THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1510 Calculus for Engineers (Fall 2021)
Suggested solutions of coursework 9

Part A

1. (a) Evaluate $\displaystyle\int \frac{1}{t^2+4t-5}\,dt$.

   (b) Using t-substitution and the result in part (a), evaluate

$$\int \frac{1}{2\sin x - 3\cos x - 2}\,dx$$

Solution:

(a) By partial fractions decomposition,

$$\frac{1}{t^2+4t-5} = \frac{1}{(t+5)(t-1)} = \frac{A}{t+5} + \frac{B}{t-1}$$

for some $A, B \in \mathbb{R}$.

$$1 = A(t-1) + B(t+5).$$

$$\begin{aligned} t \to 1: &\qquad 1 = 6B \implies B = \frac{1}{6} \\ t \to -5: &\qquad 1 = -6A \implies A = -\frac{1}{6}. \end{aligned}$$

Thus,

$$\begin{aligned} \int \frac{1}{t^2+4t-5}\,dt &= -\frac{1}{6}\int \frac{1}{t+5}\,dt + \frac{1}{6}\int \frac{1}{t-1}\,dt \\ &= -\frac{1}{6}\ln|t+5| + \frac{1}{6}\ln|t-1| + C. \end{aligned}$$

(b) Let $t = \tan\frac{x}{2}$. Then

$$\begin{aligned} \int \frac{1}{2\sin x - 3\cos x - 2}\,dx &= \int \frac{1}{2\cdot\frac{2t}{1+t^2} - 3\cdot\frac{1-t^2}{1+t^2} - 2} \cdot \frac{2}{1+t^2}\,dt \\ &= \int \frac{2}{4t - 3 + 3t^2 - 2 - 2t^2}\,dt \\ &= 2\int \frac{1}{t^2+4t-5}\,dt \\ &= -\frac{1}{3}\ln|t+5| + \frac{1}{3}\ln|t-1| + 2C && \text{(by part (a))} \\ &= -\frac{1}{3}\ln\left|\tan\frac{x}{2}+5\right| + \frac{1}{3}\ln\left|\tan\frac{x}{2}-1\right| + 2C. \end{aligned}$$

2. Evaluate the following indefinite integrals.

   (a) $\displaystyle\int x\sin(2x+1)\,dx$

   (b) $\displaystyle\int x\arccos x\,dx$

   (c) $\displaystyle\int \cos(\ln x)\,dx$

Solution:

(a)

$$\begin{aligned}\int x\sin(2x+1)\,dx &= -\frac{1}{2}\int x\,d\cos(2x+1)\\ &= -\frac{1}{2}x\cos(2x+1)+\frac{1}{2}\int\cos(2x+1)\,dx\\ &= -\frac{1}{2}x\cos(2x+1)+\frac{1}{4}\sin(2x+1)+C.\end{aligned}$$

(b)

$$\begin{aligned}I=\int x\arccos x\,dx &= \frac{1}{2}\int\arccos x\,d(x^2)\\ &= \frac{1}{2}x^2\arccos x-\frac{1}{2}\int x^2\cdot\frac{-1}{\sqrt{1-x^2}}\,dx\\ &= \frac{1}{2}x^2\arccos x+\frac{1}{2}\int\frac{x^2}{\sqrt{1-x^2}}\,dx.\end{aligned}$$

Let $x=\sin\theta \implies dx=\cos\theta\,d\theta$. Then

$$\begin{aligned}\int\frac{x^2}{\sqrt{1-x^2}}\,dx &= \int\frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}}\cdot\cos\theta\,d\theta\\ &= \int\sin^2\theta\,d\theta \qquad \Big(x\in(-1,1)\implies\theta\in(-\frac{\pi}{2},\frac{\pi}{2})\implies\cos\theta>0\Big)\\ &= \frac{1}{2}\int(1-\cos 2\theta)\,d\theta\\ &= \frac{1}{2}\theta-\frac{1}{4}\sin 2\theta+C.\end{aligned}$$

So,

$$I=\frac{1}{2}x^2\arccos x+\frac{1}{4}\arcsin x-\frac{1}{8}\sin(2\arcsin x)+\frac{1}{2}C.$$

(c) Let $x = e^u \implies dx = e^u\,du$. Then

$$\begin{aligned} I &= \int \cos(\ln x)\,dx \\ &= \int (\cos u) e^u\,du \\ &= \int \cos u\,d(e^u) \\ &= e^u \cos u + \int e^u \sin u\,du \\ &= e^u \cos u + \int \sin u\,d(e^u) \\ &= e^u \cos u + e^u \sin u - \int e^u \cos u\,du. \end{aligned}$$

Hence,

$$\begin{aligned} I &= \frac{1}{2}\left(e^u \cos u + e^u \sin u\right) + C \\ &= \frac{1}{2}\left(x \cos(\ln x) + x \sin(\ln x)\right) + C. \end{aligned}$$

Part B

3. Define $I_n = \displaystyle\int \frac{x^n}{\sqrt{2x+1}}\,dx$ for all non-negative integers n.

   (a) Evaluate I_0.

   (b) Considering that

   $$\int x^{n-1}\sqrt{2x+1}\,dx = \int x^{n-1}\frac{2x+1}{\sqrt{2x+1}}\,dx = 2\int \frac{x^n}{\sqrt{2x+1}}\,dx + \int \frac{x^{n-1}}{\sqrt{2x+1}}\,dx,$$

   show that

   $$I_n = \frac{x^n\sqrt{2x+1}}{2n+1} - \frac{n}{2n+1}I_{n-1}$$

   for all integers $n \geq 1$.

   (c) Using parts (a), (b), evaluate I_3.

Solution:

(a)

$$\begin{aligned} I_0 &= \int \frac{1}{\sqrt{2x+1}}\,dx = \frac{1}{2}\int (2x+1)^{-1/2}\,d(2x+1) \\ &= (2x+1)^{1/2} + C. \end{aligned}$$

(b) Note that

$$\begin{aligned} 2I_n + I_{n-1} &= \int x^{n-1}\sqrt{2x+1}\,dx \\ &= \frac{1}{n}\int \sqrt{2x+1}\,d(x^n) \\ &= \frac{1}{n}\left[x^n\sqrt{2x+1} - \int x^n\frac{1}{\sqrt{2x+1}}\,dx\right] \\ &= \frac{x^n\sqrt{2x+1}}{n} - \frac{1}{n}I_n, \end{aligned}$$

which implies that

$$(2n+1)I_n + nI_{n-1} = x^n\sqrt{2x+1}.$$

Hence,

$$I_n = \frac{x^n\sqrt{2x+1}}{2n+1} - \frac{n}{2n+1}I_{n-1} \qquad \text{for } n \geq 1.$$

(c) By part (b),

$$\begin{aligned} I_3 &= \frac{1}{7}x^3\sqrt{2x+1} - \frac{3}{7}I_2, \\ I_2 &= \frac{1}{5}x^2\sqrt{2x+1} - \frac{2}{5}I_1, \\ I_1 &= \frac{1}{3}x\sqrt{2x+1} - \frac{1}{3}I_0. \end{aligned}$$

By part (a), $I_0 = (2x+1)^{1/2} + C$. Hence,

$$I_3 = \frac{1}{7}x^3\sqrt{2x+1} - \frac{3}{35}x^2\sqrt{2x+1} + \frac{2}{35}x\sqrt{2x+1} - \frac{2}{35}\sqrt{2x+1} + C'.$$

4. Define $I_n = \displaystyle\int \sec^n x\, dx$ for all non-negative integers n.

   (a) By applying integration by parts to
   $$\int \sec^n x\, dx = \int \sec^{n-2} x \sec^2 x\, dx,$$
   show that
   $$I_n = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}$$
   for all integers $n \geq 2$.

   (b) Using (a), find I_4 and I_5.

Solution:

(a)
$$\begin{aligned}
I_n &= \int \sec^{n-2} x \sec^2 x\, dx \\
&= \int \sec^{n-2} x\, d(\tan x) \\
&= \sec^{n-2} x \tan x - \int \tan x\, d(\sec^{n-2} x) \\
&= \sec^{n-2} x \tan x - (n-2) \int \tan x \sec^{n-3} x \sec x \tan x\, dx \\
&= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x (\sec^2 x - 1)\, dx \\
&= \sec^{n-2} x \tan x - (n-2) \int \sec^n x\, dx + (n-2) \int \sec^{n-2} x\, dx \\
&= \sec^{n-2} x \tan x - (n-2) I_n + (n-2) I_{n-2}.
\end{aligned}$$

So,
$$I_n = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2} \qquad \text{for } n \geq 2.$$

(b) By part (a),
$$\begin{aligned}
I_4 &= \frac{\sec^2 x \tan x}{3} + \frac{2}{3} I_2, \\
I_2 &= \tan x + C.
\end{aligned}$$

Hence,
$$I_4 = \frac{\sec^2 x \tan x}{3} + \frac{2}{3} \tan x + C'.$$

By part (a),
$$\begin{aligned}
I_5 &= \frac{\sec^3 x \tan x}{4} + \frac{3}{4} I_3, \\
I_3 &= \frac{\sec x \tan x}{2} + \frac{1}{2} I_1.
\end{aligned}$$

Note that

$$I_1 = \int \sec x \, dx = \ln|\sec x + \tan x| + C.$$

Hence,

$$I_5 = \frac{\sec^3 x \tan x}{4} + \frac{3 \sec x \tan x}{8} + \frac{3}{8} \ln|\sec x + \tan x| + C''.$$