Calculus for Engineers

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Contents

Derivatives of exponential and logarithmic functions

9.1 Introduction

A well-known limit form is given in Section 9.2, $\lim\limits_{x\to+\infty}\left(1+\dfrac{1}{n}\right)^n = e$. Derivatives of exponential and logarithmic functions are studied in Section 9.3. In Section 9.4, we shall prove an important standard limit:

$$\lim_{x\to 0}\frac{\log_a(1+x)}{x} = \frac{1}{\log_e a}.$$

Using the above result, we shall provide a comprehensive derivation of derivatives of exponential function a^x and logarithmic function $\log_a x$ using the definition of derivative.

9.2 $\lim\limits_{n\to+\infty}\left(1+\dfrac{1}{n}\right)^n = e$

Let us consider the following sequence:

$$(1+1)\,,\left(1+\frac{1}{2}\right)^2,\left(1+\frac{1}{3}\right)^3,\cdots,\left(1+\frac{1}{n}\right)^n,\cdots$$

where n is the positive integers.

What happens as n gets very large? It is easy to find out if you use a scientific calculator having the function x^y. The first three terms are 2, 2.25, 2.37.

n	$\left(1+\frac{1}{n}\right)^n$ (rounding off)
10	2.59
100	2.70
1000	2.717
$10,000$	2.718
$100,000$	2.71827
$1,000,000$	2.718280

Table 9.1: Values of $\left(1+\dfrac{1}{n}\right)^n$.

As can be seen in Table 9.1, these calculations strongly suggest that as n approaches infinity, the sequence goes to a definite limit. It can be proved mathematically that does go to a limit, and this limiting value is called e. The value of e is $2.7182818283\cdots$.

Let us try to get a bit more insight into $\left(1+\frac{1}{n}\right)^n$ for large the values of n:

First, let us expand it using the binomial theorem[1]. Recall that the binomial theorem gives all the terms in $(1+x)^n$, as follows:

$$\begin{aligned}(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 \\ + \frac{n(n-1)(n-2)(n-3)}{4!}x^4 + \cdots + x^n.\end{aligned}$$

To find $\left(1+\frac{1}{n}\right)^n$, we simply need to put $x = \frac{1}{n}$, and have:

$$\begin{aligned}\left(1+\frac{1}{n}\right)^n = 1 + n\left(\frac{1}{n}\right) + \frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n}\right)^3 \\ + \frac{n(n-1)(n-2)(n-3)}{4!}\left(\frac{1}{n}\right)^4 + \cdots + \left(\frac{1}{n}\right)^n.\end{aligned}$$

We are particularly interested in what happens to this series when n gets very large, because that is when we are approaching e. In that limit,

- $\frac{n(n-1)}{n^2} = \left(1-\frac{1}{n}\right)$ tends to 1,
- as does $\frac{n(n-1)(n-2)}{n^3} = \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)$, and so forth.

So, for large enough values of n, we can ignore the n-dependence of these early terms in the series altogether!.

When we do that, the series becomes just:

$$1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

And, the larger we take n, the more accurately the terms in the binomial series can be simplified in this way, so as n goes to infinity this simple series represents the limiting value of $\left(1+\frac{1}{n}\right)^n$. Therefore, e must be simply the sum of this infinite series.

We observe that we can see immediately from this series that e is less than 3, because

- $\frac{1}{3!}$ is less than $\frac{1}{2^2}$, and
- $\frac{1}{4!}$ is less than $\frac{1}{2^3}$, and so on,

so the whole series adds up to less than $1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots \leq 3$.

Theorem 1

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e. \tag{9.1}$$

[1] see Chapter 14.

Proof. We will prove that

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$$

- Firstly, we will prove the sequence $\left\{(1+\frac{1}{n})^n\right\}$ is monotone increasing.

  Let $x_n = \left(1+\frac{1}{n}\right)^n$.

