Calculus for Engineers

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Contents

Integration by parts

24.1 Introduction

The method of integrating by parts is given in Section 24.2. The main idea behind this method is it allows us to express the integral of the product of two functions in terms of another whose evaluation may be simpler. In Section 24.3, we give a few examples of the applications of integration by substitution and integration by parts on the same integral. In Section 24.4, we present the method of integration by successive reduction using the method of integration by parts. In Section 24.5, we present a tabular integration by parts.

24.2 Integration by parts

If u and v are two functions of x, then by applying the product rule, we have

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Integrating both sides with respect to x, we get

$$uv = \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx. \tag{24.1}$$

Rearranging (24.1), we have

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx. \tag{24.2}$$

Let

$$u = f(x) \quad \text{and} \quad \frac{dv}{dx} = \phi(x).$$

Therefore

$$\frac{du}{dx} = f'(x) \text{ and } v = \int \phi(x)dx.$$

The statement (24.2) may now be rewritten as

$$\int f(x)\phi(x)dx = f(x)\int \phi(x)dx - \int \left(\left[\int \phi(x)dx\right] \cdot f'(x)\right)dx$$

or

$$\int f(x)\phi(x)dx = f(x)\int \phi(x)dx - \int \left(\int \phi(x)dx \cdot \left(\frac{d}{dx}f(x)\right)\right)dx.$$

In words, this formula states that the integral of the product of two functions is equal to first function $\times$ integral of second minus integral of (differentiation of first $\times$ integral of second.

Note 1 Let $f(x)$ be the first function and $\phi(x)$ be the second function. The success of the method depends upon choosing the first function in such a way that the second function on the RHS may be easy to evaluate. There is no general rule for choosing the first and second functions, However, in practice, the reader should keep in mind the following tips:

- Of the two functions, the one where the integral is not known should be taken as the first function.
- If the integrals of both the functions are known, then the function which vanishes by successive differentiation should be treated as the first function.
- If the integral is a single function, then 1 should be treated as the second function.
- If the integral of neither of the two functions reduces to zero by differentiating successively, then any part of the given function can treated as the first function. But if the integral on the RHS reverts to the original form, then the value of the integral can be immediately inferred by transposing the former to the LHS.

24.2.1 Worked examples

Until otherwise stated, we write $\log f(x) = \log_e f(x) = \ln f(x)$.

Example 1 Evaluate

$$\int xe^x dx.$$

Solution. Suppose we take $f(x) = x$ and $= \phi(x) = e^x$. By applying integration by parts, we get

$$\begin{aligned}\int xe^x dx &= x \cdot e^x - \int \left(\int e^x(x)dx \cdot \frac{d}{dx}x \right) dx \\ &= x \cdot e^x - \int e^x \cdot 1dx \\ &= xe^x - e^x + C,\end{aligned}$$

where C is a constant.

Note 2 Suppose we take $f(x) = e^x$ and $\phi(x) = x$. By applying integration by parts, we get

$$\begin{aligned}\int xe^x dx &= e^x \int xdx - \int \left(\int xdx \cdot \frac{d}{dx}e^x \right) dx \\ &= \frac{1}{2}x^2e^x - \frac{1}{2}\int x^2e^x dx + C,\end{aligned}$$

where C is a constant, so that the given integral $\int xe^x dx$ is converted into a comparatively more complicated integral $\int x^2 e^x dx$; the index of x having increased. Thus a proper choice of the order of factors is sometimes necessary.

□

Example 2 Evaluate

$$\int x^2 \cos x dx.$$

Solution. Suppose we have $f(x) = x^2$ and $\phi(x) = \cos x$. By applying integration by parts, we get

$$\begin{aligned}
\int x^2 \cos x dx &= x^2 \int \cos x dx - \int \left(\int \cos x dx \cdot \frac{d}{dx} x^2 \right) dx \\
&= x^2 \cdot \sin x - \int 2x \cdot \sin x dx \\
&= x^2 \cdot \sin x - 2 \left(x \cdot \int \sin x dx - \int \left(\int \sin x dx \cdot \frac{d}{dx} x \right) dx \right) \\
&= x^2 \sin x - 2 \left(x \cdot (-\cos x) - \int (-\cos x) dx \right) \\
&= x^2 \sin x + 2x \cos x - 2 \sin x + C \\
&= (x^2 - 2) \sin x + 2x \cos x + C,
\end{aligned}$$

where C is a constant.

We note that to evaluate $\int x \sin x dx$, we have again to apply the rule of integration by parts. We thus see that the *rule of integration by parts may have to be repeated several times.*

□

Example 3 Evaluate

$$\int \cos^{-1} x dx.$$

Solution. Suppose we have $f(x) = \cos^{-1} x$ and $\phi(x) = 1$. By applying integration by parts, we get

$$\begin{aligned}\int \cos^{-1} x dx &= \int \cos^{-1} x \cdot 1 dx \\ &= \cos^{-1} x \int 1 dx - \int \left(\int 1 dx \cdot \frac{d}{dx} \cos^{-1} x \right) dx \\ &= \cos^{-1} x \cdot x - \int \frac{-x}{\sqrt{1-x^2}} x dx \\ &= x \cos^{-1} x - \frac{1}{2} \int (1-x^2)^{-\frac{1}{2}} (-2x) dx \\ &= x \cos^{-1} x - \frac{1}{2} \frac{(1-x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C \\ &= x \cos^{-1} x - \sqrt{1-x^2} + C,\end{aligned}$$

where C is a constant.

□

Example 4 Evaluate

$$\int \log(x+1) dx.$$

Solution. Suppose we have $f(x) = \log(x+1)$ and $\phi(x) = 1$. By applying integration by parts, we get

$$\begin{aligned}\int \log(x+1) dx &= \log(x+1) \int 1 dx - \int \left(\int 1 dx \cdot \frac{d}{dx} \log(x+1) \right) dx \\ &= x \log(x+1) - \int \frac{x dx}{x+1} \\ &= x \log(x+1) - \int \frac{(x+1-1) dx}{x+1} \\ &= x \log(x+1) - \int dx + \int \frac{dx}{x+1} \\ &= x \log(x+1) - x + \log(x+1) + C \\ &= (x+1) \log(x+1) - x + C,\end{aligned}$$

where C is a constant.

□

Example 5 Evaluate

$$\int \cos \sqrt{x} dx.$$

Solution. If we put $x = t^2$, then we get $dx = 2t dt$. Then we have

$$\int \cos \sqrt{x} dx = 2 \int t \cos t dt.$$

Suppose we have $f(x) = t$ and $\phi(x) = \cos t$. By applying integration by parts, we get

$$
\begin{aligned}
\int \cos\sqrt{x}dx &= 2\int t\cos tdt \\
&= 2\left(t\int \cos tdt - \int\left(\left(\int \cos tdt \cdot \frac{d}{dt}t\right) dt\right)\right. \\
&= 2\left(t\sin t - \int \sin tdt\right) \\
&= 2(t\sin t + \cos t) + C \\
&= 2\left(\sqrt{x}\sin\sqrt{x} + \cos\sqrt{x}\right) + C,
\end{aligned}
$$

where C is a constant.

□

Example 6 Evaluate

$$\int \sin(\log x)dx.$$

Solution. If we put $\log x = t$, then we get $\frac{1}{x}dx = dt$ or $dx = xdt = e^t dt$. Thus

$$\int \sin(\log x)dx = \int e^t \sin tdt.$$

Suppose we have $f(t) = e^t$ and $\phi(t) = \sin t$.

$$
\begin{aligned}
I &= e^t(-\cos t) - \int(-\cos t)e^t dt \\
&= -e^t\cos t + \int e^t\cos tdt.
\end{aligned}
$$

By applying integration by parts, we get

$$-e^t\cos t + e^t\sin t - \int e^t\sin tdt = e^t(\sin t - \cos t) - I + C_1$$

or

$$2I = e^t(\sin t - \cos t) + C_1.$$

Therefore, we have

$$
\begin{aligned}
I &= \frac{1}{2}e^t(\sin t - \cos t) + \frac{C_1}{2} \\
&= \frac{1}{2}x(\sin(\log x) - \cos(\log x)) + C
\end{aligned}
$$

where $C = \frac{C_1}{2}$.

□

Example 7 Evaluate

$$\int x(\log x)^2 dx.$$

Solution. Suppose we have $f(x) = (\log x)^2$ and $\phi(x) = x$. By applying integration by parts, we get

$$\begin{aligned}
I &= (\log x)^2 \int x dx - \int \left(\int x dx \cdot \frac{d}{dx}(\log x)^2 \right) dx \\
&= \frac{x^2}{2}(\log x)^2 - \int \frac{x^2}{2} \left(2(\log x) \left(\frac{1}{x} \right) \right) dx \\
&= \frac{x^2}{2}(\log x)^2 - \int x \log x dx.
\end{aligned}$$

By applying integration by parts again, taking $\log x$ as the first function, we have

$$\begin{aligned}
I &= \frac{x^2}{2}(\log x)^2 - \left(\log x \int x dx - \int \left(\int x dx \cdot \frac{d}{dx} \log x \right) dx \right) \\
&= \frac{x^2}{2}(\log x)^2 - \left(\frac{x^2}{2} \log x - \int \frac{1}{2} x^2 \left(\frac{1}{x} \right) dx \right) \\
&= \frac{x^2}{2}(\log x)^2 - \frac{x^2}{2} \log x + \frac{1}{2} \int x dx \\
&= \frac{x^2}{2}(\log x)^2 - \frac{x^2}{2} \log x + \frac{1}{4} x^2 + C,
\end{aligned}$$

where C is a constant.

□

Example 8 Evaluate

$$\int \sec^3 x dx.$$

Solution. Let us write

$$I = \int \sec^3 x dx = \int \sec^2 x \sec x dx.$$

Suppose we have $f(x) = \sec^2 x$ and $\phi(x) = \sec x$. By applying integration by parts, we get

$$\begin{aligned}
I &= \sec^2 x \int \sec x dx - \int \left(\int \sec x dx \cdot \frac{d}{dx} \sec^2 x \right) dx \\
&= \sec x \tan x - \int \tan x (\sec x \tan x) dx \\
&= \sec x \tan x - \int \sec x \tan^2 x dx \\
&= \sec x \tan x - \int \sec x (\sec^2 x - 1) dx \\
&= \sec x \tan x - \int \sec^3 x dx + \int \sec x dx \\
&= \sec x \tan x - I + \log(\sec x + \tan x) + C_1.
\end{aligned}$$

Therefore, we have

$$2I = \sec x \tan x + \log(\sec x + \tan x) + C_1$$

or

$$I = \frac{1}{2}(\sec x \tan x + \log(\sec x + \tan x)) + C,$$

where C is a constant.

