Calculus for Engineers

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Contents

Inverse Trigonometric Functions and Their Derivatives

12.1 Introduction

In this chapter, we will study the so-called inverse trigonometric functions. Before introducing the concept of the inverse of a function, we first discuss the concept of a one-to-one function[1]. Functions that always give different outputs for different inputs are called one-to-one. Since each output of a one-to-one function comes from just one input, any one-to-one function can be reversed to turn the outputs back into the inputs from which they came. Thus, a function has an inverse if and only if it is one-to-one[2]. The function defined by reversing a one-to-one function f (which means that each ordered pair (a, b) belonging to f, is replaced by a corresponding ordered pair (b, a) in the new function) is called the inverse of f and denoted by f^{-1}.

As we shall see below, the six basic trigonometric functions, that is, $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, and $\mathrm{cosec} x$ of the real variable x, are defined and studied in this chapter. Since all these functions are periodic (and hence not one-to-one), none of them has an inverse. We can however, *restrict the domains of these functions* in a way to allow for an inverse.

12.2 Restricting the domain of a function

This following example shows us that by restricting the domain of a function appropriately, it is possible for a given formula (expression) to define a one-to-one function. This fact is mainly employed when we consider inverse trigonometric functions.

Example 1 Consider the odd function $y = x^3$. It gives different output(s) for different input(s), as shown in Figure 12.1. Hence, it is a one-to-one function.

On the other hand, the even function $y = x^2$ can give the same outputs for different inputs, as shown in Figure 12.2. For example, putting -1, 1, $-\sqrt{3}$, $\sqrt{3}$, -4 and 4, we

[1] A function $y = f(x)$ is called a one-to-one function if for each y from the range of f there exists exactly one x in the domain of f which is related to y.

[2] Both the statements are identical:

- If a function has an inverse, then it is one-to-one.
- If a function is one-to-one, then it has an inverse.

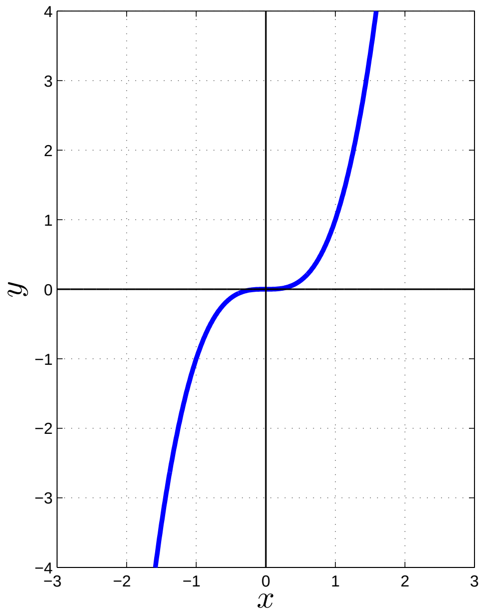


Figure 12.1: $y = x^3$.

have $y = (-1)^2 = 1^2 = 1$, $y = (-\sqrt{3})^2 = (\sqrt{3})^2 = 3$, $y = (-4)^2 = 4^2 = 16$, respectively and etc. Hence this function is *not* one-to-one because two values of x are related to one value of y (it seems to be two-to-one instead one-to-one).

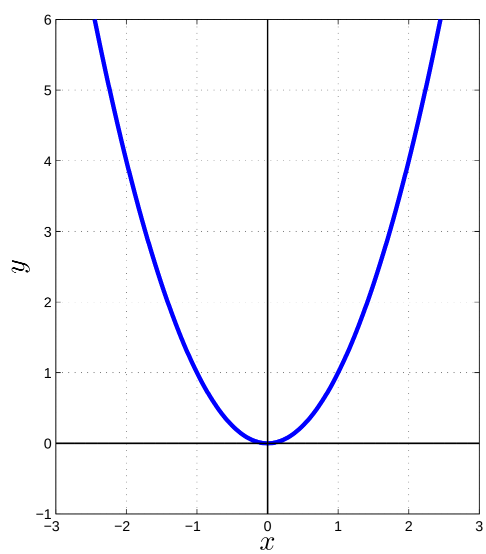


Figure 12.2: $y = x^2$.

However, if the domain of this function to *non-negative numbers* is considered, then the same expression (with restricted domain), that is, $y = x^2,\ x \geq 0$, defines a one-to-one function, as shown in Figure 12.3.

Example 2 Consider the graph of $y = f(x) = \sqrt{x}$, where $x \leq 0$, as shown in Figure 12.4.

The function $y = \sqrt{x}$ is defined for all $x \geq 0$ and its range is $y \geq 0$. For each input x_0, the function f gives a single output $y = \sqrt{x_0}$ (the green line). Since every non-negative

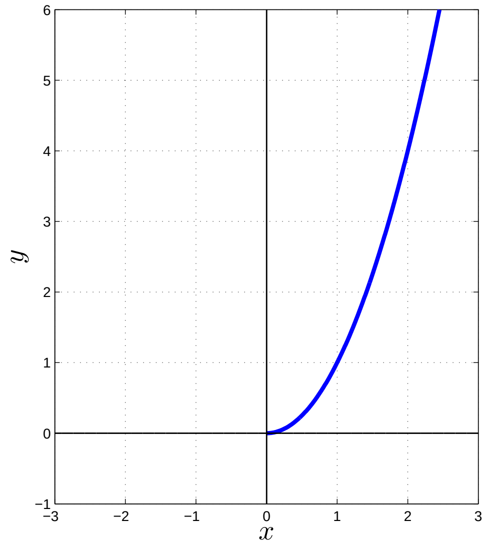


Figure 12.3: $y = x^2, \; x \geq 0$.

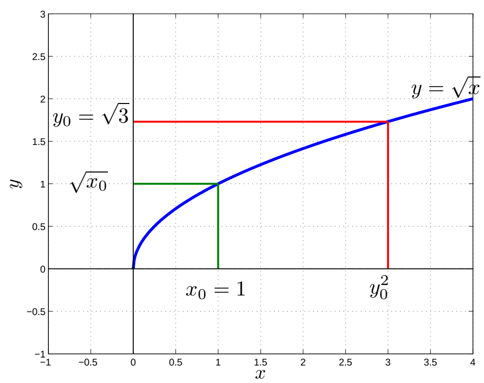


Figure 12.4: $y = f(x) = \sqrt{x}$, where $x \leq 0$.

y is the image of just one x under this function, we can reverse the construction. That is, we can start with $y \geq 0$ and then go over to the curve and down to $x = y^2$, on the x-axis (This is indicated by the red line starting from y_0 (on y-axis) and reaching (on to the x-axis) the point $x = y_0^2$.)

This construction in reverse defines the function $g(y) = y^2$, the inverse of $f(x) = \sqrt{x}$. Thus, the inverse of $y = f(x) = \sqrt{x}$ is given by $x = g(y) = y^2$ (or $x = f^{-1}(y) = y^2$).

Note 1 Each pair of inverse functions (here, f and g) behave opposite to each other in the sense that one function undoes (i.e., reverses) what the other does. The algebraic description of what we see in Figure 12.4 is

$$\left.\begin{array}{l} g(f(x)) = (\sqrt{x})^2 = x \\ f(g(y)) = (\sqrt{y})^2 = y \end{array}\right\}$$

Observe that, in the above equations f is the inverse of g. It must be noted that an inverse function associates the same pair of elements, as in the original function, but with the object and the image interchanged. In the inverse notation,

$$g = f^{-1}.$$

Note 2 Not every function has an inverse, as in the case of $y = x^2,\ x \in \mathbb{R}$.

Whenever a function

$$y = f(x) \tag{12.1}$$

has an inverse, we can write it as

$$x = f^{-1}(y) \tag{12.2}$$

provided (12.1) can be solved for x uniquely.

Both the functions at (12.1) and (12.2), if they are defined, *describe one and the same curve in the xy-plane.*

12.2.1 Two mutually inverse functions

The independent variable for the function f is x, while for the function f^{-1} the independent variable is y. If we wish to denote the argument in formula[3] (12.2) by x in a single coordinate system, we get two different graphs which are symmetric about the line $y = x$. They represent two mutually inverse functions.

The graphs of the *two mutually inverse functions* are given in Figure 12.5. The graph of a function and its inverse are (reflected) symmetric with respect to the line $y = x$ (gold).

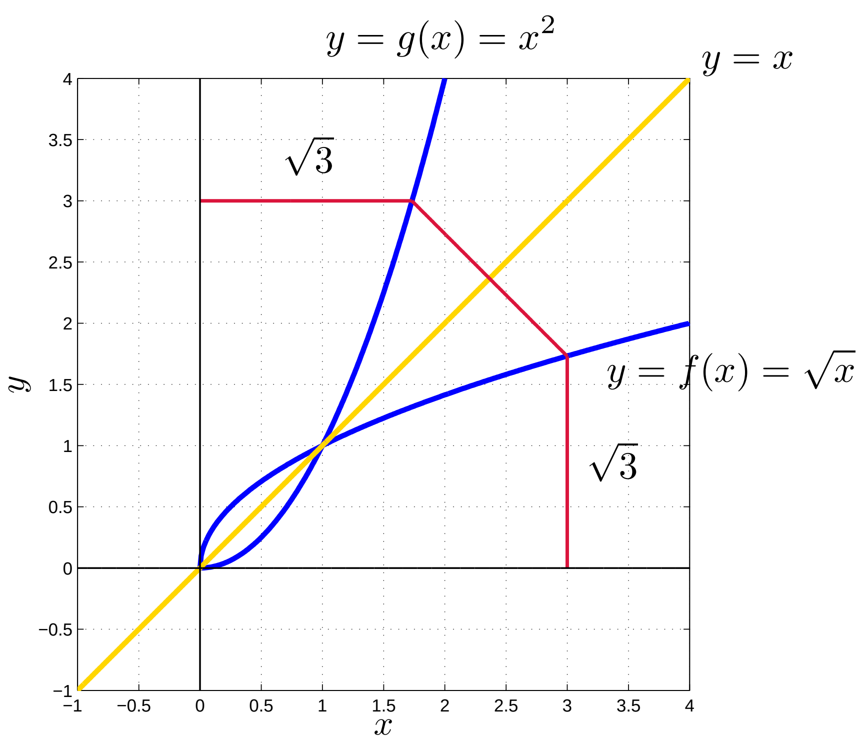


Figure 12.5: Two mutually inverse functions.

[3]That is, if we wish to write $x = f^{-1}(y)$ in the form $y = f^{-1}(x)$.

