THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1510 Calculus for Engineers (2020-2021)
Solution to Supplementary Exercise 3

Limits of Functions

1. Let $f(x) = x + 1$ and $g(x) = \dfrac{x^2 - 1}{x - 1}$.

   (a) State the domains of $f(x)$ and $g(x)$.

   (b) Fill in the blanks.
   $g(x)$ can be described in the following way:

   $$g(x) = \begin{cases} \underline{\hspace{2cm}} & \text{if} \quad x \neq 1, \\ \text{NOT defined} & \text{if} \quad x = \underline{\hspace{2cm}}. \end{cases}$$

   (c) Sketch the graphs of the functions $f(x)$ and $g(x)$.

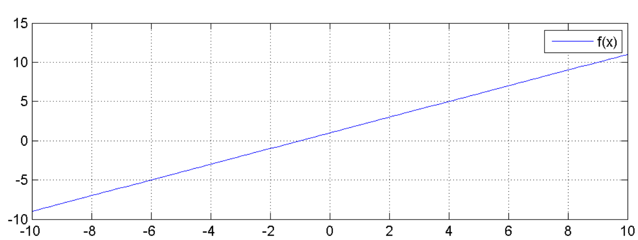
Ans:

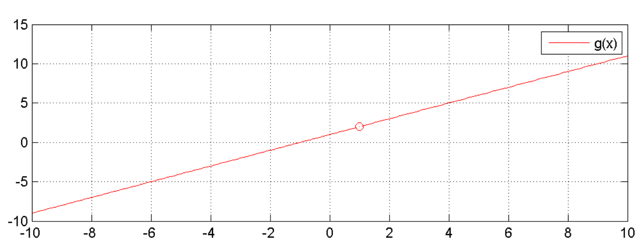
(a) Domain of $f(x) = \mathbb{R}$; Domain of $g(x) = \mathbb{R} \backslash \{1\}$.

(b) By (a), $g(x)$ can be described in the following way:

$$g(x) = \begin{cases} \underline{\quad x + 1 \quad} & \text{if} \quad x \neq 1, \\ \text{NOT defined} & \text{if} \quad x = \underline{\;\; 1 \;\;}. \end{cases}$$

(c) We have





(Remark: $f(x) = g(x)$ for all real numbers except $x = 1$.)

2. Let $f(x) = x + 1$ and $g(x) = \dfrac{x^2 - 1}{x - 1}$ for $x \neq 1$.

   Complete the following table.

x	0.9	0.99	0.999	0.9999	1	1.0001	1.001	1.01	1.1
$f(x)$									
$g(x)$									

   By observation, when x is getting closer and closer to 1, what values do $f(x)$ and $g(x)$ get closer and closer to? Hence, guess the value of $\lim_{x\to 1} f(x)$ and $\lim_{x\to 1} g(x)$.

   Ans:

x	0.9	0.99	0.999	0.9999	1	1.0001	1.001	1.01	1.1
$f(x)$	1.9	1.99	1.999	1.9999	2	2.0001	2.001	2.01	2.1
$g(x)$	1.9	1.99	1.999	1.9999	NOT defined	2.0001	2.001	2.01	2.1

   By observation, when x is getting closer and closer to 1, what both $f(x)$ and $g(x)$ get closer and closer to 2. It suggests (but not a formal proof) $\lim_{x\to 1} f(x) = 2$ and $\lim_{x\to 1} g(x) = 2$.

   Conclusion: When we consider $\lim_{x\to 1} f(x)$ and $\lim_{x\to 1} g(x)$, we only look at the values of $f(x)$ and $g(x)$ when x is close to 1, but we do NOT care whether $f(1)$ and $g(1)$ are well defined. However, $f(x) = g(x)$ everywhere except $x = 1$, so it is not surprising that $\lim_{x\to 1} f(x) = \lim_{x\to 1} g(x) = 2$.

3. Complete the following table. (Note: x is in radian)

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$							

   By observation, guess the value of $\lim_{x\to 0} \dfrac{\sin x}{x}$.

   Ans:

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	0.99833	0.99998333	0.9999998333	NOT defined	0.9999998333	0.99998333	0.99833

   By observation, it suggests (but not a formal proof) $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$.

   (Remark: $\dfrac{\sin x}{x}$ is not defined at $x = 0$ but $\lim_{x\to 0} \dfrac{\sin x}{x}$ exists.)

4. Let $f(x)$ be a function defined by

$$f(x) = \begin{cases} x + 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0 \end{cases}$$

(a) Complete the following table.

x	-0.1	-0.01	-0.001	-0.0001	0	0.0001	0.001	0.01	0.1
$f(x)$									

(b) Find $\lim_{x\to 0^-} f(x)$, $\lim_{x\to 0^+} f(x)$ and $f(0)$.

(c) Does $\lim_{x\to 0} f(x)$ exist? Why?

Ans:

(a)

x	-0.1	-0.01	-0.001	-0.0001	0	0.0001	0.001	0.01	0.1
$f(x)$	-1	-1	-1	-1	0	1.0001	1.001	1.01	1.1

(b) $\lim_{x\to 0^-} f(x) = -1$, $\lim_{x\to 0^+} f(x) = 1$ and $f(0) = 0$.

(c) $\lim_{x\to 0} f(x)$ does not exist since $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$.
(Remark: The existence of $\lim_{x\to 0} f(x)$ does not depend on $f(0)$.)

5. Let $f(x) = \dfrac{|x-1|}{x^2-1}$ for $x \neq \pm 1$.

(a) Does $\lim_{x\to 1} f(x)$ exist?

(b) Does $\lim_{x\to -1} f(x)$ exist?

(Hint: Rewrite the function $f(x)$ as a piecewise defined function.)

Ans: Rewrite the function $f(x)$ as:

$$f(x) = \begin{cases} \dfrac{x-1}{x^2-1} = \dfrac{1}{x+1} & \text{if} \quad x > 1, \\[2ex] \dfrac{-(x-1)}{x^2-1} = -\dfrac{1}{x+1} & \text{if} \quad x < 1 \text{ and } x \neq -1, \\[2ex] \text{NOT defined} & \text{if} \quad x = \pm 1. \end{cases}$$

(a) Note that $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \dfrac{1}{x+1} = \dfrac{1}{2}$ and $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} -\dfrac{1}{x+1} = -\dfrac{1}{2}$, so $\lim_{x\to 1^+} f(x) \neq \lim_{x\to 1^-} f(x)$ and $\lim_{x\to 1} f(x)$ does not exist.

(b) $\lim_{x\to -1} f(x) = \lim_{x\to -1} -\dfrac{1}{x+1}$ which goes to infinity and so it does not exist.
(In particular, $\lim_{x\to -1^+} f(x) = -\infty$ and $\lim_{x\to -1^-} f(x) = +\infty$.)

6. Let a be a real number and let $f(x)$ be a function defined by

$$f(x) = \begin{cases} x^2 & \text{if} \quad x \geq 2, \\[2ex] 3x + a & \text{if} \quad x < 2 \end{cases}$$

Given that $\lim_{x\to 2} f(x)$ exists. What is the value of a?

Ans: Since $\lim_{x\to 2} f(x)$ exists, $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$.

$$\begin{aligned}
\lim_{x\to 2^-} f(x) &= \lim_{x\to 2^+} f(x) \\
\lim_{x\to 2^-} 3x + a &= \lim_{x\to 2^+} x^2 \\
6 + a &= 4 \\
a &= -2
\end{aligned}$$

7. Let $f(x) = \dfrac{x^3}{|x|}$ for $x \neq 0$.

   (a) Find $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to 0^-} f(x)$.

   (b) Does $\lim_{x\to 0} f(x)$ exist?

   Ans: Rewrite the function $f(x)$ as:

$$f(x) = \begin{cases} \dfrac{x^3}{x} = x^2 & \text{if} \quad x > 0, \\[2ex] \dfrac{x^3}{-x} = -x^2 & \text{if} \quad x < 0 \end{cases}$$

   (a) $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2 = 0$ and $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} -x^2 = 0$

   (b) Since $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = 0$, $\lim_{x\to 0} f(x) = 0$.

8. Without using L'Hôpital rule, find the following limits.

   (a) $\lim_{x\to 1} \dfrac{x^2 - 3x + 2}{x^2 - 1}$

   Ans: $\lim_{x\to 1} \dfrac{x^2 - 3x + 2}{x^2 - 1} = \lim_{x\to 1} \dfrac{(x-2)(x-1)}{(x+1)(x-1)} = \lim_{x\to 1} \dfrac{x-2}{x+1} = -\dfrac{1}{2}$

   (Remark: When we consider the limit of the function at $x = 1$, we only care the value of the function when x is close to 1 but not 1, therefore $x - 1$ is nonzero and we can cancel it out in the second step.)

   (b) $\lim_{x\to 2} \dfrac{x^3 - 4x^2 + 5x - 2}{x^2 - 4}$

   Ans: $\lim_{x\to 2} \dfrac{x^3 - 4x^2 + 5x - 2}{x^2 - 4} = \lim_{x\to 2} \dfrac{(x-2)(x-1)^2}{(x+2)(x-2)} = \lim_{x\to 2} \dfrac{(x-1)^2}{x+2} = \dfrac{1}{4}$

   (c) $\lim_{x\to 4} \dfrac{\sqrt{x} - 2}{x - 4}$

   Ans: $\lim_{x\to 4} \dfrac{\sqrt{x} - 2}{x - 4} = \lim_{x\to 4} \dfrac{\sqrt{x} - 2}{x - 4} \cdot \dfrac{\sqrt{x} + 2}{\sqrt{x} + 2} = \lim_{x\to 4} \dfrac{x - 4}{(x-4)(\sqrt{x} + 2)} = \lim_{x\to 4} \dfrac{1}{\sqrt{x} + 2} =$
   $\dfrac{1}{4}$

(d) $\lim_{x\to 27} \frac{\sqrt[3]{x}-3}{x-27}$

Ans: $\lim_{x\to 27} \frac{\sqrt[3]{x}-3}{x-27} = \lim_{x\to 27} \frac{\sqrt[3]{x}-3}{x-27} \cdot \frac{(\sqrt[3]{x})^2+3(\sqrt[3]{x})+9}{(\sqrt[3]{x})^2+3(\sqrt[3]{x})+9} = \lim_{x\to 27} \frac{x-27}{(x-27)[(\sqrt[3]{x})^2+3(\sqrt[3]{x})+9]} =$
$\lim_{x\to 27} \frac{1}{(\sqrt[3]{x})^2+3(\sqrt[3]{x})+9} = \frac{1}{27}$

(e) $\lim_{x\to 3} \frac{x-3}{\sqrt{x+1}-2}$

Ans: $\lim_{x\to 3} \frac{x-3}{\sqrt{x+1}-2} = \lim_{x\to 3} \frac{x-3}{\sqrt{x+1}-2} \cdot \frac{\sqrt{x+1}+2}{\sqrt{x+1}+2} = \lim_{x\to 3} \frac{(x-3)(\sqrt{x+1}+2)}{x-3} =$
$\lim_{x\to 3} \sqrt{x+1}+2 = 4$

(f) $\lim_{x\to 0} \frac{\sqrt{1+x^2}-1}{x}$

Ans: $\lim_{x\to 0} \frac{\sqrt{1+x^2}-1}{x} = \lim_{x\to 0} \frac{\sqrt{1+x^2}-1}{x} \cdot \frac{\sqrt{1+x^2}+1}{\sqrt{1+x^2}+1} = \lim_{x\to 0} \frac{x^2}{x(\sqrt{1+x^2}+1)} =$
$\lim_{x\to 0} \frac{x}{\sqrt{1+x^2}+1} = \frac{0}{2} = 0$

(g) (Harder Problem) $\lim_{x\to 0} \frac{(1+x)^n-1}{x}$, where n is a positive integer.

Ans: $\lim_{x\to 0} \frac{(1+x)^n-1}{x} = \lim_{x\to 0} \frac{(1+C_1^n x + C_2^n x^2 + \cdots + C_n^n x^n)-1}{x} = \lim_{x\to 0} \frac{C_1^n x + C_2^n x^2 + \cdots + C}{x}$
$\lim_{x\to 0} C_1^n + C_2^n x + \cdots + C_n^n x^{n-1} = C_1^n = n$

(Remark: We use the binomial theorem to expand $(1+x)^n$ and $C_r^n = \frac{n!}{r!(n-r)!}$ are the binomial coefficients for $r = 0, 1, 2, \ldots, n$.)

9. By using the fact that $\lim_{x\to 0} \frac{\sin x}{x} = 1$, find the following limits.

(a) $\lim_{x\to 0} \frac{\sin 2x}{5x}$

Ans: $\lim_{x\to 0} \frac{\sin 2x}{5x} = \lim_{x\to 0} \frac{2}{5} \cdot \frac{\sin 2x}{2x} = \left(\lim_{x\to 0} \frac{2}{5}\right) \cdot \left(\lim_{x\to 0} \frac{\sin 2x}{2x}\right) = \frac{2}{5} \cdot 1 = \frac{2}{5}$

(b) $\lim_{x\to 0} \frac{\sin 5x}{\sin 7x}$

Ans: $\lim_{x\to 0} \frac{\sin 5x}{\sin 7x} = \lim_{x\to 0} \frac{5}{7} \cdot \frac{7x}{\sin 7x} \cdot \frac{\sin 5x}{5x} = \left(\lim_{x\to 0} \frac{5}{7}\right) \cdot \left(\lim_{x\to 0} \frac{7x}{\sin 7x}\right) \cdot \left(\lim_{x\to 0} \frac{\sin 5x}{5x}\right) =$
$\frac{5}{7} \cdot 1 \cdot 1 = \frac{5}{7}$

(c) $\lim_{x\to 0} \frac{\sin(x^2)}{5x^2}$

Ans: $\lim_{x\to 0} \frac{\sin(x^2)}{5x^2} = \lim_{x\to 0} \frac{1}{5} \cdot \frac{\sin(x^2)}{x^2} = \left(\lim_{x\to 0} \frac{1}{5}\right) \cdot \left(\lim_{x\to 0} \frac{\sin(x^2)}{x^2}\right) = \frac{1}{5} \cdot 1 = \frac{1}{5}$

(d) $\lim_{x\to 0} \frac{\cos ax - \cos bx}{x^2}$, where a and b are distinct real numbers.

Ans:

$$\begin{aligned}
\lim_{x\to 0}\frac{\cos ax-\cos bx}{x^2} &= \lim_{x\to 0}\frac{-2\sin(\frac{(a+b)x}{2})\sin(\frac{(a-b)x}{2})}{x^2}\\
&= \lim_{x\to 0}\frac{(a+b)(a-b)}{4}\cdot\frac{-2\sin(\frac{(a+b)x}{2})\sin(\frac{(a-b)x}{2})}{(\frac{(a+b)x}{2})(\frac{(a-b)x}{2})}\\
&= \left(\lim_{x\to 0}\frac{-(a+b)(a-b)}{2}\right)\cdot\left(\lim_{x\to 0}\frac{\sin(\frac{(a+b)x}{2})}{\frac{(a+b)x}{2}}\right)\cdot\left(\lim_{x\to 0}\frac{\sin(\frac{(a-b)x}{2})}{\frac{(a-b)x}{2}}\right)\\
&= \frac{b^2-a^2}{2}\cdot 1\cdot 1\\
&= \frac{b^2-a^2}{2}
\end{aligned}$$

10. (a) By using the fact that $\lim_{x\to 0}\frac{\sin x}{x}=1$, find $\lim_{x\to 0}\frac{\cos x-1}{x^2}$.

(b) Using (a), find $\lim_{x\to 0}\frac{\cos x-1}{x}$.

Ans:

(a)

$$\begin{aligned}
\lim_{x\to 0}\frac{\cos x-1}{x^2} &= \lim_{x\to 0}\frac{-(1-\cos(2\cdot\frac{x}{2}))}{4(\frac{x}{2})^2}\\
&= \lim_{x\to 0}\frac{-2\sin^2(\frac{x}{2})}{4(\frac{x}{2})^2}\\
&= \lim_{x\to 0}-\frac{1}{2}\cdot\frac{\sin(\frac{x}{2})}{(\frac{x}{2})}\cdot\frac{\sin(\frac{x}{2})}{(\frac{x}{2})}\\
&= \left(\lim_{x\to 0}-\frac{1}{2}\right)\cdot\left(\lim_{x\to 0}\frac{\sin(\frac{x}{2})}{(\frac{x}{2})}\right)\cdot\left(\lim_{x\to 0}\frac{\sin(\frac{x}{2})}{(\frac{x}{2})}\right)\\
&= -\frac{1}{2}\cdot 1\cdot 1\\
&= -\frac{1}{2}
\end{aligned}$$

(b)

$$\begin{aligned}
\lim_{x\to 0}\frac{\cos x-1}{x} &= \lim_{x\to 0}\frac{\cos x-1}{x^2}\cdot x\\
&= \left(\lim_{x\to 0}\frac{\cos x-1}{x^2}\right)\cdot\left(\lim_{x\to 0}x\right)\\
&= -\frac{1}{2}\cdot 0\\
&= 0
\end{aligned}$$

(Remark: This result is useful to find the derivative of sine function and cosine function later.)

Limits at Infinity

11. Let $f(x) = \dfrac{x-1}{x-2}$.

    Complete the following table.

x	10	100	1000
$f(x)$			

    By observation, guess the value of $\lim\limits_{x\to+\infty} f(x)$.

    (Remark: You may repeat the above by putting $x = -10, -100, -1000$ and guess the value of $\lim\limits_{x\to-\infty} f(x)$.)

    Ans:

x	10	100	1000
$f(x)$	1.125	1.010204	1.00010002

    By observation, it suggests (but not a formal $\lim\limits_{x\to+\infty} f(x) = 1$. Similarly, we have $\lim\limits_{x\to-\infty} f(x) = 1$.

12. The graphs of $f(x) = e^x$ (in blue) and $g(x) = \ln x$ (in red) is shown in Figure 1, while the graphs of $f(x) = e^{-x}$ (in blue) and $g(x) = \ln(1/x)$ (in red) is shown in Figure 2.

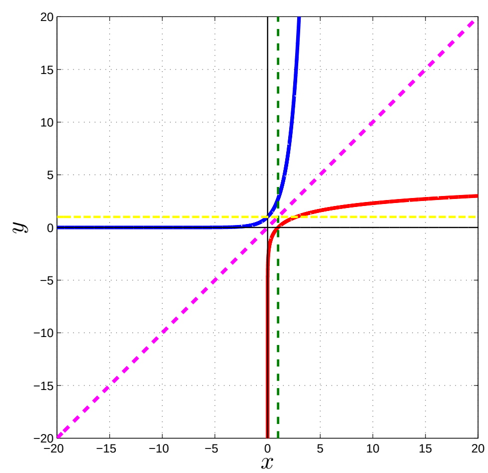


Figure 1

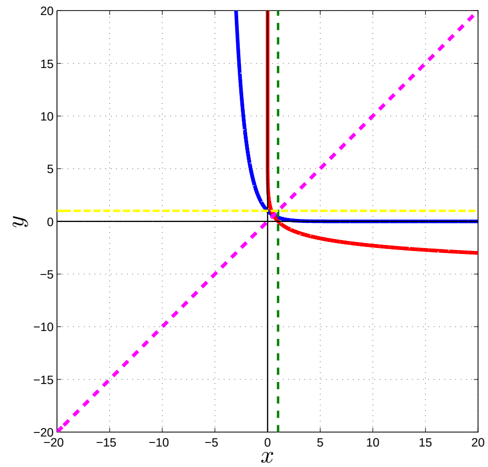


Figure 2

Without using L'Hôpital's rule, evaluate the limit. Furthermore, if the limit does not exist but diverges to $\pm\infty$, please indicate so and determine the correct sign.

(a) $\lim\limits_{x\to-\infty} e^{1+x^6}$;

Ans: When x tends to $-\infty$, $1+x^6$ tends to ∞, so e^{1+x^6} diverges to $+\infty$.

(b) $\lim\limits_{x\to+\infty} \ln\left(e^{-2x}+e^{-x}+1\right)$;

Ans: When x tends to $+\infty$, $e^{-2x}+e^{-x}+1$ tends to 1, so $\lim\limits_{x\to+\infty} \ln\left(e^{-2x}+e^{-x}+1\right) = 0$;.

(c) $\lim\limits_{x\to+\infty} \ln\left(\dfrac{e^{3x}+e^{x}}{e^{5x}+e^{2x}}\right)$;

Ans: When x tends to $+\infty$, $\dfrac{e^{3x}+e^{x}}{e^{5x}+e^{2x}} = \dfrac{1+e^{-2x}}{e^{2x}+e^{-x}}$ tends to 0 from the right hand side, so $\ln\left(\dfrac{e^{3x}+e^{x}}{e^{5x}+e^{2x}}\right)$ diverges to $-\infty$.

(d) $\lim\limits_{x\to+\infty} \ln\left(\dfrac{e^{2x+1}+2e^{-x}}{e^{2x}+e^{-x+2}}\right)$.

Ans: When x tends to $+\infty$, $\dfrac{e^{2x+1}+2e^{-x}}{e^{2x}+e^{-x+2}} = \dfrac{e\cdot e^{2x}+2e^{-x}}{e^{2x}+e^{2}\cdot e^{-x}}$ tends to e. Therefore, $\lim\limits_{x\to+\infty} \ln\left(\dfrac{e^{2x+1}+2e^{-x}}{e^{2x}+e^{-x+2}}\right) = 1$.

13. Find the following limits, if exist.

(a) $\lim\limits_{x\to+\infty} 2^x$;

Ans: As x tends to positive infinity, 2^x tends to positive infinity, therefore the limit does not exist.

(b) $\lim\limits_{x\to-\infty} 2^x$;

Ans: $\lim\limits_{x\to-\infty} 2^x = 0$

(c) $\lim_{x\to+\infty} 0.2^x$;

Ans: $\lim_{x\to+\infty} 0.2^x = 0$

(d) $\lim_{x\to-\infty} 0.2^x$;

Ans: As x tends to negative infinity, 0.2^x tends to positive infinity, therefore the limit does not exist.

(e) $\lim_{x\to+\infty} \ln\left(\dfrac{e^x+2e^{-x}}{e^x+e^{-x}}\right)$;

Ans: $\lim_{x\to+\infty} \ln\left(\dfrac{e^x+2e^{-x}}{e^x+e^{-x}}\right) = \lim_{x\to+\infty} \ln\left(\dfrac{1+2e^{-2x}}{1+e^{-2x}}\right) = \ln 1 = 0$

(f) $\lim_{x\to-\infty} \ln\left(\dfrac{e^x+2e^{-x}}{e^x+e^{-x}}\right)$.

Ans: $\lim_{x\to+\infty} \ln\left(\dfrac{e^x+2e^{-x}}{e^x+e^{-x}}\right) = \lim_{x\to+\infty} \ln\left(\dfrac{e^2x+2}{e^{2x}+1}\right) = \ln 2$

14. By using the fact that $\lim_{x\to+\infty}\left(1+\dfrac{1}{x}\right)^x = e$, find the following limits.

(a) $\lim_{x\to+\infty}\left(1+\dfrac{2}{x}\right)^{2x}$;

Ans:

$$\begin{aligned}\lim_{x\to+\infty}\left(1+\frac{2}{x}\right)^{2x} &= \lim_{x\to+\infty}\left[\left(1+\frac{1}{x/2}\right)^{x/2}\right]^4 \\ &= e^4\end{aligned}$$

(b) $\lim_{x\to+\infty}\left(1+\dfrac{1}{x+1}\right)^{x}$

Ans:

$$\begin{aligned}\lim_{x\to+\infty}\left(1+\frac{1}{x+1}\right)^{x} &= \lim_{x\to+\infty}\left(1+\frac{1}{x+1}\right)^{x+1}\left(1+\frac{1}{x+1}\right)^{-1} \\ &= \left(\lim_{x\to+\infty}\left[1+\frac{1}{x+1}\right)^{x+1}\right]\cdot\left[\lim_{x\to+\infty}\left(1+\frac{1}{x+1}\right)^{-1}\right] \\ &= e\cdot 1 \\ &= e\end{aligned}$$

(c) $\lim_{x\to+\infty}\left(\dfrac{x}{x-1}\right)^{x}$

Ans:

$$\begin{aligned}\lim_{x\to+\infty}\left(\frac{x}{x-1}\right)^x &= \lim_{x\to+\infty}\left(1+\frac{1}{x-1}\right)^x \\ &= \lim_{x\to+\infty}\left(1+\frac{1}{x-1}\right)^{x-1}\cdot\left(1+\frac{1}{x-1}\right) \\ &= e\cdot 1 \\ &= e\end{aligned}$$

15. Without using L'Hôpital rule, find the following limits, if exist.

(a) $\lim\limits_{x\to+\infty}\dfrac{x^2-3x+2}{x^2-1}$;

Ans: $\lim\limits_{x\to+\infty}\dfrac{x^2-3x+2}{x^2-1}=\lim\limits_{x\to+\infty}\dfrac{1-\frac{3}{x}+\frac{2}{x^2}}{1-\frac{1}{x^2}}=1$

(b) $\lim\limits_{x\to-\infty}\dfrac{x^3-2x}{4x^3+2x^2}$;

Ans: $\lim\limits_{x\to-\infty}\dfrac{x^3-2x}{4x^3+2x^2}=\lim\limits_{x\to-\infty}\dfrac{1-\frac{2}{x^2}}{4+\frac{2}{x}}=\dfrac{1}{4}$

(c) $\lim\limits_{x\to+\infty}\dfrac{\sqrt{x^2+4}}{x+4}$;

Ans: $\lim\limits_{x\to+\infty}\dfrac{\sqrt{x^2+4}}{x+4}=\lim\limits_{x\to+\infty}\dfrac{\sqrt{1+\frac{4}{x^2}}}{1+\frac{4}{x}}=1$

(d) $\lim\limits_{x\to+\infty}\dfrac{x}{\sqrt{9x^2+5}}$;

Ans: $\lim\limits_{x\to+\infty}\dfrac{x}{\sqrt{9x^2+5}}=\lim\limits_{x\to+\infty}\dfrac{1}{\sqrt{9+\frac{5}{x^2}}}=\dfrac{1}{3}$

(e) $\lim\limits_{x\to-\infty}\dfrac{x}{\sqrt{9x^2+5}}$;

Ans: $\lim\limits_{x\to-\infty}\dfrac{x}{\sqrt{9x^2+5}}=\lim\limits_{x\to\infty}\dfrac{1}{\frac{1}{x}\sqrt{9x^2+5}}=\lim\limits_{x\to-\infty}\dfrac{1}{-\sqrt{9+\frac{5}{x^2}}}=-\dfrac{1}{3}$

Note: if $x<0$, then $x=-\sqrt{x^2}$.

(f) $\lim\limits_{x\to+\infty}\sqrt{x+1}-\sqrt{x-1}$;

Ans: $\lim\limits_{x\to+\infty}\sqrt{x+1}-\sqrt{x-1}=\lim\limits_{x\to+\infty}(\sqrt{x+1}-\sqrt{x-1})\cdot\left(\dfrac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}}\right)=$
$\lim\limits_{x\to+\infty}\dfrac{2}{\sqrt{x+1}+\sqrt{x-1}}=0$

(g) $\lim\limits_{x\to+\infty}\sqrt{x^2+x}-x$;

Ans: $\lim_{x\to+\infty}\sqrt{x^2+x}-x=\lim_{x\to+\infty}(\sqrt{x^2+x}-x)\cdot\left(\frac{\sqrt{x^2+x}+x}{\sqrt{x^2+x}+x}\right)=\lim_{x\to+\infty}\frac{x}{\sqrt{x^2+x}+x}=$ $\lim_{x\to+\infty}\frac{1}{\sqrt{1+\frac{1}{x}}+1}=\frac{1}{2}$

(h) $\lim_{x\to-\infty}\sqrt{x^2+x}-x.$

Ans: $\lim_{x\to-\infty}\sqrt{x^2+x}-x=\lim_{x\to-\infty}(\sqrt{x^2+x}-x)\cdot\left(\frac{\sqrt{x^2+x}+x}{\sqrt{x^2+x}+x}\right)=\lim_{x\to-\infty}\frac{x}{\sqrt{x^2+x}+x}=$ $\lim_{x\to-\infty}\frac{1}{-\sqrt{1+\frac{1}{x}}+1}$ which does not exist.