Lecture Note 10

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MATH1020
General Mathematics

Vectors

A vector is a quantity that has both magnitude and direction. It is usual to represent a vector by using an arrow. The length of the arrow represents the **magnitude** of the vector, and the arrowhead indicates the **direction** of the vector.

Many quantities in physics can be represented by vectors.

Example 1 The velocity of a car can be represented by an arrow that points in the direction of movement; the length of the arrow represents speed. If the car speeds up, we lengthen the arrow; if the car changes direction, we introduce an arrow in the new direction. Based on this representation, it is not surprising that vectors and *directed line segments* are somehow related.

Geometric Vectors

If P and Q are two distinct points in the xy-plane, there is exactly one line containing both P and Q. See Figure 1.

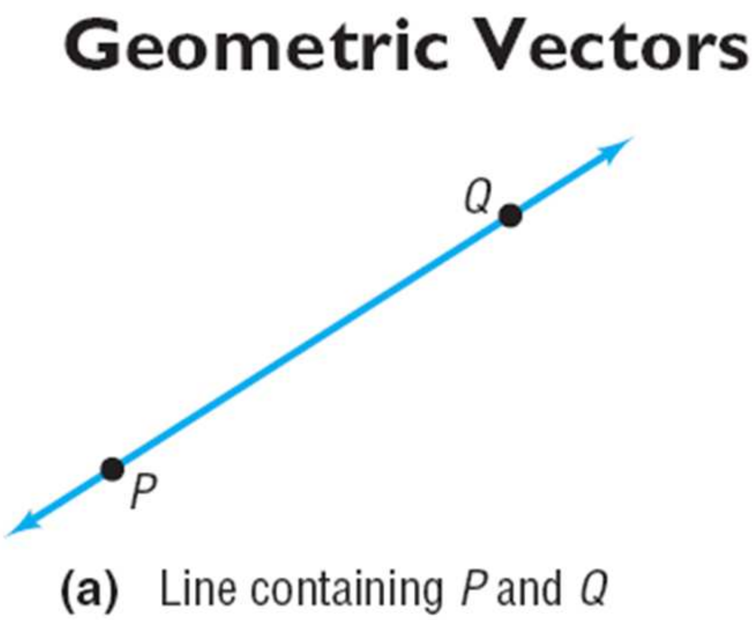

Figure 1:

The points on that part of the line that joins P and Q, including P and Q, form what is called the **line segment** $\overline{PQ}$. See Figure 2.

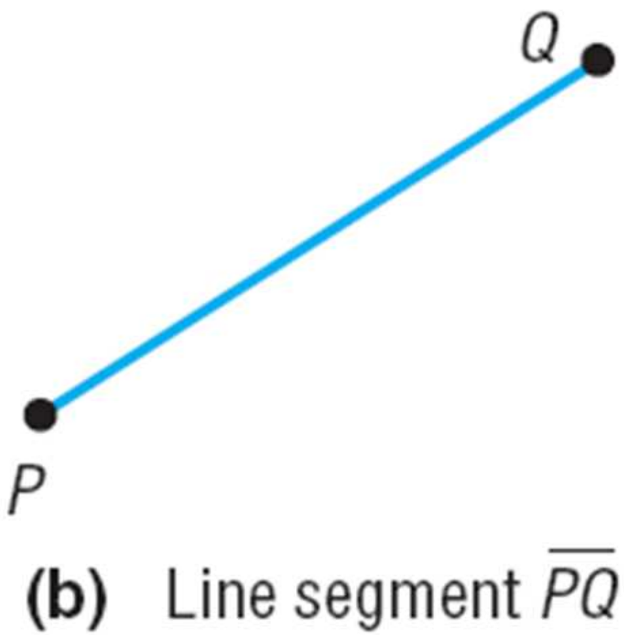

Figure 2:

If we order the points so that they proceed from P to Q, we have a **directed line segment** from P to Q, or a **geometric vector,** which we denote by $\overrightarrow{PQ}$. In a directed line segment $\overrightarrow{PQ}$, we call P the **initial point** and Q the **terminal point,** as indicated in Figure 3.

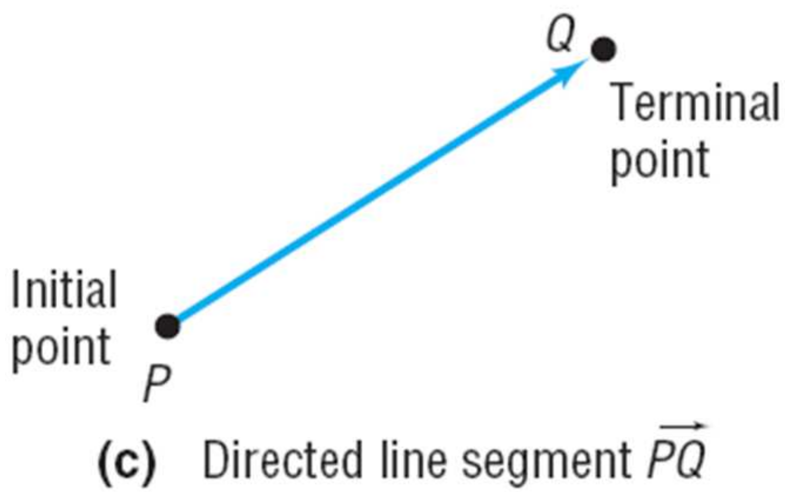

Figure 3:

The magnitude of the directed line segment $\overrightarrow{PQ}$ is the distance from the point P to the point Q; that is the length of the line segment. The direction of $\overrightarrow{PQ}$ is from P to Q. If a vector $\mathbf{v}$ has the same magnitude and the same direction as the directed line segment $\overrightarrow{PQ}$, we write

$$\mathbf{v} = \overrightarrow{PQ}.$$

Note that boldface letters will be used to denote vectors, to distinguish them from numbers.

(zero vector, 0)

The vector **v** whose magnitude is **0** is called the **zero vector, 0.** The zero vector is assigned no direction.

(**Two vectors are equal**)

Two vectors $\mathbf{v}$ and $\mathbf{w}$ are **equal**, written

$$\mathbf{v} = \mathbf{w}$$

if they have the same magnitude and the same direction.

Example 2 The three vectors shown in Figure 4 have the same magnitude and the same direction, so they are equal, even though they have different initial points and different terminal points. As a result, we find it helpful to think of a vector simply as an arrow, keeping in mind that two arrows (vectors) are equal if they have the same direction and the same magnitude (length).

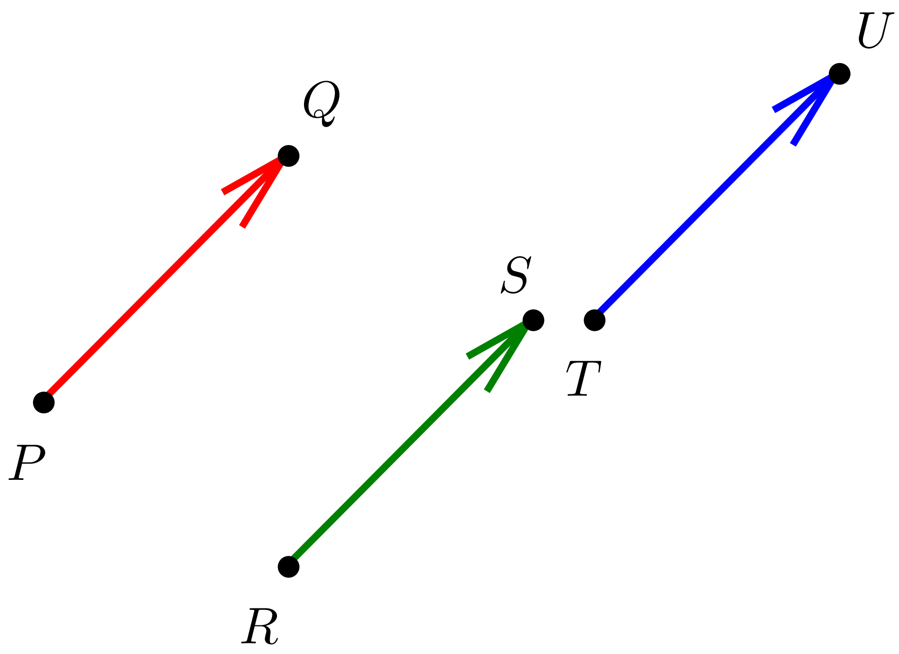

Figure 4:

Adding Vectors Geometrically

The **sum** $\mathbf{v} + \mathbf{w}$ of two vectors is defined as follows:

Let us position the vectors $\mathbf{v}$ and $\mathbf{w}$ so that the terminal point of $\mathbf{v}$ coincides with the initial point of $\mathbf{w}$, as shown in Figure 5.

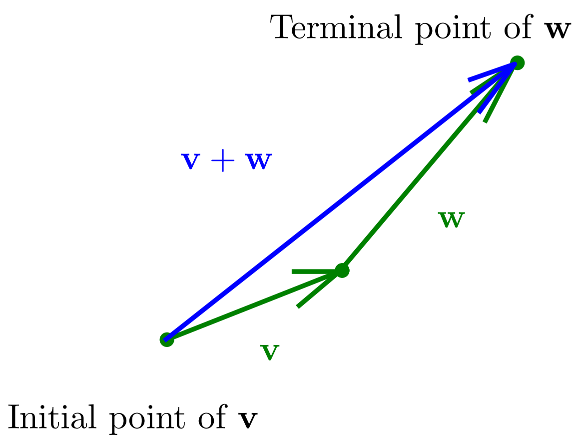

Figure 5:

The vector $\mathbf{v} + \mathbf{w}$ is then the unique vector whose initial point coincides with the initial point of $\mathbf{v}$ and whose terminal point coincides with the terminal point of $\mathbf{w}$.

Vector addition is **commutative.** That is, if $\mathbf{v}$ and $\mathbf{w}$ are any two vectors, then

$$\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}.$$

See Figure 6.

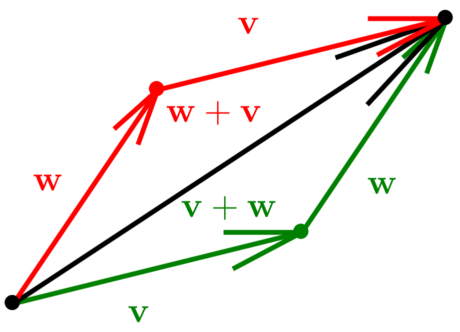

Figure 6:

Note that the commutative property is another way of saying that opposite sides of a parallelogram are equal and parallel.

Vector addition is also **associative.**

That is, if $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors, then

$$\underbrace{\mathbf{u} + \underbrace{(\mathbf{v} + \mathbf{w})}_{\text{1st}}}_{\text{2nd}} = \underbrace{\underbrace{(\mathbf{u} + \mathbf{v})}_{\text{1st}} + \mathbf{w}}_{\text{2nd}} .$$

See Figure 7.

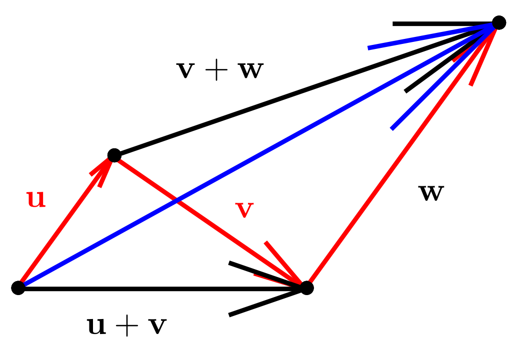

Figure 7:

The zero vector $\mathbf{0}$ has the property that

$$\mathbf{v} + \mathbf{0} = \mathbf{0} + \mathbf{v} = \mathbf{v}$$

for any vector $\mathbf{v}$.

If $\mathbf{v}$ is a vector, then $-\mathbf{v}$ is the vector having the same magnitude as $\mathbf{v}$, but whose direction is opposite to $\mathbf{v}$ (or the negative of $\mathbf{v}$).

Furthermore,

$$(-\mathbf{v}) + \mathbf{v} = \mathbf{v} + (-\mathbf{v}) = \mathbf{0}.$$

See Figure 8.

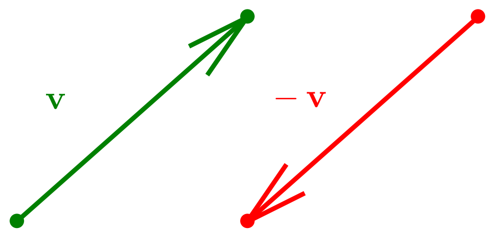

Figure 8:

If $\mathbf{v}$ and $\mathbf{w}$ are two vectors, we define the **difference** $\mathbf{v} - \mathbf{w}$ as

$$\mathbf{v} - \mathbf{w} = \mathbf{v} + (-\mathbf{w}).$$

Figure 9 illustrates the vector relationships among $\mathbf{v}$, $\mathbf{w}$, $\mathbf{v} + \mathbf{w}$, and $\mathbf{v} - \mathbf{w}$. Beware of the direction of these vectors.

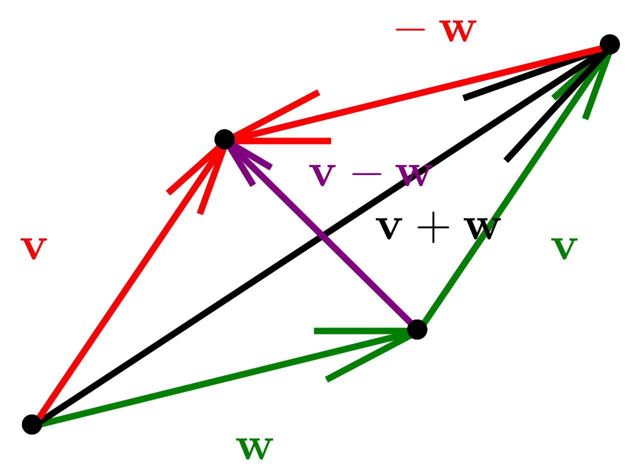

Figure 9:

Multiplying Vectors by Numbers Geometrically

When dealing with vectors, we refer to real numbers as **scalars**, $\alpha \in \mathbb{R}$. Scalars are quantities that have only magnitude.

Example 3 Examples of scalar quantities in physics include time, temperature, speed and mass.

We now define how to multiply a vector by a scalar.

Definition 1 If α is scalar and $\mathbf{v}$ is a vector, the **scalar multiple** $\alpha\mathbf{v}$ is defined as follows:

1. If $\alpha > 0$, $\alpha\mathbf{v}$ is the vector whose magnitude is α times the magnitude of $\mathbf{v}$ and whose direction is the same as $\mathbf{v}$.

2. If $\alpha < 0$, $\alpha\mathbf{v}$ is the vector whose magnitude is $|\alpha|$ times the magnitude of $\mathbf{v}$ and whose direction is opposite that of $\mathbf{v}$.

3. If $\alpha = 0$ or if $\mathbf{v} = \mathbf{0}$, then $\alpha\mathbf{v} = \mathbf{0}$.

See Figure 10.

