1. (1 point) Consider the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n (7x+5)^n}{\sqrt{n+4}}$$

Find the center and radius of convergence R. If it is infinite, type "infinity" or "inf".

Center $a =$ ________

Radius $R =$ ________

2. (1 point) Consider the power series

$$\sum_{n=1}^{\infty} \frac{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)} x^n.$$

Find the radius of convergence R. If it is infinite, type "infinity" or "inf".

Answer: $R =$ ________

3. (1 point) Find the first three **nonzero** terms of the Taylor series for the function $f(x) = \sqrt{4x - x^2}$ about the point $x = 2$. (Your answers should include the variable x when appropriate.)

$\sqrt{4x - x^2} =$ ________ + ________ + ________ + ...

4. (1 point)

Write down the first 4 nonzero terms of the Taylor series generated by the function $f(x) = \sin(3x)$ at $x = 0$.

____

5. (1 point) Let $f(x) = \frac{5}{x} + 7$.

Compute

$f(x) =$ ________		$f(1) =$ ________	
$f'(x) =$ ________		$f'(1) =$ ________	
$f''(x) =$ ________		$f''(1) =$ ________	
$f'''(x) =$ ________		$f'''(1) =$ ________	
$f^{(iv)}(x) =$ ________		$f^{(iv)}(1) =$ ________	
$f^{(v)}(x) =$ ________		$f^{(v)}(1) =$ ________	

We see that the first term does not fit a pattern, but we also see that $f^{(k)}(1) =$ ________ for $k \geq 1$.

Hence we see that the Taylor series for f centered at 1 is given by

$f(x) = 12 + \sum_{k=1}^{\infty}$ ________ $(x-1)^k$.

6. (1 point)

Find the first four nonzero terms of the Taylor series about 0 for the function $f(x) = \sqrt{1+x}\sin(6x)$. Note that you may want to find these in a manner other than by direct differentiation of the function.

Write down your answer in ascending powers of x.

$\sqrt{1+x}\sin(6x) =$ ______ + ______ + ______ + ______ $+ \cdots$

7. (1 point)

Find the Taylor series generated by the function $f(x) = \frac{1}{(2x+4)^2}$ at $x = 0$.

$f(x) = \sum_{n=0}^{\infty}$ ________ x^n

8. (1 point)

Find the Taylor series generated by the function $f(x) = \frac{1}{1+x}$ at $x = 3$.

Hint: Write $f(x) = \frac{1}{1+x}$ as $f(x) = \frac{1}{4+(x-3)} = \frac{1}{4}\left(\frac{1}{1+\frac{x-3}{4}}\right)$.

$f(x) = \sum_{n=0}^{\infty}$ ________ $(x-3)^n$

9. (1 point)

By considering $\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$, find the Taylor series generated by the function $\tan^{-1} x$ at $x = 0$.

$\tan^{-1} x = \sum_{n=0}^{\infty}$ ________ x^{2n+1}

Using the above, evaluate

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n} = 1 - \frac{1}{3 \cdot 3} + \frac{1}{5 \cdot 3^2} - \frac{1}{7 \cdot 3^3} + \cdots$$

Answer = ________

10. (1 point) Evaluate

$$\lim_{x \to 0} \frac{\ln(1-x) + x + \frac{x^2}{2}}{4x \sin(x^2)}$$

Hint: Consider power series up to degree > 3.

Answer: ________

11. (1 point) Evaluate

$$\lim_{x\to 0} \frac{\cos(2x^2) - e^{(2x^4)}}{\sin(x^4)}$$

Hint: Consider power series up to degree > 4.

Answer: ______________

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12. (1 point)

Find the Taylor polynomial $T_4(x)$ generated by the function $f(x) = e^{(-2x^2)}$ at $x = 0$.

$T_4(x) =$ _________

Using the above, approximate $\int_{-1/2}^{1/2} f(x)\,dx$.

