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香港中文大學
The Chinese University of Hong Kong

二〇一六至一七年度上學期科目考試
Course Examination 1st Term, 2016-17

科目編號及名稱 Course Code & Title : **MATH1510A/B/C/D/E/F/G/H/I Calculus for Engineers**

時間 Time allowed : 2 小時 hours 00 分鐘 minutes

學號 Student I.D. No : ________ 座號 Seat No. : ________

Please show the work with as much detail as possible for every step.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} \dfrac{2x}{1+x^2} & \text{if } x \geq 0, \\ x & \text{if } x < 0. \end{cases}$$

   (a) (3 points) Find $\lim_{x \to +\infty} f(x)$.

   (b) (4 points) Show that $f(x)$ is continuous at $x = 0$.

   (c) (3 points) Is $f(x)$ differentiable at $x = 0$? Justify your answer.

2. (a) (3 points) Find $\dfrac{dy}{dx}$ if

$$y = \frac{1}{\left(x + \dfrac{1}{x}\right)^2};$$

   (b) (3 points) Find $\dfrac{dy}{dx}$ if

$$y = \sin\left(1 + \sqrt{\cos x}\right);$$

   (c) (3 points) Find $\dfrac{dy}{dx}$ if

$$y = x^{x+1}, \text{ for } x > 0;$$

   (d) (3 points) Find

$$\frac{d}{dx}\left\{\int_{x^2}^{x^4} t^3 \sin t \, dt\right\}.$$

3. Evaluate the following integrals:

   (a) (3 points) $\displaystyle\int \left(x^{5/2} + \frac{1}{x^{7/2}} + 3^x + \frac{1}{x} \right) dx$;

   (b) (3 points) $\displaystyle\int_{5/2}^{3} (2x-5)^{10}\, dx$;

   (c) (3 points) $\displaystyle\int \frac{\sin(\sqrt{x})}{\sqrt{x}}\, dx$;

   (d) (3 points) $\displaystyle\int_0^{\pi} \sin^2 x \cos^3 x\, dx$;

   (e) (3 points) $\displaystyle\frac{1}{\pi}\int_0^{\pi} x \sin x\, dx$

4. Solve the following problems separately.

   (a) Let
   $$f(x) = x^3 - 3x^2 - 9x + 1.$$

      i. (6 points) Find all critical points of f. Then, find the interval(s) on which the f is increasing, and those on which f is decreasing.

      ii. (3 points) Determine whether each critical point is a local minimum or maximum (or neither).

   (b) (6 points) Let
   $$g(x) = (x-3)e^x + e^2.$$
   Find the critical point(s) of g and apply the Second Derivative Test (or show that it fails).

5. Solve the following problems separately.

   (a) (4 points) Find the area of the region in xy-plane bounded by the graphs of functions:
   $$\begin{cases} f(x) = x^2 - 3, \\ g(x) = x - 1. \end{cases}$$

   (b) Let $\mathcal{R}$ be the region in xy-plane bounded by the curves $y = e^x$, $x = 0$, $y = 4$.
   Express the volumes of the following solids as integrals (**You do not need to evaluate the integrals**):

      i. (2 points) The solid obtained by revolving $\mathcal{R}$ about x-axis.

      ii. (2 points) The solid obtained by revolving $\mathcal{R}$ about y-axis.

6. Solve the following problems separately. Justify your answers.

(a) Given that

$$u(x,y) = \ln\left(x^3 + y^3 - x^2y - xy^2\right).$$

i. (4 points) Find $u_x = \frac{\partial u}{\partial x}$ and $u_y = \frac{\partial u}{\partial y}$.

ii. (1 point) Show that

$$u_x + u_y = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \frac{2}{x+y}.$$

iii. (1 point) Find a constant A such that

$$u_{xx} + 2u_{xy} + u_{yy} = \frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = -\frac{A}{(x+y)^2}.$$

(b) Let

$$\phi = f(r,s,t),$$

and

$$r = 2x - 3y + \alpha, \quad s = 3y - 4z + \beta, \quad t = 4z - 2x + \gamma,$$

where f, r, s, t are assumed differentiable, x, y, z are independent variables of real numbers and α, β, γ are constants.
(4 points) Show that

$$6\phi_x + 4\phi_y + 3\phi_z = 0.$$

(c) (4 points) Compute

$$\int_0^1 \int_0^1 2x^3 e^{x^2 y}\, dy dx.$$

(d) (4 points) Let

$$w(x,y) = \frac{1}{y}.$$

Compute the double integral of $w(x,y)$ over the domain

$$\mathcal{D} = \left\{ (x,y) \middle|\ 1 \le x \le 4 \quad \text{and} \quad x \le y \le x^2 \right\}.$$

7. Solve the following problems separately.

(a) i. (2 points) Given that

$$\frac{x+1}{x^2-5x+6}=\frac{A}{x-2}+\frac{B}{x-3}$$

for some constants A, B. Find A, B.

ii. Given that

$$\frac{1}{1+x}=1-x+x^2-x^3+\cdots, \quad -1<x<1$$

and

$$\frac{1}{1-x}=1+x+x^2+x^3+\cdots, \quad -1<x<1.$$

Consider the following function

$$f(x)=\frac{x+1}{x^2-5x+6}.$$

A. (2 points) Find the Taylor series of $f(x)$ at $x=0$.

B. (2 points) Find the radius of convergence of $f(x)$.

(b) Consider the following integral:

$$\int_1^2 \frac{\sin^2 x}{x^2}\,dx.$$

i. (2 points) Find the Taylor series of

$$f(x)=\frac{1-\cos 2x}{2x^2}$$

at $x=0$.

ii. (2 points) Use the first 3 terms of the above Taylor series to approximate the integral

$$\int_1^2 \frac{\sin^2 x}{x^2}\,dx.$$

(c) (2 points) Find the first 3 terms in the Taylor series for $\sin(\sin x)$ at $x=0$. Compute

$$\lim_{x\to 0^+}\left(\frac{x-\sin(\sin x)}{x^3}\right).$$

8. Solve the following problems separately.

(a) (2 points) Evaluate the following limit:

$$\lim_{x\to 0^+}\left(\frac{e^{ax}-e^{-ax}}{\ln(1+bx)}\right)$$

in terms of a and b, where $a \neq 0$ and $b \neq 0$. Show each step of your work.

(b) Given that

$$x = a\left(\cos t + \frac{1}{2}\ln\left[\tan^2\left(\frac{t}{2}\right)\right]\right) \quad \text{and} \quad y = a\sin t$$

where a is a positive constant.

i. (1 point) Show that $\dfrac{dy}{dx} = \tan t$.

ii. (1 point) Use (i), find $\left.\dfrac{d^2y}{dx^2}\right|_{t=\pi/4}$ in terms of a.

(c) (2 points) Assume $2m-1>0$, $n>0$ and $a>0$. Verify Rolle's theorem for the following function $f(x)$ on the indicated interval:

$$f(x) = x^{2m-1}(a-x)^{2n}$$

in $(0,a)$. Your solutions should be in terms of m, n and a.

(d) (2 points) Show that, for any $x>0$,

$$1+x<e^x<1+xe^x.$$

(e) (2 points) Evaluate the following integral:

$$\int_{-\pi}^{\pi}\sin^{2015}x\cos^{2016}x\,dx.$$