THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1510 Calculus for Engineers (2020-2021)
Supplementary Exercise 4

Sandwich Theorem

1. Fill in the blanks and construct an upper bound and a lower bound of $\dfrac{3}{6+4\cos\theta}$.

$$\begin{array}{ccccc} -1 & \leq & \cos\theta & \leq & 1 \\ \_\_\_\_\_ & \leq & 4\cos\theta & \leq & 4 \\ \_\_\_\_\_ & \leq & 6+4\cos\theta & \leq & 10 \\ \dfrac{1}{10} & \leq & \dfrac{1}{6+4\cos\theta} & \leq & \_\_\_\_\_ \\ \dfrac{3}{10} & \leq & \dfrac{3}{6+4\cos\theta} & \leq & \_\_\_\_\_ \end{array}$$

2. (a) Prove that $\dfrac{1}{3} \leq \dfrac{2+\sin 3x}{3} \leq 1$.

   (b) By using (a) and the sandwich theorem, prove that $\lim\limits_{x\to+\infty} \dfrac{2+\sin 3x}{3e^x} = 0$.

3. Show that $\lim\limits_{x\to 0} x \sin\left(\dfrac{1}{e^x - e^{-x}}\right) = 0$.

4. Show that $\lim\limits_{x\to+\infty} \dfrac{e^{\cos x}}{x} = 0$.

Continuity of Functions

5. Let $f(x)$ be a function defined by

$$f(x) = \begin{cases} \dfrac{\sin x}{2x} & \text{if} \quad x \neq 0, \\ a & \text{if} \quad x = 0. \end{cases}$$

   If $f(x)$ is a continuous at $x = 0$, then what is the value of a?

6. Let $f(x)$ be a function defined by

$$f(x) = \begin{cases} \log_{10} x & \text{if} \quad x > 10, \\ a & \text{if} \quad x = 10 \\ mx - 1 & \text{if} \quad x \leq 10, \end{cases}$$

   where m, a are real numbers.

(a) Find $\lim_{x\to 10^+} f(x)$.

(b) Find $\lim_{x\to 10^-} f(x)$ in terms of m.

(c) If $f(x)$ is continuous at $x = 10$, what is the values of a and m.

7. Let a be a real number and let $f(x)$ be a function defined by

$$f(x) = \begin{cases} e^x & \text{if} \quad x > 0, \\ 1 & \text{if} \quad x = 0, \\ \cos x & \text{if} \quad x < 0. \end{cases}$$

(a) Find $\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0^+} f(x)$.

(b) Is $f(x)$ is continuous at $x = 0$?

8. Let a and b are real numbers and let $f(x)$ be a function defined by

$$f(x) = \begin{cases} x^2 & \text{if} \quad x > 3, \\ b & \text{if} \quad x = 3, \\ 2x + a & \text{if} \quad x < 3. \end{cases}$$

Given that $f(x)$ is continuous at $x = 3$. What are the values of a and b?

9. Let $f(x)$ be a function defined by

$$f(x) = \begin{cases} x^2 & \text{if} \quad x > 1, \\ ax + b & \text{if} \quad -1 \le x \le 1, \\ \sin \pi x & \text{if} \quad x < -1. \end{cases}$$

If $f(x)$ is a continuous function, find the value of a and b.

(Hint: In particular, $f(x)$ is continuous at $x = -1$ and $x = 1$.)

10. Let $f(x)$ be a function defined by

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & \text{if} \quad x \neq 0, \\ 0 & \text{if} \quad x = 0. \end{cases}$$

Prove that $f(x)$ is continuous at $x = 0$.

11. Let $f(x) = |x|$. Prove that $f(x)$ is a continuous function, i.e. $f(x)$ is continuous at every point a.

(Hint: Consider three cases, $a > 0$, $a = 0$ and $a < 0$.)

Intermediate Value Theorem

12. By using the intermediate-value theorem, show that the equation $2^x = 10 - x$ has at least one solution on $[2, 3]$, i.e. there exists $c \in [2, 3]$ such that $2^c = 10 - c$.
    (Hint: Consider the function $f(x) = 2^x + x - 10$.)

13. Show that the equation $4^x = 3^x + 2^x$ has at least one solution.