Calculus for Engineers

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Contents

Function

2.1 Motivation

Functions are the building blocks of calculus. In this chapter, the general theory of functions and their graphs will be investigated, and particular categories of functions will be studied. Emphasis is placed on the geometric interpretation of the function.

In Section 2.2.1, the concept of a set is given with its operations. In Section 2.3, we define the set of real numbers $\mathbb{R}$. Then, we will define an major subset of $\mathbb{R}$, called an interval. In Section 2.4, we review the concept and the application of functions. Several features of functions and some properties of functions are also given. Two special classes of functions, namely, composite functions and inverse functions are given in Section 2.5. The main idea behind these functions is to give a proper definition of the domain of a given function. In Section 2.6, we will discuss the special features of linear (or constant), power exponential and logarithmic functions. The concept of a polynomial is discussed in Section 2.7; a class of different polynomials is often classified by the degree. In Section 2.8, the most recognizable quotient polynomial, known as a rational function, is given, and an asymptote of a graph is introduced. In order to resolve the quotient polynomials required for obtaining these asymptotes, we use partial fractions. In Section 2.9, we give the six basic hyperbolic functions with its some interesting properties. In Section 2.10, we will discuss some interesting features of translation transformations, reflection transformations, nonrigid transformations, sequences of transformations and translations through coordinate shifts. These transformations help us better understand the graphs of functions.

2.2 Sets

2.2.1 Definition of a Set

Definition 1 A *set* is a collection of objects that is itself considered as an entity. The object may be any character so long as we know which objects are in a given set and which are not. □

Example 1 $A = \{1, 2, 3, 4, 5\}$ means A is the set consisting of numbers $1, 2, 3, 4$ and 5. □

Note 1

- We often use capital letters A, B, $\cdots$, Y, Z to denote a set, and use lower-case letters a, b, $\cdots$, y, z for the elements of the set.
- A set is usually described by using a pair of brackets $\{\ \ \}$ with words or symbols.
- We can also write the set A as

$$\{x|x \text{ is a natural number less than } 11\}.$$

  In this notation, we observe that

  1. the symbol x indicates a variable and represents a general element of the set,
  2. the vertical bar represents the words "such that", and
  3. the properties which determine the membership in the set are written to the right of the vertical bar.

If A is the set and p is an object in A, we write: $p \in A$, and say p is an element of A or p is in A. The notation $p \notin A$ indicates that p is *not* an element of A.

Example 2 Consider the following sets

$$A = \{x|\ x \text{ is an odd number},\ x < 10\}$$

and

$$B = \{2, 3, 5, 7, 15, 21\}.$$

We observe that

- $9 \in A$ but $9 \notin B$;
- $21 \in B$ but $21 \notin A$;
- $3 \in A$ and $3 \in B$;
- $6 \notin A$ and $6 \notin B$.

□

If a set has a finite element of elements, it is called a **finite** set, otherwise it is called an **infinite** set. It is possible for a set to contain no elements. This set is called an **empty** set and will be denoted by the symbol $\varnothing$.

Example 3 Let $S = \{x|\ x^2 = 9,\ x \text{ is even}\}$. Then $S = \varnothing$; that is, S is an empty set. □

If every element of a set A is also an element of a set B, then A is a subset of B, and is denoted by $A \subset B$. Two sets A and B are considered to be the same, if they have exactly the same elements; that is, $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

Example 4 An example of subset is $E = \{2, 3, 4\}$ from $F = \{1, 2, 3, 4, 5, 6\}$. That is $E \subset F$. □

2.2.2 Operations upon Sets

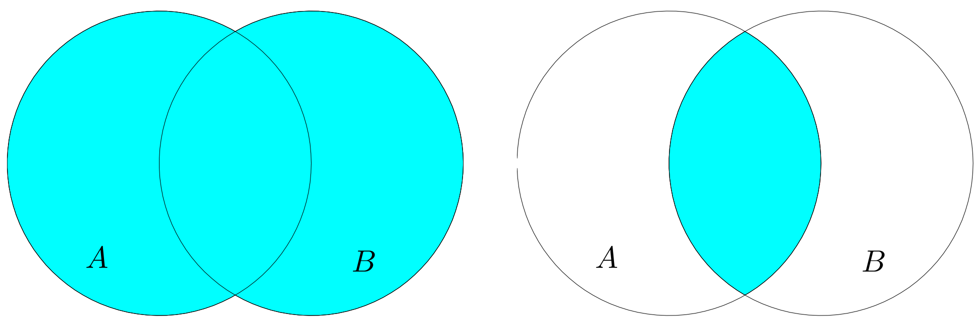
Definition 2 If A and B are sets, the union, $A \cup B$, and the intersection, $A \cap B$, are defined by

$$A \cup B = \{x|\ x \in A \text{ or } x\ \in B\}, \tag{2.1}$$
$$A \cap B = \{x|\ x \in A \text{ and } x\ \in B\}. \tag{2.2}$$

Thus,

- the union (cf. Figure 2.1(a)) consists of the elements which are either in A, in B, or in both A and B, and
- the intersection (cf. Figure 2.1(b)) consists of the elements that are in both A and B; in other words, $A \cap B$ is the common part of A and B.

$\square$



(a) $A \cup B$ (b) $A \cap B$

Figure 2.1:

Definition 3 It may happen that two sets A and B have no element in common. In such case, we say that they are disjoint, or their intersection is empty; that is, $A \cap B = \varnothing$. $\square$

Definition 4 Let S be a set and A be a subset of S. The difference $S \setminus A$ is called the complement of A. We usually employ the notation A^c or $A^{'}$, to denote this set. $\square$

By Definition 4, it is obvious that the following two statements are true:

$$A \cup A^{'} = S, \tag{2.3}$$
$$A \cap A^{'} = \varnothing. \tag{2.4}$$

Definition 5 The **difference** of A and B or the relative complement of B with respect to A, denoted by $A \setminus B$ (cf. Figure 2.2), is the set of elements which belong to A but not to B:

$$A \setminus B = \{x|\ x\ \in A, x \notin B\}. \tag{2.5}$$

$\square$

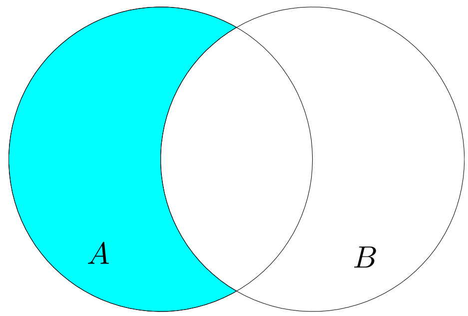


Figure 2.2: $A \setminus B$

Example 5 Let $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 6\}$, where $U = \{1, 2, 3, 4, \cdots\}$. Then

- $A \cup B = \{1, 2, 3, 4, 5, 6\}$.
- $A \cap B = \{3, 4\}$.
- $A \setminus B = \{1, 2\}$.
- $A' = \{5, 6, 7, \cdots\}$.

$\square$

2.3 The Set of Real Numbers

We use the following special symbols:

- $\mathbb{N}$ = the set of positive integers: $1, 2, 3, \cdots$;
- $\mathbb{Z}$ = the set of integers: $\cdots, -2, -1, 0, 1, 2, \cdots$;
- $\mathbb{Q}$ = the set of all quotients of the form m/n, where m is an integer and n is a natural number;
- $\mathbb{R}$ = the set of real numbers.

Thus we have $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.
Calculus is based on the real number system.

Note 2

- A real number can be depicted by a point on a line called an axis.
- As shown in Figure 2.3, an axis is a horizontal line with a positive direction, the origin (that refers to the real number 0) and a unit of distance.
- Each negative number x is represented by the point at a distance of $-x$ unit(s) to the left of the origin, and each positive number x is represented by the point at a distance of x unit(s) to the right of the origin, and

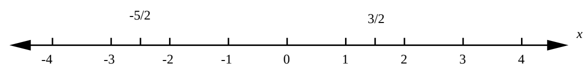


Figure 2.3: The number line.

- There is a one-to-one correspondence between $\mathbb{R}$ and the set of points on the axis.

- Hence, the points on the axis are identified with the numbers they represent, and the same symbol is usually used for both the number and the point representing that number on the axis.

- We identify $\mathbb{R}$ with the axis, and call the axis the number line.

In the following, we shall define an important subset of $\mathbb{R}$, called **intervals**.

- The set of all numbers x satisfying the inequality $a < x < b$ is called an **open interval** and is denoted by (a, b); that is,

$$(a, b) = \{x|\ a < x < b\}. \tag{2.6}$$

- The **closed interval** from a to b is the open interval (a, b) together with the two end-points a and b, and is denoted by $[a, b]$; that is,

$$[a, b] = \{x|\ a \leq x \leq b\}. \tag{2.7}$$

- The **half-open** interval $(a, b]$ and $[a, b)$ are defined by

$$(a, b] = \{x|\ a < x \leq b\} \tag{2.8}$$

  and

$$[a, b) = \{x|\ a \leq x < b\}. \tag{2.9}$$

- We use the symbols "$+\infty$" and "$-\infty$" to denote "positive infinity" and "negative infinity", respectively. It is worth noting that they are only notations and not to be interpreted as representing any real number. Infinite intervals are defined as follows:

$$(a, +\infty) = \{x|\ x > a\}, \tag{2.10}$$
$$(-\infty, b) = \{x|\ x < b\}, \tag{2.11}$$
$$[a, +\infty) = \{x|\ x \geq a\}, \tag{2.12}$$
$$(-\infty, b] = \{x|\ x \leq b\}, \tag{2.13}$$
$$(-\infty, +\infty) = \mathbb{R}. \tag{2.14}$$

2.4 Function

2.4.1 Definition

Definition 6 A **function** f from a subset D of R to a subset E of R is a correspondence that assigns each element x of D to exactly one element y of E. □

Note 3

- The element y of E is called the **value** of f at x (or the image of x under f) and is denoted by $f(x)$ if y corresponds to "an" element x of D.
- x is sometimes also called an **independent variable** and y is called a **dependent variable**.
- The set D in this definition is the **domain** of the function f, and the set E is the **co-domain** of f.
- The **range** of f is the subset of E consisting of all possible function values $f(x)$ for x in D.

Functions can usually be represented in at least three different ways:

1. by tables,
2. by graphs, or
3. by formulas (or formulae).

Example 6 If a function f is given by a formula or rule for $f(x)$, for example, $f(x) = \dfrac{1}{x}$, the domain is then assumed to be the set of all real numbers such that $f(x)$ is real. Thus, for $f(x) = \dfrac{1}{x}$, the domain of f is $(-\infty, 0) \cup (0, +\infty)$, as shown in Figure 2.4. □

Definition 7 A function is a **one-to-one**[1] function if $f(x) \neq f(y)$ whenever $x \neq y$. A special kind of one-to-one function is the **identity** function - a function I with the property that $I(x) = x$ for all x in its domain. □

Definition 8 A function is a **many-to-one** function if there are two distinct numbers x_1, x_2 in the domain with $f(x_1) = f(x_2)$. □

Definition 9 Let f be a function with domain D. We associate each x in D with its exact function value $f(x)$, and then get an ordered pair of numbers $(x, f(x))$. In the two dimensional coordinate plane, $(x, f(x))$ denotes a **unique** point in the plane. The **graph of the function** f is the set of all the points $(x, f(x))$, where x is in D; that is, the set $\{(x, f(x)) |\ x \in D\}$. □

[1] A function is **one-to-one** or **injective** if different inputs yield different outputs.

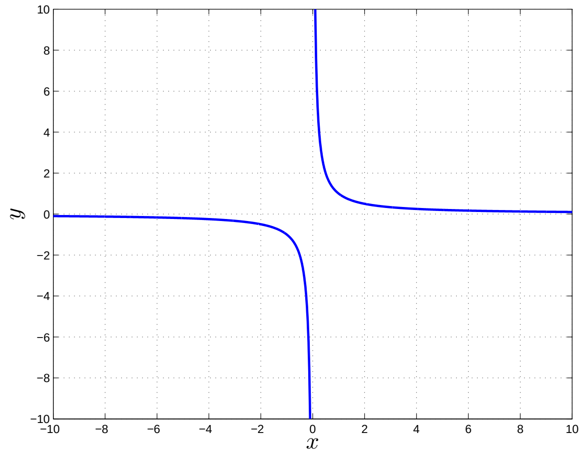


Figure 2.4: Graph of $f(x) = \dfrac{1}{x}$, where $x \in [-10, 10]$.

Example 7 Given a function $f(x) = \sqrt{9 - x^2}$, find the domain and the range of f, and sketch its graph.

Solution:

1. The domain of f is the set of all real numbers such that $f(x)$ is real. As $f(x)$ exists if and only if the radicand $9 - x^2$ is non-negative; that is, $9 - x^2 \geq 0$, or $9 \geq x^2$ equivalently, $-3 \leq x \leq 3$, the domain is $[-3, 3]$.

2. For any real number x in $[-3, 3]$, the function value $f(x)$ is given by $\sqrt{9 - x^2}$. As
$$0 \leq 9 - x^2 \leq 9, \text{ whenever } x \text{ is in } [-3, 3], \tag{2.15}$$
it follows that
$$0 \leq \sqrt{9 - x^2} \leq 3. \tag{2.16}$$
Hence, the range[2] is $[0, 3]$.

3. Figure 2.5 shows the graph of $y = f(x)$, which is the upper portion of the **unit circle**.

□

Example 8 Find the domain of the given function
$$f(x) = \frac{x^2}{x^2 + x - 6}.$$

Solution: The function
$$f(x) = \frac{x^2}{x^2 + x - 6}$$

[2] We point out that for each $y \in [0, 3]$ there is $x \in [-3, 3]$ (e.g. $x = \sqrt{9 - y^2}$, or $x = -\sqrt{9 - y^2}$) such that $f(x) = \sqrt{9 - x^2} = y$.

