THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1510 Calculus for Engineers (2020-2021)
Supplementary Exercise 1

Trigonometry

Change of Units

1. Fill in the blanks.

$$
\begin{aligned}
15^\circ &= \underline{\hspace{4cm}}\ \text{rad};\\
30^\circ &= \underline{\hspace{4cm}}\ \text{rad};\\
\underline{\hspace{4cm}} &= \frac{\pi}{4}\ \text{rad};\\
\underline{\hspace{4cm}} &= \frac{\pi}{3}\ \text{rad};\\
\underline{\hspace{4cm}} &= \frac{\pi}{2}\ \text{rad};\\
120^\circ &= \underline{\hspace{4cm}}\ \text{rad};\\
\underline{\hspace{4cm}} &= \frac{5\pi}{6}\ \text{rad};\\
180^\circ &= \underline{\hspace{4cm}}\ \text{rad};\\
270^\circ &= \underline{\hspace{4cm}}\ \text{rad};\\
360^\circ &= 2\pi\ \text{rad}\,.
\end{aligned}
$$

Trigonometric Identities

2. Find $\tan 75^\circ$ and express your answer in surd form.

3. Find $\cos 165^\circ$ and $\sin 165^\circ$ and express your answers in surd form.

4. Find $\cos^2 \dfrac{7\pi}{12}$ and $\sin^2 \dfrac{7\pi}{12}$ and express your answers in surd form.

5. By using the product to sum formula, express each of the following expressions as a sum of trigonometric functions.

   (a) $\cos 5x \cos 3x$;

   (b) $\sin 4x \sin 2x$;

   (c) $\sin 7x \cos 3x$.

6. Show that $\sin 2x \cos 3x \cos 5x = \dfrac{1}{4}(\sin 4x - \sin 6x + \sin 10x)$.

7. Show that $\sin 3x \sin 4x \cos 5x = \dfrac{1}{4}(-\cos 2x + \cos 4x + \cos 6x - \cos 12x)$.

8. Prove that $\dfrac{\cos(x+y)+\cos(x-y)}{\sin(x-y)-\sin(x+y)} = -\cot y$.

9. Prove that $\dfrac{1}{\tan(x+y)-\tan(x-y)} = \dfrac{\cos 2x}{2\sin 2y} + \dfrac{\cot 2y}{2}$.

10. Prove that $\tan\dfrac{x+y}{2} = \dfrac{\sin x+\sin y}{\cos x+\cos y}$.

11. Let $t = \tan\dfrac{x}{2}$.

    (a) By considering $\tan x = \tan\left(2\cdot\dfrac{x}{2}\right)$, show that $\tan x = \dfrac{2t}{1-t^2}$.

    (b) By using the result in (a), express $\sin x$ and $\cos x$ in terms of t.

    (Remark: The result of this question will be useful for integration of trigonometric function, called t-substitution.)

12. Prove that $\cot\dfrac{x}{2} = \dfrac{1+\cos x}{\sin x}$.

13. Prove the following identities:

    (a) $\sin^4 x = \dfrac{3-4\cos 2x+\cos 4x}{8}$;

    (b) $\sin^5 x = \dfrac{10\sin x-5\sin 3x+\sin 5x}{16}$;

    (c) $\cos^4 x = \dfrac{3+4\cos 2x+\cos 4x}{8}$;

    (d) $\cos^5 x = \dfrac{10\cos x+5\cos 3x+\cos 5x}{16}$;

    (e) $\sin^4 x\cos^4 x = \dfrac{3-4\cos 4x+\cos 8x}{128}$;

    (f) $\sin^5 x\cos^5 x = \dfrac{10\sin 2x-5\sin 6x+\sin 10x}{512}$.

14. Show that $\sin^2 x\cos^4 x = \dfrac{1}{32}(2+\cos 2x-2\cos 4x-\cos 6x)$.

15. Prove the following identites (called triple angle formula):

    (a) $\sin 3x = 3\sin x - 4\sin^3 x$;

    (b) $\cos 3x = 4\cos^3 x - 3\cos x$;

    (c) $\tan 3x = \dfrac{3\tan x-\tan^3 x}{1-3\tan^2 x}$.

16. Given that A, B, C, D are four interior angles of a quadrilateral $ABCD$.

    Prove that

$$\cos A+\cos B+\cos C+\cos D = -4\cos\frac{A+B}{2}\cos\frac{A+C}{2}\cos\frac{A+D}{2}.$$

17. If $A+B+C=\pi$, show that

    (a) $\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C$;

    (b) $\tan A+\tan B+\tan C=\tan A\tan B\tan C$;

    (c) $\cot A\cot B+\cot B\cot C+\cot C\cot A=1$.

18. Prove that for any $x\neq 2m\pi$, m is an integer,

$$1+2\cos x+2\cos 2x+2\cos 3x+\cdots+2\cos nx=\frac{\sin(n+\frac{1}{2})x}{\sin\frac{x}{2}}.$$

General Solutions of Trigonometric Equations

19. (General Solutions of Trigonometric Equations)

    - If $\sin x=p$, then let $\alpha=\sin^{-1}(p)$, then all solutions of the equation $\sin x=p$ are in form of $n\pi+(-1)^n\alpha$ where n is an integer;
    - If $\cos x=p$, then let $\alpha=\cos^{-1}(p)$, then all solutions of the equation $\cos x=p$ are in form of $2n\pi\pm\alpha$ where n is an integer;
    - If $\tan x=p$, then let $\alpha=\tan^{-1}(p)$, then all solutions of the equation $\tan x=p$ are in form of $n\pi+\alpha$ where n is an integer.

    By using the above, solve the following equations.

    (a) $\sin x=\dfrac{1}{2}$;

    (b) $\cos x=-\dfrac{\sqrt{3}}{2}$;

    (c) $\tan x=-\sqrt{3}$.

20. Solve the following equations.

    (a) $\cos 5x=\dfrac{1}{2}$, where $0\leq x<2\pi$; (Hint: $0\leq 5x<10\pi$.)

    (b) $\sin 4x=\sin 24^\circ$, where $0^\circ\leq x<180^\circ$;

    (c) $\tan 3x=1$, where $\pi\leq x<2\pi$.

21. Solve $\sin 7x-\sin x=\cos 4x$ for $0^\circ\leq x\leq 180^\circ$.

22. Solve $\sin x\sin 2x=\cos 3x\cos 4x$ for $0\leq x\leq\dfrac{\pi}{2}$.