Lecture Note 11

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MATH1020
General Mathematics

The Dot Product

The definition for a product of two vectors is somewhat awkward. However, such a product has meaning in many geometric and physical applications.

Definition 1 If $\mathbf{v} = a_1\mathbf{i} + b_1\mathbf{j}$ and $\mathbf{v} = a_2\mathbf{i} + b_2\mathbf{j}$ are two vectors, the **dot product** $\mathbf{v} \cdot \mathbf{w}$ is defined as

$$\mathbf{v} \cdot \mathbf{w} = a_1 a_2 + b_1 b_2. \tag{1}$$

Example 1 Finding Dot Product

If $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j}$ and $\mathbf{w} = 5\mathbf{i} + 3\mathbf{j}$, find:

(a) $\mathbf{v} \cdot \mathbf{w}$

(b) $\mathbf{w} \cdot \mathbf{v}$

(c) $\mathbf{v} \cdot \mathbf{v}$

(d) $\mathbf{w} \cdot \mathbf{w}$

(e) $||\mathbf{v}||$

(f) $||\mathbf{w}||$

Solution:

(a) $\mathbf{v} \cdot \mathbf{w} = 2(5) + 3(-3) = 1$

(b) $\mathbf{w} \cdot \mathbf{v} = 5(2) + 3(-3) = 1$

(c) $\mathbf{v} \cdot \mathbf{v} = 2(2) + (-3)(-3) = 13$

(d) $\mathbf{w} \cdot \mathbf{w} = 5(5) + 3(3) = 34$

(e) $||\mathbf{v}|| = \sqrt{2^2 + (-3)^2} = \sqrt{13}$

(f) $||\mathbf{w}|| = \sqrt{5^2 + 3^2} = \sqrt{34}$

Since the dot product $\mathbf{v} \cdot \mathbf{w}$ of two vectors $\mathbf{v}$ and $\mathbf{w}$ is a real number (scalar), we sometimes refer to it as the **scalar product.**

Theorem 1 Properties of the Dot Product

If $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ are vectors, then

Commutative Property

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}. \tag{2}$$

Distributive Property

$$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}. \tag{3}$$

$$\mathbf{v} \cdot \mathbf{v} = ||\mathbf{v}||^2. \tag{4}$$

$$\mathbf{0} \cdot \mathbf{v} = 0. \tag{5}$$

Proof:

To prove property (2), we let $\mathbf{u} = a_1\mathbf{i} + b_1\mathbf{j}$ and $\mathbf{v} = a_2\mathbf{i} + b_2\mathbf{j}$. Then

$$\mathbf{u} \cdot \mathbf{v} = (a_1\mathbf{i} + b_1\mathbf{j}) \cdot (a_2\mathbf{i} + b_2\mathbf{j}) = a_2a_1 + b_2b_1 = \mathbf{v} \cdot \mathbf{w}.$$

To prove property (4), we let $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$. Then

$$\mathbf{v} \cdot \mathbf{v} = a^2 + b^2 = ||\mathbf{v}||^2.$$

Example 2 Let $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$. Then

$$\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = 1$$

and

$$\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{i} = 0.$$

Example 3 Let $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$ and $\mathbf{k} = \langle 0, 0, 1 \rangle$. Then

$$\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1,$$

$$\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{i} = 0,$$

$$\mathbf{i} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0,$$

and

$$\mathbf{k} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = 0.$$

Find the Angle between Two Vectors

One use of the dot product is to calculate the angle between two vectors:

- Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors with the same initial point A.
- Then the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} - \mathbf{v}$ form a triangle.
- The angle θ at vertex A of the triangle is the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$. See Figure 1.

