THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1510 Calculus for Engineers (Fall 2021)
Homework 6
Deadline: December 11 at 23:00

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Class: MATH 1510 G

I acknowledge that I am aware of University policy and regulations on honesty in academic work, and of the disciplinary guidelines and procedures applicable to breaches of such policy and regulations, as contained in the website http://www.cuhk.edu.hk/policy/academichonesty/

David — Signature

4-12-2021 — Date

General Guidelines for Homework Submission.

- Please submit your answer to Gradescope through the centralized course MATH1510A-I in Blackboard.
- In Gradescope, for each question, please indicate exactly which page(s) its answer locates. **Answers of incorrectly matched questions will not be graded**.
- **Late submission will NOT be graded and result in zero score.** Any answers showing evidence of plagiarism will also score zero; stronger disciplinary action may also be taken.
- Points will only be awarded for answers with sufficient justifications.
- All questions in **Part A** along with some selected questions in **Part B** will be graded. Question(s) labeled with * are more challenging.

Part A:

1. Find the Maclaurin polynomials of order 4 of the following functions:

   (a)
   $$\cos(\sin x);$$

   (b)
   $$g(x) = \frac{x^2 - x + 3}{(x^2+1)(2-x)}.$$

(a) Let $u = \sin x$, we have $f(u) = \cos u$:

$\because$ $f(0) : 1$;

$f'(u) : -\sin u$, $f'(0) = 0$;

$f''(u) : -\cos u$, $f''(0) = -1$;

$f'''(u) : \sin u$, $f'''(0) = 0$;

$f^{(4)}(u) : \cos u$, $f^{(4)}(0) = 1$;

$f^{(5)}(u) : -\sin u$, $f^{(5)}(0) = 0$;

$f^{(6)}(u) : -\cos u$, $f^{(6)}(0) = -1$

$\therefore$ We have $f(u) = 1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \frac{u^6}{6!} + \dots$

$$f(\sin x) = 1 - \frac{\sin^2 x}{2} + \frac{\sin^4 x}{24} - \frac{\sin^6 x}{120} + \dots //$$

(b) $g(x) = \frac{1}{x^2+1} + \frac{1}{2-x}$

$\because$ By property of geometric sequence, we have

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$$

$$\therefore \quad \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \ldots$$

$$\& \quad \frac{1}{2-x} = \frac{1}{2}\left(\frac{1}{1-\frac{x}{2}}\right)$$

$$= \frac{1}{2}\left(1 + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{8} + \ldots\right)$$

$$= \frac{1}{2} + \frac{x}{4} + \frac{x^2}{8} + \frac{x^3}{16} + \ldots$$

$\therefore$ We have $g(x) = \left(1 - x^2 + x^4 - x^6 + \ldots\right)$

$$+ \left(\frac{1}{2} + \frac{x}{4} + \frac{x^2}{8} + \frac{x^3}{16} + \ldots\right)$$

$$= \frac{3}{2} + \frac{x}{4} - \frac{7x^2}{8} + \frac{x^3}{16} + \ldots \quad //$$

Part B:

2. For each of the following power series, find the radius of convergence and determine whether it is convergent at the given two points.

   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{n}{n+1}(x-1)^n$, at points $x = -\frac{1}{3}$, $x = \frac{3}{2}$.

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n} x^n$, at points $x = -1$, $x = \pi$.

(a) $$\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{n(n+2)}{(n+1)^2}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n^2+2n}{n^2+2n+1}\right|$$

$$= 1$$

$\because$ With centre $= 1$, the radius of convergence is 1,

the power series is convergent for $0 < x < 2$.

$\therefore$ When $x = \frac{3}{2}$, the power series is convergent.

$x = -\frac{1}{3}$, the power series is not convergent. //

(b) $$\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{n! \times (n+1)^{n+1}}{n^n \times (n+1)!}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)^n}{n^n}\right|$$

$$= \lim_{n\to\infty} \left| \frac{n^n + \sum_{k=1}^{n-1} C_k^n (n)^k + 1}{n^n} \right|$$

$$= 1$$

$\therefore$ With centre 0, we have the radius of convergence $= 1$.

the power series is convergent for $-1 < x < 1$.

$\therefore$ When $x = -1$ or TU, both of them are not convergent. //

3. Find the Maclaurin series of the following functions:

(a)

$$\sinh(x) = \frac{e^x - e^{-x}}{2};$$

(b)

$$\frac{1-x}{2+x}.$$

(a) $\because$ We have $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+1}}{(2n+1)!}$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

Let $h(x) = x$, we have:

$$\sin h(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot [h(x)]^{2n+1}}{(2n+1)!}$$

$\because$ $h(x) = \sin^{-1}\left(\frac{e^x - e^{-x}}{2}\right)$,

$\therefore$ We have $\sin h(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot \left[\sin^{-1}\left(\frac{e^x - e^{-x}}{2}\right)\right]^{2n+1}}{(2n+1)!}$ //

(b) $$\frac{1-x}{2+x} = \frac{3}{2+x} - 1$$

$$= \frac{3}{2}\left(\frac{1}{1+\frac{x}{2}}\right) - 1$$

$\because$ By property of geometric sequence, we have :

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\therefore \frac{1}{1+\frac{x}{2}} = \sum_{n=0}^{\infty} \left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^n}{2^n}$$

$$\therefore \frac{1-x}{2+x} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 3x^n}{2^{n+1}} - 1 \quad //$$

4. (**Binomial series**)

The following identity is the well-known binomial theorem

$$\begin{aligned}(1+x)^n =& \sum_{k=0}^{n} \binom{n}{k} x^k \\ =& 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots + \frac{n(n-1)\cdots 1}{n!}x^n,\end{aligned}$$

where n is a positive integer, $\binom{n}{0} = 1$ and $\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!}$.

We will consider a generalized case where n might not be positive integer.

Let $a \in \mathbb{R}$ and

$$f(x) = (1+x)^a.$$

(a) Show that the Maclaurin series of $f(x)$ is given by

$$\begin{aligned}&\sum_{k=0}^{\infty} \frac{a(a-1)\cdots(a-k+1)}{k!}x^k \\ =& 1 + ax + \frac{a(a-1)}{2!}x^2 + \cdots + \frac{a(a-1)\cdots(a-k+1)}{k!}x^k + \cdots\end{aligned}$$

(b) Write down the first 4 nonzero terms of the Maclaurin series of $(1+x)^{3/2}$. Hence, give an approximation of $(1.1)^{3/2}$ and compare your result with the value obtained from a calculator.

Remark: The Maclaurin series in part (a) is called Binomial series.

(a) $f(0) = 1$;

$f'(x) = a(1+x)^{a-1}$, $f'(0) = a$;

$f''(x) = a(a-1)(1+x)^{a-2}$, $f''(0) = a(a-1)$;

$\vdots$

$f^{(k)}(x) = \frac{a!}{k!}(1+x)^{a-k}$, $f^{(k)}(0) = \frac{a!}{k!}$

$\therefore$ At $x=0$, we have:

$$(1+x)^a = 1 + ax + \frac{a(a-1)}{2!}x^2 + \dots + \frac{a(a-1)\dots(a-k+1)}{k!}x^k + \dots$$

(b) $$(1+x)^{\frac{3}{2}} = 1 + \frac{3}{2}x + \frac{\frac{3}{2}\left(\frac{3}{2}-1\right)}{2!}x^2 + \frac{\frac{3}{2}\left(\frac{3}{2}-1\right)\left(\frac{3}{2}-2\right)}{3!}x^3 + \dots$$

$$= 1 + \frac{3}{2}x + \frac{3}{8}x^2 - \frac{1}{48}x^3 + \dots$$

$$f(0.1) = 1 + \frac{3}{2}(0.1) + \frac{3}{8}(0.1)^2 - \frac{1}{48}(0.1)^3 + \dots$$

$$\approx 1.1537291\overline{6}$$

From the calculator, we have the value $(1.1)^{\frac{3}{2}}$ is equal to $1.1536897329\dots$

The estimation is overestimated by 3.94×10^{-5}.

5. Use a Maclaurin polynomial of a suitable order to approximate $\cos(0.1)$ with error less than 10^{-5}.

$\because$ For $f(x) = \cos x$:

$f(0) = 1$;

$f'(x) = -\sin x$, $f'(0) = 0$;

$f''(x) = -\cos x$, $f''(0) = -1$;

$f'''(x) = \sin x$, $f'''(0) = 0$;

$f^{(4)}(x) = \cos x$, $f^{(4)}(0) = 1$;

$\therefore T_5(x) = T_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}$

& $\cos(0.1) \approx 1 - \frac{0.1^2}{2} + \frac{0.1^4}{24}$

$\approx 0.99500\overline{416}$

Absolute error $= \left| \frac{f^{(6)}(0)}{6!} (0.1)^6 \right|$

$= \frac{1}{6!}(0.1)^6$

$\approx 1.38 \times 10^{-9}$

$< 10^{-5}$ //

6. (a) Evaluate the following limit by using L'Hôpital's rule

$$\lim_{t\to 0}\frac{e^{2t}\cos t-(1+2t)}{t^2}.$$

(b) By considering Lagrange remainder, show that there exists some constant C such that

$$|e^{2t}\cos t-(1+2t+\frac{3}{2}t^2)|\leq Ct^3$$

for any $t\in(-0.5,0.5)$.

(c) By using part (b), evaluate the following limit

$$\lim_{t\to 0}\frac{e^{2t}\cos t-(1+2t)}{t^2}.$$

(a)
$$\lim_{t\to 0}\frac{e^{2t}\cdot\cos t-(1+2t)}{t^2}$$
$$=\lim_{t\to 0}\frac{2e^{2t}\cdot\cos t-e^{2t}\cdot\sin t-2}{2t}$$
$$=\lim_{t\to 0}\frac{4e^{2t}\cdot\cos t-2e^{2t}\cdot\sin t-2e^{2t}\cdot\sin t-e^{2t}\cdot\cos t}{2}$$
$$=\frac{3}{2}\ //$$

(b)
$$e^{2t}=1+2t+2t^2+\frac{4}{3}t^3+\frac{2}{3}t^4+\dots$$
$$\cos t=1-\frac{t^2}{2}+\frac{t^4}{24}+\dots$$

$$e^{2t}\cdot\cos t=1+2t+\frac{3}{2}t^2+\frac{1}{3}t^3-\frac{7}{24}t^4+\dots$$

Let $f(x)=e^{2t}\cdot\cos t$

$$T_2(x)=1+2t+\frac{3}{2}t^2$$

$\because$ By Taylor theorm, $f(x) = T_2(t) + \frac{f'''(x)}{3!}(t)^3$

$= T_2(t) + \frac{1}{3}t^3$,

where $\frac{1}{3}t^3$ is the absolute error (OR remainder).

$\therefore \left| e^{2t}\cdot \cos t - \left(1+2t+\frac{3}{2}t^2\right) \right|$

$\leq \frac{1}{3}t^3$, for $t \in (-0.5, 0.5)$ //

(c) $\lim_{t\to 0} \frac{e^{2t}\cdot \cos t - (1+2t)}{t^2}$

$= \lim_{t\to 0} \frac{\frac{3}{2}t^2 + \frac{1}{3}t^3 - \frac{7}{24}t^4 + \ldots}{t^2}$

$= \frac{3}{2}$ //