THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1510 Calculus for Engineers (Fall 2021)
Suggested solutions of coursework 4

Part A

1. Find $f'(x)$ if

   (a) $f(x) = \sqrt[3]{x} + \dfrac{1}{\sqrt[3]{x}}$

   (b) $f(x) = (2x+1)(3x^2 - 5x + 3)$

   (c) $f(x) = (x+1)(x+2)(x+3)(x+4)$

   (d) $f(x) = \dfrac{2x-1}{x+3}$

   (e) $f(x) = \dfrac{\sqrt{x}+1}{\sqrt{x}-1}$

   Solution:

   (a) $f'(x) = \dfrac{1}{3}x^{-\frac{2}{3}} - \dfrac{1}{3}x^{-\frac{4}{3}}$

   (b) $f'(x) = 2(3x^2 - 5x + 3) + (2x+1)(6x-5) = 18x^2 - 14x + 1$

   (c)
   $$\begin{aligned} f'(x) &= (x+2)(x+3)(x+4) + (x+1)(x+3)(x+4) \\ &\quad + (x+1)(x+2)(x+4) + (x+1)(x+2)(x+3) \end{aligned}$$

   Alternatively, let $y = (x+1)(x+2)(x+3)(x+4)$. Then

   $$\begin{aligned} \frac{d}{dx}(\ln y) &= \frac{d}{dx}\left[\ln(x+1) + \ln(x+2) + \ln(x+3) + \ln(x+4)\right] \\ \frac{1}{y}\frac{dy}{dx} &= \frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3} + \frac{1}{x+4}. \end{aligned}$$

   So, $\dfrac{dy}{dx} = \left(\dfrac{1}{x+1} + \dfrac{1}{x+2} + \dfrac{1}{x+3} + \dfrac{1}{x+4}\right)(x+1)(x+2)(x+3)(x+4)$.
   (Notice that we need to assume $x > -1$ in this approach)

   (d) $f'(x) = \dfrac{(2)(x+3) - (2x-1)(1)}{(x+3)^2} = \dfrac{7}{(x+3)^2}$

   (e) $f'(x) = \dfrac{(\frac{1}{2}x^{-\frac{1}{2}})(\sqrt{x}-1) - (\sqrt{x}+1)(\frac{1}{2}x^{-\frac{1}{2}})}{(\sqrt{x}-1)^2} = -\dfrac{1}{\sqrt{x}(\sqrt{x}-1)^2}$

Part B

2. Suppose that

$$f(x) = \begin{cases} ax + b & \text{if } x < 0; \\ \sin x + 3 & \text{if } x \geq 0, \end{cases}$$

where a and b are real numbers.

Given that f is differentiable at $x = 0$, find the values of a and b.

Solution: Note that

$$Lf'(0) = \lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^-} \frac{(ah+b) - 3}{h},$$

and thus

$$\begin{aligned} f \text{ differentiable at } 0 &\implies Lf'(0) \text{ exists} \\ &\implies b = 3. \end{aligned}$$

Then,

$$Lf'(0) = \lim_{h \to 0^-} \frac{(ah+3) - 3}{h} = a.$$

Since

$$Rf'(0) = \lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{(\sin h + 3) - 3}{h} = 1,$$

we have

$$\begin{aligned} f \text{ differentiable at } 0 &\implies Lf'(0) = Rf'(0) \\ &\implies a = 1. \end{aligned}$$

Hence, $a = 1$ and $b = 3$.

3. Let

$$f(x) = \sqrt{x^2 - 1} \quad \text{with domain } D_f = (-\infty, -1] \cup [1, \infty).$$

(a) By the first principle, find the derivative of $f(x)$ for any $x \in (-\infty, -1) \cup (1, \infty)$.

(b) Show that f is not differentiable at $x = \pm 1$.
(Hint: Show that

$$Rf'(1) = \lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} \text{ and } Lf'(-1) = \lim_{h \to 0^-} \frac{f(-1+h) - f(-1)}{h}$$

do not exist.)

Solution:

(a) When $x \in (-\infty, -1) \cup (1, \infty)$,

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{\sqrt{(x+h)^2 - 1} - \sqrt{x^2 - 1}}{h} \\
&= \lim_{h \to 0} \frac{\sqrt{(x+h)^2 - 1} - \sqrt{x^2 - 1}}{h} \cdot \frac{\sqrt{(x+h)^2 - 1} + \sqrt{x^2 - 1}}{\sqrt{(x+h)^2 - 1} + \sqrt{x^2 - 1}} \\
&= \lim_{h \to 0} \frac{((x+h)^2 - 1) - (x^2 - 1)}{h(\sqrt{(x+h)^2 - 1} + \sqrt{x^2 - 1})} \\
&= \lim_{h \to 0} \frac{2xh + h^2}{h(\sqrt{(x+h)^2 - 1} + \sqrt{x^2 - 1})} \\
&= \frac{2x}{2\sqrt{x^2 - 1}} \\
&= \frac{x}{\sqrt{x^2 - 1}}.
\end{aligned}$$

(b) Note that

$$\begin{aligned}
Rf'(1) &= \lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} \\
&= \lim_{h \to 0^+} \frac{\sqrt{(1+h)^2 - 1} - 0}{h} \\
&= \lim_{h \to 0^+} \sqrt{\frac{2}{h} + 1} \\
&= \infty \quad \text{(DNE)},
\end{aligned}$$

and

$$\begin{aligned}
Lf'(-1) &= \lim_{h \to 0^-} \frac{f(-1+h) - f(1)}{h} \\
&= \lim_{h \to 0^-} \frac{\sqrt{(-1+h)^2 - 1} - 0}{h} \\
&= \lim_{h \to 0^-} -\sqrt{-\frac{2}{h} + 1} \\
&= -\infty \quad \text{(DNE)},
\end{aligned}$$

So, f is not differentiable at $x = 1$ and $x = -1$.

4. Let $T > 0$ and let $f : \mathbb{R} \to \mathbb{R}$ be a function such that

$$f(x+T) = f(x)$$

for all $x \in \mathbb{R}$.

Show by the first principle that, if f is differentiable, then

$$f'(x+T) = f'(x)$$

for all $x \in \mathbb{R}$.

Solution: For any $x \in \mathbb{R}$,

$$\begin{aligned} f'(x+T) &= \lim_{h\to 0} \frac{f(x+T+h) - f(x+T)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= f'(x). \end{aligned}$$