1) Given a set M, a subset $X \subset \mathcal{P}(M)$ is a σ-Algebra if:

   - $\emptyset, M \in X$
   - M is closed to differences
   - M is closed to countable union

   Note that the second condition and third condition imply that M is also closed to countable intersection. For a set M along with a σ-Algebra X, we call (M, X) a measurable space and the elements of X the measurable sets (with respect to X).

   Example 0.1. *The smallest σ-Algebra is $\{\emptyset, M\}$, and the largest σ-Algebra is $\mathcal{P}(M)$ itself.*

   Given a $A \subset \mathcal{P}(M)$, there exists a σ-Algebra containing A, namely $\mathcal{P}(M)$ itself. We can then consider the *smallest* σ-Algebra containing A.

   Proposition 0.2. *The intersection of all σ-Algebras containing A is the smallest σ-Algebra that contains A.*

   Proof. Trivial. ■

   We call the smallest σ-Algebra containing A the "σ-Algebra generated by A". One generated σ-Algebra of interest is the following:

   Example 0.3. *Every Topology τ on M generates a σ-Algebra on M.*

   In particular, the Euclidean Topology on $\mathbb{R}$ generates a σ-Algebra on $\mathbb{R}$, which we call the Borel σ-Algebra on $\mathbb{R}$; every element of this σ-Algebra is called a Borel set.

2) We discuss measurable functions. Given a measurable space (M, X) and a function $f : M \to \mathbb{R}$, we see that f is X-measurable if the preimage of every Borel set is X-measurable.

   Example 0.4. *For a $E \subset M$, we let $\chi_E : M \to \mathbb{R}$ be the characteristic function of E, which evaluates to 1 on E and 0 on $M - E$.*

   Proposition 0.5. *χ_E is measurable if and only if $E \in X$.*

   Proof. If χ_E is measurable, then $E = \chi_E^{-1}(0, 2) \in X$. Conversely, if $E \in X$, then for any Borel set $B \subset \mathbb{R}$ we $\chi_E^{-1}(B)$ is either $\emptyset$, M, E, or $M - E$, which clearly means χ_E is measurable. ■

   The Borel sets as a whole are difficult to grasp, but luckily we can check measurability on a much smaller family of sets, because the Borel σ-Algebra is generated by the Euclidean topology on $\mathbb{R}$.

   Proposition 0.6. *$f : M \to \mathbb{R}$ is measurable iff $f^{-1}(O) \in X$ for every open $O \subset \mathbb{R}$.*

   Proof. One direction is clear. For the other direction, we note that the collection $\{S \in \mathcal{P}(M) : f^{-1}(S) \in X\}$ is a σ-Algebra over $\mathbb{R}$ containing the Euclidean topology (By assumption), and thus must contain the collection of Borel sets (By minimal property). ■

   Corollary 0.6.1. *$f : M \to \mathbb{R}$ is measurable iff $f^{-1}(a, b)$ is measurable for every open interval (a, b), iff the preimage of every open ray (one direction is fine) is measurable.*

Proof. The Euclidean topology is generated by the open intervals. For the second statement, note that an open interval is the intersection of two opposite open rays (and an open ray is the countable union of open intervals), the complement of an open ray is an opposite closed ray, while an opposite open ray is the countable union of opposite closed ray. ■

For a real variable function $f : B \to \mathbb{R}$ where $B \subset \mathbb{R}$ is a Borel set of $\mathbb{R}$, f is said to be Borel-measurable if the preimage of Borel sets are Borel sets. By the above, we immediately have

Proposition 0.7. *If $f : B \to \mathbb{R}$ (where $B \subset \mathbb{R}$ Borel) is continuous, then it is Borel-measurable.*

Proof. For open $O \subset \mathbb{R}$, $f^{-1}(O) = B \cap O'$, where $O' \subset \mathbb{R}$ is open. ■

Of course, the converse is false: for any Borel-measurable set $B \subset \mathbb{R}$, its characteristic function $\chi_B : \mathbb{R} \to \mathbb{R}$ is Borel-measurable, but clearly χ_B need not be continuous. Here are some more cases of Borel measurability, where $f : B \to \mathbb{R}$ for a Borel-measurable $B \subset \mathbb{R}$:

Proposition 0.8. *If f is increasing, then f is Borel-measurable.*

Proof. Given an open ray (a, ∞), we know that $f^{-1}(a, \infty)$, if non-empty, is bounded below, so letting $b = \int f^{-1}(a, \infty)$ we have $f^{-1}(a, \infty)$ equal to $(b, \infty) \cap B$ or $[b, \infty) \cap B$ (by monotonicity). ■

Proposition 0.9. *If $f : B_1 \to \mathbb{R}$ and $g : B_2 \to \mathbb{R}$ are Borel-measurable where $B_1, B_2 \subset \mathbb{R}$ are Borel sets and $f(B_1) \subset B_2$, then $g \circ f : B_1 \to \mathbb{R}$ is Borel-measuurable.*

Proof. For a Borel set $B \subset \mathbb{R}$,

$$(g \circ f)^{-1}(B) = f^{-1}(g^{-1}(B))$$

$g^{-1}(B) \subset \mathbb{R}$ is a Borel set, so $f^{-1}(g^{-1}(B)) \subset \mathbb{R}$ is a Borel set. ■

Of course the above discussion is somewhat barebones, because we want to be able to take operations of Borel-measurable functions and still get one. Therefore, we will show them below.

Proposition 0.10. *If $f : B \to \mathbb{R}$ is Borel-measurable, then $cf, f^n, \frac{1}{f}, |f|, \exp(f), \log(f)$, etc. are all Borel-measurable when well-defined.*

Proof. Compose f with an appropriate continuous (thus Borel-measurable) function. ■

Proposition 0.11. *If $f : B \to \mathbb{R}$ is Borel-measurable, then $cf, f^n, \frac{1}{f}, |f|, \exp(f), \log(f)$, etc. are all Borel-measurable when well-defined.*

Proof. Compose f with an appropriate continuous (thus Borel-measurable) function. ■

Proposition 0.12. *If $f : B \to \mathbb{R}$, $g : B \to \mathbb{R}$ are Borel-measurable then $f + g : B \to \mathbb{R}$ is Borel-measurable.*

Proof. $(f + g)^{-1}(a, \infty) = \cup_{r \in \mathbb{Q}}[f^{-1}(r, \infty) \cap g^{-1}(a - r, \infty)]$. ■

Proposition 0.13. *If $f : B \to \mathbb{R}$, $g : B \to \mathbb{R}$ are Borel-measurable then $fg, \frac{f}{g}$ are Borel-measurable.*

Proof. fg is Borel-measurable because $fg = \frac{1}{2}[(f + g)^2 - f^2 - g^2]$. $\frac{f}{g}$ is Borel-measurable because $\frac{1}{g}$ is Borel-measurable (use multiplicative property). ■

And finally (perhaps the most important), we want to show that the pointwise limit of measurable functions is measurable. To this, we will introduce the limit superior and limit inferior of a sequence of sets. For a sequence A_n of sets, we define

$$\limsup A_n := \cap_{i=1}^{\infty} \cup_{j=i}^{\infty} A_j = \{x : x \in A_j \text{ for infinite } j\}$$
$$\liminf A_n := \cup_{i=1}^{\infty} \cap_{j=i}^{\infty} A_j = \{x : x \in A_j \text{ for all but finitely many } j\}$$

Use this new set operation (noting that they preserve measurability), we can then show that the pointwise limit of measurable functions is measurable.

Proposition 0.14. *If $f_n : M \to \mathbb{R}$ is a sequenec of measurable functions that converges pointwise to $f : M \to \mathbb{R}$, then f is measurable.*

Proof. We have $f^{-1}(a, \infty) = \cup_{k=1}^{\infty} B_k$ where

$$B_k = \liminf[f_n^{-1}(a + \frac{1}{k}, \infty)]$$

The reason is: if $f(x) > a$ then we have $f_n(x) > \frac{1}{2}(a + f(x))$ for sufficiently large n, so $x \in B_k$ for large enough k, and conversely if $x \in B_k$ then $f_n(x) > a + \frac{1}{k}$ for all large enough n implies $f(x) = \lim f_n(x) \geq a + \frac{1}{k} > a$. ∎

This is a very good property, because if we went for something like Riemann integrability, then it requires uniform continuity to be preserved. Now we will also introduce functions to $[-\infty, \infty]$.

We can define Borel sets on $[-\infty, \infty]$ corresponding to a σ-Algebra on it: simply includes subsets whose intersection with $\mathbb{R}$ is a Borel set in $\mathbb{R}$; so a set of the form $B, B \cup \{-\infty\}, B \cup \{\infty\}, B \cup \{-\infty, \infty\}$ where B is a Borel set of $\mathbb{R}$. Clearly this defines a σ-Algebra on $[-\infty, \infty]$. The results described above translate easily to the $[-\infty, \infty]$ case, particularly with the following result:

Proposition 0.15. *$f : M \to [-\infty, \infty]$ is measurable iff $f^{-1}(a, \infty]$ is always measurable iff $f^{-1}[-\infty, a)$ is always measurable.*

With this we can formulate the measurability-preservation of supremum and infimum.

Proposition 0.16. *If $f_n : M \to [-\infty, \infty]$ is a measurable sequence, then $g, h : M \to [-\infty, \infty]$, the supremum/infimum of the sequence respectively, are measurable.*

Proof. For the supremum g we have $g^{-1}(a, \infty] = \cup_{n=1} f_n^{-1}(a, \infty]$. The infimum case is similar. ∎

3) We discuss measures. Given a measurable space (M, X), a measure is a function $\mu : X \to [0, \infty]$ with $\mu(\emptyset) = 0$ and satisfying disjoint-countable additivity. (M, X, μ) is then called a measure space. We will deduce a list of important properties of measure.

Proposition 0.17. *If $D, E \in X$ and $D \subset E$ then $\mu(E) = \mu(D) + \mu(E - D)$.*

Proof. E is the disjoint union of D and $E - D$, which implies $\mu(E) = \mu(D) + \mu(E - D)$. ∎

Corollary 0.17.1. *μ satisfies monotonicity.*

Corollary 0.17.2. *If $D, E \in X$ and $D \subset E$ and $\mu(D) < \infty$ then $\mu(E - D) = \mu(E) - \mu(D)$.*

Proposition 0.18. *μ satisfies countable subadditivity.*

Proof. If $A_1, A_2, ...$ is a measurable sequence, then let $E_n := A_n - \cup_{i=1}^{n-1} A_i$, which then yields us a disjoint countable sequence whose union equals the union of $A_1, A_2, ...$ (because if $x \in \cup_{n=1}^{\infty} A_n$ then $x \in E_i$ where $i = \min(k : x \in A_k)$), so

$$\begin{aligned} \mu(\cup_{n=1}^{\infty} A_n) &= \mu(\cup_{n=1}^{\infty} E_n) \\ &= \sum_{n=1}^{\infty} \mu(E_n) && \text{(Disjoint countable additivity)} \\ &\leq \sum_{n=1}^{\infty} \mu(A_n) && \text{(Monotonicity)} \end{aligned}$$

As required. ■

Proposition 0.19. *If $A_1 \subset A_2 \subset \cdots$ is an increasing measurable sequence, then their union A satisfies $\mu(A) = \lim \mu(A_n)$.*

Proof. Letting $E_n = A_n - \cup_{i=1}^{n-1} A_i$, we have $A = \cup_{n=1}^{\infty} E_n$ so by disjoint-countable additivity,

$$\begin{aligned} \mu(A) &= \sum_{i=1}^{\infty} \mu(E_i) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \mu(E_i) \\ &= \lim_{n\to\infty} \mu(A_n) && \text{(Since } A_n = \cup_{i=1}^{n} E_n\text{)} \end{aligned}$$

As required. ■

Proposition 0.20. *If $A_1 \supset A_2 \supset \cdots$ is a decreasing measurable sequence and A is their intersection, then if $\mu(A_1) < \infty$ then $\mu(A) = \lim \mu(A_n)$.*

Proof. Letting $E_n = A_1 - A_n$, we have $E_1 \cup E_2 \cup \cdots$ while $A_1 - A = \cup_{n=1}^{\infty} E_n$ so

$$\mu(A_1 - A) = \lim_{n\to\infty} \mu(E_n) = \lim_{n\to\infty} \mu(A_1 - A_n)$$

Since $\mu(A_1) < \infty$, by monotonicity we have $\mu(A), \mu(A_n) < \infty$, so the above equation simplifies to

$$\mu(A_1) - \mu(A) = \lim_{n\to\infty} (\mu(A_1) - \mu(A_n))$$

All terms are finite, so we can manipulate and rearrange. ■

The assumption that $\mu(A_1)$ is necessary to avoid the following counterexample.

Example 0.21. *Letting $A_n = [n, \infty)$, we have a decreasing sequence of Borel sets where each has ∞ Borel measure, but the intersection is $\emptyset$ whose Borel measure is 0.*

Proposition 0.22. *If $A, B \in X$ and $\mu(A \cap B) < \infty$ then $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$.*

Proof. $\mu(A \cup B) = \mu(A) + \mu(B - A)$, and $\mu(B) = \mu(A \cap B) + \mu(B - A)$. ■

4) We will show that outer measure (restricted) is a measure on the Borel σ-Algebra. Note that we only really need to show countable additivity. Since the Borel σ-Algebra is generated by the Euclidean topology, it helps to first consider simple sets: open and closed sets.

Lemma 0.23. *For a bounded open interval $A = (a, b)$, we have $|A \cup B| = |A| + |B|$ if A, B are disjoint.*

Proof. By subadditivity we have $|A \cup B| \leq |A| + |B|$, so we will show the other inequality. Since two points have no affect on outer measure, WLOG assume $a, b \notin B$.

Given $\epsilon > 0$ choose a sequence of open intervals $I_1, I_2, ...$ covering $A \cup B$ with $\sum_{n=1}^{\infty} |I_n| \leq |A \cup B| + \epsilon$. We then split these intervals into three sequences of open intervals

$$\begin{aligned} L_n &= I_n \cap (-\infty, a) \\ J_n &= I_n \cap (a, b) \\ K_n &= I_n \cap (b, \infty) \end{aligned}$$

Since $a, b \notin B$ and $(a, b) \cap B = \emptyset$, the intervals L_n and K_n cover B. Similiarly, the intervals J_n. So

$$\begin{aligned} |A \cup B| + \epsilon &\geq \sum_{n=1}^{\infty} |I_n| \\ &= (\sum_{n=1}^{\infty} |J_n|) + (\sum_{n=1}^{\infty} |L_n| + \sum_{n=1}^{\infty} |K_n|) \\ &\geq |A| + |B| \end{aligned}$$

As required. ■

Lemma 0.24. *For disjoint A, B we have $|A \cup B| = |A| + |B|$ if A is the finite union of disjoint (bounded) open intervals.*

Proof. We use induction. The $n = 1$ case is the above lemma, while the induction step is using the above lemma and the induction hypothesis. ■

Proposition 0.25. *For disjoint A, B we have $|A \cup B| = |A| + |B|$ if A is open.*

Proof. Without loss of generality, assume that $|A| < \infty$ (otherwise both sides equal ∞). Then A is the union of countable disjoint open bounded intervals $I_1, I_2, ...$, which are all disjoint from B. For any $n \in \mathbb{Z}^+$, we have

$$\begin{aligned} |A \cup B| &\geq \left|(\cup_{k=1}^{n} I_k) \cup B\right| \\ &= |\cup_{k=1}^{n} I_k| + |B| \\ &= \sum_{k=1}^{n} |I_k| + |B| \end{aligned}$$

So taking $n \to \infty$ we have $|A \cup B| \geq \sum_{k=1}^{\infty} |I_k| + |B| = |A| + |B|$. The other direction is clear. ■

Proposition 0.26. *For disjoint A, B we have $|A \cup B| = |A| + |B|$ if A is closed.*

Proof. Assume that $|A|, |B| < \infty$. Then we can choose am open set $|G| < \infty$ containing $A \cup B$. Since $G - A$ is open, we have

$$|G| = |G - A| + |A|$$

Now disjointness implies $B \subset G - A$ thus $|B| \leq |G - A|$ so $|G| \geq |A| + |B|$. Since $|A \cup B| < \infty$, we can take $|G| \leq |A \cup B| + \epsilon$ so that $|A \cup B| + \epsilon \geq |A| + |B|$. ■

The following approximation is very useful.

Proposition 0.27. *If $B \subset \mathbb{R}$ is a Borel set, then for $\epsilon > 0$ there exists a closed set $F \subset B$ such that $|B - F| < \epsilon$.*

Proposition 0.28. *For disjoint A, B we have $|A \cup B| = |A| + |B|$ if A is Borel.*

Proof. We already have $|A \cup B| \leq |A| + |B|$. For the other direction, given $\epsilon > 0$ we choose a closed $F \subset A$ with $|A - F| < \epsilon$, now by closedness we have

$$\begin{aligned}|A \cup B| &\geq |F \cup B| \\ &= |F| + |B| \\ &= |A| - |A - F| + |B| \\ &\geq |A| + |B| - \epsilon\end{aligned}$$

As required. ■

Proposition 0.29. *Outer measure (restricted) is a measure on the Borel σ-Algebra.*

5) We will now discuss Lebesgue measurable sets, which is a bigger class of sets than Borel sets. We say that $E \subset \mathbb{R}$ is a Lebesgue measurable set if $\mu(E - B) = 0$ for some Borel set $B \subset E$.

6) We discuss some theorems regarding approximation and convergence of measurable functions.

Theorem 0.30 (Egorov Theorem)**.** *If $f : M \to [-\infty, \infty]$ is measurable and $\mu(M) < \infty$, then for any sequence $f_n : M \to [-\infty, \infty]$ converging pointwise to f, and a $\epsilon > 0$, there exists a measurable $E \subset M$ such that $\mu(M - E) < \epsilon$ while the convergence is uniform on E.*

Proof. The TLDR of this proof is that for each n, we construct a set (whos measure is close to M) where eventually $|f_k - f|$ on the entire set, and then take the intersection for all n.

For $m, n \in \mathbb{Z}^+$, we define

$$A_{m,n} = \cap_{k=m}^{\infty} \{x \in M : \left|f_k(x) - f(x)\right| < \frac{1}{n}\}$$

Which is measurable, because it is the intersection of preimages (under measurable functions) of open rays. For each n, we have $A_{1,n} \subset A_{2,n} \subset \cdots$, and by pointwise convergence we also have $M = \cup_{m=1}^{\infty} A_{m,n}$; so by upward continuity, and the fact that $\mu(M) < \infty$ (**necessary!**), we choose $m_n \in \mathbb{Z}^+$ such that $\mu(M - A_{m_n,n}) < \frac{\epsilon}{357^n}$. Now we let $E = \cap_{n=1}^{\infty} A_{m_n,n}$, and the rest follows. ■

Proposition 0.31. *The assumption that $\mu(M) < \infty$ in Egorov Theorem is necessary.*

Proof. For a counterexample, consider the real line and the characteristic function of $[n, n+1)$, which converges pointwise to 0 but not uniformly on any unbounded set. ■

A function $f : M \to [-\infty, \infty]$ is simple if its image is finite. If the image points are $c_1, ..., c_n$, then we can write

$$f = \sum_{i=1}^{n} c_i \chi_{E_i}$$

Where $E_i = f^{-1}(c_i)$. Clearly, f is measurable if and only if each E_i is meausrable. The following proposition states that every measurable function can be approximated by a sequence of simple functions; this will be crucial to our study of integrals.

Proposition 0.32. *Given measurable $f : M \to [-\infty, \infty]$, there exists a simple measurable sequence $f_k : M \to [-\infty, \infty]$, converging pointwise to f, such that*

- *$|f_k|$ is k-increasing*
- *If f is bounded, then the convergence is uniform.*

Proof. The TLDR of this proof is: to construct f_k, we partition $[-k, k]$ into intervals of length 2^k, and use the output of f in this "grid" to determine what f_k outputs (and outside $[-k, k]$ we pad f_k with constant k). The specific construction is:

$$f_k(x) = \begin{cases} \frac{m}{2^k} & f(x) \in [\frac{m}{2^k}, \frac{m+1}{2^k}) \cap [0, k) \\ \frac{m+1}{2^k} & f(x) \in [\frac{m}{2^k}, \frac{m+1}{2^k}) \cap [-k, 0) \\ k & f(x) \in [k, \infty] \\ -k & f(x) \in [-\infty, k) \end{cases}$$

It is clear that f_k is measurable while $|f_k|$ is k-increasing, and that (f_k) converges pointwise to f. It is also clear that the convergence is uniform if f, since for large enough k the $f(x) \in [k, \infty]$ and $f(x) \in [-\infty, k)$ cases will disappear. To conclude that f_k is measurable, we note that the preimage of any interval (including or excluding $\pm\infty$) is measurable, since f is measurable. ■