Calculus for Engineers

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Contents

Differentials "dy" and "dx": Meanings and Applications

The simple ideas behind linear approximations are sometimes formulated in the terminology and notation of differentials. In Section 13.1, we give a definition of differentials. We will look at two different cases, namely, the differential of the dependent variable y (see Definition 1) and the differential of the independent variable (see Definition 2). The geometrical interpretation of the differential dy is also given. Applying differentials to approximate calculations is found in Section 13.2. Differentials of basic elementary functions is given in Section 13.3. Two interpretations of the notation $\frac{dy}{dx}$ are discussed in Section 13.4. Integrals in differential notation are highlighted in Section 13.5. Section 13.6 contains a number of worked examples.

13.1 What are differentials?

The concept of the *differential* is introduced, which enables us to approximate *change in function values*, where the function is differentiable. In addition, differentials are important as a conventional notation for the computation of antiderivatives, as we will learn later in the integral calculus chapters.

For a differentiable function $y = f(x)$, we have been using Leibnitz notation $\frac{dy}{dx}$ to mean the derivative of y with respect to x. Although this notation has the appearance of a quotient, it is treated as a single entity, since it is a symbol for the limit

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = f'(x)$$

if this limit exists. It is now proposed to give separate meanings to the symbols dy and dx.

The concept of the differential of a function is closely related to the derivative of the function. To understand this, we refer to Figure 13.1.

In Figure 13.1, let $y = f(x)$ be an equation of a curve. We note that

- The line PT (green) is tangent to the curve at $P(x, f(x))$, Q is the point $(x + \Delta x, f(x + \Delta x))$, and the directed distance $\overline{MQ}$ (dashed) is

  $$\Delta y = f(x + \Delta x) - f(x),$$

  which represents *the actual change in the value of f*, when x is changed to $(x + \Delta x)$.

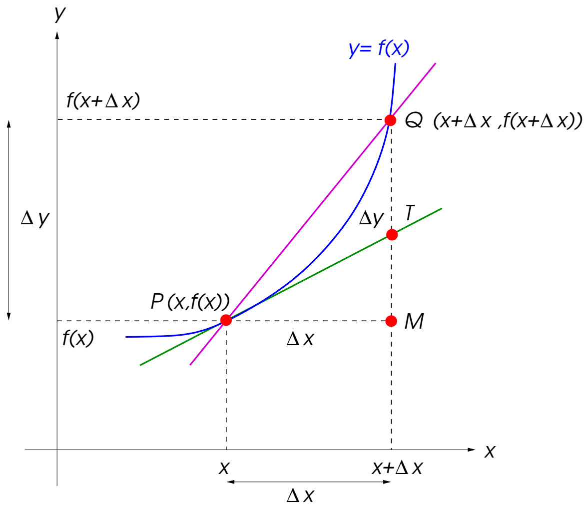

Figure 13.1: Graph of the differential of a function.

- The changes Δx and Δy are both positive; however they could be negative. For a *small value of* Δx, the slope of the secant line PQ (pink) and the slope of the tangent line (green) at P are *approximately equal;* so we can write,

  $$\frac{\Delta y}{\Delta x} \approx f'(x)$$

  or

  $$\Delta y \approx f'(x)\Delta x. \tag{13.1}$$

  The RHS of (13.1) is defined as the *differential* of y.

We give the following definition.

13.1.1 Definition of the differential of the dependent variable y

Definition 1 Let the function f be defined by the equation $y = f(x)$, then the *differential of* y is denoted by "dy" [or $df(x)$], and is given by

$$dy = f'(x)\Delta x, \tag{13.2}$$

where x is the domain of f' and Δx is an arbitrary increment to x.

Figure 13.2 is the same as Figure 13.1, except that the vertical distance segment MR is shown, where the directed line $\overline{MR} = dy$. We note that

- dy represents the change in y along the tangent line to the graph of the equation $y = f(x)$ at the point $P(x, f(x))$, when x is changed by Δx.
- $dy \neq \Delta y$, but for small values of Δx, dy is very close to Δy.

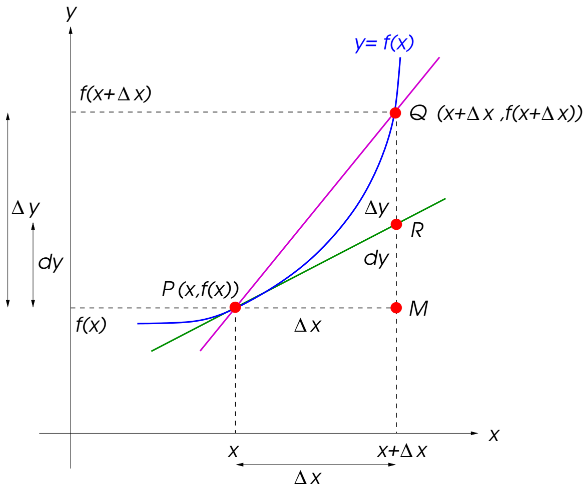

Figure 13.2: The differential of y is $dy = f'(x)\Delta x$.

- In regard to (13.2), since a variable x can be any number in the domain of f' and Δx can be any number whatsoever, the differential dy or $[df(x)]$ is a function of two variables x and Δx.

We now wish to define the differential of the independent variable or "dx". To arrive at a suitable definition consistent with the definition of dy, we give the following example.

Example 1 Let us consider the identity function denoted by $f(x) = x$. We observe that $f'(x) = 1$ and $y = x$. Thus, from (13.2), we get $dy = 1 \cdot \Delta x$, that is, if $y = x$, then, $dy = \Delta x$. For the identity function, we would want that dx be equal to dy. This permits us to write $\Delta x = dx$. This reasoning leads us to the following definition.

13.1.2 Definition of the differential of the independent variable

Definition 2 If the function f is defined by the equation $y = f(x)$, then the differential of x, denoted by dx is given by

$$dx = \Delta x.$$

where x is any number in the domain of f' and Δx is an arbitrary increment of x. The relation (13.2) can now be written as

$$dy = f'(x)dx. \tag{13.3}$$

We treat (13.3) as *the definition of the differential of* y (that is, the dependent variable). Two interesting results produced from (13.3):

1. Knowing the derivative of a function $y = f(x)$, we can readily find its differential.

2. By dividing both sides of (13.3) by dx, we can, if we wish, interpret the derivative as a quotient of two differentials, that is,

$$\frac{dy}{dx} = f'(x), \qquad \text{if } dx \neq 0. \tag{13.4}$$

This representation of the derivative, as the ratio of two differentials, is extremely important for mathematical analysis.

Remark 1 Based on the definition of the differential of a function, we have attached meanings to the symbols "dy" and "dx", and given a new meaning to $\frac{dy}{dx}$ (the derivative of y with respect to x) as a ratio of dy to dx, while retaining the meaning of the symbol $\frac{dy}{dx}$.

Remark 2 In (13.3), dx, being arbitrary, can have any (finite) magnitude big or small. Also, since the magnitude of dy depends on two variables x and Δx, it can also have any (finite) magnitude. Thus, in (13.3), dy and dx need not be small.

However, if we think of dx and dy as being small, then (13.3) proves to be very useful since it gives the approximate changes in the function values, where the function is differentiable.

Note 1 When we introduced the notation $\frac{dy}{dx}$ (for the derivative of y with respect to x), we emphasized that dy and dx had not been given independent meaning. But, now we can also treat the symbol $\frac{dy}{dx}$ as a ratio of two differentials. It is only when we think of differentials, that we can write $dy = f'(x)dx$. This permits us to write $\frac{dy}{dx} = 3x^2$ in the form $dy = 3x^2 dx$ and similarly $\frac{dy}{dx} = \cos x$ in the form $dy = \cos x dx$, and so on.

13.1.3 Geometrical interpretation of the differential dy

Let $y = f(x)$ be a differentiable function of x and consider a fixed value of x, say x_0. Then, the differential of f at x_0 is given by

$$dy = f'(x_0)dx. \tag{13.5}$$

Note that, in this case, dy is a *line function of the single variable* $dx, f'(x_0)$ being a constant.

Also, if an increment dx is given to x_0, the corresponding increment Δy in y is given by

$$f(x_0 + dx) - f(x_0) = \Delta y$$

or

$$f(x_0 + dx) = f(x_0) + \Delta y.$$

But, we know that *for small values of* dx, Δy *is very close to* dy. *Hence, replacing* Δy *by* dy, we can write,

$$\begin{aligned} f(x_0 + dx) &\approx f(x_0 + dy) \\ &\approx f(x_0) + f'(x_0)dx. \end{aligned}$$

Since f is differentiable, we may drop the subscript "0", and write the above equation as

$$f(x + dx) \approx f(x) + f'(x)dx. \tag{13.6}$$

The relation (13.6) gives us an *approximate* value of $f(x+dx)$ in terms of fully known quantities[1], where x is a number at which f is differentiable. We shall make use of this equation to estimate values of functions that are difficult or impossible to obtain exactly. The *approximation* given by this equation is *most useful* when $f(x)$ and $f'(x)$ are easy to compute. This will be clear from the solved examples which follow shortly.

Note that, *when we approximate* $f(x+dx)$ by $f(x)+dy$, *we are approximating the ordinate of the point* Q *on the curve by the ordinate of the point* R on the tangent line (see Figure 13.2).

Note 2 One should not think that the increment Δy is always greater than dy. The situation becomes clear from Figures 13.3 and 13.4. It may be noted from Figure 13.4 that $\Delta y < dy$.

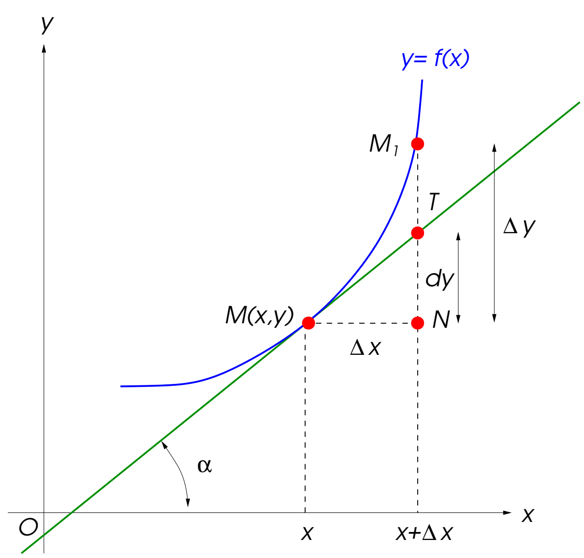

Figure 13.3: $\Delta y > dy$.

Note 3 It should also be noted that, if $f'(x) = 0$ at a point x, *the differential is equal to zero:* $dy = 0$. In this case, dy *is not compared with the increment* Δy *of the function.*

Example 2
Now, let us compute the differentials of some functions:

i Consider

$$y = x^3 + 5x^2 - 1.$$

Then,

$$dy = \underbrace{\frac{d\left(x^3 + 5x^2 - 1\right)}{dx}}_{f'(x)} \Delta x.$$

Thus,

$$dy = \left(3x^2 + 10x\right) \Delta x.$$

[1]That is, $f(x)$, $f'(x)$ and dx

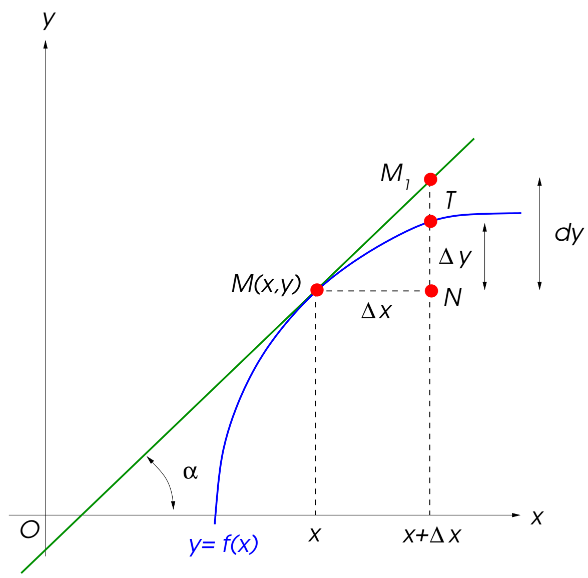

Figure 13.4: $\Delta y < dy$.

Now, if $x = 1$ and $\Delta x = 0.02$, then

$$dy = \left(3(1)^2 + 10(1)\right)(0.02) = 0.26.$$

ii Consider

$$y = \sin x.$$

Then,

$$\begin{aligned} dy &= d(\sin x) \\ &= \frac{d(\sin x)}{dx}\Delta x \\ &= \cos x \cdot \Delta x. \end{aligned}$$

iii Consider

$$y = e^{3x}.$$

Then,

$$\begin{aligned} dy &= d(e^{3x}) \\ &= \frac{d(e^{3x})}{dx} \cdot \Delta x \\ &= 3e^{3x} \cdot \Delta x. \end{aligned}$$

iv Consider

$$y = f(x) = \log_e(x^2 + 1),$$

where $\log_e(x^2+1) \equiv \ln(x^2+1)$. Then,

$$\begin{aligned} dy &= d\,(f(x)) \\ &= d\left(\log_e(x^2+1)\right) \\ &= \frac{d\,(\log_e(x^2+1))}{dx} \cdot \Delta x \\ &= \frac{1}{x^2+1} \cdot 2x \cdot \Delta x \\ &= \frac{2x \cdot \Delta x}{x^2+1}. \end{aligned}$$

$\square$

Example 3 Given $y = 4x^2 - 3x + 1$, find $\Delta y, dy$ and $\Delta y - dy$ for

(a) any x and Δx;

(b) $x = 2, \Delta x = 0.1$;

(c) $x = 2, \Delta x = 0.01$;

(d) $x = 2, \Delta x = 0.001$.

Solution: We examine each case:

(a) Consider the equation

$$y = f(x) = 4x^2 - 3x + 1.$$

Then

$$\begin{aligned} \Delta y &= f(x+\Delta x) - f(x) \\ &= 4(x+\Delta x)^2 - 3(x+\Delta x) + 1 - (4x^2 - 3x + 1) \\ &= 4x^2 + 8x \cdot \Delta x + 4(\Delta x)^2 - 3x - 3 \cdot \Delta x + 1 - 4x^2 + 3x - 1 \\ &= (8x-3) \cdot \Delta x + 4 \cdot (\Delta x)^2 \end{aligned}$$

From the definition of the differential in (13.3) above, we have

$$dy = f'(x)dx = (8x-3)\Delta x$$

since $dx = \Delta x$. Thus,

$$\Delta y - dy = 4(\Delta x)^2.$$

The results for parts **(b)**, **(c)**, and **(d)** are given in Table 13.1.

From Table 13.1, we conclude that

- the closer Δx is to zero, the smaller the difference between Δy and dy.
- for each value of Δx, the corresponding value of $\Delta y - dy$ is smaller than the value of Δx. More generally, dy is an approximation of Δy when Δx is small.

$\square$

	x	Δx	Δy	dy	$\Delta y - dy$
(b)	2	0.1	1.34	1.3	0.04
(c)	2	0.01	0.1304	0.13	0.0004
(d)	2	0.001	0.013004	0.013	0.000004

Table 13.1: The results for parts **(b)**, **(c)**, and **(d)**.

13.2 Applying differentials to approximate calculations

The application of the differential to approximate calculations is based on the replacement of the increment. Then

$$\Delta y = f(x_0 + dx) - f(x_0)$$

by the differential, $dy[= f'(x_0)dx]$, since for small values of dx we have $\Delta y \approx dy$. Therefore, we write,

$$f(x_0 + dx) - f(x_0) \approx dy = f'(x_0)dx. \tag{13.7}$$

Note that, even though the increment Δy may depend on dx in a complicated manner, the differential dy can be easily obtained by differentiation.

This *approximate equality* can be immediately used to solve the following problem.

Given the values of $f(x_0)$, $f'(x_0)$, and dx, it is required to compute an approximation to the value $f(x_0 + dx)$ of the function.

Relation (13.7) gives us the desired formula directly:

$$f(x_0 + dx) \approx f(x_0) + f'(x_0)dx.$$

Let us consider some illustrative examples. (*For brevity, we shall write x in place of x_0 and denote dx by h.)*

Example 4 Consider the function

$$y = \sqrt{x}.$$

Its differential is

$$dy = \frac{1}{2\sqrt{2}}dx.$$

Hence,

$$\sqrt{x+h} \approx \sqrt{x} + \frac{h}{2\sqrt{x}}.$$

Two cases are considered:

- In particular, for $x = 1$, we obtain,

$$\sqrt{1+h} \approx 1 + \frac{h}{2}.$$

- In the general case, for $x = a^2$ ($a > 0$), we have,

$$\sqrt{a^2 + h} \approx a + \frac{h}{2a}.$$

These approximate formulas are extremely simple and make it possible to compute square roots with sufficient accuracy when $|h|$ is small compared to a^2.

For instance, the application of these results yields

$$\sqrt{1.21} = \sqrt{1 + 0.21} \approx 1 + \frac{0.21}{(2)(1)} = 1.105.$$

The exact value of the root is equal to 1.1.

To compute the root $\sqrt{408}$ we represent it in the form $\sqrt{408} = \sqrt{20^2 + 8}$, and thus obtain $\sqrt{408} \approx 20 + (8/(2)(20)) = 20.2$.

Now, let us take $\sqrt{390}$. Here, it is convenient to put $h = -10$, then

$$\sqrt{390} = \sqrt{20^2 - 10} \approx 20 - \frac{10}{(2)(20)} = 19.75$$

- If $y = \sqrt[n]{x}$, then,

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(x^n) \\ &= \frac{1}{n}(x^{(1/n)-1}) \\ &= \frac{1}{n}x^{1/n} \cdot x^{-1} \\ &= \frac{1}{n}\frac{\sqrt[n]{x}}{x}.\end{aligned}$$

Thus,

$$\begin{aligned}dy &= \frac{1}{n}\frac{\sqrt[n]{x}}{x} \cdot dx \\ &= \frac{1}{n}\frac{\sqrt[n]{x}}{x}h\end{aligned}$$

replacing dx by h. Thus,

$$\sqrt[n]{x+h} = \sqrt[n]{x} + \frac{1}{n}\frac{\sqrt[n]{x}}{x} \cdot h. \tag{13.8}$$

For $x = 1$, this yields the approximate formula

$$\sqrt[n]{1+h} \approx 1 + \frac{h}{n}$$

- A more general formula is obtained for $x = a^n (a > 0)$:

$$\sqrt[n]{a^n + h} \approx a + \frac{h}{n \cdot a^{n-1}}.$$

□

Example 5 Let us consider the function $y = \sin x$. Its differential is $dy = \cos x dx$, and therefore

$$\sin(x + h) \approx \sin x + h \cos x.$$

In particular, for $x = 0$, we derive the formula,

$$\sin h \approx h.$$

Here are a few examples:

- We have

$$\sin \frac{\pi}{180} \approx \frac{\pi}{180} \approx 0.01745.$$

  That is, approximately,

$$\sin 1^\circ = 0.1745.$$

  This approximation is correct to the fifth decimal digit, that is, the error does not exceed 10^{-5}.

- Let us compute $\sin 31^\circ$ such that

$$\begin{aligned} \sin 31^\circ &\approx \sin 30^\circ + \frac{\pi}{180} \cos 30^\circ \\ &\approx 0.5 + \frac{\sqrt{3}}{2} \cdot (0.01745) \\ &\approx 0.5150. \end{aligned}$$

  The tabular value of $\sin 31^\circ$ correct within 10^{-4} (i.e., the fourth decimal place) is 0.5150. (The reader may account for the fact that we have obtained a major approximation of $\sin 31^\circ$.)

□

Example 6 Now, consider the function $y = \ln x$. Here we have, $dy = (1/x)dx$, and

- In $(x + h) \approx \ln x + \frac{h}{x}$.
- In particular, for $x = 1$ this yields the formula $\ln(1 + h) \approx 1$.

Take the known value $\ln 781 \approx 6.66058$. To compute $\ln 782$, we apply the above formulas

$$\begin{aligned} \ln 782 &\approx 6.66058 + \frac{1}{781} \\ &\approx 6.66186. \end{aligned}$$

The actual value of $\ln 781$ is 6.661855, so the approximated value of $\ln 781$ is correct to within 10^{-5}.

(We see that the error of our approximation is small. The reader may try to find out why in this case we have also obtained a major approximation.)

□

13.3 Differentials of basic elementary functions

Since the differential of a function is obtained as the product of the derivative by the differential of the independent variable, *we can readily write down a list of the differentials for all the basic elementary functions because their derivatives are known.* For instance,

$$\begin{aligned} d(x^n) &= nx^{n-1}dx \\ d(a^x) &= a^n \ln a dx \\ d(\ln x) &= \frac{1}{x}dx \\ d(\sin x) &= \cos x dx, \quad \text{andsoon.} \end{aligned}$$

13.3.1 Differentials of the results of arithmetical operations on functions

In accordance with the rules for finding derivatives (studied in Chapter 7), we can use the derivative formulas to write down the corresponding differentials. For example, if u and v are differentiable function of x, then the formula

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx} \tag{13.9}$$

after multiplying both the sides by dx becomes

$$d(u+v) = du + dv \tag{13.10}$$

which says that the differential of the function $(u+v)$ is the differential of the functions u plus the differential of the function v. *It is still assumed that u and v are differentiable functions, but the name of the independent variable no longer appears in the formula.* We do not need to mention it as long as we understand that (13.10) is an abbreviation for (13.9). We illustrate the major rules in Table 13.2.

	Derivative Rule	Differential Rule
1.	$\frac{d(c)}{dx} = 0$	$d(c) = 0$
2.	$\frac{d(x^n)}{dx} = nx^{n-1}$	$d(x^n) = nx^{n-1}dx$
3.	$\frac{d(cu)}{dx} = c\frac{du}{dx}$	$d(cu) = cdu$
4.	$\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}$	$d(u+v) = du + dv$
5.	$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$	$d(uv) = udv + vdu$
6.	$\frac{d(u/v)}{dx} = \frac{v(du/dx)-u(dv/dx)}{v^2}$	$d\left(\frac{u}{v}\right) = \frac{vdu-udv}{v^2}$
7.	$\frac{d(u^n)}{dx} = nu^{n-1}\frac{du}{dx}$	$d(u^n) = nu^{n-1}du$

Table 13.2: Comparison of derivative rules and differential rules.

Note 4 One must be careful to make a clear distinction between derivatives and differentials. They are not the same. When you write $\frac{dy}{dx}$ you are using a symbol for the derivative, but when you write "dy" you are dealing with a differential.

Note 5 It should be noted that a differential on the LHS of an equation (say dy), also calls for a differential (usually dx) on the RHS of the equation. Thus, we never have $dy = 3x^2$, instead we have $dy = 3x^2 dx$.

13.3.2 Differential of a composite function

While the definition of dy assumes that x is an independent variable, that assumption is not important. Let us consider the following steps:

1. Let $y = f(u)$ and $u = \phi(x)$ be two *functions of their arguments* possessing the derivatives $f'(u)$ and $\phi'(x)$ with respect to these arguments.

2. If we put

$$y = f(u) = f(\phi(x)) = F(x), \tag{13.11}$$

then, by differentiating both sides of (13.11) with respect to x, we have

$$y' = F'(x) = f'(u) \cdot \phi'(x). \tag{13.12}$$

3. After multiplying both sides of (13.12) by dx, we get

$$y' \cdot dx = f'(u) \cdot \phi'(x) \cdot dx$$

or

$$dy = f'(u) \cdot du$$

since $y' \cdot dx = dy$ and $\phi'(x)dx = du$.

4. Thus, the differential has the same form as if the magnitude of u were an independent variable.

This can be stated as follows.

The expression for the differential of a function $y = f(u)$ remains the same irrespective of whether its argument u is an independent variable or a function of another variable.

This property is referred to as the *invariance of the form of the differential.* It is because of this property that *we can write down the differential in the same form irrespective of the nature of the argument of the function.* The equality, $dy = f'(u) \cdot du$ implies

$$f'(u) = \frac{dy}{du}$$

and hence in all the cases, this equation may be looked upon as follows:

The rate of change of a function relative to its argument is equal to the ratio of the differential of the function to the differential of its argument.

Relation (13.12) can now be written as

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}. \tag{13.13}$$

The RHS of (13.13) *is obtained from the LHS by the simultaneous multiplication and division of the former by* du (if, of course, $du \neq 0$).

Hence, the arithmetical operations on differentials can be performed as if they were ordinary numbers, that is, the representation of derivatives as the ratio of the differentials. For instance, using this representation of derivatives we can readily write down the differentiation rule for inverse of a function, that is

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}.$$

Remark 3 Knowing the derivative of a function, we can find its differential and vice versa. Hence, the existence of the derivative can be taken as the condition equivalent to the differentiability of the function.

From the geometrical point of view, this condition is equivalent to the existence of the tangent to the curve $y = f(x)$, not perpendicular to the x-axis.

Definition 3 Recall that, *a function $y = f(x)$ is said to be differentiable at a point x* (in its domain) *if it possesses a derivative at that point.* Further, if a function is differentiable at every point in its domain (i.e., the derivative exists at every point in its domain) then it is called a *differentiable function.*

Now, in view of the above discussion (about the differential of a function) we can give the following definition:

A function $y = f(x)$ is said to be differentiable at a point x, *if it has a differential at that point.*

Note 6 Any problem involving differentials, (say that of finding dy when y is given as a function of x), may be handled either

(a) by finding $\frac{dy}{dx}$ and multiplying by dx

or

(b) by direct use of formulas on the differentials.

Example 7 Given a function $y = \sin\sqrt{x}$. Find dy.
Solution: Representing the given function as a composite function, we have

$$y = \sin u, \ \ u = \sqrt{x}.$$

We find,

$$\frac{dy}{dx} = \cos u \cdot \frac{du}{dx} = \cos\sqrt{x} \cdot \frac{1}{2\sqrt{x}}.$$

Thus,

$$dy = \cos\sqrt{x} \cdot \frac{1}{2\sqrt{x}} dx$$

or, we write,

$$dy = \cos u du, \quad du = (\sqrt{x})' dx = \frac{1}{2\sqrt{x}} dx.$$

Then,

$$dy = (\cos\sqrt{x})\left(\frac{1}{2\sqrt{x}} dx\right).$$

Note 7 The application of the differential of a function can also be accomplished by considering *nonuniform motion* of a particle in a straight line. Let the law of motion be expressed mathematically by

$$s = f(t), \tag{13.14}$$

where s is the distance traveled and t stands for the time taken. Then, the velocity of the particle at any instant t_1 is given by $f'(t_1)$. If now an additional time Δt passes, let the particle cover an additional distance Δs. Since the motion is nonuniform, the dependence of Δs on Δt can be complicated because the velocity of the particle varies all the time.

But if Δt is not large, the velocity will not change considerably during the period of time from t_1 to $t_1 + \Delta t$. Therefore, the motion may be regarded as "*almost uniform*" during the time interval Δt. Hence, in calculating the distance traveled, we *shall not get a serious error if we regard the motion as uniform with the constant velocity $f'(t_1)$, from the instant t_1 to $t_1 + \Delta t$.*

Thus, the (approximate) distance traveled during the interval Δt is given by $f'(t_1) \cdot \Delta t$. *This product,* as we know, *is called the differential of the distance function and is denoted by ds.* We write

$$ds = f'(t_1) \cdot \Delta t. \tag{13.15}$$

Of course, *the real distance* Δs traveled (during the interval t_1 to $t_1 + \Delta t$) differs from *the invented distance ds* given in (13.15) above.

It must be clear that the accuracy of the formula (13.15) becomes greater as Δt is decreased and vice versa. Nevertheless, it is much easier to compute ds as a distance covered by uniform motion than to evaluate the real distance Δs. This accounts for the fact that formula (13.15) is often used even when Δt is not very small.

In all such cases, *the replacement of a real change of a quantity by its differential reduces to the transition from some nonuniform processes to the uniform ones.* Such a replacement is always based upon the fact that *every process is “almost uniform” during a small interval of time.*

13.4 Two interpretations of the notation $\dfrac{dy}{dx}$

Leibniz used the suggestive notation $\dfrac{dy}{dx}$ for the instantaneous rate of change of y with respect to x. This notation suggests that the instantaneous rate comes from considering an average rate (which is indeed a quotient) and computing its limit. Thus, $\dfrac{dy}{dx}$ stands for the limit,

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}, \text{ provided the limit exists.}$$

Here, dy and dx do not have any meaning if considered separately (since $\dfrac{dy}{dx}$ is a single entity: a symbol for the limit, which we call the derivative).

Our investigation suggests that

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}, \qquad \text{provided the limit exists.}$$

It would be wrong to interpret this limiting relation in the sense that Δy tends to dy and Δx to dx, as $\Delta x \to 0$. The correct meaning is that the ratio of the increments $\dfrac{\Delta y}{\Delta x}$, as $\Delta x \to 0$, tends to the limit denoted by $\dfrac{dy}{dx}$ or $f'(x)$. On the other hand, the differential of a function $y = f(x)$ is defined by

$$dy = d\,(f(x)) = f'(x)dx,$$

where $f'(x)$ stands for the derivative of f at a point x and dx is an *arbitrary number* ($dx \neq 0$). Dividing both sides of the above expression by dx, we get

$$\frac{dy}{dx} = f'(x), \;\; dx \neq 0.$$

In this case, $\dfrac{dy}{dx}$ stands for the ratio of two quantities, namely dy and dx. In either case, we get $\dfrac{dy}{dx} = f'(x), \;\; dx \neq 0$.

Now, we can also say that the ratio of the increments $\dfrac{\Delta y}{\Delta x}$ tends to the ratio of differentials $\dfrac{dy}{dx}$, as $\Delta x \to 0$.

13.5 Integrals in differential notation

The notation of differentials allows us to express integrals in a shorthand that often proves useful. Here are a few examples:

Example 8 If u is a *differentiable function of x,* then the integral of $\dfrac{du}{dx}$ with respect to x is sometimes written simply as the integral of du:

$$\int \frac{du}{dx} \cdot dx = \int du.$$

Thus, the integral of du is required to be evaluated as

$$\int du = \int \frac{d(u)}{dx} dx = u + C$$

or simply,

$$\int du = u + C,$$

where C is a constant.

Example 9 If $u = \sin x$, then $d(\sin x) = \cos x dx$, and we can write

$$\int d(\sin x) = \sin x + C$$

which is short for

$$\int \cos x dx = \int \frac{d}{dx}(\sin x) dx = \sin x + C.$$

Thus, for a given integral $\int f(x)dx$, we have to express the differential $f(x)dx$ in the form $\frac{d}{dx}F(x)dx$ (which also stands for $d[F(x)]$). Obviously, then $\frac{d}{dx}F(x) = f(x)$. Whenever $f(x)dx$ is expressed in the form $d[F(x)]$, we say that the integrand is expressed in the standard form. Once this is done, we can immediately write down the antiderivative (or the indefinite integral), $F(x) + C$.

We will discuss these applications in greater detail in Chapter 19.

13.6 Examples

We provide a list of examples on the use of (13.7).

Example 10 Find the approximate value of $(4.01)^3$ correct to two decimal places.

Solution: We have

$$f(x + \Delta x) \approx f(x) + f'(x) \cdot \Delta x,$$

where Δx is small. Then,

$$(x + \Delta x)^3 \approx x^3 + 3x^2 \cdot \Delta x. \tag{13.16}$$

We take $x = 4$ and $\Delta x = 0.01$. Substituting these values in (13.16), we get,

$$\begin{aligned}(4 + 0.01)^3 &\approx (4)^3 + 3(4)^2 \cdot (0.01) \\ &\approx 64 + (48) \cdot (0.01) \\ &\approx 64 + 0.48 \\ &\approx (4.01)^3 \\ &\approx 64.48.\end{aligned}$$

Now, we must similarly compute $(3.97)^3$.

Here, we take $x = 4$ and $\Delta x = 0.03$. Then

$$\begin{aligned}(3.97)^3 &= (4 - 0.03)^3 \\ (4 - 0.03)^3 &\approx (4)^3 + 3 \cdot (4)^2 \cdot (-0.03) \quad \text{since} f((x + (-\Delta x)) \approx f(x) + f'(x) \cdot (-\Delta x) \\ &\approx 64 + (48) \cdot (-0.03) \\ &\approx 64 - 1.44 \\ &\approx 62.56.\end{aligned}$$

□

Example 11 Find an approximate value of $\sqrt[3]{8.05}$.
Solution: Consider the function

$$f(x) = \sqrt[3]{x} = (x)^{1/3}.$$

Then,

$$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}.$$

We have,

$$f(x + \Delta x) \approx f(x) + f'(x) \cdot \Delta x, \quad \text{where } \Delta x \text{ is small.}$$

Then,

$$(x + \Delta x)^{1/3} \approx x^{1/3} + \frac{1}{3x^{2/3}} \cdot \Delta x. \tag{13.17}$$

We take $x = 8$ and $\Delta x = 0.05$
Substituting these rules in (13.17), we get

$$\begin{aligned}(8 + 0.05)^{1/3} &\approx (8)^{1/3} + \frac{1}{3(8)^{2/3}} \cdot (0.05)\\ &\approx 2 + \frac{0.05}{(3)(4)}\\ &\approx 2 + 0.00417.\end{aligned}$$

Then,

$$\sqrt[3]{8.05} \approx 2.00417.$$

Now, if we wish to compute $\sqrt[3]{7.95}$, we get

$$\sqrt[3]{7.95} \approx 2 - 0.00417 \approx 1.99583.$$

□

Example 12 Estimate the value of $\sin 31^\circ$, assuming that $1^\circ = 0.0175$ rad, and $\cos 30^\circ = 0.8660$.
Solution: Let $f(x) = \sin x$. Then $f'(x) = \cos x$. We have,

$$f(x + \Delta x) \approx f(x) + f'(x) \cdot \Delta x,$$

where Δx is small. Then,

$$\sin(x + \Delta x) \approx \sin x + \cos x \cdot \Delta x. \tag{13.18}$$

We take $x = 30^\circ = \frac{\pi}{6}$ and $\Delta x = 1^\circ = \frac{\pi}{180} = 0.0175$. Substituting in (13.18), we get,

$$\begin{aligned}\sin 31^\circ &= \sin(30^\circ + 1^\circ)\\ &\approx \sin 30^\circ + \cos 30^\circ \cdot (0.0175)\\ &\approx \sin\left(\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{6}\right) \cdot (0.0175)\\ &\approx 0.5 + (0.8660) \cdot (0.0175)\\ &\approx 0.5 + 0.015155\\ &\approx 0.51516.\end{aligned}$$

Note 8 Assuming that $1^\circ = 0.0175$ rad and $\sin 45^\circ = 0.7071$, we can easily estimate $\cos 46^\circ$ or $\cos 44^\circ$. (Remember that $\cos 45^\circ = \sin 45^\circ = 0.7071$.) Here is an explanation:

Let $f(x) = \cos x$. Then, we have

$$f'(x) = -\sin x.$$

We have $f(x + \Delta x) \approx f(x) + f'(x)\Delta x$. Then, we have

$$\cos(x + \Delta x) = \cos x - \sin x \cdot \Delta x.$$

Thus,

$$\begin{aligned}\cos(45^\circ + 1^\circ) &\approx \cos 45^\circ - \sin 45^\circ(0.0175)\\ &\approx 0.7071 - (0.7071)\cdot(0.0175)\\ &\approx 0.7071 - 0.01237\\ &\approx 0.6947.\end{aligned}$$

and

$$\begin{aligned}\cos(45^\circ - 1^\circ) &\approx 0.7071 - (0.7071)\cdot(-0.0175)\\ &\approx 0.7071 + 0.01237\\ &\approx 0.71947.\end{aligned}$$

Example 13 Approximate $\sin\left(\dfrac{7\pi}{36}\right)$.

Solution: Applying a simple calculation gives

$$\frac{7\pi}{36} = \frac{6\pi}{36} + \frac{\pi}{36} = \frac{\pi}{6} + \frac{\pi}{36}.$$

Then, $\dfrac{7\pi}{36}$ is close to $\dfrac{\pi}{6}$.

Thus, we write

$$\sin\left(\frac{7\pi}{36}\right) = \sin\left(\frac{\pi}{6} + \frac{\pi}{36}\right).$$

Let $\dfrac{\pi}{6} = x$ and $\dfrac{\pi}{36} = \Delta x$. Then, we have

$$\sin\left(\frac{\pi}{6} + \frac{\pi}{36}\right) = \sin(x + \Delta x).$$

We have, $f(x + \Delta x) \approx f(x) + f'(x)\cdot\Delta x$. Therefore, we have

$$\sin(x + \Delta x) \approx \sin x + \cos x \cdot \Delta x$$

or

$$\begin{aligned}\sin\left(\frac{\pi}{6} + \frac{\pi}{36}\right) &\approx \sin\left(\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{6}\right)\cdot\left(\frac{\pi}{36}\right)\\ &\approx 0.5 + \left(\frac{\sqrt{3}}{2}\right)\cdot\left(\frac{\pi}{36}\right)\\ &\approx 0.5 + 0.075575 = 0.575575.\end{aligned}$$

□

Example 14 Find the value of $f(x) = 2x^3 + 7x + 5$, at $x = 2.001$.
Solution: Let $f(x) = 2x^3 + 7x + 5$. Then, we have

$$f'(x) = 6x^2 + 7.$$

We have, $f(x + \Delta x) \approx f(x) + f'(x) \cdot \Delta x$, where Δx is small. Then,

$$f(x + \Delta x) \approx (2x^3 + 7x + 5) + (6x^2 + 7) \cdot \Delta x.$$

We take $x = 2$ and $\Delta x = 0.001$. Then,

$$\begin{aligned} f(2.001) &= [2(2)^3 + 7(2) + 5] + [6(2)^2 + 7] \cdot (0.001) \\ &= (16 + 14 + 5) + (24 + 7) \cdot (0.001) \\ &= 35 + 0.031 \\ &= 35.031. \end{aligned}$$

□

Example 15 Find the approximate value of $\tan^{-1}(0.99)$.
Solution: Let $f(x) = \tan^{-1} x$. Then, we have

$$f'(x) = \frac{1}{1 + x^2}.$$

We know that,

$$f(x + \Delta x) \approx f(x) + f'(x) \cdot \Delta x,$$

where Δx is small. Then,

$$\tan^{-1}(x + \Delta x) \approx \tan^{-1} x + \left(\frac{1}{1 + x^2}\right) \cdot (\Delta x).$$

We take $x = 1$ and $\Delta x = -0.01$. Then,

$$\begin{aligned} \tan^{-1}(1 - 0.01) &\approx \tan^{-1}(1) + \left(\frac{1}{1 + (1)^2}\right) \cdot (-0.01) \\ &\approx \frac{\pi}{4} - \frac{0.01}{2} \\ &\approx \frac{\pi}{4} - 0.005 \\ &\approx 0.780. \end{aligned}$$

□

Note 9 The approximate value of $\tan^{-1}(1.001)$ is given by

$$\begin{aligned} \tan^{-1}(1 + 0.001) &\approx \tan^{-1}(1) + \left(\frac{1}{1 + 1^2}\right) \cdot (0.001) \\ &\approx \frac{\pi}{4} + \frac{1}{2} \cdot (0.001) \\ &\approx 0.7854 + 0.0005 \\ &\approx 0.7855 + 0.0005 \\ &\approx 0.7860. \end{aligned}$$

Example 16 Find the approximate value of $e^{1.002}$ taking $e = 2.71828$.

Solution: Let $f(x) = e^x$, then we know that

$$f'(x) = e^x.$$

We have, $f(x + \Delta x) \approx f(x) + f'(x) \cdot \Delta x$, where Δx is small. Then,

$$e^{x+\Delta x} \approx e^x + e^x \cdot \Delta x.$$

We take $x = 1$ and $\Delta x = 0.002$. Then,

$$\begin{aligned} e^{1.002} &\approx e^1 + e^1 \cdot (0.002) \\ &\approx 2.71828 + (2.71828) \cdot (0.002) \\ &\approx 2.71828 + 0.005437 \\ &\approx 2.7237 \text{ (correct up to four decimal places).} \end{aligned}$$

□

Example 17 Taking $\log_e 10 = 2.3026$, find the approximate value of $\log_e 101$.

Solution: Let $f(x) = \log_e x$. Then,

$$f'(x) = \frac{1}{x}.$$

We have, $f(x + \Delta x) \approx f(x) + f'(x) \cdot \Delta x$, where Δx is small. Then,

$$\log_e(x + \Delta x) \approx \log_e x + \frac{1}{x} \cdot (\Delta x).$$

We take $x = 100$ and $\Delta x = 1$. Then,

$$\begin{aligned} \log_e(100 + 1) &\approx \log_e 100 + \frac{1}{100} \cdot (1) \\ &\approx \log_e(10)^2 + 0.01 \\ &\approx 2 \cdot (2.3026) + 0.01 \\ &\approx 4.6052 + 0.01 \\ &\approx 4.6152. \end{aligned}$$

□