12.2.2 Distinguishing geometrical properties of one-to-one functions

We know that a *vertical line* can intersect the graph of a function *at one point* only. For a one-to-one function, it is also true that a horizontal line can intersect a graph at most one point. This is the situation for the one-to-one function defined by $y = x^3$ whose graph appears in Figure 12.6. On the other hand, observe in Figure 12.7 that for the function defined by $y = x^2$, which is *not one-to-one*, any horizontal line above the x-axis intersects the graph at two points. We have, therefore, the following *geometric test for determining if a function is one-to-one.*

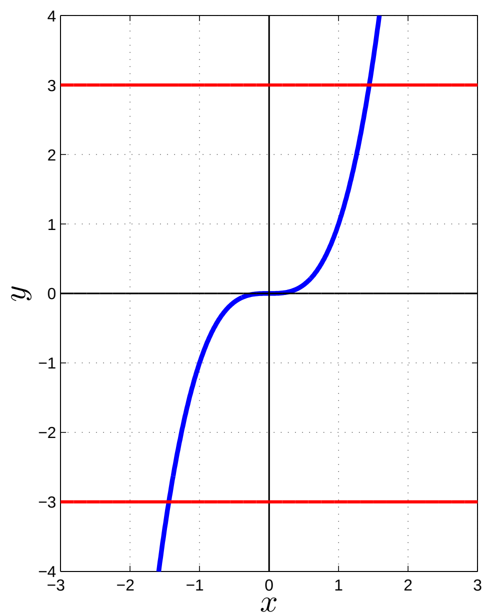


Figure 12.6: $y = x^3$.

12.2.3 Horizontal-line test

Definition 1 A function is one-to-one if and only if every horizontal line intersects the graph of a function at most one point.

Note 3 We use the terminology "*inverse functions*" only when referring to a function and its inverse.

Note 4 The criterion that a function be one-to-one, in order to have an inverse may be very hard to apply in a given situation, since it demands that we have complete knowledge of the graph. A *more practical criterion is that a function be strictly monotonic* (i.e., either strictly increasing or strictly decreasing). This is a practical result, *because we have an easy way of deciding if a function f is strictly monotonic.* We simply examine the sign of $f'(x)$. If $f'(x) > 0$ the function f is strictly increasing on its domain but if $f'(x) < 0$, f is strictly decreasing. These results are proved in Chapter 19a. Later on, in Chapter 20, it

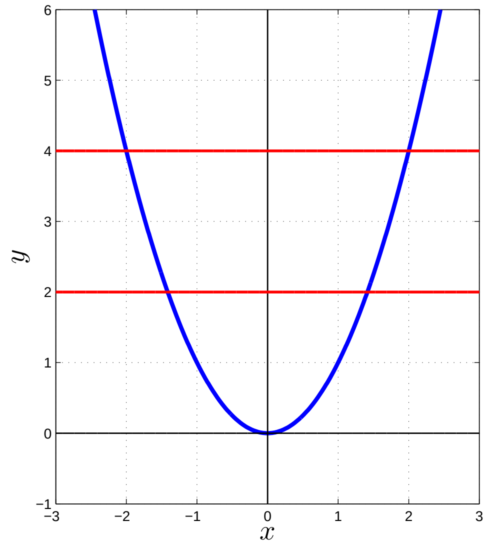


Figure 12.7: $y = x^2$.

is proved that a strictly monotonic function is one-to-one, showing that all such functions have inverses.

□

12.3 Trigonometric functions with restricted domains and their inverses

We start with the sine function, $y = \sin x$, whose graph appears in Figure 12.8.

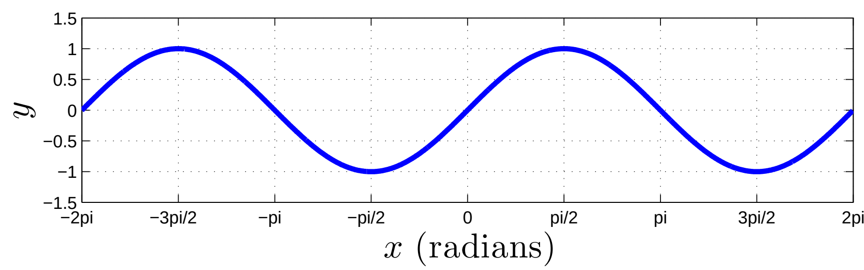


Figure 12.8: $y = \sin x$, where $x \in [-2\pi, 2\pi]$.

In Figure 12.9, we observe that the sine function is strictly increasing on the interval $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$.

Consequently, from the horizontal-line test (see Subsection 1), the function f_1, for which

$$f_1(x) = \sin x, \quad x \in \left[-\frac{1}{2}\pi, \frac{1}{2}\pi\right] \tag{12.3}$$

is one-to-one, and hence it does have an inverse in this interval. The graph of $f_1(x)$ is sketched in Figures 12.9 and 12.10. In Figure 12.10, its domain is $\left[-\frac{1}{2}\pi, \frac{1}{2}\pi\right]$ and its range is $[-1, 1]$. The inverse of this function is called *the inverse sine function.*

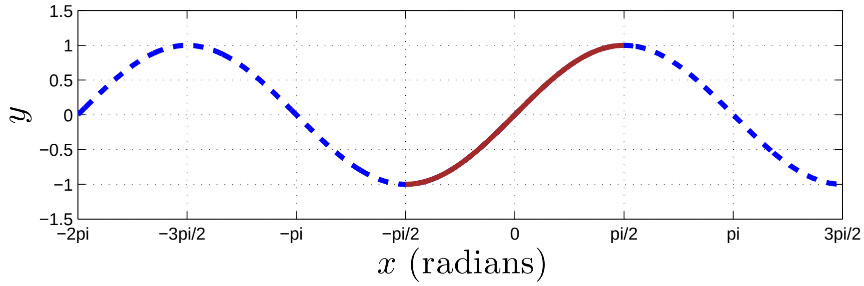


Figure 12.9: $f_1(x) = \sin x$ (brown), where $x \in [-\pi/2, \pi/2]$.

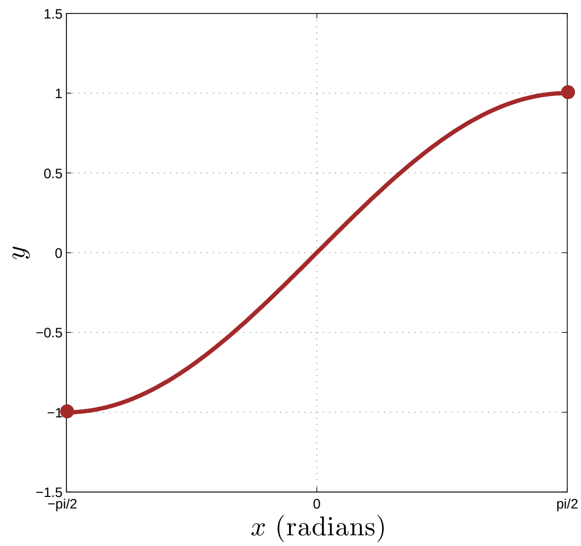


Figure 12.10: $y = f_1(x)$, where $x \in [-\pi/2, \pi/2]$ and $y \in [-1, 1]$.

12.3.1 Definition of the inverse sine function

Definition 2 The inverse sine function, denoted by $\sin^{-1}$, is defined by

$$y = \sin^{-1} x, \text{ if and only if, } x = \sin y \text{ and } y \in \left[-\frac{1}{2}\pi, \frac{1}{2}\pi\right].$$

The *domain* of $\sin^{-1} x$ is the closed interval $[-1, 1]$ and the *range* is the closed interval $\left[-\frac{1}{2}\pi, \frac{1}{2}\pi\right]$.

Example 3 Let us illustrate some applications of Definition 2. Here are a list of examples:

- $\sin^{-1}(-1) = -\frac{1}{2}\pi$, because $\sin\left(-\frac{1}{2}\pi\right) = -1$.
- $\sin^{-1}(0) = 0$, because $\sin(0) = 0$.
- $\sin^{-1}(\frac{1}{2}) = \frac{1}{6}\pi$, because $\sin\left(\frac{1}{6}\pi\right) = \frac{1}{2}$.
- $\sin^{-1}(\frac{1}{\sqrt{2}}) = \frac{1}{4}\pi$, because $\sin\left(\frac{1}{4}\pi\right) = \frac{1}{\sqrt{2}}$.

- $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right) = -\frac{1}{4}\pi$, because $\sin\left(-\frac{1}{4}\pi\right) = -\frac{1}{\sqrt{2}}$.

- $\sin^{-1}(1) = \frac{1}{2}\pi$, because $\sin\left(\frac{1}{2}\pi\right) = 1$.

Remark 1 In (12.3), the domain of $f_1(x) = \sin x$ is restricted to the closed interval $\left[-\frac{1}{2}\pi, \frac{1}{2}\pi\right]$, so that the function is strictly monotonic and therefore has an inverse function. However, the sine function on a period of 2π is (strictly) increasing on the other intervals as well, for example, $[-\frac{5}{2}\pi, -\frac{3}{2}\pi]$ and $[\frac{3}{2}\pi, \frac{5}{2}\pi]$, as shown in Figure 12.11. Also, the function is strictly decreasing on certain closed intervals, in particular the intervals $[-\frac{3}{2}\pi, -\frac{1}{2}\pi]$ and $[\frac{1}{2}\pi, \frac{3}{2}\pi]$, as shown in Figure 12.12. Any one of these intervals could just as well be chosen for the domain of the function f_1 of (12.3). The choice of the interval $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$, however, is customary because it is the largest interval containing the number 0, on which the function is (strictly) monotonic. □

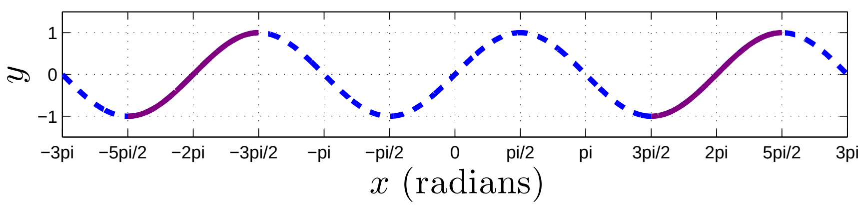


Figure 12.11: $y = f_1(x)$ (purple), where $x \in [-5\pi/2, -3\pi/2]$ or $x \in [3\pi/2, 5\pi/2]$.

