Lecture Note 27

Dr. Jeff Chak-Fu WONG

Department of Mathematics
Chinese University of Hong Kong
jwong@math.cuhk.edu.hk

MATH1510
Calculus for Engineering

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Volumes of Solids

This lecture emphasizes another geometrical use of integration, calculating volumes of solid three-dimensional objects such as those shown in Figure 1.

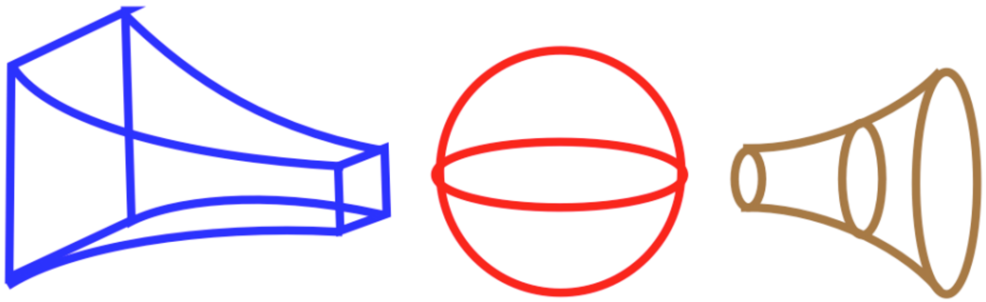


Figure 1: Volumes of solid three-dimensional objects.

Our basic approach is to cut the whole solid into thin "slices" whose volumes

- can be approximated,
- add the volumes of these "slices" together (a Riemann sum), and
- finally obtain an exact answer by taking a limit of the sums to get a definite integral.

THE BUILDING BLOCKS: RIGHT SOLIDS

A right solid is a three-dimensional shape swept out by moving a planar region A some distance h along a line perpendicular to the plane of A (Figure 2).

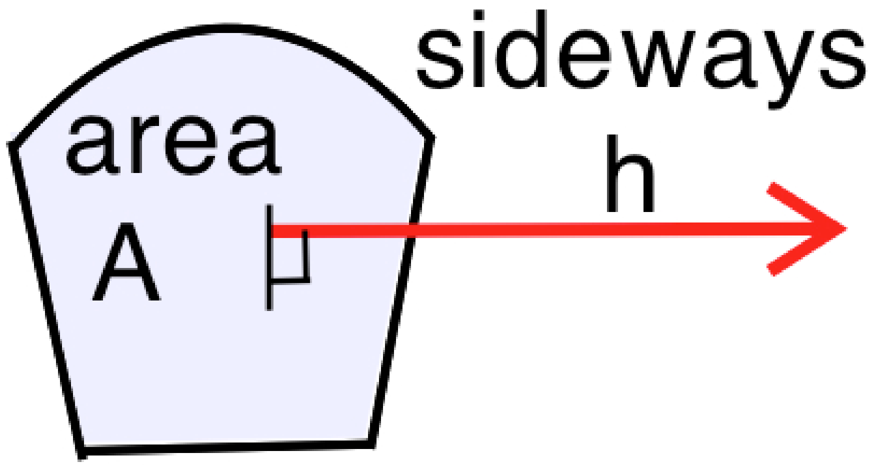


Figure 2:

The region A is called a face of the solid, and the word "right" is used to indicate that the movement is along a line perpendicular, at a right angle, to the plane of A.

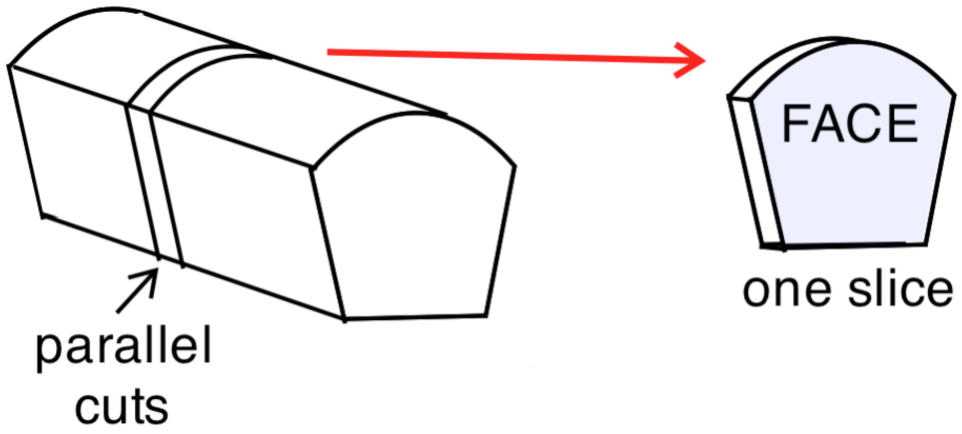


Figure 3:

Two parallel cuts produce one slice with two faces (Figure 3): a slice has volume, and a face has area.

Example 1 If A is a rectangle (Figure 4), then the “right solid” formed by moving A along the line is a $3-$dimensional solid box B.

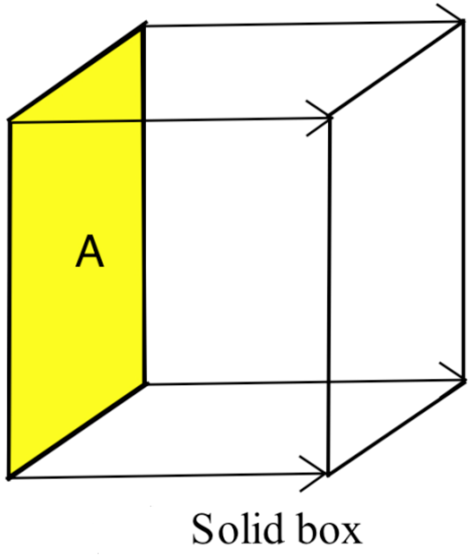


Figure 4: Solid box.