  By the binomial formula we know:

$$\begin{aligned}
x_n &= 1 + n\times\frac{1}{n} + \frac{n(n-1)}{2!}\times\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}\times\frac{1}{n^3} + \cdots \\
&\quad + \frac{n(n-1)(n-2)\cdots(n-n+1)}{n!}\times\frac{1}{n^n} \\
&= 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \cdots \\
&\quad + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right).
\end{aligned}$$

  So

$$\begin{aligned}
x_{n+1} &= 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n+1}\right) + \frac{1}{3!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right) + \cdots \\
&\quad + \frac{1}{n!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{n-1}{n+1}\right) \\
&\quad + \frac{1}{(n+1)!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{n}{n+1}\right).
\end{aligned}$$

  Comparing the expressions of x_n and x_{n+1}, we can see that

  - x_{n+1} has one more term than x_n, and also that,
  - starting from the third term, each term of x_{n+1} is bigger than the corresponding term of x_n.

  Hence the sequence $\{x_n\}$ is monotone increasing.

- Second we will prove the boundedness of the above.

  In the expression of x_n changing all the factors in the brackets of each term into 1 will make the expression larger, and thus we have,

$$\begin{aligned}
x_n &< 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} \\
&< 1 + \underbrace{1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}}_{\text{a geometric series}} \\
&= 1 + \frac{1-\frac{1}{2^n}}{1-\frac{1}{2}} \\
&< 3.
\end{aligned}$$

So that $\{x_n\}$ is the boundedness of the above. Thus $\lim\limits_{x\to\infty} x_n$ exists.

The limit of the sequence $\left\{(1+\frac{1}{n})^n\right\}$ is denoted by e, thus

$$\lim_{x\to\infty}\left(1+\frac{1}{n}\right)^n = e.$$

□

Example 1 Find the limit

$$\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^{2x}.$$

Solution. Let $t=\frac{x}{2}$, then $t\to\infty$ as $x\to\infty$. Thus

$$\begin{aligned}
\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^{2x} &= \lim_{t\to\infty}\left(1+\frac{1}{t}\right)^{4t}\\
&= \left[\lim_{t\to\infty}\left(1+\frac{1}{t}\right)^{t}\right]^4\\
&= e^4.
\end{aligned}$$

□

Example 2 Find the limit

$$\lim_{x\to\infty}\left(\frac{x+2}{x+1}\right)^{2x+1}.$$

Solution. We have

$$\begin{aligned}
\lim_{x\to\infty}\left(\frac{x+2}{x+1}\right)^{2x+1} &= \lim_{x\to\infty}\left(\frac{x+1+1}{x+1}\right)^{2x+1}\\
&= \lim_{x\to\infty}\left(1+\frac{1}{x+1}\right)^{2x+1}\\
&= \lim_{x\to\infty}\left(1+\frac{1}{x+1}\right)^{2x+2-1}\\
&= \lim_{x\to\infty}\left(1+\frac{1}{x+1}\right)^{2(x+1)-1}\\
&= \lim_{x\to\infty}\left(1+\frac{1}{x+1}\right)^{2(x+1)}\lim_{x\to\infty}\left(1+\frac{1}{x+1}\right)^{-1}\\
&= \left[\lim_{x\to\infty}\left(1+\frac{1}{x+1}\right)^{x+1}\right]^2\frac{1}{\lim\limits_{x\to\infty}\left(1+\frac{1}{x+1}\right)}\\
&= e^2\cdot 1 = e^2.
\end{aligned}$$

□

9.2.1 Another thought $\cdots$

For a positive number x the natural logarithm of x is defined as the integral:

$$\ln\, x = \int_1^x \frac{1}{t}dt.$$

Then e is the unique number such that $\ln\, e = 1$, that is,

$$\ln\, e = \int_1^e \frac{1}{t}dt.$$

The natural exponential function e^x is the function inverse to $\ln\, x$ (Why?), and all the usual properties of logarithms and exponential functions follow.

Example 3 Use a synthetic proof to prove that

$$\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n = e.$$

A synthetic proof is one that starts with statements that are already proved and progresses one step at a time until the aim is attained. A defect of synthetic proofs is that they do not explain why any step is made.

