Local Langlands and Springer Correspondences

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Abstract. This notes are the written version of a course given by the author at the workshop "Representation theory of p-adic groups" that was held at IISER Pune, India, in July 2017 (to appear in *Representations of p-adic groups: Contributions from Pune*, Progress in Math., Birkhäuser).

They give notably an overview of results obtained jointly with Ahmed Moussaoui and Maarten Solleveld on the local Langlands correspondence, focusing on the links of the latter with both the generalized Springer correspondence and the geometric conjecture, the so-called ABPS Conjecture, introduced in collaboration with Paul Baum, Roger Plymen and Maarten Solleveld.

1. Introduction

The local Langlands correspondence predicts a relation between two rather different kinds of objects: on one side irreducible representations of reductive groups over a local field F, on the other side certain analogs of Galois representations, called *Langlands parameters.*

Let G be the group of F-rational points G of a connected reductive algebraic group $\mathbf{G}$ over a non Archimedean local field F. Slightly more precisely, the local Langlands conjecture for G asserts the existence a finite to one surjection

$$\mathrm{rec}_G \colon \mathrm{Irr}(G) \to \Phi(G),$$

from the set $\mathrm{Irr}(G)$ of (isomorphism classes of) irreducible smooth representations of G to the set $\Phi(G)$ of equivalence classes (for a certain equivalence relation) of Langlands parameters for G, that satisfies a certain list of chosen, expected properties.

The (conjectural) fibers of rec_G are called L-packets for G. The irreducible representations π inside a given L-packet are said to be *L-indistinguishable.* To distinguish them, the idea (see Lusztig [Lus1], Vogan [Vog], Arthur [Ar1]) is to "enhance" ϕ by an additional datum ρ, which is an irreducible representation of a certain finite group $\mathcal{S}_\phi$ (defined in Equation (3) below). The pair (ϕ, ρ) will be called an *enhanced Langlands parameter* for G. Let $\Phi_{\mathrm{e}}(G^\sharp)$ be the collection of equivalence classes of such pairs (ϕ, ρ) with $\rho \in \mathrm{Irr}(\mathcal{S}_\phi)$. Then the LLC for G should be an injective map

$$\mathrm{Irr}(G) \to \Phi_{\mathrm{e}}(G), \tag{1}$$

which satisfies several natural properties. The map will almost never be surjective, but for every ϕ which is relevant for G the image should contain at least one pair (ϕ, ρ).

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A remarkable aspect of the Langlands conjecture [Vog] is that it is better to consider not just one reductive group G at a time, but all inner forms (more precisely all inner twists) G_ϑ of G simultaneously. Inner twists share the same Langlands dual group. The hope is that one can turn (1) into a bijection by defining a suitable equivalence relation on the set of inner twists and taking the corresponding union of the sets $\mathrm{Irr}(G_\vartheta)$ on the left hand side. Such a statement was proven for unipotent representations (also known as representations with unipotent reduction) of simple p-adic groups in [Lus5].

The main goal of these lectures is to provide an overview of the main results obtained jointly with Ahmed Moussaoui and Maarten Solleveld in [AMS1] on the local Langlands correspondence, focusing on the links of the latter with both the generalized Springer correspondence and the geometric conjecture, the so-called ABPS Conjecture, introduced in collaboration with Paul Baum, Roger Plymen and Maarten Solleveld and studied in several articles, notably [ABPS7]. We give also an account of the known results regarding the preservation (and non-preservation) of the depth by the LLC.

For maximal generality, we adhere to the setup for L-parameters used by Arthur in [Ar1]. Let W_F be the Weil group of F, let ${}^LG = G^\vee \rtimes W_F$ be the L-group of G. Let $G^\vee_{\mathrm{ad}}$ be the adjoint group of $G^\vee$, and let $G^\vee_{\mathrm{sc}}$ be the simply connected cover of the derived group of $G^\vee_{\mathrm{ad}}$. Let $\phi\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^LG$ be an L-parameter, let $\mathrm{Z}_{G^\vee_{\mathrm{ad}}}(\phi(W_F))$ be the centralizer of $\phi(W_F)$ in $G^\vee_{\mathrm{ad}}$ and let

$$\mathcal{G}_\phi := \mathrm{Z}^1_{G^\vee_{\mathrm{sc}}}(\phi(W_F)) \tag{2}$$

be its inverse image in $G^\vee_{\mathrm{sc}}$. To ϕ we associate the finite group

$$\mathcal{S}_\phi := \pi_0(\mathrm{Z}^1_{G^\vee_{\mathrm{sc}}}(\phi)), \tag{3}$$

where Z^1 is again defined via $G^\vee_{\mathrm{ad}}$. We call any irreducible representation of $\mathcal{S}_\phi$ an *enhancement* of ϕ . The group $\mathcal{S}_\phi$ coincides with the group considered by both Arthur in [Ar1] and Kaletha in [Kal1, §4.6]. A remarkable fact is that the group $\mathcal{S}_\phi$ is isomorphic to the component group

$$A_{\mathcal{G}_\phi}(u_\phi) := \pi_0(\mathrm{Z}_{\mathcal{G}_\phi}(u_\phi)), \tag{4}$$

where $u_\phi := \phi(1, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right))$. It provides a way to plug the generalized Springer correspondences for the groups $\mathcal{G}_\phi$, where ϕ runs over $\Phi(G)$, in the study of the local Langlands correspondence for G.

In particular, it allows to transfer the Lusztig notion of cuspidality for representations of the groups $A_{\mathcal{G}_\phi}(u_\phi)$ into a notion of cuspidality for enhanced L-parameters for G: an enhanced L-parameter (ϕ, ρ) for G is called *cuspidal* if u_ϕ and ρ, considered as data for the complex reductive group $\mathcal{G}_\phi$, form a *cuspidal pair*. By definition this means that the restriction of ρ from $A_{\mathcal{G}_\phi}(u_\phi)$ to $A_{\mathcal{G}_\phi^\circ}(u_\phi)$ is a direct sum of cuspidal representations in Lusztig's sense [Lus2]. Intuitively, it says that ρ or $\rho|_{A_{\mathcal{G}_\phi^\circ}(u_\phi)}$ cannot be obtained (via an appropriate notion of parabolic induction) from any pair (u', ρ') that can arise from a proper Levi subgroup of $\mathcal{G}_\phi^\circ$. It is essential to use L-parameters enhanced with a representation of a suitable component group, for cuspidality cannot be detected from the L-parameter alone.

We conjecture in [AMS1, § 6] (generalizing to G arbitrary the conjecture stated by Moussaoui when G is F-split: [Mou1, Conjecture 1.2]) that the cuspidal enhanced

Langlands parameters correspond by the LLC to the irreducible supercuspidal representations of G.

The validity of this conjecture is proved for representations with unipotent reduction of the group G of the F-rational points of any connected reductive algebraic group which splits over an unramified extension of F in [FOS, Theorem 2] (when G is simple of adjoint type it is a special case of [Lus4], [Lus5]), for the Deligne-Lusztig depth-zero supercuspidal representations (as a consequence of [DeRe]), and also for general linear groups and split classical p-adic groups (any representation) (see [Mou1]), for inner forms of linear groups and of special linear groups, and for quasi-split unitary p-adic groups (any representation) (see [AMS1, § 6]).

Contents

2. Langlands parameters

2.1. **The Weil group.** Let F be a local non-Archimedean field with finite residual field $k_F = \mathbb{F}_q$. Let F_{sep} be a fixed separable closure of F, and let Γ_F denote the Galois group of F_{sep}/F. The field F admits a unique unramified extension F_m/F of degree m and contained in F_{sep}, for each integer $m \geq 1$. The composite of all the fields F_m is the unique maximal unramified extension of F contained in F_{sep} and will be denoted by F_{ur}. The extension F_{ur} will allow us to decompose the study of Γ_F in two steps of different nature by considering separately the group $\mathrm{Gal}(F_{\mathrm{ur}}/F)$ and the group $I_F := \mathrm{Gal}(F_{\mathrm{sep}}/F_{\mathrm{ur}})$, that is called the *inertia group* of F: we have an exact sequence of topological groups

$$1 \to I_F \to \Gamma_F \to \mathrm{Gal}(F_{\mathrm{ur}}/F) \to 0. \tag{5}$$

The extension F_m/F is Galois and the group $\mathrm{Gal}(F_m/F)$ is cyclic (that is, generated by a single element). An F-automorphism of F_m is determined by its action on the residual field $k_{F_m} \simeq \mathbb{F}_{q^m}$ of F_m, and there is a unique element σ_m of $\mathrm{Gal}(F_m/F)$ which acts on k_{F_m} by the elevation at the power q. We set $\mathrm{Fr}_m := \sigma_m^{-1}$. The map $\mathrm{Fr}_m \mapsto 1 + m\mathbb{Z}$ gives a canonical isomorphism from $\mathrm{Gal}(F_m/F)$ onto $\mathbb{Z}/m\mathbb{Z}$. Taking the inverse limit over m, we get a canonical isomorphism of topological groups between $\mathrm{Gal}(F_{\mathrm{ur}}/F)$ and

$$\widehat{\mathbb{Z}} := \varprojlim_{m\geq 1} \mathbb{Z}/m\mathbb{Z},$$

and a unique element $\mathrm{Fr}_F \in \mathrm{Gal}(F_{\mathrm{ur}}/F)$ which acts on F_m as Fr_m, for all m. The element Fr_F is called the *geometric Frobenius substitution* on F_{ur} (its inverse σ_F is the *arithmetic* Frobenius substitution). An element of Γ_F is called a *geometric Frobenius element* (over F) if its image in $\mathrm{Gal}(F_{\mathrm{ur}}/F)$ is Fr_F. The Chinese Remainder Theorem gives a canonical isomorphism of topological groups $\widehat{\mathbb{Z}} \simeq \prod_\ell \mathbb{Z}_\ell$, where ℓ range over all prime numbers, and $\mathbb{Z}_\ell$ is the (additive) group of ℓ-adic integers.

We recall some properties of the *ramification groups* (with respect to the upper numbering) of Γ_F, as defined in [Ser, Remark IV.3.1]:

- $\Gamma_F^{-1} := \Gamma_F$ and $\Gamma_F^0 := I_F$, the inertia group.
- For every $r \in \mathbb{R}_{\geq 0}$, Γ_F^l is the compact subgroup of I_F that consists of all $\gamma \in \Gamma_F$ which, for every finite Galois extension E of F contained in F_{sep}, act trivially on the ring $\mathfrak{o}_E/\mathfrak{p}_E^{i(r,E)}$ (where $i(r,E) \in \mathbb{Z}_{\geq 0}$ can be found with [Ser, §IV.3]).
- $r \in \mathbb{R}_{\geq 0}$ is called a jump of the filtration if

  $$\Gamma_F^{r+} := \bigcap_{t>r} \Gamma_F^t$$

  does not equal Γ_F^r. The set of jumps of the filtration is countably infinite and need not consist of integers.

In order to formulate the Langlands correspondence we need to introduce the *Weil group* W_F, a subgroup of Γ_F. Let ${}_{\mathrm{a}}W_F$ denote the inverse image in Γ_F of the cyclic subgroup of $\mathrm{Gal}(F_{\mathrm{ur}}/F)$ generated by Fr_F. Then ${}_{\mathrm{a}}W_F$ is the dense subgroup of Γ_F generated by the geometric Frobenius elements. It is normal in Γ_F and fits into an exact sequence (of abstract groups)

$$1 \to I_F \to {}_{\mathrm{a}}W_F \to \mathbb{Z} \to 0. \tag{6}$$

The Weil group W_F of F (relative to F_{sep}) is the topological group, with underlying abstract group ${}_{\mathrm{a}}W_F$, so that I_F is an open subgroup of W_F, and the topology on I_F, as subspace of W_F, coincides with its natural topology as $\mathrm{Gal}(F_{\mathrm{sep}}/F_{\mathrm{ur}}) \subset \Gamma_F$. Thus W_F is locally profinite, and the identity map $\iota_F \colon W_F \to {}_{\mathrm{a}}W_F \subset \Gamma_F$ is continuous.

The definition of W_F does depend on the choice of F_{sep}/F, but only up to inner automorphism of Γ_F.

We will need some basic aspects of the representation theory of W_F. The group Γ_F being profinite, its smooth representations are semisimple. On the contrary, W_F has smooth representations which are not semisimple. The irreducible representations of W_F are quite closely related to those of Γ_F (in particular, they have finite dimension):

(1) if τ is an irreducible representation of Γ_F, then $\tau\circ\iota_F$ is an irreducible smooth representation of W_F,
(2) for any irreducible smooth representation σ of W_F, there is an unramified character χ of W_F such that $\chi\otimes\sigma\simeq\tau\circ\iota_F$, for some irreducible smooth representation τ of Γ_F.

2.2. **The L-group.** Let G be the group of F-points of a connected reductive algebraic group $\mathbf{G}$ defined over F. Let $\mathbf{T}$ be a maximal torus of $\mathbf{G}$ defined over F, and let $X(\mathbf{T}) := \mathrm{Hom}_F(\mathbf{T},\mathbb{G}_m)$ be the group of F-rational characters of $\mathbf{T}$. We set $X^\vee(\mathbf{T}) := \mathrm{Hom}_F(\mathbb{G}_m,\mathbf{T})$. Let $R(\mathbf{G},\mathbf{T})$ and $R^\vee(\mathbf{G},\mathbf{T})$ denote the sets of roots and coroots of $\mathbf{G}$ with respect to $\mathbf{T}$, respectively. The corresponding root datum for $\mathbf{G}$ is

$$\mathcal{R}(\mathbf{G}) := (X(\mathbf{T}), R(\mathbf{G},\mathbf{T}), X^\vee(\mathbf{T}), R(\mathbf{G},\mathbf{T})^\vee).$$

Definition 2.1. The *Langlands dual group* of $\mathbf{G}$ is the denote the reductive connected algebraic group $\mathbf{G}^\vee$, defined over $\mathbb{C}$, whose roots (resp. coroots) are the coroots (resp. roots) of $\mathbf{G}$, that is, a root datum for $\mathbf{G}^\vee$ is

$$\mathcal{R}(\mathbf{G}^\vee) := (X^\vee(\mathbf{T}), R^\vee(\mathbf{G},\mathbf{T}), X(\mathbf{T}), R(\mathbf{G},\mathbf{T})).$$

We denote by $G^\vee$ the $\mathbb{C}$-points of the group $\mathbf{G}^\vee$.

Examples 2.2. We have

- $\mathrm{GL}_n(F)^\vee = \mathrm{GL}_n(\mathbb{C})$, $\mathrm{SL}_n(F)^\vee = \mathrm{PGL}_n(\mathbb{C})$, and $\mathrm{PGL}_n(F)^\vee = \mathrm{SL}_n(\mathbb{C})$;
- $\mathrm{Sp}_{2n}(F)^\vee = \mathrm{SO}_{2n+1}(\mathbb{C})$, $\mathrm{SO}_{2n+1}(F)^\vee = \mathrm{Sp}_{2n}(\mathbb{C})$;
- $\mathrm{SO}_{2n}(F)^\vee = \mathrm{SO}_{2n}(\mathbb{C})$;
- if $\mathbf{G}$ a group of exceptional type, then $\mathbf{G}$ and $\mathbf{G}^\vee$ are of the same type (e.g. $\mathrm{G}_2(F)^\vee = \mathrm{G}_2(\mathbb{C})$).

Of special importance is the group $\mathbf{G} = \mathrm{GL}_n$. Indeed, any complex reductive group $\mathbf{G}$ may be embedded into $\mathrm{GL}_n(\mathbb{C})$ for some n.

We choose a Borel subgroup $\mathbf{B}\supset\mathbf{T}$ of $\mathbf{G}$ defined over F, and let Δ and $\Delta^\vee$denote the corresponding basis of $R(\mathbf{G},\mathbf{T})$ and $R^\vee(\mathbf{G},\mathbf{T})$. A based root datum for $\mathbf{G}$ is $\mathcal{R}_0(\mathbf{G}) := (X(\mathbf{T}),\Delta,X^\vee(\mathbf{T}),\Delta^\vee)$. We coose a pinning $(G,B,T,\{x_\alpha\}_\alpha\in\Delta)$, which induces a splitting of the exact sequence

$$1\longrightarrow \mathrm{Int}(\mathbf{G})\longrightarrow \mathrm{Aut}(G)\longrightarrow \mathrm{Aut}(\mathcal{R}_0(G))\longrightarrow 1.$$

The F structure of $\mathbf{G}$ is given by a morphism of $\theta\colon\Gamma_F\longrightarrow\mathrm{Aut}(\mathbf{G})$ which descends to $\theta\colon\Gamma_F\longrightarrow\mathrm{Aut}(\mathcal{R}_0(\mathbf{G}))$.

To obtain the L-group as a group that actually sees $G=\mathbf{G}(F)$ and not just $\mathbf{G}$, we have to work in the Galois action of Γ_F on the group $\mathbf{G}^\vee$. Since $\mathrm{Aut}(\mathcal{R}_0(\mathbf{G}))\simeq\mathrm{Aut}(\mathcal{R}_0(\mathbf{G})^\vee)\simeq\mathrm{Aut}(\mathcal{R}_0(\mathbf{G}^\vee))$, we have

$$\theta\colon\Gamma_F\longrightarrow\mathrm{Aut}(\mathcal{R}_0(\mathbf{G}))\longrightarrow\mathrm{Aut}(\mathbf{G}^\vee).$$

Definition 2.3. The (Weil form of) L-group of G is the group

$${}^LG := G^\vee\rtimes_\theta W_F.$$

Remark 2.4. In the case when G is F-split, we have a direct product, that is: ${}^LG = G^\vee \times W_F$.

Definition 2.5. Let $\mathbf{G}$, $\mathbf{H}$ be two connected reductive algebraic groups over F. An homomorphism $\eta\colon {}^LH \to {}^LG$ is called an *L-homomorphism* if

- η is continuous;
- the diagram
$$\begin{tikzcd} {}^LH \arrow[rr, "\eta"] \arrow[dr] & & {}^LG \arrow[dl] \\ & W_F & \end{tikzcd}$$
commutes;
- the restriction of η to $H^\vee$ is morphism of algebraic groups from $H^\vee$ to $G^\vee$.

General connected reductive F-groups need not be quasi-split, but they are always *forms* of split F-groups.

Definition 2.6. Two F-groups $G = \mathbf{G}(F)$ and $H = \mathbf{H}(F)$ are called *forms* of each other if $\mathbf{G}$ is isomorphic to $\mathbf{H}$ as algebraic groups, or equivalently if $\mathbf{G}(F_{\mathrm{sep}}) \cong \mathbf{H}(F_{\mathrm{sep}})$ as F_{sep}-groups.

An isomorphism $\vartheta\colon \mathbf{H} \to \mathbf{G}$ determines a 1-cocycle

$$\gamma_\vartheta\colon \begin{array}{ccc} \Gamma_F & \to & \mathrm{Aut}(\mathbf{G}) \\ \sigma & \mapsto & \vartheta\sigma\alpha^{-1}\sigma^{-1}. \end{array} \tag{7}$$

From γ_ϑ one can recover H (up to isomorphism) as

$$H \cong \{g \in \mathbf{G}(F_{\mathrm{sep}}) \,:\, (\gamma_\vartheta(\sigma) \circ \sigma)g = g \quad \forall \sigma \in \mathrm{Gal}(F_{\mathrm{sep}}/F)\}.$$

Given another form $\vartheta'\colon \mathbf{H}' \to \mathbf{G}$, the groups H and H' are F-isomorphic if and only if the 1-cocycles γ_ϑ and $\gamma_{\vartheta'}$ are cohomologous, that is, if there exists a $f \in \mathrm{Aut}(\mathbf{G})$ such that

$$\gamma_\vartheta(\sigma) = f^{-1}\gamma_{\vartheta'}(\sigma)\,\sigma f \sigma^{-1} \quad \forall \sigma \in \mathrm{Gal}(F_{\mathrm{sep}}/F). \tag{8}$$

In this way the isomorphism classes of forms of G are in bijection with the Galois cohomology group $H^1(F, \mathrm{Aut}(\mathbf{G}))$.

Definition 2.7. An F-group H is an *inner form* of G if the cocycle γ_ϑ takes values in the group of inner automorphisms $\mathrm{Inn}(\mathbf{G})$. On the other hand, if the values of γ_ϑ are not contained in $\mathrm{Inn}(\mathbf{G})$, then H is called an outer form of G.

Proposition 2.8. [Spr, §16.4] *Every connected reductive F-group G is an inner form of a unique quasi-split F-group G^*.*

Example 2.9. Let D be a division algebra with center F, of dimension d^2 over F. Then $G = \mathrm{GL}_m(D)$ is an inner form of $\mathrm{GL}_n(F) = G^*$ where $n = dm$. There is a reduced norm map $\mathrm{Nrd}\colon \mathrm{GL}_m(D) \to F^\times$ and the derived group $\mathrm{SL}_m(D) := \ker(\mathrm{Nrd}\colon \mathrm{GL}_m(D) \to F^\times)$ is an inner form of $\mathrm{SL}_n(F)$. Every inner form of $\mathrm{GL}_n(F)$ or $\mathrm{SL}_n(F)$ is isomorphic to one of this kind.

When $n = 2$, the only possibilities for d are 1 or 2, and so the inner forms are, up to isomorphism, $\mathrm{GL}_2(F)$ and $D^\times$, and $\mathrm{SL}_2(F)$ and $\mathrm{SL}_1(D)$.

Definition 2.10. An *inner twist* of G consists of a pair (H, ϑ) as above, where $H = \mathbf{H}(F)$ and $\vartheta\colon \mathbf{H} \xrightarrow{\sim} \mathbf{G}$ are such that $\mathrm{im}(\gamma_\vartheta) \subset \mathbf{G}_{\mathrm{ad}}$.

Two inner twists of G are *equivalent* if (8) holds for some $f \in \mathrm{Inn}(\mathbf{G})$. Let $\mathrm{IT}(G)$ denote the set of equivalence classes of inner twists of G.

Remark 2.11. It is quite possible that two inequivalent inner twists (H, ϑ) and (H', ϑ') share the same group $H \cong H'$. This happens precisely when γ_ϑ and γ_ϑ are in the same orbit of $\mathrm{Aut}(\mathbf{G})/\mathrm{Inn}(\mathbf{G})$ on $H^1(F, \mathbf{G}_{\mathrm{ad}})$.

Kottwitz has found an important alternative description of $H^1(F, \mathbf{G})$. Recall that the complex dual group $G^\vee = \mathbf{G}^\vee(\mathbb{C})$ is endowed with an action of $\mathrm{Gal}(F_{\mathrm{sep}}/F)$. There exists a natural isomorphism

$$\kappa_G \colon H^1(F, \mathbf{G}) \xrightarrow{\sim} \mathrm{Irr}\Big(\pi_0\big(\mathrm{Z}(G^\vee)^{W_F}\big)\Big),$$

see [Ko, Proposition 6.4]. This is particularly useful in the following way. An inner twist of G is the same thing as an inner twist of the unique quasi-split inner form $G^* = \mathbf{G}^*(F)$. Let $G^*_{\mathrm{ad}} = \mathbf{G}^*_{\mathrm{ad}}(F)$ be the adjoint group of G^*. Let $\mathbf{G}^\vee_{\mathrm{sc}}$ be the simply connected cover of the derived group $G^\vee_{\mathrm{der}}$ of $\mathbf{G}^\vee$. We have $G^\vee_{\mathrm{sc}} = (G_{\mathrm{ad}})^\vee$, and

$$\kappa_{G^*_{\mathrm{ad}}} \colon H^1(F, \mathbf{G}^*_{\mathrm{ad}}) \xrightarrow{\sim} \mathrm{Irr}\big(\mathrm{Z}(G^\vee_{\mathrm{sc}})^{W_F}\big). \tag{9}$$

The equivalence classes of inner twists of G are parametrized by the Galois cohomology group $H^1(F, \mathbf{G}^*_{\mathrm{ad}})$. For every $\vartheta \in H^1(F, \mathbf{G}^*_{\mathrm{ad}})$, we will denote by G_ϑ an inner twist of G^* which is parametrized by ϑ, and we will take G^* to be G_1.

All the inner twists of a given group G share the same L-group, because the action of W_F on $G^\vee$ is only uniquely defined up to inner automorphisms. This also works the other way round: from the Langlands dual group ${}^LG$ one can recover the inner form class of G. Hence it is natural from the point of view of the Langlands to consider all the inner twists of a given group G simultaneously.

2.3. **Definitions for Langlands parameters.** We write $W_F' := W_F \times \mathrm{SL}_2(\mathbb{C})$ (that may be viewed as version of the *Weil-Deligne group* of F).

Definition 2.12. A Langlands parameter (or L-parameter, for short) for ${}^LG$ is a smooth morphism

$$\phi \colon W_F' \to {}^LG$$

such that
$$\begin{tikzcd} W_F' \arrow[rr, "\phi"] \arrow[dr] & & {}^LG \arrow[dl] \\ & W_F & \end{tikzcd}$$
commutes, $\phi(w)$ is semisimple for each $w \in W_F$ (that is, $r(\phi(w))$ is semisimple for every finite dimensional representation r of ${}^LG$), and $\phi|_{\mathrm{SL}_2(\mathbb{C})}$ is a morphism of complex algebraic groups.

The group $G^\vee$ acts on the set $\underline{\Phi}({}^LG)$ of such ϕ's by conjugation. Let $\Phi({}^LG)$ denote the set of $G^\vee$-orbits in $\underline{\Phi}({}^LG)$

Definition 2.13. An L-parameter $\phi \colon W_F' \to {}^LG$ is said to be

- *unramified* if it has trivial restriction to I_F;
- *tame* if it has trivial restriction to the wild ramification group P_F of F.;
- *essentially tame* if $\phi(P_F)$ lies in a maximal torus of $G^\vee$;
- *discrete* (or *elliptic*) if there is no proper W_F-stable Levi subgroup $L^\vee \subset G^\vee$ such that $\phi(W_F') \subset {}^LL$.
- *bounded* if $\phi'(W_F) \subset G^\vee$ is bounded, where $\phi(w) = (\phi'(w), w)$. (This is equivalent to $\phi'(\mathrm{Frob})$ being a compact element of $G^\vee$.

2.3.1. *Relevance for Langlands parameters.* We will call a Levi factor of a parabolic subgroup of G simply a Levi subgroup of G. The bijection

$$R(\mathbf{G},\mathbf{T}) \longleftrightarrow R^\vee(\mathbf{G},\mathbf{T}) = R(G^\vee, T^\vee)$$

gives a basis $\Delta^\vee$, and provides a canonical bijection between the sets of conjugacy classes of parabolic subgroups of $\mathbf{G}$ and of $G^\vee$.

Definition 2.14. [Bor, § 3] A parabolic subgroup $P^\vee$ of $G^\vee$ is *F-relevant* if the corresponding class of parabolic subgroups of $\mathbf{G}$ contains an element $\mathbf{P}$ which is defined over F. Similarly, a Levi subgroup $L^\vee \subset G^\vee$ is *F-relevant* if it is a Levi factor of a parabolic subgroup $P^\vee \subset G^\vee$ which is F-relevant.

Definition 2.15. A parabolic subgroup $P^\vee$ of $G^\vee$ is *W_F-quasi-stable* if the projection $\mathrm{N}_{{}^LG}(P^\vee) \to W_F$ is surjective.

Remark 2.16. The W_F-quasi-stable parabolic subgroups of $G^\vee$ are precisely the neutral components of what Borel [Bor, §3] calls parabolic subgroups of ${}^LG$.

Definition 2.17. Let $\phi \in \underline{\Phi}({}^LG))$ and let $P^\vee$ be a W_F-quasi-stable parabolic subgroup of $G^\vee$ with a Levi factor $L^\vee$ such that

- the image of ϕ is contained in $\mathrm{N}_{{}^LP}(L^\vee)$, where ${}^LP := P^\vee \rtimes W_F$;
- $P^\vee$ is a minimal for this property.

The L-parameter ϕ is *G-relevant* (or is an *L-parameter for G*) if $P^\vee$ is F-relevant.

Notation 2.18. We denote by $\Phi(G)$ the subset of $\Phi({}^LG)$ of the $\mathbf{G}^\vee$-conjugacy classes of L-parameters that are G-relevant.

A Langlands classification of L-parameters for G is obtained in [SZ].

We have

$$\Phi({}^LG) = \Phi(G^*),$$

where G^* is the quasi-split inner form of G as defined in Proposition 2.8, and it is expected that in general $\Pi_\phi(G)$ is nonempty if and only if ϕ is G-relevant.

Example 2.19. Let D be a division algebra with center F over F such that $\dim_F(D) = 4$. Then $G = D^\times$ is the unique non-split inner form of $G^* = \mathrm{GL}_2(F)$.

The only Levi subgroup of $D^\times$ defined over F is $D^\times$ itself, and it corresponds to the Levi subgroup $\mathrm{GL}_2(\mathbb{C})$ on the complex side.

Let $\phi \in \underline{\Phi}(\mathrm{GL}_2(F)) = \underline{\Phi}(D^\times)$ be the embedding $W_F' \hookrightarrow \mathrm{GL}_2(\mathbb{C}) \times W_F$. No proper parabolic subgroup of $\mathrm{GL}_2(\mathbb{C})$ contains $\phi(\mathrm{SL}_2(\mathbb{C})) = \mathrm{SL}_2(\mathbb{C})$, so ϕ is relevant for both $D^\times$ and $\mathrm{GL}_2(F)$.

2.4. L-packets.

2.4.1. *For a given group G.* The local Langlands conjecture (LLC) asserts that the set $\mathrm{Irr}(G)$ of isomorphism classes of irreducible smooth representations of G can be parametrized by the set $\Phi(G)$. This parametrization is usually not a bijection. In fact, it is conjectured that each conjugacy class $\phi \in \Phi(G)$ is associated with a finite set $\Pi_\phi(G)$ of isomorphism classes of irreducible smooth representations of G, and that they give a disjoint decomposition of $\mathrm{Irr}(G)$;

$$\mathrm{Irr}(G) = \bigsqcup_{\phi\in\Phi(G)} \Pi_\phi(G). \tag{10}$$

Such finite sets are called *L-packets for G*. This parametrization is based on the belief that there should be certain arithmetic invariants (e.g., L-factors) defined on both the representation side and the parameter side so that one could match them. From this point of view, one can think that the L-packet Π_ϕ attached to some $\phi \in \Phi(G)$ consists of all irreducible smooth representations of G whose arithmetic invariants match those of ϕ.

2.4.2. *For all the inner twists of G.* The local Langlands correspondence predicts the existence of a partition of the set $\mathrm{Irr}(\mathrm{IT}(G)$ of equivalence classes of the irreducible smooth representations of all the groups G_i in $\mathrm{IT}(G)$ into finite subsets:

$$\mathrm{Irr}(\mathrm{IT}(G)) = \bigsqcup_{\phi\in\Phi({}^LG)} \Pi_\phi({}^LG), \tag{11}$$

where each *L-packet* $\Pi_\phi({}^LG)$ is the union of L-packets for the groups G_i.

Example 2.20. We keep the notation of Example 2.19. We have

$$\Pi_\phi(\mathrm{GL}_2(F)) = \{\mathrm{St}_{\mathrm{GL}_2(F)}\} \quad \text{and} \quad \Pi_\phi(D^\times) = \{\mathrm{St}_{D^\times} = \mathrm{triv}_{D^\times}\}.$$

Thus here (11) is

$$\Pi_\phi({}^LG) = \Pi_\phi(\mathrm{GL}_2(F)) \cup \Pi_\phi(D^\times) = \{\mathrm{St}_{\mathrm{GL}_2(F)}, \mathrm{St}_{D^\times}\}.$$

Let $\phi' \in \underline{\Phi}(\mathrm{GL}_2(\mathbb{C}))$ such that $\phi'(\mathrm{SL}_2(\mathbb{C})) = 1$ and $\phi'(W_F) \subset \mathrm{diag}(\mathrm{GL}_2(\mathbb{C}))\times W_F$. Then $L^\vee = \mathrm{diag}(\mathrm{GL}_2(\mathbb{C}))$ is the minimal Levi subgroup such that ${}^LL$ contains the image of ϕ_2. Thus the standard Borel subgroup $P^\vee$ of $\mathrm{GL}_2(\mathbb{C})$ satisfies the conditions in Definition 2.17. But its conjugacy class does not correspond to any parabolic subgroup of $D^\times$, so ϕ' is not relevant for $D^\times$.

Let $\mathbf{G} \subset \widetilde{\mathbf{G}}$ be a pair of quasi-split connected reductive groups, defined over F, with the same derived group. It is expected that the L-packets of $G = \mathbf{G}(F)$ are restrictions of L-packets of $\widetilde{G} := \widetilde{\mathbf{G}}(F)$ in the sense that for each L-packet $\Pi(G)$ for G there is a packet $\Pi(\widetilde{G})$ for $\widetilde{G}$ whose restriction to G is equal to Π. Typically one uses this last expectation to construct L-packets for G from the knowledge of L-packets of $\widetilde{G}$, e.g., in the cases of $\mathrm{SL}_n \subset \mathrm{GL}_n$ and $\mathrm{Sp}_{2n} \subset \mathrm{GSp}_{2n}$. In some cases, however, we wish to move in the other direction, and use the knowledge of L-packets of G to obtain structural information about the L-packets of $\widetilde{G}$, see [Xu].

3. Enhanced Langlands parameters

3.1. **The group $\mathcal{S}_\phi$.** To parametrize the irreducible representations in a given L-packet, we need more information then just the Langlands parameter itself. Let $\mathrm{Z}_{\mathbf{G}^\vee}(\phi)$ denote the centralizer of $\phi(W_F')$ in $\mathbf{G}^\vee$. This is a reductive algebraic group over $\mathbb{C}$, in general disconnected. We denote by $\mathrm{Z}_{\mathbf{G}^\vee}$ the center of $\mathbf{G}^\vee$ and we write

$$\mathcal{R}_\phi := \pi_0\big(\mathrm{Z}_{\mathbf{G}^\vee}(\phi)/\mathrm{Z}(\mathbf{G}^\vee)^{W_F}\big). \tag{12}$$

Remark 3.1. It is expected that $\Pi_\phi(G)$ is in bijection with $\mathrm{Irr}(\mathcal{R}_\phi)$ if G is quasi-split. However, for G non necessarily quasi-split, this is not always the case.

Let $\mathbf{G}^\vee_{\mathrm{ad}}$ be the quotient $\mathbf{G}^\vee/\mathrm{Z}_{\mathbf{G}^\vee}$ (that is, the group $\mathbf{G}^\vee_{\mathrm{ad}}$ is the adjoint group of $\mathbf{G}^\vee$).

Since $\mathrm{Z}_{\mathbf{G}^\vee}(\phi)\cap \mathrm{Z}(\mathbf{G}^\vee)=\mathrm{Z}(\mathbf{G}^\vee)^{W_F}$,

$$\mathrm{Z}_{\mathbf{G}^\vee}(\phi)/\mathrm{Z}(\mathbf{G}^\vee)^{W_F}\cong \mathrm{Z}_{\mathbf{G}^\vee}(\phi)\mathrm{Z}(\mathbf{G}^\vee)/\mathrm{Z}(\mathbf{G}^\vee). \tag{13}$$

The right hand side can be considered as a subgroup of the adjoint group $\mathbf{G}^\vee_{\mathrm{ad}}$. Let $\mathrm{Z}^1_{\mathbf{G}^\vee_{\mathrm{sc}}}(\phi)$ be its inverse image under the quotient map $\mathbf{G}^\vee_{\mathrm{sc}}\to\mathbf{G}^\vee_{\mathrm{ad}}$. We can also characterize it as

$$\begin{aligned}(14)\qquad \mathrm{Z}^1_{\mathbf{G}^\vee_{\mathrm{sc}}}(\phi) \;&=\; \big\{g\in\mathbf{G}^\vee_{\mathrm{sc}}\,:\, g\phi g^{-1}=\phi\,a_g \text{ for some } a_g\in B^1(W_F,\mathrm{Z}(\mathbf{G}^\vee))\big\}\\ &= \big\{g\in \mathrm{Z}_{\mathbf{G}^\vee_{\mathrm{sc}}}\big(\phi(\mathrm{SL}_2(\mathbb{C}))\big)\,:\, g\phi|_{W_F}g^{-1}=\phi|_{W_F}\,a_g \text{ for some } a_g\in B^1(W_F,\mathrm{Z}(\mathbf{G}^\vee))\big\}\\ &= \mathrm{Z}^1_{\mathbf{G}^\vee_{\mathrm{sc}}}(\phi|_{W_F})\cap \mathrm{Z}_{\mathbf{G}^\vee_{\mathrm{sc}}}\big(\phi(\mathrm{SL}_2(\mathbb{C}))\big),\end{aligned}$$

where $B^1(W_F,\mathrm{Z}(\mathbf{G}^\vee))$ denotes the set of 1-coboundaries for group cohomology, that is, the maps $W_F\to\mathrm{Z}(\mathbf{G}^\vee)$ of the form $w\mapsto zwz^{-1}w^{-1}$ with $z\in\mathrm{Z}(\mathbf{G}^\vee)$. The neutral component of $\mathrm{Z}^1_{\mathbf{G}^\vee_{\mathrm{sc}}}(\phi)$ is $\mathrm{Z}_{\mathbf{G}^\vee_{\mathrm{sc}}}(\phi)^\circ$, so it is a complex reductive group.

Remark 3.2. We have $\mathrm{Z}^1_{\mathbf{G}^\vee_{\mathrm{sc}}}(\phi)=\mathrm{Z}_{\mathbf{G}^\vee_{\mathrm{sc}}}(\phi)$ whenever $\mathrm{Z}(\mathbf{G}^\vee_{\mathrm{sc}})^{W_F}=\mathrm{Z}(\mathbf{G}^\vee_{\mathrm{sc}})$, in particular if G is an inner twist of a split group.

Following Arthur [Ar1], given ϕ, we define the group $\mathcal{S}_\phi$ as the component group of $\mathrm{Z}^1_{\mathbf{G}^\vee}(\phi)$:

$$\mathcal{S}_\phi:=\pi_0\big(\mathrm{Z}^1_{\mathbf{G}^\vee_{\mathrm{sc}}}(\phi)\big). \tag{15}$$

Via (13), the map $\mathbf{G}^\vee_{\mathrm{sc}}\to\mathbf{G}^\vee_{\mathrm{ad}}$ induces a homomorphism $\mathcal{S}_\phi\to\mathcal{R}_\phi$. We set

$$\mathcal{Z}_\phi:=\mathrm{Z}(\mathbf{G}^\vee_{\mathrm{sc}})/\mathrm{Z}(\mathbf{G}^\vee_{\mathrm{sc}})\cap\mathrm{Z}_{\mathbf{G}^\vee_{\mathrm{sc}}}(\phi)^\circ \tag{16}$$

. Then (see [ABPS7, Lemma 1.7]) $\mathcal{S}_\phi$ is a central extension of $\mathcal{R}_\phi$ by $\mathcal{Z}_\phi$:

$$1\to\mathcal{Z}_\phi\to\mathcal{S}_\phi\to\mathcal{R}_\phi\to 1. \tag{17}$$

Since $\mathbf{G}^\vee_{\mathrm{sc}}$ is a central extension of $\mathbf{G}^\vee_{\mathrm{ad}}=\mathbf{G}^\vee/\mathrm{Z}(\mathbf{G}^\vee)$, the conjugation action of $\mathbf{G}^\vee_{\mathrm{sc}}$ on itself and on $\mathcal{S}_\phi$ descends to an action of $\mathbf{G}^\vee_{\mathrm{ad}}$. Via the canonical quotient map, also $\mathbf{G}^\vee$ acts on $\mathcal{S}_\phi$ by conjugation.

We attach to a given L-parameter $\phi\in\underline{\Phi}({}^LG)$ the following (possibly disconnected) complex reductive group:

$$\mathcal{G}_\phi:=\mathrm{Z}^1_{\mathbf{G}^\vee_{\mathrm{sc}}}(\phi(W_F)). \tag{18}$$

The following proposition provides a way to link the local Langlands correspondence for G to the generalized Springer correspondence for the groups $\mathcal{G}_\phi$'s.

Proposition 3.3. [AMS1, (92)] *The group $\mathcal{S}_\phi$ is isomorphic to the group*

$$A_{\mathcal{G}_\phi}(u_\phi):=\pi_0(\mathrm{Z}_{\mathcal{G}_\phi}(u_\phi)),$$

where $u_\phi:=\phi(1,\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right))$.

3.2. **Definition of enhanced Langlands parameters.**

Definition 3.4. An *enhanced* Langlands parameter (or enhanced L-parameter) for ${}^LG$ is a pair (ϕ,ρ), where $\phi\in\underline{\Phi}({}^LG)$ and ρ is an irreducible representation of the group $\mathcal{S}_\phi$ defined in (15).

We let $\mathbf{G}^\vee$ and $\mathbf{G}^\vee_{\mathrm{sc}}$ act on the set of enhanced L-parameters for ${}^LG$ by

$$g\cdot(\phi,\rho)=(g\phi g^{-1},g\cdot\rho)\quad\text{where}\quad (g\cdot\rho)(a)=\rho(g^{-1}ag). \tag{19}$$

We note that both groups acting in (19) yield the same orbit space.

Notation 3.5. Let $\Phi_{\mathrm{e}}({}^L G)$ denote the set of $\mathbf{G}^\vee$-conjugacy classes of enhanced Langlands parameters for ${}^L G$.

We set

$$\mathcal{Z}_\phi^{W_F} := \mathrm{Z}(G_{\mathrm{sc}}^\vee)^{W_F} / \big(\mathrm{Z}(G_{\mathrm{sc}}^\vee)^{W_F} \cap Z_{G_{\mathrm{sc}}^\vee}(\phi)^\circ\big). \tag{20}$$

According to [Ar1, §4]

$$\mathrm{Z}(G_{\mathrm{sc}}^\vee) \cap \mathrm{Z}_{G_{\mathrm{sc}}^\vee}(\phi)^\circ \subset \mathrm{Z}(G_{\mathrm{sc}}^\vee)^{W_F}. \tag{21}$$

Hence $\mathcal{Z}_\phi^{W_F}$ can be regarded as a subgroup of $\mathcal{Z}_\phi$ and

$$\mathcal{Z}_\phi / \mathcal{Z}_\phi^{W_F} \cong \mathrm{Z}(G_{\mathrm{sc}}^\vee)/\mathrm{Z}(G_{\mathrm{sc}}^\vee)^{W_F}. \tag{22}$$

By Schur's lemma every enhanced Langlands parameter (ϕ, ρ) restricts to a character $\rho\big|_{\mathcal{Z}_\phi^{W_F}}$ of $\mathcal{Z}_\phi^{W_F}$. This can be inflated to a character ζ_ρ of $\mathrm{Z}(\mathbf{G}_{\mathrm{sc}}^\vee)^{W_F}$. With the Kottwitz isomorphism (9) we get an element $\vartheta := \kappa_{G_{\mathrm{ad}}^*}^{-1}(\zeta_\rho) \in H^1(F, \mathbf{G}_{\mathrm{ad}}^*)$. In this way (ϕ, ρ) determines a unique inner twist G_ϑ of G. This can be regarded as an alternative way to specify for which inner twists of G an enhanced Langlands parameter is relevant. Fortunately, it turns out that it agrees with the earlier definition of relevance of Langlands parameters.

Indeed for $\phi \in \underline{\Phi}({}^L G)$ the following are equivalent (see [ABPS7, Prop. 1.8])

(1) ϕ is relevant for G_ϑ;
(2) $Z(G_{\mathrm{sc}}^\vee)^{W_F} \cap \mathrm{Z}_{G_{\mathrm{sc}}^\vee}(\phi)^\circ \subset \ker \zeta$;
(3) there exists a $\rho \in \mathrm{Irr}(\mathcal{S}_\phi)$ such that ζ is the lift of $\rho\big|_{\mathcal{Z}_\phi^{W_F}}$ to $\mathrm{Z}(G_{\mathrm{sc}}^\vee)^{W_F}$.

Here $\zeta \in \mathrm{Irr}(\mathrm{Z}(G_{\mathrm{sc}}^\vee)^{W_F})$ and G_ϑ is the inner twist of G associated to $\vartheta = \kappa_{G_{\mathrm{ad}}^*}^{-1}(\zeta)$ via (9).

Notation 3.6. We denote by $\Phi_{\mathrm{e}}(G)$ the set of $\mathbf{G}^\vee$-equivalence classes of enhanced L-parameters that are G-relevant.

Let $(\phi, \rho) \in \Phi_{\mathrm{e}}({}^L G)$. If ρ is relevant for G, then $\zeta_\rho \in \mathrm{Irr}\big(\mathrm{Z}(G_{\mathrm{sc}}^\vee)^{W_F}\big)$. It can be extended in precisely $[\mathrm{Z}(G_{\mathrm{sc}}^\vee) : \mathrm{Z}(G_{\mathrm{sc}}^\vee)^{W_F}]$ ways to a character of $\mathrm{Z}(G_{\mathrm{sc}}^\vee)$. We choose such an extension and we denote it by ζ_G. Every $\phi \in \underline{\Phi}({}^L G)$ can be enhanced with a $\rho \in \mathrm{Irr}(\mathcal{S}_\phi)$ such that $\rho\big|_{\mathcal{Z}_\phi}$ inflates to ζ_G.

