+1}[k+1] \to R_{i,k+1} \to R_{i,k} \to 0, \quad I_{i,k+1} \in \mathrm{QCoh}(Y_{i,k})^{\heartsuit}.$$

To prove the induction step, we only have to show that

$$\lim_{i \in I} I_{i,k+1}$$

lives in cohomological degree 0.

A.3.3. First, we claim that the index category I can be chosen to be the poset $\mathbb{N}$ of natural numbers. Indeed, this follows from [GR2, Proposition 5.2.3].

Hence, it suffices to show that the inverse system

$$i \mapsto I_{i,k+1}$$

is equivalent to one given by surjective maps.

We have

$$\lim_{i\in I} I_{i,k+1} \simeq \lim_{i\in I} \lim_{j\geq i} H^0\left(I_{j,k+1} \underset{R_{j,k}}{\otimes} R_{i,k}\right).$$

Hence, it is sufficient to show that for a fixed i, the system

$$j \mapsto H^0\left(I_{j,k+1} \underset{R_{j,k}}{\otimes} R_{i,k}\right)$$

is equivalent to a constant one.

A.3.4. For a pair of indices $j \geq i$, consider

$$\operatorname{Fib}(T^*(Y_{j,k+1})|_{Y_{i,k}} \to T^*(Y_{j,k})|_{Y_{i,k}}). \tag{A.1}$$

By [GR2, Chapter 1, Lemma 5.4.3(b)], the object (A.1) lives in cohomological degrees $\leq -(k+1)$, and we we have

$$H^{-(k+1)}\left(\operatorname{Fib}(T^*(Y_{j,k+1})|_{Y_{i,k}} \to T^*(Y_{j,k})|_{Y_{i,k}})\right) \simeq H^0\left(I_{j,k+1} \underset{R_{j,k}}{\otimes} R_{i,k}\right).$$

Hence, it suffices to show that the system

$$j \mapsto H^{-(k+1)}\left(\operatorname{Fib}(T^*(Y_{j,k+1})|_{Y_{i,k}} \to T^*(Y_{j,k})|_{Y_{i,k}})\right).$$

A.3.5. Up to changing the index category, we can assume that the family $j \mapsto Y_{j,k+1}$ is obtained as

$$\tau^{\leq k+1}(Y_j)$$

for

$$\operatorname{colim}_j Y_j \simeq \mathcal{Y}.$$

Again by [GR2, Chapter 1, Lemma 5.4.3(b)], the maps

$$\operatorname{Fib}(T^*(Y_j)|_{Y_{i,k}} \to T^*(Y_{j,k})|_{Y_{i,k}}) \to \operatorname{Fib}(T^*(Y_{j,k+1})|_{Y_{i,k}} \to T^*(Y_{j,k})|_{Y_{i,k}})$$

induce an isomorphism on the cohomology in degree $-(k+1)$,

Hence, it suffices to show that the family

$$j \mapsto \operatorname{Fib}(T^*(Y_j)|_{Y_{i,k}} \to T^*(Y_{j,k})|_{Y_{i,k}})$$

is equivalent to a constant one, i.e., that the pro-object

$$\text{“}\lim_j\text{”}\ \operatorname{Fib}(T^*(Y_j)|_{Y_{i,k}} \to T^*(Y_{j,k})|_{Y_{i,k}}) \tag{A.2}$$

actually belongs to $\operatorname{QCoh}(Y_{i,k})$.

A.3.6. Note that the object (A.2) identifies with

$$\operatorname{Fib}(T^*(\mathcal{Y})|_{Y_{i,k}} \to T^*({}^{\leq k}\mathcal{Y})|_{Y_{i,k}}).$$

Now, $T^*(\mathcal{Y})|_{Y_{i,k}}$ does belong to $\operatorname{QCoh}(Y_{i,k})$, by the assumption on $\mathcal{Y}$. And $T^*({}^{\leq k}\mathcal{Y})|_{Y_{i,k}}$ also belongs to $\operatorname{QCoh}(Y_{i,k})$, since ${}^{\leq k}\mathcal{Y}$ is a formal completion of an affine scheme (by the inductive hypothesis).

Appendix B. Ind-constructible sheaves on schemes

In this section we let $\mathrm{Shv}(-)^{\mathrm{constr}}$ be a constructible sheaf theory. The algebro-geometric objects in this section will be quasi-compact schemes over k, assumed almost[27] of finite type.

B.1. **The left completeness theorem.**

B.1.1. Recall that for a (quasi-compact) scheme Y we define

$$\mathrm{Shv}(Y) := \mathrm{Ind}(\mathrm{Shv}(Y)^{\mathrm{constr}}).$$

The goal of this subsection is to prove Theorem 1.1.4. The proof will be obtained as a combination of the following two statements:

Theorem B.1.2. *The canonical functor*

$$D^b(\mathrm{Perv}(Y)) \to \mathrm{Shv}(Y)^{\mathrm{constr}}$$

is an equivalence.

Theorem B.1.3. *Let $\mathcal{A}$ be a small abelian category of bounded cohomological dimension. Then the DG category* $\mathrm{Ind}(D^b(\mathcal{A}))$ *is left-complete in its t-structure.*

Theorem B.1.2 is a theorem of A. Beilinson, and it is proved in [Be1]. The rest of this subsection is devoted to the proof of Theorem B.1.3.

B.1.4. Recall that for a DG category $\mathbf{C}$ equipped with a t-structure, we denote by $\mathbf{C}^\wedge$ its left completion, i.e.,

$$\mathbf{C}^\wedge := \lim_n \mathbf{C}^{\geq -n}.$$

We have the tautological functor

$$\mathbf{C} \to \mathbf{C}^\wedge \tag{B.1}$$

and its right adjoint given by

$$\{\mathbf{c}^n \in \mathbf{C}^{\geq -n}\} \mapsto \mathit{lim}_n\, \mathbf{c}^n, \tag{B.2}$$

where the limit is taken in $\mathbf{C}$.

We shall say that $\mathbf{C}$ has *convergent Postnikov towers* if (B.1) is fully faithful. Equivalently, if for $\mathbf{c} \in \mathbf{C}$, the natural map

$$\mathbf{c} \to \mathit{lim}_n\, \tau^{\geq -n}(\mathbf{c})$$

is an isomorphism.

We claim:

Proposition B.1.5. *Let $\mathbf{C}$ be generated by compact objects of finite cohomological dimension. Then $\mathbf{C}$ has convergent Postnikov towers. Furthermore, the right adjoint to* (B.1) *is continuous.*

Proof. It is enough to show that for every $\mathbf{c}_0 \in \mathbf{C}^c$, the functor

$$\mathbf{C}^\wedge \overset{\text{right adjoint to (B.1)}}{\longrightarrow} \mathbf{C} \overset{\mathcal{H}om_{\mathbf{C}}(\mathbf{c}_0,-)}{\longrightarrow} \mathrm{Vect}_{\mathsf{e}} \tag{B.3}$$

is continuous, and that its precomposition with (B.1) is isomorphic to $\mathcal{H}om_{\mathbf{C}}(\mathbf{c}_0, -)$.

Now, the above functor sends

$$\{\mathbf{c}^n \in \mathbf{C}^{\geq -n}\} \in \mathbf{C}^\wedge$$

to

$$\mathit{lim}_n\, \mathcal{H}om_{\mathbf{C}}(\mathbf{c}_0, \mathbf{c}^n),$$

while in the above limit, each individual cohomology group stabilizes due to the assumption on $\mathbf{c}_0$.

This implies both claims.

[27]Since we are dealing with $\mathrm{Shv}(-)^{\mathrm{constr}}$, we lose nothing by only considering classical schemes, i.e., derived algebraic geometry over k will play no role.

□

B.1.6. Applying Proposition B.1.5 to $\mathrm{Ind}(D^b(\mathcal{A}))$, we obtain that it has convergent Postnikov towers. It remains to show that the functor (B.2) is fully faithful.

We have:

Lemma B.1.7. *Assume that* $\mathbf{C}^{\leq 0}$ *is closed under filtered limits. Then the functor* (B.2) *is fully faithful.*

Proof. Let $\{\mathbf{c}^n \in \mathbf{C}^{\geq -n}\}$ be an object of $\mathrm{Ind}(D^b(\mathcal{A}))^\wedge$, and set

$$\mathbf{c} := \underset{n}{lim}\, \mathbf{c}^n.$$

We need to show that the natural maps

$$\tau^{\geq -n}(\mathbf{c}) \to \mathbf{c}^n$$

are isomorphisms.

We have an exact triangle

$$\lim_{m\geq n} \tau^{<-n}(\mathbf{c}^m) \to \mathbf{c} \to \lim_{m\geq n} \tau^{\geq -n}(\mathbf{c}^m),$$

where the left-most term belongs to $\mathbf{C}^{<-n}$ by assumption, and the right-most term is $\mathbf{c}^n$.

This implies the assertion.

□

B.1.8. Let us show that Lemma B.1.7 is applicable to $\mathrm{Ind}(D^b(\mathcal{A}))$. Since $\mathrm{Ind}(D^b(\mathcal{A}))$ is compactly generated by objects from $\mathcal{A}$, it suffices to show the following:

Lemma B.1.9. *Let* $\mathbf{c} \in \mathrm{Ind}(D^b(\mathcal{A}))$ *be an object such that*

$$\mathcal{H}om_{\mathrm{Ind}(D^b(\mathcal{A}))}(a, \mathbf{c}) \in \mathrm{Vect}_{\mathsf{e}}^{\leq 0} \text{ for all } a \in \mathcal{A} \subset \mathrm{Ind}(D^b(\mathcal{A})).$$

Then $\mathbf{c} \in \mathrm{Ind}(D^b(\mathcal{A}))^{\leq 0}$.

Proof. Suppose that $\tau^{>0}(\mathbf{c}) \neq 0$. Let k be the smallest integer such that $H^k(\mathbf{c}) \neq 0$. We can find an object $a \in \mathcal{A}$ and a non-zero morphism

$$a \to H^k(\mathbf{c}).$$

We claim that we can find a surjection $a' \to a$ such that the above morphism lifts to a morphism

$$a'[-k] \to \mathbf{c},$$

which would be a contradiction.

First, we claim that for any n, we can find a surjection $a' \to a$ such that the above map

$$a[-k] \to \tau^{\geq k}(\mathbf{c})$$

can be lifted to a map

$$a'[-k] \to \tau^{\geq -n}(\mathbf{c}).$$

This follows from the definition of the derived category: every non-trivial $\mathrm{Ext}^i(a, -)$ can be annihilated by a surjection $a' \to a$.

Let d be the cohomological dimension of $\mathcal{A}$, and take $n \geq d$. Then the above map

$$a'[-k] \to \tau^{\geq -n}(\mathbf{c})$$

automatically lifts to a map

$$a'[-k] \to \mathbf{c}.$$

□

B.2. **Categorical** $K(\pi, 1)$**'s.**

B.2.1. Recall the subcategories

$$\mathrm{Lisse}(Y) \subset \mathrm{IndLisse}(Y) \subset \mathrm{QLisse}(Y).$$

Definition B.2.2. *We shall say that Y is a* categorical $K(\pi, 1)$ *if the naturally defined functor*

$$D^b(\mathrm{Lisse}(Y)^{\heartsuit}) \to \mathrm{Lisse}(Y)$$

is an equivalence.

Note that from Theorem B.1.3 we obtain:

Corollary B.2.3. *If X is a categorical $K(\pi, 1)$, then the inclusion*

$$\mathrm{IndLisse}(Y) \subset \mathrm{QLisse}(Y) \tag{B.4}$$

is an equality.

B.2.4. An easy example of X, which is *not* a categorical $K(\pi, 1)$ is $Y = \mathbb{P}^1$. We claim that in this example, the functor (B.4) is *not* an equivalence.

Indeed, the category $\mathrm{IndLisse}(Y)$ is generated by one object, namely, $\mathsf{e}_{\mathbb{P}^1}$, whose algebra of endomorphisms is

$$A := \mathsf{e}[\eta]/\eta^2 = 0, \ \deg(\eta) = 2.$$

Hence,

$$\mathrm{IndLisse}(Y) \simeq A\text{-mod}.$$

By Koszul duality, we have

$$A\text{-mod} \simeq B\text{-mod}_0,$$

where

$$B = \mathsf{e}\langle\xi\rangle, \ \deg(\xi) = -1$$

is the free *associative* algebra on one generator in degree -1, and

$$B\text{-mod}_0 \subset B\text{-mod} \tag{B.5}$$

is the full subcategory consisting of objects on which ξ acts nilpotently.

