Lecture 2

Time Value of Money I
Single Cash Flow

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Cash Flows and Assets

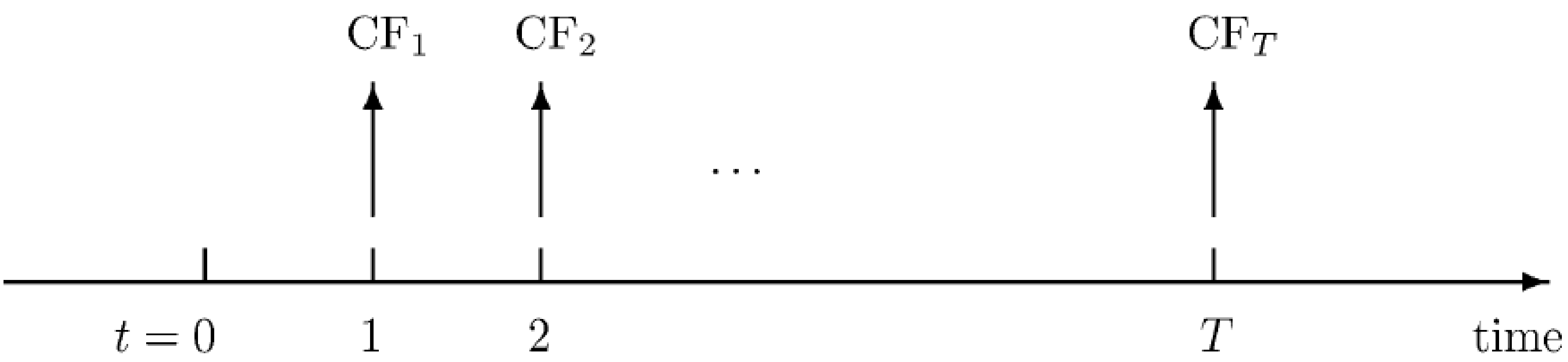
- Key question: what is an "*Asset*"?
  - Property, plant, and equipment
  - Patents, R&D
  - Stocks, bonds, options, ...
  - Knowledge, reputation, opportunities, etc.

- From a business perspective, an asset is a sequence of cash-flows

$$Asset_t = \{CF_t, CF_{t+1}, CF_{t+2}, \ldots\}$$

Cash Flows and Assets

- Valuing an asset requires valuing a sequence of cash flows
- Sequences of cash flows are the "basic building blocks" of finance



- Value of the asset is $V_t\{CF_t, CF_{t+1}, CF_{t+2}, \ldots\}$
- What is V_t?

The Present Value Operator

Key: cash flows at different dates are different "currencies"

- consider manipulating foreign currencies

$$\$150 + £300 =??450$$

- cannot add currencies without converting into common currency

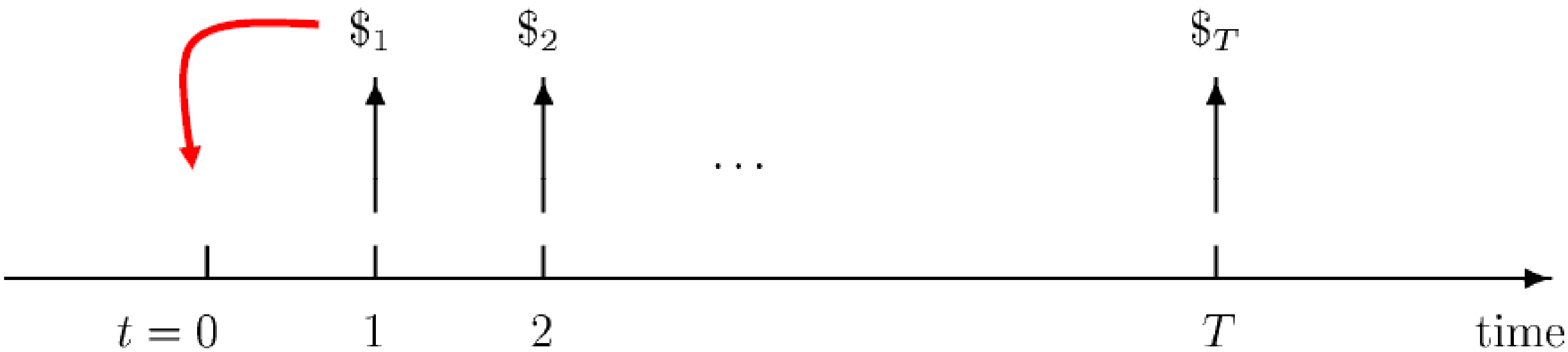
$$\$150 + £300 \times (\frac{\$}{£}10) = \$3{,}150$$

$$\$150 \times (\frac{£}{\$}0.10) + £300 = £315$$

- given exchange rates, either currency can be used as "numeraire"
- same idea for cash flows of different dates?