□

Example 9 Evaluate

$$\int \left(\cos^{-1} x\right)^2 dx.$$

Solution. Suppose we have $f(x) = \left(\cos^{-1} x\right)^2$ and $\phi(x) = 1$. By applying integration by parts, we get

$$\begin{aligned}
I &= \left(\cos^{-1} x\right)^2 \int 1 dx - \int \left(\int 1 dx \cdot \frac{d}{dx} \left(\cos^{-1} x\right)^2 \right) dx \\
&= x(\cos^{-1} x)^2 - \int x \left(2(\cos^{-1} x) \left(\frac{-1}{\sqrt{1-x^2}} \right) \right) dx \\
&= x(\cos^{-1} x)^2 + \int \frac{2x}{\sqrt{1-x^2}} \cos^{-1} x dx \\
&= x(\cos^{-1} x)^2 + I_1,
\end{aligned}$$

where

$$I_1 = \int \frac{2x}{\sqrt{1-x^2}} \cos^{-1} x dx.$$

Suppose we have $f(x) = \cos^{-1} x$ and $\phi(x) = \frac{2x}{\sqrt{1-x^2}}$. By applying integration by parts, we get

$$\begin{aligned} I_1 &= \cos^{-1} x \int \frac{2x}{\sqrt{1-x^2}}dx - \int \left(\int \frac{2x}{\sqrt{1-x^2}}dx \cdot \frac{d}{dx}\cos^{-1} x \right) dx \\ &= -2\sqrt{1-x^2}\cos^{-1} x - \int \left(-2\sqrt{1-x^2}\right)\left(-\frac{1}{\sqrt{1-x^2}}\right) dx \\ &= -2\sqrt{1-x^2}\cos^{-1} x - 2x + C. \end{aligned}$$

Therefore, we have

$$I = x(\cos^{-1} x)^2 - 2\sqrt{1-x^2}\cos^{-1} x - 2x + C,$$

where C is a constant.

□

24.2.2 Evaluation of the integral

Consider

$$\int e^x[f(x) + f'(x)]dx.$$

By applying integration by parts, we get

$$\int e^x f(x)dx = e^x f(x) - \int e^x f'(x)dx$$

or

$$\int e^x(f(x) + f'(x))dx = e^x f(x) + C, \tag{24.3}$$

where C is a constant.

This form of integral is quite important.

24.2.3 Worked Examples

Example 10 Evaluate

1. $$\int \frac{xe^x}{(x+1)^2}dx,$$

2. $$\int e^x \frac{1-\sin x}{1-\cos x}dx.$$

Solution.

1. **Solution 1:** We have

$$
\begin{aligned}
\int \frac{xe^x}{(x+1)^2}dx &= \int \frac{x+1-1}{(x+1)^2}e^x dx, \\
&= \int \left(\frac{1}{x+1} - \frac{1}{(x+1)^2}\right) e^x dx.
\end{aligned}
$$

By letting $f(x) = \dfrac{1}{x+1}$, we have $f'(x) = -\dfrac{1}{(x+1)^2}$. The function to be integrated is of the form (24.3). Hence, the integral is

$$
\begin{aligned}
\int \frac{xe^x}{(x+1)^2}dx &= \int e^x(f(x)+f'(x))dx \\
&= e^x f(x) + C \\
&= \frac{e^x}{x+1} + C,
\end{aligned}
$$

where C is a constant.

Solution 2: Suppose we have $f(x) = \dfrac{1}{x+1}$ and $\phi(x) = e^x$. By applying integration by parts, we get

$$
\begin{aligned}
\int \frac{1}{x+1}e^x dx &= \frac{1}{x+1}\int e^x dx - \int \left(\left(\int e^x dx \cdot \frac{d}{dx}\left(\frac{1}{x+1}\right)\right)\right) dx \\
&= \frac{1}{x+1}e^x - \int -\frac{1}{(x+1)^2}e^x dx,
\end{aligned}
$$

or

$$
\int \left(\frac{1}{x+1} - \frac{1}{(x+1)^2}\right) e^x dx = \frac{1}{x+1}e^x + C,
$$

where C is a constant.

2. We have

$$
\begin{aligned}
e^x\left(\frac{1-\sin x}{1-\cos x}\right) &= e^x\left(\frac{1-2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}\right) \\
&= e^x\left(\frac{1}{2}\mathrm{cosec}^2\frac{x}{2} - \cot\frac{x}{2}\right)
\end{aligned}
$$

so that $f(x) = \cot\frac{x}{2}$. We see that the integrand is of the form

$$
e^x(f(x)+f'(x)).
$$

By applying integration by parts, we get

$$
\int e^x \cot\frac{x}{2}dx = e^x\cot\frac{x}{2} - \int -\frac{1}{2}\mathrm{cosec}^2 x dx.
$$

Hence, we have

$$\int e^x \left(\cot\frac{x}{2} - \frac{1}{2}\mathrm{cosec}^2\frac{x}{2}\right) dx = e^x \cot\frac{x}{2},$$

or

$$\int e^x \left(\frac{1}{2}\mathrm{cosec}^2\frac{x}{2} - \cot\frac{x}{2}\right) dx = -e^x \cot\frac{x}{2}.$$

