Calculus for Engineers

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Contents

Limits

3.1 Introduction

In this chapter we shall study the concept of limits. Emphasis is placed upon the term "approaching" in a limit sense. Using the $\epsilon - \delta$ definition for analyzing a list of the limit form results is presented. Each concept is explained and illustrated by worked examples.

3.2 Limit of a function

The concepts of limit and continuity are basic cornerstones to the study of calculus. The concept of a limit helps us to describe the behaviour of $f(x)$ when x *is close to but not equal to a particular value*[1] c. We shall consider an intuitive approach to this concept as shown in the following example.

Example 1 Consider the function f defined by

$$f(x) = \frac{x^2 - 4}{x - 2}.$$

In Figure 3.1, the function is defined for all values of x except $x = 2$.[2] If $x = 2$, we can divide the numerator and the denominator by $(x - 2)$ to obtain

$$f(x) = x + 2, \quad x \neq 2.$$

We shall investigate the values of the function, when x is close to 2, but not equal to 2.

x	1.5	1.8	1.9	1.999	$\rightarrow 2 \leftarrow$	2.001	2.01	2.1	2.2	2.5
$f(x)$	3.5	3.8	3.9	3.999	$\rightarrow 4 \leftarrow$	4.001	4.01	4.1	4.2	4.5

Table 3.1: Values of f.

In Table 3.1, x is approaching 2 through values less than 2 and $f(x)$ is approaching the value 4, while x is approaching 2 through values greater than 2 and $f(x)$ is approaching the same value 4. Thus, $f(x)$ approaches the value of 4 as x approaches the value 4.

Table 3.1 clearly shows that the closer x is to 2 the closer $f(x)$ is to 4. If x differs from 2 by 0.001, then $f(x)$ differs 4 by ± 0.001. Hence we can make $f(x)$ as close to 4 as we

[1] "x is close to but not equal to a particular value c" is identical to "x approaches c".

[2] The value of this function at $x = 2$ is of the form $\frac{0}{0}$ which is meaningless.

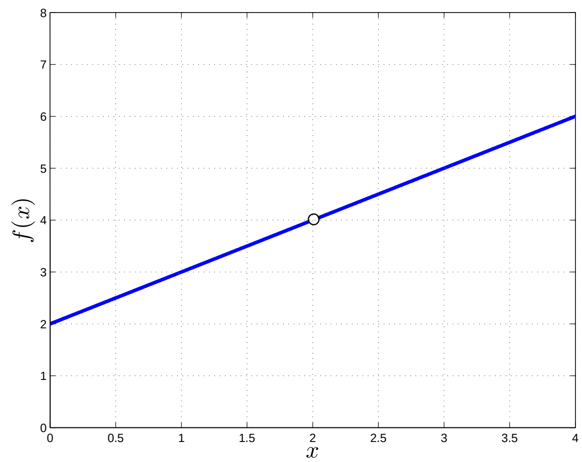

Figure 3.1: Graph of $f(x) = \dfrac{x^2 - 4}{x - 2}$, where $x \neq 2$ and $x \in [0, 4]$.

please by making x close enough to 2. In other words, we can make the absolute value of the difference between $f(x)$ and 4 as small as we please by making the difference between x and 2 small enough. To express this mathematically, we use the symbols ϵ and δ. Thus, $|f(x) - 4|$ will be less than ϵ whenever $|x - 2| < \delta$ and $|x - 2| \neq 0$. Still another way of expressing the same idea is:

Given any positive number ϵ, there exists a small positive number δ such that

$$|f(x) - 4| < \epsilon \quad \text{whenever} \quad 0 < |x - 2| < \delta.$$

From Table 3.2, we see that

$$|f(x) - 4| = 0.2 \text{ when } |x - 2| = 0.2.$$

So, given $\epsilon = 0.2$, we take $\delta = 0.2$ and state that

$$|f(x) - 4| < 0.2 \quad \text{whenever} \quad 0 < |x - 2| < 0.2.$$

Also

$$|f(x) - 4| = 0.001 \text{ when } |x - 2| = 0.001.$$

Hence, if $\epsilon = 0.001$, we take $\delta = 0.001$ and state that

$$|f(x) - 4| < 0.001 \quad \text{whenever} \quad 0 < |x - 2| < 0.001.$$

We could go on and give ϵ any small positive value and find a suitable positive value for δ such that

$$|f(x) - 4| < \epsilon \quad \text{whenever} \quad 0 < |x - 2| < \delta.$$

It is important to note that the value of δ depends on the value of ϵ. Now, because for any $\epsilon > 0$, however, small, we can find a positive number δ such that

$$|f(x) - 4| < \epsilon \quad \text{whenever} \quad 0 < |x - 2| < \delta,$$

we say that the limit of $f(x)$, as x approaches 2, is equal to 4. In symbols,

x	1.5	1.8	1.9	1.99	1.999
$2-x$	0.5	0.2	0.1	0.01	0.001
$f(x)$	3.5	3.8	3.9	3.99	3.999
$4-f(x)$	0.5	0.2	0.1	0.01	0.001
x	2.001	2.01	2.1	2.2	2.5
$x-2$	0.001	0.01	0.1	0.2	0.5
$f(x)$	4.001	4.01	4.1	4.2	4.5
$f(x)-4$	0.001	0.01	0.1	0.2	0.5

Table 3.2:

$$\lim_{x\to 2} f(x) = 4.$$

$\square$

We shall define the limit of a function in general.

