Homework Assignment 1
Fall 2021

Question 1

(a) The cost of a new automobile is $25,000. If the interest rate is 6%, how much would you have to set aside now to provide this sum in five years? Assume the interests are compounded annually and the cost of the car does not appreciate over those five years.

(b) How much will you have at the end of 30 years if you invest $200 today at 12% annually compounded? How much will you have if you invested at 12% continuously compounded?

Solution

(a) PV $= \frac{25{,}000}{(1+0.06)^5} = \$18{,}681.45.$

(b) With annual compounding: FV $= \$200 \times 1.12^{30} = \$5{,}991.98.$
With continuous compounding: FV $= \$200 \times e^{0.12\times 30} = \$7{,}319.65.$

Question 2

The Murphy family are planning to buy a new home. The house costs $800,000, but the family have $235,000 in savings that they can use as down-payment. The remainder is to be financed by a mortgage. Their bank offers a 30-year loan, with fixed interest at 9.6% (per annum), with fixed monthly payments.

(a) What are the monthly payments that the Murphy's will have to make so that the entire mortgage is paid off in 30 years?

(b) If the Murphy's can afford to pay $4,500 per month, can they afford the house? If not, how much additional down-payment would they require?

(c) Now suppose the family pays $5,600 per month instead. Can they pay off their mortgage in 20 years without increasing their down-payment?

Solution

As they can make $235,000 down-payment and the house costs $800,000, the Murphy family will need to raise $565,000 through the mortgage.

(a) To find the monthly payment, we need to find the amount C so that the present value of the corresponding annuity (12 × 30 payments of C, discounted at 9.6%) covers the amount the family need to borrow, $275,000. We must solve:

$$565{,}000 = C \times PV_A\left(\frac{0.096}{12}; 12 \times 30\right) = C \times \frac{1}{0.008} \times \left(1 - \frac{1}{1.008^{360}}\right)$$

This above can now easily be solved for $\mathrm{C} = \frac{\$565{,}000}{117.90} = \$4{,}792.10.$

(b) Clearly the answer is "no": as we have just calculated, to borrow the required amount the family would have to make monthly payments of $4,792.10, hence payments of $4,500 would not suffice. More precisely, for payments of $4,500, the amount they could borrow is: $4,500 × 117.90 = $530,560.71, which is $34,439.29 short of the desired $565,000.

(c) For a term of 20 years, the corresponding unit annuity factor is:

$$PV_A\left(\frac{0.096}{12}; 12 \times 20\right) = \frac{1}{0.008} \times \left(1 - \frac{1}{1.008^{240}}\right) = 106.53$$

Hence, the Murphy can borrow $5,600 × 106.53 = $596,588.14. In other words, with the higher monthly payments, they can not only pay off the mortgage in 20 years, they can even reduce their down-payment by more than $30,000.

Question 3

The Murphy family, whose acquaintance we have made earlier, are setting up a retirement plan. They will make fixed monthly contributions to a pension fund, until Mr and Mrs Murphy retire 30 years from now. After retirement, the family are planning to withdraw a fixed amount "C" each month for the next 20 years. Assume that the fund earns a fixed 7.2% return.

(a) If the Murphy's plan to withdraw $2,500 each month, how much would they have to pay into the fund each month before they retire?

(b) How much can the family withdraw each month after retirement, if they can only afford to contribute $300 each month to the fund now?

Solution

The basic principle we apply here is this: we adjust the deposit and/or withdrawal amounts such that:

FV (at time of retirement) of all deposits= PV (at time of retirement) of all withdrawals

To do this, we calculate the unit annuity factors (at the rate of 6%) corresponding to the deposit and withdrawal periods, respectively:

Deposit:

$$FV_A\left(\frac{0.072}{12}; 12 \times 30\right) = \frac{12}{0.072} \times \left(\left(1 + \frac{0.072}{12}\right)^{12\times30} - 1\right) = 1{,}269.23$$

Withdraw:

$$PV_A\left(\frac{0.072}{12}; 12 \times 20\right) = \frac{12}{0.072} \times \left(1 - \frac{1}{\left(1 + \frac{0.072}{12}\right)^{12\times20}}\right) = 127.01$$

If we denote by C_D the amount that the family deposit each month prior to retirement, and by C_W the amount they withdraw thereafter, we must solve:

$$C_D \times 1{,}269.23 = C_W \times 127.01.$$

(a) For $C_W = \$2{,}500$, the above imply $C_D = \$2{,}500 \times (127.01 / 1{,}269.23) = \$250.17$.

(b) For $C_D = \$300$, we get $C_W = \$2{,}998.18$.

Note: This example again shows the power of compounding/discounting. Reducing the deposit amount by only $19 means the family lose $155 per month of retirement income. Compounding (future value) amplifies small changes in cash flows, while discounting (present value) reduces the magnitude of cash flow variations.