Calculus for Engineers

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Contents

Indeterminate Forms

18.1 Motivation

Let us consider the function

$$h(x) = \frac{f(x)}{g(x)}$$

for which

$$\lim_{x \to a} h(x)$$

is desired. It can be obtained by dividing the limit of the function in numerator with that of denominator.

But if

$$\lim_{x \to a} f(x) \quad \text{and} \quad \lim_{x \to a} g(x)$$

are zero, then

$$\lim_{x \to a} h(x)$$

cannot be equal to

$$\frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$$

which is a $\frac{0}{0}$ form and is consequently meaningless. The form $\frac{0}{0}$ is often called *an indeterminate form.*

Other types of indeterminate forms are

$$\frac{\infty}{\infty},\ 0 \times \infty,\ \infty - \infty,\ 0^0,\ 1^\infty \text{ and } \infty^0.$$

In this chapter, the methods of finding the limits leading to indeterminate forms shall be discussed in detail. One thing which should be addressed is that we are not going to find out the value of $\frac{0}{0}$ or any of the indeterminate forms. So we shall discuss an appropriate method of evaluating the limits in such and similar other cases.

The techniques we adopt here are based on Chapters 2, 3, and 4. The reader should work out each example carefully.

18.2 L'Hôpital's Rule

Theorem 1 Let $f(x)$ and $g(x)$ be functions of x, which are capable of being expanded by Taylor's theorem in the neighbourhood of $x = a$, and let $f(a)$ and $g(a)$ both equal zero, then

$$\lim_{x\to a} h(x) = \lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Proof.

Expanding by Taylor's theorem, we have

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + R_1}{g(a) + (x-a)g'(a) + \frac{(x-a)^2}{2!}g''(a) + \cdots + R_2},$$

where R_1 and R_2 are the corresponding values of the remainders and may be given as

$$R_1 = \frac{(x-a)^n}{n!} f^{(n)}(a + \theta_1(x-a)), \quad 0 < \theta_1 < 1$$

and

$$R_2 = \frac{(x-a)^n}{n!} g^{(n)}(a + \theta_2(x-a)), \quad 0 < \theta_2 < 1.$$

Now as stated in the theorem, $f(a)$ and $g(a)$ are zero, therefore,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{(x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + R_1}{(x-a)g'(a) + \frac{(x-a)^2}{2!}g''(a) + \cdots + R_2}.$$

Dividing the numerator and denominator throughout by $(x-a)$, we can obtain

$$\begin{aligned}\lim_{x\to a} \frac{f(x)}{g(x)} &= \lim_{x\to a} \frac{f'(a) + (x-a)\left(\frac{1}{2!}f''(a) + \cdots\right)}{g'(a) + (x-a)\left(\frac{1}{2!}g''(a) + \cdots\right)} \\ &= \frac{f'(a)}{g'(a)} \\ &= \lim_{x\to a} \frac{f'(x)}{g'(x)}.\end{aligned}$$

Hence the proposition holds.

Let us examine a few particular cases:

In the case when

$$f'(a), f''(a), \cdots, f^{(n-1)}(a)$$

and

$$g'(a), g''(a), \cdots, g^{(n-1)}(a)$$

are all zero, but neither $f^{(n)}(a)$ nor $g^{(n)}(a)$ are not zero, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f^{(n)}(x)}{g^{(n)}(x)}.$$

The above rule is true even when $x \to a$ is replaced by $x \to \infty$, for if

$$\lim_{x\to\infty} f(x) = 0 \quad \text{and} \quad \lim_{x\to\infty} g(x) = 0,$$

then by putting $x = \dfrac{1}{y}$, we have

$$\begin{aligned}
\lim_{x\to\infty}\frac{f(x)}{g(x)} &= \lim_{y\to 0}\frac{f\left(\frac{1}{y}\right)}{g\left(\frac{1}{y}\right)} \\
&= \lim_{y\to 0}\frac{f'\left(\frac{1}{y}\right)\left(-\frac{1}{y^2}\right)}{g'\left(\frac{1}{y}\right)\left(-\frac{1}{y^2}\right)} \\
&= \lim_{y\to 0}\frac{f'\left(\frac{1}{y}\right)}{g'\left(\frac{1}{y}\right)},
\end{aligned}$$

which can also be written as

$$\lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

□

Note 1 When applying Theorem 1, the numerator $f(x)$ and the denominator $g(x)$ should be differentiated simultaneously but separately until the occurrence of the indeterminate form disappears and then the limit should be evaluated. It is worth mentioning that the readers are not supposed to differentiate $\frac{f(x)}{g(x)}$ as a function.

Note 2 Finding out the value of $\frac{0}{0}$ is not our main concern. Instead, we are simply finding out the limit of the combination of the functions which has assumed the $\frac{0}{0}$ form, when the limits of the functions are taken separately. A similar proposition holds for other indeterminate forms.

An alternative version of L'Hôpital's rule can be stated as follows (the proof is based on Cauchy's theorem (see Theorem **??**)):

Theorem 2 Let $f(a) = g(a) = 0$ and both $f'(x)$ and $g'(x)$ exist near a (but not necessarily at a), then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

Proof.

Since $f'(x)$ and $g'(x)$ exist in an interval around a, $f(x)$ and $g(x)$ are continuous in the neighbourhood of a. Also $g'(a) \neq 0$. Therefore the conditions of Cauchy's theorem are satisfied in an interval $[a, x]$. Hence, we have

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(\xi)}{g'(\xi)}, \quad a < \xi < x.$$

As $x \to a$, $\xi \to a$, and

$$\frac{f'(\xi)}{g'(\xi)} \to \frac{f'(a)}{g'(a)},$$

then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

Suppose that in L'Hôpital's's rule, $f'(a) = 0$ and $g'(a) \neq 0$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{g'(a)} = 0.$$

If $f'(a) \neq 0$ and $g'(a) = 0$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{0} \to \infty.$$

But if $f'(a) = 0$, $g'(a) = 0$, and both $f''(a)$ and $g''(a)$ exist, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} = \lim_{x \to a} \frac{f''(x)}{g''(x)}$$

provide $g''(a) \neq 0$. This can be generalized further as follows.

If $f(x)$ and $g(x)$ and their first $(n-1)$ derivatives vanish at $x = a$ and $f^{(n)}(a)$ and $g^{(n)}(a)$ both exist, with $g^n(a) \neq 0$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f^{(n)}(x)}{g^{(n)}(x)}.$$

By Taylor's theorem, we have

$$f(a + h) = f(a) + \frac{h^n}{n!} f^{(n)}(a + \theta_1 h), \quad 0 < \theta_1 < 1$$

since $f'(a) = f''(a) = \cdots = f^{(n-1)}(a) = 0$.

