Calculus for Engineers

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Contents

Polar Coordinates

11.1 Introduction

In Section 11.2, the definition of polar coordinates is given. Converting from Cartesian coordinates to polar coordinates is a key point. In Section 11.3, we present a list of polar curves. Emphasis is placed upon the variation of the angle θ. In Section 11.4, with the aid of the implicit differentiation technique, finding the slope of the tangent line to a curve at a point is examined.

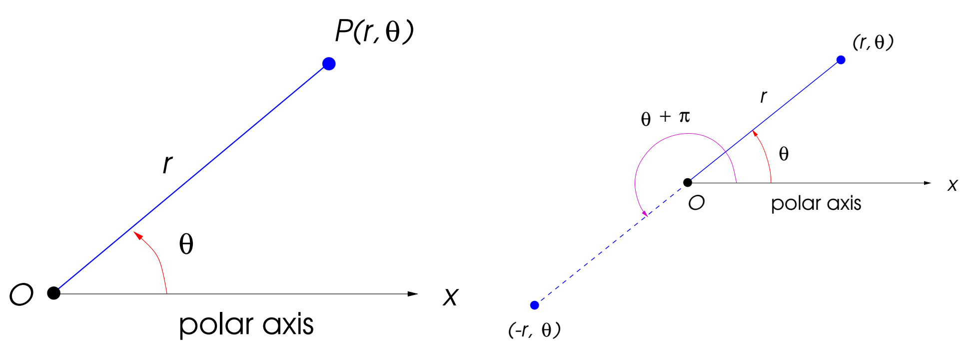
11.2 What are polar coordinates?

Definition 1 The polar coordinate system is a two-dimensional coordinate system in which each point P on a plane is determined by a distance r from a fixed point O that is called the pole (or origin) and an angle θ from a fixed direction. The point P is represented by the ordered pair (r, θ) which are called **polar coordinates**.

Note 1 We extend the meaning of polar coordinates (r, θ) to the case in which r is negative by agreeing that the points (r, θ) and $(-r, \theta)$ lie on the same line through O and at the same distance $|r|$ from O; but on opposite sides of O. Two cases are summarized:

- If $r > 0$, the point (r, θ) lies in the same quadrant as θ.
- If $r < 0$, it lies in the quadrant on the opposite side of the pole.

Figure 11.1 illustrates these two cases.



(a) $r > 0$ (b) $r < 0$

Figure 11.1: Graphs of (r, θ) when $r > 0$ and $r < 0$.

Example 1 Plot the points whose polar coordinates are given:

1. $(1, 5\pi/4)$;
2. $(2, 3\pi)$;
3. $(2, -2\pi/3)$;
4. $(-3, 3\pi 4)$.

Solution. Exercise $\square$

Note 2 In the Cartesian coordinate system every point has only one representation, but in the polar coordinate system each point has many representations. For instance, the point $(1, 5\pi/4)$ in Example 1 could be written as $(1, -3\pi/4)$ or $(1, 13\pi/4)$ or $(-1, \pi/4)$.

Example 2 Find all the polar coordinates of the point $P(2, \pi/6)$ and sketch the corresponding figure(s).

Solution. We sketch the initial ray of the coordinate system, draw the ray from the origin that makes an angle of $\pi/6$ radians with the initial ray, and mark the point $(2, \pi/6)$. We then find the angles for the other coordinate pairs of P in which $r = 2$ and $r = -2$.

For $r = 2$, the complete list of angles is

$$\frac{\pi}{6}, \frac{\pi}{6} \pm 2\pi, \frac{\pi}{6} \pm 4\pi, \frac{\pi}{6} \pm 6\pi, \cdots .$$

For $r = -2$, the complete list of angles is

$$-\frac{5\pi}{6}, -\frac{5\pi}{6} \pm 2\pi, -\frac{5\pi}{6} \pm 4\pi, -\frac{5\pi}{6} \pm 6\pi, \cdots .$$

The corresponding coordinate pairs of P are

$$\left(2, \frac{\pi}{6} \pm 2n\pi\right), \quad n = 0, \pm 1, \pm 2, \cdots$$

and

$$\left(-2, -\frac{5\pi}{6} \pm 2n\pi\right), \quad n = 0, \pm 1, \pm 2, \cdots .$$

When $n = 0$, the formulas give $(2, \pi/6)$ and $(-2, 5\pi/6)$. When $n = 1$, they give $(2, 13\pi/6)$ and $(-2, 7\pi/6)$, and so on.

$\square$

11.2.1 Connection between polar and Cartesian coordinates

The connection between polar and Cartesian coordinates can be seen from Figure 11.2 and are described by the following formulae:

$$
\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta.
\end{aligned}
$$

and

$$
\begin{aligned}
r^2 &= x^2 + y^2 \\
\tan\theta &= \frac{y}{x}
\end{aligned}
$$

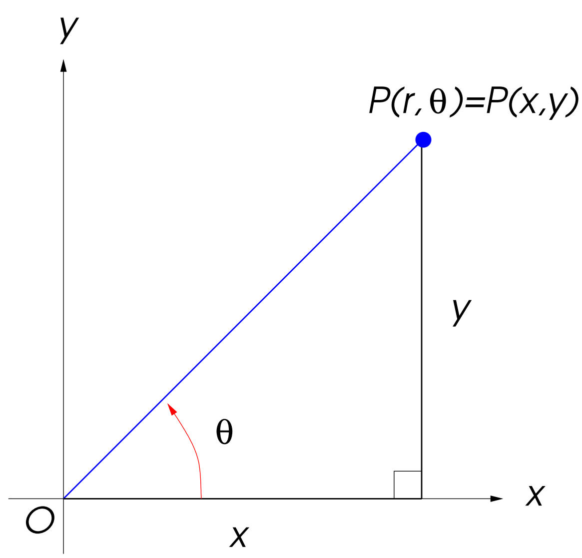


Figure 11.2: Connection between polar and Cartesian coordinates.

Example 3 Answer the following questions:

1. Convert the point $(2, \pi/3)$ from polar to Cartesian coordinates.
2. Represent the point with Cartesian coordinates $(1, -1)$ in terms of polar coordinates.

Solution.

1. We have

$$
\begin{cases}
x = r\cos\theta = 2\cdot\cos\left(\dfrac{\pi}{3}\right) = 2\cdot\dfrac{1}{2} = 1; \\
y = r\sin\theta = 2\cdot\sin\left(\dfrac{\pi}{3}\right) = 2\cdot\dfrac{\sqrt{3}}{2} = \sqrt{3}.
\end{cases}
$$

Therefore, the point is $(1, \sqrt{3})$ in Cartesian coordinates.

2. If we choose r to be positive, then

$$\begin{cases} r = \sqrt{x^2 + y^2} = \sqrt{1^2 + (-1)^2} = \sqrt{2} \\ \tan\theta = \frac{y}{x} = -1. \end{cases}$$

Since the point $(1, -1)$ lies in the fourth quadrant, we can choose $\theta = -\pi/4$ or $\theta = 7\pi/4$. Thus one possible answer is $(\sqrt{2}, -\pi/4)$, while another is $(\sqrt{2}, 7\pi/4)$.

□

Example 4 Express the equation $x = 1$ in polar coordinates.

Solution. We use the formula $x = r\cos\theta$. Then

$$\begin{aligned} x &= 1 \\ r\cos\theta &= 1 \\ r &= \sec\theta. \end{aligned}$$

□

Example 5 Express the equation $x^2 = 4y$ in polar coordinates.

Solution.
We use the formula $x = r\cos\theta$ and $y = r\sin\theta$. Then

$$\begin{aligned} x^2 &= 4y \\ (r\cos\theta)^2 &= 4r\sin\theta \\ r^2\cos^2\theta &= 4r\sin\theta \\ r &= 4\cdot\frac{\sin\theta}{\cos^2\theta} = 4\sec\theta\tan\theta. \end{aligned}$$

□

11.3 What are polar curves?

The graph of a polar equation $r = f(\theta)$ or more generally $F(r, \theta) = 0$ consists of all points P that have at least one polar representation (r, θ) whose coordinates satisfy the equation.

Example 6 Sketch the polar curve $\theta = 1$.

Solution.
This curve consists of all points (r, θ) such that the polar angle θ is 1 radian. It is the straight line that passes through O and makes an angle of 1 radian with the polar axis. Notice that the points $(r, 1)$ on the line with $r > 0$ are in the first quadrant, whereas those with $r < 0$ are in the third quadrant. Figure 11.3 illustrates all these cases.

□

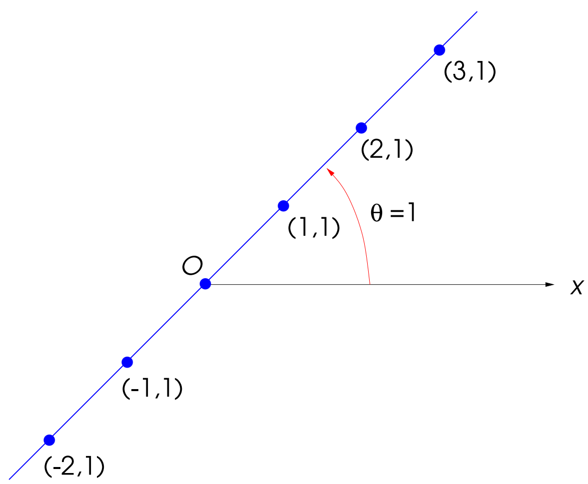


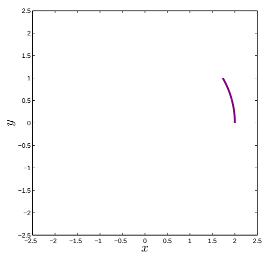
Figure 11.3: Graph of the polar curve $\theta = 1$.

Example 7 Sketch the following curves:

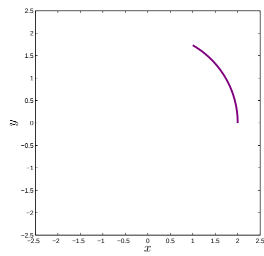
1. $r = 2$, $0 \le \theta \le 2\pi$.
2. $r = 2\cos\theta$, $0 \le \theta \le \pi$.