Solution:

Let t be any number in an interval $\left[1, 1+\frac{1}{n}\right]$. Then

$$\frac{1}{1+\frac{1}{n}} \le \frac{1}{t} \le 1.$$

Therefore, we have:

$$\int_1^{1+1n} \frac{1}{1+\frac{1}{n}}\, dt \int_1^{1+1n} \frac{1}{t}\, dt \le \int_1^{1+1n} 1\, dt.$$

Let us evaluate each integral individually:

$$\begin{aligned}
\int_1^{1+1n} \frac{1}{1+\frac{1}{n}}\, dt &= \frac{1}{1+\frac{1}{n}} \int_1^{1+1n}\, dt \\
&= \frac{1}{1+\frac{1}{n}}\, t\Big|_1^{1+\frac{1}{n}} \\
&= \frac{1}{1+\frac{1}{n}}\left(1+\frac{1}{n}-1\right) = \frac{1}{n+1}.
\end{aligned}$$

$$\begin{aligned}
\int_1^{1+1n} \frac{1}{t}\, dt &= \ln\, |t|\Big|_1^{1+\frac{1}{n}} \\
&= \ln\, t\Big|_1^{1+\frac{1}{n}} \\
&= \ln\left(1+\frac{1}{n}\right) - \ln\, (1) \\
&= \ln\left(1+\frac{1}{n}\right).
\end{aligned}$$

$$\begin{aligned}\int_1^{1+1n} 1\,dt &= t\big|_1^{1+\frac{1}{n}} \\ &= \left(1+\frac{1}{n}-1\right) \\ &= \frac{1}{n}.\end{aligned}$$

Therefore, we obtain:

$$\frac{1}{n+1} \leq \ln\left(1+\frac{1}{n}\right) \leq \frac{1}{n}.$$

Exponentiating, we find:

$$\exp\left(\frac{1}{n+1}\right) \leq \exp\left(\ln\left(1+\frac{1}{n}\right)\right) \leq \exp\left(\frac{1}{n}\right)$$

or

$$e^{\frac{1}{n+1}} \leq e^{\ln\left(1+\frac{1}{n}\right)} \leq e^{\frac{1}{n}}$$

or

$$e^{\frac{1}{n+1}} \leq 1+\frac{1}{n} \leq e^{\frac{1}{n}}.$$

Taking the $(n+1)^{\text{st}}$ power of the left inequality gives us

$$\left(e^{\frac{1}{n+1}}\right)^{n+1} \leq \left(1+\frac{1}{n}\right)^{n+1}$$

or

$$e \leq \left(1+\frac{1}{n}\right)^{n+1}$$

while taking the nth power of the right inequality gives us

$$\left(1+\frac{1}{n}\right)^{n} \leq \left(e^{\frac{1}{n}}\right)^{n}$$

or

$$\left(1+\frac{1}{n}\right)^{n} \leq e.$$

Together, they give us these important bounds on the value of e :

$$\left(1+\frac{1}{n}\right)^{n} \leq e \leq \left(1+\frac{1}{n}\right)^{n+1}.$$

Dividing the right inequality by $1+\frac{1}{n}$ gives

$$\frac{e}{1+\frac{1}{n}} \leq \frac{\left(1+\frac{1}{n}\right)^{n+1}}{1+\frac{1}{n}}$$

or

$$\frac{e}{1+\frac{1}{n}} \leq \left(1+\frac{1}{n}\right)^n$$

which we combine with the left inequality to get

$$\frac{e}{1+\frac{1}{n}} \leq \left(1+\frac{1}{n}\right)^n \leq e.$$

But both

$$\lim_{n\to+\infty} \frac{e}{1+\frac{1}{n}} = e$$

and

$$\lim_{n\to+\infty} e = e,$$

so by the Squeeze theorem, we have

$$\lim_{n\to+\infty} \left(1+\frac{1}{n}\right)^n = e.$$

9.3 Derivatives of exponential and logarithmic functions

We know that

(a)

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots .$$

(b)

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots .$$ [2]

(c)

$$\frac{d}{dx}\left(e^x\right) = e^x.$$

Clearly, this result is obtained by differentiating both sides of the result in equation **(b)**[3].

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[2]See Chapter 17.

[3] Taking the derivative of e^x with respect to x, we get

$$\begin{aligned}\frac{d}{dx}\left(e^x\right) &= \frac{d}{dx}\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)\\ &= 0 + \frac{1}{1!} + \frac{2x}{2\cdot 1} + \frac{3x^2}{3\cdot 2\cdot 1} + \frac{4x^3}{4\cdot 3\cdot 2\cdot 1} + \cdots\\ &= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots .\end{aligned}$$

Now, to find the derivative of the logarithmic function $y = \log_e x$, we use the following two results:

1. The logarithmic function $y = \log_e x$ and the exponential function $x = e^y$ are mutually inverse functions, of which we know the derivative of the exponential function e^y.

2. For any pair of mutually inverse functions $y = f(x)$ and $x = f^{-1}(y)$, their derivatives are related by the condition

$$\frac{dy}{dx} = \frac{1}{(dx/dy)}, \qquad \text{provided } \frac{dx}{dy} \neq 0. \tag{9.2}$$

9.3.1 Finding the derivative of the logarithmic function

To find the derivative of the logarithmic function $y = \log_e x$, let us consider the following equation

$$y = \log_e x. \tag{9.3}$$

We transform (9.3) into its equivalent exponential form. We have,

$$x = e^y, \tag{9.4}$$

where x stands for the function (that is, the dependent variable) and y for the independent variable.

Hence, differentiating both sides of (9.4) with respect to y, we get,

$$\frac{dx}{dy} = \frac{d}{dy}(e^y) = e^y. \tag{9.5}$$

The derivative of e^y with respect to y is the original function unchanged.

Now, to compute the derivative of $y = \log_e x$, we use the formula

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{(dx/dy)} && \text{provided } \frac{dx}{dy} \neq 0 \\ &= \frac{1}{e^y} && \text{Since } \frac{dx}{dy} = e^y, \text{ by (9.5)} \\ &= \frac{1}{x}. && \text{Since } e^y = x, \text{ by (9.4)} \end{aligned}$$

Therefore, for the function $y = \log_e x$, we get

$$\frac{dy}{dx} = \frac{d}{dx}(\log_e x) = \frac{1}{x} = x^{-1}. \tag{9.6}$$

This is a very interesting result. Note that, x^{-1} is a result that we obtained by differentiating the function $\log_e x$ with respect to x, and that we could never have got it by differentiating the power functions, as can be seen from the following results:

$$\frac{d}{dx}\left(\frac{x^3}{3}\right) = x^2;$$
$$\frac{d}{dx}\left(\frac{x^2}{2}\right) = x^1;$$
$$\frac{d}{dx}(x) = x^0 = 1;$$
$$\frac{d}{dx}(????) = x^{-1} = \frac{1}{x};$$
$$\frac{d}{dx}(-x^{-1}) = x^{-2};$$
$$\frac{d}{dx}\left(\frac{x^{-2}}{2}\right) = x^{-3}.$$

From the above list of derivatives, we note that by differentiating any power function, we can never get the result x^{-1}. Thus, we can say that if there exists any function whose derivative is $(1/x)$, then such a function must be a new function other than a power function. We ask the question:

Is there any function whose derivative is $(1/x)$?

Note that, we have obtained the function $\log_e x$ whose derivative is $(1/x)$. Thus, $\log_e x$ is the *desired (new) function* that fills up the gap noticed above, i.e.,

$$\frac{d}{dx}(\log_e x) = \frac{1}{x}.$$

We call it the *natural logarithm function.*

Recall that the definition of logarithmic function was encountered in algebra and it was based on exponents. The properties of logarithms were then proved from the corresponding properties of exponents.

Definition 1 The *natural logarithmic function* denoted by $\ln$ (or $\log_e$) is defined by

$$\ln x = \log_e x = \int_1^x \frac{1}{t}dt, \quad x > 0.$$

The properties of logarithms can be proved by means of this definition. However, to understand this definition, we have to study the properties of definite integrals and the first fundamental theorem of calculus. These topics are discussed in later chapter.

