Calculus for Engineers

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August 2015

Contents

Some Applications of Integrals

26.1 Introduction

In Section 26.2, we will find the total area of a region by the graph of f and $x-$axis on a clsoed interval. In Section 26.3, we will find the area between two continuous functions using definite integrals. In Section 26.4, two mean value theorems of integral calculus are given. We first recognize that one of the uses of the definite integral is (that it can help us) to find the average value of a function on an interval $[a, b]$, referred to as the first mean value theorem for integrals. It us helpful that every continuous function attains its average value at least once on an interval. Then we will study the generalized mean value theorem for integrals. In Section 26.5, we will present the so-called second mean value theorem for integrals.

26.2 Total Area

Suppose the function $y = f(x)$ is continuous on the interval $[a, b]$ and that $f(x) < 0$ on $[a, c]$ and $f(x) \geq 0$ on $[c, d]$.

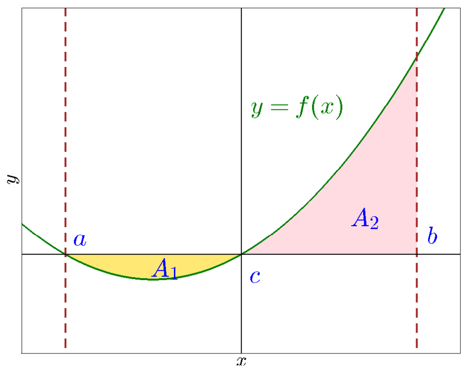


Figure 26.1: The definite integral of f on $[a, b]$ is not area.

The total are is the area of the region bounded by the graph of f, the $x-$axis, and the vertical lines $x = a$ and $x = b$, as shown in Figure 26.2.

To find this area we employ the absolute value of the function $y = |f(x)|$, which is non-negative for all x in $[a, b]$.

Recall, $|f(x)|$ is defined in a piece manner. For the function f shown in Figure **??**,

$f(x) < 0$ on the interval $[a, c)$ and $f(x) \geq 0$ on the interval $[c, b]$. Thus

$$|f(x)| = \begin{cases} -f(x), & \text{for } f(x) < 0; \\ f(x), & \text{for } f(x) \geq 0. \end{cases} \tag{26.1}$$

As shown in Figure 26.2, the graph of $y = |f(x)|$ on the interval $[a, c)$ is obtained by reflecting the portion of the graph of $y = f(x)$ through the $x-$axis.

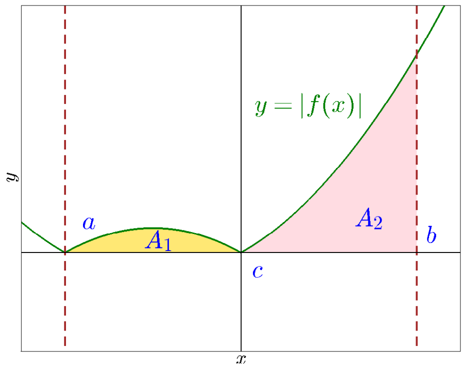


Figure 26.2: The definite integral of $|f|$ on $[a, b]$ is area. Total area is $A = A_1 + A_2$.

On the interval $[c, b]$, where $f(x) \geq 0$, the graphs of $y = f(x)$ and $y = |f(x)|$ are the same.

To find the total area

$$A = A_1 + A_2$$

shown in Figure 26.2, we use the additive interval property of the definite integral along with Equation (26.1):

$$\begin{aligned} \int_a^b |f(x)|dx &= \int_a^c |f(x)|dx + \int_c^b |f(x)|dx \\ &= \int_a^c (-f(x))dx + \int_c^b f(x)dx \\ &= A_1 + A_2 \end{aligned}$$

Definition 1 Definition of Total Area

If $y = f(x)$ is continuous on $[a, b]$, then the total area A bounded by its graph and the $x-$axis on the integral is given by

$$A = \int_a^b |f(x)|dx. \tag{26.2}$$

Example 1 Find the total area bounded by the graph of $y = x^3$ and the $x-$axis on $[-2, 1]$.

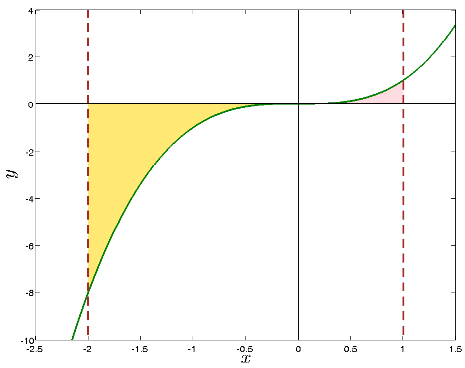


Figure 26.3: Graph of Example 1 of $y = f(x)$.

Solution:
From Equation (26.2), we have

$$A = \int_{-2}^{1} |x^3| dx.$$

In Figure 1, we have compared with the graph of $y = x^3$ with the graph of $y = |x^3|$. Since $x^3 < 0$ for $x < 0$, we have on $[-2, 1]$,

$$|f(x)| = \begin{cases} -x^3, & \text{for} \quad -2 \leq x < 0; \\ x^3, & \text{for} \quad 0 \leq x \leq 1. \end{cases}$$

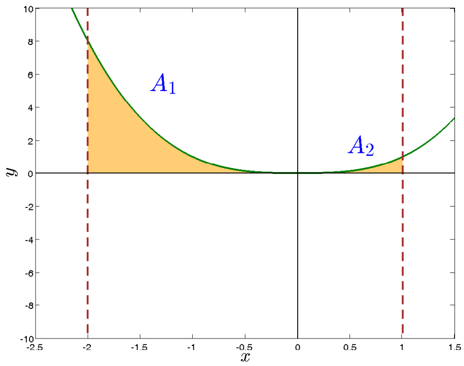


Figure 26.4: Graph of Example 1 of $y = |f(x)|$.

Thus Equation (26.2) of Definition 1 the desired area is

$$\begin{aligned} A &= \int_{-2}^{1} |x^3| dx \\ &= \int_{-2}^{0} |x^3| dx + \int_{0}^{1} |x^3| dx \\ &= \int_{-2}^{0} (-x^3) dx + \int_{0}^{1} x^3 dx \\ &= -\frac{1}{4} x^4 \Big|_{0}^{-2} + \frac{1}{4} x^4 \Big|_{1}^{0} \\ &= 0 - \left(-\frac{16}{4}\right) + \frac{1}{4} - 0 = \frac{17}{4}. \end{aligned}$$

Example 2 Find the total area bounded by the graph of $y = x^2 + 2x$ and the $x-$axis on $[-2, 2]$.
Solution:
From Equation (26.2), we have

$$A = \int_{-2}^{2} |x^2 + 2x| dx.$$

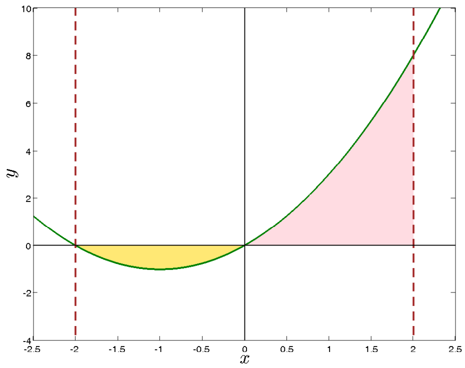


Figure 26.5: Graph of Example 2 of $y = f(x)$.

The graphs of $y = x^2 + 2x$ and $y = |x^2 + 2x|$ are shown in Figure 2. We see that on $[-2, 2]$

$$|f(x)| = \begin{cases} -(x^2 + 2x), & \text{for} \quad -2 \leq x < 0; \\ x^2 + 2x, & \text{for} \quad 0 \leq x \leq 2. \end{cases}$$

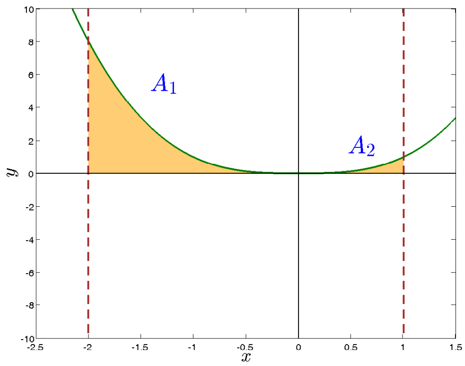


Figure 26.6: Graph of Example 2 of $y = |f(x)|$.

Thus Equation (26.2) of Definition 1 the desired area is

$$\begin{aligned} A &= \int_{-2}^{2} |x^2 + 2x| dx \\ &= \int_{-2}^{0} |x^2 + 2x| dx + \int_{0}^{2} |x^2 + 2x| dx \\ &= \int_{-2}^{0} (-x^2 + 2x) dx + \int_{0}^{2} x^2 + 2x dx \\ &= \left(-\frac{1}{3}x^3 - x^2\right)\Big|_{0}^{-2} + \left(\frac{1}{3}x^3 + x^2\right)\Big|_{2}^{0} \\ &= 0 - \left(\frac{8}{3} - 4\right) + \left(\frac{8}{3} + 4\right) - 0 = 8. \end{aligned}$$

26.3 Areas between f and g

We have already used integrals to find the area between the graph of a function and the horizontal axis. Integrals can also be used to find the area between two graphs. If $f(x) \geq g(x)$ for all x in $[a, b]$, then we can approximate the area between f and g by partitioning the interval [a,b] and forming a Riemann sum, as shown in Figure 26.3.

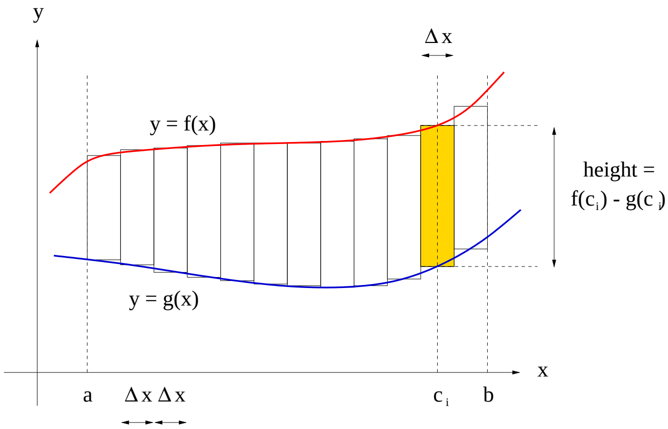


Figure 26.7: Approximate the area between f (in red) and g (in blue).

The height of each rectangle is

$$f(c_i) - g(c_i)$$

so the area of the ith rectangle is

$$(\text{height}) \cdot (\text{base}) = (f(c_i) - g(c_i)) \cdot \Delta x_i.$$

This approximation of the total area $\mathcal{A}$ as a Riemann sum is

$$\mathcal{A} \approx \sum_{i=1}^{n} (f(c_i) - g(c_i)) \cdot \Delta x_i.$$

The limit of this Riemann sum, as the mesh of the partitions approaches 0, is the definite integral

$$\int_a^b f(x)dx.$$

We will sometimes use an arrow to indicate "the limit of the Riemann sum as the mesh of the partitions approaches zero," and will write

$$\sum_{i=1}^{n} (f(c_i) - g(c_i)) \cdot \Delta x_i \longrightarrow \int_a^b f(x)dx.$$

Hence, we have following result:

Theorem 1 If $f(x) \geq g(x)$ for all x in $[a, b]$, then

$$\mathcal{A} = \left\{ \begin{array}{c} \text{area bounded by the graphs} \\ \text{of } f \text{ and } g \text{ and vertical} \\ \text{lines at } x = a \text{ and } x = b \end{array} \right\} = \int_a^b f(x)dx.$$

26.3.1 Worked examples

Example 3 Find the area of the region bounded above by $y = \dfrac{1}{x}$, bounded below by the $x-$axis, and bounded on the sides by $x = 1$ and $x = 2$, as shown in Figure 26.8.

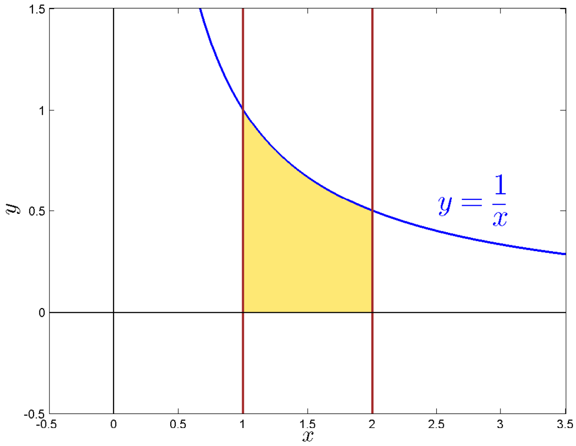


Figure 26.8: Graph of Example 3.

Solution. Two suggested solutions are given:

Solution 1 : Since the function $y = \dfrac{1}{x}$ is positive on $[1, 2]$, we have

$$\begin{aligned}\mathcal{A} = \int_1^2 \frac{1}{x}dx &= (\ln|x|)|_1^2 \\ &= \ln 2 - \ln 1 \\ &= \ln 2 - 0 \\ &= \ln 2.\end{aligned}$$

Solution 2 : We can use Theorem 1 with $f(x) = \dfrac{1}{x}$ and $g(x) = 0$, since $f(x) > g(x)$ on $[1, 2]$. We have

$$\begin{aligned}\mathcal{A} = \int_a^b (f(x) - g(x))\, dx &= \int_a^b \left(\frac{1}{x} - 0\right) dx \\ &= \int_1^2 \frac{1}{x}dx \\ &= (\ln|x|)|_1^2 \\ &= \ln 2 - \ln 1 \\ &= \ln 2 - 0 \\ &= \ln 2.\end{aligned}$$

□

Example 4 Find the area of the region bounded above by the $x-$axis, bounded below by $y = x^3$, and bounded on the sides by $x = -1$ and $x = 0$, as shown in Figure 26.9.

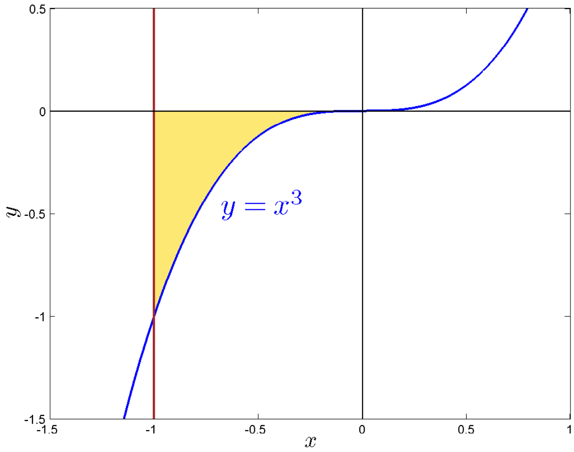


Figure 26.9: Graph of Example 4.

Solution. Two suggested solutions are given:

Solution 1 : Since the function $y = x^3 \leq 0$ on $[-1, 0]$, we have

$$\begin{aligned}\mathcal{A} = -\int_{-1}^{0} x^3 dx &= \left(-\frac{x^4}{4}\right)\Big|_{-1}^{0} \\ &= -\frac{(0)^4}{4} - \left(-\frac{(-1)^4}{4}\right) \\ &= \frac{1}{4}.\end{aligned}$$

Solution 2 : We can use Theorem 1 with $f(x) = 0$ and $g(x) = x^3$, since $f(x) \geq g(x)$ on $[-1, 0]$. We have

$$\begin{aligned}\mathcal{A} = \int_{a}^{b} (f(x) - g(x))\, dx &= \int_{-1}^{0} \left(0 - x^3\right) dx \\ &= -\int_{-1}^{0} x^3 dx \\ &= \left(-\frac{x^4}{4}\right)\Big|_{-1}^{0} \\ &= -\frac{(0)^4}{4} - \left(-\frac{(-1)^4}{4}\right) \\ &= \frac{1}{4}.\end{aligned}$$

$\square$

Example 5 Find the area of the region bounded by $y = x$, $y = \cos x$, $x = 0$, and $x = \frac{\pi}{6}$, as shown in Figure 26.10.

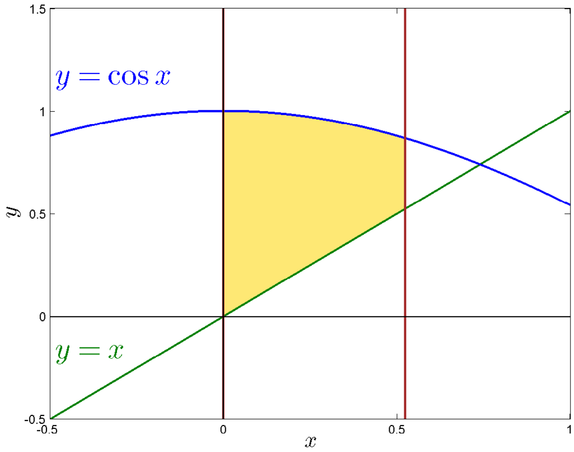


Figure 26.10: Graph of Example 5.

Solution. We can use Theorem 1 with $f(x) = \cos x$ (in blue) and $g(x) = x$ (in green), since $f(x) > g(x)$ on $\left[0, \frac{\pi}{6}\right]$. We have

$$\begin{aligned}
\mathcal{A} = \int_a^b (f(x) - g(x))\,dx &= \int_0^{\pi/6} (\cos x - x)\,dx \\
&= \left(\sin x - \frac{x^2}{2}\right)\Bigg|_0^{\pi/6} \\
&= \left(\sin\left(\frac{\pi}{6}\right) - \frac{(\pi/6)^2}{2}\right) - \left(\sin(0) - \frac{(0)^2}{2}\right) \\
&= \frac{1}{2} - \frac{\pi^2}{72}.
\end{aligned}$$

$\square$

Example 6 Find the area of the region bounded by $y = 9 - x^2, y = x + 1, x = -1$, and$x = 2$, as shown in Figure 26.11.

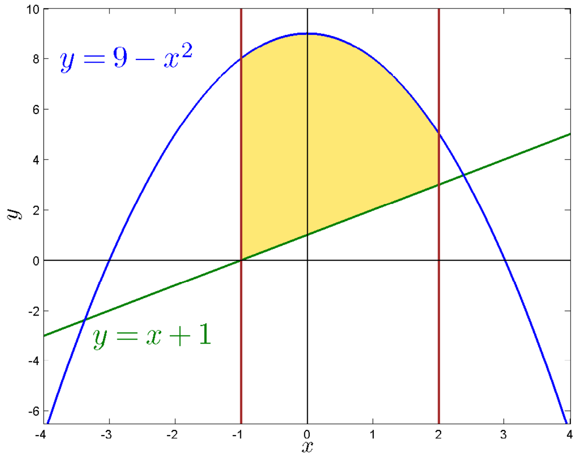


Figure 26.11: Graph of Example 6.

Solution. We can use Theorem 1 with $f(x) = 9 - x^2$ (in blue) and $g(x) = x + 1$ (in green), since $f(x) > g(x)$ on $[-1, 2]$. We have

$$\begin{aligned}
\mathcal{A} = \int_a^b (f(x) - g(x))\,dx &= \int_{-1}^{2} \left((9 - x^2) - (x + 1)\right) dx \\
&= \int_{-1}^{2} \left(8 - x - x^2\right) dx \\
&= \left(8x - \frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_{-1}^{2} \\
&= \left(16 - 2 - \frac{8}{3}\right) - \left(-8 - \frac{1}{2} + \frac{1}{3}\right) \\
&= 22 - 3 + \frac{1}{2} \\
&= \frac{39}{2}.
\end{aligned}$$

□

Example 7 Find the area of the region bounded by $y = x^2 - 2x$ and $y = x + 4$, as shown in Figure 26.12.

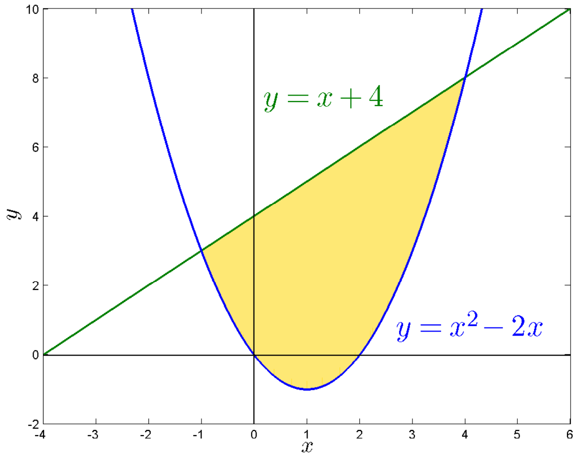


Figure 26.12: Graph of Example 7.

Solution. Two curves intersect when

$$x^2 - 2x = x + 4$$

or

$$x^2 - 3x - 4 = 0$$

or

$$(x + 1)(x - 4) = 0,$$

so $x = -1$ or $x = 4$. We can use Theorem 1 with $f(x) = x+4$ (in green) and $g(x) = x^2-2x$ (in blue), since $f(x) \geq g(x)$ on $[-1, 4]$. We have

$$\begin{aligned}
\mathcal{A} = \int_a^b (f(x) - g(x))\, dx &= \int_{-1}^{4} \big((x+4) - \big(x^2 - 2x\big)\big)\, dx \\
&= \int_{-1}^{4} \big(-x^2 + 3x + 4\big)\, dx \\
&= \left(-\frac{x^3}{3} + \frac{3x^2}{2} + 4x\right)\Bigg|_{-1}^{4} \\
&= \left(-\frac{64}{3} + 24 + 16\right) - \left(\frac{1}{3} + \frac{2}{3} - 4\right) \\
&= \frac{125}{6}.
\end{aligned}$$

$\square$

Example 8 Find the area of the region bounded by $y = \sqrt{x+3}$ and $y = \dfrac{x+3}{2}$, as shown in Figure 26.13.

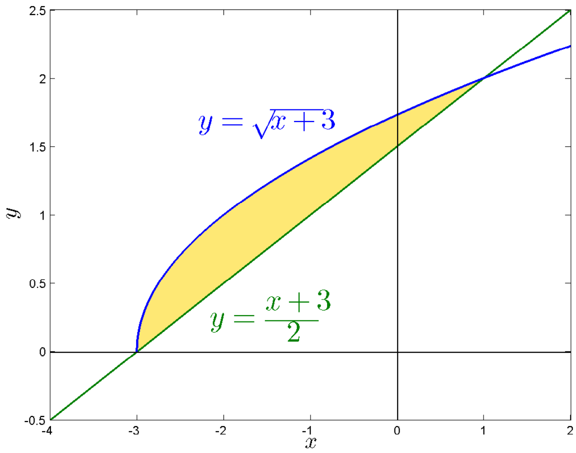


Figure 26.13: Graph of Example 8.

Solution. Two curves intersect when

$$\sqrt{x+3} = \frac{x+3}{2}$$

or

$$\left(\sqrt{x+3}\right)^2 = \left(\frac{x+3}{2}\right)^2$$

or

$$x + 3 = \frac{1}{4}(x+3)^2$$

or

$$4(x+3) - (x+3)^2 = 0$$

or

$$(x+3)(1-x) = 0$$

so $x = -3$ or $x = 1$. We can use Theorem 1 with $f(x) = \sqrt{x+3}$ (in blue) and $g(x) = \dfrac{x+3}{2}$ (in green), since $f(x) \geq g(x)$ on $[-3, 1]$. We have

$$\begin{aligned}
\mathcal{A} = \int_a^b (f(x) - g(x))\,dx &= \int_{-3}^{1} \left(\sqrt{x+3} - \frac{x+3}{2}\right) dx \\
&= \left(\frac{2}{3}(x+3)^{3/2} - \frac{(x+3)^2}{4}\right)\Bigg|_{-3}^{1} \\
&= \left(\frac{16}{3} - 4\right) - (0 - 0) \\
&= \frac{4}{3}.
\end{aligned}$$

□

Example 9 Find the area of the region bounded by $y = \sqrt{x}$, $y = \dfrac{x}{2}$ and $x = 5$, as shown in Figure 26.14.

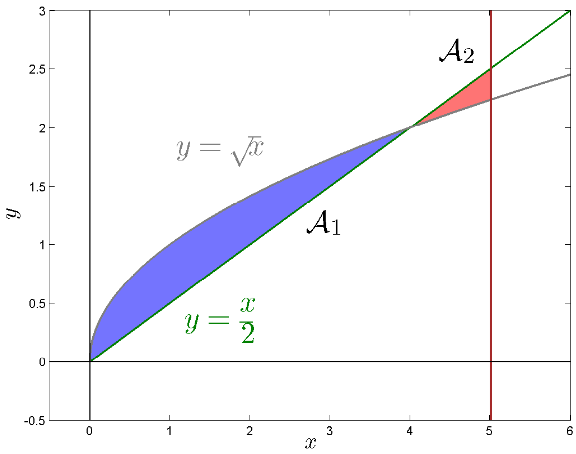


Figure 26.14: Graph of Example 9.

Solution. Two curves intersect when

$$\sqrt{x} = \frac{x}{2}$$

or

$$\left(\sqrt{x}\right)^2 = \left(\frac{x}{2}\right)^2$$

or

$$x = \frac{1}{4}x^2$$

or

$$x^2 - 4x = 0$$

or

$$x(x-4) = 0$$

so $x = 0$ or $x = 4$. We can use Theorem 1 with $f(x) = \sqrt{x}$ (in grey) and $g(x) = \dfrac{x}{2}$ (in green), since $f(x) \geq g(x)$ on $[0,4]$ and $g(x) \geq f(x)$ on $[4,9]$. We have

$$\begin{aligned}
\mathcal{A} &= \mathcal{A}_1 + \mathcal{A}_2 \\
&= \int_0^4 \left(\sqrt{x} - \frac{x}{2}\right) dx + \int_4^5 \left(\frac{x}{2} - \sqrt{x}\right) dx \\
&= \left(\frac{2}{3}x^{3/2} - \frac{x^2}{4}\right)\Bigg|_0^4 + \left(\frac{x^2}{4} - \frac{2}{3}x^{3/2}\right)\Bigg|_0^4 \\
&= \frac{107 - 40\sqrt{5}}{12}.
\end{aligned}$$

□

Note 1 If $f(x) \geq g(x)$, we can use the simpler argument that the area of region $\mathcal{A}$ is $\int_a^b f(x)dx$ and the area of region $\mathcal{B}$ is $\int_a^b g(x)dx$, so the area of region $\mathcal{C}$, the area between f and g , is

$$\begin{aligned}\text{area of } \mathcal{C} &= (\text{area of } \mathcal{A}) - (\text{area of } \mathcal{B}) \\ &= \int_a^b f(x)dx - \int_a^b g(x)dx \\ &= \int_a^b (f(x) - g(x))\, dx.\end{aligned}$$

If the same function is not always greater, then we need to be very careful and find the intervals where $f \geq g$ and the intervals where $g \geq f$ (see Examples 10 and 11).

Example 10 Find the area of the region bounded by $y = \sin x$, $y = \cos x$, $x = 0$ and $x = \pi/2$, as shown in Figure 26.15.

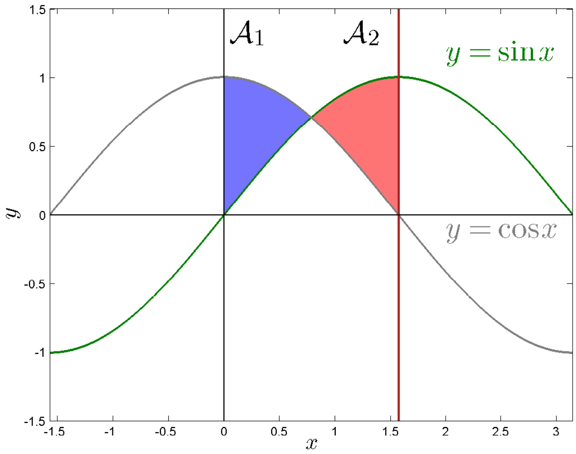


Figure 26.15: Graph of Example 10.

Solution. The points of intersection occur when $\sin x = \cos x$, that is, when $x = \pi/4$ (since $0 \le x \le \pi/2$). We observe that $\cos x \ge \sin x$ when $0 \le x \le \pi/4$ but $\sin x \ge \cos x$ when $\pi/4 \le x \le \pi/2$. Therefore the required area is

$$
\begin{aligned}
\mathcal{A} &= \int_0^{\pi/2} |\cos x - \sin x| \, dx \\
&= \int_0^{\pi/4} (\cos x - \sin x) \, dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x) \, dx \quad (= \mathcal{A}_1 + \mathcal{A}_2) \\
&= (\sin x + \cos x)|_0^{\pi/4} + (-\cos x - \sin x)|_{\pi/4}^{\pi/2} \\
&= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 0 - 1 \right) + \left(-0 - 1 \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right) \\
&= 2\sqrt{2} - 2.
\end{aligned}
$$

□

Example 11 Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

Solution. By solving the two equations we find that the points of intersection are $(-1, -2)$ and $(5, 4)$. Two suggested solutions are given:

Solution 1 : We solve the equation of the parabola for x and notice from Figure 26.16 that the left and right boundary curves are

$$x_L = \frac{1}{2}y^2 - 3 \qquad \text{and} \qquad x_R = y + 1.$$

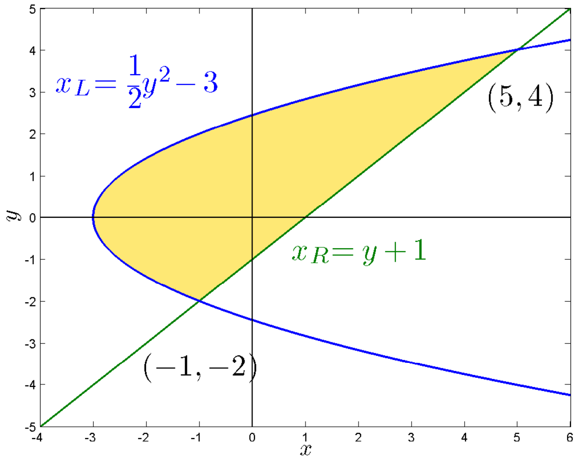


Figure 26.16: Graph of Example 11.

We must integrate between the appropriate $y-$values, $y = -2$ and $y = 4$. Thus

$$\begin{aligned}
\mathcal{A} &= \int_{-2}^{4} (x_R - x_L)\, dx \\
&= \int_{-2}^{4} \left((y+1) - \left(\frac{1}{2}y^2 - 3 \right) \right) dx \\
&= \int_{-2}^{4} \left(-\frac{1}{2}y^2 + y + 4 \right) dx \\
&= \left(-\frac{1}{2} \cdot \frac{y^3}{3} + \frac{y^2}{2} + 4y \right) \Big|_{4}^{-2} \\
&= \left(-\frac{64}{6} + 8 + 16 \right) - \left(-\frac{4}{3} + 2 - 8 \right) \\
&= 18.
\end{aligned}$$

Solution 2 : In Figure 26.17, we can find the area by integrating with respect to x instead of y, but the calculation is much more involved:

$$\begin{aligned}
\mathcal{A} &= \mathcal{A}_1 + \mathcal{A}_2 \\
&= \int_{-3}^{-1} \left(\sqrt{2x+6} - \left(-\sqrt{2x+6}\right)\right) dx + \int_{-1}^{5} \left(\sqrt{2x+6} - (x-1)\right) dx \\
&= 2\int_{-3}^{-1} \sqrt{2x+6}dx + \int_{-1}^{5} \left(\sqrt{2x+6} - (x-1)\right) dx \\
&= 2\left(\frac{1}{3}(2x+6)^{3/2}\right)\Bigg|_{-3}^{-1} + \left(\frac{1}{3}(2x+6)^{3/2} - \frac{x^2}{2} + x\right)\Bigg|_{-1}^{5} \\
&= 18.
\end{aligned}$$

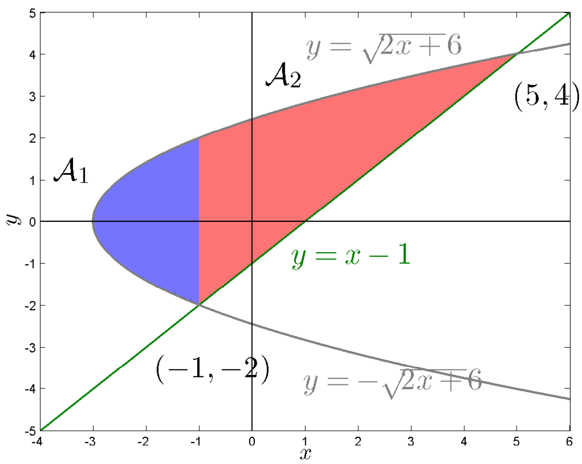


Figure 26.17: Graph of Example 11.

$\square$

Example 12 Two objects start from the same location and travel along the same path with velocities (in meters per second):

$$v_A(t) = t + 3$$

and

$$v_B(t) = t^2 - 4t + 3$$

as shown in Figure 26.18. How far ahead is A after 3 seconds? After 5 seconds?

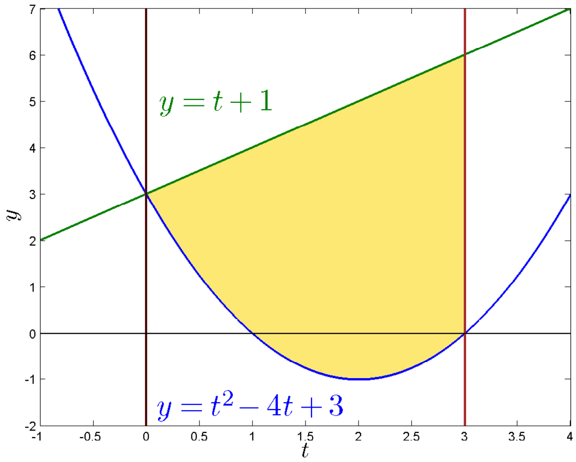


Figure 26.18: Graph of Example 12.

Solution. We can use Theorem 1 with $v_A(x) = t + 3$ (in grey) and $v_B(x) = t^2 - 4t + 3$ (in green), since $v_A(t) \geq v_B(t)$ on $[0, 3]$. Let $\mathcal{A}$ be the area between the graphs of v_A and v_B which represents the distance between the objects.

After 3 seconds, the distance apart is

$$\begin{aligned}
\mathcal{A} &= \int_0^3 (v_A(t) - v_B(t))\, dt \\
&= \int_0^3 \big((t+3) - (t^2 - 4t + 3)\big)\, dt \\
&= \int_0^3 \left(5t - t^2\right) dt \\
&= \left(\frac{5}{2}t^2 - \frac{t^3}{3}\right)\Bigg|_0^3 \\
&= \frac{27}{2} \text{ meters.}
\end{aligned}$$

After 5 seconds, the distance apart is

$$\begin{aligned}\mathcal{A} &= \int_0^5 (v_A(t) - v_B(t))\, dt \\ &= \left(\frac{5}{2}t^2 - \frac{t^3}{3}\right)\Bigg|_0^5 \\ &= \frac{125}{6} \text{ meters.}\end{aligned}$$

□

26.4 The mean value theorem of integral calculus

Two mean value theorems for integrals are given.

26.4.1 First mean value theorem for integrals

Our aim here is to find the average value of positive f that is the height of the rectangle whose area is the same as the area under f.

Theorem 2 (The First Mean Value Theorem for Integrals): If $f(x)$ is continuous on $[a, b]$, then there exists a number c in $[a, b]$ such that

$$f(c) = f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)dx \tag{26.3}$$

or

$$\int_a^b f(x)dx = f(c) \cdot (b - a).$$

Proof. Since $f(x)$ is continuous on $[a, b]$, $f(x)$ has minimum value m and maximum value M on $[a, b]$, and by Proposition 8 of integrals in Chapter 21,

$$m(b-a) \le \int_a^b f(x)dx \le M(b-a). \tag{26.4}$$

Now, dividing Inequality (26.4) through by $(b - a)$ gives

$$m \le \frac{1}{b-a}\int_a^b f(x)dx \le M. \tag{26.5}$$

and one can apply the Intermediate Value Theorem to Inequality (26.5). Since $f(x)$ assumes the values m and M, it must assume all values in between m and M, including the particular value

$$\frac{1}{b-a}\int_a^b f(x)dx.$$

In other words: there is at least one number c on the interval that satisfies (26.3).

□

Note 2 Let us interpret the geometrical meaning of Theorem 2. One imagines that the area under f (Figure 26.19) is a green liquid that can "leak" through the graph to form a purple rectangle with the same area (Figure 26.20).

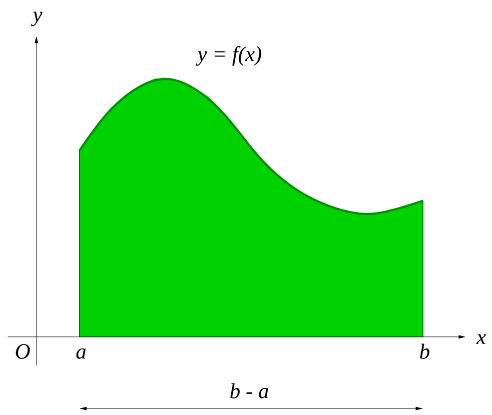


Figure 26.19: Illustration of Theorem 2.

In addition, the point $f(c)$ is called the average value of $f(x)$ on $[a, b]$, as shown in Figure 26.20.

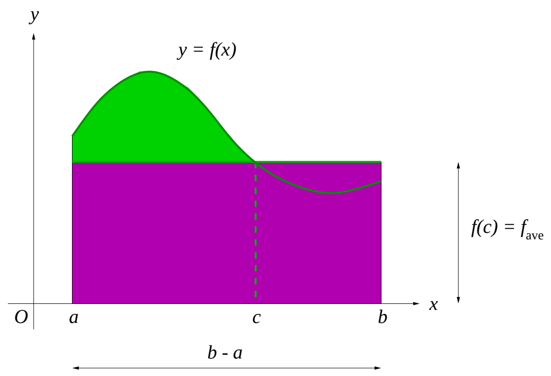


Figure 26.20: Illustration of Theorem 2.

26.4.2 Worked examples

Example 13 Find the average value of the function $f(x) = \sqrt{x}$ on the interval $[1, 4]$, as shown in Figure 26.21.

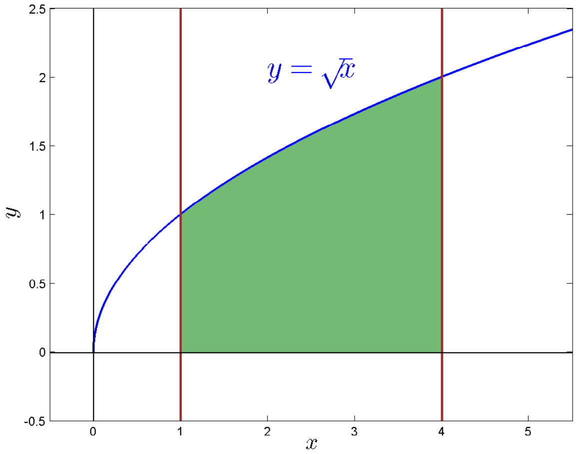


Figure 26.21: Graph of Example 13.

Solution. We have

$$\begin{aligned}
f_{\text{ave}} &= \frac{1}{4-1}\int_1^4 \sqrt{x}dx \\
&= \frac{1}{3}\int_1^4 x^{\frac{1}{2}}dx \\
&= \frac{1}{3}\left(\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right)\Bigg|_1^4 \\
&= \frac{1}{3}\left(\frac{4^{\frac{1}{2}+1}}{\frac{1}{2}+1}-\frac{1^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right) \\
&= \frac{14}{9}.
\end{aligned}$$

In Figure 26.22, we now find c :

$$f(c) = f_{\text{ave}} = \frac{14}{9}.$$

Then

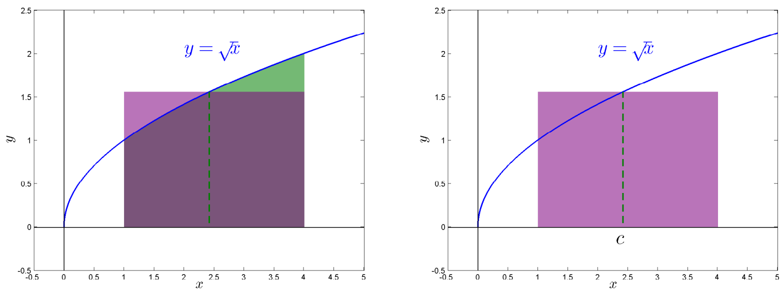
$$\sqrt{c} = \frac{14}{9}$$

or

$$c = \frac{196}{81}.$$

□

Example 14 Find the average value of the function $f(x) = 1+x^2$ on the interval $[-1, 2]$, as shown in Figure 26.23.



(a) (b)

Figure 26.22: Graph of Example 13.

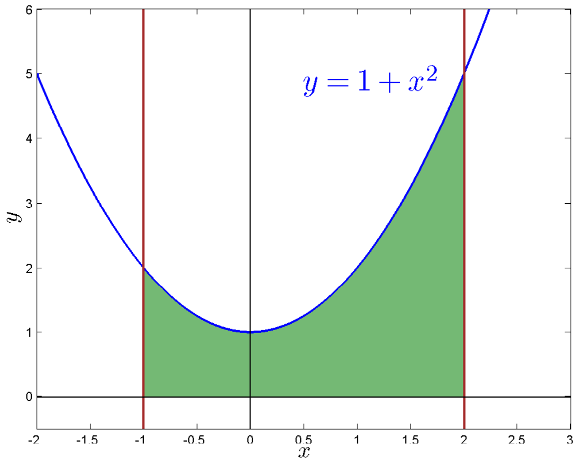


Figure 26.23: Graph of Example 14.

Solution. We have

$$\begin{aligned} f_{\text{ave}} &= \frac{1}{2-(-1)} \int_{-1}^{2} \left(1+x^2\right) dx \\ &= \frac{1}{3}\left(x + \frac{x^3}{3}\right)\Big|_{-1}^{2} \\ &= 2. \end{aligned}$$

In Figure 26.24, we now find c :

$$f(c) = f_{\text{ave}} = 2.$$

Then

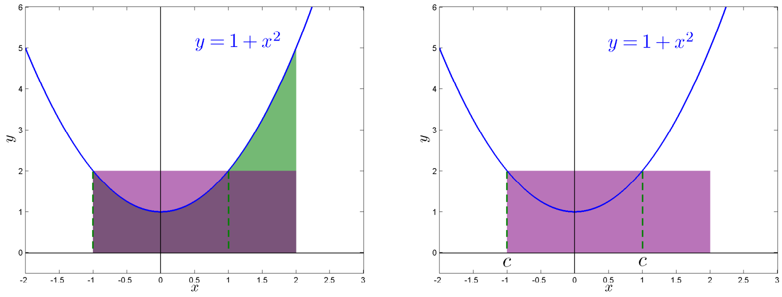
$$1 + c^2 = 2$$

or

$$c^2 = 1$$

or

$$c = \pm 1.$$



(a) (b)

Figure 26.24: Graph of Example 14.

□

Example 15 Find the average value of the function $f(x) = \sqrt{4 - x^2}$ on the interval $[-2, 2]$, as shown in Figure 26.25.

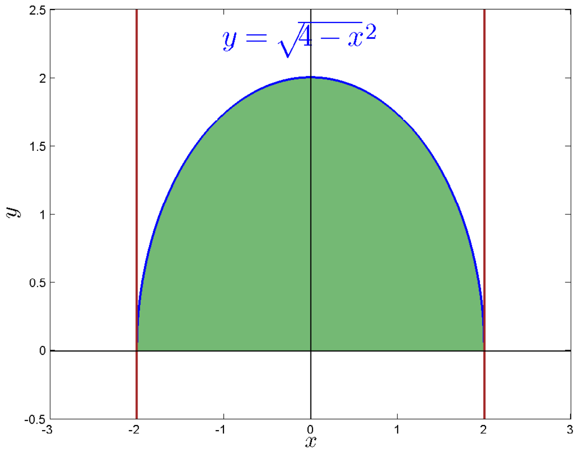


Figure 26.25: Graph of Example 15.

Solution. We have

$$f_{\text{ave}} = \frac{1}{2-(-2)} \int_{-2}^{2} \left(\sqrt{4 - x^2}\right) dx$$
$$= \frac{1}{4} \cdot \left(\frac{\pi \cdot 2^2}{2}\right)$$
$$= \frac{\pi}{2}.$$

In Figure 26.26, we now find c :

$$f(c) = f_{\text{ave}} = \frac{\pi}{2}.$$

Then

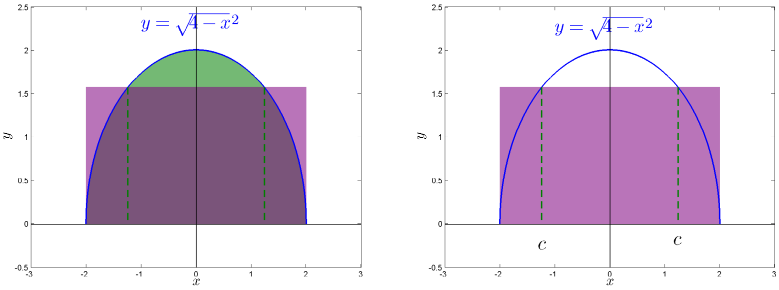
$$\sqrt{4-c^2} = \frac{\pi}{2}$$

or

$$4 - c^2 = \frac{\pi^2}{4}$$

or

$$c = \pm\sqrt{4 - \frac{\pi^2}{4}} \approx \pm 1.23798.$$



(a) (b)

Figure 26.26: Graph of Example 15.

□

Example 16 Assume that at different altitudes in the earth's atmosphere, sound travels at different speeds. The speed of sound $s(x)$ (in meters per second) can be modeled by

$$s(x) = \begin{cases} -4x + 341, & \text{if } 0 \le x < 11.5, \\ 295, & \text{if } 11.5 \le x < 22, \\ \frac{3}{4}x + 278.5, & \text{if } 22 \le x < 32 \\ \frac{3}{2}x + 254.5, & \text{if } 32 \le x < 50 \\ -\frac{3}{2}x + 404.5, & \text{if } 50 \le x < 80. \end{cases} \tag{26.6}$$

which is the altitude in kilometers. What is the average speed of sound over the interval $[0, 80]$?

Solution. By integrating over the 5 subintervals, we have

$$\int_{0}^{11.5} s(x)dx = \int_{0}^{11.5} (-4x + 341)\, dx = 3657$$

$$\int_{11.5}^{22} s(x)dx = \int_{11.5}^{22} (295)\, dx = 3097.5$$

$$\int_{22}^{32} s(x)dx = \int_{22}^{32} \left(\frac{3}{4}x + 278.5\right) dx = 2987.5$$

$$\int_{32}^{50} s(x)dx = \int_{32}^{50} \left(\frac{3}{2}x + 254.5\right) dx = 5688$$

$$\int_{50}^{80} s(x)dx = \int_{50}^{80} \left(-\frac{3}{2}x + 404.5\right) dx = 9210$$

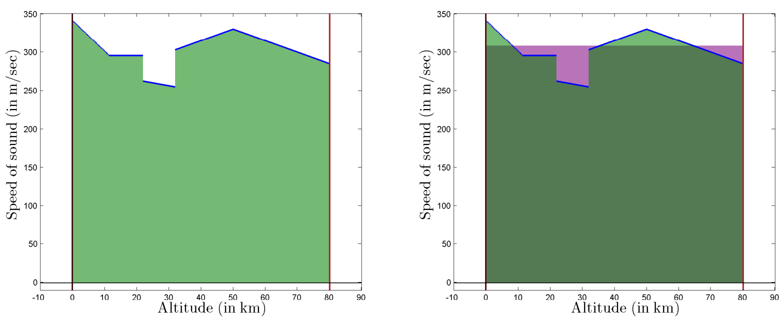
By adding the values of the five integrals, we have

$$\int_{0}^{11.5} s(x)dx = 24,640.$$

Therefore, the average speed of sound from an altitude of 0 kilometers to 80 kilometers is

$$\text{Average speed} = \int_{0}^{11.5} s(x)dx = \frac{24,640}{80} = 308 \text{ meters per second,}$$

as shown in Figure 26.27.



(a) (b)

Figure 26.27: Graph of Example 16.

□

26.4.3 Generalized mean value theorem for integrals

Theorem 3 (The Generalized Mean Value Theorem for Integrals): If $f(x)$ and $g(x)$ are integrable on $[a, b]$ and $g(x)$ keeps the same sign over $[a, b]$, then there exists a number μ lying between the bounds of $f(x)$ such that there then exists a number c in $[a, b]$ such that

$$\int_a^b f(x)g(x)dx = \mu \int_a^b g(x)dx. \tag{26.7}$$

Proof. Let $g(x)$ be positive (that is, $g(x) \geq 0$) over $[a, b]$. If m and M are the bounds of $f(x)$, for all $x \in [a, b]$, then we have

$$m \leq f(x) \leq M. \tag{26.8}$$

Now, multiplying Inequality (26.8) through by $g(x)$ gives

$$mg(x) \leq f(x)g(x) \leq Mg(x).$$

Thus

$$m \int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq M \int_a^b g(x)dx, \quad \text{when } b \geq a. \tag{26.9}$$

Let μ be a number lying between m and M. Therefore

$$\int_a^b f(x)g(x)dx = \mu \int_a^b g(x)dx. \tag{26.10}$$

$\square$

Corollary 4 If in addition to the conditions of Theorem 3, $f(x)$ is continuous on $[a, b]$, then there exists a number $\xi \in [a, b]$ such that

$$\int_a^b f(x)g(x)dx = f(\xi) \int_a^b g(x)dx. \tag{26.11}$$

Note 3 The number $f(\xi)$ is called the $g(x)$-weighted average of $f(x)$ on the interval $[a, b]$.

Note 4 Two notes are:

1. If $g(x) \leq 0$, then the sign of the inequality changes, but (26.9), (26.10) and (26.11) remain unchanged.

2. If $b \leq a$, then the sign of the inequality changes, but (26.9), (26.10) and (26.11) remain unchanged.

Example 17 Assume a metal rod is a function of position and non-homogenous, where $x \in [a, b]$. Find the center of mass of a one-dimensional object.

Solution. First, it is easy to see that if the object is homogeneous and lying on the $x-$axis from $x = a$ to $x = b$, then its center of mass is simply the midpoint

$$\frac{a+b}{2}.$$

If, on the other hand, the object is non-homogeneous with $\rho(x)$ being the density function, then the total mass $\mathcal{M}$ is

$$\mathcal{M} = \int_a^b \rho(x)dx. \tag{26.12}$$

Applying Corollary 4 in (26.12), the density-weighted average x_c is defined by

$$\int_a^b x\rho(x)dx = x_c \int_a^b \rho(x) = x_c\mathcal{M}$$

or equivalently

$$x_c = \frac{1}{\mathcal{M}} \int_a^b x\rho(x).$$

Here the point x_c is the center of mass of the object. □

26.5 Second mean value theorem for integrals

One of the significant applications of the generalized mean value theorem for integrals extends to the so-called second mean theorem for integrals. As we shall see below, integration by parts plays a key role in the construction of Theorem 5.

Theorem 5 If f is monotonic and f, f' and g are all continuous in $[a, b]$, then there exists $\xi \in [a, b]$ such that

$$\int_a^b f(x)g(x)dx = f(a)\int_a^\xi g(x)dx + f(b)\int_\xi^b g(x)dx.$$

Proof. Let

$$G(x) = \int_a^x g(t)dt.$$

Clearly $G(a) = 0$, and under the given conditions, $G(x)$ is differentiable and $G'(x) = g(x)$. Using integration by parts, we have

$$\int_a^b f(x)g(x)dx = \int_a^b f(x)G'(x)dx \tag{26.13}$$

$$= f(x)G(x)|_b^a - \int_a^b G(x)f'(x)dx. \tag{26.14}$$

Since $G(x)$, being continuous, is integrable and $f(x)$ is also continuous on $[a, b]$, therefore, upon using the generalized mean value theorem for integrals, there exists $\xi \in [a, b]$ such that

$$\begin{aligned}\int_a^b f(x)g(x)dx &= f(b)G(b) - f(a)\underbrace{G(a)}_{=0} - G(\xi)\int_a^b f'(x)dx \\ &= f(b)G(b) - G(\xi)\left(f(b) - f(a)\right) \\ &= f(b)\left(G(b) - G(\xi)\right) + f(a)G(\xi) \\ &= f(b)\int_\xi^b g(x)dx + f(a)\int_a^\xi g(x)dx.\end{aligned}$$

$\square$