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香港中文大學
The Chinese University of Hong Kong

二〇一七至一八年度上學期科目考試
Course Examination 1st Term, 2017-18

科目編號及名稱 Course Code & Title : **MATH1510A/B/C/D/E/F/G/H/I Calculus for Engineers**
時間 Time allowed : 2 小時 hours 00 分鐘 minutes
學號 Student I.D. No : ____________ 座號 Seat No. : ____________

- Answer all questions. Please show the work with as much detail as possible for every step.
- There are a total of 100 points and 11 questions.
- Answer Questions 1 and 10 in the question paper.
- Answer Questions 2 - 9, and 11 in the examination answer book. If you need extra room to answer questions, raise your hands.
- You must return your question paper and examination answer book(s) with your scratch paper at the end of the examination.

Please answer all questions:

1. Write your answer only inside the box. No justification is needed.

   (a) (1 point) The derivative of $f(x)$ with respect to x, $f'(x)$, is defined as

$$f'(x) = \lim_{\underline{\qquad\qquad}} \underline{\qquad\qquad\qquad\qquad}.$$

(b) (1 point) **(Chain Rule)** If g is differentiable at x and f is differentiable at $y = g(x)$, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and the derivative is given by

$$\frac{d}{dx}\Big((f \circ g)(x) \Big) = \underline{\hspace{6cm}}.$$

(c) (1 point) The derivative of $\cot x$ is

(d) (1 point) The derivative of $\sin^{-1}(x)$, $-1 < x < 1$, is

(e) (1 point) **(Mean Value Theorem)** Suppose $f(x)$ satisfies:

i. $f(x)$ is continuous on the closed interval $[a, b]$.

ii. $f(x)$ is differentiable on the open interval (a, b).

Then there exists c satisfying $a < c < b$ such that

$$f'(\underline{\hspace{1.5cm}}) = \underline{\hspace{6cm}}.$$

(f) (1 point) Let $a > 1$. Then $\int a^x \, dx$ is

(g) (1 point) Let $b > 0$. Then $\int \sin(bx) \, dx$ is

(h) (1 point) The Maclaurin series (or Taylor series at $x = 0$) of e^x is

$$e^x = \sum_{\underline{\qquad}}^{\overline{\qquad}} \underline{\qquad\qquad\qquad\qquad}.$$

(i) (1 point) Suppose that f is a function of two independent variables x and y. The partial derivative of f with respect to x is defined as

$$\frac{\partial f(x,y)}{\partial x} = \lim_{\rule{3cm}{0.4pt}} \rule{5cm}{0.4pt}.$$

Question 1 is worth 9 points. You must return your question paper and examination answer book(s) with your scratch paper at the end of the examination.

2. Let

$$f(x) = \begin{cases} \dfrac{x^2 - x - 6}{x - 3} & \text{if } x \neq 3; \\ c & \text{if } x = 3. \end{cases}$$

(a) (3 points) Find $\lim_{x \to 3} f(x)$;

(b) (2 points) Find the value of c for which $f(x)$ is continuous at $x = 3$.

3. (3 points) Let

$$f(x) = \begin{cases} 4 & \text{if } x < 2; \\ 2x & \text{if } x \geq 2. \end{cases}$$

Use the definition of derivative to determine whether $f(x)$ is differentiable at $x = 2$ or not.

4. (a) (2 points) Find $\dfrac{dy}{dx}$ if

$$y = \left(\frac{1}{x} + \frac{1}{x^2} \right)^{-2};$$

(b) (2 points) Find $\dfrac{dy}{dx}$ if

$$y = \cos\left(1 + \sin x\right);$$

(c) (2 points) Find $\dfrac{dy}{dx}$ if

$$y = x^{\sqrt{x}}, \text{ for } x > 0;$$

(d) (2 points) Evaluate $\left.\dfrac{dy}{dx}\right|_{(x,y)=(1,1)}$ if

$$y^3x^2 - yx + 2y^2 = 2x;$$

(e) (2 point) Find

$$\frac{d}{dx} \left\{ \int_{x^2}^{x^3} \sin^2 t \, dt \right\}.$$

5. Evaluate the following integrals:

(a) (2 points) $\int_0^3 2x\, e^{x^2}\, dx$;

(b) (2 points) $\int_4^9 \frac{2}{x-3}\, dx$;

(c) (2 points) $\int (1-\cos x)^5 \sin x\, dx$;

(d) (2 points) $\int x \cos x\, dx$;

(e) (2 points) $\int \frac{1}{\sqrt{x^2-1}}\, dx$, where $x > 1$.

6. Solve the following problems separately.

   (a) Let

   $$f(x) = x^3 + 3x^2 - 24x + 7.$$

   i. (4 points) Find all critical point(s)(or stationary point(s)) of f. Then, find the interval(s) on which f is increasing, and those on which f is decreasing.

   ii. (2 points) Determine whether each critical point is a local minimum or maximum, or neither.

