THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1510 Calculus for Engineers (Fall 2021)
Suggested solutions of coursework 6

Part A

1. Let $f(x) = \sqrt[3]{x}$.

   (a) Find the equation of the tangent of $f(x)$ at $x = 1000$. Express your answer in form of $y = mx + c$.

   (b) Using the fact that
   $$y = L(x) = mx + c$$
   is close to $f(x)$ around the point $x = 1000$, give an approximation of $\sqrt[3]{999}$.

Solution:

(a) The equation of the tangent of $f(x)$ at $x = 1000$ is

$$\begin{aligned} y &= f(1000) + f'(1000)(x - 1000) \\ &= \sqrt[3]{1000} + \frac{1}{3}(1000)^{-\frac{2}{3}}(x - 1000) \\ &= 10 + \frac{1}{300}(x - 1000) \\ &= \frac{1}{300}x + \frac{20}{3}. \end{aligned}$$

(b)

$$\begin{aligned} \sqrt[3]{999} = f(999) &\approx L(999) \\ &= \frac{1}{300}(999) + \frac{20}{3} \\ &= \frac{2999}{300}. \end{aligned}$$

2. (a) Let $0 < a < b$. Show that

$$\frac{b-a}{b} < \ln b - \ln a < \frac{b-a}{a}.$$

(b) Using the result obtained in part (a), show that

$$\frac{1}{21} < \ln 1.05 < \frac{1}{20}.$$

Solution:

(a) Let $f(x) = \ln x$, which is differentiable over $(0, \infty)$.
By Lagrange Mean Value Theorem on f over $[a, b]$,

$$\begin{aligned} \frac{f(b) - f(a)}{b-a} &= f'(c) \qquad \text{for some } c \in (a,b) \\ \frac{\ln b - \ln a}{b-a} &= \frac{1}{c}. \end{aligned}$$

Since $\dfrac{1}{x}$ is a strictly decreasing function over $(0, \infty)$,

$$\begin{aligned} &\frac{1}{b} < \frac{1}{c} < \frac{1}{a} \\ \Longrightarrow\quad &\frac{1}{b} < \frac{\ln b - \ln a}{b-a} < \frac{1}{a} \\ \Longrightarrow\quad &\frac{b-a}{b} < \ln b - \ln a < \frac{b-a}{a}. \end{aligned}$$

(b) Using $a = 1$, $b = 1.05$,in part (a),

$$\begin{aligned} \frac{1.05-1}{1.05} &< \ln 1.05 - \ln 1 < \frac{1.05-1}{1} \\ \frac{1}{21} = \frac{0.05}{1.05} &< \ln 1.05 < 0.05 = \frac{1}{20}. \end{aligned}$$

Part B

3. (a) Let $0 \le a < b$. Show that

$$\frac{1}{1+b^2} < \frac{\tan^{-1} b - \tan^{-1} a}{b-a} < \frac{1}{1+a^2}.$$

(b) Using the result obtained in part (a), show that $2 < \pi < 4$.

Solution:

(a) Let $f(x) = \tan^{-1} x$, which is differentiable over $\mathbb{R}$.
By Lagrange Mean Value Theorem on f over $[a, b]$,

$$\begin{aligned} \frac{f(b)-f(a)}{b-a} &= f'(c) \qquad \text{for some } c \in (a,b) \\ \frac{\tan^{-1} b - \tan^{-1} a}{b-a} &= \frac{1}{1+c^2}. \end{aligned}$$

Note that

$$\begin{aligned} & 0 \le a < c < b \\ \Longrightarrow\quad & 1+a^2 < 1+c^2 < 1+b^2 \\ \Longrightarrow\quad & \frac{1}{1+a^2} > \frac{1}{1+c^2} > \frac{1}{1+b^2}. \end{aligned}$$

Thus,

$$\frac{1}{1+b^2} < \frac{\tan^{-1} b - \tan^{-1} a}{b-a} < \frac{1}{1+a^2}.$$

(b) Using $a = 0$, $b = 1$ in part (a),

$$\begin{aligned} \Longrightarrow\quad & \frac{1}{1+1^2} < \frac{\tan^{-1} 1 - \tan^{-1} 0}{1-0} < \frac{1}{1+0^2} \\ \Longrightarrow\quad & \frac{1}{2} < \frac{\pi}{4} < 1 \\ \Longrightarrow\quad & 2 < \pi < 4. \end{aligned}$$

4. Let $f(x) = \dfrac{1}{\sqrt{4x - x^2}}$.

 (a) Prove that
$$(4x - x^2)f'(x) = (x - 2)f(x)$$

 (b) Prove that for any positive integer n,
$$(4x - x^2)f^{(n+1)}(x) = (2n + 1)(x - 2)f^{(n)}(x) + n^2 f^{(n-1)}(x),$$
 where $f^{(0)}(x) = f(x)$.

Solution:

(a)
$$\begin{aligned}
\text{LHS} &= (4x - x^2)f'(x) \\
&= (4x - x^2)\left(-\frac{1}{2}\right)(4x - x^2)^{-\frac{3}{2}}(4 - 2x) \\
&= (x - 2)(4x - x^2)^{-\frac{1}{2}} = \text{RHS}.
\end{aligned}$$

(b) When $n = 1$,
$$\begin{aligned}
\frac{d}{dx}\left((4x - x^2)f'(x)\right) &= \frac{d}{dx}\left((x - 2)f(x)\right) \\
(4x - x^2)f''(x) + (4 - 2x)f'(x) &= (x - 2)f'(x) + f(x) \\
(4x - x^2)f''(x) &= 3(x - 2)f'(x) + f(x).
\end{aligned}$$

For any $n \geq 2$,
$$\begin{aligned}
&\frac{d^n}{dx^n}\left((4x - x^2)f'(x)\right) = \frac{d^n}{dx^n}\left((x - 2)f(x)\right) \\
&(4x - x^2)f^{(n+1)}(x) + C^n_1(4 - 2x)f^{(n)}(x) + C^n_2(-2)f^{(n-1)}(x) \\
&\qquad = (x - 2)f^{(n)}(x) + C^n_1 f^{(n-1)}(x) \\
&(4x - x^2)f^{(n+1)}(x) = [n(2x - 4) + (x - 2)]f^{(n)}(x) + [n(n - 1) + n]f^{(n-1)}(x) \\
&\qquad = (2n + 1)(x - 2)f^{(n)}(x) + n^2 f^{(n-1)}(x).
\end{aligned}$$