THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1510 Calculus for Engineers (Fall 2021)
Suggested solutions of coursework 1

Part A

1. (a) Given that

$$f(x) = \frac{1}{x} \quad \text{and} \quad g(x) = \sqrt{x-2},$$

write down the function $f \circ g$ explicitly. Find the domain of $f \circ g$ and express your answer in interval notation.

(b) Suppose

$$f(x) = \frac{x^2-1}{|x-1|}.$$

i. Rewrite the function $f(x)$ as a piecewise function in terms of polynomials in the following form.

$$f(x) = \begin{cases} \underline{\hspace{3cm}} & \text{if } \ x > 1, \\ \text{undefined} & \text{if } \ x = 1, \\ \underline{\hspace{3cm}} & \text{if } \ x < 1. \end{cases}$$

ii. Find $f(-100000)$ and $f(100000)$.

Solution:

(a) $(f \circ g)(x) = f(g(x)) = \dfrac{1}{\sqrt{x-2}}$.

$D_f = \{x \in \mathbb{R} \mid x \neq 0\}$ and $D_g = [2, \infty)$. Hence

$$\begin{aligned} D_{f\circ g} &= \{x \in D_g \mid g(x) \in D_f\} \\ &= \{x \in [2,\infty) \mid \sqrt{x-2} \neq 0\} \\ &= \{x \in [2,\infty) \mid x \neq 2\} \\ &= (2, \infty). \end{aligned}$$

(b) i. If $x > 1$, $f(x) = \dfrac{x^2-1}{x-1} = x+1$.

If $x < 1$, $f(x) = \dfrac{x^2-1}{-(x-1)} = -x-1$.

ii. $f(-100000) = -(-100000) - 1 = 99999$.

$f(100000) = (100000) + 1 = 100001$.

2. For the sequence

$$a_n = \left\{ \sqrt{n^2 + n} - \sqrt{n^2 + (-1)^n} \right\}, \qquad \text{for } n \geq 1$$

fill the following table (correct to 4 decimal places) and guess the value of a_n when n gets very large (approaches ∞).

n	100	1000	10000
a_n			

Solution:

n	1	100	10000
a_n	0.4938	0.4994	0.4999

So, a_n should be $\frac{1}{2}$ when n gets very large.

Part B

3. Let $f(x) = x^2 + 2x + 2$ and $g(x) = \ln x$.

   (a) By completing square, find the minimum value of $f(x)$.

   (b) Find the range of $g \circ f$. Express your answer in interval notation.

Solution:

(a) Note $f(x) = (x + 1)^2 + 1$.
So, the minimum value of $f(x)$ is 1.

(b) Note $R_f = [1, \infty)$.
Since $\ln x$ is increasing, we have $R_{g \circ f} = [\ln 1, \infty) = [0, \infty)$.

4. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Given that $f(x) = \sqrt{x}$ for $x \geq 0$, sketch the graph of $f(x)$ if

   i. f is an even function;

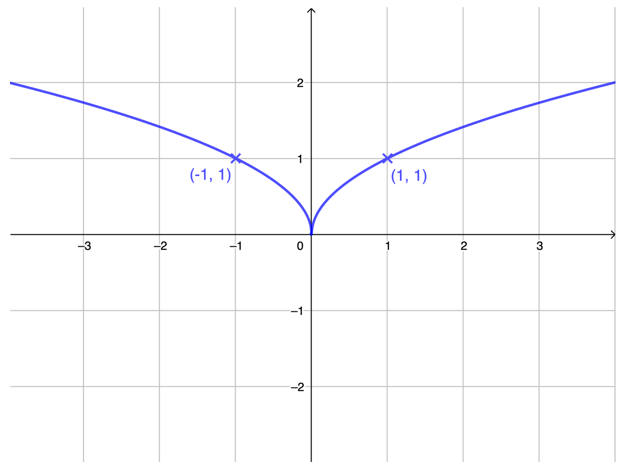
   ii. f is an odd function.

   (b) Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x) = 0$ when $-1 \leq x \leq 0$ and $f(x) = x$ when $0 < x < 1$.

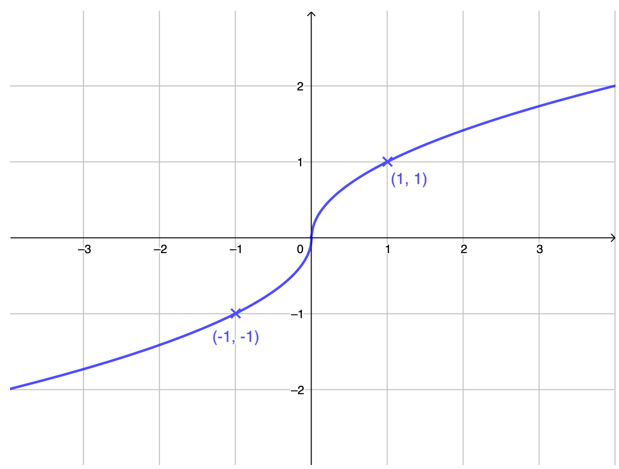
   Suppose that f is a periodic function with period 2. Sketch the graph of $f(x)$.

Solution:

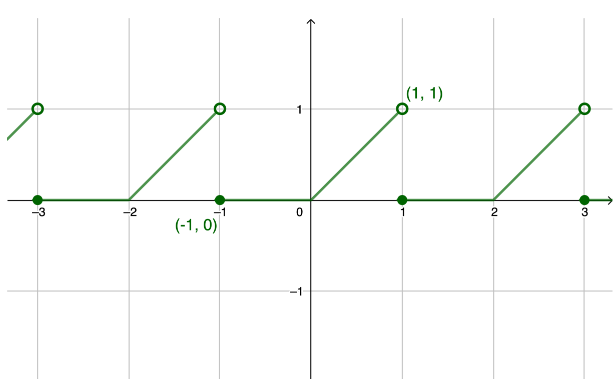
(a) i.



ii.



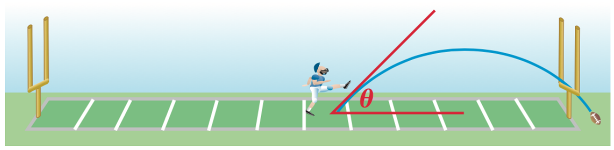
(b)



5. The path traveled by an object that is projected at an initial height of h_0 feet, an initial speed of v feet per second, and an initial angle θ is given by

$$y = -\frac{16}{v^2 \cos^2 \theta} x^2 + (\tan \theta) x + h_0$$

where x and y are the horizontal distance and vertical distance respectively.
(This model takes $g = 32\,\mathrm{ft/s}^2$ and neglects air resistance.)



If a football is kicked from the ground level with speed v,

(a) Show that the total horizontal distance traveled is $\dfrac{v^2 \sin\theta \cos\theta}{16}$.

(b) With what angle θ will the total horizontal distance traveled be maximized? (Hint: Consider the double angle formula: $\sin 2x = 2 \sin x \cos x$)

Solution:

(a) As the ball is kicked from the ground level, $h_0 = 0$. The ball hit the ground again when $y = 0$, hence

$$0 = -\frac{16}{v^2 \cos^2 \theta} x^2 + (\tan \theta) x + 0 = x(\tan \theta - \frac{16}{v^2 \cos^2 \theta} x),$$

which implies that

$$x = 0 \text{ (rejected)} \qquad \text{or} \qquad x = \frac{v^2 \tan\theta \cos^2\theta}{16} = \frac{v^2 \sin\theta \cos\theta}{16}.$$

Therefore the total horizontal distance travelled is $\dfrac{v^2 \sin\theta \cos\theta}{16}$.

(b) Note that

$$\text{Horizontal distance travelled} = \frac{v^2 \sin\theta \cos\theta}{16} = \frac{v^2 \sin 2\theta}{32}, \qquad 0^\circ \le \theta \le 90^\circ,$$

and $\sin 2\theta$ achieves its maximum when $2\theta = 90^\circ$. Hence the total horizontal distance travelled will be maximized when $\theta = 45^\circ$.