Calculus for Engineers

Jeff Chak-Fu WONG[1]

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Derivatives and Differentials of Higher Orders

14.1 Introduction

Different notations for the high-order derivatives of a function $y = f(x)$ are summarized in Section 14.2. The process of differentiating the same function again and again is called successive differentiation. The derivatives obtained during this process are called successive derivatives. Factorial calculation becomes a useful tool when simplifying the coefficients of successive derivatives. High order derivatives for implicit functions are given in Section 14.3. High order derivatives for parametric functions are given in Section 14.4. High order derivatives for a product of two functions are presented in Section 14.5. Discussion on differentials of higher orders is given in Section 14.6.

14.2 Notations

We have studied several methods for finding the derivatives of differentiable functions. If $y = f(x)$ is a *differentiable* function of x, then its derivative is denoted by

$$\frac{dy}{dx} \quad \text{or} \quad f'(x) \quad \text{or} \quad y' \quad \text{or} \quad y_1$$

(See Chapter 6 and Chapter 7). The notation $f'(x)$ suggests that the derivative of $f(x)$ is also a function of x. If the function $f'(x)$ is in turn differentiable, its derivative is called the second derivative (or the second order derivative) of the original function $f(x)$ and is denoted by $f''(x)$. This leads us to the concept of high order derivatives.

$$f''(x) = [f'(x)]' = \lim_{\Delta x \to 0} \frac{f'(x + \Delta x) - f'(x)}{\Delta x}.$$

We write,

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} \quad \text{or} \quad \left[\frac{d(f'(x))}{dx} = f''(x) \quad \text{or} \quad y'' \quad \text{or} \quad y_2\right].$$

Similarly, we can find the derivative of d^2y/dx^2 *provided it exists*, and it is denoted by d^3y/dx^3 [or $f'''(x)$ or y''' or y_3], called the third derivative of $y = f(x)$ and so on.

Order of Derivative	Prime Notation ($'$)	Leibniz Notation	y-Notation	D-Notation
1st	y' or $f'(x)$	dy/dx	y_1	Df
2nd	y'' or $f''(x)$	d^2y/dx^2	y_2	D^2f
3rd	y''' or $f'''(x)$	d^3y/dx^3	y_3	D^3f
4th	y^{iv} or $f^{iv}(x)$	d^4y/dx^4	y_4	D^4f
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
nth	$y^{(n)}$ or $f^{(n)}(x)$	d^ny/dx^n	y_n	D^nf

14.2.1 Notations for derivatives of $y = f(x)$

Table 14.2.1 shows different notations for high-order derivatives of $y = f(x)$.

Example 1 If $y = f(x) = 2x^5 - x^2 + 3$, then

$$\begin{aligned}
y_1 &= 10x^4 - 2x,\\
y_2 &= 40x^3 - 2,\\
y_3 &= 120x^2,\\
y_4 &= 240x,\\
y_5 &= 240,\\
y_6 &= 0,\\
&\vdots\\
y_n &= 0.
\end{aligned}$$

Note that, for a polynomial function $f(x)$ of degree 5, $f^{(n)}(x) = 0$ for $n \geq 6$. More generally, the $(n+1)^{\text{th}}$ and all higher derivatives of any polynomial of degree n are equal to 0.

□

Note 1 There are functions, like $\sin x$, $\cos x$, e^x, $\log_e x$, and their extended forms, that is, $\sin(ax+b)$, $\cos(ax+b)$, e^{ax}, $\log_e(ax+b)$, or more general ones like $\sin(f(x))$, $e^{f(x)}$, and $\log_a(f(x))$ that can be differentiated any number of times and $f^{(n)}(x)$ is clearly never 0.

Note 2 In Chapter 6, we note that the most important derivatives in physical applications are the first and the second, and these have different special meanings. For example, if x represents time and y the distance, then $\dfrac{dy}{dx}$ represents *velocity* v. In this case, the rate of change of velocity, that is, $\dfrac{dv}{dx} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d^2y}{dx^2}$ is called the *acceleration.*

Note 3 In addition, the second-order derivative has other special interpretations, depending on the meaning of the related variables x and y. When the relation between x and y is graphed, then one interpretation of $\dfrac{d^2y}{dx^2}$ is associated with the curvature of the graph (See in Chapter 15 and Chapter 16).

Note 4 Some real-life applications of the generation of successive derivatives are found as follows:

- A railroad engineer has to employ second derivatives to calculate the curvature of the line he constructs. He needs a precise measure of the curvature to find the exact degree of banking required to prevent trains from overturning.
- An automobile designer utilizes the third derivative in order to test the ride quality of the car he designs and the structural engineer has to go to the fourth derivative in order to measure the elasticity of the beam and the strength of the columns. As well, we will see that the high order derivatives are needed to expand functions (to the desired degree of accuracy) in the form of polynomials.

Example 2 Let us find the nth derivatives of the following functions:

1. x^n
2. e^x
3. a^x
4. $\sin x$
5. $\cos x$
6. $\dfrac{1}{x}$
7. $\log_e x$

Solutions:

1. Let $y = x^n$. Then

$$\begin{aligned}
y_1 &= nx^{n-1}, \\
y_2 &= n(n-1)x^{n-2}, \\
y_3 &= n(n-1)(n-2)x^{n-3}, \\
y_4 &= n(n-1)(n-2)(n-3)x^{n-4}, \\
&\vdots \\
y_n &= n(n-1)(n-2)(n-3)\cdots 2\cdot 1\cdot x^{n-n} \\
&= n(n-1)(n-2)(n-3)\cdots 2\cdot 1 \\
&= n!.
\end{aligned}$$

Remark 1 $y_{n+1} = 0$ since, $y_n = n! =$ constant.

2. Let $y = e^x$. Then

$$\begin{aligned} y_1 &= e^x, \\ y_2 &= e^x, \\ y_3 &= e^x, \\ &\vdots \\ y_n &= e^x. \end{aligned}$$

3. Let $y = a^x$. Then

$$y_1 = a^x \log_e a = a^x \cdot k,$$

where $k = \log_e a$ =constant. Now

$$\begin{aligned} y_2 &= k \cdot a^x \log_e a = k^2 \cdot a^x, \\ y_3 &= k^3 \cdot a^x, \\ &\vdots \\ y_n &= k^n \cdot a^x = (\log_e a)^n \cdot a^x. \end{aligned}$$

4. Let $y = \sin x$. Then

$$\begin{aligned} y_1 &= \cos x = \sin\left(\frac{\pi}{2} + x\right) && \text{because } \sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta, \\ y_2 &= \cos\left(\frac{\pi}{2} + x\right) = \sin\left(\frac{\pi}{2} + \left(\frac{\pi}{2} + x\right)\right) = \sin\left(2 \cdot \frac{\pi}{2} + x\right), \\ y_3 &= \cos\left(2 \cdot \frac{\pi}{2} + x\right) = \sin\left(\frac{\pi}{2} + \left(2 \cdot \frac{\pi}{2} + x\right)\right) = \sin\left(3 \cdot \frac{\pi}{2} + x\right), \\ &\vdots \\ y_n &= \sin\left(n \cdot \frac{\pi}{2} + x\right). \end{aligned}$$

5. Let $y = \cos x$. Then

$$\begin{aligned} \frac{dy}{dx} &= y_1 \\ &= -\sin x \\ &= \cos\left(\frac{\pi}{2} + x\right) && \text{because} \quad \cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta. \end{aligned}$$

Now, it is easy to show that,

$$y_n = \cos\left(n \cdot \frac{\pi}{2} + x\right).$$

6. Let $y = \dfrac{1}{x} = x^{-1}$. Then

$$\begin{aligned}
y_1 &= -1 \cdot x^{-2} = \frac{(-1)}{x^2},\\
y_2 &= (-1)(-2)x^{-3} = \frac{(-1)^2 \cdot 1 \cdot 2}{x^3} = \frac{(-1)^2 \cdot 2!}{x^3},\\
y_3 &= (-1)(-2)(-3) \cdot x^{-4} = \frac{(-1)^3 \cdot 1 \cdot 2 \cdot 3}{x^4} = \frac{(-1)^3 \cdot 3!}{x^4},\\
&\vdots\\
y_n &= \frac{(-1)^n \cdot n!}{x^{n+1}}.
\end{aligned}$$

7. Let $y = \log_e x$. Then

$$\begin{aligned}
y_1 &= \frac{1}{x} = x^{-1},\\
y_2 &= (-1)x^{-2} = \frac{(-1)}{x^2},\\
&\vdots\\
y_n &= \frac{(-1)(-2)(-3)\cdots(-n+1)}{x^n} = \frac{(-1)^{n-1}(n-1)!}{x^n}.
\end{aligned}$$

The reader may compare this result with the nth derivative of $\dfrac{1}{x}$ in Example 2 part 6.

□

Note 5 The higher derivatives with respect to the *extended forms* of the above functions are given below. The reader may easily prove these results. It is useful to remember them since they will be needed for solving problems.

1. Let $y = (ax+b)^r$. Then

   Then

$$\begin{aligned}
y_1 &= ra(ax+b)^{r-1},\\
y_2 &= r(r-1)a^2(ax+b)^{r-2},\\
y_3 &= r(r-1)(r-2)a^3(ax+b)^{r-3},\\
&\vdots\\
y_n &= r(r-1)(r-2)\cdots(r-(n-1)a^n(ax+b)^{r-n}.
\end{aligned}$$

If r is a positive integer, the above result can be written in a compact form using the factorial notation. Therefore

$$\begin{aligned}
y_n &= \frac{r(r-1)(r-2)\cdots(r-(n-1)(r-n)(r-n-1)\cdots 2\cdot 1}{(r-n)(r-n-1)\cdots 2\cdot 1}a^n(ax+b)^{r-n}\\
&= \frac{r!}{(r-n)!}a^n(ax+b)^{r-n}.
\end{aligned}$$

There are few special cases for the above result:

(a) When $r = n$, then we shall have

$$y_n = \frac{n!}{(n-n)!}a^n(ax+b)^{n-n} = \frac{n!}{0!}a^n(ax+b)^0 = n!a^n$$

since $(ax+b)^0 = 1$ and $0! = 1$.

(b) When $r < n$, then

$$y_n = 0.$$

(c) If $r = -p$, where p is a positive integer, then

$$\begin{aligned} y_n &= (-p)(-p-1)(-p-2)\cdots(-p-(n-1)a^n(ax+b)^{-p-n}) \\ &= (-1)^n(p)(p+1)(p+2)\cdots(p+n-1)a^n(ax+b)^{-p-n}) \\ &= (-1)^n\frac{(p+n-1)!}{(p-1)!}a^n(ax+b)^{-p-n}. \end{aligned}$$

Hence for $p = 1$ or $r = -1$, then

$$y_n = (-1)^n n!a^n(ax+b)^{-n-1}.$$

2. Let $y = \dfrac{1}{(ax+b)}$. Then

$$\begin{aligned} y_1 &= (-1)\cdot a\cdot(ax+b)^{-2}, \\ y_2 &= (-1)(-2)\cdot a^2\cdot(ax+b)^{-3}, \\ y_3 &= (-1)(-2)(-3)\cdot a^3\cdot(ax+b)^{-4}, \\ &\vdots \\ y_n &= (-1)^n n!a^n(ax+b)^{-(n+1)} \\ &= \frac{(-1)^n n!a^n}{(ax+b)^{n+1}}. \end{aligned}$$

3. Let $y = \log(ax+b)$. Then

$$\begin{aligned} y_1 &= \frac{a}{(ax+b)}, \\ y_2 &= \frac{(-1)a^2}{(ax+b)^2}, \\ y_3 &= \frac{(-1)(-2)a^3}{(ax+b)^3}, \\ &\vdots \\ y_n &= \frac{(-1)(-2)(-3)\cdots(-n+1)a^n}{(ax+b)^n} \\ &= \frac{(-1)^{n-1}(1)(2)(3)\cdots(n-1)a^n}{(ax+b)^n} \\ &= \frac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n}. \end{aligned}$$

4. Let $y = e^{ax}$. Then

$$\begin{aligned} y_1 &= ae^{ax}, \\ y_2 &= a^2 e^{ax}, \\ y_3 &= a^3 e^{ax}, \\ &\vdots \\ y_n &= a^n e^{ax}. \end{aligned}$$

5. Let $y = a^{kx}$. Then

$$y_n = a^{kx} \cdot k^n (\log_e a)^k$$

or

$$y_n = k^n \cdot a^{kx} \cdot (\log_e a)^{k^{(1)}}.$$

6. If $y = \sin(ax + b)$. Then

$$\begin{aligned} y_1 &= a\cos(ax+b) = a\sin\left(ax + b + \frac{\pi}{2}\right), \\ y_2 &= a^2 \cos\left(ax + b + \frac{\pi}{2}\right) = a^2 \sin\left(ax + b + \frac{\pi}{2} + \frac{\pi}{2}\right), \\ y_3 &= a^3 \sin\left(ax + b + 3 \cdot \frac{\pi}{2}\right), \\ &\vdots \\ y_n &= a^n \sin\left(ax + b + n \cdot \frac{\pi}{2}\right). \end{aligned}$$

Two special cases are:

(a) If $y = \sin ax$, then $y_n = a^n \sin\left(ax + n \cdot \frac{\pi}{2}\right)$.

(b) If $y = \sin x$, then $y_n = \sin\left(x + n \cdot \frac{\pi}{2}\right)$.

7. If $y = \cos(ax + b)$. Then

$$y_n = a^n \cos\left(ax + b + n \cdot \frac{\pi}{2}\right).$$

Two special cases are:

(a) If $y = \cos ax$, then $y_n = a^n \cos\left(ax + n \cdot \frac{\pi}{2}\right)$.

(b) If $y = \cos x$, then $y_n = \sin\left(x + n \cdot \frac{\pi}{2}\right)$.

14.3 High order derivatives of implicit functions

If y is an implicit function, its higher derivatives are found by differentiating the required number of times the equation linking with x and y, bearing in mind that y and all its derivatives are functions of the independent variable x.

Example 3 The second derivative of the function y specified by the circle equation

$$x^2 + y^2 = 1 \tag{14.1}$$

is found by differentiating (14.1) twice. We get

$$2x + 2yy' = 0$$

or

$$x + yy' = 0 \tag{14.2}$$

and

$$(x') + yy' + y'y' = 0$$

or

$$1 + (y')^2 + yy'' = 0. \tag{14.3}$$

From (14.2) and (14.3), we have

$$y' = -\frac{x}{y}$$

and

$$y'' = -\frac{1 + (y')^2}{y}.$$

Therefore, using (14.1), we have

$$y'' = -\frac{1 + \left(-\frac{x}{y}\right)^2}{y} = -\frac{x^2 + y^2}{y^3} = -\frac{1}{y^3}.$$

□

14.4 High order derivatives of parametric functions

In order to find a high order derivative of a function specified by parametric equations, we differentiate the expression of the preceding derivative considering it as a composite function of the independent variable.

Let $x = \phi(t)$ and $y = f(t)$. Then, we have,

$$\frac{dy}{dx} = \frac{f'(t)}{\phi'(t)} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}},$$

where $\frac{dx}{dt} \neq 0$. Also the function $x = \phi(t)$ has an *inverse function* $t = \phi^{-1}(x)$.

Furthermore,

$$
\begin{aligned}
y'' &= \frac{d}{dx}\left(\frac{f'(t)}{\phi'(t)}\right) \\
&= \frac{d}{dt}\left(\frac{f'(t)}{\phi'(t)}\right)\cdot\frac{dt}{dx} && \text{Using the property, } \frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx} \\
&= \frac{\phi'(t)\cdot f''(t)-f'(t)\cdot\phi''(t)}{[\phi'(t)]^2}\cdot\frac{dt}{dx}. && \text{By the quotient rule}
\end{aligned}
$$

From the inverse function $t=\phi^{-1}(x)$, we obtain

$$\frac{dt}{dx}=\frac{1}{\phi'(x)}.$$

and obtain the expression

$$y''=\frac{\phi'(t)\cdot f''(t)-f'(t)\cdot\phi''(t)}{[\phi'(t)]^3},$$

where $\phi'(t)\neq 0$, $\forall t\in\mathbb{R}$. The differentiation of the last relation with respect to x leads to the expression for the third derivative, and so on.

Example 4 Let us find the derivatives y' and y'' of the function specified by the equations $x=a\cos t$ and $y=b\sin t$.
Solution: Differentiating the given equations with respect to t twice, we obtain

$$
\begin{cases}
\dfrac{dx}{dt}=-a\sin t, & \dfrac{d^2x}{d^2t}=-a\cos t \\
\dfrac{dy}{dt}=b\cos t, & \dfrac{d^2y}{d^2t}=-b\sin t
\end{cases}
$$

Therefore

$$y'=\frac{dy}{dx}=\frac{b\cos t}{-a\sin t}=-\frac{b}{a}\cot t$$

because

$$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

Therefore

$$
\begin{aligned}
y'' &= \frac{d}{dx}\left(\frac{dy}{dx}\right) \\
&= \frac{(-a\sin t)(-\sin t)-(b\cos t)(-a\cos t)}{(-a\sin t)^2}\cdot\left(\frac{dt}{dx}\right) \\
&= \frac{ab\sin^2 t+ab\cos^2 t}{a^2\sin^2 t}\cdot\frac{1}{(-a\sin t)} \\
&= -\frac{b}{a^2\sin^3 t}.
\end{aligned}
$$

Therefore

$$y' = -\frac{b}{a}\cot t$$

and

$$y'' = -\frac{b}{a^2 \sin^3 t},$$

where $a \neq 0$ and $\sin t \neq 0$, $\forall t \in \mathbb{R}$. □

14.5 High order derivatives of product of two functions

Let us find the nth derivative of the product of two functions.

Theorem 1 (Leibnitz's theorem) Let $u(x)$ and $v(x)$ be functions of x, possessing derivatives of nth order, and $y = u \cdot v$. Then,

$$y_n = (uv)_n =^n C_0 u_n v_0 +^n C_1 u_{n-1} v_1 +^n C_2 u_{n-2} v_2 + \cdots +^n C_r u_{n-r} v_r + \cdots +^n C_n u_0 v_n,$$

where[1]

$$^nC_r = \frac{n!}{(n-r)!r!}.$$

Proof. We shall prove this theorem by mathematical induction.

Let $y = uv$. Then we have

$$\begin{aligned} y_1 = (uv)_1 &= u_1 v_0 + u_0 v_1 \\ &=^1 C_0 u_1 v_0 +^1 C_1 u_0 v_1, \end{aligned} \tag{14.4}$$

where $^1C_0 =^1 C_1 = 1$ and $0! = 1$. From (14.4) we see that the theorem is true for $n = 1$.

Differentiating (14.4) on both sides with respect to x, we have

$$\begin{aligned} y_2 = (uv)_2 &= u_2 v_0 + u_1 v_1 + u_1 v_1 + u_0 v_2 \\ &= u_2 v_0 + 2u_1 v_1 + u_0 v_2 \\ &=^2 C_0 u_2 v_0 +^2 C_1 u_1 v_1 +^2 C_2 u_0 v_2. \end{aligned} \tag{14.5}$$

From (14.5) we see that the theorem is true for $n = 2$.

Now let us assume that the theorem is true for a particular value of n, say m, that is

$$y_m = (uv)_m =^m C_0 u_m v_0 +^m C_1 u_{m-1} v_1 +^m C_2 u_{m-2} v_2 + \cdots +^m C_r u_{m-r} v_r + \cdots +^m C_m u_0 v_m. \tag{14.6}$$

[1]Other ways of writing the notation ^nC_r include $\binom{n}{r}$, $C(n,r)$, $_nC_r$ and $C\binom{n}{r}$. The C represents combinations or choices and is often called the choose function.

Differentiating (14.6) on both sides with respect to x, we have

$$\begin{aligned}
y_{m+1} = (uv)_{m+1} &=^{m+1} C_0 u_{m+1} v_0 +^m C_0 u_m v_1 +^m C_1 (u_m v_1 + u_{m-1} v_2) \\
&\quad +^m C_2 (u_{m-1} v_2 + u_{m-2} v_3) + \cdots \\
&\quad +^m C_r (u_{m-r+1} v_r + u_{m-r} v_{r+1}) + \cdots \\
&\quad +^{m+1} C_{m+1} (u_1 v_m + u_0 v_{m+1}) \\
&=^{m+1} C_0 u_{m+1} v_0 + (^m C_0 +^m C_1) u_m v_1 + \\
&\quad + (^m C_1 +^m C_2) u_{m-1} v_2 + (^m C_2 +^m C_3) u_{m-2} v_3 + \cdots \\
&\quad + (^m C_{m-r+1} +^m C_r) u_{m-r+1} v_r + \cdots \\
&\quad +^{m+1} C_{m+1} u_0 v_{m+1}.
\end{aligned} \tag{14.7}$$

But we know that[2] $^m C_1 +^m C_{r+1} =^{m+1} C_{r+1}$. Therefore, we have

$$1 +^m C_1 =^m C_0 +^m C_1 =^{m+1} C_1, \quad ^m C_1 +^m C_2 =^{m+1} C_2, \quad \text{and so on.}$$

Hence, (14.7) gives

$$\begin{aligned}
y_{m+1} &= (uv)_{m+1} \\
&=^{m+1} C_0 u_{m+1} v_0 +^{m+1} C_1 u_m v_1 +^{m+1} C_2 u_{m-1} v_2 + \cdots +^{m+1} C_r u_{m-r+1} v_r \\
&\quad + \cdots +^{m+1} C_{m+1} u_0 v_{m+1}.
\end{aligned} \tag{14.8}$$

which shows that the theorem is true for $n = m + 1$, if it is true for $n = m$.

From (14.8) we see that if the theorem is true for any value of n, it is also true for the next value of n. But we have already seen that the theorem is true for $n = 1$. Hence is must be true for $n = 2$ and for $n = 3$, and so on. Thus Leibnitz's theorem is true for all positive integral values of n. □

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[2]Show that the well-known Pascal's rule is valid

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}.$$

Proof.

$$\begin{aligned}
\binom{n}{k} + \binom{n}{k+1} &= \frac{n!}{(n-k)!k!} + \frac{n!}{(n-(k+1))!(k+1)!} \\
&= \frac{n!}{(n-k)!k!} \cdot \frac{k+1}{k+1} + \frac{n!}{(n-(k+1))!(k+1)!} \cdot \frac{n-k}{n-k} \\
&= \frac{n!(k+1+n-k)}{(n-k)!(k+1)!} \\
&= \frac{(n+1)!}{(n+1-(k+1))!(k+1)!} \\
&= \binom{n+1}{k+1}.
\end{aligned}$$

□

Remark 2 This formula can be formally obtained if we take *Newton's binomial formula*[3] for the expansion of $(u+v)^n$ and then replace the powers of u and v by the derivatives of the corresponding orders of u and v (and put $u_0 = u, v_0 = v$).

---

[3]Newton's binomial formula is given as

$$\begin{aligned}(a+b)^n &= \sum_{k=0}^{n} {}^{n}C_r a^{n-k} b^k \\ &= {}^{n}C_0 a^n b^0 + {}^{n}C_1 a^{n-1} b + {}^{n}C_2 a^{n-2} b^2 + \cdots + {}^{n}C_n a^0 b^n\end{aligned}$$

Proof of Newton's binomial formula using mathematical induction is given below. Assume the variables a and b are real numbers and the variables m and n are non-negative integers, that is, $m, n \geq 0$.

When $n = 1$, we have

$$(a+b)^1 = a + b = \sum_{k=0}^{1} a^{1-k} b^k.$$

For the inductive step, assume the theorem holds when the exponent is m. We have

$$\begin{aligned}(a+b)^m = &\binom{m}{0} a^m \\ &+ \left[\binom{m}{1} a^{m-1} b + \binom{m}{2} a^{m-2} b^2 + \binom{m}{3} a^{m-3} b^3 + \cdots + \binom{m}{m-2} a^2 b^{m-2} + \binom{m}{m-1} a b^{m-1}\right] \\ &+ \binom{m}{m} b^m.\end{aligned}$$

Then consider the case $n = m + 1$. By the inductive hypothesis, we have

$$\begin{aligned}(a+b)^{m+1} &= (a+b)(a+b)^m \\ &= (a+b)\left(a^m + b^m + \sum_{k=1}^{m-1} \binom{m}{k} a^{m-k} b^k\right) \\ &= a^{m+1} + b^{m+1} + ab^m + ba^m + \sum_{k=1}^{m-1} \binom{m}{k} a^{m-k+1} b^k + \sum_{k=1}^{m-1} \binom{m}{k} a^{m-k} b^{k+1} \\ &= a^{m+1} + b^{m+1} + \sum_{k=1}^{m} \binom{m}{k} a^{m-k+1} b^k + \sum_{k=0}^{m-1} \binom{m}{k} a^{m-k} b^{k+1}.\end{aligned}$$

Let $j = k + 1$ in the second term, then we have

$$(a+b)^{m+1} = a^{m+1} + b^{m+1} + \sum_{k=1}^{m} \binom{m}{k} a^{m-k+1} b^k + \sum_{j=1}^{m} \binom{m}{j-1} a^{m+1-k} b^j.$$

Let $j = i$ and $k = i$, then we have

$$(a+b)^{m+1} = a^{m+1} + b^{m+1} + \sum_{i=1}^{m} \binom{m}{i} a^{m-i+1} b^i + \sum_{i=1}^{m} \binom{m}{i-1} a^{m+1-i} b^i.$$

Combining the sums and using the Pascal rule, then we have

$$\begin{aligned}(a+b)^{m+1} &= a^{m+1} + b^{m+1} + \sum_{i=1}^{m} \left(\binom{m}{i} + \binom{m}{i-1}\right) a^{m-i+1} b^i \\ &= a^{m+1} + b^{m+1} + \sum_{i=1}^{m} \binom{m+1}{i} a^{m-i+1} b^i.\end{aligned}$$

Note 6 When one of the functions in the above theorem is of the form x^m, then we should choose it as (the second function) v, and the other as (the first function) u, because x^m shall have only m derivatives (and not more).

Note 7 From the expression for ${}^nC_r$, we get

$$\begin{aligned}
{}^nC_1 &= n \\
{}^nC_2 &= \frac{n(n-1)}{2!} = \frac{n(n-1)}{1\cdot 2} \\
{}^nC_3 &= \frac{n(n-1)(n-2)}{3!} = \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}, \\
&\vdots
\end{aligned}$$

Example 5 If $y = e^{ax}x^2$, find y_n.

Solution: Let $u_0 = e^{ax}$ and $v_0 = x^2$. Then

$$\begin{aligned}
u_1 &= ae^{ax}, & v_1 &= 2x \\
u_2 &= a^2e^{ax}, & v_2 &= 2 \\
u_n &= a^ne^{ax}, & v_3 &= 0 = v_4 = v_5 = \cdots .
\end{aligned}$$

We have

$$y_n = a^n e^{ax} x^2 + na^{n-1} e^{ax} 2x + \frac{n(n-1)}{1\cdot 2} a^{n-2} e^{ax} \cdot 2$$

or

$$y_n = e^{ax}\left(a^n x^2 + 2na^{n-1}x + n(n-1)a^{n-2}\right).$$

□

Example 6 Let us compute the 100th derivative of the function $y = x^2 \sin x$.

Solution: We have

$$\begin{aligned}
y_{100} &= (\sin x \cdot x^2)_{100} \\
&= (\sin x)_{100} \cdot x^2 +^{100} C_1 (\sin x)_{99} \cdot (2x) +^{100} C_2 (\sin x)_{98} \cdot (2) \\
&= x^2 \cdot (\sin x)_{100} + 200x \cdot (\sin x)_{99} + \frac{100\cdot 99}{2} \cdot (\sin x)_{98} \cdot (2).
\end{aligned}$$

All the subsequent terms are omitted here since they are all equal to zero. Therefore, we have

$$\begin{aligned}
y_{100} &= x^2 \cdot \sin\left(x + 100\frac{\pi}{2}\right) + 200x \cdot \sin\left(x + 99\frac{\pi}{2}\right) + 9900 \cdot \sin\left(x + 98\frac{\pi}{2}\right) \\
&= x^2 \sin x - 200x \cos x - 9900 \sin x.
\end{aligned}$$

□

This means that the statement is true for $n = m + 1$ as desired. Thus, Newton's binomial formula is proven by mathematical induction. □

Example 7 Differentiate the following equation n times

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0$$

Here, each term is differentiated n times.

Solution: We have

$$D^n(y_2x^2)=y_{n+2}\cdot x^2+^nC_1y_{n+1}\cdot(2x)+^nC_2y_n\cdot(2)$$

$$D^n(y_2x^2)=x^2y_{n+2}+n(2x)y_{n+1}+\frac{n(n-1)}{2!}(2)\cdot y_n \tag{14.9}$$

$$D^n(y_1x)=x\cdot y_{n+1}+n\cdot y_n \tag{14.10}$$

$$D^n(y)=+y_n. \tag{14.11}$$

Adding (14.9), (14.10), and (14.11), we get

$$\begin{aligned}0&=x^2\cdot y_{n+2}+(2n+1)\times y_{n+1}+[n(n-1)+n+1]y_n\\0&=x^2\cdot y_{n+2}+(2n+1)\times y_{n+1}+[n^2+1]y_n.\end{aligned}$$

□

Example 8 Let $y=\sin(m\sin^{-1}x)$. Prove

$$(1-x^2)-xy_1+m^2y=0$$

and deduce

$$(1-x^2)y_{n+2}-(2n+1)\times y_{n+1}-(n^2-m^2)y_n=0.$$

Solution: We have

$$y=\sin(m\sin^{-1}x). \tag{14.12}$$

Then, using the chain rule, we have

$$y_1=\cos(m\sin^{-1}x)\cdot\frac{m}{\sqrt{1-x^2}}$$

or

$$\sqrt{1-x^2}\cdot y_1=m\cos(m\sin^{-1}x).$$

Squaring both sides, we have

$$\begin{aligned}(1-x^2)\cdot y_1^2&=m^2\cos^2(m\sin^{-1}x)\\&=m^2[1-\sin^2(m\cdot\sin^{-1}x)] && \text{since }\cos^2\theta=1-\sin^2\theta\\&=m^2[1-y^2].\end{aligned}$$

Therefore

$$1 - x^2 \cdot y_1^2 = m^2[1 - y^2]. \tag{14.13}$$

Differentiating both sides of (14.13) with respect to x, we have

$$(1 - x^2)2y_1y_2 + y_1^2(-2x) = m^2[-2yy_1]$$

or

$$(1 - x^2)y_2(2y_1) - x(2y_1^2) = -m^2 \cdot y(2y_1).$$

Canceling the factor $(2y_1)$ from both sides, we get

$$(1 - x^2)y_2 - xy_1 + m^2y = 0. \tag{14.14}$$

Now, in order to prove the second relation, we shall differentiate each term of (14.14) n times using Leibniz theorem.

$$\begin{aligned} D^n(1 - x^2)y_2 &= y_{n+2}(1 - x^2) + ny_{n+1}(-2x) + \frac{n(n-1)}{2!}y_n(-2) \\ &= y_{n+2}(1 - x^2) - ny_{n+1}(2x) - y_n \cdot n(n-1). \end{aligned} \tag{14.15}$$

Now

$$\begin{aligned} D^n(-xy_1) &= y_{n+1}(-x) +^n C_1y_n(-1) \\ &= -xy_{n+1} - ny_n. \end{aligned} \tag{14.16}$$

Now

$$D^n(m^2y) = m^2y_n. \tag{14.17}$$

Adding (14.15), (14.16) and (14.17), we get

$$0 = (1 - x^2)y_{n+2} + (-2xn - x)y_{n+1} + [-n(n-1) - n + m^2]y_n$$

or

$$(1 - x^2)y_{n+2} - (2n+1)y_{n+1} - (n^2 - m^2)y_n = 0.$$

□

The following results can be easily proved:

- If $y = e^{ax} \cdot \sin bx$, then,

  $$y_n = (a^2 + b^2)^{n/2} \sin(bx + n\alpha) \cdot e^{ax},$$

  where $\alpha = \tan^{-1}(b/a)$.

- If $y = e^{ax} \sin(bx + c)$, then

  $$y_n = (a^2 + b^2)^{n/2} \sin(bx + c + n\alpha) \cdot e^{ax},$$

  where, $\alpha = \tan^{-1}(b/a)$.

- In particular, if $y = e^x \sin x$ (here $a = 1, b = 1, c = 0$), then

$$\begin{aligned} y_n &= (1^2 + 1^2)^{n/2} \sin(x + n\alpha) \cdot e^x \\ &= 2^{n/2} \cdot e^x \cdot \sin(x + n \tan^{-1} 1/1) \\ &= 2^{n/2} \cdot e^x \cdot \sin(x + n \cdot \pi/4). \end{aligned}$$

- Similarly, if $y = e^x \cos x$, then

$$y_n = 2^{n/2} \cdot e^x \cdot \cos(x + n \cdot \pi/4).$$

- Now, if $y = e^{2x} \cdot \sin x$ (here $a = 2, b = 1, c = 0$), then

$$\begin{aligned} y_n &= (2^2 + 1^2)^{n/2} \sin(x + n\alpha) \\ &= 5^{n/2} \sin\left(x + n \tan^{-1} \frac{1}{2}\right). \end{aligned}$$

- If $y = e^{ax} \cdot \cos bx$, then

$$y_n = (a^2 + b^2)^{n/2} \cos(bx + n\alpha) e^{ax},$$

  where $\alpha = \tan^{-1}(b/a)$.

- If $y = e^{ax} \cos(bx + c)$, then

$$y_n = (a^2 + b^2)^{n/2} \cdot \cos(bx + c + n\alpha) \cdot e^{ax},$$

  where $\alpha = \tan^{-1}(b/a)$.

14.6 Differentials of higher orders

Consider a function $y = f(x)$, where x is the independent variable. The differential of this function is denoted by

$$dy = f'(x)dx$$

which depends on *two arguments*, namely, the independent variable x and its differential dx. Here, it is important to remember that the differential dx of the independent variable x has a magnitude independent of x: for any given value of x, the value of dx can be chosen quite arbitrarily.

This means that dy must be looked upon as a function of x alone and that we have the right to speak of the differential of this function. The differential of the differential of a function, that is, $d[df(x)]$, is called the second differential (or the second order differential) of the function $f(x)$ and is denoted by

$$d^2y: \qquad d^2y = d(dy).$$

By virtue of the general definition of a differential, we have,

$$d^2y = [f'(x)dx]'dx$$

which is a function of x (for an arbitrary but fixed value of dx independent of x). Since dx is independent of x, dx is taken outside the sign of the derivative upon differentiation, and we get

$$d^2y = f''(x)(dx)^2.$$

Note 8 When writing the degree of the differential, it is common to drop the brackets; in place of $(dx)^2$, we write dx^2, and so on. Therefore

$$d^2y = f''(x)dx^2.$$

Note 9 To unify the terminology, we call the differential $df(x)[= f'(x)dx]$ of the function $f(x)$ the differential of the first order (or the first differential).

Similarly, the third differential (or the third-order differential) of a function is the differential of its second differential.

$$d^3y = d(d^2y) = [f''(x)dx^2]dx = f'''(x)dx^3.$$

Analogously, for the nth order differential, we arrive at the formula

$$d^ny = f^{(n)}(x)dx^n$$

where dx^n is the nth power of dx. Thus, the differential of nth order is equal to the product of the nth derivative with respect to the independent variable by the nth power of the differential of the independent variable.

We have seen (in Chapter 13) that, if $y = f(x)$, then $dy = f'(x)dx$ irrespective of whether the argument x is an independent variable or a function of another argument. [Recall that if $y = f(u)$, where $u = \phi(x)$, then $dy = f'(u)\phi'(x)dx = f'(u)du$]. In the general case, this property (i.e., the invariance property of the first derivative) is not possessed by the higher order differentials.

Indeed, suppose that x is no longer an independent variable like before, but a function of a new independent variable t, that is, $x = \phi(t)$. Then, dx also becomes function of t, and therefore it is not allowable to regard dx as a constant when the first differential is differentiated. This leads to a new expression of d^2y different from the one above. Computing the differential of dy by applying the differentiation rule for a product, we find

$$d^2y = d[f'(x)dx].$$

Now, treating $f'(x)dx$ as a product of functions, we get,

$$\begin{aligned} d^2y &- d[f'(x)]dx + f'(x)d(dx) \\ &= [f''(x)]dx + f'(x)d^2x \\ &= f''(x)dx^2 + f'(x)d^2x. \end{aligned}$$

Observe that there the additional term $f'(x)d^2x$ appears. If x is an independent variable, the first term is retained but the second one vanishes, since,

$$d^2x = (x)''dx^2 = 0 \cdot dx^2 = 0.$$

The expression for the third differential in the case when x depends on t is still more complicated. Thus, when finding a higher order differential, we take into account the nature of the function and distinguish between the cases of when it is an independent variable or when it depends on some other variable.