Solution.

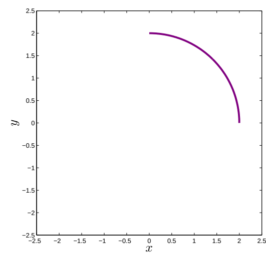
1. We have



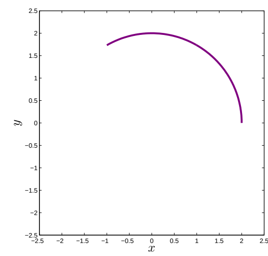
(a) $r = 2$, $\theta = \pi/6$.



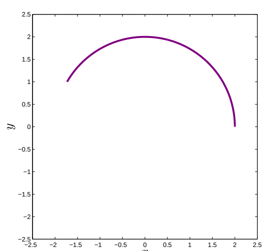
(b) $r = 2$, $\theta = 2\pi/6$.



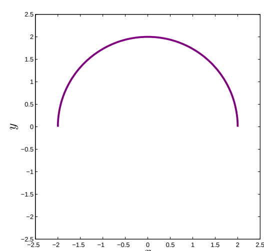
(c) $r = 2$, $\theta = 3\pi/6$.



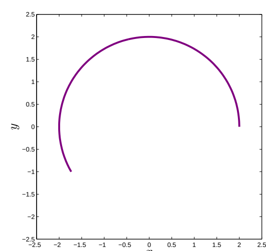
(d) $r = 2$, $\theta = 4\pi/6$.



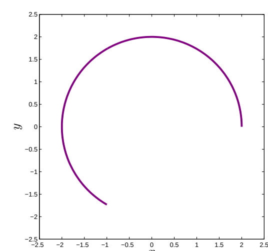
(e) $r = 2$, $\theta = 5\pi/6$.



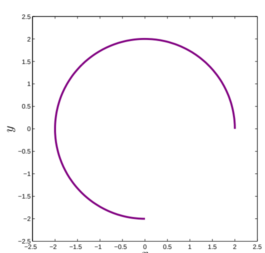
(f) $r = 2$, $\theta = 6\pi/6$.



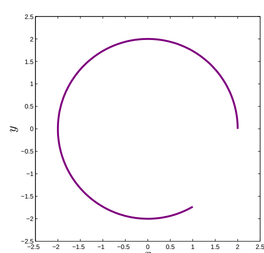
(g) $r = 2$, $\theta = 7\pi/6$.



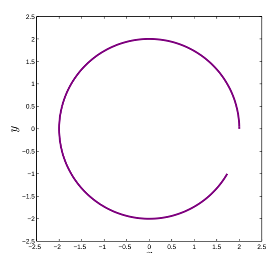
(h) $r = 2$, $\theta = 8\pi/6$.



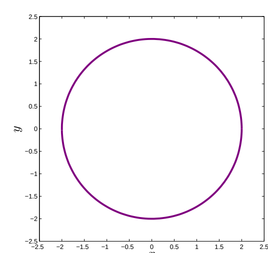
(i) $r = 2$, $\theta = 9\pi/6$.



(j) $r = 2$, $\theta = 10\pi/6$.



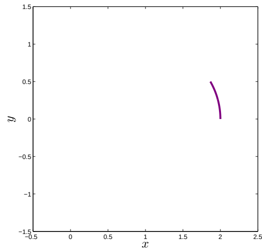
(k) $r = 2$, $\theta = 11\pi/6$.



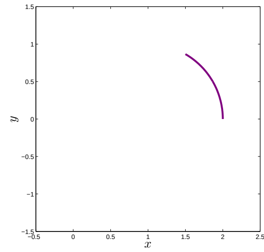
(l) $r = 2$, $\theta = 12\pi/6$.

Figure 11.4: Solutions for Example 7-1.

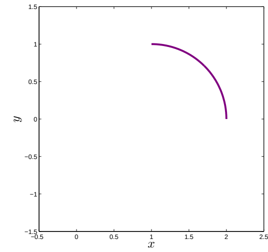
2. We have



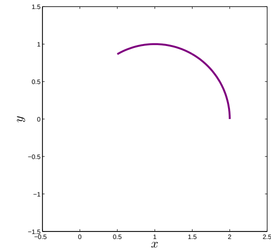
(a) $r = 2\cos\theta$, $\theta = \pi/12$.



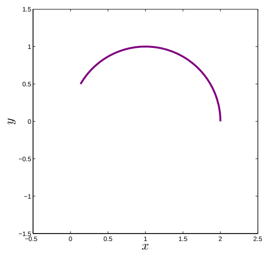
(b) $r = 2\cos\theta$, $\theta = 2\pi/12$.



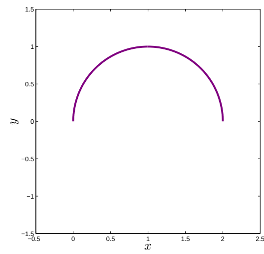
(c) $r = 2\cos\theta$, $\theta = 3\pi/12$.



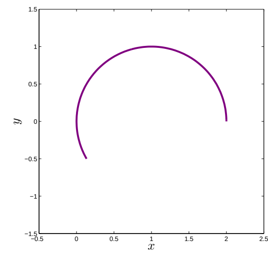
(d) $r = 2\cos\theta$, $\theta = 4\pi/12$.



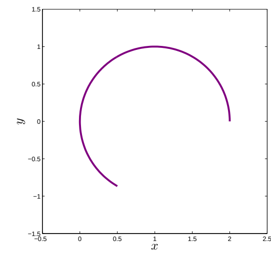
(e) $r = 2\cos\theta$, $\theta = 5\pi/12$.



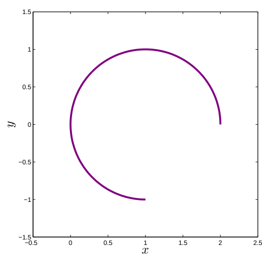
(f) $r = 2\cos\theta$, $\theta = 6\pi/12$.



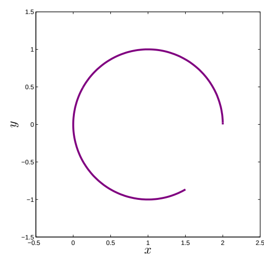
(g) $r = 2\cos\theta$, $\theta = 7\pi/12$.



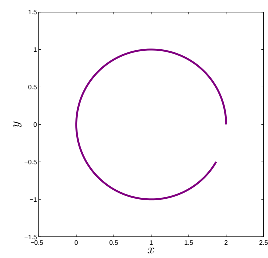
(h) $r = 2\cos\theta$, $\theta = 8\pi/12$.



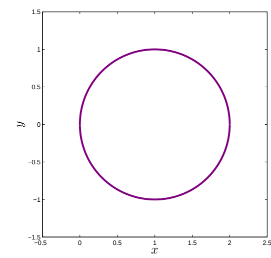
(i) $r = 2\cos\theta$, $\theta = 9\pi/12$.



(j) $r = 2\cos\theta$, $\theta = 10\pi/12$.



(k) $r = 2\cos\theta$, $\theta = 11\pi/12$.



(l) $r = 2\cos\theta$, $\theta = 12\pi/12$.

Figure 11.5: Solutions for Example 7-2.

□

Example 8 Express the polar equation $r = 2\cos\theta$ in rectangular coordinates.

Solution.

We use the formulas $r^2 = x^2 + y^2$ and $x = r\cos\theta$. We have

$$\begin{aligned} r &= 2\cos\theta \\ r^2 &= 2r\cos\theta \\ x^2 + y^2 &= 2x \\ x^2 - 2x + y^2 &= 0 \\ x^2 - 2x + 1 + y^2 &= 1 \\ (x-1)^2 + y^2 &= = 1. \end{aligned}$$

□

Example 9 Express the polar equation in rectangular coordinates. If possible, determine the graph of the equation from its rectangular form.

1. $r = 5\sec\theta$;
2. $r = 2\sin\theta$;
3. $r = 2 + 2\cos\theta$.

Solution. We use the formulae $r^2 = x^2 + y^2$, $x = r\cos\theta$ and $y = r\cos\theta$.

1. We have

$$\begin{aligned} r &= 5\sec\theta \\ r\cos\theta &= 5 \\ x &= 5. \end{aligned}$$

2. We have

$$\begin{aligned} r &= 2\sin\theta \\ r^2 &= 2r\sin\theta \\ x^2 + y^2 &= 2y \\ x^2 + y^2 - 2y &= 0 \\ x^2 + y^2 - 2y + 1 &= 1 \\ x^2 + (y-1)^2 &= 1. \end{aligned}$$

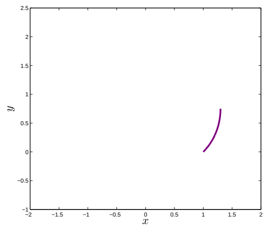
3. We have

$$\begin{aligned} r &= 2 + 2\cos\theta \\ r^2 &= 2r + 2r\cos\theta \\ x^2 + y^2 &= 2r + 2x \\ x^2 - 2x + y^2 &= 2r \\ \left(x^2 - 2x + y^2\right)^2 &= (2r)^2 \\ \left(x^2 - 2x + y^2\right)^2 &= 4\left(x^2 + y^2\right). \end{aligned}$$

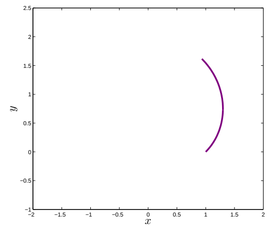
$\square$

Example 10 Sketch the curve $r = 1 + \sin\theta$, $0 \le \theta \le 2\pi$ (cardioid).

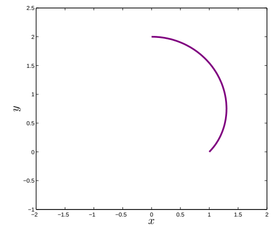
Solution. We have



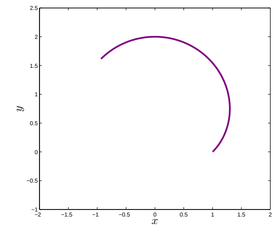
(a) $r = 1 + \sin\theta$, $\theta = \pi/6$.