The t-structure on $\mathrm{IndLisse}(Y)$ corresponds to the usual t-structure on B-mod, for which the forgetful functor to $\mathrm{Vect}_{\mathsf{e}}$ is t-exact.

Now it is easy to see that the embedding (B.5) realizes B-mod as the left completion of $B\text{-mod}_0$.

B.2.5. We now claim:

Theorem B.2.6. *All algebraic curves other than $\mathbb{P}^1$ are categorical $K(\pi, 1)$'s.*

The proof will be based on the following observation:

Lemma B.2.7. *Let $\mathbf{C}_0$ be a small DG category equipped with a* bounded *t-structure, such that*

$$\mathrm{Hom}_{\mathbf{C}_0}(a, a'[k]) = 0 \textit{ for } k > 2 \textit{ for all } a, a' \in \mathbf{C}_0^{\heartsuit}.$$

Then

$$D^b(\mathbf{C}_0^{\heartsuit}) \to \mathbf{C}$$

is an equivalence if and only if for every a, a' as above, the map

$$\mathrm{Ext}^2_{\mathbf{C}_0^{\heartsuit}}(a, a') \to \mathrm{Hom}_{\mathbf{C}_0}(a, a'[2])$$

is surjective.

Note that the above lemma immediately implies the assertion of Theorem B.2.6 when X is affine, as in this case

$$\mathrm{Hom}_{\mathrm{QLisse}(Y)}(E, E'[2]) = 0$$

for any pair of local systems E and E'.

B.2.8. We will now consider the case of complete curves.

Proof of Theorem B.2.6. With no restriction of generality, it suffices to show that for an arbitrary local system E and the trivial local system E_0, any class $\alpha \in H^2(X,E)$ can be written as a cup product of an element in $H^1(X,\mathsf{e})$ and a class in $H^1(X,E)$.

Write

$$0 \to E'' \to E \to E' \to 0,$$

where E' is a direct sum of copies of the trivial local system, and E'' does not have trivial quotients. Note that the map

$$H^2(X,E) \to H^2(X,E')$$

is an isomorphism, since $H^2(X,E'')=0$. Let α' be the image of α in $H^2(X,E')$

Since the pairing $H^1(X,\mathsf{e}) \otimes H^1(X,\mathsf{e}) \to H^2(X,\mathsf{e})$ is non-degenerate, we can write α' as a cup product of $\beta \cup \gamma'$, with $\beta \in H^1(X,\mathsf{e})$ and $\gamma' \in H^1(X,E')$.

It suffices to show that γ' can be lifted to an element $\gamma \in H^1(X,E)$. However, the obstruction to such a lift lies in $H^2(X,E'')$, which vanishes.

□

B.3. **The dual of** QLisse(Y).

B.3.1. We give the following definition:

Definition B.3.2. *We shall say that* QLisse(Y) *is Verdier-compatible if the functor*

$$\text{QLisse}(Y) \otimes \text{QLisse}(Y) \to \text{Vect}_{\mathsf{e}}, \quad E_1, E_2 \mapsto \mathrm{C}^{\cdot}(Y, E_1 \overset{!}{\otimes} E_2). \tag{B.6}$$

is the counit of a self-duality.

B.3.3. We claim:

Proposition B.3.4. *Let Y be smooth, and assume that* IndLisse(Y) $\to$ QLisse(Y) *is an equivalence. Then* QLisse(Y) *is Verdier-compatible.*

Proof. Since IndLisse(Y) $\to$ QLisse(Y) is an equivalence, the category QLisse(Y) is compactly generated by Lisse(Y). Now, *naive* duality defines a contravariant equivalence

$$\text{Lisse}(Y) \simeq \text{Lisse}(Y)^{\text{op}}.$$

If Y is smooth, the above naive duality coincides with Verdier duality, up to a shift. Hence, the latter defines an identification

$$\text{QLisse}(Y) \simeq \text{QLisse}(Y)^{\vee}.$$

Its counit is given by (B.6) by definition.

□

Remark B.3.5. Note that the above argument shows that for any smooth Y, the pairing

$$\mathcal{F}_1, \mathcal{F}_2 \mapsto \mathrm{C}^{\cdot}(Y, \mathcal{F}_1 \overset{!}{\otimes} \mathcal{F}_2)$$

defines a self-duality on IndLisse(Y).

B.3.6. We claim:

Corollary B.3.7. *If X is a smooth curve, then* QLisse(X) *is Verdier-compatible.*

Proof. The case of curves different from $\mathbb{P}^1$ follows from Theorem B.2.6 and Proposition B.3.4.

The case of $\mathbb{P}^1$ follows by direct inspection: in terms of the equivalence

$$\mathrm{QLisse}(\mathbb{P}^1) \simeq B\text{-mod}$$

(see Sect. B.2.4), the pairing (B.6) corresponds (up to a shift) to the canonical pairing

$$B\text{-mod} \otimes B^{\mathrm{op}}\text{-mod} \to \mathrm{Vect}_{\mathsf{e}},$$

corresponding to the isomotophism

$$B \simeq B^{\mathrm{op}}, \quad \xi \mapsto -\xi.$$

□

B.4. **Specifying singular support.**

B.4.1. Let Y be a scheme and $\mathcal{N}$ a conical Zariski-closed subset of $T^*(Y)$. In this case we have a well-defined full subcategory

$$\mathrm{Perv}_{\mathcal{N}}(Y) \subset \mathrm{Perv}(Y).$$

Consider the abelian category

$$\mathrm{Ind}(\mathrm{Perv}_{\mathcal{N}}(Y)) \subset \mathrm{Ind}(\mathrm{Perv}(Y)) \simeq \mathrm{Shv}(Y)^{\heartsuit}.$$

We let

$$\mathrm{Shv}_{\mathcal{N}}(Y) \subset \mathrm{Shv}(Y)$$

be the full subcategory consisting of objects whose cohomologies belong to $\mathrm{Ind}(\mathrm{Perv}_{\mathcal{N}}(Y))$.

By Theorem 1.1.4, the category $\mathrm{Shv}_{\mathcal{N}}(Y)$ is left-complete in its t-structure.

B.4.2. Set

$$\mathrm{Shv}_{\mathcal{N}}(Y)^{\mathrm{constr}} := \mathrm{Shv}_{\mathcal{N}}(Y) \cap \mathrm{Shv}(Y)^{\mathrm{constr}} \subset \mathrm{Shv}(Y).$$

This is the full subcategory of $\mathrm{Shv}(Y)^{\mathrm{constr}}$ consisting of objects whose cohomologies belong to $\mathrm{Perv}_{\mathcal{N}}(Y)$.

Set

$$\mathrm{Shv}_{\mathcal{N}}(Y)^{\mathrm{access}} := \mathrm{Ind}(\mathrm{Shv}_{\mathcal{N}}(Y)^{\mathrm{constr}}).$$

The ind-extension of the tautological embedding

$$\mathrm{Shv}_{\mathcal{N}}(Y)^{\mathrm{constr}} \hookrightarrow \mathrm{Shv}_{\mathcal{N}}(Y)$$

defines a fully faithful functor

$$\mathrm{Shv}_{\mathcal{N}}(Y)^{\mathrm{access}} \hookrightarrow \mathrm{Shv}_{\mathcal{N}}(Y). \tag{B.7}$$

From the fact that $\mathrm{Shv}_{\mathcal{N}}(Y)$ is left-complete, we obtain that the functor (B.7) realizes $\mathrm{Shv}_{\mathcal{N}}(Y)$ realizes as a left completion of $\mathrm{Shv}_{\mathcal{N}}(Y)^{\mathrm{access}}$.

B.4.3. *Example.* Take $\mathcal{N} = \{0\}$. Then $\mathrm{Shv}_{\mathcal{N}}(Y)$ is what we have previously denoted by QLisse(Y) and $\mathrm{Shv}_{\mathcal{N}}(Y)^{\mathrm{access}} = \mathrm{IndLisse}(Y)$.

Remark B.4.4. Note that the process of left completion in (B.7) is in general non-trivial, i.e., the category $\mathrm{Shv}_{\mathcal{N}}(Y)^{\mathrm{access}}$ is not necessarily left-complete, see Sect. B.2.4.

Remark B.4.5. Our conventions are different from those of [GKRV]. In *loc.cit.* we denoted by $\mathrm{Shv}_{\mathcal{N}}(Y)$ what we denote here by $\mathrm{Shv}_{\mathcal{N}}(Y)^{\mathrm{access}}$.

B.5. **The tensor product theorems.** In this subsection we will discuss several variants of the tensor product result [GKRV, Theorem A.3.8].

B.5.1. As was remarked in Sect. 11.5.1, for a pair of schemes, the external tensor product functor

$$\mathrm{Shv}(Y_1)\otimes\mathrm{Shv}(Y_2)\to\mathrm{Shv}(Y_1\times Y_2)$$

is t-exact, sends compacts to compacts, and is fully faithful, but *not an equivalence* (unless one of the schemes is a disjoint union of set-theoretic points).

From here, it follows formally that for $\mathcal{N}_i\subset T^*(Y_i)$, the functor

$$\mathrm{Shv}_{\mathcal{N}_1}(Y_1)\otimes\mathrm{Shv}_{\mathcal{N}_2}(Y_2)\to\mathrm{Shv}_{\mathcal{N}_1\times\mathcal{N}_2}(Y_1\times Y_2)$$

is t-exact, sends compacts to compacts, and is fully faithful.

First, we have the following result, which is [GKRV, Theorem A.3.8].

Theorem B.5.2. *Assume that X is smooth and proper. Let $\mathcal{N}\subset T^*(\mathcal{Y})$ be half-dimensional. Then the resulting functor*

$$\mathrm{IndLisse}(X)\otimes\mathrm{Shv}_{\mathcal{N}}(Y)^{\mathrm{access}}\to\mathrm{Shv}_{\{0\}\times\mathcal{N}}(X\times Y)^{\mathrm{access}} \tag{B.8}$$

is an equivalence.

Remark B.5.3. It is natural to ask whether the functor

$$\mathrm{QLisse}(X)\otimes\mathrm{Shv}_{\mathcal{N}}(Y)\to\mathrm{Shv}_{\{0\}\times\mathcal{N}}(X\times Y)$$

is an equivalence.

Unfortunately, we do not have an answer to this, except in the case covered by Theorem B.5.8 below. Namely, we did not find a way to control when the tensor product

$$\mathrm{QLisse}(X)\otimes\mathrm{Shv}_{\mathcal{N}}(Y)$$

is left-complete.

B.5.4. Next, we claim:

Theorem B.5.5. *Let X be smooth and proper. Then the essential image of*

$$\mathrm{QLisse}(X)\otimes\mathrm{Shv}(Y)\to\mathrm{Shv}(X\times Y) \tag{B.9}$$

consists of objects of whose perverse cohomologies have singular support contained in $\{0\}\times T^*(Y)$.

The proof will use the following variant of Theorem B.1.3 (the proof is given in Sect. B.5.9):

Theorem B.5.6. *Let $\mathcal{A}$ be a small abelian category of bounded cohomological dimension. Let $\mathbf{C}$ be a DG category equipped with a t-structure in which it is left-compete. Then*

$$\mathrm{Ind}(D^b(\mathcal{A}))\otimes\mathbf{C}$$

is left-complete in its t-structure.

Proof of Theorem B.5.5. First, we claim that every bounded below object in $\mathrm{Shv}(X\times Y)$ with specified singular support in contained in the essential image of (B.9).

Indeed, by devissage we can assume that the object in question is also bounded above, then is in the heart of the t-structure, and then is contained in $\mathrm{Perv}(X\times Y)$, with specified singular support.

Let $\mathcal{F}\neq 0$ be such an object in $\mathrm{Perv}(X\times Y)$. The condition on $\mathrm{SingSupp}(\mathcal{F})$ implies that $\mathcal{F}$ cannot be supported over a proper closed subscheme of Y. Hence, $\mathrm{SingSupp}(\mathcal{F})$ projects surjectively onto $\{0\}\subset T^*(X)$. Now, by [Be1], singular support of perverse sheaves is half-dimensional. Hence, $\mathrm{SingSupp}(\mathcal{F})$ is contained in a subset of the form $\{0\}\times\mathcal{N}$, where $\mathcal{N}\subset T^*(Y)$ is half-dimensional. But then such object is contained in the essential image of (B.8), by Theorem B.5.2.

