Calculus for Engineers

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Implicit Functions and Their Differentiation

10.1 Introduction

Some functions can be described by expressing one variable explicitly in terms of another variable, that is, $y = f(x)$. Some functions, however, are defined implicitly by a relation between x and y. In some problems it is possible to solve such an equation for x or for y; but sometimes it is impossible. In Section 10.2, we will show how to find derivatives of implicitly defined functions. This process is often called implicit differentiation techniques. Implicit differentiation is nothing more than a special case of the chain rule for derivatives. In Section 10.3, we will study one application of implicit differentiation using a tangent line example. In Sections 10.4 and 10.5 , however, there is another application that is often referred to as the logarithmic differentiation problems. A wider class of implicit problems can be examined and studied.

10.2 What is implicit differentiation?

The difference between *explicit* and *implicit functions* is as follows. Functions of the form, $y = f(x)$, in which y (along) is directly expressed in terms of the function(s) of x, are called *explicit functions*.

Example 1

- $y = x^2 + 3x - 2$;
- $y = \sin x + 2e^x$;
- $y = (x + 3)/(1 + x^2)$;
- $y = \cos x + \log_e(1 + x^2)$
- and so on.

Not all functions, however, can be defined by equations of this type. For example, we cannot solve the following equations for y (alone) in terms of the functions of x. Here are a list of examples.

Example 2

- $x^3 + y^3 = 2xy$;
- $(x^2 + y^2)^4 = (x^2 - y^2)^2$;
- $x^2 + y^2 = 49$;
- $\sin y = x \sin(a + y)$;
- $y^5 + 3y^2 - 2x^2 = -4$
- and so on

Figure 10.1 shows all the cases.

Such relations which depend on x and y are called implicit relations. An implicit relation (in x and y) may represent jointly two or more functions of x.

As an example, the relation $x^2 + y^2 = 36$ jointly represents the following two functions:

$$y = \sqrt{36 - x^2} \quad \text{and} \quad y = -\sqrt{36 - x^2}.$$

Remark 1 Every explicit function

$$y = f(x)$$

can also be represented as an implicit function. For example, we may write the above equation in the form

$$y - f(x) = 0$$

and call it an implicit function of x. Thus, the terms *explicit function and implicit function do not characterize the nature of a function* but only the way a function is defined.

Note 1 Implicit functions may be expressed in the form

$$f = \{(x, y)| \ y = f(x)\}.$$

Note 2 In the case of an implicit function in the form,

$$y - f(x) = 0,$$

it is quite simple to compute the derivative $\dfrac{dy}{dx}$ since it is as good as if we are handling an explicit function. Hence, from here onward we shall only consider the implicit functions such as those given in Example 2 above.

Note 3 *It is assumed that an implicit relation defines y as at least one differentiable function of x.* With this *assumption*, the derivative of y with respect to x can be found *without converting it into the explicit form.*

This assumption is crucial because certain relations in x and y may not represent any function. For example, the relation $x^2 + y^2 = -36$ does not represent any function.

Note 4 The technique of implicit differentiation is based on the chain rule. Example 3 illustrates this idea.

Example 3 Consider the equation

$$y^3 + 7y = x^3. \tag{10.1}$$

Differentiating both sides of (10.1) with respect to x, treating y as a function of x, we get (via the *rule for differentiating a composite function*)

$$\begin{aligned}
\frac{d}{dx}\left(y^3 + 7y\right) &= \frac{d}{dx}\left(x^3\right); \\
\frac{d\left(y^3\right)}{dy}\frac{dy}{dx} + 7\frac{dy}{dx} &= 3x^2; \\
3y^2\frac{dy}{dx} + 7\frac{dy}{dx} &= 3x^2.
\end{aligned} \tag{10.2}$$

Now solving (10.2) for $\frac{dy}{dx}$, we get

$$\frac{dy}{dx}(3y^2 + 7) = 3x^2$$

Thus,

$$\frac{dy}{dx} = \frac{3x^2}{3y^2 + 7}.$$

Note that, the above expression for dy/dx involves both x and y. If it is required to find the value of the derivative of an implicit function for a given value of x, then we have to first find the corresponding value of y, using the given relation (such as in (10.1)). This will help in computing the value of dy/dx (or the slope of the curve) at those points that lie on the graph of the given equation.

For example, the point $(2, 1)$ satisfies equation (10.1); hence, it must be on its graph. At $(2, 1)$, we have

$$\left.\frac{dy}{dx}\right|_{(2,1)} = \frac{3(2)^2}{3(1)^2 + 7} = \frac{12}{10} = \frac{6}{5}.$$

Thus, the slope of the curve at $(2, 1)$ is $6/5$. □

Example 4 On the other hand, finding gradient at the point $(1, 1)$ of the curve $x^2 + y^2 - 3x + 4y - 3 = 0$ is a simpler situation. It can be seen that $\frac{dy}{dx} = \frac{-2x+3}{2y+4} = \frac{1}{6}$ at $(1, 1)$. (Exercise) □

Now, we pose the following questions:

- Is the method of implicit differentiation legitimate?
- Does it give the right answer?