The volume of B is

$$(\text{area of } A) \cdot (\text{distance along the line}) = (\text{base}) \cdot (\text{height}) \cdot (\text{width}).$$

Example 2 If A is a circle with radius r meters (Figure 5), then the "right solid" formed by moving A along the line h meters is a right circular cylinder with volume equal to

$$\begin{aligned}(\text{area of } A) \cdot (\text{distance along the line}) &= (\pi(r \text{ ft})^2) \cdot (h \text{ ht}) \\ &= (\pi r^2 \text{ ft}^2) \cdot (h \text{ ht}) \\ &= \pi r^2 h \text{ ft}^3.\end{aligned}$$

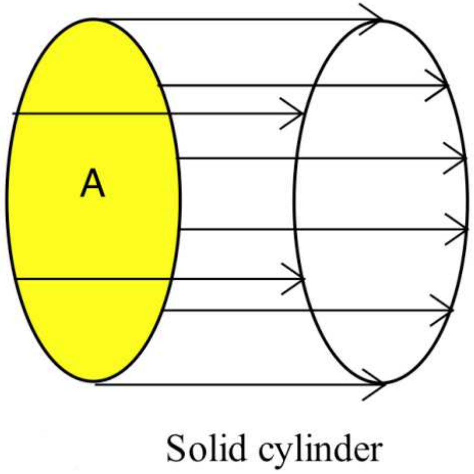


Figure 5: Circular cylinder.

If we cut a right solid perpendicular to its axis (like cutting a loaf of bread), then each face (cross section) has the same two-dimensional shape and area.

Figure 6: Loafbread

In general, if a $3-$dimensional right solid B is formed by moving a $2-$dimensional shape A along a line perpendicular to A, then the volume of B is defined to be

$$\text{volume of } B = (\text{area of } A) \cdot (\text{distance moved along the line perpendicular to } A).$$

Example 3 Calculate the volumes of the right solids in Figure 7.

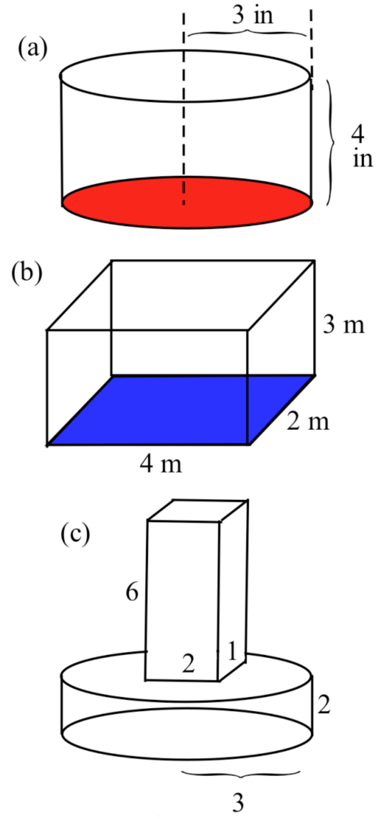


Figure 7:

1. This cylinder is formed by moving the circular base

$$\text{Area} = \pi r^2 = 9\pi \text{ in}^2$$

along a line perpendicular to the base for 4 inches, so the volume is

$$(9\pi \text{ in}^2) \cdot (4 \text{ in}) = 36\pi \text{ in}^3.$$

2.

$$\text{volume} = (\text{base area}) \cdot (\text{distance along the line}) = (8 \text{ m}^2) \cdot (3 \text{ m}) = 24 \text{ m}^3$$

3. This shape is composed to two easy right solids with volumes

$$V_1 = (\pi 3^2) \cdot (2) = 18\pi \text{ cm}^3$$

and

$$V_1 = (6)(1) \cdot (2) = 12\pi \text{ cm}^3,$$

so the total volume is $(18\pi + 12) \text{ cm}^3$ or approximately 68.5 cm^3.

Volumes of General Solids

- A general solid can be cut into slices which are almost right solids.
- An individual slice may not be exactly a right solid since its cross sections may have different areas.
- However, if the cuts are close together, then the cross sectional areas will not change much within a single slice.
- Each slice will be almost a right solid, and its volume will be almost the volume of a right solid.

