THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1510 Calculus for Engineers (2020-2021)
Supplementary Exercise 3

Limits of Functions

1. Let $f(x) = x + 1$ and $g(x) = \dfrac{x^2 - 1}{x - 1}$.
   (a) State the domains of $f(x)$ and $g(x)$.
   (b) Fill in the blanks.
   $g(x)$ can be described in the following way:
   $$g(x) = \begin{cases} \underline{\qquad\qquad} & \text{if} \quad x \neq 1, \\ \text{NOT defined} & \text{if} \quad x = \underline{\qquad\qquad}. \end{cases}$$
   (c) Sketch the graphs of the functions $f(x)$ and $g(x)$.

2. Let $f(x) = x + 1$ and $g(x) = \dfrac{x^2 - 1}{x - 1}$ for $x \neq 1$.
   Complete the following table.

x	0.9	0.99	0.999	0.9999	1	1.0001	1.001	1.01	1.1
$f(x)$									
$g(x)$									

By observation, when x is getting closer and closer to 1, what values do $f(x)$ and $g(x)$ get closer and closer to? Hence, guess the value of $\lim_{x\to 1} f(x)$ and $\lim_{x\to 1} g(x)$.

3. Complete the following table. (Note: x is in radian)

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$							

By observation, guess the value of $\lim_{x\to 0} \dfrac{\sin x}{x}$.

4. Let $f(x)$ be a function defined by
   $$f(x) = \begin{cases} x + 1 & \text{if} \quad x > 0, \\ 0 & \text{if} \quad x = 0, \\ -1 & \text{if} \quad x < 0 \end{cases}$$
   (a) Complete the following table.

x	-0.1	-0.01	-0.001	-0.0001	0	0.0001	0.001	0.01	0.1
$f(x)$									

(b) Find $\lim_{x\to 0^-} f(x)$, $\lim_{x\to 0^+} f(x)$ and $f(0)$.

(c) Does $\lim_{x\to 0} f(x)$ exist? Why?

5. Let $f(x) = \dfrac{|x-1|}{x^2-1}$ for $x \neq \pm 1$.

   (a) Does $\lim_{x\to 1} f(x)$ exist?

   (b) Does $\lim_{x\to -1} f(x)$ exist?

   (Hint: Rewrite the function $f(x)$ as a piecewise defined function.)

6. Let a be a real number and let $f(x)$ be a function defined by

$$f(x) = \begin{cases} x^2 & \text{if } \ x \geq 2, \\ 3x + a & \text{if } \ x < 2 \end{cases}$$

   Given that $\lim_{x\to 2} f(x)$ exists. What is the value of a?

7. Let $f(x) = \dfrac{x^3}{|x|}$ for $x \neq 0$.

   (a) Find $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to 0^-} f(x)$.

   (b) Does $\lim_{x\to 0} f(x)$ exist?

8. Without using L'Hôpital rule, find the following limits.

   (a) $\lim_{x\to 1} \dfrac{x^2-3x+2}{x^2-1}$

   (b) $\lim_{x\to 2} \dfrac{x^3-4x^2+5x-2}{x^2-4}$

   (c) $\lim_{x\to 4} \dfrac{\sqrt{x}-2}{x-4}$

   (d) $\lim_{x\to 27} \dfrac{\sqrt[3]{x}-3}{x-27}$

   (e) $\lim_{x\to 3} \dfrac{x-3}{\sqrt{x+1}-2}$

   (f) $\lim_{x\to 0} \dfrac{\sqrt{1+x^2}-1}{x}$

   (g) (Harder Problem) $\lim_{x\to 0} \dfrac{(1+x)^n-1}{x}$, where n is a positive integer.

9. By using the fact that $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$, find the following limits.

   (a) $\lim_{x\to 0} \dfrac{\sin 2x}{5x}$

(b) $\lim_{x\to 0}\frac{\sin 5x}{\sin 7x}$

(c) $\lim_{x\to 0}\frac{\sin(x^2)}{5x^2}$

(d) $\lim_{x\to 0}\frac{\cos ax-\cos bx}{x^2}$, where a and b are distinct real numbers.

10. (a) By using the fact that $\lim_{x\to 0}\frac{\sin x}{x}=1$, find $\lim_{x\to 0}\frac{\cos x-1}{x^2}$.

    (b) Using (a), find $\lim_{x\to 0}\frac{\cos x-1}{x}$.

Limits at Infinity

11. Let $f(x)=\frac{x-1}{x-2}$.

    Complete the following table.

x	10	100	1000
$f(x)$			

By observation, guess the value of $\lim_{x\to+\infty} f(x)$.

(Remark: You may repeat the above by putting $x=-10,-100,-1000$ and guess the value of $\lim_{x\to-\infty} f(x)$.)

12. The graphs of $f(x)=e^x$ (in blue) and $g(x)=\ln x$ (in red) is shown in Figure 1, while the graphs of $f(x)=e^{-x}$ (in blue) and $g(x)=\ln(1/x)$ (in red) is shown in Figure 2.

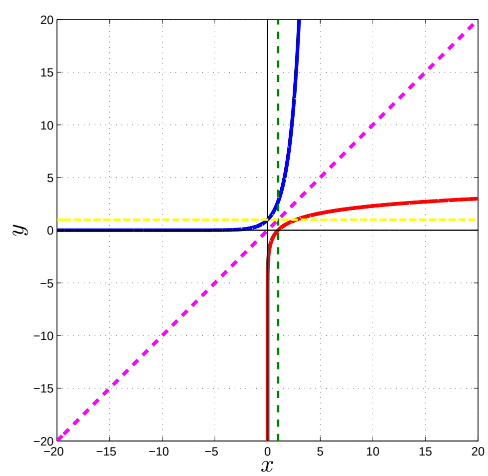

Figure 1

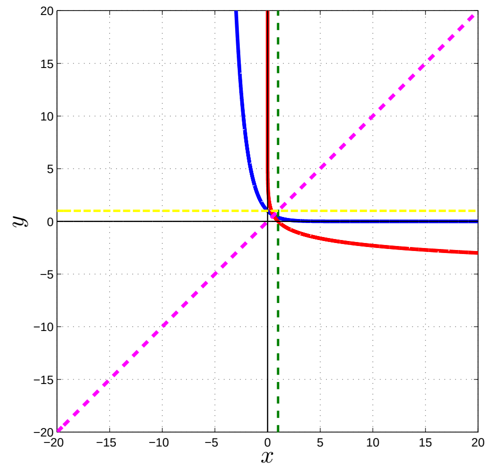

Figure 2

Without using L'Hôpital's rule, evaluate the limit. Furthermore, if the limit does not exist but diverges to $\pm\infty$, please indicate so and determine the correct sign.

(a) $\displaystyle\lim_{x\to-\infty} e^{1+x^6}$;

(b) $\displaystyle\lim_{x\to+\infty} \ln\left(e^{-2x}+e^{-x}+1\right)$;

(c) $\displaystyle\lim_{x\to+\infty} \ln\left(\frac{e^{3x}+e^{x}}{e^{5x}+e^{2x}}\right)$;

(d) $\displaystyle\lim_{x\to+\infty} \ln\left(\frac{e^{2x+1}+2e^{-x}}{e^{2x}+e^{-x+2}}\right)$.

13. Find the following limits, if exist.

(a) $\displaystyle\lim_{x\to+\infty} 2^x$;

(b) $\displaystyle\lim_{x\to-\infty} 2^x$;

(c) $\displaystyle\lim_{x\to+\infty} 0.2^x$;

(d) $\displaystyle\lim_{x\to-\infty} 0.2^x$;

(e) $\displaystyle\lim_{x\to+\infty} \ln\left(\frac{e^{x}+2e^{-x}}{e^{x}+e^{-x}}\right)$;

(f) $\displaystyle\lim_{x\to-\infty} \ln\left(\frac{e^{x}+2e^{-x}}{e^{x}+e^{-x}}\right)$.

14. By using the fact that $\displaystyle\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x = e$, find the following limits.

(a) $\displaystyle\lim_{x\to+\infty}\left(1+\frac{2}{x}\right)^{2x}$;

(b) $\displaystyle\lim_{x\to+\infty}\left(1+\frac{1}{x+1}\right)^{x}$

(c) $\lim_{x \to +\infty} \left(\frac{x}{x-1} \right)^x$

15. Without using L'Hôpital rule, find the following limits, if exist.

(a) $\lim_{x \to +\infty} \frac{x^2 - 3x + 2}{x^2 - 1}$;

(b) $\lim_{x \to -\infty} \frac{x^3 - 2x}{4x^3 + 2x^2}$;

(c) $\lim_{x \to +\infty} \frac{\sqrt{x^2 + 4}}{x + 4}$;

(d) $\lim_{x \to +\infty} \frac{x}{\sqrt{9x^2 + 5}}$;

(e) $\lim_{x \to -\infty} \frac{x}{\sqrt{9x^2 + 5}}$;

(f) $\lim_{x \to +\infty} \sqrt{x+1} - \sqrt{x-1}$;

(g) $\lim_{x \to +\infty} \sqrt{x^2 + x} - x$;

(h) $\lim_{x \to -\infty} \sqrt{x^2 + x} - x$.