Periods & L-functions

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G split red$/\mathbb{Z}$ $\quad$ k: global field $\quad$ $\mathbb{A}$: adeles

$$[G] = G(k)\backslash G(\mathbb{A}) \Big(/ G(\mathcal{O}) = \prod_v G(\mathcal{O}_v) \Big) \quad \Big(= \mathrm{Bun}_G \Big)$$

§ Automorphic L-fns

$\pi \hookrightarrow C^\infty([G])$ with Langl. param.: $W_k \xrightarrow{\varphi_\pi} \check{G}$

$\rho : \check{G} \longrightarrow GL(V)$ $\quad$ $\mathbb{Z}$-graded v.sp

$$L(\pi, \rho) := \prod_v \mathrm{tr}(\mathrm{Frob}_v, \Lambda^\bullet V_{\rho\circ\varphi})^{-1}$$

$V_{\rho\varphi}$ $\quad$ $W_k \longrightarrow \check{G} \longrightarrow GL(V)$ twisting Frob_v by $q_v^{-\frac{i}{2}}$ on i-th graded space

Over function fields: $k = \mathbb{F}_q(C)$

$\rho \circ \varphi_\pi$ defines a local system $V_{\rho,\varphi}$ over C

$L(\pi, \rho) = tr(Frob_q, S^\bullet H^\bullet(C_{\bar{\mathbb{F}}_q}, V_{\rho,\varphi}))$ $(\ast)$

super v.s.

This is the L fn on the "spectral" side (i.e. in terms of Langl. param φ_π).

<u>Q:</u> What is the meaning of $L(\pi, \rho)$ on autom. side?

Before: Reinterpret $L(\pi, \rho)$ as follows

Loc: Moduli of $\check{G}$-local systems on $C_{\bar{\mathbb{F}}_q}$

$V^{\ast fix} \xrightarrow{p} Loc$: parametrizes pairs (P, σ)
$P \in Loc$, $\sigma: C \longrightarrow P \overset{\check{G}}{\times} V^\ast$

(<u>Naively:</u> $\sigma \in H^0(C_{\bar{\mathbb{F}}_q}, V^\ast_{\rho,\varphi})$ but higher $H^\bullet$.)

So $S^\bullet H^\bullet(V_{\rho,\varphi})$ = the fiber at P_φ of $p_\ast \mathcal{O}_{V^{\ast fix}}$

In de Rham setting:

$$\boxed{\begin{array}{l} V^{*}\text{fix} = \Gamma(G_{dR}, V^*/\check{G}) \\ \quad p\downarrow \\ \text{Loc} = \Gamma(G_{dR}, pt/\check{G}) \end{array}}$$

$$\underset{\text{tr}(\text{Frob}, \ldots \text{ at } P_q)}{p_* \mathcal{O}_{V^* \text{fix}}} \longleftrightarrow L(V_{\rho, q})$$

§ : Periods

Various L-fns appear as integrals of autom forms

$$f \in \pi^K \hookrightarrow C^\infty([G])^{K = G(\mathcal{O})}$$

Ham. G-space $M \to \mathfrak{g}^*$ (e.g. T^*X)

Local Quantization $\underline{\mathcal{O}}$ (eg. $S(X(\mathbb{A}))$) $\ni \Phi$

Autom. fnl $\downarrow$ $C^\infty([G])$ $\qquad \Phi \mapsto \Sigma\Phi := \sum_{\gamma \in X(k)} \Phi(\gamma \bullet)$

