1) Recall that we discussed integrals where the codomain is $[0, \infty)$, but now we will discuss them in full generality where the codomain is $\mathbb{R}$. Given a function $f : \mathbb{R} \to \mathbb{R}$, we have a positive/negative parts $f_+, f_- : \mathbb{R} \to [0, \infty)$, so we formulate measurability and integrability in terms of these parts.

   - We say that f is measurable is f_+, f_- are measurable.
   - We say that f is integrable if f is measurable while $\int f_+, f_-$ are integrable, and define

   $$\int f := \int f_+ - \int f_-$$

   Which is finite because $\int f_+, \int f_-$ are finite by integrability.

   We will quickly prove the linear of integrals in the general case.

   Proposition 0.1. *If $f, g : \mathbb{R} \to \mathbb{R}$ are integrable, then $f + g$ is integrable with $\int f + g = \int f + \int g$.*

   Proof. By assumption f_+, g_+, f_-, g_- are integrable, thus $(f + g)_+ \leq f_+ + g_+$ is integrable, and the same applyies for $(f + g)_-$. Now

   $$(f_+ - f_-) + (g_+ - g_-) = (f + g)_+ - (f + g)_-$$

   Therefore

   $$(f + g)_+ + f_- + g_- = (f + g)_- + f_+ + g_+$$

   And then we can integrate both sides (Both sides are functions to $[0, \infty)$), use additivity in the special case, then rearrange (every term involved is finite). ∎

   Proposition 0.2. *If $f : \mathbb{R} \to \mathbb{R}$ is integrable, then cf is integrable with $\int cf = c \int f$.*

   Proof. Obvious. ∎

   The above shows that the set of integrable functions $\mathbb{R} \to \mathbb{R}$ can be given a linear structure. In particular the space of functions of f where $|f|$ is integrable is a linear space, which we denote as the L^1 space. The integral operator is a linear map $L^1 \to \mathbb{R}$.

2) We introduce the Fourier transform. Given $f \in L^1(\mathbb{R}^d, \mathbb{C})$, we define the Fourier Transform $\hat{f} : \mathbb{R}^d \to \mathbb{C}$ with

   $$\hat{f}(\zeta) = \int f(x) e^{-2\pi i x} dx$$

   First, we need to verify that $\hat{f}$ is well defined.

   Proposition 0.3. *If f is in L^1 then $x \mapsto f(x)e^{-2\pi i x}$ is integrable.*

   Proof. The product of measurable functions is measurable. Now $\left|f(x)e^{-2\pi i x}\right| = \left|f(x)\right|$, where $\int |f| < \infty$ by assumption. ∎

   In the above proposition, we have shown that $\left|\hat{f}\right| \leq \int |f| < \infty$. We immediately see that the Fourier transform operation is linear in L^1, and we will now show that $\hat{f}$ is continuous.

   Proposition 0.4. *Given $f \in L^1$, $\hat{f}$ is continuous.*

Proof. Suppose we have a sequence $\zeta_n \to \zeta$. We have a sequence of measurable functions $f_n(x) = f(x)e^{-2\pi i\zeta_n x}$, which is integrable due to domination from $|f|$, while

$$\hat{f}(\zeta_n) = \int f_n.$$

Now note that $f_n(x)$ converges everywhere to $f(x)e^{-2\pi i}\zeta x$ (exponential is continuous), who is also dominated by $|f|$. Therefore we apply the DCT to get

$$\hat{f}(\zeta_n) = \int f_n \to \int f(x)e^{-2\pi i\zeta x}dx = \hat{f}(\zeta)$$

As required. ∎

Since $\hat{f}$ is continuous, it is naturally measurable. But is it also in L^1 (from which we would have a Fourier inverse transform $f(x) = \int \hat{f}(\zeta)e^{2\pi i\zeta}d\zeta$)? Unfortunately the answer is no, unless we consider a smaller space of L^1, the Schwarz space.

3) A very dumbed down explanation of a Schwarz space is the space of functions that are very flat and decrease very quickly. A more formal way of saying this is that f is $\mathcal{C}^\infty$ while

$$\sup_{\mathbb{R}^d}\left|x^\beta D^\alpha f\right| < \infty$$

For every $\alpha, \beta \in \mathbb{Z}_{\geq 0}$, where D^α encompasses all α-th partial derivatives of f. We let $\mathcal{S}$ denote the set of all such functions $\mathbb{R}_d \to \mathbb{C}$, and call it the Schwarz space. We immediately have the following:

Proposition 0.5. *The following hold:*

1) *$\mathcal{S}$ is a complex linear space.*
2) *$\mathcal{S} \subset L^1$, so $\mathcal{S}$ is a complex linear subspace of L^1.*
3) *$\mathcal{S}$ is closed to multiplication.*
4) *$x^\beta f, D^\alpha f \in \mathcal{S}$ for $f \in \mathcal{S}$.*

Proof. We have

1) Clear from the triangle inequality (Note that $0 \in \mathcal{S}$).
2) Considering $\beta = 69$ and $\alpha = 0$, we have $|f| < \frac{A}{x^{69}}$ for $|x| > 1$, while $\int_{|x|>1} \frac{1}{x^{69}} < \infty$, and clearly $|f|$ is bounded on $|x| < 1$, by considering $\beta = 0$ and $\alpha = 0$.
3) Note that all derivatives of $f, g \in \mathcal{S}$ must be bounded (by considering $\alpha = 0$), and that differentiation obeys the product rule. Then use induction.
4) Straightforward.

Done. ∎

To make the discussion more concrete, here are some examples of functions in $\mathcal{S}$.

Example 0.6. *The space of compactly supported smooth functions, $\mathcal{C}_c^\infty$, is contained in $\mathcal{S}$.*

Proof. A continuous function on a compact set admits a maximum and minimum. ∎

Example 0.7. $e^{-x^2} \in \mathcal{S}$

Proof. Too lazy to type it out. ■

Since $\mathcal{S}$ is contained in L^1, it then makes sense to consider the Fourier transform operator on $\mathcal{S}$. The punchline will be that this is a complex linear isomorphism $\mathcal{S} \to \mathcal{S}$. We first list some basic properties regarding Fourier transform on $\mathcal{S}$:

Proposition 0.8. *The following hold:*

1) $\widehat{f(\cdot + h)}(\zeta) = e^{2\pi i h\zeta}\hat{f}(\zeta)$

2) $\hat{f}(\zeta + h) = (\widehat{e^{-2\pi i hx} f})(\zeta)$

3) For $\delta > 0$, $\widehat{f(\delta \cdot)}(\zeta) = \frac{1}{\delta}\hat{f}(\frac{1}{\delta}\zeta)$

Proof. We calculate

1) Letting $g(x + h) = f(x)$, we have

$$\hat{g}(\zeta) = \int g(x)e^{-2\pi i x\zeta} = e^{2\pi i h\zeta}\int g(x)e^{-2\pi i(x+h)\zeta} = e^{2\pi i h\zeta}\hat{f}(\zeta)$$

2) Letting $g(x) = e^{-2\pi i hx}f(x)$, we have

$$\hat{g}(\zeta) = \int g(x)e^{-2\pi i x\zeta} = \int f(x)e^{-2\pi i x(\zeta+h)} = \hat{f}(\zeta + h)$$

3) Letting $g(x) = f(\delta x)$, we have

$$\begin{aligned} \hat{g}(\zeta) &= \int g(x)e^{-2\pi i x\zeta} \\ &= \frac{1}{\delta}\int f(y)e^{-2\pi i \frac{y}{\delta}\zeta} && \text{(COV, } y = \delta x) \\ &= \frac{1}{\delta}\hat{f}(\frac{1}{\delta}\zeta) \end{aligned}$$

As required. ■

We will then list more involved properties, regarding differentiation.

Proposition 0.9. *The following hold:*

1) $\widehat{(f')}(\zeta) = 2\pi i\zeta\hat{f}(\zeta)$

2) $(\hat{f})'(\zeta) = (\widehat{-2\pi i x f})(\zeta)$

Proof. We calculate

1) $\widehat{(f')}(\zeta) = \int f'(x)e^{-2\pi i\zeta x} = 0 - \int[f(x)\frac{d}{dx}e^{-2\pi i\zeta x}] = 2\pi i\zeta\hat{f}(\zeta)$

2) Let $g(x) = -2\pi i x f(x)$. We have

$$(\hat{f})'(\zeta) = \lim_{\delta \to 0} \frac{\hat{f}(\zeta + \delta) - \hat{f}(\zeta)}{\delta}$$

Where

$$\frac{\hat{f}(\zeta + \delta) - \hat{f}(\zeta)}{\delta} = \int f(x) \frac{e^{-2\pi i x(\zeta+\delta)} - e^{-2\pi i x \zeta}}{\delta} = \int f(x) e^{-2\pi i x \zeta} (\frac{e^{-2\pi i x \delta} - 1}{\delta})$$

Here we can apply the DCT, because for sufficiently

TBD ■

These two properties clearly imply that the Fourier transform maps $\mathcal{S} \to \mathcal{S}$.

Corollary 0.9.1. *The fourier transform operator is a linear map $\mathcal{S} \to \mathcal{S}$.*