□

24.2.4 Integrals of $e^{ax}\cos(bx+c)$ and $e^{ax}\sin(bx+c)$

Two special integrals are studied:

1. $$\int e^{ax}\cos(bx+c)dx,$$

2. $$\int e^{ax}\sin(bx+c)dx.$$

There are two forms of solutions:

1. Applying the rule of integration by parts, we obtain

$$\begin{aligned}\int e^{ax}\cos(bx+c)dx &= \cos(bx+c)\int e^{ax}dx - \int\left(\int e^{ax}dx\cdot\frac{d}{dx}\cos(bx+c)\right)dx\\ &= \frac{e^{ax}}{a}\cos(bx+c) - \int -\frac{e^{ax}}{a}b\sin(bx+c)dx\\ &= \frac{e^{ax}}{a}\cos(bx+c) + \frac{b}{a}\int e^{ax}\sin(bx+c)dx. \end{aligned}\tag{24.4}$$

Similarly, we have

$$\begin{aligned}\int e^{ax}\sin(bx+c)dx &= \sin(bx+c)\int e^{ax}dx - \int\left(\int e^{ax}dx\cdot\frac{d}{dx}\sin(bx+c)\right)dx\\ &= \frac{e^{ax}}{a}\sin(bx+c) - \int\frac{e^{ax}}{a}\cdot b\cos(bx+c)dx\\ &= \frac{e^{ax}}{a}\sin(bx+c) - \frac{b}{a}\int e^{ax}\cos(bx+c)dx. \end{aligned}\tag{24.5}$$

If the value of $\int e^{ax}\cos(bx+c)dx$ is required, we substitute the RHS of (24.5) for the last term of (24.4) and if the value of $\int e^{ax}\sin(bx+c)dx$ is required, we substitute the RHS of 24.4) for the last term of (24.5). In the former case, we have

$$
\begin{aligned}
&\int e^{ax}\cos(bx+c)dx \\
&= \frac{e^{ax}}{a}\cos(bx+c) + \frac{b}{a}\left(\sin(bx+c)\int e^{ax})dx - \int\left(\left(\int e^{ax}dx\cdot\frac{d}{dx}\sin(bx+c)\right)dx\right)\right. \\
&= \frac{e^{ax}}{a}\cos(bx+c) + \frac{b}{a^2}e^{ax}\sin(bx+c) - \frac{b^2}{a^2}\int e^{ax}\cos(bx+c)dx.
\end{aligned}
$$

So

$$\left(1+\frac{b^2}{a^2}\right)\int e^{ax}\cos(bx+c)dx = e^{ax}\frac{a\cos(bx+c)+b\sin(bx+c)}{a^2},$$

or

$$\int e^{ax}\cos(bx+c)dx = e^{ax}\frac{a\cos(bx+c)+b\sin(bx+c)}{a^2+b^2}.$$

Similarly, we have

$$\int e^{ax}\sin(bx+c)dx = e^{ax}\frac{a\sin(bx+c)-b\cos(bx+c)}{a^2+b^2}.$$

2. To put the results in another form, we determine two numbers r and α such that

$$a = r\cos\alpha \quad \text{and} \quad b = r\sin\alpha.$$

These give

$$r = \sqrt{a^2+b^2}, \quad \text{and} \quad \alpha = \tan^{-1}\left(\frac{b}{a}\right).$$

Therefore, we have

$$
\begin{aligned}
\int e^{ax}\cos(bx+c)dx &= e^{ax}\frac{r\cos(bx+c-\alpha)}{a^2+b^2} \\
&= e^{ax}\frac{\cos\left(bx+c-\tan^{-1}\left(\frac{b}{a}\right)\right)}{\sqrt{a^2+b^2}}.
\end{aligned}
$$

Similarly, we have

$$\int e^{ax}\sin(bx+c)dx = e^{ax}\frac{\sin\left(bx+c-\tan^{-1}\left(\frac{b}{a}\right)\right)}{\sqrt{a^2+b^2}}.$$

24.2.5 Worked examples

Example 11 Evaluate

1.

$$\int e^{3x}\sin 4x dx,$$

2.
$$\int e^{4x} \cos 2x \cos 4x dx,$$

3.
$$\int x e^{2x} \cos x dx.$$

Solution.

1. Applying the rule of integration by parts, we obtain
$$\begin{aligned}\int e^{3x} \sin 4x dx &= \frac{e^{3x}}{\sqrt{3^2+4^2}} \sin\left(4x - \tan^{-1}\frac{4}{3}\right) \\ &= \frac{e^{3x}}{5} \sin\left(4x - \tan^{-1}\frac{4}{3}\right) + C,\end{aligned}$$
where C is a constant.

2. Using the identity
$$\cos 2x \cos 4x = \frac{1}{2}(2\cos 2x \cos 4x) = \frac{1}{2}(\cos 6x + \cos 2x),$$
the integral becomes
$$\begin{aligned}&\int e^{4x} \cos 2x \cos 4x dx \\ &= \frac{1}{2}\int e^{4x} \cos 6x dx + \frac{1}{2}\int e^{4x} \cos 2x dx \\ &= \frac{1}{2}\frac{e^{4x}}{\sqrt{4^2+6^2}} \cos\left(6x - \tan^{-1}\left(\frac{6}{4}\right)\right) + \frac{1}{2}\frac{e^{4x}}{\sqrt{4^2+2^2}} \cos\left(2x - \tan^{-1}\left(\frac{2}{4}\right)\right) \\ &= \frac{e^{4x}}{2}\left[\frac{1}{\sqrt{52}} \cos\left(6x - \tan^{-1}\left(\frac{3}{2}\right)\right) + \frac{1}{\sqrt{20}} \cos\left(2x - \tan^{-1}\left(\frac{1}{2}\right)\right)\right] + C,\end{aligned}$$
where C is a constant.

3. To evaluate $\int x e^{2x} \cos x dx$, we apply the rule of integration by parts. Taking x and $e^{2x} \cos x$ as two factors, we have
$$\int x e^{2x} \cos x dx = x \frac{e^{2x}}{\sqrt{5}} \cos\left(x - \tan^{-1}\frac{1}{2}\right) - \int 1 \frac{e^{2x}}{\sqrt{5}} \cos\left(x - \tan^{-1}\frac{1}{2}\right) dx. \tag{24.6}$$
By performing integration by parts again, the RHS of (24.6) for the last term becomes
$$\int e^{2x} \cos\left(x - \tan^{-1}\frac{1}{2}\right) dx = \frac{e^{2x}}{\sqrt{5}} \cos\left(x - 2\tan^{-1}\frac{1}{2}\right)$$

Therefore, we have

$$\int xe^{2x}\cos x dx = e^{2x}\left[\frac{x}{\sqrt{5}}\cos\left(x-\tan^{-1}\frac{1}{2}\right)-\frac{1}{5}\cos\left(x-2\tan^{-1}\frac{1}{2}\right)\right]+C,$$

where C is a constant.

$\square$

24.3 Combination of integration by substitution and integration by parts

Sometimes both methods of integration have to be applied in the same question. We will illustrate the procedure through two examples.

Example 12 Evaluate

$$\int \frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx.$$

Solution. By applying the substitution rule, we put $x = \sin\theta$, then $dx = \cos\theta d\theta$. Therefore, the integral becomes

$$\begin{aligned}\int \frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx &= \int \frac{\sin\theta\cdot\theta}{\cos\theta}\cos\theta d\theta \\ &= \int \theta\sin\theta d\theta.\end{aligned}$$

To evaluate $\int \theta\sin\theta d\theta$, we apply the rule of integration by parts and obtain

$$\begin{aligned}\int \theta\sin\theta d\theta &= \theta\int\sin\theta dx - \int\left(\int\sin\theta dx\cdot\frac{d}{dx}\theta\right)dx \\ &= (-\theta\cos\theta) - \int -\cos\theta\cdot 1 d\theta \\ &= -\theta\cos\theta + \int\cos\theta d\theta \\ &= -\theta\cos\theta + \sin\theta + C.\end{aligned}$$

Thus, we have

$$\begin{aligned}\int \frac{x\sin^{-1}x}{\sqrt{1-x^2}} &= -\theta\cos\theta+\sin\theta+C \\ &= -\sqrt{1-x^2}\sin^{-1}x + x + C,\end{aligned}$$

where C is a constant. $\square$

Example 13 Evaluate

$$\int \tan^{-1}\left(\sqrt{\frac{1-x}{1+x}}\right) dx.$$

Solution. By applying the substitution rule, we put $x = \cos\theta$, then $dx = -\sin\theta d\theta$. Therefore, the integral becomes

$$\begin{aligned}
\tan^{-1}\sqrt{\frac{1-x}{1+x}} &= \tan^{-1}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \\
&= \tan^{-1}\sqrt{\frac{2\sin^2(\theta/2)}{2\cos^2(\theta/2)}} \\
&= \tan^{-1}\left(\tan\frac{\theta}{2}\right) \\
&= \frac{\theta}{2}.
\end{aligned}$$

By performing integration by parts, we obtain

$$\begin{aligned}
\int \tan^{-1}\sqrt{\frac{1-x}{1+x}} dx &= \int \frac{\theta}{2}(-\sin\theta) d\theta \\
&= -\frac{1}{2}\int \theta\sin\theta d\theta \\
&= -\frac{1}{2}\left(\theta\int \sin\theta dx - \int\left(\left(\int \sin\theta dx \cdot \frac{d}{dx}\theta\right) dx\right)\right. \\
&= -\frac{1}{2}(-\theta\cos\theta + \sin\theta) \\
&= -\frac{1}{2}\left(-x\cos^{-1}x + \sqrt{1-x^2}\right).
\end{aligned}$$

$\square$

24.4 Reduction formulae

A formula which connects an integral with another in which the integrand is of the same type, but is of lower degree or is otherwise easier to integrate, is called a reduction formula.

Usually the reduction formula has to be used repeated to arrive at the integral of the given function. This method of integration is called integration by successive reduction.

Reduction formulae are usually obtained by applying the rule of integration by parts and are useful when the integral cannot be otherwise immediately obtained.

24.4.1 Worked examples

Example 14 Establish a reduction formula for $\int x^n e^{ax} dx$ and apply it to evaluate $\int x^3 e^{ax} dx$.

Solution. Suppose we take $f(x) = x^n$ and $\phi(x) = e^{ax}$.

By using integrating by parts, we have

$$\begin{aligned}\int x^n e^{ax} dx &= x^n \int e^{ax} dx - \int \left(\int e^{ax} dx \cdot \frac{d}{dx} x^n \right) dx \\ &= x^n \frac{e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax} dx + C,\end{aligned}$$

which is the desired reduction formula.

Putting $n = 3, 2, 1$ successively in it, we obtain

$$\int x^3 e^{ax} dx = \frac{x^3 e^{ax}}{a} - \frac{3}{a} \int x^2 e^{ax} dx \tag{24.7}$$

$$\int x^2 e^{ax} dx = \frac{x^2 e^{ax}}{a} - \frac{2}{a} \int x e^{ax} dx \tag{24.8}$$

$$\int x e^{ax} dx = \frac{x e^{ax}}{a} - \frac{1}{a} \int e^{ax} dx = \frac{x e^{ax}}{a} - \frac{e^{ax}}{a^2}. \tag{24.9}$$

From (24.8) and (24.9), we get

$$\int x^2 e^{ax} dx = \frac{x^2 e^{ax}}{a} - \frac{2x e^{ax}}{a^2} + \frac{2e^{ax}}{a^3}. \tag{24.10}$$

Again from (24.7) and (24.10), we obtain

$$\begin{aligned}\int x^3 e^{ax} dx &= \frac{x^3 e^{ax}}{a} - \frac{3x^2 e^{ax}}{a^2} + \frac{6x e^{ax}}{a^3} - \frac{6e^{ax}}{a^4} + C \\ &= \frac{e^{ax}}{a^4}(a^3 x^3 - 3a^2 x^2 + 6ax - 6) + C,\end{aligned}$$

where C is a constant.

□

Example 15 Obtain a reduction formula for $\int x^m \sin nx dx$.

Solution. We have, integrating by parts,

$$\begin{aligned}\int x^m \sin nx dx &= x^m \int \sin n dx - \int \left(\int \sin n dx \cdot \frac{d}{dx} x^m \right) dx \\ &= -\frac{x^m \cos nx}{n} + \frac{m}{n} \int x^{m-1} \cos nx dx. \qquad (24.11)\end{aligned}$$

Again,

$$\begin{aligned}\int x^{m-1} \cos nx dx &= x^{m-1} \int \cos nx dx - \int \left(\int \cos nx dx \cdot \frac{d}{dx} x^{m-1} \right) dx \\ &= \frac{x^{m-1} \sin nx}{n} - \frac{m-1}{n} \int x^{m-2} \sin nx dx. \qquad (24.12)\end{aligned}$$

Combining (24.11) and (24.12), we get

$$\begin{aligned}\int x^m \sin nx dx = -\frac{x^m \cos nx}{n} &+ \frac{m x^{m-1} \sin nx}{n^2} \\ &- \frac{m(mn-1)}{n^2} \int x^{m-2} \sin nx dx,\end{aligned}$$

which is the required reduction formula.

□

Example 16 Obtain a reduction formula for $\int x^m (\log x)^n dx$ and apply it to evaluate $\int_0^1 x^4 (\log x)^3 dx$.

Solution. Integrating by parts, we have

$$\begin{aligned}&\int x^m (\log x)^n dx \\ &= (\log x)^n \int x^m dx - \int \left(\int x^m dx \cdot \frac{d}{dx} (\log x)^n \right) dx \\ &= \frac{x^{m+1}}{m+1} (\log x)^n - \frac{n}{m+1} \int x^{m+1} (\log x)^{n-1} \cdot \frac{1}{x} dx \\ &= \frac{x^{m+1}}{m+1} (\log x)^n - \frac{n}{m+1} \int x^m (\log x)^{n-1} dx,\end{aligned}$$

as the required reduction formula.

Putting $m = 4$, we get

$$\int x^4 (\log x)^n dx = \frac{x^5}{5} (\log x)^n - \frac{n}{5} \int x^4 (\log x)^{n-1} dx.$$

Putting $n = 3, 2, 1$ successively, we get

$$\int x^4(\log x)^3dx = \frac{x^5}{5}(\log x)^3 - \frac{3}{5}\int x^4(\log x)^2dx, \tag{24.13}$$

$$\int x^4(\log x)^2dx = \frac{x^5}{5}(\log x)^2 - \frac{2}{5}\int x^4 \log x dx \tag{24.14}$$

and

$$\begin{aligned}\int x^4 \log x dx =& \frac{x^5}{5}\log x - \frac{1}{5}\int x^4 dx \\ =& \frac{x^5}{5}\log x - \frac{x^5}{25}.\end{aligned}$$

From these, we obtain

$$\int x^4(\log x)^3dx = \frac{x^5}{5}(\log x)^3 - \frac{3x^5}{25}(\log x)^2 + \frac{6x^5}{125}\log x - \frac{6}{625}x^5.$$

□

24.5 Tabular integration by parts

The key concept of (24.2), that is,

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

or

$$\int udv = uv - \int vdudx. \tag{24.15}$$

is applicable when the following criteria are met:

- v' or dv is easy to integrate.
- u is easy to differentiate.
- $\displaystyle\int vu'dx$ is easier to compute than $\displaystyle\int uv'dx$.

Often, we need to do integration by parts several times to obtain an antiderivative. Equation (24.15) can be put into a table, known as *tabular integration by parts* or simply the tabular method. It is based on the following theorem and table.

Theorem 1 (High order integration by parts) Suppose that $u_0(x)$ and $v^0(x)$ are given functions. Now define a sequence of derivatives of u_0 by

$$u_1(x) = u_0'(x), \quad u_2(x) = u_1'(x), \quad , \cdots , u_n(x) = u_{n-1}'(x)$$

and define a sequence of antiderivative of $v^0(x)$ by

$$v^1(x) = \int v^0(x)dx, \quad v^2(x) = \int v^1(x)dx, \quad , \cdots , v^2(x) = \int v^1(x)dx.$$

Then the $n-$th order integration by parts formula is

$$\int u_0v^0dx = u_0v^1 - u_1v^2 + u_2v^3 + \cdots + (-1)^{n-1}u_{n-1}v^n + (-1)^n \int u_nv^n(x)dx \tag{24.16}$$

for any integer $n > 0$.

Theorem 1 can be visualized in a table as

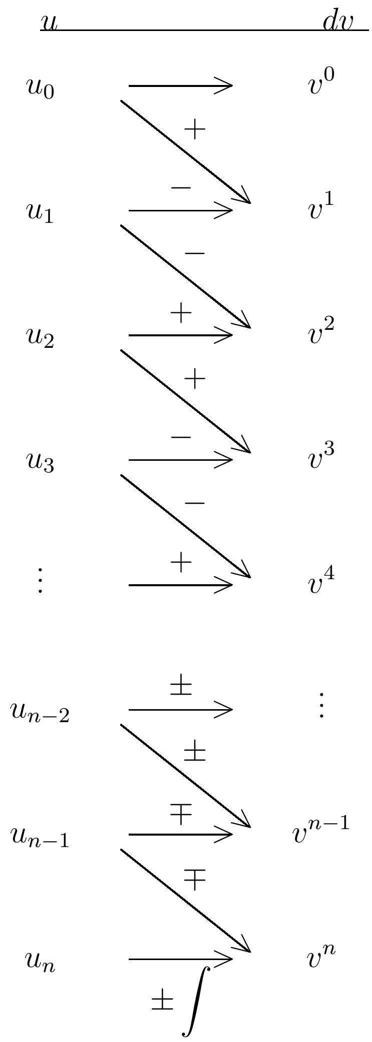

Figure 24.1: Table form of (24.16).

In Table 24.1, each diagonal arrow means that the two functions should be multiplied with the appropriate sign (+ or $-$), while the horizontal arrow that the product of the two functions should be integrated (with the appropriate sign).

This technique is often used when the integrand is a product of two different classes of functions. We have seen in Chapter 1 that there are several classes of elementary functions, for example, algebraic, trigonometric, inverse trigonometric, exponential and logarithmic functions. To use (24.16), we have to choose u and dv wisely. Two useful rules of thumb are:

u : The best choice is to set u equal to the first expression present from a class of elementary functions.

dv**:** This should be the most "complicated" expression that one can integrate, even if one must use a substitution to do so. If there is absolutely nothing one can integrate, simply set $dv = 1dx$.

One places u and dv at the top of the appropriate columns, then one writes each new row in the table by differentiating the u column and integrating the dv column.

Three main cases will be discussed to illustrate a practice.

Case 1 : The u column eventually goes to 0, that is, the function u belongs to a class of algebraic functions, where the algebraic functions are those without trigonometric, inverse trigonometric, exponential or logarithmic expressions, so essentially they are combinations of power functions. Once the u column arrives at 0, then the antiderivative can be read from the table. The integral is of the form:

$$\int \underbrace{(\text{polynomial})}_{u} \cdot \underbrace{\underbrace{(\text{something easy to integrate})}_{v'}\, dx}_{dv}.$$

Example 17 Evaluate

$$\int (x^3 + 2x^2) \cos x dx.$$

Solution. Let $u_0(x) = x^3 + 2x^2$ and $v^0(x) = \cos x$. Using the tabular method, we

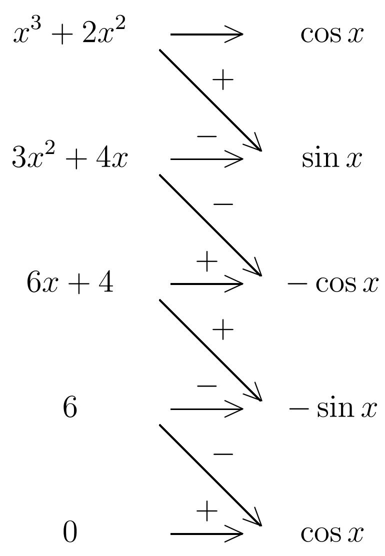

Figure 24.2: Example 17.

have the results depicted in Figure 24.2, where the fourth order derivative of $u_0(x)$ becomes 0 and $v^0(x)$ is integrated four times. Hence, we have

$$\begin{aligned}
&\int (x^3 + 2x^2) \cos x dx \\
&= (x^3 + 2x^2) \cdot \sin x - (3x^2 + 4x) \cdot \cos x + (-\sin x) \cdot (6x + 4) - 6 \cdot \cos x \\
&\qquad + \int 0 \cdot \cos x dx \\
&= (x^3 + 2x^2) \cdot \sin x - (3x^2 + 4x) \cdot \cos x - (6x + 4) \cdot \sin x - 6 \cdot \cos x + C,
\end{aligned}$$

where C is a constant.