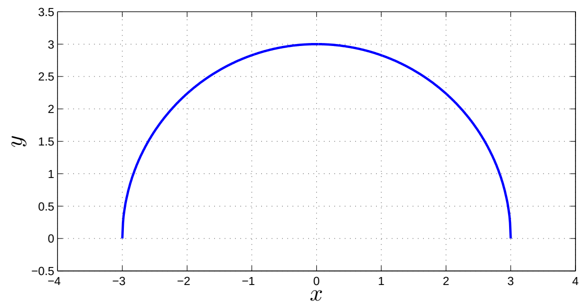


Figure 2.5: Graph of $f(x) = \sqrt{9 - x^2}$, where $x \in [-3, 3]$.

is defined for all x except when $0 = x^2 + 6 - x = (x+3)(x-2) \iff x = -3$ or 2, so the domain is $\{x \in \mathbb{R} |\ x \neq -3, 2\}$. Figure 2.6 illustrates the graph of $y = f(x)$. $\square$

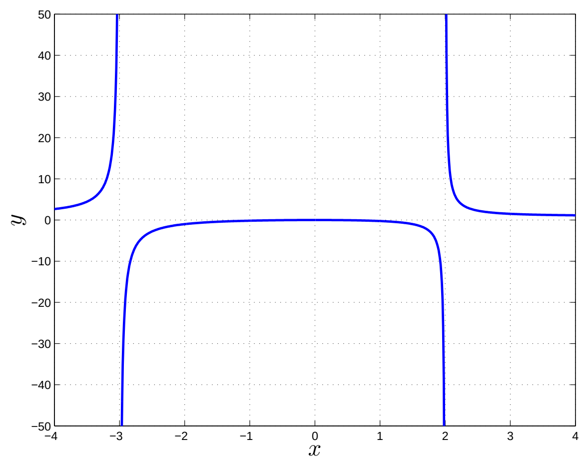


Figure 2.6: Graph of $f(x) = \dfrac{x^2}{x^2 + x - 6}$, where $x \in [-4, 4]$.

A function that is defined by different rules on different parts of its domain is called **a piecewise-defined function.**

Example 9 Sketch the graph of the function defined as follows:

$$f(x) = \begin{cases} x+1, & \text{if } x < 0, \\ x^2, & \text{if } 0 \leq x \leq 2, \\ 1, & \text{if } x > 2. \end{cases} \tag{2.17}$$

Solution:

1. If $x < 0$, then $f(x) = x + 1$, and the graph of f is part of the line $y = x + 1$, as shown in Figure 2.7. The open circle indicates that the point $(0, 1)$ is not on the graph.

2. If $0 \leq x \leq 2$, then $f(x) = x^2$, and the graph of f is part of the parabola $y = x^2$.

3. If $x > 2$, the function values are a constant number 1, and the graph of f is horizontal half-line without the end-point $(2, 1)$.

$\square$

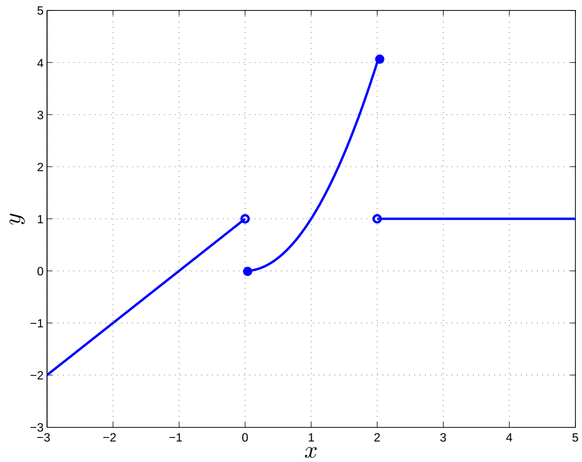


Figure 2.7: Graph of Example 9.

In Example 9, we observe that the graph of a piecewise-defined function is usually composed of several disconnected pieces. Another function with this property is the **greatest integer function** (or floor function) f defined by $f(x) = [x]$, where $[x]$ is the greatest integer function, for instance, $[4] = 4$, $[4.8] = 4$, $[\pi] = 3$, $[-\frac{1}{2}] = -1$.

Example 10 Find all function values of $f(x) = [x]$ for x in the interval $[-2, 2]$, and sketch the graph of the greatest integer function.

Solution: According to the definition of the greatest integer function, we list the x and its corresponding function value $f(x)$, where x is in $[-2, 2]$, in Table 2.1. Whenever x is between successive integers, the corresponding part of the graph is a line segment. Part of the graph of the greatest integer function $f(x) = [x]$ is shown in Figure 2.8. The graph continues infinitely to the right and to the left.

value of x	$f(x) = [x]$
$-2 \le x < -1$	-2
$-1 \le x < 0$	-1
$0 \le x < 1$	0
$1 \le x < 2$	1
$x = 2$	2

Table 2.1: Values of $f(x) = [x]$.

$\square$

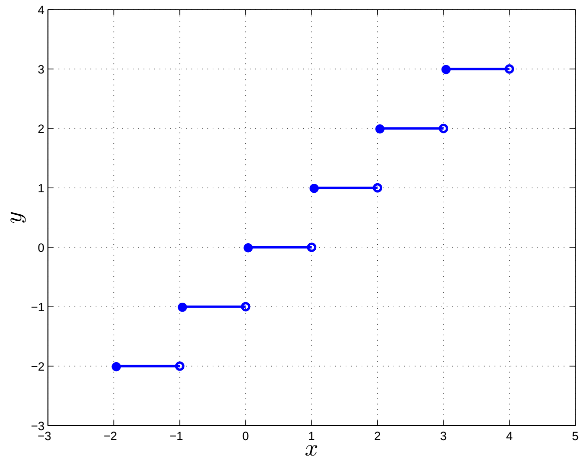


Figure 2.8: Graph of $f(x) = [x]\ \forall x \in [-2, 2]$.

2.4.2 Some Properties of Functions

Four basic properties of functions are considered:

1. symmetry,
2. boundedness,
3. monotonicity, and
4. periodicity.

Symmetry

Definition 10 Let f be a function with domain D which is symmetric with respect to the origin (i.e., $x \in D$ implies that $-x \in D$).

- If $f(-x) = f(x)$ for all $x \in D$, then f is called an **even function**.
- If $f(-x) = -f(x)$ for all $x \in D$, then f is called an **odd function**.

□

If f is an even function, the graph of f is symmetric with respect to the y-axis, since if a point $(x, f(x))$ is on the graph, the point $(-x, f(x))$ is also on the graph.

A typical example of even functions is the function $f(x) = x^2$, since $f(-x) = (-x)^2 = x^2 = f(x)$ for every $x \in \mathbb{R}$.

How about $f(x) = x^4$, or $f(x) = x^6$? After studying Figures 2.9 and 2.10, what can you say about the values of $f(x) = x^2$, $f(x) = x^4$ and $f(x) = x^6$?

If f is an odd function, the graph of f is symmetric with respect to the origin as for any x in the domain of f, points $(x, f(x))$ and $(-x, -f(x))$ are both on the graph.

The example $f(x) = x^3$ is an example of an odd function according to the definition of an odd function.

How about $f(x) = x^5$ or $f(x) = x^7$? After studying Figures 2.11 and 2.12, what can you say about the values of $f(x) = x^3$, $f(x) = x^5$ and $f(x) = x^7$?

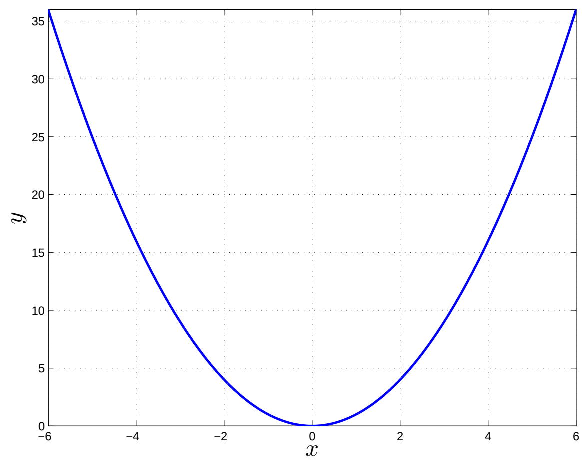
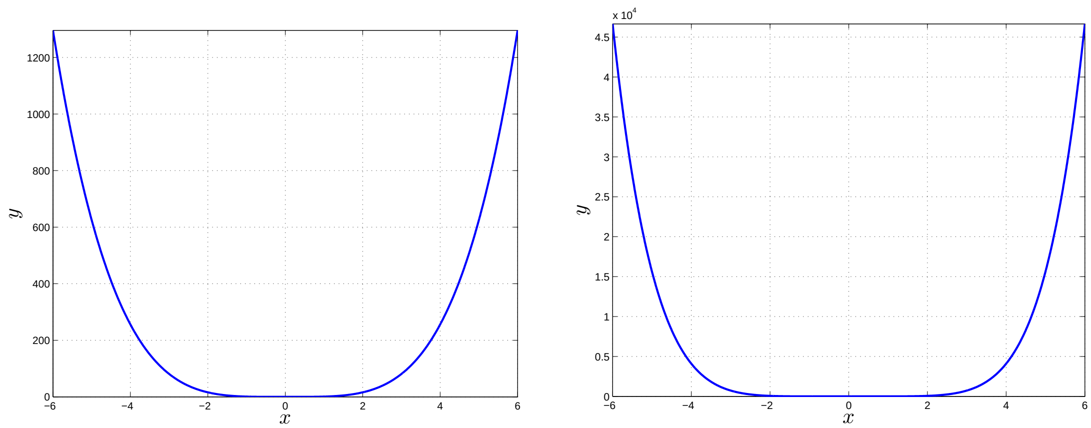


Figure 2.9: Graph of $f(x) = x^2$, where $x \in [-6, 6]$.



(a) $f(x) = x^4$

(b) $f(x) = x^6$

Figure 2.10: Values of $f(x) = x^4$ and $f(x) = x^6$, where $x \in [-6, 6]$.

Boundedness

For a function f, if there exists a number M such that $f(x) \leq M$ for every x in S, where S is a subset of the domain of f, then f is said to be **bounded above** in S. The number M is called an **upper boundary** of f.

If there exists a number m such that $f(x) \geq m$ for every x in S, f is said to be **bounded below** in S, and the number m is a **lower boundary** of f. A function f is **bounded** if it is bounded above and below in its domain.

Example 11 The function $y = \sin(x)$ is a bounded function, because we can find a number 1 such that $|\sin(x)| \leq 1$; i.e., $-1 \leq \sin(x) \leq 1$, for every real number x. Hence $y = \sin(x)$ has an upper boundary 1 and a lower boundary -1. Actually, any number greater than 1 could be an upper boundary of $y = \sin(x)$ and any number less than -1 could be a lower boundary. The graph of a bounded function lies between two horizontal lines, as shown in Figure 2.13. $\square$

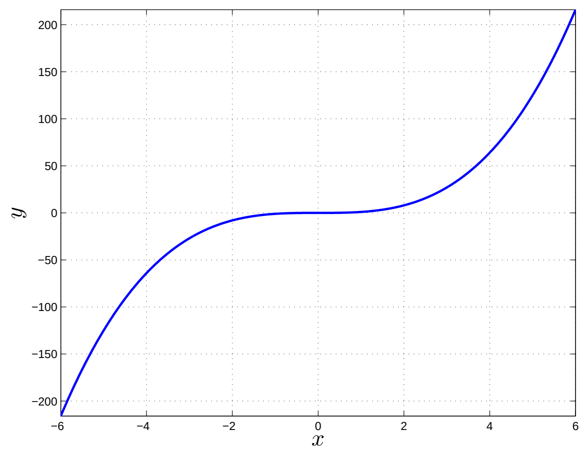
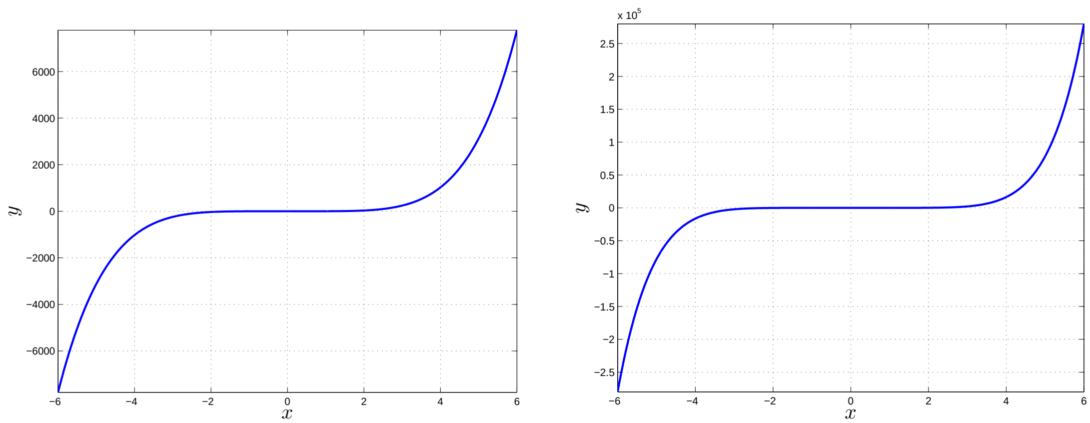


Figure 2.11: Graph of $f(x) = x^3$, where $x \in [-6, 6]$.



(a) $f(x) = x^5$ (b) $f(x) = x^7$

Figure 2.12: Values of $f(x) = x^5$ and $f(x) = x^7$, where $x \in [-6, 6]$.

Monotonicity

Let f be a function and S be a subset of the domain of f.

- f is **increasing** (**strictly increasing**) in S if it satisfies the property that $f(x_1) \leq f(x_2)$ ($f(x_1) < f(x_2)$) whenever $x_1 < x_2$, $x_1 \in S$ and $x_2 \in S$.

- f is **decreasing** (**strictly decreasing**) in S if it satisfies the property that $f(x_1) \geq f(x_2)$ ($f(x_1) > f(x_2)$) whenever $x_1 < x_2$, $x_1 \in S$ and $x_2 \in S$.

Example 12 The function $y = x^2$ is increasing in the interval $[0, \infty)$ and decreasing in the interval $(-\infty, 0]$. But it is neither increasing nor decreasing in its domain. □

Example 13 The function $y = x^3$ is increasing in its domain R. In general, if a function is increasing in its domain, it is called an **increasing** function; if a function is decreasing in its domain, it is called a **decreasing** function. □

As shown in Figure 2.14, the graph of an increasing function arises gradually as x moves along the positive direction of the x-axis, while the graph of a decreasing function

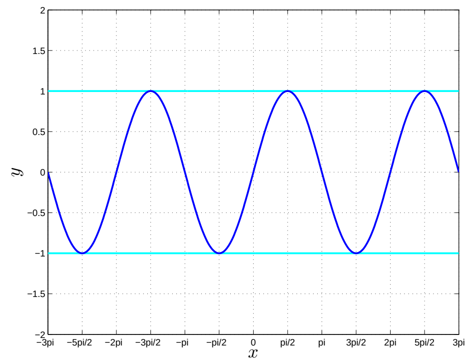
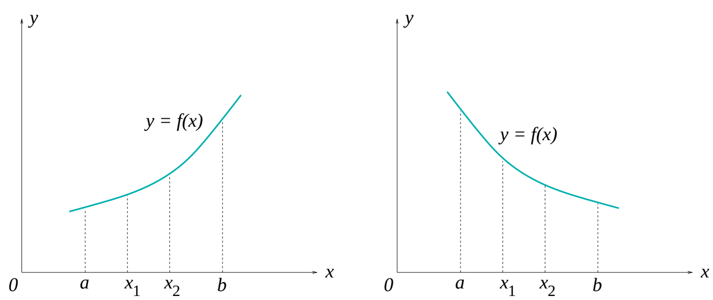


Figure 2.13: Graph of $f(x) = \sin(x)$.



(a) Increasing Function (b) Decreasing Function

Figure 2.14: Increasing function and decreasing function.

falls as x moves along the positive direction of the x-axis. If c is in the closed interval $[a, b]$, then $f(c)$ is called the minimum value of $f(x)$ on $[a, b]$ if $f(c) \leq f(x)$ for all x in $[a, b]$. Similarly, if d is in $[a, b]$, then $f(d)$ is called the maximum value of $f(x)$ on $[a, b]$ if $f(d) \geq f(x)$ for all x in $[a, b]$.

Periodicity

Definition 11 A function f is said to be periodic if there exists a real number T such that whenever x is in the domain of f, then $x+T$ is also in the domain of f and $f(x+T) = f(T)$. The lowest positive number T is called the **period** of f. □

Note 4 Clearly, if T is a period of f, so are $2T, 3T, 4T$, etc.

Example 14 The function $y = \cos(x)$ is a periodic function and 2π is its period, as shown in Figure 2.15. □

Example 15 The functions shown in Figure 2.16 have a period of 3 and 6, respectively. □

2.5 Composite Functions and Inverse Functions

In calculus, we often build functions from simpler functions by combining them in various ways. For example, if f and g are functions, we define

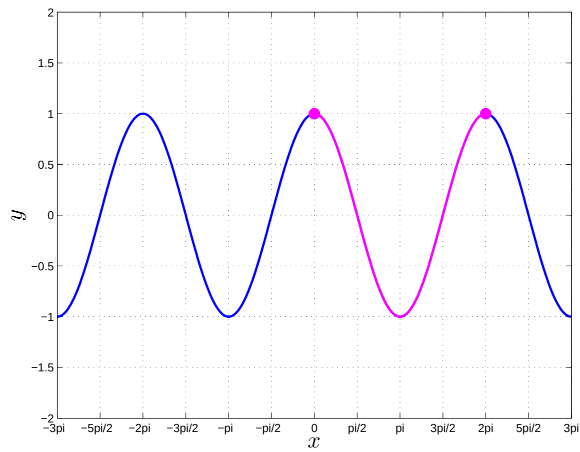


Figure 2.15: $f(x) = \cos(x)$.

- the sum $f + g$,
- the difference $f - g$,
- the product fg, and
- the quotient $\dfrac{f}{g}$

by using arithmetic operations. We shall develop a new way to form a function.

2.5.1 Composite Functions

First let us consider the function $(x + 10)^5$. This function is made up of the 5th power function and the function $x + 10$. Namely, given a number of x, we first add 10 to it, and then take the 5th power.

Let $g(x) = x + 10$ and let f be the 5th power function. Then we can take the value of f at $x + 10$, namely

$$f(x + 10) = (x + 10)^5 = f(g(x)).$$

Next let us consider the function $5x^3 - 2$. If we let $g(x) = 5x^3 - 2$ and f be the square root function, then

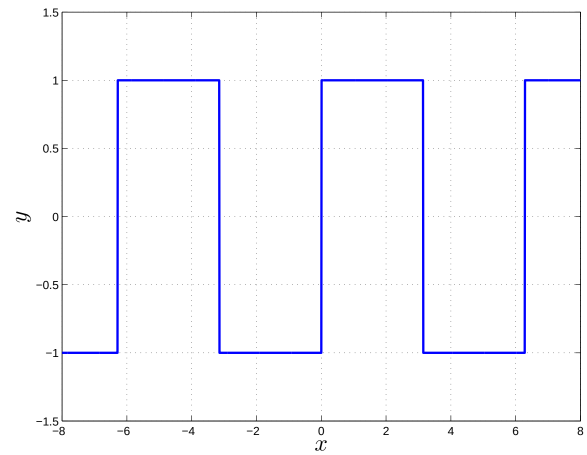
$$f(g(x)) = \sqrt{5x^3 - 2} = (5x^3 - 2)^{1/2}.$$

We may write $f(u) = u^{1/2}$, and $u = g(x) = 5x^3 - 2$.

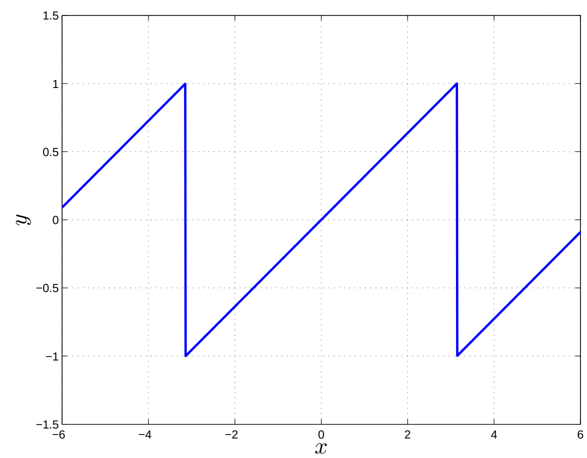
Definition 12 Given two functions f and g, the composite function $f \circ g$ (also called the composition of f and g) is defined by

$$(f \circ g)(x) = f(g(x)).$$

The domain of $f \circ g$ is the set x in the domain of g such $g(x)$ is in the domain of f. □



(a) Square wave



(b) Sawtooth wave

Figure 2.16: Two examples of periodic functions.

- "$f \circ g$".
- In general, $f \circ g \neq g \circ f$.
- $f \circ g$ means that the function g is applied first and then f is applied second.

Example 16 Consider functions f and g defined by $f(x) = x^2$ and $g(x) = x - 3$. Show that $f \circ g \neq g \circ f$ and find their domains.

Solution: We have

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) = f(x-3) = (x-3)^2;\\ (g \circ f)(x) &= g(f(x)) = g(x^2) = x^2 - 3.\end{aligned}$$

Therefore, $f \circ g \neq g \circ f$.

The domains of both $f \circ g$ and $g \circ f$ are $\mathbb{R}$ (the set of all real numbers). $\square$

2.5.2 Inverse Functions

For a given function f, suppose that the input x produces the output y, that is, $y = f(x)$.

We pose a simple question:

Is there a function g (which depends on the given f) such that

- Domain of g = Range of f, and that
- $g(f(x)) = x$ (that is, if $y = f(x)$, then $g(y) = x$) for $x \in$ the domain of f?

If such a function g exists, we call this the inverse function of f and write $g = f^{-1}$. In this case, we have

$$\text{Domain of } g = \text{Range of } f^{-1}$$

and

$$\text{Domain of } f^{-1} = \text{Range of } f.$$

Theorem 1 If the function f is one-to-one, the inverse of f exists so that

$$f^{-1}(f(x)) = x, \quad \forall x \in \text{Domain of } f$$

and

$$f\left(f^{-1}(y)\right) = y, \quad \forall y \in \text{Domain of } f^{-1}.$$

□

Example 17 Find the inverse function of $f(x) = \sqrt{x} + 1,\ 0 \leq x \leq 3$.

Solution: First, let us write $y = \sqrt{x} + 1$ and solve x in terms of y. We obtain

$$x = (y-1)^2$$

from which we see that the given f is one-to-one. The range of f is clearly $1 \leq y \leq \sqrt{3}+1$. Therefore,

$$f^{-1}(y) = (y-1)^2, \quad 1 \leq y \leq \sqrt{3} + 1.$$

□

Note 5 If one wishes to have x rather than y as the independent variable, we can replace y by x in Example 17 to get another form of the solution:

$$f^{-1}(x) = (x-1)^2, 1 \leq x \leq \sqrt{3} + 1.$$

Example 18 Consider the function f defined by $f(x) = x^2, \ -3 \leq x \leq 3$. Show that this function is many-to-one and hence has no inverse.

Solution: If we solve $y = x^2$ for x, we obtain two results

$$x = \sqrt{y} \qquad \text{and} \qquad x = -\sqrt{y}.$$

Since $\sqrt{y} \neq -\sqrt{y}$ if $y \neq 0$, we see that there are two different $x-$values taking the same nonzero $y-$value, This shows that the function is many-to-one.

□

Note 6 However, if we restrict the domain to say the interval $[0, 3]$, the function f becomes a new function h which is one-to-one and whose range is $[0, 9]$.

2.6 Elementary Functions

We will discuss the special features of linear (or constant), power exponential, logarithmic and trigonometric functions below.

2.6.1 Constant Functions

A **constant** function has the form $f(x) = c$, where c is a real number. The domain of f is $\mathbb{R}$ and its range consists of the single number c. Its graph is a horizontal line with y-intercept c. The graph of $f(x) = 3$ is shown in Figure 2.17.

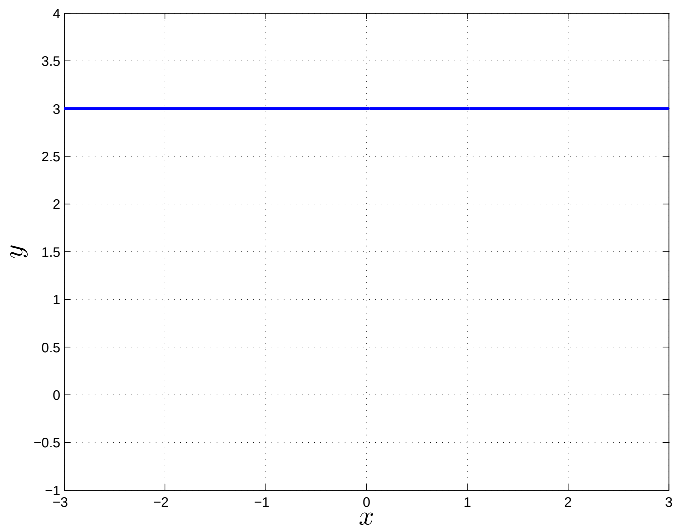


Figure 2.17: Graph of $y = 3$.

2.6.2 Power Functions

A function of the form $f(x) = x^a$, where a is a constant, is called a **power** function. The domain of $f(x) = x^a$ depends on the value of the number a. However the interval $(0, +\infty)$ is included in the domain of $f(x) = x^a$ whatever a is. The graph of $f(x) = x^a$ passes through the point $(1, 1)$. Figure 2.18 shows the graphs of $f(x) = x^a$ for $a = -1$, 2 and 3.

2.6.3 Exponential Functions

Consider a function of the form $f(x) = a^x$, where $a > 0$. Such a function is called an exponential function. Three different cases, where $a = 1, 0 < a < 1$ and $a > 1$ are studied.

Cases I If $a = 1$ then $f(x) = 1^x = 1$. So this just gives us the constant function $f(x) = 1$.

Cases II What happens if $a > 1$? To examine this case, let us take a numerical example. Suppose that $a = 2$. We have

$$f(x) = 2^x.$$

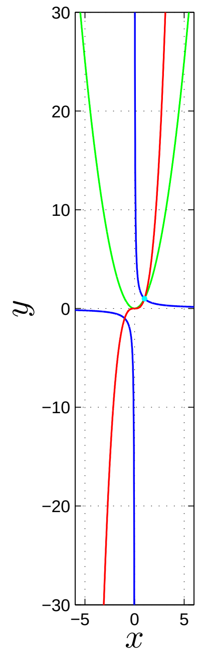


Figure 2.18: Graph of $f(x) = x^a$ for $a = -1$ in blue, 2 in green and 3 in red.

We can put these results into Table 2.2, and plot a graph of the function in Figure 2.19.

x	-3	-2	-1	0	1	2	3
$f(x)$	$f(-3) = \frac{1}{8}$	$f(-2) = \frac{1}{4}$	$f(-1) = \frac{1}{2}$	$f(0) = 1$	$f(1) = 2$	$f(2) = 4$	$f(3) = 8$

Table 2.2: Values of $f(x) = 2^x$.

This example demonstrates the general shape for graphs of functions of the form $f(x) = a^x$ when $a > 1$, as illustrated in Figure 2.20.

The important properties of the graphs of these types of functions are:

- $f(0) = 1$ for all values of a. This is because $a^0 = 1$ for any value of a.
- $f(x) > 0$ for all values of a. This is because $a > 0$ implies $a^x > 0$.

Cases III What happens if $0 < a < 1$? To examine this case, let us take another numerical example. Suppose that $a = \frac{1}{2}$. We have

$$f(x) = \left(\frac{1}{2}\right)^x.$$

We can put these results into Table 2.3, and plot a graph of the function in Figure 2.21.

What is the effect of varying a? Again we can see the effect by looking at sketches of a few graphs of similar functions, as illustrated in Figure 2.22.

The important properties of the graphs of these types of functions are:

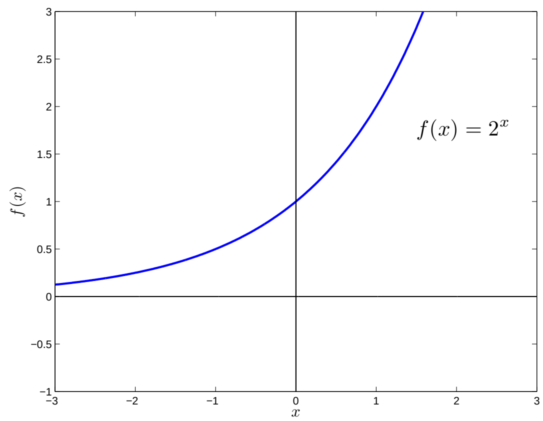


Figure 2.19: Graph of $f(x) = 2^x$.

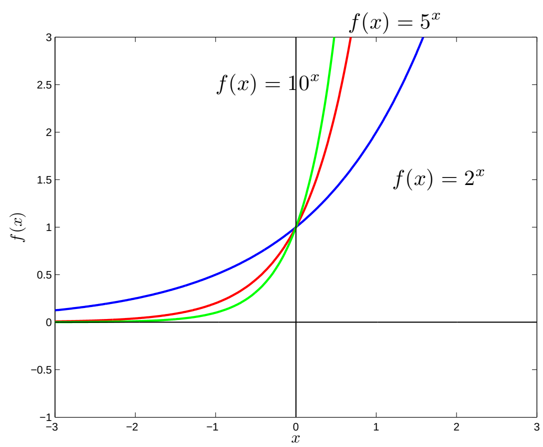


Figure 2.20: Graph of $f(x) = a^x$ for $a = 2$ in blue, $a = 5$ in red and $a = 10$ in green.

- $f(0) = 1$ for all values of a. This is because $a^0 = 1$ for any value of a.
- $f(x) > 0$ for all values of a. This is because $a > 0$ implies $a^x > 0$.

These properties are the same as when $a > 1$. One interesting thing that you might have spotted is that $f(x) = \left(\frac{1}{2}\right)^x = 2^{-x}$ is a reflection of $f(x) = 2^x$ in the $f(x)$ axis, and that $f(x) = \left(\frac{1}{5}\right)^x = 5^{-x}$ is a reflection of $f(x) = 5^x$ in the $f(x)$ axis, as illustrated in Figure 2.23.

In general, $f(x) = \left(\frac{1}{a}\right)^x = a^{-x}$, $(a > 0)$ is a reflection of $f(x) = a^x$ in the $f(x)$-axis.

A particularly important example of an exponential function arises when $a = e$. You might recall that the number e is approximately equal to 2.718. The function $f(x) = e^x$ is often called 'the' exponential function. Since $e > 1$and $\frac{1}{e} < 1$, we can sketch the graphs of the exponential functions $f(x) = e^x$ and $f(x) = e^{-x} = \left(\frac{1}{e}\right)^x$, as illustrated in Figure 2.24.

- A function of the form $f(x) = a^x$ (where $a > 0$) is called an exponential function.
- The function $f(x) = 1^x$ is just the constant function $f(x) = 1$.

x	-3	-2	-1	0	1	2	3
$f(x)$	$f(-3) = 8$	$f(-2) = 4$	$f(-1) = 2$	$f(0) = 1$	$f(1) = \frac{1}{2}$	$f(2) = \frac{1}{4}$	$f(3) = \frac{1}{8}$

Table 2.3: Values of $f(x) = 2^{-x}$.

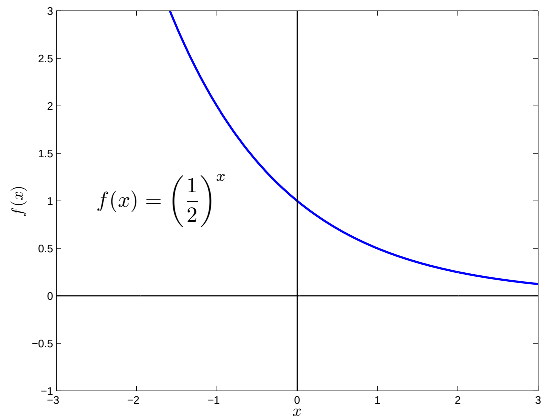


Figure 2.21: Graph of $f(x) = 2^{-x}$.

- The function $f(x) = a^x$ for $a > 1$ has a graph which is close to the $x-$axis for negative x and increases rapidly for positive x.

- The function $f(x) = a^x$ for $0 < a < 1$ has a graph which is close to the $x-$axis for positive$x$ and increases rapidly for decreasing negative x.

- For any value of a, the graph always passes through the point $(0, 1)$.

- The graph of $f(x) = \left(\frac{1}{a}\right)^x = a^{-x}$ is a reflection, in the vertical axis, of the graph of $f(x) = a^x$.

- A particularly important exponential function is $f(x) = e^x$, where $e = 2.718281828459046 \cdots$. This is often called the exponential function.

2.6.4 Logarithm Functions

We shall now look at logarithm functions. These are functions of the form $f(x) = \log_a x$ where $a > 0$. We do not consider the case $a = 1$, as this will not give us a valid function.

What happens if $a > 1$? To examine this case, let us take a numerical example. Suppose that $a = 2$. Then

$$f(x) = \log_2 x \quad \text{means} \quad 2^{f(x)} = x.$$

An important point to note here is that, regardless of the argument, $2^{f(x)} > 0$. So we shall consider only positive arguments.

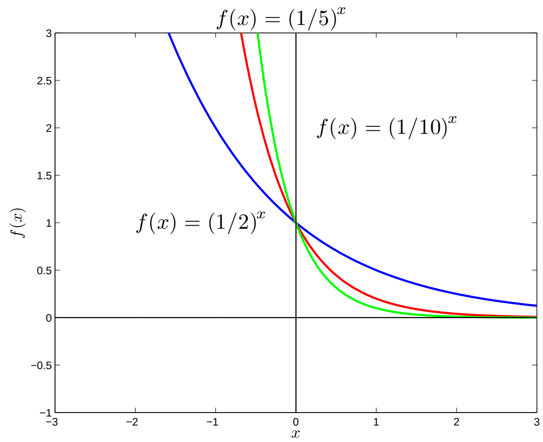


Figure 2.22: Graph of $f(x) = a^{-x}$ for $a = 2$ in blue, $a = 5$ in red and $a = 10$ in green.

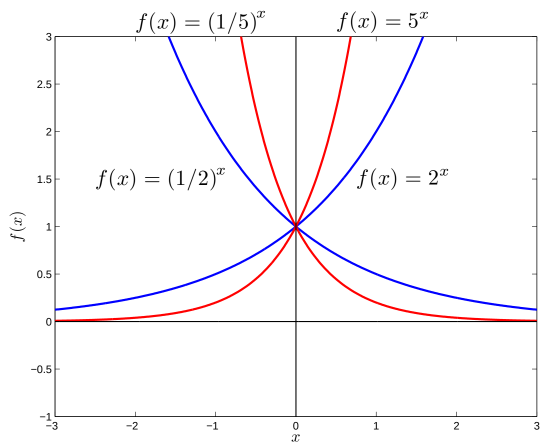


Figure 2.23:

$f(1) = \log_2 1$	means	$2^{f(1)} = 1$	so	$f(1) = 0.$
$f(2) = \log_2 2$	means	$2^{f(2)} = 2$	so	$f(2) = 1.$
$f(4) = \log_2 3$	means	$2^{f(4)} = 4$	so	$f(4) = 2.$
$f\left(\frac{1}{2}\right) = \log_2\left(\frac{1}{2}\right)$	means	$2^{f\left(\frac{1}{2}\right)} = \left(\frac{1}{2}\right)$	so	$f\left(\frac{1}{2}\right) = -1.$
$f\left(\frac{1}{4}\right) = \log_2\left(\frac{1}{4}\right)$	means	$2^{f\left(\frac{1}{4}\right)} = \left(\frac{1}{4}\right)$	so	$f\left(\frac{1}{4}\right) = -2.$

This example demonstrates the general shape for graphs of functions of the form $f(x) = \log_a x$ when $a > 1$. What is the effect of varying a? We can see the effect by looking at sketches of a few graphs of similar functions. For the special case where $a = e$, we often write $\ln x$ instead of $\log_e x$.

The important properties of the graphs of these types of functions are:

- $f(1) = 0$ for all values of a;
- we must have $x > 0$ for all values of a.

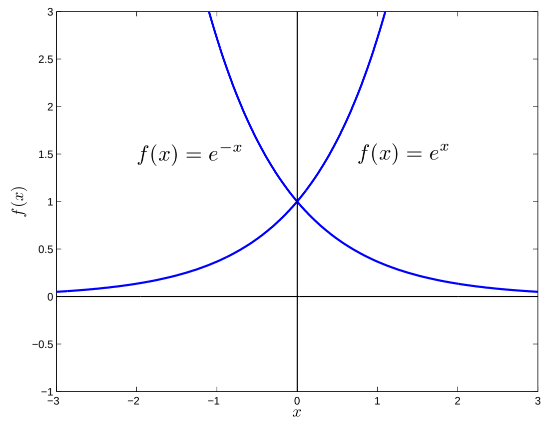


Figure 2.24: Graphs of $f(x) = e^x$ and $f(x) = e^{-x}$.

What happens if $0 < a < 1$? To examine this case, take another numerical example. Suppose that $a = \frac{1}{2}$. Then

$$f(x) = \log_{\frac{1}{2}} x \quad \text{means} \quad \left(\frac{1}{2}\right)^{f(x)} = x.$$

An important point to note here is that, regardless of the argument, $\left(\frac{1}{2}\right)^{f(x)} > 0$ So we shall consider only positive arguments:

$$\begin{array}{lllll}
f(1) = \log_{\frac{1}{2}} 1 & \text{means} & \left(\frac{1}{2}\right)^{f(1)} = 1 & \text{so} & f(1) = 0. \\
f(2) = \log_{\frac{1}{2}} 2 & \text{means} & \left(\frac{1}{2}\right)^{f(2)} = 2 & \text{so} & f(2) = -1. \\
f(4) = \log_{\frac{1}{2}} 4 & \text{means} & \left(\frac{1}{2}\right)^{f(4)} = 4 & \text{so} & f(4) = -2. \\
f\left(\frac{1}{2}\right) = \log_{\frac{1}{2}} \left(\frac{1}{2}\right) & \text{means} & \left(\frac{1}{2}\right)^{f\left(\frac{1}{2}\right)} = \left(\frac{1}{2}\right) & \text{so} & f\left(\frac{1}{2}\right) = 1. \\
f\left(\frac{1}{4}\right) = \log_{\frac{1}{2}} \left(\frac{1}{4}\right) & \text{means} & \left(\frac{1}{2}\right)^{f\left(\frac{1}{4}\right)} = \left(\frac{1}{4}\right) & \text{so} & f\left(\frac{1}{4}\right) = 2.
\end{array}$$

2.7 Polynomials

A polynomial is a function of the form

$$P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

where $a_0, a_1, a_2, \cdots, a_n$ are given constants (called the coefficients) and x is the independent variable. The domain of $P(x)$ is $\mathbb{R}$. If $a_n \neq 0$, then n is the degree of $P(x)$. We sometimes write $\deg P$ for the degree of $P(x)$.

If all the coefficients $a_0, a_1, a_2, \cdots, a_n$ are zero, the polynomial reduces to the zero polynomial, that is,

$$P(x) = 0 + 0x + 0x^2 + \cdots + 0x^n.$$

The degree of the zero polynomial is regarded as 0 in this note. A zero of $P(x)$ is a root (or a solution) of the equation $P(x) = 0$.

Polynomial	Degree	Name
a_0	0	constant
$a_0 + a_1 x, (a_1 \neq 0)$	1	linear
$a_0 + a_1 x + a_2 x^2, (a_2 \neq 0)$	2	quadratic
$a_0 + a_1 x + a_2 x^2 + a_3 x^3, (a_3 \neq 0)$	3	cubic

Table 2.4:

The graphs of polynomials are formed by continuous curves with different degrees, as shown in Figures 2.25 and 2.26.

The following two theorems are basic:

Theorem 2 (Remainder theorem) If we divide a polynomial $P(x)$ by $x-a$, then the remainder is $P(a)$. □

Theorem 3 (Fundamental theorem of algebra) If $P(z)$ is a nonzero polynomial of degree n (with real or complex coefficients), then the equation $P(z)$ has exactly n roots (counting real roots, complex roots and their multiplicities). □

2.8 Rational Functions

A rational function $f(x)$ is the quotient of polynomials:

$$f(x) = \frac{P(x)}{Q(x)},$$

where the denominator $Q(x)$ is a nonzero polynomial. The function $f(x)$ is not defined at $x = a$ if $Q(a) = 0$.

The graph of a rational function is formed by continuous curves broken at the zeros of the denominator, as shown in Figure 2.27.

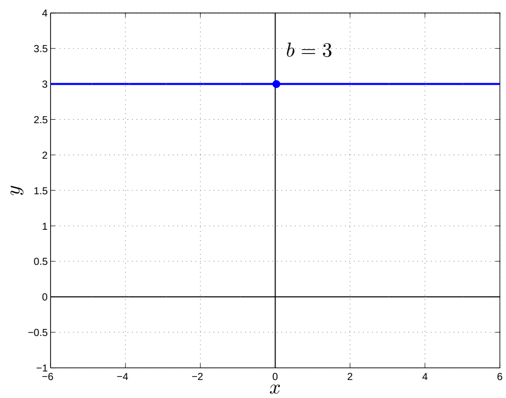
A rational function is proper if the degree of the numerator is less than that of the denominator. Otherwise, $f(x)$ is improper.

By a direct division calculation, we can write a given improper rational function in the form:

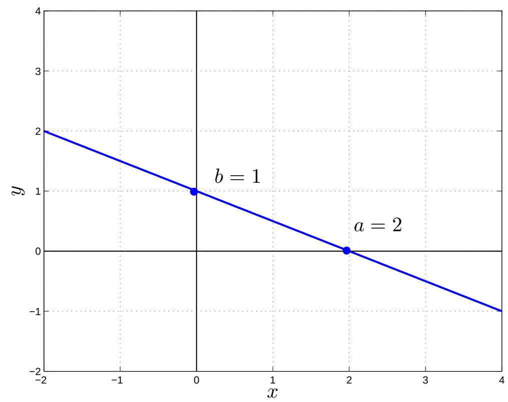
$$\begin{pmatrix} \text{An improper} \\ \text{rational function} \end{pmatrix} = \text{a polynomial} + \begin{pmatrix} \text{a improper} \\ \text{rational function} \end{pmatrix}. \tag{2.18}$$

2.8.1 Rational Functions

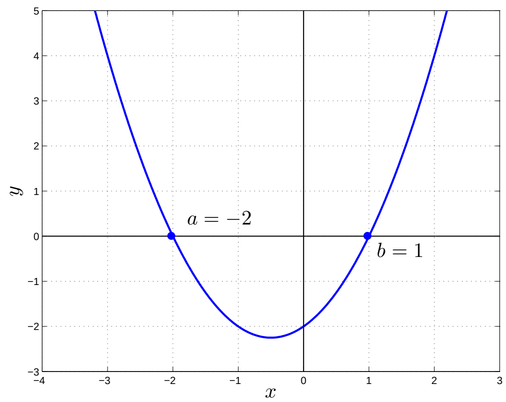
In Figure 2.27, we see that the graph of a rational function consists of two or more continuous branches. Each of these branches (green) approaches a straight line (drawn as a dashed line in Figure 2.27) as the point on the branch moves towards infinity in a certain direction. Such a straight line is called **an asymptote of the graph**. The equations of the asymptotes of a given rational function can be found using the following theorem:



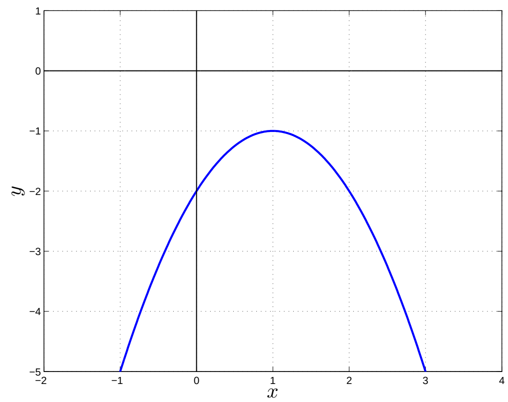
(a) Degree $0: y = b$, with $b = 3$



(b) Degree $1 : \dfrac{x}{a} + \dfrac{y}{b} = 1$, with $a = 2$ and $b = 1$



(c) Degree $2 : y = (x-a)(x-b)$, with $a = -2$ and $b = 1$



(d) Degree $2 : y = -x^2 - 2$.

Figure 2.25: Examples of polynomials.

Theorem 4 Let $f(x) = \dfrac{P(x)}{Q(x)}$ be a rational function where $P(x)$ and $Q(x)$ have no common factor. Suppose $(x - c_1), (x - c_2)$, etc. are factors of $Q(x)$, where c_1, c_2, etc. are distinct real constants.

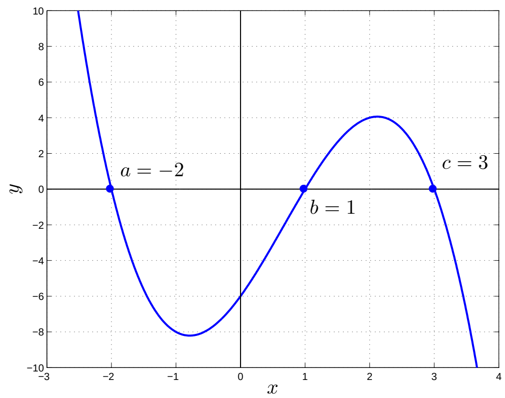
- Then the vertical lines $x = c_1, x = c_2$, etc. are asymptotes of the graph of $f(x)$.
- Furthermore, if $\deg P \leq \deg Q + 1$ so that $f(x)$ can be resolved in the following special form of (2.18):

$$f(x) = ax + b + \frac{S(x)}{Q(x)}, \quad \deg S \leq \deg Q,$$

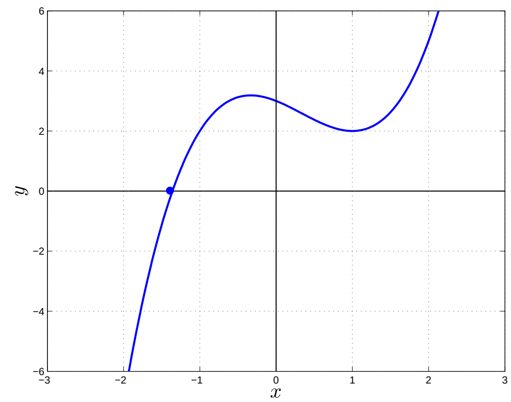
then the line $y = ax + b$ is also an asymptote of the graph.

□

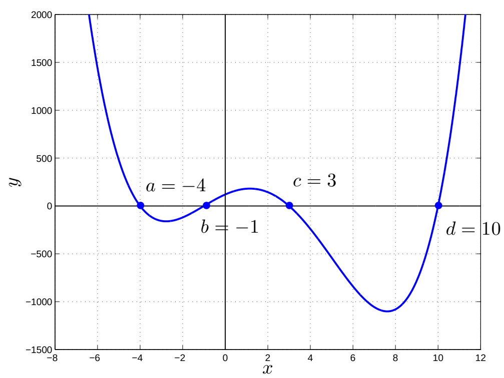
Note 7 In the first part of Theorem 4, the asymptotes are vertical. In the second part of Theorem 4, if $a \neq 0$, then the asymptote is oblique, while if $a = 0$, then the asymptote is horizontal. For a proper rational function ($\deg P < \deg Q$), we have $a = b = 0$ and therefore the $x-$axis ($y = 0$) is an asymptote of the graph.



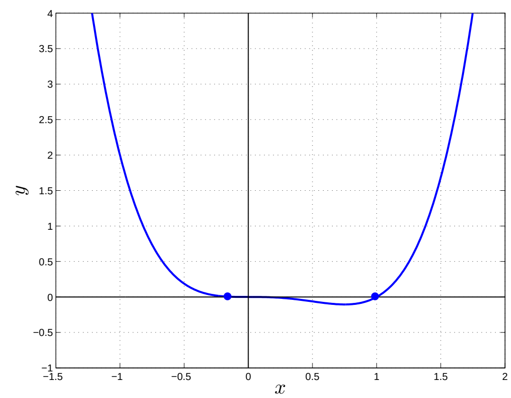
(a) Degree 3 : $y=(x-a)(x-b)(x-c)$, with $a=-2$, $b=1$ and $c=3$



(b) Degree 3 : $y=x^3-x^2-x+3$



(c) Degree 4 : $y=(x-a)(x-b)(x-c)(x-d)$, with $a=-4$, $b=-1$, $c=3$ and $d=10$



(d) Degree 4 : $y=x^4-x^3$.

Figure 2.26: Examples of polynomials.

Example 19 Find the asymptotes of the rational function

$$f(x)=\frac{x^3+2x^2+1}{(x-1)(x+2)}.$$

Solution: By long division, we have

$$f(x)=x+1+\frac{x+3}{(x-1)(x+2)}.$$

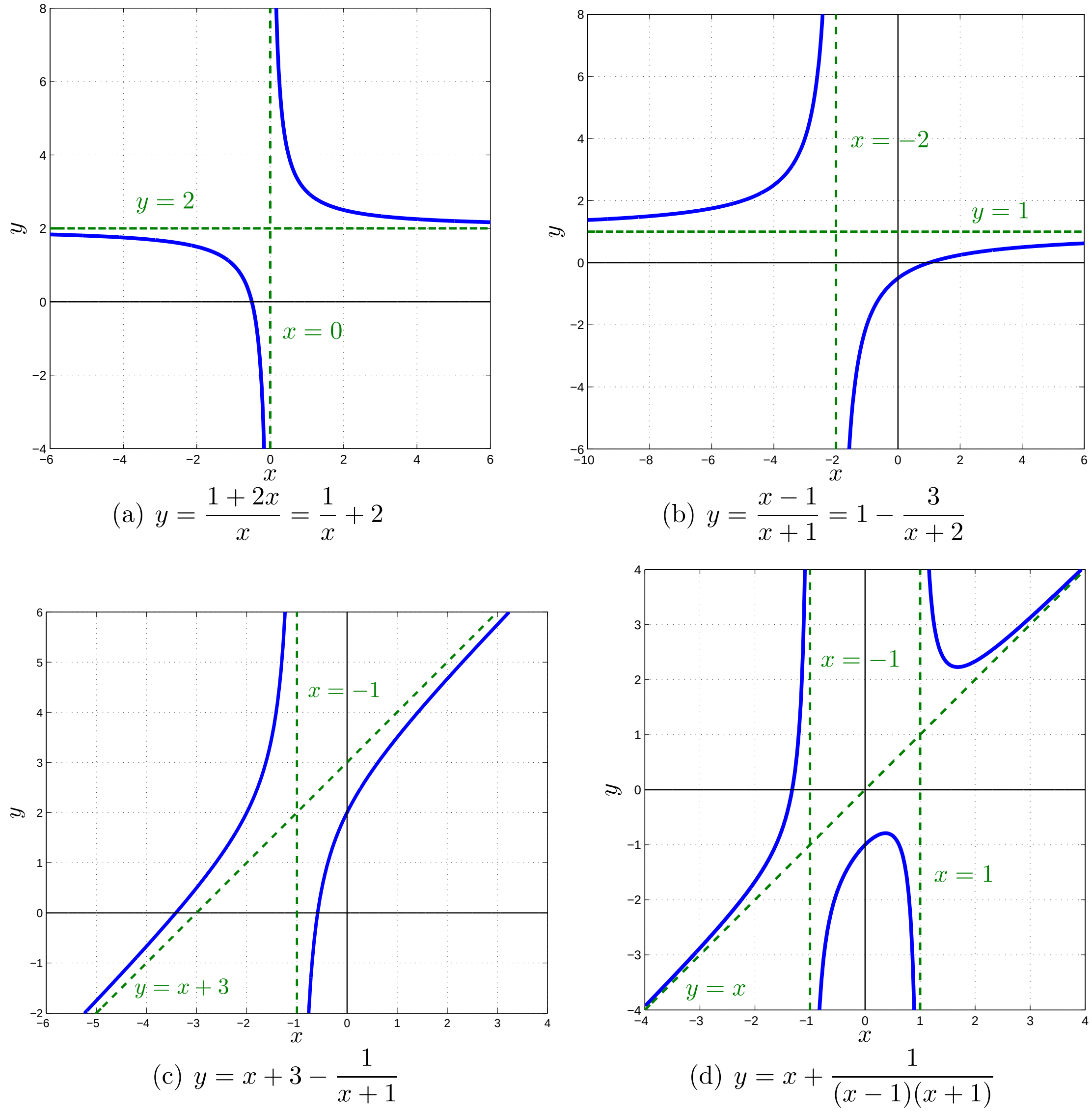
Therefore, the asymptotes are the lines

$$x=1, \quad x=-2 \quad \text{and} \quad y=x+1.$$

$\square$

2.8.2 Partial Fractions

A proper rational function, with real coefficients, can sometimes be expressed as a sum of two or more proper rational functions, with coefficients, called partial fractions. For



(a) $y = \dfrac{1+2x}{x} = \dfrac{1}{x} + 2$ (b) $y = \dfrac{x-1}{x+1} = 1 - \dfrac{3}{x+2}$

(c) $y = x + 3 - \dfrac{1}{x+1}$ (d) $y = x + \dfrac{1}{(x-1)(x+1)}$

Figure 2.27: Examples of rational functions.

example,

$$\frac{x-3}{(2x-1)(x^2+1)} = \frac{-2}{2x-1} + \frac{x+1}{x^2+1}.$$

In a later chapter, we will resolve a rational function into partial fractions this way to do integration.

Each factor of the denominator of a given rational function, is associated with a partial fraction or a sum of partial fractions. The rule of association is shown in Table 2.5 for a linear factor and an irreducible quadratic factor (that cannot be factorized into a product of real linear factors).

Example 20 Resolve

$$f(x) = \frac{x+3}{(x-1)(x-3)}$$

into partial fractions.

Rule	Factor of denominator	Form of the partial fractions
1	$ax+b$	$\dfrac{A_1}{ax+b}$
2	$(ax+b)^2$	$\dfrac{A_1}{ax+b}+\dfrac{A_2}{(ax+b)^2}$
3	$(ax+b)^3$	$\dfrac{A_1}{ax+b}+\dfrac{A_2}{(ax+b)^2}+\dfrac{A_3}{(ax+b)^3}$
4	ax^2+bx+c	$\dfrac{A_1x+B_1}{ax^2+bx+c}$
5	$(ax^2+bx+c)^2$	$\dfrac{A_1x+B_1}{ax^2+bx+c}+\dfrac{A_1x+B_1}{(ax^2+bx+c)^2}$

Table 2.5:

Solution: First we observe that the given $f(x)$ is a proper rational function. Next we consider each factor of the denominator of $f(x)$. There are two linear factors $x-1$ and $x-3$. By Rule 1 of Table 2.5, we can assume partial fractions of the forms

$$\frac{A}{x-1} \quad \text{and} \quad \frac{B}{x-3},$$

where A and B are real constants, and get the identity

$$\frac{x+3}{(x-1)(x-3)} \equiv \frac{A}{x-1}+\frac{B}{x-3}.$$

To get the constants A and B, we remove the denominators and obtain

$$x+3 \equiv A(x-3)+B(x-1).$$

Comparing the coefficient of x and the constant term, we get two equations

$$\begin{aligned} x: &\quad 1 = A+B \\ \text{constant term}: &\quad 3 = -3A-B. \end{aligned}$$

Solving these equations by the elimination method, we get $A=-2$ and $B=3$. Therefore, we have

$$\frac{x+3}{(x-1)(x-3)} \equiv \frac{-2}{x-1}+\frac{3}{x-3}.$$

□

The above method for finding the coefficients A and B is called **the method of undetermined coefficients.**

Example 21 Resolve

$$f(x)=\frac{7x+5}{(x+1)^2(x-1)}$$

into partial fractions.

Solution: First we observe that the given $f(x)$ is a proper rational function. Next we consider each factor of the denominator of $f(x)$. There are two linear factors $x+1$ (with power 2) and $x-1$. By Rule 1 and Rule 2 of Table 2.5, we can assume partial fractions of the forms

$$\frac{A}{x+1}+\frac{B}{(x+1)^2} \quad \text{and} \quad \frac{C}{x-1},$$

where A, B and C are real constants, and get the identity

$$\frac{7x+5}{(x+1)^2(x-1)} \equiv \frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{x-1}.$$

To get the constants A and B, we remove the denominators and obtain

$$7x+5 \equiv A(x+1)(x-1)+B(x-1)+C(x+1)^2.$$

Comparing the coefficients of x and x^2, and the constant term, we get three equations

$$\begin{aligned} x^2: &\quad 0 = A + C \\ x: &\quad 7 = B + 2C \\ \text{constant term}: &\quad 5 = -A - B + C. \end{aligned}$$

Solving these equations by the elimination method, we get $A=-3$, $B=1$ and $C=3$. Therefore, we have

$$\frac{7x+5}{(x+1)^2(x-1)} \equiv \frac{-3}{x+1}+\frac{1}{(x+1)^2}+\frac{3}{x-1}.$$

□

Example 22 Resolve

$$f(x)=\frac{x-3}{(2x-1)(x^2+1)}$$

into partial fractions.

Solution: The denominator has two factors: one is $2x-1$ and the other x^2+1. By Rule 1 and Rule 4 of Table 2.5, $f(x)$ has partial fractions in the forms

$$\frac{A}{2x-1} \quad \text{and} \quad \frac{Bx+C}{x^2+1},$$

where A, B and C are real constants, and get the identity

$$\frac{x-3}{(2x-1)(x^2+1)} \equiv \frac{A}{2x-1}+\frac{Bx+C}{x^2+1}.$$

To get the constants A and B, we remove the denominators and obtain

$$x-3 \equiv A(x^2+1)+(Bx+C)(2x-1).$$

Comparing the coefficients of x and x^2, and the constant term, we get three equations

$$\begin{aligned} x^2: &\quad 0 = A + 2B \\ x: &\quad 1 = -B + 2C \\ \text{constant term}: &\quad -3 = A - C. \end{aligned}$$

Solving these equations by the method of elimination, we get $A = -2$, $B = 1$ and $C = 1$. Therefore, we have

$$\frac{x-3}{(2x-1)(x^2+1)} \equiv \frac{-2}{2x-1} + \frac{x+1}{x^2+1}.$$

□

2.9 Hyperbolic Functions

2.9.1 Definitions

The hyperbolic cosine function, written $\cosh x$, is defined for all real values of x by the relation

$$\cosh x = \frac{e^x + e^{-x}}{2}. \tag{2.19}$$

The name of the function is pronounced "cosh".

Similarly, the hyperbolic sine function, written $\sinh x$, is defined by

$$\sinh x = \frac{e^x - e^{-x}}{2}. \tag{2.20}$$

The name of the function is pronounced "shine".

Example 23 Show that

$$\cosh x + \sinh x = e^x$$

and

$$\cosh x - \sinh x = e^{-x}.$$

Solution Adding (2.19) and (2.20), we have

$$\begin{aligned} \cosh\ x + \sinh\ x &= \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2} \\ &= e^x. \end{aligned}$$

and subtracting (2.20) from (2.19), we have

$$\begin{aligned} \cosh\ x - \sinh\ x &= \frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2} \\ &= e^{-x}. \end{aligned}$$

□

Example 24 Show that

$$\cosh^2 x - \sinh^2 x = 1. \tag{2.21}$$

Solution By multiplying the expression for $(\cosh\ x + \sinh\ x)$ and $(\cosh\ x - \sinh\ x)$ together, we have

$$\begin{aligned}(\cosh\ x + \sinh\ x)(\cosh\ x - \sinh\ x) &= e^x e^{-x} \\ &= 1.\end{aligned}$$

□

Example 25 Show that

$$\cosh(x-y) = \cosh x\ \cosh y - \sinh x\ \sinh y.$$

Solution Let us compute the RHS of the above identity:

$$\begin{aligned}\cosh x\ \cosh y &= \left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^y + e^{-y}}{2}\right) \\ &= \frac{e^{x+y} + e^{x-y} + e^{-(x+y)} + e^{-(x-y)}}{4}. \\ \sinh x\ \sinh y &= \left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^y - e^{-y}}{2}\right) \\ &= \frac{e^{x+y} - e^{x-y} + e^{-(x+y)} - e^{-(x-y)}}{4}.\end{aligned}$$

Substraction produces

$$\begin{aligned}\cosh x\ \cosh y - \sinh x\ \sinh y &= 2\left(\frac{e^{x-y} - e^{-(x-y)}}{4}\right) \\ &= \cosh(x-y).\end{aligned}$$

□

2.9.2 Further Functions

Corresponding to the trigonometric functions, $\tan\ x, \cot\ x, \sec\ x$ and $\operatorname{cosec} x$, we define

Hyperbolic tangent	$\tanh x = \dfrac{\sinh x}{\cosh x}$
Hyperbolic cotangent	$\coth x = \dfrac{1}{\tanh x} = \dfrac{\cosh x}{\sinh x}$
Hyperbolic secant	$\operatorname{sech} x = \dfrac{1}{\cosh x}$
Hyperbolic cosecant	$\operatorname{csch} x = \dfrac{1}{\sinh x}$

We summarize that

- The name of the hyperbolic tangent is pronounced "than".
- The name of the hyperbolic cotangent is pronounced "coth".
- The name of the hyperbolic secant is pronounced "shek".
- The name of the hyperbolic cosecant is pronounced "coshek".

Note 8

- Dividing (2.21) throughout by $\cosh^2 x$ gives the identity
$$1 - \tanh^2 x = \text{sech}^2 x.$$
- Dividing (2.21) throughout by $\sinh^2 x$ gives the identity
$$\coth^2 x - 1 = \text{cosech}^2 x.$$

Example 26 Suppose
$$\sinh x = \frac{3}{4}.$$
Find the exact value of x.

Solution:

Method 1 Using the identity
$$\cosh^2 x = 1 + \sinh^2 x$$
and considering $\cosh x$ to always be positive, then when $\sinh x = \frac{3}{4}$, $\cosh x = \frac{5}{4}$. Using Example 23, we have
$$\sinh\ x + \cosh\ x = e^x$$
So
$$e^x = \frac{3}{4} + \frac{5}{4} = 2$$
and hence
$$x = \ln 2.$$

Method 2 One can write
$$\sinh x = \frac{e^x + e^{-x}}{2}.$$
So $\sinh x = \frac{3}{4}$ means
$$\frac{e^x + e^{-x}}{2} = \frac{3}{4}$$
or
$$2e^x - 3 - 2e^{-x} = 0$$

and multiplying by e^x, one gets

$$2e^{2x} - 3e^x - 2 = 0$$

or

$$(e^x - 2)(2e^x + 1) = 0$$

which implies that

$$e^x = 2 \quad \text{or} \quad e^x = -\frac{1}{2}.$$

However, e^x is always positive so $e^x = 2$ implies $x = \ln 2$.

□

Example 27 Solve the equation

$$2\cosh 2x + 10\sinh 2x = 5.$$

giving your answer in terms of a natural logarithm.

Solution: Using the identities

$$\cosh 2x = \frac{e^{2x} + e^{-2x}}{2} \quad \text{and} \quad \sinh 2x = \frac{e^{2x} - e^{-2x}}{2},$$

one gets

$$2\left(\frac{e^{2x} + e^{-2x}}{2}\right) + 10\left(\frac{e^{2x} - e^{-2x}}{2}\right) = e^{2x} + e^{-2x} + 5e^{2x} - 5e^{-2x} = 5$$

or

$$6e^{2x} - 5 - 4e^{-2x} = 0$$

or

$$6e^{4x} - 5e^{2x} - 4 = 0$$

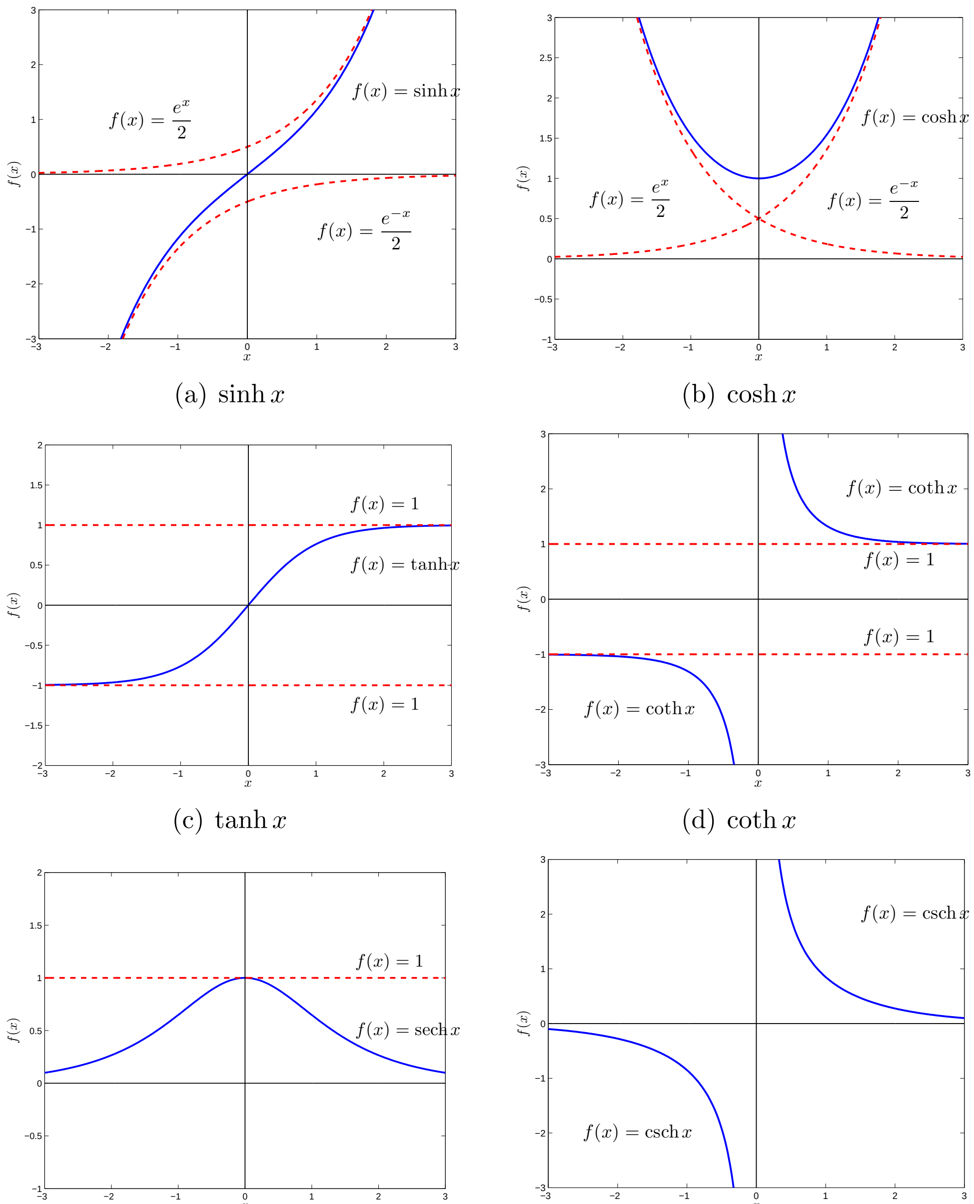
which implies that

$$\left(3e^{2x} - 4\right)\left(2e^{2x} + 1\right) = 0$$

$$e^{2x} = \frac{4}{3} \quad \text{or} \quad e^{2x} = -\frac{1}{2}.$$

The only real solution occurs when $e^{2x} > 0$. So $2x = \ln \frac{4}{3}$ implies that $x = \frac{1}{2}\ln\frac{4}{3}$.

□



(a) $\sinh x$ (b) $\cosh x$

(c) $\tanh x$ (d) $\coth x$

(e) $\operatorname{sech} x$ (f) $\operatorname{csch} x$

Figure 2.28: The six basic hyperbolic functions.

2.10 Transformations

2.10.1 Translations ("Shifts")

Translations ("shifts") are transformations that move a graph without changing its shape or orientation.

Definition 13 Let G be the graph of $y = f(x)$ and k be a positive real number.

- Vertical Translations ("shifts")
  - The graph of $y = f(x) + k$ is G shifted up by k units. (Simply increase the $y-$coordinate by k units.)
  - The graph of $y = f(x) - k$ is G shifted down by k units.
- Horizontal Translations ("shifts")
  - The graph of $y = f(x - k)$ is G shifted right by k units.
  - The graph of $y = f(x + k)$ is G shifted left by k units.

□

Example 28 Let $f(x) = \sqrt{x}$.

1. Let $f(x) = \sqrt{x}$. Graph $y = f(x) + 2$, $y = f(x) - 2$, $y = f(x - 2)$ and $y = f(x + 2)$.
2. Explain how these translations work.

x	$f(x) = \sqrt{x}$	$f(x) + 2$	$f(x) - 2$	$f(x - 2)$	$f(x + 2)$
-3	undefined	undefined	undefined	undefined	undefined
-2	undefined	undefined	undefined	undefined	0
-1	undefined	undefined	undefined	undefined	1
0	0	2	-2	undefined	$\sqrt{2}$
1	1	3	-1	undefined	$\sqrt{3}$
2	$\sqrt{2}$	$\sqrt{2} + 2$	$\sqrt{2} - 2$	0	2
3	$\sqrt{3}$	$\sqrt{3} + 2$	$\sqrt{3}$ - 2	1	$\sqrt{5}$

Table 2.6:

3. Find the domain and the range of Table 2.6.

Solution:

1. Figure 2.29 shows the four cases.
2. Refer to Table 2.7.
3. Refer to Table 2.8.

□

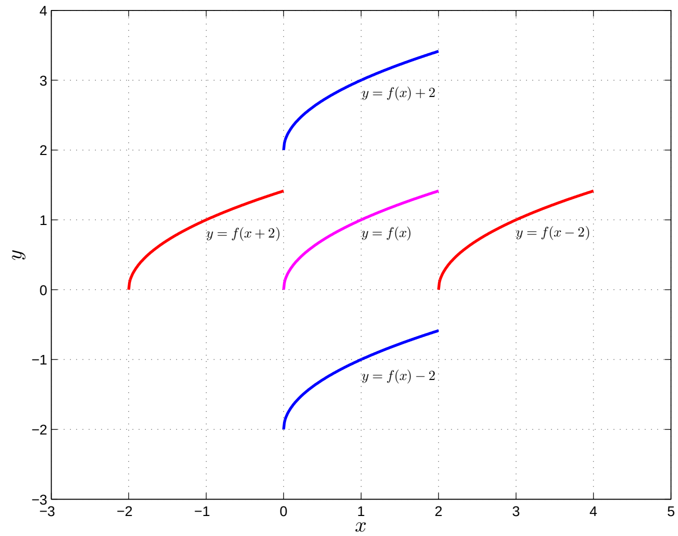


Figure 2.29: $f(x) = \sqrt{x}$

	$f(x)+2$	$f(x)-2$	$f(x-2)$	$f(x+2)$
How points change	$y-$coordinates increase 2 units	$y-$coordinates decrease 2 units	$x-$coordinates increase 2 units	$x-$coordinates decrease 2 units
G moves	up $\uparrow$	down $\downarrow$	right $\rightarrow$	left $\leftarrow$

Table 2.7:

$g(x)$	$g(x) = f(x)+2$	$g(x) = f(x)-2$	$g(x) = f(x-2)$	$g(x) = f(x+2)$
domain of $g(x)$	$[0, \infty)$	$[0, \infty)$	$[2, \infty)$	$[-2, \infty)$
range of $g(x)$	$[2, \infty)$	$[-2, \infty)$	$[0, \infty)$	$[0, \infty)$

Table 2.8:

2.10.2 Reflections

Definition 14 Let G be the graph of $y = f(x)$.

- The graph of $y = -f(x)$ is G reflected about the $x-$axis.
- The graph of $y = f(-x)$ is G reflected about the $y-$axis.
- The graph of $y = -f(-x)$ is G reflected about the origin.

Example 29 Let $f(x) = \sqrt{x}$.

1. Let $f(x) = \sqrt{x}$. Graph $y = -f(x)$, $y = f(-x)$ and $y = -f(-x)$.
2. Explain how these translations work.

x	$f(x) = \sqrt{x}$	$-f(x)$	$f(-x)$	$-f(-x)$
-3	undefined	undefined	$\sqrt{3}$	$-\sqrt{3}$
-2	undefined	undefined	$\sqrt{2}$	$-\sqrt{2}$
-1	undefined	undefined	1	-1
0	0	0	0	0
1	1	-1	undefined	undefined
2	$\sqrt{2}$	$-\sqrt{2}$	undefined	undefined
3	$\sqrt{3}$	$-\sqrt{3}$	undefined	undefined

Table 2.9:

3. Find the domain and the range of Table 2.9.

Solution:

1. Figure 2.30 shows $y = f(x)$ in pink, $y = -f(x)$ in green, $y = f(-x)$ in red and $y = -f(-x)$ in blue.
2. Refer to Table 2.10.

	$-f(x)$	$f(-x)$	$-f(-x)$
Points are reflected about	$x-$axis	$y-$axis	Both or origin

Table 2.10:

3. Refer to Table 2.11.

$g(x)$	$g(x) = -f(x)$	$g(x) = f(-x)$	$g(x) = -f(-x)$
domain of $g(x)$	$[0, \infty)$	$(-\infty, 0]$	$(-\infty, 0]$
range of $g(x)$	$(-\infty, 0]$	$[0, \infty)$	$(-\infty, 0]$

Table 2.11:

$\square$

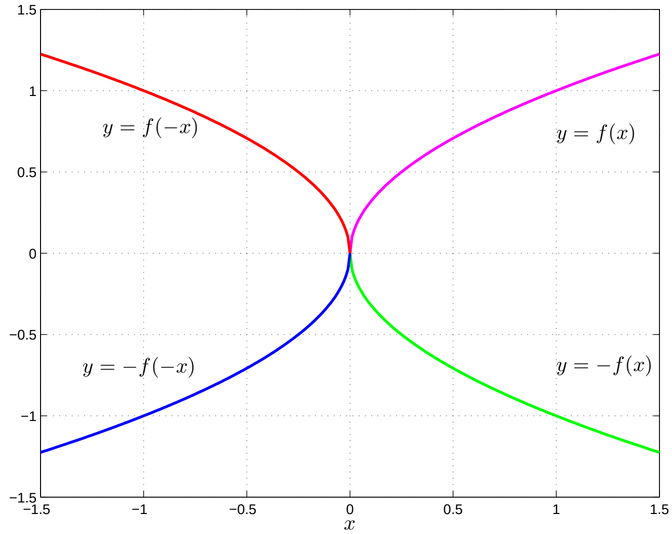


Figure 2.30: $f(x) = \sqrt{x}$

2.10.3 Nonrigid Transformations

Nonrigid transformations can change the shape of a graph beyond simply reorientation, perhaps by stretching or squeezing, unlike rigid transformations such as translations, reflections, and rotations.

Definition 15 If f is a function, and k is a real number, then kf is called a constant (or scalar) multiple of f. □

Definition 16 Let $y = f(x)$ be a function and k be a real number.

- The graph of $y = kf(x)$ is:
  - a vertically stretching version of G if $k > 1$;
  - a vertically shrinking version of G if $0 < k < 1$.
- The graph of $y = f(kx)$ is:
  - a horizontally shrinking version of G if $k > 1$;
  - a horizontally stretching version of G if $0 < k < 1$.

□

If $k < 0$, then the function y performs the corresponding reflection either before or after the vertical or horizontal stretching or shrinking (or squeezing).

Example 30 Let $f(x) = k\sqrt{x}$ with $k = \frac{1}{2}, 1, 2$. Examine each k.

Solution: Figure 2.31 shows $y = f(x)$ in pink, $y = \frac{1}{2}f(x)$ in green and $y = 2f(x)$ in red. We conclude that

- For any $x-$value in $[0, \infty)$, such as 1, the corresponding $y-$coordinate for the $y = \sqrt{x}$ graph is doubled to obtain the $y-$coordinate for the $y = 2\sqrt{x}$ graph. This is why there is vertical stretching.

- Similarly, the graph of $y = \frac{1}{2}\sqrt{x}$ exhibits vertical shrinking, because the $y-$coordinates have been halved.

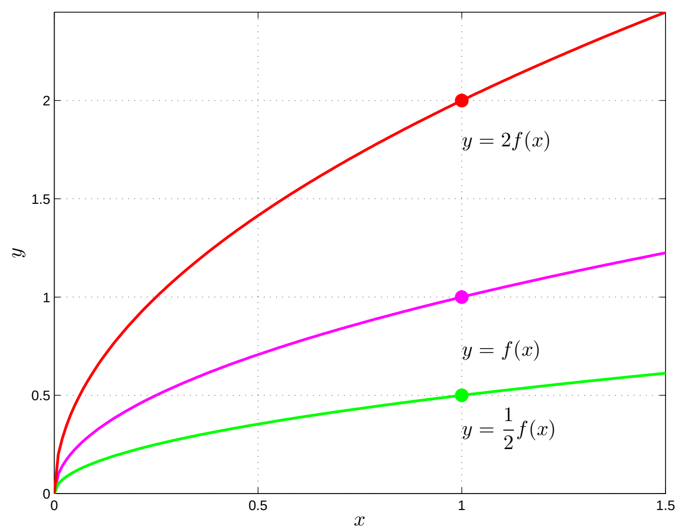


Figure 2.31: Graph of $f(x) = \sqrt{x}$.

□

Example 31 Let $f(x) = \sqrt{kx}$ with $k = \frac{1}{4}, 1, 4$. Examine each k.

Solution: In Figure 2.32, we conclude that

- The graph of $y = f(4x)$ is the graph of $y = 2\sqrt{x}$ (in red), because $f(4x) = \sqrt{4x} = 2\sqrt{x}$. The vertical stretching we described in Example 30 may now be interpreted as a horizontal squeezing. (This is not true of all functions.) The function value we got at $x = 1$ we now get at $x = \frac{1}{4}$.

- The graph of $y = f\left(\frac{1}{4}x\right)$ is the graph of $y = \frac{1}{2}\sqrt{x}$ (in green), because $f\left(\frac{1}{4}x\right) = \sqrt{\frac{1}{4}x} = \frac{1}{2}\sqrt{x}$. The vertical squeezing we described in Example 30 may now be interpreted as a horizontal stretching. The function value we got at $x = 1$ we now get at $x = 4$.

□

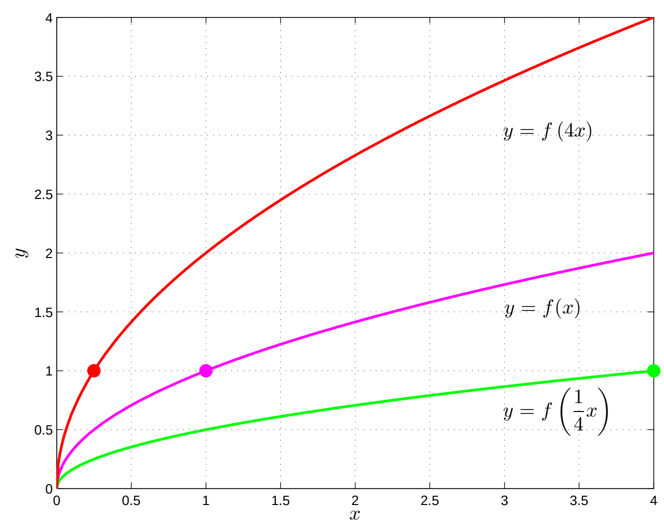


Figure 2.32: Graph of $f(x) = \sqrt{x}$.

2.10.4 Sequences of Transformation

Example 32 Graph $y = f(x) = 2 - \sqrt{x+3}$.

Solution:

- Let us rewrite the equation as $y = -\sqrt{x+3}+2$ to more clearly indicate the vertical shift.
- We will "build up" the right-hand side step-by-step. Along the way, we transform the corresponding function and its graph.
- We start with a basic function with a known graph. (Point-plotting should be a last resort.) Here, it is a square root function $y = \sqrt{x}$. Let us study a sequence of transformations as follows:
  - $y = \sqrt{x}$; $f_1(x) = \sqrt{x}$, as illustrated in Figure 2.33(a).
  - $y = \sqrt{x+3}$; $f_2(x) = f_1(x+3)$, where the graph $f_1(x)$ is shifted to the left by 3 units, as illustrated in Figure 2.33(b).
  - $y = -\sqrt{x+3}$; $f_3(x) = -f_2(x)$, where the graph $f_2(x)$ is reflected above the $x-$axis, as illustrated in Figure 2.33(c).
  - $y = -\sqrt{x+3}+2$; $f_4(x) = f_3(x) + 2$, where the graph $f_3(x)$ is shifted up by 2 units, as illustrated in Figure 2.33(d).

□

Example 33 Find an equation for the transformed basic graph below (Strategy 1, Strategy 2, and Strategy 3):

Solution: There are different strategies that can lead to a correct equation.

1. Strategy 1 : (First raise, then reflect)

   (a) Shift graph up by 1 unit: $f_2(x) = f_1(x) + 1$, where $f_1(x) = |x|$.

   (b) Reflect graph about the $x-$axis: $f(x) = -f_2(x)$.

   One of the possible answers is $f(x) = -(|x| + 1)$.

2. Strategy 2: (First reflect, then drop)

   (a) Reflect graph about the $x-$axis: $f_2(x) = -f_1(x)$, where $f_1(x) = |x|$.

   (b) Shift graph down by 1 unit: $f(x) = f_2(x) - 1$.

   One of the possible answers is $f(x) = -|x| - 1$ that is same as Strategy 1.

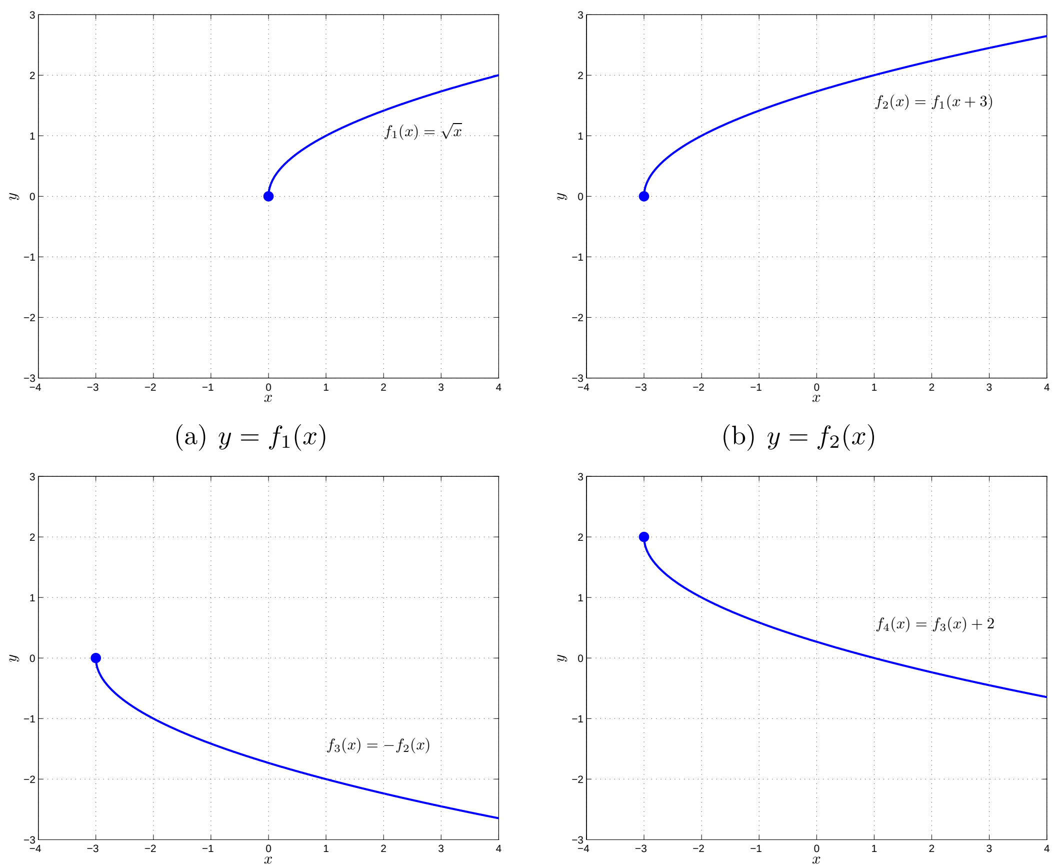
3. Strategy 3 : (Swap the order in Strategy 2, but this fails!)

   (a) Shift graph down by 1 unit: $f_2(x) = f_1(x) - 1$, where $f_1(x) = |x|$.

   (b) Reflect graph about the $x-$axis: $f_3(x) = -f_2(x)$.

   We note that $f_3(x) = -(|x| - 1)$ is not identical to our previous answers.

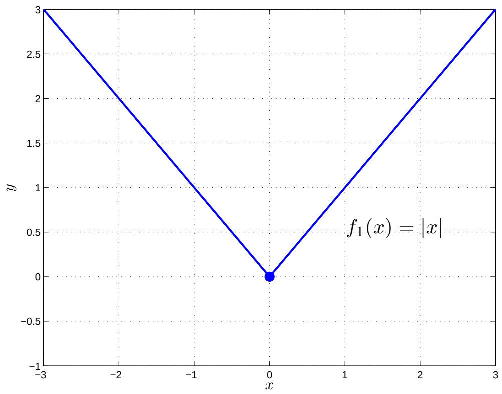
□



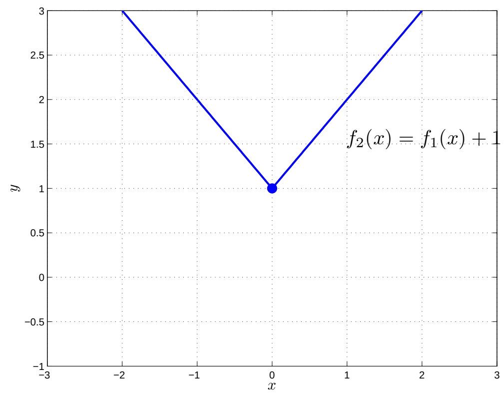
(a) $y = f_1(x)$ (b) $y = f_2(x)$

(c) $y = f_3(x)$ (d) $y = f_4(x)$

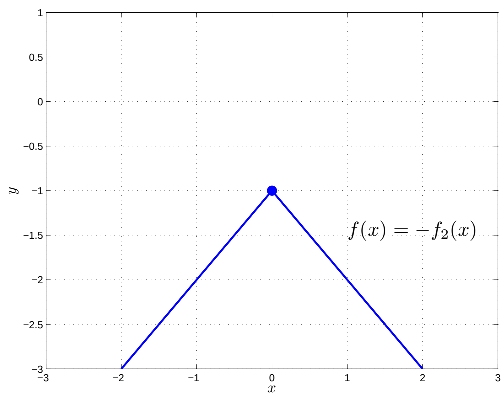
Figure 2.33: Graph of $y = f(x) = 2 - \sqrt{x+3}$.



(a) $f_1(x) = |x|$

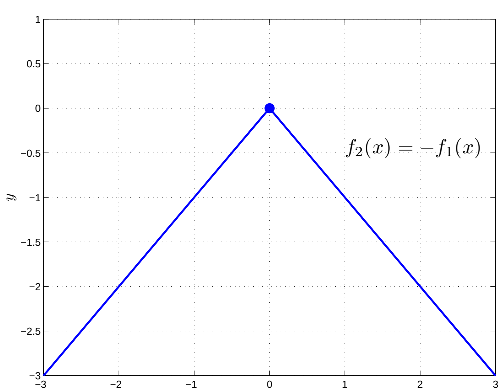


(b) $y = f_2(x)$

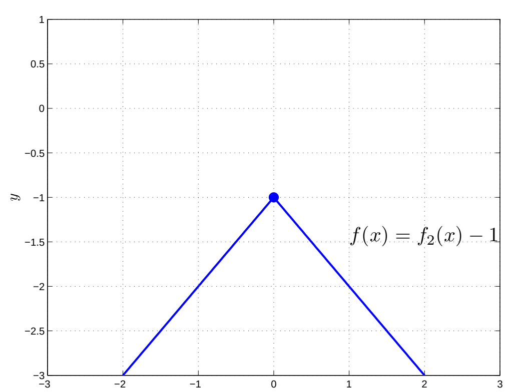


(c) $y = f(x)$

Figure 2.34: Graph of $y = f(x) = -(|x| + 1)$.

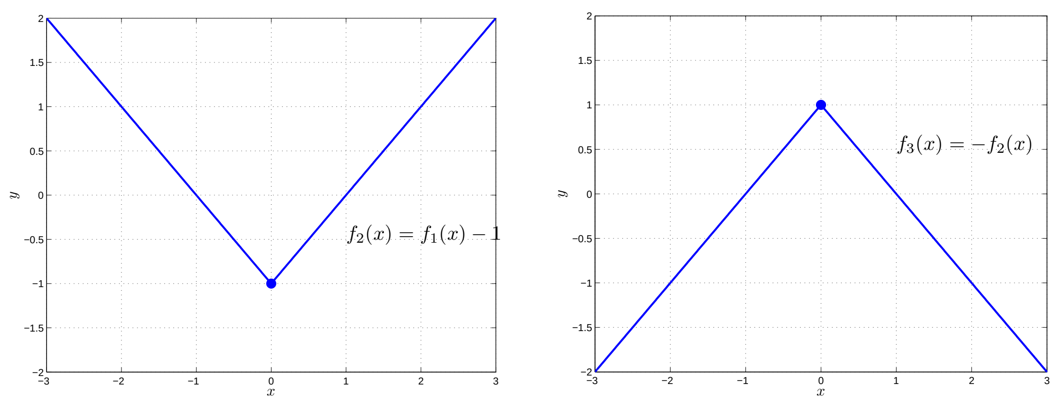


(a) $y = f_2(x)$



(b) $y = f(x)$

Figure 2.35: Graph of $y = f(x) = -|x| - 1$.



(a) $y = f_2(x)$ (b) $y = f_3(x)$

Figure 2.36: Graph of $y = f_3(x) = -(|x| - 1)$.

2.10.5 Translations Through Coordinate Shifts

Definition 17 Let h and k be positive real numbers.

- A graph G in the $xy-$plane is shifted h units horizontally and k units vertically.
  - If $h < 0$, then G is shifted left by $|h|$ units.
  - If $k < 0$, then G is shifted down by $|k|$ units.
- To obtain an equation for the new graph, take an equation for G and:
  - Replace all occurrences of x with $(x - h)$, and
  - Replace all occurrences of y with $(y - k)$.

□

Example 34 We want to translate the circle, with radius 3 and center $(0, 0)$, in the $xy-$plane so that its new center is at $(-2, 1)$. Find the standard form of the equation of the new circle.

Solution: We begin with the equation $x^2 + y^2 = 9$ for the original (blue) circle and:

- Replace x with $(x - (-2))$, or $(x + 2)$, and
- Replace y with $(y - 1)$.

This is because we need to shift the blue circle left 2 units and up 1 unit to obtain the new (red) circle. The equation is:

$$(x + 2)^2 + (y - 1)^2 = 9.$$

□

Definition 18 Consider the graph of $y = f(x)$. A coordinate shift of h units horizontally and k units vertically yields an equation that is equivalent to one we would have obtained from our previous approach:

$$y - k = f(x - h) \quad \text{if and only if} \quad y = f(x - h) + k.$$

□

Example 35 In Figure 2.38, we shift the blue curve right 2 units and up 1 unit to obtain the new red curve.

□

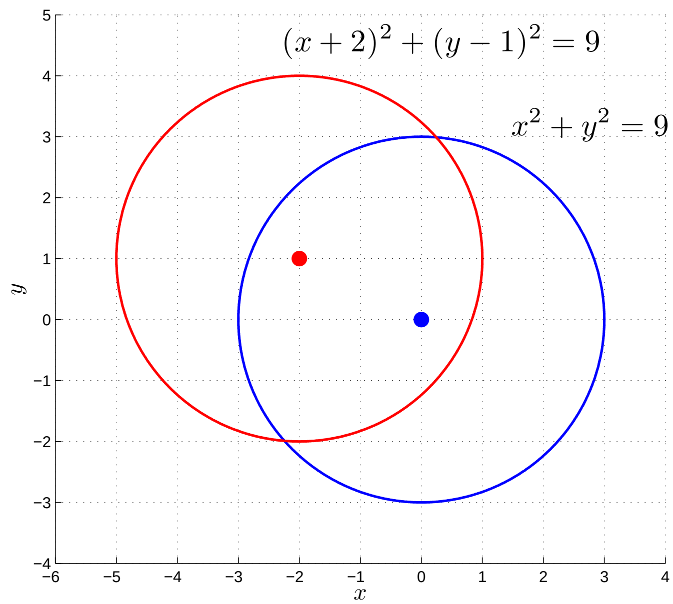


Figure 2.37:

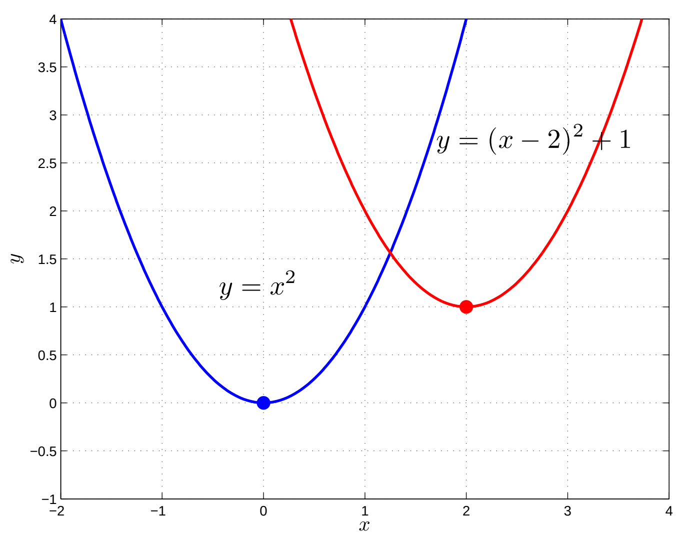


Figure 2.38: