## Partial Fractions

1. Resolve the following expressions into partial fractions.

   (a) $\dfrac{5}{x^2 + x - 6}$  (Hint: $\dfrac{5}{x^2 + x - 6} \equiv \dfrac{A}{x + 3} + \dfrac{B}{x - 2}$)

   **Ans:** $\dfrac{5}{x^2 + x - 6} = \dfrac{1}{x - 2} - \dfrac{1}{x + 3}$

   (b) $\dfrac{1}{x(x^2 + 1)}$  (Hint: $\dfrac{1}{x(x^2 + 1)} \equiv \dfrac{A}{x} + \dfrac{Bx + C}{x^2 + 1}$)

   **Ans:** $\dfrac{1}{x(x^2 + 1)} = \dfrac{1}{x} - \dfrac{x}{x^2 + 1}$

   (c) $\dfrac{5x^2 - 3x + 4}{(x + 1)(x^2 - 2x + 6)}$

   **Ans:** $\dfrac{5x^2 - 3x + 4}{(x + 1)(x^2 - 2x + 6)} = \dfrac{4}{3(x + 1)} + \dfrac{11x - 12}{3(x^2 - 2x + 6)}$

2. Resolve the following expressions into partial fractions.

   (a) $\dfrac{x^2 + 3x}{x^2 + 3x + 2}$

   **Ans:** $\dfrac{x^2 + 3x}{x^2 + 3x + 2} = 1 - \dfrac{2}{x + 1} + \dfrac{2}{x + 2}$

   (b) $\dfrac{x^4 + 2x + 4}{(2x^2 + 3)(x - 2)}$

   **Ans:** $\dfrac{x^4 + 2x + 4}{(2x^2 + 3)(x - 2)} = \dfrac{x}{2} + 1 + \dfrac{24}{11(x - 2)} - \dfrac{41x + 38}{22(2x^2 + 3)}$

   (c) $\dfrac{2x^5}{(x^2 - 1)(x^2 - 4)}$

   **Ans:** $\dfrac{2x^5}{(x^2 - 1)(x^2 - 4)} = 2x - \dfrac{1}{3(x - 1)} - \dfrac{1}{3(x + 1)} + \dfrac{16}{3(x - 2)} + \dfrac{16}{3(x + 2)}$

3. Resolve the following expressions into partial fractions.

   (a) $\dfrac{x^3 + 1}{(x - 2)^4}$  (Hint: $\dfrac{x^3 + 1}{(x - 2)^4} \equiv \dfrac{A}{x - 2} + \dfrac{B}{(x - 2)^2} + \dfrac{C}{(x - 2)^3} + \dfrac{D}{(x - 2)^4}$)

   **Ans:** $\dfrac{x^3 + 1}{(x - 2)^4} = \dfrac{9}{(x - 1)^4} + \dfrac{12}{(x - 2)^3} + \dfrac{6}{(x - 2)^2} + \dfrac{1}{x - 2}$

   (b) $\dfrac{2x^2 + 1}{x^2(x^2 + 1)^2}$  (Hint: $\dfrac{2x^2 + 1}{x^2(x^2 + 1)^2} \equiv \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{Cx + D}{x^2 + 1} + \dfrac{Ex + F}{(x^2 + 1)^2}$)

   **Ans:** $\dfrac{2x^2 + 1}{x^2(x^2 + 1)^2} = \dfrac{1}{x^2} - \dfrac{1}{x^2 + 1} + \dfrac{1}{(x^2 + 1)^2}$

**Indefinite Integration**

4. (a) Resolve $\dfrac{x^5 + 3x^2 + 1}{(x-1)(x^2+4)}$ into partial fractions.

     **Ans:** $\dfrac{x^5 + 3x^2 + 1}{(x-1)(x^2+4)} = x^2 + x - 3 + \dfrac{1}{x-1} + \dfrac{-x+15}{x^2+4}$

  (b) Hence, evaluate $\displaystyle\int \dfrac{x^5 + 3x^2 + 1}{(x-1)(x^2+4)}\,dx$

     **Ans:** $\displaystyle\int \dfrac{x^5 + 3x^2 + 1}{(x-1)(x^2+4)}\,dx = \dfrac{x^3}{3} + \dfrac{x^2}{2} - 3x - \dfrac{1}{2}\ln(x^2+4) + \ln|x-1| +$
     $\dfrac{15}{2}\tan^{-1}(\dfrac{x}{2}) + C$

5. **(Integration by substitutions)**
   Evaluate the following integrals.

  (a) $\displaystyle\int (2x-1)^{10}\,dx$

     **Ans:** $\dfrac{1}{22}(2x-1)^{11} + C$

  (b) $\displaystyle\int \dfrac{1}{\sqrt{5x+7}}\,dx$

     **Ans:** $\dfrac{2}{5}\sqrt{5x+7} + C$

  (c) $\displaystyle\int \dfrac{x}{\sqrt{1+x^2}}\,dx$

     **Ans:** $\sqrt{x^2+1} + C$

  (d) $\displaystyle\int x^2\sqrt{x^3+2}\,dx$

     **Ans:** $\dfrac{2}{9}(x^3+2)^{3/2} + C$

  (e) $\displaystyle\int e^x \sin e^x\,dx$

     **Ans:** $-\cos e^x + C$

  (f) $\displaystyle\int \dfrac{(\ln x)^4}{x}\,dx$

     **Ans:** $\dfrac{(\ln x)^5}{5} + C$

  (g) $\displaystyle\int \dfrac{\cos x}{\sqrt{\sin^3 x}}\,dx$

     **Ans:** $-\dfrac{2}{\sqrt{\sin x}} + C$

  (h) $\displaystyle\int \dfrac{1}{\sqrt{x}+\sqrt{x+1}}\,dx$

     **Ans:** $\dfrac{2}{3}\left((x+1)^{3/2} - x^{3/2}\right) + C$

  (i) $\displaystyle\int \dfrac{e^{3x}+1}{e^x+1}\,dx$

     **Ans:** $\dfrac{e^{2x}}{2} - e^x + x + C$

  (j) $\displaystyle\int \dfrac{1}{\sqrt{x}(1+\sqrt{x})^2}\,dx$

     **Ans:** $-\dfrac{2}{\sqrt{x}+1} + C$

  (k) $\displaystyle\int \sec 2x \tan 2x\,dx$

     **Ans:** $\dfrac{1}{2}\sec 2x + C$

  (l) $\displaystyle\int \left(1-\cos\dfrac{x}{2}\right)^2 \sin\dfrac{x}{2}\,dx$

     **Ans:** $\dfrac{2}{3}\left(1-\cos\dfrac{x}{2}\right)^3 + C$

  (m) $\displaystyle\int \dfrac{1}{x^2}\cos\left(\dfrac{1}{x}\right)\,dx$

     **Ans:** $-\sin\left(\dfrac{1}{x}\right) + C$

6. **(Integration by parts)**
   Evaluate the following integrals by using integration by parts.

(a) $\displaystyle\int x \sin\frac{x}{2}\, dx$

**Ans:** $-2x\cos\dfrac{x}{2} + 4\sin\dfrac{x}{2} + C$

(b) $\displaystyle\int x \ln x\, dx$

**Ans:** $\dfrac{x^2}{2}\ln x - \dfrac{x^2}{4} + C$

(c) $\displaystyle\int xe^{3x}\, dx$

**Ans:** $\dfrac{x}{3}e^{3x} - \dfrac{1}{9}e^{3x} + C$

(d) $\displaystyle\int \tan^{-1} x\, dx$

**Ans:** $x\tan^{-1} x - \dfrac{1}{2}\ln(x^2 + 1) + C$

(e) $\displaystyle\int \sin^{-1} x\, dx$

**Ans:** $x\sin^{-1} x + \sqrt{1 - x^2} + C$

(f) $\displaystyle\int x \sec^2 x\, dx$

**Ans:** $x\tan x + \ln|\cos x| + C$

(g) $\displaystyle\int x^3 e^x\, dx$

**Ans:** $e^x(x^3 - 3x^2 + 6x - 6) + C$

(h) $\displaystyle\int e^x \sin x\, dx$

**Ans:** $\dfrac{1}{2}e^x(\sin x - \cos x) + C$

(i) $\displaystyle\int e^{-x} \cos x\, dx$

**Ans:** $\dfrac{1}{2}e^{-x}(\sin x - \cos x) + C$

7. **(Powers of trigonometric functions)**
   Evaluate the following integrals.

(a) $\displaystyle\int \cos^3 x \sin x\, dx$

**Ans:** $-\dfrac{1}{4}\cos^4 x + C$

(b) $\displaystyle\int \sin^4 x \cos x\, dx$

**Ans:** $\dfrac{1}{5}\sin^5 x + C$

(c) $\displaystyle\int \sin^3 x\, dx$

**Ans:** $\dfrac{1}{12}\cos 3x - \dfrac{3}{4}\cos x + C$

(d) $\displaystyle\int \cos^3 x\, dx$

**Ans:** $\dfrac{1}{12}\sin 3x + \dfrac{3}{4}\sin x + C$

(e) $\displaystyle\int \cos^4 x \sin^2 x\, dx$

**Ans:** $\dfrac{1}{16}x + \dfrac{1}{64}\sin 2x - \dfrac{1}{64}\sin 4x - \dfrac{1}{192}\sin 6x + C$

(f) $\displaystyle\int \sec^2 x \tan x\, dx$

**Ans:** $\dfrac{\sec^2 x}{2} + C$

(g) $\displaystyle\int \sec^3 x \tan x\, dx$

**Ans:** $\dfrac{\sec^3 x}{3} + C$

(h) $\displaystyle\int \sec^4 x \tan^2 x \, dx$

**Ans:** $-\dfrac{2}{15} \tan x - \dfrac{1}{15} \sec^2 x \tan x + \dfrac{1}{5} \sec^4 x \tan x + C$

8. **(Products of sines and cosines)**
   Evaluate the following integrals.

   (a) $\displaystyle\int \cos 3x \sin 2x \, dx$

   **Ans:** $\dfrac{1}{2} \cos x - \dfrac{1}{10} \cos 5x + C$

   (b) $\displaystyle\int \sin^3 x \sin 3x \, dx$

   **Ans: Ans:** $-\dfrac{x}{8} + \dfrac{3}{16} \sin 2x - \dfrac{3}{32} \sin 4x + \dfrac{1}{48} \sin 6x + C$

   (c) $\displaystyle\int \cos x \cos 7x \, dx$

   **Ans:** $\dfrac{1}{12} \sin 6x + \dfrac{1}{16} \sin 8x + C$

   (d) $\displaystyle\int \sin^2 x \cos 3x \, dx$

   **Ans:** $\dfrac{2}{5} \sin^3 x \cos 2x - \dfrac{1}{15} \sin^3 x + C$

   (e) $\displaystyle\int \cos^3 x \sin 2x \, dx$

   **Ans:** $-\dfrac{2}{5} \cos^5 x + C$

   (f) $\displaystyle\int \sin x \sin 2x \sin 3x \, dx$

   **Ans: Ans:** $-\dfrac{1}{8} \cos 2x - \dfrac{1}{16} \cos 4x + \dfrac{1}{24} \cos 6x + C$

9. **(Trigonometric substitutions)**
   Evaluate the following integrals.

   (a) $\displaystyle\int \sqrt{25 - x^2} \, dx$

   **Ans:** $\dfrac{x}{2} \sqrt{25 - x^2} + \dfrac{25}{2} \sin^{-1} \dfrac{x}{5} + C$

   (b) $\displaystyle\int \dfrac{1}{8 + 2x^2} \, dx$

   **Ans:** $\dfrac{1}{4} \tan^{-1} \dfrac{x}{2} + C$

   (c) $\displaystyle\int \dfrac{1}{\sqrt{4 + x^2}} \, dx$

   **Ans:** $\ln(x + \sqrt{4 + x^2}) + C$

(d) $\int \dfrac{x^2}{\sqrt{9-x^2}} \, dx$

**Ans:** $-\dfrac{x}{2}\sqrt{9-x^2} + \dfrac{9}{2}\arcsin\dfrac{x}{3} + C$

(e) $\int \dfrac{1}{x^2\sqrt{x^2-1}} \, dx$

**Ans:** $\dfrac{\sqrt{x^2-1}}{x} + C$

(f) $\int \dfrac{x^2}{4+x^2} \, dx$

**Ans:** $x - 2\tan^{-1}\dfrac{x}{2} + C$

(g) $\int \dfrac{2}{x^3\sqrt{x^2-1}} \, dx$

**Ans:** $\dfrac{\sqrt{x^2-1}}{x^2} + \tan^{-1}\sqrt{x^2-1} + C$

10. **(Integration of rational functions by partial fractions)**
Evaluate the following integrals.

(a) $\int \dfrac{x+4}{x^2+5x-6} \, dx$

**Ans:** $\dfrac{5}{7}\ln|x-1| - \dfrac{2}{7}\ln|x+6| + C$

(b) $\int \dfrac{x+3}{2x^3-8x} \, dx$

**Ans:** $\dfrac{5}{16}\ln|x-2| + \dfrac{1}{16}\ln|x+2| - \dfrac{3}{8}\ln|x| + C$

(c) $\int \dfrac{x^3}{x^2+2x+1} \, dx$

**Ans:** $\dfrac{x^2}{2} - 2x + 3\ln|x+1| + \dfrac{1}{x+1} + C$

(d) $\int \dfrac{x^2}{(x-1)(x+1)^2} \, dx$

**Ans:** $\dfrac{1}{4}\ln|x-1| + \dfrac{3}{4}\ln|x+1| + \dfrac{1}{2(x+1)} + C$

(e) $\int \dfrac{1}{(x+1)(x^2+1)} \, dx$

**Ans:** $\dfrac{1}{2}\ln|x+1| - \dfrac{1}{4}\ln(x^2+1) + \dfrac{1}{2}\tan^{-1}x + C$

(f) $\int \dfrac{x^2}{x^4-1} \, dx$

**Ans:** $\dfrac{1}{4}\ln\left|\dfrac{x-1}{x+1}\right| + \dfrac{1}{2}\tan^{-1}x + C$

(g) $\displaystyle\int \frac{x^4}{x^2 - 4}\, dx$

**Ans:** $\displaystyle\frac{x^3}{3} + 4x + 4\ln\left|\frac{x-2}{x+2}\right| + C$

11. **(Integration by t-substitution)**

(a) Let $t = \tan\dfrac{x}{2}$, show that $\dfrac{dt}{dx} = \dfrac{1}{2}(1 + t^2)$.

**Ans:**

$$
\begin{aligned}
\frac{dt}{dx} &= \frac{1}{2}\sec^2\frac{x}{2} \\
&= \frac{1}{2}\left(1 + \tan^2\frac{x}{2}\right) \\
&= \frac{1}{2}(1 + t^2).
\end{aligned}
$$

(b) Express $\sin x$ and $\cos x$ in terms of $t$.

**Ans:** $\sin x = \dfrac{2t}{1 + t^2}$ and $\cos x = \dfrac{1 - t^2}{1 + t^2}$.

(c) By considering the substitution $t = \tan\dfrac{x}{2}$, evaluate the following integrals.

(i) $\displaystyle\int \frac{1}{2 + \sin x}\, dx$

**Ans:** $\dfrac{2}{\sqrt{3}}\tan^{-1}\left(\dfrac{2\tan\frac{x}{2} + 1}{\sqrt{3}}\right) + C$

(ii) $\displaystyle\int \frac{1}{3 - 2\cos x}\, dx$

**Ans:** $\dfrac{2}{\sqrt{5}}\tan^{-1}\left(\sqrt{5}\tan\dfrac{x}{2}\right) + C$

(iii) $\displaystyle\int \frac{1}{2 + \sin x + \cos x}\, dx$

**Ans:** $\sqrt{2}\tan^{-1}\left(\dfrac{\tan\frac{x}{2} + 1}{\sqrt{2}}\right) + C$

(iv) $\displaystyle\int \frac{1}{(2 + \cos x)\sin x}\, dx$

**Ans:** $\dfrac{1}{3}\ln\left|\tan\dfrac{x}{2}\left(\tan^2\dfrac{x}{2} + 3\right)\right| + C$