2021R1-MATH1510 HW 6

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TOTAL POINTS

9.5 / 20

QUESTION 1

1 Q1 **2 / 5**

(a)

✓ **- 1 pts** Incorrect method

✓ **- 0.5 pts** Incorrect constant term

✓ **- 0.5 pts** Incorrect coefficient of x^2

✓ **- 0.5 pts** Incorrect coefficient of x^4

(b)

✓ **- 0.5 pts** Incorrect coefficient of x^4

QUESTION 2

2 Q2 **2.5 / 4**

(a)

✓ **- 0 pts** Correct

(b)

✓ **- 1 pts** Incorrect radius of convergence.

✓ **- 0.5 pts** Incorrect conclusion on x=-1.

QUESTION 3

3 Q3b **3 / 3**

✓ **- 0 pts** Correct

QUESTION 4

4 Q6 **2 / 8**

(b)

✓ **- 1 pts** Incorrect f''': $$2e^{2t}\cos(t) - 11e^{2t}\sin(t)$$

✓ **- 1 pts** Incorrect Lagrange remainder

✓ **- 1 pts** Incorrect bound: $$13e/6 |t|^3$$

(c)

✓ **- 1 pts** Incorrect application of result from (b)

✓ **- 1 pts** Incorrect limits of the lower bound and upper bound

✓ **- 1 pts** Did not apply the squeeze theorem

Part A:

1. Find the Maclaurin polynomials of order 4 of the following functions:

   (a)
   $$\cos(\sin x);$$

   (b)
   $$g(x) = \frac{x^2 - x + 3}{(x^2+1)(2-x)}.$$

(a) Let $u = \sin x$, we have $f(u) = \cos u$:

$\because$ $f(0) = 1$;

$f'(u) = -\sin u$, $f'(0) = 0$;

$f''(u) = -\cos u$, $f''(0) = -1$;

$f'''(u) = \sin u$, $f'''(0) = 0$;

$f^{(4)}(u) = \cos u$, $f^{(4)}(0) = 1$;

$f^{(5)}(u) = -\sin u$, $f^{(5)}(0) = 0$;

$f^{(6)}(u) = -\cos u$, $f^{(6)}(0) = -1$

$\therefore$ We have $f(u) = 1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \frac{u^6}{6!} + \dots$

$$f(\sin x) = 1 - \frac{\sin^2 x}{2} + \frac{\sin^4 x}{24} - \frac{\sin^6 x}{120} + \dots //$$

(b) $g(x) = \frac{1}{x^2+1} + \frac{1}{2-x}$

$\because$ By property of geometric sequence, we have

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

$$\therefore \quad \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots$$

$$\& \quad \frac{1}{2-x} = \frac{1}{2}\left(\frac{1}{1-\frac{x}{2}}\right)$$

$$= \frac{1}{2}\left(1 + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{8} + \dots\right)$$

$$= \frac{1}{2} + \frac{x}{4} + \frac{x^2}{8} + \frac{x^3}{16} + \dots$$

$\therefore$ We have $g(x) = \left(1 - x^2 + x^4 - x^6 + \dots\right)$

$$+ \left(\frac{1}{2} + \frac{x}{4} + \frac{x^2}{8} + \frac{x^3}{16} + \dots\right)$$

$$= \frac{3}{2} + \frac{x}{4} - \frac{7x^2}{8} + \frac{x^3}{16} + \dots \quad //$$

1 Q1 **2 / 5**

(a)

✓ **- 1 pts** Incorrect method

✓ **- 0.5 pts** Incorrect constant term

✓ **- 0.5 pts** Incorrect coefficient of x^2

✓ **- 0.5 pts** Incorrect coefficient of x^4

(b)

✓ **- 0.5 pts** Incorrect coefficient of x^4

Part B:

2. For each of the following power series, find the radius of convergence and determine whether it is convergent at the given two points.

   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{n}{n+1}(x-1)^n$, at points $x = -\frac{1}{3}$, $x = \frac{3}{2}$.

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n} x^n$, at points $x = -1$, $x = \pi$.

(a) $\displaystyle\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{n(n+2)}{(n+1)^2}\right|$

$\displaystyle= \lim_{n\to\infty}\left|\frac{n^2+2n}{n^2+2n+1}\right|$.

$= 1$

$\because$ With centre $= 1$, the radius of convergence is 1,

the power series is convergent for $0 < x < 2$.

$\therefore$ When $x = \frac{3}{2}$, the power series is convergent.

$x = -\frac{1}{3}$, the power series is not convergent. //

(b) $\displaystyle\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{n! \times (n+1)^{n+1}}{n^n \times (n+1)!}\right|$

$\displaystyle= \lim_{n\to\infty}\left|\frac{(n+1)^n}{n^n}\right|$

$$= \lim_{n \to \infty} \left| \frac{n^n + \sum_{k=1}^{n-1} C_k^n (n)^k + 1}{n^n} \right|$$

$$= 1$$

$\therefore$ With centre 0. we have the radius of convergence $= 1$.

the power series is convergent for $-1 < x < 1$.

$\therefore$ When $x = -1$ or TU, both of them are not convergent. //

3. Find the Maclaurin series of the following functions:

(a)

$$\sinh(x) = \frac{e^x - e^{-x}}{2};$$

(b)

$$\frac{1-x}{2+x}.$$

(a) $\because$ We have $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+1}}{(2n+1)!}$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

Let $h(x) = x$, we have :

$$\sin h(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot [h(x)]^{2n+1}}{(2n+1)!}$$

$\because h(x) = \sin^{-1}\left(\frac{e^x - e^{-x}}{2}\right)$,

$\therefore$ We have $\sin h(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot \left[\sin^{-1}\left(\frac{e^x - e^{-x}}{2}\right)\right]^{2n+1}}{(2n+1)!}$ //

2 Q2 2.5 / 4

(a)

✓ - 0 pts Correct

(b)

✓ - 1 pts Incorrect radius of convergence.

✓ - 0.5 pts Incorrect conclusion on x=-1.

$$= \lim_{n\to\infty} \left| \frac{n^n + \sum_{k=1}^{n-1} C_k^n (n)^k + 1}{n^n} \right|$$

$$= 1$$

$\therefore$ With centre 0, we have the radius of convergence $= 1$.
the power series is convergent for $-1 < x < 1$.

$\therefore$ When $x = -1$ or TU, both of them are not convergent. //

3. Find the Maclaurin series of the following functions:

(a)

$$\sinh(x) = \frac{e^x - e^{-x}}{2};$$

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(a) $\because$ We have $\sinh x = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+1}}{(2n+1)!}$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

Let $h(x) = x$, we have:

$$\sinh h(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot [h(x)]^{2n+1}}{(2n+1)!}$$

$\because h(x) = \sinh^{-1}\left(\frac{e^x - e^{-x}}{2}\right)$,

$\therefore$ We have $\sinh h(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot \left[\sinh^{-1}\left(\frac{e^x - e^{-x}}{2}\right)\right]^{2n+1}}{(2n+1)!}$ //

(b) $$\frac{1-x}{2+x} = \frac{3}{2+x} - 1$$

$$= \frac{3}{2}\left(\frac{1}{1+\frac{x}{2}}\right) - 1$$

$\because$ By property of geometric sequence, we have:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\therefore \frac{1}{1+\frac{x}{2}} = \sum_{n=0}^{\infty} \left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^n}{2^n}$$

$$\therefore \frac{1-x}{2+x} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 3x^n}{2^{n+1}} - 1 \quad //$$

3 Q3b 3 / 3

✓ - 0 pts Correct

6. (a) Evaluate the following limit by using L'Hôpital's rule

$$\lim_{t\to 0} \frac{e^{2t}\cos t - (1+2t)}{t^2}.$$

(b) By considering Lagrange remainder, show that there exists some constant C such that

$$|e^{2t}\cos t - (1 + 2t + \frac{3}{2}t^2)| \le Ct^3$$

for any $t \in (-0.5, 0.5)$.

(c) By using part (b), evaluate the following limit

$$\lim_{t\to 0} \frac{e^{2t}\cos t - (1+2t)}{t^2}.$$

(a) $$\lim_{t\to 0} \frac{e^{2t}\cdot\cos t - (1+2t)}{t^2}$$

$$= \lim_{t\to 0} \frac{2e^{2t}\cdot\cos t - e^{2t}\cdot\sin t - 2}{2t}$$

$$= \lim_{t\to 0} \frac{4e^{2t}\cdot\cos t - 2e^{2t}\cdot\sin t - 2e^{2t}\cdot\sin t - e^{2t}\cdot\cos t}{2}$$

$$= \frac{3}{2} \; //$$

(b) $e^{2t} = 1 + 2t + 2t^2 + \frac{4}{3}t^3 + \frac{2}{3}t^4 + \dots$

$\cos t = 1 - \frac{t^2}{2} + \frac{t^4}{24} + \dots$

$e^{2t}\cdot\cos t = 1 + 2t + \frac{3}{2}t^2 + \frac{1}{3}t^3 - \frac{7}{24}t^4 + \dots$

Let $f(x) = e^{2t}\cdot\cos t$

$T_2(x) = 1 + 2t + \frac{3}{2}t^2$

$\because$ By Taylor theorm, $f(x) = T_2(t) + \frac{f'''(x)}{3!}(t)^3$

$= T_2(t) + \frac{1}{3}t^3$,

where $\frac{1}{3}t^3$ is the absolute error (OR remainder).

$\therefore \left| e^{2t} \cdot \cos t - \left(1 + 2t + \frac{3}{2}t^2\right) \right|$

$\leq \frac{1}{3}t^3$, for $t \in (-0.5, 0.5)$ //

(c) $\lim_{t \to 0} \frac{e^{2t} \cdot \cos t - (1 + 2t)}{t^2}$

$= \lim_{t \to 0} \frac{\frac{3}{2}t^2 + \frac{1}{3}t^3 - \frac{7}{24}t^4 + \ldots}{t^2}$

$= \frac{3}{2}$ //

4 Q6 **2 / 8**

(b)

✓ **- 1 pts** **Incorrect f''': $$2e^{2t}\cos(t) - 11e^{2t}\sin(t)$$**

✓ **- 1 pts** **Incorrect Lagrange remainder**

✓ **- 1 pts** **Incorrect bound: $$13e/6 |t|^3$$**

(c)

✓ **- 1 pts** **Incorrect application of result from (b)**

✓ **- 1 pts** **Incorrect limits of the lower bound and upper bound**

✓ **- 1 pts** **Did not apply the squeeze theorem**