Now, the assertion of the theorem follows, as both sides are left-complete in their respective t-structures, the right-hand side by construction, and the left-hand side by Theorem B.5.6.

□

B.5.7. Finally, we claim:

Theorem B.5.8. *Let X be smooth and proper. Assume also that* QLisse(X) *Verdier-compatible (see Sect. B.3). Let $\mathcal{N} \subset T^*(Y)$ be half-dimensional. Then the resulting functor*

$$\text{QLisse}(X) \otimes \text{Shv}_{\mathcal{N}}(Y) \to \text{Shv}_{\{0\}\times\mathcal{N}}(X \times Y) \tag{B.10}$$

is an equivalence.

Proof. Since QLisse(X) is dualizable, the functor

$$\text{QLisse}(X) \otimes \text{Shv}_{\mathcal{N}}(Y) \to \text{QLisse}(X) \otimes \text{Shv}(Y) \tag{B.11}$$

is fully faithful.

Given Theorem B.5.5, it suffices to show that any object

$$\mathcal{F} \in \text{QLisse}(X) \otimes \text{Shv}(Y)$$

whose image $\mathcal{F}' \in \text{Shv}(X \times Y)$ has singular support in $\{0\} \times \mathcal{N}$, belongs to the essential image of (B.11).

Since QLisse(X) Verdier-compatible, it suffices to show that for any $E \in \text{QLisse}(X)$, the object

$$(\mathrm{C}^\cdot(X, -) \otimes \text{Id})(E \overset{!}{\otimes} \mathcal{F}) \in \text{Shv}(Y)$$

belongs to $\text{Shv}_{\mathcal{N}}(Y)$.

However, the latter object is the same as

$$(p_Y)_*(p_X^!(E) \otimes \mathcal{F}'),$$

where p_X and p_Y are the two projections from $X \times Y$ to X and Y, respectively.

The latter object indeed belongs to $\text{Shv}_{\mathcal{N}}(Y)$, due to the assumption on the singular support of $\mathcal{F}'$ and the fact that X is proper.

□

B.5.9. *Proof of Theorem B.5.6.* The proof repeats the argument of Theorem B.1.3, using the following variants of Proposition B.1.5 and Lemma B.1.9, respectively:

Proposition B.5.10. *Let $\mathbf{C}$ be compactly generated by compact objects of finite cohomological dimension. Then for any $\mathbf{C}_1$ equipped with a t-structure in which it is left-complete, the functor*

$$\mathbf{C} \otimes \mathbf{C}_1 \to (\mathbf{C} \otimes \mathbf{C}_1)^\wedge \tag{B.12}$$

is fully faithful and its right adjoint is continuous.

Lemma B.5.11. *Let $\mathcal{A}$ be as in Theorem B.5.6, and let $\mathbf{C}_1$ be equipped with a t-structure. Let $\mathbf{c} \in \text{Ind}(D^b(\mathcal{A})) \otimes \mathbf{C}_1$ be an object satisfying*

$$(\mathcal{H}om_{\text{Ind}(D^b(\mathcal{A}))}(a, -) \otimes \text{Id})(\mathbf{c}) \in (\mathbf{C}_1)^{\leq 0} \textit{ for all } a \in \mathcal{A} \subset \text{Ind}(D^b(\mathcal{A})).$$

Then $\mathbf{c} \in (\text{Ind}(D^b(\mathcal{A})) \otimes \mathbf{C}_1)^{\leq 0}$.

Both these statements are proved in a way mimicking the original arguments.

Appendix C. Constructible sheaves on an algebraic stack

As in Sect. B, in this section we let $\text{Shv}(-)^{\text{constr}}$ be a constructible sheaf theory. All algebro-geometric objects will be assumed (locally) of finite type over the ground field k.

C.1. Generalities.

C.1.1. Let $\mathcal{Y}$ be a prestack. Recall that we define

$$\mathrm{Shv}(\mathcal{Y}) := \lim_{S} \mathrm{Shv}(S),$$

where the index category is that of affine schemes equipped with a map to $\mathcal{Y}$, and the transition functors are given by !-pullback.

Since we are in the constructible context, !-pullback admits a left adjoint, given by !-pushforward, so we can write

$$\mathrm{Shv}(\mathcal{Y}) := \operatorname{colim}_{S} \mathrm{Shv}(S), \tag{C.1}$$

where the transition functors are given by !-pushforward.

In particular, we obtain that $\mathrm{Shv}(\mathcal{Y})$ is compactly generated.

Suppose for a moment that $\mathcal{Y}$ is an algebraic stack. Then the above index category can be replaced by its non-full subcategory, where we allow as objects affine schemes that are smooth over $\mathcal{Y}$, and as morphisms smooth maps between those.

C.1.2. Recall that for a quasi-compact scheme Y, Verdier duality defines a contravariant equivalence

$$(\mathrm{Shv}(S)^{\mathrm{constr}})^{\mathrm{op}} \overset{\mathbb{D}}{\to} \mathrm{Shv}(S)^{\mathrm{constr}}.$$

Since

$$\mathrm{Shv}(Y) := \mathrm{Ind}(\mathrm{Shv}(Y)^{\mathrm{constr}}),$$

we obtain that the category $\mathrm{Shv}(Y)$ is canonically self-dual with the counit

$$\mathrm{Shv}(Y) \otimes \mathrm{Shv}(Y) \to \mathrm{Vect}_{\mathsf{e}}$$

given by

$$\mathcal{F}_1, \mathcal{F}_2 \mapsto \mathrm{C}^{\cdot}(Y, \mathcal{F}_1 \overset{!}{\otimes} \mathcal{F}_2).$$

C.1.3. In particular, by [DrGa2, Proposition 1.8.3] and (C.1), we obtain that for a prestack $\mathcal{Y}$, the category $\mathrm{Shv}(\mathcal{Y})$ is dualizable, and

$$\mathrm{Shv}(\mathcal{Y})^{\vee} \simeq \operatorname{colim}_{S} \mathrm{Shv}(S),$$

where the transition functors are given by *-pushforward.

Remark C.1.4. Note that there is no a priori reason for $\mathrm{Shv}(\mathcal{Y})^{\vee}$ to be equivalent to the original $\mathrm{Shv}(\mathcal{Y})$.

We will see that there is a canonical such equivalence when $\mathcal{Y}$ is a quasi-compact algebraic stack (at least when $\mathcal{Y}$ is locally a quotient). However, for more general $\mathcal{Y}$ (e.g., for non-quasi-compact algebraic stacks) such an equivalence would reflect a particular feature of $\mathcal{Y}$, for example its property of being *miraculous*, see [Ga3, Sect. 6.7].

C.2. **Constructible vs compact.**

C.2.1. Let $\mathcal{Y}$ be an algebraic stack with an affine diagonal. Let

$$\mathrm{Shv}(\mathcal{Y})^{\mathrm{constr}} \subset \mathrm{Shv}(\mathcal{Y})$$

be the full subcategory consisting of objects that pullback to an object of

$$\mathrm{Shv}(S)^{\mathrm{constr}} = \mathrm{Shv}(S)^{c} \subset \mathrm{Shv}(S)$$

for any affine scheme S mapping to $\mathcal{Y}$.

It is easy to see that this condition is enough to test on smooth maps $S \to \mathcal{Y}$. In the latter case, we can use either !- or *- pullback, as they differ by a cohomological shift.

C.2.2. Using the definition of the constructible subcategory via *-pullbacks along smooth maps, we obtain that we have an inclusion

$$\mathrm{Shv}(\mathcal{Y})^c \subset \mathrm{Shv}(\mathcal{Y})^{\mathrm{constr}}.$$

Indeed, for $f : S \to \mathcal{Y}$, the functor f^* sends compacts to compacts, since its right adjoint, namely f_*, is continuous.

However, the above inclusion is typically not an equality. For example, the constant sheaf

$$\underline{\mathsf{e}}_{\mathcal{Y}} \in \mathrm{Shv}(\mathcal{Y})^{\mathrm{constr}}$$

is *not* compact for $\mathcal{Y} = B(\mathbb{G}_m)$.

C.2.3. Verdier duality defines a contravariant equivalence

$$(\mathrm{Shv}(\mathcal{Y})^{\mathrm{constr}})^{\mathrm{op}} \overset{\mathbb{D}}{\to} \mathrm{Shv}(\mathcal{Y})^{\mathrm{constr}}.$$

If $\mathcal{Y}$ is not quasi-compact, the functor $\mathbb{D}$ will typically *not* send $\mathrm{Shv}(\mathcal{Y})^c$ to $\mathrm{Shv}(\mathcal{Y})^c$.

C.2.4. Assume that $\mathcal{Y}$ is quasi-compact. We will say that $\mathcal{Y}$ is *duality-adapted* if the functor $\mathbb{D}$ sends $\mathrm{Shv}(\mathcal{Y})^c$ to $\mathrm{Shv}(\mathcal{Y})^c$.

Based in [DrGa1, Corollary 8.4.2], we conjecture:

Conjecture C.2.5. *Any quasi-compact algebraic stack with an affine diagonal is duality-adapted.*

We are going to prove:

Theorem C.2.6. *Let $\mathcal{Y}$ be such that it can be covered by open subsets each of which has the form S/G, where S is a quasi-compact scheme and G is an algebraic group. Then $\mathcal{Y}$ is duality-adapted.*

C.2.7. *Proof of Theorem C.2.6, reduction step.* Let us reduce the assertion to the case when $\mathcal{Y}$ is globally a quotient, i.e., is of the form S/G.

Indeed, suppose $\mathcal{Y}$ can be covered by open substacks $\mathcal{U}_i \overset{j_i}{\hookrightarrow} \mathcal{Y}$, such that each $\mathcal{U}_i$ is duality-adapted. We will show that $\mathcal{Y}$ is duality-adapted.

Since $\mathcal{Y}$ was assumed quasi-compact, we can assume that the above open cover is finite. Now the assertion follows from the fact that an object $\mathcal{F} \in \mathrm{Shv}(\mathcal{Y})$ is compact if and only if all $j_i^*(\mathcal{F})$ are compact.

Indeed, the implication

$$\mathcal{F} \in \mathrm{Shv}(\mathcal{Y}) \;\Rightarrow\; j_i^*(\mathcal{F}) \in \mathrm{Shv}(\mathcal{U}_i)^c$$

follows from the fact that j_i^* admits a continuous right adjoint, namely, $(j_i)_*$.

The opposite implication follows from the fact that $\mathcal{H}om_{\mathrm{Shv}(\mathcal{Y})}(\mathcal{F}, -)$ can be expressed as a finite limit in terms of $\mathcal{H}om_{\mathrm{Shv}(\mathcal{U}_i)}(j_i^*(\mathcal{F}), -)$ and finite intersections of these opens.

C.2.8. *Proof of Theorem C.2.6, explicit generators for a global quotient.* Thus, we can assume that $\mathcal{Y}$ has the form S/G.

It suffices to find a system of compact generators of $\mathrm{Shv}(\mathcal{Y})$ that are sent to compact objects by the functor $\mathbb{D}$.

Let π_S denote the map

$$S \to S/G = \mathcal{Y}.$$

Let p_S/G denote the map $S/G \to \mathrm{pt}/G$.

Note that for any $\mathcal{F} \in \mathrm{Shv}(S/G)^{\mathrm{constr}}$, the object

$$(\pi_S)_! \circ (\pi_S)^*(\mathcal{F})$$

is compact. Hence, it suffices to show that:

(I) Such objects generate $\mathrm{Shv}(S/G)$;

(II) They are sent to compact objects by Verdier duality.

C.2.9. *Proof of Theorem C.2.6, verifying the properties.* Consider the map

$$\pi_{\mathrm{pt}} : \mathrm{pt} \to \mathrm{pt}/G$$

and the objects

$$(\pi_{\mathrm{pt}})_*(\mathsf{e}), (\pi_{\mathrm{pt}})_!(\mathsf{e}) \in \mathrm{Shv}(\mathrm{pt}/G).$$

Note that that

$$(\pi_{\mathrm{pt}})^* \circ (\pi_{\mathrm{pt}})_*(\mathsf{e}) \simeq \mathrm{C}^\cdot(G).$$

Note also that

$$(\pi_{\mathrm{pt}})_*(\mathsf{e}) \simeq (\pi_{\mathrm{pt}})_!(\mathsf{e})[d], \tag{C.2}$$

where for

$$1 \to G_{\mathrm{unip}} \to G \to G_{\mathrm{red}} \to 1,$$

we have

$$d = 2\dim(G_{\mathrm{unip}}) + \dim(G_{\mathrm{red}}).$$

For any $\mathcal{F}' \in \mathrm{Shv}(S/G)$ we have:

$$(\pi_S)_! \circ (\pi_S)^*(\mathcal{F}') \simeq \mathcal{F}' \overset{*}{\otimes} (p_S/G)^*((\pi_{\mathrm{pt}})_!(\mathsf{e}))$$

and

$$(\pi_S)_* \circ (\pi_S)^!(\mathcal{F}') \simeq \mathcal{F}' \overset{!}{\otimes} (p_S/G)^!((\pi_{\mathrm{pt}})_*(\mathsf{e})) \simeq \mathcal{F}' \overset{*}{\otimes} (p_S/G)^*((\pi_{\mathrm{pt}})_*(\mathsf{e}))[2\dim(G)] \simeq$$
$$\simeq \mathcal{F}' \overset{*}{\otimes} (p_S/G)^*((\pi_{\mathrm{pt}})_!(\mathsf{e}))[2\dim(G)+d] \simeq (\pi_S)_! \circ (\pi_S)^*(\mathcal{F}')[2\dim(G)+d]$$

Hence,

$$\mathbb{D}((\pi_S)_! \circ (\pi_S)^*(\mathcal{F})) \simeq (\pi_S)_* \circ (\pi_S)^!(\mathbb{D}(\mathcal{F})) \simeq (\pi_S)_! \circ (\pi_S)^*(\mathbb{D}(\mathcal{F}))[2\dim(G)+d].$$

This proves (II).

To prove (I), let $\mathcal{F}'$ be a non-zero object of $\mathrm{Shv}(S/G)$, and let us find $\mathcal{F} \in \mathrm{Shv}(S/G)^{\mathrm{constr}}$ so that

$$\mathcal{H}om_{\mathrm{Shv}(S/G)}((\pi_S)_! \circ (\pi_S)^*(\mathcal{F}), \mathcal{F}') \neq 0.$$

We have

$$\mathcal{H}om_{\mathrm{Shv}(S/G)}((\pi_S)_! \circ (\pi_S)^*(\mathcal{F}), \mathcal{F}') \simeq \mathcal{H}om_{\mathrm{Shv}(S/G)}(\mathcal{F}, (\pi_S)_* \circ (\pi_S)^!(\mathcal{F}'))$$

Now,

$$\mathcal{F}' \neq 0 \Rightarrow \pi_S^*(\mathcal{F}') \neq 0 \Rightarrow \pi_S^*(\mathcal{F}') \otimes \mathrm{C}^\cdot_c(G) \neq 0 \Rightarrow \pi_S^*(\mathcal{F}' \overset{*}{\otimes} (p_S/G)^*(\pi_{\mathrm{pt}})_!(\mathsf{e})) \neq 0 \Rightarrow$$
$$\Rightarrow \mathcal{F}' \overset{*}{\otimes} (p_S/G)^*(\pi_{\mathrm{pt}})_!(\mathsf{e}) \neq 0 \Rightarrow (\pi_S)_* \circ (\pi_S)^!(\mathcal{F}') \neq 0.$$

Hence, we can find $\mathcal{F} \in \mathrm{Shv}(S/G)^c \subset \mathrm{Shv}(S/G)^{\mathrm{constr}}$ such that

$$\mathcal{H}om_{\mathrm{Shv}(S/G)}(\mathcal{F}, (\pi_S)_* \circ (\pi_S)^!(\mathcal{F}')) \neq 0,$$

as required.

□[Theorem C.2.6]

C.3. **Duality-adapted stacks.** In this subsection $\mathcal{Y}$ will be a duality-adapted quasi-compact algebraic stack.

C.3.1. The assumption on $\mathcal{Y}$ implies that the Verdier duality functor defines a contravariant equivalence

$$(\mathrm{Shv}(\mathcal{Y})^c)^{\mathrm{op}} \to \mathrm{Shv}(\mathcal{Y})^c.$$

Hence, we obtain a canonical identification

$$\mathrm{Shv}(\mathcal{Y})^\vee \simeq \mathrm{Shv}(\mathcal{Y}).$$

By construction, the corresponding pairing

$$\mathrm{Shv}(\mathcal{Y})^c \times \mathrm{Shv}(\mathcal{Y})^c \to \mathrm{Vect}_{\mathsf{e}}$$

sends

$$\mathcal{F}_1, \mathcal{F}_2 \to \mathrm{C}^\cdot(\mathcal{Y}, \mathcal{F}_1 \overset{!}{\otimes} \mathcal{F}_2).$$

C.3.2. Let

$$\mathrm{C}^\cdot_\blacktriangle(\mathcal{Y}, -) : \mathrm{Shv}(\mathcal{Y}) \to \mathrm{Vect}$$

be the functor dual to the functor

$$\mathrm{Vect}_{\mathsf{e}} \to \mathrm{Shv}(\mathcal{Y}), \quad \mathsf{e} \mapsto \omega_{\mathcal{Y}},$$

see [DrGa1, Sect. 9.1]. This functor is the ind-extension of the restriction of the functor

$$\mathrm{C}^\cdot(\mathcal{Y}, -) : \mathrm{Shv}(\mathcal{Y}) \to \mathrm{Vect}$$

to $\mathrm{Shv}(\mathcal{Y})^c \subset \mathrm{Shv}(\mathcal{Y})$.

In particular, we have a natural transformation

$$\mathrm{C}^\cdot_\blacktriangle(\mathcal{Y}, -) \to \mathrm{C}^\cdot(\mathcal{Y}, -), \tag{C.3}$$

which is an equivalence when evaluated on compact objects.

Furthermore, the duality pairing on all of $\mathrm{Shv}(\mathcal{Y}) \otimes \mathrm{Shv}(\mathcal{Y})$ can be written as

$$\mathcal{F}_1, \mathcal{F}_2 \mapsto \mathrm{C}^\cdot_\blacktriangle(\mathcal{Y}, \mathcal{F}_1 \overset{!}{\otimes} \mathcal{F}_2).$$

We have a map

$$\mathrm{C}^\cdot_\blacktriangle(\mathcal{Y}, \mathcal{F}_1 \overset{!}{\otimes} \mathcal{F}_2) \to \mathrm{C}^\cdot(\mathcal{Y}, \mathcal{F}_1 \overset{!}{\otimes} \mathcal{F}_2),$$

which is an isomorphism when one of the objects $\mathcal{F}_1$ or $\mathcal{F}_2$ is compact.

C.3.3. We observe:

Lemma C.3.4. *For $\mathcal{F} \in \mathrm{Shv}(\mathcal{Y})^c$ and $\mathcal{F}' \in \mathrm{Shv}(\mathcal{Y})^{\mathrm{constr}}$, both*

$$\mathcal{F} \overset{*}{\otimes} \mathcal{F}' \text{ and } \mathcal{F} \overset{!}{\otimes} \mathcal{F}'$$

are compact.

Proof. The assertion for $\mathcal{F} \overset{*}{\otimes} \mathcal{F}'$ follows from the fact that

$$\mathcal{H}om(\mathcal{F} \overset{*}{\otimes} \mathcal{F}', \mathcal{F}'') \simeq \mathcal{H}om(\mathcal{F}, \mathbb{D}(\mathcal{F}') \overset{!}{\otimes} \mathcal{F}'').$$

The assertion for $\mathcal{F} \overset{!}{\otimes} \mathcal{F}'$ follows by Verdier duality. □

For future reference, we record the following properties of duality-adapted prestacks, borrowed from [DrGa1, Theorem 10.2.9] (we will omit the proof as it repeats verbatim the arguments from *loc. cit.*):

Proposition C.3.5. *The following properties of an object $\mathcal{F} \in \operatorname{Shv}(\mathcal{Y})^{\mathrm{constr}}$ are equivalent:*

(i) *$\mathcal{F}$ is compact;*

(i') *$\mathbb{D}(\mathcal{F})$ is compact;*

(ii) *$\mathcal{F}$ belongs to the smallest (non cocomplete) subcategory of $\operatorname{Shv}(\mathcal{Y})$ closed under taking direct summands that contains objects of the form $f_!(\mathcal{F}_S)$, where $f : S \to \mathcal{Y}$ with S an affine scheme and $\mathcal{F}_S \in \operatorname{Shv}(S)^c$;*

(ii') *$\mathcal{F}$ belongs to the smallest (non cocomplete) subcategory of $\operatorname{Shv}(\mathcal{Y})$ closed under taking direct summands that contains objects of the form $f_*(\mathcal{F}_S)$, where $f : S \to \mathcal{Y}$ with S an affine scheme and $\mathcal{F}_S \in \operatorname{Shv}(S)^c$;*

(iii) *The functor*

$$\mathcal{F}' \mapsto \mathrm{C}^\cdot(\mathcal{Y}, \mathcal{F} \overset{!}{\otimes} \mathcal{F}'), \quad \operatorname{Shv}(\mathcal{Y}) \to \mathrm{Vect}_{\mathsf{e}}$$

is continuous;

(iv) *The functor*

$$\mathcal{F}' \mapsto \mathrm{C}^\cdot(\mathcal{Y}, \mathcal{F} \overset{!}{\otimes} \mathcal{F}'), \quad \operatorname{Shv}(\mathcal{Y}) \to \mathrm{Vect}_{\mathsf{e}}$$

is cohomologically bounded on the right;

(v) *The functor*

$$\mathcal{F}' \mapsto \mathrm{C}^\cdot_c(\mathcal{Y}, \mathcal{F} \overset{*}{\otimes} \mathcal{F}'), \quad \operatorname{Shv}(\mathcal{Y}) \to \mathrm{Vect}_{\mathsf{e}}$$

is cohomologically bounded on the left;

(vi) *The functor*

$$\mathcal{F}' \mapsto \mathrm{C}^\cdot_\blacktriangle(\mathcal{Y}, \mathcal{F} \overset{!}{\otimes} \mathcal{F}'), \quad \operatorname{Shv}(\mathcal{Y}) \to \mathrm{Vect}_{\mathsf{e}}$$

is cohomologically bounded on the left;

(vii) *The natural transformation*

$$\mathrm{C}^\cdot_\blacktriangle(\mathcal{Y}, \mathcal{F} \overset{!}{\otimes} \mathcal{F}') \to \mathrm{C}^\cdot(\mathcal{Y}, \mathcal{F} \overset{!}{\otimes} \mathcal{F}'), \quad \mathcal{F}' \in \operatorname{Shv}(\mathcal{Y})$$

is an isomorphism;

(ix) *For any schematic quasi-compact morphism $g : \mathcal{Y}' \to \mathcal{Y}$ and $f : \mathcal{Y}' \to S$ where S is a scheme, the object $g_* \circ f^!(\mathcal{F})$ is cohomologically bounded above;*

(ix') *For any schematic quasi-compact morphism $g : \mathcal{Y}' \to \mathcal{Y}$ and $f : \mathcal{Y}' \to S$ where S is a scheme, the object $g_! \circ f^*(\mathcal{F})$ is cohomologically bounded below;*

(x) *Same as* (ix) *but g is a finite étale map onto a locally closed substack of $\mathcal{Y}$;*

(x') *Same as* (ix') *but g is a finite étale map onto a locally closed substack of $\mathcal{Y}$.*

C.3.6. For future reference, we note that given a morphism $f : \mathcal{Y} \to \mathcal{Y}'$, where $\mathcal{Y}$ is duality-adapted, we can define a functor

$$f_\blacktriangle : \operatorname{Shv}(\mathcal{Y}) \to \operatorname{Shv}(\mathcal{Y}')$$

to be the ind-restriction of the (a priori discontinuous) functor f_* to $\operatorname{Shv}(\mathcal{Y})^c \subset \operatorname{Shv}(\mathcal{Y}')$.

By construction, we have a natural transformation

$$f_\blacktriangle \to f_*,$$

which is an isomorphism when evaluated on compact objects.