In Example 5, we can provide *evidence for the correctness of the method through examples*, which can be solved in two ways.

Example 5 Let us find $\dfrac{dy}{dx}$, if

$$4x^2y - 3y = x^3 - 1.$$

Solution:

Method 1**:** We first write

$$y = \frac{x^3 - 1}{4x^2 - 3},$$

which defines y explicitly. Applying the quotient rule gives

$$\begin{aligned}\frac{dy}{dx} &= \frac{(4x^2-3)\frac{d}{dx}(x^2-1) - (x^2-1)\frac{d}{dx}(4x^2-3)}{(4x^2-3)^2} \\ &= \frac{4x^4 - 9x^2 + 8x}{(4x^2-3)^2}. \qquad (10.3)\end{aligned}$$

Method 2**:** (Implicit Differentiation)

Now, after using the product rule in the LHS and the power rule on the RHS, we obtain

$$4x^2\frac{dy}{dx} + y \cdot 8x - 3\frac{dy}{dx} = 3x^2.$$

Thus,

$$\frac{dy}{dx} = \frac{3x^2 - 8xy}{4x^2 - 3}. \qquad (10.4)$$

This answer looks different from the one obtained at (10.3). However, if we substitute $y = (x^3-1)/(4x^2-3)$ in (10.4), we get the same expression for $\dfrac{dy}{dx}$, that is,

$$\frac{dy}{dx} = \frac{3x^2\left(\dfrac{4x^2-3}{4x^2-3}\right) - 8x\left(\dfrac{x^3-1}{4x^2-3}\right)}{4x^2-3},$$

as in (10.3).

Thus, we observe that *if* an equation in x and y determines a function $y = f(x)$ and *if this function is differentiable, then the method of implicit differentiation* will yield a correct expression for $\dfrac{dy}{dx}$.

(Note the two "*ifs*" in this statement.)

10.3 What type of difficulty does "implicit differentiation" involve?

Two worked examples are:

Example 6 The equation $x^2 + y^2 = -1$ has no solution (why?) and, therefore, does not determine a function. □

Example 7 Consider the relation

$$x^2 + y^2 = 25. \tag{10.5}$$

Which represents a circle with center at the origin and radius 5 units, as shown in Figure 10.2. It does not represent any function of x.

For each x in the open interval $(-5, 5)$, there are two corresponding values of y, namely,

$$y = \sqrt{25 - x^2} \quad \text{and} \quad y = -\sqrt{25 - x^2}$$

They represent two functions, in the interval $(-5, 5)$, given by

$$y = f(x) = \sqrt{25 - x^2} \tag{10.6}$$

and

$$y = g(x) = -\sqrt{25 - x^2}. \tag{10.7}$$

Their graphs are the upper and lower semicircles, respectively, as shown in Figures 10.3 and 10.4.

It may be noted that both functions are differentiable in the open interval $(-5, 5)$, but not at $x = \pm 5$ (since their graphs have vertical tangents at those (end) points). Let us find their derivatives.

First, consider $f(x) = \sqrt{25 - x^2}$. It satisfies $x^2 + [f(x)]^2 = 25$, where

$$f(x) = y.$$

When we differentiate $f(x)$ implicitly and solve for $f'(x)$, we obtain

$$2x + 2f(x)f'(x) = 0.$$

Thus,

$$f'(x) = \frac{-2x}{2f(x)} = -\frac{x}{\sqrt{25 - x^2}}.$$

A completely similar treatment of $g(x)$ yields

$$g'(x) = \frac{-x}{g(x)} = \frac{-x}{-\sqrt{25 - x^2}} = \frac{x}{\sqrt{25 - x^2}}.$$

For practical purposes, *we can obtain both these results simultaneously by the implicit differentiation* of $x^2 + y^2 = 25$. We get

$$2x + 2y\frac{dy}{dx} = 0.$$

Thus

$$\frac{dy}{dx} = \frac{-x}{y} = \begin{cases} -x/\sqrt{25-x^2}, & \text{if } y = f(x) \\ -x/-\sqrt{25-x^2}, & \text{if } y = g(x) \end{cases}$$

It is enough to know that $\frac{dy}{dx} = -\frac{x}{y}$. Suppose, we want to know the slope of the tangent line to the circle $x^2 + y^2 = 25$, when $x = 3$. The corresponding y-values are 4 and -4. The slope at $(3, 4)$ is $-3/4$, and at $(3, -4)$ it is $3/4$.