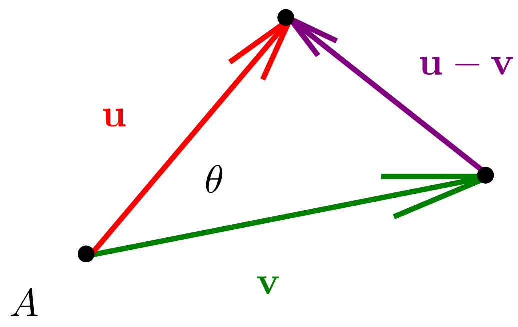


Figure 1:

We wish to find a formula for calculating the angle θ.

The sides of the triangle have lengths $||\mathbf{u}||$, $||\mathbf{v}||$, and $||\mathbf{u}-\mathbf{v}||$, and θ is the included angle between the sides of the length $||\mathbf{u}||$ and $||\mathbf{v}||$.

The law of Cosines can be used to find the cosine of the included angle.

$$||\mathbf{u}-\mathbf{v}||^2 = ||\mathbf{u}|| + ||\mathbf{v}|| - 2||\mathbf{u}||||\mathbf{v}||\cos\theta.$$

Now we use property (4) to rewrite this equation in terms of dot products.

$$(\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) = \mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} - 2||\mathbf{u}||||\mathbf{v}||\cos\theta. \tag{6}$$

Then we apply the distributive property (3) twice on the left side of (6) to obtain to

$$\left\{\begin{array}{rcll} (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) & = & \mathbf{u}\cdot(\mathbf{u}-\mathbf{v}) - \mathbf{v}\cdot(\mathbf{u}-\mathbf{v}) & \\ & = & \mathbf{u}\cdot\mathbf{u} - \mathbf{u}\cdot\mathbf{v} - \mathbf{v}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} & \\ & = & \mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} - 2\mathbf{u}\cdot\mathbf{v} & \text{Property (2).} \end{array}\right. \tag{7}$$

Combining Equations (6) and (7), we have

$$\mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} - 2\mathbf{u}\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} - 2||\mathbf{u}||||\mathbf{v}||\cos\theta$$

or

$$\cancel{\mathbf{u}\cdot\mathbf{u}}+\cancel{\mathbf{v}\cdot\mathbf{v}}-2\mathbf{u}\cdot\mathbf{v}=\cancel{\mathbf{u}\cdot\mathbf{u}}+\cancel{\mathbf{v}\cdot\mathbf{v}}-2||\mathbf{u}||||\mathbf{v}||\cos\theta$$

or

$$\mathbf{u}\cdot\mathbf{v}=||\mathbf{u}||||\mathbf{v}||\cos\theta.$$

Theorem 2 If $\mathbf{u}$ and $\mathbf{v}$ are two nonzero vectors, the angle $\theta, 0 \leq \theta \leq \pi$, between $\mathbf{u}$ and $\mathbf{v}$ is determined by the formula

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}||||\mathbf{v}||}. \tag{8}$$

Example 4 Finding the Angle θ between Two Vectors

Find the angle between $\mathbf{u} = 4\mathbf{i} - 5\mathbf{j}$ and $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j}$

Solution:

We compute the quantities $\mathbf{u} \cdot \mathbf{v}$, $||\mathbf{u}||$, and $||\mathbf{v}||$:

$$\begin{aligned}
\mathbf{u} \cdot \mathbf{v} &= (4\mathbf{i} - 5\mathbf{j}) \cdot (2\mathbf{i} + 5\mathbf{j}) \\
&= 4(2)\mathbf{i} \cdot \mathbf{i} + 4(5)\mathbf{i} \cdot \mathbf{j} + (-5)(2)\mathbf{i} \cdot \mathbf{j} + (-3)(5)\mathbf{j} \cdot \mathbf{j} \\
&= 4(2) + (-3)(5) = -7; \\
||\mathbf{u}|| &= \sqrt{4^2 + (-3)^2} = 5; \\
||\mathbf{v}|| &= \sqrt{2^2 + 5^2} = \sqrt{29}.
\end{aligned}$$

By Formula (8), if θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}|| \cdot ||\mathbf{v}||} = \frac{-7}{5\sqrt{29}} \approx -0.26.$$

We find that $\theta \approx 105^\circ$. See Figure 2.

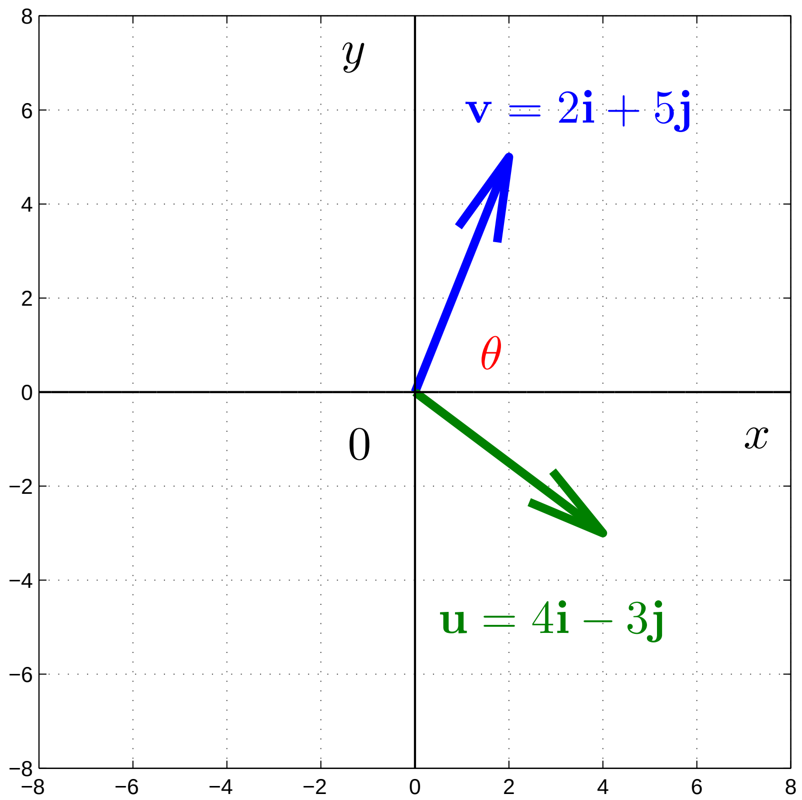

8
6
4
2
0
−2
−4
−6
−8
−8
−6
−4
−2
0
2
4
6
8
y
$\mathbf{v} = 2\mathbf{i} + 5\mathbf{j}$
θ
0
x
$\mathbf{u} = 4\mathbf{i} - 3\mathbf{j}$

Figure 2:

Determine Whether Two Vectors Are Parallel

Two vectors $\mathbf{v}$ and $\mathbf{w}$ are said to be **parallel** if there is a nonzero scalar α so that $\mathbf{v} = \alpha\mathbf{w}$. In this case, the angle θ between $\mathbf{v}$ and $\mathbf{w}$ is 0 or π.

Example 5 Determining Whether Two Vectors Are Parallel

The vectors $\mathbf{v} = 3\mathbf{i} - \mathbf{j}$ and $\mathbf{w} = 6\mathbf{i} - 2\mathbf{j}$ are parallel, since $\mathbf{v} = \frac{1}{2}\mathbf{w}$. Furthermore, since

$$\cos\theta = \frac{\mathbf{v} \cdot \mathbf{w}}{||\mathbf{v}||||\mathbf{w}||} = \frac{18+2}{\sqrt{10}\sqrt{40}} = \frac{20}{\sqrt{400}} = 1$$

the angle θ between $\mathbf{v}$ and $\mathbf{w}$ is 0. See Figure 3.