The Present Value Operator

- Key: cash flows at different dates are different "currencies"
- past and future cannot be combined without converting them
- once "exchange rates" are given, combining cash flows is trivial



- a numeraire date should be picked, typically t = 0 or "today“
- cash flows can then be converted to present value
- $V_0\{CF_t, CF_{t+1}, CF_{t+2}, \ldots\} = \left(\frac{\$_0}{\$_1}\right) \times CF_1 + \left(\frac{\$_0}{\$_2}\right) \times CF_2 + \cdots$

Time Value of Money

Concept:

- The time value of money received today is more than the value of same amount of money received after a certain period

Time preference for money:

Options of time period for receivables

- *Immediate*
- Later

Rationale for Time Preference for Money

- **Uncertainty and loss**: future is uncertain and it involves risk, hence one prefers to receive cash today instead in the future
  - example
    - bird in your hand or two birds in the bush
    - which one do your prefer?

- **To satisfy present need**: most people prefer to use the present money for satisfying the present needs

- **Investment opportunity:** there may exist investment opportunities through which people can earn additional cash

Lecture Outline

- Single Cash Flow
  - Future Value
    - Simple interest
    - Compound interest
      - Multiple compounding
      - Continues compounding
  - Present Value

- Financial Calculator

Interest Rates

This lecture is all about **interest rates**

- Why a borrower pays interest to the lender?
- What determines the amount of interest the lender will demand?

Answers from economic theory:

- because the lender incurs an opportunity cost for not being able to use the money

  interest = compensation for these cost

- the amount of interest charged depends on many factors:
  - term (length) of loan
  - credit standing of borrower
  - availability / tangibility of collateral

The term *interest rate* is also referred as the *discount rate*, the *opportunity cost of capital* and the *required rate of return*

Notation and Terminology

terminology:	**notes**
• principal = amount relative to which interest is computed	= amount initially invested
• interest rate = amount of interest per year as percentage of principal	expressed as % per annum even if accrual period < 1 year Note: 6% is written as .06 not as 6
• accrual period = length of time period for which interest is calculated	expressed as fraction of 1 year for example, one month = 1 / 12
• payment date = date on which interest is paid	= end of accrual period

Interest Amount

terminology:

- **principal** **denoted "P"**
  = amount relative to which interest is computed

- **interest rate** **denoted "r"**
  = amount of interest per year as percentage of principal

- **accrual period** **denoted "Δ"**
  = length of time period for which interest is calculated

amount of interest paid:

$$\text{Interest} = \mathrm{P} \times r \times \Delta$$

Fundamental Equation

fundamental relation between

future and present value:

$$FV = PV + P \times r \times \Delta$$

interest

present value (PV)
= initial investment

future value (FV)
= total value at end of accrual period

amount of interest paid:

$$\text{Interest} = P \times r \times \Delta$$

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Simple Interest: Future Value

investing for a single interval:

principal "P" = initial investment "PV"

$$\underbrace{FV}_{\text{future value (FV)}} = \underbrace{PV}_{\text{present value (PV)}} + \underbrace{P \times r \times \Delta}_{\text{interest}}$$

P = PV

interest

present value (PV)
= initial investment

future value (FV)
= total value at end of accrual period

future value

with simple interest:

$$\boxed{FV = PV \times (1 + r \times \Delta)}$$

= the ***future value*** of

- *an* ***amount PV***
- *invested* ***today***
- *for a* ***period*** Δ *(as fraction of 1 year)*
- *earning* ***simple interest at (annual) rate r***

Simple Interest Example

Suppose you invest $100 in a fixed deposit that pays 5 percent simple interest. How much will you have at the end of five years?