We denote the set of equivalence classes of such $(\phi, \rho) \in \underline{\Phi}_{\mathrm{e}}({}^L G)$ by $\Phi_{\mathrm{e},\zeta_G}({}^L G)$. Of course we pick $\zeta_G = \mathrm{triv}$ when $G = G^*$. We have

$$\Phi_{\mathrm{e},\zeta_G}({}^L G) = \Phi_{\mathrm{e}}(G),$$

and, in particular $\Phi_{\mathrm{e},\mathrm{triv}}({}^L G) = \Phi_{\mathrm{e}}(G^*)$.

3.3. **Desiderata for the local Langlands correspondence.** We are ready to formulate the version of the conjectural local Langlands correspondence stated in [ABPS7]. It is inspired by many sources, in particular [Bor, §10], [Vog, §4], [Ar1, §3] and [Hai, §5.2].

Conjecture 3.7. *Let (G, ϑ) be an inner twist of a quasi-split F-group G^*. There exists a surjection*

$$\Phi_{\mathrm{e}}({}^L G) \longrightarrow \mathrm{Irr}(G) \, : \, (\phi, \rho) \mapsto \pi_{\phi,\rho},$$

which becomes bijective when restricted to $\Phi_{\mathrm{e},\zeta_G}(G) = \Phi_{\mathrm{e}}(G)$. *We write its inverse as*

$$\mathrm{Irr}(G) \longrightarrow \Phi_{\mathrm{e}}(G) : \pi \mapsto (\phi_\pi, \rho_\pi).$$

Then the map $\mathrm{Irr}(G) \to \Phi(G) : \pi \mapsto \phi_\pi$ *is canonical. These maps satisfy the properties (1) – (7) listed below.*

Remark 3.8. The above bijection becomes more elegant if one considers the union over inner twists, then it says that there exists a surjection

$$\{(\phi, \rho) : \phi \in \Phi(G^*), \rho \in \mathrm{Irr}(\mathcal{S}_\phi)\} \to \{(G, \vartheta, \pi) \,:\, (G, \vartheta) \text{ inner twist of } G^*, \pi \in \mathrm{Irr}(G)\}$$

whose fibers have exactly $[\mathrm{Z}(G^\vee_{\mathrm{sc}}) : \mathrm{Z}(G^\vee_{\mathrm{sc}})^{W_F}]$ elements.

Properties:

(1) The central character of π equals the character of $\mathrm{Z}(G)$ constructed from ϕ_π in [Bor, §10.1].
(2) Let $z \in H^1_c(W_F, \mathrm{Z}(\mathbf{G}^\vee))$ be a class in continuous group cohomology, and let $\chi_z \colon G \to \mathbb{C}^\times$ be the character associated to it in [Bor, §10.2]. Thus $z\phi_\pi \in \tilde{\Phi}(G)$ and $\mathcal{S}_{z\phi_\pi} = \mathcal{S}_{\phi_\pi}$. Then the LLC should satisfy $(z\phi_\pi, \rho_\pi) = (\phi_{\chi_z\pi}, \rho_{\chi_z\pi})$.
(3) π is essentially square-integrable if and only if ϕ_π is discrete.
(4) π is tempered if and only if ϕ_π is bounded.
(5) Let P be a parabolic subgroup of G with Levi factor L. Suppose that $g \in \mathrm{N}_G(L)$ and $\check{g} \in \mathrm{N}_{G^\vee}(L^\vee)$ are such that $\mathrm{Ad}(g) \colon L \to L$ and $\mathrm{Ad}(\check{g}) \colon L^\vee \to L^\vee$ form a corresponding pair of homomorphisms, in the sense of [Bor, §2]. Then

$$(\phi_{g\cdot\pi}, \rho_{g\cdot\pi}) = (\mathrm{Ad}(\check{g})\phi_\pi, \check{g} \cdot \rho_\pi) \quad \text{for all} \quad \pi \in \mathrm{Irr}(L).$$

(6) Suppose that $(\phi^L, \rho^L) \in \Phi_{\mathrm{e}}(L)$ is bounded. Then

$$\{\pi_{\phi,\rho} \colon \phi = \phi^L \text{ composed with } {}^LL \to {}^LG, \rho\big|_{\mathcal{S}^L_\phi} \text{ contains } \rho^L\} \tag{23}$$

equals the set of irreducible constituents of the parabolically induced representation $\mathrm{I}_P^G(\pi_{\phi^L,\rho^L})$.
(7) If ϕ^L is discrete but not necessarily bounded, then (23) is the set of Langlands constituents of $\mathrm{I}_P^G(\pi_{\phi^L,\rho^L})$, as in [ABPS1, p. 30].

3.3.1. *Inner forms of* $\mathrm{GL}_n(F)$. The local Langlands correspondence for supercuspidal representations of $\mathrm{GL}_n(F)$ was established first for F of positive chacteristic in [LRS], and later for F of characteristic zero in [HaTa], [Hen2] and [Scho], independently.

Together with the Jacquet–Langlands correspondence this provides the LLC for essentially square-integrable representations of inner forms $G = \mathrm{GL}_m(D)$ of $\mathrm{GL}_n(F)$. It is extended to all irreducible G-representations via the Zelevinsky classification [Zel], see [HiSa], and [ABPS4]. For these groups every L-packet is a singleton and the LLC is a canonical bijective map

$$\mathrm{rec}_{\mathrm{GL}_m(D)} \colon \mathrm{Irr}(\mathrm{GL}_m(D)) \to \Phi(\mathrm{GL}_m(D)). \tag{24}$$

3.3.2. *Inner forms of* $\mathrm{SL}_n(F)$. The local Langlands correspondence for $\mathrm{SL}_m(D)$ was established [HiSa] (for F of characteristic zero) and [ABPS4] (for F of positive characteristic). It is derived from the LLC for $\mathrm{GL}_m(D)$, in the sense that every L-packet for $\mathrm{SL}_m(D)$ consists of the irreducible constituents of

$$\mathrm{Res}_{\mathrm{SL}_m(D)}^{\mathrm{GL}_m(D)}(\Pi_\phi(\mathrm{GL}_m(D))),$$

with $\phi \in \underline{\Phi}(\mathrm{GL}_m(D))$. Of course these L-packets have more than one element in general (as showed Example 2.20).

3.3.3. *The local Langlands conjecture for tempered representation.* Let $\mathrm{Irr}^{\mathrm{t}}(G)$ denote the subset of $\mathrm{Irr}(G)$ consisting of all irreducible smooth representations π whose matrix coefficients are $L^{2+\epsilon}$ on G modulo the center $\mathrm{Z}(G)$, and let $\Phi^{\mathrm{bd}}(G)$ be the subset of $\Phi(G)$ consisting of $\mathbf{G}^\vee$-conjugacy classes of L-parameters ϕ such that the $\phi(W_F \times \mathrm{SU}_2)$ is bounded modulo the center.

The local Langlands conjecture for tempered representation asserts the existence of a surjective map

$$\mathrm{rec}_G^{\mathrm{t}} \colon \mathrm{Irr}^{\mathrm{t}}(G) \to \Phi^{\mathrm{bd}}(G), \tag{25}$$

the fibers of it are called the *tempered L-packets*. For each $\phi \in \Phi^{\mathrm{bd}}(G)$, the corresponding tempered L-packet $\Pi_\phi^{\mathrm{t}}(G)$ is expected to be parametrized by $\mathrm{Irr}(S_\phi)_{\zeta_G}$, and $\mathrm{Irr}^{\mathrm{t}}(G)$ to be a disjoint union of the $\Pi_\phi^{\mathrm{t}}(G)$. In the case when the characteristic of F is zero, this conjecture has been proved by Arthur in [Ar2] for quasi-split orthogonal groups and symplectic groups, by Mok in [Mok] for quasi-split unitary groups, and by Kaletha, Minguez, Shin and White for non in [KMSW]. Note that the unitary group $\mathrm{U}_{E/F}(n)$ admits a non-quasi-split inner form exactly when n is even.

The LLC was also proved for the groups $\mathrm{GSp}_4(F)$ [GaTak1] and [Ga], its inner form [GaTan], the group $\mathrm{Sp}_4(F)$ [GaTak2] and its inner form [Ch], and for the groups GSpin_4, GSpin_6, and their inner forms [AsCh].

In [GaVa], Ganapathy and Varma used Arthur's results to lift the LLC for split symplectic and special orthogonal groups on a non Archimedean field of odd positive characteristic, but using a "Gan-Takeda type" characterization instead of the theory of endoscopy. They proved (see [GaVa, Theorem 13.6.1]) that given a tempered representation π of G, there exists a unique bounded Langlands parameter ϕ_π, defined by suitable compatibility conditions on Langlands-Shahidi L-functions and γ-factors, and on Plancherel measures, together with the requirement that ϕ_π is discrete if π belongs to the discrete series.

For nice and precise states of art of the local Langlands conjecture the reader should consult [Kal2], [JN] and [Lom], the latter including global aspects and the link with the Ramanujan conjecture. A survey on other aspects of the Langlands correspondance may be found in [Au] and the references herein.

3.4. **Depth.** Another invariant that makes sense on both sides of the LLC is the *depth*. The *depth* $d(\pi)$ of an irreducible smooth representation π of a reductive p-adic group G was defined by Moy and Prasad [MoPr] in terms of filtrations $G_{x,r}$ (with x a point in the Bruhat-Tits buiding of G and $r \in \mathbb{R}_{\geq 0}$) of its parahoric subgroups $G_{x,0}$.

The depth of a Langlands parameter ϕ is defined to be the smallest number $d(\phi) \geq 0$ such that ϕ is trivial on Γ^r for all $r > d(\phi)$.

Yu [Yu, §7.10] proved that the depth is preserved by the LLC for unramified tori. Recently, Mishra and Patanayak proved that it is not preserved for wildly ramified tori [MiPa].

3.4.1. *Inner forms of general linear groups.* Let $\mathrm{GL}_m(D)$ an inner form of $\mathrm{GL}_n(F)$. Let $k_D = \mathfrak{o}_D/\mathfrak{p}_D$ be the residual field of D. Let $\mathfrak{A}$ be a hereditary $\mathfrak{o}_F$-order in $\mathrm{M}_m(D)$. The Jacobson radical of $\mathfrak{A}$ will be denoted by $\mathfrak{P}$. Let $r = e_D(\mathfrak{A})$ and $e = e_F(\mathfrak{A})$ denote the integers defined by $\mathfrak{p}_D\mathfrak{A} = \mathfrak{P}^r$ and $\mathfrak{p}_F\mathfrak{A} = \mathfrak{P}^e$, respectively. We have $e_F(\mathfrak{A}) = d\, e_D(\mathfrak{A})$. The normalizer in G of $\mathfrak{A}^\times$ will be denoted by

$$\mathfrak{K}(\mathfrak{A}) := \left\{g \in G \,:\, g^{-1}\mathfrak{A}^\times g = \mathfrak{A}^\times\right\}.$$

Define a sequence of compact open subgroups of $G = \mathrm{GL}_m(D)$ by

$$U^0(\mathfrak{A}) := \mathfrak{A}^\times, \quad \text{and} \quad U^j(\mathfrak{A}) := 1 + \mathfrak{P}^j, \;\; j \geq 1.$$

Then $\mathfrak{A}^\times$ is a parahoric subgroup of G and $U^1(\mathfrak{A})$ is its pro-unipotent radical. We define the *normalized level* of an irreducible representation π of G to be

$$\ell(\pi) := \min\left\{j/e_F(\mathfrak{A})\right\}, \tag{26}$$

where $(j, \mathfrak{A})$ ranges over all pairs consisting of an integer $j \geq 0$ and a hereditary $\mathfrak{o}_F$-order $\mathfrak{A}$ in $\mathrm{M}_m(D)$ such that π contains the trivial character of $U^{j+1}(\mathfrak{A})$. Then (see for instance [ABPS3, Proposition 2.5]) the normalized level of $\pi \in \mathrm{Irr}(G)$ equals its Moy–Prasad depth:

$$\ell(\pi) = d(\pi).$$

.

Let ψ be a nontrivial character of F and let $c(\psi)$ be the largest integer c such that $\mathfrak{p}_F^{-c} \subset \ker\psi$. The ϵ factor of ϕ (and ψ) was defined in [Tat]. It takes the form

$$\epsilon(s, \phi, \psi) = \epsilon(0, \phi, \psi) q^{-(a(\phi)+nc(\psi))s} \text{ with } \epsilon(0, \phi, \psi) \in \mathbb{C}^\times. \tag{27}$$

Here $a(\phi) \in \mathbb{Z}_{\geq 0}$ is the Artin conductor of ϕ (called $f(\phi)$ in [Ser, §VI.2]). For any elliptic $\phi \in \Phi(GL_n(F))$, we have (see [ABPS3, Lemma 2.3])

$$d(\phi) = \begin{cases} 0 & \text{if } I_F \subset \ker(\phi), \\ \dfrac{a(\phi)}{n} - 1 & \text{otherwise.} \end{cases} \tag{28}$$

Let π be an irreducible representation of $\mathrm{GL}_m(D)$. Let $\epsilon(s, \pi, \psi)$ denote the Godement–Jacquet local constant [GoJa]. It takes the form

$$\epsilon(s, \pi, \psi) = \epsilon(0, \pi, \psi)\, q^{-f(\pi,\psi)s}, \quad \text{where } \epsilon(0, \pi, \psi) \in \mathbb{C}^\times. \tag{29}$$

A representation of $D^\times$ is called *unramified* if it is trivial on $\mathfrak{o}_D^\times$. An unramified representation of $D^\times$ is a character and has depth zero.

Let π be a supercuspidal irreducible representation of G. We have (see [ABPS3, Proposition 2.6]:

$$f(\pi, \psi) \;=\; \begin{cases} n\,(c(\psi) + 1) - 1 & \text{if } m = 1 \text{ and } \pi \text{ is unramified,} \\ n\;(d(\pi) + 1 + c(\psi)) & \text{otherwise.} \end{cases} \tag{30}$$

We set

$$f(\pi) := f(\pi, \psi) - nc(\psi). \tag{31}$$

Theorem 3.9. [ABPS3, Theorem 2.7]
The depth $d(\pi)$ and the conductor $f(\pi)$ of each essentially square-integrable irreducible representation π of $\mathrm{GL}_m(D)$ are linked by the following relation:

$$d(\pi) = \begin{cases} 0 & \text{if } \pi \text{ is an unramified twist of } \mathrm{St}_{\mathrm{GL}_m(D)} \\ \dfrac{f(\pi)-n}{n} & \text{otherwise.} \end{cases}, \tag{32}$$

In particular

$$d(\pi) = \max\left\{\frac{f(\pi)-n}{n}, 0\right\}. \tag{33}$$

Theorem 3.9 is a key ingredient in the proof of the following result:

Theorem 3.10. [ABPS3, Theorem 2.9]
The LLC *for $G = \mathrm{GL}_m(D)$ preserves the depth, that is:*

$$d(\pi) = d(\phi_\pi), \quad \text{where } \phi_\pi = \mathrm{rec}_G(\pi).$$

3.4.2. *Inner forms of special general linear groups.* The situation is different for $\mathrm{SL}_m(D)$. All the irreducible representations in a given L-packet Π_ϕ have the same depth, so the depth is an invariant of the L-packet, say $d(\Pi_\phi)$. We have $d(\Pi_\phi) = d(\varphi)$ where φ is a lift of ϕ which has minimal depth among the lifts of ϕ, and the following holds:

$$d(\phi) \leq d(\Pi_\phi) \tag{34}$$

for any Langlands parameter ϕ for $\mathrm{SL}_m(D)$ [ABPS3, Proposition 3.4 and Corollary 3.4]. Moreover (34) is an equality *if* ϕ is essentially tame (in the terminology of Definition 2.13).

Remark 3.11. The notion of essentially tameness is consistent with the usual notion for Langlands parameters for $\mathrm{GL}_n(F)$. Indeed, any lift $\varphi\colon W_F \to \mathrm{GL}_n(\mathbb{C})$ of ϕ, is called essentially tame if its restriction to P_F is a direct sum of characters. We denote by $t(\varphi)$ the *torsion number* of φ, that is, the number of unramified characters χ of W_F such $\varphi\chi \cong \varphi$. Then ϕ and φ are essentially tame if and only if the residual characteristic p of F does not divide $n/t(\varphi)$ [BuHe, Appendix].

However, let F be a local non-archimedean field of characteristic 2, that is F is of the form $F = \mathbb{F}_q((t))$, the field of Laurent series with coefficients in $\mathbb{F}_q$, with $q = 2^f$. This case is particularly interesting because there are countably many quadratic extensions of $\mathbb{F}_q((t))$. Then equality holds in (34) *only if* ϕ is essentially tame (*i.e.*, $t(\varphi) = 2$), as proved in[AMPS].

In the case when G is a classical group, the characteristic of F is zero (that is, F is a finite extension of $\mathbb{Q}_p$) and p is odd, Ganapathy and Varma proved in [GaVa, Lemma 8.2.3] that the following inequality holds

$$d(\phi_\pi) \leq \lfloor d(\pi) \rfloor + 1, \tag{35}$$

where ϕ the Langlands parameter attached to π by Arthur. If moreover p is sufficiently large with respect to G, then it is shown in [GaVa] that

$$d(\pi) \leq d(\phi_\pi). \tag{36}$$

Very recently, it was proved in [Oi], for a quasi-split classical group over F, with F of characteristic equal to zero and p sufficiently large, that the depth of representations in each L-packet equals the depth of the corresponding L-parameter, and that, for quasi-split unitary groups, the depth is constant in each L-packet.

4. Generalized Springer correspondence

4.1. **Cuspidal enhanced unipotent classes.** Let $\mathcal{G}$ be a complex (possibly disconnected) reductive group. Let $\mathcal{G}^\circ$ be its identity component. For $u \in \mathcal{G}$ unipotent, we denote by $A_{\mathcal{G}}(u)$ the component group of the centralizer of u in $\mathcal{G}$. We denote by $\mathrm{U}(\mathcal{G})$ the unipotent variety of $\mathcal{G}$.

Definition 4.1. The *enhancement* of $\mathrm{U}(\mathcal{G})$ is the set $\mathrm{U}_{\mathrm{e}}(\mathcal{G})$ of $\mathcal{G}$-conjugacy classes of pairs (u,ρ), with $u \in \mathcal{G}$ unipotent and $\rho \in \mathrm{Irr}(A_{\mathcal{G}}(u)$. We call a pair (u,ρ) an *enhanced unipotent class.*

Let $D^b_{\mathcal{G}}(\mathrm{U}(\mathcal{G}))$ denote the constructible $\mathcal{G}$-equivariant derived category on $\mathrm{U}(\mathcal{G})$defined by Bernstein and Lunts in [BeLu], and let $\mathrm{Perv}_{\mathcal{G}}(\mathrm{U}(\mathcal{G}))$ be its subcategory of $\mathcal{G}$-equivariant perverse sheaves.

By a $\mathcal{P}$-resolution of an algebraic variety X we mean a variety Y endowed with a free $\mathcal{P}$-action and a smooth $\mathcal{P}$-equivariant morphism $Y \to X$.

Definition 4.2. The *integration functor* is the functor

$$\gamma_{\mathcal{P}}^{\mathcal{G}}\colon D^b_{\mathcal{P}}(\mathrm{U}(\mathcal{G})) \to D^b_{\mathcal{G}}(\mathrm{U}(\mathcal{G})),$$

defined by

$$(\gamma_{\mathcal{P}}^{\mathcal{G}} A)(Y) := (q_Y)_! A(Y)[2\dim \mathcal{G}/\mathcal{P}],$$

for A any object of $D^b_{\mathcal{P}}(\mathrm{U}(\mathcal{G}))$ and Y a $\mathcal{G}$-resolution of $\mathrm{U}(\mathcal{G})$, where $q_Y\colon \mathcal{P}\backslash Y \to \mathcal{G}\backslash Y$ is the quotient functor and $A(Y)$ is defined by regarding Y as a $\mathcal{P}$-resolution of $\mathrm{U}(\mathcal{G})$.

Definition 4.3. Let $\mathcal{P}^\circ = \mathcal{L}^\circ\mathcal{U}$ be a parabolic subgroup of $\mathcal{G}^\circ$. Let

$$m\colon \mathrm{U}(\mathcal{P}^\circ) \hookrightarrow \mathrm{U}(\mathcal{G}^\circ) \quad \text{and} \quad p\colon \mathrm{U}(\mathcal{P}^\circ) \to \mathrm{U}(\mathcal{L}^\circ)$$

denote inclusion and projection, respectively. The *parabolic induction functor* is the functor

$$\mathrm{i}_{\mathcal{L}^\circ\subset\mathcal{P}^\circ}^{\mathcal{G}^\circ} := \gamma_{\mathcal{P}^\circ}^{\mathcal{G}^\circ} \circ m_! \circ p^* : \mathrm{Perv}_{\mathcal{L}^\circ}(\mathrm{U}(\mathcal{L}^\circ)) \to \mathrm{Perv}_{\mathcal{G}^\circ}(\mathrm{U}(\mathcal{G}^\circ)).$$

Remarks 4.4.
(1) The functor $\mathrm{i}_{\mathcal{L}^\circ\subset\mathcal{P}^\circ}^{\mathcal{G}^\circ}$ commutes with Verdier duality. It is left adjoint to $\mathrm{r}_{\mathcal{L}^\circ\subset\mathcal{P}^\circ}^{\mathcal{G}^\circ} := p_! \circ m^*$ and right adjoint to ${}'\mathrm{r}_{\mathcal{L}^\circ\subset\mathcal{P}^\circ}^{\mathcal{G}^\circ} := p_* \circ m^!$. These functors are exchanged by Verdier duality.
(2) If $\mathcal{F}_L$ is a simple object in $\mathrm{Perv}_{\mathcal{L}^\circ}(\mathrm{U}(\mathcal{L}^\circ))$, then $\mathrm{i}_{\mathcal{L}^\circ\subset\mathcal{P}^\circ}^{\mathcal{G}^\circ}(\mathcal{F}_{\mathcal{L}^\circ})$ is semisimple.

Definition 4.5. A simple object $\mathcal{F}$ in $\mathrm{Perv}_{\mathcal{G}^\circ}(\mathrm{U}(\mathcal{G}^\circ))$ is *cuspidal* if for any simple object $\mathcal{F}_{\mathcal{L}^\circ}$ in $\mathrm{Perv}_{\mathcal{L}^\circ}(\mathrm{U}(\mathcal{L}^\circ))$, $\mathcal{F}$ does not occur in $\mathrm{i}_{\mathcal{L}^\circ\subset\mathcal{P}^\circ}^{\mathcal{G}^\circ}(\mathcal{F}_{\mathcal{L}^\circ})$ (equivalently, if $\mathrm{r}_{\mathcal{L}^\circ\subset\mathcal{P}^\circ}^{\mathcal{G}^\circ}(\mathcal{F}) = 0$, resp, ${}'\mathrm{r}_{\mathcal{L}^\circ\subset\mathcal{P}^\circ}^{\mathcal{G}^\circ}(\mathcal{F}) = 0$) for any proper parabolic subgroup $\mathcal{P}^\circ$ of $\mathcal{G}^\circ$, with Levi factor $\mathcal{L}^\circ$.

Remarks 4.6.
(1) Cuspidality is preserved by Verdier duality (see for instance [AHJR, Remark 2.3]).
(2) The above definition of cuspidality is inspired by [Lus3] (see also [AHJR]). It is equivalent to the original definition of cuspidality given by Lusztig in 1984 in [Lus2] (as shown in [Lus6, 23.2. (b)]).

For u a given unipotent element in $\mathcal{G}^\circ$, let $A_{\mathcal{G}^\circ}(u)$ denote the component group of its centralizer $\mathrm{Z}_{\mathcal{G}^\circ}(u)$ of u in $\mathcal{G}^\circ$. Let $\mathcal{O}_u = (u)_{\mathcal{G}^\circ}$ be the $\mathcal{G}^\circ$-conjugacy class of u. We set $A_{\mathcal{G}^\circ}(\mathcal{O}_u) := A_{\mathcal{G}^\circ}(u)$. We will denote by

$$\rho \mapsto \mathcal{E}_\rho \tag{37}$$

the bijection between $\mathrm{Irr}(A_{\mathcal{G}^\circ}(u))$ and the irreducible $\mathcal{G}^\circ$-equivariant local systems $\mathcal{E}$ on $\mathcal{O}_u$. We denote by

$$\mathcal{E} \mapsto \rho_{\mathcal{E}} \tag{38}$$

the inverse bijection.

Definition 4.7.

(1) A character $\rho \in \mathrm{Irr}(A_{\mathcal{G}^\circ}(u))$ is *cuspidal* if the perverse sheaf $\mathrm{IC}(\mathcal{O}, \mathcal{E}_\rho)$ is cuspidal.
(2) An *enhanced unipotent class* $(\mathcal{O}, \rho)$ in $\mathcal{G}^\circ$ is *cuspidal* if ρ is cuspidal.

Proposition 4.8. [Lus2, Proposition 2.8] *If $(\mathcal{O}_u, \rho)$ is cuspidal, then u is a distinguished unipotent element in $\mathcal{G}^\circ$ (i.e., u does not meet $\mathrm{U}(\mathcal{L}^\circ)$ for any proper Levi subgroup $\mathcal{L}$ of $\mathcal{G}^\circ$).*

Remark 4.9. In general not every distinguished unipotent element supports a cuspidal representation.

Example 4.10. For $\mathcal{G} = \mathrm{SL}_n(\mathbb{C})$, the unipotent classes in $\mathcal{G}$ are in bijection with the partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ of n: the corresponding $\mathcal{G}$-conjugacy class $\mathcal{O}_\lambda$ consists of unipotent matrices with Jordan blocks of sizes λ_1, λ_2, $\ldots$,λ_r. We identify the center $\mathrm{Z}(\mathcal{G})$ with the group μ_n of complex n-roots of unity. For $u \in \mathcal{O}_\lambda$, the natural homomorphism $\mathrm{Z}(\mathcal{G}) \to A_{\mathcal{G}}(u)$ is surjective with kernel $\mu_{n/\gcd(\lambda)}$, where $\gcd(\lambda) := \gcd(\lambda_1, \lambda_2, \ldots, \lambda_r)$. Hence the irreducible $\mathcal{G}$-equivariant local systems on $\mathcal{O}_\lambda$ all have rank one, and they are distinguished by their central characters, which range over those $\chi \in \mathrm{Irr}(\mu_n)$ such $\gcd(\lambda)$ is a multiple of the order of χ. We denote these local systems by $\mathcal{E}_{\lambda,\chi}$. The unique distinguished unipotent class in $\mathcal{G}$ is the regular unipotent class $\mathcal{O}_{(n)}$, consisting of unipotent matrices with a single Jordan block. The cuspidal irreducible G-equivariant local systems are supported on $\mathcal{O}_{(n)}$ (by Proposition 4.8) and are of the form $\mathcal{E}_{(n),\chi}$, with $\chi \in \mathrm{Irr}(\mu_n)$ of order n (see [Lus2, (10.3.2)]).

We will now extend the above notion of cuspidality from $\mathcal{G}^\circ$ to $\mathcal{G}$:

Definition 4.11. An enhanced unipotent class $(\mathcal{O}, \rho)$ with $\mathcal{O} = (u)_{\mathcal{G}}$ and $\rho \in \mathrm{Irr}(A_{\mathcal{G}}(u))$ is *cuspidal* if the restriction of ρ to $A_{\mathcal{G}^\circ}(u)$ is a direct sum of cuspidal irreducible representations of $A_{\mathcal{G}^\circ}(u)$.

Notation 4.12. We set

$$\mathrm{Irr}_{\mathrm{cusp}}(A_{\mathcal{G}}(u)) := \{\epsilon \in \mathrm{Irr}(A_{\mathcal{G}}(u)) \text{ such that } \epsilon \text{ is cuspidal}\}.$$

Definition 4.13. A *quasi Levi subgroup* of $\mathcal{G}$ is a subgroup $\mathcal{M}$ of the form $\mathcal{M} = \mathrm{Z}_{\mathcal{G}}(\mathrm{Z}(\mathcal{L})^\circ)$, with $\mathcal{L}$ a Levi subgroup of $\mathcal{G}^\circ$. The group $\mathcal{M}$ is said to be *cuspidal* if there exists a cuspidal enhanced unipotent pair in $\mathcal{M}$.

Remark 4.14. A Levi subgroup of $\mathcal{G}$ is a quasi-Levi subgroup, and in the case when $\mathcal{G}$ connected, both notions coincide.

Notation 4.15. Let $\mathfrak{B}(\mathrm{U_e}(\mathcal{G}))$ be the set of $\mathcal{G}$-conjugacy classes of pairs $(\mathcal{M}, (\mathcal{O}, \epsilon))$, where $\mathcal{M}$ is a cuspidal quasi-Levi subgroup of $\mathcal{G}$, and $(\mathcal{O}, \epsilon)$ is a cuspidal enhanced unipotent pair in $\mathcal{M}$.

4.2. **A partition of the enhancement of the unipotent variety of $\mathcal{G}^\circ$.** The purpose of this section is to describe a *theory à la Harish-Chandra for* $\mathrm{U_e}(\mathcal{G}^\circ)$, the first step being to define a *cuspidal support map* for enhanced unipotent classes of $\mathcal{G}$.

The enhancement $\mathrm{U_e}(\mathcal{G}^\circ)$ of $\mathrm{U}(\mathcal{G}^\circ)$ parametrizes the isomorphism classes of simple objects of $\mathrm{Perv}_{\mathcal{G}^\circ}(\mathrm{U}(\mathcal{G}^\circ))$. Indeed, the simple objects in $\mathrm{Perv}_{\mathcal{G}^\circ}(\mathrm{U}(\mathcal{G}^\circ))$ are the $\mathrm{IC}(\mathcal{O}, \mathcal{E})$, where $\mathcal{O}$ is a unipotent class in $\mathcal{G}^\circ$ and $\mathcal{E}$ is an irreducible $\mathcal{G}^\circ$-equivariant $\overline{\mathbb{Q}}_\ell$-local system on $\mathcal{O}$.

Let $(\mathcal{O}, \rho) \in \mathrm{U_e}(\mathcal{G}^\circ)$ be an arbitrary enhanced unipotent class, and set $\mathcal{F}_\rho := \mathrm{IC}(\mathcal{O}, \mathcal{E}_\rho)$. Then $\mathcal{F}_\rho$ occurs as a summand of $\mathrm{i}_{\mathcal{L}^\circ \subset \mathcal{P}^\circ}(\mathrm{IC}(\mathcal{O}_0, \mathcal{E}_0))$, for some quadruple $(\mathcal{P}^\circ, \mathcal{L}^\circ, \mathcal{O}_0, \mathcal{E}_0)$, where $\mathcal{P}^\circ$ is a parabolic subgroup of $\mathcal{G}^\circ$ with Levi subgroup $\mathcal{L}^\circ$ and $(\mathcal{O}_0, \mathcal{E}_0)$ is a cuspidal enhanced unipotent class in $\mathcal{L}^\circ$ (see [Lus2, § 6.2] and [AHJR, Cor. 2.7]) and, moreover, $(\mathcal{P}^\circ, \mathcal{L}^\circ, \mathcal{O}_0, \mathcal{E}_0)$ is unique up to $\mathcal{G}^\circ$-conjugation (see [Lus2, Prop. 6.3]). We set $\epsilon := \rho_{\mathcal{E}_0}$ using the bijection (38), and we denote by $\mathfrak{t}^\circ := (\mathcal{L}^\circ, (\mathcal{O}_0, \epsilon))_{\mathcal{G}^\circ}$, the $\mathcal{G}^\circ$-conjugacy class of $(\mathcal{L}^\circ, (\mathcal{O}_0, \epsilon))$ and we call it the *cuspidal support* of the enhanced unipotent class $(\mathcal{O}, \rho)$.

The center $\mathrm{Z}(\mathcal{G}^\circ)$ of $\mathcal{G}^\circ$ maps naturally to $A_{\mathcal{G}^\circ}(\mathcal{O})$ and to $A_{\mathcal{L}^\circ}(\mathcal{O}_0)$. By construction [Lus2, Theorem 6.5.a]

$$\rho \text{ and } \epsilon \text{ have the same } \mathrm{Z}(\mathcal{G}^\circ)\text{-character.} \tag{39}$$

Definition 4.16. The *cuspidal support map* for $\mathrm{U_e}(\mathcal{G}^\circ)$ is the map

$$\Psi_{\mathcal{G}^\circ} \colon \mathrm{U_e}(\mathcal{G}^\circ) \to \mathfrak{B}(\mathrm{U_e}(\mathcal{G}^\circ)),$$

where $\mathfrak{B}(\mathrm{U_e}(\mathcal{G}^\circ))$ is as in Notation 4.15, which sends the $\mathcal{G}^\circ$-conjugacy class of $(\mathcal{O}, \rho)$ to its cuspidal support $\mathfrak{t}^\circ = (\mathcal{L}^\circ, (\mathcal{O}_0, \epsilon))_{\mathcal{G}^\circ}$.

By (39) the map $\Psi_{\mathcal{G}^\circ}$ preserves the $\mathrm{Z}(\mathcal{G}^\circ)$-characters of the involved representations.

Notation 4.17. Let $\mathfrak{t}^\circ \in \mathfrak{B}(\mathrm{U_e}(\mathcal{G}^\circ))$.

(1) We denote by $\mathrm{U_e}(\mathcal{G}^\circ)^{\mathfrak{t}^\circ}$ denotes the fiber of $\mathfrak{t}$ under the map $\Psi_{\mathcal{G}^\circ}$.
(2) Let $W_{\mathcal{L}^\circ} := \mathrm{N}_{\mathcal{G}^\circ}(\mathcal{L}^\circ)/\mathcal{L}^\circ$, and let $W_{\mathfrak{t}^\circ} := \mathrm{N}_{\mathcal{G}^\circ}(\mathfrak{t}^\circ)/\mathcal{L}^\circ$.

Theorem 4.18. [Lus2]

(1) The group $W_{\mathcal{L}^\circ}$ is a Weyl group and it coincides with $W_{\mathfrak{t}^\circ}$ for every $\mathfrak{t}^\circ \in \mathfrak{B}(\mathrm{U_e}(\mathcal{G}^\circ))$.
(2) We have

$$\mathrm{U_e}(\mathcal{G}^\circ) = \bigsqcup_{\mathfrak{t}^\circ \in \mathfrak{B}(\mathrm{U_e}(\mathcal{G}^\circ))} \mathrm{U_e}(\mathcal{G}^\circ)^{\mathfrak{t}^\circ},$$

and $\mathrm{U_e}(\mathcal{G}^\circ)^{\mathfrak{t}^\circ}$ *is in bijection with* $\mathrm{Irr}(W_{\mathcal{L}^\circ})$.

The goal of the next two subsections is to generalize the previous notation and terminology from $\mathcal{G}^\circ$ to $\mathcal{G}$.

4.3. **Twisted group algebras.** Throughout this section Γ is a finite group and K is an algebraically closed field whose characteristic does not divide the order of Γ. Suppose that $\kappa\colon \Gamma\times\Gamma\to K^\times$ is a 2-cocycle, that is,

$$\kappa(\gamma_1,\gamma_2\gamma_3)\kappa(\gamma_2,\gamma_3)=\kappa(\gamma_1,\gamma_2)\kappa(\gamma_1\gamma_2,\gamma_3)\quad \forall\gamma_1,\gamma_2,\gamma_3\in\Gamma. \tag{40}$$

The κ-twisted group algebra of Γ is defined to be the K-vector space $K[\Gamma,\kappa]$ with basis $\{T_\gamma:\gamma\in\Gamma\}$ and multiplication rules

$$T_\gamma T_{\gamma'}=\kappa(\gamma,\gamma')T_{\gamma\gamma'}\quad \gamma,\gamma'\in\Gamma. \tag{41}$$

Its representations can be considered as projective Γ-representations. Schur showed (see [CuRe, Theorem 53.7]) that there exists a finite central extension $\tilde\Gamma$ of Γ, such that

- $\mathrm{char}(K)$ does not divide $|\tilde\Gamma|$,
- every irreducible projective Γ-representation over K lifts to an irreducible K-linear representation of $\tilde\Gamma$.

Then $K[\Gamma,\kappa]$ is a direct summand of $K[\tilde\Gamma]$, namely the image of a minimal idempotent in $K[\ker(\tilde\Gamma\to\Gamma)]$. The condition on $\mathrm{char}(K)$ ensures that $K[\tilde\Gamma]$ is semisimple, so $K[\Gamma,\kappa]$ is also semisimple.

4.4. **A partition of the enhancement of the unipotent variety of $\mathcal{G}$.**

Notation 4.19. Let $\mathfrak{t}=(\mathcal{M},(\mathcal{O}_0,\epsilon))_\mathcal{G}\in\mathfrak{B}(\mathrm{U_e}(\mathcal{G}))$. We set

$$W_\mathfrak{t}:=\mathrm{N}_\mathcal{G}(\tau)/\mathcal{M}\quad\text{and}\quad W_\mathfrak{t}^\circ:=\mathrm{N}_{\mathcal{G}^\circ}(\mathcal{M}^\circ)/\mathcal{M}^\circ.$$

Theorem 4.20. [AMS1, §4] *Let $(\mathcal{O},\rho)\in\mathrm{U_e}(\mathcal{G})$. There exists a 2-cocycle*

$$\kappa_\mathfrak{t}\colon W_\mathfrak{t}/W_\mathfrak{t}^\circ\times W_\mathfrak{t}/W_\mathfrak{t}^\circ\to\mathbb{C}^\times$$

and a map, called the cuspidal support map for $\mathrm{U_e}(\mathcal{G})$

$$\Psi_\mathcal{G}\colon \mathrm{U_e}(\mathcal{G})\to\mathfrak{B}(\mathrm{U_e}(\mathcal{G}))$$

(which coincides to the map $\Psi_{\mathcal{G}^\circ}$ defined above in the case when $\mathcal{G}$ is connected), such that

$$\mathrm{U_e}(\mathcal{G})=\bigsqcup_{\mathfrak{t}\in\mathfrak{B}(\mathrm{U_e}(\mathcal{G}))}\mathrm{U_e}(\mathcal{G})^\mathfrak{t}, \tag{42}$$

in which the fiber $\mathrm{U_e}(\mathcal{G})^\mathfrak{t}$ of $\mathfrak{t}$ under the map $\Psi_\mathcal{G}$ is isomorphic to the $\kappa_\mathfrak{t}$-twisted version $\mathbb{C}[W_\mathfrak{t},\kappa_\mathfrak{t}]$ of the group algebra $\mathbb{C}(W_\mathfrak{t}]$ of the finite group $W_\mathfrak{t}$.

Notation 4.21. Let $\Sigma_\mathfrak{t}$ denote the bijection

$$\mathrm{U_e}(\mathcal{G})^\mathfrak{t}\longrightarrow\mathbb{C}[W_\mathfrak{t},\kappa_\mathfrak{t}]$$

mentioned in Theorem 4.20.

Remark 4.22. When $\mathcal{G}$ is connected, the cocycle $\kappa_\mathfrak{t}$ is trivial. When $\mathcal{G}$ is disconnected, the cocycle $\kappa_\mathfrak{t}$ is not always trivial.

An intuitive way of thinking of Theorem 4.20 is to view the set $\mathfrak{B}(\mathrm{U_e}(\mathcal{G}))$ as a “palette of colors” and the map $\Psi_\mathcal{G}$ as a way to paint the elements of the fiber $\mathrm{U_e}(\mathcal{G})^\mathfrak{t}$ of $\mathfrak{t}$ under $\Psi_\mathcal{G}$ in the same color as $\mathfrak{t}$. Moreover, for each color, the subset of elements of $\mathrm{U_e}(\mathcal{G})$ with that color has a nice structure (here, that of the twisted group algebra).

5. Cuspidality for enhanced L-parameters: definition and conjecture

Definition 5.1. An enhanced L-parameter $(\phi, \rho) \in \Phi(G)_{\mathrm{e}}$ is called *cuspidal* if ϕ is discrete and (u_ϕ, ρ) is a cuspidal enhanced unipotent class in $\mathcal{G}_{\lambda_\phi}$ in the terminology of Definition 4.11.

Recall that an irreducible smooth complex representation of the group G is called *supercuspidal* if it does not appear in any G-representation induced from a proper Levi subgroup of G. Bernstein [BeDe, §2] realized that an irreducible G-representation is supercuspidal if and only if it is compact. Here compact means that the representation behaves like one of a compact group, in the sense that all its matrix coefficients have compact support modulo the centre of G.

Conjecture of cuspidality 5.2. [AMS1, § 6]
The cuspidal enhanced Langlands parameters correspond by the LLC *to the irreducible supercuspidal representations of G.*

Conjecture 5.2 is known to be true in the following cases:

- for general linear groups and split classical groups (any representation), with F of characteristic equal to 0: [Mou1],
- for inner forms of linear groups and of special linear groups, quasi-split unitary groups (any representation) with F of characteristic equal to 0, and for Deligne-Lusztig depth-zero representations: [AMS1, § 6],
- for the representations with unipotent reduction of the group G of the F-rational points of a connected reductive algebraic group which splits over an unramified extension of F: [FOS, Theorem 2] (when G is simple of adjoint type it is a special case of [Lus4], [Lus5]).

6. A partition of the set of enhanced Langlands parameters

We shall define a similar partition of the set of enhanced Langlands parameters by plugging the above construction into the framework of the Langlands correspondence. Let $(\phi, \rho) \in \Phi(^L G)$. We take for $\mathcal{G}$ the group $\mathcal{G}_\phi$ defined in (18) and set

$$[\mathcal{M}_\phi, (\mathcal{O}_0, \epsilon)]_{\mathcal{G}_\phi} := \Psi_{\mathcal{G}_\phi}(u_\phi, \rho) \in \mathfrak{B}(\mathrm{U}_{\mathrm{e}}(\mathcal{G}_\phi)), \tag{43}$$

where the map $\Psi_{\mathcal{G}_\phi}$ is that of Theorem 4.20. Hence $\mathcal{M}_\phi$ is a quasi Levi subgroup of $\mathcal{G}_\phi$ and $(\mathcal{O}_0, \epsilon)$ is a cuspidal enhanced unipotent class in $\mathcal{M}_\phi$. For simplicity, we will often refer to the unipotent class $\mathcal{O}_0$ in $\mathcal{M}_\phi$ by a unipotent element v in it.

The idea is very similar as above: using the data defined in (43), we will construct a set $\mathfrak{B}^\vee(G)$ which will play the role of the palette of colors for $\Phi_{\mathrm{e}}(^L G)$, and we will define a "way to paint" the elements in $\Phi_{\mathrm{e}}(^L G)$ with this set of colors (that is, we will construct a cuspidal support map) such there is a decomposition

$$\Phi_{\mathrm{e}}(^L G) = \bigsqcup_{\mathfrak{s}^\vee \in \mathfrak{B}^\vee(\mathbf{G})} \Phi_{\mathrm{e}}(^L G)^{\mathfrak{s}^\vee}, \tag{44}$$

where, for each $\mathfrak{s}^\vee$ in $\mathfrak{B}^\vee(\mathbf{G})$, the subset $\Phi_{\mathrm{e}}(^L G)^{\mathfrak{s}^\vee}$ of elements with color $\mathfrak{s}^\vee$ is related to a finite collection[1] of generalized affine Hecke algebras with possibly unequal parameters.

[1]This reflects the fact that we consider simultaneously all the inner twists of a given group G.

Definition 6.1. Let $\mathbf{T}^\vee \subset \mathbf{G}^\vee$ be a torus such that the projection $\mathrm{Z}_{{}^LG}(T) \to W_F$ is surjective. Then we call $\mathrm{Z}_{{}^LG}(T)$ a Levi L-subgroup of ${}^LG$.

Remark 6.2. In [Bor] such groups are called Levi subgroups of ${}^LG$, however we prefer to stick to the connectedness of Levi subgroups.

Remark 6.3. Choose a W_F-stable pinning for $\mathbf{G}^\vee$. This defines the notion of standard Levi subgroups of $\mathbf{G}^\vee$. An alternative characterization of the Levi L-subgroups of ${}^LG$ is as follows [AMS1, Lem]: Let $\mathrm{Z}_{{}^LG}(T)$ be a Levi L-subgroup of ${}^LG$. There exists a W_F-stable standard Levi subgroup $\mathbf{L}^\vee$ of $\mathbf{G}^\vee$ such that $\mathrm{Z}_{{}^LG}(T)$ is $\mathbf{G}^\vee$-conjugate to $\mathbf{L}^\vee \rtimes W_F =: {}^LL$ and $\mathrm{Z}_{{}^LG}(T) \cap \mathbf{G}^\vee$ is conjugate to $\mathbf{L}^\vee$.

Conversely, every $\mathbf{G}^\vee$-conjugate of this ${}^LL$ is a Levi L-subgroup of ${}^LG$.

Notation 6.4. In the sequel, we will use the (slightly abusive) notation ${}^LL$ for an arbitrary L-Levi of G.

Definition 6.5. Let ${}^LL$ be a Levi L-subgroup of ${}^LG$.

(1) A *Langlands parameter for* ${}^LL$ is a group homomorphism $\phi\colon W_F' \to {}^LL$ satisfying the requirements of Definition 2.12.
(2) An *enhancement of* ϕ is an irreducible representation ρ of $\pi_0(\mathrm{Z}^1_{\mathbf{L}^\vee_{\mathrm{sc}}}(\phi))$, where $\mathbf{L}^\vee_{\mathrm{sc}}$ is the simply connected cover of the derived group of $\mathbf{L}^\vee = {}^L\mathbf{L}^\vee \cap \mathbf{G}^\vee$.

The group $\mathbf{L}^\vee$ acts on the collection of enhanced L-parameters for ${}^LL$ by (19).

Definition 6.6. We say that an enhanced L-parameter (ϕ,ρ) for ${}^LL$ is *cuspidal* if ϕ is discrete for ${}^LL$ and $\big(u_\phi = \phi\big(1, \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)\big), \rho\big)$ is a cuspidal pair for $\mathrm{Z}^1_{\mathbf{L}^\vee_{\mathrm{sc}}}(\phi|_{W_F})$.

Notation 6.7. We denote the set of $\mathbf{L}^\vee$-orbits by $\Phi_{\mathrm{e}}({}^LL)$ and the subset of cuspidal $\mathbf{L}^\vee$-orbits by $\Phi_{\mathrm{e,cusp}}({}^LL)$.

Let $\mathbf{L}^\vee$ be a Levi subgroup of $\mathbf{G}^\vee$, and let $\mathbf{L}^\vee_{\mathrm{c}}$ denote the pre-image of $L^\vee$ under under $\mathbf{G}^\vee_{\mathrm{sc}} \to \mathbf{G}^\vee$. The derived group of $\mathbf{L}^\vee_{\mathrm{c}}$ is the simply connected cover of $\mathbf{L}^\vee_{\mathrm{der}}$, and we identify $\mathbf{L}^\vee_{\mathrm{sc}}$ with the inverse image of $\mathbf{L}^\vee_{\mathrm{der}}$ under $\mathbf{G}^\vee_{\mathrm{sc}} \to \mathbf{G}^\vee$.

Definition 6.8. A cuspidal datum for ${}^LG$ is a triple $({}^LL, \phi, \rho)$ where ${}^LL$ is a Levi L-subgroup of ${}^LG$, such that (ϕ,ρ) is cuspidal for ${}^LL$. It is relevant for G if

- $\rho = \zeta$ on $\mathbf{L}^\vee_{\mathrm{sc}} \cap \mathrm{Z}(\mathbf{G}^\vee_{\mathrm{sc}})^{W_F}$, where $\zeta \in \mathrm{Irr}(\mathrm{Z}(\mathbf{G}^\vee_{\mathrm{sc}})^{W_F})$ parametrizes the inner twist G of G^* via the Kottwitz isomorphism (9):
- $\rho = 1$ on $\mathbf{L}^\vee_{\mathrm{sc}} \cap \mathrm{Z}(\mathbf{L}^\vee_{\mathrm{c}})^\circ$.

We will give now the technical details of the construction. Upon replacing (ϕ,ρ) by a $\mathbf{G}^\vee$-conjugate, there exists a Levi subgroup L of G such that $(\phi_{|W_F}, v, \epsilon)$ is a cuspidal enhanced Langlands parameter for L, and

$$ {}^LL := \mathbf{L}^\vee \rtimes W_F = \mathrm{Z}_{{}^LG}(\mathrm{Z}^\circ_{\mathcal{M}_\phi}). \tag{45} $$

Then we set

$$ {}^L\Psi(\phi,\rho) := [{}^LL, (\phi_{|W_F}, v, \epsilon)]_{\mathbf{G}^\vee}. \tag{46} $$

The group of unramified characters of L is naturally isomorphic to $(\mathrm{Z}_{\mathbf{L}^\vee \rtimes I_F})^\circ_{W_F}$. Denote the latter by $X_{\mathrm{nr}}({}^LL)$. Given $(\varphi,\epsilon) \in \Phi_{\mathrm{e}}(\mathbf{L}^\vee, F)$, and $\xi \in X_{\mathrm{nr}}({}^LL)$, define $(\xi\varphi, \varrho) \in \Phi_{\mathrm{e}}(\mathbf{L}^\vee, F)$ by $\xi\varphi := \varphi$ on $I_F \times \mathrm{SL}_2(\mathbb{C})$ and $(\xi\varphi)(\mathrm{Fr}_F) := \tilde{\xi}\varphi(\mathrm{Fr}_F)$, where $\tilde{\xi} \in \mathrm{Z}^\circ_{L^\vee \rtimes I_F}$ represents z.

Definition 6.9. We denote by $\mathfrak{s}^\vee$ the $\mathbf{G}^\vee$-conjugacy class of $({}^L L, X_{\mathrm{nr}}({}^L L)\cdot(\varphi,\epsilon))$, where L is a Levi subgroup of G, and (φ,ϵ) is a cuspidal enhanced Langlands parameter for L. We write

$$\mathfrak{s}^\vee = \mathfrak{s}^\vee_G = [{}^L L, (\varphi,\epsilon)]_{G^\vee}.$$

We call $\mathfrak{s}^\vee$ an *inertial class* for $\Phi_{\mathrm{e}}(\mathbf{G}^\vee, F)$ and denote by $\mathfrak{B}^\vee(\mathbf{G})$ the set of such $\mathfrak{s}^\vee$.

Then the subset

$$\Phi_{\mathrm{e}}({}^L G)^{\mathfrak{s}^\vee} := ({}^L\Psi)^{-1}(\mathbf{L}^\vee \rtimes W_F, \mathfrak{s}^\vee_L)$$

of $\Phi_{\mathrm{e}}({}^L G)$ is in bijection with the simple modules of a finite collection of twisted extended affine Hecke algebra $\mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}})_\vartheta$ with $\vartheta \in H^1(F, \mathbf{G}^*_{\mathrm{ad}})$: it is the object of [AMS2] and [AMS3].

6.1. **A generalized Springer correspondence for enhanced L-parameters.** Let (ϕ,ρ) be an enhanced L-parameter for G and write as above $\Psi_{\mathcal{G}_\phi}(u_\phi,\rho) =: [\mathcal{M}_\phi, v, \epsilon]_{\mathcal{G}_\phi}$. Up to $\mathcal{G}_\phi$-conjugacy there exists a unique $\gamma_v\colon \mathrm{SL}_2(\mathbb{C}) \to \mathcal{M}_\phi^\circ$ adapted to ϕ. Moreover the cocharacter

$$\chi_{\phi,v}\colon z \mapsto \phi\big(1, \left(\begin{smallmatrix} z & 0 \\ 0 & z^{-1}\end{smallmatrix}\right)\big)\gamma_v\left(\begin{smallmatrix} z^{-1} & 0 \\ 0 & z\end{smallmatrix}\right)$$

has image in $\mathrm{Z}(\mathcal{M}_\phi)^\circ$.

Fix a G-relevant cuspidal datum $({}^L L, \phi_v, \epsilon)$ for ${}^L G$, and write

$$\mathfrak{t}_\phi := [\mathcal{G}_\phi \cap \mathbf{L}^\vee_{\mathrm{c}}, v, \epsilon]_{\mathcal{G}_\phi}, \quad \mathfrak{t}^\circ_\phi := [\mathcal{G}^\circ_\phi \cap \mathbf{L}^\vee_{\mathrm{c}}, \mathcal{O}_v^{G^\circ\cap \mathbf{L}^\vee_{\mathrm{c}}}, \mathcal{E}]_{\mathcal{G}^\circ_\phi}. \tag{47}$$

The next result may be viewed as a version of the generalized Springer correspondence for enhanced L-parameters instead of enhaced unipotent classes.

Proposition 6.10. [AMS1, Proposition 9.1]

(*a*) *There is a bijection*

$$\begin{array}{cccc} {}^L\Sigma_{\mathfrak{t}_\phi}\colon & {}^L\Psi^{-1}({}^L L, \phi_v, q\epsilon) & \longleftrightarrow & \mathrm{Irr}(\mathbb{C}[W_{\mathfrak{t}_\phi}, \kappa_{\mathfrak{t}_\phi}]) \\ & (\phi,\rho) & \mapsto & \Sigma_{\mathfrak{t}_\phi}(u_\phi,\rho) \\ & (\phi|_{\mathbf{W}_F}, \Sigma^{-1}_{\mathfrak{t}_\phi}(E)) & \leftarrow & E, \end{array}$$

where $\Sigma_{\mathfrak{t}_\phi}$ is the bijection defined in Notation 4.21.

(*b*) *The canonical bijection $\Sigma_{\mathfrak{t}^\circ_\phi}$ between $\Psi^{-1}_{\mathcal{G}^\circ}(\mathfrak{t}^\circ_\phi) \subset \mathrm{U}_{\mathrm{e}}(\mathcal{G}^\circ)$ and $\mathrm{Irr}_{\mathbb{C}}(W_{\mathfrak{t}^\circ})$ relates to part (a) by*

$${}^L\Sigma_{\mathfrak{t}_\phi}(\phi,\rho)|_{W_{\mathfrak{t}^\circ_\phi}} = \bigoplus_i \Sigma_{\mathfrak{t}^\circ_\phi}(u_\phi,\rho_i),$$

where $\rho = \bigoplus_i \rho_i$ is a decomposition into irreducible $A_{\mathcal{G}^\circ_\phi}(u_\phi)$-subrepresentations.

(*c*) *The $\mathbf{G}^\vee$-conjugacy class of $(\phi|_{W_F}, u_\phi, \rho_i)$ is determined by any irreducible $\mathbb{C}[W_{\mathfrak{t}^\circ_\phi}]$-subrepresentation of ${}^L\Sigma_{\mathfrak{t}_\phi}(\phi,\rho)$.*

7. A Galois version of the ABPS Conjecture

7.1. **Twisted extended quotients.** Let Γ be a group acting on a topological space X, and let $\natural$ be a given function which assigns to each $x \in X$ a 2-cocycle

$$\natural_x\colon \Gamma_x \times \Gamma_x \to \mathbb{C}^\times, \text{ where } \Gamma_x = \{\gamma \in \Gamma \,:\, \gamma x = x\}.$$

We assume that $\natural_{\gamma x}$ and $\gamma_*\natural_x$ define the same class in $H^2(\Gamma_{\gamma x}, \mathbb{C}^\times)$, where $\gamma_*\colon \Gamma_x \to \Gamma_{\gamma x}$ sends α to $\gamma\alpha\gamma^{-1}$. Let $\mathbb{C}[\Gamma_x, \natural_x]$ be the corresponding twisted algebra as defined in Section 4.3. We set

$$\widetilde{X}_\natural := \{(x, E) \,:\, x \in X, E \in \mathrm{Irr}\,\mathbb{C}[\Gamma_x, \natural_x]\}.$$

and we topologize it by decreeing that a subset of $\widetilde{X}_\natural$ is open if and only if its projection to the first coordinate is open in X.

We require, for every $(\gamma, x) \in \Gamma \times X$, a definite algebra isomorphism

$$\phi_{\gamma,x} \colon \mathbb{C}[\Gamma_x, \natural_x] \to \mathbb{C}[\Gamma_{\gamma x}, \natural_{\gamma x}]$$

such that:

- if $\gamma x = x$, then $\phi_{\gamma,x}$ is conjugation by an element of $\mathbb{C}[\Gamma_x, \natural_x]^\times$;
- $\phi_{\gamma',\gamma x} \circ \phi_{\gamma,x} = \phi_{\gamma'\gamma,x}$ for all $\gamma', \gamma \in \Gamma, x \in X$.

Then we can define a Γ-action on $\widetilde{X}_\natural$ by

$$\gamma \cdot (x, E) := (\gamma x, E \circ \phi_{\gamma,x}^{-1}).$$

We form the *twisted extended quotient*

$$(X/\!/\Gamma)_\natural := \widetilde{X}_\natural / \Gamma.$$

Remark 7.1. Furthermore we note that $(X/\!/\Gamma)_\natural$ reduces to the extended quotient of the second kind $(X/\!/\Gamma)_2$ from [ABPS6, §2] if $\natural_x$ is trivial for all $x \in X$ and $\phi_{\gamma,x}$ is conjugation by γ.

The extended quotient of the second kind is an extension of the ordinary quotient in the sense that it keeps track of the duals of the isotropy groups. Namely, in $(X/\!/\Gamma)_2$ every point $x \in X/\Gamma$ has been replaced by the set $\mathrm{Irr}(\Gamma_x)$.

7.2. **The Bernstein decomposition of the category of smooth representations.**

Let $\mathrm{Rep}(G)$ denote the category of smooth representations of G. Let P be a parabolic subgroup of G and let L be a Levi factor of P. Let σ be a supercuspidal irreducible representation of L. We call (L, σ) a cuspidal pair, and we consider such pairs up to *inertial equivalence*: this is the equivalence relation generated by:

- unramified twists, $(L, \sigma) \sim (L, \sigma \otimes \chi)$ for $\chi \in X_{\mathrm{nr}}(L)$, where $X_{\mathrm{nr}}(L)$ is the group of unramified (not necessarily unitary) characters $L \to \mathbb{C}^\times$;
- G-conjugation, $(L, \sigma) \sim (gLg^{-1}, g \cdot \sigma)$ for $g \in G$.

We denote a typical inertial equivalence class by $\mathfrak{s} = [L, \sigma]_G$. In particular

$$\mathfrak{s}_L := [L, \sigma]_L = \{\sigma \otimes \chi \in \mathrm{Irr}(L) \,:\, \chi \in X_{\mathrm{nr}}(L)\}.$$

Bernstein attached to every $\mathfrak{s}$ a block in the category $\mathrm{Rep}(G)$, in the following way. Denote the normalized parabolic induction functor by I_P^G. We define

$$\mathrm{Irr}(G)^{\mathfrak{s}} = \{\pi \in \mathrm{Irr}(G) \,:\, \pi \text{ is a constituent of } \mathrm{I}_P^G(\sigma \otimes \chi) \text{ for some } \sigma \in \mathfrak{s}_L\},$$

$$\mathrm{Rep}(G)^{\mathfrak{s}} = \{\pi \in \mathrm{Rep}(G) :\ \text{every irreducible constituent of } \pi \text{ belongs to } \mathrm{Irr}(G)^{\mathfrak{s}}\}.$$

We denote the set of all inertial equivalence classes for G by $\mathfrak{B}(G)$.

Theorem 7.2. [BeDe, Proposition 2.10]
The category $\mathrm{Rep}(G)$ *decomposes as*

$$\mathrm{Rep}(G) = \prod\nolimits_{\mathfrak{s} \in \mathfrak{B}(G)} \mathrm{Rep}(G)^{\mathfrak{s}}.$$

The space of irreducible G-representations is a disjoint union

$$\mathrm{Irr}(G) = \bigsqcup\nolimits_{\mathfrak{s} \in \mathfrak{B}(G)} \mathrm{Irr}(G)^{\mathfrak{s}}.$$

Notation 7.3. Let $\mathrm{Irr}_{\mathrm{cusp}}(L)$ be the set of isomorphism classes of supercuspidal irreducible smooth representations of L.

For $\sigma \in \mathrm{Irr}_{\mathrm{cusp}}(L)$ (and in fact for every irreducible L-representation) the group

$$X_{\mathrm{nr}}(L,\sigma) := \{\chi \in X_{\mathrm{nr}}(L) \,:\, \sigma \otimes \chi \cong \sigma\}$$

is finite. Thus there is a bijection

$$X_{\mathrm{nr}}(L)/X_{\mathrm{nr}}(L,\sigma) \to \mathrm{Irr}(L)^{\mathfrak{s}_L} \,:\, \chi \mapsto \sigma \otimes \chi, \tag{48}$$

which endows $\mathrm{Irr}(L)^{\mathfrak{s}_L}$ with the structure of a complex torus. Up to isomorphism this torus depends only on $\mathfrak{s}$, and it is known as the *Bernstein torus* $T_{\mathfrak{s}}$ attached to $\mathfrak{s}$. We note that $T_{\mathfrak{s}}$ is only an algebraic variety, it is not endowed with a natural multiplication map. In fact it does not even possess an unambigous "unit", because in general there is no preferred choice of an element $\sigma \in \mathfrak{s}_L$.

The group $W(G,L) := \mathrm{N}_G(L)/L$ acts on $\mathrm{Irr}(L)$ by

$$w \cdot \pi = [\bar{w} \cdot \pi \,:\, l \mapsto \pi(\bar{w}^{-} l \bar{w})] \quad \text{for any lift } \bar{w} \in \mathrm{N}_G(L) \text{ of } w \in W(G,L). \tag{49}$$

Bernstein also associated to $\mathfrak{s}$ the finite group

$$W_{\mathfrak{s}} := \{w \in W(G,L) \,:\, w \cdot \mathrm{Irr}(L)^{\mathfrak{s}_L} = \mathrm{Irr}(L)^{\mathfrak{s}_L}\}. \tag{50}$$

It acts naturally on $T_{\mathfrak{s}}$, by automorphisms of algebraic varieties.

Closely related to the Bernstein decomposition is the theory of the Bernstein center. By [BeDe, Théorème 2.13] the categorical centre of the Bernstein block $\mathrm{Rep}^{\mathfrak{s}}(G)$ is

$$\mathrm{Z}(\mathrm{Rep}(G)^{\mathfrak{s}}) \cong \mathcal{O}(T_{\mathfrak{s}})^{W_{\mathfrak{s}}} = \mathcal{O}(T_{\mathfrak{s}}/W_{\mathfrak{s}}). \tag{51}$$

Here $\mathcal{O}$ stands for the regular functions on an affine variety. Moreover the map

$$\mathbf{sc}\colon \mathrm{Irr}(G)^{\mathfrak{s}} \to T_{\mathfrak{s}}/W_{\mathfrak{s}} \tag{52}$$

induced by (51) is surjective and has finite fibers [BeDe, §3]. Theorem 7.2 implies that every $\pi \in \mathrm{Irr}(G)$ is a constituent of $I_P^G(\sigma)$, where $[L,\sigma]_G$ is uniquely determined. By (51) the supercuspidal L-representation $\sigma \in T_{\mathfrak{s}}$ is in fact uniquely determined up to $W_{\mathfrak{s}}$. The map $\pi \mapsto W_{\mathfrak{s}}\sigma$ is just $\mathbf{sc}$, and for this reason it is called the cuspidal support map. Via this map $\mathrm{Irr}^{\mathfrak{s}}(G)$ can be regarded as a non-separated algebraic variety lying over $T_{\mathfrak{s}}/W_{\mathfrak{s}}$.

7.3. **The ABPS Conjecture.** Let $\mathfrak{s} = [L,\sigma]_G$ be an inertial equivalence class for G. Let $W_{\mathfrak{s},t}$ be the stabilizer in $W_{\mathfrak{s}}$ of a point $t \in T_{\mathfrak{s}}$.

The ABPS conjecture from [ABPS2, §15] and [ABPS7, Conjecture 2] in its roughest form asserts that there exists a family of 2-cocycles

$$\natural_t\colon W_{\mathfrak{s},t} \times W_{\mathfrak{s},t} \to \mathbb{C}^\times \qquad t \in T_{\mathfrak{s}},$$

and a bijection

$$\mathrm{Irr}(G)^{\mathfrak{s}} \longleftrightarrow (T_{\mathfrak{s}}/\!/W_{\mathfrak{s}})_\natural \tag{53}$$

such that:

- it restricts to a bijection between tempered representations and the unitary part of the extended quotient (as explained below);
- it is canonical up to permutations within L-packets, that is, for any $\phi \in \Phi(G)$, the image of $\Pi_\phi(G) \cap \mathrm{Irr}^{\mathfrak{s}}(G)$ is canonically defined (assuming a LLC for G exists).

The set $\mathrm{Irr}_{\mathrm{cusp}}(L)$ of supercuspidal L-representations is stable under the $W(G,L)$-action (49). The definitions of $W_{\mathfrak{s}}$ and of extended quotients imply that for a fixed Levi subgroup L of G there is a natural bijection

$$\bigsqcup\nolimits_{\mathfrak{s}=[L,\sigma]_G} (T_{\mathfrak{s}}/\!/W_{\mathfrak{s}})_\natural \to \Big(\mathrm{Irr}_{\mathrm{cusp}}(L)/\!/W(G,L)\Big)_\natural . \tag{54}$$

In view of Theorem 7.2, the ABPS-Conjecture can also be formulated in terms of a bijection

$$\mathrm{Irr}(G) \longleftrightarrow \bigsqcup\nolimits_L \Big(\mathrm{Irr}_{\mathrm{cusp}}(L)/\!/W(G,L)\Big)_\natural , \tag{55}$$

where L runs through a set of representatives for the G-conjugacy classes of Levi subgroups of G. In this version, our conjecture asserts that $\mathrm{Irr}(G)$ is determined by a much smaller set of data, namely the supercuspidal representations of Levi subgroups L of G, and the actions of the Weyl groups $W(G,L)$ on those.

It is expected that the group cohomology classes $\natural_t \in H^2(W_{\mathfrak{s},t},\mathbb{C}^\times)$ reflect the character of $\mathrm{Z}(G^\vee_{\mathrm{sc}})^{W_F}$ which via the Kottwitz isomorphism determines how G is an inner twist of a quasi-split group. In particular $\natural$ should be trivial whenever G is quasi-split. The simplest known example of a nontrivial cocycle involves a non-split inner form of $\mathrm{SL}_{10}(F)$ [ABPS5, Example 5.5]. That example also shows that it is sometimes necessary to use *twisted* extended quotients in the ABPS Conjecture.

7.4. **A version for enhanced L-parameters.** The main goal of this section is to state an analogue of (53) and (55) for enhanced Langlands parameters.

We will recall first the statement of [AMS1, Lemma. 7.4]. There exists a character $\zeta_G \in \mathrm{Irr}(\mathrm{Z}(\mathbf{G}^\vee_{\mathrm{sc}}))$ such that: $\zeta_G|_{\mathrm{Z}(\mathbf{G}^\vee_{\mathrm{sc}})^{W_F}}$ parametrizes the inner twist G via the Kottwitz isomorphism (9), and $\zeta_G = 1$ on $\mathrm{Z}(\mathbf{G}^\vee_{\mathrm{sc}}) \cap \mathrm{Z}(\mathbf{L}^\vee_{\mathrm{c}})^\circ$, for every Levi subgroup L of G. We set

$$\Phi_{\mathrm{e},\zeta_G} := \Phi_{\mathrm{e}}(G), \tag{56}$$

where $\Phi_{\mathrm{e}}(G)$ is as in 3.6. Let $L \subset G$ be a Levi subgroup and let $\phi\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L L$ be a Langlands parameter for L. There exists a natural injection $\mathcal{R}^{\mathcal{L}}_\phi \to \mathcal{R}_\phi$. We extend the character ζ_G to a character of $\mathrm{Z}(\mathbf{G}^\vee_{\mathrm{sc}})\mathrm{Z}(L^\vee_{\mathrm{c}})^\circ$ which is trivial on $\mathrm{Z}(L^\vee_{\mathrm{c}})^\circ$, and denote by ζ^L_G the restriction of the latter character to $\mathrm{Z}(\mathbf{L}^\vee_{\mathrm{sc}})$.

The following result is proved in [Mou2, Theorem 3.3] for G a split classical group, and in [AMS1, Theorem 9.3] for general G.

Theorem 7.4. [AMS1, Theorem 9.3]

(*a*) *Let $\mathfrak{s}^\vee_L = [{}^L L, \phi_v, q\epsilon]_{{}^L L}$ be an $\mathcal{H}$-relevant inertial equivalence class for the Levi L-subgroup ${}^L L$ of ${}^L\mathcal{H}$ and recall the notations* (47)*. The maps ${}^L\Sigma_{\mathfrak{t}}$ from Proposition 6.10.a combine to a bijection*

$$\begin{array}{ccc} \Phi_{\mathrm{e}}({}^L G)^{\mathfrak{s}^\vee} & \longleftrightarrow & \big(\Phi_{\mathrm{e}}({}^L L)^{\mathfrak{s}^\vee_L}/\!/W_{\mathfrak{s}^\vee}\big)_\kappa \\ (\phi,\rho) & \mapsto & \big({}^L\Psi(\phi,\rho), \Sigma_{\mathfrak{t}}(u_\phi,\rho)\big) \\ \big(\phi_v|_{W_F}, \Sigma_{\mathfrak{t}}^{-1}(\tau)\big) & \leftarrow\!\shortmid & ({}^L L, \phi_v, \varepsilon, \tau). \end{array}$$

(*b*) *The bijection from part (a) has the following properties:*
- *It preserves boundedness of (enhanced) L-parameters.*
- *The restriction of τ to $W_{\mathfrak{t}^\circ}$ canonically determines the (non-enhanced) L-parameter in ${}^L\Sigma_{\mathfrak{t}}(\tau)$.*

- *Let* $z, z' \in X_{\mathrm{nr}}({}^L L)$ *and let* $\Gamma \subset W_{\mathfrak{s}^\vee, z\phi_v,\epsilon}$ *be a subgroup. Suppose that* $\Gamma = \overline{\Gamma}/L \cong \overline{\Gamma}_{\mathrm{c}}/L_{\mathrm{c}}$, *where*

$$\overline{\Gamma} \subset \mathrm{N}_{\mathbf{G}^\vee}({}^L L) \cap \mathrm{Z}^1_{\mathbf{G}^\vee}(z'\phi|_{W_F}) \quad \textit{with preimage} \quad \overline{\Gamma}_c \subset \mathrm{Z}_{\mathbf{G}^\vee_{\mathrm{sc}}}(z'\phi(W_F))^\circ.$$

Then the 2*-cocycle* $\kappa_{\mathfrak{s}^\vee, z\phi_v,\varepsilon}$ *is trivial on* Γ.

(*c*) *Let* $\zeta_G \in \mathrm{Irr}(\mathrm{Z}(\mathbf{G}^\vee_{\mathrm{sc}}))$. *We write*

$$\Phi_{\mathrm{e},\zeta_G}(G,L) := \{(\phi,\rho) \in \Phi_{e,\zeta_G}(G) \,:\, \mathbf{Sc}(\phi,\rho) \in \Phi_{\mathrm{cusp}}(L)\}.$$

The bijections from part (a) give a bijection

$$\Phi_{\mathrm{e},\zeta_G}(G,L) \;\longleftrightarrow\; \big(\Phi_{\mathrm{cusp},\zeta_G^L}(L)/\!/W(G,L)\big)_\kappa.$$

(*d*) *Let* $\mathfrak{Lev}(G)$ *be a set of representatives for the conjugacy classes of Levi subgroups of* G. *The maps from part (c) combine to a bijection*

$$\Phi_{\mathrm{e},\zeta_G}(G) \;\longleftrightarrow\; \bigsqcup_{L\in\mathfrak{Lev}(G)} \big(\Phi_{\mathrm{cusp},\zeta_G^L}(L)/\!/W(G,L)\big)_\kappa.$$

(*e*) *Assume that* $\mathrm{Z}(\mathbf{L}^\vee_{\mathrm{sc}})$ *is fixed by* W_F *for every Levi subgroup* $L \subset G$ *(for instance,* G *is an inner twist of a split group). Recall that* G_ϑ *is the inner twist of* G *determined by* $\vartheta \in H^1(F, G_{\mathrm{ad}}) \cong \mathrm{Irr}_{\mathbb{C}}\big(\mathrm{Z}(\mathbf{G}^\vee_{\mathrm{sc}})^{W_F}\big)$. *The union of part (d) for all such* ϑ *is a bijection*

$$\Phi_{\mathrm{e}}({}^L G) \;\longleftrightarrow\; \bigsqcup_{\vartheta\in H^1(F,G_{\mathrm{ad}})} \;\bigsqcup_{L_\vartheta\in\mathfrak{Lev}(G_\vartheta)} \big(\Phi_{\mathrm{cusp}}(L_\vartheta)/\!/W(G_\vartheta,L_\vartheta)\big)_\kappa.$$

7.5. **A conjectural diagram.** The above result leads to the following conjecture stated in [AMS1]:

Conjecture 7.5. *There exists a commutative bijective diagram*

$$\begin{array}{ccc}
\mathrm{Irr}(G) & \longleftrightarrow & \Phi_{\mathrm{e},\zeta_G}(G) \\
\updownarrow & & \updownarrow \\
\bigsqcup_{L\in\mathfrak{Lev}(G)} \big(\mathrm{Irr}_{\mathrm{cusp}}(L)/\!/W(G,L)\big)_\kappa & \longleftrightarrow & \bigsqcup_{L\in\mathfrak{Lev}(G)} \big(\Phi_{\mathrm{cusp},\zeta_G}(L)/\!/W(G,L)\big)_\kappa
\end{array}$$

with the following maps:

- *the right hand side is Theorem 7.4,*
- *the upper horizontal map is a local Langlands correspondence for* G,
- *the lower horizontal map is obtained from local Langlands correspondences for* $\mathrm{Irr}_{\mathrm{cusp}}(L)$ *by applying* $(\cdot/\!/W(G,L))_\kappa$,
- *the left hand side is the bijection in the ABPS conjecture* [ABPS7, § 2].

With this conjecture one can reduce the problem of finding a LLC for G to that of finding local Langlands correspondences for supercuspidal representations of its Levi subgroups. Conjecture 7.5 is currently known in the following cases:

- inner forms of $\mathrm{GL}_n(F)$ [ABPS4, Theorem 5.3],
- inner forms of $\mathrm{SL}_n(F)$ [ABPS4, Theorem 5.6],
- split classical groups [Mou1, §5.3],
- principal series representations of split groups [ABPS6, § 16].

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