THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1510 Calculus for Engineers (2020-2021)
Solution to Supplementary Exercise 5

Differentiability of Functions (First Principle)

1. A function f is said to be differentiable at a point $x = c$ if the limit

$$\lim_{h\to 0} \frac{f(c+h)-f(c)}{h} \tag{1}$$

exists. The value of the limit is said to be the derivative of the function f at the point $x = c$, which is denoted by $f'(c)$. The above definition is also called the first principle.

Compute the derivative of each of the following function at the given point by using the definition (i.e. the first principle).

(a) $f(x) = x^2 + 1$ at the point $x = 2$;

Ans:

$$\begin{aligned} \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} &= \lim_{h\to 0} \frac{[(2+h)^2+1]-[2^2+1]}{h} \\ &= \lim_{h\to 0} \frac{4h+h^2}{h} \\ &= \lim_{h\to 0} 4+h \\ &= 4 \end{aligned}$$

Therefore, $f'(2) = 4$.

(b) $f(x) = \dfrac{1}{x}$ at the point $x = 3$;

Ans:

$$\begin{aligned} \lim_{h\to 0} \frac{f(3+h)-f(3)}{h} &= \lim_{h\to 0} \frac{\frac{1}{3+h}-\frac{1}{3}}{h} \\ &= \lim_{h\to 0} \frac{-h}{3h(3+h)} \\ &= \lim_{h\to 0} -\frac{1}{3(3+h)} \\ &= -\frac{1}{9} \end{aligned}$$

Therefore, $f'(3) = -\dfrac{1}{9}$.

(c) $f(x) = \cos x$ at the point $x = \dfrac{\pi}{2}$;

Ans:

$$\begin{aligned}
\lim_{h\to 0}\frac{f(\frac{\pi}{2}+h)-f(\frac{\pi}{2})}{h} &= \lim_{h\to 0}\frac{\cos(\frac{\pi}{2}+h)-\cos(\frac{\pi}{2})}{h}\\
&= \lim_{h\to 0}\frac{[\cos(\frac{\pi}{2})\cos h-\sin(\frac{\pi}{2})\sin h]-\cos(\frac{\pi}{2})}{h}\\
&= \lim_{h\to 0}\frac{-\sin h-0}{h}\\
&= \lim_{h\to 0}-\frac{\sin h}{h}\\
&= -1
\end{aligned}$$

Therefore, $f'(\dfrac{\pi}{2})=-1$.

(d) (Harder Problem) $f(x)=x^n$, where n is a natural number, at the point $x=2$.

Ans:

$$\begin{aligned}
\lim_{h\to 0}\frac{f(2+h)-f(2)}{h} &= \lim_{h\to 0}\frac{(2+h)^n-2^n}{h}\\
&= \lim_{h\to 0}\frac{(2^n+C_1^n 2^{n-1}h+C_2^n 2^{n-2}h^2+\cdots+C_{n-1}^n 2h^{n-1}+h^n)-2^n}{h}\\
&\quad \text{(By binomial theorem)}\\
&= \lim_{h\to 0}C_1^n 2^{n-1}+C_2^n 2^{n-2}h+\cdots+C_{n-1}^n 2h^{n-2}+h^{n-1}\\
&= C_1^n 2^{n-1}\\
&= n2^{n-1}
\end{aligned}$$

Therefore, $f'(2)=n2^{n-1}$.

2. The **left derivative** of a function $f(x)$ at $x=c$ is by definition:

$$Lf'(c)=\lim_{h\to 0^-}\frac{f(c+h)-f(c)}{h};$$

and the **right derivative** at $x=c$ is:

$$Rf'(c)=\lim_{h\to 0^+}\frac{f(c+h)-f(c)}{h}.$$

From the definition (see (1)), the function $f(x)$ is differentiable at $x=c$ if and only if

$$\lim_{h\to 0^-}\frac{f(c+h)-f(c)}{h}=\lim_{h\to 0^+}\frac{f(c+h)-f(c)}{h},$$

i.e. $Lf'(c)=Rf'(c)$. In this case, $f'(c)$ equals to their common value.

Suppose

$$f(x)=\begin{cases} 3-\sin x & \text{if } \ x<0,\\ a & \text{if } \ x=0,\\ bx+c & \text{if } \ x>0,\end{cases}$$

where a,b are some real numbers. Given that $f(x)$ is continuous at $x=0$.

(a) What is the values of a and c?

(b) Find $Lf'(0)$.

(c) Find $Rf'(0)$ (in terms of b).

(d) For what value of b is the function $f(x)$ differentiable at 0?

Ans:

(a) Since $f(x)$ is continuous at $x = 0$, we have $f(0) = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x)$. Therefore, $a = \lim_{x\to 0^+} bx + c = \lim_{x\to 0^-} 3 - \sin x$ and we have $a = c = 3$.

(b) $Lf'(0) = \lim_{h\to 0^-} \dfrac{f(0+h) - f(0)}{h} = \lim_{h\to 0^-} \dfrac{(3 - \sin h) - 3}{h} = \lim_{h\to 0^-} -\dfrac{\sin h}{h} = -1.$

(c) $Rf'(0) = \lim_{h\to 0^+} \dfrac{f(0+h) - f(0)}{h} = \lim_{h\to 0^+} \dfrac{(bh + 3) - 3}{h} = \lim_{h\to 0^-} b = b.$

(d) If $f(x)$ is differentiable at $x = 0$, then $Lf'(0) = Rf'(0)$ and so $b = -1$.

3. Let us study the derivative as a function. The function $f'(x)$ is still defined as a limit, but the fixed number c in the definition (see (1)) is replaced by the variable x:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}. \tag{2}$$

If $y = f(x)$, we also write y' or $y'(x)$ for $f'(x)$. The domain of $f'(x)$ consists of all values of x in the domain of $f(x)$ for which the limit in equation (2) exists. If f is differentiable at every point in the domain, then f is said to be a differentiable function.

Using Equation (2), determine the domain of f', then give a formula describing $f'(x)$ where

$$f(x) = \sqrt{2 - x} \quad \text{with domain } D_f = (-\infty, 2].$$

Ans: For $x < 2$, we have

$$\begin{aligned}
\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h\to 0} \frac{\sqrt{2 - x - h} - \sqrt{2 - x}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{2 - x - h} - \sqrt{2 - x}}{h} \cdot \frac{\sqrt{2 - x - h} + \sqrt{2 - x}}{\sqrt{2 - x - h} + \sqrt{2 - x}} \\
&= \lim_{h\to 0} \frac{-1}{\sqrt{2 - x - h} + \sqrt{2 - x}} \\
&= -\frac{1}{2\sqrt{2 - x}}
\end{aligned}$$

However, for $x = 2$, we can see that $f(2 + h) = \sqrt{-h}$ is undefined when $h > 0$. Therefore, the domain of f' is $(-\infty, 2)$ and $f'(x) = -\dfrac{1}{2\sqrt{2 - x}}$ when $x < 2$.

4. Compute the derivative function of each of the following functions by using the definition (i.e. the first principle).

(a) $f(x) = x^2 + 1$;

Ans:

$$\begin{aligned}
\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0} \frac{[(x+h)^2+1]-[x^2+1]}{h} \\
&= \lim_{h\to 0} \frac{2xh+h^2}{h} \\
&= \lim_{h\to 0} 2x+h \\
&= 2x
\end{aligned}$$

Therefore, $f'(x) = 2x$.

(b) $f(x) = \dfrac{1}{x}$, for $x \neq 0$;

Ans: For $x \neq 0$,

$$\begin{aligned}
\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0} \frac{\frac{1}{x+h}-\frac{1}{x}}{h} \\
&= \lim_{h\to 0} \frac{-h}{xh(x+h)} \\
&= \lim_{h\to 0} -\frac{1}{x(x+h)} \\
&= -\frac{1}{x^2}
\end{aligned}$$

Therefore, $f'(x) = -\dfrac{1}{x^2}$.

(c) $f(x) = \cos x$;

Ans:

$$\begin{aligned}
\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0} \frac{\cos(x+h)-\cos x}{h} \\
&= \lim_{h\to 0} \frac{[\cos x\cos h - \sin x \sin h]-\cos x}{h} \\
&= \lim_{h\to 0} \cos x \cdot \frac{\cos h - 1}{h} - \sin x \cdot \frac{\sin h}{h} \\
&= -\sin x
\end{aligned}$$

Therefore, $f'(x) = -\sin x$.

(Recall: $\displaystyle\lim_{h\to 0} \frac{\cos h - 1}{h} = 0$.)

(d) (Harder Problem) $f(x) = x^n$, where n is a natural number.

Ans:

$$\begin{aligned}
\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{(x+h)^n-x^n}{h}\\
&= \lim_{h\to 0}\frac{(x^n+C_1^n x^{n-1}h+C_2^n x^{n-2}h^2+\cdots+C_{n-1}^n xh^{n-1}+h^n)-x^n}{h}\\
&\quad \text{(By binomial theorem)}\\
&= \lim_{h\to 0} C_1^n x^{n-1}+C_2^n x^{n-2}h+\cdots+C_{n-1}^n xh^{n-2}+h^{n-1}\\
&= C_1^n x^{n-1}\\
&= nx^{n-1}
\end{aligned}$$

Therefore, $f'(x)=nx^{n-1}$.

5. Let $f(x)=|x|$. $f(x)$ can be described as the following

$$f(x)=\begin{cases} x & \text{if } x>0,\\ 0 & \text{if } x=0,\\ -x & \text{if } x<0.\end{cases}$$

(a) Write down $\dfrac{f(0+h)-f(0)}{h}$ explicitly for the cases $h>0$ and $h<0$.

(b) By using the result in (a), find left derivative, i.e.

$$Lf'(0)=\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h},$$

and the right derivative, i.e.

$$Rf'(0)=\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h},$$

at the point $x=0$.

(Hint: You have to use the first expression in (a) to find the left hand limit and the second expression to compute the right hand limit.)

(c) Does $f'(0)$ exist?

(d) Find $f'(x)$ by using the first principle for the cases $x>0$ and $x<0$. Hence, write down the domain of $f'(x)$.

Ans:

(a) If $h>0$, $\dfrac{f(0+h)-f(0)}{h}=\dfrac{h-0}{h}=1$;

If $h<0$, $\dfrac{f(0+h)-f(0)}{h}=\dfrac{-h-0}{h}=-1$.

(b) $Lf'(0)=\displaystyle\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^-}-1=-1$ and

$Rf'(0)=\displaystyle\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^+}1=1$

(Remark: We do not care $\dfrac{f(0+h)-f(0)}{h}$ when $h=0$.)

(c) Since $\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} \neq \lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h}$ (i.e. $Lf'(0) \neq Rf'(0)$), $\lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$ does not exist.
Therefore, $f(x) = |x|$ is not differentiable at $x = 0$.

(d) When $x > 0$,

$$\begin{aligned}
\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0} \frac{(x+h)-x}{h} \\
&= \lim_{h\to 0} 1 \\
&= 1
\end{aligned}$$

(Note that $x > 0$, when h is sufficiently close to 0, we have $x + h > 0$, so $f(x+h) = x + h$.)
Therefore, $f'(x) = 1$ when $x > 0$.

When $x < 0$,

$$\begin{aligned}
\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0} \frac{[-(x+h)]-(-x)}{h} \\
&= \lim_{h\to 0} -1 \\
&= -1
\end{aligned}$$

(Note that $x < 0$, when h is sufficiently close to 0, we have $x + h < 0$, so $f(x+h) = -(x+h)$.)
Therefore, $f'(x) = -1$ when $x < 0$.

From the above, we can see $f'(x)$ is defined only when $x \neq 0$, i.e. Domain of f $= (-\infty, 0) \cup (0, \infty) = \mathbb{R}/\{0\}$.

(Remark: As you can see, $f(x)$ is continuous at $x = 0$ but NOT differentiable at $x = 0$.)

6. Let $f(x)$ be a function defined by

$$f(x) = \begin{cases} x^2 & \text{if} \quad x \geq 0, \\ 0 & \text{if} \quad x < 0. \end{cases}$$

(a) Is $f(x)$ differentiable at $x = 0$? If yes, find $f'(0)$.

(b) Compute $f'(x)$ for the cases $x > 0$ and $x < 0$.

(c) Is $f'(x)$ differentiable at $x = 0$?

Ans:

(a) $Lf'(0) = \lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{0-0}{h} = 0$ and

$Rf'(0) = \lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{h^2-0}{h} = \lim_{h\to 0^+} h = 0.$

Since $\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h} = 0$,
we have $\lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = 0$.
Therefore, $f(x)$ is differentiable at $x = 0$ and $f'(0) = 0$.

(b) If $x > 0$, $f'(x) = 2x$; If $x < 0$, $f'(x) = 0$.
Hence $f'(x)$ can be described as the following:

$$f'(x) = \begin{cases} \underline{\quad 2x \quad} & \text{if } x > 0, \\ \underline{\quad 0 \quad} & \text{if } x = 0, \\ \underline{\quad 0 \quad} & \text{if } x < 0. \end{cases}$$

(c) $\lim_{h\to 0^-}\frac{f'(0+h)-f'(0)}{h} = \lim_{h\to 0^-}\frac{0-0}{h} = 0$ and $\lim_{h\to 0^+}\frac{f'(0+h)-f'(0)}{h} = \lim_{h\to 0^+}\frac{2h-0}{h} = 2$.
Since $\lim_{h\to 0^-}\frac{f'(0+h)-f'(0)}{h} \neq \lim_{h\to 0^+}\frac{f'(0+h)-f'(0)}{h}$, $\lim_{h\to 0}\frac{f'(0+h)-f'(0)}{h}$ does not exist. Therefore, $f'(x)$ is not differentiable at $x = 0$.

(Remark: Even a function is differentiable at a point, it may not be twice differentiable at that point.)

Derivatives

7. Find the first derivatives of the following functions.

(a) $y = 2x^3 - 4x + 2$
Ans: $6x^2 - 4$

(b) $y = 5x^3 - 4x^2 + 7$
Ans: $15x^2 - 8x$

(c) $y = e^{3x}$
Ans: $3e^{3x}$

(d) $y = \cos 2x$
Ans: $-2\sin 2x$

(e) $y = \sin 5x$
Ans: $5\cos 5x$

(f) $y = -\tan 3x$
Ans: $-3\sec^2 3x$

(g) $y = \sqrt{x}$
Ans: $\frac{1}{2\sqrt{x}}$

(h) $y = \ln(1 + x^2)$
Ans: $\frac{2x}{1+x^2}$

8. Find the first derivatives of the following functions.

(a) $y = 4\sqrt{x} + \dfrac{2}{\sqrt{x}}$

Ans: $2x^{-1/2} - x^{-3/2}$

(b) $y = x^3 e^{-2x}$

Ans: $e^{-2x}x^2(3-2x)$

(c) $y = \sin x \ln x$

Ans: $\dfrac{\sin x}{x} + (\ln x)(\cos x)$

(d) $y = \sec x - 3\tan x$

Ans: $\sec^2 x(\sin x - 3)$

(e) $y = x\csc x$

Ans: $\csc x(1 - x\cot x)$

(f) $y = \dfrac{3x-4}{x+2}$

Ans: $\dfrac{10}{(x+2)^2}$

(g) $y = \dfrac{x^2+1}{x+1}$

Ans: $\dfrac{x^2+2x-1}{(x+1)^2}$

(h) $y = \dfrac{\sin x}{x}$

Ans: $\dfrac{x\cos x - \sin x}{x^2}$

(i) $y = (3x^2-4)^{10}$

Ans: $60x(3x^2-4)^9$

(j) $y = \sqrt{x^3+1}$

Ans: $\dfrac{3x^2}{2\sqrt{x^3+1}}$

(k) $y = \ln(\ln x)$

Ans: $\dfrac{1}{x\ln x}$

(l) $y = e^{\cot x}$

Ans: $-\csc^2 x e^{\cot x}$

(m) $y = \ln(x + \sqrt{x})$

Ans: $\dfrac{2\sqrt{x}+1}{2x(\sqrt{x}+1)}$

9. Let $\mathcal{C}$ be the graph of the function $y = 4e^x(x+1)$.

(a) Show that $A = (0,4)$ is a point lies on $\mathcal{C}$.

(b) Find the equations of tangent and normal of $\mathcal{C}$ at the point A.

Ans:

(a) Put $x = 0$ into the the function, then $y = 4e^0(0+1) = 4$. Therefore, $A = (0,4)$ is a point lying on $\mathcal{C}$.

(b) The slope of the tangent $= \left.\dfrac{dy}{dx}\right|_{x=0} = 8$, so the equation of tangent is $8x - y + 4 = 0$.

Recall that the normal is perpendicular to the tangent, so the slope of the normal is $-\dfrac{1}{8}$ and the equation of it is $x + 8y - 32 = 0$.

10. Let $\mathcal{C}$ be the graph of the function $y = \sin x + \cos x$.

(a) Show that $A = (\dfrac{\pi}{2}, 1)$ is a point lies on $\mathcal{C}$.

(b) Find the equations of tangent and normal of $\mathcal{C}$ at the point A.

Ans:

(a) Put $x = \frac{\pi}{2}$ into the the function, then $y = \sin\frac{\pi}{2} + \cos\frac{\pi}{2} = 1$. Therefore, $A = (\frac{\pi}{2}, 1)$ is a point lying on $\mathcal{C}$.

(b) tangent: $x + y - \frac{\pi}{2} - 1 = 0$; normal: $x - y - \frac{\pi}{2} + 1 = 0$

11. By using the logarithmic differentiation, find the first derivative of the following functions.

(a) $y = (2x+1)^3(x-1)^4\sqrt{(3x+2)^5}$

Ans: $\frac{dy}{dx} = (2x+1)^3(x-1)^4\sqrt{(3x+2)^5}\left(\frac{6}{2x+1} + \frac{15}{2(3x+2)} + \frac{4}{x-1}\right)$

(b) $y = \frac{e^{2x}}{(x-1)^4}$

Ans: $\frac{dy}{dx} = \frac{e^{2x}}{(x-1)^4}\left(2 - \frac{4}{x-1}\right)$

(c) $y = x^x$

Ans: $\frac{dy}{dx} = x^x(\ln x + 1)$

(d) $y = (\sin x)^{(\cos x)}$

Ans: $\frac{dy}{dx} = (\sin x)^{(\cos x)}\left(\frac{\cos^2 x}{\sin x} - \sin x \ln(\sin x)\right)$

12. Find $\frac{dy}{dx}$ in terms of x and y for the following implicit functions.

(a) $x^2 + y^2 = 9$

Ans: $\frac{dy}{dx} = -\frac{x}{y}$

(b) $x^3y + xy^2 = 1$

Ans: $\frac{dy}{dx} = -\frac{y(3x^2+y)}{x^3+2xy}$

(c) $x^3 + y^3 = 2xy$

Ans: $\frac{dy}{dx} = \frac{3x^2-2y}{2x-3y^2}$

(d) $ye^{xy} = 1$

Ans: $\frac{dy}{dx} = -\frac{y^2}{xy+1}$

13. Let $\mathcal{C}$ be the curve given by the equation $x^3 + xy + y^3 = 11$.

(a) Show that $A = (1, 2)$ is a point lies on $\mathcal{C}$.

(b) Find the equation of tangent of $\mathcal{C}$ at the point A.

Ans:

(a) Put $x = 1$ and $y = 2$, then LHS $= (1)^3 + (1)(2) + (2)^3 = 11 =$ RHS. Therefore, $A = (1, 2)$ is a point lying on $\mathcal{C}$.

(b) tangent: $5x + 13y - 31 = 0$

14. If $y = x^2e^x$, show that $\dfrac{d^2y}{dx^2} = 2\dfrac{dy}{dx} - y + 2e^x$.

Ans: $\dfrac{dy}{dx} = e^x x(x + 2)$ and $\dfrac{d^2y}{dx^2} = e^x(x^2 + 4x + 2)$.
The result follows by putting them into LHS and RHS.

15. Let $\mathcal{C} : (x(t), y(t)) = (\sqrt{2}\cos t, \sqrt{2}\sin t)$, for $t \in \mathbb{R}$, be a curve defined on $\mathbb{R}^2$.

(a) Find $\dfrac{dy}{dx}$ in terms of t.

(b) Find the equation of tangent of $\mathcal{C}$ at the point $(x(\frac{\pi}{4}), y(\frac{\pi}{4})) = (1, 1)$.

Ans:

(a) By chain rule, $\dfrac{dy}{dt} = \dfrac{dy}{dx} \cdot \dfrac{dx}{dt}$, therefore we have

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}.$$

Now, $\dfrac{dy}{dt} = \sqrt{2}\cos t$ and $\dfrac{dx}{dt} = -\sqrt{2}\sin t$. Therefore, $\dfrac{dy}{dx} = -\dfrac{\cos t}{\sin t}$.

(b) When $t = \dfrac{\pi}{4}$, $\dfrac{dy}{dx} = -1$. Therefore, the equation of the tangent at $(1, 1)$ is $x + y - 2 = 0$.

16. Let $\mathcal{C} : (x(t), y(t)) = (t^2, t^3)$, for $t \in \mathbb{R}$, be a curve defined on $\mathbb{R}^2$.

(a) Find $\dfrac{dy}{dx}$ in terms of t.

(b) Find the equation of tangent of $\mathcal{C}$ at the point $(x(1), y(1) = (1, 1)$.

Ans:

(a) By chain rule, $\dfrac{dy}{dt} = \dfrac{dy}{dx} \cdot \dfrac{dx}{dt}$, therefore we have

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}.$$

Now, $\dfrac{dy}{dt} = 3t^2$ and $\dfrac{dx}{dt} = 2t$. Therefore, $\dfrac{dy}{dx} = \dfrac{3t}{2}$.

(b) When $t = 1$, $\dfrac{dy}{dx} = \dfrac{3}{2}$. Therefore, the equation of the tangent at $(1, 1)$ is $3x - 2y - 1 = 0$.

Linearization

17. Let $f(x)$ be a function which is differentiable at $x = a$.

   (a) Show that the equation of tangent at the point $x = a$ is

$$y = L(x) = f'(a)(x - a) + f(a).$$

   (b) The function $L(x)$ obtained in (a) is a linear function (i.e. a polynomial function of degree 1) which is called the linearization of $f(x)$ at the point $x = a$ (also called the Talyor polynomial of degree 1 generated by $f(x)$ at the point $x = a$, which is commonly denoted by $T_1(x)$).

   Now, suppose that $f(x) = \sqrt{x}$. By stekching the graphs of $f(x)$ and $L(x)$, one can observe that when x is close to 9, $L(x)$ approximately equals $f(x)$.
   By using the result in (a), approximate the value of $\sqrt{9.1}$.
   (Remark: Compare your approximated value obtained and the value obtained by using calculator.)

Ans:

   (a) Note that the slope of the tangent is $f'(a)$ and the tangent passes through the point $(a, f(a))$. Therefore, the required equation is

$$\begin{aligned} \frac{y - f(a)}{x - a} &= f'(a) \\ y &= f'(a)(x - a) + f(a) \end{aligned}$$

   (b) From (a), the linearization of $f(x) = \sqrt{x}$ at the point $x = 9$ is

$$L(x) = f'(9)(x - 9) + f(9) = \frac{1}{6}x + \frac{3}{2}.$$

   Then,

$$\sqrt{9.1} = f(9.1) \approx L(9.1) = (\frac{1}{6})(\frac{91}{10}) + \frac{3}{2} = \frac{181}{60} \approx 3.016667$$

   (Remark: By using the calculator $\sqrt{9.1} \approx 3.016621$.)

18. Approximate the value of $e^{0.1}$ by linearizing an appropriately chosen function at an appropriately chosen point.

   Ans: Let $f(x) = e^x$. Then, the linearization of $f(x)$ at the point $x = 0$ is

$$L(x) = f'(0)(x - 0) + f(9) = x + 1.$$

   Then,

$$e^{0.1} = f(0.1) \approx L(0.1) = 1.1$$

   (Remark: By using the calculator $e^{0.1} \approx 1.105171$. If we try to linearize the function $f(x) = e^x$ at a point $x = a$ other than 0, we have to compute $f(a) = e^a$ and $f'(a) = e^a$ which may not be done by hand.)