Lecture Note 29

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MATH1510
Calculus for Engineers

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What is a double integral?

Recall that a single integral is something of the form

$$\int_a^b f(x)dx.$$

A double integral is something of the form

$$\iint\limits_R f(x,y)dxdy$$

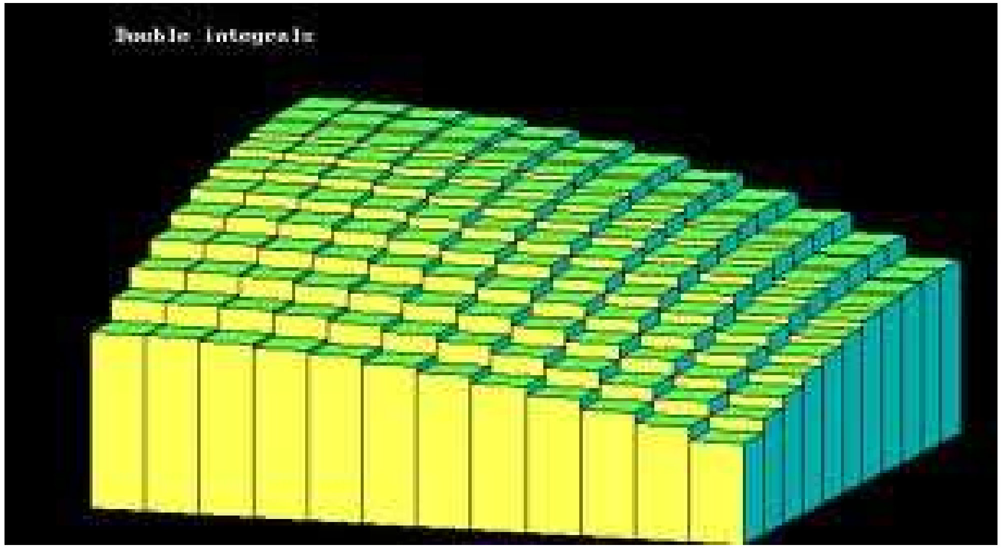
where R is called **the region of integration** and is a region in the (x,y) plane. The double integral gives us the volume under the surface $z = f(x,y)$, just as a single integral gives the area under a curve.

Double Integrals Over Rectangles

Theorem 1

If $f(x, y) \geq 0$ and f is integrable over the rectangle R, then the volume V of the solid that lies above R and under the surface $z = f(x, y)$ is

$$V = \iint\limits_R f(x, y) dA.$$


Double integrals

Properties of Double Integrals:

1. $$\iint\limits_R (f(x,y)+g(x,y))dA = \iint\limits_R f(x,y)dA + \iint\limits_R g(x,y)dA.$$

2. $$\iint\limits_R kf(x,y)dA = k\iint\limits_R f(x,y)dA.$$

3. If $f(x,y) \geq g(x,y)$ for all (x,y) in R, then
$$\iint\limits_R f(x,y)dA \geq \iint\limits_R g(x,y)dA.$$

To evaluate

$$V = \iint_R f(x, y)dxdy$$

proceed as follows:

- Always work from the inside out: first evaluate the inside integral.
- For $\displaystyle\int f(x, y)dx$ integrate with respect to x and "treat y as a constant".
- For $\displaystyle\int f(x, y)dy$ integrate with respect to y and "treat x as a constant".

Example 1 Evaluate

$$\int_0^3 \int_1^2 x^2 y dy dx.$$

Solution. $\int_0^3 \int_1^2 x^2 y dy dx$ means $\int_0^3 \left\{ \int_1^2 x^2 y dy \right\} dx$ so first we evaluate the inside integral

$$\left\{ \int_1^2 x^2 y dy \right\}$$

treating x as a constant:

$$\int_1^2 x^2 y dy = \frac{1}{2} x^2 y^2 \Big|_{y=1}^2 = \cdots = \frac{3}{2} x^2.$$

Then

$$\int_0^3 \left\{ \int_1^2 x^2 y dy \right\} dx = \int_0^1 \left\{ \frac{3}{2} x^2 \right\} dx = \frac{1}{2} x^3 \Big|_{x=0}^3 = \frac{27}{2}$$

Example 2 Evaluate

$$\int_1^2 \int_0^3 x^2 y dx dy.$$

Solution. $\int_1^2 \int_0^3 x^2 y dx dy$ means $\int_1^2 \left\{ \int_0^3 x^2 y dx \right\} dy$ so first we evaluate the inside integral

$$\left\{ \int_0^3 x^2 y dx \right\}$$

treating y as a constant:

$$\int_0^3 x^2 y dx = \frac{1}{3} x^3 y \Big|_{x=0}^3 = \cdots = 9y.$$

Then

$$\int_1^2 \left\{ \int_0^3 x^2 y dx \right\} dy = \int_1^2 \{9y\} dy = \frac{9}{2} y^2 \Big|_{y=1}^2 = \frac{27}{2}.$$

It was not an accident that the answers to Example 1 and Example 2 were the same.