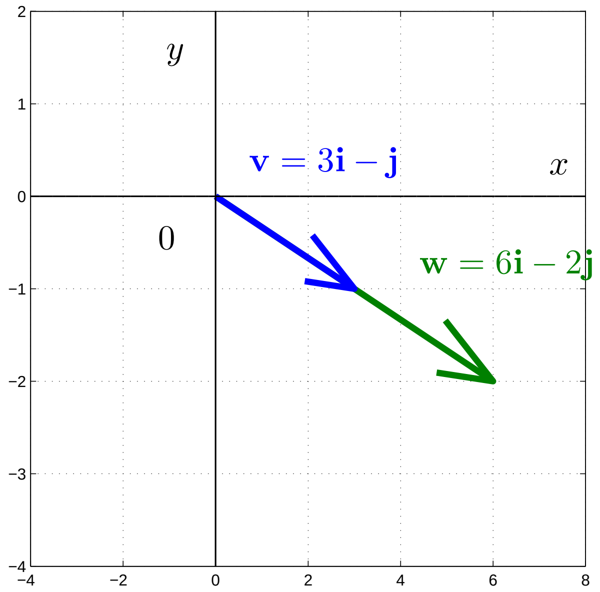


Figure 3:

Determining Whether Two Vectors Are Orthogonal

If the angle θ between two nonzero vectors $\mathbf{v}$ and $\mathbf{w}$ is $\dfrac{\pi}{2}$, the vectors $\mathbf{v}$ and $\mathbf{w}$ are called **orthogonal**. See Figure 4.

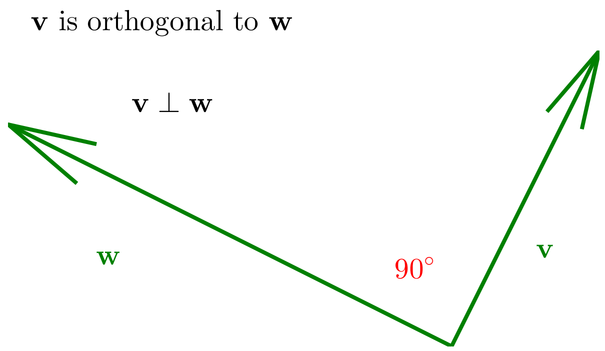


Figure 4:

Since $\cos \dfrac{\pi}{2} = 0$, it follows formula (8) that if $\mathbf{v}$ and $\mathbf{w}$ are orthogonal then $\mathbf{v} \cdot \mathbf{w} = 0$.

On the other hand, if $\mathbf{v} \cdot \mathbf{w} = 0.$, then either $\mathbf{v} = 0$ or $\mathbf{w} = 0$ or $\cos\theta = 0$.

In the latter case, $\theta = \dfrac{\pi}{2}$, and $\mathbf{v}$ and $\mathbf{w}$ are orthogonal.

If $\mathbf{v}$ or $\mathbf{w}$ is the zero vector, then, since the zero vector has no specific direction, we adopt the convention that the zero vector is orthogonal to every vector.

Remark 1 If $\mathbf{u}$ and $\mathbf{v}$ are two nonzero vectors, then Theorem 2 implies that

1. $\mathbf{u} \cdot \mathbf{v} > 0$ if and only if $\cos\theta$ is acute,
2. $\mathbf{u} \cdot \mathbf{v} < 0$ if and only if $\cos\theta$ is obtuse, and
3. $\mathbf{u} \cdot \mathbf{v} = 0$ if and only if $\cos\theta = 0$.

But in the last case, the only number in $[0, 2\pi]$ for which $\cos\theta = 0$ is $\theta = \pi/2$. When $\theta = \pi/2$, we say that the vectors are orthogonal or perpendicular. Then we are led to the following result.

Theorem 3 Two vectors $\mathbf{v}$ and $\mathbf{w}$ are orthogonal if and only if

$$\mathbf{v} \cdot \mathbf{w} = 0.$$

Since $\mathbf{0} \cdot \mathbf{u} = 0$ for every vector $\mathbf{v}$, the zero vector is regarded to be orthogonal to every vector.

Note that orthogonal, perpendicular, and normal are all terms that mean "meet at a right angle." It is customary to refer to two vectors as being orthogonal, two lines as being perpendicular, and a line and a plane or a vector and a line as being normal.

