THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1510 Calculus for Engineers (Fall 2021)
Suggested solutions of coursework 3

Part A

1. Without using L'Hôpital's rule, evaluate the following limits. Furthermore, if the limit does not exist but diverges to $\pm\infty$, please indicate so and determine the correct sign.

   (a) $\displaystyle\lim_{x\to 9}\frac{\sqrt{x+7}-4}{x-9}$

   (b) $\displaystyle\lim_{x\to\infty} x(x-\sqrt{x^2+1})$

   Solution:

   (a)

$$
\begin{aligned}
\lim_{x\to 9}\frac{\sqrt{x+7}-4}{x-9} &= \lim_{x\to 0}\frac{\sqrt{x+7}-4}{x-9}\cdot\frac{\sqrt{x+7}+4}{\sqrt{x+7}+4}\\
&= \lim_{x\to 9}\frac{(x+7)-16}{(x-9)\sqrt{x+7}+4)}\\
&= \lim_{x\to 9}\frac{1}{\sqrt{x+7}+4}\\
&= \frac{1}{8}.
\end{aligned}
$$

   (b)

$$
\begin{aligned}
\lim_{x\to\infty} x(x-\sqrt{x^2+1}) &= \lim_{x\to\infty} x(x-\sqrt{x^2+1})\cdot\frac{x+\sqrt{x^2+1}}{x+\sqrt{x^2+1}}\\
&= \lim_{x\to\infty}\frac{x(x^2-(x^2+1))}{x+\sqrt{x^2+1}}\\
&= \lim_{x\to\infty}\frac{-x}{x+\sqrt{x^2+1}}\\
&= \lim_{x\to\infty}\frac{x}{x}\cdot\frac{-1}{1+\sqrt{1+\frac{1}{x^2}}}\\
&= -\frac{1}{2}.
\end{aligned}
$$

2. Suppose that

$$f(x) = \begin{cases} 1-x & \text{if } x < 1; \\ 2 & \text{if } x = 1; \\ \ln x & \text{if } x > 1. \end{cases}$$

(a) Find $\lim_{x\to 1^-} f(x)$ and $\lim_{x\to 1^+} f(x)$.

(b) Determine if f is continuous at $x = 1$.

Solution:

(a)

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(1-x) = 0.$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \ln x = \ln 1 = 0.$$

(b) Note that

$$\lim_{x\to 1} f(x) = 0 \neq 2 = f(1).$$

So, f is not continuous at $x = 1$.

Part B

3. Without using L'Hôpital's rule, evaluate the following limits. Furthermore, if the limit does not exist but diverges to $\pm\infty$, please indicate so and determine the correct sign.

(a) $\lim_{x\to 0} \dfrac{\tan 4x}{\sin 2x}$

(b) $\lim_{x\to\infty} \left(1 + \dfrac{1}{3x-1}\right)^{2x+1}$

(c) $\lim_{x\to\frac{\pi}{2}} \dfrac{\cos 3x}{\frac{\pi}{2} - x}$

(Hint: Let $y = \dfrac{\pi}{2} - x$. You may use the formula

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta)$$

Solution:

(a)

$$\begin{aligned} \lim_{x\to 0} \frac{\tan 4x}{\sin 2x} &= \lim_{x\to 0} \left(\frac{1}{\cos 4x}\right)\left(\frac{\sin 4x}{4x}\right)\left(\frac{2x}{\sin 2x}\right)\left(\frac{4x}{2x}\right) \\ &= (1)(1)(1)\left(\frac{4}{2}\right) \\ &= 2. \end{aligned}$$

(b)

$$\begin{aligned}
\lim_{x\to\infty}\left(1+\frac{1}{3x-1}\right)^{2x+1} &= \lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{2\left(\frac{y+1}{3}\right)+1} && \text{(Let } y=3x-1\text{)}\\
&= \lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{\frac{2}{3}y+\frac{5}{3}}\\
&= \left(\lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{y}\right)^{\frac{2}{3}}\cdot\left(\lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{\frac{5}{3}}\right)\\
&= e^{\frac{2}{3}}.
\end{aligned}$$

(c)

$$\begin{aligned}
\lim_{x\to\frac{\pi}{2}}\frac{\cos 3x}{\frac{\pi}{2}-x} &= \lim_{y\to 0}\frac{\cos\left(3\left(\frac{\pi}{2}-y\right)\right)}{y} && \text{(Let } y=\frac{\pi}{2}-x\text{)}\\
&= \lim_{y\to 0}\frac{\cos\left(\frac{3\pi}{2}-3y\right)}{y}\\
&= \lim_{y\to 0}\frac{\cos\left(\frac{3\pi}{2}\right)\cos 3y+\sin\left(\frac{3\pi}{2}\right)\sin 3y}{y}\\
&= \lim_{y\to 0}\left(-\frac{\sin 3y}{3y}\right)\cdot 3\\
&= -3.
\end{aligned}$$

4. Suppose that

$$f(x)=\begin{cases} 2+e^{\frac{1}{x}} & \text{if } x<0;\\ ax+2 & \text{if } 0\le x<1;\\ x^2 & \text{if } x\ge 1.\end{cases}$$

where a is a real number.

(a) Show that f is continuous at $x=0$ for any real number a.

(b) Given that f is continuous at $x=1$, find the value(s) of a.

Solution:

(a) For any $a\in\mathbb{R}$,

$$\begin{aligned}
\lim_{x\to 0^-}f(x) &= \lim_{x\to 0^-}\left(2+e^{\frac{1}{x}}\right)=2,\\
\lim_{x\to 0^+}f(x) &= \lim_{x\to 0^+}(ax+2)=2,
\end{aligned}$$

So,

$$\lim_{x\to 0}f(x)=2=f(0)$$

and f is continuous at $x=0$.

(b)

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (ax+2) = 2+a,$$
$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \left(x^2\right) = 1.$$

$$\begin{aligned} & f \text{ continuous at } x = 1 \\ \implies & \lim_{x\to 1^-} f(x) = 2+a = 1 = \lim_{x\to 1^+} f(x) \\ \implies & a = -1. \end{aligned}$$

5. Show that the equation $4^x = 3^x + 2^x$ has at least one real solution.

   (Hint: Consider the function $f(x) = 4^x - 3^x - 2^x$.)

   Solution:

   Let $f(x) = 4^x - 3^x - 2^x$, which is continuous over its domain $D_f = \mathbb{R}$.

   $$f(0) = -1 < 0,$$
   $$f(2) = 16 - 9 - 4 = 3 > 0.$$

   Since f is continuous over $[0,2]$ and $f(0)$, $f(2)$ have opposite signs, by Bolzano's Theorem,

   $$f(c) = 0 \qquad \text{for some } c \in (0,2).$$

   Hence, $4^c - 3^c - 2^c = 0$, that is, $4^c = 3^c + 2^c$.