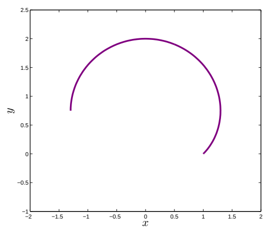
(b) $r = 1 + \sin\theta$, $\theta = 2\pi/6$.



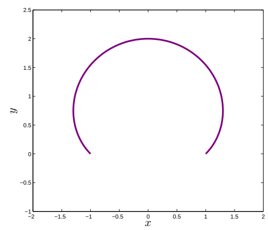
(c) $r = 1 + \sin\theta$, $\theta = 3\pi/6$.



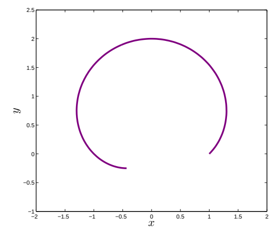
(d) $r = 1 + \sin\theta$, $\theta = 4\pi/6$.



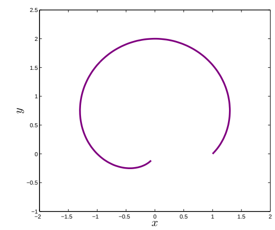
(e) $r = 1 + \sin\theta$, $\theta = 5\pi/6$.



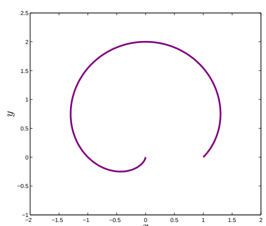
(f) $r = 1 + \sin\theta$, $\theta = 6\pi/6$.



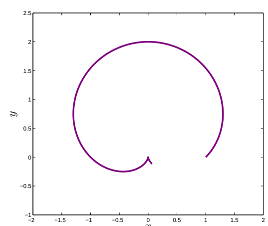
(g) $r = 1 + \sin\theta$, $\theta = 7\pi/6$.



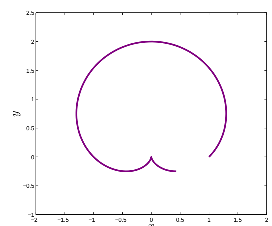
(h) $r = 1 + \sin\theta$, $\theta = 8\pi/6$.



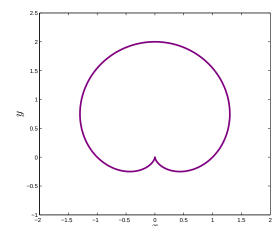
(i) $r = 1 + \sin\theta$, $\theta = 9\pi/6$.



(j) $r = 1 + \sin\theta$, $\theta = 10\pi/6$.



(k) $r = 1 + \sin\theta$, $\theta = 11\pi/6$.



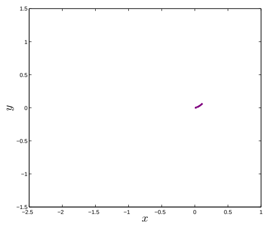
(l) $r = 1 + \sin\theta$, $\theta = 12\pi/6$.

Figure 11.6: Solutions for Example 10.

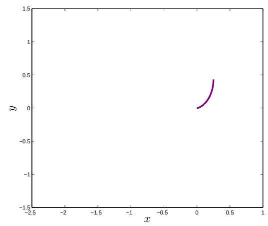
$\square$

Example 11 Sketch the curve $r = 1 - \cos\theta$, $0 \le \theta \le 2\pi$ (cardioid).

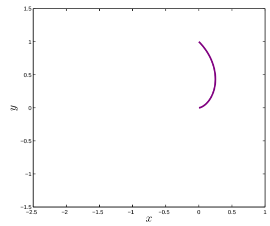
Solution. We have



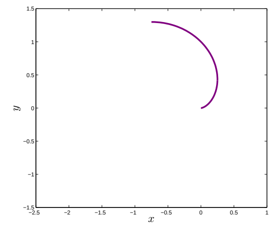
(a) $r = 1 - \cos\theta$, $\theta = \pi/6$.



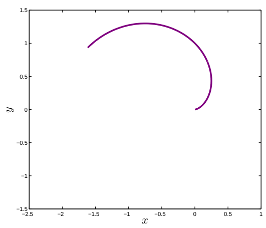
(b) $r = 1 - \cos\theta$, $\theta = 2\pi/6$.



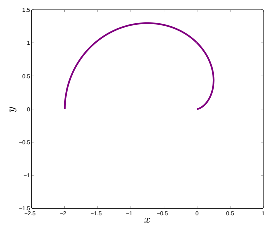
(c) $r = 1 - \cos\theta$, $\theta = 3\pi/6$.



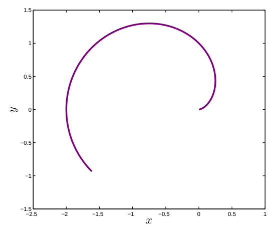
(d) $r = 1 - \cos\theta$, $\theta = 4\pi/6$.



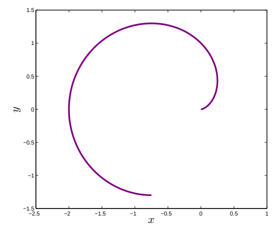
(e) $r = 1 - \cos\theta$, $\theta = 5\pi/6$.



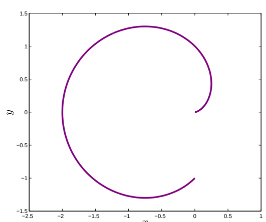
(f) $r = 1 - \cos\theta$, $\theta = 6\pi/6$.



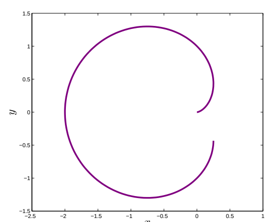
(g) $r = 1 - \cos\theta$, $\theta = 7\pi/6$.



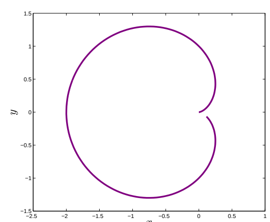
(h) $r = 1 - \cos\theta$, $\theta = 8\pi/6$.



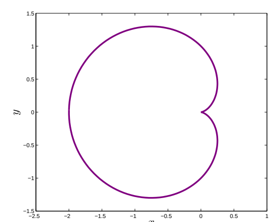
(i) $r = 1 - \cos\theta$, $\theta = 9\pi/6$.



(j) $r = 1 - \cos\theta$, $\theta = 10\pi/6$.



(k) $r = 1 - \cos\theta$, $\theta = 11\pi/6$.



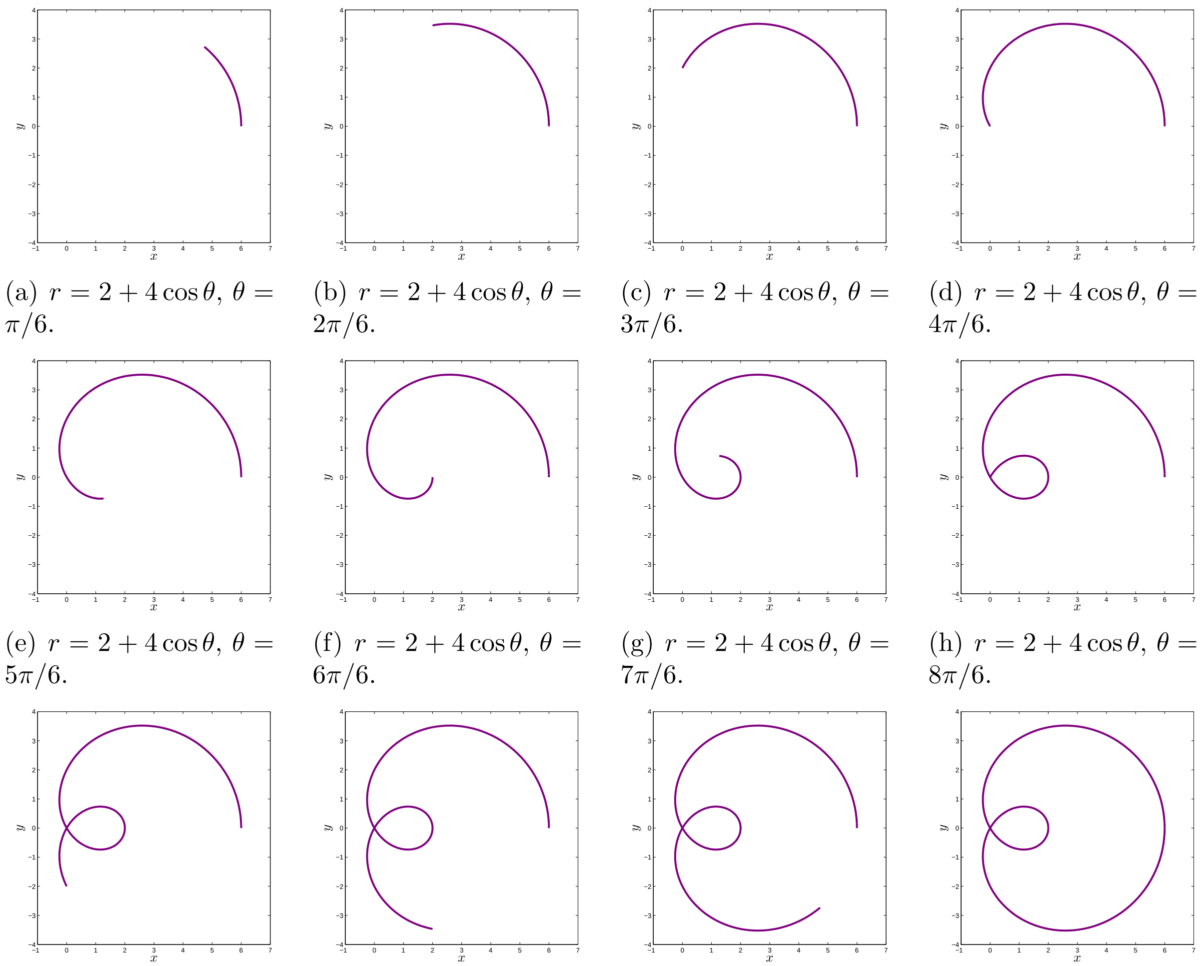
(l) $r = 1 - \cos\theta$, $\theta = 12\pi/6$.

Figure 11.7: Solutions for Example 11.

$\square$

Example 12 Sketch the curve $r = 2 + 4\cos\theta$, $0 \le \theta \le 2\pi$.

Solution. We have



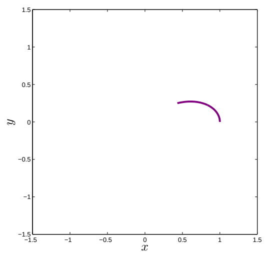
(a) $r = 2 + 4\cos\theta$, $\theta = \pi/6$.
(b) $r = 2 + 4\cos\theta$, $\theta = 2\pi/6$.
(c) $r = 2 + 4\cos\theta$, $\theta = 3\pi/6$.
(d) $r = 2 + 4\cos\theta$, $\theta = 4\pi/6$.
(e) $r = 2 + 4\cos\theta$, $\theta = 5\pi/6$.
(f) $r = 2 + 4\cos\theta$, $\theta = 6\pi/6$.
(g) $r = 2 + 4\cos\theta$, $\theta = 7\pi/6$.
(h) $r = 2 + 4\cos\theta$, $\theta = 8\pi/6$.
(i) $r = 2 + 4\cos\theta$, $\theta = 9\pi/6$.
(j) $r = 2 + 4\cos\theta$, $\theta = 10\pi/6$.
(k) $r = 2 + 4\cos\theta$, $\theta = 11\pi/6$.
(l) $r = 2 + 4\cos\theta$, $\theta = 12\pi/6$.

Figure 11.8: Solutions for Example 12.

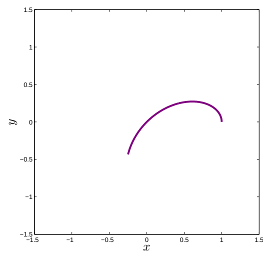
□

Example 13 Sketch the curve $r = \cos(2\theta)\,,\ 0 \le \theta \le 2\pi$ (four-leaved flower).

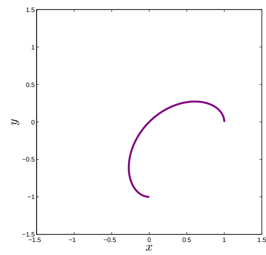
Solution. We have



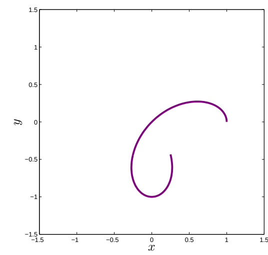
(a) $r = \cos(2\theta)\,,\ \theta = \pi/6$.



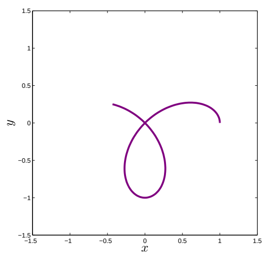
(b) $r = \cos(2\theta)\,,\ \theta = 2\pi/6$.



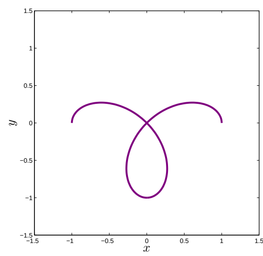
(c) $r = \cos(2\theta)\,,\ \theta = 3\pi/6$.



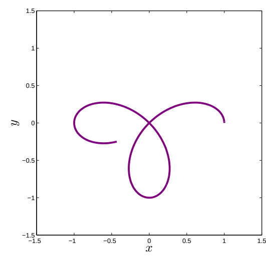
(d) $r = \cos(2\theta)\,,\ \theta = 4\pi/6$.



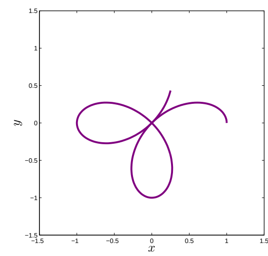
(e) $r = \cos(2\theta)\,,\ \theta = 5\pi/6$.



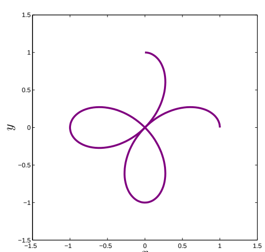
(f) $r = \cos(2\theta)\,,\ \theta = 6\pi/6$.



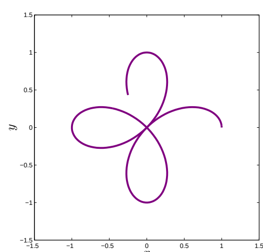
(g) $r = \cos(2\theta)\,,\ \theta = 7\pi/6$.



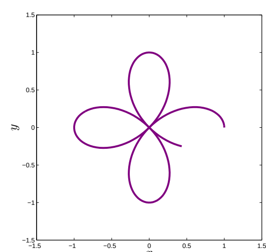
(h) $r = \cos(2\theta)\,,\ \theta = 8\pi/6$.



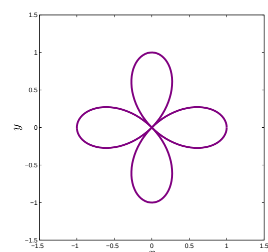
(i) $r = \cos(2\theta)\,,\ \theta = 9\pi/6$.



(j) $r = \cos(2\theta)\,,\ \theta = 10\pi/6$.



(k) $r = \cos(2\theta)\,,\ \theta = 11\pi/6$.



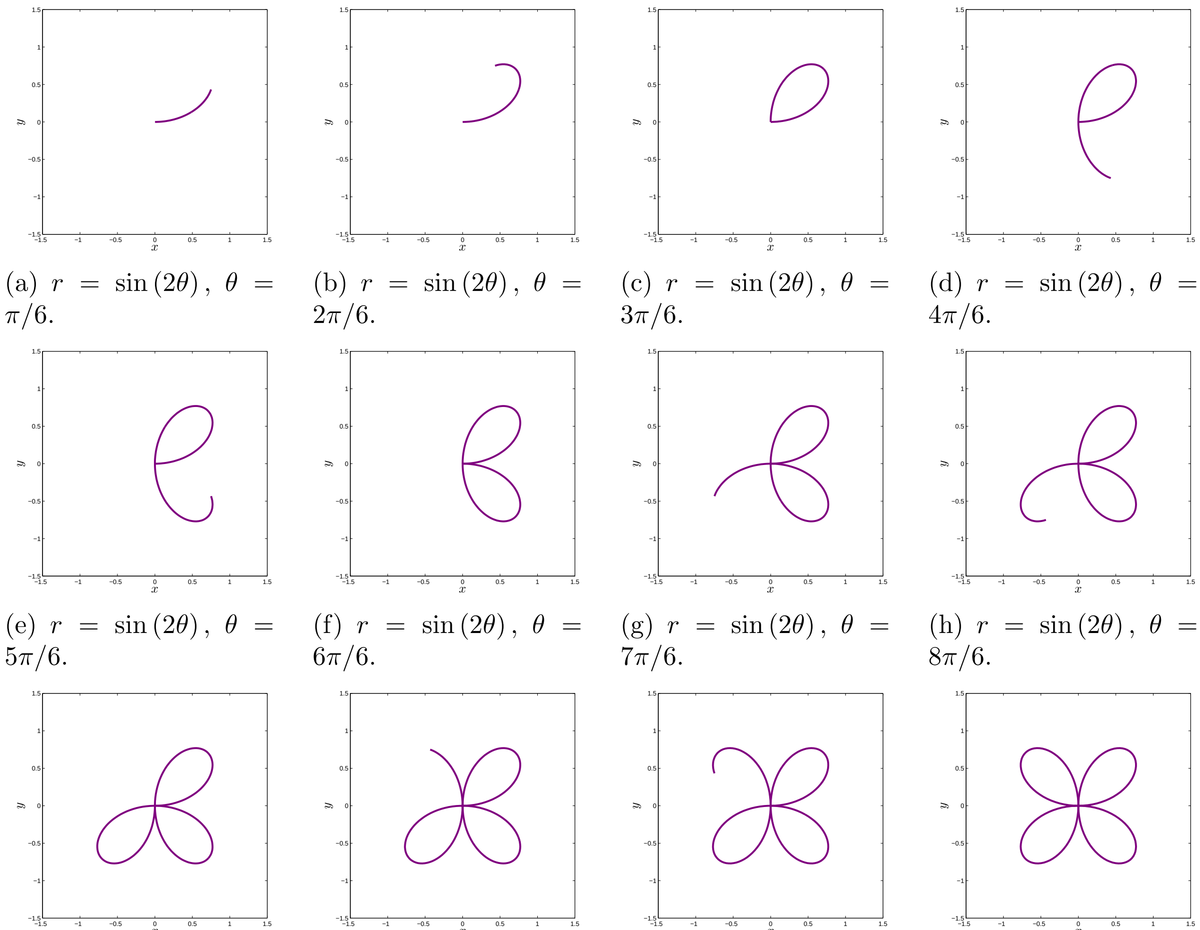
(l) $r = \cos(2\theta)\,,\ \theta = 12\pi/6$.

Figure 11.9: Solutions for Example 13.

□

Example 14 Sketch the curve $r = \sin(2\theta)$, $0 \le \theta \le 2\pi$ (four-leaved flower).

Solution. We have



(a) $r = \sin(2\theta)$, $\theta = \pi/6$.

(b) $r = \sin(2\theta)$, $\theta = 2\pi/6$.

(c) $r = \sin(2\theta)$, $\theta = 3\pi/6$.

(d) $r = \sin(2\theta)$, $\theta = 4\pi/6$.

(e) $r = \sin(2\theta)$, $\theta = 5\pi/6$.

(f) $r = \sin(2\theta)$, $\theta = 6\pi/6$.

(g) $r = \sin(2\theta)$, $\theta = 7\pi/6$.

(h) $r = \sin(2\theta)$, $\theta = 8\pi/6$.

(i) $r = \sin(2\theta)$, $\theta = 9\pi/6$.

(j) $r = \sin(2\theta)$, $\theta = 10\pi/6$.

(k) $r = \sin(2\theta)$, $\theta = 11\pi/6$.

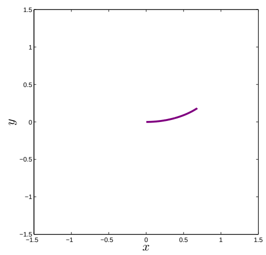
(l) $r = \sin(2\theta)$, $\theta = 12\pi/6$.

Figure 11.10: Solutions for Example 14.

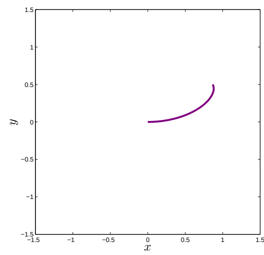
$\square$

Example 15 Sketch the curve $r = \sin(3\theta)$, $0 \le \theta \le \pi$ (three-leaved flower).

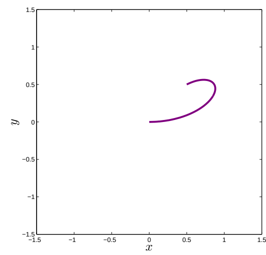
Solution. We have



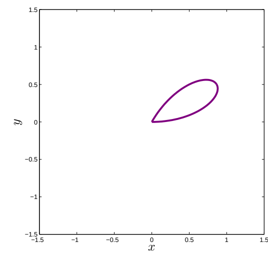
(a) $r = \sin(3\theta)$, $\theta = \pi/6$.



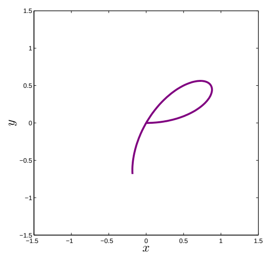
(b) $r = \sin(3\theta)$, $\theta = 2\pi/6$.



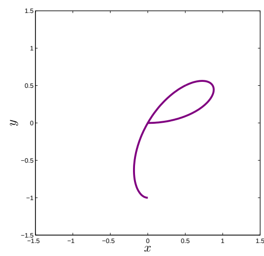
(c) $r = \sin(3\theta)$, $\theta = 3\pi/6$.



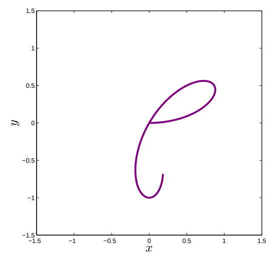
(d) $r = \sin(3\theta)$, $\theta = 4\pi/6$.



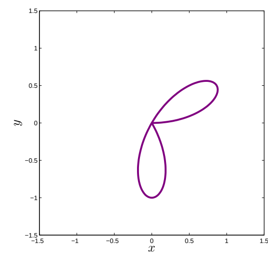
(e) $r = \sin(3\theta)$, $\theta = 5\pi/6$.



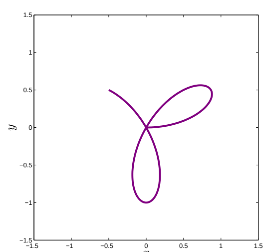
(f) $r = \sin(3\theta)$, $\theta = 6\pi/6$.



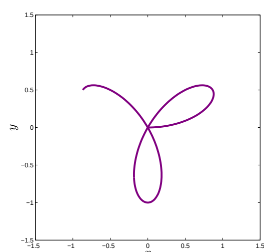
(g) $r = \sin(3\theta)$, $\theta = 7\pi/6$.



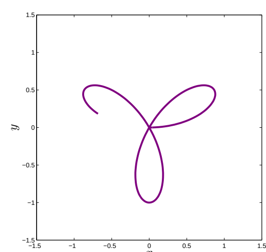
(h) $r = \sin(3\theta)$, $\theta = 8\pi/6$.



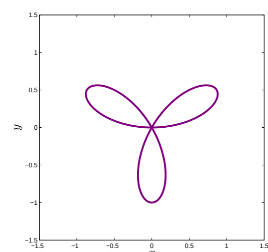
(i) $r = \sin(3\theta)$, $\theta = 9\pi/6$.



(j) $r = \sin(3\theta)$, $\theta = 10\pi/6$.



(k) $r = \sin(3\theta)$, $\theta = 11\pi/6$.



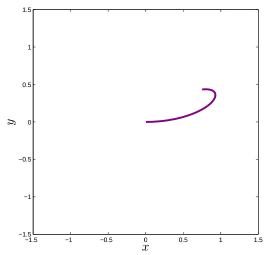
(l) $r = \sin(3\theta)$, $\theta = 12\pi/6$.

Figure 11.11: Solutions for Example 15.

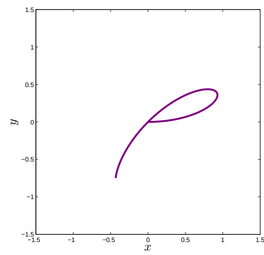
$\square$

Example 16 Sketch the curve $r = \sin(4\theta)$, $0 \le \theta \le 2\pi$ (eight-leaved flower).

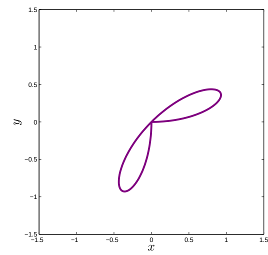
Solution. We have



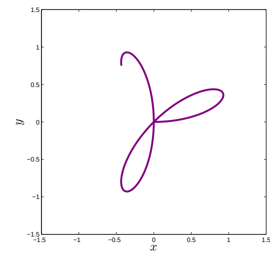
(a) $r = \sin(4\theta)$, $\theta = \pi/6$.



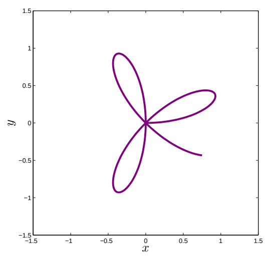
(b) $r = \sin(4\theta)$, $\theta = 2\pi/6$.



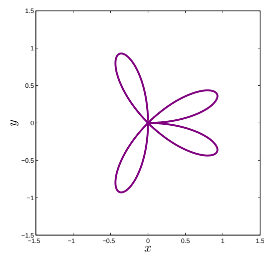
(c) $r = \sin(4\theta)$, $\theta = 3\pi/6$.



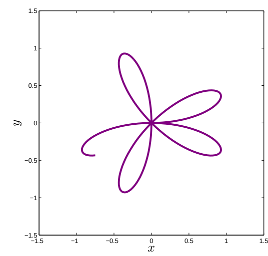
(d) $r = \sin(4\theta)$, $\theta = 4\pi/6$.



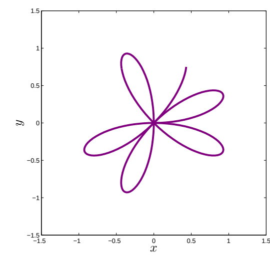
(e) $r = \sin(4\theta)$, $\theta = 5\pi/6$.



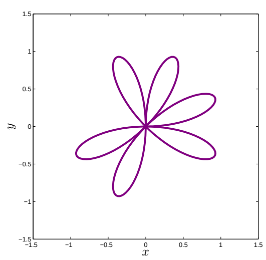
(f) $r = \sin(4\theta)$, $\theta = 6\pi/6$.



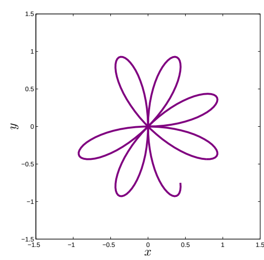
(g) $r = \sin(4\theta)$, $\theta = 7\pi/6$.



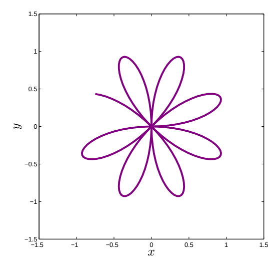
(h) $r = \sin(4\theta)$, $\theta = 8\pi/6$.



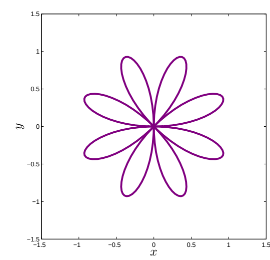
(i) $r = \sin(4\theta)$, $\theta = 9\pi/6$.



(j) $r = \sin(4\theta)$, $\theta = 10\pi/6$.



(k) $r = \sin(4\theta)$, $\theta = 11\pi/6$.



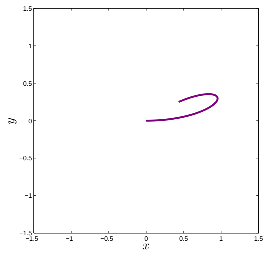
(l) $r = \sin(4\theta)$, $\theta = 12\pi/6$.

Figure 11.12: Solutions for Example 16.

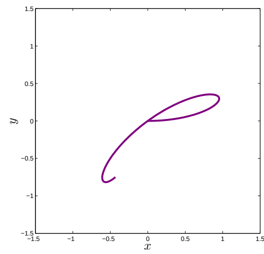
$\square$

Example 17 Sketch the curve $r = \sin(5\theta)$, $0 \le \theta \le 2\pi$ (five-leaved flower).

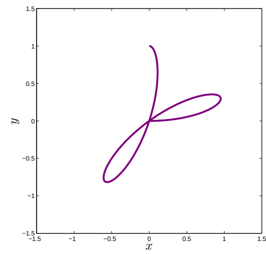
Solution. We have



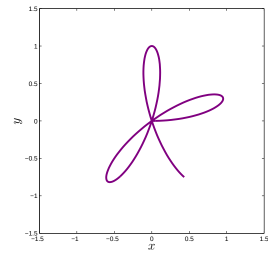
(a) $r = \sin(5\theta)$, $\theta = \pi/6$.



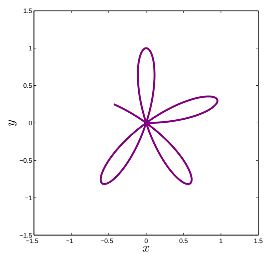
(b) $r = \sin(5\theta)$, $\theta = 2\pi/6$.



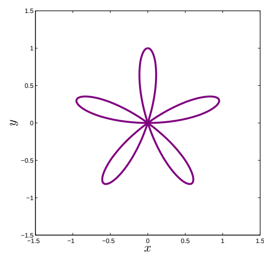
(c) $r = \sin(5\theta)$, $\theta = 3\pi/6$.



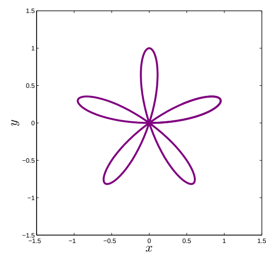
(d) $r = \sin(5\theta)$, $\theta = 4\pi/6$.



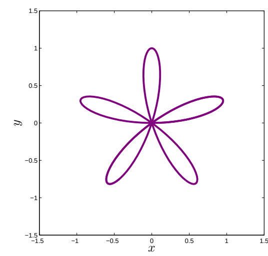
(e) $r = \sin(5\theta)$, $\theta = 5\pi/6$.



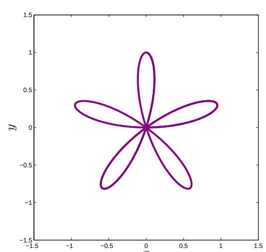
(f) $r = \sin(5\theta)$, $\theta = 6\pi/6$.



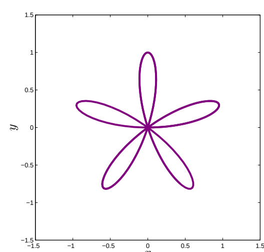
(g) $r = \sin(5\theta)$, $\theta = 7\pi/6$.



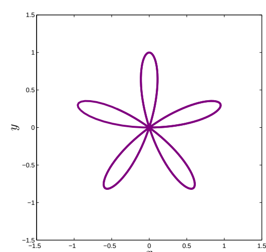
(h) $r = \sin(5\theta)$, $\theta = 8\pi/6$.



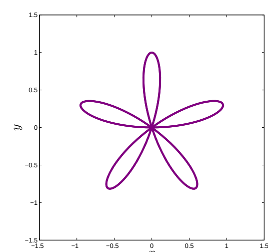
(i) $r = \sin(5\theta)$, $\theta = 9\pi/6$.



(j) $r = \sin(5\theta)$, $\theta = 10\pi/6$.



(k) $r = \sin(5\theta)$, $\theta = 11\pi/6$.



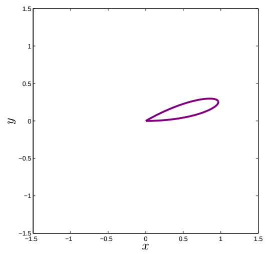
(l) $r = \sin(5\theta)$, $\theta = 12\pi/6$.

Figure 11.13: Solutions for Example 17.

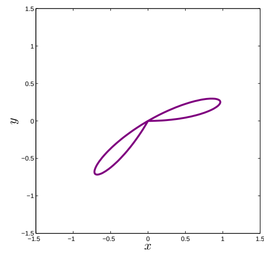
□

Example 18 Sketch the curve $r = \sin(6\theta)\,,\ 0 \le \theta \le 2\pi$ (twelve-leaved flower).

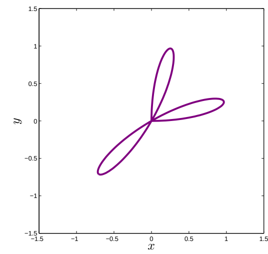
Solution. We have



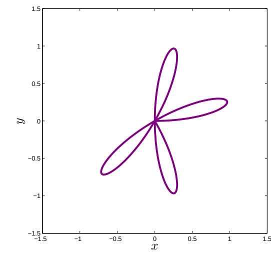
(a) $r = \sin(6\theta)\,,\ \theta = \pi/6$.



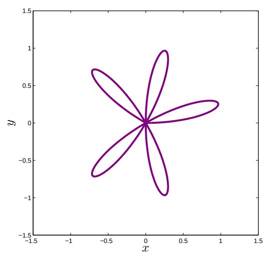
(b) $r = \sin(6\theta)\,,\ \theta = 2\pi/6$.



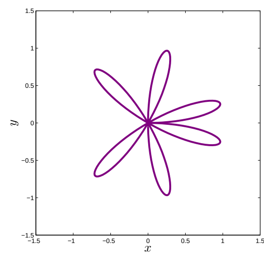
(c) $r = \sin(6\theta)\,,\ \theta = 3\pi/6$.



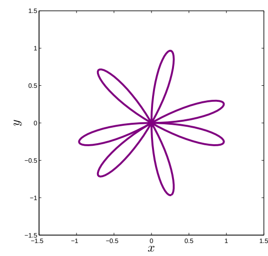
(d) $r = \sin(6\theta)\,,\ \theta = 4\pi/6$.



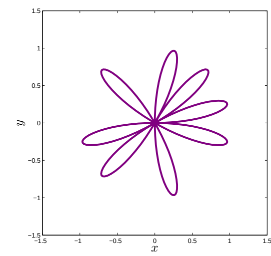
(e) $r = \sin(6\theta)\,,\ \theta = 5\pi/6$.



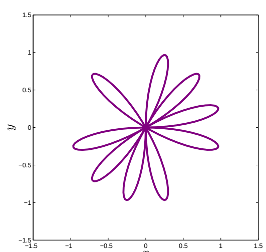
(f) $r = \sin(6\theta)\,,\ \theta = 6\pi/6$.



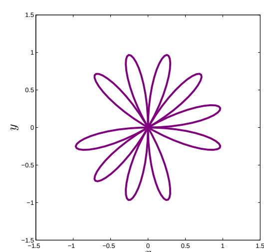
(g) $r = \sin(6\theta)\,,\ \theta = 7\pi/6$.



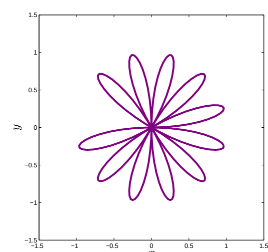
(h) $r = \sin(6\theta)\,,\ \theta = 8\pi/6$.



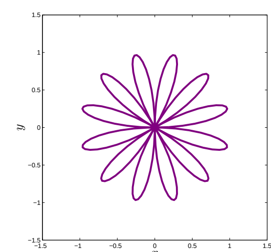
(i) $r = \sin(6\theta)\,,\ \theta = 9\pi/6$.



(j) $r = \sin(6\theta)\,,\ \theta = 10\pi/6$.



(k) $r = \sin(6\theta)\,,\ \theta = 11\pi/6$.



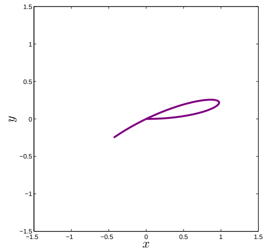
(l) $r = \sin(6\theta)\,,\ \theta = 12\pi/6$.

Figure 11.14: Solutions for Example 18.

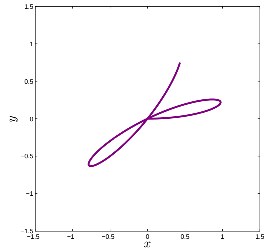
□

Example 19 Sketch the curve $r = \sin(7\theta)\,,\ 0 \le \theta \le 2\pi$ (seven-leaved flower).

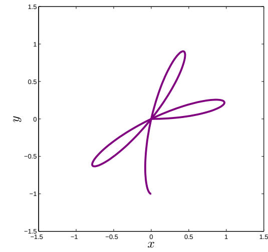
Solution. We have



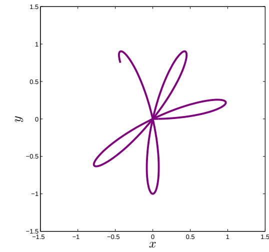
(a) $r = \sin(7\theta)\,,\ \theta = \pi/6$.



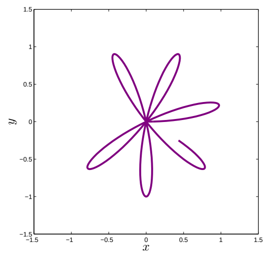
(b) $r = \sin(7\theta)\,,\ \theta = 2\pi/6$.



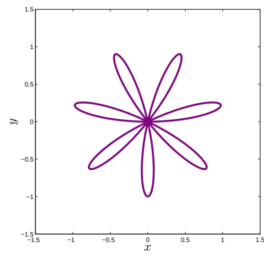
(c) $r = \sin(7\theta)\,,\ \theta = 3\pi/6$.



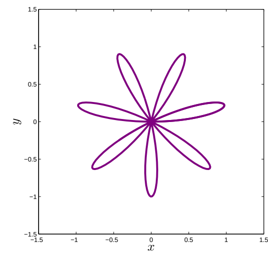
(d) $r = \sin(7\theta)\,,\ \theta = 4\pi/6$.



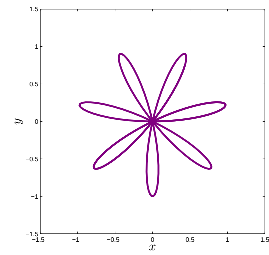
(e) $r = \sin(7\theta)\,,\ \theta = 5\pi/6$.



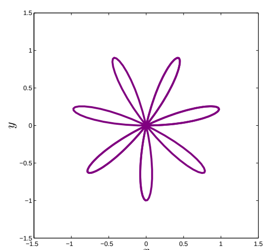
(f) $r = \sin(7\theta)\,,\ \theta = 6\pi/6$.



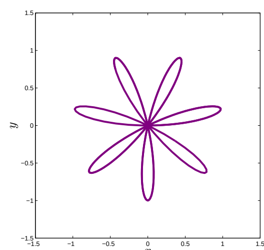
(g) $r = \sin(7\theta)\,,\ \theta = 7\pi/6$.



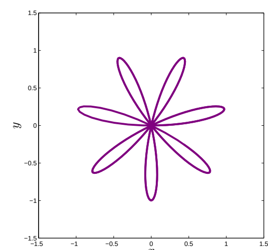
(h) $r = \sin(7\theta)\,,\ \theta = 8\pi/6$.



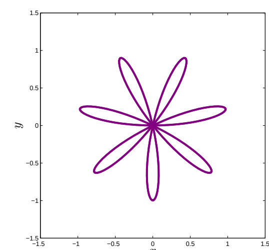
(i) $r = \sin(7\theta)\,,\ \theta = 9\pi/6$.



(j) $r = \sin(7\theta)\,,\ \theta = 10\pi/6$.



(k) $r = \sin(7\theta)\,,\ \theta = 11\pi/6$.



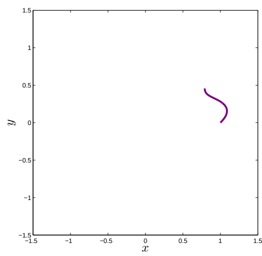
(l) $r = \sin(7\theta)\,,\ \theta = 12\pi/6$.

Figure 11.15: Solutions for Example 20.

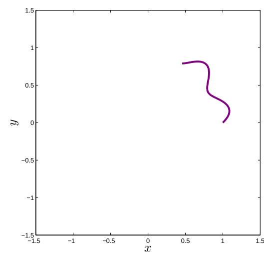
$\square$

Example 20 Sketch the curve $r = 1 + \dfrac{1}{10}\sin(10\theta)$, $0 \le \theta \le 2\pi$.

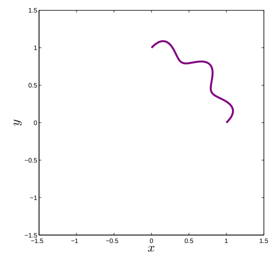
Solution. We have



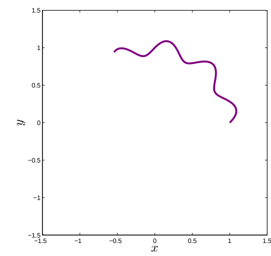
(a) $r = 1 + \dfrac{1}{10}\sin(10\theta)$, $\theta = \pi/6$.



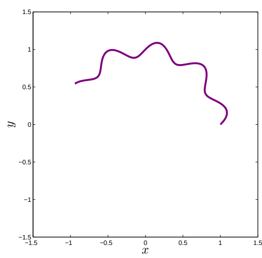
(b) $r = 1 + \dfrac{1}{10}\sin(10\theta)$, $\theta = 2\pi/6$.



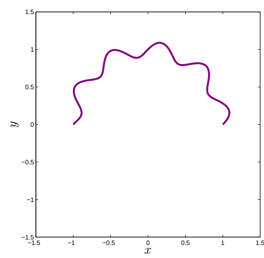
(c) $r = 1 + \dfrac{1}{10}\sin(10\theta)$, $\theta = 3\pi/6$.



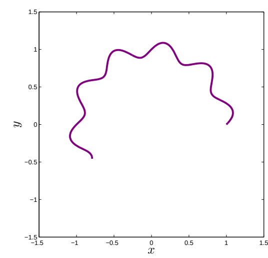
(d) $r = 1 + \dfrac{1}{10}\sin(10\theta)$, $\theta = 4\pi/6$.



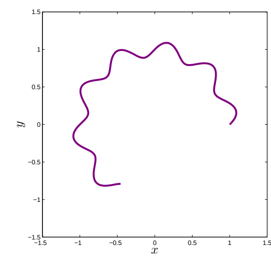
(e) $r = 1 + \dfrac{1}{10}\sin(10\theta)$, $\theta = 5\pi/6$.



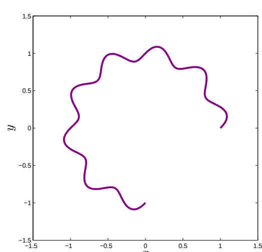
(f) $r = 1 + \dfrac{1}{10}\sin(10\theta)$, $\theta = 6\pi/6$.



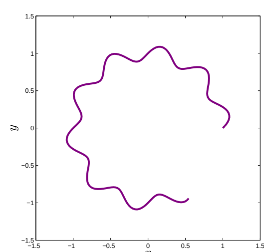
(g) $r = 1 + \dfrac{1}{10}\sin(10\theta)$, $\theta = 7\pi/6$.



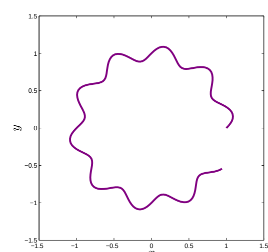
(h) $r = 1 + \dfrac{1}{10}\sin(10\theta)$, $\theta = 8\pi/6$.



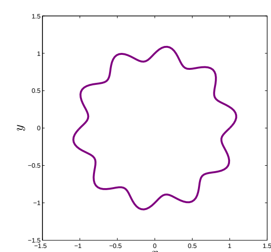
(i) $r = 1 + \dfrac{1}{10}\sin(10\theta)$, $\theta = 9\pi/6$.



(j) $r = 1 + \dfrac{1}{10}\sin(10\theta)$, $\theta = 10\pi/6$.



(k) $r = 1 + \dfrac{1}{10}\sin(10\theta)$, $\theta = 11\pi/6$.



(l) $r = 1 + \dfrac{1}{10}\sin(10\theta)$, $\theta = 12\pi/6$.

Figure 11.16: Solutions for Example 20.

$\square$

Example 21 Match the polar equations with the graphs labeled Figure 11.17 (a) - (f):

1. $r = \sin\left(\dfrac{\theta}{2}\right)$, $-4\pi \le \theta \le 4\pi$;
2. $r = \sin\left(\dfrac{\theta}{4}\right)$, $-4\pi \le \theta \le 4\pi$;
3. $r = \sin\theta + \sin^3\left(\dfrac{5\theta}{4}\right)$, $-4\pi \le \theta \le 4\pi$;
4. $r = \theta \sin\theta$, $-8\pi \le \theta \le 8\pi$;
5. $r = 1 + 4\cos(5\theta)$, $-\pi \le \theta \le \pi$;
6. $r = \dfrac{1}{\sqrt{\theta}}$, $0 \le \theta \le 6\pi$.

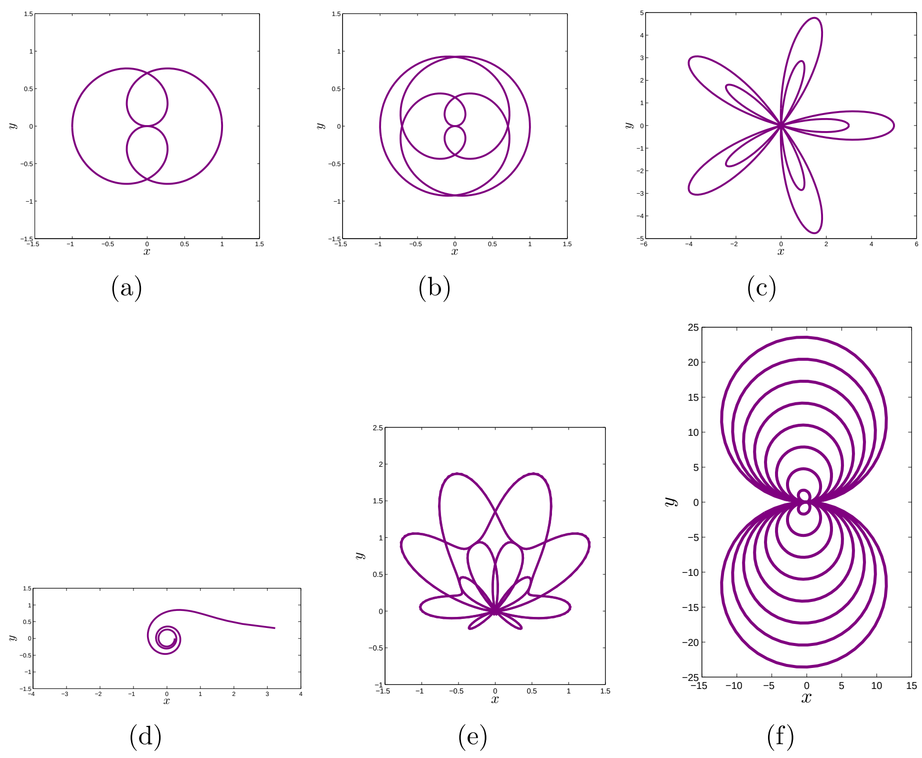


Figure 11.17: Solutions for Example 21.

11.4 Tangents to polar curves

In Cartesian coordinates you've learned

$$\frac{dy}{dx}$$

is the slope of the tangent line to a curve at a point. But what about $r = f(\theta)$?

At first you might think

$$\frac{dr}{d\theta}$$

is the slope of the tangent line to the curve but consider $r =$ constant e.g. $r = 1$ which is a circle.

$$\frac{dr}{d\theta} = 0$$

at every point which is not obviously the slope of the tangent line.

What does $\frac{dr}{d\theta}$ compute? It is the rate at which the distance from the origin to the curve changes with respect to a change in θ . It makes sense for $r = 1$ then that

$$\frac{dr}{d\theta} = 0$$

since the distance doesn't change as we move along a circle.

Example 22 Examine $\frac{dr}{d\theta}$, where $r = 1 + \cos\theta$.

Solution. We have

$$\frac{dr}{d\theta} = \sin\theta.$$

In Figure 11.18, we conclude that

- $\frac{dr}{d\theta} < 0$ for $0 < \theta < \pi$, that is, r decreases for $0 < \theta < \pi$, and
- $\frac{dr}{d\theta} > 0$ for $\pi < \theta < 2\pi$, that is, r increases for $\pi < \theta < 2\pi$.

□

This brings us to the question:

How do we calculate the slope of the tangent line to a polar curve?

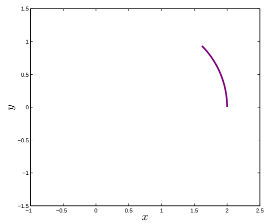
The answer relies heavily on parametric equations.

Let us recall that if

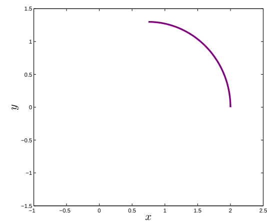
$$\begin{cases} x = x(t); \\ y = y(t), \end{cases}$$

then

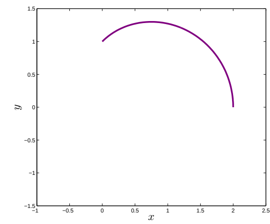
$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$



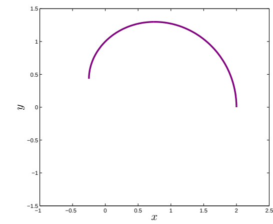
(a) $r = 1 + \cos\theta$, $\theta = \pi/6$.



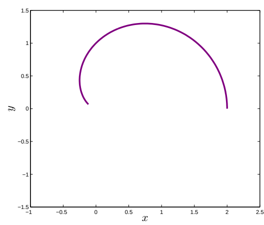
(b) $r = 1 + \cos\theta$, $\theta = 2\pi/6$.



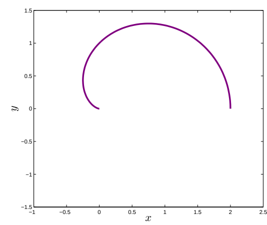
(c) $r = 1 + \cos\theta$, $\theta = 3\pi/6$.



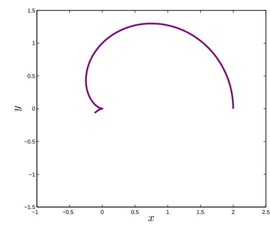
(d) $r = 1 + \cos\theta$, $\theta = 4\pi/6$.



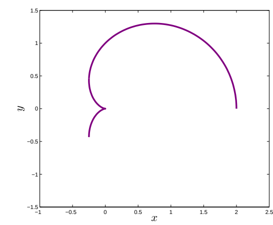
(e) $r = 1 + \cos\theta$, $\theta = 5\pi/6$.



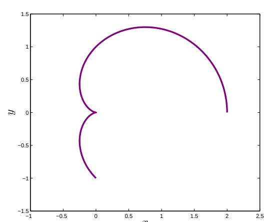
(f) $r = 1 + \cos\theta$, $\theta = 6\pi/6$.



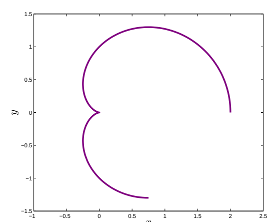
(g) $r = 1 + \cos\theta$, $\theta = 7\pi/6$.



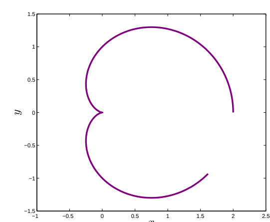
(h) $r = 1 + \cos\theta$, $\theta = 8\pi/6$.



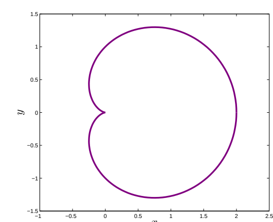
(i) $r = 1 + \cos\theta$, $\theta = 9\pi/6$.



(j) $r = 1 + \cos\theta$, $\theta = 10\pi/6$.



(k) $r = 1 + \cos\theta$, $\theta = 11\pi/6$.



(l) $r = 1 + \cos\theta$, $\theta = 12\pi/6$.

Figure 11.18: Solutions for Example 22.

In polar coordinates, we have

$$\begin{cases} x = r\cos\theta; \\ y = r\sin\theta \end{cases}$$

But so that $r = f(\theta)$, using the product rule, we have

$$\frac{dy}{dx} = \frac{\dfrac{dy}{d\theta}}{\dfrac{dx}{d\theta}} = \frac{\dfrac{r}{d\theta}\cdot\sin\theta + r\cdot\cos\theta}{\dfrac{dr}{d\theta}\cdot\cos\theta - r\cdot\sin\theta}.$$

In other words, the slope of the tangent line to a polar graph is

$$m_{\tan} = \frac{\dfrac{r}{d\theta}\cdot\sin\theta + r\cdot\cos\theta}{\dfrac{dr}{d\theta}\cdot\cos\theta - r\cdot\sin\theta}. \tag{11.1}$$

Example 23 Use (11.1) to find the slope of the tangent line to a polar graph for $r = 1$.

Solution. We have

$$m_{\tan} = \frac{\dfrac{r}{d\theta} \cdot \sin\theta + r \cdot \cos\theta}{\dfrac{dr}{d\theta} \cdot \cos\theta - r \cdot \sin\theta} = \frac{0 \cdot \sin\theta + r \cdot \cos\theta}{0 \cdot \cos\theta - r \cdot \sin\theta} = -\frac{x}{y}.$$

$\square$

Example 24 In Cartesian coordinates, use implicit differentiation to find the slope of the tangent line to the circle equation $x^2 + y^2 = 1$.

Solution. Differentiating the circle equation on both sides with respect to x, we have

$$\begin{aligned} \frac{d}{dx}\left(x^2 + y^2\right) &= \frac{d}{dx}(1) \\ 2x + 2y \cdot \frac{dy}{dx} &= 0. \end{aligned}$$

Now, we have

$$\frac{dy}{dx} = -\frac{x}{y}$$

which is the same as the result in Example 23.

$\square$

Example 25 Use (11.1) to find the slope of the tangent line to a polar graph for $r = 1 + \cos\theta$.

Solution. We have

$$\begin{aligned} m_{\tan} = \frac{\dfrac{r}{d\theta} \cdot \sin\theta + r \cdot \cos\theta}{\dfrac{dr}{d\theta} \cdot \cos\theta - r \cdot \sin\theta} &= \frac{\dfrac{(1+\cos\theta)}{d\theta} \cdot \sin\theta + (1+\cos\theta) \cdot \cos\theta}{\dfrac{d(1+\cos\theta)}{d\theta} \cdot \cos\theta - (1+\cos\theta) \cdot \sin\theta} \\ &= \frac{-\sin\theta \cdot \sin\theta + (1+\cos\theta) \cdot \cos\theta}{-\sin\theta \cdot \cos\theta - (1+\cos\theta) \cdot \sin\theta} \\ &= \frac{2 \cdot \cos^2\theta + \cos\theta - 1}{-2 \cdot \sin\theta \cdot \cos\theta - \sin\theta}. \end{aligned}$$

We want to consider in particular where $m_{\tan}$ is either 0 or undefined to determine where we have horizontal tangents, vertical tangents or cusps.

Let us examine the following cases:

- The numerator is 0 for $2 \cdot \cos^2\theta + \cos\theta - 1 = 0$ which yields

  $$(2\cos\theta - 1)(\cos\theta + 1) = 0$$

  which yields $\theta = \pi, \pi/3$, and $5\pi/3$.

- The denominator is 0 for $-2 \cdot \sin\theta \cdot \cos\theta - \sin\theta$ which yields $\sin\theta = 0$ or $\cos\theta = -\frac{1}{2}$. This yields $\theta = 0, \pi, 2\pi/3$, and $4\pi/3$

We conclude that

- horizontal tangents are obtained at $\theta = \pi, \pi/3$, and $5\pi/3$.
- vertical tangents are obtained at $\theta = 0, \pi, 2\pi/3$, and $4\pi/3$.
- a cusp is obtained at $\theta = \pi$.

□

Next, this brings us to the question:

How do we plot the tangent line to a polar graph?

Let's start with the tangent line in Cartesian coordinates.

$$y = m(x - x_0) + y_0,$$

where (x_0, y_0) is the point where we want the tangent line.

In polar coordinates, (x_0, y_0) is replaced by $(r(\theta_0), \theta_0)$ which is a specific point and and m is replaced by

$$m_{\tan} = \frac{\dfrac{dr}{d\theta} \cdot \sin\theta_0 + r \cdot \cos\theta_0}{\dfrac{dr}{d\theta} \cdot \cos\theta_0 - r \cdot \sin\theta_0} \tag{11.2}$$

or

$$r \cdot \sin\theta = m_{\tan}\left(r \cdot \cos\theta - r(\theta_0) \cdot \cos\theta_0\right) + r(\theta_0) \cdot \sin\theta_0.$$

Solving for r, we have

$$r \cdot (\sin\theta - m_{\tan} \cdot \cos\theta) = -m_{\tan} r(\theta_0) \cdot \cos\theta_0 + r(\theta_0) \cdot \sin\theta_0.$$

or

$$r = \frac{-m_{\tan} r(\theta_0) \cdot \cos\theta_0 + r(\theta_0) \cdot \sin\theta_0}{\sin\theta - m_{\tan} \cdot \cos\theta},$$

where

$$m_{\tan} = \frac{\dfrac{dr}{d\theta} \cdot \sin\theta_0 + r \cdot \cos\theta_0}{\dfrac{dr}{d\theta} \cdot \cos\theta_0 - r \cdot \sin\theta_0}.$$

Example 26 Compute the polar form of $r = 1 + \cos\theta$ of the tangent line at $\pi/6$.

Solution. We have

$$\begin{aligned} m_{\tan} &= \frac{2 \cdot \cos^2\theta + \cos\theta - 1}{-2 \cdot \sin\theta \cdot \cos\theta - \sin\theta} \\ &= \frac{2 \cdot \cos^2\left(\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{6}\right) - 1}{-2 \cdot \sin\left(\frac{\pi}{6}\right) \cdot \cos\left(\frac{\pi}{6}\right) - \sin\left(\frac{\pi}{6}\right)} = -1 \end{aligned}$$

and

$$r = \frac{-(-1)\left(1+\cos\left(\frac{\pi}{6}\right)\right)\cdot\cos\left(\frac{\pi}{6}\right)+\left(1+\cos\left(\frac{\pi}{6}\right)\right)\cdot\sin\left(\frac{\pi}{6}\right)}{\sin\theta-(-1)\cdot\cos\theta} = \frac{-0.683}{\sin\theta+\cos\theta}.$$

$\square$