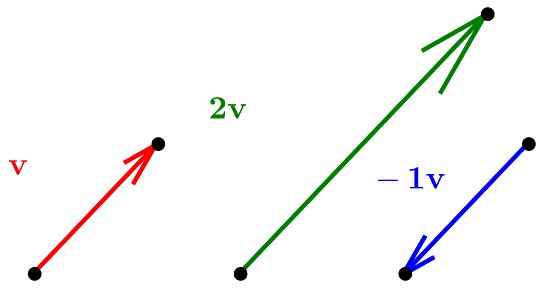

Figure 10:

Example 4 In accordance with Newton's second law, $\mathbf{F} = m\mathbf{a}$, where $\mathbf{F}$ is a force, m is a scalar, and $\mathbf{a}$ is an acceleration. When we multiply a scalar times a vector, the result is a vector.

Scalar multiples have the following properties:

$$0\mathbf{v} = \mathbf{0}$$

$$1\mathbf{v} = \mathbf{v}$$

$$-1\mathbf{v} = -\mathbf{v}$$

$$(\alpha + \beta)\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}$$

$$\alpha(\mathbf{v} + \mathbf{w}) = \alpha\mathbf{v} + \alpha\mathbf{w}$$

$$\alpha(\beta\mathbf{v}) = (\alpha\beta)\mathbf{v}$$

Magnitudes of Vectors

If $\mathbf{v}$ is a vector, we use the symbol $||\mathbf{v}||$ to represent the **magnitude** of $\mathbf{v}$. Since $||\mathbf{v}||$ equals the length of a directed line segment, it follows that $||\mathbf{v}||$ has the following properties:

Theorem 1 **Properties of $||\mathbf{v}||$**

If $\mathbf{v}$ is a vector and if α is a scalar, then

$$
\begin{array}{ll}
(a) & ||\mathbf{v}|| \geq 0 \\
(b) & ||\mathbf{v}|| = 0 \quad \text{if and only if} \quad \mathbf{v} = \mathbf{0} \\
(c) & ||-\mathbf{v}|| = ||\mathbf{v}|| \\
(d) & ||\alpha\mathbf{v}|| = |\alpha|||\mathbf{v}||.
\end{array}
$$

Property (a) is a consequence of the fact that distance is a nonnegative number.

Property (b) follows, because the length of the directed line segment $\overrightarrow{PQ}$ is positive unless P and Q are the same point, in which case the length is 0.

Property (c) follows because the length of the line segment $\overline{PQ}$ equals the length of the line segment $\overline{PQ}$.

Property (d) is a direct consequence of the line of the definition of a scalar multiple.

Definition 2 A vector $\mathbf{u}$ for which $||\mathbf{u}|| = 1$ is called a **unit vector**.

Find a Position Vector

To complete the magnitude and direction of a vector, we need an algebraic way of representing vectors.

Definition 3 An **algebraic vetor** $\mathbf{v}$ is represented as

$$\mathbf{v} = \langle a, b \rangle,$$

where a and b are real numbers (scalers) called the **components** of the vector $\mathbf{v}$.

A rectangular coordinate system to represent algebraic vectors in the plane $\mathbb{R}^2$ is used. If $\mathbf{v} = \langle a, b \rangle$ is an algebraic vector whose initial point is the origin, then $\mathbf{v}$ is called a position vector. See Figure 11.

Note that the terminal point of the position vector $\mathbf{v} = \langle a, b \rangle$ is $P = (a, b)$.

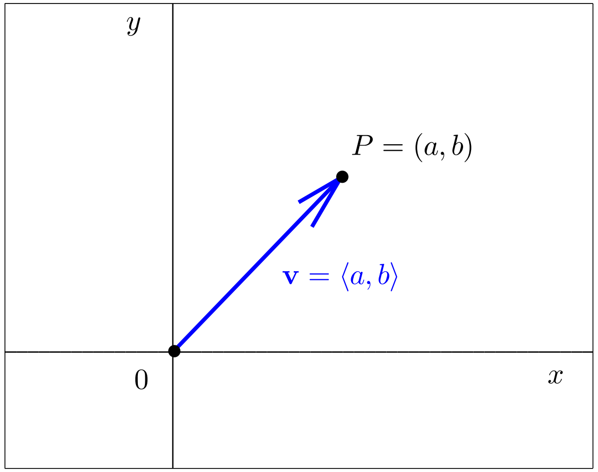

Figure 11:

The next result states that any vector whose initial point is not at the origin is equal to a unique position vector.

Theorem 2 Suppose that $\mathbf{v}$ is a vector with initial point $P_1 = (x_1, y_1)$, not necessarily the origin, and terminal point $P_2 = (x_2, y_2)$. If $\mathbf{v} = \overrightarrow{PQ}$, then $\mathbf{v}$ is equal to the position vector

$$\mathbf{v} = \langle x_2 - x_1, y_2 - y_1 \rangle. \tag{1}$$

To see why this is true, look at Figure 12.

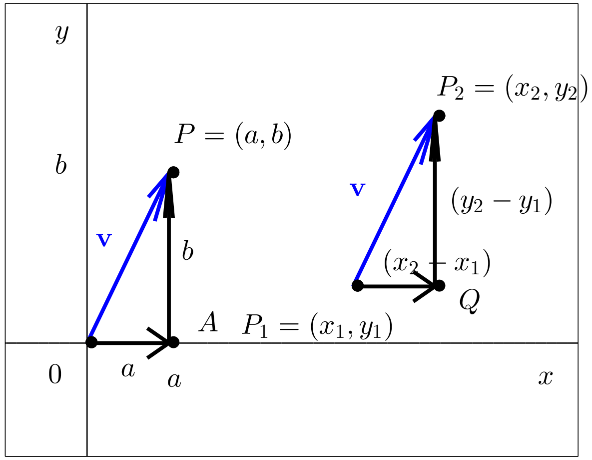

Figure 12:

Example 5 Show that triangle OPA and triangle P_1P_2Q are congruent.