   (b) (4 points) Let

   $$g(x) = 2\sin^2 x - x, \ 0 < x < \pi.$$

   Find all the critical point(s) of $g(x)$ on the interval $(0, \pi)$ and apply the Second Derivative Test for them.

   (c) (3 points) Let

   $$h(x) = e^{2x} + 16e^x - 10x^2.$$

   Find all the inflection point(s) of $h(x)$.

7. Solve the following problems separately.

   (a) (4 points) Find the area of the region in the xy-plane bounded by the curves C_1 and C_2:

   $$\begin{cases} C_1: & y = x, \\ C_2: & x^2 - y = 2. \end{cases}$$

   (b) Let $\mathcal{R}$ be the region in the xy-plane bounded by

   the curves $y = x^3$, $x = 0$, and $y = 1$.

   Express the volumes of the following solids as integrals (**You do not need to evaluate the integrals**):

   i. (2 points) The solid obtained by revolving $\mathcal{R}$ about the x-axis.

   ii. (2 points) The solid obtained by revolving $\mathcal{R}$ about the y-axis.

8. Solve the following problems separately.

(a) Given that

$$u(x,y) = x \ln y.$$

i. (2 points) Find $u_x = \dfrac{\partial u}{\partial x}$ and $u_y = \dfrac{\partial u}{\partial y}$.

ii. (2 points) Let $u_{yx} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial y}\right)$ and $u_{xy} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial x}\right)$. Show that

$$u_{yx} = u_{xy}.$$

iii. (1 point) Find $u_{xx}\,u_{yy} - u_{yx}\,u_{xy}$.

(b) (3 points) Let

$$z = f(r,s),$$

and

$$r = e^{x} + e^{-y}, \quad s = e^{-x} - e^{y}$$

where f, r and s are assumed differentiable, and x and y are independent variables of real numbers.

Show that

$$\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = r\frac{\partial z}{\partial r} - s\frac{\partial z}{\partial s}.$$

(c) (3 points) Compute

$$\int_0^1 \int_0^1 \cos(2x+y)\,dy\,dx.$$

(d) (3 points) Find the volume of the solid under the graph of the function

$$w(x,y) = xe^{y^2}$$

over the region

$$\mathcal{D} = \{(x,y) \mid 0 \le x \le 1, x^2 \le y \le 1\}.$$

9. Solve the following problems.

(a) (3 points) Recall that the Taylor polynomial of degree n about $x = a$ for the function $f(x)$ is given by

$$T_n(x) = \sum_{i=0}^{n} \left. \frac{f^{(i)}(x)}{n!} \right|_{x=a} (x-a)^i.$$

Find the Taylor polynomial of degree 3 for $f(x) = \sqrt{x}$ at $x = 1$, i.e., $T_3(x)$.

(b) i. (2 points) Find constants A and B such that

$$\frac{-2x+7}{x^2-7x+12} = \frac{A}{x-3} + \frac{B}{x-4}.$$

ii. (2 points) Given that

$$\frac{1}{1-x} = \sum_{k=0}^{+\infty} x^k = 1 + x + x^2 + x^3 + \cdots, \quad -1 < x < 1.$$

Find the Taylor series at $x = 0$ for

$$f(x) = \frac{-2x+7}{x^2-7x+12}.$$

iii. (2 points) Find the radius of convergence for the above Taylor series for $f(x)$.

(c) (2 points) Given that

$$f(x) = x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} \cdots,$$

$$g(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

Find the first 3 nonzero terms in the Taylor series for $f(x)g(x)$.

10.

True or False Questions
Not to be provided
P.10 - P.11
(Question 10)

11. Solve the following problems separately.

(a) (2 point) Suppose

$$x^x y^y z^z = 7$$

determines a function $z = z(x, y)$. Use the technique of logarithmic differentiation to show that at $x = y = z$,

$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{1}{x(1 + \ln x)}.$$

(b) i. (2 point) Let $f(x) = x^{\frac{1}{3}} - \frac{1}{3}x - \frac{2}{3}$ for $x > 0$. Show that $f(x) \leq 0$ for all $x > 0$.

ii. (1 point) By using the previous result in i, or otherwise, show that for any $a, b > 0$,

$$a^{\frac{1}{3}} b^{\frac{2}{3}} \leq \frac{1}{3}a + \frac{2}{3}b.$$

(c) (2 point) Define $I_n = \int (\ln x)^n \, dx$ for any nonnegative integer n. Show that for any positive integer n,

$$I_n = x (\ln x)^n - nI_{n-1}.$$

Hence, evaluate I_3.

(d) (2 point) Let $a \in \mathbb{R}$ and let $f : [0, a] \to \mathbb{R}$ be a continuous function. Show that

$$\int_0^a f(x) \, dx = \int_0^a f(a - x) \, dx.$$

Hence, find

$$\int_0^{\pi/2} \frac{\cos^3 x}{\sin x + \cos x} \, dx.$$

(e) (2 point) Write the first 2 non-zero terms in the Taylor series of $\sin(x^2)$, and compute

$$\lim_{x \to 0} \frac{\sin(x^2) - x^2}{x^6}.$$

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