Example 6 Determining Whether Two Vectors Are Orthogonal

The vectors $\mathbf{v} = 2\mathbf{i} - \mathbf{j}$ and $\mathbf{w} = 3\mathbf{i} + \mathbf{j}$ are orthogonal,, since

$$\mathbf{v} \cdot \mathbf{w} = 6 - 6 = 0.$$

See Figure 5.

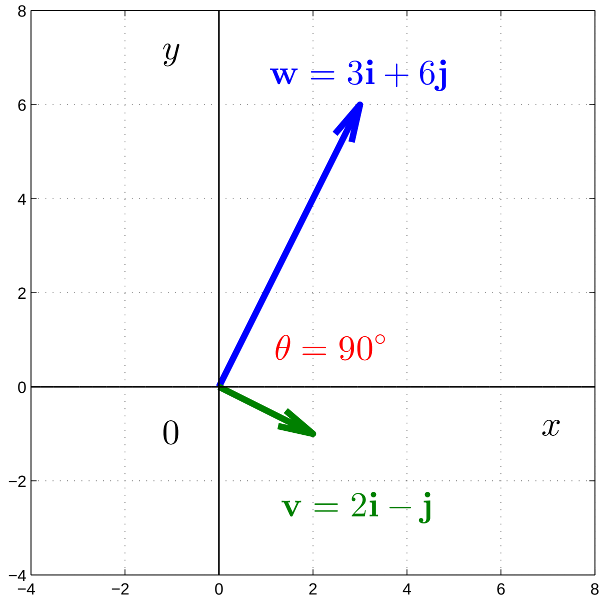


Figure 5:

Decompose a Vector into Two Orthogonal Vectors

Let us find "how much" of a vector that is applied in a given direction. See Figure 6.

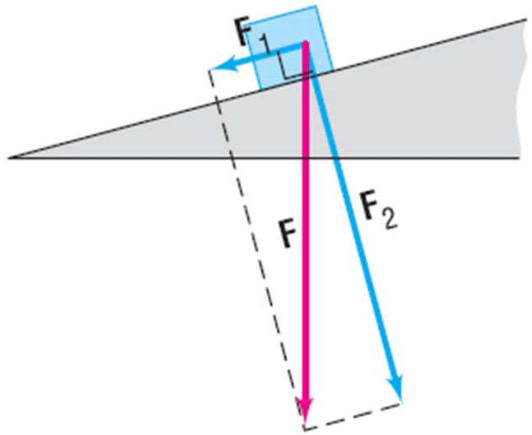


Figure 6:

The force **F** due to gravity is pulling straight down (toward the center of Earth) on the block.

To study the effect if gravity on the block, it is necessary to determine how much of **F** is actually pushing the block down the incline ($\mathbf{F}_1$) and how much is pressing the block against the incline ($\mathbf{F}_2$), at a right angle to the incline.

Knowing the **decomposition** of **F** often will allow us to determine when friction (the force holding the block in place on the incline) is overcome and the block will slide down the incline.

Suppose that $\mathbf{v}$ and $\mathbf{w}$ are two nonzero vectors with the same initial point P. We seek to decompose $\mathbf{v}$ into two vectors: $\mathbf{v}_1$, which is parallel to $\mathbf{w}$ and $\mathbf{v}_2$, which is orthogonal to $\mathbf{w}$.

See Figure 7.

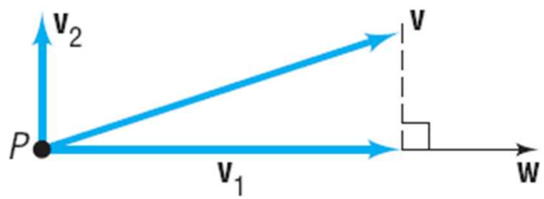


Figure 7:

See Figure 8.