Proof:

The line segments have magnitude, so $d(O, P) = d(P_1, P_2)$; and they have the same direction, so $\angle POA = \angle P_2P_1Q$. Since the triangles are right triangles, we have angle-side-angle.

It follows that corresponding sides are equal. As a result, $x_2 - x_1 = a$ and

$y_2 - y_1 = b$, so $\mathbf{v}$ may be written as

$$\mathbf{v} = \langle x_2 - x_1, y_2 - y_1 \rangle.$$

Because of this result, we can replace any algebraic vector by a unique position vector, and vice versa. This flexibility is one of the main reasons for the side use of vectors.

■

Theorem 3 Equality of Vectors

Let $\mathbf{v},\ \mathbf{w} \in \mathbb{R}^2$. Two vectors $\mathbf{v}$ and $\mathbf{w}$ are equal if and only if their corresponding components are equal. That is,

If $\mathbf{v} = \langle a_1, b_1 \rangle$ and $\mathbf{w} = \langle a_2, b_2 \rangle$
then $\mathbf{v} = \mathbf{w}$ if and only if $a_1 = a_2$ and $b_1 = b_2$.

We now present an alternative representation of a vector in the plane that is common in the physical sciences.

Let $\mathbf{i}$ denote the unit vector whose direction is along the positive x-axis; let $\mathbf{j}$ denote the unit vector whose direction is along the positive y-axis. Then $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$, as shown in Figure 13.

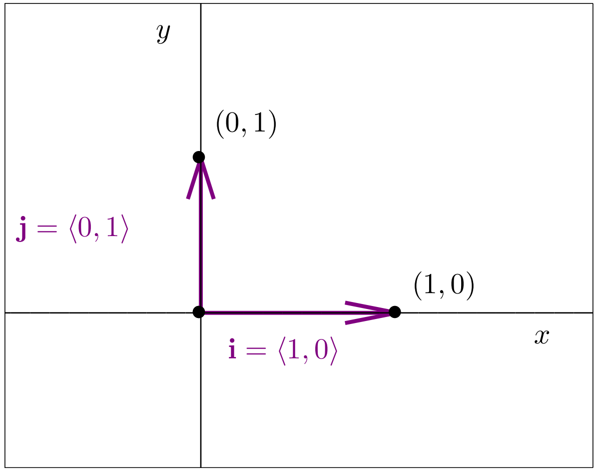

Figure 13:

Any vector $\mathbf{v} = \langle a, b \rangle$ can be written using the unit vectors $\mathbf{i}$ and $\mathbf{j}$ as follows:

$$\mathbf{v} = \langle a, b \rangle = a\langle 1, 0 \rangle + b\langle 0, 1 \rangle = a\mathbf{i} + b\mathbf{j}.$$

We call a and b the **horizontal** and **vertical components** of **v**, respectively.

Example 6 If $\mathbf{v} = \langle 5, 4 \rangle = 5\langle 1, 0 \rangle + 4\langle 0, 1 \rangle = 5\mathbf{i} + 4\mathbf{j}$, then 5 is the horizontal component and 4 is the vertical component.

Add and Subtract Vectors Algebraically

We define addition, subtraction, scalar multiple, and magnitude of algebraic vector in terms of their components.

Definition 4 Let $\mathbf{v} = a_1\mathbf{i} + a_1\mathbf{j}$ and $\mathbf{w} = a_2\mathbf{i} + a_2\mathbf{j}$ be two vectors, and let α be a scalar. Then

$$\mathbf{v} + \mathbf{w} = (a_1 + a_2)\mathbf{i} + (b_1 + b_2)\mathbf{j} = \langle a_1 + a_2, b_1 + b_2 \rangle. \tag{2}$$

$$\mathbf{v} - \mathbf{w} = (a_1 - a_2)\mathbf{i} + (b_1 - b_2)\mathbf{j} = \langle a_1 - a_2, b_1 - b_2 \rangle. \tag{3}$$

$$\alpha\mathbf{v} = (\alpha a_1)\mathbf{i} + (\alpha b_1)\mathbf{j} = \langle \alpha a_1, \alpha b_1 \rangle. \tag{4}$$

$$\mathbf{v} = \sqrt{a_1^2 + b_1^2}. \tag{5}$$

These definitions are compatible with the geometric definitions given earlier in this note. See Figures 14 - 16.

$$\mathbf{v} + \mathbf{w} = (a_1 + a_2)\mathbf{i} + (b_1 + b_2)\mathbf{j} = \langle a_1 + a_2, b_1 + b_2 \rangle.$$

$$\mathbf{v} - \mathbf{w} = (a_1 - a_2)\mathbf{i} + (b_1 - b_2)\mathbf{j} = \langle a_1 - a_2, b_1 - b_2 \rangle.$$

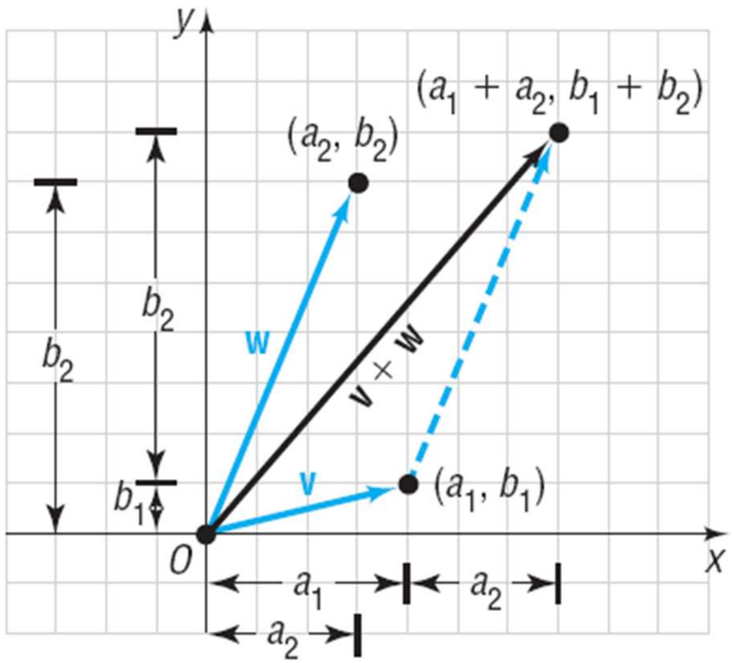

Figure 14:

$$\alpha \mathbf{v} = (\alpha a_1)\mathbf{i} + (\alpha b_1)\mathbf{j} = \langle \alpha a_1, \alpha b_1 \rangle.$$

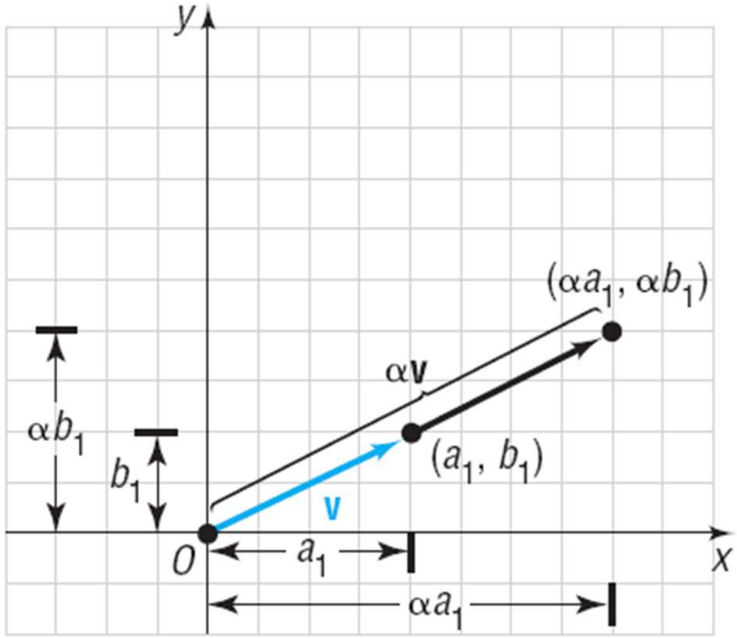

Figure 15:

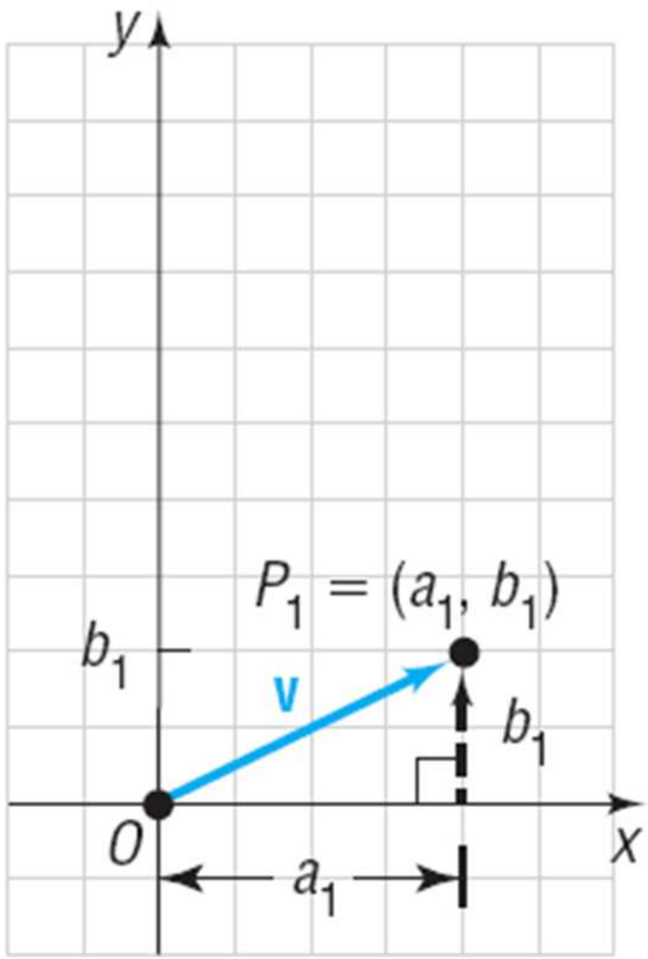

Figure 16:

Find a Unit Vector

Recall that a unit vector for which $\mathbf{u}$ is a vector which $||\mathbf{u}|| = 1$. In many practical applications, it is useful to find a unit vector $\mathbf{u}$ that has the same direction as a given vector $\mathbf{v}$.

