Calculus for Engineers

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Contents

Tangent Lines and Rates of Change

6.1 Introduction

In Section 6.2, we make a clear distinction between secant lines and tangent lines. A secant line is a straight line joining two points on a function, whereas a tangent line is a straight line that touches a function at only one point. As various examples show, we recognize that a secant line is identical to the average rate of change or simply the slope between two points (See Definition 1), while the tangent line represents the instantaneous rate of change of the function at that one point, and the slope of the tangent line at a point on the function is equivalent to the derivative of the function at the same point (See Definition 2). In Section 6.3, we study the so-called free-falling body problems in order to illustrate the concepts of velocity and acceleration linked with the derivatives.

6.2 The tangent line at a point

In our first example we will calculate a series of functions whose graphs are secant lines to the graph of a given function f and use them to infer an equation of the tangent line at a point.

Example 1 Let $f(x) = 2x^3 - 2x + 2$.

1. Find an equation of the line that passes through the points $\left(\frac{1}{2}, \frac{5}{4}\right)$ and $(1, 2)$ and sketch the graph of both f and the secant line.
2. Use the points $\left(\frac{1}{2}, \frac{5}{4}\right)$ and $\left(\frac{3}{4}, \frac{43}{32}\right)$ to find an equation of the secant line.
3. Fill in the blank in the following table and from this table guess what the equation is for the tangent line of the function $f(x) = 2x^3 - 2x + 2$ at $\left(\frac{1}{2}, \frac{1}{4}\right)$. Explain your conclusion.

Solution:

1. Using the point-slope form, the equation of the secant line is

$$y = \frac{3}{2}x + \frac{1}{2}.$$

Δx	$(x, f(x))$	$(\Delta x, f(x+\Delta x))$	Equation of secant line
0.5	$(0.5, 1.25)$		
0.25	$(0.5, 1.25)$		
0.125	$(0.5, 1.25)$		
0.0625	$(0.5, 1.25)$		
0.03125	$(0.5, 1.25)$		
0.015625	$(0.5, 1.25)$		

Table 6.1:

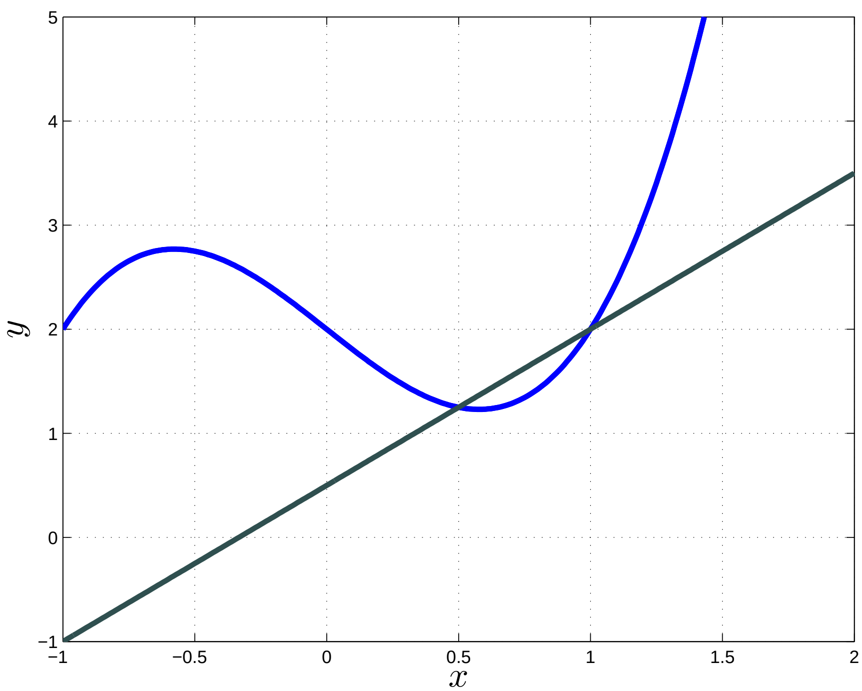


Figure 6.1: $y = f(x) = 2x^3 - 2x + 2$ (Blue) and $y = \dfrac{3}{2}x + \dfrac{1}{2}$ (Dark green).

Figure 6.1 shows the graphs of the curve f and its secant line.

2. Using the point-slope form, the equation of the secant line is

$$y = \frac{3}{8}x + \frac{17}{16}.$$

Figure 6.2 shows the graphs of the curve f and its secant line.

3. We infer that the tangent line is

$$y = -\frac{1}{2}x + \frac{1}{2}.$$

Figure 6.3 shows a family of equations of secant lines.

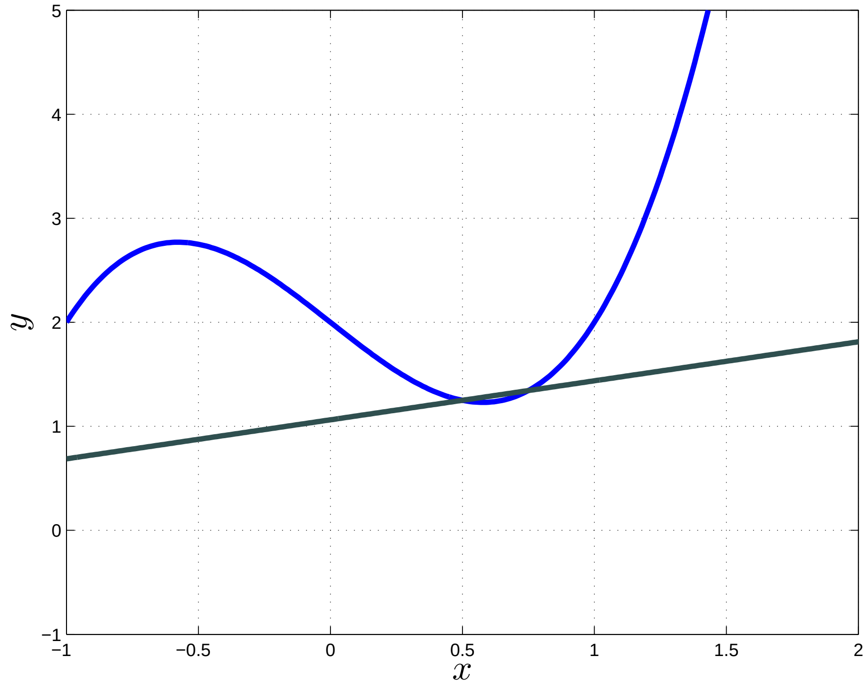
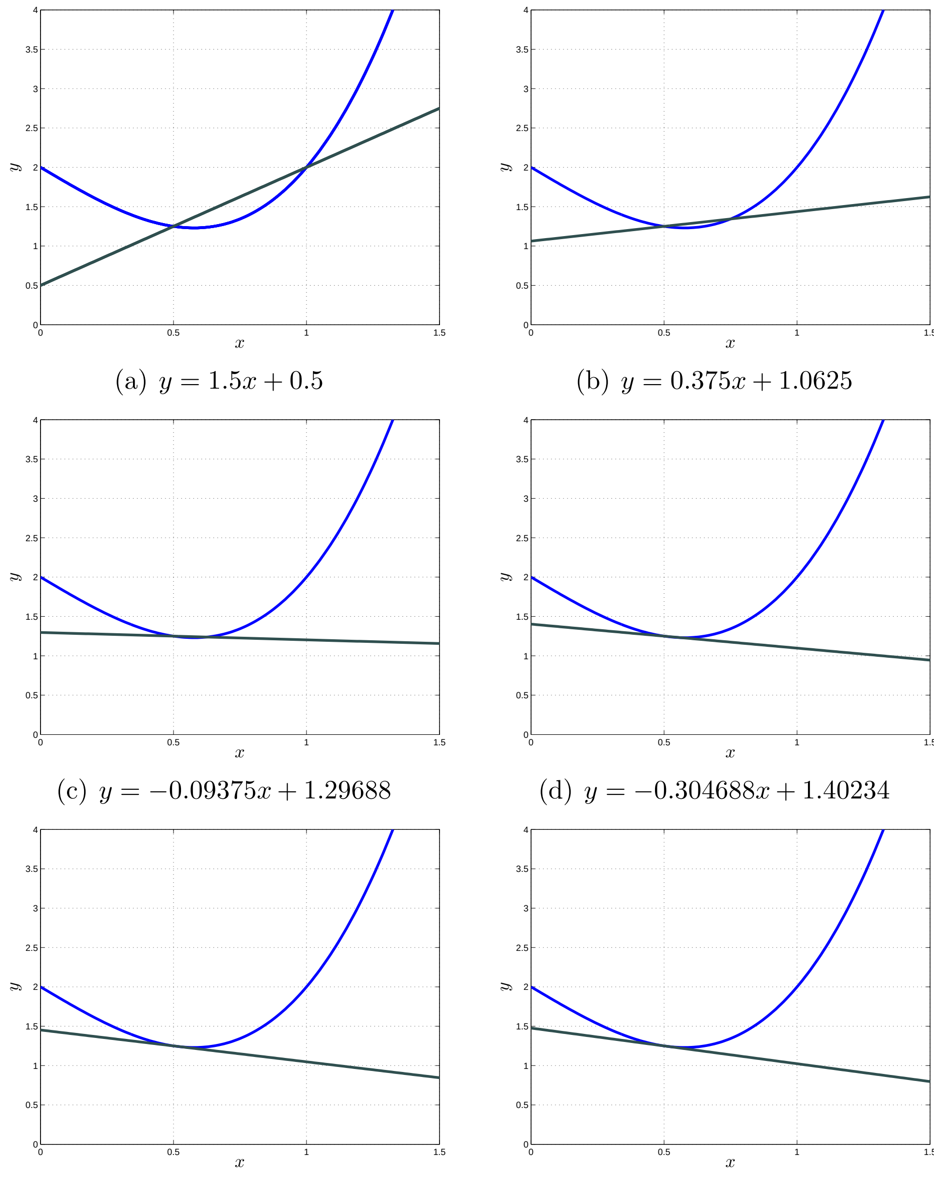


Figure 6.2: $y = f(x) = 2x^3 - 2x + 2$ (Blue) and $y = \dfrac{3}{8}x + \dfrac{17}{16}$ (Dark green).

Δx	$(x, f(x))$	$(\Delta x, f(x + \Delta x))$	Equation of secant line
0.5	$(0.5, 1.25)$	$(1, 2)$	$y = 1.5x + 0.5$
0.25	$(0.5, 1.25)$	$(0.75, 1.34375)$	$y = 0.375x + 1.0625$
0.125	$(0.5, 1.25)$	$(0.625, 1.23828)$	$y = -0.09375x + 1.29688$
0.0625	$(0.5, 1.25)$	$(0.5625, 1.23096)$	$y = -0.304688x + 1.40234$
0.03125	$(0.5, 1.25)$	$(0.53125, 1.23737)$	$y = -0.404297x + 1.45215$
0.015625	$(0.5, 1.25)$	$(0.515625, 1.24293)$	$y = -0.452637x + 1.47632$

Table 6.2:



(a) $y = 1.5x + 0.5$

(b) $y = 0.375x + 1.0625$

(c) $y = -0.09375x + 1.29688$

(d) $y = -0.304688x + 1.40234$

(e) $y = -0.09375x + 1.29688$

(f) $y = -0.452637x + 1.47632$

Figure 6.3: Equation of secant lines (Dark green).

Theorem 1 If $f'(a)$ exists then an equation of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$ is $y - f(a) = f'(a)(x - a)$.

Example 2 Find the equations of the tangent lines to the curve

$$y = \frac{x-1}{x+1}.$$

that are parallel to the line $x - 2y = 1$.

Solution:

The line $x - 2y = 1$ has slope $m = \frac{1}{2}$ and we use this with the derivative of $y = \dfrac{x-1}{x+1}$ to find the unknown x. Since

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(\frac{x-1}{x+1}\right)\\ &= \frac{(x+1)\frac{d}{dx}(x-1) - (x-1)\frac{d}{dx}(x+1)}{(x+1)^2}\\ &= \frac{2}{(x+1)^2},\end{aligned}$$

we have

$$\frac{1}{2} = \frac{2}{(x+1)^2}.$$

Solving $(x+1)^2 = 4$ for x, we get $x = 1$ and $x = -3$. Therefore, the points of tangency are at $(1, 0)$ and $(-3, 2)$. The tangent lines are found by using $y = \dfrac{1}{2}x + b$, where $b = y - \dfrac{1}{2}x$ with $(1, 0)$ and $(-3, 2)$. We find $b = -\dfrac{1}{2}$ and $b = \dfrac{7}{2}$, respectively. Therefore, the equations of the tangent lines are

$$y = \frac{1}{2}x - \frac{1}{2}$$

and

$$y = \frac{1}{2}x + \frac{7}{2}.$$

Figure 6.4 shows the graphs of the curve y and its tangent lines.

Example 3 How many tangent lines to the curve $y = \dfrac{x}{x+1}$ pass through the point $(1, 2)$? At which points do these tangent lines touch the curve?

Solution:

All tangent lines through $(1, 2)$ have the form $y - 2 = m(x - 1)$, where

$$\begin{aligned}m = f'(x) &= \frac{d}{dx}\left(\frac{x}{x+1}\right)\\ &= \frac{(x+1)\frac{d}{dx}(x) - x \cdot \frac{d}{dx}(x+1)}{(x+1)^2}\\ &= \frac{1}{(x+1)^2}.\end{aligned}$$

Since we are looking for the intersection (point of tangency), we eliminate y as follows:

$$y = \frac{x}{x+1} = \frac{1}{(x+1)^2}(x-1) + 2.$$

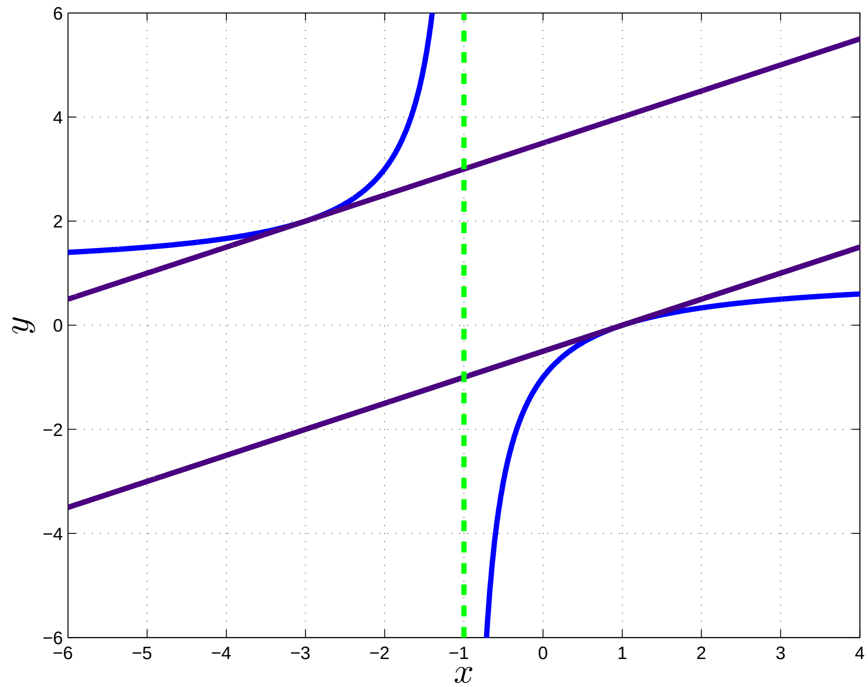


Figure 6.4: $y = \dfrac{x-1}{x+1}$ (Blue), $y = \dfrac{1}{2}x - \dfrac{1}{2}$ (Indigo) and $y = \dfrac{1}{2}x + \dfrac{7}{2}$ (Indigo).

Solving for x, we obtain, $x = -2 + \pm\sqrt{3}$. Thus there are two tangent lines and they are tangent at the point

$$\left(-2 + \pm\sqrt{3}, \frac{-2 + \pm\sqrt{3}}{-2 + \pm\sqrt{3} + 1}\right).$$

□

Example 4 Find equations of both tangent lines through the point $(2, -3)$ that are tangent to the parabola $y = x^2 + x$.

Solution:

All tangent lines through $(2, -3)$ have the form $y + 3 = m(x - 2)$, where

$$\begin{aligned} m = f'(x) &= \frac{d}{dx}\left(x^2 + x\right) \\ &= 2x + 1. \end{aligned}$$

Since we are looking for the intersection (point of tangency) we eliminate y as follows:

$$y = x^2 + x = (2x + 1)(x - 2) - 3.$$

Solving for x we obtain, $x = -1$ and $x = 5$. Thus there are two tangent lines and they are tangent at the points $(-1, 0)$ and $(5, 30)$. The two tangent lines are

$$y = -x - 1$$

and

$$y = 11x - 25.$$

□

Figure 6.5 shows the graphs of the curve y and its tangent lines.

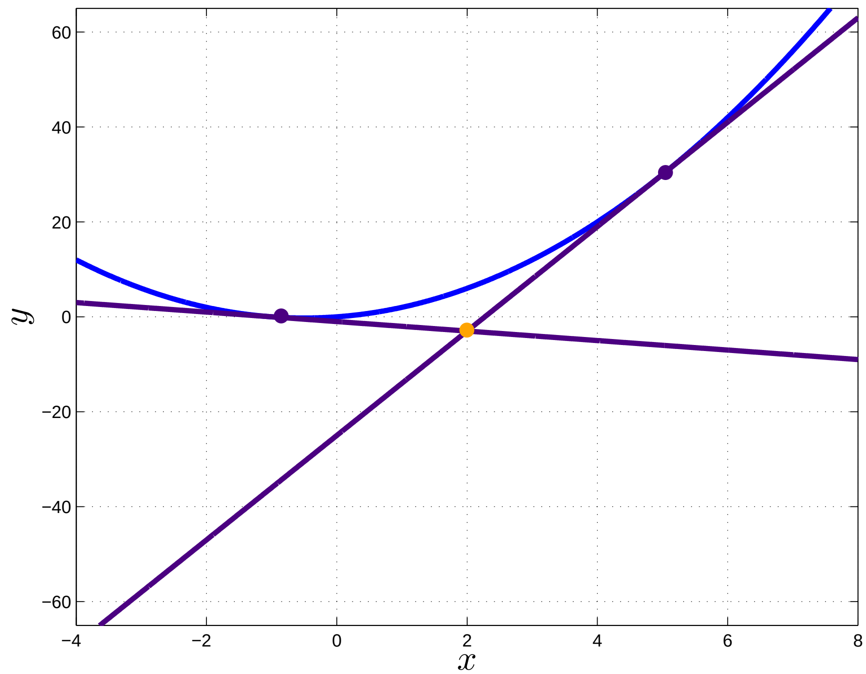


Figure 6.5: $y = x^2 + x$ (Blue), $y = -x - 1$ (Indigo) and $y = 11x - 25$ (Indigo).

Theorem 2 If $f'(a) = 0$ then the equation of the tangent line to the curve $y = f(x)$ at the point $(a, 0)$ is $y = f(a)$ and f is said to have a horizontal tangent line at $x = a$.

Example 5 For what values of x does the graph of $f(x) = 2x^3 - 3x^2 - 6x + 87$ have a horizontal tangent?

Solution:

According to Theorem 2, to find the horizontal tangent lines, we first find where the derivative of f is 0. Then we compute,

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(2x^3 - 3x^2 - 6x + 87\right) \\ &= 6x^2 - 6x - 6. \end{aligned}$$

Solving $6x^2 - 6x - 6 = 0$ or $x^2 - x - 1 = 0$ using the quadratic formula, we have

$$x = \frac{1}{2}\left(1 + \pm\sqrt{5}\right).$$

Thus, the values of x where the tangent lines are horizontal are

$$x = \frac{1}{2}\left(1 + \pm\sqrt{5}\right).$$

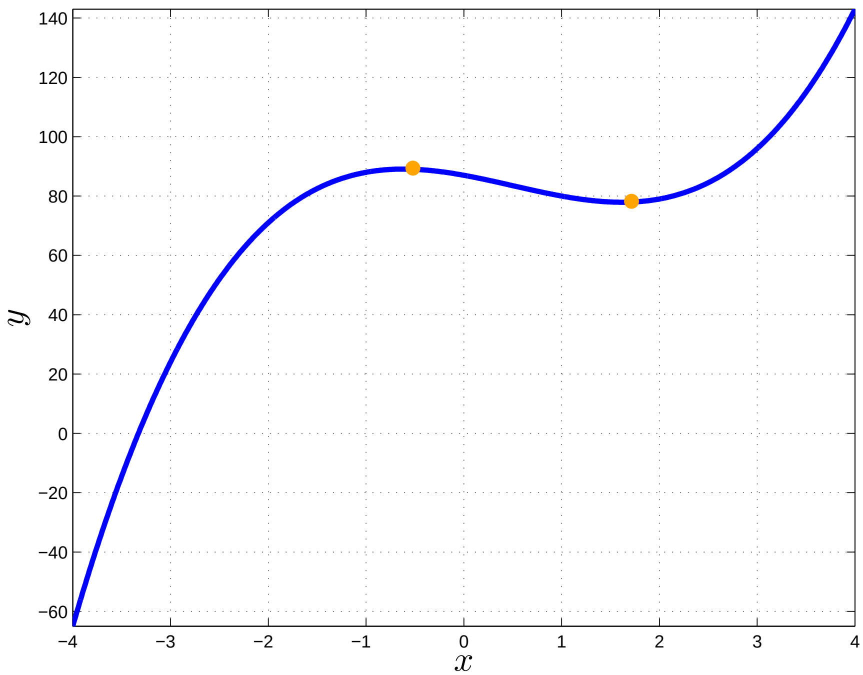


Figure 6.6: $y = f(x) = 2x^3 - 3x^2 - 6x + 87$ (Blue).

□

Figure 6.6 shows the graphs of the curve f.

Example 6 Find the points on the curve $y = x^3 - x^2 - x + 1$ where the tangent line is horizontal.

Solution:

According to Theorem 2, to find the horizontal tangent lines, we first find where the derivative of f is 0. Then we compute,

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(x^3 - x^2 - x + 1\right) \\ &= 3x^2 - 2x - 1. \end{aligned}$$

Solving $3x^2 - 2x - 1 = 0$ using the quadratic formula, we have $x = -\frac{1}{3}$ and $x = 1$. Thus, the values of x where the tangent lines are horizontal are $x = -\frac{1}{3}$ and $x = 1$. □

Figure 6.7 shows the graphs of the curve y.

Definition 1 Suppose y is a function of x; say $y = f(x)$. When a change in the variable is made from x to $x+\Delta x$, there is a corresponding change in y, namely $\Delta y = f(x+\Delta x) - f(x)$. The **average rate of change** of y with respect to x is

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

and is also known as the **difference quotient.**

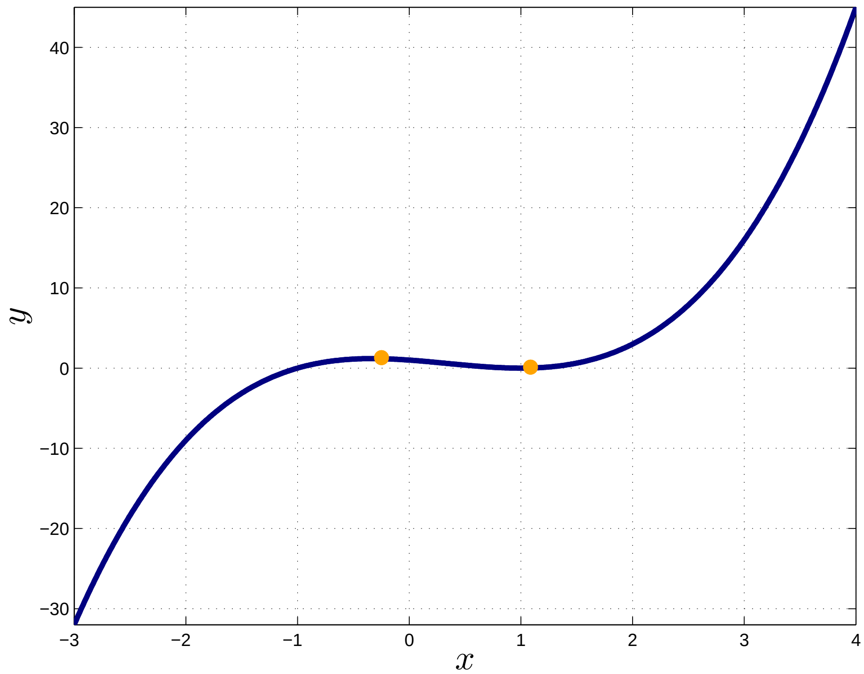


Figure 6.7: $y = x^3 - x^2 - x + 1$ (Blue).

Example 7 Let $f(x) = \sqrt{x^2 - 9}$. Find the average rate of change from $x = 3$ to $x = 5$.

Solution: Using Definition 1, we find the average rate of change of f from $x = 3$ to $x = 5$ is given by

$$\begin{aligned}\frac{f(x+\Delta x) - f(x)}{\Delta x} &= \frac{f(5) - f(3)}{5 - 3} \\ &= \frac{\sqrt{5^2 - 9} - \sqrt{3^2 - 9}}{5 - 3} \\ &= 2\end{aligned}$$

which is also the slope of the secant line through $(3, 0)$ and $(5, 4)$.

□

In general, suppose an object moves along a straight line according to an equation of motion

$$s = f(t),$$

where s is the displacement (directed distance) of the object from the origin at time t. The function f that describes the motion is called the position function of the object. In the time interval from $t = a$ to $t = a + h$ the change in position is $f(a + h) - f(a)$ and the average velocity over this time interval is

$$\frac{f(x+h) - f(x)}{h}$$

which is the same as the slope of the secant line through these two points.

Example 8 If a billiard is dropped from a height of feet 200, its height s at time t is given by the position function

$$s = -9t^2 + 200,$$

where s is measured in feet and t is measured in seconds. Find the average velocity over the intervals $[2, 2.7]$ and $[2, 2.8]$.

Solution: Two cases are considered as follows:

1. For the interval $[2, 2.7]$, the object falls from a height of feet

$$s(2) = -9(2)^2 + 200 = 164$$

to a height of feet

$$s(2.7) = -9(2.7)^2 + 200 = 134.39.$$

The average velocity in the interval $[2, 2.7]$ is

$$\begin{aligned}\frac{\text{distance traveled}}{\text{elapsed time}} &= \frac{s(2.7) - s(2)}{2.7 - 2} \\ &= \frac{134.39 - 164}{0.7} = -42.3 \text{ feet.}\end{aligned}$$

2. For the interval $[2, 2.8]$, the object falls from a height of feet

$$s(2) = -9(2)^2 + 200 = 164$$

to a height of feet

$$s(2.8) = -9(2.8)^2 + 200 = 129.44.$$

The average velocity in the interval $[2, 2.8]$ is

$$\begin{aligned}\frac{\text{distance traveled}}{\text{elapsed time}} &= \frac{s(2.8) - s(2)}{2.8 - 2} \\ &= \frac{129.44 - 164}{0.8} = -43.2 \text{ feet.}\end{aligned}$$

Note that the average velocities are negative indicating that the object is moving downward.

□

The difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

is the average rate of change of y with respect to x over the interval $[x_1, x_2]$ and can be interpreted as the slope of the secant line. Its limit as $\Delta x \to 0$ is the derivative at $x = x_1$ and is denoted by $f'(x_1)$.

We interpret the limit of the average rate of change as the interval becomes smaller and smaller to be the instantaneous rate of change. Often, different fields of science have specific interpretations of the derivative.

Definition 2 The average rate of change approaches the **instantaneous rate for change**[1]; that is,

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x)$$

and is also known as the derivative of f at x.

Note 1 In Definition 2, the following expressions also valid:

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(x)$$

or

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(x).$$

Figure 6.8 illustrates these cases with $y = f(x)$.

Example 9 Find the instantaneous rate of change of

$$s = -9t^2 + 200$$

at $t = 2$.

Solution: Since

$$s'(t) = -18t,$$

the instantaneous rate of change of

$$s = -9t^2 + 200$$

at $t = 2$ is given by

$$s'(2) = -18 \cdot 2 = -36.$$

□

Note 2 In Example 9, the following results are also correct.

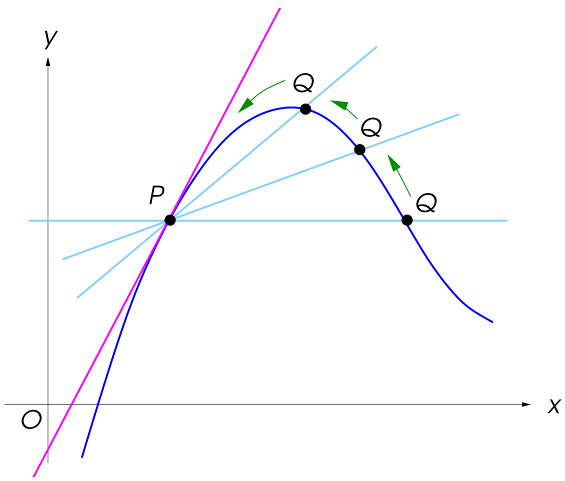
- The instantaneous rate of change of

  $$s = -9t^2 + 200$$

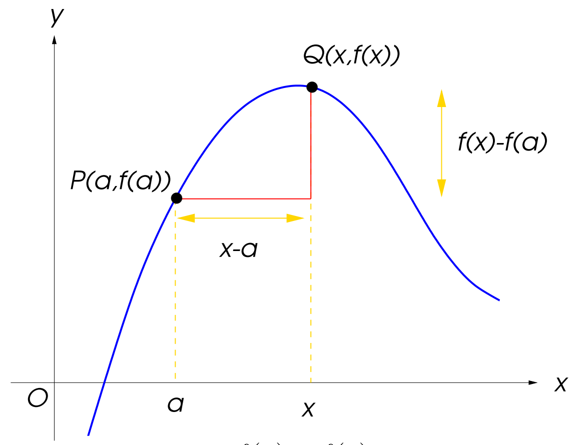
  at $t = a$ is:

$$\begin{aligned}
\lim_{t \to a} \frac{s(t) - s(a)}{t - a} &= \lim_{t \to a} \frac{(-9t^2 + 200) - (-9a^2 + 200)}{t - a} \\
&= \lim_{t \to a} \frac{9(-t^2 + a^2)}{t - a} \\
&= \lim_{t \to a} \frac{9(a-t)(a+t)}{-(a-t)} \\
&= -9 \lim_{t \to a}(a+t) \\
&= -9(a+a) = -18a.
\end{aligned}$$

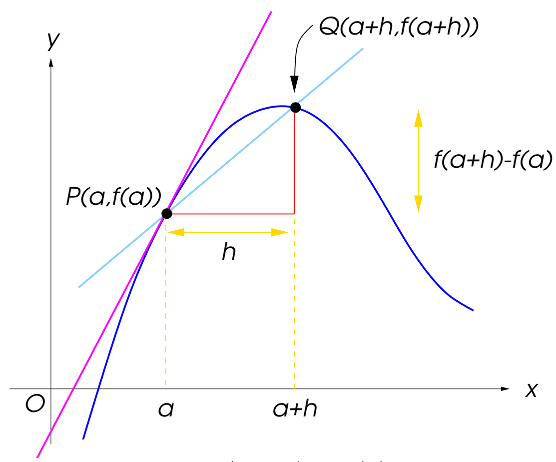
[1] It is conventional to use the word *instantaneous* even when x does not represent time, although the word is frequently omitted. When we talk about rates of change we are talking about instantaneous rates of change.



(a) Tangent line (pink) and secant lines (light blue)



(b) $\lim_{x\to a} \dfrac{f(x)-f(a)}{x-a} = f'(x)$



(c) $\lim_{h\to 0} \dfrac{f(a+h)-f(a)}{h} = f'(x)$

Figure 6.8: Instantaneous rate of change of f.

- The instantaneous rate of change of

$$s = -9t^2 + 200$$

at $t = a$ is:

$$
\begin{aligned}
\lim_{h\to 0}\frac{s(a+h)-s(a)}{h} &= \lim_{h\to 0}\frac{(-9(a+h)^2+200)-(-9a^2+200)}{h}\\
&= \lim_{h\to 0}\frac{(-9(a^2+2ah+h^2)+200)-(-9a^2+200)}{h}\\
&= \lim_{h\to 0}\frac{-9(2ah+h^2)}{h}\\
&= -9\lim_{h\to 0}(2a+h)\\
&= -18a.
\end{aligned}
$$

Next we illustrate the importance of the relative rate of change, as compared to the difference between the absolute rate of change and the average rate of change.

The absolute change is not the same as the average rate of change. Namely, the absolute change is just the differences in the values of f at the boundary of the interval $[x, x+\Delta x]$, namely $f(x+\Delta x)-f(x)$, whereas the average rate of change is the absolute change divided by the size of the interval:

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{\Delta x}.$$

Definition 3 Let $y = f(x)$, then the absolute change of f at the boundary of the interval $[x, x+\Delta x]$ is the difference

$$f(x+\Delta x)-f(x).$$

Definition 4 Let $y = f(x)$, then the relative rate of change of f at $x = x_0$ is the ratio

$$\frac{f'(x_0)}{f(x_0)}.$$

We can use these three ways to describe the change in f. The following example illustrates these types of change.

Example 10 Let $f(x) = 2x^2 - 3x + 5$.

1. Find the average rate of change of f from $x = 2$ to $x = 4$.
2. Find the instantaneous rate of change of f at x = 2.
3. Find the relative rate of change of f at $x = 2$.

Solution:

1. The average rate of change of f from $x = 2$ to $x = 4$ is given by

$$
\begin{aligned}
\frac{f(x+\Delta x)-f(x)}{\Delta x} &= \frac{f(4)-f(2)}{4-2}\\
&= \frac{(2(4)^2-3(4)+5)-(2(2)^2-3(2)+5)}{2}\\
&= 9.
\end{aligned}
$$

2. Since $f'(x) = 4x - 3$, the instantaneous rate of change of f at $t = 2$ is given by $f'(2) = 4(2) - 3 = 5$.

3. The relative rate of change of f at $x = 2$ is

$$\frac{f'(2)}{f(2)} = \frac{4(2) - 3}{2(2)^2 - 3(2) + 5} = \frac{5}{7} \approx 0.714286 \text{ or } 71\%.$$

□

6.3 Free-falling body problem

Rectilinear motion often refers to the motion of an object that can be modeled along a straight line; and the so-called falling body problems are a special type of rectilinear motion where the motion of an object is that it is falling (or being propelled) in a vertical direction. Another type of rectilinear motion is the free- falling body problem.

The position of a free-falling body[2] (neglect air resistance) under the influence of gravity can be represented by the function

$$s(t) = \frac{1}{2}gt^2 + v_0 t + s_0,$$

where g is the acceleration due to gravity[3] (on earth $g \approx 32$ ft$/s^2$) and s_0 and v_0 are the initial height and velocity of the object (when $t = 0$), respectively.

Example 11 A ball is thrown vertically upward from the ground with an initial velocity of 120 ft$/s$.

1. When will it hit the ground?

2. With what velocity will the ball hit the ground?

3. When will the ball reach its maximum height and what is that maximum height?

Solution:

1. We can determine when the ball will hit the ground by solving

$$s(t) = \frac{1}{2}gt^2 + v_0 t + s_0 = 0$$

for t.

Using $g = 32$, $v_0 = 120$ and $s_0 = 0$. We find

$$s(t) = -\frac{1}{2}32t^2 + 120t + (0) = 0$$

[2] A free-falling object is an object that is falling under the only influence of gravity.

[3] In terms of SI units, the value of g is 9.8 m/s^2.

or

$$-16t^2 + 120t = 0$$

when $t = 0$ and $t = 7.5$. Thus the ball will hit the ground 7.5 seconds after it is thrown upwards.

2. The velocity of the ball at time t is given by the first derivative of s, namely

$$v(t) = s'(t) = gt + v_0.$$

When $t = 7.5$, we find

$$v(7.5) = s'(7.5) = -(32)(7.5) + 120 = -120$$

and so the velocity of the ball is -120 ft$/s$ when it hits the ground.

3. The ball reaches it's maximum height when the velocity is zero, thus we solve

$$v(t) = s'(t) = -gt + v_0 = 0$$

yielding $t = -v_0/g = -120/(-32) = 3.75$. Its position at $t = 3.75$ is the maximum height which is

$$s(3.75) = -\frac{1}{2}32(3.75)^2 + 120(3.75) + (0) = 225.$$

Therefore, 3.75 seconds after the ball is thrown, the ball reaches it's maximum height of 225 ft.

$\square$

Definition 5 An object that moves along a straight line with position $s(t)$ has **velocity** $v(t) = \dfrac{ds}{dt}$ and **acceleration** $a(t) = \dfrac{dv}{dt} = \dfrac{d^2s}{dt^2}$ when these derivatives exist. The **speed** of an object at time t is $|v(t)|$.

Example 12 A particle moving along the $x-$axis has position

$$x(t) = 2t^3 + 3t^2 - 36t + 40$$

after an elapsed time of$t$ seconds.

1. Find the velocity of the particle at time t.

2. Find the acceleration at time t.

3. What is the total distance travelled by the particle during the first 3 seconds?

Solution:

1. The velocity is given by

$$v(t) = x'(t) = 6t^2 + 6t - 36.$$

2. The acceleration is given by

$$a(t) = v'(t) = x''(t) = 12t + 6.$$

3. Since $v(t) = 0$, when

$$6t^2 + 6t - 36 = 6(t-2)(t+3) = 0$$

then $t = 2, -3$ but -3 is not on $[0, 3]$. Therefore, the total distance travelled is

$$|x(2) - x(0)| + |x(3) - x(2)| = |-4 - 40| + |13 - (-4)| = 61.$$

$\square$