香 港 中 文 大 學
The Chinese University of Hong Kong

二〇一三至一四年度上學期科目考試
Course Examination 1st Term, 2013-14

科目編號及名稱 Course Code & Title : **MATH1510A/B/C/D/E/F Calculus for Engineers**

時間 Time allowed : 2 小時 hours 00 分鐘 minutes

學號 Student I.D. No. : ____________ 座號 Seat No.: ____________

SHOW ALL NECESSARY WORK TO GET CREDIT FOR SOLUTIONS
NOTE: THERE ARE 100 POINTS TOTAL ON THIS EXAMINATION.

1. Solve each of the following problems separately.
   Justify your solution carefully.

   (a) Determine whether each of the following limits exists.
   If so, compute the limit.

   i. (4 points) $\displaystyle\lim_{x\to 1}\frac{x^2-1}{x+1}$

   ii. (4 points) $\displaystyle\lim_{x\to +\infty}\frac{(x-2)^2}{x^2-2x-8}$

   iii. (4 points) $\displaystyle\lim_{x\to 0}\frac{\sqrt{x+4}-2}{x}$

   iv. (4 points) $\displaystyle\lim_{x\to 0} x\cos\left(\frac{50\pi}{x}\right)$

   (b) (4 pts) Suppose that

$$f(x)=\begin{cases}-x^4+3, & x\le 2;\\ x^2+9, & x>2.\end{cases}$$

   Is f continuous everywhere? Justify your conclusion.

2. Find the derivative y' $\left(\text{or } \dfrac{dy}{dx}\right)$ of each function y below.

(a) (5 points) $\quad y = \sqrt[3]{x} + \dfrac{1}{\sqrt{x}} + e^{\pi}$

(b) (5 points) $\quad y = 2^{x}\left(\sin x + \cos x\right)$

(c) (5 points) $\quad y = \dfrac{1 - e^{-x}}{1 + e^{x}}$

(d) (5 points) $\quad y = \sqrt{\sin(\ln x)}$

3. Solve each of the following problems separately.
Justify your solution carefully.

(a) (5 points) Find all local (that is, relative) maxima and minima of the function:

$$f(x) = x^3 - 3x^2, \quad \text{if any.}$$

(b) Let f be the function defined as follows:

$$f(x) = (x^2 - x)e^{-x}, \quad x \in (-\infty, \infty).$$

i. (1 point) Verify that $f'(x) = -(x-\alpha)(x-\beta)e^{-x}$ or not, where

$$\alpha = \frac{3-\sqrt{5}}{2} \qquad \text{and} \qquad \beta = \frac{3+\sqrt{5}}{2}.$$

ii. (5 points) Determine the intervals on which f is increasing or decreasing, and find all local (that is, relative) maxima and minima, if any. Substantiate your results by determining the sign of the first derivative in the neighborhood of the stationary points (or the critical values), that is, those x at which $f'(x) = 0$.

iii. (4 points) Determine the intervals on which f is concave up or concave down (or convex down/up), and find all inflection points, if any.

4. Evaluate the following integrals.

(a) (5 points) $\displaystyle\int \left(\frac{2}{x^{3/4}} - \sqrt[3]{x}\right)\, dx$

(b) (5 points) $\displaystyle\int 2x(x^2+1)^{23}\, dx$

(c) (5 points) $\displaystyle\int_{\pi/8}^{\pi/4} \cos^2(x)\, dx$

(d) (5 points) $\displaystyle\int_1^e x \ln x\, dx$

5. Solve each of the following problems separately.
Justify your solution carefully.

(a) (3 points) Find the area of the region enclosed by the parabolas $y^2 = x$ and $y = x - 2$.

(b) (3 points) Calculate the volume of the solid that results when the region enclosed by the given curves is revolved about the x-axis, where

$$y = x^2 \qquad \text{and} \qquad y = x^3.$$

6. Solve each of the following problems separately.
   Justify your solution carefully.

   (a) Consider
   $$w(x,y) = \sin(xy).$$

      i. (2 points) Compute w_{xy} and w_{yx}.

      ii. (1 point) What can be said about w_{xxy} and w_{xyx}?

   (b) (3 points) Does the function
   $$f(x,t) = t^{-1/2}e^{-x^2/(4ct)}$$
   satisfy the following equation
   $$c\frac{\partial^2 f}{\partial x^2} - \frac{\partial f}{\partial t} = 0\,?$$
   where c is a constant. Justify your answer.

7. Solve each of the following problems separately.
   Justify your solution carefully.

   (a) Evaluate each of the following limits:

      i. (2 points) $\displaystyle\lim_{x\to 1^+} (\ln x)^{x-1}$

      ii. (2 points) $\displaystyle\lim_{x\to +\infty} e^{-x}\ln x$

      iii. (2 points) $\displaystyle\lim_{x\to +\infty} \left(\sqrt{x^2+x} - \sqrt{x^2-x}\right)$

      iv. (1 point) $\displaystyle\lim_{x\to +\infty} \frac{\int_1^x \ln(e^{4t-1})dt}{x}$

   (b) (1 point) Use the Taylor series expansion to compute the following limit:
   $$\lim_{x\to 0} \frac{\ln(\cos x)}{x^2}.$$

8. Solve each of the following problems separately.
Justify your solution carefully.

(a) Consider the following differential equation:

$$xydx = e^{-x^2}(1-y^2)dy,$$

with $y(0) = 1$.

Answer the following questions:

i. (1 point) First find the general solution (involving a constant C) for the given differential equation.

ii. (1 point) Then solve the differential equation subject to the given condition.

(b) (1 point) Let

$$G(x) = \int_0^x s \int_0^s f(t)dtds,$$

where f is continuous for all real t. Find $G'(0)$ and $G''(x)$, if any.

(c) (1 point) Given $z = f(x, y)$, $x = r\cos v$, and $y = r\sin v$. Show that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial v^2}.$$

(d) (1 point) Find the following limit by first identifying it with a definite integral:

$$\lim_{n\to+\infty} \frac{1}{n^{3/2}} \sum_{k=1}^{n} \sqrt{k}.$$

THE END