Theorem 4 **Unit Vector in the Direction of $\mathbf{v}$**

For any nonzero vector $\mathbf{v}$, the vector

$$\mathbf{u} = \frac{\mathbf{v}}{||\mathbf{v}||}$$

is a unit vector that has the same direction as $\mathbf{v}$.

Proof:

Let $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$. Then $||\mathbf{v}|| = \sqrt{a^2 + b^2}$ and

$$\mathbf{u} = \frac{\mathbf{v}}{||\mathbf{v}||} = \frac{a\mathbf{i} + b\mathbf{j}}{\sqrt{a^2 + b^2}} = \frac{a}{\sqrt{a^2 + b^2}}\mathbf{i} + \frac{b}{\sqrt{a^2 + b^2}}\mathbf{j}.$$

The vector $\mathbf{u}$ is in the same direction as $\mathbf{v}$, since $||\mathbf{v}|| > 0$. Furthermore,

$$||\mathbf{u}|| = \sqrt{\frac{a^2}{a^2 + b^2} + \frac{b^2}{a^2 + b^2}} = 1.$$

That is, $\mathbf{u}$ is a unit vector in the direction of $\mathbf{v}$.

■

As a consequence of Theorem 4, if $\mathbf{u}$ is a unit vector in the same direction as a vector $\mathbf{v}$, then $\mathbf{v}$ may be expressed as

$$\mathbf{v} = ||\mathbf{v}||\mathbf{u}. \tag{6}$$

This way of expressing a vector is useful in many applications.

Example 7 Finding a Unit Vector

Find a unit vector in the same direction as $\mathbf{v} = 4\mathbf{i} + 3\mathbf{j}$.

Solution:

We find $||\mathbf{v}||$ first:

$$||\mathbf{v}|| = ||4\mathbf{i} + 3\mathbf{j}|| = \sqrt{16+9} = 5.$$

Now we multiply $\mathbf{v}$ by the scalar $\dfrac{1}{||\mathbf{v}||} = \dfrac{1}{5}$. A unit vector in the direction as $\mathbf{v}$ is

$$\frac{\mathbf{v}}{||\mathbf{v}||} = \frac{4\mathbf{i} + 3\mathbf{j}}{5} = \frac{4}{5}\mathbf{i} + \frac{3}{5}\mathbf{j}.$$

See Figure 17.

Check: This vector is, in fact, a unit vector because

$$\left(\frac{4}{5}\right)^2 + \left(\frac{3}{5}\right)^2 = \frac{16}{25} + \frac{9}{25} = \frac{25}{25} = 1.$$

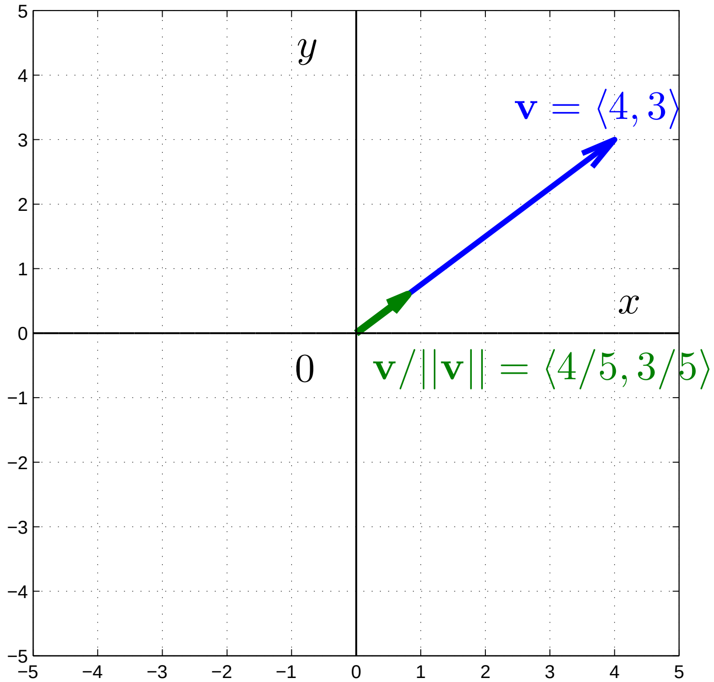
y
$\mathbf{v} = \langle 4, 3 \rangle$
x
0
$\mathbf{v}/||\mathbf{v}|| = \langle 4/5, 3/5 \rangle$
5
4
3
2
1
0
−1
−2
−3
−4
−5
−5
−4
−3
−2
−1
0
1
2
3
4
5

Figure 17:

Find a Vector from Its Direction and Magnitude

If a vector represents the speed and direction of an object, it is called a **velocity vector.** If a vector represents the direction and amount of a force acting on an object, it is called a **force vector.** In many practical applications, a vector is described in terms of its magnitude and direction, rather than in terms of its components.

