Calculus for Engineers

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Contents

The Fundamental Theorem of Calculus

22.1 Introduction

The Fundamental Theorem of Calculus allows us to think of integrals or antiderivatives as net areas. There are two parts of the Fundamental Theorem:

- The first part of the theorem allows us to think of indefinite integrals as an area function, that is, the derivative with respect to the upper limit of the integral of a function is the function itself. Theorem 1 says that every continuous function has an antiderivative and shows how to differentiate a function defined as an integral.
- The second part of the theorem, which deals with the definite integral, shows us how to find net areas for defined regions. Theorem 3 shows how to evaluate the definite integral of any function if we know an antiderivative of that function.

It is worth to noticing that calculus is the study of derivatives and integrals, their meanings and their applications. The Fundamental Theorem of Calculus shows that differentiation and integration are closely related and that integration is really antidifferentiation, or the inverse of differentiation.

22.2 Part I of The Fundamental Theorem Of Calculus

Before answering the following questions, "What is the derivative of the integral?" and "What is $\frac{d}{dx}\left(\int\limits_a^x f(t)dt\right)$?," let us study the following examples.

Example 1 If f is a function whose graph is shown below and

$$g(x) = \int\limits_0^x f(t)dt,$$

find the values of $g(0), g(1), g(2), g(3), g(4)$, and $g(5)$. Then sketch a rough graph of g.

Solution. We evaluate each case as follows, as shown in Figure 22.3:

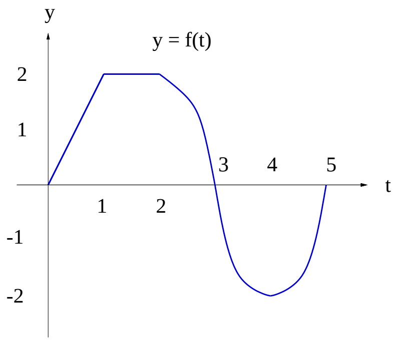

Figure 22.1: Example 1.

- First we notice that
$$g(0) = \int_0^0 f(t)dt = 0.$$

- From the figure above we see that $g(1)$ is the area of a triangle:
$$g(1) = \int_0^1 f(t)dt = \frac{1}{2}(1 \cdot 2) = 1.$$

- To find $g(2)$ we add to $g(1)$ the area of a rectangle:
$$g(2) = \int_0^2 f(t)dt = \int_0^1 f(t)dt + \int_1^2 f(t)dt = 1 + (1 \cdot 2) = 3.$$

- We estimate that the area under f from 2 to 3 is about 1.3, so
$$g(3) = g(2) + \int_2^3 f(t)dt \approx 3 + 1.3 = 4.3.$$

- For $t > 3$, $f(t)$ is negative and so we start subtracting areas:
$$g(4) = g(3) + \int_3^4 f(t)dt \approx 4.3 + (-1.3) = 3.$$

- And
$$g(5) = g(4) + \int_4^5 f(t)dt \approx 3 + (-1.3) = 1.7.$$

We use these values from Table 22.1 to sketch the graph of g, as shown in Figure 22.3.

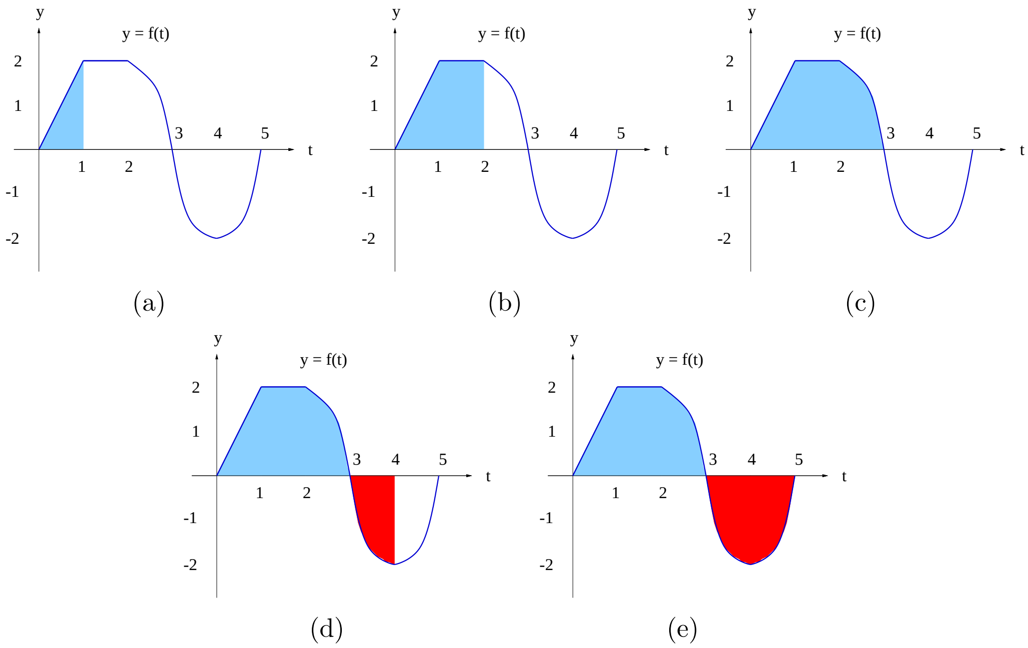

Figure 22.2: Solutions for Example 1.

value of x	$g(x)$
0	$g(0) = 0$
1	$g(1) = 1$
2	$g(2) = 3$
3	$g(3) \approx 4.3$
4	$g(4) \approx 3$
5	$g(5) \approx 1.7$

Table 22.1: Values of $g(x)$.

Example 2 If

$$g(x) = \int_a^x f(t)dt,$$

where $a = 0$ and $f(t) = \sin t$, find a formula for $g(x)$ and calculate $g'(x)$.

Solution. By the definition of the definite integral, we have

$$\begin{aligned} g(x) = \int_a^x f(t)dt &= -\cos t|_a^x \\ &= -\cos x - (-\cos 0) \\ &= -\cos x + 1. \end{aligned}$$

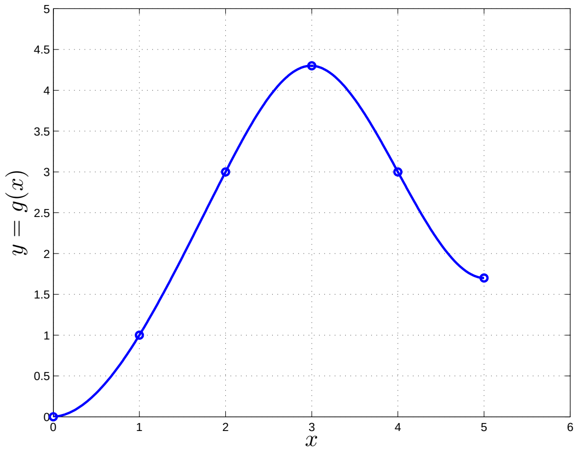

Figure 22.3: Graph of $g(x)$, where we generate a coarse curve and interpolate over a finer abscissa using the cubic spline interpolation.

Then, we have

$$g'(x) = \frac{d}{dx}(-\cos x + 1) = \sin x.$$

Note 1 To see why this might be generally true we consider a continuous function f with $f(x) \geq 0$. Then

$$g(x) = \int_a^x f(t)dt.$$

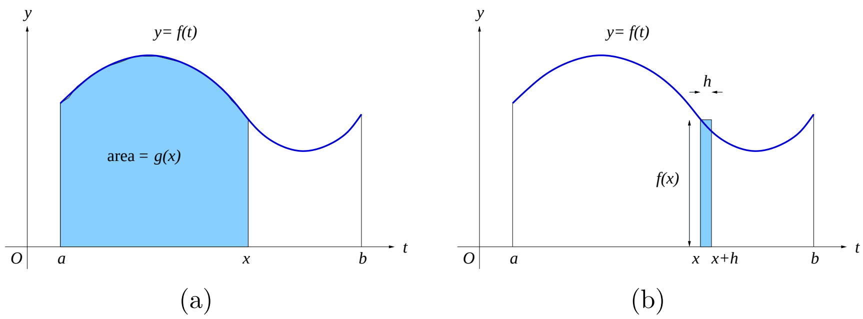

Figure 22.4: Illustration of $g(x) = \int_a^x f(t)dt$.

To compute $g'(x)$ using the definition of a derivative we first observe that, for $h > 0$,

$$g(x+h) - g(x)$$

is obtained by subtracting areas, so it is the area under the graph of f from x to $x+h$ (the blue area, as shown in Figure 22.4). For small h you can see that this area is approximately equal to the area of the rectangle with height $f(x)$ and width h :

$$g(x+h) - g(x) \approx hf(x).$$

Then[1]

$$\frac{g(x+h)-g(x)}{h} \approx f(x).$$

Intuitively, we therefore expect that

$$g'(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = f(x).$$

The fact that this is true, even when f is not necessarily positive, is the first part of the Fundamental Theorem of Calculus (Exercise).

Theorem 1 (Part I of The Fundamental Theorem Of Calculus): If f is continuous on $[a,b]$, then the function g defined by

$$g(x) = \int_a^x f(t)dt, \quad a \le x \le b$$

is an antiderivative of f, that is

$$\frac{d}{dx}\left(\int_a^x f(t)dt\right) = g'(x) = f(x) \quad \text{for } a < x < b.$$

Proof. If x and $x+h$ are in the open interval (a,b), then using properties of the definite integral

$$\begin{aligned} g(x+h)-g(x) &= \int_a^{x+h} f(t)dt - \int_a^x f(t)dt \\ &= \left(\int_a^x f(t)dt + \int_x^{x+h} f(t)dt\right) - \int_a^x f(t)dt \\ &= \int_x^{x+h} f(t)dt \end{aligned}$$

and so, for $h \neq 0$, we have

$$\frac{g(x+h)-g(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)dt. \tag{22.1}$$

For now let us assume that $h > 0$. Since f is continuous on $[x, x+h]$, the Extreme Value Theorem says that there are numbers u and v in $[x, x+h]$ such that $f(u) = m$ and

[1] In Figure 22.4, this is the difference between two partially overlapping ranges of integration. Hence, the result must be only over the range that is not common to both, that is, x to $x+h$.

$f(v) = M$, where m and M are the absolute minimum and maximum values of f on $[x, x+h]$, respectively. By Proposition 8 of integrals in Chapter 21, we have

$$mh \leq \int_x^{x+h} f(t)dt \leq Mh.$$

Then

$$f(u)h \leq \int_x^{x+h} f(t)dt \leq f(v)h.$$

Since $h > 0$, we can divide this inequality by h:

$$f(u) \leq \frac{1}{h} \int_x^{x+h} f(t)dt \leq f(v).$$

Now we use (22.1) to replace the middle part of this inequality:

$$f(u) \leq \frac{g(x+h) - g(x)}{h} \leq f(v). \tag{22.2}$$

Inequality (22.2) can be proved in a similar manner for the case $h < 0$. Now let $h \to 0$. Then $u \to x$ and $v \to x$, since u and v lie between x and $x + h$. Thus

$$\lim_{h \to 0} f(u) = \lim_{u \to x} f(u) = f(x)$$

and

$$\lim_{h \to 0} f(v) = \lim_{v \to x} f(v) = f(x)$$

because f is continuous at x. We conclude, from (22.2) and the squeeze theorem, that

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)dt = f(x). \tag{22.3}$$

If $x = a$ or b, then (22.3) can be interpreted as a one-side-limit. We know that if f is differentiable at a, then f is continuous at a. If we adopt this theorem for one-sided limits, we find that g is continuous on $[a, b]$.

Note 2 Every continuous function has an antiderivative, even those non-differentiable functions with "corners" such as absolute value.

Example 3 Let $A(x) = \int_0^x f(t)dt$ for the function $f(x)$ shown in Figure 22.5. Evaluate $A(x)$ and $A'(x)$ for $x = 1, 2, 3$, and 4.

Solution. We have

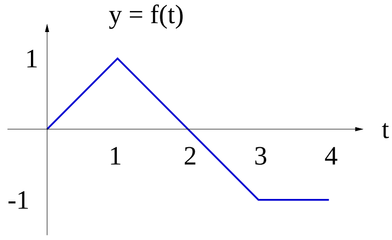

Figure 22.5: Example 3.

- $A(1) = \int_0^1 f(t)dt = \frac{1}{2}$,
- $A(2) = \int_0^2 f(t)dt = 1$,
- $A(3) = \int_0^3 f(t)dt = \frac{1}{2}$, and
- $A(4) = \int_0^4 f(t)dt = -\frac{1}{2}$.

Since $f(x)$ is continuous, $A'(x) = f(x)$. Thus

- $A'(1) = f(1) = 1$,
- $A'(2) = f(2) = 0$,
- $A'(3) = f(3) = -1$, and
- $A'(4) = f(4) = -1$.

Example 4 Let $A(x) = \int_0^x f(t)dt$ for the function $f(x)$ shown in Figure 22.6. For which value of x is $A(x)$ maximum? For which x is the rate of change of A maximum?

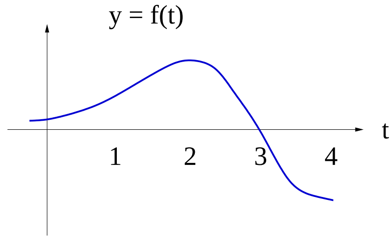

Figure 22.6: Example 4.

Solution. We have

- Since A is differentiable, the only critical points are where $A'(x) = 0$ or at endpoints. Then, $A'(x) = f(x) = 0$ at $x = 3$, and A has a maximum at $x = 3$.
- Notice that the values of $A(x)$ increase as $x$goes from 0 to 3 and then the A values decrease.
- The rate of change of $A(x)$ is $A'(x) = f(x)$, and $f(x)$ appears to have a maximum at $x = 2$ so the rate of change of $A(x)$ is maximum when $x = 2$.
- When x is close to 2, a slight increase in the value of x yields the maximum increase in the value of $A(x)$.

Example 5 Find the derivative of the function

$$g(x) = \int_3^x t^4 dt.$$

Solution. Since $f(t) = t^4$ is continuous, Part I of the Fundamental Theorem of Calculus gives

$$\frac{d}{dx} g(x) = x^4.$$

□

Example 6 Find the derivative of the function

$$g(x) = \int_{-1}^x e^{t^2} dt.$$

Solution. Since $f(t) = e^{t^2}$ is continuous, Part I of the Fundamental Theorem of Calculus gives

$$\frac{d}{dx} g(x) = e^{x^2}.$$

□

22.2.1 Leibniz' rule for differentiating integrals

If the endpoint of an integral is a function of x rather than simply x, then we need to use the Chain Rule together with Part I of the Fundamental Theorem of Calculus to calculate the derivative of the integral. According to the Chain Rule, if

$$\frac{d}{dx} g(x) = f(x),$$

then

$$\frac{d}{dx} g(x^3) = f(x^3) \cdot 3x^2,$$

and, applying the Chain Rule to the derivative of the integral,

$$\frac{d}{dx}\left(\int_a^{h(x)} f(t)dt\right) = \frac{d}{dx}g(h(x)) = f(h(x)) \cdot h'(x).$$

Theorem 2 If $f(x)$ is continuous and $g(x) = \int_a^x f(t)dt$, then

$$\frac{d}{dx}\left(\int_a^x f(t)dt\right) = g'(x) = f(x) \qquad \text{(Part I of Fundamental Theorem of Calculus)}$$

and, if $h(x)$ is differentiable,

$$\frac{d}{dx}\left(\int_a^{h(x)} f(t)dt\right) = \frac{d}{dx}g(h(x)) = f(h(x)) \cdot h'(x) \qquad \text{(Leibniz' rule).}$$

□

Example 7 Find the derivative of the function

$$g(x) = \int_{-1}^{x^2} e^{t^2} dt.$$

Solution. Since $f(t) = e^{t^2}$ is continuous and x^2 is differentiable, Leibniz' rule and Part I of the Fundamental Theorem of Calculus give

$$\begin{aligned}\frac{d}{dx}\int_{-1}^{x^2} e^{t^2} dt &= \frac{d}{dx}\int_{-1}^{u} e^{t^2} dt \\ &= \frac{d}{du}\left(\int_{-1}^{u} e^{t^2} dt\right)\frac{du}{dx} \\ &= e^{u^2}\frac{du}{dx} \\ &= e^{\left(x^2\right)^2} \cdot 2x \\ &= 2xe^{x^4}.\end{aligned}$$

□

Example 8 Find the derivative of the function

$$g(x) = \int_{2}^{\sin x} \sqrt[3]{t}dt.$$

Solution. Since $f(t) = \sqrt[3]{t}$ is continuous and $\sin x$ is differentiable, Leibniz' rule and Part I of the Fundamental Theorem of Calculus give

$$\begin{aligned}
\int_2^{\sin x} \sqrt[3]{t}dt &= \frac{d}{dx}\int_2^u \sqrt[3]{t}dt \\
&= \frac{d}{du}\left(\int_2^u \sqrt[3]{t}dt\right)\frac{du}{dx} \\
&= \sqrt[3]{u}\frac{du}{dx} \\
&= \sqrt[3]{\sin x} \cdot \cos x.
\end{aligned}$$

□

Example 9 Find the derivative of the function

$$g(x) = \int_{x^4}^{1} \sec t dt.$$

Solution. Since $f(t) = \sec t$ is continuous and x^4 is differentiable, Leibniz' rule and Part I of the Fundamental Theorem of Calculus give

$$\begin{aligned}
\int_{x^4}^{1} \sec t dt &= -\frac{d}{dx}\int_1^{x^4} \sec t dt \\
&= -\frac{d}{dx}\int_1^u \sec t dt \\
&= -\frac{d}{du}\left(\int_1^u \sec t dt\right)\frac{du}{dx} \\
&= -\sec u\frac{du}{dx} \\
&= -\sec\left(x^4\right) \cdot 4x^4.
\end{aligned}$$

□

Example 10 Find the derivative of the function

$$g(x) = \int_{\sin x}^{\cos x} t^2 dt.$$

Solution. We have

$$\int_{\sin x}^{\cos x} t^2 dt = \int_{\sin x}^{0} t^2 dt + \int_{0}^{\cos x} t^2 dt = -\int_{0}^{\sin x} t^2 dt + \int_{0}^{\cos x} t^2 dt.$$

Since $f(t) = t^2$ is continuous, and $\cos x$ and $\sin x$ are differentiable, Leibniz' rule and Part I of the Fundamental Theorem of Calculus give

$$\begin{aligned}
\frac{d}{dx}\int_{0}^{\sin x} t^2 dt &= \frac{d}{dx}\int_{0}^{u} t^2 dt \\
&= \frac{d}{du}\left(\int_{0}^{u} t^2 dt\right)\frac{du}{dx} \\
&= u^2\frac{du}{dx} \\
&= \sin^2 x \cdot \cos x
\end{aligned}$$

and

$$\begin{aligned}
\frac{d}{dx}\int_{0}^{\cos x} t^2 dt &= \frac{d}{dx}\int_{0}^{u} t^2 dt \\
&= \frac{d}{du}\left(\int_{0}^{u} t^2 dt\right)\frac{du}{dx} \\
&= u^2\frac{du}{dx} \\
&= \cos^2 x \cdot (-\sin x).
\end{aligned}$$

Therefore

$$\frac{d}{dx}\int_{\sin x}^{\cos x} t^2 dt = -\sin^2 x \cdot \cos x - \cos^2 x \cdot \sin x.$$

$\square$

22.3 Part II of The Fundamental Theorem Of Calculus

If we know and can evaluate some antiderivative of a function, then we can evaluate any definite integral of that function.

Theorem 3 (Part II of The Fundamental Theorem Of Calculus): If f is continuous on $[a, b]$, then

$$\int_a^b f(x)dx = F(x)|_a^b = F(b) - F(a),$$

where F is any antiderivative of f, that is $F'(x) = f(x)$.

Proof. Let

$$g(x) = \int_a^x f(t)dt.$$

By Part I of the Fundamental Theorem Of Calculus, $g(x)$ is an antiderivative of $f(x)$. Therefore any other antiderivative $F(x)$ of $f(x)$ can be written as

$$F(x) = g(x) + C = \int_a^x f(t)dt + C.$$

It follows that

$$F(a) = \int_a^a f(t)dt + C = 0 + C = C.$$

Then

$$F(b) = \int_a^b f(t)dt + C = \int_a^b f(t)dt + F(a).$$

Therefore

$$F(b) - F(a) = \int_a^b f(t)dt.$$

□

Note 3 The definite integral of a continuous function $f(x)$ can be found by finding an antiderivative of $f(x)$ (any antiderivative of $f(x)$ will work) and then doing some arithmetic with this antiderivative. Theorem 3 does not tell us how to find an antiderivative of $f(x)$, and it does not tell us how to find the definite integral of a discontinuous function. It is possible to evaluate definite integrals of some discontinuous functions, but the Fundamental Theorem of Calculus can not be used to do so.

Example 11 Evaluate $\displaystyle\int_0^2 \left(t^2 - 1\right) dt$.

Solution. Clearly, $F(t) = \frac{t^3}{3} - t$ is an antiderivative of $f(t) = t^2 - 1$. Then

$$\int_0^2 \left(t^2 - 1\right) dt = \left(\frac{t^3}{3} - t\right)\Big|_0^2$$
$$= \left(\frac{2^3}{3} - 2\right) - \left(\frac{0^3}{3} - 0\right)$$
$$= \frac{2}{3} - 0$$
$$= \frac{2}{3}.$$

If one had picked a different antiderivative of $t^2 - 1$, say, $F(t) = \frac{t^3}{3} - t + 5$, then the calculations would be slightly different but the result would be the same:

$$\int_0^2 \left(t^2 - 1\right) dt = \left(\frac{t^3}{3} - t + 5\right)\Big|_0^2$$
$$= \left(\frac{2^3}{3} - 2 + 5\right) - \left(\frac{0^3}{3} - 0 + 5\right)$$
$$= \frac{2}{3} - 0$$
$$= \frac{2}{3}.$$

□

Note 4 In general, "the derivative of the integral is the original function" and "the integral of the derivative of a function is the original function" do not commute with each other. The reason for this simple. In Example 11, two functions $F(t)$ differ by a constant. Since the derivative of a constant is zero, their derivatives are the same. Therefore, given only the derivative, we cannot know which one was the original function. We know the integral only to within an additive constant; there is an arbitrary additive constant to be solved onto the integration process when we think of integration as the antiderivative.

Example 12 Evaluate $\int_{1.5}^{2.7} [x] dx$, where $[x]$ is the largest integer less than or equal to x, as shown in Figure 22.7.

Solution. The given function $f(x) = [x]$ is not continuous at $x = 2$ in the interval $[1.5, 2.7]$ so the Fundamental Theorem of Calculus can not be used. We can, however, use

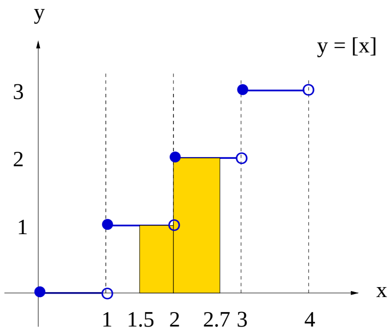

Figure 22.7: Example 12.

our understanding of the meaning of an integral to get

$$\begin{aligned}\int_{1.5}^{2.7} [x]dx &= (\text{area for } x \text{ between } 1.5 \text{ and } 2) + (\text{area for } x \text{ between } 2 \text{ and } 2.7)\\ &= (\text{base}) \cdot (\text{height}) + (\text{base}) \cdot (\text{height})\\ &= (.5) \cdot (1) + (.7) \cdot (2) = 1.9.\end{aligned}$$

$\square$