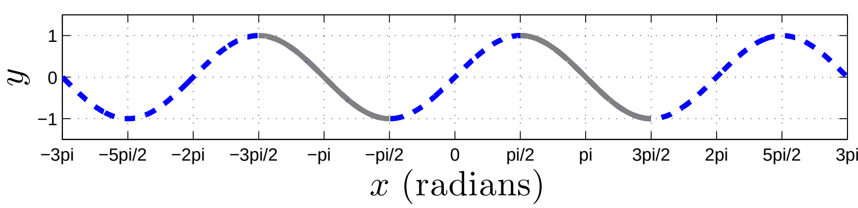


Figure 12.12: $y = f_1(x)$ (grey), where $x \in [-3\pi/2, -\pi/2]$ or $x \in [\pi/2, 3\pi/2]$.

Note 5 The use of the symbol "-1" *to represent the inverse sine function* makes it necessary *to denote the reciprocal of* $\sin x$ *by* $(\sin x)^{-1}$, to avoid confusion, i.e.,

$$\sin^{-1} x \neq (\sin x)^{-1}.$$

A similar convention is applied when using any negative exponent with a trigonometric function. For example, $1/(\tan x) = \dfrac{1}{\tan x} = (\tan x)^{-1}$, $1/(\cos^2 x) = \dfrac{1}{\cos^2 x} = (\cos x)^{-2}$, and so on. □

Note 6 The terminology *arcsine* is sometimes used *in place of inverse sine*, and the notation *arcsine* is then used *instead of* $\sin^{-1} x$. This notation probably stems from the fact that, *if* $t = \arcsin u$, *then* $\sin t = u$, *and* t *units is the length of the arc on the unit circle for which the sine is* u.

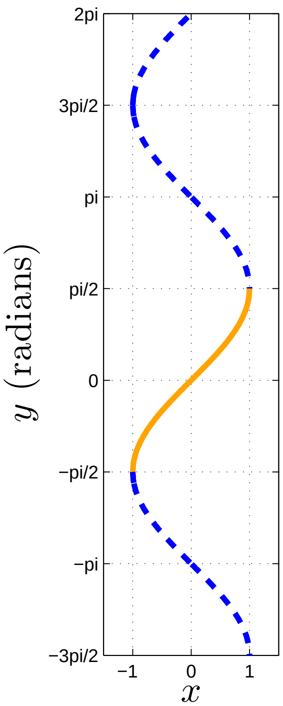


Figure 12.13: $y = \sin^{-1} x$ (orange), where $x \in [-1, 1]$.

In this note, we shall be using the symbol "-1" (rather than the word arc) and thus writing $\sin^{-1} x, \cos^{-1} x$, and so on (instead of $\arcsin x, \arccos x$, etc.). (This symbol is consistent with the general notation for inverse functions.)

□

We can sketch *the graph of the inverse sine function* by locating some points from the values of $\sin^{-1} x$ such as those given in Table 12.1. The graph appears in Figures 12.13 and 12.14.

x	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1
$\sin^{-1} x$	$-\frac{1}{2}\pi$	$-\frac{1}{3}\pi$	$-\frac{1}{6}\pi$	0	$\frac{1}{6}\pi$	$\frac{1}{3}\pi$	$\frac{1}{2}\pi$

Table 12.1:

From the definition of the inverse sine function (see Definition 2), we have

$$\sin(\sin^{-1} x) = x \quad \text{for } x \in [-1, 1]$$
$$\sin^{-1}(\sin y) = y \quad \text{for } y \in \left[-\frac{1}{2}\pi, \frac{1}{2}\pi\right]$$

Caution: Observe that $\sin(\sin^{-1} x) = x$ is *valid for all real values of* x. It must be noted that $\sin^{-1}(\sin y) \neq y$, if y is not in the interval $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$.

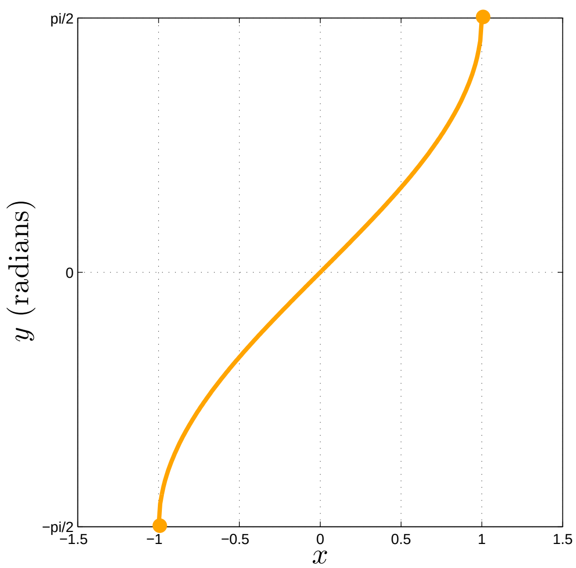


Figure 12.14: $y = \sin^{-1} x$, where $x \in [-1, 1]$ and $y \in [-\frac{1}{2}\pi, \frac{1}{2}\pi]$.

Example 4 Evaluate $\sin^{-1}(\sin \frac{5}{6}\pi)$.

Solution: First we use the fact that

$$\sin\left(\frac{5}{6}\pi\right) = \sin\left(\pi - \frac{1}{6}\pi\right) = \sin\left(\frac{1}{6}\pi\right) = \frac{1}{2}.$$

Applying $\sin^{-1}$ to both sides of the above expression gives

$$\sin^{-1}\left(\sin\frac{5}{6}\pi\right) = \sin^{-1}\left(\frac{1}{2}\right).$$

We know that $\sin\left(\frac{1}{6}\pi\right) = \frac{1}{2}$, so it follows that $\sin^{-1}\left(\frac{1}{2}\right) = \frac{1}{6}\pi$. Thus,

$$\sin^{-1}\left(\sin\frac{5}{6}\pi\right) = \frac{1}{6}\pi.$$

Note that, $\sin^{-1}\left(\sin\frac{1}{6}\pi\right) \neq \frac{5\pi}{6}$, since $\frac{5\pi}{6} \notin \left[-\frac{1}{2}\pi, \frac{1}{2}\pi\right]$.

Similarly,

$$\sin^{-1}\left(\sin\frac{3}{4}\pi\right) = \frac{1}{4}\pi,$$

where we have

$$\sin^{-1}\left(\sin\frac{3}{4}\pi\right) = \sin^{-1}\left(\frac{1}{2}\right) = \frac{1}{4}\pi$$

and

$$\sin^{-1}(\sin\frac{7}{4}\pi) = -\frac{1}{4}\pi,$$

where we have

$$\sin^{-1}\left(\sin\frac{7}{4}\pi\right) = \sin^{-1}\left(-\frac{1}{\sqrt{2}}\right) = -\frac{1}{4}\pi.$$

□

Example 5 Find

(a) $\cos\left(\sin^{-1}\left(-\frac{1}{2}\right)\right)$;

(b) $\sin^{-1}\left(\cos\frac{2}{3}\pi\right)$.

Solution:

We know that *the range of the inverse sine function* is $\left[-\frac{1}{2}\pi, \frac{1}{2}\pi\right]$.

Further,

$$\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{1}{6}\pi.$$

(a)

$$\cos\left(\sin^{-1}\left(-\frac{1}{2}\right)\right) = \cos\left(-\frac{1}{6}\pi\right) = \frac{\sqrt{3}}{2}.$$

(b)

$$\sin^{-1}\left(\cos\frac{2}{3}\pi\right) = \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{1}{6}\pi.$$

12.3.2 Derivative of the inverse sine function

We now obtain the formula for the derivative of the inverse sine function by applying the rule that deals with the differentiation of inverse functions[4].

Example 6 Derive

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}. \tag{12.4}$$

Solution:

Let $y = \sin^{-1}x$, which is equivalent to

$$x = \sin y \quad \text{and} \quad y \in \left[-\frac{1}{2}\pi, \frac{1}{2}\pi\right]. \tag{12.5}$$

[4] If $y = f(x)$ is a derivable function of x such that the inverse function $x = f^{-1}(y)$ is defined and $dy/dx, dx/dy$ both exist, then the derivative of the inverse function is given by $dx/dy = 1/(dy/dx)$, provided $dy/dx \neq 0$.]

Differentiating both the sides of (12.5) with respect to y, we obtain

$$\frac{dx}{dy} = \cos y \quad \text{and} \quad y \in \left[-\frac{1}{2}\pi, \frac{1}{2}\pi\right]. \tag{12.6}$$

If y is in $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$, then $\cos y$ is non-negative.

We know that the derivative of an inverse function is equal to the reciprocal of the derivative of its given function. Applying (12.6) gives

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\cos y}. \tag{12.7}$$

Here, we have to express the RHS of (12.7) in terms of x. Since, $\sin y = x$, we have[5]

$$\cos y = \pm\sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$$

Of these two values for $\cos y$, we should take

$$\cos y = \sqrt{1 - x^2}, \tag{12.8}$$

since y lies between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$. Thus,

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(\sin^{-1} x) && \text{By } y = \sin^{-1} x \\ &= \frac{1}{\cos y} && \text{By (12.8)} \\ &= \frac{1}{\sqrt{1 - x^2}}. \end{aligned}$$

Thus,

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}.$$

□

Theorem 1 If u is a differentiable function of x, then

$$\frac{d}{dx}(\sin^{-1} u) = \frac{1}{\sqrt{1 - u^2}} \cdot \frac{du}{dx} \qquad \text{By the Chain Rule.}$$

Example 7 Find $f'(x)$, if $f(x) = \sin^{-1} x^4$.
Solution: From Theorem 1, letting $u = x^4$, one gets

$$\begin{aligned} f'(x) &= \frac{1}{\sqrt{1 - (x^4)^2}} \frac{d}{dx}\left(x^4\right) \\ &= \frac{1}{\sqrt{1 - (x^4)^2}} \cdot 4x^3 \\ &= \frac{4x^3}{\sqrt{1 - x^8}}. \end{aligned}$$

□

[5] A trigonometric identity, $\sin^2 y + \cos^2 y = 1$, is used.

12.4 The inverse cosine function

To obtain the inverse cosine function, we perform the same procedure as we did with the inverse sine function. We restrict the cosine to an interval on which the function is (strictly) monotonic. We select the interval $[0, \pi]$ *on which the cosine is decreasing*, as shown by the graph of the cosine in Figure 12.15.

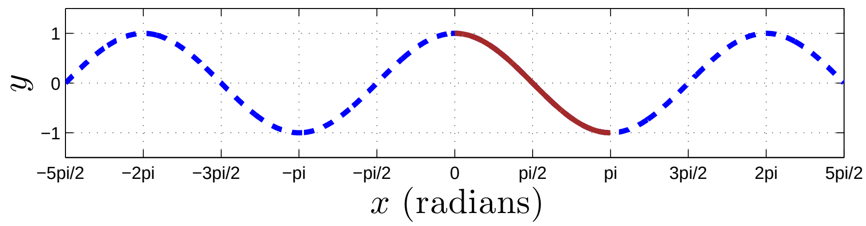


Figure 12.15: $y = \cos x$, where $x \in \left[\frac{-5\pi}{2}, \frac{5\pi}{2}\right]$.

Let us consider the function $f_2(x)$ defined by $f_2(x) = \cos x, x \in [0, \pi]$. The graph of $f_2(x)$ appears in Figure 12.16.

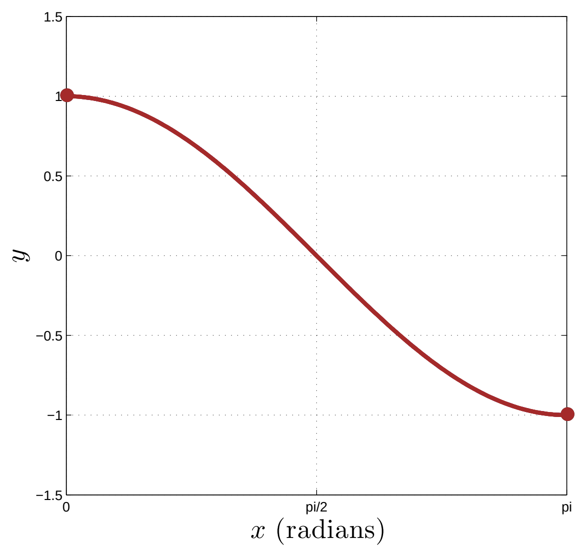


Figure 12.16: $y = f_2(x)$ (brown), where $x \in [0, \pi]$ and $y \in [-1, 1]$.

The *domain of* $f_2(x)$ is the closed interval $[0, \pi]$ and *the range* is the closed interval $[-1, 1]$. Because $f_2(x)$ is *continuous and decreasing on its domain, it has an inverse*, which we will now define.

12.4.1 Definition of the inverse cosine function

Definition 3 The inverse cosine function, denoted by $\cos^{-1}$, is defined by

$$y = \cos^{-1} x, \text{ if and only if, } x = \cos y \text{ and } y \in [0, \pi] .$$

The domain of $\cos^{-1}$ is the closed interval $[-1, 1]$ and the range is the closed interval $[0, \pi]$.

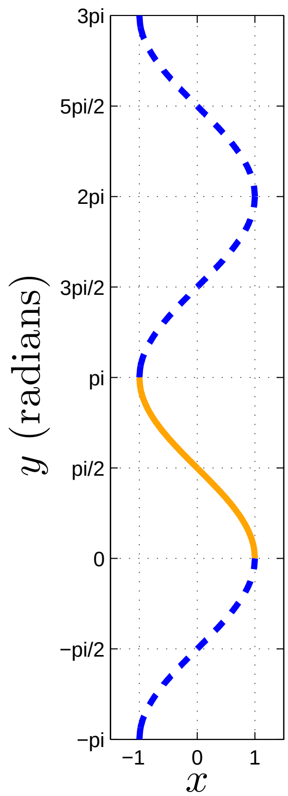


Figure 12.17: $y = \cos^{-1} x$ (orange), where $x \in [-1, 1]$.

The graph of the inverse cosine function appears in Figures 12.17 and 12.18. From Definition 3, we have

$$\begin{cases} \cos(\cos^{-1} x) = x, & \text{for } x \in [-1, 1]; \\ \cos^{-1}(\cos y) = y, & \text{for } y \in [0, \pi]. \end{cases}$$

Note 7 Observe that there is again a restriction on y (with the choice of a specific interval) in order to have the equality. $\square$

Example 8 Consider

$$\cos^{-1}(\cos y) = y,$$

because $(3/4)\pi$ is in $[0, \pi]$. We have

$$\cos^{-1}\left(\cos \frac{3}{4}\pi\right) = \frac{3}{4}\pi$$

However, $\cos^{-1}(\cos \frac{5}{4}\pi) = \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \frac{3}{4}\pi$, and

$$\cos^{-1}\left(\cos \frac{7}{4}\pi\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{4}\pi.$$

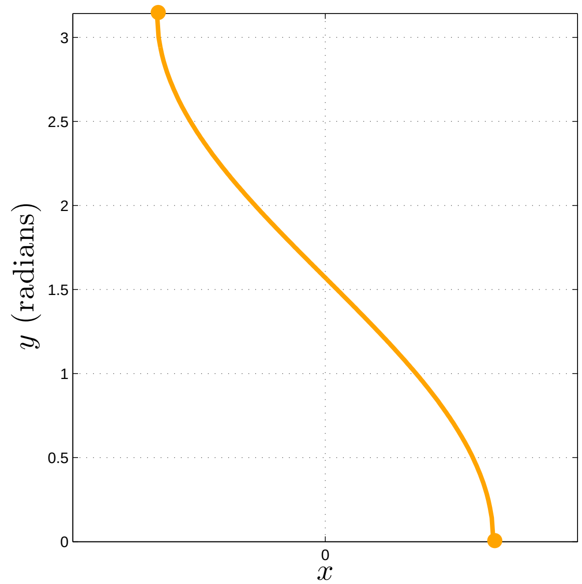


Figure 12.18: $y = \cos^{-1} x$, where $x \in [-1, 1]$ and $y \in [0, \pi]$.

12.4.2 Formula for the derivative of the inverse cosine function

Example 9 Derive

$$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}. \tag{12.9}$$

Solution: Let $y = \cos^{-1} x$, which is equivalent to

$$x = \cos y \qquad \text{and} \qquad y \in [0, \pi]. \tag{12.10}$$

Differentiating both sides of (12.10) with respect to y, we have

$$\frac{dx}{dy} = -\sin y, \qquad \text{and} \qquad y \in [0, \pi]. \tag{12.11}$$

If y is in $[0, \pi]$, $\sin y$ is non-negative, making the above term on the RHS negative. But,

$$\frac{dy}{dx} = \frac{1}{dx/dy} = \frac{-1}{\sin y}. \tag{12.12}$$

Here, we have to write the RHS of (12.12) in terms of x. Since $\cos y = x$, we have

$$\sin y = \pm\sqrt{1 - \cos^2 x} = \pm\sqrt{1-x^2}$$

Of these two values for $\sin y$, we should take $\sin y = \sqrt{1-x^2}$, since y lies between 0 and π. Thus

$$\frac{dy}{dx} = \frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sin y} = \frac{-1}{\sqrt{1-x^2}}.$$

The derivative of the inverse cosine becomes

$$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}.$$

$\square$

12.4.3 Important identities involving inverse trigonometric functions

The following identities involving inverse trigonometric functions are very important.

Example 10 Show that

1. $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$.
2. $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$, $x \geq 0$.
3. $\sec^{-1} x + \mathrm{cosec}^{-1} x = \frac{\pi}{2}$.

Solution:

1. Let
$$\sin^{-1} x = t. \tag{12.13}$$
Then
$$x = \sin t = \cos\left(\frac{\pi}{2} - t\right).$$
Thus
$$\frac{\pi}{2} - t = \cos^{-1} x. \tag{12.14}$$
Adding (12.13) and (12.14), we get
$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}.$$

2. Let
$$\cot^{-1} x = t. \tag{12.15}$$
Then
$$x = \cot t = \tan\left(\frac{\pi}{2} - t\right).$$
Thus
$$\frac{\pi}{2} - t = \tan^{-1} x. \tag{12.16}$$
Adding (12.15) and (12.16), we get
$$\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}.$$

3. Let

$$\operatorname{cosec}^{-1} x = t. \tag{12.17}$$

Then

$$x = \operatorname{cosec} t = \sec\left(\frac{\pi}{2} - t\right).$$

Thus

$$\frac{\pi}{2} - t = \sec^{-1} x. \tag{12.18}$$

Adding (12.17) and (12.18), we get

$$\sec^{-1} x + \operatorname{cosec}^{-1} x = \frac{\pi}{2}.$$

□

Example 11 Using the identity at Example 10(1) above and the result,

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}},$$

show that

$$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}.$$

Solution: Consider the identity

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}. \tag{12.19}$$

Differentiating both sides of (12.19) with respect to x, we get,

$$\frac{d}{dx}(\sin^{-1} x) + \frac{d}{dx}(\cos^{-1} x) = 0.$$

or

$$\begin{aligned} \frac{d}{dx}(\cos^{-1} x) &= -\frac{d}{dx}(\sin^{-1} x) \qquad \text{Using } \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \\ &= \frac{-1}{\sqrt{1-x^2}}. \end{aligned}$$

□

Theorem 2 If u is a differentiable function of x, then

$$\frac{d}{dx}(\cos^{-1} u) = \frac{-1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}.$$

Example 12 Find $\dfrac{dy}{dx}$ if $y = \cos^{-1} e^{4x}$.

Solution: Given $y = \cos^{-1} e^{4x}$. From Theorem 2, we obtain

$$\begin{aligned}\frac{dy}{dx} &= \frac{-1}{\sqrt{1-(e^{4x})^2}}\frac{d}{dx}(e^{4x}) \\ &= \frac{-1}{\sqrt{1-(e^{4x})^2}}\frac{d(e^{4x})}{d(4x)}\frac{d(4x)}{dx} \\ &= \frac{-4e^{4x}}{\sqrt{1-(e^{4x})^2}} \\ &= \frac{-4e^{4x}}{\sqrt{1-e^{8x}}}.\end{aligned}$$

□

12.5 The inverse tangent function

To develop the *inverse tangent function*, observe from the graph in Figure 12.19, that the tangent function is *continuous* and (*(strictly) increasing on the open interval* $\left(-\frac{1}{2}\pi, \frac{1}{2}\pi\right)$. We restrict the tangent function to this interval, denote it by f_3 and define it by $f_3(x) = \tan x$ and $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$.

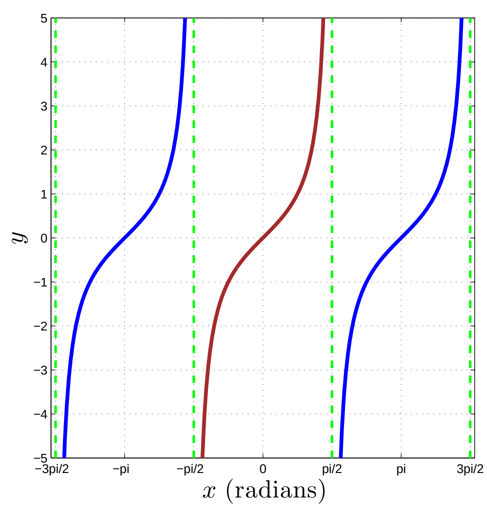


Figure 12.19: $y = \tan x$, where $x \in (-3\pi/2, -\pi/2) \cup (-\pi/2, \pi/2) \cup (\pi/2, 3\pi/2)$.

The *domain* of $f_3(x)$ is the open interval $\left(-\frac{1}{2}\pi, \frac{1}{2}\pi\right)$ and the range is the set $\mathbb{R}$ of real numbers. The graph of $f_3(x)$, where $x \in \left(-\frac{1}{2}\pi, \frac{1}{2}\pi\right)$, is given in Figure 12.20. This function has an inverse called *the inverse tangent function.*

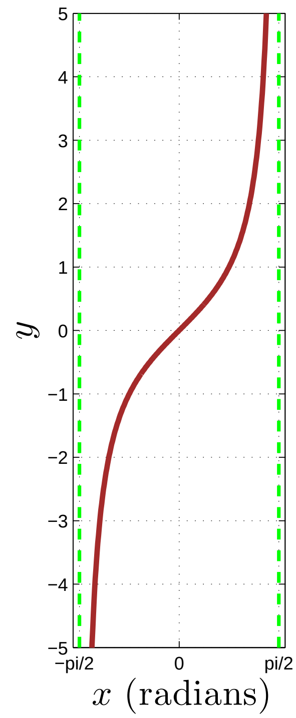


Figure 12.20: $y = f_3(x)$ (brown), where $x \in \left(-\frac{1}{2}\pi, \frac{1}{2}\pi\right)$.

12.5.1 Definition of the inverse tangent function

Definition 4 The *inverse tangent function*, denoted by $\tan^{-1}$, is defined by

$$y = \tan^{-1} x, \text{ if and only if, } x = \tan y \text{ and } -\frac{1}{2}\pi < y < \frac{1}{2}\pi.$$

The domain of $\tan^{-1}$ is the set $\mathbb{R}$ of real numbers and the range is the open interval $\left(-\frac{1}{2}\pi, \frac{1}{2}\pi\right)$.

The graph of the inverse tangent function is shown in Figures 12.21 and 12.22. From Definition 4, we have

$$\begin{aligned} \tan(\tan^{-1} x) &= x, \quad \text{for } x \text{ in } (-\infty, +\infty) \\ \tan^{-1}(\tan y) &= y, \quad \text{for } y \text{ in } \left(\frac{-1}{2}\pi, \frac{1}{2}\pi\right) \end{aligned}$$

The restrictions on y are discussed through the following examples.

Example 13 Given $\tan^{-1}(\tan \frac{1}{4}\pi) = \frac{1}{4}\pi$ and $\tan^{-1}\left(\tan(-\frac{1}{4}\pi)\right) = -\frac{1}{4}\pi$
However,

$$\tan^{-1}\left(\tan \frac{3}{4}\pi\right) = \tan^{-1}(-1) = -\frac{1}{4}\pi$$

and

$$\tan^{-1}\left(\tan \frac{5}{4}\pi\right) = \tan^{-1}(1) = -\frac{1}{4}\pi.$$

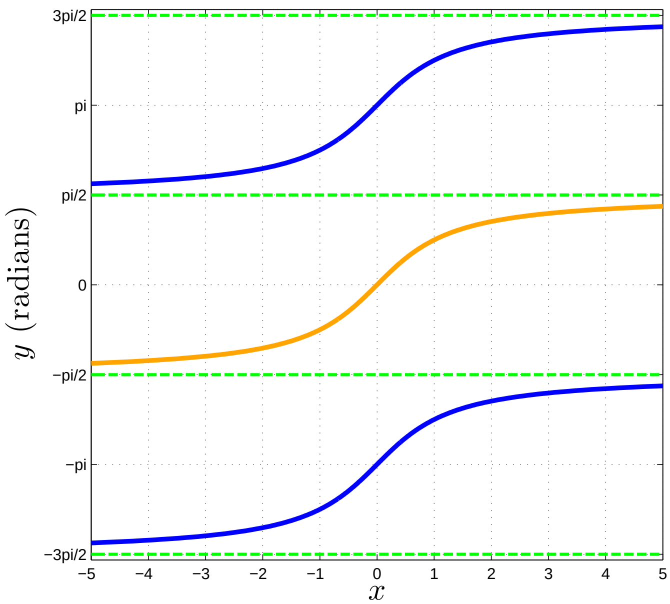


Figure 12.21: $y = \tan^{-1} x$, where $x \in \mathbb{R}$.

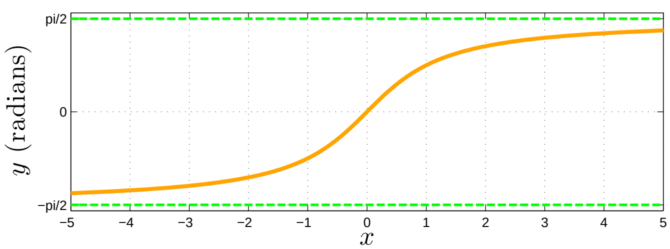


Figure 12.22: $y = \tan^{-1} x$, where $x \in \mathbb{R}$ and $y \in \left(-\frac{1}{2}\pi, \frac{1}{2}\pi\right)$.

12.5.2 Formula for the derivative of the inverse tangent function

Example 14

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}. \tag{12.20}$$

Solution: Let $y = \tan^{-1} x$. Then,

$$x = \tan y \quad \text{and} \quad y \in \left(-\frac{1}{2}\pi, \frac{1}{2}\pi\right). \tag{12.21}$$

Differentiating both the sides of (12.21) with respect to y, we obtain

$$\frac{dx}{dy} = \sec^2 y \quad \text{and} \quad y \in \left(-\frac{1}{2}\pi, \frac{1}{2}\pi\right). \tag{12.22}$$

From the identity $\sec^2 y = 1 + \tan^2 y$, and replacing $\tan y$ by x, we have

$$\sec^2 y = 1 + x^2$$

But

$$\frac{dy}{dx} = \frac{1}{(dx/dy)}$$

or

$$\frac{dy}{dx} = \frac{1}{1+x^2}.$$

Thus

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}.$$

The domain of the derivative of the inverse tangent function is the set $\mathbb{R}$ of real numbers.
□

Theorem 3 If u is a differentiable function of x, then

$$\frac{d}{dx}(\tan^{-1} u) = \frac{1}{1+u^2} \cdot \frac{du}{dx}.$$

Example 15 Find $f'(x)$, if $f(x) = \tan^{-1} \dfrac{1}{x+2}$.

Solution: From Theorem 3, we get

$$f'(x) = \frac{1}{1+(1/(x+2)^2)} \cdot \frac{d}{dx}\left(\frac{1}{x+2}\right)$$

or

$$\begin{aligned} f'(x) &= \frac{1}{1+(1/(x+2)^2)} \cdot \frac{-1}{(x+2)^2} \\ &= \frac{-1}{(x+2)^2+1} = \frac{-1}{x^2+4x+5}. \end{aligned}$$

□

Example 16 Differentiate $\tan^{-1} \log x$.

Solution:

$$\begin{aligned} \frac{d}{dx}\left(\tan^{-1}(\log x)\right) &= \frac{1}{1+(\log x)^2} \cdot \frac{d}{dx}(\log x) \\ &= \frac{1}{1+(\log x)^2} \cdot \frac{1}{x} \\ &= \frac{1}{x\,(1+(\log x)^2)}. \end{aligned}$$

□

12.6 Definition of the inverse cotangent function

To define the *inverse cotangent function*, we use the *identity* $\tan^{-1} x + \cot^{-1} x = \pi/2$, (see Subsection 10) where x is any real number.

Definition 5 The inverse cotangent function, denoted by $\cot^{-1}$, is defined by

$$y = \cot^{-1} x = \frac{1}{2}\pi - \tan^{-1} x \text{ where } x \text{ is any real number.} \tag{12.23}$$

The domain of $\cot^{-1}$ is the set $\mathbb{R}$ of real numbers and the range is the open interval $(0, \pi)$.

To see how the range is obtained, we write (12.23) in Definition 5 as

$$\tan^{-1} x = \frac{1}{2}\pi - \cot^{-1} x. \tag{12.24}$$

We know that,

$$-\frac{1}{2}\pi < \tan^{-1} x < \frac{1}{2}\pi. \tag{12.25}$$

Using (12.24) in (12.25), we get

$$-\frac{1}{2}\pi < \frac{1}{2}\pi - \cot^{-1} x < \frac{1}{2}\pi.$$

Subtracting $(1/2)\pi$ from each part, we get

$$-\pi < -\cot^{-1} x < 0.$$

Now, multiplying each part by -1, we get

$$\pi > \cot^{-1} x > 0.$$

Reversing the direction of inequality signs, we obtain

$$0 < \cot^{-1} x < \pi.$$

The range of the inverse cotangent function is therefore the open interval $(0, \pi)$.
As compared with Figure 12.23, its graph is sketched in Figure 12.24.

Example 17 Applying Definition 5 gives

1. $\tan^{-1}(1) = \frac{1}{4}\pi$.
2. $\tan^{-1}(-1) = -\frac{1}{4}\pi$.
3. $\cot^{-1}(1) = \frac{1}{2}\pi - \tan^{-1}(1) = \frac{1}{2}\pi - \frac{1}{4}\pi = \frac{1}{4}\pi$.
4. $\cot^{-1}(-1) = \frac{1}{2}\pi - \tan^{-1}(-1) = \frac{1}{2}\pi - (\frac{-1}{4}\pi) = \frac{3}{4}\pi$.

□

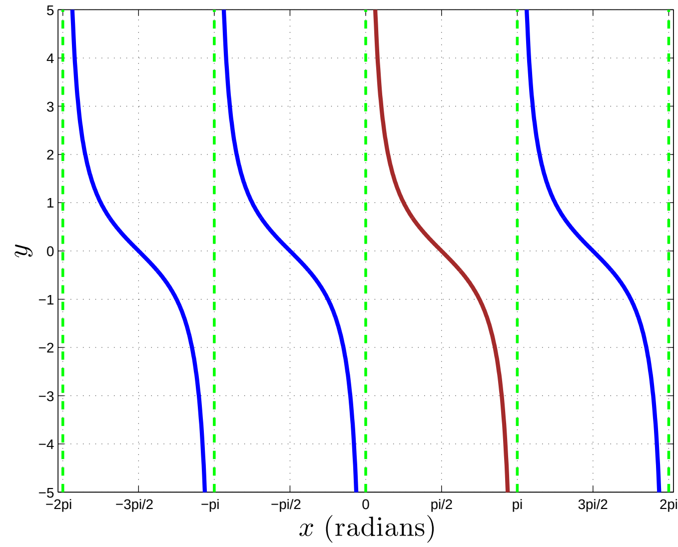


Figure 12.23: $y = \cot x$, where $x \in (-2\pi, -\pi) \cup (-\pi, 0) \cup (0, \pi) \cup (\pi, 2\pi)$.

12.6.1 Formula for the derivative of the inverse cotangent function

Example 18

$$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}. \tag{12.26}$$

Solution: From Definition 5, we have

$$\cot^{-1} x = \frac{1}{2}\pi - \tan^{-1} x. \tag{12.27}$$

Differentiating both sides of (12.27) with respect to x, we get

$$\frac{d}{dx}\cot^{-1} x = \frac{d}{dx}\left(\frac{1}{2}\pi - \tan^{-1} x\right).$$

Thus,

$$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}.$$

□

Theorem 4 If u is a differentiable function of x, then

$$\frac{d}{dx}(\cot^{-1} u) = -\frac{1}{1+u^2} \cdot \frac{du}{dx}.$$

Before we define the inverse secant and the inverse cosecant functions, let us again look at the graphs of basic trigonometric functions and inverse trigonometric functions.

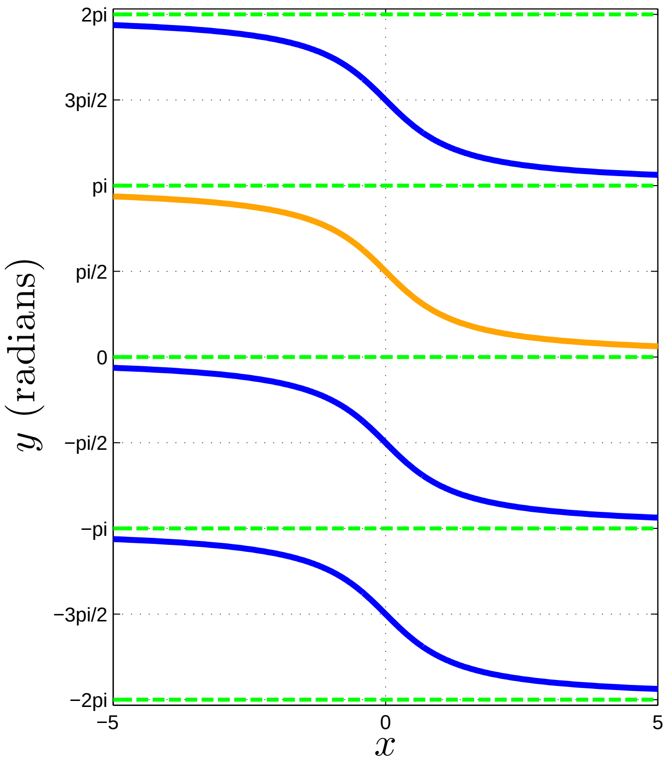


Figure 12.24: $y = \tan^{-1} x$ (orange), where $x \in \mathbb{R}$ and $y \in (0, \pi)$.

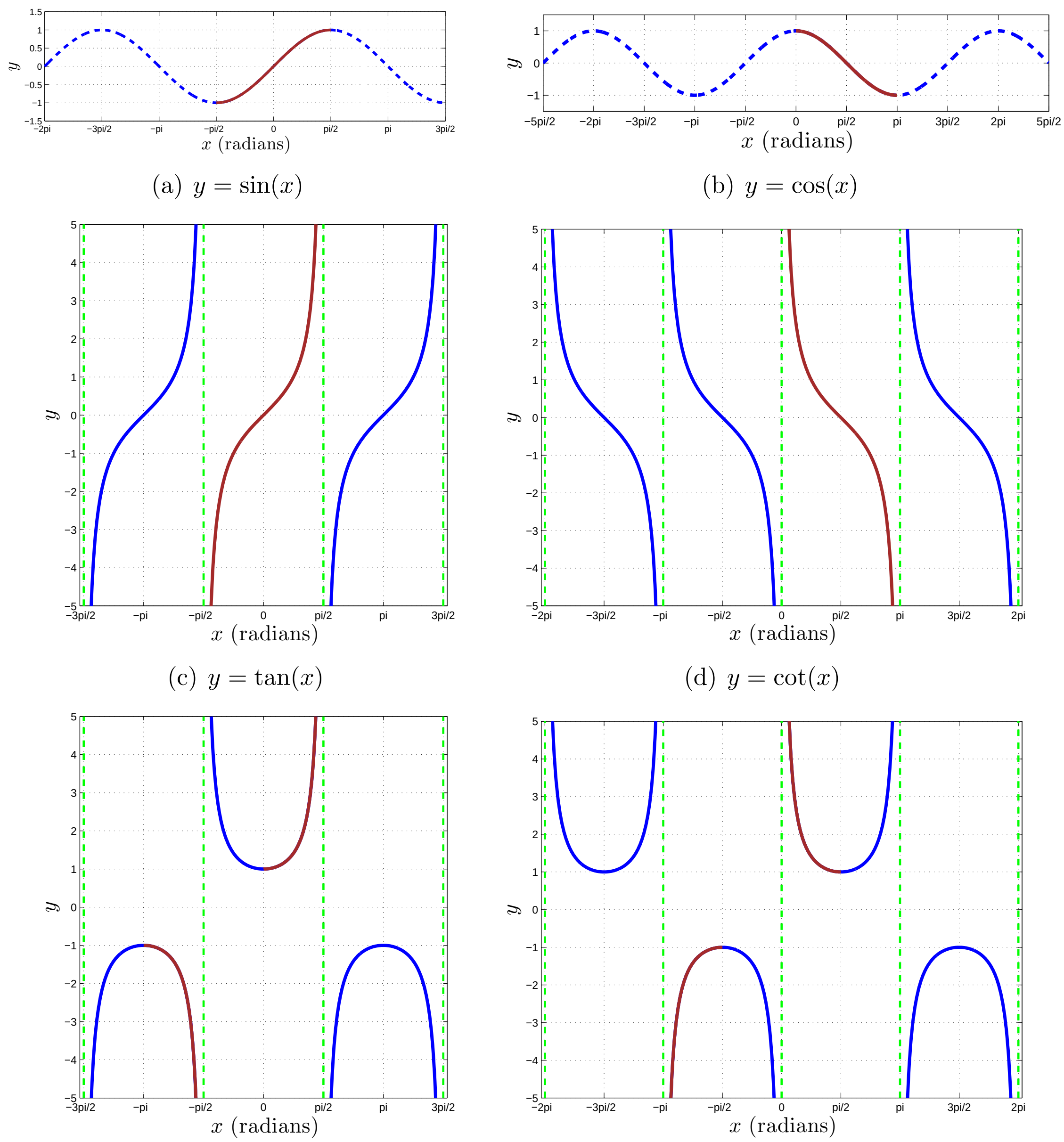
The graphs of six trigonometric functions are shown in Figure 12.25. None of these functions has an inverse, since a horizontal line $y = c$ may cross each graph at more than one point.

Now consider the six functions ($f_1 \;-\; f_6$), which its graphs have heavily colored portions (brown) of the six trigonometric functions in the same graph (Figure 12.25). (In fact, these portions of the graph define the respective trigonometric functions with restricted domain.) Each of these graphs represents a new function, which has the same range as the corresponding trigonometric function, and each new function has an inverse. We call them the *principal branches of the basic trigonometric functions.*

By abuse of terminology, the inverses of $f_1, f_2, \cdots, f_6$ are called the inverse trigonometric functions, so that f_1^{-1} is the inverse sine, denoted by $x = \sin^{-1} y$, f_2^{-1} is the inverse cosine, denoted by $x = \cos^{-1} y$, and so on. Similar notations are used for the remaining four inverse trigonometric functions. The graphs of the inverse trigonometric functions as functions of the independent variable x are shown in Figures 12.26 and 12.27, in the orange colored portions of the functions in the graph.

Note 8 As can be seen from the graph of $\sec x$ and cosec$x$ (Figure 12.25), it is impossible to choose "branches" of these functions so that the inverse functions become continuous. The branches of $\sec^{-1} x$ and cosec^{-1}x (Figure 12.27(c) and (d)) are chosen to make the formulas for the derivatives of these functions come out nicely, without ambiguity to sign. Now, the derivatives of $\sec^{-1} x$ and cosec^{-1}x can easily be found just as we found the derivatives in other cases.

$\square$



(a) $y = \sin(x)$

(b) $y = \cos(x)$

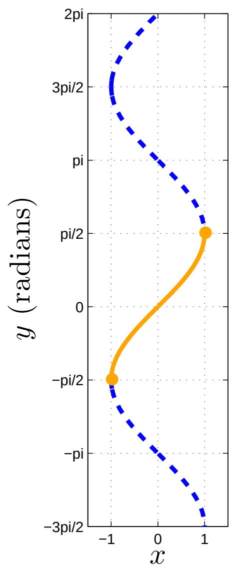
(c) $y = \tan(x)$

(d) $y = \cot(x)$

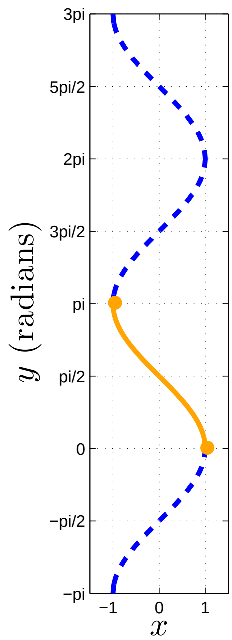
(e) $y = \sec(x)$

(f) $y = \mathrm{cosec}(x)$

Figure 12.25: Six trigonometric functions with restricted domains.

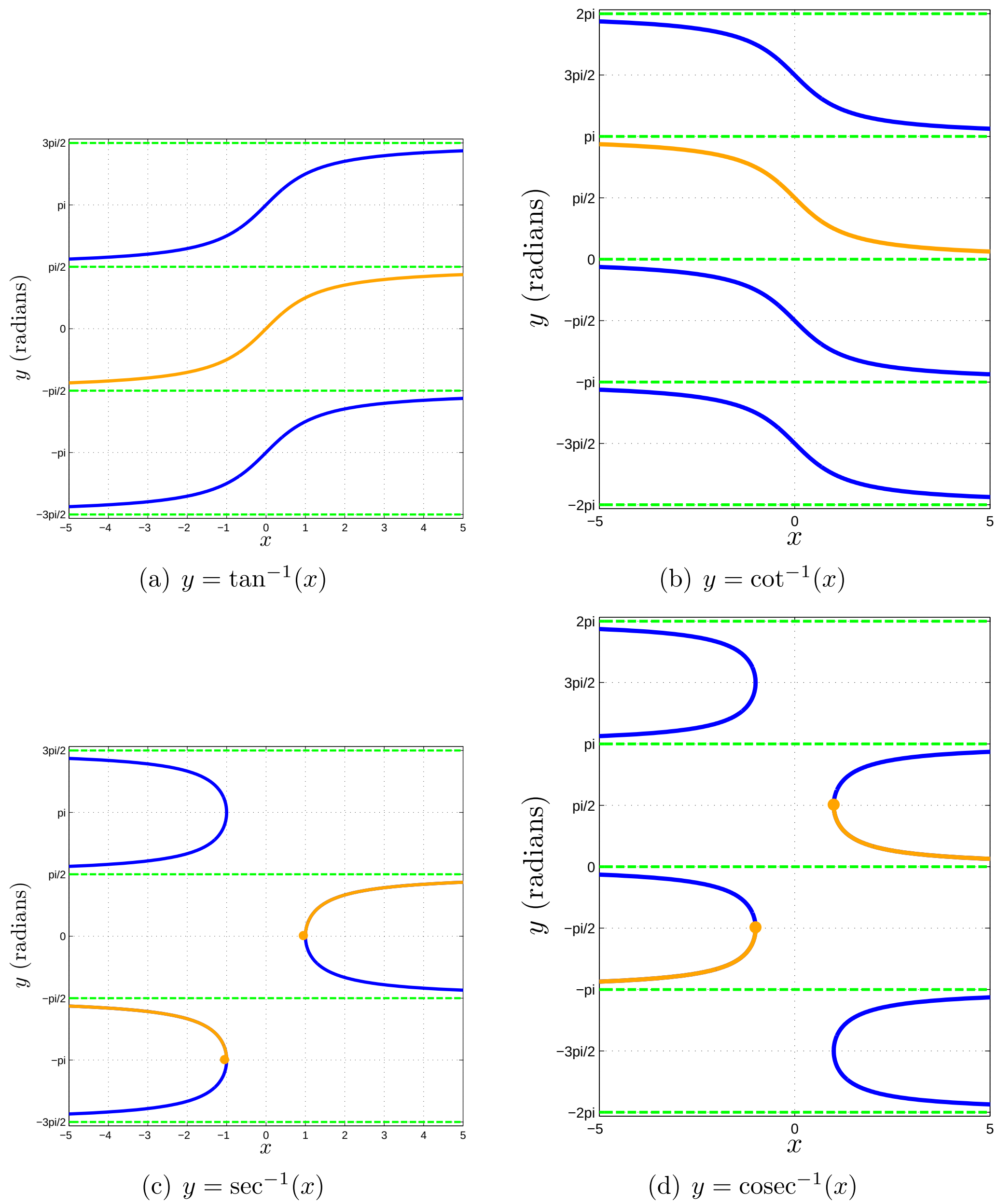


(a) $y = \sin^{-1}(x)$



(b) $y = \cos^{-1}(x)$

Figure 12.26:



(a) $y = \tan^{-1}(x)$

(b) $y = \cot^{-1}(x)$

(c) $y = \sec^{-1}(x)$

(d) $y = \text{cosec}^{-1}(x)$

Figure 12.27:

12.6.2 Formula for the derivative of the inverse secant function

Example 19

$$\frac{d(\sec^{-1} x)}{dx} = \frac{1}{x\sqrt{x^2-1}}, \quad |x| > 1. \tag{12.28}$$

Solution: Let

$$y = \sec^{-1} x.$$

Now

$$x = \sec y. \tag{12.29}$$

Differentiating both sides of (12.29) with respect to y, we get

$$\begin{aligned}\frac{dx}{dy} &= \sec y \cdot \tan y \\ &= \sec y \sqrt{\tan^2 y} \\ &= \sec y \sqrt{\sec^2 y - 1} \\ &= x\sqrt{x^2-1}.\end{aligned}$$

Now

$$\frac{dy}{dx} = \frac{1}{(dy/dx)} = \frac{1}{x\sqrt{x^2-1}}.$$

Thus

$$\frac{d(\sec^{-1} x)}{dx} = \frac{1}{x\sqrt{x^2-1}}, \quad |x| > 1.$$

□

Theorem 5 If y is a differentiable function of x, then

$$\frac{d}{dx}(\sec^{-1} u) = \frac{1}{u\sqrt{u^2-1}} \cdot \frac{du}{dx}, \quad |u| > 1.$$

12.6.3 Formula for the derivative of the inverse cosecant function

Example 20 Derive

$$\frac{d(\mathrm{cosec}^{-1} x)}{dx} = \frac{-1}{x\sqrt{x^2-1}}, \quad |x| > 1. \tag{12.30}$$

Solution: Let

$$y = \mathrm{cosec}^{-1} x.$$

Then,

$$x = \text{cosec} y. \tag{12.31}$$

Differentiating both sides of (12.31) with respect to y, we get,

$$\begin{aligned}\frac{dx}{dy} &= -\text{cosec} y \cdot \cdot y = -\text{cosec} y\sqrt{\text{cosec}^2 y - 1} \\ &= -\text{cosec} y\sqrt{\text{cosec}^2 y - 1} \\ &= -x\sqrt{x^2 - 1}.\end{aligned}$$

Now,

$$\frac{dy}{dx} = \frac{1}{(dy/dx)} = \frac{-1}{x\sqrt{x^2 - 1}}, \quad |x| > .1$$

Thus,

$$\frac{d(\text{cosec}^{-1} x)}{dx} = \frac{-1}{x\sqrt{x^2 - 1}}, \quad |x| > 1.$$

□

Theorem 6 If u is a differentiable function of x, then

$$\frac{d}{dx}(\text{cosec}^{-1} u) = \frac{-1}{u\sqrt{u^2 - 1}} \cdot \frac{du}{dx}, \quad |u| > 1.$$

Table 12.2 summarizes the data that we should remember regarding inverse trigonometric functions.

Function	Domain	Range	Derivative
$\sin^{-1} x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$	$\frac{-1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$x \in \mathbb{R}$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$	$\frac{1}{1+x^2}$
$\cot^{-1} x$	$x \in \mathbb{R}$	$0 < y < \pi$	$\frac{-1}{1+x^2}$
$\sec^{-1} x$	$x \leq -1$ or $x \geq 1$	$-\pi \leq y < -\frac{\pi}{2}$ or $0 \leq y < \frac{\pi}{2}$	$\frac{1}{x\sqrt{x^2-1}}$
$\text{cosec}^{-1} x$	$x \leq -1$ or $x \geq 1$	$-\pi < y \leq -\frac{\pi}{2}$ or $0 < y \leq \frac{\pi}{2}$	$\frac{-1}{x\sqrt{x^2-1}}$

Table 12.2:

Source: Calculus with Analytic Geometry by John B. Fraleigh (p. 263), Addison-Wesley.

From the theorems stated at 1 - 6 above, we know that if u is a function of the independent variable x, then we may write the formulas for the derivatives of the inverse trigonometric functions of u, using the chain rule. For example,

$$\frac{d}{dx}\sin^{-1} u = \frac{1}{1 - u^2} \cdot \frac{du}{dx},$$

and so on.

These results may also be written as

$$\frac{d}{dx}\sin^{-1}[f(x)] = \frac{f'(x)}{\sqrt{1-[f(x)]^2}},$$

$$\frac{d}{dx}\cos^{-1}[f(x)] = \frac{-f'(x)}{\sqrt{1-[f(x)]^2}},$$

$$\frac{d}{dx}\tan^{-1}[f(x)] = \frac{f'(x)}{1+[f(x)]^2},$$

and, so on.

These formulas are primarily important for the evaluation of certain definite integrals. In fact, this is the main reason for studying the calculus of inverse trigonometric functions.

Note 9 Figures 12.28, 12.29 and 12.30 show the graphical symmetry of trigonometric functions and their inverse functions.

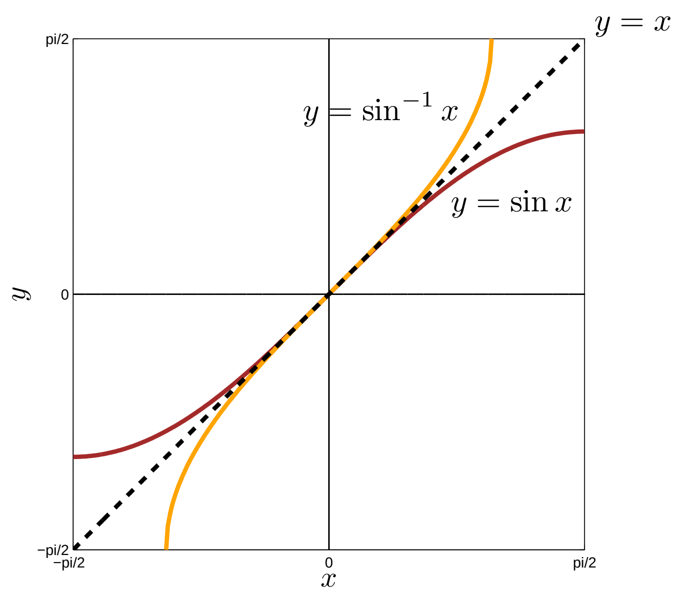


Figure 12.28: Graphs of $\sin x$ and $\sin^{-1} x$.

Note 10 In some textbooks, the inverse secant function, denoted by $\sec^{-1} x$, is defined to be the inverse of the restricted secant function:

$$\sec x, \quad x \in [0, \pi/2) \cup [\pi, 3\pi/2) \quad \text{or} \quad x \in [0, \pi/2) \cup (\pi/2, \pi].$$

Similar cases of the definition of the inverse cosecant function can been found elsewhere.

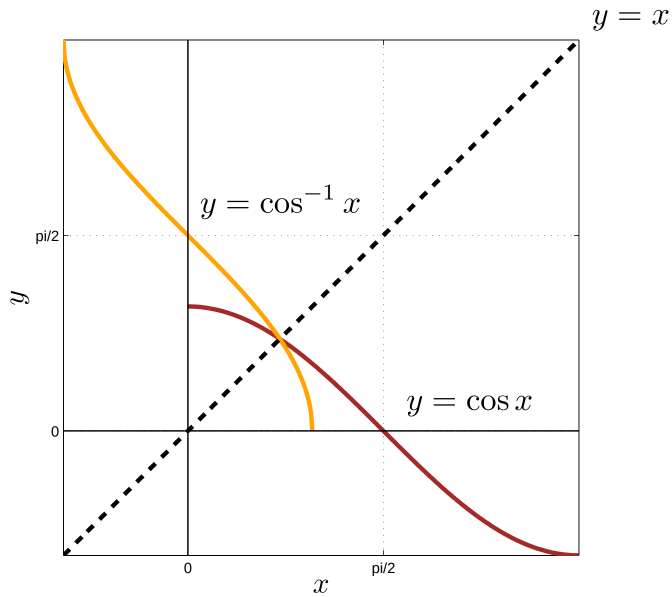


Figure 12.29: Graphs of $\cos x$ and $\cos^{-1} x$.

12.7 Important sets of results and their applications

The following sets of results [**Set (1)** to **Set (5)**] connecting trigonometric (circular) functions and inverse trigonometric functions are useful in simplifying certain inverse trigonometric functions for computing their derivatives.

In the above results (or formulas) *it is assumed that we are dealing with the principal branch(es) of the functions and their appropriate domain(s).* Their applications are given below:

Set(1)

$$\sin^{-1}(\sin x)) = x$$
$$\cos^{-1}(\cos x)) = x$$
$$\tan^{-1}(\tan x)) = x$$
and so on.

Set(2)

$$\sin\left(\sin^{-1} x\right) = x$$
$$\cos\left(\cos^{-1} x\right) = x$$
$$\tan\left(\tan^{-1} x\right) = x$$
and so on.

Example 21 Applications of **Set (1)** and **Set (2)** (differentiate (????) with respect to x, where (????) is taken from **Set (1)** and **Set (2)**) present as follows:

Consider

$$y = \sin^{-1}(\sin 3x).$$

Putting $3x = t$, we have

$$y = \sin^{-1}(\sin t) = t$$

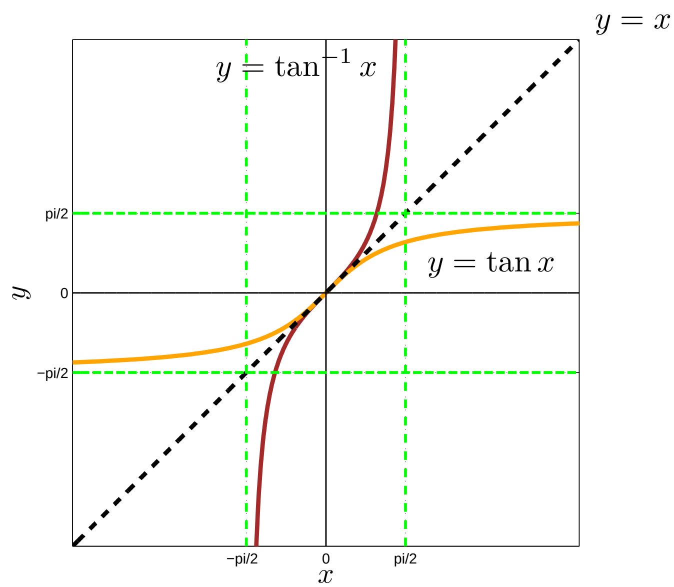


Figure 12.30: Graphs of $\tan x$ and $\tan^{-1} x$.

or putting $y = 3x$, we have

$$\frac{dy}{dx} = \frac{d}{dx}(3x) = 3.$$

□

Set(3)

$$\begin{aligned}
&\sin^{-1}(\cos x) = \sin^{-1}\left(\sin\left(\tfrac{\pi}{2} - x\right)\right) = \tfrac{\pi}{2} - x\\
&\cos^{-1}(\sin x) = \cos^{-1}\left(\cos\left(\tfrac{\pi}{2} - x\right)\right) = \tfrac{\pi}{2} - x\\
&\tan^{-1}(\cot x) = \tan^{-1}\left(\tan\left(\tfrac{\pi}{2} - x\right)\right) = \tfrac{\pi}{2} - x
\end{aligned}$$

and so on.

Example 22 Application of **Set (3)** presents as follows:

Consider

$$y = \sin^{-1}(\cos 5x).$$

Then

$$y = \sin^{-1}\left(\sin\left(\frac{\pi}{2} - 5x\right)\right) = \frac{\pi}{2} - 5x.$$

Thus,

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{2} - 5x\right) = 0 - 5 = -5.$$

Set(4)

$$\begin{aligned}
\tan^{-1} x + \tan^{-1} y &= \tan^{-1}\left(\frac{x+y}{1-xy}\right)\\
\tan^{-1} x - \tan^{-1} y &= \tan^{-1}\left(\frac{x-y}{1+xy}\right)
\end{aligned}$$

These results are very useful as can be seen from the solved examples (it is proposed to prove these results at the end of this chapter).

Note 11 Note that the expression $\dfrac{x+y}{1-xy}$ can be converted to the form $\tan(p+q)$ by proper substitution and similarly $\dfrac{x-y}{1+xy}$ can be converted to the form $\tan(p-q)$. Thus, in any expression of the type $\tan^{-1}(f(x))$, if it is possible to break up $f(x)$ in any of the two above forms, then the given function $\tan^{-1}(f(x))$ can be simplified for the purpose of the differentiation as will be clear from the following solved examples. □

Applications of Set (4) (differentiate the following with respect to x) present as follows:

Example 23 Let

$$y = \tan^{-1}\left(\frac{5x}{1-6x^2}\right).$$

Then

$$\begin{aligned} y &= \tan^{-1}\left(\frac{3x+2x}{1-(3x)\cdot(2x)}\right) \\ &= \tan^{-1}(3x) + \tan^{-1}(2x). \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{1+(3x)^2}\cdot\frac{d}{dx}(3x) + \frac{1}{1+(2x)^2}\cdot\frac{d}{dx}(2x) \\ &= \frac{3}{1+9x^2} + \frac{2}{1+4x^2}. \end{aligned}$$

□

Example 24 Let

$$y = \tan^{-1}\left(\frac{\sin 7x - \cos 7x}{\sin 7x + \cos 7x}\right).$$

Dividing numerator and denominator by $\cos 7x$, we have

$$\begin{aligned} y &= \tan^{-1}\left(\frac{\tan 7x - 1}{\tan 7x + 1}\right) \\ &= \tan^{-1}\left(\frac{\tan 7x - 1}{1 + \tan 7x}\right) \\ &= \tan^{-1}\left(\frac{\tan 7x - \tan(\pi/4)}{1 + \tan 7x \cdot \tan(\pi/4)}\right) && \text{because } \tan(\pi/4) = 1 \\ &= \tan^{-1}(\tan(7x)) - \tan^{-1}\left(\tan\left(\frac{\pi}{4}\right)\right) \end{aligned}$$

$$= 7x - \frac{\pi}{4} \quad .$$

Thus,

$$\frac{dy}{dx} = 7.$$

□

Set(5)

$$\begin{aligned}
&\sin^{-1} x = \text{cosec}^{-1}\left(\tfrac{1}{x}\right)\\
&\cos^{-1} x = \sec^{-1}\left(\tfrac{1}{x}\right)\\
&\tan^{-1} x = \cot^{-1}\left(\tfrac{1}{x}\right)\\
&\cot^{-1} x = \tan^{-1}\left(\tfrac{1}{x}\right)\\
&\sec^{-1} x = \cos^{-1}\left(\tfrac{1}{x}\right)\\
&\text{cosec}^{-1} x = \sin^{-1}\left(\tfrac{1}{x}\right)
\end{aligned}$$

Applications of **Set (5)** (differentiate the following with respect to x) present as follows:

Example 25 Let $y = \sin\left(\text{cosec}^{-1}(\tfrac{1}{x})\right)$. Then,

$$\begin{aligned}
y &= \sin(\sin^{-1} x)\\
&= x.
\end{aligned}$$

Therefore,

$$\frac{dy}{dx} = 1.$$

□

Example 26 Let $y = \sec\left(\cos^{-1}\left(\tfrac{2}{5x}\right)\right)$. Thus,

$$y = \sec\left(\sec^{-1}\frac{5x}{2}\right) = \frac{5x}{2}.$$

Thus,

$$\frac{dy}{dx} = \frac{5}{2}.$$

□

Example 27 Let $y = \cot^{-1}\left(\dfrac{3 - 2\tan x}{2 + 3\tan x}\right)$. Note that, using the formula

$$\cot^{-1} x = \tan^{-1}(\frac{1}{x}),$$

we can write,

$$y = \tan^{-1}\left(\frac{2 + 3\tan x}{3 - 2\tan x}\right).$$

Observe that the expression on the RHS can be simplified *if the denominator is expressed in the form* $(1 - k\tan x)$. This can be done by dividing the numerator and denominator by 3. We then get,

$$\begin{aligned}
y &= \tan^{-1}\left(\frac{(2/3) + \tan x}{1 - (2/3)\tan x}\right)\\
&= \tan^{-1}\left(\frac{2}{3}\right) + \tan^{-1}(\tan x) \qquad \text{because } \tan^{-1}\left(\frac{a+b}{1 - a\cdot b}\right) = \tan^{-1} a + \tan^{-1} b\\
&= \tan^{-1}\left(\frac{2}{3}\right) + x.
\end{aligned}$$

Therefore,

$$\frac{dy}{dx} = 0 + 1 = 1.$$

□

Note 12 It is normally preferred to express $\cot^{-1} x, \sec^{-1} x$, and $\text{cosec}^{-1} x$ in the forms $\tan^{-1} t, \cos^{-1} t$ and $\sin^{-1} t$, respectively, where t stands for $(1/x)$.

Example 28 Let $y = \cot^{-1}\left(\frac{5+4x}{5x-4}\right)$. Then

$$\cot^{-1}\left(\frac{5+4x}{5x-4}\right) = \tan^{-1}\left(\frac{5x-4}{5+4x}\right)$$

because

$$\cot^{-1} t = \tan^{-1} \frac{1}{t}.$$

Dividing numerator and denominator by 5, we get

$$\begin{aligned} y &= \tan^{-1}\left(\frac{x-(4/5)}{1+(4x/5)}\right) \\ &= \tan^{-1}\left(\frac{x-(4/5)}{1+x\cdot(4/5)}\right) \\ &= \tan^{-1} x - \tan^{-1}\left(\frac{4}{5}\right). \end{aligned}$$

Therefore,

$$\frac{dy}{dx} = \frac{1}{1+x^2} - 0 = \frac{1}{1+x^2}.$$