Solution:

Future value at the end of year 5 = Original invested amount

+ Interest earned in year 1
+ Interest earned in year 2
+ Interest earned in year 3
+ Interest earned in year 4
+ Interest earned in year 5

=$100

+ $100 x 0.05 + $100 x 0.05 + $100 x 0.05 + $100 x 0.05 + $100 x 0.05

= $125

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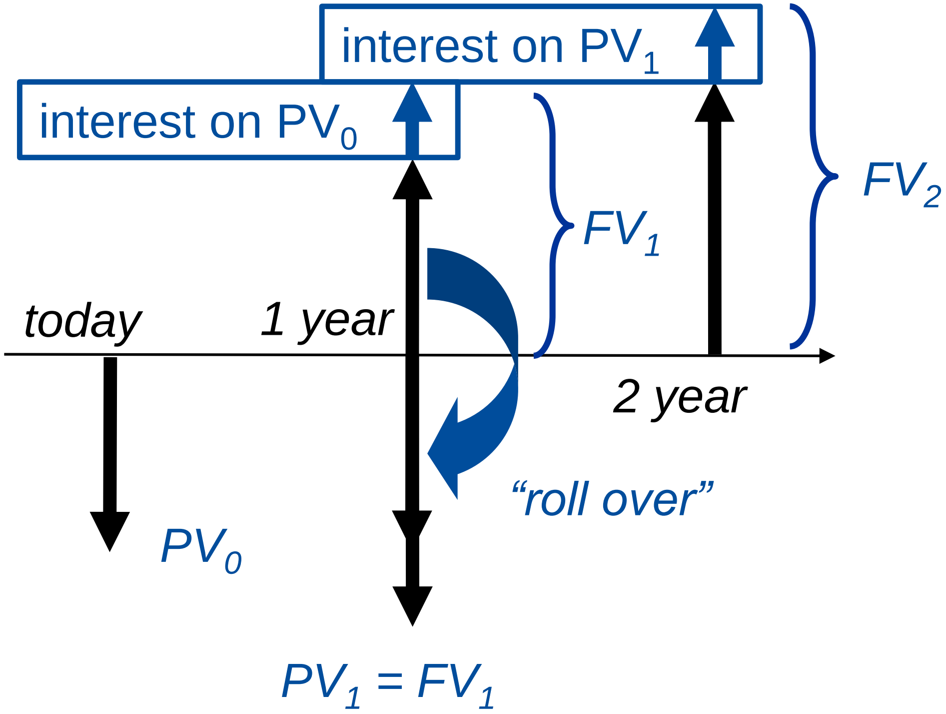
Compound Interest

- ***Simple Interest***: interest earned only on the *original investment*.
- ***Compound Interest***: in addition to interest earned on the *original investment*, interest is also earned on *interest previously received* (on the original investment).



Compound Interest Example

Suppose you invest \$100 in a fixed deposit that pays 5 percent compound interest. How much will you have at the end of five years?

Solution:

Future value at the end of year 5 = Original invested amount

+ Interest earned in year 1 + Interest earned in year 2

+ Interest earned in year 3 + Interest earned in year 4

+ Interest earned in year 5

$= \$100 + \$100 \times 0.05$

$+ \$100 \times (1+ 0.05) \times 0.05$

$+ \$100 \times (1+ 0.05)^2 \times 0.05$

$+ \$100 \times (1+ 0.05)^3 \times 0.05$

$+ \$100 \times (1+ 0.05)^4 \times 0.05$

$= \$100 \times (1+ 0.05)^5 = \$127.63$

Compound Interest: Future Value

The **future value** of an amount PV invested today after a total of T years is

$$\boxed{FV = PV \times (1 + r)^T}$$

- the (annual) interest rate is *r*, compounded annually

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Multiple Compounding

investing for multiple periods

suppose interest is paid in several regular intervals per year

For example

- semi-annually N = 2
- quarterly N = 4
- monthly N = 12
- etc …

We define the **compounding frequency** "N"

N is number of accrual periods per year

amount of interest paid per period:

$$\text{Interest} = P \times r \times \Delta = P \times \frac{r}{N}$$

each of the N periods has a length $\Delta = \dfrac{1}{N}$

Multiple Compounding

for example suppose

today: invest amount PV_0 at rate r with semi-annual compounding

after 6 months: you have:

