Calculus for Engineers

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Continuity

4.1 Introduction

In the phrase "drawing without lifting your pencil, no break, no hold or no kink", the concept of the continuity of the function is examined in this chapter. Each concept is explained and illustrated by worked examples.

4.2 Continuity

In this section, we shall consider the following cases which show the relation between the limit of a function and the value of the function at a point c:

(a) $\lim_{x\to c} f(x) = k$, $f(c)$ is not defined.

(b) $\lim_{x\to c} f(x) = k$, $f(c)$ exists but $k \neq f(c)$.

(c) $\lim_{x\to c} f(x) = k$, $f(c)$ is defined and $k = f(c)$.

(d) $\lim_{x\to c} f(x)$ does not exist, $f(c)$ is defined.

(e) $\lim_{x\to c} f(x)$ does not exist, $f(c)$ is not defined.

These cases can be shown in Figure 4.1. In all the above cases, except in the case of (c), the graph has a break at the point $x = c$. The function in (c) is said to be continuous at $x = c$. In all the other cases, the function is said to be discontinuous at $x = c$.

To be more precise, we have:

Definition 1 A function f is said to be continuous at $x = c$ provided the following conditions are satisfied:

1. $f(c)$ is defined.
2. $\lim_{x\to c} f(x)$ exists; and
3. $\lim_{x\to c} f(x) = f(c)$.

If one or more of these conditions are not satisfied by the function $x = c$ then it is said to be discontinuous at $x = c$.

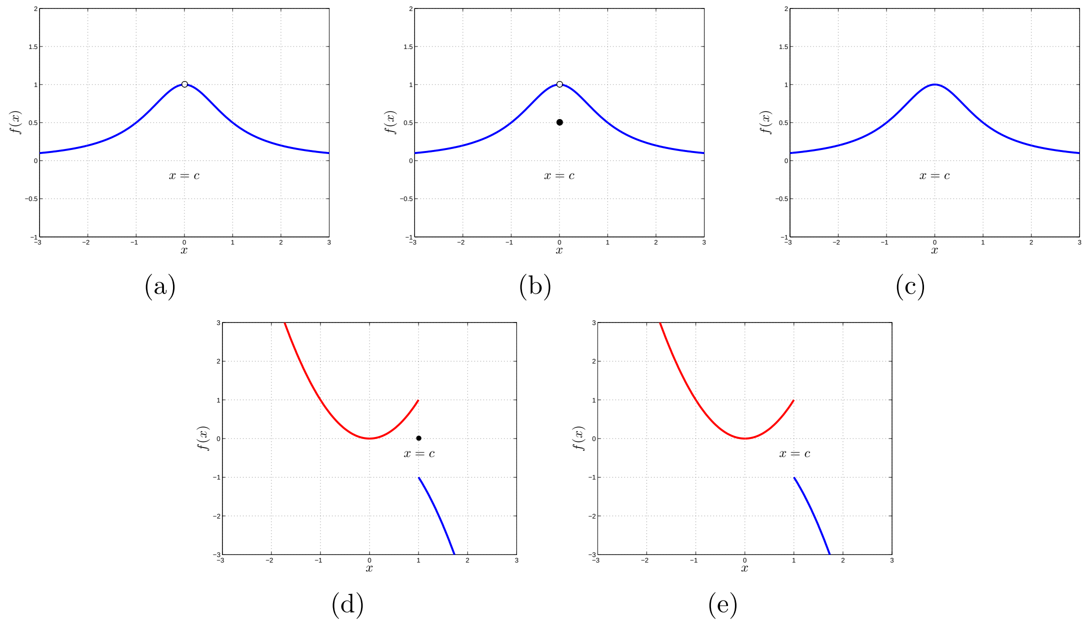

Figure 4.1

Using the definition of the limit, we can define the continuity of a function at a point $x = c$ as follows:

Definition 2 Given an $\epsilon > 0$, however small, there exists a $\delta > 0$, such that

$$|f(x) - f(c)| < \epsilon \text{ whenever } 0 < |x - c| < \delta.$$

That is,

$$\lim_{x \to c} f(x) = f(c).$$

Note 1 If the limit of the function as x tends to c exists, then $\lim_{x \to c^-} f(x)$ exists, $\lim_{x \to c^+} f(x)$ exists and both these limits are equal.

Definition 3 A function $f(x)$ defined in an interval $[a, b]$ is said to be continuous in this interval if it is continuous at every point in the interval.

Note 2 In the above interval $[a, b]$, $f(x)$ is not defined for $x < c$ and hence $\lim_{x \to a^-} f(x)$ does not exist. Similarly, $\lim_{x \to b^+} f(x)$ does not exist. Hence if $f(x)$ is continuous in $[a, b]$, then it is right continuous at a and left continuous at b and continuous at all other points in this interval.

Example 1 Let

$$f(x) = \begin{cases} \dfrac{\sin 5x}{x}, & x \neq 0; \\ 1, & x = 0. \end{cases}$$

Is the function continuous at $x = 0$?

Solution. We shall investigate the three conditions to be satisfied by $f(x)$ for its continuity at $x = 0$, as illustrated in Figure 4.2.

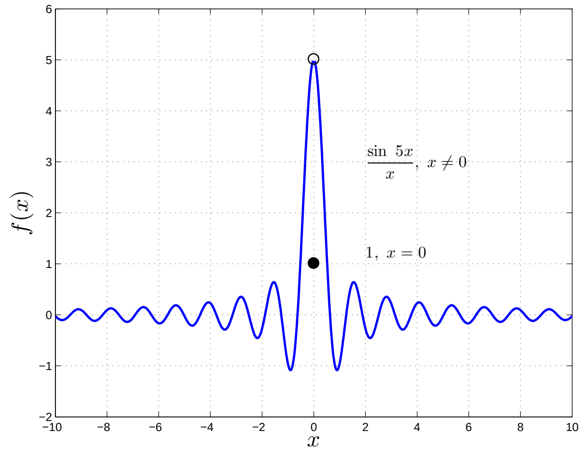

Figure 4.2: $f(x)$

1.
$$\begin{aligned}\lim_{x\to 0}\frac{\sin 5x}{x} &= \lim_{x\to 0}\frac{\sin 5x}{5x}\cdot 5\\ &= 5\cdot\lim_{x\to 0}\frac{\sin 5x}{5x}\\ &= 5\cdot 1 = 5.\end{aligned}$$

   Condition 1 is satisfied.

2. $f(0)$ is defined to be equal to 1. Hence Condition 2 is satisfied.

3.
$$\lim_{x\to 0} f(x) \neq f(0) = 1.$$

   Condition 3 is not satisfied.

Hence the function is discontinuous at $x = 0$. $\square$

Example 2 The signum function (or sign function), denoted by sgn, is defined by

$$\mathsf{sgn}\ x = \begin{cases} -1, & x < 0;\\ 0, & x = 0;\\ 1, & x > 0.\end{cases}$$

Solution. It is clear that

$$\lim_{x\to 0^-} f(x) = -1 \quad \neq \quad \lim_{x\to 0^+} f(x) = 1,$$

Hence $\lim_{x\to 0} f(x)$ does not exist.

The first condition for the continuity of $f(x)$ at $x = 0$ is not satisfied. Even though the second condition is satisfied, the function is discontinuous at $x = 0$. $\square$

Example 3 Show that $f(x) = \sin x$ is continuous for all values of x.

Proof.

Let $x = c$ be any point. Therefore, $f(c) = \sin c$.
Let $\epsilon > 0$ be given.
Consider

$$\begin{aligned}
|f(x) - f(c)| &= |\sin x - \sin c| \\
&= 2\left|\cos\left(\frac{x+c}{2}\right)\sin\left(\frac{x-c}{2}\right)\right| \\
&= 2\left|\cos\left(\frac{x+c}{2}\right)\right|\left|\sin\left(\frac{x-c}{2}\right)\right| \\
&\leq 2\left|\sin\left(\frac{x-c}{2}\right)\right| \qquad \text{because } |\cos\theta| \leq 1.
\end{aligned}$$

But $|\sin\theta| \leq |\theta|$. Therefore, we have

$$|f(x) - f(c)| \leq 2\left|\frac{x-c}{2}\right|.$$

That is ,

$$|f(x) - f(c)| \leq |x - c|.$$

Hence, $|f(x) - f(c)| < \epsilon$ gives $|x - c| < \epsilon$.
Taking $\epsilon = \delta$, we get

$$|f(x) - f(c)| < \epsilon \text{ whenever } 0 < |x - c| < \delta = c.$$

Hence, $\sin x$ is continuous at $x = c$. □

Example 4 Let

$$f(x) = \begin{cases} 3 + x, & x \leq 1 \\ 3 - x, & x > 1. \end{cases}$$

Solution. Figure 4.3 depicts the graph of the function $f(x)$. We have

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1}(3 + x) = 4.$$

Similarly, we have

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1}(3 - x) = 2.$$

The first condition is not satisfied because

$$\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x).$$

Hence, the function is discontinuous at $x = 1$. □

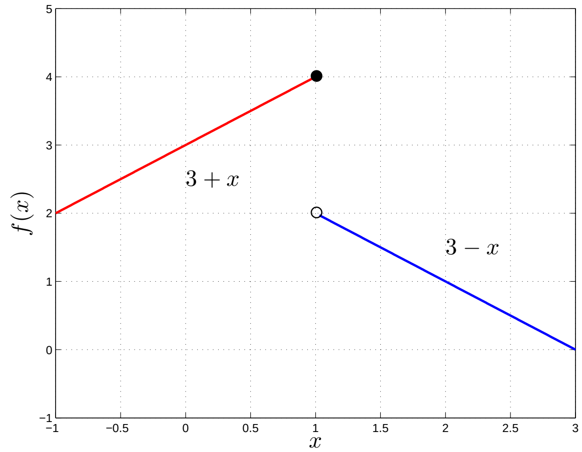

Figure 4.3: $f(x)$

Example 5 A function $f(x)$ is defined by

$$f(x) = \begin{cases} 1 - x, & x \leq -2; \\ 2 - x, & -2 < x \leq 2; \\ 2x - 4, & 2 < x. \end{cases}$$

Examine the function for continuity at $x = -2, 2$.

Solution. Figure 4.4 depicts the function $f(x)$.

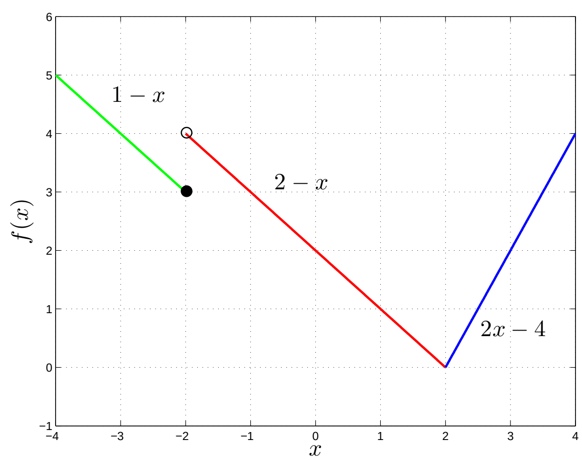

Figure 4.4: $f(x)$

- At $x = -2$, we have

$$\lim_{x \to -2^-} f(x) = \lim_{x \to -2} (1 - x) = 1 + 2 = 3.$$

Similarly, we have

$$\lim_{x \to -2^+} f(x) = \lim_{x \to -2} (2 - x) = 2 + 2 = 4.$$

The first condition is not satisfied because

$$\lim_{x\to -2^-} f(x) \neq \lim_{x\to -2^+} f(x).$$

Therefore, the function is discontinuous at $x = -2$.

- At $x = 2$, we have

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2}(2 - x) = 2 - 2 = 0.$$

Similarly, we have

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2}(2x - 4) = 2 \cdot 2 - 4 = 0.$$

The first condition is satisfied because

$$\lim_{x\to 2^-} f(x) = 0 = \lim_{x\to 2^+} f(x).$$

Therefore, $\lim_{x\to 2} f(x)$ exists and at $x = 2$, $f(2) = 0$.

Since the limit of the function as $x \to 2$ is equal to the value of the function at $x = 2$, the function is continuous at $x = 2$. □

Example 6 Show that the function

$$f(x) = x \sin\left(\frac{1}{x}\right)$$

for $x \neq 0$, $f(0)$, is continuous.

Solution. As x approaches zero either from the left or from the right, y approaches the origin. From the graph, it is clear that the limit of the function is zero as x tends to zero. We shall prove this analytically.

Let $\epsilon = \dfrac{1}{10} = 0.1$. We shall find $\delta > 0$ corresponding to the given ϵ.
Consider

$$|f(x) - 0| = \left| x \sin\left(\frac{1}{x}\right) - 0 \right| < \frac{1}{10}.$$

But,

$$\left| x \sin\left(\frac{1}{x}\right) \right| < |x|$$

since

$$\left| \sin\left(\frac{1}{x}\right) \right| < 1 \ (x \neq 0).$$

Thus, if $|x| < \frac{1}{10}$, then

$$\left|\sin\left(\frac{1}{x}\right)\right| < \frac{1}{10}.$$

Therefore, for the given $\epsilon = \frac{1}{10}$, $\delta = \frac{1}{10}$ is such that $\left|x\sin\left(\frac{1}{x}\right) - 0\right| < \frac{1}{10}$ whenever $|x - 0| < \frac{1}{10}$.

Instead of $\epsilon = \frac{1}{10}$, we may take $\epsilon = \frac{1}{100}, \frac{1}{1000}$, etc. For each of these values of ϵ, there exists a δ satisfying the inequality $\left|x\sin\left(\frac{1}{x}\right) - 0\right| < \frac{1}{10}$ whenever $|x - 0| < \frac{1}{10}$.

But $f(0) = 0$. Hence, the function $f(x)$ is continuous. □

4.3 Properties of continuous functions

Theorem 1 If f and g are two functions which are continuous at a point c, then

1. $f + g$ is continuous at c;
2. $f - g$ is continuous at c;
3. $f \cdot g$ is continuous at c;
4. $\frac{f}{g}$ is continuous at c, provided $g(c) \neq 0$.

Proof.

From the theorem on the properties of limits, we have

$$\lim_{x\to c}(f(x) + g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x). \tag{4.1}$$

Since $f(x)$ and $g(x)$ are continuous, we have

$$\lim_{x\to c} f(x) = f(c)$$

and

$$\lim_{x\to c} g(x) = g(c).$$

Therefore, (4.1) becomes

$$\lim_{x\to c}(f(x) + g(x)) = f(c) + g(c),$$

which is the value of $f(x) + g(x)$ at $x = c$.

Hence $f + g$ is continuous at $x = c$. □

Remark 1 The sum, difference, product and quotient of two continuous functions at any point $x = c$ in their domain is continuous at the same point $x = c$, provided in the case of the quotient, the divisor (the function in denominator) at the same point $x = c$ is not zero, i.e. if $f(x)$ and $g(x)$ are two functions continuous at any point $x = c$, then the functions $f(x) + g(x)$, $f(x) - g(x)$, $f(x) \cdot g(x)$ $\dfrac{f(x)}{g(x)}$, $g(c) \neq 0$, are also continuous at the same point $x = c$.

Corollary 2 A polynomial function

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n$$

is continuous for all values of x.

Proof.

Let $x = c$ be any point. We have

$$\lim_{x\to c} a_n = a_n, \ \lim_{x\to c} a_0a^n = a_0c^n \text{ and so on}$$

for all values of n.

From the above theorem, it follows that

$$\lim_{x\to c} f(x) = a_0c^n + a_1c^{n-1} + a_2c^{n-2} + \cdots + a_n = f(c).$$

Hence, the polynomial function $f(x)$ is continuous. □

Theorem 3 If f is a function which is continuous at a point c, then

1. kf is continuous at c, where $k \in \mathbb{R}$;
2. $\dfrac{1}{f}$ is continuous at c, provided $f(c) \neq 0$.

Remark 2 The scalar multiple and reciprocal of a function that are continuous at a point in their domain are continuous at the same point, provided in the case of the reciprocal of a function, the function in the denominator is not zero at the point where the continuity of y is required to be tested, i.e., if $f(x)$ is a function continuous at a point $x = c$ and k is any constant, then the functions $y = kf(x)$ and $y = \dfrac{1}{f(x)}$, $f(c) \neq 0$ are also continuous at the same point $x = c$.

Theorem 4 If $\lim_{x\to c} g(x) = b$ and if the function f is continuous at b, then $\lim_{x\to c} f(g(x)) = f(b)$. That is

$$\lim_{x\to c} f(g(x)) = f\left(\lim_{x\to c} g(x)\right).$$

This equality remains valid if $\lim_{x\to c}$ is replaced everywhere by one of $\lim_{x\to c^+}$, $\lim_{x\to c^-}$ or $\lim_{x\to +\infty}$, or $\lim_{x\to -\infty}$.

Theorem 5

1. If the function g is continuous at c, and the function f is continuous at $g(c)$, then the composition $f \circ g$ is continuous at c.

2. If the function g is continuous everywhere and the function f is continuous everywhere, then the composition $f \circ g$ is continuous everywhere.

4.4 Types of discontinuities

If a function is discontinuous at a point, then that point is said to be a point of discontinuity. By investigating various possibilities, the types of discontinuity for $f(x)$ at $x = c$ are classified.

4.4.1 Removable discontinuity

If $\lim_{x \to c} f(x)$ exists and is not equal to $f(c)$, then c is said to be a point of removable discontinuity.

In this case, the function f can be made to be continuous at c by defining

$$f(c) = \lim_{x \to c} f(x).$$

Example 7 Let

$$f(x) = \begin{cases} \dfrac{\sin(x-5)}{(x-5)}, & x \neq 5; \\ 5, & x = 5. \end{cases}$$

$f(x)$ is discontinuous for $x = 5$ since

$$\lim_{x \to 5} f(x) \neq f(5).$$

The function has a removable discontinuity. In fact, we can redefine $f(5) = 1$ so that f becomes continuous at $x = 5$, as illustrated in Figure 4.5. □

4.4.2 Jump discontinuity

If $\lim_{x \to c^-} f(x)$ and $\lim_{x \to c^+} f(x)$ both exist but are unequal, then c is said to be a point of jump discontinuity.

The point c is said to be a discontinuity of $f(x)$ from the left or from the right depending on whether

$$\lim_{x \to c^-} f(x) \neq f(a)$$

or

$$\lim_{x \to c^+} f(x) \neq f(a).$$

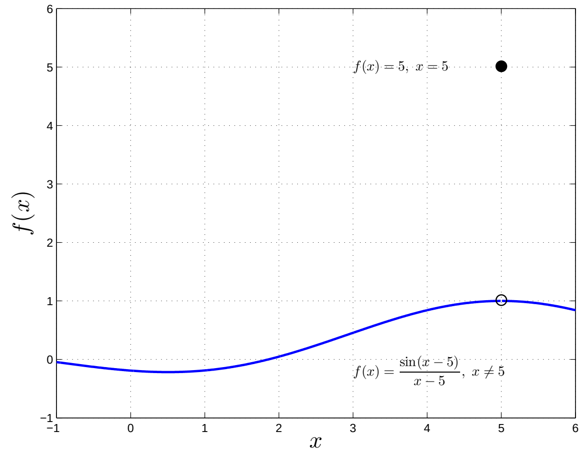

Figure 4.5: $f(x)$, where $x \in [-1, 6]$.

Example 8 A function f is defined as follows:

$$f(x) = \begin{cases} 5x+9, & x > 1; \\ 14x-9, & x < 1; \\ 14, & x = 1. \end{cases}$$

Examine the type of discontinuity at $x = 1$.

Solution. We have

$$\lim_{x\to 1^+} f(x) = 14,\ f(1) = 14 \text{ and } \lim_{x\to 1^-} f(x) = 5.$$

Here, we have

$$\lim_{x\to 1^-} f(x) \neq f(1) = \lim_{x\to 1^+} f(x).$$

Therefore, f has a jump discontinuity from the left at $x = 1$. $\square$

Example 9 A function f is defined as follows:

$$f(x) = \begin{cases} \frac{1}{1-e^{-1/x}}, & x \neq 0; \\ 0, & x = 0. \end{cases}$$

Examine the type of discontinuity at $x = 0$.

Solution. We have

$$\begin{aligned} \lim_{x\to 0^-} f(x) &= \lim_{h\to 0} \frac{1}{1-e^{-1/(0-h)}},\ h > 0 \\ &= \lim_{h\to 0} \frac{1}{1-e^{1/h}} \\ &= \lim_{h\to 0} \frac{1}{1-e^{1/h}} \cdot \frac{e^{-1/h}}{e^{-1/h}} \\ &= \lim_{h\to 0} \frac{e^{-1/h}}{e^{-1/h}-1} \\ &= 0. \end{aligned}$$

But, we have

$$\begin{aligned}\lim_{x\to 0^+} f(x) &= \lim_{h\to 0} \frac{1}{1-e^{-1/(0+h)}}, \quad h>0 \\ &= \lim_{h\to 0} \frac{1}{1-e^{-1/h}} \\ &= 1.\end{aligned}$$

Here, we have

$$\lim_{x\to 0^-} f(x) = f(0) \neq \lim_{x\to 0^+} f(x).$$

Therefore, f has a jump discontinuity from the right at $x = 0$. □

4.4.3 Infinite discontinuity

If $f(x)$ is said to have an infinite (or essential) discontinuity at $x = c$ from the left or from the right depending on whether

$$\lim_{x\to c^-} f(x)$$

or

$$\lim_{x\to c^+} f(x)$$

does not exist.

If neither of these two limits exists then c is also said to be a point of infinite discontinuity.

Example 10 Consider

$$f(x) = \sin\left(\frac{1}{x}\right)$$

at $x = 0$.

We have seen that

$$\lim_{x\to 0^-} f(x)$$

and

$$\lim_{x\to 0^+} f(x)$$

do not exist. Therefore, f has an infinite discontinuity at $x = 0$. □

4.5 Theorems on continuous functions

4.5.1 Bounds of a function

Let f be a function defined over a closed interval $[a, b]$. As x varies in this interval, $f(x)$ assumes varying values. If these are two numbers m and M such that $m \leq f(x) \leq M$ for every value of x in the interval, then m and M are, respectively, called the lower and upper bounds of f over $[a, b]$.

For any given small positive number ϵ, if there is at least one value of x in the interval such that $f(x) < m + \epsilon$ and at least one value of x such that $f(x) > M - \epsilon$, then we say that m and M are the greatest lower and least upper bounds (g.l.b. and l.u.b.) of the function in the interval $[a, b]$. M and m are also called supermum and infinmum respectively.

Example 11 The function $\sin x$ defined on the interval $(0, 2\pi)$ has a least upper bound $= -1$ and a greatest lower bound $= 1$. □

Example 12 Consider f defined on $[0, 1]$ by

$$f(x) = \begin{cases} \frac{1}{x}, & x \neq 0; \\ 0, & x = 0. \end{cases}$$

The function f is not bounded in $[0, 1]$. □

Example 13 Consider f defined on $[0, \infty)$ by

$$f(x) = \frac{x}{x+1}, \quad x \in [0, \infty).$$

The function f has a least upper bound $= 1$ and a greatest lower bound 0. Hence, it is bounded. □

4.5.2 Continuity and boundaries

Theorem 6 Let f be continuous in the closed interval $[a, b]$. Then f is bounded in $[a, b]$.

Proof.

Suppose f is not bounded in $[a, b]$.

- Let us divide $[a, b]$ into two equal intervals $[a, c]$ and $[c, b]$, where $c = \frac{a+b}{2}$. Then by our assumption, it follows that f is not bounded in at least one of these intervals. Choose one such interval in which f is not bounded and call it $[a_1, b_1]$.

  We now have an interval $[a_1, b_1]$ such that

  1. $a \leq a_1, \ b \leq b_1$
  2. $b_1 - a_1 = \frac{b-a}{2}$
  3. f is not bounded in $[a_1, b_1]$.

- Let us divide $[a_1, b_1]$ into two equal intervals $[a_1, c_1]$ and $[c_1, b_1]$, where $c_1 = \frac{a_1+b_1}{2}$. As before choose the interval in which f is not bounded and call it $[a_2, b_2]$.

  We now have an interval $[a_2, b_2]$ such that

  1. $a_1 \leq a_2, \ b_1 \leq b_2$
  2. $b_2 - a_2 = \frac{b_1-a_1}{2} = \frac{b-a}{2^2}$
  3. f is not bounded in $[a_2, b_2]$.

- Proceeding in this manner, we construct what is called a nest of intervals

$$[a, b], [a_1, b_1], [a_2, b_2], \cdots, [a_n, b_n], \cdots$$

with the following properties

1. $$a \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq a_{n+1} \leq \cdots \leq b,$$

$$b \geq b_1 \geq b_2 \geq \cdots \geq b_n \geq b_{n+1} \geq \cdots \geq a;$$

2. $b_n - a_n = \frac{b-a}{2^n}$,
3. f is not bounded in $[a_n, b_n]$,

for $n = 1, 2, 3, \cdots$.

Now $a_1, a_2, \cdots, a_n, \cdots$ is a monotonic increasing sequence which is bounded as above.

Hence the sequence tends to a limit as n tends to infinity. Similarly, the sequence $\{b_n\}$ tends to a limit as n tends to infinity. But

$$\lim_{n\to\infty} (b_n - a_n) = \lim_{n\to\infty} \frac{b-a}{2^n} = 0.$$

Therefore, we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n.$$

If this limit is x_0 where $x_0 \in [a, b]$, then $x_0 \in [a_n, b_n]$ for all n.

Now, f is continuous at x_0 since it is continuous in $[a, b]$. Therefore, given a positive number ϵ we can find a $\delta > 0$ such that

$$|f(x) - f(x_0)| < \epsilon \text{ whenever } |x - x_0| < \delta.$$

This implies that

$$f(x_0) - \epsilon < f(x) < f(x_0) + \epsilon \text{ whenever } x_0 - \delta < x < x_0 + \delta.$$

Hence, f is bounded in $x_0 - \delta, x_0 + \delta$. Since a_n tends to x_0 and b_n tends to x_0, we can find an interval $[a_n, b_n]$ contained in $x_0 - \delta, x_0 + \delta$. It follows that f is bounded in $[a_n, b_n]$. This contradicts the property 3 that f is not bounded in $[a_n, b_n]$. Hence our assumption that f is not bounded in $[a, b]$ is wrong. Therefore, f is bounded in $[a, b]$.

□

Example 14 Let $f(x) = \frac{1}{x}$ in the open interval $(0, 1)$ and f is continuous in $(0, 1)$. As $x \to 0^+$, $\frac{1}{x} \to +\infty$. Therefore, f is not bounded in $(0, 1)$. □

Example 15 Let $f(x) = \log x$, $x \in (0, \infty)$ and f is continuous in $(0, \infty)$. However, f is not bounded. $\square$

Example 16 The function $f(x) = \sin x$ is a bounded function for all $x \in \mathbb{R}$ since $-1 \leq \sin x \leq 1$ for all $x \in \mathbb{R}$. $\square$

Theorem 7 A function which is continuous in the closed interval $[a, b]$ attains its bounds at least once in the interval.

Proof.

We know that f is bounded. Let M and m be its bounds. It has to be shown that there exist points α and $\beta \in [a, b]$ such that $f(\alpha) = M$ and $f(\beta) = m$.

Let us assume that $f(x)$ does not attain the value of M for any $x \in [a, b]$.

Therefore, we have

$$M - f(x) \neq 0 \quad \text{for} \quad x \in [a, b] \tag{4.2}$$

Also, $M - f(x)$ is continuous and hence $\frac{1}{M-f(x)}$ is also continuous in $[a, b]$ and therefore bounded in this interval.

Let ϵ be any given number. Since M is the supermum of f, there exists a $\xi \in [a, b]$ such that

$$f(\xi) > M - \frac{1}{\epsilon}.$$

This implies that there exists $\xi \in [a, b]$ such that

$$\frac{1}{M - f(\xi)} > \epsilon$$

Since ϵ is any given number, this shows that $\frac{1}{M-f(x)}$ is not bounded.

Thus we have two contradictory statements. Therefore, our assumption that f does not attain its supermum is wrong. It may similarly be shown that f attains its infimum at least once. $\square$

Corollary 8 If f is continuous in $[a, b]$, then the range of f is a subset of $[m, M]$.

Example 17 The function $y = x$ is continuous on $\mathbb{R}$ but it is unbounded on every infinite interval. $\square$

Example 18 Let $f(x) = \dfrac{x^2}{x^2 + 1}$. The function $f(x)$ is continuous and bounded on $\mathbb{R}$. It attains its infimum 0 but not the supermum 1 on any interval. Here, $f(x)$ is not continuous on the closed interval ($\mathbb{R}$ is open.) Hence the above theorem is not applicable here. $\square$

Theorem 9 If f is continuous from the left to right at $x = c$ and $f(c) \neq 0$, there exists $\delta > 0$ such that $f(x)$ keeps the sign of $f(c)$ according to either $x \in (c - \delta, c]$ or $x \in [c, c + \delta)$

Proof.

Let f be continuous from the left at $x = c$ and $f(c) > 0$.

Therefore for $\epsilon = f(c) > 0$ there exists $\delta > 0$ such that for $x \in (c - \delta, c]$

$$|f(x) - f(c)| < \frac{f(c)}{2}$$

which implies

$$f(c) - \frac{f(c)}{2} < f(x) < f(c) + \frac{f(c)}{2}.$$

That is,

$$\frac{f(c)}{2} < f(x) < \frac{3f(c)}{2}, \quad x \in x \in (c - \delta, c].$$

Considering

$$\frac{f(c)}{2} < f(x),$$

we get

$$f(x) > \frac{f(c)}{2} > 0 \text{ for every } x \in (c - \delta, c].$$

That is, $f(x)$ has the same sign as $f(c)$ in $(c - \delta, c]$

The other cases can be similarly proved. □

If $f(c) > 0$, the point $[c, f(c)]$ lies above the $x-$axis if we consider the graph of $f(x)$. By the property of continuity there must be an interval $(c - \delta, c + \delta)$ in which the entire graph is above the $x-$axis.

Corollary 10 If $f(x)$ is continuous at $x = c$ and $f(c) \neq 0$ there exists $\delta > 0$ such that $f(x)$ has the same sign as $f(c)$ for every $x \in (c - \delta, c + \delta)$.

4.5.3 Intermediate Value Theorem

Theorem 11 If f is continuous in $[a, b]$ and $f(a)$ and $f(b)$ have opposite signs, there exists a $\xi \in [a, b]$ such that $f(\xi) = 0$.

Proof.

Let $f(a) > 0$ and $f(b) < 0$.

The set of points $x \in [a, b]$ such that $f(x) > 0$ is infinite. By the previous theorem, there exists an interval $[a, a + \delta)$ for every point x for which $f(x) > 0$. Let this set be denoted by S. If ξ is the supermum of S, we shall show that $f(\xi) = 0$.

Let $f(\xi) < 0$. There exists a $\delta > 0$ such that

$$f(x) < 0 \text{ for } x \in (\xi - \delta, \xi].$$

Also, since ξ is the supermum of S, there exists $\eta \in S$ such that $\xi - \delta < \eta \leq \xi$.

Now $\eta \in S$ implies $f(\eta) > 0$.

Also, $\eta \in (\xi - \delta, \xi]$ implies $f(\eta) < 0$.

Hence, we arrive at a contradiction thus proving that $f(\xi)$ is not less than 0.

Now, suppose $f(\xi) > 0$. Surely $\xi \neq b, \xi \neq a$.
There exists a $\delta > 0$ such that

$$f(x) > 0 \text{ for } x \in (\xi - \delta, \xi + \delta).$$

But, $\xi + \delta \in S$ is a contradiction of ξ being the supermum of S.
Therefore, $f(\xi) = 0$. □

Geometrically speaking, a curve which is starts below the $x-$axis and then goes above it must cross the $x-$axis at some point.

A more general form of the intermediate value theorem is

Theorem 12 If f is continuous in $[a, b]$ and $f(a) \neq f(b)$, then f assumes every value between $f(a)$ and $f(b)$.

Proof.

Let ξ be a number between $f(a)$ and $f(b)$.
Consider a function $g(x)$ such that

$$g(x) = f(x) - \xi.$$

Now, $g(x)$ is continuous in $[a, b]$ and

$$g(a) = f(a) - \xi < 0$$

and

$$g(b) = f(b) - \xi > 0.$$

That is, $g(a)$ and $g(b)$ are of opposite signs.
Therefore, there exists a point $c \in [a, b]$ such that $g(c) = 0$. That is,

$$f(c) - \xi = 0$$

or

$$f(c) = \xi.$$

Hence, the theorem is proved. □

Corollary 13 We have seen that if f is continuous in $[a, b]$, then the range of f is a subset of $[m, M]$. From this theorem it follows that $[m, M]$ is itself the range.

Note 3 The Intermediate Value Theorem states that as x changes from a to b, the continuous function f takes on every value between $f(a)$ and $f(b)$. If we think the graph of the continuous function f has no holes or breaks or kinks, as illustrated in Figure 4.6, then for any number ξ between $f(a)$ and $f(b)$, the horizontal line with $y-$intercept ξ intersects the graph at least one point P. The $x-$coordinate c (i.e., c_1, c_2, c_3) of P (i.e., P_1, P_2, P_3) is a number such that $f(c) = \xi$.

It is important to mention that the function f in Theorem 12 is continuous. The Intermediate Value Theorem is not true in general for discontinuous functions. Figure 4.7 shows this case.

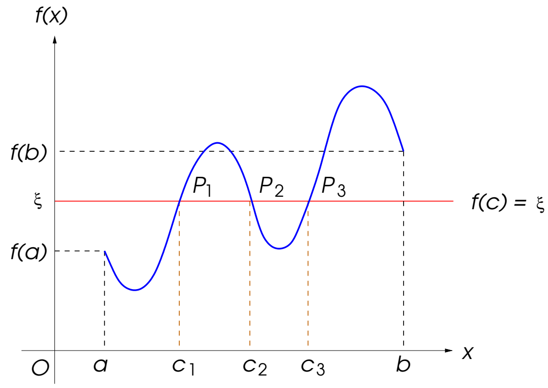

Figure 4.6: $f(x)$

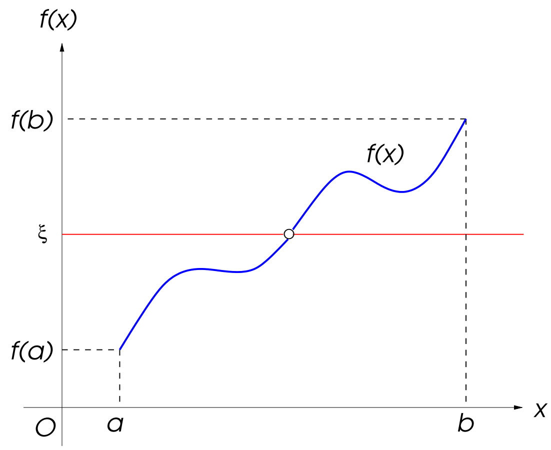

Figure 4.7: $f(x)$

A nice application of the Intermediate Value Theorem is to prove the existence of roots of equations as the following example shows.

Example 19 Show that there is a root of the equation

$$x^5 + x^2 - 10 = 0$$

between 1 and 2.

Solution. Let $f(x) = x^5 + x^2 - 10 = 0$. We are seeking a solution for the given equation; that is, we are looking for a number c between 1 and 2 such that $f(c) = 0$. Therefore we take $a = 1, b = 2$, and $\xi = 0$ in Theorem 11.

Since

$$f(1) = (1)^5 + (1)^2 - 10 = -8 < 0,$$
$$f(2) = (2)^5 + (2)^2 - 10 = 26 > 0,$$

then ξ is a number between $f(1)$ and $f(2)$. f is continuous on $[1, 2]$ since it is a polynomial. By the intermediate value theorem (Theorem 12), there is a number c between 1 and 2 such that $f(c) = 0$. That is to say there is a root of the given equation between 1 and 2.
□