Example 4 Differentiate $y = \log_e(x + a)$.
Solution: Consider $y = \log_e(x + a)$. Then

$$x + a = e^y. \tag{9.7}$$

Differentiating both sides of (9.7) with respect to y, we get

$$\frac{d}{dy}(x + a) = e^y$$

since

$$\frac{d}{dy}\left(e^y\right) = e^y.$$

This gives

$$\frac{dx}{dy} = e^y = x + a.$$

Now, for reverting to the original function, we use the formula

$$\lim_{x\to 0} \frac{1}{x} \cdot \log_a(1+x).$$

We get

$$\frac{dy}{dx} = \frac{1}{x+a}.$$

Note that $\frac{dx}{dy} = x + a = e^y \neq 0$. Thus, for $y = \log_e(x+a)$, we have

$$\frac{dy}{dx} = \frac{1}{x+a}.$$

□

Example 5 Differentiate $y = \log_a x$.
Solution: First, we must change $\log_a x$ to natural logarithms (why?). We get

$$\begin{aligned} y &= \log_e x \cdot \log_a e && \text{where } \log_a e \text{ is a constant} \\ &= \log_e x \cdot \frac{1}{\log_e a}. \end{aligned} \tag{9.8}$$

Thus, differentiating both sides of (9.8) with respect to y, we get

$$\frac{dy}{dx} = \frac{1}{x} \cdot \frac{1}{\log_e a} \qquad \text{where } 1/\log_e a \text{ is constant.}$$

□

Example 6 Find the derivative of $y = \log_{10} x$.
Solution: We have

$$\frac{dy}{dx} = \frac{d}{dx}(\log_{10} x) = \frac{1}{x} \cdot \frac{1}{\log_e 10} = \frac{0.4343}{x}.$$

because

$$\frac{1}{\log_e 10} = \log_{10} e = 0.4343.$$

□

9.3.2 Finding the derivative of the exponential function

To find the derivative of the exponential function

$$y = a^x, \quad (a > 0,\ a \neq 1) \tag{9.9}$$

is not a very trivial task.

Taking the natural logarithm of both the sides of (9.9), we get

$$\log_e y = x \log_e a.$$

Thus,

$$x = \log_e y \cdot \frac{1}{\log_e a}. \tag{9.10}$$

Note that, here the independent variable is y; hence, differentiating both the sides of (9.10) with respect to y, we get

$$\frac{dx}{dy} = \frac{1}{y} \cdot \frac{1}{\log_e a} = \frac{1}{a^x} \cdot \frac{1}{\log_e a}.$$

Now, reverting to the original function, we get

$$\frac{dy}{dx} = \frac{1}{(dx/dy)} = a^x \cdot \log_e a.$$

Thus, for the function $y = a^x$ $(a > 0, a \neq 1)$, we have

$$\frac{dy}{dx} = \frac{d}{dx}(a^x) = a^x \cdot \log_e a.$$

Remark 1 We have obtained the following results:

$$\frac{d}{dx}(e^x) = e^x \tag{9.11}$$

$$\frac{d}{dx}(a^x) = a^x \cdot \log_e a \tag{9.12}$$

$$\frac{d}{dx}(\log_e x) = \frac{1}{x} \tag{9.13}$$

$$\frac{d}{dx}(\log_a x) = \frac{1}{x} \cdot \log_e a \tag{9.14}$$

From the above results (9.11) - (9.14), observe that the derivatives of exponential and logarithmic functions assume simplest forms, if the number e is chosen as the base. Also, note that for any constant

$$\frac{d}{dx}(e^{kx}) = ke^{kx}.$$

and

$$\begin{aligned}
\frac{d}{dx}(a^{kx}) &= \frac{d}{dx}(b^x) && \text{where } b = a^k \\
&= b^x \log_e b && \text{Using (9.2)} \\
&= a^{kx} \cdot \log_e a^k && \\
&= ka^{kx} \cdot \log_e a. && \text{Using } \log_e a^k = k \log_e a
\end{aligned}$$

Note 1 We have obtained the derivatives of exponential functions (e^x and a^x) and those of logarithmic functions ($\log_e x$ and $\log_a x$) using the special property that $(d/dx)(e^x) = e^x$ and (the relationship) that the functions a^t and $\log_a t$ (with the same base) are mutually inverse. In fact, this is an indirect approach by which we could obtain their derivatives. Subsequently, we shall obtain the derivatives of these functions by applying the *definition of derivative*, see Chapter 11.

Note 2 Let us derive the change of base formula for logarithms:

$$\log_a x = \frac{\log_b x}{\log_b a}. \tag{9.15}$$

Solution: Let us consider the logarithmic function

$$\log_a x = y.$$

Now we write that in exponential form by means of raising the base a to the exponent on each side of the above function:

$$x = a^y.$$

Our aim here is solve this equation for y, using only base b logs, not base a logs. To accomplish this, we take the log of each side:

$$\log_b x = \log_b a^y.$$

Now we simplify the right side:

$$\log_b x = y \log_b a,$$

where the exponent y brings down times $\log_b a$.

To get y by itself, we just have to divide both sides by $\log_b a$, where $\log_b a \neq 0$:

$$\frac{\log_b x}{\log_b a} = y.$$

Substituting $\log_a x$ back in for y, we have:

$$\log_a x = \frac{\log_b x}{\log_b a}.$$

$\square$

9.4 An important standard limit

To prove the following result,

$$\lim_{x\to 0} \frac{a^x - 1}{x} = \log_e a,$$

we shall prove the following prerequisite results:

Theorem 2

$$\lim_{x\to 0}\frac{\log_a(1+x)}{x}=\frac{1}{\log_e a}.$$

Proof: Consider

$$\begin{aligned}
\lim_{x\to 0}\frac{\log_a(1+x)}{x} &= \lim_{x\to 0}\frac{1}{x}\cdot\log_a(1+x) \\
&= \lim_{x\to 0}\log_a(1+x)^{1/x} \\
&= \log_a\left(\lim_{x\to 0}(1+x)^{1/x}\right) \\
&= \log_a e && \text{Using (9.1)} \\
&= \frac{1}{\log_e a}. && \text{By change of base}
\end{aligned}$$

□

Remark 2 From (9.15), by letting $x=e$ and $b=e$,, we have

$$\log_a e=\frac{\log_e e}{\log_e a}=\frac{1}{\log_e a}.$$

Corollary 3

$$\lim_{x\to 0}\frac{\log_e(1+x)}{x}=\log_e e=1.$$

Theorem 4

$$\lim_{x\to 0}\frac{a^x-1}{x}=\log_e a$$

Proof: Putting $a^x-1=y$, we have

$$a^x=1+y$$

or from the definition of logarithm, we have

$$x=\log_a(1+y).$$

Also, as $x\to 0, y\to 0$ (because as $x\to 0, a^x-1=y\to 0$), we have

$$\begin{aligned}
\lim_{x\to 0}\frac{a^x-1}{x} &= \lim_{y\to 0}\frac{y}{\log_a(1+y)}=\frac{1}{\lim\limits_{y\to 0}\dfrac{\log_a(1+y)}{y}} && \text{By Theorem 2} \\
&= \frac{1}{\log_a e}=\log_e a. && \text{By change of base}
\end{aligned}$$

Thus,

$$\lim_{x\to 0}\frac{a^x-1}{x}=\log_e a. \tag{9.16}$$

□

Corollary 5

$$\lim_{x\to 0}\frac{e^x-1}{x}=\log_e e=1.$$

Example 7 Two examples are given as follows:

- $$\lim_{x\to 0}\frac{7^x-1}{x}=\log_e 7.$$
- $$\lim_{x\to 0}\frac{e^x-1}{x}=\log_e e=1.$$

□

Remark 3 Note that if $f(x)\to 0$, as $x\to 0$ and $k\neq 0$, then $t=k\cdot f(x)\to 0$ as $x\to 0$. Thus,

$$\lim_{x\to 0}\frac{a^{kf(x)}-1}{kf(x)}=\lim_{t\to 0}\frac{a^t-1}{t}=\log_e a.$$

9.4.1 Derivative of the exponential function a^x using the definition of derivative

Theorem 6

$$\frac{d}{dx}(a^x)=a^x\cdot\log_e a,\quad a>0,\ a\neq 1.$$

Proof: Let

$$f(x)=a^x,\ a>0,\ a\neq 1.$$

Then

$$f(x+h)=a^{x+h}.$$

Now,

$$\begin{aligned}
\frac{d}{dx}(a^x)&=\frac{d}{dx}f(x)\\
&=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &&\text{(Definition of derivative)}\\
&=\lim_{h\to 0}\frac{a^{x+h}-a^x}{h}\\
&=\lim_{h\to 0}\frac{a^x(a^h-1)}{h}\\
&=a^x\lim_{h\to 0}\frac{(a^h-1)}{h}\\
&=a^x\cdot\log_e a. &&\text{By (9.16)}
\end{aligned}$$

Thus,

$$\frac{d}{dx}(a^x)=a^x\cdot\log_e a. \tag{9.17}$$

□

Example 8 Let $f(x) = 2^x$. Then

$$\frac{d}{dx}(2^x) = 2^x \cdot \log_e 2.$$

Example 9 Let $f(x) = e^x$. Then

$$\begin{aligned} \frac{d}{dx}(e^x) &= e^x \cdot \log_e e && \text{Use } \log_e e = 1 \\ &= e^x. \end{aligned}$$

Recall that earlier we had proved this result by differentiating both sides of the result.

Alternative proof:

Using

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots ,$$

the result (9.17) can also be obtained as follows:

We have

$$\begin{aligned} \frac{d}{dx}(a^x) &= \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} && \text{Provided the limit exists} \\ &= a^x \lim_{h\to 0} \frac{(a^h - 1)}{h} \end{aligned}$$

Now, putting $a^h = e^{h\cdot\log_e a}$, we obtain

$$\begin{aligned} \frac{d}{dx}(a^x) &= a^x \lim_{h\to 0} \frac{(e^{h\log_e a} - 1)}{h} \\ &= a^x \lim_{h\to 0} \frac{1}{h} \cdot \left[\left(1 + h\log_e a + \frac{(h\log_e a)^2}{2!} + \frac{(h\log_e a)^3}{3!} + \cdots \right) - 1 \right] \\ &= a^x [\log_e a + 0 + 0 + 0 + \cdots] && \text{Take } \lim_{h\to 0} \\ &= a^x \cdot \log_e a. \end{aligned}$$

□

Note 3 By expressing a^h in the form $e^{h\log_e a}$, we can expand it by using the *exponential series.*

9.4.2 Derivative of the logarithmic function $\log_a x$ using the definition of derivative

Corollary 7

$$\frac{d}{dx}(\log_a x) = \frac{1}{x} \cdot \frac{1}{\log_e a}.$$

Proof: Using the definition of derivative, we have

$$\begin{aligned}
\frac{d}{dx}(\log_a x) &= \lim_{h\to 0}\frac{\log_a(x+h)-\log_a x}{h} && \text{Provided the limit exists}\\
&= \lim_{h\to 0}\left[\frac{1}{h}\cdot\log_a\left(\frac{x+h}{x}\right)\right]\\
&= \lim_{h\to 0}\left[\frac{1}{h}\cdot\frac{x}{x}\cdot\log_a\left(1+\frac{h}{x}\right)\right]\\
&= \lim_{h\to 0}\left[\frac{1}{x}\log_a\left(1+\frac{h}{x}\right)^{x/h}\right].
\end{aligned}$$

Put $(h/x)=t$. Therefore, as $h\to 0, t\to 0$, we get

$$\begin{aligned}
\frac{d}{dx}(\log_a x) &= \frac{1}{x}\cdot\lim_{t\to 0}\left[\log_a(1+t)^{1/t}\right]\\
&= \frac{1}{x}\cdot\log_a e && \text{Because } \lim_{t\to 0}(1+t)^{1/t}=e\\
&= \frac{1}{x}\cdot\frac{1}{\log_e a}. && \text{By change of base}
\end{aligned}$$

Thus,

$$\frac{d}{dx}(\log_a x) = \frac{1}{x}\cdot\frac{1}{\log_e a}.$$

□

Example 10 Let $a=e$. Then,

$$\frac{d}{dx}(\log_e x) = \frac{1}{x}\cdot\frac{1}{\log_e e} = \frac{1}{x}$$

because $\log_e e = 1$. □

Corollary 8 If $y=\log_a[f(x)]$ $(a>0, a\neq 1)$. Then,

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\log_a[f(x)]\\
&= \frac{1}{f(x)\log_e a}\cdot f'(x). && \text{By the chain rule}
\end{aligned}$$

Example 11 Let $y=\log_e[f(x)]$. Then,

$$\frac{dy}{dx} = \frac{1}{f(x)\log_e e}\cdot f'(x) = \frac{f'(x)}{f(x)}$$

because $\log_e e = 1$.