The functor $f_\blacktriangle$ satisfies the projection formula

$$f_\blacktriangle(\mathcal{F}) \overset{!}{\otimes} \mathcal{F}' \simeq f_\blacktriangle(\mathcal{F} \overset{!}{\otimes} f^!(\mathcal{F}')).$$

From here, one deduces that $f_\blacktriangle$ satisfies base change: for a Cartesian diagram

$$\begin{tikzcd} \mathcal{Y}_1 \arrow[r, "g"] \arrow[d, "f'"'] & \mathcal{Y}_2 \arrow[d, "f_2"] \\ \mathcal{Y}'_1 \arrow[r, "g'"] & \mathcal{Y}'_2 \end{tikzcd}$$

we have a canonical isomorphism

$$(g')^! \circ (f_2)_\blacktriangle \simeq (f_1)_\blacktriangle \circ g^!. \tag{C.4}$$

Furthermore, if $\mathcal{Y}'$ is also duality-adapted and we have a morphism $g : \mathcal{Y}' \to \mathcal{Y}''$, we have a canonical isomorphism

$$g_\blacktriangle \circ f_\blacktriangle \simeq (g \circ f)_\blacktriangle.$$

C.4. **The renormalized category of sheaves.**

C.4.1. Let $\mathcal{Y}$ be a quasi-compact algebraic stack.

We define the renormalized version of the category of sheaves on $\mathcal{Y}$, denoted $\mathrm{Shv}(\mathcal{Y})^{\mathrm{ren}}$ to be

$$\mathrm{Ind}(\mathrm{Shv}(\mathcal{Y})^{\mathrm{constr}}).$$

This construction mimics the construction of how one defines $\mathrm{IndCoh}(S)$ for an eventually coconnective affine scheme, and shares its formal properties:

- Ind-extension of the tautological embedding $\mathrm{Shv}(\mathcal{Y})^{\mathrm{constr}} \hookrightarrow \mathrm{Shv}(\mathcal{Y})$ defines a functor
$$\mathrm{unren} : \mathrm{Shv}(\mathcal{Y})^{\mathrm{ren}} \to \mathrm{Shv}(\mathcal{Y}).$$
- Ind-extension of the tautological embedding $\mathrm{Shv}(\mathcal{Y})^c \hookrightarrow \mathrm{Shv}(\mathcal{Y})^{\mathrm{constr}}$ defines a fully faithful functor
$$\mathrm{ren} : \mathrm{Shv}(\mathcal{Y}) \to \mathrm{Shv}(\mathcal{Y})^{\mathrm{ren}},$$
which is the left adjoint of unren.
- Ind-extension of the t-structure on $\mathrm{Shv}(\mathcal{Y})^{\mathrm{constr}}$ defines a t-structure on $\mathrm{Shv}(\mathcal{Y})^{\mathrm{ren}}$. The functor unren is t-exact and induces an equivalence on eventually coconnective (a.k.a. bounded below) subcategories.
- The functor ren realizes $\mathrm{Shv}(\mathcal{Y})$ as the co-localization of $\mathrm{Shv}(\mathcal{Y})^{\mathrm{ren}}$ with respect to the subcategory consisting of objects all of whose cohomologies with respect to the above t-structure vanish.
- The operation of *-tensor product makes $\mathrm{Shv}(\mathcal{Y})^{\mathrm{ren}}$ into a symmetric monoidal category, and $\mathrm{Shv}(\mathcal{Y})$ into a module category over it (see Lemma C.3.4). The same is true for the !-tensor product provided that $\mathcal{Y}$ is duality-adapted.

C.4.2. Note that Verdier duality

$$\mathbb{D} : (\mathrm{Shv}(\mathcal{Y})^{\mathrm{constr}})^{\mathrm{op}} \to \mathrm{Shv}(\mathcal{Y})^{\mathrm{constr}}$$

defines an identification

$$\mathrm{Shv}(\mathcal{Y})^{\mathrm{ren}} \simeq (\mathrm{Shv}(\mathcal{Y})^{\mathrm{ren}})^\vee.$$

Assume for a moment that $\mathcal{Y}$ is duality-adapted. In particular, we have also the identification

$$\mathrm{Shv}(\mathcal{Y}) \simeq \mathrm{Shv}(\mathcal{Y})^\vee.$$

The functors ren and unren are mutually dual with respect to these identifications.

C.4.3. Let us consider the example of $\mathcal{Y} = \mathrm{pt}/G$. In this case $\mathrm{Shv}(\mathrm{pt}/G)^{\mathrm{ren}}$ is compactly generated by the object $\underline{\mathsf{e}}_{\mathrm{pt}/G}$. Hence, we obtain a canonical equivalence

$$\mathrm{Shv}(\mathrm{pt}/G)^{\mathrm{ren}} \simeq \mathrm{C}^\cdot(\mathrm{pt}/G)\text{-mod}. \tag{C.5}$$

Under this equivalence, the symmetric monoidal structure on $\mathrm{Shv}(\mathrm{pt}/G)^{\mathrm{ren}}$ given by *-tensor product corresponds to the usual symmetric monoidal structure on the category of modules over a commutative algebra.

Recall that $\mathrm{C}^\cdot(\mathrm{pt}/G)$ is isomorphic to a polynomial algebra on generators in even degrees. The canonical point

$$\pi_G : \mathrm{pt} \to \mathrm{pt}/G$$

defines an augmentation module

$$\mathsf{e} \in \mathrm{C}^\cdot(\mathrm{pt}/G)\text{-mod}.$$

Note that under the equivalence (C.5), we have

$$\mathsf{e} \in \mathrm{C}^\cdot(\mathrm{pt}/G)\text{-mod} \;\leftrightarrow\; \mathrm{ren}((\pi_G)_*(\mathsf{e})) \in \mathrm{Shv}(\mathrm{pt}/G)^{\mathrm{ren}}.$$

Hence, under (C.5), the (isomorphic) essential image of the functor ren corresponds to the full subcategory

$$\mathrm{C}^\cdot(\mathrm{pt}/G)\text{-mod}_0 \subset \mathrm{C}^\cdot(\mathrm{pt}/G)\text{-mod}$$

be the full subcategory generated by the the augmentation module e.

C.4.4. Let $\mathcal{Y}$ be of the form S/G, where S is a quasi-compact scheme. The functor of *- (resp., !-) pullback

$$\mathrm{Shv}(S/G)^{\mathrm{ren}} \to \mathrm{Shv}(S/G)$$

has a natural symmetric monoidal structure with respect to the *- (resp., !-) tensor product operation.

We claim:

Proposition C.4.5. *The colocalization*

$$\mathrm{Shv}(\mathcal{Y})^{\mathrm{ren}} \leftrightarrows \mathrm{Shv}(\mathcal{Y})$$

identifies with the colocalization

$$\mathrm{Shv}(\mathcal{Y})^{\mathrm{ren}} \simeq \mathrm{Shv}(\mathcal{Y})^{\mathrm{ren}} \underset{\mathrm{Shv}(\mathrm{pt}/G)^{\mathrm{ren}}}{\otimes} \mathrm{Shv}(\mathrm{pt}/G)^{\mathrm{ren}} \leftrightarrows \mathrm{Shv}(\mathcal{Y})^{\mathrm{ren}} \underset{\mathrm{Shv}(\mathrm{pt}/G)^{\mathrm{ren}}}{\otimes} \mathrm{Shv}(\mathrm{pt}/G)$$

*(for either *- or !- monoidal structures).*

Proof. The functor

$$\mathrm{unren} : \mathrm{Shv}(\mathcal{Y})^{\mathrm{ren}} \to \mathrm{Shv}(\mathcal{Y})$$

clearly factors as

$$\mathrm{Shv}(\mathcal{Y})^{\mathrm{ren}} \to \mathrm{Shv}(\mathcal{Y})^{\mathrm{ren}} \underset{\mathrm{Shv}(\mathrm{pt}/G)^{\mathrm{ren}}}{\otimes} \mathrm{Shv}(\mathrm{pt}/G) \to \mathrm{Shv}(\mathcal{Y}).$$

Hence, to prove the proposition it suffices to show that the essential image of

$$\mathrm{Shv}(\mathcal{Y})^{\mathrm{ren}} \underset{\mathrm{Shv}(\mathrm{pt}/G)^{\mathrm{ren}}}{\otimes} \mathrm{Shv}(\mathrm{pt}/G) \to \mathrm{Shv}(\mathcal{Y})^{\mathrm{ren}}$$

is contained in that of

$$\mathrm{Shv}(\mathcal{Y}) \overset{\mathrm{ren}}{\hookrightarrow} \mathrm{Shv}(\mathcal{Y})^{\mathrm{ren}}.$$

For that end it suffices to show that for $\mathcal{F} \in \mathrm{Shv}(\mathcal{Y})^{\mathrm{constr}}$, we have

$$\mathcal{F} \overset{*}{\otimes} (p_S/G)^*((\pi_G)_*(\mathsf{e})) \in \mathrm{Shv}(\mathcal{Y})^c.$$

However, that has been established in Sect. C.2.9.

□

C.4.6. Let now $\mathcal{Y}$ be a not necessarily quasi-compact algebraic stack. We let

$$\mathrm{Shv}(\mathcal{Y})^{\mathrm{ren}} := \lim_{\mathcal{U}} \mathrm{Shv}(\mathcal{U})^{\mathrm{ren}}, \tag{C.6}$$

where the limit is taken over the index category of quasi-compact open substacks $\mathcal{U} \subset \mathcal{Y}$, and the transition functors are given by restriction.

The properties and structures listed in Sect. C.4.1 for the opens $\mathcal{U}$ induce the corresponding properties and structures on $\mathcal{Y}$. In particular, we have an adjunction

$$\mathrm{unren} : \mathrm{Shv}(\mathcal{Y})^{\mathrm{ren}} \leftrightarrow \mathrm{Shv}(\mathcal{Y}) : \mathrm{ren},$$

with ren fully faithful, a t-structure on $\mathrm{Shv}(\mathcal{Y})^{\mathrm{ren}}$, etc.

Note also that the transition functors in forming the limit (C.6) admit left adjoints, given by !-extension. Hence, we can rewrite $\mathrm{Shv}(\mathcal{Y})^{\mathrm{ren}}$ as

$$\operatorname*{colim}_{\mathcal{U}} \mathrm{Shv}(\mathcal{U})^{\mathrm{ren}},$$

where the transition functors are given by !-extension.

In particular, we obtain that $\mathrm{Shv}(\mathcal{Y})^{\mathrm{ren}}$ is compactly generated by objects of the form $j_!(\mathcal{F})$, where

$$j : \mathcal{U} \hookrightarrow \mathcal{Y}$$

with $\mathcal{U}$ quasi-compact and $\mathcal{F} \in \mathrm{Shv}(U)^{\mathrm{constr}}$.

C.5. **Singular support on stacks.**

C.5.1. Let $\mathcal{Y}$ be an algebraic stack. Let $\mathcal{N}$ be a conical Zariski-closed subset in $T^*(\mathcal{Y})$, see [GKRV, Sect. A.3.4] for what this means.

We define the full subcategory

$$\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y}) \subset \mathrm{Shv}(\mathcal{Y})$$

to consist of those $\mathcal{F} \in \mathrm{Shv}(\mathcal{Y})$ whose pullback under any smooth map $S \to \mathcal{Y}$ (with S a scheme) belong to

$$\mathrm{Shv}_{\mathcal{N}_S}(S).$$

where

$$\mathcal{N}_S := \mathcal{N} \underset{\mathcal{Y}}{\times} S \subset T^*(\mathcal{Y}) \underset{\mathcal{Y}}{\times} S \subset T^*(S).$$

Remark C.5.2. The category $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})$ defined above is different from the category denoted by the same symbol in [GKRV]. In our current notations, the category in *loc.cit.* is

$$\lim_{S} \mathrm{Shv}_{\mathcal{N}_S}(S)^{\mathrm{access}}$$

(the limit taken over the category of affine schemes S smooth over $\mathcal{Y}$ and smooth maps between such).

Since for schemes, the functor $\mathrm{Shv}_{\mathcal{N}_S}(S)^{\mathrm{access}} \to \mathrm{Shv}_{\mathcal{N}_S}(S)$ is fully faithful, the category in [GKRV] embeds fully faithfully into our $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})$.

C.5.3. Assume that $\mathcal{Y}$ is quasi-compact. We will consider the following three variants of the category of sheaves on $\mathcal{Y}$ with singular support in $\mathcal{N}$:

One is $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})$ as defined above.

Set

$$\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{constr}} := \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y}) \cap \mathrm{Shv}(\mathcal{Y})^{\mathrm{constr}}$$

and define

$$\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{ren}} := \mathrm{Ind}(\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{constr}}).$$

Note that we have a tautologically defined functor

$$\mathrm{unren}_{\mathcal{N}} : \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{ren}} \to \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y}). \tag{C.7}$$

The third category, denoted $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{access}}$ is the full subcategory in $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})$ generated by the essential image of the functor $\mathrm{unren}_{\mathcal{N}}$ above.

C.5.4. Thus, we have the functors

$$\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{ren}} \overset{\mathrm{unren}_{\mathcal{N}}}{\longrightarrow} \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{access}} \hookrightarrow \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y}). \tag{C.8}$$

All three categories carry natural t-structures and the functors in (C.9) are t-exact. Furthermore, it is easy to see that the functor

$$\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{access}} \subset \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y}) \tag{C.9}$$

realizes $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})$ as the left completion of $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{access}}$ with respect to its t-structure.

C.5.5. We give the following definitions:

Definition C.5.6. *We shall say that the pair* $(\mathcal{Y}, \mathcal{N})$ *is* renormalization-adapted *if the category* $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{access}}$ *is generated by objects that are compact as objects of* $\mathrm{Shv}(\mathcal{Y})$*.*

Definition C.5.7. *We shall say that the pair* $(\mathcal{Y}, \mathcal{N})$ *is* constraccessible *if the inclusion* (C.9) *is an equality.*

Some remarks are in order:

(I) If $\mathcal{Y} = S$ is a scheme, it is tautologically renormalization-adapted.

(II) If $(\mathcal{Y}, \mathcal{N})$ is renormalization-adapted, the functor $\mathrm{unren}_{\mathcal{N}}$ admits a left adjoint, to be denoted $\mathrm{ren}_{\mathcal{N}}$, and the resulting adjunction

$$\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{ren}} \leftrightarrows \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{access}}$$

has the same formal properties as the ones listed in Sect. C.4.1.

(III) If $(\mathcal{Y}, \mathcal{N})$ is renormalization-adapted and $\mathcal{Y}$ is duality-adapted, then the category $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{access}}$ is naturally self-dual, with the pairing

$$\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{access}} \otimes \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{access}} \to \mathrm{Vect}_{\mathsf{e}}$$

given by

$$\mathcal{F}_1, \mathcal{F}_2 \mapsto \mathrm{C}^{\cdot}_{\blacktriangle}(\mathcal{Y}, \mathcal{F}_1 \overset{!}{\otimes} \mathcal{F}_2).$$

(IV) A pair $(\mathcal{Y}, \mathcal{N})$ *may not be* constraccessible even if $\mathcal{Y} = S$ is a scheme and $\mathcal{N} = \{0\}$ (see Remark B.4.4).

(V) The pair $(\mathcal{Y}, \mathcal{N})$ is both renormalization-adapted and constraccessible if and only if $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})$ is generated by objects that are compact in $\mathrm{Shv}(\mathcal{Y})$. (In particular, if $\mathcal{N}$ is all of $T^*(\mathcal{Y})$, then $(\mathcal{Y}, \mathcal{N})$ is both renormalization-adapted and constraccessible.)

C.5.8. We claim:

Proposition C.5.9. *Suppose that* $\mathcal{Y}$ *can be covered by open substacks* $\mathcal{U}$*, such that:*

(i) *Each* $\mathcal{U}$ *is a global quotient, i.e.,* $\mathcal{U} = S/G$*, where* S *is a scheme and* G *an algebraic group.*

(ii) *The !- (equivalently, *-) pushforward functor* $\mathrm{Shv}(\mathcal{U})^{\mathrm{constr}} \to \mathrm{Shv}(\mathcal{Y})^{\mathrm{constr}}$ *sends* $\mathrm{Shv}_{\mathcal{N}}(\mathcal{U})$ *to* $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})$*.*

Then $(\mathcal{Y}, N)$ *is renormalization-adapted for any* $\mathcal{N}$*.*

Proof. The proof repeats the argument from (I) in the proof of Theorem C.2.6. □

Based on the above proposition, we propose:

Conjecture C.5.10. *For any quasi-compact algebraic stack with an affine diagonal, and any* $\mathcal{N} \subset T^*(\mathcal{Y})$*, the pair* $(\mathcal{Y}, \mathcal{N})$ *is renormalization-adapted.*

C.5.11. Let now $\mathcal{Y}$ be not necessarily quasi-compact. By definition,

$$\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y}) \simeq \lim_{\mathcal{U}} \mathrm{Shv}_{\mathcal{N}}(U),$$

where the index category is the poset of quasi-compact open substacks of $\mathcal{Y}$, and the transition functors used in forming the limit are given by restriction.

We define the categories

$$\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{ren}} \text{ and } \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{access}} \tag{C.10}$$

similarly:

$$\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{ren}} := \lim_{\mathcal{U}} \mathrm{Shv}_{\mathcal{N}}(U)^{\mathrm{ren}} \text{ and } \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{access}} := \lim_{\mathcal{U}} \mathrm{Shv}_{\mathcal{N}}(U)^{\mathrm{access}}.$$

It follows formally that $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})$ identifies with the left completion of $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{access}}$ with respect to its t-structure.

We shall say that $(\mathcal{Y}, \mathcal{N})$ is renormalization-adapted if all of its quasi-compact open substacks have this property (equivalently, if $\mathcal{Y}$ can be covered by such).

We shall say that $(\mathcal{Y}, \mathcal{N})$ is constraccessible if all of its quasi-compact open substacks have this property (equivalently, if $\mathcal{Y}$ can be covered by such). This is equivalent to the requirement that the inclusion

$$\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{access}} \subset \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})$$

be an equality.

C.5.12. Assume now that $\mathcal{Y}$ has the following property: $\mathcal{Y}$ can be covered by quasi-compact open substacks $\mathcal{U}_i$, such that for every i and the corresponding open embedding

$$\mathcal{U}_i \overset{j_i}{\hookrightarrow} \mathcal{Y},$$

the functor

$$(j_i)_! : \mathrm{Shv}(\mathcal{U}_i)^{\mathrm{constr}} \to \mathrm{Shv}(\mathcal{Y})^{\mathrm{constr}}$$

sends

$$\mathrm{Shv}_{\mathcal{N}}(\mathcal{U}_i)^{\mathrm{constr}} \to \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{constr}}.$$

In this case, the transition functors used in forming the limits

$$\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y}),\ \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{ren}} \text{ and } \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{access}}$$

admit left adjoints, given by !-extension.

Hence, these limits can be rewritten as colimits, where we use !-extensions as transition functors.

In particular, in this case, $(\mathcal{Y}, \mathcal{N})$ is renormalization-adapted and constraccessible if and only if $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})$ is generated by objects that are compact in the ambient category $\mathrm{Shv}(\mathcal{Y})$.

C.6. **Product theorems for stacks.** In this subsection we will prove versions of Theorems B.5.2, B.5.5 and B.5.8 for stacks.

C.6.1. Let $\mathcal{Y}_1$ and $\mathcal{Y}_2$ be a pair of algebraic stacks. First, by [GKRV, Proposition A.2.10], the external tensor product functor

$$\mathrm{Shv}(\mathcal{Y}_1) \otimes \mathrm{Shv}(\mathcal{Y}_2) \to \mathrm{Shv}(\mathcal{Y}_1 \times \mathcal{Y}_2) \tag{C.11}$$

is fully faithful. It is also easy to see that it is t-exact.

An argument parallel to *loc. cit.* shows that the functor

$$\mathrm{Shv}(\mathcal{Y}_1)^{\mathrm{ren}} \otimes \mathrm{Shv}(\mathcal{Y}_2)^{\mathrm{ren}} \to \mathrm{Shv}(\mathcal{Y}_1 \times \mathcal{Y}_2)^{\mathrm{ren}} \tag{C.12}$$

is also fully faithful and t-exact.

C.6.2. Let $\mathcal{N} \subset T^*(\mathcal{Y})$ a Zariski-closed conical subset. From the fact that (C.12) is fully faithful, we formally obtain that for a scheme X, the functor

$$\text{IndLisse}(X) \otimes \text{Shv}_{\mathcal{N}}(\mathcal{Y})^{\text{ren}} \to \text{Shv}_{\{0\}\times\mathcal{N}}(X \times \mathcal{Y})^{\text{ren}} \tag{C.13}$$

is also fully faithful (if $\mathcal{Y}$ is not quasi-compact we use the fact that $\text{IndLisse}(X)$ is dualizable and hence the operation $\text{IndLisse}(X) \otimes -$ commutes with limits).

From now on let us assume that $\mathcal{N}$ is half-dimensional.

First, we claim:

Theorem C.6.3. *Let X be smooth and proper. Then the functor* (C.13) *is an equivalence.*

The proof repeats the argument of [GKRV, Sect. A.5].

C.6.4. Next, we claim:

Theorem C.6.5. *Let X be smooth and proper. Then the functor*

$$\text{IndLisse}(X) \otimes \text{Shv}_{\mathcal{N}}(\mathcal{Y})^{\text{access}} \to \text{Shv}_{\{0\}\times\mathcal{N}}(X \times \mathcal{Y})^{\text{access}} \tag{C.14}$$

is an equivalence.

Proof. Since $\text{IndLisse}(X)$ is dualizable, we reduce to the case when $\mathcal{Y}$ is quasi-compact (see above). We have a commutative diagram

$$\begin{tikzcd}
\text{IndLisse}(X) \otimes \text{Shv}_{\mathcal{N}}(\mathcal{Y})^{\text{access}} \arrow[r] \arrow[d] & \text{Shv}_{\{0\}\times\mathcal{N}}(X \times \mathcal{Y})^{\text{access}} \arrow[d] \\
\text{IndLisse}(X) \otimes \text{Shv}_{\mathcal{N}}(\mathcal{Y}) \arrow[d] & \text{Shv}_{\{0\}\times\mathcal{N}}(X \times \mathcal{Y}) \arrow[d] \\
\text{IndLisse}(X) \otimes \text{Shv}(\mathcal{Y}) \arrow[r] & \text{Shv}(X) \otimes \text{Shv}(\mathcal{Y})
\end{tikzcd}$$

with vertical arrows being fully faithful (for the left column this again relies on the fact that $\text{IndLisse}(X)$ is dualizable). The bottom horizontal arrow is also fully faithful (because $\text{Shv}(\mathcal{Y})$) is dualizable. This implies that the top horizontal arrow, i.e., our functor (C.14), is fully faithful.

Hence, it suffices to show that the right adjoint of (C.14) is conservative (this right adjoint is continuous since the functor (C.14) sends compacts to compacts). Let us describe this right adjoint explicitly.

Recall that the category $\text{IndLisse}(X)$ is canonically self-dual, see Remark B.3.5. In terms of this self-duality, the right adjoint to (C.14) corresponds to the functor

$$\text{IndLisse}(X) \otimes \text{Shv}_{\{0\}\times\mathcal{N}}(X \times \mathcal{Y})^{\text{access}} \to \text{Shv}_{\mathcal{N}}(\mathcal{Y})^{\text{access}}$$

given by

$$E, \mathcal{F} \mapsto (p_Y)_*(p_X^!(E) \otimes \mathcal{F}).$$

where p_X and p_Y are the two projections from $X \times Y$ to X and Y, respectively.

This description of the right adjoint that it commutes with pullbacks for smooth maps $\mathcal{Y}' \to \mathcal{Y}$. This allows to replace $\mathcal{Y}$ by an affine scheme covering it. In the latter case, the assertion that right adjoint to (C.14) is conservative follows from Theorem B.5.2.

□

C.6.6. Finally, we claim:

Theorem C.6.7. *Let X be smooth and proper. Assume also that* $\mathrm{QLisse}(X)$ *Verdier-compatible (see Sect. B.3). Let $\mathcal{N} \subset T^*(\mathcal{Y})$ be half-dimensional. Then the resulting functor*

$$\mathrm{QLisse}(X) \otimes \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y}) \to \mathrm{Shv}_{\{0\}\times\mathcal{N}}(X \times \mathcal{Y})$$

is an equivalence.

Proof. Since $\mathrm{QLisse}(X)$ is assumed dualizable, the assertion reduces to the case of schemes. In the latter case, it is given by Theorem B.5.8.

□

By a similar logic, from Theorem B.5.5 we obatin:

Theorem C.6.8. *Let X be smooth and proper. Assume that* $\mathrm{QLisse}(X)$ *is dualizable as a DG category. Then the functor*

$$\mathrm{QLisse}(X) \otimes \mathrm{Shv}(\mathcal{Y}) \to \mathrm{Shv}(X \times \mathcal{Y})$$

is an equivalence into the full subcategory of $\mathrm{Shv}(X \times \mathcal{Y})$ *that consists of objects of whose perverse cohomologies have singular support contained in* $\{0\} \times T^*(Y)$.

C.7. **Compactness in the Betti situation.** The goal of this subsection is to prove Lemma 10.4.4.

C.7.1. We start by quoting the following:

Proposition C.7.2. *Let Y be a scheme and let $\mathcal{N} \subset T^*(Y)$ be a Zariski-closed conical Largrangian subset. Then the category* $\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y)$ *is self-dual and* smooth*, i.e., the unit object*

$$u_{\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y)} \subset \mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y) \otimes \mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y)$$

is compact.

For completeness, we will supply a proof, see below.

C.7.3. *Proof of Lemma 10.4.4, the case of schemes.* We first consider the case when $\mathcal{Y} = Y$ is a quasi-compact scheme. The argument below was explained to us by D. Nadler, see [Na].

Choose a μ-stratification on Y, so that $\mathcal{N}$ is contained in the closure of the union of conormals to the strata (this is possible thanks to [KS, Corolllary 8.3.22]).

Choose a point s_i on each connected component of each stratum. For every i, the functor of !-fiber at s_i commutes with limits (on all of $\mathrm{Shv}^{\mathrm{all}}(Y)$ and hence on $\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y)$ as the latter is closed under limits). Hence, when considered as a functor on $\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y)$, it is corepresentable. Denote the corepresenting object by P_i.

Now, objects from $\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y)$ are locally constant along the strata; hence the above functor is continuous on $\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y)$. Hence, the objects $P_i \in \mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y)$ are compact. They also generate $\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y)$: every object of this category has a non-zero !-restriction to at least one stratum.

Thus, the objects P_i form a *finite* collection of compact generators of $\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y)$.

Let now $\mathcal{F}$ be an object from

$$\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(S) \cap \mathrm{Shv}(S)^{\mathrm{constr}}.$$

Then for each i, the vector space

$$\mathcal{H}om_{\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(S)}(P_i, \mathcal{F}) \in \mathrm{Vect}_{\mathsf{e}}$$

is finite-dimensional.

The assertion of the lemma follows now from the next general observation:

Lemma C.7.4. *Let $\mathbf{C}$ be a dualizable DG category that admits a finite number of compact generators $\mathbf{c}_i$. Let $\mathbf{c} \in \mathbf{C}$ be an object such that all $\mathcal{H}om_{\mathbf{C}}(\mathbf{c}_i, \mathbf{c})$ are finite-dimensional. Assume now that $\mathbf{C}$ is smooth, i.e., the unit object in $\mathbf{C} \otimes \mathbf{C}^\vee$ is compact. Then $\mathbf{c}$ is compact.*

C.7.5. *Proof of Lemma 10.4.4, the case of stacks.* Since compact objects in $\mathrm{Shv}(\mathcal{Y})$ are !-extensions from quasi-compact substacks, we can assume that $\mathcal{Y}$ is quasi-compact.

Choose a smooth cover $f : S \to \mathcal{Y}$ with S a quasi-compact scheme. We claim that for $\mathcal{F} \in \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{constr}}$, the functor

$$\mathcal{F}' \mapsto \mathcal{H}om_{\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(\mathcal{Y})}(\mathcal{F}, \mathcal{F}')$$

commutes with colimits for

$$\mathcal{F}' \in \mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(\mathcal{Y})^{\geq -n}$$

for any fixed n.

Indeed, this follows from the assertion of the lemma in the case of schemes, using a standard argument with truncated totalizations.

Hence, to prove the lemma, it suffices to show that for $\mathcal{F} \in \mathrm{Shv}(\mathcal{Y})^c$ the functor

$$\mathcal{F}' \mapsto \mathcal{H}om_{\mathrm{Shv}^{\mathrm{all}}(\mathcal{Y})}(\mathcal{F}, \mathcal{F}'), \quad \mathcal{F}' \in \mathrm{Shv}^{\mathrm{all}}(\mathcal{Y})$$

the functor has a finite cohomological amplitude.

We can assume that $\mathcal{F}$ has the form $f_!(\mathcal{F}_S)$ for (f, S) as above and $\mathcal{F}_S \in \mathrm{Shv}(S)^{\mathrm{constr}}$. Hence, the assertion reduces to the case when $\mathcal{Y} = S$ is a quasi-compact scheme. However, in this case the assertion follows from the fact that the category $\mathrm{Shv}^{\mathrm{all}}(S)$ has a cohomological dimension bounded by $2\dim(S)$. $\square$[Lemma 10.4.4]

C.7.6. The rest of this subsection is devoted to the proof of Proposition C.7.2.

First, we recall some facts:

(I) The functor

$$\mathrm{Shv}^{\mathrm{all}}(Y) \otimes \mathrm{Shv}^{\mathrm{all}}(Y) \to \mathrm{Shv}^{\mathrm{alll}}(Y \times Y),$$

given by external tensor product, is an equivalence (see [Lu1, Theorem 7.3.3.9, Proposition 7.3.1.11] and [Lu2, Proposition 4.8.1.17]).

(II) The functors

$$\mathrm{Shv}^{\mathrm{all}}(Y) \otimes \mathrm{Shv}^{\mathrm{all}}(Y) \overset{\Delta_Y^*}{\to} \mathrm{Shv}^{\mathrm{all}}(Y) \overset{\mathrm{C}^{\cdot}_c(Y,-)}{\longrightarrow} \mathrm{Vect}_{\mathsf{e}} \tag{C.15}$$

and

$$\mathrm{Vect}_{\mathsf{e}} \overset{\mathsf{e}_Y}{\longrightarrow} \mathrm{Shv}^{\mathrm{all}}(Y) \overset{(\Delta_Y)_!}{\longrightarrow} \mathrm{Shv}^{\mathrm{all}}(Y \times Y) \simeq \mathrm{Shv}^{\mathrm{all}}(Y) \otimes \mathrm{Shv}^{\mathrm{all}}(Y)$$

define a self-duality on $\mathrm{Shv}^{\mathrm{all}}(Y)$.

(III) The embedding

$$\iota : \mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y) \hookrightarrow \mathrm{Shv}^{\mathrm{all}}(Y)$$

admits a left adjoint (since the essential image is closed under limits).

(IV) The (discontinuous) Verdier duality functor

$$\mathbb{D} : (\mathrm{Shv}^{\mathrm{all}}(Y))^{\mathrm{op}} \to \mathrm{Shv}^{\mathrm{all}}(Y), \quad \mathcal{H}om_{\mathrm{Shv}^{\mathrm{all}}(Y)}(\mathcal{F}', \mathbb{D}(\mathcal{F})) = \mathcal{H}om_{\mathrm{Shv}^{\mathrm{all}}(Y)}(\mathcal{F} \overset{*}{\otimes} \mathcal{F}', \omega_Y)$$

sends $(\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y))^{\mathrm{op}} \to \mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y)$.

Note that the last property has the following corollary:

Corollary C.7.7. *For $\mathcal{F} \in \mathrm{Shv}^{\mathrm{all}}(Y)$ and $E \in \mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y)$, the natural map*

$$\mathrm{C}^{\cdot}_c(Y, \mathcal{F} \overset{*}{\otimes} E) \to \mathrm{C}^{\cdot}_c(Y, \iota^L(\mathcal{F}) \overset{*}{\otimes} E)$$

is an isomorphism.

C.7.8. We define the pairing

$$\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y) \otimes \mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y) \to \mathrm{Vect}_{\mathsf{e}}$$

to be induced by the pairing (C.15).

We set

$$u_{\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y)} = (\iota^L \otimes \iota^L)(u_{\mathrm{Shv}^{\mathrm{all}}(Y)}).$$

It follows from Corollary C.7.7 and (II) above that these data define a self-duality on $\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y)$. Note that $u_{\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y)}$ is actually isomorphic to

$$(\mathrm{Id} \otimes \iota^L)(u_{\mathrm{Shv}^{\mathrm{all}}(Y)}) \simeq (\iota^L \otimes \mathrm{Id})(u_{\mathrm{Shv}^{\mathrm{all}}(Y)}).$$

C.7.9. Finally, let us show that $u_{\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y)}$ is compact as an object of $\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y)$. I.e., we have to check that the functor

$$E_1, E_2 \mapsto \mathcal{H}om_{\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y) \otimes \mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y)}(u_{\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y)}, E_1 \boxtimes E_2)$$

is continuous.

We rewrite the above expression as

$$\mathcal{H}om((\Delta_Y)_!(\underline{\mathsf{e}}_Y), E_1 \boxtimes E_2) \simeq \mathrm{C}^{\cdot}(Y, E_1 \overset{!}{\otimes} E_2),$$

and the assertion follows from the fact that the latter functor is indeed continuous on the category $\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y) \otimes \mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y)$ (but not on all $\mathrm{Shv}^{\mathrm{all}}(Y) \otimes \mathrm{Shv}^{\mathrm{all}}(Y)$).

□[Proposition C.7.2]

Appendix D. Proof of Theorem 13.1.2

In this section, we will prove Theorem 13.1.2 in the context of ℓ-adic sheaves.

D.1. **Strategy of proof.**

D.1.1. We no restriction of generality, we can assume that $\mathcal{Y}_2$ is a smooth scheme (and hence $\mathcal{Y}_1$ is a scheme as well, since f was assumed schematic).

The assumption on $\mathcal{F}_1$ is local on $\mathcal{Y}_1$ around the point y_1. Hence, we can assume that f is proper: indeed choose a relative compactification of f

$$\mathcal{Y}_1 \overset{j}{\hookrightarrow} \overline{\mathcal{Y}}_1 \overset{\overline{f}}{\to} \mathcal{Y}_2,$$

and replace the initial $\mathcal{F}_1$ by $j_!(\mathcal{F}_1)$.

Hence, we can assume that f is proper.

D.1.2. We will now describe the framework in which we will prove Theorem 13.1.2. In fact, we will formulate an assertion of local nature with respect to $\mathcal{Y}_1$.

Fix a Zariski-closed subset $\mathcal{N}_1 \subset T^*(\mathcal{Y}_1)$. Denote

$$\mathcal{N}_{1,2} := (df^*)^{-1}(\mathcal{N}_1) \subset T^*(\mathcal{Y}_2) \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1.$$

Let $y_1 \in \mathcal{Y}_1$, $y_2 = f(y_1)$ and $\xi_2 \in T^*_{y_2}(\mathcal{Y}_2)$ be points such that

$$(\xi_2, y_1) \in T^*(\mathcal{Y}_2) \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1$$

belongs to and is isolated in the intersection

$$\mathcal{N}_{1,2} \cap (\{\xi_2\} \times f^{-1}(y_2)) \subset T^*(\mathcal{Y}_2) \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1. \tag{D.1}$$

Let $\mathcal{N}_{1,2,y_1} \subset \mathcal{N}_{1,2}$ be the union of irreducible components that contain the element (ξ_2, y_1). Let $\mathcal{N}_{2,y_1}$ be the image of $\mathcal{N}_{1,2,y_1}$ along

$$T^*(\mathcal{Y}_2) \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1 \to T^*(\mathcal{Y}_2). \tag{D.2}$$

We will assume that $\mathcal{N}_{2,y_1}$ has dimension $\dim(\mathcal{Y}_2)$ at ξ_2.

D.1.3. In what follows we will use the following notation. Given $\mathcal{Y}$ and a conical Zariski-closed subset $\mathcal{N} \subset \mathcal{Y}$ and $Z \subset \mathcal{N}$ we will let $\mathrm{Shv}_{\mathcal{N},(Z)}(\mathcal{Y})$ denote the quotient of $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})$ by all

$$\mathrm{Shv}_{\mathcal{N}'}(\mathcal{Y}) \subset \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})$$

for $\mathcal{N}' \subset \mathcal{N}$ with $\mathcal{N}' \cap Z = \emptyset$.

For $\mathcal{N}$ being all of $T^*(\mathcal{Y})$ we will simply write $\mathrm{Shv}_{(Z)}(\mathcal{Y})$.

In the applications, we will be mostly interested in the case when Z is a single point ξ for $\xi \in T^*_y(\mathcal{Y})$ for some $y \in \mathcal{Y}$.

D.1.4. We will construct a category $\mathbf{C}$, equipped with a t-structure and a t-exact functor

$$\Phi_2 : \mathrm{Shv}_{(\xi_2)}(\mathcal{Y}_2) \to \mathbf{C}$$

with the following properties:

- The composite
$$\mathrm{Shv}_{\mathcal{N}_1}(\mathcal{Y}_1) \to \mathrm{Shv}(\mathcal{Y}_2) \twoheadrightarrow \mathrm{Shv}_{(\xi_2)}(\mathcal{Y}_2) \overset{\Phi_2}{\to} \mathbf{C}$$
admits a canonically defined direct summand that factors as
$$\mathrm{Shv}_{\mathcal{N}_1}(\mathcal{Y}_1) \twoheadrightarrow \mathrm{Shv}_{\mathcal{N}_1,(\xi_2,y_1)}(\mathcal{Y}_1) \overset{\Phi_1}{\to} \mathbf{C}.$$
- The functor Φ_1 is t-exact and conservative on $\mathrm{Shv}_{\mathcal{N}_1,(\xi_2,y_1)}(\mathcal{Y}_1)^{\heartsuit}$.

Clearly, the existence of such a triple $(\mathbf{C}, \Phi_2, \Phi_1)$ implies the assertion of Theorem 13.1.2.

D.2. **Construction of** $(\mathbf{C}, \Phi_2, \Phi_1)$**.**

D.2.1. Let $\mathcal{Y}$ be a smooth scheme and $\mathcal{N} \subset T^*(\mathcal{Y})$ a conical Zariski-closed subset.

Let $g : \mathcal{Y} \to \mathbb{A}^1$ be a function with non-vanishing differential. We shall say that g is $\mathcal{N}$-*characteristic* at $y \in \mathcal{Y}$ is if the element $dg_y \in T^*_y(\mathcal{Y})$ belongs to $\mathcal{N}$.

We shall say that g is *non-characteristic with respect to* $\mathcal{N}$ if it is not $\mathcal{N}$-characteristic at all $y \in \mathcal{Y}$.

We shall say that a point $y \in \mathcal{Y}$ is an *isolated* point for the pair $(\mathcal{N}, g)$ if:

- $g(y) = 0$ and g is $\mathcal{N}$-characteristic at y;
- There exists a Zariski neighborhood $y \in \mathcal{U} \subset \mathcal{Y}$ such that g is non-characteristic with respect to $\mathcal{N}$ on $U - \{y\}$.

We will use the following assertion:

Proposition D.2.2. *Assume that* $\dim(\mathcal{N}) \leq \dim(\mathcal{Y})$. *Then for any non-zero vector* $\xi \in T^*_y(\mathcal{Y}) \cap \mathcal{N}$ *there exists a function* g *defined on a Zariski neighborhood of* y*, such that* $dg_y = \xi$ *and* y *is an isolated point for the pair* $(\mathcal{N}, g)$*.*

The proof is given in Sect. D.3.

D.2.3. Let $y \in \mathcal{Y}$ be isolated for $(\mathcal{N}, g)$. Replacing $\mathcal{Y}$ by a Zariski neighborhood of y, let us assume that g is defined on all of $\mathcal{Y}$ and is non-characteristic with respect to $\mathcal{N}$ on $\mathcal{Y} - \{y\}$.

Consider the Vanishing Cycles functor

$$\Phi_g : \mathrm{Shv}(\mathcal{Y}) \to \mathrm{Shv}(\mathcal{Y}_0),$$

where $\mathcal{Y}_0 := \mathcal{Y} \underset{\mathbb{A}^1}{\times} \{0\}$.

Note that the condition that $y \in \mathcal{Y}$ is an isolated point for the pair $(\mathcal{N}, g)$ implies that the objects in the image of Φ_g on $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})$ are supported on

$$\{y\} \subset \mathcal{Y}_0.$$

Thus, we obtain that Φ_g is a functor

$$\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y}) \to \mathrm{Vect}_{\mathsf{e}}\,.$$

Furthermore, it is t-exact and factors as

$$\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y}) \twoheadrightarrow \mathrm{Shv}_{\mathcal{N},(\xi)}(\mathcal{Y}) \to \mathrm{Vect}_{\mathsf{e}},$$

see Sect. D.1.3 for the notation.

We have the following fundamental result:

Theorem D.2.4. *The above functor* $\mathrm{Shv}_{\mathcal{N},(\xi)}(\mathcal{Y}) \to \mathrm{Vect}_{\mathsf{e}}$ *is conservative on* $\mathrm{Shv}_{\mathcal{N},(\xi)}(\mathcal{Y})^{\heartsuit}$.

For ℓ-adic sheaves, this theorem follows [Sai, Theorem 5.9].

D.2.5. We are now ready to perform the construction from Sect. D.1.4.

Up to shrinking $\mathcal{Y}_2$ around y_2, applying Proposition D.2.2, choose a function g such that y_2 is an isolated point for $(\mathcal{N}_{2,y_1}, g)$.

Set

$$\mathcal{Y}_{2,0} := \mathcal{Y}_2 \underset{\mathbb{A}^1}{\times} \{0\}.$$

Let

$$\mathbf{C} := \underset{y_2 \in U \subset \mathcal{Y}_{2,0}}{\mathrm{colim}}\ \mathrm{Shv}(U),$$

where the colimit is taken over the poset of Zariski neighborhoods of y_2 in $\mathcal{Y}_{2,0}$.

Consider the functor

$$\mathrm{Shv}(\mathcal{Y}_2) \overset{\Phi_g}{\to} \mathrm{Shv}(\mathcal{Y}_{2,0}) \to \mathbf{C}.$$

By construction, this functor factors through the quotient

$$\mathrm{Shv}(\mathcal{Y}_2) \twoheadrightarrow \mathrm{Shv}_{(\xi_2)}(\mathcal{Y}_2) \to \mathbf{C},$$

and we let Φ_2 be the resulting functor $\mathrm{Shv}_{(\xi_2)}(\mathcal{Y}_2) \to \mathbf{C}$.

D.2.6. Consider the function $g \circ f : \mathcal{Y}_1 \to \mathbb{A}^1$. Set

$$\mathcal{Y}_{1,0} := \mathcal{Y}_1 \underset{\mathbb{A}^1}{\times} \{0\}.$$

Note now that the assumption that y_2 is isolated for $(\mathcal{N}_{2,y_1}, g)$, combined with the assumption that y_1 is isolated in (D.1) implies that y_1 is isolated for $(\mathcal{N}_1, g \circ f)$.

This implies that the functor

$$\Phi_{g\circ f} : \mathrm{Shv}_{\mathcal{N}_1}(\mathcal{Y}_1) \to \mathrm{Shv}(\mathcal{Y}_{1,0})$$

splits canonically as a direct sum

$$\Phi_{g\circ f} \simeq \Phi'_{g\circ f} \oplus \Phi''_{g\circ f},$$

where objects in the image of $\Phi'_{g\circ f}$ are supported at $\{y_1\} \subset \mathcal{Y}_{1,0}$ and objects in the image of $\Phi''_{g\circ f}$ are supported on a closed subset of $\mathcal{Y}_{1,0}$ disjoint from $\{y_1\}$.

Note that we can regard $\Phi'_{g\circ f}$ as a functor with values in $\mathrm{Vect}_{\mathsf{e}}$, and it naturally factors as

$$\mathrm{Shv}_{\mathcal{N}_1}(\mathcal{Y}_1) \to \mathrm{Shv}_{\mathcal{N}_1,(\xi_2,y_1)}(\mathcal{Y}_1) \to \mathrm{Vect}_{\mathsf{e}}\,.$$

D.2.7. Let Φ_1 be the composite functor

$$\mathrm{Shv}_{\mathcal{N}_1,(\xi_2,y_1)}(\mathcal{Y}_1) \to \mathrm{Vect}_{\mathsf{e}} \to \mathbf{C},$$

where the second arrow corresponds to direct image along

$$\mathrm{pt} \overset{y_2}{\hookrightarrow} \mathcal{Y}_2.$$

We claim that Φ_1 is a direct summand of

$$\mathrm{Shv}_{\mathcal{N}_1}(\mathcal{Y}_1) \to \mathrm{Shv}(\mathcal{Y}_2) \twoheadrightarrow \mathrm{Shv}_{(\xi_2)}(\mathcal{Y}_2) \overset{\Phi_2}{\to} \mathbf{C}.$$

Indeed, let f_0 denote the map $\mathcal{Y}_{1,0} \to \mathcal{Y}_{2,0}$. Since f was assumed proper, we have

$$(f_0)_* \circ \Phi_{g\circ f} \simeq \Phi_g \circ f_* : \mathrm{Shv}(\mathcal{Y}_1) \to \mathrm{Shv}(\mathcal{Y}_{2,0}),$$

which implies the required assertion.

D.2.8. Finally, we have to show that Φ_1 is conservative on $\mathrm{Shv}_{\mathcal{N}_1,(\xi_2,y_1)}(\mathcal{Y}_1)^{\heartsuit}$. However, this follows from Theorem D.2.4.

D.3. **Proof of Proposition D.2.2.**

D.3.1. The proof proceeds by induction on $\dim(\mathcal{Y})$. The base of induction is when $\dim(\mathcal{Y}) = 0$, and there is nothing to prove.

We may assume that $\mathcal{Y}$ admits a smooth map

$$f : \mathcal{Y} \to \mathcal{Y}'$$

of relative dimension 1. We may assume that $\mathcal{Y}$ and $\mathcal{Y}'$ are affine.

For a function h on $\mathcal{Y}$ let shr_h denote the automorphism of $T^*(\mathcal{Y})$ that acts by translation by dh_y in $T^*_y(\mathcal{Y})$.

Let τ_h denote the composite map

$$T^*(\mathcal{Y}') \underset{\mathcal{Y}'}{\times} \mathcal{Y} \hookrightarrow T^*(\mathcal{Y}) \xrightarrow{\mathrm{shr}_h} T^*(\mathcal{Y}).$$

We claim:

Lemma D.3.2. *There exists a function h on $\mathcal{Y}$ such that:*
- $\dim(\tau_h^{-1}(\mathcal{N})) \leq \dim(\mathcal{Y}) - 1$*;*
- $\xi \in \mathrm{Im}(\tau_h)$.

The proof of the lemma is given below. Let us assume it, and perform the induction step.

D.3.3. Denote $y' = f(y)$. Let h be as in Lemma D.3.2. Let ξ' be the unique element in $T^*_{y'}(\mathcal{Y}')$ such that

$$\tau_h(\xi', y) = \xi.$$

Consider the projection

$$(\mathrm{id} \times f) : T^*(\mathcal{Y}') \underset{\mathcal{Y}'}{\times} \mathcal{Y} \to T^*(\mathcal{Y}').$$

Let $\mathcal{N}'$ be the closure of the image along $(\mathrm{id} \times f)$ of $\tau_h^{-1}(\mathcal{N})$. By construction, $\xi' \in T^*_{y'}(\mathcal{Y}') \in \mathcal{N}'$.

Applying the induction hypothesis to $\mathcal{N}'$ and ξ', we can find a function g' on $\mathcal{Y}'$ such that the point y' is isolated for $(\mathcal{N}', g')$.

Take $g := g' \circ f + h$. Then the point y is isolated for $(\mathcal{N}, g)$ by construction.

$\square$[Proposition D.2.2]

D.3.4. *Proof of Lemma D.3.2.* Consider the relative cotangent bundle $T^*(\mathcal{Y}/\mathcal{Y}')$ along with the projection

$$r : T^*(\mathcal{Y}) \to T^*(\mathcal{Y}/\mathcal{Y}');$$

it is smooth of relative dimension $n-1$.

For each irreducible component $\mathcal{N}_i$ of $\mathcal{N}$, the image $r(\mathcal{N}_i)$ is *not* finite. Therefore, we can find a point $\eta_i \in r(\mathcal{N}_i)$ with $\eta_i \neq r(\xi)$.

For a function h on $\mathcal{Y}$ consider the composite map

$$\mathcal{Y} \overset{dh}{\to} T^*(\mathcal{Y}) \overset{r}{\to} T^*(\mathcal{Y}/\mathcal{Y}').$$

We can find h so that $r(\xi)$ belongs to the image of the above map, but none of the points η_i do.

Such h satisfies the requirements of the lemma.

$\square$[Lemma D.3.2]

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