Calculus for Engineers

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Contents

Trigonometric limits and its derivatives

8.1 Introduction

In Section 8.2, a few basic trigonometric limits are given. The so-called squeeze theorem is presented in Section 8.3. Hence, an important limit form $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$ is easily obtained. Formulas for finding the derivative of the trigonometric functions are presented in Section 8.4. We assume that the trigonometric functions are functions of real numbers (angles measured in radians) since the trigonometric differentiation formulas rely on limit formulas that become more complicated and require more works if the degree measurement is used instead of the radian measure.

8.2 Basic trigonometric limits

The following two examples tell us that both sine and cosine functions are continuous at any point "a" in their domain.

Example 1 Prove that

$$\lim_{h \to 0} \sin(a + h) = \sin a,$$

where a is a fixed number.

Solution:

Using the trigonometric identity

$$\sin(a + h) = \sin a \cdot \cos h + \cos a \cdot \sin h,$$

we note that $\sin a$ and $\cos a$ are constants. Now,

$$\begin{aligned}
\lim_{h \to 0} \sin(a + h) &= \lim_{h \to 0} (\sin a \cdot \cos h + \cos a \cdot \sin h) \\
&= \sin a \left(\lim_{h \to 0} \cos h \right) + \cos a \left(\lim_{h \to 0} \sin h \right) \\
&= \sin a \cdot 1 + \cos a \cdot 0 \\
&= \sin a.
\end{aligned}$$

$\square$

Example 2 Prove that

$$\lim_{h\to 0} \cos(a+h) = \cos a,$$

where a is a fixed number.

Solution:

Using the trigonometric identity

$$\cos(a+h) = \cos a \cdot \cos h - \sin a \cdot \sin h,$$

we note that $\sin a$ and $\cos a$ are constants. Now,

$$\begin{aligned}
\lim_{h\to 0} \cos(a+h) &= \lim_{h\to 0} (\cos a \cdot \cos h - \sin a \cdot \sin h) \\
&= \cos a \left(\lim_{h\to 0} \cos h\right) - \sin a \left(\lim_{h\to 0} \sin h\right) \\
&= \cos a \cdot 1 + \sin a \cdot 0 \\
&= \cos a.
\end{aligned}$$

□

Example 3 Prove that

$$\lim_{x\to a} \tan x = \tan a,$$

where a is a fixed number and belongs to the domain of the tangent function.

Solution:

Using the limit rules, we have

$$\begin{aligned}
\lim_{x\to a} \tan x &= \lim_{x\to a} \frac{\sin x}{\cos x} \\
&= \frac{\lim_{x\to a} \sin x}{\lim_{x\to a} \cos x} \\
&= \frac{\sin a}{\cos a} \\
&= \tan a.
\end{aligned}$$

□

Note 1 We note that

$$\lim_{x\to 0} \tan x = 0.$$

8.3 The Squeeze theorem

As $x \to c$, the left and right parts of x approach c. The squeeze (or sandwich) theorem indicates that the middle part is forced to approach c, as well. The middle part is "squeezed" or "sandwiched" between the left and right parts, and it must approach the same limit that the other two are approaching.

Theorem 1 If the following conditions are satisfied:

1. $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c except possibly at c itself;

2. $$\lim_{x \to c} g(x) = A = \lim_{x \to c} h(x)$$

then

$$\lim_{x \to c} f(x) = A.$$

Proof:

Let $\epsilon > 0$ be given. Our aim is to find a $\delta > 0$ such that

$$|f(x) - A| < \epsilon \text{ whenever } 0 < |x - c| < \delta.$$

Since

$$\lim_{x \to c} g(x) = A,$$

by the definition of limits, there exists some $\delta_1 > 0$ such that

$$|g(x) - A| < \epsilon \text{ for all } 0 < |x - c| < \delta_1.$$

Thus,

$$-\epsilon < g(x) - A < \epsilon \text{ for all } 0 < |x - c| < \delta_1.$$

so

$$A - \epsilon < g(x) < A + \epsilon \text{ for all } 0 < |x - c| < \delta_1. \tag{8.1}$$

Similarly, since

$$\lim_{x \to c} h(x) = A,$$

by the definition of limits, there exists some $\delta_2 > 0$ such that

$$A - \epsilon < h(x) < A + \epsilon \text{ for all } 0 < |x - c| < \delta_2. \tag{8.2}$$

Additionally, since $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c, there exists some $\delta_3 > 0$ such that

$$g(x) \leq f(x) \leq h(x) \text{ for all } 0 < |x - c| < \delta_3. \tag{8.3}$$

Now, we choose $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Then by (8.1), (8.2) and (8.3), we have

$$A - \epsilon < g(x) \leq f(x) \leq h(x) < A + \epsilon \text{ for all } 0 < |x - c| < \delta.$$

Therefore, $-\epsilon < f(x) - A < \epsilon$ for all $0 < |x - c| < \delta$, so

$$|f(x) - A| < \epsilon \text{ for all } 0 < |x - c| < \delta_1.$$

Hence, by the definition of limits,

$$\lim_{x \to c} f(x) = A.$$

□

Example 4 Use the squeeze theorem to verify that

$$\lim_{x\to 0} x^2 \cos\left(\frac{1}{x}\right) = 0.$$

Solution:

For any real number θ, the values of the cosine function are bounded between -1 and 1, inclusive of the end points, that is, $-1 \leq \cos(\theta) \leq 1$. Letting $\theta = \dfrac{1}{x}$ for all $x \neq 0$, it follows that

$$-1 \leq \cos(\theta) \leq 1.$$

Hence, $\cos(\theta)$ is bounded.

Noting that $x^2 > 0$ for $x \neq 0$, each term in this inequality is multiplied by x^2 such that

$$-x^2 \leq x^2 \cos(\theta) \leq x^2.$$

These inequalities are illustrated in Figure 8.1. Because

$$\lim_{x\to 0} x^2 = \lim_{x\to 0} (-x)^2 = 0,$$

the squeeze theorem implies that

$$\lim_{x\to 0} x^2 \cos\left(\frac{1}{x}\right) = 0.$$

$\square$

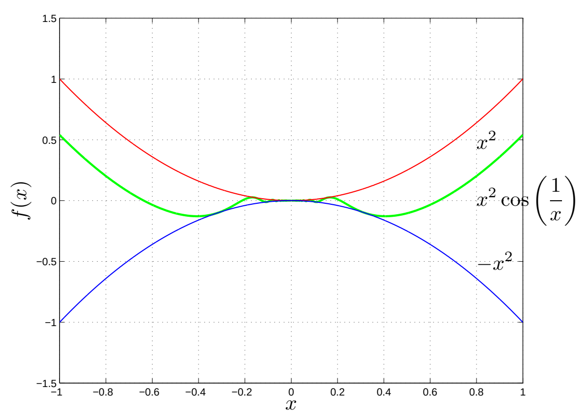


Figure 8.1: $f(x)$, where $x \in [-1, 1]$.

Example 5 Use the squeeze theorem to verify that

$$\lim_{x\to 0} x^3 \cos\left(\frac{1}{\sqrt[3]{x}}\right) = 0.$$

Solution:

Let us analyze the right-hand and left-hand limits of $x^3 \cos\left(\dfrac{1}{\sqrt[3]{x}}\right)$, separately.

1. By assuming $x > 0$, we consider $x \to 0^+$. Now, we write

$$\lim_{x \to 0^+} x^3 \cos\left(\frac{1}{\sqrt[3]{x}}\right).$$

For any real number θ, the values of the sine function are bounded between -1 and 1, inclusive of the end points, that is, $-1 \leq \sin(\theta) \leq 1$. Letting $\theta = \dfrac{1}{\sqrt[3]{x}}$ for all $x > 0$, it follows that

$$-1 \leq \sin(\theta) \leq 1.$$

Hence, $\sin(\theta)$ is bounded.

Noting that $x^3 > 0$ for $x > 0$, each term in this inequality is multiplied by x^3 such that

$$-x^3 \leq x^3 \sin(\theta) \leq x^3.$$

These inequalities are illustrated in Figure 8.2. Because

$$\lim_{x \to 0^+} x^3 = \lim_{x \to 0^+} (-x)^3 = 0,$$

the squeeze theorem implies that

$$\lim_{x \to 0^+} x^3 \sin\left(\frac{1}{\sqrt[3]{x}}\right) = 0.$$

2. By assuming $x < 0$, we consider $x \to 0^-$. Now, we write

$$\lim_{x \to 0^-} x^3 \cos\left(\frac{1}{\sqrt[3]{x}}\right).$$

For any real number θ, the values of the sine function are bounded between -1 and 1, inclusive of the end points, that is, $-1 \leq \sin(\theta) \leq 1$. Letting $\theta = \dfrac{1}{\sqrt[3]{x}}$ for all $x < 0$, it follows that

$$-1 \leq \sin(\theta) \leq 1.$$

Hence, $\sin(\theta)$ is bounded.

Noting that $x^3 < 0$ for $x < 0$, each term in this inequality is multiplied by x^3. Therefore, we must reverse the $\leq$ inequality symbols

$$-x^3 \geq x^3 \sin(\theta) \geq x^3.$$

Reversing the compound inequality will make it easier to read:

$$x^3 \leq x^3 \sin(\theta) \leq -x^3.$$

These inequalities are illustrated in Figure 8.2. Because

$$\lim_{x \to 0^-} x^3 = \lim_{x \to 0^-} (-x)^3 = 0,$$

the squeeze theorem implies that

$$\lim_{x \to 0^-} x^3 \sin\left(\frac{1}{\sqrt[3]{x}}\right) = 0.$$

Combining these two results, $\lim_{x \to 0^+} x^3 \sin\left(\frac{1}{\sqrt[3]{x}}\right) = 0$ and $\lim_{x \to 0^-} x^3 \sin\left(\frac{1}{\sqrt[3]{x}}\right) = 0,$, so we have

$$\lim_{x \to 0} x^3 \sin\left(\frac{1}{\sqrt[3]{x}}\right) = 0.$$

□

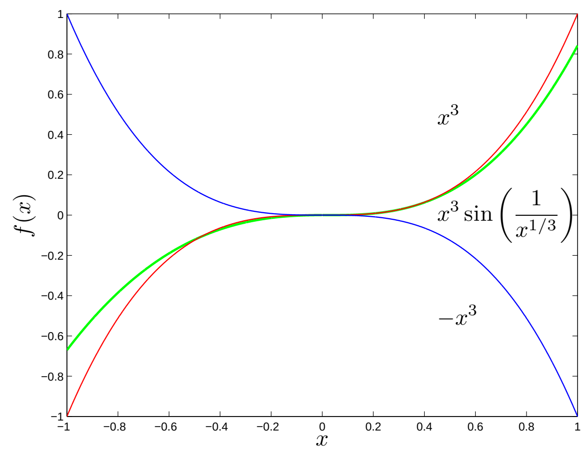


Figure 8.2: $f(x)$, where $x \in [-1, 1]$.

8.3.1 $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$

To investigate limits of trigonometric functions, we start with Figure 8.3, which shows an angle θ with its vertex at the origin, its initial side along the positive x-axis, and its terminal side intersecting the unit circle at the point P.

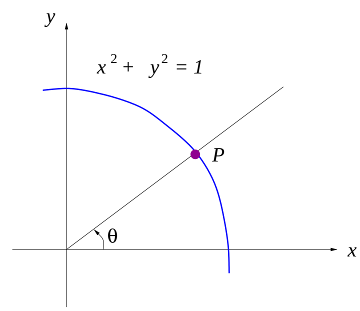


Figure 8.3: An angle θ.

- By the definition of the sine and cosine functions, the coordinates of P are $P(\cos\theta, \sin\theta)$.
-
  - In Figure 8.3, we observe that, as $\theta \to 0$, the point $P(\cos\theta, \sin\theta)$ approaches the point $R(1, 0)$. Hence $\cos\theta \to 1$ and $\sin\theta \to 0$ as $\theta \to 0$ through positive values.
  - The results for negative values of θ are similar, and we see from Examples 1 and 3 that

$$\lim_{\theta \to 0} \cos\theta = 1 \quad \text{and} \quad \lim_{\theta \to 0} \sin\theta = 0 \tag{8.4}$$

- (8.4) says that the limits of the functions $\cos\theta$ and $\sin\theta$ as $\theta \to 0$ are equal to their respective values at $\theta = 0$ such that

$$\cos 0 = 1 \quad \text{and} \quad \sin 0 = 0.$$

-
  - In calculus, the limit of the quotient $\frac{\sin\theta}{\theta}$ as $\theta \to 0$ plays a special role with regard to trigonometric functions.
  - For example, it is needed to find the slopes of lines tangent to trigonometric graphs such as $y = \cos x$ and $y = \sin x$.
-
  - Note that the value of the quotient $\frac{\sin\theta}{\theta}$ is not defined when $\theta = 0$ (why not?).
  - But a calculator set in radian mode provides us the numerical evidence shown in Table 8.1.
  - This table strongly suggests that the limit of $\frac{\sin\theta}{\theta}$ is 1 as $\theta \to 0$.

θ	$\dfrac{\sin\theta}{\theta}$
± 1.0	0.84147
± 0.5	0.95885
± 0.1	0.99833
± 0.05	0.99958
± 0.01	0.99998
± 0.005	1.00000
± 0.001	1.00000
$\vdots$	$\vdots$
$\downarrow$	$\downarrow$
0	1

Table 8.1: The numerical data suggest that $\lim_{\theta\to 0} \dfrac{\sin\theta}{\theta} = 1$.

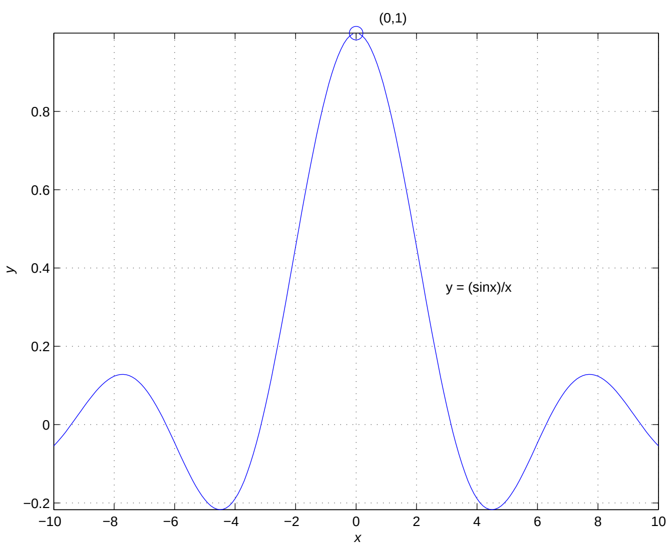


Figure 8.4: $y = \frac{\sin x}{x}$ for $x \neq 0$.

- This conclusion is supported by the graph of $y = \dfrac{\sin x}{x}$ shown in Figure 8.4, where it appears that the point (x, y) on the curve is near $(0, 1)$ when x is near zero.
- We provide a proof, see Theorem 2.

The following example constitutes a warning:

The results of a numerical investigation can be misleading unless they are interpreted with care.

Example 6 The numerical data shown in Table 8.2 suggest that the limit

$$\lim_{x\to 0} \sin\frac{\pi}{x} \tag{8.5}$$

has a value of <u>zero</u>.

x	$\sin \frac{\pi}{x}$
1.0	0
0.5	0
0.1	0
0.05	0
0.01	0
0.005	0
0.001	0

Table 8.2: Do you think that $\lim_{x \to 0} \sin \frac{\pi}{x} = 0$?

- But it appears in the graph of $y = \sin \frac{\pi}{x}$ (for $x \neq 0$), shown in Figure 8.5, that the value of $\sin \frac{\pi}{x}$ oscillates infinitely often between $+1$ and -1 as $x \to 0$.
- Indeed, this fact follows from the periodicity of the sine function, because $\frac{\pi}{x}$ increases without bound as $x \to 0$.
- Hence $\sin \frac{\pi}{x}$ can not approach zero (or any other number) as $x \to 0$.
- Therefore the limit in (8.5) does not exist.

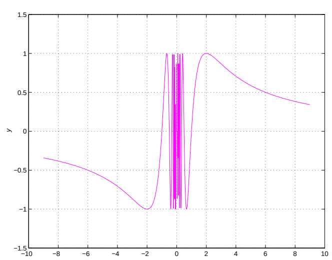


Figure 8.5: The graph of $y = \sin \frac{\pi}{x}$ shows infinite oscillation as $x \to 0$.

- We can explain the potentially misleading results tabulated in Table 8.2 as follows: Each value of x shown there just happens to be of the form $\frac{1}{n}$, the reciprocal of an integer. Therefore,
$$\sin \frac{\pi}{x} = \sin \frac{\pi}{\frac{1}{n}} = \sin n\pi = 0$$
for every nonzero integer n.
- But with a different selection of "trial values" of x, we might have obtained the results shown in Table 8.3, which immediately suggest the nonexistence of the limit in (8.5).

Theorem 2
$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1.$$

x	$\sin\frac{\pi}{x}$
$\frac{2}{9}$	$+1$
$\frac{2}{11}$	-1
$\frac{2}{101}$	$+1$
$\frac{2}{103}$	-1
$\frac{2}{1001}$	$+1$
$\frac{2}{1003}$	-1

Table 8.3: Verify the entries in the second column.

Proof:

- First suppose $0 < \theta < \frac{\pi}{2}$.
- Figure 8.6 shows a section (or sector) with center O, central angle θ and radius 1.

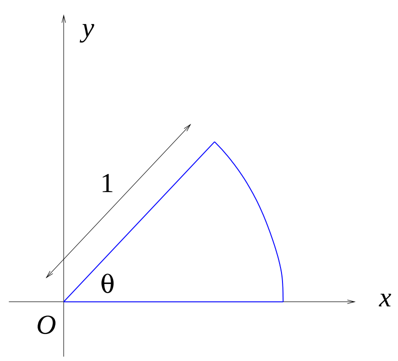


Figure 8.6: A sector.

- From Figure 8.7, we have

$$MP = \sin\theta \quad \text{and} \quad AQ = \tan\theta.$$

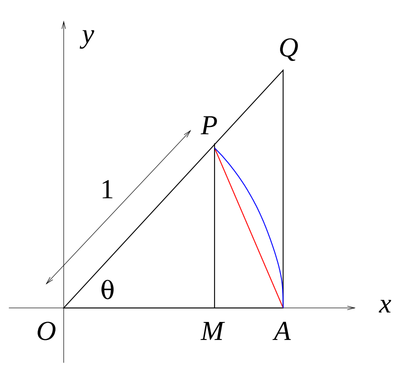


Figure 8.7:

- In Figure 8.7, we note that

$$\text{Area of } \Delta AOP < \text{Area of sector } AOP < \text{Area of } \Delta AOQ.$$

- In Figure 8.8, by geometry and trigonometry, we have

$$\text{Area of } \Delta AOP = \frac{1}{2}(1)(MP) = \frac{1}{2}\sin\theta,$$
$$\text{Area of section } AOP = \frac{1}{2}(1)^2\theta = \frac{1}{2}\theta,$$
$$\text{Area of } \Delta AOQ = \frac{1}{2}(1)(AQ) = \frac{1}{2}\tan\theta.$$

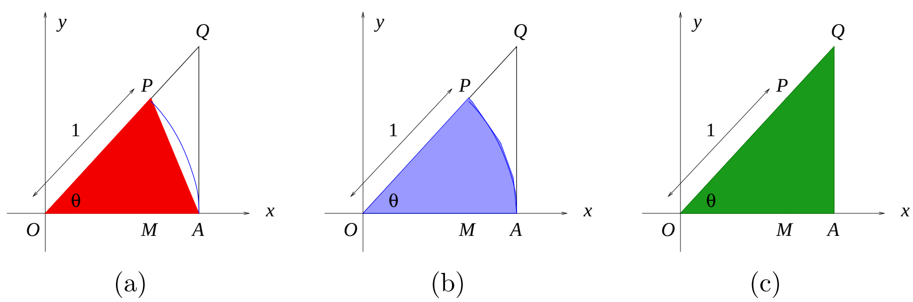


Figure 8.8:

- The preceding inequality may be written as

$$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta.$$

- Since $0 < \theta < \frac{\pi}{2}$, $\sin\theta > 0$.
- Dividing each term of the inequality by $\frac{1}{2}\sin\theta$, we obtain the following equivalent inequalities:

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta} \qquad \text{or} \qquad \cos\theta < \frac{\sin\theta}{\theta} < 1.$$

- Up to now, the variable θ is in the interval $(0, \frac{\pi}{2})$. If we replace θ by $(-\theta)$, then

$$\cos(-\theta) < \frac{\sin(-\theta)}{-\theta} = \frac{-\sin(\theta)}{-\theta} = \frac{\sin(\theta)}{\theta} < 1.$$

That is,

$$\cos\theta < \frac{\sin\theta}{\theta} < 1.$$

is valid for all $\theta \in (-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2})$.

- The inequality appears to have the same form as the Squeeze theorem. Applying the squeeze theorem together with the fact that

$$\lim_{\theta\to 0}\cos\theta = \cos 0 = 1,$$

we get

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1.$$

□

Note 2 We stress that the result

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1.$$

holds only if the angle θ in $\sin \theta$ is expressed in radians.

Example 7 Evaluate

$$\lim_{\theta \to 0} \frac{\sin \theta^\circ}{\theta},$$

where $\theta^\circ = \dfrac{\pi\theta}{180}$ radians.

Solution:

Firstly, we note that the degree measure of an angle is a linear function of the radian measure x. Then, we replace θ° by the number $\dfrac{\pi\theta}{180}$ and adjust the denominator suitably so that we can apply the result

$$\lim_{h \to 0} \frac{\sin h}{h} = 1.$$

Now,

$$\begin{aligned}
\lim_{\theta \to 0} \frac{\sin \theta^\circ}{\theta} &= \lim_{\theta \to 0} \frac{\sin\left(\frac{\pi\theta}{180}\right)}{\theta} \\
&= \lim_{\theta \to 0} \frac{\sin\left(\frac{\pi\theta}{180}\right)}{\theta} \cdot \frac{\left(\frac{\pi}{180}\right)}{\left(\frac{\pi}{180}\right)} \\
&= \frac{\pi}{180} \cdot \lim_{\theta \to 0} \frac{\sin\left(\frac{\pi\theta}{180}\right)}{\frac{\pi\theta}{180}} && \text{Because } x \to 0,\ \frac{\pi\theta}{180} \to 0 \\
&= \frac{\pi}{180} \cdot 1 \\
&= \frac{\pi}{180} \neq 1.
\end{aligned}$$

Example 8 Show that

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0. \tag{8.6}$$

Solution:

We multiply the numerator and denominator in (8.6) by the "conjugate" $1 + \cos x$ of the numerator $1 - \cos x$. Then we apply the identity $1 - \cos^2 x = \sin^2 x$.

This gives

$$\begin{aligned}\lim_{x\to 0}\frac{1-\cos x}{x} &= \lim_{x\to 0}\frac{1-\cos x}{x}\cdot\left(\frac{1+\cos x}{1+\cos x}\right)\\ &= \lim_{x\to 0}\frac{\sin^2 x}{x(1+\cos x)}\\ &= \left(\lim_{x\to 0}\frac{\sin x}{x}\right)\cdot\left(\lim_{x\to 0}\frac{\sin x}{1+\cos x}\right)\\ &= 1\cdot\frac{0}{1+1}=0.\end{aligned}$$

□

Example 9 Find

$$\lim_{x\to 0}\frac{\sin 5x}{3x}.$$

Solution: In order to apply Theorem 2, we first rewrite the function as follows:

$$\frac{\sin 5x}{3x}=\frac{5}{3}\cdot\frac{\sin 5x}{5x}$$

We note that as $x\to 0$, we have $5x\to 0$, and so, by Theorem 2 with $\theta = 5x$,

$$\lim_{x\to 0}\frac{\sin 5x}{5x}=1$$

Therefore

$$\lim_{x\to 0}\frac{\sin 5x}{3x}=\lim_{x\to 0}\frac{5}{3}\cdot\frac{\sin 5x}{5x}=\frac{5}{3}\lim_{x\to 0}\frac{\sin 5x}{5x}=\frac{5}{3}.$$

□

8.4 Derivatives of trigonometric functions

Example 10 Use the definition of the derivative to show that

$$\frac{d}{dx}\sin x=\cos x. \tag{8.7}$$

Solution: Let $f(x)=\sin x$. Then

$$\begin{aligned}\frac{d}{dx}\sin x &= \lim_{\Delta x\to 0}\frac{\sin(x+\Delta x)-\sin x}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{(\sin x\cdot\cos\Delta x+\cos x\cdot\sin\Delta x)-\sin x}{\Delta x}\\ &= \lim_{\Delta x\to 0}\left[\frac{\sin x(\cos\Delta x-1)}{\Delta x}+\frac{\cos x\sin\Delta x}{\Delta x}\right]\\ &= \sin x\cdot\lim_{\Delta x\to 0}\left(\frac{\cos\Delta x-1}{\Delta x}\right)+\cos x\cdot\lim_{\Delta x\to 0}\left(\frac{\sin\Delta x}{\Delta x}\right) \quad \text{Use Example 8.6 and Theorem 2}\\ &= -\sin x\cdot 0+\cos x\cdot 1\\ &= \cos x,\quad \forall x\in\mathbb{R}.\end{aligned}$$

$\square$

Example 11 Use the definition of the derivative to show that

$$\frac{d}{dx}\cos x = -\sin x. \tag{8.8}$$

Solution: Let $f(x) = \cos x$. Then

$$\begin{aligned}
\frac{d}{dx}\cos x &= \lim_{\Delta x \to 0} \frac{\cos(x+\Delta x) - \cos x}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(\cos x \cdot \cos \Delta x - \sin x \cdot \sin \Delta x) - \sin x}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left[\frac{\cos x(\cos \Delta x - 1)}{\Delta x} - \frac{\sin x \sin \Delta x}{\Delta x} \right] \\
&= \cos x \cdot \lim_{\Delta x \to 0} \left(\frac{\cos \Delta x - 1}{\Delta x} \right) - \sin x \cdot \lim_{\Delta x \to 0} \left(\frac{\sin \Delta x}{\Delta x} \right) \quad \text{Use Example 8.6 and Theorem 2} \\
&= -\sin x \cdot 0 - \sin x \cdot 1 \\
&= -\sin x, \quad \forall x \in \mathbb{R}.
\end{aligned}$$

$\square$

Example 12 Use the definition of the derivative to show that

$$\frac{d}{dx}\tan x = \sec^2 x. \tag{8.9}$$

Solution: Let $f(x) = \tan x$. Then

$$\begin{aligned}
\frac{d}{dx}\tan x &= \lim_{\Delta x \to 0} \frac{\tan(x+\Delta x) - \tan x}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \left(\frac{\sin(x+\Delta x)}{\cos(x+\Delta x)} - \frac{\sin(x)}{\cos(x)} \right) \\
&= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \left(\frac{\sin(x+\Delta x)\cos x - \cos(x+\Delta x)\sin x}{\cos(x+\Delta x)\cos x} \right) \\
&= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \left(\frac{\sin(x+\Delta x - x)}{\cos(x+\Delta x)\cos x} \right) \\
&= \lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x} \left(\frac{1}{\cos(x+\Delta x)\cos x} \right) \\
&= \lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x} \cdot \lim_{\Delta x \to 0} \left(\frac{1}{\cos(x+\Delta x)\cos x} \right) \\
&= 1 \cdot \frac{1}{\cos x \cos x} = \frac{1}{\cos^2 x} = \sec^2 x.
\end{aligned}$$

$\square$

Let us use formal rules of differentiation, trigonometric identities and derivatives of $\sin x$ and $\cos x$ to derive the following results. We call it *a simpler alternative method*.

Example 13 Use a simpler alternative method to show that

$$\frac{d}{dx}\cos x = -\sin x. \tag{8.10}$$

Solution:

First, we recognize that

$$\cos x = \sin\left(x + \frac{\pi}{2}\right).$$

Now,

$$\begin{aligned}
\frac{d}{dx}\cos x &= \frac{d}{dx}\sin\left(x + \frac{\pi}{2}\right) && \text{Use Example 8.7} \\
&= \cos\left(x + \frac{\pi}{2}\right) && \text{Because } \cos\left(x + \frac{\pi}{2}\right) = -\sin x \\
&= -\sin x.
\end{aligned}$$

□

The derivatives of trigonometric functions, like $\tan x, \cot x, \sec x$, and $\text{cosec}x$ can be obtained by using the quotient rule for differentiation along with $\sin x$ and $\cos x$. Here are a few examples:

Example 14 Use the formula $\tan x = \dfrac{\sin x}{\cos x}$ to find the derivative of the tangent function:

$$\frac{d}{dx}\tan x = \sec^2 x. \tag{8.11}$$

Solution:

$$\begin{aligned}
\frac{d}{dx}\tan x &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \\
&= \frac{\cos x \cdot \frac{d}{dx}(\sin x) - \sin x \cdot \frac{d}{dx}(\cos x)}{\cos^2 x} \\
&= \frac{\cos x \cdot \cos x - \sin x \cdot - \sin x}{\cos^2 x} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} = \sec^2 x.
\end{aligned}$$

□

Example 15 Use the formula $\cot x = \dfrac{\cos x}{\sin x}$ to find the derivative of the cotangent function:

$$\frac{d}{dx}\cot x = -\mathrm{cosec}^2 x. \tag{8.12}$$

Solution:

$$\begin{aligned}
\frac{d}{dx}\cot x &= \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) \\
&= \frac{\sin x \cdot \dfrac{d}{dx}(\cos x) - \cos x \cdot \dfrac{d}{dx}(\sin x)}{\sin^2 x} \\
&= \frac{\sin x \cdot - \sin x - \cos x \cdot \cos x}{\sin^2 x} \\
&= -\frac{\sin^2 x + \cos^2 x}{\sin^2 x} \\
&= -\frac{1}{\sin^2 x} = -\mathrm{cosec}^2 x.
\end{aligned}$$

□

Before finding the derivative of $\sec x$ and $\mathrm{cosec} x$, we need the formula (8.13).

Example 16 Find the derivative of the following function:

$$\frac{d}{dx}\left(\frac{1}{g(x)}\right) = -\frac{\frac{d}{dx}g(x)}{(g(x))^2}, \quad g(x) \neq 0, \tag{8.13}$$

where x is in the domain of $g(x)$.

Solution:

$$\begin{aligned}
\frac{d}{dx}\left(\frac{1}{g(x)}\right) &= \frac{g(x) \cdot \dfrac{d}{dx}(1) - 1 \cdot \dfrac{d}{dx}g(x)}{(g(x))^2} \\
&= \frac{-\dfrac{d}{dx}g(x)}{(g(x))^2}.
\end{aligned}$$

Example 17 Use the formula $\sec x = \dfrac{1}{\cos x}$ to find the derivative of the secant function:

$$\frac{d}{dx}\sec x = \sec x \tan x. \tag{8.14}$$

Solution:

$$\begin{aligned}\frac{d}{dx}\sec x &= \frac{d}{dx}\left(\frac{1}{\cos x}\right)\\ &= \frac{-\frac{d}{dx}\cos x}{(\cos x)^2}\\ &= \frac{\sin x}{\cos^2 x} = \frac{\sin x}{\cos x}\frac{1}{\cos x} = \sec x \tan x.\end{aligned}$$

□

Example 18 Use the formula $\mathrm{cosec} x = \dfrac{1}{\sin x}$ to find the derivative of the cosecant function:

$$\frac{d}{dx}\mathrm{cosec} x = -\mathrm{cosec} x \cot x. \tag{8.15}$$

Solution:

$$\begin{aligned}\frac{d}{dx}\mathrm{cosec} x &= \frac{d}{dx}\left(\frac{1}{\sin x}\right)\\ &= \frac{-\frac{d}{dx}\sin x}{(\sin x)^2}\\ &= \frac{-\cos x}{\sin^2 x} = -\frac{\cos x}{\sin x}\frac{1}{\sin x} = -\mathrm{cosec} x \cot x.\end{aligned}$$

□

Example 19 Find the derivative of $\sin u(x)$, where $u(x)$ is a differentiable function of x.

Solution: Applying the chain rule gives

$$\begin{aligned}\frac{d}{dx}\sin u(x) &= \frac{d\sin u(x)}{du(x)}\frac{du(x)}{dx}\\ &= \cos u(x)\cdot\frac{du(x)}{dx}.\end{aligned}$$

□

Example 20 Find the derivative of $\sin x^5$.

Solution: Applying the chain rule and the power rule give

$$\begin{aligned}\frac{d}{dx}\sin x^5 &= \frac{d\sin x^5}{dx^5}\frac{dx^5}{dx}\\ &= \cos x^5\cdot 5x^4 = 5x^4\cdot\cos x^5.\end{aligned}$$

□

Example 21 Find the derivative of $\cos\sqrt{x}$, where $x > 0$.

Solution: Applying the chain rule and the power rule give

$$\begin{aligned}\frac{d}{dx}\cos\sqrt{x} &= \frac{d\cos\sqrt{x}}{d\sqrt{x}}\frac{d\sqrt{x}}{dx}\\ &= -\sin\sqrt{x}\cdot\frac{1}{2}x^{-1/2} = -\frac{\sin\sqrt{x}}{2\sqrt{x}}.\end{aligned}$$

□

Example 22 Find the derivative of $\cos x^\circ$, where $x^\circ = \dfrac{\pi x}{180}$ radians.

Solution: Applying the chain rule gives

$$\begin{aligned}\frac{d}{dx}\cos x^\circ &= \frac{d}{dx}\cos\left(\frac{\pi x}{180}\right)\\ &= \frac{d\cos\left(\frac{\pi x}{180}\right)}{d\left(\frac{\pi x}{180}\right)}\frac{d\left(\frac{\pi x}{180}\right)}{dx}\\ &= -\frac{\pi}{180}\cdot\sin\left(\frac{\pi x}{180}\right).\end{aligned}$$

□

Example 23 Find the derivative of $y = \dfrac{\sin x + \cos x}{\sin x - \cos x}$.

Solution: Applying the quotient rule, the derivative rules for sine and cosine, and a few trigonometric identities give

$$\frac{dy}{dx} = \frac{-2}{1-\sin 2x}. \qquad \text{Exercise !}$$

□

Figures 8.9 and 8.10 show graphs of derivatives of trigonometric functions. We note that since the trigonometric functions are differentiable functions on their domains they are also continuous functions on their domains.

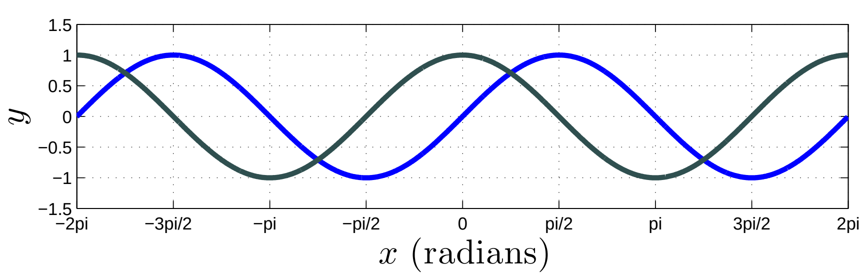
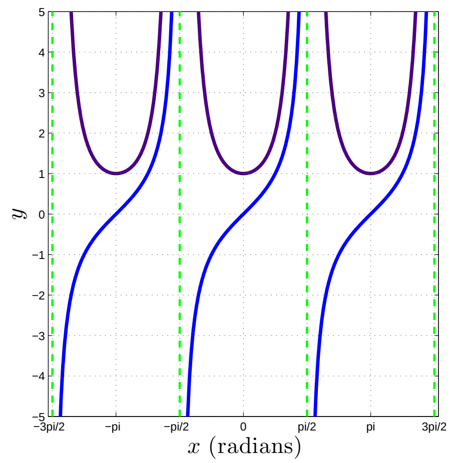
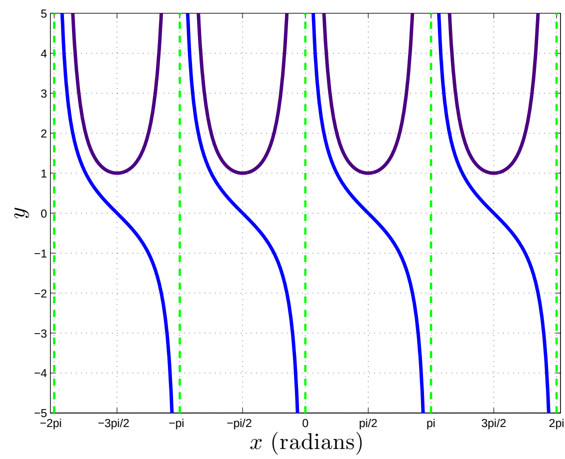


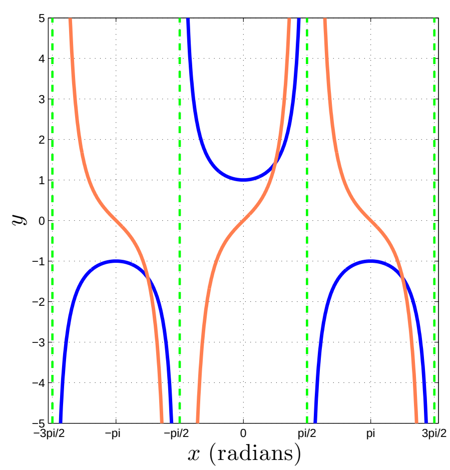
Figure 8.9: The graphs of $y = \sin x$ (blue) and $y = \cos x$ (dark green), where $x \in [-2\pi, 2\pi]$.



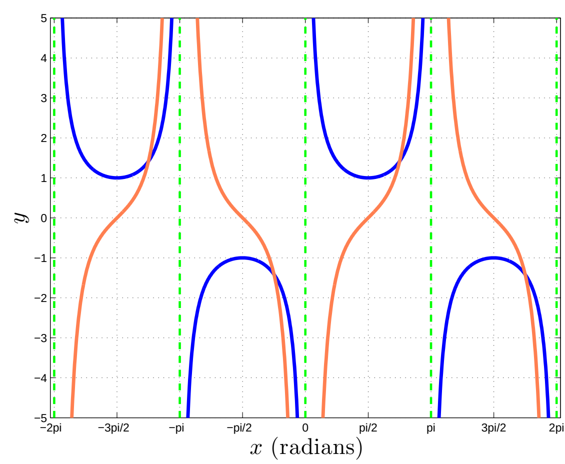
(a) graphs of $y = \tan x$ (blue) and $y = \sec^2 x$ (indigo).



(b) graphs of $y = \cot x$ (blue) and $y = -\text{cosec}^2 x$ (indigo).



(c) graphs of $y = \sec x$ (blue) and $y = \sec x \tan x$ (orange).



(d) graphs of $y = \text{cosec} x$ (blue) and $y = -\text{cosec} x \cot x$ (orange).

Figure 8.10: Graphs of derivatives of trigonometric functions.