THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1510 Calculus for Engineers (2020-2021)
Solution to Supplementary Exercise 1

Trigonometry

Change of Units

1. Fill in the blanks.

   Ans:

$$
\begin{aligned}
15^\circ &= \underline{\ \frac{\pi}{12}\ } \text{ rad};\\
30^\circ &= \underline{\ \frac{\pi}{6}\ } \text{ rad};\\
\underline{\ 45^\circ\ } &= \frac{\pi}{4} \text{ rad};\\
\underline{\ 60^\circ\ } &= \frac{\pi}{3} \text{ rad};\\
\underline{\ 90^\circ\ } &= \frac{\pi}{2} \text{ rad};\\
120^\circ &= \underline{\ \frac{2\pi}{3}\ } \text{ rad};\\
\underline{\ 150^\circ\ } &= \frac{5\pi}{6} \text{ rad};\\
180^\circ &= \underline{\ \pi\ } \text{ rad};\\
270^\circ &= \underline{\ \frac{3\pi}{2}\ } \text{ rad};\\
360^\circ &= 2\pi \,\text{rad}.
\end{aligned}
$$

Trigonometric Identities

2. Find $\tan 75^\circ$ and express your answer in surd form.

   Ans: $\tan 75^\circ = \tan(30^\circ + 45^\circ) = \dfrac{\tan 30^\circ + \tan 45^\circ}{1 - \tan 30^\circ \tan 45^\circ} = \dfrac{\frac{\sqrt{3}}{3} + 1}{1 - \frac{\sqrt{3}}{3}} = 2 + \sqrt{3}.$

3. Find $\cos 165^\circ$ and $\sin 165^\circ$ and express your answers in surd form.

   Ans: $\cos 165^\circ = \cos(120^\circ + 45^\circ) = \cos 120^\circ \cos 45^\circ - \sin 120^\circ \sin 45^\circ = -\dfrac{1+\sqrt{3}}{2\sqrt{2}}.$

   Similarly $\sin 165^\circ = \sin(120^\circ + 45^\circ) = \sin 120^\circ \cos 45^\circ + \cos 120^\circ \sin 45^\circ = \dfrac{\sqrt{3}-1}{2\sqrt{2}}.$

4. Find $\cos^2 \dfrac{7\pi}{12}$ and $\sin^2 \dfrac{7\pi}{12}$ and express your answers in surd form.

Ans: $\cos\dfrac{7\pi}{12} = \cos\left(\dfrac{\pi}{3}+\dfrac{\pi}{4}\right) = \cos\dfrac{\pi}{3}\cos\dfrac{\pi}{4} - \sin\dfrac{\pi}{3}\sin\dfrac{\pi}{4} = \dfrac{\sqrt{2}-\sqrt{6}}{4}$

$\Rightarrow \cos^2\dfrac{7\pi}{12} = \dfrac{2-\sqrt{3}}{4}$ and $\sin^2\dfrac{7\pi}{12} = 1-\cos^2\dfrac{7\pi}{12} = \dfrac{2+\sqrt{3}}{4}$.

5. By using the product to sum formula, express each of the following expressions as a sum of trigonometric functions.

   (a) $\cos 5x \cos 3x$;

   Ans: $\cos 5x \cos 3x = \dfrac{1}{2}(\cos 2x + \cos 8x)$.

   (b) $\sin 4x \sin 2x$;

   Ans: $\sin 4x \sin 2x = \dfrac{1}{2}(\cos 2x - \cos 6x)$.

   (c) $\sin 7x \cos 3x$.

   Ans: $\sin 7x \cos 3x = \dfrac{1}{2}(\sin 4x + \sin 10x)$.

6. Show that $\sin 2x \cos 3x \cos 5x = \dfrac{1}{4}(\sin 4x - \sin 6x + \sin 10x)$.

   Ans: By product-to-sum identities we have

$$\begin{aligned}\sin 2x \cos 3x \cos 5x &= \frac{1}{2}\sin 2x(\cos 2x + \cos 8x) = \frac{1}{2}(\sin 2x \cos 2x + \sin 2x \cos 8x)\\ &= \frac{1}{4}(\sin 4x - \sin 6x + \sin 10x).\end{aligned}$$

7. Show that $\sin 3x \sin 4x \cos 5x = \dfrac{1}{4}(-\cos 2x + \cos 4x + \cos 6x - \cos 12x)$.

   Ans: By product-to-sum identities we have

$$\begin{aligned}\sin 3x \sin 4x \cos 5x &= \frac{1}{2}(\cos x - \cos 7x)\cos 5x = \frac{1}{2}(\cos x \cos 5x - \cos 7x \cos 5x)\\ &= \frac{1}{2}\left(\frac{\cos 4x + \cos 6x}{2} - \frac{\cos 2x + \cos 12x}{2}\right)\\ &= \frac{1}{4}(-\cos 2x + \cos 4x + \cos 6x - \cos 12x).\end{aligned}$$

8. Prove that $\dfrac{\cos(x+y)+\cos(x-y)}{\sin(x-y)-\sin(x+y)} = -\cot y$.

   Ans: $\dfrac{\cos(x+y)+\cos(x-y)}{\sin(x-y)-\sin(x+y)} = \dfrac{2\cos x \cos y}{-2\cos x \sin y} = -\dfrac{\cos y}{\sin y} = -\cot y$.

9. Prove that $\dfrac{1}{\tan(x+y)-\tan(x-y)} = \dfrac{\cos 2x}{2\sin 2y} + \dfrac{\cot 2y}{2}$.

   Ans: Using product-to-sum and sum-to-product identities, we have

$$\frac{1}{\tan(x+y)-\tan(x-y)} = \frac{1}{\frac{\sin(x+y)}{\cos(x+y)} - \frac{\sin(x-y)}{\cos(x-y)}}$$

$$
\begin{aligned}
&= \frac{\cos(x+y)\cos(x-y)}{\sin(x+y)\cos(x-y) - \cos(x+y)\sin(x-y)} \\
&= \frac{\frac{1}{2}\{\cos[(x+y)+(x-y)] + \cos[(x+y)-(x-y)]\}}{\sin[(x+y)-(x-y)]} \\
&= \frac{\cos 2x + \cos 2y}{2\sin 2y} \\
&= \frac{\cos 2x}{2\sin 2y} + \frac{\cot 2y}{2}
\end{aligned}
$$

10. Prove that $\tan\dfrac{x+y}{2} = \dfrac{\sin x + \sin y}{\cos x + \cos y}$.

    Ans: $\dfrac{\sin x + \sin y}{\cos x + \cos y} = \dfrac{2\sin\frac{x+y}{2}\cos\frac{x-y}{2}}{2\cos\frac{x+y}{2}\cos\frac{x-y}{2}} = \dfrac{\sin\frac{x+y}{2}}{\cos\frac{x+y}{2}} = \tan\dfrac{x+y}{2}$.

11. Let $t = \tan\dfrac{x}{2}$.

    (a) By considering $\tan x = \tan\left(2\cdot\dfrac{x}{2}\right)$, show that $\tan x = \dfrac{2t}{1-t^2}$.

    Ans: $\tan x = \tan\left(2\cdot\dfrac{x}{2}\right) = \dfrac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}} = \dfrac{2t}{1-t^2}$.

    (b) By using the result in (a), express $\sin x$ and $\cos x$ in terms of t.

    Ans: $\sin x = 2\sin\dfrac{x}{2}\cos\dfrac{x}{2} = 2\tan\dfrac{x}{2}\cos^2\dfrac{x}{2} = \dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \dfrac{2t}{1+t^2}$, and $\cos x = \dfrac{\sin x}{\tan x} = \dfrac{1-t^2}{1+t^2}$.

    (Remark: The result of this question will be useful for integration of trigonometric function, called t-substitution.)

12. Prove that $\cot\dfrac{x}{2} = \dfrac{1+\cos x}{\sin x}$.

    Ans: $\dfrac{1+\cos x}{\sin x} = \dfrac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}} = \dfrac{\cos\frac{x}{2}}{\sin\frac{x}{2}} = \cot\dfrac{x}{2}$.

    Alternative method: $\dfrac{1+\cos x}{\sin x} = \dfrac{1+(1-\tan^2\frac{x}{2})/(1+\tan^2\frac{x}{2})}{2\tan\frac{x}{2}/(1+\tan^2\frac{x}{2})} = \dfrac{2}{2\tan\frac{x}{2}} = \cot\dfrac{x}{2}$.

13. Prove the following identities:

    (a) $\sin^4 x = \dfrac{3 - 4\cos 2x + \cos 4x}{8}$;

    Ans: Use double-angle formula twice and we have

$$
\begin{aligned}
\sin^4 x = (\sin^2 x)^2 &= \left(\frac{1-\cos 2x}{2}\right)^2 = \frac{1 - 2\cos 2x + \cos^2 2x}{4} \\
&= \frac{1 - 2\cos 2x + \frac{1+\cos 4x}{2}}{4}
\end{aligned}
$$

$$= \frac{3 - 4\cos 2x + \cos 4x}{8}.$$

(b) $\sin^5 x = \dfrac{10\sin x - 5\sin 3x + \sin 5x}{16}$;

Ans: Make use of the identity in (a) and thus

$$\begin{aligned} 16\sin^5 x &= 2\sin x \cdot 8\sin^4 x = 2\sin x(3 - 4\cos 2x + \cos 4x) \\ &= 6\sin x - 8\sin x\cos 2x + 2\sin x\cos 4x \\ &= 6\sin x - 4(\sin 3x - \sin x) + (\sin 5x - \sin 3x) \\ &= 10\sin x - 5\sin 3x + \sin 5x. \end{aligned}$$

Therefore, $\sin^5 x = \dfrac{10\sin x - 5\sin 3x + \sin 5x}{16}$.

(c) $\cos^4 x = \dfrac{3 + 4\cos 2x + \cos 4x}{8}$;

Ans: Similar to (a),

$$\begin{aligned} \cos^4 x &= (\cos^2 x)^2 = \left(\frac{1 + \cos 2x}{2}\right)^2 = \frac{1 + 2\cos 2x + \cos^2 2x}{4} \\ &= \frac{1 + 2\cos 2x + \frac{1+\cos 4x}{2}}{4} \\ &= \frac{3 + 4\cos 2x + \cos 4x}{8}. \end{aligned}$$

(d) $\cos^5 x = \dfrac{10\cos x + 5\cos 3x + \cos 5x}{16}$;

Ans: Similar to (b), we use the identity in (c) to derive that

$$\begin{aligned} 16\cos^5 x &= 2\cos x \cdot 8\cos^4 x = 2\cos x(3 + 4\cos 2x + \cos 4x) \\ &= 6\cos x + 8\cos x\cos 2x + 2\cos x\cos 4x \\ &= 6\cos x + 4(\cos 3x + \cos x) + (\cos 5x + \cos 3x) \\ &= 10\cos x + 5\cos 3x + \cos 5x. \end{aligned}$$

Therefore, $\cos^5 x = \dfrac{10\cos x + 5\cos 3x + \cos 5x}{16}$.

(e) $\sin^4 x\cos^4 x = \dfrac{3 - 4\cos 4x + \cos 8x}{128}$;

Ans: Since $\sin^4 x\cos^4 x = (\sin x\cos x)^4 = \left(\dfrac{1}{2}\sin 2x\right)^4 = \dfrac{1}{16}\sin^4 2x$, we replace x by $2x$ in the identity of (a) to get

$$\sin^4 x\cos^4 x = \frac{1}{16}\cdot\frac{3 - 4\cos 4x + \cos 8x}{8} = \frac{3 - 4\cos 4x + \cos 8x}{128}.$$

(f) $\sin^5 x \cos^5 x = \dfrac{10\sin 2x - 5\sin 6x + \sin 10x}{512}$.

Ans: Using the same technique as in (e), we start from the identity in (b) to obtain

$$\begin{aligned}\sin^5 x \cos^5 x &= \frac{1}{32}\sin^5 2x = \frac{1}{32}\cdot\frac{10\sin 2x - 5\sin 6x + \sin 10x}{16}\\ &= \frac{10\sin 2x - 5\sin 6x + \sin 10x}{512}.\end{aligned}$$

14. Show that $\sin^2 x\cos^4 x = \dfrac{1}{32}(2 + \cos 2x - 2\cos 4x - \cos 6x)$.

Ans: Decomposition $\cos^4 x$ into two parts equally and then we can compute that

$$\begin{aligned}\sin^2 x\cos^4 x &= \sin^2 x\cos^2 x\cdot\cos^2 x = \left(\frac{1}{2}\sin 2x\right)^2\cos^2 x\\ &= \frac{1}{4}\sin^2 2x\cdot\frac{\cos 2x + 1}{2} = \frac{1-\cos 4x}{8}\cdot\frac{\cos 2x + 1}{2}\\ &= \frac{1}{16}(\cos 2x + 1 - \cos 4x\cos 2x - \cos 4x)\\ &= \frac{1}{16}\left(\cos 2x + 1 - \frac{\cos 6x + \cos 2x}{2} - \cos 4x\right)\\ &= \frac{1}{32}(2 + \cos 2x - 2\cos 4x - \cos 6x).\end{aligned}$$

15. Prove the following identites (called triple angle formula):

(a) $\sin 3x = 3\sin x - 4\sin^3 x$;

Ans:

$$\begin{aligned}\sin 3x &= \sin(2x + x)\\ &= \sin 2x\cos x + \cos 2x\sin x\\ &= (2\sin x\cos x)\cos x + (1 - 2\sin^2 x)\sin x\\ &= 2\sin x(1-\sin^2 x) + (1 - 2\sin^2 x)\sin x\\ &= 3\sin x - 4\sin^3 x\end{aligned}$$

(b) $\cos 3x = 4\cos^3 x - 3\cos x$;

Ans:

$$\begin{aligned}\cos 3x &= \cos(2x + x)\\ &= \cos 2x\cos x - \sin 2x\sin x\\ &= (2\cos^2 x - 1)\cos x - (2\sin x\cos x)\sin x\\ &= (2\cos^2 x - 1)\cos x - 2(1-\cos^2 x)\cos x\\ &= 4\cos^3 x - 3\cos x\end{aligned}$$

(c) $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$.

Ans:

$$\begin{aligned}
\tan 3x &= \tan(2x + x) \\
&= \frac{\tan 2x + \tan x}{1 - \tan 2x \tan x} \\
&= \frac{\left(\frac{2\tan x}{1-\tan^2 x}\right) + \tan x}{1 - \left(\frac{2\tan x}{1-\tan^2 x}\right)\tan x} \\
&= \frac{\left(\frac{3\tan x - \tan^3 x}{1-\tan^2 x}\right)}{\left(\frac{1-3\tan^2 x}{1-\tan^2 x}\right)} \\
&= \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}
\end{aligned}$$

16. Given that A, B, C, D are four interior angles of a quadrilateral $ABCD$. Prove that

$$\cos A + \cos B + \cos C + \cos D = -4\cos\frac{A+B}{2}\cos\frac{A+C}{2}\cos\frac{A+D}{2}.$$

Ans: We have that $D = 2\pi - (A + B + C)$ and thus

$$\begin{aligned}
&\cos A + \cos B + \cos C + \cos D \\
&= \cos A + \cos B + \cos C + \cos(A + B + C) \\
&= 2\cos\frac{A+B}{2}\cos\frac{A-B}{2} + 2\cos\frac{A+B}{2}\cos\frac{A+B+2C}{2} \\
&= 2\cos\frac{A+B}{2}\left(\cos\frac{A-B}{2} + \cos\frac{A+B+2C}{2}\right) \\
&= 4\cos\frac{A+B}{2}\cos\frac{A+C}{2}\cos\frac{B+C}{2} \\
&= 4\cos\frac{A+B}{2}\cos\frac{A+C}{2}\cos\left(\pi - \frac{A+D}{2}\right) \\
&= -4\cos\frac{A+B}{2}\cos\frac{A+C}{2}\cos\frac{A+D}{2}.
\end{aligned}$$

17. If $A + B + C = \pi$, show that

(a) $\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$;

Ans:

$$\begin{aligned}
\sin 2A + \sin 2B + \sin 2C &= 2\sin(A + B)\cos(A - B) + \sin 2C \\
&= 2\sin(\pi - C)\cos(A - B) + \sin 2C \\
&= 2\sin C\cos(A - B) + 2\sin C\cos C \\
&= 2\sin C[\cos(A - B) + \cos C]
\end{aligned}$$

$$
\begin{aligned}
&= 2\sin C \cdot 2\cos\frac{A-B+C}{2}\cos\frac{A-B-C}{2} \\
&= 4\sin C\cos\frac{\pi-2B}{2}\cos\frac{2A-\pi}{2} \\
&= 4\sin A\sin B\sin C.
\end{aligned}
$$

(b) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$;

Ans:

$$
\begin{aligned}
\tan A + \tan B + \tan C &= \tan A + \tan B + \tan(\pi - A - B) \\
&= \tan A + \tan B - \tan(A+B) \\
&= (1 - \tan A\tan B)\tan(A+B) - \tan(A+B) \\
&= -\tan A\tan B\tan(A+B) \\
&= \tan A\tan B\tan C.
\end{aligned}
$$

(c) $\cot A\cot B + \cot B\cot C + \cot C\cot A = 1$.

Ans: Recall that $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$ and we have

$$
\begin{aligned}
&\cot A\cot B + \cot B\cot C + \cot C\cot A \\
&= \frac{1}{\tan A\tan B} - \frac{1}{\tan(A+B)}\left(\frac{1}{\tan B} + \frac{1}{\tan A}\right) \\
&= \frac{1}{\tan A\tan B} - \frac{1}{\tan(A+B)}\frac{\tan A + \tan B}{\tan A\tan B} \\
&= \frac{1}{\tan A\tan B}\left[1 - \frac{\tan A + \tan B}{\tan(A+B)}\right] \\
&= 1.
\end{aligned}
$$

18. Prove that for any $x \neq 2m\pi$, m is an integer,

$$
1 + 2\cos x + 2\cos 2x + 2\cos 3x + \cdots + 2\cos nx = \frac{\sin(n+\frac{1}{2})x}{\sin\frac{x}{2}}.
$$

Ans: Note that $2\sin\dfrac{x}{2}\cos kx = \sin\dfrac{(2k+1)x}{2} - \sin\dfrac{(2k-1)x}{2}$ for all $k \geq 1$.

Therefore,

$$
\begin{aligned}
&\sin\frac{x}{2}(1 + 2\cos x + 2\cos 2x + 2\cos 3x + \cdots + 2\cos nx) \\
&= \sin\frac{x}{2} + \sum_{k=1}^{n} 2\sin\frac{x}{2}\cos kx \\
&= \sin\frac{x}{2} + \sum_{k=1}^{n}\left[\sin\frac{(2k+1)x}{2} - \sin\frac{(2k-1)x}{2}\right]
\end{aligned}
$$

$$= \sin\frac{(2n+1)x}{2} = \sin\left(n+\frac{1}{2}\right)x.$$

Hence $1 + 2\cos x + 2\cos 2x + 2\cos 3x + \cdots + 2\cos nx = \dfrac{\sin(n+\frac{1}{2})x}{\sin\frac{x}{2}}$.

General Solutions of Trigonometric Equations

19. (General Solutions of Trigonometric Equations)

- If $\sin x = p$, then let $\alpha = \sin^{-1}(p)$, then all solutions of the equation $\sin x = p$ are in form of $n\pi + (-1)^n\alpha$ where n is an integer;
- If $\cos x = p$, then let $\alpha = \cos^{-1}(p)$, then all solutions of the equation $\cos x = p$ are in form of $2n\pi \pm \alpha$ where n is an integer;
- If $\tan x = p$, then let $\alpha = \tan^{-1}(p)$, then all solutions of the equation $\tan x = p$ are in form of $n\pi + \alpha$ where n is an integer.

By using the above, solve the following equations.

(a) $\sin x = \dfrac{1}{2}$;

Ans: Note that $\alpha = \dfrac{\pi}{6}$ satisfies the above equation. Thus,
$x = n\pi + (-1)^n\dfrac{\pi}{6} = \dfrac{6n + (-1)^n}{6}\pi$, where n is an integer.

(b) $\cos x = -\dfrac{\sqrt{3}}{2}$;

Ans: Note that $\alpha = \dfrac{5\pi}{6}$ satisfies the above equation. Thus,
$x = 2n\pi \pm \dfrac{5\pi}{6} = \dfrac{12n \pm 5}{6}\pi$, where n is an integer.

(c) $\tan x = -\sqrt{3}$.

Ans: Note that $\alpha = \dfrac{2\pi}{3}$ satisfies the above equation. Thus,
$x = n\pi + \dfrac{2\pi}{3} = \dfrac{3n+2}{3}\pi$, where n is an integer.

20. Solve the following equations.

(a) $\cos 5x = \dfrac{1}{2}$, where $0 \le x < 2\pi$; (Hint: $0 \le 5x < 10\pi$.)

Ans: From the previous question we have
$5x = 2n\pi \pm \dfrac{\pi}{3}$, where n is an integer. Since $0 \le 5x < 10\pi$,
$5x = 2n\pi + \dfrac{\pi}{3}$ for $n = 0, 1, 2, 3, 4$ or $5x = 2n\pi - \dfrac{\pi}{3}$ for $n = 1, 2, 3, 4, 5$.
Hence, $x = \dfrac{\pi}{15}, \dfrac{\pi}{3}, \dfrac{7\pi}{15}, \dfrac{11\pi}{15}, \dfrac{13\pi}{15}, \dfrac{17\pi}{15}, \dfrac{19\pi}{15}, \dfrac{23\pi}{15}, \dfrac{5\pi}{3}, \dfrac{29\pi}{15}$.

(b) $\sin 4x = \sin 24°$, where $0° \leq x < 180°$;

Ans: $4x = n\pi + (-1)^n \frac{2}{15}\pi$, where n is an integer.

Since $0 \leq 4x < 4\pi$, $n = 0, 1, 2, 3$.

Hence, $x = \frac{\pi}{30}, \frac{13\pi}{60}, \frac{8\pi}{15}, \frac{43\pi}{60}$ or expressed in degree as $x = 6°, 39°, 96°, 129°$.

(c) $\tan 3x = 1$, where $\pi \leq x < 2\pi$.

Ans: $3x = n\pi + \frac{\pi}{4}$,where n is an integer.

Since $3\pi \leq 3x < 6\pi$, $n = 3, 4, 5$.

Hence, $x = \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{7\pi}{4}$.

21. Solve $\sin 7x - \sin x = \cos 4x$ for $0° \leq x \leq 180°$.

Ans: Note that $\sin 7x - \sin x = 2 \sin 3x \cos 4x$ and thus

$$2 \sin 3x \cos 4x = \cos 4x \Longrightarrow \sin 3x = \frac{1}{2} \quad \text{or} \quad \cos 4x = 0.$$

By solving the equations, we have $x = \frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{18}; \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}$.

In other words, $x = 10°, 50°, 130°, 170°; 22.5°, 67.5°, 112.5°, 157.5°$.

22. Solve $\sin x \sin 2x = \cos 3x \cos 4x$ for $0 \leq x \leq \frac{\pi}{2}$.

Ans: Note that $2 \sin x \sin 2x = \cos x - \cos 3x$ and $2 \cos 3x \cos 4x = \cos 7x + \cos x$.

Therefore, we have

$$\begin{aligned}
\cos x - \cos 3x &= \cos 7x + \cos x \\
0 &= \cos 7x + \cos 3x \\
0 &= 2 \cos 5x \cos 2x
\end{aligned}$$

Then $\cos 5x = 0$ or $\cos 2x = 0$.

By solving the equation, we have $x = \frac{\pi}{10}, \frac{\pi}{4}, \frac{3\pi}{10}, \frac{\pi}{2}$.