THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1510 Calculus for Engineers (2020-2021)
Solution to Supplementary Exercise 8

Fundamental Theorem of Calculus

1. Find $\dfrac{d}{dx}\displaystyle\int_3^x \cos(e^t - e^{-t})\,dt$ by using the fundamental theorem of calculus.

   Ans: $\cos(e^x - e^{-x})$

2. Find $\dfrac{dy}{dx}$ if

   (a) $y = \displaystyle\int_1^x \cos(t^2)\,dt$

   Ans: $\cos(x^2)$

   (b) $y = \displaystyle\int_1^{x^2+1} \cos(t^2)\,dt$

   Ans: $2x\cos\left((x^2+1)^2\right)$

   (c) $y = \displaystyle\int_x^{x^2+1} \cos(t^2)\,dt$

   (Hint: $\displaystyle\int_x^{x^2+1} \cos(t^2)\,dt = \int_1^{x^2+1} \cos(t^2)\,dt - \int_1^x \cos(t^2)\,dt.$)

   Ans: $2x\cos\left((x^2+1)^2\right) - \cos(x^2)$

3. Find $\displaystyle\lim_{h\to 0}\frac{1}{h^3}\int_0^h \sin(t^2)\,dt$.

   (Hint: Using L'Hôpital rule.)

   Ans: $\dfrac{1}{3}$

Area Bounded by Graphs

4. Consider the curves

$$\begin{aligned} C_1 :\quad & y = 1 - x^2 \quad \text{with } x \in \mathbb{R} \\ C_2 :\quad & y = x + 1 \quad \text{with } x \in \mathbb{R} \end{aligned}$$

   (a) Find the intersection(s) of C_1 and C_2.

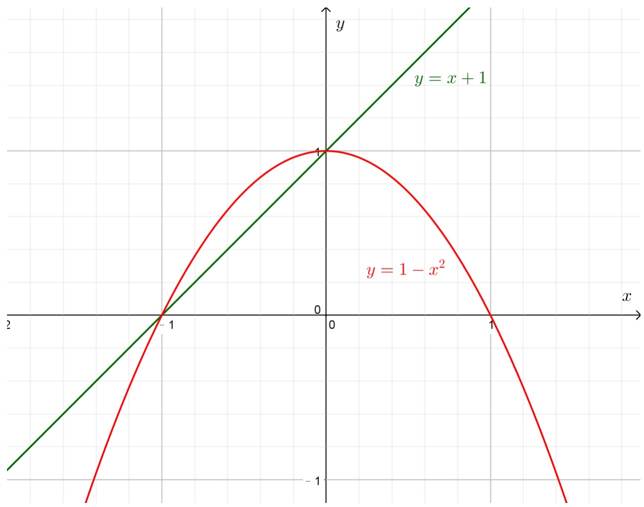
   (b) Sketch the graphs of C_1 and C_2. Make sure to include their intersection(s) in your graphs.

   (c) Find the area of the region bounded by C_1 and C_2.

   Ans:

(a) $(-1,0)$ and $(0,1)$ are intersection points of C_1 and C_2.

(b)



(c) Area $= \displaystyle\int_{-1}^{0} [(1-x^2)-(x+1)]\,dx = \int_{-1}^{0} -x - x^2\,dx = \left[-\frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{0} = \frac{1}{6}$

5. Express the area of the region bounded by the curves:

$$y = \frac{2}{x+1}$$

$$y = 2 - x^2$$

as a definite integral (or a sum of definite integrals) along:

(a) the x-axis;

Ans: Area $= \displaystyle\int_0^1 (2-x^2) - \frac{2}{x+1}\,dx$

(b) the y-axis.

Ans: Area $= \displaystyle\int_1^2 \sqrt{2-y} - \left(\frac{2}{y} - 1\right)\,dy$

6. Find the area enclosed by the curve $y = x^2$, the x-axis and the line $x = 2$.

Ans: Area $= \displaystyle\int_0^2 x^2\,dx = \frac{8}{3}$

7. Find the area enclosed by the curve $y = \sin x$ for $0 \le x \le \pi$ and the x-axis.

Ans: Area $= \displaystyle\int_0^{\pi} \sin x\,dx = 2$

8. Find the area enclosed by the curves $y = x$ and $y = x^2$.

Ans: Area $= \displaystyle\int_0^1 x - x^2\,dx = \frac{1}{6}$

9. Find the area enclosed by the curves $y = 2^x$, $y = 1 - x$ and $y = 4x - 4$.

Ans: Area $= \displaystyle\int_0^1 2^x - (1-x)\,dx + \int_1^2 2^x - (4x-4)\,dx = \left(\frac{1}{\ln 2} - \frac{1}{2}\right) + \left(\frac{2}{\ln 2} - 2\right) = \frac{3}{\ln 2} - \frac{5}{2}$

10. Let $f(x) = |x|$. Recall that $f(x)$ can be expressed as

$$f(x) = \begin{cases} x & \text{if } \ x \geq 0, \\ -x & \text{if } \ x < 0. \end{cases}$$

By writing $\displaystyle\int_{-3}^{2} |x|\,dx = \int_{-3}^{0} |x|\,dx + \int_{0}^{2} |x|\,dx = \int_{-3}^{0} -x\,dx + \int_{0}^{2} x\,dx$,

evaluate $\displaystyle\int_{-3}^{2} |x|\,dx$.

Ans: $\dfrac{13}{2}$

11. (a) Solve $x^2 - 5x + 6 > 0$.

(b) Let $f(x) = |x^2 - 5x + 6|$. Then, $f(x)$ can be expressed as

$$f(x) = \begin{cases} \_\_\_\_\_\_ & \text{if } \quad x > \_\_\_\_\_\_, \\ \_\_\_\_\_\_ & \text{if } \ \_\_\_\_ \leq x \leq \_\_\_\_, \\ \_\_\_\_\_\_ & \text{if } \quad x < \_\_\_\_. \end{cases}$$

(c) Evaluate $\displaystyle\int_0^5 |x^2 - 5x + 6|\,dx$.

Ans:

(a) $x < 2$ or $x > 3$

(b) $f(x)$ can be expressed as

$$f(x) = \begin{cases} \underline{\quad x^2 - 5x + 6 \quad} & \text{if } \quad x > \underline{\ 3\ }, \\ \underline{\ -(x^2 - 5x + 6) \ } & \text{if } \ \underline{\ 2\ } \leq x \leq \underline{\ 3\ }, \\ \underline{\quad x^2 - 5x + 6 \quad} & \text{if } \quad x < \underline{\ 2\ }. \end{cases}$$

(c)

$$\begin{aligned} &\int_0^5 |x^2 - 5x + 6|\,dx \\ &= \int_0^2 (x^2 - 5x + 6)\,dx + \int_2^3 -(x^2 - 5x + 6)\,dx + \int_3^5 (x^2 - 5x + 6)\,dx \\ &= \frac{14}{3} + \frac{1}{6} + \frac{14}{3} \\ &= \frac{19}{2}. \end{aligned}$$

12. Evaluate $\int_{-2}^{3} |2x - x^2|\, dx$.

    Ans: $\frac{28}{3}$

13. Find the area enclosed by the curve $y^2 = -x + 6$, the x-axis and the line $y = x$ for $x \geq 0$.

    Ans: $\int_0^2 x\, dx + \int_2^6 \sqrt{-x+6}\, dx = \frac{22}{3}$

14. Find the area bounded by the lower semi-circle defined by $x^2 + y^2 = 25$ $(y < 0)$ and the parabola $x^2 + 2y - 1 = 0$.

    Ans: $\int_{-3}^{3} \frac{1-x^2}{2} - (-\sqrt{25 - x^2})\, dx = 6 + 25 \sin^{-1}(\frac{3}{5}) \approx 22.088$

15. Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a, b > 0$.

    Ans: $2 \int_{-a}^{a} b\sqrt{1 - \frac{x^2}{a^2}}\, dx = ab\pi$

    (Remark: The integral is the area of the upper half of the ellipse.)