Suppose that we are given the magnitude $||\mathbf{v}||$ of a nonzero vector $\mathbf{v}$ and the angle α, $0^\circ \leq \alpha < 360^\circ$, between $\mathbf{v}$ and $\mathbf{i}$. To express $\mathbf{v}$ in terms of $||\mathbf{v}||$ and α, we first find the unit vector $\mathbf{u}$ having the seme direction as $\mathbf{v}$.

$$\mathbf{u} = \frac{\mathbf{v}}{||\mathbf{v}||} \qquad \text{or} \qquad \mathbf{v} = ||\mathbf{v}||\mathbf{u}. \tag{7}$$

See Figure 18. The coordinates of the terminal point of $\mathbf{u}$ are $(\cos\alpha, \sin\alpha)$. Then $\mathbf{u} = \cos\alpha\mathbf{i} + \sin\alpha\mathbf{j}$ and, from (7),

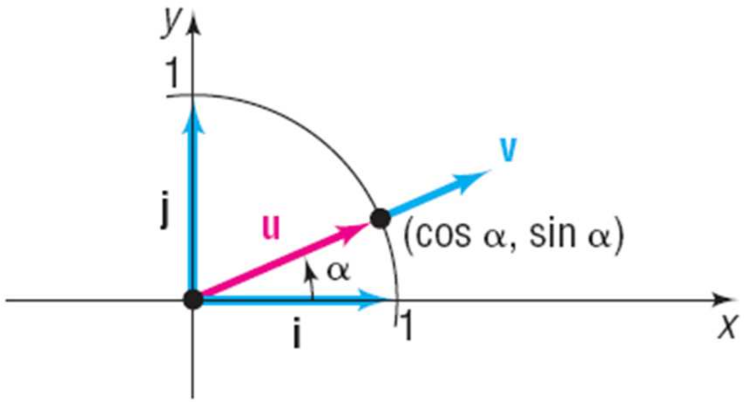

Figure 18:

$$\mathbf{u} = ||\mathbf{u}||(\cos\alpha\mathbf{i} + \sin\alpha\mathbf{j}). \tag{8}$$

where α is the angle between $\mathbf{v}$ and $\mathbf{i}$.

Exercise 1 Writing a Vector When Its Magnitude and Direction Are Given

A ball is thrown with an initial speed of 30 miles per hour in a direction that makes an angle of 60° with the positive x-axis. Express the velocity vector $\mathbf{v}$ in terms of $\mathbf{i}$ and $\mathbf{j}$. What is the initial speed in the horizontal direction? What is the initial speed in the vertical direction?

Analyze Objects in Static Equilibrium

When two forces combine, we simply add these two vectors because forces can be represented by vectors.

If $\mathbf{F_1}$ and $\mathbf{F_1}$ are two forces simultaneously acting on an object, the vector sum $\mathbf{F}_1 + \mathbf{F}_2$ is the **resultant force**. The resultant force products the same effect on the object as that obtained when the two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ act on the object. See Figure 19. An application of this concept is *static equilibrium.* An object is said to be in **static equilibrium** if

1. the object is at rest and

2. the sum of all forces acting on the object is zero, that is, if the resultant force is $\mathbf{0}$.

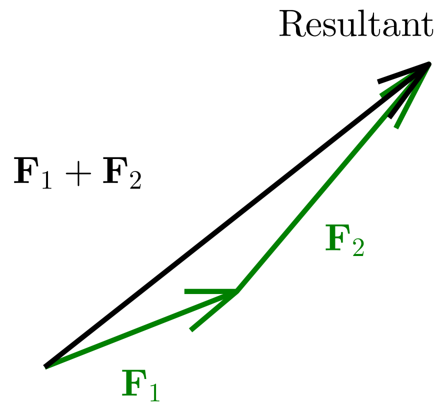

Figure 19:

Exercise 2 An Object in Static Equilibrium

A box of supplies (in yellow) that weighs 1000 pounds is suspended by two cables attached to the ceiling, as shown in Figure 20, where $\alpha = 45°$ and $\beta = 30°$. What are the tensions in the two cables?

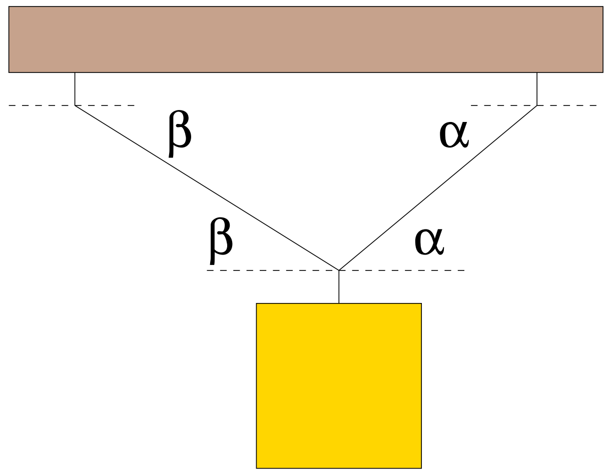

Figure 20: