THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1510 Calculus for Engineers (Fall 2021)
Midterm examination

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Class: MATH1510 G1

I acknowledge that I am aware of University policy and regulations on honesty in academic work, and of the disciplinary guidelines and procedures applicable to breaches of such policy and regulations, as contained in the website http://www.cuhk.edu.hk/policy/academichonesty/

Signature　　Date 23-10-2021

- The testing time of this exam is 90 minutes.
- There are a total of 150 points and 25 questions.
- Please go to the class indicated by your registered course code via the CUSIS system. Failure to comply will result in a **5-point deduction** of the final score.
- Only calculators without graphing, calculus capabilities are allowed. The teachers have the final discretion on any disputes.
- For the short questions, write down the answers in the given boxes. No justification is required. Partial credit is available for some of the questions.
- For the long questions, points will only be awarded for answers with sufficient justifications.
- Please submit your answer to Gradescope through the centralized course MATH1510A-I in Blackboard.
- In Gradescope, for each question, please indicate exactly which page(s) its answer locates. **There will be penalty for answers of incorrectly matched questions.**

$x > \frac{7}{2}$ $x > -\frac{5}{3}$

$2x-7>0$ & $3x+5>0$

or $2x-7<0$ & $3x+5<0$

$x < \frac{7}{2}$ $x < -\frac{5}{3}$

Short Questions

Each of question 1-18 is worth 3 points.

1. Find the domain of the function

$$f(x) = \ln((2x-7)(3x+5))$$

Answer: $\boxed{(-\infty, -\frac{5}{3}) \cup (\frac{7}{2}, \infty)}$

2. Find the range of the function

$$f(x) = x^2 - 1$$

with domain $[-2, 6]$. $=(x-1)(x+1)$

Answer: $\boxed{[-1, 35]}$

3. Which of the following functions have minimum value on the specified intervals? If none of them has minimum value, write NONE.

(a) $f(x) = \dfrac{1}{x}$ on $[1, \infty)$. X

(b) $g(x) = (x-3)e^x \sin x$ on $[-3, 7]$. ✓

(c) $h(x) = x$ on $(0, 4]$. X

Answer: $\boxed{(b),}$

4. Let

$$f(x) = \begin{cases} |x| & \text{if } |x| \geq 1; \\ 2x-3 & \text{if } 0 \leq x < 1; \\ x^2 & \text{if } -1 < x < 0. \end{cases}$$

Write down all the point(s) on $\mathbb{R}$ where $f(x)$ is not continuous. If there is no such point, write NONE.

Answer: $\boxed{\text{NONE}}$

$\cancel{\lim_{x \to 1^-}} = 2-3 = -1$ $\cancel{\lim_{x \to 1^+}} = 1$ $\lim_{x \to 0^-} = 0$ $\lim_{x \to 0^+} 0-3 = -3$

$\lim_{x\to 1^-} = 1-a \qquad \lim_{x\to 1^+} = a+2$

5. Let

$$f(x) = \begin{cases} ax+2 & \text{if } x \geq 1; \\ x-a & \text{if } x < 1. \end{cases}$$

$1-a = a+2$

$-1 = 2a$

Find all the value(s) of a so that $f(x)$ is continuous on $\mathbb{R}$.

Answer: $a = -\frac{1}{2}$

6. Find $\frac{dy}{dx}$ if

$$y = \frac{x\sin x}{2-\cos x}$$

$\frac{(\sin x + x\cos x)(2-\cos x) - x\sin^2 x}{(2-\cos x)^2}$

Answer: $\frac{2\sin x - \sin x\cos x + 2x\cos x - x}{(2-\cos x)^2}$

7. Find $\frac{dy}{dx}$ if

$$y = \sin(\cos(\sin x))$$

$\frac{dy}{dx} = \cos(\cos(\sin x))\ x - \sin(\sin x)\ x \cos x$

Answer: $-(\cos x)(\sin(\sin x))[\cos(\cos(\sin x))]$

8. Find $\frac{dy}{dx}$ if

$$y = \arctan x + e^x \arcsin x, \quad \text{where } x \in (-1,1)$$

Answer: $\frac{1}{1+x^2} + e^x\left(\arcsin x + \frac{1}{\sqrt{1-x^2}}\right)$

9. Find $\frac{dy}{dx}$ if

$$y = e^{\sin x}\ln x$$

Answer: $\frac{e^{\sin x}}{x} + \ln x \cdot (\cos x) e^{\sin x}$

$\sin^{-1} x = y$

$x = \sin y$

$1 = \cos y \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

$f'(x) = -\sin x$ $\cancel{f^{(4)}}$

$f''(x) = -\cos x$

$f'''(x) = \sin x$ $f^{(4)}(x) = \cos x$

10. Let $f(x) = \cos x$. Find $f^{(1510)}\left(\frac{\pi}{6}\right)$. $\rightarrow -\cos\left(\frac{\pi}{6}\right)$

Answer: $-\frac{\sqrt{3}}{2}$

11. Let $f(x) = (x^2 + x + 1)e^{ax}$ for some $a > 0$. Find $f^{(8)}(x)$.

Answer: $(x^2+x+1)a^8e^{ax} + (16x+8)a^7e^{ax} + 56a^6a^{ax}$

12. Evaluate

$$\lim_{t\to\infty} \frac{\log_3(t+1)}{\ln(3t)}$$

$$\frac{\lim_{t\to\infty} \frac{1}{(t+1)\ln 3}}{\lim_{t\to\infty}\left(\frac{1}{t}\right)} = \frac{t}{(t+1)\ln 3} = \frac{1}{\ln 3\,(1)}$$

Answer: $\frac{1}{\ln 3}$

13. Evaluate

$$\lim_{x\to 1} \left(\tan\left(\frac{\pi x}{2}\right)\ln x\right)$$

$$\frac{\ln x}{\cot\left(\frac{\pi x}{2}\right)} = \frac{\frac{1}{x}}{-\csc^2\left(\frac{\pi x}{2}\right)\left(\frac{\pi}{2}\right)} = \frac{2\sin^2\left(\frac{\pi x}{2}\right)}{-\pi x} = \frac{2}{\pi}$$

Answer: $-\frac{2}{\pi}$

14. Evaluate

$$\lim_{x\to\infty} \frac{ae^x + e^{-ax}}{be^x + e^{-bx}},$$

where $a, b > 0$.

$$\frac{a + e^{-2ax}}{b + e^{-2bx}}$$

Answer: $\frac{a}{b}$ //

15. Evaluate

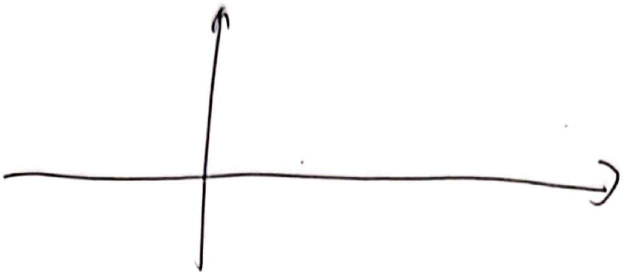
$$\lim_{x\to\infty} \frac{(ax)^2 + x\sin(\pi x)}{bx^2 + c},$$

where $a, b, c \neq 0$.

$$\frac{a^2x^2 + x\sin(\pi x)}{bx^2 + c}$$

Answer:

$$= \frac{2a^2 + \pi\cos(\pi x) + \pi\cos(\pi x) - x\pi^2\sin(\pi x)}{2b} = \frac{2a^2x + \sin(\pi x) + x\pi\cos(\pi x)}{2bx}$$



16. Given that

$$\lim_{x\to\infty} f(x) = a, \quad \lim_{x\to-\infty} f(x) = b, \quad \lim_{x\to 0^+} f(x) = c, \quad \lim_{x\to 0^-} f(x) = d$$

for some $a, b, c, d > 0$, evaluate

$$\lim_{x\to 0} f\left(\left|\frac{1}{x}\right|\right) = \lim_{x\to\infty} f(|x|)$$

Answer: $\boxed{a}$

17. Apply linearization of the function

$$f(x) = \sqrt{x}$$

at $x = 16$ to approximate $\sqrt{17}$.

$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$

$= \frac{1}{2\sqrt{x}}$

$x=16,\ y=4$

Answer: $\boxed{\frac{\sqrt{17}}{8} + 2 \approx 2.5154 \text{ (4 d.p.)}}$ //

$y = \frac{1}{8}x + 2$.

18. Let

$$C : x^3 + 5xy - 3y^2 = 3$$

be a curve. Find the equation of the tangent of C at the point $(1, 1)$.

Answer: $\boxed{y = 8x - 7}$

$3x^2 + 5y + 5x\left(\frac{dy}{dx}\right) - 6y\left(\frac{dy}{dx}\right) = 0$

$\frac{dy}{dx}(5x - 6y) = -3x^2 - 5y$

$\text{slope} = 8 \quad \frac{-3x^2 - 5y}{5x - 6y} = \frac{dy}{dx}$

$(1) \quad (1) \quad (-7)$

$y = 8x \cancel{+} -7$

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Long Questions

19. (12 points) Suppose that

$$f(x) = \begin{cases} \tan x - 1 & \text{if } x \in (-\frac{\pi}{2}, 0] \\ x^2 + ax + \cancel{b} & \text{if } x > 0, \end{cases}$$

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where a and b are real numbers.

Given that f is differentiable at $x = 0$, without using L'Hôpital's rule, find the values of a and b.

$\because f$ is differentiable at $x = 0$,

$\therefore f$ is continuous at $x = 0$:

$\therefore$ We have $\lim_{x\to 0^-} f(x) = \lim_{x \to 0^+} f(x)$

$0 - 1 = b$

$b = -1$ //

Also, Let $y(x) = \tan x - 1$ & $z(x) = x^2 + ax - 1$.

We have $y'(0) = z'(0)$

$\frac{1}{\cos^2(0)} \rightarrow$ $\sec^2(0) = 2(0) + a$

$a = 1$ //

20. (14 points) Let C be the curve defined by the equation

$$y^4 - x = xy + \cos x$$

Given that $A = (0, 1)$ is a point on C,

(a) Find $\left.\dfrac{dy}{dx}\right|_A$

(b) Find $\left.\dfrac{d^2y}{dx^2}\right|_A$

(a) By implicit differentiation,

$$4y^3\left(\frac{dy}{dx}\right) - 1 = y + x\left(\frac{dy}{dx}\right) - \sin x$$

$$\left(\frac{dy}{dx}\right)(4y^3 - x) = 1 + y - \sin x$$

$$\frac{dy}{dx} = \frac{1+y-\sin x}{4y^3 - x}$$

$$\left.\frac{dy}{dx}\right|_A = \frac{1+1-0}{4-0}$$

$$= \frac{1}{2} \quad //$$

(b)

$$\frac{d^2y}{dx^2} = \frac{\left(\frac{dy}{dx} - \cos x\right)(4y^3 - x) - (1+y-\sin x)\left(12y^2\frac{dy}{dx} - 1\right)}{(4y^3 - x)^2}$$

$$\left.\frac{d^2y}{dx^2}\right|_A = \frac{\left(\frac{1}{2} - 1\right)(4-0) - (1+1-0)\left(12\cdot\frac{1}{2} - 1\right)}{(4-0)^2}$$

$$= \frac{-2 - 2\times 5}{16}$$

$$= -\frac{3}{4} \quad //$$

21. (10 points) Show that the function

$$f(x) = x^4 - 5x + 1$$

has at least two real roots.

$\because$ $f(x)$ is a continuous function for a $\forall x \to \mathbb{R}$.

① $f(1) = -3$

$f(2) = 7$

By Intermediate value Theorem,

there exists a real value $c \in (1, 2)$ where

$f(c) = 0$.

② $f(0) = 1$

$f(1) = -3$

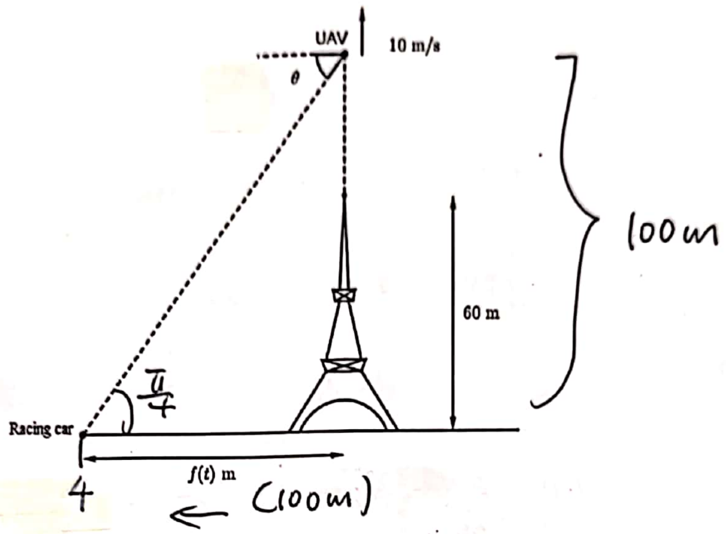
By Intermediate value Theorem,

there exists a real value $d \in (0, 1)$

where $f(d) = 0$.

From the above, the function has at least two real roots which are a c and d. //

22. (16 points) A racing car moves away from a tower, and the horizontal distance between the racing car and the tower after $t > 0$ seconds is given by a differentiable function $f(t)$ meter. To monitor the racing car, when it starts to move, an unmanned aerial vehicle (UAV) flies vertically upwards with constant speed 10 m/s from the top of a tower 60 meters in height.



Given that at $t = 4$, the angle of depression θ from the UAV to the racing car is $\frac{\pi}{4}$ radian.

(a) Find $f(4)$.

(b) At $t = 4$, the angle of depression θ decreases at a rate of 0.15 radian/s. Find the speed of the racing car at that moment.

(a) $\tan\frac{\pi}{4} = \frac{60 + 10\times 4}{f(4)}$

$f(4) = 100$ //

(b) $\frac{d}{d\theta}\tan\theta = \frac{10f(t) - (60+10t)f'(t)}{f'(t)}$

Sub $\theta = -0.15$, $t=4$ and $f(4) = 100$:

$\sec^2(-0.15) = \frac{10\times100 - 100\times f'(4)}{f'(4)}$

$f'(4)\sec^2(-0.15) = 1000 - 100f'(4)$

$$f'(4)\left(100 + \sec^2(-0.15)\right) = 1000$$

$$f'(4) = 9.8988\ m/s$$

(4 dps)

$$\frac{-2c(\frac{1}{})}{x^2}$$

23. (14 points) Suppose

$$f(x) = \begin{cases} 2x - \sin x & \text{if } x \le 0 \\ x^2 \cos\left(\frac{2}{x}\right) & \text{if } x > 0 \end{cases}$$

Find $D_{f'}$ (the domain of f') and $f'(x)$ for any $x \in D_{f'}$.

$$f'(x) = \begin{cases} 2 - \cos x & \text{if } x < 0 \\ \text{Undefined} & \text{if } x = 0 \\ 2x\cos\left(\frac{2}{x}\right) + x^2 \sin\left(\frac{2}{x}\right) \cdot \frac{2}{x^2} & \text{if } x > 0. \end{cases}$$

$$= \begin{cases} 2 - \cos x & \text{if } x < 0 \\ \text{Undefined} & \text{if } x = 0 \\ 2x\cos\left(\frac{2}{x}\right) + 2\sin\left(\frac{2}{x}\right) & \text{if } x > 0 \end{cases}$$

$\therefore D_{f'} = (-\infty, 0) \cup (0, \infty)$ //

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24. (20 points)

(a) Let k be a positive integer. Show that

$$\frac{1}{k+1} < \ln(k+1) - \ln(k) < \frac{1}{k}$$

(b) Let n be a positive integer. Show that

$$\ln(n+1) < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < 1 + \ln n$$

(c) By part (b), evaluate

$$\lim_{n\to\infty} \frac{1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}}{\ln n}.$$

(a) Let $f(x) = \ln x$

$f'(x) = \frac{1}{x}$.

By the mean value theorem, there exists a real value c where $c \in (a, b)$ and

$$f'(c) = \frac{f(b) - f(a)}{b-a},$$

given that $f(x)$ is continuous & differentiable.

$\therefore$ Sub $b = k+1$ & $a = k$,

we have: $\frac{1}{c} = \frac{\ln(k+1) - \ln k}{k+1-k}$

$\frac{1}{c} = \ln(k+1) - \ln k$

$\therefore$ Also, we have: $\frac{1}{b} < \frac{1}{c} < \frac{1}{a}$

$\therefore \frac{1}{k+1} < \ln(k+1) - \ln(k) < \frac{1}{k}$ //

(b) Sub n into k for (a).

$$\frac{1}{n+1} < \ln(n+1) - \ln(n) < \frac{1}{n}$$

$<$ $<$

$<$ $<$

(c) $\lim_{n \to \infty}$

25. (10 points) Given that

$$\lim_{x\to 0}(\sin(x^3)f(ax)) = \lim_{x\to 0}(\sin^3(x)g(bx)) = 1$$

for some $a, b \neq 0$, evaluate

$$\lim_{x\to 0}\frac{f(x)}{g(x)}$$

$$\lim_{x\to 0} \frac{\sin(x^3) f(ax)}{x^3} \times x^3 = 1$$

$\lim_{x\to 0} \frac{\sin x}{x} = 1 \rightarrow$

$$\lim_{x\to 0} f(ax) \cdot x^3 = 1$$

$$f(ax) = 0$$

$$f(x) = 0.$$

$$\lim_{x\to 0} \frac{\sin^3(x)\ g(bx) \cdot x^3}{x^3} = 1$$

$$\lim_{x\to 0} g(bx) \cdot x^3 = 1$$

$$g(bx) = 0$$

$$g(x) = 0$$

$$\therefore \lim_{x\to 0} \frac{f(x)}{g(x)} = \frac{0}{0}$$

= ONE //