THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1510C/G Calculus for Engineers 2021-2022
Supplementary Notes on Partial Fractions

Theorem 1. *If $Q(x)$ is a polynomial of degree $n \geq 1$, then $Q(x)$ can be factorized as a product of linear and quadratic factors, i.e. $Q(x)$ can be factorized as*

$$Q(x) = A(x-\alpha_1)^{p_1} \cdots (x-\alpha_k)^{p_k}(x^2+b_1x+c_1)^{m_1} \cdots (x^2+b_rx+c_r)^{m_r},$$

where $p_1 + \cdots + p_k + 2(m_1 + \cdots + m_r) = n$.

If $P(x)$ and $Q(x)$ are polynomials, then the fractions $\dfrac{P(x)}{Q(x)}$ can be expressed as sum of simpler fractions according to certain rules, the fraction $\dfrac{P(x)}{Q(x)}$ is said to be resolved in partial fractions. The Rules are listed:

(Rule 0) If $\deg P(x) \geq \deg Q(x)$, by performing long division $P(x) = a(x)Q(x) + b(x)$ where $\deg b(x) < \deg Q(x)$, then

$$\frac{P(x)}{Q(x)} = a(x) + \frac{b(x)}{Q(x)}.$$

Therefore, we only focus on the case that $\deg P(x) < \deg Q(x)$.

(Rule 1) If there is a non-repeating factor $x-a$ of $Q(x)$, then there will be a corresponding partial fraction $\dfrac{A}{x-a}$.

(Rule 2) If there is a non-repeating factor x^2+px+q which cannot be further factorized, then there will be a corresponding partial fraction $\dfrac{Bx+C}{x^2+px+q}$.

1. Resolve the following expressions into partial fractions.

   (a) $\dfrac{5}{x^2+x-6}$ (Hint: $\dfrac{5}{x^2+x-6} \equiv \dfrac{A}{x+3} + \dfrac{B}{x-2}$)

   (b) $\dfrac{1}{x(x^2+1)}$ (Hint: $\dfrac{1}{x(x^2+1)} \equiv \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1}$)

   (c) $\dfrac{5x^2-3x+4}{(x+1)(x^2-2x+6)}$

2. Resolve the following expressions into partial fractions.

   (a) $\dfrac{x^2+3x}{x^2+3x+2}$

   (b) $\dfrac{x^4+2x+4}{(2x^2+3)(x-2)}$

   (c) $\dfrac{2x^5}{(x^2-1)(x^2-4)}$

(Rule 3) If $x-a$ is a repeating factor of $Q(x)$ with multiplicity k (i.e $Q(x) = (x-a)^kQ_1(x)$, where $k > 1$ and $(x-a)$ is not a factor of $Q_1(x)$), then there will be corresponding partial fractions

$$\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_k}{(x-a)^k}.$$

(Rule 4) If $x^2 + px + q$, which cannot be further factorized, is a repeating factor of $Q(x)$ with multiplicity k, then there will be corresponding partial fractions

$$\frac{B_1x + C_1}{x^2 + px + q} + \frac{B_2x + C_2}{(x^2 + px + q)^2} + \cdots + \frac{B_kx + C_k}{(x^2 + px + q)^k}.$$

3. Resolve the following expressions into partial fractions.

   (a) $\dfrac{x^3+1}{(x-2)^4}$ (Hint: $\dfrac{x^3+1}{(x-2)^4} \equiv \dfrac{A}{x-2} + \dfrac{B}{(x-2)^2} + \dfrac{C}{(x-2)^3} + \dfrac{D}{(x-2)^4}$)

   (b) $\dfrac{2x^2+1}{x^2(x^2+1)^2}$ (Hint: $\dfrac{2x^2+1}{x^2(x^2+1)^2} \equiv \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{Cx+D}{x^2+1} + \dfrac{Ex+F}{(x^2+1)^2}$)

4. (a) Resolve $\dfrac{x^5+3x^2+1}{(x-1)(x^2+4)}$ into partial fractions.

   (b) Hence, evaluate $\displaystyle\int \frac{x^5+3x^2+1}{(x-1)(x^2+4)}\,dx$

5. Evaluate $\displaystyle\int \frac{-2x^2+6x-8}{(x-1)(x^4-1)}\,dx$.

Solution/Hint:

1. (a) $\dfrac{5}{x^2+x-6} = \dfrac{1}{x-2} - \dfrac{1}{x+3}$

   (b) $\dfrac{1}{x(x^2+1)} = \dfrac{1}{x} - \dfrac{x}{x^2+1}$

   (c) $\dfrac{5x^2-3x+4}{(x+1)(x^2-2x+6)} = \dfrac{4}{3(x+1)} + \dfrac{11x-12}{3(x^2-2x+6)}$

2. (a) $\dfrac{x^2+3x}{x^2+3x+2} = 1 - \dfrac{2}{x+1} + \dfrac{2}{x+2}$

   (b) $\dfrac{x^4+2x+4}{(2x^2+3)(x-2)} = \dfrac{x}{2} + 1 + \dfrac{24}{11(x-2)} - \dfrac{41x+38}{22(2x^2+3)}$

   (c) $\dfrac{2x^5}{(x^2-1)(x^2-4)} = 2x - \dfrac{1}{3(x-1)} - \dfrac{1}{3(x+1)} + \dfrac{16}{3(x-2)} + \dfrac{16}{3(x+2)}$

3. (a) $\dfrac{x^3+1}{(x-2)^4} = \dfrac{9}{(x-1)^4} + \dfrac{12}{(x-2)^3} + \dfrac{6}{(x-2)^2} + \dfrac{1}{x-2}$

   (b) $\dfrac{2x^2+1}{x^2(x^2+1)^2} = \dfrac{1}{x^2} - \dfrac{1}{x^2+1} + \dfrac{1}{(x^2+1)^2}$

4. (a) $\dfrac{x^5+3x^2+1}{(x-1)(x^2+4)} = x^2 + x - 3 + \dfrac{1}{x-1} + \dfrac{-x+15}{x^2+4}$

   (b) $\displaystyle\int \frac{x^5+3x^2+1}{(x-1)(x^2+4)}\,dx = \frac{x^3}{3} + \frac{x^2}{2} - 3x - \frac{1}{2}\ln(x^2+4) + \ln|x-1| + \frac{15}{2}\tan^{-1}(\frac{x}{2}) + C$

5. $\displaystyle\int \frac{-2x^2+6x-8}{(x-1)(x^4-1)}\,dx = \frac{1}{x-1} + 2\ln|x-1| - 2\ln|x+1| - 3\tan^{-1}x + C.$