Definition 1 Let f be a function defined in some open interval containing c, except possibly at c itself. Then the limit of $f(x)$ as x approaches (or as x tends to) c is A, written as

$$\lim_{x\to c} f(x) = A.$$

If for any $\epsilon > 0$, however small, there exists a number $\delta > 0$ such that

$$|f(x) - A| < \epsilon \quad \text{whenever} \quad 0 < |x-c| < \delta.$$

Note 1 It is not necessary that the function be defined at $x = c$ for the limit to exist. That is

$$\lim_{x\to c} f(x) \neq f(a).$$

We will see examples of this later.

3.2.1 Geometrical significance of ϵ and δ

Another way of stating this is: $f(x)$ on the vertical axis can be restricted to lie between $4-\epsilon$ and $4+\epsilon$ by restricting x on the $x-$axis to lie between $2-\delta$ and $2+\delta$. The condition is equivalent to

$$k-\epsilon < f(x) < k+\epsilon \quad \text{whenever} \quad c-\delta < x < c+\delta.$$

The above definition is now applied to find δ corresponding to a specific ϵ in some simple cases.

Example 2 Consider

$$\lim_{x\to 2}(4x-5) = 3; \quad \epsilon = 0.002.$$

Given

$$\lim_{x\to 2} f(x) = 3,$$

find a number δ for $\epsilon = 0.002$ such that

$$|f(x) - 3| < \epsilon \quad \text{whenever} \quad 0 < |x - 2| < \delta.$$

We have

$$\begin{aligned} |f(x) - 3| &= |4x - 5 - 3| \\ &= |4x - 8| \\ &= 4|x - 2|. \end{aligned}$$

Therefore, we want

$$4|x - 2| < 0.002 \quad \text{whenever} \quad 0 < |x - 2| < \delta.$$

If we take $\delta = 0.0005$, then

$$|(4x - 5) - 3| < 0.002 \quad \text{whenever} \quad 0 < |x - 2| < 0.0005.$$

It is important to note that any number less than 0.0005 can be used in place of 0.0005 for δ. □

Example 3 Consider

$$\lim_{x \to \frac{1}{3}} \frac{9x^2 - 1}{3x - 1} = 2; \quad \epsilon = 0.01.$$

Given

$$\lim_{x \to 2} f(x) = 3,$$

find a number δ for $\epsilon = 0.002$ such that

$$|f(x) - 3| < \epsilon \quad \text{whenever} \quad 0 < |x - 2| < \delta.$$

We have

$$\begin{aligned} |f(x) - 2| &= \left| \frac{9x^2 - 1}{3x - 1} - 2 \right| \\ &= |3x + 1 - 2| \\ &= |3x - 1| \\ &= 3\left| x - \frac{1}{3} \right|. \end{aligned}$$

We have to find a δ such that

$$3\left| x - \frac{1}{3} \right| < 0.01 \quad \text{whenever} \quad 0 < \left| x - \frac{1}{3} \right| < \delta.$$

That is,

$$\left| x - \frac{1}{3} \right| < 0.0033 \quad \text{whenever} \quad 0 < \left| x - \frac{1}{3} \right| < \delta.$$

If we take $\delta = 0.0033$, then the definition of the limit holds true. □

In the following example, we shall establish the limit by using Definition 1, that is, for any $\epsilon > 0$, find a $\delta > 0$ such that

$$|f(x) - A| < \epsilon \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

Example 4 Show that

$$\lim_{x \to 1} x^2 = 1.$$

Solution: We must show that for any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|x^2 - 1| < \epsilon \quad \text{whenever} \quad 0 < |x - 1| < \delta.$$

Consider

$$\begin{aligned} |x^2 - 1| &= |(x-1)(x+1)| \\ &= |x-1||x+1|. \end{aligned}$$

Since x is approaching 1, the maximum value of x is 2 and $|x-1| < 1$. Therefore, we have

$$|x - 1| \text{ is close to } 2, \text{ that is, } |x^2 - 1| < 2.$$

Hence

$$|x^2 - 1| = |x - 1||x + 1| < 2|x - 1|,$$

whenever $|x - 1| < 1$.

Now we want

$$2\,|x - 1| < \epsilon \quad \text{whenever} \quad |x - 1| < \epsilon/2.$$

Thus, if we choose δ to be smaller of 2 and $\epsilon/2$, then whenever $|x - 1| < \delta$, it follows

$$|x^2 - 1| < \epsilon/2.$$

Therefore, we conclude that $|x^2 - 1| < \epsilon$ whenever $0 < |x - 1| < \delta$, where $\delta = \min\{1, \epsilon/2\}$. $\square$

Example 5 Show that

$$\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0.$$

Solution: We must show that for any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - 0| < \epsilon \text{ whenever} |x - 0| < \delta.$$

Consider

$$\left| x \sin\left(\frac{1}{x}\right) - 0 \right| = |x| \left| \sin\left(\frac{1}{x}\right) \right|.$$

Since $\left|\sin\left(\frac{1}{x}\right)\right| \le 1$, we have

$$\left| x \sin\left(\frac{1}{x}\right) - 0 \right| \le |x|.$$

Now we want

$$\left|x\sin\left(\frac{1}{x}\right)-0\right| \leq |x| < \epsilon \text{ implies } |x-0| < \epsilon.$$

If we take $\delta = \epsilon$, then the existence of δ is proven. Therefore, we have

$$\lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0.$$

□

Example 6 Show that

$$\lim_{x\to 6}\frac{x}{x-3} = 2.$$

Solution: We must show that for any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\left|\frac{x}{x-3}-2\right| < \epsilon \text{ whenever} 0 < |x-6| < \delta.$$

Consider

$$\begin{aligned}\left|\frac{x}{x-3}-2\right| &= \left|\frac{x-2x+6}{x-3}\right| \\ &= \left|\frac{-x+6}{x-3}\right| \\ &= |x-6|\left|\frac{1}{x-3}\right|. \end{aligned} \tag{3.1}$$

We want to show that $\left|\frac{x}{x-3}-2\right|$ is small when x is close to 6. We shall find a small upper bound for $\frac{1}{x-3}$. Since x is approaching to 6, $|x-6| < 1$ which is equivalent to

$$5 < x < 7.$$

That is,

$$5-3 < x-3 < 7-3.$$

That is,

$$2 < x-3 < 4,$$

or

$$2 < |x-3| < 4.$$

Therefore,

$$|x-3| > 2,$$

or

$$\frac{1}{|x-3|} < \frac{1}{2}.$$

Therefore, (3.1) becomes

$$\left|\frac{x}{x-3} - 2\right| = |x-6|\left|\frac{1}{x-3}\right| < |x-6|\frac{1}{2}.$$

If $\frac{1}{2}|x-6| < \epsilon$, then $|x-6| < 2\epsilon$ whenever $|x-6| < 1$.

If we take δ to be the smaller of 1 and 2ϵ, then

$$\left|\frac{x}{x-3} - 2\right| < \epsilon \quad \text{whenever} \quad 0 < |x-6| < \delta.$$

Therefore, we have

$$\lim_{x\to 6} \frac{x}{x-3} = 2.$$

□

Example 7 Show that

$$\lim_{x\to 4} \sqrt{x+5} = 3.$$

Solution: We must show that for any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\left|\sqrt{x+5} - 3\right| < \epsilon \quad \text{whenever} \quad 0 < |x-4| < \delta.$$

Consider

$$\begin{aligned} \left|\sqrt{x+5} - 3\right| &= \left|\frac{(\sqrt{x+5}-3)(\sqrt{x+5}+3)}{(\sqrt{x+5}+3)}\right| \\ &= \left|\frac{x+5-9}{\sqrt{x+5}+3}\right| \\ &= |x-4|\left|\frac{1}{\sqrt{x+5}+3}\right| \end{aligned} \tag{3.2}$$

We shall find a small upper bound for $\frac{1}{x-3}$. Since x is tending to 6, $|x-4| < 1$ which is equivalent to

$$3 < x < 5.$$

That is,

$$3+5 < x+5 < 5+5,$$

or,

$$8 < x+5 < 10.$$

That is,

$$\sqrt{8} < \sqrt{x+5} < \sqrt{10}.$$

Therefore,

$$\sqrt{8}+3 < \sqrt{x+5}+3 < \sqrt{10}+3.$$

That is,

$$\sqrt{8}+3 < |\sqrt{x+5}+3| < \sqrt{10}+3.$$

Therefore, we have

$$|\sqrt{x+5}+3| > \sqrt{8}+3$$

implying

$$\frac{1}{|\sqrt{x+5}+3|} < \frac{1}{\sqrt{8}+3}.$$

Hence (3.2) becomes

$$\left|\sqrt{x+5}-3\right| < |x-4|\,\frac{1}{\sqrt{8}+3} \quad \text{whenever} \quad |x-4|<1.$$

We want

$$|x-4|\,\frac{1}{\sqrt{8}+3} < \epsilon$$

or equivalently

$$|x-4| < \left(\sqrt{8}+3\right)\epsilon.$$

Therefore, we take $\delta = \min\{1, \left(\sqrt{8}+3\right)\epsilon\}$ which gives

$$\left|\sqrt{x+5}-3\right| < \epsilon \quad \text{whenever} \quad 0<|x-4|<\delta.$$

Since we have proved the existence of δ for any $\epsilon > 0$, it follows that

$$\lim_{x\to 4}\sqrt{x+5} = 3.$$

□

The following theorem proves the uniqueness of the limit, if it exists.

Theorem 1 If $\lim\limits_{x\to c} f(x) = A_1$ and $\lim\limits_{x\to c} f(x) = A_2$, then $A_1 = A_2$.

Proof.

Let $k_1 \neq k_2$.
Since $\lim\limits_{x\to c} f(x) = A_1$, given an $\epsilon > 0$, there exists a $\delta_1 > 0$ such that

$$|f(x)-A_1| < \epsilon \quad \text{whenever} \quad 0<|x-c|<\delta_1.$$

Similarly, for the same $\epsilon > 0$, there exists a $\delta_2 > 0$ such that

$$|f(x)-A_2| < \epsilon \quad \text{whenever} \quad 0<|x-c|<\delta_2.$$

Consider

$$\begin{aligned} |A_1 - A_2| &= |A_1 - f(x) + f(x) - A_2| \\ &\leq |A_1 - f(x)| + |f(x) - A_2| && \text{By triangle inequality} \\ &< \epsilon + \epsilon, \end{aligned}$$

wherever $0 < |x-c| < \delta_1$ and $0 < |x-c| < \delta_2$.
If $\delta = \min\{\delta_1, \delta_2\}$, then the above inequality can be written as

$$|A_1 - A_2| < 2\epsilon \quad \text{whenever} \quad 0 < |x-c| < \delta.$$

Choose ϵ such that $\epsilon = \frac{1}{2}|A_1 - A_2|$. Therefore,

$$|A_1 - A_2| < 2 \cdot \frac{|A_1 - A_2|}{2} \quad \text{whenever} \quad 0 < |x-c| < \delta.$$

That is,

$$|A_1 - A_2| < |A_1 - A_2|$$

which is a contraction. Therefore, our assumption that $A_1 \neq A_2$ is false.
Hence $A_1 = A_2$ holds, i.e., $\lim_{x \to c} f(x)$ is unique. □

Note 2 It is important to note that the limit of a function may not exist.

If the limit of a function exists, it is necessary and sufficient that corresponding to every positive number ϵ, it is possible to find a δ. If corresponding to a given ϵ, a δ does not exist, then the function will not approach a limit or the limit will not exist. We shall consider examples for which the limit does not exist.

Example 8 Show that

$$\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$$

does not exist.

Solution:

Using the $\epsilon - \delta$ definition, we show that $f(x)$ does not approach any value as x tends to zero. It is enough to find one $\epsilon > 0$ for which the condition $|f(x) - 0| < \epsilon$ cannot be guaranteed, no matter how small we take $|x|$ to be (that is, we show that $f(x)$ does not tend to zero.)

Let $\epsilon = 1/2$, and A be an $x-$interval containing zero. There is some number

$$x = \frac{1}{90 + 360n} \quad \text{(denominator in degrees)}$$

which is in this interval and for which $f(x) = 1$.

Hence for this x, we have

$$|f(x) - 0| = |1 - 0| < 1/2 \quad \text{(the value of } \ \epsilon).$$

That is, $f(x)$ does not approach zero as x approaches zero.

Let $f(x) \to k$ as $x \to 0$ whenever $k \neq 0$.

We have to prove that $|f(x) - A| < \epsilon$ whenever $|x| < \delta$. To prove that this is not true, let $\epsilon = 1/2$. As before there exists an

$$x_1 = \frac{1}{90 + 360n}$$

in the interval containing zero such that $f(x_1) = 1$ and also some

$$x_2 = \frac{1}{270 + 360n}$$

in the interval containing zero such that $f(x_2) = -1$.

But the interval from $k - \frac{1}{2}$ to $k + \frac{1}{2}$ cannot contain both -1 and $+1$. Therefore, we cannot have $|1 - k| < \frac{1}{2}$ and $|-1 - k| < \frac{1}{2}$ no matter what k is. Therefore,

$$\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$$

does not exist. $\square$

3.3 Algebra of limits of functions

Theorem 2 If $\lim_{x\to c} f(x) = A$ and $\lim_{x\to c} g(x) = B$, then

1. $$\lim_{x \to c} (f(x) + g(x)) = A + B = \lim_{x \to c} f(x) + \lim_{x \to c} g(x).$$

2. $$\lim_{x \to c} (f(x) - g(x)) = A - B = \lim_{x \to c} f(x) - \lim_{x \to c} g(x).$$

3. $$\lim_{x \to c} f(x) \cdot g(x) = A \cdot B = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x).$$

4. $$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{A}{B} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}, \quad \text{provided } B \neq 0.$$

Proof. We shall prove two of these results and leave the other two as exercises for the reader.

- Proof. for the first result:

  Let $\epsilon > 0$ be given.

  Consider

$$\begin{aligned} |f(x) + g(x) - (A + B)| &= |f(x) - A + g(x) - B| \\ &\leq |f(x) - A| + |g(x) - B|. \quad \text{By triangle inequality} \end{aligned} \tag{3.3}$$

  Since $\lim_{x\to c} f(x) = A$, given an $\epsilon > 0$, however small, there exists a $\delta_1 > 0$ such that

$$|f(x) - A| < \epsilon/2 \quad \text{whenever} \quad 0 < |x - c| < \delta_1. \tag{3.4}$$

Similarly, for the same $\epsilon > 0$, however small, there exists a $\delta_2 > 0$ such that

$$|f(x) - B| < \epsilon/2 \quad \text{whenever} \quad 0 < |x - c| < \delta_2. \tag{3.5}$$

If $\delta = \min\{\delta_1, \delta_2\}$, then (3.4) and (3.5) hold true for $0 < |x - c| < \delta$.
Using (3.4) and (3.5) in (3.3), we have

$$|f(x) + g(x) - (A + B)| < \epsilon/2 + \epsilon/2 \quad \text{whenever} \quad 0 < |x - c| < \delta. \tag{3.6}$$

That is,

$$|f(x) + g(x) - (A + B)| < \epsilon \quad \text{whenever} \quad 0 < |x - c| < \delta, \tag{3.7}$$

which proves that

$$\lim_{x \to c} (f(x) + g(x)) = A + B = \lim_{x \to c} f(x) + \lim_{x \to c} g(x).$$

- Proof. for the third result:

$$\lim_{x \to c} f(x) \cdot g(x) = A \cdot B = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x).$$

Consider

$$\begin{aligned} |f(x) \cdot g(x) - AB| &= |f(x)g(x) - g(x)A + g(x)A - AB| \\ &= |g(x)(f(x) - A) + A(g(x) - B)| \\ &\leq |g(x)(f(x) - A)| + |A(g(x) - B)| \\ &= |g(x)||f(x) - A| + |A||g(x) - B|. \end{aligned} \tag{3.8}$$

Since $\lim_{x \to c} g(x)$ exits, $g(x)$ is bounded, that is, there exists a real number K such that for

$$0 < |x - c| < \delta_1, \quad |g(x)| \leq K. \tag{3.9}$$

Since $\lim_{x \to c} f(x) = A$, given an $\epsilon > 0$, there exists a $\delta_2 > 0$ such that for

$$0 < |f(x) - A| < \frac{\epsilon}{2K}, \quad 0 < |x - c| < \delta_2. \tag{3.10}$$

Similarly, for the same $\epsilon > 0$ there exists a $\delta_3 > 0$ such that

for

$$0 < |f(x) - A| < \frac{\epsilon}{2|A|}, \quad 0 < |x - c| < \delta_3, \tag{3.11}$$

If $\delta = \min\{\delta_1, \delta_2, \delta_3\}$, then (3.9), (3.10) and (3.11) hold true for $0 < |x - c| < \delta$ so that (3.8) becomes

$$\begin{aligned} |f(x) \cdot g(x) - AB| &< K\frac{\epsilon}{2K} + |A|\frac{\epsilon}{2|A|} && \text{for } |x - c| < \delta \\ &< \epsilon && \text{whenever } 0 < |x - c| < \delta. \end{aligned}$$

Therefore, we have

$$\lim_{x\to c} f(x)\cdot g(x) = A\cdot B = \lim_{x\to c} f(x)\cdot \lim_{x\to c} g(x).$$

$\square$

Theorem 3

1. $$\lim_{x\to c} x^n = c^n.$$

2. $$\lim_{x\to c} b = b.$$

3. $$\lim_{x\to c}(mx+b) = mc+b.$$

Example 9 Find the limits:

1. $$\lim_{x\to 1} 4x^2-2x+5.$$

2. $$\lim_{x\to -1} \frac{2x+1}{x^2-2x+1}.$$

3. $$\lim_{x\to 0} \frac{1-\sqrt{x+1}}{x}.$$

Solution:

1. $$\begin{aligned}\lim_{x\to 1} 4x^2-2x+5 &= 4\lim_{x\to 1} x^2 - 2\lim_{x\to 1} x+5\\ &= 4\cdot 1^2-2\cdot 1+5\\ &= 4-2+5=7.\end{aligned}$$

2. $$\begin{aligned}\lim_{x\to -1} \frac{2x+1}{x^2-2x+1} &= \frac{2\lim\limits_{x\to -1} x+1}{\lim\limits_{x\to -1} x^2-2\lim\limits_{x\to -1} x+1}\\ &= \frac{2\cdot(-1)+1}{(-1)^2-2(-1)+1}\\ &= \frac{-1}{4}.\end{aligned}$$

3.

$$
\begin{aligned}
\lim_{x\to 0}\frac{1-\sqrt{x+1}}{x} &= \lim_{x\to 0}\frac{1-\sqrt{x+1}}{x}\cdot\frac{1+\sqrt{x+1}}{1+\sqrt{x+1}}\\
&= \lim_{x\to 0}\frac{1-(x+1)}{x(1+\sqrt{x+1})}\\
&= -\lim_{x\to 0}\frac{x}{x}\cdot\frac{1}{1+\sqrt{x+1}}\\
&= \frac{1}{1+\lim_{x\to 0}\sqrt{x+1}}\\
&= -\frac{1}{2}.
\end{aligned}
$$

□

Theorem 4

Suppose $\lim_{x\to c} f(x) = A$.

1. If n is a positive integer, then

$$\lim_{x\to c}(f(x))^n = \left(\lim_{x\to c} f(x)\right)^n = A^n.$$

2. If n is a positive integer, and either
   - n is odd, or
   - n is even, and $A > 0$,

   then

$$\lim_{x\to c}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to c} f(x)} = \sqrt[n]{A}.$$

3. If b is a real constant, then

$$\lim_{x\to c}(bf(x)) = b\cdot\lim_{x\to c} f(x) = b\cdot A.$$

Note 3 From the above theorem, we notice that

1. the limit of a power equals the power of the limit.
2. the limit of a root equals the root of the limit.
3. the limit of a constant multiple equals the constant multiple of the limit.

3.4 One-sided limits

Consider a function f defined by

$$f(x) = \sqrt{1-x}.$$

Here $f(x)$ does not exist for x greater than 1. Therefore, f is not defined for any open interval containing 1. Hence we cannot define the limit of this function as x tends to 1. But, if we consider the values of the function for x less than 1, then the value of the function can be made to approach zero. In such a case, we consider the limit on only one side. In the above case, the limit is said to be *right-handed* as $x \to 0^+$ (we will see examples of this later). Thus, we have the following one-sided limits.

3.4.1 Left-hand limit

Definition 2 Let f be a function defined for values $x < c$. Then the limit of $f(x)$ as x approaches c through values less than c, that is, as x approaches from the left, that is, the left-hand limit L of $f(x)$ is defined as follows:

Given an $\epsilon > 0$, however small, there exists a $\delta > 0$ such that

$$0 < |f(x) - L| < \epsilon \text{ whenever } \quad 0 < c - x < \delta.$$

It is written as

$$\lim_{x \to c^-} f(x) = L \text{ or } \lim_{x \to c-0} f(x) = L.$$

Note 4 To find the left-hand limit, replace x by $c - h$ $(h > 0)$ and make h tend to zero.

3.4.2 Right-hand limit

Definition 3 Let f be a function defined for values $x > c$. Then the limit of $f(x)$ as x approaches c from the right, that is, the right-hand limit R of $f(x)$ is defined as follows:

Given an $\epsilon > 0$, however small, there exists a $\delta > 0$ such that

$$0 < |f(x) - R| < \epsilon \text{ whenever } \quad 0 < x - c < \delta.$$

It is written as

$$\lim_{x \to c^+} f(x) = R \text{ or } \lim_{x \to c+0} f(x) = R.$$

Note 5 To find the right-hand limit, replace x by $c + h$ $(h > 0)$ and make h tend to zero.

Theorem 5 The limit of $f(x)$ as x tends to exists if and only if

$$\lim_{x \to c^-} f(x) \quad \text{and} \quad \lim_{x \to c^+} f(x)$$

exist and are equal.

Example 10 A function f is defined as follows:

$$f(x) = \begin{cases} x^2, & \text{if } x \leq 2; \\ 8 - 2x, & \text{if } x > 2. \end{cases}$$

Find

1. $\lim_{x \to 2^-} f(x)$;
2. $\lim_{x \to 2^+} f(x)$;
3. $\lim_{x \to 2} f(x)$ if it exists.

Solution:

Figure 3.2 displays the graph of the function f.

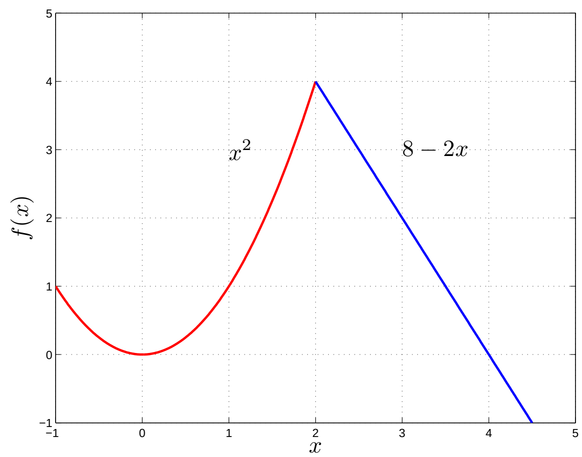

Figure 3.2: Graph of $f(x)$, where $x \in [-1, 5]$.

1. We have
$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2} x^2 = 4.$$
Since $\lim_{x \to 2^-} f(x)$ means x is tending towards 2 through values less than 2 and the values of the function for these values of x are defined by $f(x) = x^2$.

2. We have
$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2} 8 - 2x = 8 - 4 = 4.$$

3. Since
$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x),$$
we have
$$\lim_{x \to 2} f(x) = 4.$$

□

Example 11 A function f is defined as follows:

$$f(x) = \frac{|x|}{x}.$$

Find

1. $\lim_{x \to 0^-} f(x)$;
2. $\lim_{x \to 0^+} f(x)$;
3. $\lim_{x \to 0} f(x)$ if it exists.

Solution:

Figure 3.3 displays the graph of the function f.

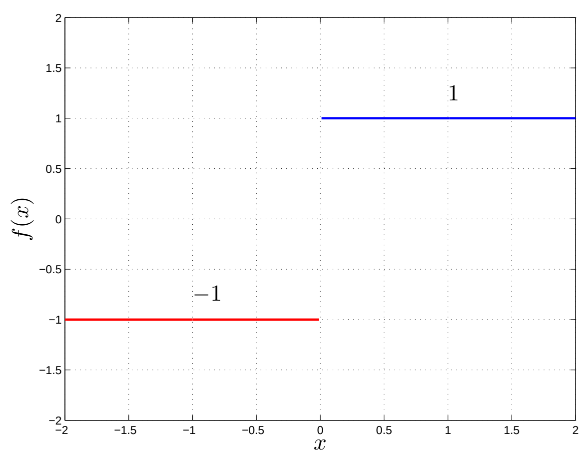

Figure 3.3: Graph of $f(x)$, where $x \in [-2, 2]$.

1. We have
$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{|x|}{x} = \lim_{h \to 0^-} \frac{|0-h|}{0-h} = \lim_{h \to 0} \frac{h}{-h} = -1.$$
2. We have
$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{|x|}{x} = \lim_{h \to 0^+} \frac{|0+h|}{0+h} = \lim_{h \to 0} \frac{h}{h} = 1.$$
3. Since
$$\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x),$$
the limit
$$\lim_{x \to 0} f(x)$$
does not exist.

□

Example 12 A function f is defined as follows:

$$f(x) = [x].$$

Find

1. $\lim_{x \to c^-} f(x)$;
2. $\lim_{x \to c^+} f(x)$;

where c is any integer.

Solution:

1. We have
$$\lim_{x \to c^-} f(x) = \lim_{h \to 0}(c - h) = \lim_{h \to 0}(c - 1 - h) = c - 1.$$
2. We have
$$\lim_{x \to c^+} f(x) = \lim_{h \to 0}(c + h) = \lim_{h \to 0} c = c.$$

□

Example 13 A function f is defined as follows:

$$f(x) = \sqrt{x}.$$

Find

1. $\lim_{x \to -1} f(x)$;
2. $\lim_{x \to 0^+} f(x)$;
3. $\lim_{x \to 0^-} f(x)$.

Solution:

1. $\lim_{x \to -1} f(x)$ does not exist as a real number.
2. $\lim_{x \to 0^+} f(x) = 0$.
3. $\lim_{x \to 0^-} f(x)$ does not exist.

It is impossible to approach 0 from the left while staying in the domain of x. A limit can only exist if you can approach through the domain set $x \in [0, \infty)$.

Therefore, $\lim_{x \to 0} f(x)$ does not exist as a real number. □

Let us have a further discussion on the limit of a root.

Theorem 6 Assume for now that n is a positive even integer.
Three cases are summarized:

1. $\lim_{x \to c} f(x) = A$, then Theorem 4-2 applies if $A > 0$.
2. If $A < 0$, then $\lim_{x \to c} f(x) = A$ does not exist as a real number.
3. If $A = 0$, then $\lim_{x \to c} \sqrt[n]{f(x)} = 0$ if and only if $f(x)$ stays nonnegative in value as x approaches c.

More precisely, the limit is 0 if and only if $f(x) \geq 0$ on some open $x-$interval that contains c, while possibly excluding c itself. Change this to an interval of the form (c, d) for a right-hand limit, and the form (d, c) for c left-hand limit, where d is a real constant. Otherwise, the limit does not exist.

Example 14 Evaluate

$$\lim_{t \to 1^+} \sqrt{3t^2 - 3}.$$

Solution:

As $t \to 1^+$, $3t^2 - 3 \to 0$.
As $t \to 1^+$, $t > 1$.
Now, we have

$$\begin{aligned} t &> 1 \\ t^2 &> 1 \\ 3t^2 &> 3 \\ 3t^2 - 3 &> 0. \end{aligned}$$

Therefore,

$$\lim_{t \to 1^+} \sqrt{3t^2 - 3} = 0.$$

□

Example 15 Evaluate

$$\lim_{t \to 1^+} \left(7\sqrt{3t^2 - 3} + 5\right).$$

Solution:

$$\begin{aligned} \lim_{t \to 1^+} \left(7\sqrt{3t^2 - 3} + 5\right) &= \lim_{t \to 1^+} 7\sqrt{3t^2 - 3} + \lim_{t \to 1^+} 5 \\ &= 7\left(\lim_{t \to 1^+} \sqrt{3t^2 - 3}\right) + 5 \\ &= 7(0) + 5 && \text{see Example (14)} \\ &= 5. \end{aligned}$$

□

3.5 Limits at infinity

3.5.1 Limit of $f(x)$ as x tends to $+\infty$

Definition 4 Let f be a function which is defined for all $x > c$. The limit of $f(x)$ as x tends to infinity is k if given an $\epsilon > 0$, however small, there exist an $N > 0$ such that

$$|f(x) - A| < \epsilon \quad \text{whenever} \quad x > N.$$

Example 16 Show

$$\lim_{x\to\infty} \frac{2x+1}{5x-2} = \frac{2}{5}$$

using the $\epsilon-$definition.

Solution:

Consider

$$\begin{aligned} \left|\frac{2x+1}{5x-2} - \frac{2}{5}\right| &= \left|\frac{10x+5-10x+4}{5(5x-2)}\right| \\ &= \frac{9}{5} \cdot \frac{1}{|5x-2|} \\ &= \frac{9}{5} \cdot \frac{1}{5x-2}. \end{aligned}$$

As x is increasing without bound for $x > 2/5$, $(5x-2)$ is positive. Therefore,

$$|5x-2| = 5x-2.$$

Now, $|f(x) - A| < \epsilon$ gives

$$\frac{9}{5(5x-2)} < \epsilon.$$

That is,

$$\frac{5(5x-2)}{9} > \frac{1}{\epsilon},$$

or

$$5x - 2 > \frac{9}{5\epsilon}.$$

Therefore,

$$5x > \frac{9}{5\epsilon} + 2.$$

That is,

$$x > \frac{9}{25\epsilon} + \frac{2}{5}.$$

If $N = \dfrac{9}{25\epsilon} + \dfrac{2}{5}$, then we have

$$\left|\frac{2x+1}{5x-2} - \frac{2}{5}\right| < \epsilon \text{ whenever } x > N.$$

$\square$

Note 6 Any real number greater than $N = \dfrac{9}{25\epsilon} + \dfrac{2}{5}$ can also be taken as N.

Theorem 7 If r is any positive integer, then

$$\lim_{x\to\infty} \frac{1}{x^r} = 0.$$

Proof. We have to show that given an $\epsilon > 0$, there exists an $N > 0$, such that

$$\left|\frac{1}{x^r} - 0\right| < \epsilon \quad \text{whenever} \quad x > N.$$

Consider

$$\left|\frac{1}{x^r} - 0\right| < \epsilon.$$

That is ,

$$\left|\frac{1}{x}\right| < \epsilon^r,$$

or ,

$$|x| < \frac{1}{\epsilon^r}.$$

If we take $N = \dfrac{1}{\epsilon^r}$, then we have

$$|f(x) - A| < \epsilon \quad \text{whenever} \quad x > \frac{1}{\epsilon^r} = N.$$

By assuming x to be positive and by letting $x \to \infty$, we have

$$\lim_{x\to\infty} \frac{1}{x^r} = 0.$$

□

Example 17 Find the limit:

$$\lim_{x\to\infty} \frac{2x^2 - 3x + 5}{9x^2 + 8x + 7}.$$

Solution:

$$\begin{aligned}
\lim_{x\to\infty} \frac{2x^2 - 3x + 5}{9x^2 + 8x + 7} &= \lim_{x\to\infty} \frac{\dfrac{2x^2 - 3x + 5}{x^2}}{\dfrac{9x^2 + 8x + 7}{x^2}} \\
&= \lim_{x\to\infty} \frac{2 - \frac{3}{x} + \frac{5}{x^2}}{9 + \frac{8}{x} + \frac{7}{x^2}} \\
&= \frac{2 - 3\lim\limits_{x\to\infty} \frac{1}{x} + 5\lim\limits_{x\to\infty} \frac{1}{x^2}}{9 + 8\lim\limits_{x\to\infty} \frac{1}{x} + 7\lim\limits_{x\to\infty} \frac{1}{x^2}} \\
&= \frac{2 - 0 + 0}{9 + 0 + 0} \\
&= \frac{2}{9}.
\end{aligned}$$

$\square$

3.5.2 Limit of $f(x)$ as x tends to $-\infty$.

Definition 5 A function $f(x)$ defined for all $x < c$ is said to tend to a limit k as x tends to $-\infty$ if given an $\epsilon > 0$, there exists an $N < 0$, such that

$$|f(x) - A| < \epsilon \quad \text{whenever} \quad x < N.$$

Note 7 If n is a positive integer, then

$$\lim_{x \to -\infty} \frac{1}{x^n} = 0.$$

Example 18 A function f is defined as follows:

$$f(x) = \frac{x}{\sqrt{x^2 + 1}}.$$

Find

1. $\lim_{x \to +\infty} f(x)$;

2. $\lim_{x \to -\infty} f(x)$.

Solution:

Figure 3.4 displays the graph of the function f.

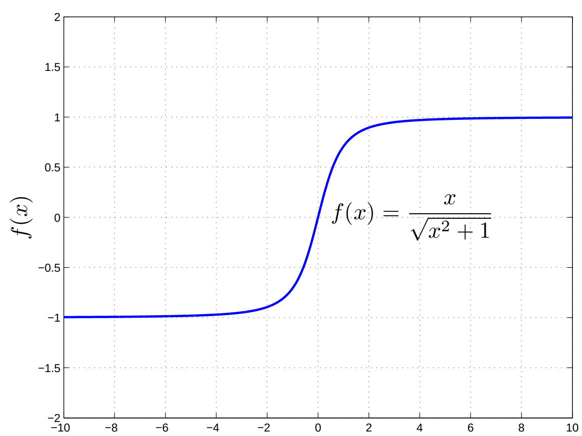

Figure 3.4: Graph of $f(x)$, where $x \in [-10, 10]$.

1. We write $x = \sqrt{x^2}$ ($x > 0$, because $x \to +\infty$). Then

$$\begin{aligned}
\lim_{x\to+\infty} f(x) &= \lim_{x\to+\infty} \frac{\sqrt{x^2}}{\sqrt{x^2+1}} \\
&= \lim_{x\to+\infty} \left(\frac{x^2}{x^2+1}\right)^{\frac{1}{2}} \\
&= \lim_{x\to+\infty} \left(\frac{\frac{x^2}{x^2}}{\frac{x^2+1}{x^2}}\right)^{\frac{1}{2}} \\
&= \lim_{x\to+\infty} \left(\frac{1}{1+\frac{1}{x^2}}\right)^{\frac{1}{2}} \\
&= \lim_{x\to+\infty} \frac{1}{\sqrt{1+\frac{1}{x^2}}}.
\end{aligned}$$

Therefore, we have

$$\lim_{x\to+\infty} \frac{x}{\sqrt{x^2+1}} = 1.$$

2. We write $x = -\sqrt{x^2}$ ($x < 0$, because $x \to -\infty$). Then

$$\begin{aligned}
\lim_{x\to-\infty} f(x) &= \lim_{x\to+\infty} \frac{-\sqrt{x^2}}{\sqrt{x^2+1}} \\
&= \lim_{x\to-\infty} -\left(\frac{x^2}{x^2+1}\right)^{\frac{1}{2}} \\
&= \lim_{x\to-\infty} -\left(\frac{\frac{x^2}{x^2}}{\frac{x^2+1}{x^2}}\right)^{\frac{1}{2}} \\
&= \lim_{x\to-\infty} -\left(\frac{1}{1+\frac{1}{x^2}}\right)^{\frac{1}{2}} \\
&= \lim_{x\to-\infty} \frac{-1}{\sqrt{1+\frac{1}{x^2}}}.
\end{aligned}$$

Therefore, we have

$$\lim_{x\to-\infty} \frac{x}{\sqrt{x^2+1}} = -1.$$

$\square$

3.6 Infinite limits

Definition 6 Let f be a function defined in some open interval containing c, except possibly at c itself. Then the limit of $f(x)$, as x tends to c, is $+\infty$ written as

$$\lim_{x\to c} f(x) = +\infty$$

if for any $N > 0$, there exists $\delta > 0$ such that an $\epsilon > 0$, there exists an $N < 0$, such that

$$f(x) > N \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

Example 19 Show that

$$\lim_{x \to 0^+} \frac{1}{x} = +\infty.$$

Proof.

Let $N > 0$ be given.
Consider $\frac{1}{x} > N$ which implies $x < \frac{1}{N}$.
That is,

$$|x - 0| < \frac{1}{N}.$$

By taking $\delta = \frac{1}{N}$, the result is valid. □

Definition 7 Let f be a function defined in an open interval containing c, except possibly at c itself. Then the limit of $f(x)$, as x tends to c, is $-\infty$ written as

$$\lim_{x \to c} f(x) = -\infty$$

if for any $N < 0$, there exists $\delta > 0$ such that an $\epsilon > 0$, there exists an $N < 0$, such that

$$f(x) < N \quad \text{whenever} \quad 0 < |x - c| < \delta.$$