THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1510 Calculus for Engineers (Fall 2021)
Suggested solutions of homework 1
Deadline: September 25 at 23:00

Part A:

1. Without using L'Hôpital's rule, evaluate the following limits of sequences. Furthermore, if the limit does not exist but diverges to $\pm\infty$, please indicate so and determine the correct sign.

   (a) $\lim_{n\to\infty}(\sqrt{n^2+n}-\sqrt{n^2-1})$

   (b) $\lim_{n\to\infty}\dfrac{\sin(n)+\cos(n^2)}{n-100}$

   Solution:

   (a)
$$\begin{aligned}
&\lim_{n\to\infty}(\sqrt{n^2+n}-\sqrt{n^2-1})\\
&=\lim_{n\to\infty}(\sqrt{n^2+n}-\sqrt{n^2-1})\cdot\frac{\sqrt{n^2+n}+\sqrt{n^2-1}}{\sqrt{n^2+n}+\sqrt{n^2-1}}\\
&=\lim_{n\to\infty}\frac{n+1}{\sqrt{n^2+n}+\sqrt{n^2-1}}\\
&=\lim_{n\to\infty}\frac{1+\frac{1}{n}}{\sqrt{1+\frac{1}{n}}+\sqrt{1-\frac{1}{n^2}}}\\
&=\frac{1+0}{1+1}\\
&=\frac{1}{2}.
\end{aligned}$$

   (b) Note that for all positive integers n, we have $-1\le\sin(n)\le 1$ and $-1\le\cos(n^2)\le 1$. So, for $n>100$,
$$-2\le\sin(n)+\cos(n^2)\le 2$$
$$-\frac{2}{n-100}\le\frac{\sin(n)+\cos(n^2)}{n-100}\le\frac{2}{n-100}.$$
   Furthermore, $\lim_{n\to\infty}-\frac{2}{n-100}=\lim_{n\to\infty}\frac{2}{n-100}=0.$

   By sandwich theorem, we have $\lim_{n\to\infty}\dfrac{\sin(n)+\cos(n^2)}{n-100}=0.$

2. Let

$$f(x) = \begin{cases} \dfrac{1}{x}\tan\dfrac{x}{2} & \text{if } -1 < x < 0; \\ \dfrac{|x-1|}{2x-2} & \text{if } 0 < x < 1; \\ \dfrac{x^2-4x+3}{x^2+2x-3} & \text{if } x > 1. \end{cases}$$

Then find each of the following limits or state that it does not exist. Furthermore, if the limit does not exist but diverges to $\pm\infty$, please indicate so, and determine the correct sign.

(a) $\lim_{x\to 0^-} f(x)$;

(b) $\lim_{x\to 0^+} f(x)$;

(c) $\lim_{x\to 0} f(x)$.

(d) $\lim_{x\to 1} f(x)$;

Solution:

(a) $\displaystyle\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{1}{x}\tan\frac{x}{2} = \lim_{x\to 0^-} \frac{\sin\frac{x}{2}}{\frac{x}{2}} \cdot \frac{1}{2\cos\frac{x}{2}} = 1\cdot\frac{1}{2} = \frac{1}{2}$

(b) $\displaystyle\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{|x-1|}{2x-2} = \frac{|0-1|}{0-2} = -\frac{1}{2}$

(c) Since $\lim_{x\to 0^+} f(x) \neq \lim_{x\to 0^-} f(x)$, $\lim_{x\to 0} f(x)$ does not exist (DNE)

(d) $\displaystyle\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{|x-1|}{2x-2} = \lim_{x\to 1^-} \frac{-(x-1)}{2(x-1)} = \lim_{x\to 1^-} \frac{-1}{2} = -\frac{1}{2}.$

$\displaystyle\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{x^2-4x+3}{x^2+2x-3} = \lim_{x\to 1^+} \frac{(x-1)(x-3)}{(x-1)(x+3)} = \lim_{x\to 1^+} \frac{x-3}{x+3} = -\frac{1}{2}.$

Since $\displaystyle\lim_{x\to 1^-} f(x) = -\frac{1}{2} = \lim_{x\to 1^+} f(x)$, we have $\displaystyle\lim_{x\to 1} f(x) = -\frac{1}{2}.$

Part B:

3. (a) Let $f(x) = \dfrac{1}{\sqrt{5-4x-x^2}}$. Express the domain and range of f in interval notation.

   (b) Let $f(x) = x^2 - 1$, $g(x) = \dfrac{1}{3}\log_2 x$. Express the domain and range of $g \circ f$ in interval notation.

Solution:

(a) Note that $f(x)$ is well-defined if and only if

$$\begin{aligned} 5 - 4x - x^2 &> 0 \\ x^2 + 4x - 5 &< 0 \\ (x+5)(x-1) &< 0 \\ -5 < x &< 1. \end{aligned}$$

Hence, the domain of f is $(-5, 1)$.
For $x \in D_f = (-5, 1)$, $5-4x-x^2 = 9-(x+2)^2$ is positive and has a maximum value of 9. Hence, the range of f is

$$\left[\frac{1}{\sqrt{9}}, \infty\right) = \left[\frac{1}{3}, \infty\right).$$

(b) $(g \circ f)(x) = \dfrac{1}{3}\log_2(x^2 - 1)$ is well-defined if and only if

$$\begin{aligned} x^2 - 1 &> 0 \\ (x+1)(x-1) &> 0 \\ x < -1 \quad \text{or} \quad x &> 1 \end{aligned}$$

Hence, the domain of $g \circ f$ is $(-\infty, -1) \cup (1, \infty)$.
Since $(g \circ f)(x) = \dfrac{1}{3}\log_2(x^2 - 1)$, and $x^2 - 1$ can be any positive real number for $x \in D_{g\circ f}$, so the range of $g \circ f$ is $(-\infty, \infty)$.

4. Let $\gamma(t) = (x(t), y(t)) = (3\cos 2t - 1, 3\sin 2t + 2)$, $t \in \mathbb{R}$ be a curve.

   (a) Write down an equation of the curve in terms of x and y without t.

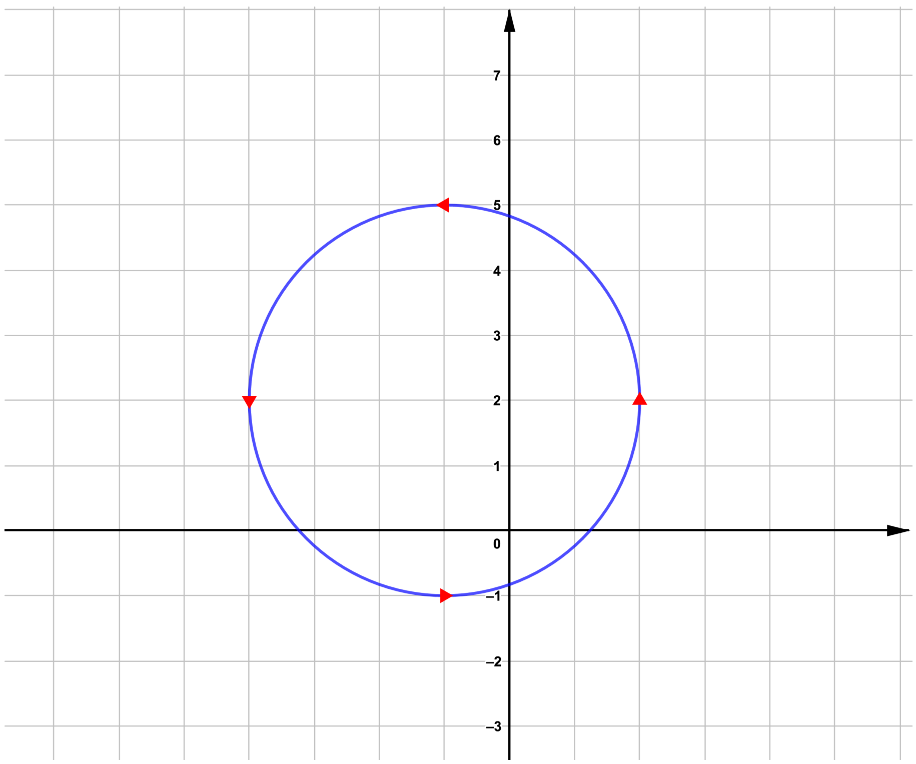
   (b) Sketch the curve in xy-plane, and indicate the direction for t increasing with an arrow.

Solution:

(a) We have $x + 1 = 3\cos 2t$ and $y - 2 = 3\sin 2t$, then $(x+1)^2 = 9\cos^2 2t$ and $(y-2)^2 = 9\sin^2 2t$. By adding them up, we have $(x+1)^2 + (y-2)^2 = 9$.

(b) The curve γ is the circle centered at $(-1, 2)$ with radius 3.

(Remark: If $(x(t), y(t))$ describes a moving point, then as t increases, the point is moving along the circle in counter-clockwise direction.)



5. Without using L'Hôpital's rule, evaluate the following limits. Furthermore, if the limit does not exist but diverges to $\pm\infty$, please indicate so and determine the correct sign.

   (a) $\displaystyle\lim_{x\to 3}\frac{\sqrt{x+1}-2}{4-\sqrt{5x+1}}$;

   (b) $\displaystyle *\lim_{x\to 8}\frac{x^2-7x-8}{\sqrt[3]{x}-2}$;

Solution:

(a)

$$\begin{aligned}\lim_{x\to 3}\frac{\sqrt{x+1}-2}{4-\sqrt{5x+1}} &= \lim_{x\to 3}\frac{\sqrt{x+1}-2}{4-\sqrt{5x+1}}\cdot\frac{\sqrt{x+1}+2}{\sqrt{x+1}+2}\cdot\frac{4+\sqrt{5x+1}}{4+\sqrt{5x+1}}\\ &= \lim_{x\to 3}\frac{(x-3)(4+\sqrt{5x+1})}{-5(x-3)(\sqrt{x+1}+2)}\\ &= \lim_{x\to 3}\frac{4+\sqrt{5x+1}}{-5(\sqrt{x+1}+2)}\\ &= \frac{4+\sqrt{5(3)+1}}{-5(\sqrt{3+1}+2)}\\ &= -\frac{2}{5}.\end{aligned}$$

(b)

$$\begin{aligned}\lim_{x\to 8}\frac{x^2-7x-8}{\sqrt[3]{x}-2} &= \lim_{x\to 8}\frac{(x-8)(x+1)}{\sqrt[3]{x}-2}\cdot\frac{(\sqrt[3]{x})^2+2\sqrt[3]{x}+2^2}{(\sqrt[3]{x})^2+2\sqrt[3]{x}+2^2}\\ &= \lim_{x\to 8}\frac{(x-8)(x+1)[(\sqrt[3]{x})^2+2\sqrt[3]{x}+2^2]}{x-8}\\ &= \lim_{x\to 8}(x+1)[(\sqrt[3]{x})^2+2\sqrt[3]{x}+2^2]\\ &= (8+1)[(\sqrt[3]{8})^2+2\sqrt[3]{8}+2^2]\\ &= 108.\end{aligned}$$

6. Without using L'Hôpital's rule, evaluate the following limits. Furthermore, if the limit does not exist but diverges to $\pm\infty$, please indicate so and determine the correct sign.