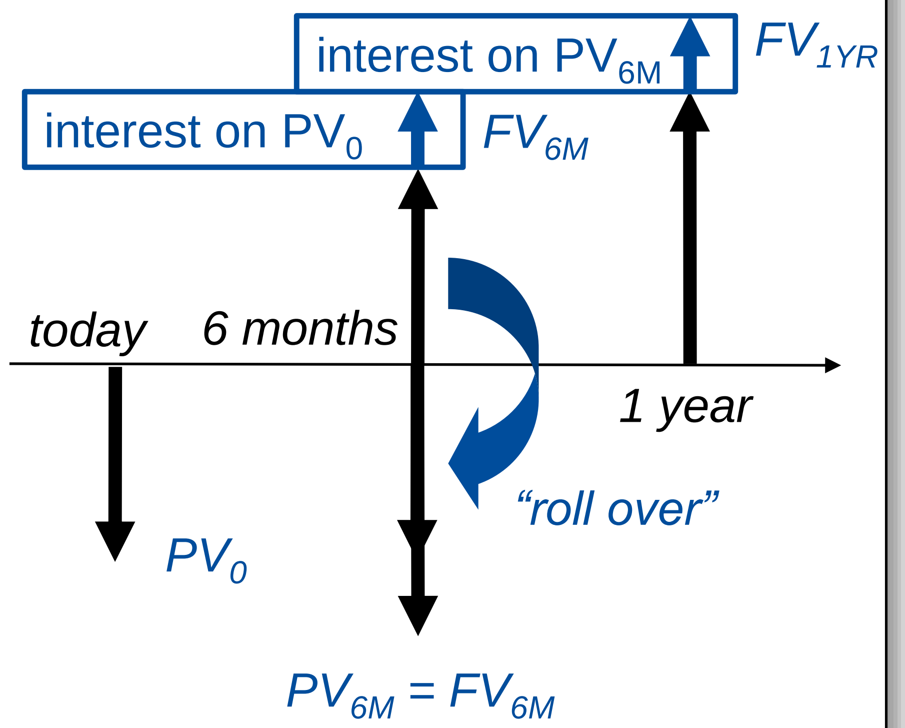
$$FV_{6M} = PV_0 \times \left(1 + \frac{r}{2}\right)$$

roll over $PV_{6M} = FV_{6M}$ into the next 6 months

at the end of 1 year: you have:

$$FV_{1YR} = PV_{6M} \times \left(1 + \frac{r}{2}\right) = PV_0 \left(1 + \frac{r}{2}\right)^2$$

cash flows:



Multiple Compounding

for example suppose

today: invest amount PV_0 at rate *r* with semi-annual compounding
after 6 months: you have:

$$FV_{6M} = PV_0 \times \left(1 + \frac{r}{2}\right)$$

roll over $PV_{6M} = FV_{6M}$ into the next 6 months
at the end of 1 year: you have:

$$FV_{1YR} = PV_{6M} \times \left(1 + \frac{r}{2}\right) = PV_0 \left(1 + \frac{r}{2}\right)^2$$

generic principle:

- invest PV_0 today
- with interest paid
  - at frequency N per year
  - at (annual) interest rate *r*
- roll over any interest proceeds

at the end of 1 year, you will have:

$$FV_{1YR} = PV_0 \times \left(1 + \frac{r}{N}\right)^N$$

Multiple Compounding: Summary

multiple compounding:

*the **future value** of an amount PV invested today after a total of **T** years is:*

$$FV = PV \times \left(1 + \frac{r}{N}\right)^{N \times T}$$

- the (annual) interest rate is *r*
- paid at frequency *N* per annum

rearrange:

To find the implied rate of return on the investment:

$$\left(1 + \frac{r}{N}\right)^{N \times T} = \frac{FV}{PV}$$

$$r = \left(\sqrt[N \times T]{\frac{FV}{PV}} - 1\right) \times N$$

To find the years of the investment:

$$N \times T \times \ln\left(1 + \frac{r}{N}\right) = \ln\left(\frac{FV}{PV}\right)$$

$$T = {\ln\left(\frac{FV}{PV}\right)} \Big/ {\ln\left(1 + \frac{r}{N}\right) \times N}$$

Multiple Compounding: Future Value

example:

Suppose 10 years ago, you invested $100,000 in a savings account. The bank guaranteed an annual interest rate of 6%, paid monthly. How much do you have today?

solution:

Your account balance is:

- $FV = PV \times \left(1 + \frac{r}{N}\right)^{N \times T}$

$$= \$100{,}000 \times \left(1 + \frac{0.06}{12}\right)^{12 \times 10} = \$181{,}940$$

Multiple Compounding: Implied Interest Rate

example:

You are looking at an investment that will pay \$1,200 in 5 years if you invest \$1,000 today. What is the implied rate of interest, compounded annually?

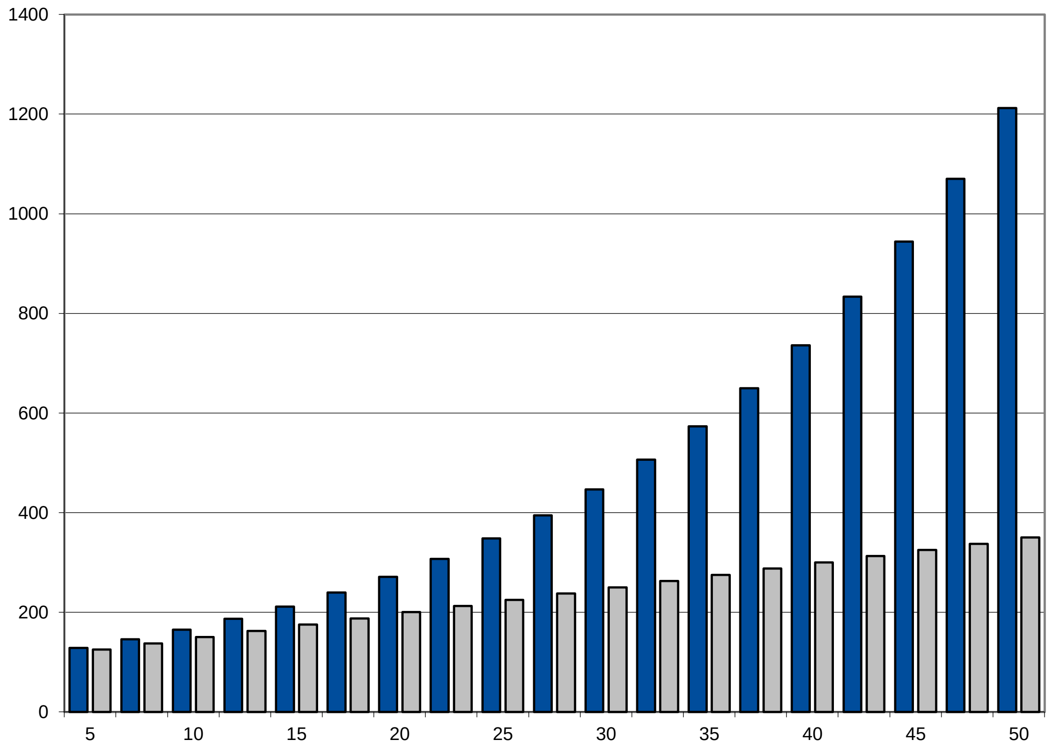
solution:

In this example, interest is compounded annually, so compounding frequency N=1

- $r = \left(\sqrt[N\times T]{\frac{FV}{PV}} - 1\right) \times N = \sqrt[5]{\frac{1{,}200}{1{,}000}} - 1 = 0.03714 = 3.714\%$

The Power of Compounding

example: the power of compounding

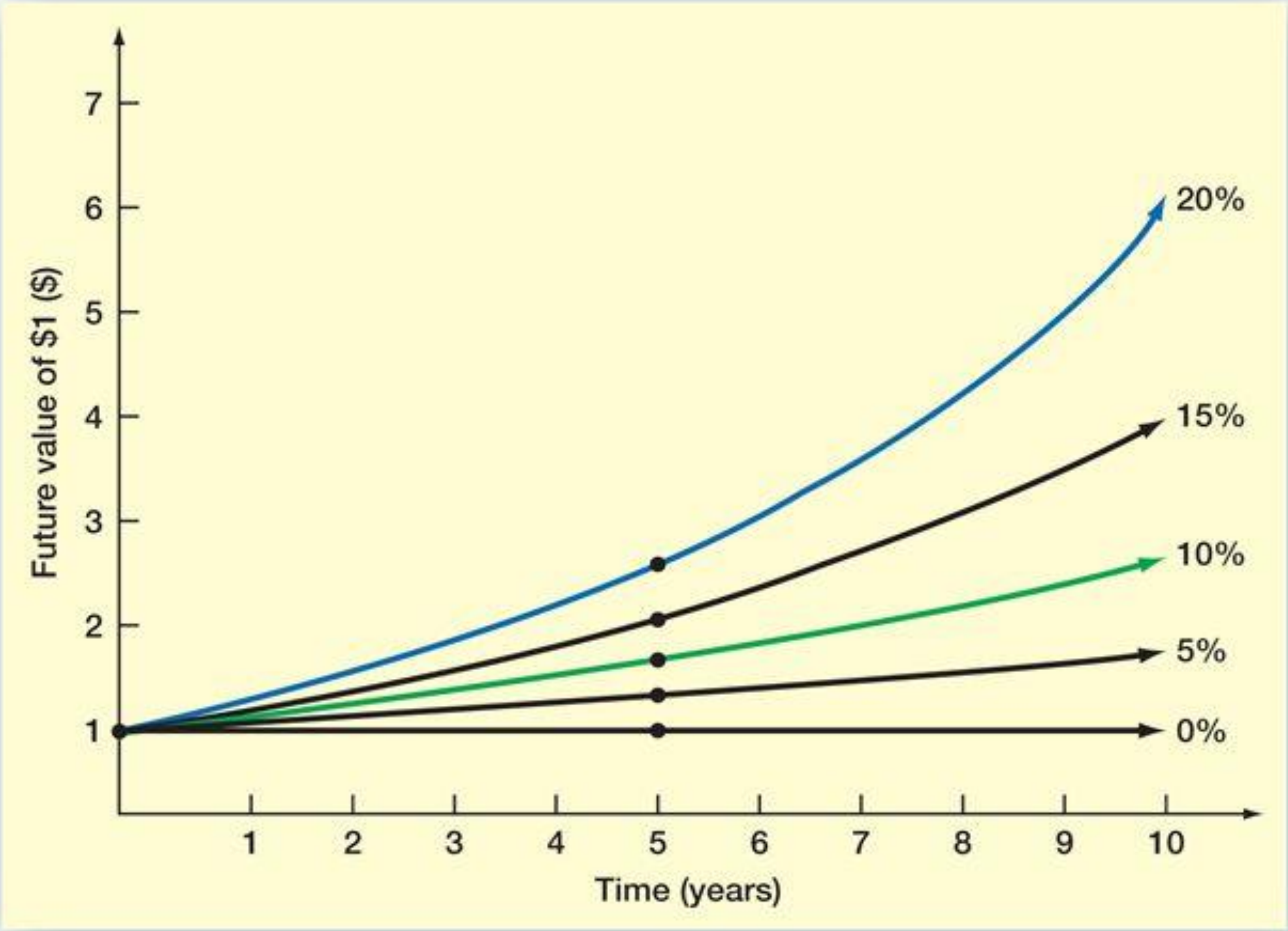


Blue bar = principle + compound interest

Grey bar = principle + simple interest

future value of $100 invested at 5% interest (paid monthly)

Future Value of $1 with Annual Compounding



Multiple computing paid annually

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In the limit: Continuous Compounding

Consider

- the future value after 10 years of $100 invested at 5%:

$$FV = PV \times \left(1 + \frac{r}{N}\right)^{N \times T}$$

frequency	*N*	*FV*
Annual	1	162.89
semi-annual	2	163.86
Quarterly	4	164.36
monthly	12	164.70
daily	365	164.87

FV increases with compounding frequency!

Continuous compounding:

- as frequency N increases

$$FV = PV \times \lim_{N \to \infty} \left(1 + \frac{r}{N}\right)^{N \times T} = PV \times e^{r \times T}$$

"exponential function"

intuition:

- multiple compounding is like a dripping tap:
  each drop is one interest payment
- continuous compounding is like a "flow" of interest payments:
  water flows from the tap

What is the Natural Number "e"?

- The number *e* is a ***mathematical constant*** that is the base of the natural logarithm: the unique number whose natural logarithm is equal to one.

- It is approximately equal to 2.71828

Continuous Compounding

Properties:

of the exponential function:

$$e^X \times e^{-X} = 1$$

Or

$$\frac{1}{e^X} = e^{-X}$$

$$e^{(\ln(X))} = X$$

And

$$\text{In}^{(e^X)} = X$$

Continuous compounding:

- future value:

$$FV = PV \times e^{r \times T}$$

- present value:

$$PV = FV \times e^{-r \times T}$$

- implied rate:

$$r = \frac{1}{T} \times \ln\left(\frac{FV}{PV}\right)$$

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Discounting: Present Value

future value:

- from before we know the future value
- $FV = PV \times \boxed{\left(1+\frac{r}{N}\right)^{N\times T}}$

Compound factor (future value factor)

present value:

- the present value of an amount FV to be received in T year's time from now is:
- $PV = FV \times \boxed{\frac{1}{\left(1+\frac{r}{N}\right)^{N\times T}}}$

Discount factor (present value factor)

Discounting: Present Value

example:

You own an insurance policy which pays \$100,000 in 10 years. Your bank pays you 6% interest per annum, compounded monthly. You are offered \$50,000 if you sell the policy.
Should you accept the offer?

Solution:

the value of the policy is:

$$PV = \frac{\$100{,}000}{\left(1+\frac{0.06}{12}\right)^{12\times 10}} = \boxed{\$54{,}963}$$

$\Rightarrow$ reject the offer

present value:

- the present value of an amount FV to be received in T year's time from now is:

$$\boxed{PV = \frac{FV}{\left(1+\frac{r}{N}\right)^{N\times T}}}$$

- where the (annual) interest rate is *r* paid at frequency N per annum

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Example

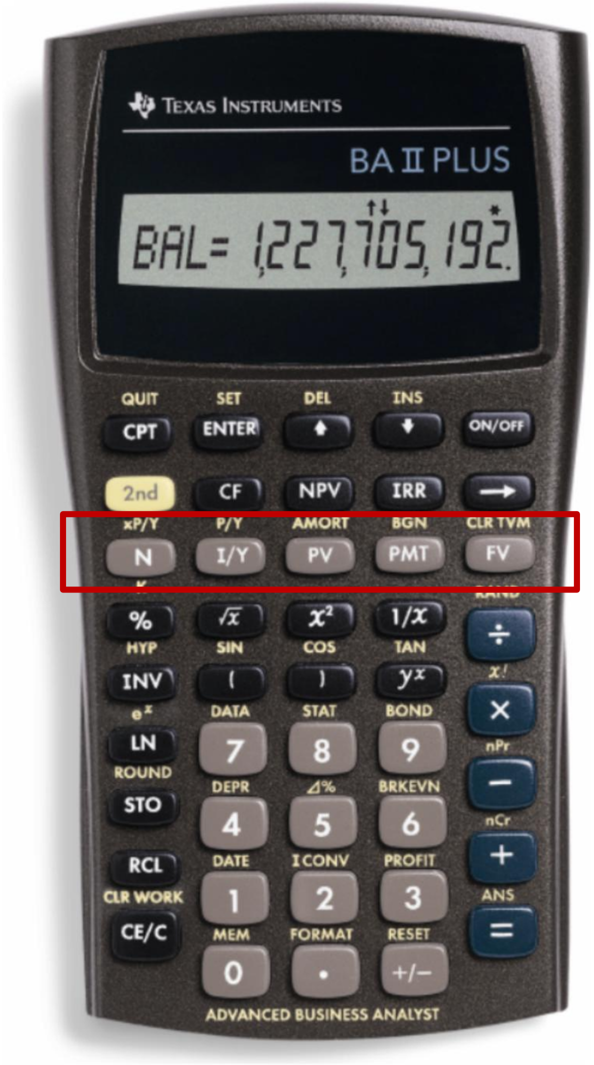
Singapore Airlines paid a cash dividend of $0.46 per share for the year ended March 2014. You believe that the dividend will increase by 10 percent every year indefinitely. How big will the dividend be in 2020?

Solution

$$FV_t = PV \times (1 + r)^t = \$0.46 \times (1 + 10\%)^6 = \$0.81$$

Financial Calculator Solution

- $FV_t = PV \times (1 + r)^t$
- There are 4 variables. If 3 are known, the calculator will solve for the 4th
- N: number of periods
- I/Y: interest rate per period
- PV: present value
- PMT: payment per period
- FV: future value

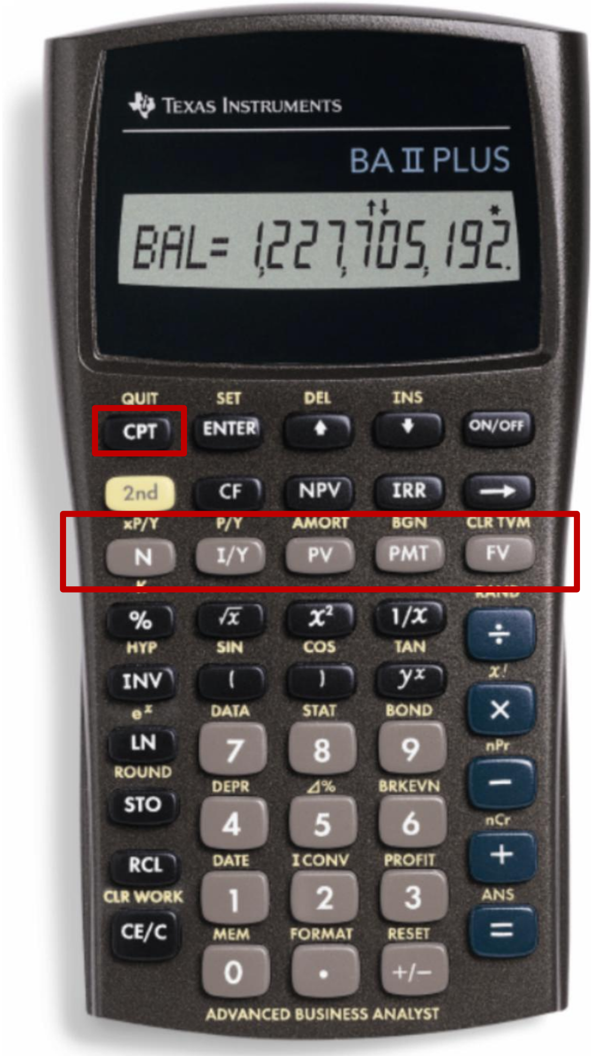


Financial Calculator Solution

- N: 6 periods (enter as 6)
- I/Y: 10% interest rate per period (*enter as 10 NOT 0.1*)
- PV: −0.46 (enter as negative to get positive FV here)
- PMT: not relevant in this situation (enter as 0)
- FV: compute (resulting answer is positive)

Financial Calculator Solution

Press:	
6	N
10	1/Y
-0.46	PV
0	PMT
CPT	FV



Example

You would like to buy a new automobile. You have \$50,000, but the car costs \$68,500. If you can earn 9 percent compounded annually, how much do you have to invest today to buy the car in two years? Do you have enough? Assume the price will stay the same.

$$PV_t = FV_t/(1+r)^t = 68{,}500/(1+0.09)^2 = \$57{,}655.08$$

You’re still about \$7,655 short, even if you are willing to wait two years.

Financial Calculator Solution

- N: 2 periods (enter as 2)
- I/Y: 9% interest rate per period (enter as 9 NOT 0.09)
- FV: 685,000
- PMT: not relevant in this situation (enter as 0)
- PV: compute (resulting answer is negative)

Summary: Three Rules of Time Travel

- Financial decisions often require combining cash flows or comparing values. Three rules govern these processes.

Rule 1	Only values at the same point in time can be compared or combined.	
Rule 2	To move a cash flow forward in time, you must compound it.	Future Value of a Cash Flow $FV_n = C \times (1 + r)^n$
Rule 3	To move a cash flow backward in time, you must discount it.	Present Value of a Cash Flow $PV = C \div (1 + r)^n = \dfrac{C}{(1 + r)^n}$

- Converting future value to present value is called discounting
- Converting present value to future value is called compounding

Appendix: Proof of Continuous Compounding Factor

$$\lim_{N\to\infty}\left(1+\frac{r}{N}\right)^{N\times T}=e^{r\times T}$$

- Using the fact: $x=e^{ln(x)}$, we have $\left(1+\frac{r}{N}\right)^{N\times T}=e^{\ln\left[\left(1+\frac{r}{N}\right)^{N\times T}\right]}=e^{N\times T\ln\left(1+\frac{r}{N}\right)}$
- Then $\lim_{N\to\infty}\left(1+\frac{r}{N}\right)^{N\times T}=\lim_{N\to\infty}e^{ln\left[\left(1+\frac{r}{N}\right)^{N\times T}\right]}=\lim_{N\to\infty}e^{N\times T\ln\left(1+\frac{r}{N}\right)}=e^{\lim_{N\to\infty}\left[N\times T\ln\left(1+\frac{r}{N}\right)\right]}$
- According to the L'Hospital's rule:

$$\lim_{N\to\infty}\left[N\times T\ln\left(1+\frac{r}{N}\right)\right]=\lim_{N\to\infty}\frac{\ln\left(1+\frac{r}{N}\right)}{\frac{1}{N\times T}}=\lim_{N\to\infty}\frac{\frac{N}{r+N}\left(-\frac{r}{N^2}\right)}{-\frac{1}{(N\times T)^2}T}=\lim_{N\to\infty}\frac{-\frac{r}{N(N+r)}}{-\frac{1}{N^2T}}$$

$$=\lim_{N\to\infty}\left(r\times T\frac{N}{N+r}\right)=r\times T$$

- So, $\lim_{N\to\infty}\left(1+\frac{r}{N}\right)^{N\times T}=e^{r\times T}$

You will not be held responsible for mathematical derivations, but you are expected to know the formula in the exam.