1) Letting (X, S, μ) be a measure space, we define the integral of a measurable non-negative function $f : X \to [0, \infty]$. A $\mathcal{S}$-partition of X is a finite collection of disjoint $A_1, ..., A_m \in \mathcal{S}$ such that $X = \cup_{i=1}^m A_i$. This partition can be denoted $\mathcal{P}$, and we denote the lower sum by

$$\mathcal{L}(f, \mathcal{P}) := \sum_{i=1}^{m} \mu(A_i) \inf_{A_i} f$$

We now define the integral of f as

$$\int f := \sup(\mathcal{L}(f, \mathcal{P}) : \mathcal{P} \text{ a } \mathcal{S}\text{-partition of } X)$$

We immediately obtain the following results:

Proposition 0.1. *If $E \in \mathcal{S}$ then $\int \chi_E = \mu(E)$.*

Proof. $E, X - E$ form a partition of X, and the lower sum is $\mu(E)$. To show that this is indeed the largest lower sum, we consider an arbitrary partition $A_1, ..., A_m$. $\inf_{A_i} f$ equals 1 if $A_i \subset E$, and equals 0 otherwise. Therefore $\mu(A_i) \inf_{A_i} f \leq \mu(A_i \cap E)$ so

$$\begin{aligned} \sum_{i=1}^{m} \mu(A_i) \inf_{A_i} f &\leq \sum_{i=1}^{m} \mu(A_i \cap E) \\ &= \mu(E) && \text{(Disjointness)} \end{aligned}$$

As required. ■

Proposition 0.2. *If $f, g : X \to [0, \infty]$ are measurable with $f \leq g$ then $\int f \leq \int g$.*

Proof. Clear. ■

Proposition 0.3. *If $f : X \to [0, \infty]$ is measurable then $\int cf = c \int f$.*

Proof. Clear. ■

Now we examine the integrals of simple functions, because we can use them to approximate the integral of any measurable function. To start, we note the independence of representation of sets:

Proposition 0.4. *If $\sum_{i=1}^m a_i \chi_{E_i} = \sum_{j=1}^n b_j \chi_{D_j}$, then $\sum_{i=1}^m a_i \mu(E_i) = \sum_{j=1}^n b_j \mu(D_j)$.*

Proof. Very tedious proof so omitted. ■

We now see that the integral of a simple function takes a simple form.

Proposition 0.5. *If $E_1, ..., E_m \in \mathcal{S}$ are disjoint, then $\int \sum_{i=1}^m c_i \chi_{E_i} = \sum_{i=1}^m c_i \mu(E_i)$.*

Proof. Assume that $E_1, ..., E_m$ form a $\mathcal{S}$-partition of P, since we can add the complement of the union along with the constant 0. Since $E_1, ..., E_m$ form a $\mathcal{S}$-partition, the corresponding lower sum is $\sum_{i=1}^m c_i \mu(E_i)$. Now letting $A_1, ..., A_n$ be another $\mathcal{S}$-partition, the corresponding lower sum is

$$\sum_{j=1}^{n} \mu(A_j) \inf_{A_j} f = \sum_{j=1}^{n} \sum_{i=1}^{m} \mu(A_j \cap E_i) \inf_{A_j} f$$

$$\leq \sum_{j=1}^{n} \sum_{i=1}^{m} \mu(A_j \cap E_i) c_i \qquad \text{(True if } A_j \cap E_i \neq \emptyset \text{, and if } A_j \cap E_i = \emptyset \text{ then just 0)}$$
$$= \sum_{i=1}^{m} c_i \mu(E_i)$$

As required. ■

Proposition 0.6. *If $E_1, ..., E_m \in \mathcal{S}$, then $\int \sum_{i=1}^m c_i \chi_{E_i} = \sum_{i=1}^m c_i \mu(E_i)$.*

Proof. Write $\sum_{i=1}^m c_i \chi_{E_i}$ in standard form, then use the disjoint case identity and the independence of decomposition. ■

The above establishes the additivity of integrals for simple functions. Here is an alternative characterization of integrals, in terms of simple functions.

Proposition 0.7. *If $f : X \to [0, \infty]$ is measurable, then $\int f = \sup(\int T : T \text{ simple}, T \leq f)$.*

Proof. For every simple $T \leq f$, we have $\int T \leq \int f$. Therefore $\int f \geq \sup(\int T : T \text{ simple}, T \leq f)$. Thus we want to show the other direction.

Suppose that for every $\mu(A) > 0$ we have $\inf_A f < \infty$. Then for any $\mathcal{S}$-partition $A_1, ..., A_m$, letting $c_i = \inf_{A_i} f$ if $\mu(A_i) > 0$ and $c_i = 0$ if $\mu(A_i) = 0$, the corresponding lower sum is equal to the integral of $\sum_{i=1}^m c_i \chi_{A_i}$ whic is simple and $\leq f$. Therefore, we have $\int f \leq \sup(\int T : T \text{ simple}, T \leq f)$.

Now suppose there exists $\mu(A) > 0$ such that $\inf_A f = \infty$ (so f is ∞ on A). Then clearly $\sup(\int T : T \text{ simple}, T \leq f) = \infty$, so the required inequality holds. ■

Finally, we will prove the Monotone Convergence Theorem, which combined with our knowledge of simple functions proves the additivity of integrals. It'll be a bit bashy, but yeah some hard work will be needed to prove all those juicy results.

Theorem 0.8 (Monotone Convergence Theorem)**.** *If $f_k : X \to [0, \infty]$ is a measurable sequence converging pointwise increasing to $f : X \to [0, \infty]$, then $\int f = \lim \int f_n$.*

Proof. Note that f is measurable, because the pointwise limit of measurable functions is measurable. Now fix a $\alpha \in (0, 1)$ and a simple function $\varphi : X \to [0, \infty]$ with $\varphi \leq f$ and $\varphi = \sum_{i=1}^m c_i \chi_{E_i}$. If we let $A_k = \{x \in X : f_k(x) \geq \alpha\varphi(x)\}$, then:

- By monotonicity, $A_1 \subset A_2 \subset A_3 \subset \cdots$
- A_k is measurable because it is the preimage of $[0, \infty]$ under $f_k - \alpha\varphi$
- By pointwise convergence and the fact that $\alpha < 1$, we have $X = \cup_{k=1}^{\infty} A_k$. So we can use upward measure continuity.

We have $f_k \geq \alpha \sum_{i=1}^m c_i \chi_{E_i \cap A_k}$, so integrating both sides,

$$\int f_k \geq \alpha \sum_{i=1}^{m} c_i \mu(E_i \cap A_k)$$

By upward measure continuity we have $\mu(E_i \cap A_k) \to \mu(E_i)$ as $k \to \infty$, therefore taking $k \to \infty$,

$$\lim_{k\to\infty} \int f_k \geq \alpha \sum_{i=1}^{m} c_i \mu(E_i)$$

Since $\alpha \in (0,1)$ was arbitrary, taking $\alpha \to 1$ we have

$$\lim_{k\to\infty} \int f_k \geq \sum_{i=1}^{m} c_i \mu(E_i) = \int \varphi$$

Since $\varphi \leq f$ was an arbitrary simple function, this shows that $\lim_{k\to\infty} \int f_k \geq \int f$. The other direction is clear, so we have equality. ∎

We are now ready to prove additivity of integrals.

Proposition 0.9. *If $f, g : X \to [0, \infty]$ are measurable, then $\int f + g = \int f + \int g$.*

Proof. Let $\varphi_k : X \to [0, \infty]$ be a sequence of measurable simple functions converging upward to f, and let $\psi_k : X \to [0, \infty]$ be the same for g. Then $\varphi_k + \psi_k : X \to [0, \infty]$ is a sequence of measurable simple functions converging upward to $f + g$. By the MCT, we have

$$\begin{aligned}
\int f + g &= \lim \int (\varphi_k + \psi_k) \\
&= \lim(\int \varphi_k + \int \psi_k) && \text{(Additivity clearly holds for simple functions)} \\
&= \lim \int \varphi_k + \lim \int \psi_k \\
&= \int f + \int g && \text{(MCT)}
\end{aligned}$$

As required. ∎

2) We extend integration codomain to $[-\infty, \infty]$. Given a meausurable $f : X \to [-\infty, \infty]$, we can consider the positive and negative parts $f^+, f^- : X \to [0, \infty]$. Clearly, f^+, f^- are both measurable by measurability of f. If at least one of $\int f^+, f^-$ are finite, then we can define the integral of f:

$$\int f := \int f^+ - \int f^- \in [-\infty, \infty]$$

Since $|f| = f^+ + f^-$, we see that $\int |f| < \infty$ if and only if $\int f^+$ and $\int f^-$ are both finite. This characterization is of interest when we discuss L^p spaces later.

Proposition 0.10. *If $f : X \to [-\infty, \infty]$ is a measurable function such that $\int f$ is defined, then cf has a defined integral, with $\int cf = c \int f$.*

Proof. Clear. ∎

Proposition 0.11. *If $f, g : X \to [-\infty, \infty]$ are L^1, then $f + g$ is L^1 with $\int f + g = \int f + \int g$.*

Proof. MAT357. ∎

Proposition 0.12. *If $f, g : X \to [-\infty, \infty]$ have defined integral and $f \geq g$, then $\int f \leq \int g$.*

Proof. Suppose that f, g are L^1 (the other cases are clear). Then

$$\int g - \int f = \int (f - g) \geq 0$$

Since $f - g$ is non-negative. ■

Proposition 0.13. *If $f : X \to [-\infty, \infty]$ is measurable, then $\left|\int f\right| \leq \int |f|$.*

Proof. We have $|f| = f^+ + f^-$, so

$$\int |f| = \int f^+ + \int f^-$$

Now $\int f^+ + \int f^- \geq \int f^+ - \int f^-$, and $\int f^+ + \int f^- \geq -\int f^+ + \int f^-$. Thus $\int |f| \pm \int f$, done. ■

3) We now aim to prove the Dominated Convergence Theorem.

Proposition 0.14. *Suppose $f : X \to [-\infty, \infty]$ is measurable with a defined integral. Then $\left|\int_E f\right| \leq \mu(E) \sup_E |f|$.*

Proof. Letting $c = \sup_E |f|$, we have

$$\left|\int_E f\right| = \left|\int \chi_E f\right| \leq \int |\chi_E f| \leq \int c\chi_E \leq c\mu(E).$$

As required. ■

We will prove below a special case of the Dominated Convergence Theorem.

Proposition 0.15. *Suppose $\mu(X) < \infty$ and $f_k : X \to [-\infty, \infty]$ is a measurable sequence converging pointwise to $f : X \to [-\infty, \infty]$, where $|f_k| < c$ for some constant $c \in (0, \infty)$. Then $\int f = \lim \int f_n$.*

Proof. By Egorov theorem choose a $E \in \mathcal{S}$ with $\mu(X - E) <$ and such that f_k converges uniformly to f on E. We have

$$\begin{aligned}
\left|\int f - \int f_k\right| &\leq \int |f - f_k| \\
&= \int_E |f - f_k| + \int_{X-E} |f - f_k| \\
&\leq \mu(X) \sup_E |f - f_k| + \frac{\epsilon}{4c} \sup_X |f - f_k| \\
&\leq \mu(X) \sup_E |f - f_k| + \frac{\epsilon}{2}
\end{aligned}$$

By uniform convergence, we have $\sup_E |f - f_k| \to 0$ as $k \to \infty$. ■

The following proposition will provide a simplication in the proof of the DCT.