Similarly, we have

$$g(a + h) = g(a) + \frac{h^n}{n!} g^{(n)}(a + \theta_2 h), \quad 0 < \theta_2 < 1$$

since $g'(a) = g''(a) = \cdots = g^{(n-1)}(a) = 0$. Therefore, we have

$$\begin{aligned}
\lim_{x\to a} \frac{f(x)}{g(x)} &= \lim_{h\to 0} \frac{f(a+h)}{g(a+h)} \\
&= \lim_{h\to 0} \frac{\frac{h^n}{n!} f^{(n)}(a+\theta_1 h)}{\frac{h^n}{n!} g^{(n)}(a+\theta_2 h)} \\
&= \frac{f^{(n)}(a)}{g^{(n)}(a)} \\
&= \lim_{x\to a} \frac{f^{(n)}(x)}{g^{(n)}(a)}
\end{aligned}$$

provided $g^{(n)}(a) \neq 0$.

$\square$

Examples

Example 1 Evaluate

$$\lim_{t\to 0}\frac{t-\sin t}{t^3}.$$

Solution.

Method I

$$\begin{aligned}
\lim_{t\to 0}\frac{t-\sin t}{t^3} &= \lim_{t\to 0}\frac{\dfrac{d(t-\sin t)}{dt}}{\dfrac{d\left(t^3\right)}{dt}} && \text{By L'Hôpital's Rule}\\
&= \lim_{t\to 0}\frac{1-\cos t}{3t^2} && \frac{0}{0}\text{ form}\\
&= \lim_{t\to 0}\frac{\dfrac{d(1-\cos t)}{dt}}{\dfrac{d\left(3t^2\right)}{dt}} && \text{By L'Hôpital's Rule}\\
&= \lim_{t\to 0}\frac{\sin t}{6t} && \frac{0}{0}\text{ form}\\
&= \lim_{t\to 0}\frac{\dfrac{d(\sin t)}{dt}}{\dfrac{d\left(6t\right)}{dt}} && \text{By L'Hôpital's Rule}\\
&= \frac{\lim_{t\to 0}\cos t}{6} = \frac{1}{6}.
\end{aligned}$$

Method II

$$\begin{aligned}
\lim_{t\to 0}\frac{t-\sin t}{t^3} &= \lim_{t\to 0}\frac{\dfrac{d(t-\sin t)}{dt}}{\dfrac{d\left(t^3\right)}{dt}} && \text{By L'Hôpital's Rule}\\
&= \lim_{t\to 0}\frac{1-\cos t}{3t^2} && \frac{0}{0}\text{ form}\\
&= \lim_{t\to 0}\frac{\dfrac{d(1-\cos t)}{dt}}{\dfrac{d\left(3t^2\right)}{dt}} && \text{By L'Hôpital's Rule}\\
&= \lim_{t\to 0}\frac{1}{6}\frac{\sin t}{t}\\
&= \frac{1}{6}\lim_{t\to 0}\frac{\sin t}{t} && \text{Use } \lim_{t\to 0}\frac{\sin t}{t}=1\\
&= \frac{1}{6}.
\end{aligned}$$

Method III

$$
\begin{aligned}
\lim_{t\to 0}\frac{t-\sin t}{t^3} &= \lim_{t\to 0}\frac{t-\left(t-\dfrac{t^3}{3!}+\dfrac{t^5}{5!}-\cdots\right)}{t^3} \\
&= \lim_{t\to 0}\frac{\dfrac{t^3}{3!}-\dfrac{t^5}{5!}+\cdots}{t^3} && \text{Divide out common term } t^3 \\
&= \lim_{t\to 0}\left(\frac{1}{3!}-\frac{t^2}{5!}+\cdots\right) \\
&= \frac{1}{6}.
\end{aligned}
$$

$\square$

Example 2 Evaluate

$$\lim_{x\to 0}\frac{x\cos t-\ln(1+x)}{x^2}.$$

Solution.

Method I

$$
\begin{aligned}
\lim_{x\to 0}\frac{x\cos x-\ln(1+x)}{x^2} &= \lim_{x\to 0}\frac{\dfrac{d\left(x\cos x-\ln(1+x)\right)}{dx}}{\dfrac{d\left(x^2\right)}{dx}} && \text{By L'Hôpital's Rule} \\
&= \lim_{x\to 0}\frac{\cos x-x\sin x-\dfrac{1}{1+x}}{2x} && \frac{0}{0}\text{ form} \\
&= \lim_{x\to 0}\frac{\dfrac{d\left(\cos x-x\sin x-\dfrac{1}{1+x}\right)}{dx}}{\dfrac{d\left(2x\right)}{dx}} && \text{By L'Hôpital's Rule} \\
&= \lim_{x\to 0}\frac{-\sin x-\sin x-x\cos x+\dfrac{1}{(1+x)^2}}{2} \\
&= \frac{1}{2}.
\end{aligned}
$$

Method II

$$
\begin{aligned}
&\lim_{x\to 0} \frac{x\cos x - \ln(1+x)}{x^2} \\
&= \lim_{x\to 0} \frac{x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) - \left(x - \frac{x^2}{2!} + \frac{x^3}{3!} - \cdots\right)}{x^2} \\
&= \lim_{x\to 0} \frac{\frac{x^2}{2} - x^3\left(\frac{1}{2} + \frac{1}{3}\right) + \frac{x^4}{4} + \cdots}{x^2} && \text{Divide out common term } x^2 \\
&= \lim_{x\to 0} \left(\frac{1}{2} - x\left(\frac{5}{6}\right) + \frac{x^2}{4} + \cdots\right) \\
&= \frac{1}{2}.
\end{aligned}
$$

$\square$

Example 3 Evaluate

$$\lim_{x\to 0}\frac{\cos x-\ln(1+x)+\sin x-1}{e^x-(1+x)}.$$

Solution.

Method I

$$\begin{aligned}
\lim_{x\to 0}\frac{\cos x-\ln(1+x)+\sin x-1}{e^x-(1+x)} &= \lim_{x\to 0}\frac{\dfrac{d(\cos x-\ln(1+x)+\sin x-1)}{dx}}{\dfrac{d\left(e^x-(1+x)\right)}{dx}} && \text{By L'Hôpital's Rule}\\
&= \lim_{x\to 0}\frac{-\sin x+\dfrac{1}{x+1}-\cos x}{e^x-1} && \frac{0}{0}\text{ form}\\
&= \lim_{x\to 0}\frac{\dfrac{d\left(-\sin x+\dfrac{1}{x+1}-\cos x\right)}{dx}}{\dfrac{d\left(e^x-1\right)}{dx}} && \text{By L'Hôpital's Rule}\\
&= \lim_{x\to 0}\frac{-\cos x+\dfrac{1}{(1+x)^2}-\sin x}{e^x}\\
&= 0.
\end{aligned}$$

