2021R1-MATH1510 HW 3

Cho Kit CHAN

TOTAL POINTS

16 / 20

QUESTION 1

1 Q1 **6 / 6**

✓ **- 0 pts** Correct

QUESTION 2

2 Q2 **3 / 3**

✓ **- 0 pts** Correct

QUESTION 3

3 Q3 **0 / 3**

RHS

✓ **- 1 pts** incorrect

LHS

✓ **- 2 pts** incorrect

QUESTION 4

4 Q4 **5 / 5**

✓ **- 0 pts** Correct

QUESTION 5

5 Q5 **2 / 3**

5 (a)

✓ **- 0 pts** correct

✓ **- 0.5 pts** incorrect value of v(t)

✓ **- 0.5 pts** incorrect value of a(t)

Click here to replace this description.

✓ **- 0 pts** correct

Part A:

1. **Procedure for graphing functions using Calculus**

Step 1: Pre-calculus analysis:

(a) Find the domain of the function.
(b) Find the $x-$ and $y-$ intercepts.
(c) Test for symmetry with respect to the $y-$axis and the origin. (Verify whether the function is even or odd or neither or both).

Step 2: Calculus analysis:

(a) Use the first derivative to find the critical points and to find out where the graph is increasing and decreasing.
(b) Test the critical points for local maxima and minima.
(c) Use the second derivative to find out where the graph is concave upward and concave downward, and to locate inflection points.
(d) Find all asymptotes (horizontal, vertical), if any.

Step 3: Plot all critical points, inflection points, and $x-$ and $y-$ intercepts.

Step 4: Sketch the graph.

Sketch the graph of

$$f(x) = \frac{x}{(x-1)^2}$$

following the above procedure.

Domain of $f(x)$: $x \neq 1$

$= (-\infty, 1) \cup (1, \infty)$

Sub $f(x) = 0$: $x = 0$:

x-intercept: $(0,0)$ = y-intercept

$\because f(-x) = \frac{-x}{(-x-1)^2}$

$= -\frac{x}{(x+1)^2}$

$\neq f(x)$ & $-f(x)$

$\therefore$ The function is not a even or odd function. //

$$f'(x) = \frac{(x-1)^2 - x(2)(x-1)}{(x-1)^4}$$

$$= \frac{x-1-2x}{(x-1)^3}$$

$$= -\frac{x+1}{(x-1)^3}$$

$\therefore$ We have:

x		-1		1	
	$\searrow$		$\nearrow$		$\searrow$
$f'(x)$	$-$ve	0	$+$ve	Undefined	$-$ve

$$f''(x) = -\frac{(x-1)^3 - (x+1)(3)(x-1)^2}{(x-1)^6}$$

$$= -\frac{(x-1) - 3(x+1)}{(x-1)^4}$$

$$= \frac{3x+3 - x+1}{(x-1)^4}$$

$$= \frac{2(x+2)}{(x-1)^4}$$

$\therefore$ We have:

x		-2		1	
	$\cap$		$\cup$		$\cup$
$f''(x)$	$-$ve	0	$+$ve	Undefined	$+$ve

$\because \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{x}{(x-1)^2}$

$= \frac{1}{0}$

$= +\infty$

$\therefore$ $x=1$ is the vertical asysptote of $f(x)$. //

$\because \lim_{x=\infty} f(x) = \lim_{x\to\infty} \frac{x}{(x-1)^2}$

$= \lim_{x\to\infty} \frac{\frac{1}{x}}{1-\frac{2}{x}+\frac{1}{x^2}}$

$= \frac{0}{1}$

$= 0$

$\therefore$ $y=0$ is the horizontal asysptote of $f(x)$. //

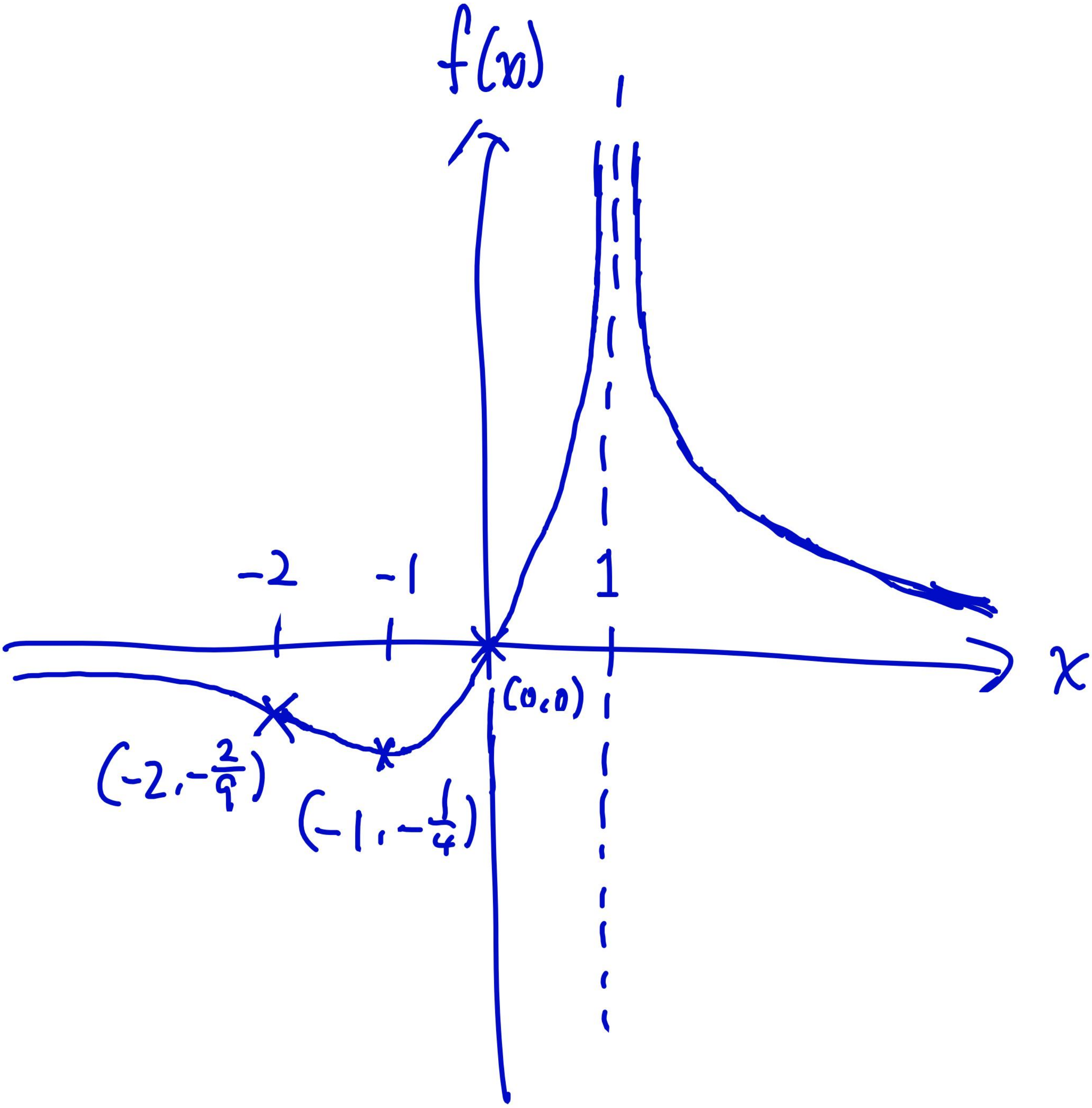
$\because f(-2) = -\frac{2}{9}$

$f(-1) = -\frac{1}{4}$

$\therefore$ The function passes through $(-2, -\frac{2}{9})$,

$(-1, -\frac{1}{4})$ //

$\therefore$ Curve-sketching :



1 Q1 6 / 6

✓ - 0 pts Correct

Part B:

$$\frac{dV}{dt} = 100\pi$$

2. A spherical balloon is inflated with helium at the rate of $100\pi \text{ft}^3/\text{min}$. How fast is the balloon's radius increasing at the instant the radius is 5ft? How fast is the surface area increasing? Recall that

$$V(t) = \frac{4}{3}\pi r(t)^3, S(t) = 4\pi r(t)^2,$$

where $V(t)$ and $S(t)$ are the volume and surface of the sphere where the radius is given by $r(t)$.