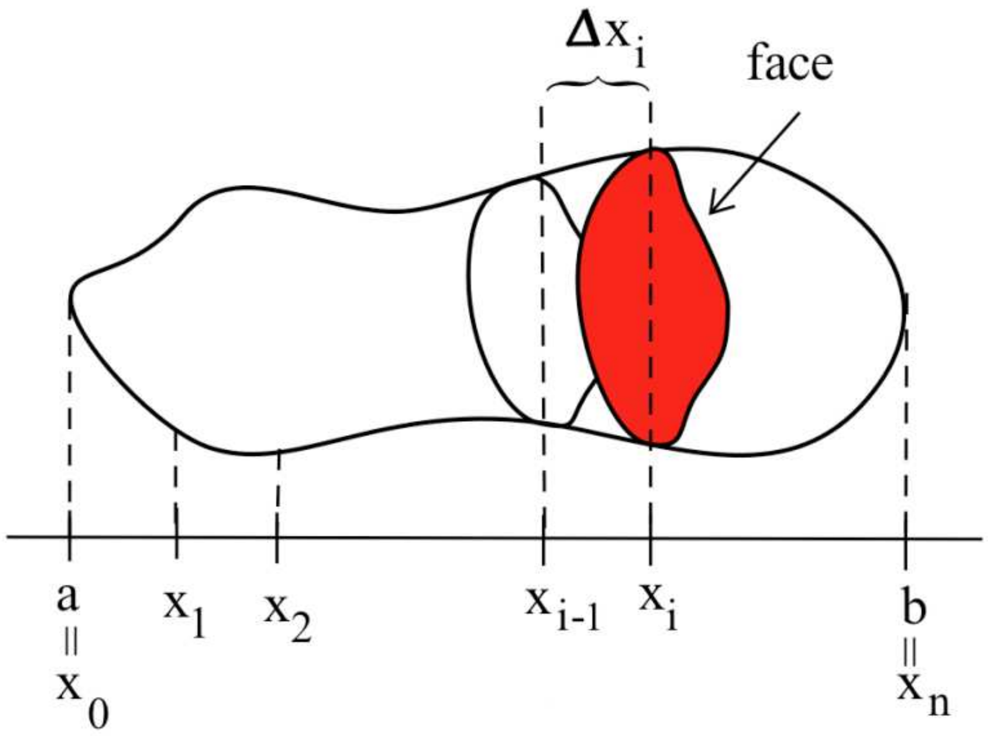
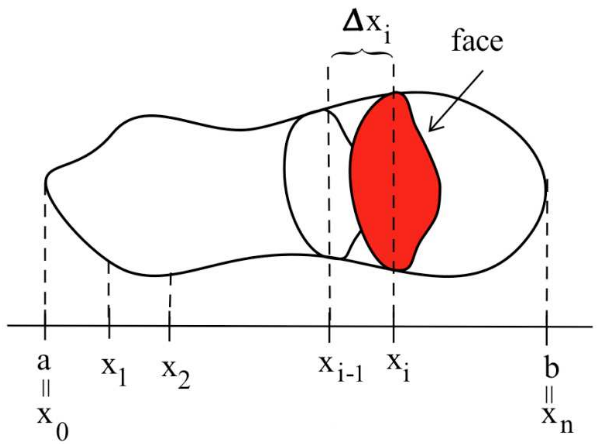


Figure 8:

Suppose an $x-$axis is positioned below the solid shape (Figure 8), and let $A(x)$ be the area of the face formed when the solid is cut at x perpendicular to the $x-$axis.



If

$$P = \{x_0 = a, x_1, x_2, \cdots, x_n = b\}$$

is a partition of $[a, b]$, and the solid is cut at each x_i, then each slice of the solid is almost a right solid, and the volume of each slice is approximately

$$\text{(area of a face of the slice)} \cdot \text{(thickness of the slice)} \approx A(x_i)\Delta x_i.$$

The total volume V of the solid is approximately the sum of the volumes of the slices:

$$V = \sum \{\text{volume of each slice}\} \approx \sum A(x_i)\Delta x_i$$

which is a Riemann sum.

The limit, as the mesh of the partition approaches 0 (taking thinner and thinner slices), of the Riemann sum is the definite integral of $A(x)$:

$$V \approx \sum A(x_i)\Delta x_i \longrightarrow \int_a^b A(x)dx.$$

Volume By Slices Formula

Theorem 1 If S is a solid and $A(x)$ is the area of the face formed by a cut at x and perpendicular to the $x-$axis, then the volume V of the part of S above the interval $[a, b]$ is

$$V = \int_a^b A(x)dx.$$

If S is a solid (Figure 9), and $A(y)$ is the area of a face formed by a cut at y perpendicular to the $y-$axis, then the volume of a slice with thickness Δy_i is approximately $A(y_i) \cdot \Delta y_i$.

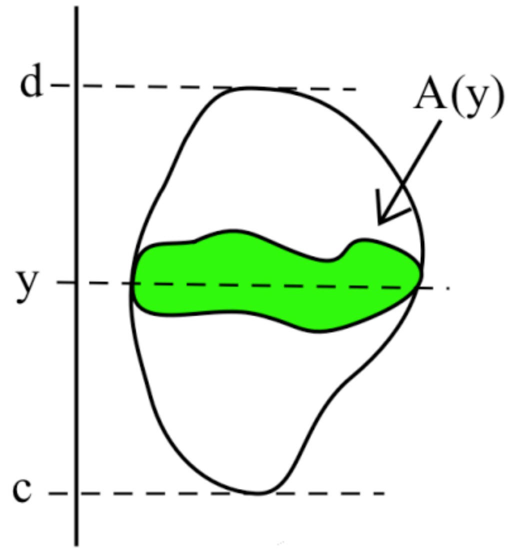


Figure 9:

The volume of the part of S between cuts at c and d on the $y-$axis is

$$V = \int_c^d A(y)dy.$$

Example 4 For the solid in Figure 10, the face formed by a cut at x is a rectangle with a base of 2 inches and a height of $\cos(x)$ inches.

1. Write a formula for the approximate volume of the slice between x_{i-1} and x_i.
2. Write and evaluate an integral for the volume of the solid for x between 0 and $\pi/2$.

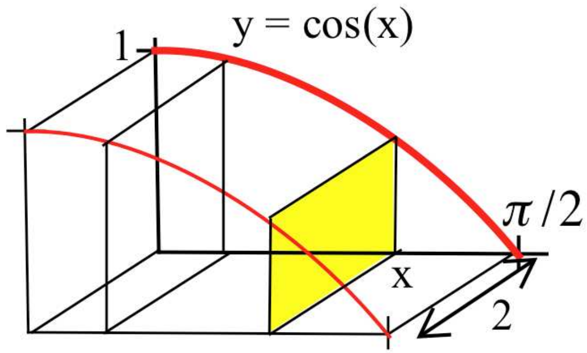
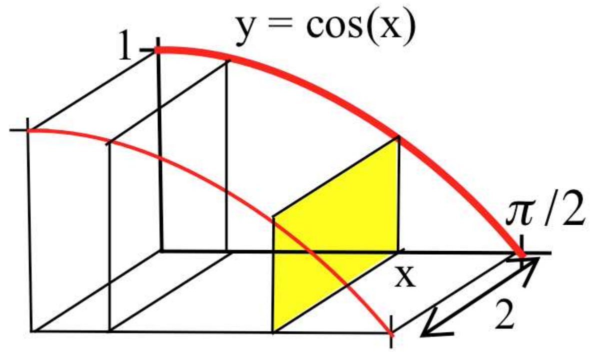


Figure 10:

Solution

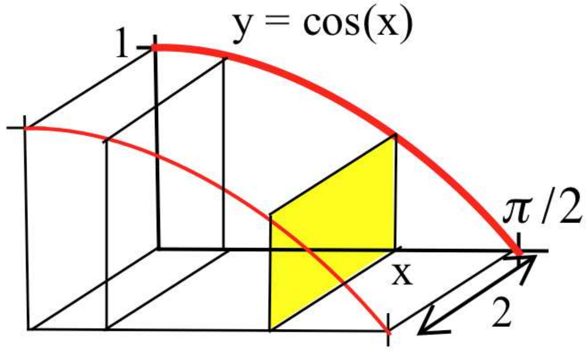
1. Write a formula for the approximate volume of the slice between x_{i-1} and x_i.

$$\begin{aligned}\text{The volume of the slice} &\approx (\text{area of the face})\cdot(\text{thickness})\\ &= (\text{base})\cdot(\text{height})\cdot(\text{thickness})\\ &= (2\text{ in})\cdot(\cos(x_i)\text{ in})\cdot(\Delta x_i\text{ in})\\ &= 2\cos(x_i)\Delta x_i\text{ in}^3\end{aligned}$$



2. Write and evaluate an integral for the volume of the solid for x between 0 and $\pi/2$.

$$
\begin{aligned}
\text{Volume} &= \int_a^b A(x)dx \\
&= \int_0^{\pi/2} 2\cos x dx \\
&= 2\ \sin(x)|_0^{\pi/2} \\
&= 2\sin(\pi/2) - 2\sin(0) = 2\ \text{in}^3.
\end{aligned}
$$



Example 5 For the solid in Figure 11, each face formed by a cut at x is a circle with diameter $\sqrt{x}$.

1. Write a formula for the approximate volume of the slice between x_{i-1} and x_i.

2. Write and evaluate an integral for the volume of the solid for x between 1 and 4.

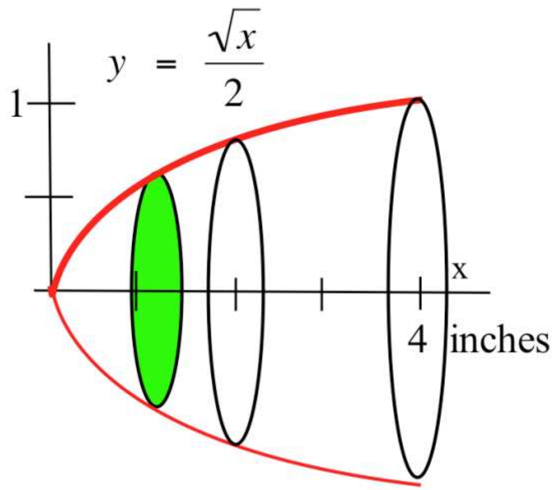


Figure 11:

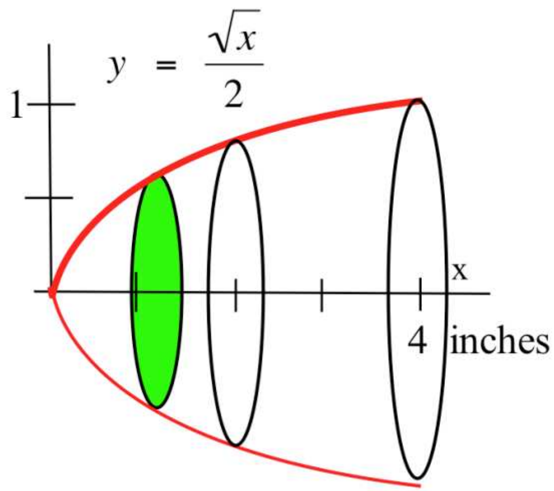
Solution

1. Each face is a circle with diameter $\sqrt{x_i}$, and the area of the circle is

$$A(x_i) = \pi(\text{radius})^2 = \pi(1/2\ \text{radius})^2 = \pi(1/2\ \sqrt{x_i})^2 = \pi x_i/4$$

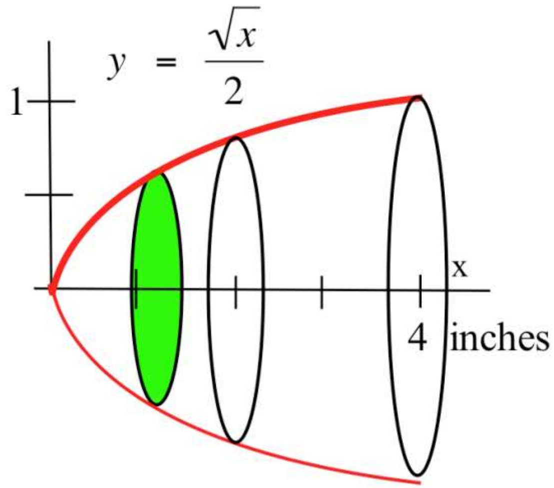
The volume of the slice

$$\approx (\text{area of the face}) \cdot (\text{thickness}) = (\pi x_i/4) \cdot (\Delta x_i).$$



2.

$$
\begin{aligned}
\text{Volume} &= \int_a^b A(x)dx \\
&= \int_1^4 \frac{\pi x}{4} dx \\
&= \frac{\pi}{4} \left. \frac{x^2}{2} \right|_1^4 \\
&= \frac{15\pi}{8} \approx 5.89 \text{ in}^3.
\end{aligned}
$$



Example 6 Find the volume of the square-based pyramid in Figure 12.

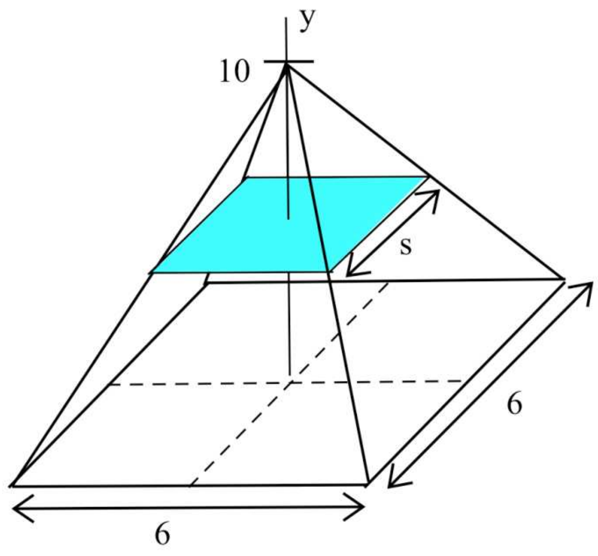


Figure 12:

Solution

Each cut perpendicular to the $y-$axis yields a square face, but in order to find the area of each square we need a formula for the length of one side s of the square as a function of y, the location of the cut.

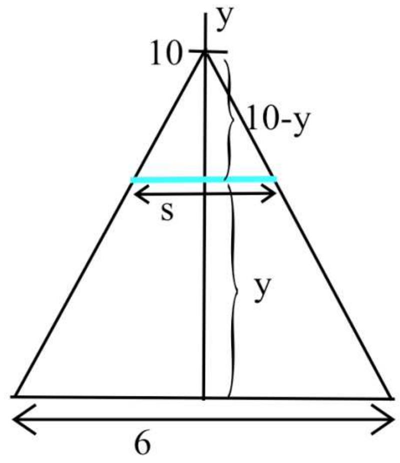


Figure 13:

Using similar triangles (Figure 13), we know that

$$\frac{s}{10-y} = \frac{6}{10}$$

so

$$s = \frac{6}{10}(10-y).$$

The rest of the solution is straightforward.

$$A(y) = (\text{side})^2 = \left(\frac{3}{5}(10-y)\right) = \frac{9}{25}(100 - 20y + y^2)$$

and

$$\begin{aligned}
\text{Volume} &= \int_0^{10} A(x)dx \\
&= \int_0^{10} \frac{9}{25}(100 - 20y + y^2)dx \\
&= \frac{9}{25}\left(100y - 10y^2 + \frac{y^3}{3}\right)\Big|_0^{10} \\
&= \frac{9}{25}\frac{1000}{3} \approx 120 \text{ ft}^3.
\end{aligned}$$

Example 7 A solid is built between the graphs of $f(x) = x + 1$ and $g(x) = x^2$ by building squares with heights (sides) equal to the vertical distance between the graphs of f and g (Figure 14).

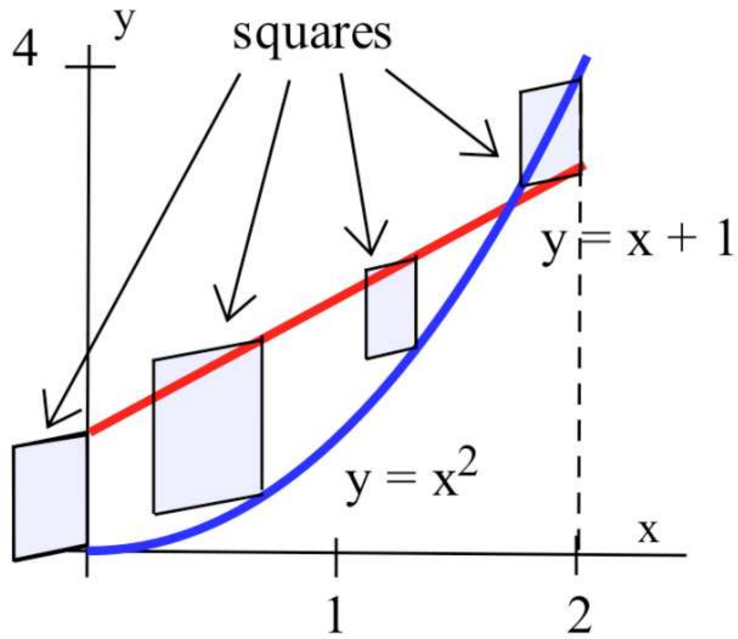
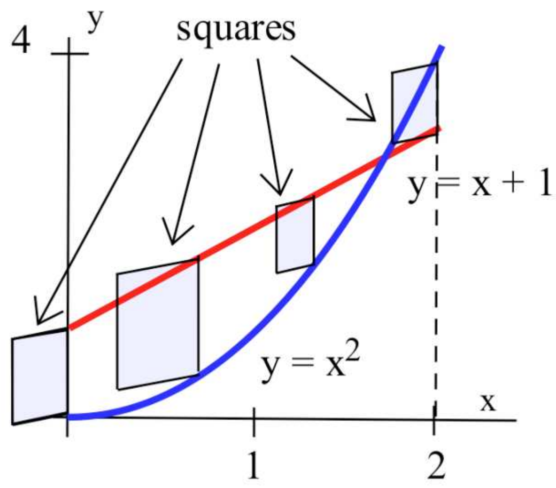


Figure 14:

Find the volume of this solid for $0 \leq x \leq 2$.



Solution

The area of a square face is

$$A(x) = (\text{side})^2,$$

and the length of a side is either $f(x) - g(x)$ or $g(x) - f(x)$, depending on which function is higher at x. Fortunately, the side is squared in the area formula so it does not matter which is taller, and

$$A(x) = (f(x) - g(x))^2.$$

Then

$$
\begin{aligned}
\text{Volume} &= \int_a^b A(x)dx \\
&= \int_0^2 (f(x) - g(x))^2 dx \\
&= \int_0^2 ((x+1) - x^2)^2 dx \\
&= \int_0^2 (1 + 2x - x^2 - 2x^3 + x^4) dx \\
&= \left(x + x^2 - \frac{x^3}{3} - \frac{x^4}{2} + \frac{x^5}{5} \right)\Bigg|_0^2 \\
&= \frac{26}{15}.
\end{aligned}
$$

We saw earlier that areas can have non-geometric interpretations such as distance and total accumulation. Similarly, volumes can have non-geometric interpretations.

If x represents an age in years, and $f(x)$ is the number of females in a population with age exactly equal to x, then the "area",

$$\int_a^b f(x)dx$$

is the total number of females with ages between a and b (Figure 15).

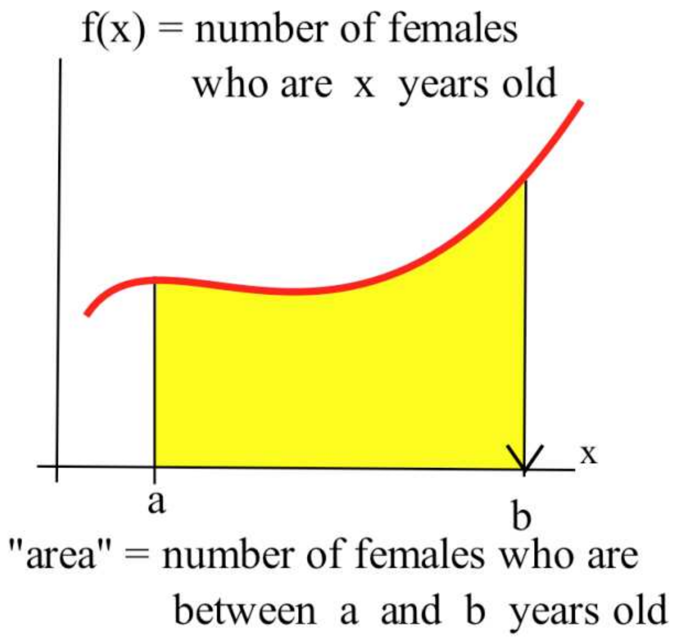


Figure 15:

If the birth rate for females of age x is $r(x)$, with units "births per female per year",

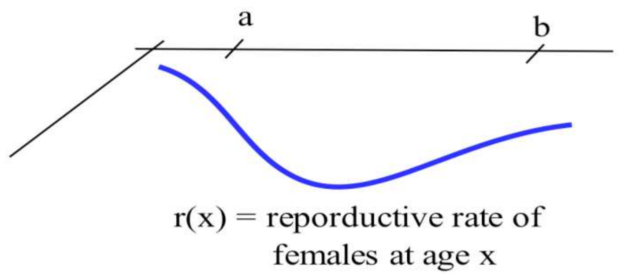


Figure 16:

then the "volume" of the solid in Figure 17 is

$$C = \int_a^b r(x)f(x)dx.$$

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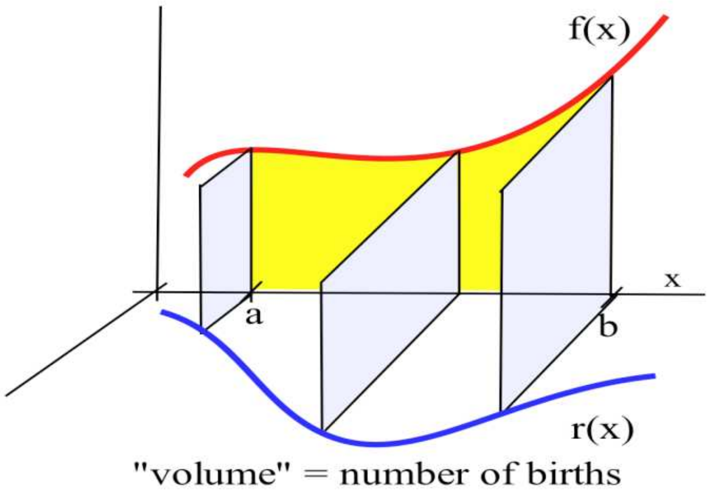


Figure 17:

C is the number of births during a year to females between the ages a and b, and the units of C will be “births.”

Volumes of Revolved Regions

When a region is revolved around a line (Figure 18) a right solid is formed. The face of each slice of the revolved region is a circle so the formula for the area of the face is easy:

$$A(x) = \text{area of a circle} = \pi(\text{radius})^2,$$

where the radius is often a function of the location x.

Finding a formula for the changing radius requires care.

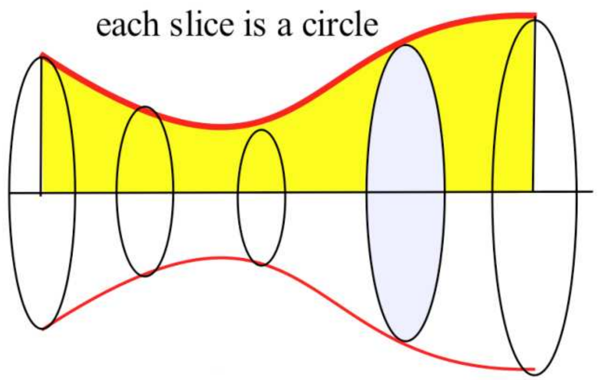


Figure 18:

Example 8 For $0 \leq x \leq 2$, the area between the graph of $f(x) = x^2$ and the horizontal line $y = 1$ is revolved about the horizontal line $y = 1$ to form a solid (Figure 19). Calculate the volume of the solid.

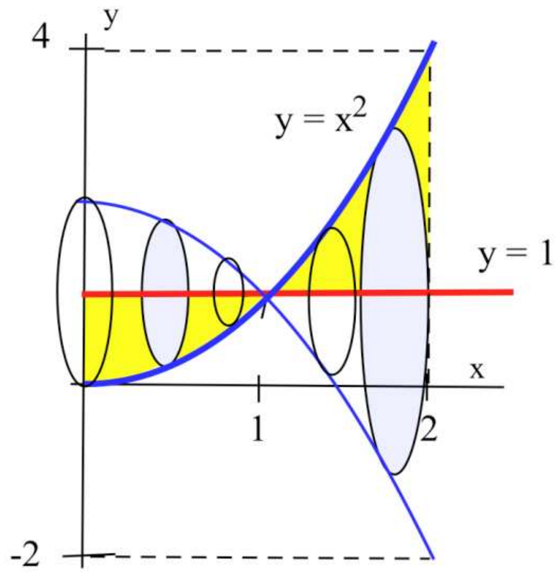
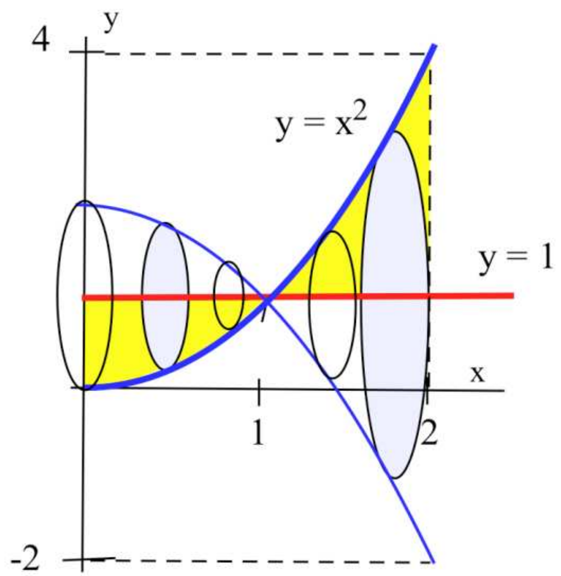


Figure 19:

Solution

The radius function is shown in the figure for several values of x.



If $0 \leq x \leq 1$, then $r(x) = 1 - x^2$, and if $1 \leq x \leq 2$ then $r(x) = x^2 - 1$.

Fortunately, however, $A(x) = \pi(r(x))^2$ always uses the square of $r(x)$ and the squares of $1 - x^2$ and $x^2 - 1$ are equal.

$$A(x) = \pi(r(x))^2 = \pi(x^2 - 1)^2 = \pi(x^4 - 2x^2 + 1)$$

and

$$V = \int_0^2 \pi(x^4 - 2x^2 + 1)dx = \pi \left(\frac{x^5}{5} - \frac{2}{3}x^3 + x \right) \Big|_0^2 = \frac{46}{15}\pi \approx 9.63.$$

Volumes of Revolved Regions ("Disks")

Theorem 2 If the region formed between f, the horizontal line $y = L$, and the interval $[a, b]$ is revolved about the horizontal line $y = L$, (Figure 20) then the volume is

$$V = \int_a^b A(x)dx = \int_a^b \pi(\text{radius})^2 dx = \int_a^b \pi(f(x) - L)^2 dx.$$

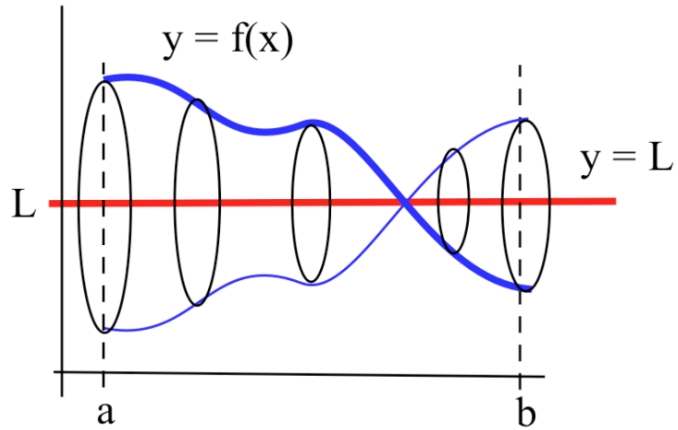


Figure 20:

This is called the “disk” method because the shape of each thin slice is a circular disk.

If the region between f and the $x-$axis $(L = 0)$ is revolved about the $x-$axis, then the previous formula reduces to

$$V = \int_a^b \pi(f(x))^2 dx.$$

Example 9 Find the volume generated when the region between one arch of the sine curve $(0 \le x \le \pi)$ and the $x-$axis is revolved about

1. the $x-$axis and
2. the line $y = 1/2$.

Solution

1.

$$\begin{aligned} V &= \int_a^b \pi(\text{radius})^2 dx = \int_0^\pi \pi(\sin x)^2 dx = \frac{\pi}{2}\int_0^\pi (1-\cos(2x))dx \\ &= \frac{\pi}{2}\left(x - \frac{\sin(2x)}{2}\right)\Big|_0^\pi = \frac{\pi}{2}(\pi - 0) = \frac{\pi^2}{2}. \end{aligned}$$

2.

$$\begin{aligned} V &= \int_a^b \pi(\text{radius})^2 dx = \int_0^\pi \pi(\sin x - 1/2)^2 dx = \pi\int_0^\pi \left(\sin^2(x) - \sin x + \frac{1}{4}\right) dx \\ &= \pi\left(\frac{\pi}{2} - 2 + \frac{\pi}{4}\right)\Big|_0^\pi \approx 1.12. \end{aligned}$$

Example 10 Given that

$$\int_1^5 f(x)dx = 4 \qquad \text{and} \qquad \int_1^5 (f(x))^2 dx = 7.$$

Represent the volumes of the solids 1., 2. and 3. in (Figure 21) as definite integrals and evaluate the integrals.

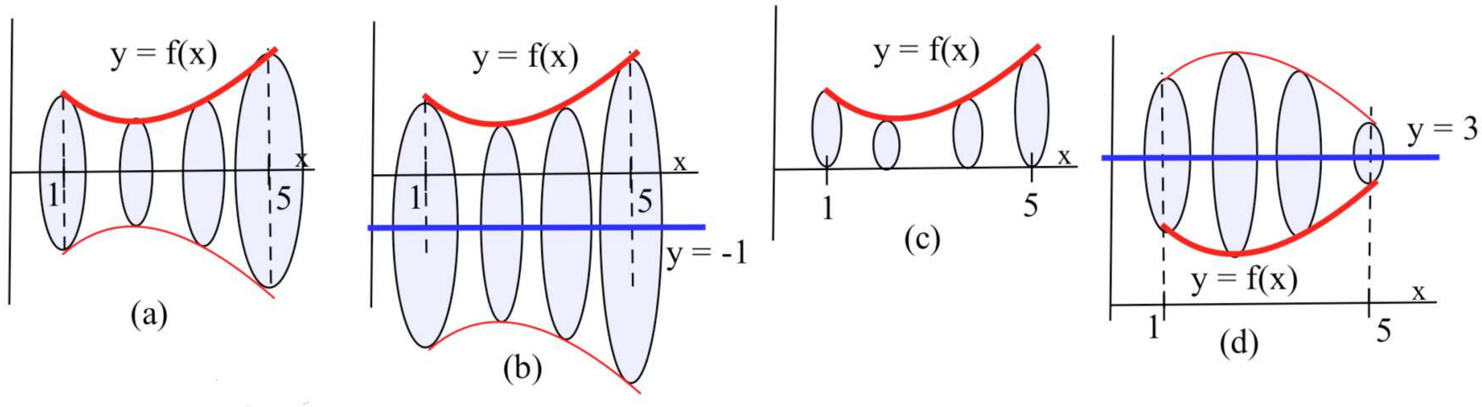


Figure 21:

Solution

1.

$$V = \int_1^5 \pi(\text{radius})^2 dx = \int_1^5 \pi(f(x))^2 dx = \pi \int_1^5 f^2(x) dx = 7\pi.$$

2.

$$V = \int_1^5 \pi(\text{radius})^2 dx = \int_1^5 \pi(f(x) - (-1))^2 dx = \pi \int_1^5 \left(f^2(x) + 2f(x) + 1\right) dx$$
$$= \pi \left(\int_1^5 f^2(x) dx + 2 \int_1^5 f(x) dx + \int_1^5 1 dx \right) = \pi(7 + 2 \cdot 4 + 4) = 19\pi.$$

3.

$$V = \int_1^5 \pi(\text{radius})^2 dx = \int_1^5 \pi(f(x)/2)^2 dx = \frac{\pi}{4} \int_1^5 f^2(x) dx = \frac{7\pi}{4}.$$

Solids With Holes

The previous ideas and techniques can also be used to find the volumes of solids with holes in them.

If $A(x)$ is the area of the face formed by a cut at x, then it is still true that the volume is

$$\int_a^b A(x)dx.$$

However, if the solid has holes, then some of the faces will also have holes and a formula for $A(x)$ may be more complicated.

Sometimes it is easier to work with two integrals and then subtract:

(i) calculate the volume S of the solid without the hole,

(ii) calculate the volume H of the hole, and

(iii) subtract H from S.

Example 11 Calculate the volume of the solid in Figure 22:

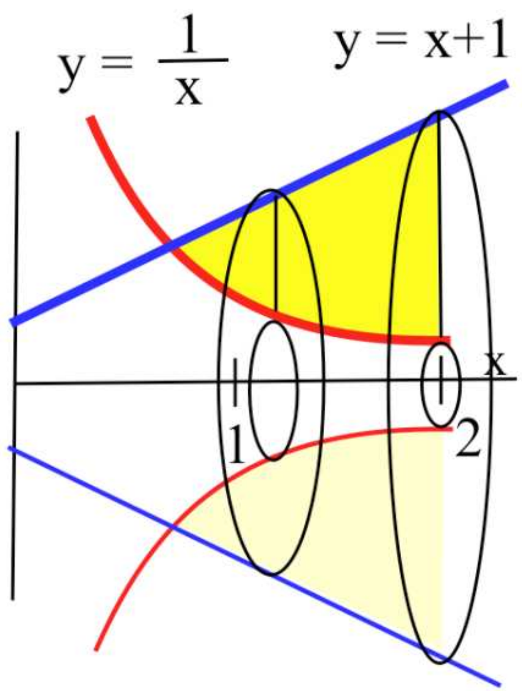


Figure 22:

Solution

The face for a slice at x, has area

$$\begin{aligned} A(x) &= (\text{area of large circle}) - (\text{area of small circle}) \\ &= \pi(\text{large radius})^2 - \pi(\text{small radius})^2 \\ &= \pi(x+1)^2 - \pi(1/x)^2 \\ &= \pi\left(x^2 + 2x + 1 - \frac{1}{x^2}\right). \end{aligned}$$

Then

$$\begin{aligned} \text{Volume} = \int_a^b A(x)dx &= \int_1^2 \pi\left(x^2 + 2x + 1 - \frac{1}{x^2}\right) dx \\ &= \pi\left(\frac{1}{3}x^3 + x^2 + x + \frac{1}{x}\right)\Bigg|_1^2 \approx 18.33. \end{aligned}$$

Alternately, the volume of the solid with the large circular faces is

$$\int_1^2 \pi\left(x^2 + 2x + 1\right) dx = \frac{19\pi}{3} \approx 19.9$$

and the volume of the hole is

$$\int_1^2 \pi \left(\frac{1}{x^2}\right) dx = \frac{\pi}{2} \approx 1.57$$

so the volume we want is $19.90 - 1.57 = 18.33$.