Proposition 0.16. *If $f : X \to [0, \infty]$ is measurable and $\int f < \infty$, then for $\epsilon > 0$ there exists $E \in \mathcal{S}$ such that $\mu(E) < \infty$ and $\int_{X-E} f < \epsilon$.*

Proof. Choose a $\mathcal{S}$-partition $A_1, ..., A_m$ whose lower sum is $> \int f - \epsilon$. If we let E be the union of all A_i such that $\inf_{A_i} f > 0$, then $\mu(E) < \infty$ (because the lower sum is $< \infty$). We then have $\int_E f > \int f - \epsilon$. ■

We are now ready to prove the DCT.

Theorem 0.17 (Dominated Convergence Theorem). *Suppose $f_k : X \to [-\infty, \infty]$ is a measurable sequence (integral exists) converging pointwise to $f : X \to [-\infty, \infty]$. If there exists a measurable $g : X \to [0, \infty]$ with $|f_k| \leq g$ and $\int g < \infty$, then $\int f = \lim \int f_n \in \mathbb{R}$.*

Proof. By the previous proposition, we can assume $\mu(X) < \infty$. Since $\int g < \infty$, there exists $\delta > 0$ such that $\mu(B) < \delta$ implies $\int_B g < \delta$. By Egorov theorem choose a measurable E such that $\mu(X - E) < \frac{\delta}{4}$ while (f_k) converges uniformly on E. Due to the integrable denominator g, we have $\int |f| < \infty$ and $\int |f_k| < \infty$, so

$$\begin{aligned}\left|\int f - \int f_k\right| &\leq \int |f - f_k| \\ &= \int_E |f - f_k| + \int_{X-E} |f - f_k| \\ &\leq \mu(X) \sup_E |f - f_k| + 2\int_{X-E} g \\ &\leq \mu(X) \sup_E |f - f_k| + \frac{\epsilon}{2}\end{aligned}$$

The left term is eventually $< \frac{\epsilon}{2}$ by $\mu(X) < \infty$ and uniform convergence on E. ■

4) We will introduce the L^p space, where further discussion is to happen later.

Proposition 0.18. *For measurable $f : X \to [0, \infty]$, $\int f = 0$ iff f vanishes almost everywhere.*

Proof. Suppose $\int f = 0$, then we want to show that $\mu(f^{-1}(0, \infty]) = 0$. Since $f^{-1}(0, \infty] = \cup_{q \in \mathbb{Q}_{>0}} f^{-1}(q, \infty]$, it suffices to show that $\mu(f^{-1}(q, \infty]) = 0$. It we suppose for contradiction that this is false, then denoting $E = f^{-1}(q, \infty]$ (we have $\mu(E) > 0$), the two sets $E, X - E$ form a $\mathcal{S}$-partition of X, and its upper sum is $\geq \mu(E)q > 0$, a contradiction.

Suppose f vanishes almost everywhere; then $E = f^{-1}(0, \infty]$ has measure 0. For any $\mathcal{S}$-partition $A_1, ..., A_m$, without loss of generality assuming $\mu(A_i) > 0$, we must have $\inf_{A_i} f = 0$ so the corresponding lower sum is 0. ■

Corollary 0.18.1. *For measurable $f : X \to [-\infty, \infty]$, $\int |f| = 0$ iff $f = 0$ almost everywhere.*

Proof. Clear. ■

Corollary 0.18.2. *For measurable $f : X \to [-\infty, \infty]$, $(\int |f|^p)^{1/p} = (\int |g|^p)^{1/p}$ iff $f = g$ almost everywhere.*

Proof. Clear. ■

The above Corollary allows us to talk about an L^p space: if we consider the set of measurable functions with $(\int |f|^p)^{1/p} < \infty$, quotiented by the equivalence of being equal almost everywhere, we can define a p-norm $\|\cdot\|_p$ with

$$\|f\|_p = (\int |f|^p)^{1/p}$$

All norm properties except the triangle inequality are clear (note that we've quotiented the equivalence of almost-everywhere equality, so only the zero function has zero norm). For the triangle equality, the $p = 1$ case is clear, and we will show the general case later.