Method II

$$\begin{aligned}
&\lim_{x\to 0}\frac{\cos x-\ln(1+x)+\sin x-1}{e^x-(1+x)}\\
&= \lim_{x\to 0}\frac{\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots\right)-\left(x-\frac{x^2}{2!}+\frac{x^3}{3!}-\cdots\right)+\left(2-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\right)-1}{1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots-1-x}\\
&= \lim_{x\to 0}\frac{-x^3\frac{1}{2}+\frac{x^4}{24}+\cdots}{\frac{x^2}{2}+\frac{x^3}{6}+\cdots}\\
&= \lim_{x\to 0}\frac{-x+\frac{x^2}{12}+\cdots}{1+\frac{x}{3}+\cdots}\\
&= 0.
\end{aligned}$$

$\square$

Example 4 Evaluate

$$\lim_{x\to 0}\frac{e^x+\ln(1-x)-1}{\tan x-x}.$$

Solution.

$$
\begin{aligned}
&\lim_{x\to 0}\frac{e^x+\ln(1-x)-1}{\tan x - x} \\
&= \lim_{x\to 0}\frac{\dfrac{d(e^x+\ln(1-x)-1)}{dx}}{\dfrac{d\,(\tan x - x)}{dx}} && \text{By L'Hôpital's Rule} \\
&= \lim_{x\to 0}\frac{e^x-\dfrac{1}{1-x}}{\sec^2 x - 1} && \frac{0}{0}\text{ form} \\
&= \lim_{x\to 0}\frac{e^x(1-x)-1}{(1-x)\tan^2 x} && \text{Rewrite} \\
&= \lim_{x\to 0}\frac{\dfrac{d(e^x(1-x)-1)}{dx}}{\dfrac{d\,((1-x)\tan^2 x)}{dx}} && \text{By L'Hôpital's Rule} \\
&= \lim_{x\to 0}\frac{-e^x+(1-x)e^x}{2(1-x)\tan x\sec^2 x-\tan^2 x} && \frac{0}{0}\text{ form} \\
&= \lim_{x\to 0}\frac{\dfrac{d(-e^x+(1-x)e^x)}{dx}}{\dfrac{d\,(2(1-x)\tan x\sec^2 x-\tan^2 x)}{dx}} && \text{By L'Hôpital's Rule} \\
&= \lim_{x\to 0}\frac{(1-x)e^x-2e^x}{4(1-x)\tan^2 x\sec^2 x+2(1-x)\sec^4 x-4\tan x\sec^2 x} \\
&= \frac{1}{2}.
\end{aligned}
$$

The above question can also be done using Taylor's theorem. It is being left as an exercise for the reader. □

Example 5 Evaluate

$$\lim_{x\to 0}\frac{\ln\left(1+\alpha x^2\right)}{1-\cos x}.$$

Solution.

$$\begin{aligned}
\lim_{x\to 0}\frac{\ln(1+\alpha x^2)}{1-\cos x} &= \lim_{x\to 0}\frac{\dfrac{d(\ln(1+\alpha x^2))}{dx}}{\dfrac{d(1-\cos x)}{dx}} && \text{By L'Hôpital's Rule}\\
&= \lim_{x\to 0}\frac{\dfrac{2\alpha x}{1+\alpha x^2}}{\sin x} = \lim_{x\to 0}\frac{2\alpha x}{(1+\alpha x^2)\sin x} && \frac{0}{0}\text{ form}\\
&= \lim_{x\to 0}\frac{\dfrac{d(2\alpha x)}{dx}}{\dfrac{d((1+\alpha x^2)\sin x)}{dx}} && \text{By L'Hôpital's Rule}\\
&= \lim_{x\to 0}\frac{2\alpha}{(1+kx^2)\cos x+2\alpha x\sin x}\\
&= 2\alpha.
\end{aligned}$$

The above question can also be done using algebra. It is being left as an exercise for the reader.

□

Example 6 Evaluate

$$\lim_{x\to 0}\frac{x^{1/2}\tan x}{(e^x-1)^{3/2}}.$$

Solution.

Clearly, $x^{1/2}$ cannot be expanded by Taylor's theorem in the neighbourhood of $x=0$. But even then

$$\begin{aligned}
\lim_{x\to 0}\frac{x^{1/2}\tan x}{(e^x-1)^{3/2}} &= \lim_{x\to 0}\frac{x^{1/2}\tan x}{\left(\left(1+x+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\cdots\right)-1\right)^{3/2}}\\
&= \lim_{x\to 0}\frac{x^{1/2}\tan x}{x^{3/2}\left(1+\dfrac{x}{2!}+\cdots\right)^{3/2}}\\
&= \lim_{x\to 0}\frac{\tan x}{x}\frac{1}{\left(1+\dfrac{x}{2!}+\cdots\right)^{3/2}}\\
&= 1.
\end{aligned}$$

□

Example 7 Evaluate

$$\lim_{x\to 0}\frac{5\sin x-7\sin 2x+3\sin 3x}{\tan x-x}.$$

Solution.

$$\begin{aligned}
&\lim_{x\to 0}\frac{5\sin x-7\sin 2x+3\sin 3x}{\tan x-x}\\
&=\lim_{x\to 0}\frac{\dfrac{d(5\sin x-7\sin 2x+3\sin 3x)}{dx}}{\dfrac{d\,(\tan x-x)}{dx}} && \text{By L'Hôpital's Rule}\\
&=\lim_{x\to 0}\frac{5\cos x-14\cos 2x+9\cos 3x}{\sec^2 x-1} && \frac{0}{0}\text{ form}\\
&=\lim_{x\to 0}\frac{\dfrac{d(5\cos x-14\cos 2x+9\cos 3x)}{dx}}{\dfrac{d\,(\sec^2 x-1)}{dx}} && \text{By L'Hôpital's Rule}\\
&=\lim_{x\to 0}\frac{-5\sin x+28\sin 2x-27\sin 3x}{2\tan x\sec^2 x} && \frac{0}{0}\text{ form}\\
&=\lim_{x\to 0}\frac{\dfrac{d(-5\sin x+28\sin 2x-27\sin 3x)}{dx}}{\dfrac{d\,(2\tan x\sec^2 x)}{dx}} && \text{By L'Hôpital's Rule}\\
&=\lim_{x\to 0}\frac{-5\cos x+56\cos 2x-81\cos 3x}{4\sec^2 x\tan^2 x+2\sec^4 x}\\
&=-15.
\end{aligned}$$

□

18.3 The form $\frac{\infty}{\infty}$

Theorem 3 Let us consider the function

$$h(x)=\lim_{x\to a}\frac{f(x)}{g(x)},$$

where both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ are infinite, then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

Proof.

Here we shall convert it into the $\frac{0}{0}$ form so that L'Hôpital's's Rule discussed therein could be applicable.

We can write

$$\begin{aligned}
\lim_{x\to a}\frac{f(x)}{g(x)} &= \lim_{x\to a}\frac{\dfrac{1}{g(x)}}{\dfrac{1}{f(x)}} \\
&= \lim_{x\to a}\frac{-\dfrac{g'(x)}{(g(x))^2}}{\dfrac{f'(x)}{(f(x))^2}} \\
&= \lim_{x\to a}\left(\frac{f(x)}{g(x)}\right)^2\frac{g'(x)}{f'(x)}
\end{aligned}$$

Thus

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{g'(x)}{f'(x)}\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right)^2. \tag{18.1}$$

Now let us suppose

$$\frac{f(x)}{g(x)} = \alpha \tag{18.2}$$

which gives rise to following three cases:

1. When α is neither zero nor infinite. From (18.1) we have

$$\alpha = \alpha^2\lim_{x\to a}\frac{g'(x)}{f'(x)}$$

or

$$\alpha^{-1} = \lim_{x\to a}\frac{g'(x)}{f'(x)},$$

which can also be written as

$$\alpha = \lim_{x\to a}\frac{f'(x)}{g'(x)},$$

which proves the theorem.

2. When $\alpha = 0$

Adding 1 to both sides of (18.2), we have

$$\begin{aligned}
\alpha + 1 &= \lim_{x\to a}\frac{f(x)+g(x)}{g(x)} \\
&= \lim_{x\to a}\frac{f'(x)+g'(x)}{g'(x)} \\
&= \lim_{x\to a}\frac{f'(x)}{g'(x)} + 1
\end{aligned}$$

so that

$$\alpha = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Hence the proposition holds.

3. When α is infinite

In this case we may write

$$\begin{aligned}\frac{1}{\alpha} &= \frac{1}{\lim\limits_{x\to a} \dfrac{f(x)}{g(x)}} = \lim_{x\to a} \frac{g(x)}{f(x)} \\ &= \lim_{x\to a} \frac{g'(x)}{f'(x)}.\end{aligned}$$

Therefore

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

which establishes the theorem.

Therefore, in general,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

when $\lim\limits_{x\to a} f(x) = \infty$ and $\lim\limits_{x\to a} g(x) = \infty$. □

Note 3 We have observed that the methods for evaluating the limits of the quotients of the two functions are similar in each of the two cases when they assume the $\frac{0}{0}$ form or the $\frac{\infty}{\infty}$ form. It is advisable for the reader to change any form to $\frac{0}{0}$ so that the limit may be obtained more quickly. Suppose we have the $\frac{1}{x}$ form in the numerator or the denominator, and the limit as x tends to zero is to be sought, then the process of differentiation would not terminate, since it would involve $\frac{1}{x}, \frac{1}{x^2}, \cdots$, etc., which would all tend to ∞ as x tends to 0. Hence we should change it to the $\frac{0}{0}$ form at an appropriate stage.

Examples

Example 8 Evaluate

$$\lim_{x\to 0}\frac{\ln x}{\cot x}.$$

Solution.

This limit leads to the indeterminate form $\dfrac{\infty}{\infty}$.

$$\begin{aligned}
\lim_{x\to 0}\frac{\ln x}{\cot x} &= \lim_{x\to 0}\frac{\frac{d(\ln x)}{dx}}{\frac{d(\cot x)}{dx}} && \text{By L'Hôpital's Rule}\\
&= \lim_{x\to 0}\frac{\frac{1}{x}}{-\mathrm{cosec}^2 x} = -\lim_{x\to 0}\frac{\sin^2 x}{x} && \frac{0}{0}\text{ form}\\
&= -\lim_{x\to 0}\frac{\frac{d(\sin^2 x)}{dx}}{\frac{d(x)}{dx}} && \text{By L'Hôpital's Rule}\\
&= -\lim_{x\to 0}\frac{2\sin x\cos x}{1}\\
&= 0.
\end{aligned}$$

□

Example 9 Evaluate

$$\lim_{x\to 0}\frac{\ln\sin 2x}{\ln\sin x}.$$

Solution.

This limit leads to the indeterminate form $\dfrac{\infty}{\infty}$.

$$\begin{aligned}
\lim_{x\to 0}\frac{\ln\sin 2x}{\ln\sin x} &= \lim_{x\to 0}\frac{\frac{d(\ln\sin 2x)}{dx}}{\frac{d(\ln\sin x)}{dx}} && \text{By L'Hôpital's Rule}\\
&= \lim_{x\to 0}\frac{\frac{2\cos 2x}{\sin 2x}}{\frac{\cos x}{\sin x}}\\
&= \lim_{x\to 0}\frac{\cos 2x}{\cos^2 x}\\
&= \lim_{x\to 0}\frac{\cos^2 x-\sin^2 x}{\cos^2 x}\\
&= \lim_{x\to 0}\left(1-\tan^2 x\right)\\
&= 1.
\end{aligned}$$

□

Example 10 Find the value of

$$\lim_{x\to 0}\frac{\ln\tan x}{\ln x}.$$

Solution.

This limit leads to the indeterminate form $\dfrac{\infty}{\infty}$.

$$\begin{aligned}
\lim_{x\to 0}\frac{\ln\tan x}{\ln x} &= \lim_{x\to 0}\frac{\dfrac{d(\ln\tan x)}{dx}}{\dfrac{d\,(\ln x)}{dx}} && \text{By L'Hôpital's Rule}\\
&= \lim_{x\to 0}\frac{\dfrac{1}{\tan x}\sec^2 x}{\dfrac{1}{x}}\\
&= \lim_{x\to 0}\frac{x}{\sin x\cos x} && \frac{0}{0}\text{ form}\\
&= \lim_{x\to 0}\frac{\dfrac{d(x)}{dx}}{\dfrac{d\,(\sin x\cos x)}{dx}} && \text{By L'Hôpital's Rule}\\
&= \lim_{x\to 0}\frac{1}{\cos^2 x-\sin^2 x}\\
&= 1.
\end{aligned}$$

□

18.4 The form $0\times\infty$

Theorem 4 Let $h(x)=\lim\limits_{x\to a} f(x)g(x)$, such that $\lim\limits_{x\to a} f(x)=0$ and $\lim\limits_{x\to a} g(x)=\infty$, then we can write

$$h(x)=\lim_{x\to a} f(x)g(x)=\lim_{x\to a}\frac{f(x)}{\dfrac{1}{g(x)}}$$

which assumes the $\frac{0}{0}$ form and so can be evaluated by Theorem 1.

The function $\lim\limits_{x\to a} f(x)g(x)$ can be converted in the $\frac{0}{0}$ form as

$$h(x)=\lim_{x\to a} f(x)g(x)=\lim_{x\to a}\frac{f(x)}{\dfrac{1}{g(x)}}$$

and so can be evaluated by Theorem 3.

Example 11 Find the value of

$$\lim_{x\to 0}(1-x)\tan\left(\frac{\pi x}{2}\right).$$

Solution.

This limit leads to the indeterminate form $0 \times \infty$.

Let us transform the $0 \times \infty$ form to the $\dfrac{0}{0}$ form.

$$\begin{aligned}
\lim_{x\to 0}(1-x)\tan\left(\frac{\pi x}{2}\right) &= \lim_{x\to 0}\frac{1-x}{\cot\left(\frac{\pi x}{2}\right)} && \frac{0}{0} \text{ form}\\
&= \lim_{x\to 0}\frac{\frac{d(1-x)}{dx}}{\frac{d\left(\cot\left(\frac{\pi x}{2}\right)\right)}{dx}} && \text{By L'Hôpital's Rule}\\
&= \lim_{x\to 0}\frac{-1}{-\frac{\pi}{2}\operatorname{cosec}^2\left(\frac{\pi x}{2}\right)}\\
&= \frac{2}{\pi}.
\end{aligned}$$

□

Example 12 Find the value of

$$\lim_{x\to 0}\sec\left(\frac{\pi}{2x}\right)\ln x.$$

Solution.

This limit leads to the indeterminate form $\infty \times 0$.

Let us transform the $\infty \times 0$ form to the $\dfrac{0}{0}$ form.

$$\begin{aligned}
\lim_{x\to 0}\sec\left(\frac{\pi}{2x}\right)\ln x &= \lim_{x\to 0}\frac{\ln x}{\cos\left(\frac{\pi}{2x}\right)} && \frac{0}{0} \text{ form}\\
&= \lim_{x\to 0}\frac{\frac{d(\ln x)}{dx}}{\frac{d\left(\cos\left(\frac{\pi}{2x}\right)\right)}{dx}} && \text{By L'Hôpital's Rule}\\
&= \lim_{x\to 0}\frac{\frac{1}{x}}{\sin\left(\frac{\pi}{2x}\right)\cdot\frac{\pi}{2}\cdot\frac{1}{x^2}}\\
&= \frac{2}{\pi}.
\end{aligned}$$

□

18.5 The form $\infty - \infty$

This type of expression can also be reduced to the $\frac{0}{0}$ form or the $\frac{\infty}{\infty}$ form.

Theorem 5 Let us consider the function

$$h(x) = \lim_{x \to a} \left(f(x) - g(x) \right)$$

such that $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = \infty$, then writing

$$f(x) - g(x) = \frac{\dfrac{1}{g(x)} - \dfrac{1}{f(x)}}{\dfrac{1}{f(x)g(x)}}.$$

It is evident that the RHS of the function has taken the $\frac{0}{0}$ form.

Examples

Example 13 Find the value of

$$\lim_{x \to 0} \left(\frac{1}{x^2} - \operatorname{coses}^2 x \right).$$

Solution.

This limit leads to the indeterminate form $\infty - \infty$.

Let us transform the $\infty - \infty$ form to the $\dfrac{0}{0}$ form.

$$
\begin{aligned}
\lim_{x\to 0}\left(\frac{1}{x^2} - \text{coses}^2 x\right) &= \lim_{x\to 0}\frac{\sin^2 x - x^2}{x^2\sin^2 x} && \frac{0}{0}\text{ form}\\
&= \lim_{x\to 0}\frac{\frac{d(\sin^2 x - x^2)}{dx}}{\frac{d\left(x^2\sin^2 x\right)}{dx}} && \text{By L'Hôpital's Rule}\\
&= \lim_{x\to 0}\frac{\sin 2x - 2x}{x^2\sin 2x + 2x\sin^2 x} && \frac{0}{0}\text{ form}\\
&= \lim_{x\to 0}\frac{\frac{d(\sin 2x - 2x)}{dx}}{\frac{d\left(x^2\sin 2x + 2x\sin^2 x\right)}{dx}} && \text{By L'Hôpital's Rule}\\
&= \lim_{x\to 0}\frac{2\cos 2x - 2}{2x^2\cos 2x + 4x\sin 2x + 2\sin^2 x} && \frac{0}{0}\text{ form}\\
&= \lim_{x\to 0}\frac{\frac{d(2\cos 2x - 2)}{dx}}{\frac{d\left(2x^2\cos 2x + 4x\sin 2x + 2\sin^2 x\right)}{dx}} && \text{By L'Hôpital's Rule}\\
&= \lim_{x\to 0}\frac{-4\sin 2x}{-4x^2\sin 2x + 12x\cos 2x + 6\sin 2x} && \frac{0}{0}\text{ form}\\
&= \lim_{x\to 0}\frac{\frac{d(-4\sin 2x)}{dx}}{\frac{d\left(-4x^2\sin 2x + 12x\cos 2x + 6\sin 2x\right)}{dx}} && \text{By L'Hôpital's Rule}\\
&= \lim_{x\to 0}\frac{8\cos 2x}{-8x^2\cos 2x - 32\sin 2x + 24\cos 2x}\\
&= -\frac{8}{24} = -\frac{1}{3}.
\end{aligned}
$$

$\square$

Example 14 Find the value of

$$\lim_{x\to 0}\left(\frac{1}{x^2} - \cot^2 x\right).$$

Solution.

This limit leads to the indeterminate form $\infty - \infty$.

Let us transform the $\infty - \infty$ form to the $\frac{0}{0}$ form.

$$\begin{aligned}
\lim_{x\to 0}\left(\frac{1}{x^2}-\cot^2 x\right) &= \lim_{x\to 0}\frac{\sin^2 x - x^2\cos^2 x}{x^2\sin^2 x} \qquad \frac{0}{0} \text{ form}\\
&= \lim_{x\to 0}\frac{\left(x-\frac{x^3}{3!}+\cdots\right)^2 - x^2\left(1-\frac{x^2}{2!}+\cdots\right)^2}{x^2\left(x-\frac{x^3}{3!}+\cdots\right)^2}\\
&= \lim_{x\to 0}\frac{\left(x^2-2\frac{x^4}{3!}+\cdots\right) - x^2\left(1-\frac{x^2}{2!}+\cdots\right)}{x^2\left(x^2-\cdots\right)}\\
&= \lim_{x\to 0}\frac{\left(1-\frac{2}{6}\right)x^4+\cdots}{x^4}\\
&= \frac{2}{3}.
\end{aligned}$$

□

18.6 The forms 0^0, 1^∞ and ∞^0

Each of these forms can be transformed into the $0\times\infty$ form and ultimately either to $\frac{0}{0}$ or to $\frac{\infty}{\infty}$. Using the method of transformation, it will be evident from the following explanation:

Theorem 6 Let

$$h(x) = f(x)^{g(x)}, \tag{18.3}$$

then

$$\ln h(x) = g(x)\ln\ f(x). \tag{18.4}$$

The following three cases shall be discussed:

1. If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, so that $h(x)$ defined in (18.3) is of the 0^0 form, then expression in (18.4) shall have the $0\times\infty$ form.
2. If $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \infty$, which means that (18.3) has the 1^∞ form, then (18.4) shall have the $\infty\times 0$ form.
3. If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = 0$, so that $f(x)$ given in (18.3) has the ∞^0 form, then(18.4) shall have the $0\times\infty$ form.

Here, if the limit of the function defined in (18.4) turns out to be ∞, then the limit of the given function shall be e^∞.

Examples

Example 15 Find the value of

$$\lim_{x\to 1}\left(1-x^2\right)^{\frac{1}{\ln(1-x)}}.$$

Solution.

Let us first transform the 0^0 form to the $\infty\times 0$ form, and then convert it to $\dfrac{\infty}{\infty}$.

Let

$$f(x)=\lim_{x\to 1}\left(1-x^2\right)^{\frac{1}{\ln(1-x)}}\quad\left[\,0^0\text{ form}\right], \tag{18.5}$$

then taking the natural logarithm of both sides of (18.5) yields

$$\begin{aligned}
\ln f(x) &= \lim_{x\to 1}\frac{1}{\ln(1-x)}\ln\left(1-x^2\right) && \infty\times 0\text{ form}\\
&= \lim_{x\to 1}\frac{\ln\left(1-x^2\right)}{\ln(1-x)} && \frac{\infty}{\infty}\text{ form}\\
&= \lim_{x\to 1}\frac{\dfrac{d(\ln\left(1-x^2\right))}{dx}}{\dfrac{d\left(\ln(1-x)\right)}{dx}} && \text{By L'Hôpital's Rule}\\
&= \lim_{x\to 1}\frac{\dfrac{-2x}{(1-x^2)}}{\dfrac{-1}{(1-x)}} && \text{Simplify}\\
&= \lim_{x\to 1}\frac{2x}{1+x}\\
&= 1.
\end{aligned} \tag{18.6}$$

Since $\ln f(x)=1$, taking the exponent of each side of (18.6) yields $e^{\ln f(x)}=e^1=1$. Therefore, we have

$$f(x)=\lim_{x\to 1}\left(1-x^2\right)^{\frac{1}{\ln(1-x)}}=e.$$

□

Example 16 Find the value of

$$\lim_{x\to 0}x^x.$$

Solution.

Let us first transform the 0^0 form to the$0\times\infty$ form, and then convert it to $\dfrac{\infty}{\infty}$.

Let

$$f(x)=\lim_{x\to 0}x^x\quad\left[0^0\text{ form }\right], \tag{18.7}$$

then taking the natural logarithm of both sides of (18.7) yields

$$
\begin{aligned}
\ln f(x) &= \lim_{x\to 0} x\ln x && 0\times\infty \text{ form}\\
&= \lim_{x\to 0}\frac{\ln x}{\frac{1}{x}} && \frac{\infty}{\infty}\text{ form}\\
&= \lim_{x\to 0}\frac{\frac{d(\ln x)}{dx}}{\frac{d\left(\frac{1}{x}\right)}{dx}} && \text{By L'Hôpital's Rule}\\
&= \lim_{x\to 0}\frac{\frac{1}{x}}{\frac{-1}{x^2}} && \text{Simplify}\\
&= \lim_{x\to 0} -x\\
&= 0. && (18.8)
\end{aligned}
$$

Since $\ln f(x) = 0$, taking the exponent of each side of (18.8) yields $e^{\ln f(x)} = e^0 = 1$. Therefore, we have

$$f(x) = \lim_{x\to 0} x^x = 1.$$

□

Example 17 Find the value of

$$\lim_{x\to 0}(\cos x)^{\cot x}.$$

Solution.

Let us first transform the 1^∞ form to the $\infty\times 0$ form, and then convert it to $\frac{0}{0}$. Let

$$f(x) = \lim_{x\to 0}(\cos x)^{\cot x}\ \left[1^\infty \text{ form}\right], \tag{18.9}$$

then taking the natural logarithm of both sides of (18.9) yields

$$
\begin{aligned}
\ln f(x) &= \lim_{x\to 0}\cot x\ln(\cos x) && \infty\times 0\text{ form}\\
&= \lim_{x\to 0}\frac{\ln(\cos x)}{\tan x} && \frac{0}{0}\text{ form}\\
&= \lim_{x\to 0}\frac{\frac{d(\ln(\cos x))}{dx}}{\frac{d(\tan x)}{dx}} && \text{By L'Hôpital's Rule}\\
&= \lim_{x\to 0}\frac{-\tan x}{\sec^2 x} && \text{Simplify}\\
&= \lim_{x\to 0}(-\sin x\cos x)\\
&= 0. && (18.10)
\end{aligned}
$$

Since $\ln f(x) = 0$, taking the exponent of each side of (18.10) yields $e^{\ln f(x)} = e^0 = 1$. Therefore, we have

$$f(x) = \lim_{x\to 0}(\cos x)^{\cot x} = 1.$$

□

Example 18 Evaluate the limit

$$\lim_{x\to\frac{\pi}{2}}(\tan x)^{\cos x}.$$

Solution.

Let us first transform the ∞^0 form to the $0\times\infty$ form, and then convert it to $\dfrac{\infty}{\infty}$.

Let

$$f(x) = \lim_{x\to\frac{\pi}{2}}(\tan x)^{\cos x} \quad [1^\infty \text{ form}], \tag{18.11}$$

then taking the natural logarithm of both sides of (18.11) yields

$$\begin{aligned}
\ln f(x) &= \lim_{x\to\frac{\pi}{2}} \cos x \ln(\tan x) && \infty\times 0 \text{ form}\\
&= \lim_{x\to\frac{\pi}{2}} \frac{\ln(\tan x)}{\sec x} && \frac{0}{0} \text{ form}\\
&= \lim_{x\to\frac{\pi}{2}} \frac{\dfrac{d(\ln(\tan x))}{dx}}{\dfrac{d(\sec x)}{dx}} && \text{By L'Hôpital's Rule}\\
&= \lim_{x\to\frac{\pi}{2}} \frac{\dfrac{\sec^2 x}{\tan x}}{\sec x\tan x} && \text{Simplify}\\
&= \lim_{x\to\frac{\pi}{2}} \frac{\sec x}{\tan^2 x}\\
&= \lim_{x\to\frac{\pi}{2}} \frac{\cos x}{\sin^2 x}\\
&= 0.
\end{aligned} \tag{18.12}$$

Since $\ln f(x) = 0$, taking the exponent of each side of (18.12) yields $e^{\ln f(x)} = e^0 = 1$. Therefore, we have

$$f(x) = \lim_{x\to\frac{\pi}{2}}(\tan x)^{\cos x} = 1.$$

□

18.7 Compound indeterminate forms

If a function is the product of several factors, where the limit of each corresponds to a simple indeterminate form for the same value of x, then the limit of such a function will be identical to the product of the limits of the factors provided the product is not itself an indeterminate form. A similar rule is applicable in the cases of a sum, difference, quotient or power.

The expansions of the functions involved are sometimes used to evaluate the given limits which assume the indeterminate forms. Here is a list of some important algebraic expansions:

$$(x+a)^n = x^n + nx^{n-1}a + \frac{n(n-1)}{2!}x^{n-2}a^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}a^3 + \cdots + nxa^{n-1} + a^n$$

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \cdots$$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \cdots$$

$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \cdots$$

$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots$$

$$(1+x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + \cdots$$

$$(1-x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + \cdots$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots$$

$$\tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \cdots$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$$

$$\ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - \cdots$$

$$\ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - \cdots$$

$$\sin^{-1} x = x + 1^2 \cdot \frac{x^3}{3!} + 3^2 \cdot 1^2 \cdot \frac{x^5}{5!} + 5^2 \cdot 3^2 \cdot 1^2 \cdot \frac{x^7}{7!} + \cdots$$

Examples

Example 19 Evaluate the limit

$$\lim_{x\to\frac{\pi}{2}} \frac{\tan 5x}{\tan x}.$$

Solution.

If we differentiate the numerator and denominator, we would have the $\frac{\infty}{\infty}$ form. Hence, we change it to the $\frac{0}{0}$ form, that is,

$$\lim_{x\to\frac{\pi}{2}} \frac{\tan 5x}{\tan x} \quad \left[\frac{\infty}{\infty} \text{ form}\right] = \lim_{x\to\frac{\pi}{2}} \frac{\cot x}{\cot 5x} \quad \left[\frac{0}{0} \text{ form}\right].$$

Dividing the numerator and denominator by $\tan x \tan 5x$, we have

$$\begin{aligned}
\lim_{x\to\frac{\pi}{2}} \frac{\tan 5x}{\tan x} &= \lim_{x\to\frac{\pi}{2}} \frac{\cot x}{\cot 5x} \\
&= \lim_{x\to\frac{\pi}{2}} \frac{\frac{\cot x}{\tan x \tan 5x}}{\frac{\cot 5x}{\tan x \tan 5x}} \\
&= \lim_{x\to\frac{\pi}{2}} \frac{-\mathrm{cosec}^2 x}{-5\mathrm{cosec}^2 5x} && \text{By L'Hôpital's Rule} \\
&= \frac{1}{5}.
\end{aligned}$$

□

Example 20 Evaluate the limit

$$\lim_{x\to 0} \frac{\ln x}{\cot x}.$$

Solution.

$$
\begin{aligned}
&\lim_{x\to 0}\frac{\ln x}{\cot x} && \frac{\infty}{\infty}\text{ form}\\
&=\lim_{x\to 0}\frac{\frac{d(\ln x)}{dx}}{\frac{d\,(\cot x)}{dx}} && \text{By L'Hôpital's Rule}\\
&=\lim_{x\to 0}\frac{\frac{1}{x}}{-\text{cosec}^2 x} && \text{Simplify}\\
&=\lim_{x\to 0}\frac{-\sin^2 x}{x}\\
&=\lim_{x\to 0}(-\sin x)\left(\frac{\sin x}{x}\right) && \text{Use }\lim_{x\to 0}\frac{\sin x}{x}=1\\
&=\lim_{x\to 0}-\sin x\cdot\lim_{x\to 0}\frac{\sin x}{x}\\
&=(0)(1)\\
&=0
\end{aligned}
$$

□

Example 21 Evaluate the limit

$$\lim_{x\to 0}\frac{\ln\sin 2x}{\ln\sin x}.$$

Solution.

$$
\begin{aligned}
&\lim_{x\to 0}\frac{\ln\sin 2x}{\ln\sin x} && \frac{\infty}{\infty}\text{ form}\\
&=\lim_{x\to 0}\frac{\frac{d(\ln\sin 2x)}{dx}}{\frac{d\left(\ln\sin x\right)}{dx}} && \text{By L'Hôpital's Rule}\\
&=\lim_{x\to 0}\frac{\frac{\cos 2x}{\sin 2x}}{\frac{\cos x}{\sin x}} && \text{Simplify}\\
&=\lim_{x\to 0}\frac{2\sin x\cos 2x}{\cos x\sin 2x}\\
&=\lim_{x\to 0}\frac{2\sin x\cos 2x}{\cos x 2\sin x\cos x}\\
&=\lim_{x\to 0}\frac{\cos 2x}{\cos^2 x}\\
&=\lim_{x\to 0}\left(\frac{\cos^2 x-\sin^2 x}{\cos^2 x}\right)\\
&=\lim_{x\to 0}\left(1-\tan^2 x\right)\\
&=1.
\end{aligned}
$$

□

Example 22 Evaluate the limit

$$
\lim_{x\to 0}\left(\frac{1}{x^2}-\cot^2 x\right).
$$

Solution.

$$
\begin{aligned}
&\lim_{x\to 0}\left(\frac{1}{x^2}-\cot^2 x\right) && \infty-\infty\text{ form}\\
&=\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{\cos^2 x}{\sin^2 x}\right)\\
&=\lim_{x\to 0}\left(\frac{\sin^2 -x^2\cos^2 x}{x^2\sin^2 x}\right) && \frac{0}{0}\text{ form}\\
&=\lim_{x\to 0}\left(\frac{\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\right)^2-x^2\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots\right)^2}{x^2\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\right)^2}\right)\\
&=\lim_{x\to 0}\frac{\frac{2}{3}x^4+\text{ terms containing higher powers of x}}{x^4+\text{ terms containing higher powers of x}}\\
&=\frac{2}{3}.
\end{aligned}
$$

□

Example 23 Evaluate the limit

$$\lim_{x\to 1} \sec\left(\frac{\pi x}{2}\right) \ln x.$$

Solution.

$$\begin{aligned}
&\lim_{x\to 1} \sec\left(\frac{\pi x}{2}\right) \ln x && \infty \times 0 \text{ form} \\
&= \lim_{x\to 1} \frac{\ln x}{\cos\left(\frac{\pi x}{2}\right)} && \frac{0}{0} \text{ form} \\
&= \lim_{x\to 1} \frac{\dfrac{d(\ln x)}{dx}}{\dfrac{d\left(\cos\left(\frac{\pi x}{2}\right)\right)}{dx}} && \text{By L'Hôpital's Rule} \\
&= \lim_{x\to 1} \frac{\frac{1}{x}}{-\sin\left(\frac{\pi x}{2}\right)\cdot\frac{\pi}{2}} \\
&= \frac{2}{\pi}.
\end{aligned}$$

□

Example 24 Evaluate the limit

$$\lim_{x\to 0} \left(\frac{\tan x}{x}\right)^{1/x^3}.$$

Solution.

$$\begin{aligned}
&\lim_{x\to 0} \left(\frac{\tan x}{x}\right)^{1/x^3} \\
&= \lim_{x\to 0} \left(\frac{x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots}{x}\right)^{1/x^3} \\
&= \lim_{x\to 0} \left(1 + \frac{1}{3}x^2 + \frac{2}{15}x^4 + \cdots\right)^{1/x^3} \\
&= \lim_{x\to 0} \left(1 + \frac{1}{3}x^2\right)^{1/x^3} \\
&= \lim_{x\to 0} \left[\left(1 + \frac{1}{3}x^2\right)^{\frac{3}{x^2}}\right]^{\frac{1}{3x}} \\
&= \lim_{\substack{x\to 0\\ y\to 0}} \left[(1+y)^{\frac{1}{y}}\right]^{\frac{1}{3x}} && \text{Put } y = \frac{1}{3}x^2 \\
&= \lim_{x\to 0} e^{\frac{1}{3x}} \\
&= e^{\infty} \\
&= \infty.
\end{aligned}$$

□

Example 25 Evaluate the limit

$$\lim_{x\to 0}\left(\frac{1}{x}\right)^{\tan x}.$$

Solution.

Let us first transform the ∞^0 form to the $0\times\infty$ form, then convert $0\times\infty$ to $\dfrac{\infty}{\infty}$ and finally change $\dfrac{\infty}{\infty}$ to $\dfrac{0}{0}$.

Let

$$f(x)=\lim_{x\to 0}\left(\frac{1}{x}\right)^{\tan x}\quad\left[\infty^0\text{ form}\right], \tag{18.13}$$

then taking the natural logarithm of both sides of (18.13) yields

$$\begin{aligned}
\ln f(x) &= \lim_{x\to 0}\tan x\ln\left(\frac{1}{x}\right) && 0\times\infty\text{ form}\\
&= \lim_{x\to 0}\frac{\ln\left(\frac{1}{x}\right)}{\cot x} && \frac{\infty}{\infty}\text{ form}\\
&= \lim_{x\to 0}\frac{\dfrac{d\ln\left(\frac{1}{x}\right)}{dx}}{\dfrac{d\left(\cot x\right)}{dx}} && \text{By L'Hôpital's Rule}\\
&= \lim_{x\to 0}\frac{-\frac{1}{x}}{-\operatorname{cosec}^2 x} && \text{Simplify}\\
&= \lim_{x\to 0}\frac{\sin^2 x}{x} && \frac{0}{0}\text{ form}\\
&= \lim_{x\to 0}\frac{\sin x}{x}\lim_{x\to 0}\sin x && \text{Use }\lim_{x\to 0}\frac{\sin x}{x}=1\\
&= 1\cdot 0\\
&= 0 && (18.14)
\end{aligned}$$

Since $\ln f(x)=0$, taking the exponent of each side of (18.14) yields $e^{\ln f(x)}=e^0=1$. Therefore, we have

$$f(x)=\lim_{x\to 0}\left(\frac{1}{x}\right)^{\tan x}=1.$$

□

Example 26 Evaluate the limit

$$\lim_{x\to 0}\left[\frac{2(\cosh x-1)}{x^2}\right]^{\frac{1}{x^2}}.$$

Solution.
Let

$$f(x) = \lim_{x\to 0}\left[\frac{2(\cosh x - 1)}{x^2}\right]^{\frac{1}{x^2}}, \tag{18.15}$$

then taking the natural logarithm of both sides of (18.15) yields

$$\begin{aligned}
\ln f(x) &= \lim_{x\to 0}\frac{1}{x^2}\ln\left(\frac{2(\cosh x - 1)}{x^2}\right)\\
&= \lim_{x\to 0}\frac{1}{x^2}\ln\left(\frac{2}{x^2}\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots-1\right)\right)\\
&= \lim_{x\to 0}\frac{1}{x^2}\ln\left(1+\frac{x^2}{12}+\cdots\right)\\
&= \lim_{x\to 0}\frac{1}{x^2}\left[\left(\frac{x^2}{12}+\cdots\right)-\frac{1}{2}\left(\frac{x^2}{12}+\cdots\right)^2+\cdots\right].\\
&= \frac{1}{12}
\end{aligned} \tag{18.16}$$

Since $\ln f(x) = 12$, taking the exponent of each side of (18.16) yields $e^{\ln f(x)} = e^{12}$. Therefore, we have

$$f(x) = \lim_{x\to 0}\left[\frac{2(\cosh x - 1)}{x^2}\right]^{\frac{1}{x^2}} = e^{12}.$$

□

Example 27 Evaluate the limit

$$\lim_{x\to 0}\frac{(1+x)^{1/x}-e}{x}.$$

Solution.
Let

$$y = (1+x)^{1/x}, \tag{18.17}$$

then taking the natural logarithm of both sides of (18.17) yields

$$\begin{aligned}
\ln y &= \ln(1+x)^{1/x}\\
&= \frac{1}{x}\ln(1+x)\\
&= \frac{1}{x}\left(x-\frac{x^2}{3}+\frac{x^3}{2}-\cdots\right)\\
&= 1-\frac{x}{2}+\frac{x^2}{3}-\cdots.
\end{aligned} \tag{18.18}$$

Since $\ln y = 1 - \frac{x}{2} + \frac{x^2}{3} - \cdots$, taking the exponent of each side of (18.18) yields

$$\begin{aligned}
e^{\ln y} = y &= e^{1-\frac{x}{2}+\frac{x^2}{3}-\cdots} \\
&= e \cdot e^{-\frac{x}{2}+\frac{x^2}{3}-\cdots} \\
&= e \cdot \left[1 + \left(-\frac{x}{2} + \frac{x^2}{3} - \cdots\right) + \frac{1}{2!}\left(-\frac{x}{2} + \frac{x^2}{3} - \cdots\right)^2 + \cdots\right] \\
&= e \cdot \left[1 - \frac{x}{2} + x^2\left(\frac{1}{3} + \frac{1}{8}\right) + \cdots\right] \\
&= e \cdot \left[1 - \frac{x}{2} + \frac{11}{28}x^2 + \cdots\right]. \qquad (18.19)
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\lim_{x\to 0} \frac{(1+x)^{1/x} - e}{x} && \frac{0}{0} \text{ form} \\
&= \lim_{x\to 0} \frac{e \cdot \left[1 - \frac{x}{2} + \frac{11}{28}x^2 + \cdots\right] - e}{x} && \text{By (18.19)} \\
&= -\frac{e}{2}.
\end{aligned}$$

□