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$$

$$100\pi = 4\pi r^2 \times \frac{dr}{dt}$$

$$\left.\frac{dr}{dt}\right|_{r=5} = \frac{\cancel{25}}{\cancel{25}}$$

$$= 1 \text{ ft/min} //$$

$$\frac{dS}{dt} = \frac{dS}{dr} \times \frac{dr}{dt}$$

$$\frac{dS}{dt} = 8\pi r \times \frac{dr}{dt}$$

$$\left.\frac{dS}{dt}\right|_{r=5} = 40\pi \times 1$$

$$= 40\pi \text{ ft}^2/\text{min} //$$

2 Q2 3 / 3

✓ - 0 pts Correct

3. Show that

$$\frac{2}{\pi}x < \sin x < x, \quad x \in (0, \frac{\pi}{2}).$$

Let $f(x) = -\cos x$.

$\because$ The function of $f(x)$ is continuous & differentiable when $0 \leq x \leq \frac{\pi}{2}$.

$\therefore$ By the Mean Value Theorm, there exists

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad \text{for } c \in (a, b).$$

when $a < b$. //

Let $b = \frac{\pi}{2}$, $a = 0$:

$$\sin c = \frac{0 + 1}{\frac{\pi}{2} - 0}$$

$$= \frac{2}{\pi}$$

$$c = \sin^{-1}\left(\frac{2}{\pi}\right) > \sin c$$

$\because$ c and $\sin c$ are smaller than 1,

$$c \sin c = \frac{2}{\pi} c$$

$$< \sin c$$

$\therefore$ We have $\frac{2}{\pi}x < \sin x < x$ //

3 Q3 **0 / 3**

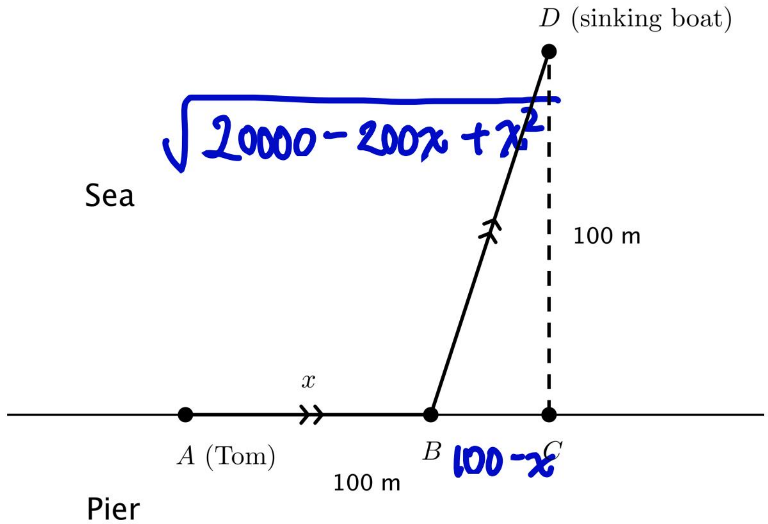
RHS

✓ **- 1 pts** incorrect

LHS

✓ **- 2 pts** incorrect

4. Tom, a lifeguard stationed at point A, spotted a sinking boat at point D:



To get to point D as soon as possible, he decided to run from A to B and swim from B to D. Suppose his running and swimming speeds are 9 and 1.8 respectively, and $|AC| = |CD| = 100$. Denote $|AB| = x,\ x \in [0, 100]$.

(a) Express the total time taken of the trip T as a function of x.

(b) Find all the critical points of $T(x)$ over the interval $(0, 100)$.

(c) Find the value(s) of x that minimizes the total time taken and the corresponding T (correct to 2 d.p.).

(a) $$|BD| = \sqrt{100^2 + (100-x)^2}$$

$$= \sqrt{20000 - 200x + x^2}$$

$\therefore$ Total time taken: $$\frac{x}{9} + \frac{\sqrt{20000-200x+x^2}}{1.8}$$

$$= T(x)$$

(b) $$T'(x) = \frac{1}{9} + \frac{5}{9} \times \frac{1}{2}(20000-200x+x^2)^{-\frac{1}{2}}(2x-200)$$

$$= \frac{1}{9} + \frac{5(x-100)}{9\sqrt{20000-200x+x^2}}$$

$T'(x) = 0$:

$$-\frac{1}{9} = \frac{5(x-100)}{9\sqrt{20000-200x+x^2}}$$

$$\sqrt{20000-200x+x^2} = 5(100-x)$$

$$20000-200x+x^2 = 25(10000-200x+x^2)$$

$$24x^2-4800x+230000 = 0$$

$$x = 79.58758548\ldots \text{ or } 120.4124145\ldots \text{ (rejected.)}$$

$\therefore$ The critical point $(79.58758548\ldots, 65.54421651\ldots)$

$\approx (79.59, 65.54)$ (2.dp.) //

(c) We have $x_1 = 79.58758548\ldots$

x	$0 \le x < x_1$	$x = x_1$	$x_1 < x \le 100$
$T(x)$	$\searrow$		$\nearrow$
$T'(x)$	$-$ve	0	$+$ve

$\therefore$ The value of x: 79.59 (2.dp.);

value of T: 65.54 (2.dp.) //

4 Q4 **5 / 5**

✓ **- 0 pts** Correct

5. In physics, if the displacement of an object is described by a function $x(t)$, then its velocity, denoted by $v(t)$, and its acceleration, denoted by $a(t)$, are given by $x'(t) = \dfrac{dx}{dt}$ and $x''(t) = \dfrac{d^2x}{dt^2}$ respectively.

Ideally, an object attached to a spring oscillates in simple harmonic motion. Its displacement from the equilibrium position would then be a function of time t, given by

$$x(t) = A\cos(\omega t - \varphi),$$

where m is the mass of the object, k is the spring constant, $\omega = \sqrt{\dfrac{k}{m}}$ and A, φ are two constants determined by the initial situation.

(a) Find the velocity $v(t)$ and acceleration $a(t)$ of the object as a function of time.

(b) Find the maximum velocity and acceleration in magnitude and the value(s) of t achieving them.

(c) The kinetic and potential energy of the object are given by

$$K(t) = \frac{1}{2}m(v(t))^2 \quad \text{and} \quad U(t) = \frac{1}{2}k(x(t))^2$$

respectively. Show that the total mechanical energy, i.e. the sum of kinetic energy and potential energy, is independent of time t.

(a) $v(t) = -A\sin(wt - \varphi)(w)$

$= -Aw\sin(wt - \varphi)$ //

$a(t) = -Aw^2\cos(wt - \varphi)$ //

(b) $\because -1 \leq \sin(wt - \varphi) \leq 1$

To attain the maximum of $v(t)$,

$\sin(wt - \varphi) = -1$

$wt - \varphi = \dfrac{3\pi}{2}$, for $[0, 2\pi]$

$t = \dfrac{3\pi - 2\varphi}{2w}$

$\because \quad -1 \leq \cos(\omega t - \varphi) \leq 1$

To attain the maximum of $a(t)$,

$$\cos(\omega t - \varphi) = -1$$

$$\omega t - \varphi = \pi \qquad , \text{for } [0, 2\pi]$$

$$t = \frac{\pi + \varphi}{\omega}$$

(c) Total mechanic energy: $\underline{\underline{K(t) + U(T)}}$:

$$\frac{1}{2} m \left(-A\omega \sin(\omega t - \varphi)\right)^2 + \frac{1}{2} k \left(A \cos(\omega t - \varphi)\right)^2$$

$$= \frac{1}{2} m (A^2) \left(\frac{k}{m}\right) \sin^2(\omega t - \varphi)$$

$$+ \frac{1}{2} k (A^2) \cos^2(\omega t - \varphi)$$

$$= \frac{1}{2} k (A^2) \left[\sin^2(\omega t - \varphi) + \cos^2(\omega t - \varphi)\right]$$

$$= \frac{1}{2} k A^2$$

$\because$ The variable of t is absent in the answer

$\therefore$ The total mechanic energy is independent of time t. //

5 Q5 2 / 3

5 (a)

✓ - 0 pts correct

✓ - 0.5 pts incorrect value of v(t)

✓ - 0.5 pts incorrect value of a(t)

Click here to replace this description.

✓ - 0 pts correct