Lecture 3

Time Value of Money II
Multiple Cash Flows

Instructor: Prof. Chen (Alison) Yao
CUHK Business School

Last Lecture

- Single Cash Flow
  - Future Value
    - Simple interest
    - Compound interest
      - Multiple compounding
      - Continues compounding
  - Present Value

Lecture Outline

- Multiple Cash Flows
  - Future and Present Values of Multiple Cash Flows
  - Valuing Level Cash Flows
    - Annuities
      - Ordinary annuities
      - Annuity due
    - Perpetuities

- Using financial calculator

- How Interest Rates are Presented

Multiple Cash Flows – Future Value

- Suppose you plan to deposit $100 in Year 1, $200 in Year 2 and $300 Year 3. How much will be in the account in five years if the interest rate is 7%? Assume interest is compounded annually.

Notes about cash flow timing:

- It is implicitly assume that cash flows occur at the *end* of each period
- We are at Time 0

Multiple Cash Flows – Present Value

- You are offered an investment that will pay you $200 in one year, $400 the next year, $600 the next year and $800 at the end of the fourth year. You can earn 12 percent on very similar investments. Assume interest is compounded annually. What is the most you should pay for this one?

Lecture Outline

- Multiple Cash Flows
  - Future and Present Values of Multiple Cash Flows
  - Valuing Level Cash Flows
    - Annuities
      - Ordinary annuities
      - Annuity due
    - Perpetuities

- Using financial calculator

- How Interest Rates are Presented

Annuities and Perpetuities Defined

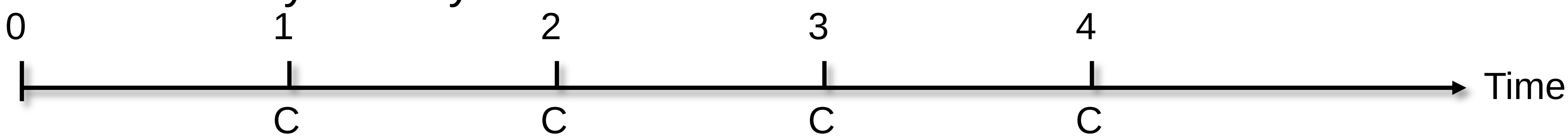
- Annuity – finite series of equal payments that occur at regular intervals
  - If the first payment occurs at the end of the period, it is called an ordinary annuity
  - If the first payment occurs at the beginning of the period, it is called an annuity due

- Perpetuity – infinite series of equal payments

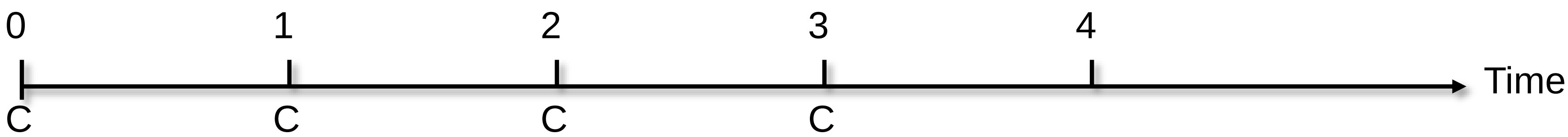
- Growing Perpetuity- infinite series of payments, payment in each period grows at rate *g*

Comparison of Cash Flow Timing

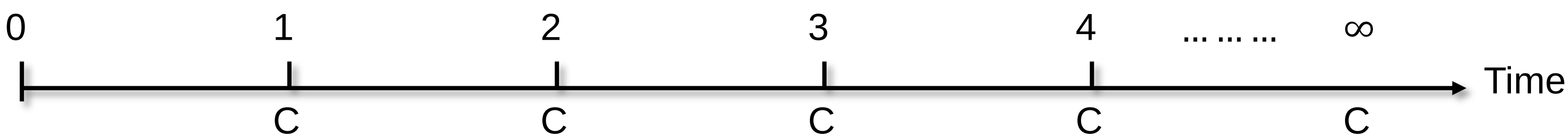
- **Annuity**
  - ordinary annuity



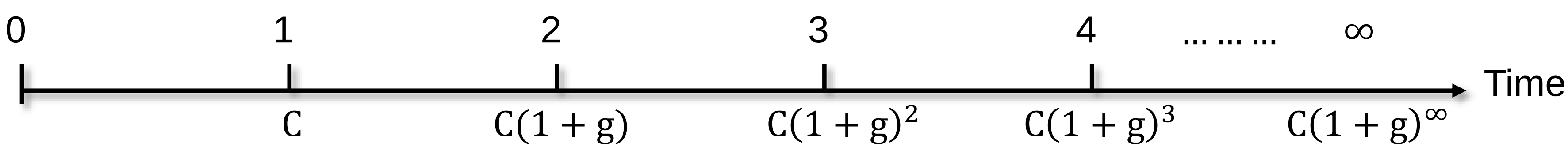
  - annuity due



- **Perpetuity**



- **Growing Perpetuity**



Unit Ordinary Annuity

Assume that cash flow C equals one in each period.

$$PV = \frac{1}{(1+r)} + \frac{1}{(1+r)^2} + \frac{1}{(1+r)^3} + \cdots + \frac{1}{(1+r)^t}$$

- t is the number of payments.
- Multiplying PV by $(1+r)$, this gives

$$PV \times (1+r) = 1 + \frac{1}{(1+r)} + \frac{1}{(1+r)^2} + \cdots + \frac{1}{(1+r)^{t-1}}$$

- Now subtract PV from $PV(1+r)$. Most of the terms cancel and we are left with:

$$PV \times (1+r) - PV = 1 - \frac{1}{(1+r)^t}$$

- When $t \neq \infty$, then

$$PV \times r = 1 - \frac{1}{(1+r)^t}$$

- Which is:

$$PV = \frac{1}{r}\left[1 - \frac{1}{(1+r)^t}\right]$$

PV for Ordinary Annuity

Ordinary Annuity

- the present value of identical cash flows C paid at the end of each period for a total of *t* periods with interest rate *r* per period is

$$PV = C \times \frac{1}{r}\left[1 - \frac{1}{(1+r)^t}\right] = C \times PV_A(r;t)$$

- $PV_A(r;t) = \frac{1}{r}\left[1 - \frac{1}{(1+r)^t}\right]$ is the present value of a unit annuity which pays $1 per period

Notes:

- If there are *N* periods per year:
  - replace *r* with *r / N* and replace *t* with *T* × *N*
  - where T = number of years

PV Annuity: Example

Mrs Smith just won the grand prize in a lottery. The lottery company offers her the choice to either receive £175,000 in one lump sum payment, or get £12,000 every year for the rest of her life. Mrs Smith estimates that she will live for at least another 35 years. On their savings account, the Smith's earn on average 6% interest, paid annually. Which option should she choose?

PV Annuity: Example Cont'd

Continuing the Mrs Smith lottery example, now assume that the lottery company offers monthly payments of £1,000, and the bank pays interest at 6% per annum at monthly frequency. Does this change the decision?

FV for Ordinary Annuity

- the future value of identical cash flows *C* paid at the end of each period for a total of *t* periods with interest rate *r* per period is

$$FV = C \times \frac{(1+r)^t - 1}{r} = C \times FV_A(r;t)$$

- $FV_A(r;t) = \frac{(1+r)^t - 1}{r}$ is the future value of a unit annuity which pays $1 per period
- The expression for the future value is just the present value compounded up $FV = PV(1+r)^t$

Notes:

- If there are *N* periods per year:
  - replace *r* with *r / N* and replace *t* with *T* × *N*
  - where T = number of years

FV Annuity: Example

The Smith family just had their first child, Adam. They plan to make a deposit into an endowment fund to finance Adam's college education once he turns 18. the Smith's plan to make monthly payments into the fund. The fund pays interest at 6% as before, compounded monthly. How much must the Smith's pay into the fund each month to make sure that in 18 year's time they have the £200,000 they need to finance Adam's education?

Lecture Outline

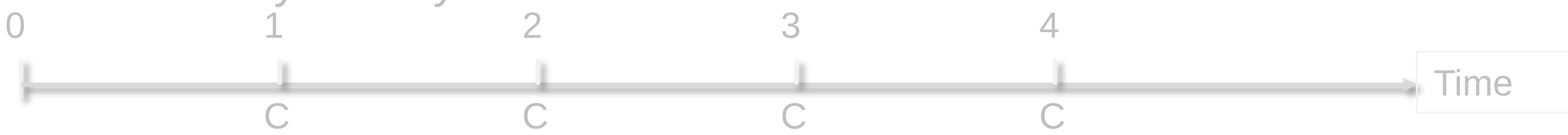
- Multiple Cash Flows
  - Future and Present Values of Multiple Cash Flows
  - Valuing Level Cash Flows
    - Annuities
      - Ordinary annuities
      - Annuity due
    - Perpetuities

- Using financial calculator

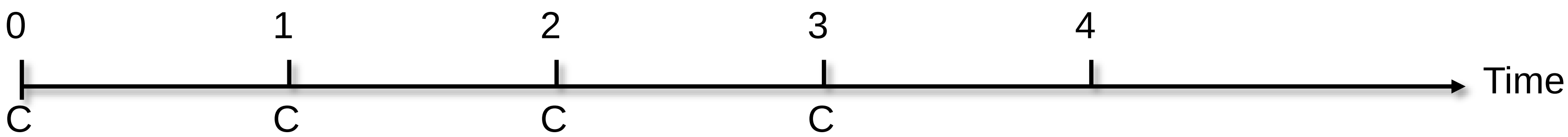
- How Interest Rates are Presented

Comparison of Cash Flow Timing

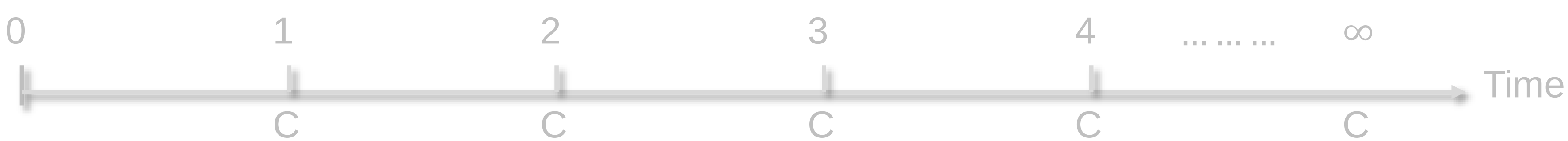
- **Annuity**
  - ordinary annuity



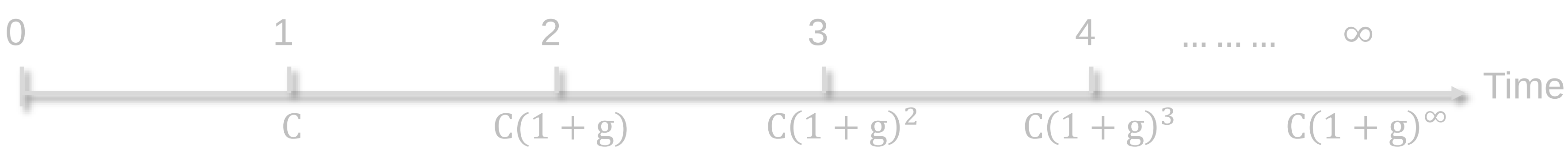
  - annuity due



- **Perpetuity**

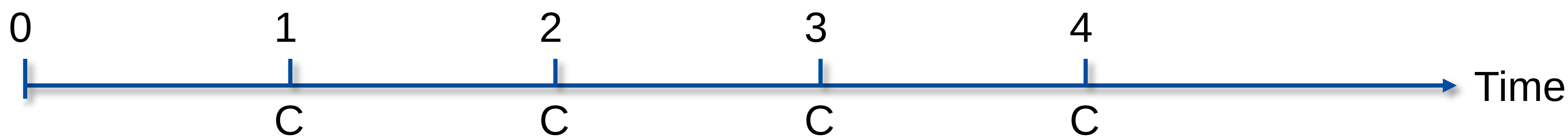


- **Growing Perpetuity**

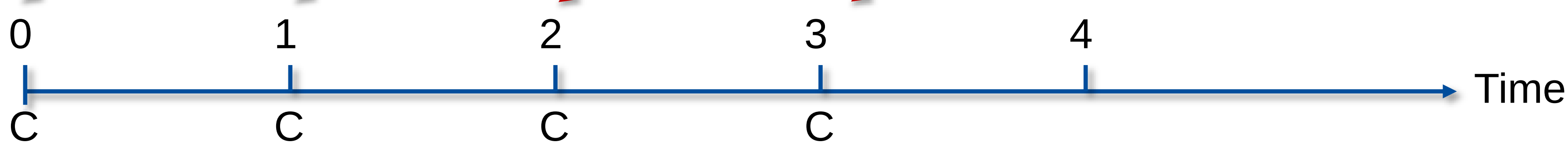


Annuity due

- Ordinary Annuity



- Annuity Due



- PV of Annuity Due $= \text{PV of Ordinary Annuity} \times (1+r)$

$$= C \times PV_A(r;t) \times (1+r) = C \times \frac{1}{r} \times \left[1 - \frac{1}{(1+r)^t}\right] \times (1+r)$$

Annuity Due: Example

Assume that you have $50,423 in your saving account. You will need to draw upon this for your university education in four equal annual payments. You would like to have the first payment now. Assume that the interest rate is 8 percent compounded annually. How large will each of the payment be?

Lecture Outline

- Multiple Cash Flows
  - Future and Present Values of Multiple Cash Flows
  - Valuing Level Cash Flows
    - Annuities
      - Ordinary annuities
      - Annuity due
    - Perpetuities
- Using financial calculator
- How Interest Rates are Presented

Comparison of Cash Flow Timing

- **Annuity**
  - ordinary annuity

0	1	2	3	4	Time
	C	C	C	C	

  - annuity due

0	1	2	3	4	Time
C	C	C	C		

- **Perpetuity**

0	1	2	3	4	… … …	∞	Time
	C	C	C	C		C	

- **Growing Perpetuity**

0	1	2	3	4	… … …	∞	Time
	C	$C(1+g)$	$C(1+g)^2$	$C(1+g)^3$		$C(1+g)^{\infty}$	

Unit Perpetuity

Assume that cash flow C equals one in each period.

$$PV = \frac{1}{(1+r)} + \frac{1}{(1+r)^2} + \frac{1}{(1+r)^3} + \cdots + \frac{1}{(1+r)^t}, \qquad t \to \infty$$

This present value PV is the sum of a geometric series, and it has a simple solution. Here is how to get it:

- Multiplying PV by $(1+r)$, this gives

$$PV \times (1+r) = 1 + \frac{1}{(1+r)} + \frac{1}{(1+r)^2} + \cdots + \frac{1}{(1+r)^{t-1}}$$

- Now subtract PV from $PV \times (1+r)$. Most of the terms cancel and we are left with:

$$PV \times (1+r) - PV = 1 - \frac{1}{(1+r)^t}$$

- If t is large, $\frac{1}{(1+r)^t}$ disappears and the solution is:

$$PV = \frac{1}{r}$$

PV for Perpetuity

Perpetuity

- the present value of a perpetual stream of identical cash flows C paid at the end of each period with interest rate *r* per period is

$$PV = C \times \frac{1}{r} = C \times PV_P(r)$$

- $PV_P(r) = \frac{1}{r}$ is the present value of a unit perpetuity which pays $1 per period

Notes:

- If there are *N* periods per year:
  - replace *r* with *r / N* and replace *t* with *T × N*
  - where T = number of years

Perpetuity: Example

An investment offers a perpetual cash flow of $500 every year staring from the end of the period. The return you require on such an investment is 8 percent. What is the present value of this investment?

Comparison of Cash Flow Timing

- **Annuity**
  - ordinary annuity

0	1	2	3	4	Time
	C	C	C	C	

  - annuity due

0	1	2	3	4	Time
C	C	C	C		

- **Perpetuity**

0	1	2	3	4	… … …	∞	Time
	C	C	C	C		C	

- **Growing Perpetuity**

0	1	2	3	4	… … …	∞	Time
	C	$C(1+g)$	$C(1+g)^2$	$C(1+g)^3$		$C(1+g)^{\infty}$	

Unit Growing Perpetuity

Imagine that the dividend grows at the rate g every year. Instead of PV above, we would have:

$$PV_{GP} = \frac{1}{(1+r)} + \frac{(1+g)}{(1+r)^2} + \frac{(1+g)^2}{(1+r)^3} + \cdots + \frac{(1+g)^{t-1}}{(1+r)^t}, \qquad t \to \infty$$

This also has a neat solution. This is the way that we get it:

- First multiply PV_{GP} by $(1+r)$ as we did before to obtain

$$PV_{GP} \times (1+r) = 1 + \frac{(1+g)}{(1+r)} + \frac{(1+g)^2}{(1+r)^2} + \cdots + \frac{(1+g)^{t-1}}{(1+r)^{t-1}}$$

- In addition, on this occasion, we separately multiply PV_{GP} by $(1+g)$ to obtain:

$$PV_{GP} \times (1+g) = \frac{(1+g)}{(1+r)} + \frac{(1+g)^2}{(1+r)^2} + \frac{(1+g)^3}{(1+r)^3} \ldots + \frac{(1+g)^t}{(1+r)^t}$$

- Now subtracts $PV_{GP} \times (1+g)$ from $PV_{GP} \times (1+r)$:

$$PV_{GP} \times (1+r) - PV_{GP} \times (1+g) = 1 - \frac{(1+g)^t}{(1+r)^t}$$

- If the growth rate g is less than the rate of interest r, $\frac{(1+g)^t}{(1+r)^t}$ will disappear for large t and we have:

$$PV_{GP} = \frac{1}{r-g}$$

PV for Growing Perpetuity

Growing perpetuity

- the present value of a perpetual stream of cash flows C_t (t = 1, 2, 3 …) where $C_{t+1} = C_t \times (1 + g)$ is paid at the end of each period with interest rate *r* per period is

$$PV = C \times \frac{1}{r-g} = C \times PV_{GP}(r;g)$$

- $PV_{GP}(r;g) = \frac{1}{r-g}$ is the present value of a unit growing perpetuity which pays $1 per period

Notes:

- If there are *N* periods per year:
  - replace *r* with *r* / *N* and replace *t* with *T* × *N*
  - where T = number of years

Lecture Outline

- Multiple Cash Flows
  - Future and Present Values of Multiple Cash Flows
  - Valuing Level Cash Flows
    - Annuities
      - Ordinary annuities
      - Annuity due
    - Perpetuities

- Using financial calculator

- How Interest Rates are Presented

Example

Suppose we are examining an asset that promises to pay $500 at the *end* of each of the next three years. The cash flows from this asset are in the form of a three-year, $500 annuity. If we want to earn 10 percent on our money, how much would we offer for this annuity?

$$PV = \frac{500}{1.1} + \frac{500}{1.1^2} + \frac{500}{1.1^3} = \$1,243.43$$

Financial Calculator Solution

- N: 3 periods (enter as 3)
- I/Y: 10% interest rate per period (*enter as 10 NOT 0.1*)
- FV: 0 (no lump sum FV)
- PMT: 500 annuity (enter as 500)
- PV: compute (resulting answer is negative)

Example

You ran a little short on your spring break vacation, so you put $1,000 on your credit card. You can afford only the minimum payment of $20 per month. The interest rate on the credit card is 1.5 percent per month. How long will you need to pay off the $1,000?

$$Annuity\ PV = C \times \frac{1}{r}\left[1 - \frac{1}{(1+r)^t}\right]$$

$$1{,}000 = 20 \times \frac{1}{0.015}\left[1 - \frac{1}{(1+0.015)^t}\right]$$

$t = 93.11$ months

Financial Calculator Solution

- N: number of *month* to pay back
- I/Y: interest rate per *month*
- PV: 1,000 (you *receive* $1,000 today)
- PMT: −20 (you *pay* $20 per month)
- FV:0 (you have paid off the loan)

Lecture Outline

- Multiple Cash Flows
  - Future and Present Values of Multiple Cash Flows
  - Valuing Level Cash Flows
    - Annuities
      - Ordinary annuities
      - Annuity due
    - Perpetuities

- Using financial calculator

- How Interest Rates are Presented

Two Ways Interest Rates Are Presented

1). Annual Percentage Rate (APR)

- APR the annual rate that is quoted by law, and the rate *r* we have discussed so far

- APR is the *quoted* rate, not the actual interests in one year

- The amount of actual interests in one year also depends on the compounding frequency

- By definition, APR = period rate × the number of periods per year

- Consequently, to get the period rate we rearrange the APR equation:
  - Period rate = APR / number of periods per year

Example

You place $125,000 on deposit with a bank today, for a total period of 15 years. The bank guarantees a fixed rate of interest of 3% for the entire tenor. How much will you have 15 years from now, if interest is paid

a) with annual compounding?

b) with monthly compounding?

c) with continuous compounding?

*Unless otherwise specified, the stated interest rate represents the **APR***

We have $r = 3\%$, and $T = 15$ for all three cases

a) For annual compounding, N=1

$$FV = PV \times \left(1 + \frac{r}{N}\right)^{N\times T} = 125{,}000 \times (1 + 0.03)^{15} = 194{,}746$$

b) For monthly compounding, N=12

$$FV = PV \times \left(1 + \frac{r}{N}\right)^{N\times T} = 125{,}000 \times (1 + 0.03/12)^{15\times 12} = 195{,}928$$

b) For continuous compounding, N is infinitely (arbitrarily large)

$$FV = PV \times \lim_{N\to\infty} \left(1 + \frac{r}{N}\right)^{N\times T} = PV \times e^{r\times T} = 125{,}000 \times e^{0.03\times 15} = 196{,}038$$

Credit Card APR

- Credit card interest rate is quoted on APR
  - most credit card issuers calculate and charge interest on a periodic basis, e.g. daily, monthly

Eg.

29.90 percent!!!

Annual Percentage Rate (APR) for Purchases	0% introductory APR for one year (which is a daily periodic rate of 0%). After that, the APR will be 29.9% (which is a daily periodic rate of .08191%). This APR will vary with the market based on the Prime Rate plus a margin of 26.65 percentage points.
APR for Cash Advances and Balance Transfers*	29.9% (which is a daily periodic rate of .08191%). This APR will vary with the market based on the Prime Rate plus a margin of 26.65 percentage points.
Paying Interest	Your due date is at least 21 days after the close of each billing cycle. We will not charge you any interest on purchases if you pay your entire balance by the due date each month. We will begin charging interest on cash advances and balance transfers on the transaction date.

Daily periodic rate (interests are compounded daily)

- $\text{Daily Periodic Rate}(DPR) = \frac{APR}{365} = 0.08191\%$
- Actual annual interests you pay:

$$(1 + 0.08191\%)^{365} - 1 = 34.8\%$$

Two Ways Interest Rates Are Presented

2). Effective Annual Rate (EAR)

This is the actual rate paid (or received) after accounting for compounding that occurs during the year

Example

You place $125,000 on deposit with a bank today, for a total period of 15 years. The bank guarantees a fixed rate of interest of 3% for the entire tenor. If interest is paid monthly, what is the effective annual rate?

Things to Remember

- You ALWAYS need to make sure that the interest rate and the time period match.
  - If you are looking at annual periods, you need an annual rate.
  - If you are looking at monthly periods, you need a monthly rate.

- If you have an APR based on monthly compounding, you have to use monthly periods, or adjust the interest rate appropriately if you have payments other than monthly

Summary

- Future value and present value (single cash flow)
  - $FV_t = PV_t \times (1 + r)^t$

- Multiple Cash Flows (equal payments of C):
  - **Ordinary Annuity**: equal payments for a set number of periods, the first cash flow occurs one period from now.
    - $Annuity\ PV = C \times PV_A(r;t) = C \times \left[\frac{1}{r} - \frac{1}{r(1+r)^t}\right]$
    - $Annuity\ FV = C \times FV_A(r;t) = C \times \left[\frac{(1+r)^t}{r} - \frac{1}{r}\right]$
  - **Annuity Due**: the first cash flow occurs immediately.
    - $Annuity\ Due\ PV = Annuity\ PV \times (1 + r)$
    - $Annuity\ Due\ FV = Annuity\ FV \times (1 + r)$
  - **Perpetuity**: equal payments that are paid forever.
    - $Perpetuity\ PV = PV_P(r) = C \times \frac{1}{r}$