$\square$

Example 18 Evaluate

$$\int x^2 e^{-\frac{x}{2}}\,dx.$$

Solution. Let $u_0(x) = x^2$ and $v^0(x) = e^{-\frac{x}{2}}$. Using the tabular method, we have the

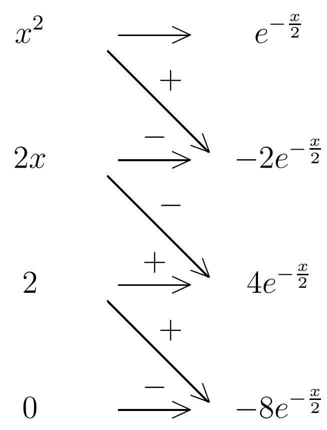

Figure 24.3: Example 18.

results depicted in Figure 24.3, where the third order derivative of $u_0(x)$ becomes 0 and $v^0(x)$ is integrated three times. Hence, we have

$$\begin{aligned}\int x^2 e^{-\frac{x}{2}}\,dx &= x^2 \cdot \left(-2e^{-\frac{x}{2}}\right) - 2x \cdot \left(4e^{-\frac{x}{2}}\right) + 2 \cdot \left(-8e^{-\frac{x}{2}}\right) - \int 0 \cdot \left(-8e^{-\frac{x}{2}}\right) dx \\ &= -2x^2 e^{-\frac{x}{2}} - 8xe^{-\frac{x}{2}} - 16e^{-\frac{x}{2}} + C,\end{aligned}$$

where C is a constant.

$\square$

Case 2 : The u column eventually becomes algebraic, that is, the function u belongs to a class of either logarithmic or inverse trigonometric functions. Two tips are:

u : In the u column, cross out the last entry and write 1 beneath it.

dv**:** In the dv column, regarding the last entry, write the product of the last u and dv terms.

The integral is of the form:

$$\int \underbrace{(\text{logarithmic})}_{u} \cdot \underbrace{\underbrace{(\text{algebraic})}_{v'}\,dx}_{dv}$$

or

$$\int \underbrace{(\text{inverse trigonometric})}_{u} \cdot \underbrace{\underbrace{(\text{algebraic})}_{v'}\, dx}_{dv}.$$

Example 19 Evaluate

$$\int x^3 \ln x dx.$$

Solution. Let $u_0(x) = \ln x$ and $v^0(x) = x^3$. Using the tabular method, we have

$$\begin{tikzcd} \ln x \arrow[r] \arrow[dr, "+"] & x^3 \\ \frac{1}{x} \arrow[r, "-"] & \frac{1}{4}x^4 \end{tikzcd}$$

Figure 24.4: Example 19.

the results depicted in Figure 24.4 and write

$$\begin{aligned} \int x^3 \ln x dx &= \ln x \cdot \left(\frac{1}{4}x^4\right) - \int \frac{1}{x} \cdot \left(\frac{1}{4}x^4\right) dx \\ &= \frac{1}{4}x^4 \ln x - \frac{1}{4}\int x^3 dx \\ &= \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C, \end{aligned}$$

where C is a constant.

$\square$

Case 3 : Both the u and dv columns are periodic (ever repeating), that is, the function u belongs to a class of either trigonometric or exponential functions. Four procedures are:

1. When two row computations are finished, one stops. Here the entry in the u column should match one's original choice of u up to a multiple term.
2. The user extracts the integral equation from the table.
3. One uses basic algebra to solve for the unknown integral; one simply moves it to the LHS and divides out the multiple term at the front of the integral.
4. If it is an indefinite integral, please add the constant C.

The integral is of the form:

$$\int \underbrace{(\text{sine/cosine})}_{u} \cdot \underbrace{\underbrace{(\text{exponential})}_{v'} \, dx}_{dv}$$

or

$$\int \underbrace{(\text{sine/cosine})}_{u} \cdot \underbrace{\underbrace{(\text{sine/cosine})}_{v'} \, dx}_{dv}.$$

Example 20 Evaluate

$$\int x^2 \sin x dx.$$

Solution. Let $u_0(x) = x^2$ and $v^0(x) = \sin x$. Using the tabular method, we have

$$\begin{array}{ccc} e^x & \longrightarrow & \sin x \\ & \searrow^{+} & \\ e^x & \xrightarrow{-} & -\cos x \\ & \searrow^{-} & \\ e^x & \xrightarrow{+} & -\sin x \end{array}$$

Figure 24.5: Example 20.

the results depicted in Figure 24.5, where the third order derivative of $u_0(x)$ is found and $v^0(x)$ is integrated three times. We observe that $\int e^x \sin x dx$ appears on the bottom. Hence, we have

$$\begin{aligned} &\int x^2 e^{-\frac{x}{2}} dx \\ &= e^x \cdot (-\cos x) - e^x \cdot (-\sin x) + \int e^x \cdot (-\sin x) dx \\ &= -e^x \cos x + e^x \sin x - \int e^x \sin x dx. \end{aligned}$$

Therefore, we have

$$\int x^2 e^{-\frac{x}{2}} dx = \frac{1}{2} \left(e^x \sin x - e^x \cos x \right) + C,$$

where C is a constant.

$\square$