THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1510 Calculus for Engineers (Fall 2021)
Homework 5
Deadline: November 27 at 23:00

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Class: MATH1510G

I acknowledge that I am aware of University policy and regulations on honesty in academic work, and of the disciplinary guidelines and procedures applicable to breaches of such policy and regulations, as contained in the website http://www.cuhk.edu.hk/policy/academichonesty/

David — Signature

20-11-2021 — Date

General Guidelines for Homework Submission.

- Please submit your answer to Gradescope through the centralized course MATH1510A-I in Blackboard.
- In Gradescope, for each question, please indicate exactly which page(s) its answer locates. **Answers of incorrectly matched questions will not be graded**.
- **Late submission will NOT be graded and result in zero score.** Any answers showing evidence of plagiarism will also score zero; stronger disciplinary action may also be taken.
- Points will only be awarded for answers with sufficient justifications.
- All questions in **Part A** along with some selected questions in **Part B** will be graded. Question(s) labeled with * are more challenging.

Part A:

1. Evaluate the following definite integrals.

   (a) $\int_0^2 e^{\sqrt{x}} dx.$

   (b) $\int_{2/\sqrt{3}}^2 \frac{\sqrt{x^2-1}}{x} dx;$

(a) Let $u = \sqrt{x}$, $du = \frac{1}{2\sqrt{x}} dx$

$$\int_0^{\sqrt{2}} 2u e^u \, du$$

$$= 2\left[ue^u\right]_0^{\sqrt{2}} - 2\int_0^{\sqrt{2}} e^u \, du$$

$$= 2\left(\sqrt{2}e^{\sqrt{2}}\right) - 2\left(e^{\sqrt{2}} - 1\right)$$

$$= 2e^{\sqrt{2}}\left(\sqrt{2}-1\right) + 2 \quad //$$

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(b) Let $u = \sqrt{x^2 - 1}$, $du = \frac{1}{2\sqrt{x^2-1}}(2x)dx$

$$= \frac{x}{\sqrt{x^2-1}} dx$$

$$\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{u^2}{u^2+1} du$$

$$= \left[u \right]_{1/\sqrt{3}}^{\sqrt{3}} - \left[\tan^{-1}(u) \right]_{1/\sqrt{3}}^{\sqrt{3}}$$

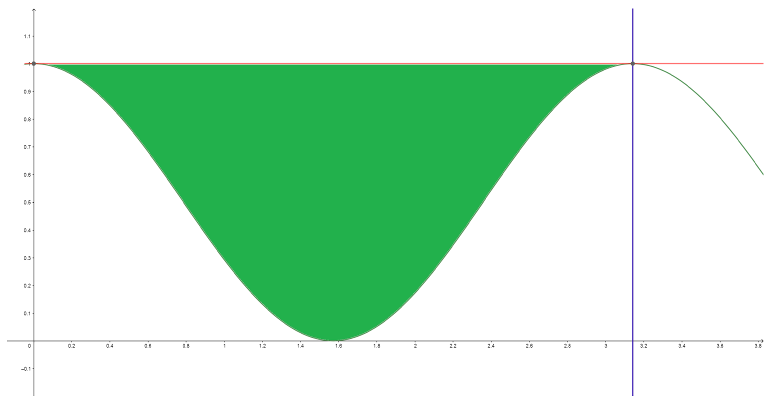
$$= \sqrt{3} - \frac{1}{\sqrt{3}} - \frac{\pi}{3} + \frac{\pi}{6}$$

$$= \sqrt{3} - \frac{1}{\sqrt{3}} - \frac{\pi}{6} \quad //$$

2. Let R be the region bounded between the curves $y = 1$ and $y = \cos^2 x$ for $0 \le x \le \pi$.

   (a) Find the volume of the solid generated by rotating the region R about the x-axis.

   (b) Find the volume of the solid generated by rotating the region R about the line $y = 1$.



(a) $$\int_0^\pi \pi (1)^2\,dx - \int_0^\pi \pi (\cos^2 x)^2\,dx$$

$$= \pi \big[x\big]_0^\pi - \pi \int_0^\pi \left(\frac{1+\cos(2x)}{2}\right)^2 dx$$

$$= \pi \big[x\big]_0^\pi - \frac{\pi}{4}\int_0^\pi \left[1 + 2\cos(2x) + \cos^2(2x)\right] dx$$

$$= \pi^2 - \frac{\pi}{4}\int_0^\pi \left[1 + 2\cos(2x) + \frac{1+\cos(4x)}{2}\right] dx$$

$$= \pi^2 - \frac{\pi}{8}\int_0^\pi \left[3 + 4\cos(2x) + \cos(4x)\right] dx$$

$$= \pi^2 - \frac{\pi}{8}\left[3x + 2\sin(2x) + \frac{1}{4}\sin(4x)\right]_0^\pi$$

$$= \frac{5\pi^2}{8} \quad //$$

(b) $\int_0^{\pi} \pi (1-1)^2 \, dx - \int_0^{\pi} \pi (\cos^2 x - 1)^2 \, dx$

$= \pi \int_0^{\pi} (\sin^2 x) \, dx$

$= \pi \int_0^{\pi} \left(\frac{1 - \cos 2x}{2} \right) dx$

$= \frac{\pi}{2} \int_0^{\pi} (1 - \cos 2x) \, dx$

$= \frac{\pi}{2} \left[x - \frac{1}{2} \sin 2x \right]_0^{\pi}$

$= \frac{\pi^2}{2}$ //

$$-6y^2+7y=0$$

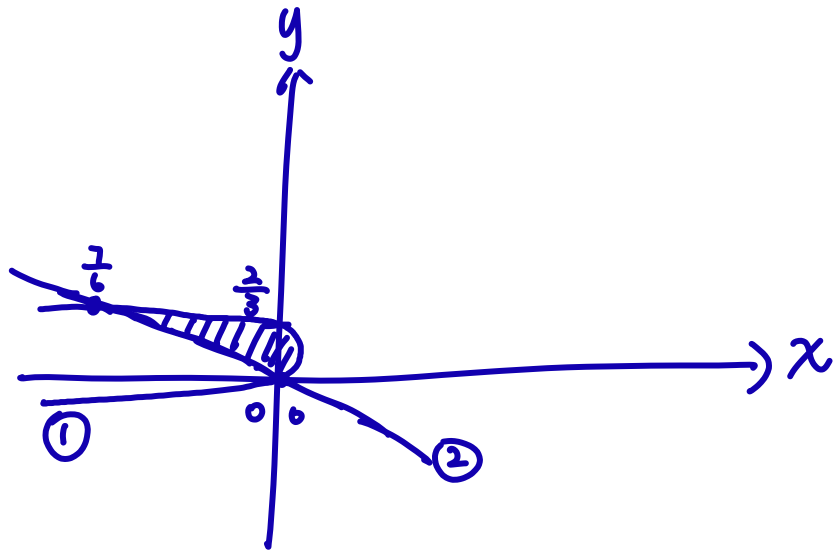
Part B:

3. Let R be the region bounded by curve $x=-6y^2+4y$ and the line $x+3y=0$ on the $xy-$plane. Find the area of R.

Required area :

$$\int_0^{\frac{7}{6}}(-6y^2+4y)\,dy$$

$$-\int_0^{\frac{7}{6}}(-3y)\,dy$$



$$=\int_0^{\frac{7}{6}}(7y-6y^2)\,dy$$

$$=\left[\frac{7}{2}y^2-2y^3\right]_0^{\frac{7}{6}}$$

$$=\frac{343}{216}\quad //$$

4. A particle moves in a straight line with speed $v(t) = t^2 + 2t$, where $t \in [0, 9]$ is the time.

   (a) Find the average speed v^* of the particle between $t = 0$ and $t = 9$.

   (b) Find the time $t^* \in [0, 9]$ when the particle moves in the average speed v^*.

(a) Average speed v^*: $\frac{1}{9}\int_0^9 (t^2+2t)\,dt$

$$= \frac{1}{9}\left[\frac{1}{3}t^3 + t^2\right]_0^9$$

$$= 36 \quad //$$

(b) The required time t^*: $t^2 + 2t = 36$

$$t^2 + 2t - 36 = 0$$

$$t = \frac{-2 \pm \sqrt{4-(4)(-36)}}{2}$$

$$= -1 + \sqrt{37} \quad //$$

or $-1 - \sqrt{37}$ (rejected.)

5. Evaluate

$$\lim_{x\to 0} \frac{\int_0^{2x} \sin(e^t - e^{-t})\,dt}{x\sin x}.$$

$\frac{?}{0}$

By L'hôpital's Rule, we have:

$$\lim_{x\to 0} \frac{\frac{d}{dx}\left[\int_0^{2x} \sin(e^t - e^{-t})\,dt\right]}{\sin x + x\cos x}$$

$$= \lim_{x\to 0} \frac{2\sin(e^{2x} - e^{-2x})}{\sin x + x\cos x}$$

$$= \lim_{x\to 0} \frac{2\cos(e^{2x} - e^{-2x}) \times (2e^{2x} + 2e^{-2x})}{2\cos x - x\sin x}$$

$$= \frac{2\times(2+2)}{2}$$

$$= 4 \quad //$$

6. By considering Riemann sum of a suitable integral, evaluate each of the following limits.

   (a) $\displaystyle\lim_{n\to\infty}\left(\frac{1}{n}e^{1/n}+\frac{1}{n}e^{2/n}+\cdots+\frac{1}{n}e^{n/n}\right)$

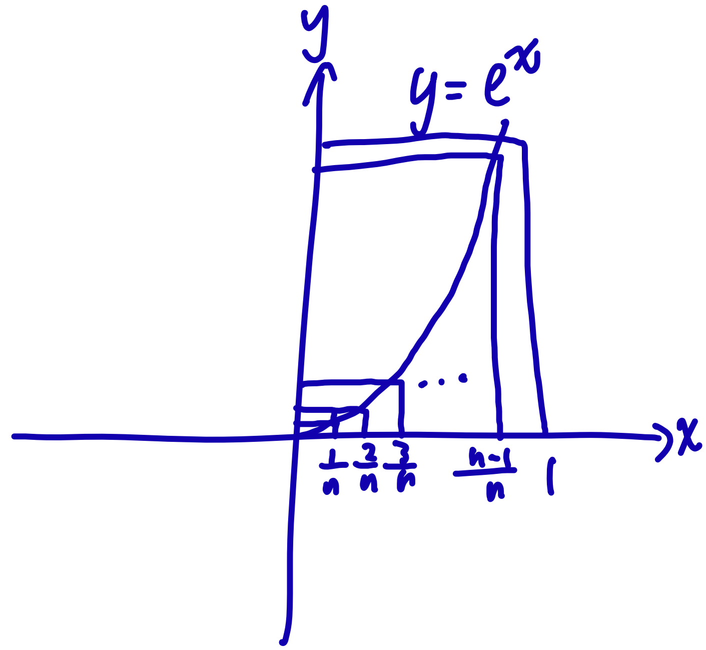
   (b) $\displaystyle\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+n}\right)$

(a) $\displaystyle\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{1}{n}\right)e^{\frac{k}{n}}$

$$= \int_0^1 (e^x)\,dx$$

$$= \left[e^x\right]_0^1$$

$$= e-1 \;//$$

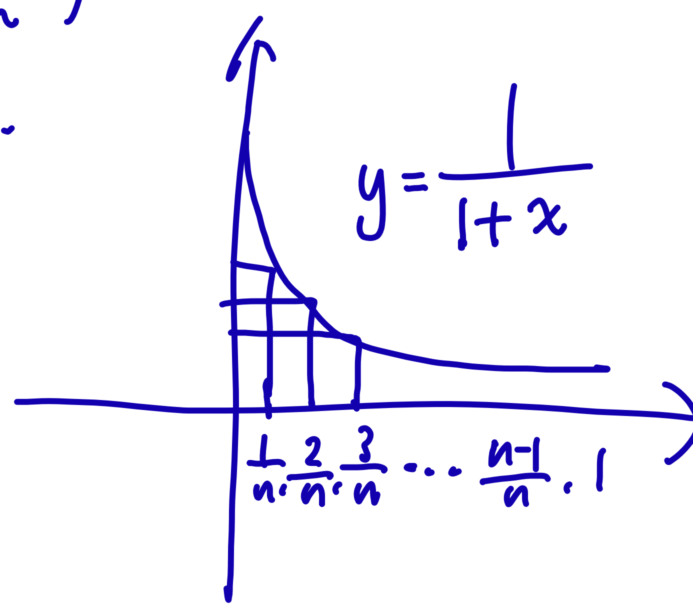


(b) $\displaystyle\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{1}{n}\right)\left(\frac{1}{1+\frac{k}{n}}\right)$

$$= \int_0^1 \left(\frac{1}{1+x}\right)dx$$

$$= \Big[\ln|1+x|\Big]_0^1$$

$$= \ln 2 \;//$$



7. Evaluate the following indefinite integrals and improper integrals.

(a) $\displaystyle\int \frac{1}{x^2+3x+2}\,dx$, and $\displaystyle\int_0^\infty \frac{1}{x^2+3x+2}\,dx$.

(b) $\displaystyle\int \frac{x}{\sqrt{1-x^2}}\,dx$, and $\displaystyle\int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx$.

(a) $\displaystyle\int \frac{-1}{x+2} + \frac{1}{x+1}\,dx$

$\displaystyle = \ln\left|\frac{x+1}{x+2}\right| + \text{Constant}$ //

$\displaystyle\int_0^\infty \frac{1}{x^2+3x+2}\,dx = \left[\ln\left|\frac{x+1}{x+2}\right|\right]_0^\infty$

$\displaystyle = 0 - \ln\frac{1}{2}$

$\displaystyle = -\ln\frac{1}{2}$ //

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(b) $\displaystyle\int \frac{x}{\sqrt{1-x^2}}\,dx$

Let $u=\sqrt{1-x^2}$, $du = \dfrac{-x}{\sqrt{1-x^2}}\,dx$

$\therefore -\displaystyle\int 1 \, du$

$= -\sqrt{1-x^2} + \text{Constant}$ //

$\displaystyle\int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx$

$= -\left[\sqrt{1-x^2}\right]_0^1$

$= 1$ //

8. * Let $f(x)$ be continuous on $\mathbb{R}$ and $a \in \mathbb{R}$ be a given point.

Suppose $\int_{-\infty}^{a} f(x)dx$ and $\int_{a}^{+\infty} f(x)dx$ both converge. Prove that for any point $b \in \mathbb{R}$, $\int_{-\infty}^{b} f(x)dx$ and $\int_{b}^{+\infty} f(x)dx$ both converge, and

$$\int_{-\infty}^{b} f(x)dx + \int_{b}^{+\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_{a}^{+\infty} f(x)dx.$$

Remark: This problem implies that the improper integral $\int_{-\infty}^{+\infty} f(x)dx$ defined by

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{b} f(x)dx + \int_{b}^{+\infty} f(x)dx$$

is independent of the choice of b. So for convenience, we can just choose $b = 0$.

$\because$ $\int_{-\infty}^{a} f(x)\,dx$ & $\int_{a}^{+\infty} f(x)\,dx$ both converge for all $a \in \mathbb{R}$.

$\therefore$ Radius of convergence of them : $+\infty$

$\because$ $\int_{-\infty}^{+\infty} f(x)\,dx = \underbrace{\int_{a}^{+\infty} f(x)\,dx}_{\text{Converges}} + \underbrace{\int_{-\infty}^{a} f(x)\,dx}_{\text{Converges}}$

$\therefore$ $\int_{-\infty}^{+\infty} f(x)\,dx = \int_{b}^{+\infty} f(x)\,dx + \int_{-\infty}^{b} f(x)\,dx$,

for any $b \in \mathbb{R}$.

where $\int_{b}^{+\infty} f(x)\,dx$ & $\int_{-\infty}^{b} f(x)\,dx$ both converge

for all $b \in \mathbb{R}$. //