□

10.3.1 Under what condition can dy/dx be evaluated?

When an equation of the form

$$\phi(x, y) = 0$$

is differentiated implicitly, we get dy/dx in the form of a *quotient*. At certain points (x, y) on the curve, *the denominator of this quotient, representing dy/dx, may become zero.* Indeed, these are the points where the tangent line is vertical and hence the slope of the curve (i.e., dy/dx) is not defined.

Example 8 Find dy/dx, if

$$y^5 + 3y^2 - 2x^2 = -4. \tag{10.8}$$

Solution:

Differentiating both sides of (10.8) with respect to x (using the chain rule), we obtain

$$5y^4\frac{dy}{dx} + 6y\frac{dy}{dx} - 4x = 0.$$

We now solve for dy/dx, obtaining

$$\frac{dy}{dx} = \frac{4x}{5y^4 + 6y}.$$

This formula gives dy/dx *at any point* (x, y) on the curve *where the denominator* $5y^4 + 6y$ *is nonzero.*

For instance, it is easily seen that the point $(2, 1)$ satisfies $y^5 + 3y^2 - 2x^2 = -4$, and therefore it lies on the curve. Then

$$\left.\frac{dy}{dx}\right|_{(2,1)} = \left.\frac{4x}{5y^4 + 6y}\right|_{(2,1)} = \frac{8}{11}.$$

Two intersecting curves are said to be *orthogonal* to each other if the tangent lines at the point of their intersection are perpendicular. □

Example 9 Let us show that the curve $y - x^2 = 0$ is orthogonal to the curve $x^2 + 2y^2 = 3$, at the point of intersection $(1, 1)$.
Solution: The given curve is $y = x^2$. The slope of the tangent line to this curve is given by

$$\frac{dy}{dx} = 2x.$$

Thus,

$$\left.\frac{dy}{dx}\right|_{(1,1)} = 2 = m_1 \text{ (say)}.$$

The other curve is $x^2 + 2y^2 = 3$.

Differentiating both sides of the curve *implicitly* with respect to x, we get

$$2x + 4y\frac{dy}{dx} = 0$$

or

$$\frac{dy}{dx} = \frac{-2x}{4y} = \frac{-x}{2y}$$

or

$$\left.\frac{dy}{dx}\right|_{(1,1)} = \frac{-1}{2} = -\frac{1}{2} = m_2 \text{ (say)}.$$

Since $m_1 \cdot m_2 = -1$, the curve $y = x^2$ is orthogonal to the curve $x^2+2y^2 = 3$, at the point of their intersection $(1, 1)$. □

As a check, Figure 10.5 shows the curve $y-x^2 = 0$ is orthogonal to the curve $x^2+2y^2 = 3$, at the point of intersection $(1, 1)$.

Remark 2 Whenever it is required to find the value of dy/dx at a particular point on a given curve, we can *easily check that the point in question lies on the curve.*

Note 5 Implicit differentiation is useful in computing *related rates.*

Example 10 If

$$x^3 + y^3 = 3axy, \tag{10.9}$$

find dy/dx.

Solution: We have $x^3 + y^3 = 3axy$.

Differentiating both sides of (10.9) with respect to x, we get

$$3x^2 + 3y^2\frac{dy}{dx} = 3a\left(x\frac{dy}{dx} + y \cdot 1\right) = 3ax\frac{dy}{dx} + 3ay$$

or

$$(3y^2 - 3ax)\frac{dy}{dx} = 3ay - 3x^2.$$

Thus,

$$\frac{dy}{dx} = \frac{3(ay - x^2)}{3(y^2 - ax)} = \frac{ay - x^2}{y^2 - ax}.$$

□

Example 11 If $x^y = e^{x-y}$, show that $dy/dx = (\log_e x)/(1+\log_e x)^2$.

Solution: We have $x^y = e^{x-y}$. Then, using the definition of logarithm, we have

$$x - y = \log_e x^y = y \log_e x$$

or

$$x = y + y \log_e x = y(1 + \log_e x)$$

or

$$y = \frac{x}{1+\log_e x}. \tag{10.10}$$

Differentiating both sides of (10.10) with respect to x, we get

$$\begin{aligned}\frac{dy}{dx} &= \frac{(1+\log_e x)(1) - x(0+1/x)}{(1+\log_e x)^2}\\ &= \frac{1+\log_e x - 1}{(1+\log_e x)^2}\\ &= \frac{\log_e x}{(1+\log_e x)^2}.\end{aligned}$$

□

10.4 The method of logarithmic differentiation

The method of logarithmic differentiation is used for (complicated) functions such as general exponential functions and other expressions involving products, quotients, and powers of functions.

Recall that to find the derivative $d(x^n)/dx$, we use the *power rule:*

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

Also, we get

$$\frac{d}{dx}[f(x)^n] = n[f(x)]^{n-1} f'(x).$$

using the power rule and the chain rule.

But, we cannot use the power rule to find $\frac{d(e^x)}{dx}$. Thus, $d(e^x)/dx \neq x \cdot e^{x-1}$.

Recall that, $\frac{d(a^x)}{dx} = a^x \log_e a$, which is the differentiation formula for the exponential function.

Thus, we get

$$\frac{d}{dx}(e^x) = e^x \log_e e = e^x \qquad \text{Since} \log_e e = 1$$

and

$$\frac{d}{dx}[a^{f(x)}] = a^{f(x)} \log_e a \cdot f'(x)$$

using the differentiation formula for exponential function and the chain rule.

10.4.1 What is $d(x^x)/dx$?

Now, we ask the question:

- what can we write for $d(x^x)/dx$?

Of course, it would be complete nonsense to write $d(x^x)/dx = x \cdot x^{x-1}$.

It is for these types of functions, and *more generally for functions of the type* $y = [f(x)]^{g(x)}$, where both $f(x)$ and $g(x)$ are *differentiable functions* of x, that we can use *the technique of logarithmic differentiation* for computing their derivatives.

This technique is also used *to simplify differentiation of many (complicated) functions* involving *products, quotients, and powers of different functions.*

We list below the right technique for differentiating each of the following forms of functions:

$$\begin{aligned}
[f(x)]^n &\rightarrow \text{Power rule} \\
y = a^{f(x)} &\rightarrow \text{Differentiation formula for exponential functions} \\
[f(x)]^{g(x)} &\rightarrow \text{Logarithmic differentiation}
\end{aligned}$$

Remark 3 The technique of logarithmic differentiation is so powerful that it can be used for each of these forms.

10.5 Procedure of logarithmic differentiation

Basically the procedure is to take the *natural logarithm* of both sides, then use implicit differentiation to differentiate both sides, and solve for dy/dx. The usefulness of the process is due to the fact that the differentiation of the product of the functions is reduced to that of a sum; of their quotients to that of a difference; and of the general exponential to that of the product of simpler functions.

The following solved examples will illustrate the process of *logarithmic differentiation.* First, we start with the differentiation of certain (complicated) functions involving *products, quotients, and powers of functions.*

Example 12 If $y = e^{5x} \sin 2x \cos x$, find dy/dx.

Solution:

We take

$$y = e^{5x} \sin 2x \cos x. \tag{10.11}$$

(i) Taking the natural logarithm of both sides of (10.11), we get

$$\log_e y = \log_e e^{5x} + \log_e \sin 2x + \log_e \cos x. \tag{10.12}$$

(ii) Differentiating both sides of (10.12) with respect to x, we get

$$\begin{aligned}
\frac{1}{y}\frac{dy}{dx} &= \frac{1}{e^{5x}}\frac{d}{dx}(e^{5x}) + \frac{1}{\sin 2x}\frac{d}{dx}(\sin 2x) + \frac{1}{\cos x}\frac{d}{dx}(\cos x) \\
&= \frac{1}{e^{5x}}e^{5x} \cdot 5 + \frac{1}{\sin 2x}\cos 2x \cdot 2 + \frac{1}{\cos x}(-\sin x) \\
&= 5 + 2\cot 2x - \tan x.
\end{aligned}$$

(iii) Thus,

$$\begin{aligned}\frac{dy}{dx} &= y[5 + 2\cot 2x - \tan x] \\ &= e^{5x}\sin 2x\cos x[5 + 2\cot 2x - \tan x].\end{aligned}$$

□

Example 13 If $y = e^{4x}\sin^2 x\tan^3 x$, find dy/dx.
Solution:
We have

$$y = e^{4x}\sin^2 x\tan^3 x. \tag{10.13}$$

(i) Taking the natural logarithm of both sides of (10.13), we get

$$\begin{aligned}\log_e y &= \log_e e^{4x} + \log_e \sin^2 x + \log_e \tan^3 x \\ &= 4x + 2\log_e \sin x + 3\log_e \tan x.\end{aligned} \tag{10.14}$$

(ii) Differentiating both sides of (10.14) with respect to x, we get

$$\begin{aligned}\frac{1}{y}\frac{dy}{dx} &= 4 + 2\frac{1}{\sin x}\cos x + 3\frac{1}{\tan x}\sec^2 x \\ &= 4 + 2\cot x + \frac{3}{\sin x\cdot\cos x}.\end{aligned}$$

(iii) Thus,

$$\begin{aligned}\frac{dy}{dx} &= y\left[4 + 2\cot x + \frac{3}{\sin x\cdot\cos x}\right] \\ &= e^{4x}\sin^2 x\tan^3 x\left[4 + 2\cot x + \frac{3}{\sin x\cdot\cos x}\right].\end{aligned}$$

□

Example 14 If $y = \sqrt{\dfrac{(1+x)(2+x)}{(1-x)(2-x)}}$, find $\frac{dy}{dx}$.

Solution:
We have,

$$y = \sqrt{\frac{(1+x)(2+x)}{(1-x)(2-x)}}. \tag{10.15}$$

(i) Taking the natural logarithm of both sides of (10.15), we get

$$\begin{aligned}\log_e y &= \frac{1}{2}[\log_e(1+x)(2+x) - \log_e(1-x)(2-x)] \\ &= \frac{1}{2}[\log_e(1+x) + \log_e(2+x) - \log_e(1-x) - \log_e(2-x)].\end{aligned} \tag{10.16}$$

(ii) Differentiating both sides of (10.16) with respect to x, we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{1+x} + \frac{1}{2+x} - \frac{1}{1-x}(-1) - \frac{1}{2-x}(-1)\right].$$

(iii) Thus,

$$\begin{aligned}\frac{dy}{dx} &= \frac{y}{2}\left[\frac{1}{1+x} + \frac{1}{2+x} + \frac{1}{1-x} + \frac{1}{2-x}\right] \\ &= \frac{y}{2}\left[\frac{(1-x)+(1+x)}{(1+x)(1-x)} + \frac{(2-x)+(2+x)}{(2+x)(2-x)}\right] \\ &= \frac{y}{2}\left[\frac{2}{1-x^2} + \frac{4}{4-x^2}\right] \\ &= y\left[\frac{1}{1-x^2} + \frac{2}{4-x^2}\right] \\ &= y\left[\frac{4-x^2+2-2x^2}{(1-x^2)(4-x^2)}\right] \\ &= \sqrt{\frac{(1+x)(2+x)}{(1-x)(2-x)}}\left[\frac{6-3x^2}{(1-x^2)(4-x^2)}\right].\end{aligned}$$

□

Now, we consider functions of the type $[f(x)]^{g(x)}$. Here, it may be mentioned that such functions do not occur naturally. However, to demonstrate the power of the technique of logarithmic differentiation, we solve the following examples.

Example 15 If $y = 5^{\tan x}$, find dy/dx.

Solution:
We have,

$$y = 5^{\tan x}. \tag{10.17}$$

(i) Taking the natural logarithm of both sides of (10.17), we get

$$\log_e y = \tan x \cdot \log_e 5. \tag{10.18}$$

(ii) Differentiating both sides of (10.18) with respect to x, we get

$$\frac{1}{y}\frac{dy}{dx} = \sec^2 x \cdot \log_e 5.$$

(iii) Thus,

$$\begin{aligned}\frac{dy}{dx} &= y[\sec^2 x \cdot \log_e 5] \\ &= 5^{\tan x}[\sec^2 x \cdot \log_e 5].\end{aligned}$$

□

Example 16 If $y = x^x$, find dy/dx.

Solution:

We have,

$$y = x^x. \tag{10.19}$$

(i) Taking the natural logarithm of both sides of (10.19), we get

$$\log_e y = x \log_e x. \tag{10.20}$$

(ii) Differentiating both sides of (10.20) with respect to x, we have

$$\begin{aligned}\frac{1}{y}\frac{dy}{dx} &= x\left(\frac{1}{x}\right) + (\log_e x)(1)\\ &= 1 + \log_e x.\end{aligned}$$

(iii) Thus,

$$\frac{dy}{dx} = y(1 + \log_e x) = x^x(1 + \log_e x). \tag{10.21}$$

□

Example 17 If $y = x^{x^x}$, find dy/dx.

Solution:

Method I We have,

$$y = (x)^{x^x}. \tag{10.22}$$

(i) Taking the natural logarithm of each side, we get

$$\log_e y = x^x \log_e x. \tag{10.23}$$

(ii) Differentiating both sides of (10.23) with respect to x, we get

$$\begin{aligned}\frac{1}{y}\frac{dy}{dx} &= x^x \frac{d}{dx}\log_e x + \log_e x \frac{d}{dx}(x^x)\\ &= x^x \frac{1}{x} + \log_e x \frac{d}{dx}(x^x)\\ &= x^{x-1} + \log_e x[x^x(1 + \log_e x)] \quad \text{Because } \frac{d}{dx}(x^x) = x^x(1 + \log_e x), \text{ from (10.21)}\\ &= x^{x-1} + x^x \log_e x(1 + \log_e x).\end{aligned}$$

(iii) Thus,

$$\begin{aligned}\frac{dy}{dx} &= y[x^{x-1} + x^x \log_e x(1 + \log_e x)] \\ &= x^{x^x}[x^{x-1} + x^x \log_e x(1 + \log_e x)].\end{aligned}$$

Method II The process requires taking the natural logarithm twice.

Taking the natural logarithm of each side, we get

$$\log_e y = x^x \log_e x. \tag{10.24}$$

ii Taking the natural logarithm of each side, we get

$$\log_e(\log_e y) = x \log_e x + \log_e(\log_e x). \tag{10.25}$$

iii Differentiating both sides of (10.25) with respect to x, we get

$$\frac{1}{\log_e y}\frac{1}{y}\frac{dy}{dx} = \left(x\frac{1}{x} + \log_e x(1)\right) + \frac{1}{\log_e x}\frac{1}{x}.$$

iv Thus,

$$\begin{aligned}\frac{dy}{dx} &= y \log_e y \left[1 + \log_e x + \frac{1}{x \log_e x}\right] \\ &= x^{x^x} \cdot x^x \log_e x \left[1 + \log_e x + \frac{1}{x \log_e x}\right]. \qquad \text{Use (10.21).}\end{aligned}$$

□

Example 18 If $y = (x^x)^x$, then find dy/dx.

Solution:
We have

$$y = (x^x)^x = x^{x \cdot x} = x^{x^2}. \tag{10.26}$$

(i) Taking the natural logarithm on both sides of (10.26), we get

$$\log_e y = x^2 \log_e x. \tag{10.27}$$

(ii) Differentiating both sides of (10.26) with respect to x, we get

$$\begin{aligned}\frac{1}{y}\frac{dy}{dx} &= x^2\frac{1}{x} + (\log_e x)(2x) \\ &= x + 2x \log_e x.\end{aligned}$$

(iii) Thus,

$$\begin{aligned}\frac{dy}{dx} &= y[x + 2x\log_e x] \\ &= x^{x^2} \cdot x[1 + 2\log_e x] \\ &= x^{x^2+1}[1 + 2\log_e x].\end{aligned}$$

□

Example 19 If $y = (\log_e x)^x$, find dy/dx.

Solution:

We have

$$y = (\log_e x)^x. \tag{10.28}$$

(i) Taking the natural logarithm on both sides of (10.26), we get

$$\log_e y = x^2 \log_e x. \tag{10.29}$$

(ii) Differentiating both sides of (10.29) with respect to x, we get

$$\begin{aligned}\frac{1}{y}\frac{dy}{dx} &= x\frac{d}{dx}[\log_e(\log_e x)] + \log_e(\log_e x)\frac{d}{dx}(x) \\ &= x\frac{1}{\log_e x}\frac{1}{x} + \log_e(\log_e x) \cdot 1 \\ &= \frac{1}{\log_e x} + \log_e(\log_e x).\end{aligned}$$

(iii) Thus,

$$\begin{aligned}\frac{dy}{dx} &= y\left[\frac{1}{\log_e x} + \log_e(\log_e x)\right] \\ &= (\log_e x)^x\left[\frac{1}{\log_e x} + \log_e(\log_e x)\right].\end{aligned}$$

□

Example 20 If $y = (\cos x)^{\sin x}$, find dy/dx.

Solution:

We have

$$y = (\cos x)^{\sin x}. \tag{10.30}$$

(i) Taking the natural logarithm on both sides of (10.30), we get

$$\log_e y = \sin x \cdot \log_e \cos x. \tag{10.31}$$

(ii) Differentiating both sides of (10.31) with respect to x, we get

$$\begin{aligned}\frac{1}{y}\frac{dy}{dx} &= \sin x\left[\frac{1}{\cos x}(-\sin x)\right] + (\log_e \cos x)(\cos x)\\ &= -\frac{\sin^2 x}{\cos x} + \cos x \cdot \log_e \cos x.\end{aligned}$$

(iii) Thus,

$$\begin{aligned}\frac{dy}{dx} &= y\left[\cos x \cdot \log_e \cos x - \frac{\sin^2 x}{\cos x}\right]\\ &= (\cos x)^{\sin x}\left[\cos x \cdot \log_e \cos x - \frac{\sin^2 x}{\cos x}\right].\end{aligned}$$

□

Example 21 If $y = (\tan x)^{\log_e x}$, find dy/dx.

Solution:
We have

$$y = (\tan x)^{\log_e x}. \tag{10.32}$$

(i) Taking the natural logarithm on both sides of (10.32), we get

$$\log_e y = \log_e x \cdot \log_e(\tan x). \tag{10.33}$$

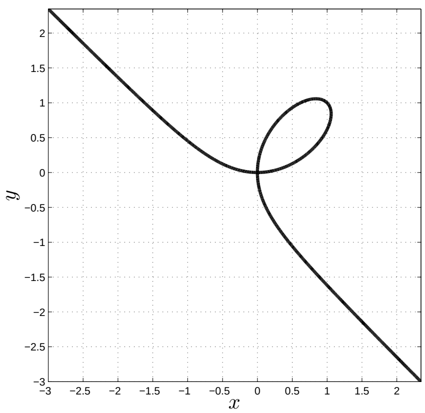
(ii) Differentiating both sides of (10.33) with respect to x, we get

$$\begin{aligned}\frac{1}{y}\frac{dy}{dx} &= \log_e x \cdot \frac{1}{\tan x} \cdot \sec^2 x + \log_e(\tan x) \cdot \frac{1}{x}\\ &= \log_e x \cdot \frac{\cos x}{\sin x} \cdot \frac{1}{\cos^2 x} + \frac{\log_e(\tan x)}{x}.\end{aligned}$$

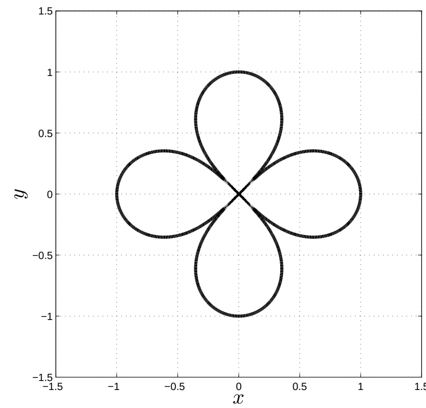
(iii) Thus,

$$\begin{aligned}\frac{dy}{dx} &= y\left[\frac{\log_e x}{\sin x \cos x} + \frac{\log_e(\tan x)}{x}\right]\\ &= (\tan x)^{\log_e x}\left[\frac{\log_e x}{\sin x \cos x} + \frac{\log_e(\tan x)}{x}\right].\end{aligned}$$

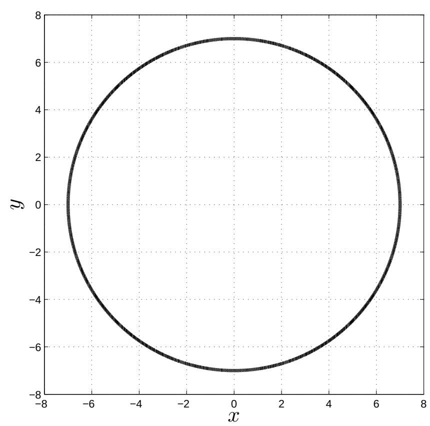
□



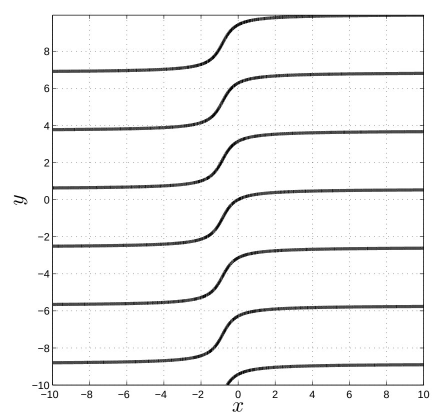
(a) $x^3+y^3=2xy$, with $-3 \le x, y \le 3$.



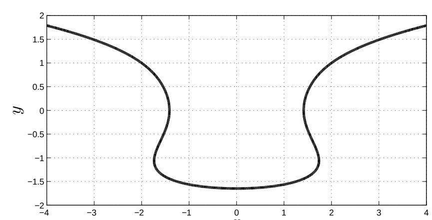
(b) $(x^2+y^2)^4 = (x^2-y^2)^2$, with $-1.5 \le x, y \le 1.5$.



(c) $x^2+y^2=49$, with $-6 \le x, y \le 6$.



(d) $\sin y = x\sin(1+y)$, with $a=1$ and $-10 \le x, y \le 10$.



(e) $y^5+3y^2-2x^2=-4$, with $-4 \le x \le 4$ and $-2 \le y \le 2$.

Figure 10.1: Graphs of the functions in Example 2.

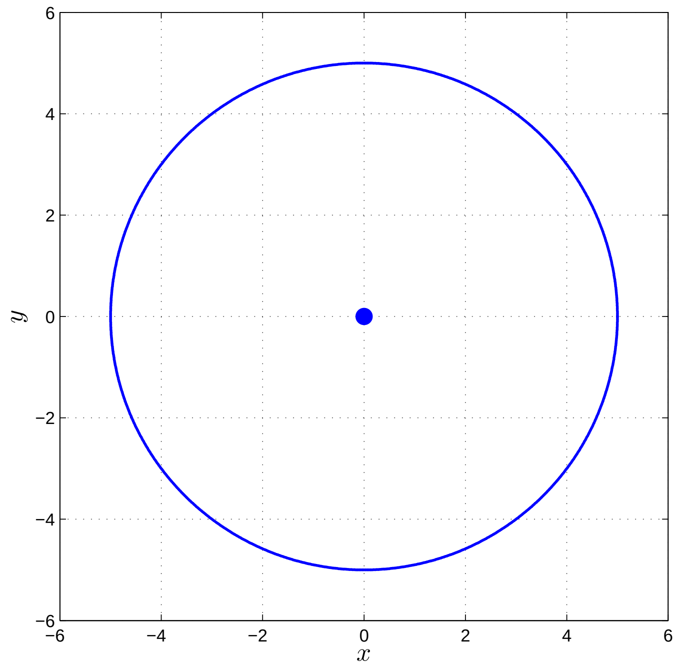


Figure 10.2: $y = f(x) = x^2 + y^2 = (5)^2 = 25$.

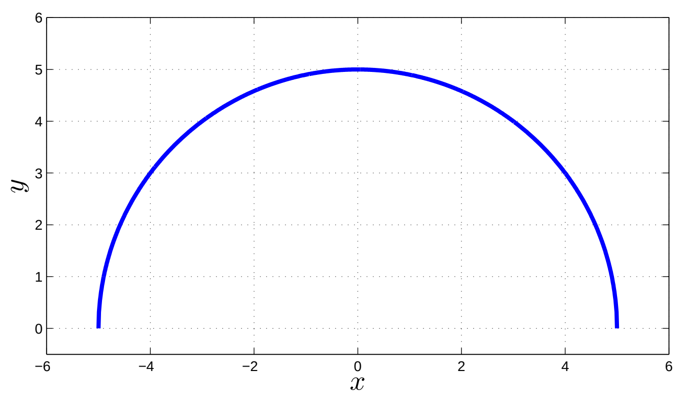


Figure 10.3: $y = f(x) = \sqrt{25 - x^2}$.

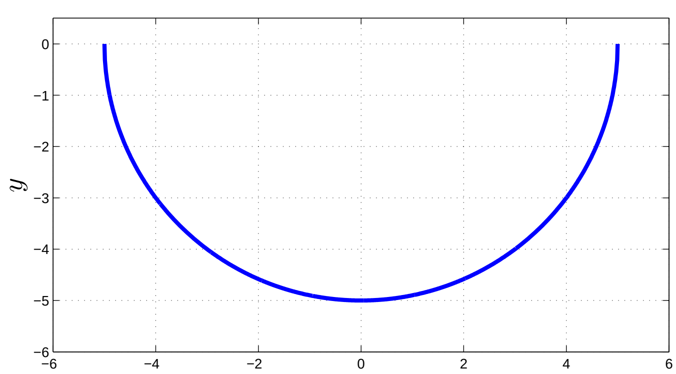


Figure 10.4: $y = g(x) = -\sqrt{25 - x^2}$.

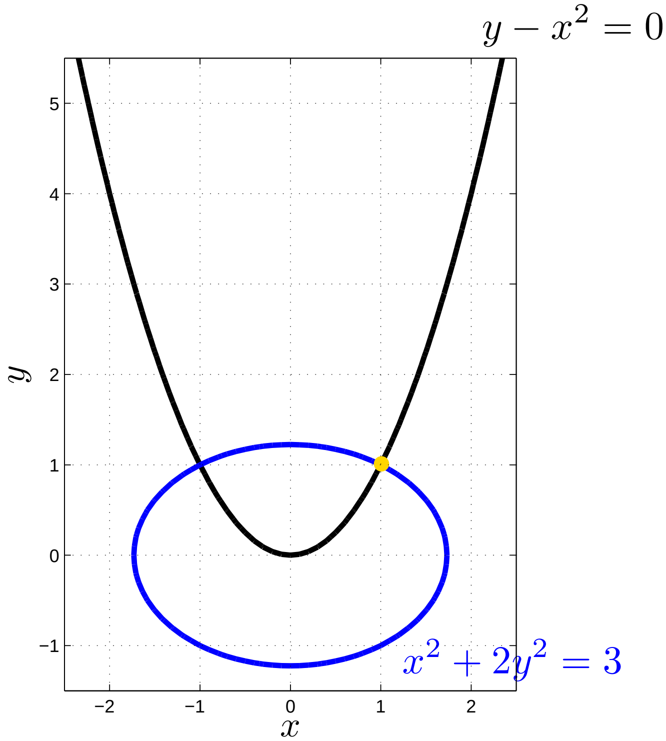


Figure 10.5: $y - x^2 = 0$ (black) and $x^2 + 2y^2 = 3$ (blue).