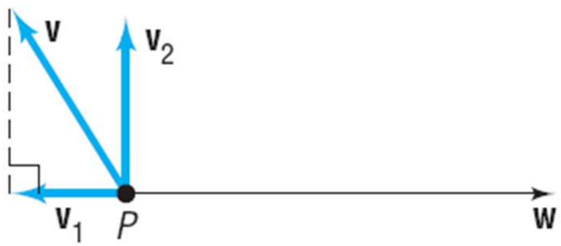


Figure 8:

The $\mathbf{v}_1$ is called the **vector projection of $\mathbf{v}$ onto $\mathbf{w}$.**

- The vector $\mathbf{v}_1$ is obtained as follows:
- From the terminal point of $\mathbf{v}$, drop a perpendicular to the line containing $\mathbf{w}$. The vector $\mathbf{v}_1$ is the vector from P to the foot of this perpendicular.
- The vector $\mathbf{v}_2$ is given by $\mathbf{v}_2 = \mathbf{v} - \mathbf{v}_1$.
- Note that $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$, the vector $\mathbf{v}_1$ is parallel to $\mathbf{w}$, and the vector $\mathbf{v}_2$ is orthogonal to $\mathbf{w}$.
- This is the decomposition of $\mathbf{v}$ and that we wanted.

Now we seek a formula for $\mathbf{v}_1$ that is based on a knowledge of the vectors $\mathbf{v}$ and $\mathbf{w}$. Since $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$, we have

$$\mathbf{v} \cdot \mathbf{w} = (\mathbf{v}_1 + \mathbf{v}_2) \cdot \mathbf{w} = \mathbf{v}_1 \cdot \mathbf{w} + \mathbf{v}_2 \cdot \mathbf{w}. \tag{9}$$

Since $\mathbf{v}_2$ is orthogonal to $\mathbf{w}$, i.e., $\mathbf{v}_2 \perp \mathbf{w}$, we have $\mathbf{v}_2 \cdot \mathbf{w} = 0$. Since $\mathbf{v}_1$ is parallel to $\mathbf{w}$, i.e., $\mathbf{v}_1 \parallel \mathbf{w}$, we have $\mathbf{v}_1 = \alpha\mathbf{w}$ for some scalar α. Equation (9) can be written as

$$\begin{aligned}
\mathbf{v} \cdot \mathbf{w} &= \mathbf{v}_1 \cdot \mathbf{w} + \overset{= 0}{\cancel{\mathbf{v}_2 \cdot \mathbf{w}}} \\
\mathbf{v} \cdot \mathbf{w} &= \alpha\mathbf{w} \cdot \mathbf{w} \qquad\qquad (\mathbf{v}_1 = \alpha\mathbf{w};\ \mathbf{v}_2 \cdot \mathbf{w} = 0) \\
&= \alpha||\mathbf{w}||^2 \\
\alpha &= \frac{\mathbf{v} \cdot \mathbf{w}}{||\mathbf{w}||^2}.
\end{aligned}$$

Then

$$\mathbf{v}_1 = \alpha\mathbf{w} = \frac{\mathbf{v} \cdot \mathbf{w}}{||\mathbf{w}||^2}\mathbf{w}.$$

Theorem 4 If $\mathbf{v}$ and $\mathbf{w}$ are two nonzero vectors, the vector projection of $\mathbf{v}$ onto $\mathbf{w}$ is

$$\mathbf{v}_1 = \frac{\mathbf{v} \cdot \mathbf{w}}{||\mathbf{w}||^2}\mathbf{w}. \tag{10}$$

The decomposition of $\mathbf{v}$ into $\mathbf{v}_1$ and $\mathbf{v}_2$ where $\mathbf{v}_1$ is parallel to $\mathbf{w}$, i.e., $\mathbf{v}_1 \parallel \mathbf{w}$ and $\mathbf{v}_2$ is orthogonal to $\mathbf{w}$, i.e., $\mathbf{v}_2 \perp \mathbf{w}$, is

$$\begin{cases} \mathbf{v}_1 = \dfrac{\mathbf{v} \cdot \mathbf{w}}{||\mathbf{w}||^2}\mathbf{w} \\ \mathbf{v}_2 = \mathbf{v} - \mathbf{v}_1. \end{cases} \tag{11}$$