   (a) $\displaystyle\lim_{x\to+\infty}\left(1+\frac{2}{4x-1}\right)^x$

   (b) $\displaystyle\lim_{x\to+\infty}\left(\frac{2x-1}{2x+1}\right)^x$

   Solution:

   (a)
$$\begin{aligned}
\lim_{x\to+\infty}\left(1+\frac{2}{4x-1}\right)^x &= \lim_{x\to+\infty}\left(1+\frac{1}{2x-\frac{1}{2}}\right)^x \\
&= \lim_{y\to+\infty}\left(1+\frac{1}{y}\right)^{\frac{1}{2}y+\frac{1}{4}} && (\text{letting } y = 2x-\frac{1}{2}) \\
&= \lim_{y\to+\infty}\left(\left(1+\frac{1}{y}\right)^y\right)^{\frac{1}{2}} \cdot \lim_{y\to+\infty}\left(1+\frac{1}{y}\right)^{\frac{1}{4}} \\
&= e^{\frac{1}{2}}\cdot 1 \\
&= e^{\frac{1}{2}}.
\end{aligned}$$

   (b)
$$\begin{aligned}
\lim_{x\to+\infty}\left(\frac{2x-1}{2x+1}\right)^x &= \lim_{x\to+\infty}\left(\frac{2x+1}{2x-1}\right)^{-x} \\
&= \lim_{x\to+\infty}\left(1+\frac{2}{2x-1}\right)^{-x} \\
&= \lim_{x\to+\infty}\left(1+\frac{1}{x-\frac{1}{2}}\right)^{-x} \\
&= \lim_{x\to+\infty}\left(\left(1+\frac{1}{x-\frac{1}{2}}\right)^{x-\frac{1}{2}}\right)^{-1}\left(1+\frac{1}{x-\frac{1}{2}}\right)^{-\frac{1}{2}} \\
&= e^{-1}\cdot 1 \\
&= \frac{1}{e}.
\end{aligned}$$

7. * Let $\{a_n\}$ be the sequence defined by the recursive relation

$$a_1 = \sqrt{6} \quad \text{and} \quad a_{n+1} = \sqrt{6 + a_n} \text{ for all postive integer } n.$$

Given that $\lim_{n\to\infty} a_n$ exists and equals c. Find the value of c.

Solution: Since

$$a_{n+1}^2 = 6 + a_n$$

for any positive integer n, we have

$$\begin{aligned} \lim_{n\to\infty} a_{n+1}^2 &= \lim_{n\to\infty} (6 + a_n) \\ c^2 &= 6 + c \\ c^2 - c - 6 &= 0 \\ (c-3)(c+2) &= 0 \\ c = 3 \quad &\text{or} \quad c = -2. \end{aligned}$$

Note $a_n > 0$ for all positve integer n. Hence, $c \geq 0$ and so, $c = 3$.

8. * The following statements are both false. Give one counterexample for each of them.

   (a) If $\lim_{n\to+\infty} a_n = 0$, $\lim_{n\to+\infty} b_n = +\infty$, then $\lim_{n\to+\infty} a_n b_n = 0$.

   (b) If $f(x) > 0$ for all $x \in \mathbb{R}$ and $\lim_{x\to 0} f(x)$ exists,
   then $\lim_{x\to 0} f(x) > 0$.

Solution:

(a) Let $a_n = \dfrac{1}{n}$ and $b_n = n$ for each positive integer n.
Then $\lim_{n\to+\infty} a_n = 0$ and $\lim_{n\to+\infty} b_n = +\infty$.
However, $\lim_{n\to+\infty} a_n b_n = \lim_{n\to+\infty} 1 = 1 \neq 0$.

(b) Let

$$f(x) = \begin{cases} x^2 & \text{if } x \neq 0 \\ 1 & \text{if x} = 0. \end{cases}$$

Then $f(x) > 0$ for all $x \in \mathbb{R}$.
However, $\lim_{x\to 0} f(x) = \lim_{x\to 0} x^2 = 0$.