THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1510 Calculus for Engineers (2020-2021)
Supplementary Exercise 2

Set Notations

1. Describe the elements in the following sets.

   (a) $\{2,4\}$;

   (b) $(2,4)$;

   (c) $[2,4]$.

2. Describe the elements in the following sets.

   (a) $\mathbb{R}\backslash[2,4]$;

   (b) $\mathbb{R}\backslash\{2,4\}$;

   (c) $(-\infty,2)\cup(4,\infty)$;

   (d) $\mathbb{Z}^+\cap(5,\infty)$;

   (e) $\mathbb{Z}^+\cap[5,\infty)$.

   Remark: Here we use $\mathbb{Z}$ to denote the set of all integers and $\mathbb{Z}^+$ to denote the set of all positive integers.

3. Describe the elements in the following sets.

   (a) $\{x\in\mathbb{R} : x\geq 3\}$;

   (b) $\{n\in\mathbb{Z}^+ : n\geq 3\}$;

   (c) $\{m\in\mathbb{Z} : -5<m<5\}$;

   (d) $\{2m-1 : m\in\mathbb{Z}^+\}$;

   (e) $\{3n : n\in\mathbb{Z}^+\}$.

4. Using set notations to describe the following sets.

   (a) the set of all real numbers except -1 and 1;

   (b) the set of all positive real numbers x such that $x<1$ or $x>6$;

   (c) the set of all positive even integers;

   (d) the set of all integers which are divisible by 5.

   (Remark: The way of describing a set is not unique.)

Functions

5. Describe the domain and range of each of the following functions.

(a) $f(x) = \sqrt{x-1}$;

(b) $f(x) = \dfrac{1}{x^2}$;

(c) $g(x) = \sin x$;

(d) $g(x) = 2 + 3\cos x^2$;

(e) $h(x) = \log_2 x$;

(f) $h(x) = 3^x$.

6. Describe the domain of each of the following functions.

   (a) $f(x) = \dfrac{1}{x^2 - 4x - 12}$;

   (b) $f(x) = \dfrac{1}{\sqrt{4 - x^2}}$;

7. Consider the following functions:

$$f(x) = \sqrt{x} \qquad \text{and} \qquad g(x) = x + 5.$$

   Find the formulas explicitly describing $f+g$, fg, $f \circ g$ and $g \circ f$; and state the domains of the functions. Furthermore, state the range of $f \circ g$ and $g \circ f$.

8. Consider the function $f(x)$ defined by

$$f(x) = \begin{cases} x+1 & \text{if} \quad x \geq 0, \\ 0 & \text{if} \quad x < 0. \end{cases}$$

   Find the value of $f(-1)$, $f(0)$ and $f(1)$.

9. Consider the function $f(x)$ defined by

$$f(x) = \begin{cases} \sqrt{x} & \text{if} \quad x \geq 4, \\ \dfrac{1}{x-4} & \text{if} \quad x < 4. \end{cases}$$

   Find the value of $f(0)$, $f(4)$ and $f(9)$.

10. Fill in the blanks.

   (a) Consider the function $f(x) = |x|$. The function can be described explicitly by

$$f(x) = \begin{cases} \underline{\hspace{4cm}} & \text{if} \quad x \geq 0, \\ \underline{\hspace{4cm}} & \text{if} \quad x < 0. \end{cases}$$

   (b) Consider the function $f(x) = |x^2 - 9|$. The function can be described explicitly by

$$f(x) = \begin{cases} \underline{\hspace{4cm}} & \text{if} \quad x \geq 3, \\ \underline{\hspace{4cm}} & \text{if} \quad -3 < x < 3, \\ \underline{\hspace{4cm}} & \text{if} \quad x \leq -3 \end{cases}$$

Graphs of Functions

11. Sketch the graph of $y = f(x) = a^x$ if

    (a) $a > 1$;

    (b) $a = 1$;

    (c) $0 < a < 1$.

12. Let $f(x) = e^x$. Sketch the graphs of the following functions.

    (a) $y = 3f(x)$;

    (b) $y = -3f(x)$;

    (c) $y = f(x+3)$;

    (d) $y = f(x-3)$;

    (e) $y = f(x) + 3$;

    (f) $y = f(x) - 3$;

    (g) $y = f(3x)$.

    (Remark: What is the relation between each of the graph and the graph of $f(x)$?)

Summation Notation

13. Write down the expansion of the following expressions.

    (e.g.) $\displaystyle\sum_{i=1}^{5} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2$;

    (a) $\displaystyle\sum_{i=1}^{4} (2i+3)^2$;

    (b) $\displaystyle\sum_{i=2}^{5} (i^2+3)$;

    (c) $\displaystyle\sum_{r=0}^{5} 2^r$;

    (d) $\displaystyle\sum_{r=0}^{7} \left(-\frac{1}{2}\right)^r$.

14. Write down the expansion of the following expressions.

    (e.g.) $\displaystyle\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots$;

    (a) $\displaystyle\sum_{i=5}^{n} \left(\frac{1}{3}\right)^i$;

(b) $\displaystyle\sum_{r=0}^{4} \frac{x^r}{r!}$;

(c) $\displaystyle\sum_{r=0}^{\infty} (-1)^r \frac{x^{2r+1}}{(2r+1)!}$;

(Recall: If n is a positive integer, $n! = 1 \times 2 \times 3 \times \cdots \times n$ and we define $0! = 1$.)

(d) $\displaystyle\sum_{r=0}^{n} (-1)^r \frac{x^{2r}}{(2r)!}$;

(e) $\displaystyle\sum_{r=1}^{3} \frac{1}{r^2} \sin rx$;

(f) $\displaystyle\sum_{r=0}^{n} \frac{(-1)^r}{r!} \cos(2r+1)x$.

Parametrized Curves

15. Let $(x(t), y(t)) = (\cos t, \sin t)$, for $t \in \mathbb{R}$, be a curve defined on $\mathbb{R}^2$.
    (a) Write down the equation of the curve in x and y only.
    (b) What is the curve?

16. Let $A = (x_1, y_1)$ and $B = (x_2, y_2)$ be two distinct points on $\mathbb{R}^2$.
    Let $(x(t), y(t)) = t(x_2, y_2) + (1-t)(x_1, y_1) = (x_1 + t(x_2 - x_1), y_1 + t(y_2 - y_1))$, for $t \in [0, 1]$, be a curve defined on $\mathbb{R}^2$.
    (a) Find the endpoints $(x(0), y(0))$ and $(x(1), y(1))$ of the curve.
    (b) Write down the equation of the curve in x and y only.
    (c) What is the curve?

17. Let $(x(t), y(t)) = (3\cos t - 2, 3\sin t + 1)$, for $t \in \mathbb{R}$, be a curve defined on $\mathbb{R}^2$.
    (a) Write down the equation of the curve in x and y only.
    (b) What is the curve?

18. Let $(x(t), y(t)) = (t^2, t^3)$, for $t \in \mathbb{R}$, be a curve defined on $\mathbb{R}^2$. Write down the equation of the curve in x and y only.

19. Let $(x(t), y(t)) = (a\cos t, b\sin t)$, for $t \in \mathbb{R}$, $a, b > 0$, be a curve defined on $\mathbb{R}^2$. Write down the equation of the curve in x and y only.

Sequences

20. A sequence $\{a_n\}$ is defined recursively by the following equations:
    $$\begin{cases} a_1 = 2 \\ a_{n+1} = a_n^2 + 1 \text{ for } n \geq 1 \end{cases}$$
    Find the first 4 terms of the sequence.

21. A sequence $\{a_n\}$ is defined recursively by the following equations:

$$\begin{cases} a_1 = 1 \text{ and } a_2 = 2 \\ a_n = 2a_{n-1} + a_{n-2} \text{ for } n \geq 3 \end{cases}$$

Find a_4.

22. Let $\{a_n\}$ be a sequence defined by $a_n = \dfrac{2n+1}{n+3}$ for any positive integer n.

Complete the following table.

n	10	100	1000	10000
a_n				

By observation, when n is getting bigger and bigger, what value does a_n get closer and closer to? Hence, guess the value of $\lim\limits_{n\to\infty} a_n$.

23. For each of the following sequences, find $\lim\limits_{n\to\infty} a_n$, if it exists.

(a) $a_n = \left(\dfrac{1}{3}\right)^n$;

(b) $a_n = (-1)^n$;

(c) $a_n = 3^n$;

(d) $a_n = \dfrac{n^2 - n + 3}{3n^2 + 2n}$;

(e) $a_n = \dfrac{6n+3}{2n^2 + 9n - 5}$;

(f) $a_n = \dfrac{n^2 + n}{n + 7}$;

(g) $a_n = \dfrac{\sqrt{4n^2+3}}{2n+7}$;

(h) $a_n = \cos\dfrac{n\pi}{2}$;

(i) $a_n = \dfrac{\sin n}{n}$. (Hint: Use the sandwich theorem.)

24. Prove that $\lim\limits_{n\to\infty} \dfrac{\sin n + 100}{2n + (-1)^n} = 0$.

25. (Challenge) Let $\alpha > 0$. Prove that $\lim\limits_{n\to\infty} \dfrac{\alpha^n}{n!} = 0$.