Example 7 Decomposing a Vector into Two Orthogonal Vector

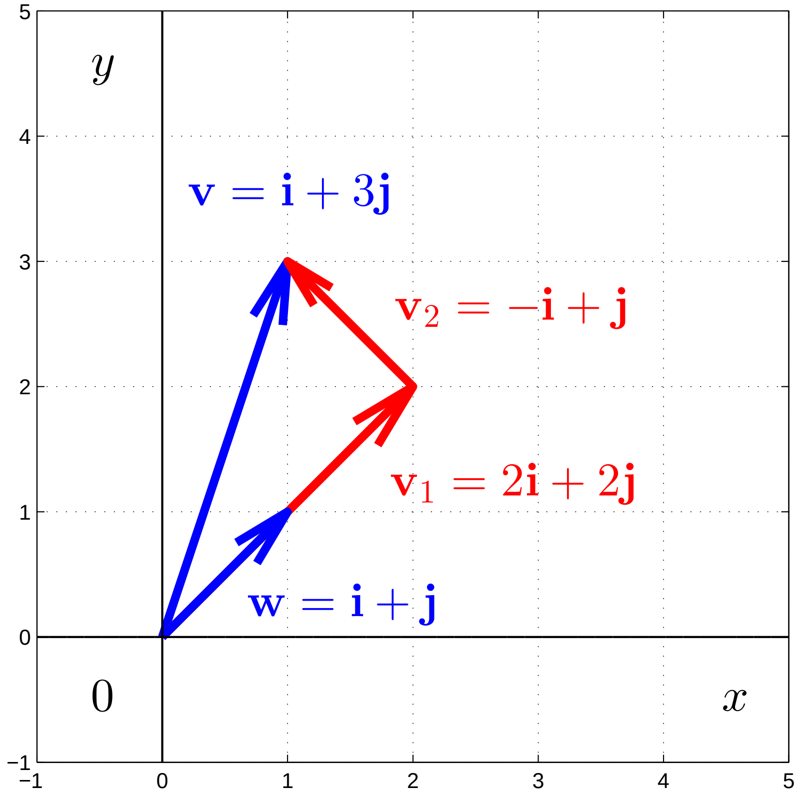
Find the vector projection of $\mathbf{v} = \mathbf{i} + 3\mathbf{j}$ onto $\mathbf{w} = \mathbf{i} + 3\mathbf{j}$. Decompose $\mathbf{v}$ into two vectors, $\mathbf{v}_1$ and $\mathbf{v}_2$, where $\mathbf{v}_1$ is parallel to $\mathbf{w}$ and $\mathbf{v}_2$ is orthogonal to $\mathbf{w}$.

Solution:

We use formulas (10) and (11).

$$\begin{aligned}\mathbf{v}_1 &= \frac{\mathbf{v}\cdot\mathbf{w}}{||\mathbf{w}||^2}\mathbf{w} = \frac{1+3}{\sqrt{2}^2}\mathbf{w} = 2\mathbf{w} = 2(\mathbf{i}+\mathbf{j});\\ \mathbf{v}_2 &= \mathbf{v} - \mathbf{v}_1 = \mathbf{i} + 3\mathbf{j} - 2(\mathbf{i}+\mathbf{j}) = -\mathbf{i}+\mathbf{j}.\end{aligned}$$

See Figure 9.



y
$\mathbf{v} = \mathbf{i} + 3\mathbf{j}$
$\mathbf{v}_2 = -\mathbf{i} + \mathbf{j}$
$\mathbf{v}_1 = 2\mathbf{i} + 2\mathbf{j}$
$\mathbf{w} = \mathbf{i} + \mathbf{j}$
0
x

Figure 9: