THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1510 Calculus for Engineers (Fall 2021)
Suggested solutions of homework 5
Deadline: November 27 at 23:00

Part A:

1. Evaluate the following definite integrals.

(a) $\displaystyle\int_0^2 e^{\sqrt{x}} dx.$

(b) $\displaystyle\int_{2/\sqrt{3}}^2 \frac{\sqrt{x^2-1}}{x} dx;$

Solution:

(a) Let $t = \sqrt{x}$, or $x = t^2$, then $dx = 2t\,dt$,

$$\begin{aligned}\int_0^2 e^{\sqrt{x}}\,dx &= \int_0^{\sqrt{2}} e^t \cdot 2t\,dt = 2\int_0^{\sqrt{2}} t\,de^t = 2te^t\Big|_0^{\sqrt{2}} - 2\int_0^{\sqrt{2}} e^t\,dt \\ &= 2\sqrt{2}e^{\sqrt{2}} - 2e^t\Big|_0^{\sqrt{2}} = 2\sqrt{2}e^{\sqrt{2}} - 2e^{\sqrt{2}} + 2.\end{aligned}$$

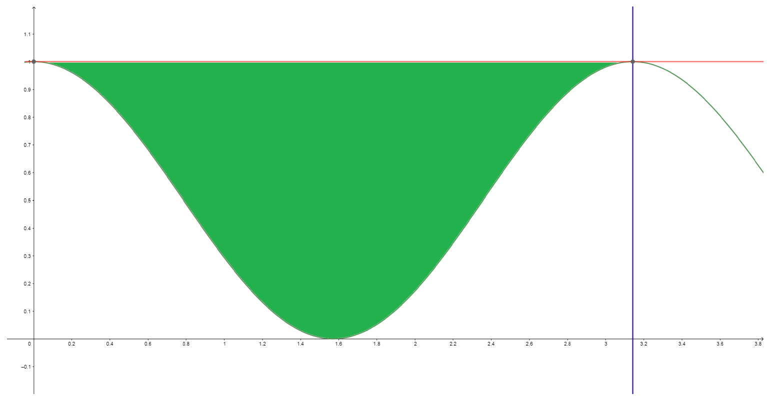
(b) Let $x = \sec t,\ t \in [\pi/6, \pi/3]$, then $dx = \sec t \tan t\,dt$,

$$\begin{aligned}\int_{2/\sqrt{3}}^2 \frac{\sqrt{x^2-1}}{x} dx &= \int_{\pi/6}^{\pi/3} \frac{\tan t}{\sec t} \cdot \sec t \tan t\,dt = \int_{\pi/6}^{\pi/3} \tan^2 t\,dt \\ &= \int_{\pi/6}^{\pi/3} (\sec^2 t - 1)\,dt = \tan t\Big|_{\pi/6}^{\pi/3} - \left(\frac{\pi}{3} - \frac{\pi}{6}\right) \\ &= \frac{2\sqrt{3}}{3} - \frac{\pi}{6}.\end{aligned}$$

2. Let R be the region bounded between the curves $y = 1$ and $y = \cos^2 x$ for $0 \le x \le \pi$.

   (a) Find the volume of the solid generated by rotating the region R about the x-axis.

   (b) Find the volume of the solid generated by rotating the region R about the line $y = 1$.



Solution:

(a)

$$\begin{aligned}
\text{Volume} &= \int_0^{\pi} \left(\pi(1)^2 - \pi(\cos^2 x)^2\right)\, dx \\
&= \pi^2 - \pi \int_0^{\pi} \cos^4 x\, dx \\
&= \pi^2 - \pi \int_0^{\pi} \left(\frac{1+\cos 2x}{2}\right)^2\, dx \\
&= \pi^2 - \frac{\pi}{4} \int_0^{\pi} \left(1 + 2\cos 2x + \cos^2 2x\right)\, dx \\
&= \pi^2 - \frac{\pi}{4} \int_0^{\pi} \left(1 + 2\cos 2x + \frac{1+\cos 4x}{2}\right)\, dx \\
&= \pi^2 - \frac{\pi}{4} \left(\frac{3x}{2} + \sin 2x + \frac{1}{8}\sin 4x\right)\Bigg|_0^{\pi} \\
&= \pi^2 - \frac{3\pi^2}{8} = \frac{5\pi^2}{8}.
\end{aligned}$$

(b)

$$
\begin{aligned}
\text{Volume} &= \int_0^\pi \pi \left(1-\cos^2 x\right)^2\, dx \\
&= \pi \int_0^\pi \sin^4 x\, dx \\
&= \pi \int_0^\pi \left(\frac{1-\cos 2x}{2}\right)^2\, dx \\
&= \ \frac{\pi}{4} \int_0^\pi \left(1 - 2\cos 2x + \cos^2 2x\right)\, dx \\
&= \ \frac{\pi}{4} \int_0^\pi \left(1 - 2\cos 2x + \frac{1+\cos 4x}{2}\right)\, dx \\
&= \frac{\pi}{4} \left(\frac{3x}{2} - \sin 2x + \frac{1}{8}\sin 4x\right)\Big|_0^\pi \\
&= \frac{3\pi^2}{8}.
\end{aligned}
$$

Part B:

3. Let R be the region bounded by curve $x = -6y^2 + 4y$ and the line $x + 3y = 0$ on the $xy-$plane. Find the area of R.

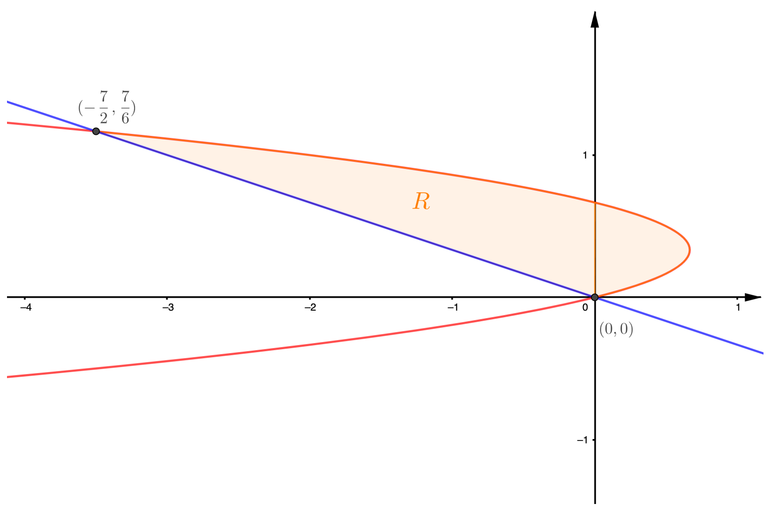
Solution: Solving

$$\begin{cases} x = -6y^2 + 4y \\ x + 3y = 0, \end{cases}$$

we have

$$\begin{aligned} -3y &= -6y^2 + 4y \\ 6y^2 - 7y &= 0 \\ y(6y - 7) &= 0. \end{aligned}$$

Hence, the intersections are $(0, 0)$ and $(-\frac{7}{2}, \frac{7}{6})$.



Thus,

$$\begin{aligned} \text{Area of } R &= \int_0^{\frac{7}{6}} ((-6y^2 + 4y) - (-3y))\, dy \\ &= \int_0^{\frac{7}{6}} (-6y^2 + 7y)\, dy \\ &= \left(-2y^3 + \frac{7}{2}y^2\right)\Bigg|_0^{\frac{7}{6}} \\ &= \frac{343}{216}. \end{aligned}$$

(It is more convenient to use y instead of x.)

4. A particle moves in a straight line with speed $v(t) = t^2 + 2t$, where $t \in [0, 9]$ is the time.

   (a) Find the average speed v^* of the particle between $t = 0$ and $t = 9$.

   (b) Find the time $t^* \in [0, 9]$ when the particle moves in the average speed v^*.