If X is smooth, $\Phi = \mathbb{1}_{X(\mathcal{O})}$, general: IC_{L^+X}

	X	$\check G_X$		$\check M = V \times^{\check G_X} \check G$
Whittaker	$N,\psi \backslash G$	$\check G$	$\int_{[N]} f(n)\psi(n)\,dn = 1$ (normalization)	$pt \times^{\check G} \check G = pt$
Group	$H \backslash H\times H = H$	$\check H$	$\int_{[H]} f(h)\overline{f(h)}\,dh = L(\pi, Ad^*, 1)$ $\longleftrightarrow$	$\check{\mathfrak h}^* \times^{\check H} (\check H \times \check H) = T^*\check H$
Point	$G \backslash G = pt$	1	$\int_{[G]} 1 = L(M_G^*(1)) = L(\mathfrak t/\!/W)$	$(\mathfrak t/\!/W) \times \check G = T^*(N,\psi \backslash \check G)$
Hecke	$G_m \backslash PGL_2$	$\check G$	$\int f\begin{pmatrix} a & \\ & 1\end{pmatrix} d^\times a = L(\pi, Std, \frac{1}{2})$	
Waldspurger	$T_\eta \backslash PGL_2$	$\check G$	$\left\lvert \int_{[T]} f \right\rvert^2 = L(\pi, Std, \frac{1}{2})\, L(\pi\otimes\eta, Std, \frac{1}{2})$	
Gross-Prasad	$SO_n \backslash SO_n \times SO_{n+1}$	$\check G$	$\left\lvert \int_{[H]} f \right\rvert^2 = L(\pi_1 \otimes \pi_2, \otimes, \frac{1}{2})$	
Tate	$\mathbb A^1 \circlearrowleft G_m$	$\check G$	$\mathcal S(\mathbb A^1) \ni \Phi = 1\!\!1_{\mathcal O}$, $f = \chi$, $\int (\Sigma\Phi)\chi = L(\chi, 0)$	
Theta (Rallis IPF)	$\omega_\psi\vert_{SO_{2n}\times Sp_{2n}}$	SO_{2n}	$\mathcal S(\ell) \ni \Phi = 1\!\!1_{\ell(\mathcal O)}$, $f = f_1 \otimes \theta(f_1)$, $\left\lvert \int (\Sigma\Phi) f_1\, \theta(f_1) \right\rvert^2 = L(\pi, Std, 1)$	

Unified formalism: $\Theta = \mathcal S(X(\mathbb A)) \ni \Phi \longmapsto \sum_{\gamma \in X(k)} \Phi(\gamma g) \in C^\infty([G])$

$\otimes \qquad \longrightarrow \langle \Sigma\Phi, f \rangle \in \mathbb C$

$\pi \ni f$

If X is smooth (+affine): $\Phi = 1\!\!1_{X(\mathcal O)}$, else: $\Phi = IC_{\mathcal L^+ X}$ the "IC function"

Mult. free Hamiltonian $-M \longleftrightarrow \check{M}$ smooth (affine)
in fld

(e.g. if $M = T^*X$ iff X is spherical)

$M \longrightarrow \mathfrak{g}^*$ general coadj. ob. has fin # G-orbits over it

$\Sigma \Phi$ has geom. interpretation:

$$\begin{array}{c} \mathrm{Bun}_X = \mathrm{Maps}(C, X/G) \\ \downarrow p \\ \mathrm{Bun}_G = \mathrm{Maps}(C, pt/G) \end{array}$$

$\sum \Phi$ "is" $p_! \, IC_{\mathrm{Bun}_X} \in D(\mathrm{Bun}_G)$
period integral

$\rho : \check{G}_X \longrightarrow GL(V)$

$$\begin{array}{c} V^{*}{}^{fix} : \mathrm{Map}(C_{dR}, V^*/\check{G}_X) \\ \downarrow p \\ \mathrm{Loc}_{\check{G}} = \mathrm{Map}(C_{dR}, pt/\check{G}) \end{array}$$

$p_* \, \mathcal{O}_{V^{*}{}^{fix}} \in QC(\mathrm{Loc})$

L-fn

Natural to conjecture: Under geom. Langlands

$p_! \, IC_{\mathrm{Bun}_X}$ is sent to $p_* \, \mathcal{O}_{\check{M}^{fix}}$
(slightly imprecise)

$$M \longleftrightarrow \check{M} = V \times^{\check{G}_X} \check{G}$$

(e.g. $M = T^* X$) $\quad$ (G-N $\check{G}_X \subset \check{G}$, $\rho \searrow V$)

(Correction: $p_* \mathcal{O}_{\check{M}^{fix}}$ corresponds to "absolute value square" of period.

Should be replaced by $p_* \mathcal{O}_{\check{X}^{fix}}$ when $\check{M} = T^* \check{X}$, some "spectral quantization" of $\check{M}$, in general)

Given $M = T^*X$, how to compute $\check{M}$?

$$\check{M} = V \times^{\check{G}_X} \check{G} \ ?$$

Boils down to the local Planch. formula

for $L^2(X(F))^{G(\mathcal{O})}$
local

$\Phi = IC_{L^+X}$ supp. on $X(\mathcal{O})$

Q: What is the spectral decomp?

Weyl determinant

$$\|\Phi\|^2 = \int_{\varphi \in \check{A}_X / W_X} \underbrace{L(\varphi, \rho)}\ \mu_{\check{G}_X}(\varphi)$$

$\exists$ geometric formulation.

$L(\varphi, \rho)$ has been computed in a # of cases

$$\rho : \check{G}_X \longrightarrow GL(V)$$

• When $X = H\backslash G$ affine (H: reductive)

[S. — "spherical fns ...."]

• When $X = \overset{\rightsquigarrow \text{ L-monoid}}{H_\lambda} \supset H \underset{\lambda}{\overset{\det}{\rightleftarrows}} \mathbb{G}_m$ (e.g. $H = GL_n \subset H_\lambda = \mathrm{Mat}_n$)

$\lambda \leftrightarrow V_\lambda$ of $\check{H}$

[Bouthier – Ngô – S.]

• Vast generalization (w. Jonathan Wang):

X arbitrary affine spherical variety (with $\check{G} = \check{G}_X$)

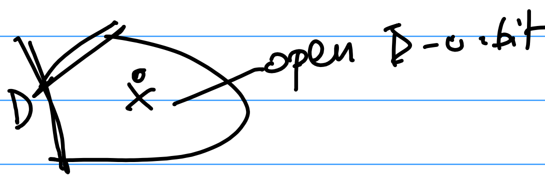
Decomp $\|IC\|^2_{X(O)} = \int_{A_X/W_X} L(\varphi,\rho)\, \phi(\varphi)$

Basic input to compute $\rho : \check{G}_X \longrightarrow GL(V)$:

$\cup$

$\check{A}_X$ - - - weights $\vec{\nu}_X$

What are ν_X? When $X = \overline{H\backslash G}^{\text{aff}}$



B-stable divisors D : "colors"

$M = T^*X$

each D $\rightsquigarrow$ valuation $v_D\big|_{k(X)^{(B)}} \in \mathrm{Hom}(A_X^\vee \to \mathbb{G}_m)$

Ex: $X = H$, D: Bruhat divisors

$v_D = \check{\alpha}$ (simple coroot) $\rightsquigarrow \rho : \check{H} \xrightarrow{Ad^*} GL(\check{\mathfrak{h}}^*)$

$$\| IC \|^2 = tr(Frob, End(IC)).$$

Bezrukavnikov–Finkelberg: $D(X(F)/G(O)) = QC_{perf}(\check{M}/\check{G})$

$SO_{2n} \xrightarrow{\Delta} SO_{2n} \times SO_{2n+1}$

$\check{G} = \check{G}_X = SO_{2n} \times Sp_{2n}$

$\rho = Ad \otimes Std$

$GL_n \to GL_n \times GL_{n+1}$

$\check{G} = \check{G}_X = GL_n \times GL_{n+1}$

$\check{M}$: $\rho = Std \otimes Std \oplus Std^\vee \otimes Std^\vee$