$\int_{-1/2}^{1/2} f(x)\,dx \approx$ _________

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13. (1 point)

Let $f(x) = \ln(1+x)$.

By considering the Taylor polynomial $T_2(x)$ of degree 2 generated by $f(x)$ at the point $x = 0$, approximate $\ln(1.4)$ by $T_2(0.4)$.

$\ln(1.4) \approx$ _________

By Taylor's theorem, there exists $c \in (0, 0.4)$ such that the error of the above approximation is $\frac{f'''(c)}{3!}(0.4-0)^3$.

Hence, estimate the absolute error by giving an upper bound without c.

The absolute value $= \left|\frac{f'''(c)}{3!}(0.4-0)^3\right| \leq$ _________

Remark: Instead of putting a large upper bound, You should give the upper bound as sharp as you can.

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14. (1 point)

Let $f(x) = \sqrt{x}$.

By considering the Taylor polynomial $T_2(x)$ of degree 2 generated by $f(x)$ at the point $x = 100$, approximate $\sqrt{103}$ by $T_2(103)$.

$\sqrt{103} \approx$ _________

By Taylor's theorem, there exists $c \in (100, 103)$ such that the error of the above approximation is $\frac{f'''(c)}{3!}(103-100)^3$.

Hence, estimate the absolute error by giving an upper bound without c.

The absolute value $= \left|\frac{f'''(c)}{3!}(103-100)^3\right| \leq$ _________

Remark: Instead of putting a large upper bound, You should give the upper bound as sharp as you can.

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15. (1 point)

Let 'n' be a positive integer. Let 'f(x)' be a function whose 'n'-th derivative is continuous at 'x = 0'. Let:

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^n.$$

Show that:

$$\lim_{x\to 0} \frac{f(x) - P_n(x)}{x^n} = 0$$

by dragging all relevant statements below into the **right column** in an appropriate order.

(Leave all irrelevant or incorrect statements in the left column.)

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0. by L'Hopital's Rule we have: $\lim_{x\to 0} \frac{f^{(n-1)}(x) - P_n^{(n-1)}(x)}{x}$ $= \lim_{x\to 0} \frac{f^{(n)}(x) - P_n^{(n)}(x)}{1} = 0.$
1. Since $\lim_{x\to 0} f^{(n)}(x) = 0$,
2. Hence, the limit $\lim_{x\to 0} \frac{f^{(k)}(x) - P_n^{(k)}(x)}{x^k}$ corresponds to the indeterminate form $\frac{0}{0}$.
3. Since the limit $\lim_{x\to 0} \frac{f^{(n)}(x) - P_n^{(n)}(x)}{1} = 0$ exists,
4. For all $k > n$, we have $P_n^{(k)}(0) = 0$.
5. Applying L'Hopital's Rule again, we have: $\lim_{x\to 0} \frac{f^{(n-2)}(x) - P_n^{(n-2)}(x)}{\frac{1}{2}x^2} = \lim_{x\to 0} \frac{f^{(n-1)}(x) - P_n^{(n-1)}(x)}{x}$ $= 0.$
6. Repeating the process, we eventually obtain: $\lim_{x\to 0} \frac{f(x) - P_n(x)}{\frac{1}{n!}x^n} = \lim_{x\to 0} \frac{f'(x) - P_n'(x)}{\frac{1}{(n-1)!}x^{(n-1)}} = \cdots = 0,$
7. which implies that $\lim_{x\to 0} \frac{f(x) - P_n(x)}{x^n} = 0.$
8. First, observe that for $1 \leq k \leq n$, we have $\lim_{x\to 0} P_n^{(k)}(x) =$ $P_n^{(k)}(0) = f^{(k)}(0) = \lim_{x\to 0} f^{(k)}(x).$

Does the statement remain true if 'f(x)' is only 'n'-time differentiable at 'x = 0', but not necessarily continuous there?