THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1510 Calculus for Engineers (Fall 2021)
Suggested solutions of coursework 2

Part A

1. Without using L'Hôpital's rule, evaluate the following limits. Furthermore, if the limit does not exist but diverges to $\pm\infty$, please indicate so and determine the correct sign.

   (a) $\lim\limits_{x\to 9} \dfrac{x-9}{\sqrt{x}-3}$

   (b) $\lim\limits_{x\to -\infty} \dfrac{|x+1|}{x-3}$

   Solution:

   (a)
$$\begin{aligned}
\lim_{x\to 9} \frac{x-9}{\sqrt{x}-3} &= \lim_{x\to 9} \frac{x-9}{\sqrt{x}-3} \cdot \frac{\sqrt{x}+3}{\sqrt{x}+3} \\
&= \lim_{x\to 9} \frac{(x-9)(\sqrt{x}+3)}{x-9} \\
&= \lim_{x\to 9} (\sqrt{x}+3) \\
&= 6.
\end{aligned}$$

   (b)
$$\begin{aligned}
\lim_{x\to -\infty} \frac{|x+1|}{x-3} &= \lim_{x\to -\infty} \frac{-(x+1)}{x-3} \\
&= \lim_{x\to -\infty} \frac{x}{x} \cdot \frac{-1-\frac{1}{x}}{1-\frac{3}{x}} \\
&= -1.
\end{aligned}$$

2. Let $f(x) = \dfrac{|x^2-3x+2|}{x-2}$

   Evaluate the following limits. Furthermore, if the limit does not exist but diverges to $\pm\infty$, please indicate so and determine the correct sign.

   (a) $\lim\limits_{x\to 2^-} f(x)$

   (b) $\lim\limits_{x\to 2^+} f(x)$

   (c) $\lim\limits_{x\to 2} f(x)$

Solution: Note that
$$f(x) = \frac{|(x-1)(x-2)|}{x-2}.$$

(a)
$$\begin{aligned}\lim_{x\to 2^-} f(x) &= \lim_{x\to 2^-} \frac{-(x-1)(x-2)}{x-2} \\ &= \lim_{x\to 2^-} -(x-1) \\ &= -1.\end{aligned}$$

(b)
$$\begin{aligned}\lim_{x\to 2^+} f(x) &= \lim_{x\to 2^+} \frac{(x-1)(x-2)}{x-2} \\ &= \lim_{x\to 2^+} (x-1) \\ &= 1.\end{aligned}$$

(c) Since $\lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x)$, $\lim_{x\to 2} f(x)$ does not exist (DNE).

Part B

3. Let $f(x) = \dfrac{2^{x+1} - 2^{-x}}{2^x + 2^{-x}}$.

Without using L'Hôpital's rule, evaluate the following limits. Furthermore, if the limit does not exist but diverges to $\pm\infty$, please indicate so and determine the correct sign.

(a) $\lim_{x\to 0} f(x)$

(b) $\lim_{x\to \infty} f(x)$

(c) $\lim_{x\to -\infty} f(x)$

Solution:

(a)
$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{2^{x+1} - 2^{-x}}{2^x + 2^{-x}} = \frac{2-1}{1+1} = \frac{1}{2}.$$

(b)
$$\lim_{x\to \infty} f(x) = \lim_{x\to \infty} \frac{2^{x+1}}{2^x} \cdot \frac{1 - 2^{-2x-1}}{1 + 2^{-2x}} = 2 \cdot \frac{1-0}{1+0} = 2.$$

(c)
$$\lim_{x\to -\infty} f(x) = \lim_{x\to -\infty} \frac{2^{-x}}{2^{-x}} \cdot \frac{2^{2x+1} - 1}{2^{2x} + 1} = 1 \cdot \frac{0-1}{0+1} = -1.$$

4. Let $f(x) = \dfrac{x^{1510} + x^{1509} + \cdots + x^{1020}}{1510x^{1510} + 1509x^{1509} + \cdots + 1020x^{1020}}$.
Without using L'Hôpital's rule, evaluate the following limits. Furthermore, if the limit does not exist but diverges to $\pm\infty$, please indicate so and determine the correct sign.

(a) $\displaystyle\lim_{x\to 0} f(x)$

(b) $\displaystyle\lim_{x\to\infty} f(x)$

Solution:

(a)

$$\begin{aligned}
\lim_{x\to 0} f(x) &= \lim_{x\to 0} \frac{x^{1510} + x^{1509} + \cdots + x^{1020}}{1510x^{1510} + 1509x^{1509} + \cdots + 1020x^{1020}} \\
&= \lim_{x\to 0} \frac{x^{1020}}{x^{1020}} \cdot \frac{x^{490} + x^{489} + \cdots + 1}{1510x^{490} + 1509x^{489} + \cdots + 1020} \\
&= \frac{1}{1020}.
\end{aligned}$$

(b)

$$\begin{aligned}
\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{x^{1510} + x^{1509} + \cdots + x^{1020}}{1510x^{1510} + 1509x^{1509} + \cdots + 1020x^{1020}} \\
&= \lim_{x\to\infty} \frac{x^{1510}}{x^{1510}} \cdot \frac{1 + x^{-1} + \cdots + x^{-490}}{1510 + 1509x^{-1} + \cdots + 1020x^{-490}} \\
&= \frac{1}{1510}.
\end{aligned}$$

5. Without using L'Hôpital's rule, evaluate the following limits. Furthermore, if the limit does not exist but diverges to $\pm\infty$, please indicate so and determine the correct sign.

(a) $\displaystyle\lim_{x\to-\infty} \frac{\sqrt{4x^2+x}}{2x+1}$

(b) $\displaystyle\lim_{x\to 0^+} x\sin\left(\frac{1}{\sqrt{x}}\right)$

(c) $\displaystyle\lim_{n\to\infty} \frac{\sin n + 2\cos n}{n}$

Solution:

(a) Let $y = -x$. Then

$$\begin{aligned}
\lim_{x\to-\infty} \frac{\sqrt{4x^2+x}}{2x+1} &= \lim_{y\to\infty} \frac{\sqrt{4y^2-y}}{-2y+1} \\
&= \lim_{y\to\infty} \frac{y}{y} \cdot \frac{\sqrt{4-\frac{1}{y}}}{-2+\frac{1}{y}} \\
&= \frac{\sqrt{4}}{-2} \\
&= -1.
\end{aligned}$$

(b) For any $x > 0$, we have

$$0 \leq \left| x \sin\left(\frac{1}{\sqrt{x}}\right) \right| \leq |x|.$$

Note that $\lim\limits_{x \to 0^+} 0 = 0 = \lim\limits_{x \to 0^+} |x|$.

By squeeze theorem, $\lim\limits_{x \to 0^+} \left| x \sin\left(\frac{1}{\sqrt{x}}\right) \right| = 0$.

Hence, $\lim\limits_{x \to 0^+} x \sin\left(\frac{1}{\sqrt{x}}\right) = 0$.

(c) For any positive integer n,

$$0 \leq \left| \frac{\sin n + 2\cos n}{n} \right| \leq \frac{|\sin n| + 2|\cos n|}{n} \leq \frac{3}{n}.$$

Note that $\lim\limits_{n \to \infty} 0 = 0 = \lim\limits_{n \to \infty} \frac{3}{n}$.

By squeeze theorem, $\lim\limits_{n \to \infty} \left| \frac{\sin n + 2\cos n}{n} \right| = 0$.

Hence, $\lim\limits_{n \to \infty} \frac{\sin n + 2\cos n}{n} = 0$.