THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1510 Calculus for Engineers (2020-2021)
Solution to Supplementary Exercise 6

Extrema, Inflection Points and Graphing

1. (a) Factor theorem states that if $P(x)$ is a polynomial and $P(a) = 0$, then $x - a$ is a factor of $P(x)$.
   By using factor theorem, factorize the following polynomials.
   (i) $x^3 + 2x^2 - 5x - 6$
   Ans: $(x+1)(x-2)(x+3)$
   (ii) $2x^3 - 3x^2 + 1$
   Ans: $(x-1)^2(2x+1)$
   (iii) $3x^3 - x^2 - x - 1$
   Ans: $(x-1)(3x^2 + 2x + 1)$

   (b) State the domain of the function $f(x) = \dfrac{1}{x^3 + 2x^2 - 5x - 6}$.
   Ans: The function $f(x)$ is undefined when the denominator equals to 0, i.e. $x^3 + 2x^2 - 5x - 6 = 0$. By (a)(i), if $(x+1)(x-2)(x+3) = 0$, then $x = -3, -1$ or 2. Therefore, the domain of $f(x)$ is $\mathbb{R}\backslash\{-3, -1, 2\}$.

2. Let $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 3$.

   (a) Find $f'(x)$. By using the factor theorem or otherwise, show that $f'(x) = 4(x-1)(x-2)(x-3)$.
   Ans: $f'(x) = 4x^3 - 24x^2 + 44x - 24$.
   Note that $f'(1) = f'(2) = f'(3) = 0$, so by the factor theorem, $x - 1$, $x - 2$ and $x - 3$ are factors of $f'(x)$ and $f(x) = A(x-1)(x-2)(x-3)$ for some real number A.
   Furthermore, the coefficient of x^3 of $f'(x)$ is 4, so $A = 4$ and $f'(x) = 4(x-1)(x-2)(x-3)$.

   (b) In the following table, fill in the signs of the factors in the corresponding intervals.
   Ans:

	$x<1$	$x=1$	$1<x<2$	$x=2$	$2<x<3$	$x=3$	$x>3$
$x-1$	$-$	0	$+$	$+$	$+$	$+$	$+$
$x-2$	$-$	$-$	$-$	0	$+$	$+$	$+$
$x-3$	$-$	$-$	$-$	$-$	$-$	0	$+$
$f'(x)$	$-$	0	$+$	0	$-$	0	$+$

   (c) Solve $f'(x) > 0$ and $f'(x) < 0$.
   Hence, find the extreme points of the graph $y = f(x)$.
   Ans: $f'(x) > 0$ when $1 < x < 2$ or $x > 3$; $f'(x) < 0$ when $x < 1$ or $2 < x < 3$.
   Therefore, $(1, f(1)) = (1, -6)$ and $(3, f(3)) = (3, -6)$ are minimum points and $(2, f(2)) = (2, -5)$ is a maximum point.

3. Let $f(x) = x^2 \ln x$ for $x > 0$.

   Find $f'(x)$ and $f''(x)$. Hence, determine the extreme point(s) of the function.

   Ans: $f'(x) = 2x \ln x + x$ and $f''(x) = 2 \ln x + 3$.

   $f'(x) = 0$ when $x = \dfrac{1}{\sqrt{e}}$.

   Then $f''(\dfrac{1}{\sqrt{e}}) = 2 > 0$.

   Therefore, $(\dfrac{1}{\sqrt{e}}, f(\dfrac{1}{\sqrt{e}})) = (\dfrac{1}{\sqrt{e}}, -\dfrac{1}{2e})$ is a minimum point.

4. Find the greatest and least values of the following functions on the given closed interval:

   (a) $f(x) = x - 2\sqrt{x}$ on $[0, 9]$;

   Ans: $f'(x) = 1 - \dfrac{1}{\sqrt{x}}$ and so $f'(x) = 0$ when $x = 1$.

   Note that $f'(x) < 0$ when $0 < x < 1$ and $f'(x) > 0$ when $1 < x < 9$. Therefore, $f(x)$ attains absolute minimum at $x = 1$ and $f(1) = -1$

   For the boundary points of the interval, we have $f(0) = 0$ and $f(9) = 3$.

   Therefore, the greatest and least values of $f(x)$ are 3 and -1 respectively.

   (b) $f(x) = x^4 - 8x^2 + 2$ on $[-1, 3]$;

   Ans: the greatest value = 11; the least value = -14;

   (c) $f(x) = e^x \ln x$ on $[1, 2]$.

   Ans: the greatest value = $e^2 \ln 2$; the least value = 0;

5. Let $f(x) = \dfrac{x^2 + 3x}{x - 1}$.

   (a) Find $f'(x)$.

   Ans: $f'(x) = \dfrac{(x-3)(x+1)}{(x-1)^2}$

   (b) Determine the values of x for each of the following cases:

   (i) $f'(x) = 0$; (ii) $f'(x) > 0$; (iii) $f'(x) < 0$.

   Ans:

   (i) $f'(x) = 0$ when $x = -1$ or 3

   (ii) $f'(x) > 0$ when $x < -1$ or $x > 3$

   (iii) $f'(x) < 0$ when $-1 < x < 3$ and $x \neq 1$

   (c) Find all relative extrema of $f(x)$.

   Ans: maximum point: $(-1, 1)$; minimum point: $(3, 9)$.

   (d) Find all asymptotes of $f(x)$.

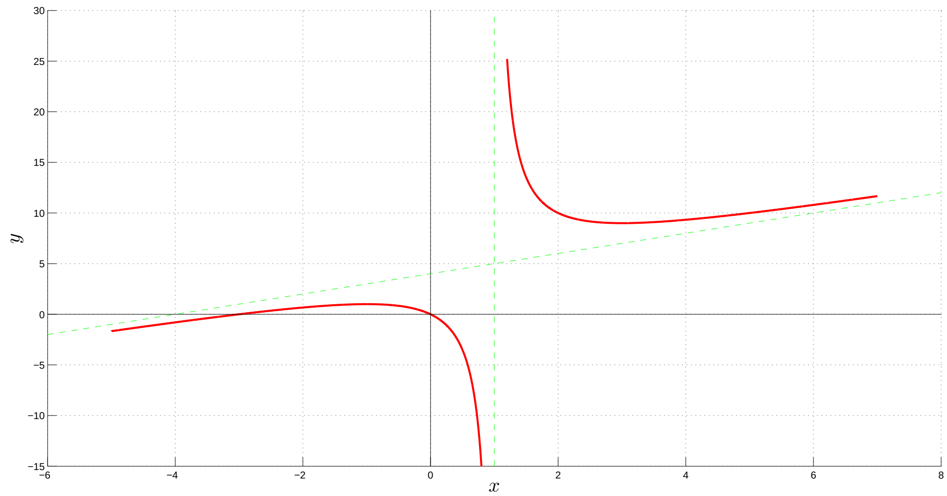
   Ans: Vertical asymptote: $x = 1$;

   By long division, we can rewrite $f(x)$ as $f(x) = \dfrac{x^2 + 3x}{x - 1} = x + 4 + \dfrac{4}{x - 1}$.

   Oblique asymptote: $y = x + 4$.

(e) Sketch the graph of $f(x)$.

Ans:



6. Let $f(x) = xe^{-x^2}$.

(a) Find $f'(x)$ and $f''(x)$.

Ans: $f'(x) = e^{-x^2}(1 - 2x^2)$ and $f''(x) = 2e^{-x^2}x(2x^2 - 3)$

(b) Determine the values of x for each of the following cases:

(i) $f'(x) = 0$;	(iii) $f'(x) < 0$;	(v) $f''(x) > 0$;
(ii) $f'(x) > 0$;	(iv) $f''(x) = 0$;	(vi) $f''(x) < 0$.

Ans:

(i) $f'(x) = 0$ when $x = -\dfrac{1}{\sqrt{2}}$ or $x = \dfrac{1}{\sqrt{2}}$

(ii) $f'(x) > 0$ when $-\dfrac{1}{\sqrt{2}} < x < \dfrac{1}{\sqrt{2}}$

(iii) $f'(x) < 0$ when $x < -\dfrac{1}{\sqrt{2}}$ or $x > \dfrac{1}{\sqrt{2}}$

(iv) $f''(x) = 0$ when $x = -\sqrt{\dfrac{3}{2}}$, 0 or $\sqrt{\dfrac{3}{2}}$

(v) $f''(x) > 0$ when $-\sqrt{\dfrac{3}{2}} < x < 0$ or $x > \sqrt{\dfrac{3}{2}}$

(vi) $f''(x) < 0$ when $x < -\sqrt{\dfrac{3}{2}}$ or $0 < x < \sqrt{\dfrac{3}{2}}$

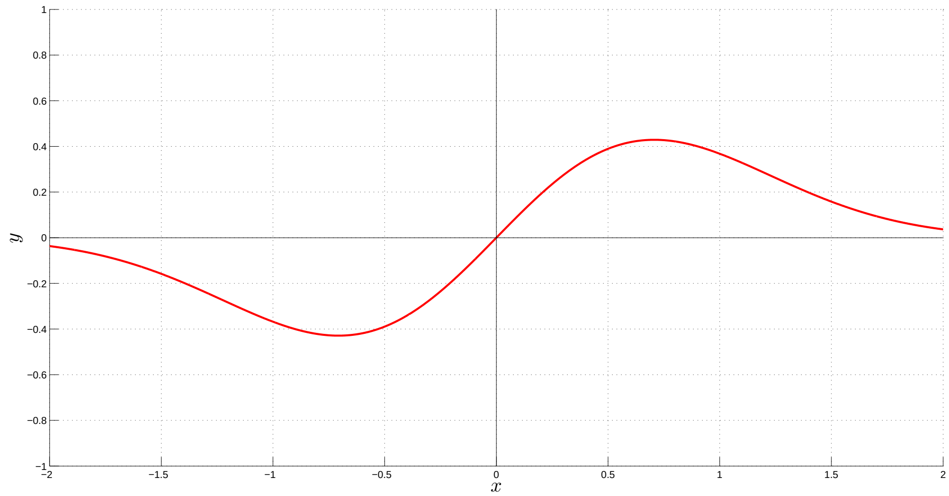
(c) Find all relative extrema and points of inflexion of $f(x)$.

Ans: maximum point: $(\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2e}})$; minimum point: $(-\dfrac{1}{\sqrt{2}}, -\dfrac{1}{\sqrt{2e}})$;

points of inflexion: $(-\sqrt{\dfrac{3}{2}}, -\sqrt{\dfrac{3}{2}}e^{-3/2})$, $(0, 0)$, $(\sqrt{\dfrac{3}{2}}, \sqrt{\dfrac{3}{2}}e^{-3/2})$

(d) Sketch the graph of $f(x)$.

Ans:



7. Let $f(x) = \dfrac{e^x}{x^e}$, for $x > 0$.

   (a) Solving $f'(x) > 0$ and $f'(x) < 0$. Hence, find the least value of $f(x)$.

   (b) Show that $e^\pi > \pi^e$.

   Ans:

   (a) $f'(x) = e^x x^{-e-1}(x - e)$.
   $f'(x) > 0$ when $x > e$; $f'(x) < 0$ when $0 < x < e$. Therefore, the least value of $f(x)$ is $f(e) = 1$.

   (b) By (a), $f(x) > f(e)$ for all $x > 0$ and $x \neq e$.
   In particular, put $x = \pi$, we have

$$\begin{array}{rcl} f(\pi) & > & f(e) \\ \dfrac{e^\pi}{\pi^e} & > & 1 \\ e^\pi & > & \pi^e \end{array}$$

Mean Value Theorem

8. By considering the function $f(x) = \sin x$ on $[0, 1]$ and applying the mean value theorem, show that $\sin 0.1 \leq 0.1$.

   Ans: Since $f(x)$ is continuous on $[0, 0.1]$ and differentiable on $(0, 0.1)$ ‡, by the mean value theorem, there exists $c \in (0, 1)$ such that

$$\begin{array}{rcl} \dfrac{f(0.1) - f(0)}{0.1 - 0} & = & f'(c) \\ \dfrac{\sin 0.1 - \sin 0}{0.1 - 0} & = & \cos c \;\leq\; 1 \\ \sin 0.1 & \leq & 0.1 \end{array}$$

(‡ You must check the conditions before applying the mean value theorem.)

9. By using the mean value theorem, prove that for all $x, y \in \mathbb{R}$,

$$|\cos x - \cos y| \leq |x - y|.$$

Ans: If $x = y$, the inequality is trivially true.

Now, suppose that $x > y$. Note that cosine function is differentiable everywhere (which implies that it is continuous everywhere), in particular it is continuous on $[y, x]$ and differentiable on (y, x). By the mean value theorem, there exists $c \in (y, x)$ such that

$$\begin{aligned}
\frac{\cos x - \cos y}{x - y} &= -\sin c \\
\left|\frac{\cos x - \cos y}{x - y}\right| &= |\sin c| \\
&\leq 1
\end{aligned}$$

The result follows. By switching the role of x and y, we can show that the inequality is also true for $y > x$.

10. By using the mean value theorem, prove that for all $x > 0$,

$$1 + x < e^x < 1 + xe^x.$$

Ans: Let $f(x) = e^x$ which is differentiable everywhere. Let $x > 0$, by the mean value theorem, there exists $c \in (0, x)$ such that

$$\begin{aligned}
\frac{f(x) - f(0)}{x - 0} &= f'(c) \\
\frac{e^x - 1}{x} &= e^c
\end{aligned}$$

Note that $0 < c < x$ implies that $1 = e^0 < e^c < e^x$. Therefore,

$$\begin{array}{ccccc}
1 & < & \frac{e^x-1}{x} & < & e^x \\
x & < & e^x - 1 & < & xe^x \\
x + 1 & < & e^x & < & 1 + xe^x
\end{array}$$

L'Hôpital Rule

11. By using L'Hôpital rule, find the following limits.

(a) $\displaystyle\lim_{x \to 0} \frac{\ln(\cos x)}{x^2}$

Ans: $-\dfrac{1}{2}$

(b) $\displaystyle\lim_{x \to 0} \frac{e^x + e^{-x} - 2}{1 - \cos 2x}$

Ans: $\dfrac{1}{2}$

(c) $\lim_{x\to\pi^+} \dfrac{\sin x}{\sqrt{x-\pi}}$
Ans: 0

(d) $\lim_{x\to 0^+} \dfrac{\ln(\cos 3x)}{\ln(\cos 2x)}$
Ans: $\dfrac{9}{4}$

12. By using L'Hôpital rule, find the following limits.

(a) $\lim_{x\to 0^+} \dfrac{\ln(\sin x)}{\ln(\tan x)}$
Ans: 1

(b) $\lim_{x\to 0^+} \dfrac{\ln(\tan 2x)}{\ln(\tan 3x)}$
Ans: 1

(c) $\lim_{x\to \frac{\pi}{2}^-} \dfrac{4\tan x}{1+\sec x}$
Ans: 4

(d) $\lim_{x\to\infty} x^n e^{-ax}$, where n is a natural number and a is a positive real number.
Ans: 0

13. By using L'Hôpital rule, find the following limits.

(a) $\lim_{x\to 0^+} x^2 \ln x$
Ans: 0

(b) $\lim_{x\to \frac{\pi}{2}} (2x-\pi)\sec x$
Ans: -2

(c) $\lim_{x\to 1^+} (x^2-1)\tan \dfrac{\pi x}{2}$
Ans: $-\dfrac{4}{\pi}$

(d) $\lim_{x\to\infty} x(\dfrac{\pi}{2} - \tan^{-1} x)$
Ans: 1

14. By using L'Hôpital rule, find the following limits.

(a) $\lim_{x\to 0} \left(\dfrac{1}{e^x-1} - \dfrac{1}{x}\right)$
Ans: $-\dfrac{1}{2}$

(b) $\lim_{x\to 0} \left(\dfrac{1}{\sin^2 x} - \dfrac{1}{x^2}\right)$
Ans: $\dfrac{1}{3}$

15. By using L'Hôpital rule, find the following limits.

(a) $\lim_{x\to 0} x^x$

Ans: 1

(b) $\lim_{x\to\infty} (e^{3x} - 5x)^{1/x}$

Ans: e^3

(c) $\lim_{x\to 0} (\cos x)^{\frac{1}{x^2}}$

Ans: $\dfrac{1}{\sqrt{e}}$

(d) $\lim_{x\to 0} \sin x \ln(\sin x)$

Ans: 0