Theorem 2 Fubini's Theorem

If f is integrable over the rectangle
$R = \{\,(x, y)|\, a \leq x \leq b$ and $c \leq y \leq d\} = [a, b] \times [c, d]$, then

$$\iint_R f(x, y)dA = \int_a^b \int_c^d f(x, y)dydx = \int_c^d \int_a^b f(x, y)dxdy.$$

Fubini's Theorem says that we can integrate in either order and still get the same result - sometimes one order of integration is much easier than the other order.

Example 3 Evaluate

$$\iint\limits_R y\sin(xy)dA,$$

where $R = [1,2] \times [0,\pi]$.

Solution.

The notation $R = [1,2] \times [0,\pi]$ means the rectangle $1 \le x \le 2$ and $0 \le y \le \pi$.

By Fubini's Theorem we have a choice of evaluating

$$\int_0^\pi \int_1^2 y\sin(xy)dxdy \quad \text{or} \quad \int_1^2 \int_0^\pi y\sin(xy)dydx.$$

1.

$$\begin{aligned}
\int_0^\pi \int_1^2 y\sin(xy)dxdy &= \int_0^\pi \left\{\int_1^2 y\sin(xy)dx\right\} dy \qquad \text{Let } u = xy \\
&= \int_0^\pi \left\{-\cos(xy)|_{x=1}^2\right\} dy \\
&= \int_0^\pi \{-\cos(2y) + \cos(y)\} dy \\
&= -\frac{1}{2}\sin(2y) + \sin(y)\bigg|_{y=0}^{\pi} \\
&= 0.
\end{aligned}$$

2. $\displaystyle\int_1^2 \int_0^\pi y\sin(xy)dydx = \int_1^2 \left\{\int_0^\pi y\sin(xy)dy\right\} dx$ so first we need to evaluate $\displaystyle\int_0^\pi y\sin(xy)dy$ and that requires Integration by Parts, a more difficult situation than in part (a). Exercise!

Example 4 Find the volume of the solid bounded above by the plane $z = 4 - x - y$ and below by the rectangle $R = \{(x, y)\ 0 \le x \le 1,\ 0 \le y \le 2\}$.

Solution: The volume under any surface $z = f(x, y)$ and above a region R is given by

$$V = \iint\limits_{R} f(x, y) dx dy.$$

In our case

$$\begin{aligned} V = \int_0^2 \int_0^1 (4 - x - y) dx dy &= \int_0^2 \left\{ 4x - \frac{1}{2}x^2 - yx \right\} \Big|_{x=0}^{1} dy \\ &= \int_0^2 \left\{ 4 - \frac{1}{2} - y \right\} dy \\ &= \frac{7y}{2} - \frac{y^2}{2} \Big|_{y=0}^{2} \\ &= 5. \end{aligned}$$

The double integrals in the above examples are the easiest types to evaluate because they are examples in which all four limits of integration are constants.

This happens when the region of integration is rectangular in shape.

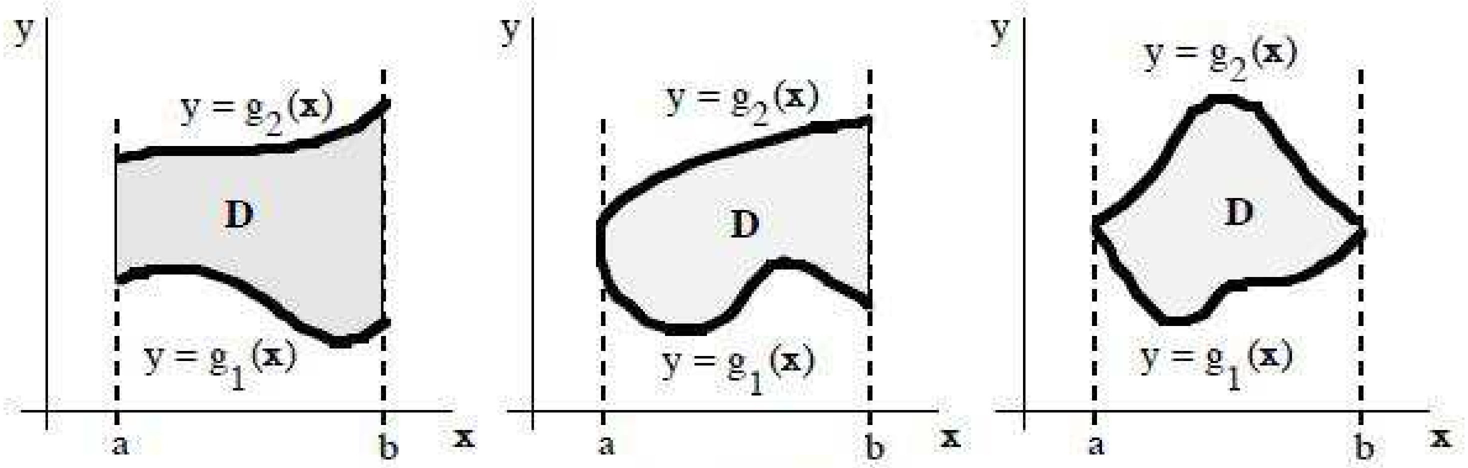
In non-rectangular regions of integration the limits are not all constant so we have to get used to dealing with non-constant limits.

Double Integrals Over General Regions

Sometimes we need the double integral of a function $z = f(x, y)$ over a domain D (in the $xy-$plane) that is not a rectangle, and in those cases we must chose the endpoints of the integrals very carefully.

A plane region D uses Vertical Slices if it lies between the graphs of two continuous functions of x :

$D = \{(x, y) : a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x)\}$ as in the figure.



Some Vertical Slice Regions

In the Vertical Slice situation, it is usually easier to evaluate the double integral over D as

$$\iint_D f(x, y) dA = \int_{x=a}^{b} \int_{y=g_1(x)}^{g_2(x)} f(x, y) dy dx.$$

Example 5 Evaluate

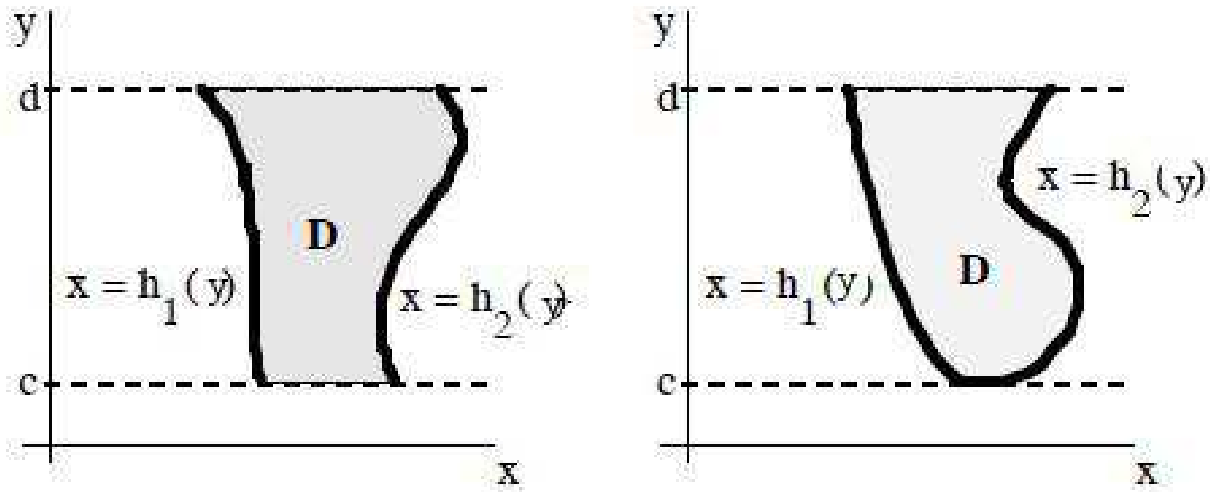
$$\iint_D (x + 2y)dA,$$

where D is the region bounded by $2x^2 \le y \le 1 + x^2$ for $-1 \le x \le 1$.

Solution.

$$\begin{aligned}
\iint_D (x + 2y)dA &= \int_{x=-1}^{1} \int_{y=2x^2}^{1+x^2} (x + 2y)dydx \\
&= \int_{x=-1}^{1} \left.(xy + y^2)\right|_{y=2x^2}^{1+x^2} dx \\
&= \int_{x=-1}^{1} \left(\{(x(1 + x^2) + (1 + x^2)^2\} - \{(x(2x^2) + (2x^2)^2\}\right) dx \\
&= \int_{x=-1}^{1} \left(-3x^4 - x^3 + 2x^2 + x + 1\right) dx \\
&= \frac{32}{15}.
\end{aligned}$$

Sometimes it is more useful to use Horizontal Slices and to integrate first with respect to x and then with respect to y (see the Horizontal Slice figure below).



Some **Horizontal Slice** Regions

$$\iint_D f(x,y)dA = \int_{y=c}^{d} \int_{x=h_1(y)}^{h_2(y)} f(x,y)dxdy.$$

Example 6 Evaluate

$$\iint_D (x^2 + y^2)dA,$$

where D is the region bounded by $0 \le y \le 4$ for $\dfrac{y}{2} \le x \le \sqrt{y}$.

Solution.

$$\begin{aligned}
\iint_D (x^2 + y^2)dA &= \int_{y=0}^{4} \int_{x=\frac{y}{2}}^{\sqrt{y}} (x^2 + y^2)dxdy \\
&= \int_{y=0}^{4} \frac{1}{3}x^3 + xy^2 \Big|_{x=\frac{y}{2}}^{\sqrt{y}} dy \\
&= \int_{y=0}^{4} (\{\frac{1}{3}(\sqrt{y})^3 + (\sqrt{y})y^2\} - \{\frac{1}{3}(y/2)^3 + (y/2)y^2\})dy \\
&= \int_{y=0}^{4} \left(\frac{y^{3/2}}{3} + y^{5/2} - \frac{y^3}{24} - \frac{y^3}{2} \right) dy \\
&= \cdots = \frac{216}{35}.
\end{aligned}$$

Example 7 Evaluate

$$\int_0^2 \int_{x^2}^x y^2 x dy dx,$$

Solution.

$$\int_0^2 \int_{x^2}^x y^2 x dy dx = \cdots = -\frac{128}{15}.$$

Example 8 Evaluate

$$\int_{\pi/2}^{\pi} \int_0^{x^2} \frac{1}{x} \cos\left(\frac{y}{x}\right) dy dx,$$

Solution.

$$\int_{\pi/2}^{\pi} \int_0^{x^2} \frac{1}{x} \cos\left(\frac{y}{x}\right) dy dx = \cdots = 1.$$

Example 9 Evaluate

$$\int_1^4 \int_0^{\sqrt{y}} e^{x/\sqrt{y}} dxdy,$$

Solution.

$$\int_1^4 \int_0^{\sqrt{y}} e^{x/\sqrt{y}} dxdy = \cdots = \frac{14}{3}(e-1).$$

Changing variables in a double integral

We know how to change variables in a single integral:

$$\int_a^b f(x)dx = \int_A^B f(x(u))\frac{dx}{du}du$$

where A and B are the new limits of integration.

For double integrals the rule is more complicated. Suppose we have

$$\int\int_D f(x,y)dxdy$$

and want to change the variables to u and v given by $x = x(u,v)$, $y = y(u,v)$. The change of variables formula is

$$\int\int_D f(x,y)dxdy = \int\int_{D*} f(x(u,v),y(u,v))|J|dudv \qquad (1)$$

where J is the Jacobian, given by

$$J = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

and D^* is the new region of integration, in the (u,v) plane.

Transforming a double integral into polars

A very commonly used substitution is conversion into polars. This substitution is particularly suitable when the region of integration D is a circle or an annulus (i.e. region between two concentric circles). Polar coordinates r and θ are defined by

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

The variables u and v in the general description above are r and θ in the polar coordinates context and the Jacobian for polar coordinates is

$$\begin{aligned} J &= \frac{\partial x}{\partial r}\frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial r} \\ &= (\cos\theta)(r\cos\theta) - (-r\sin\theta)(\sin\theta) \\ &= r(\cos^2\theta + \sin^2\theta) = r. \end{aligned}$$

So $|J| = r$ and the change of variables rule (2) becomes

$$\int\int_D f(x,y)dxdy = \int\int_{D*} f(r\cos\theta, r\sin\theta)rdrd\theta. \tag{2}$$

Example 10 Use polar coordinates to evaluate

$$\int\int_D xydxdy$$

where D is the portion of the circle centre 0, radius 1, that lies in the first quadrant

Solution:

For the portion in the first quadrant we need $0 \leq r \leq 1$ and $0 \leq \theta \leq \pi/2$. These inequalities give us the limits of integration in the r and θ variables, and these limits will all be constants.

With $x = r\cos\theta$, $y = r\sin\theta$ the integral becomes

$$
\begin{aligned}
\int\int_D xy\,dx\,dy &= \int_0^{\pi/2}\int_0^1 r^2\cos\theta\sin\theta\, r\,dr\,d\theta \\
&= \int_0^{\pi/2}\left\{\frac{r^4}{4}\cos\theta\sin\theta\right\}\Bigg|_{r=0}^{1} d\theta \\
&= \int_0^{\pi/2}\frac{1}{4}\sin\theta\cos\theta\, d\theta = \int_0^{\pi/2}\frac{1}{8}\sin 2\theta\, d\theta \\
&= \frac{1}{8}\left[-\frac{\cos 2\theta}{2}\right]\Bigg|_0^{\pi/2} \\
&= \frac{1}{8}.
\end{aligned}
$$

Application of double integrals: centres of gravity

We will show how double integrals may be used to find the location of the centre of gravity of a two-dimensional object.

Mathematically speaking, a plate is a thin $2-$dimensional distribution of matter considered as a subset of the (x, y) plane.

Let

$$\sigma = \text{mass per unit area.}$$

- This is the definition of density for two-dimensional objects.
- If the plate is all made of the same material (a sheet of metal, perhaps) then σ would be a constant, the value of which would depend on the material of which the plate is made.
- However, if the plate is not all made of the same material then σ could vary from point to point on the plate and therefore be a function of x and y, $\sigma(x, y)$.
- For some objects, part of the object may be made of one material and part of it another (some currencies have coins that are like this).
- But $\sigma(x, y)$ could quite easily vary in a much more complicated way (a pizza is a simple example of an object with an uneven distribution of

matter).

The intersection of the two thin strips defines a small rectangle of length δx and width δy. Thus

$$\text{mass of little rectangle} = (\text{mass per unit area})(\text{area}) = \sigma(x, y)dxdy.$$

Therefore the total mass of the plate D is

$$M = \int\int_D \sigma(x, y)dxdy.$$

Suppose you try to balance the plate D on a pin. The centre of mass of the plate is the point where you would need to put the pin. It can be shown that the coordinates (x, y) of the centre of mass are given by

$$\bar{x} = \frac{\iint_D x\sigma(x, y)dA}{\iint_D \sigma(x, y)dA} \quad \text{and} \quad \bar{y} = \frac{\iint_D y\sigma(x, y)dA}{\iint_D \sigma(x, y)dA}. \tag{3}$$

Example 11 A homogeneous triangle with vertices $(0,0)$, $(1,0)$ and $(1,3)$. Find the coordinates of its centre of mass. ['Homogeneous' means the plate is all made of the same material which is uniformly distributed across it, so that $\sigma(x,y) = \sigma$, a constant.]

Solution: A diagram of the triangle would be useful. With σ constant, we have

$$\begin{aligned}
\bar{x} &= \frac{\iint_D x\sigma(x,y)dA}{\iint_D \sigma(x,y)dA} = \frac{\sigma \int_0^1 \int_0^{3x} xdydx}{\sigma \int_0^1 \int_0^{3x} dydx} \\
&= \frac{\int_0^1 \{xy\}|_{y=0}^{y=3x}\, dx}{\int_0^1 \{y\}|_{y=0}^{y=3x}\, dx} = \frac{\int_0^1 3x^2 dx}{\int_0^1 3x dx} \\
&= \frac{1}{3/2} = \frac{2}{3}
\end{aligned}$$

and

$$\bar{y} = \frac{\iint_D y\sigma(x,y)dA}{\iint_D \sigma(x,y)dA} = \frac{\sigma \int_0^1 \int_0^{3x} ydydx}{\sigma \int_0^1 \int_0^{3x} dydx}$$

$$= \frac{\int_0^1 \{\frac{y^2}{2}\}\Big|_{y=0}^{y=3x} dx}{\int_0^1 \{y\}|_{y=0}^{y=3x} dx} = \frac{\int_0^1 \frac{9x^2}{2} dx}{\int_0^1 3xdx}$$

$$= \frac{3/2}{3/2} = 1.$$

So the centre of mass is at $(\bar{x}, \bar{y}) = \left(\frac{2}{3}, 1\right)$.