   Solution:

   (a) The average speed of the particle between $t = 0$ and $t = 9$ is

$$\begin{aligned} v^* &= \frac{1}{9-0} \int_0^9 v(t)\, dt \\ &= \frac{1}{9} \int_0^9 (t^2 + 2t)\, dt \\ &= \frac{1}{9} \left(\frac{1}{3} t^3 + t^2 \right) \Bigg|_0^9 \\ &= 36. \end{aligned}$$

   (b) Solving

$$\begin{aligned} v(t) &= v^* \\ t^2 + 2t &= 36 \\ t^2 + 2t - 36 &= 0 \\ t &= -1 \pm \sqrt{37}. \end{aligned}$$

   Since only $-1 + \sqrt{37} \in [0, 9]$, we have $t^* = -1 + \sqrt{37}$.

5. Evaluate
$$\lim_{x\to 0} \frac{\int_0^{2x} \sin(e^t - e^{-t})\,dt}{x\sin x}.$$

Solution: By the fundamental theorem of calculus,
$$\frac{d}{dx}\int_0^{2x} \sin(e^t - e^{-t})\,dt = 2\sin(e^{2x} - e^{-2x}).$$

Hence, by L'Hôpital's rule,
$$\begin{aligned}
&\lim_{x\to 0} \frac{\int_0^{2x} \sin(e^t - e^{-t})\,dt}{x\sin x} && \left(\frac{0}{0}\right)\\
&= \lim_{x\to 0} \frac{\frac{d}{dx}\int_0^{2x} \sin(e^t - e^{-t})\,dt}{\frac{d}{dx}(x\sin x)}\\
&= \lim_{x\to 0} \frac{2\sin(e^{2x} - e^{-2x})}{\sin x + x\cos x} && \left(\frac{0}{0}\right)\\
&= \lim_{x\to 0} \frac{2\cos(e^{2x} - e^{-2x})(2e^{2x} + 2e^{-2x})}{2\cos x - x\sin x}\\
&= \frac{2(1)(4)}{2-0}\\
&= 4.
\end{aligned}$$

6. By considering Riemann sum of a suitable integral, evaluate each of the following limits.

   (a) $\lim_{n\to\infty}\left(\frac{1}{n}e^{1/n}+\frac{1}{n}e^{2/n}+\cdots+\frac{1}{n}e^{n/n}\right)$

   (b) $\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+n}\right)$

   Solution:

   (a)

$$\begin{aligned}\lim_{n\to\infty}\left(\frac{1}{n}e^{1/n}+\frac{1}{n}e^{2/n}+\cdots+\frac{1}{n}e^{n/n}\right) &= \lim_{n\to\infty}\sum_{k=1}^{n} e^{k/n}\frac{1}{n}\\ &= \int_0^1 e^x\,dx\\ &= e^x\Big|_0^1\\ &= e-1.\end{aligned}$$

   (b)

$$\begin{aligned}\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+n}\right) &= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+k}\\ &= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{1+\frac{k}{n}}\cdot\frac{1}{n}\\ &= \int_0^1\frac{1}{1+x}\,dx\\ &= \ln|1+x|\Big|_0^1\\ &= \ln 2.\end{aligned}$$

7. Evaluate the following indefinite integrals and improper integrals.

(a) $\displaystyle\int \frac{1}{x^2+3x+2}\,dx$, and $\displaystyle\int_0^{\infty} \frac{1}{x^2+3x+2}\,dx$.

(b) $\displaystyle\int \frac{x}{\sqrt{1-x^2}}\,dx$, and $\displaystyle\int_0^{1} \frac{x}{\sqrt{1-x^2}}\,dx$.

Solution:

(a) Note that

$$\frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}.$$

For $x \neq -1$ and $x \neq -2$, we have

$$\begin{aligned}\int \frac{1}{x^2+3x+2}\,dx &= \int \left(\frac{1}{x+1} - \frac{1}{x+2}\right) dx \\ &= \ln|x+1| - \ln|x+2| + C \\ &= \ln\left|\frac{x+1}{x+2}\right| + C.\end{aligned}$$

For $N > 0$, we have

$$\int_0^N \frac{1}{x^2+3x+2}\,dx = \ln\left|\frac{x+1}{x+2}\right|_0^N = \ln\left|\frac{N+1}{N+2}\right| - \ln\frac{1}{2}.$$

Since $\displaystyle\lim_{N\to+\infty} \ln\left(\frac{N+1}{N+2}\right) = \lim_{N\to+\infty} \ln\left(\frac{1+1/N}{1+2/N}\right) = \ln(1) = 0$, we have

$$\int_0^{\infty} \frac{1}{x^2+3x+2}\,dx = \lim_{N\to+\infty} \int_0^N \frac{1}{x^2+3x+2}\,dx = -\ln\frac{1}{2} = \ln 2.$$

(b) For $-1 < x < 1$, we can take $u = 1-x^2$, then $\dfrac{du}{dx} = -2x$.

$$\begin{aligned}\int \frac{x}{\sqrt{1-x^2}}\,dx &= \int \frac{-1}{2\sqrt{u}}\,du \\ &= -\sqrt{u} + C \\ &= -\sqrt{1-x^2} + C.\end{aligned}$$

For $0 < L < 1$, we have

$$\begin{aligned}\int_0^L \frac{x}{\sqrt{1-x^2}}\,dx &= -\sqrt{1-x^2}\Big|_0^L \\ &= 1 - \sqrt{1-L^2}.\end{aligned}$$

Since $\displaystyle\lim_{L\to1^-} \sqrt{1-L^2} = 0$, we have

$$\int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx = \lim_{L\to1^-} \int_0^L \frac{x}{\sqrt{1-x^2}}\,dx = 1.$$

8. * Let $f(x)$ be continuous on $\mathbb{R}$ and $a \in \mathbb{R}$ be a given point.

   Suppose $\int_{-\infty}^{a} f(x)dx$ and $\int_{a}^{+\infty} f(x)dx$ both converge. Prove that for any point $b \in \mathbb{R}$, $\int_{-\infty}^{b} f(x)dx$ and $\int_{b}^{+\infty} f(x)dx$ both converge, and

$$\int_{-\infty}^{b} f(x)dx + \int_{b}^{+\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_{a}^{+\infty} f(x)dx.$$

   Remark: This problem implies that the improper integral $\int_{-\infty}^{+\infty} f(x)dx$ defined by

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{b} f(x)dx + \int_{b}^{+\infty} f(x)dx$$

   is independent of the choice of b. So for convenience, we can just choose $b = 0$.

   Solution: Since $\int_{-\infty}^{a} f(x)dx$ converges,

$$\begin{aligned}\lim_{M \to -\infty} \int_{M}^{b} f(x)dx &= \lim_{M \to -\infty} \left(\int_{M}^{a} f(x)dx + \int_{a}^{b} f(x)dx \right) \\ &= \int_{-\infty}^{a} f(x)dx + \int_{a}^{b} f(x)dx.\end{aligned}$$

   Hence $\int_{-\infty}^{b} f(x)dx$ converges, and $\int_{-\infty}^{b} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_{a}^{b} f(x)dx$.

   Similarly, since $\int_{a}^{+\infty} f(x)dx$ converges,

$$\begin{aligned}\lim_{N \to +\infty} \int_{b}^{N} f(x)dx &= \lim_{N \to +\infty} \left(\int_{a}^{N} f(x)dx + \int_{b}^{a} f(x)dx \right) \\ &= \int_{a}^{+\infty} f(x)dx - \int_{a}^{b} f(x)dx.\end{aligned}$$

   Hence $\int_{b}^{+\infty} f(x)dx$ converges, and $\int_{b}^{+\infty} f(x)dx = \int_{a}^{+\infty} f(x)dx - \int_{a}^{b} f(x)dx$.

   Moreover,

$$\begin{aligned}&\int_{-\infty}^{b} f(x)dx + \int_{b}^{+\infty} f(x)dx \\ &= \left(\int_{-\infty}^{a} f(x)dx + \int_{a}^{b} f(x)dx \right) + \left(\int_{a}^{+\infty} f(x)dx - \int_{a}^{b} f(x)dx \right) \\ &= \int_{-\infty}^{a} f(x)dx + \int_{a}^{+\infty} f(x)dx.\end{aligned}$$