THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1510 Calculus for Engineers (2020-2021)
Solution to Supplementary Exercise 4

Sandwich Theorem

1. Fill in the blanks and construct an upper bound and a lower bound of $\dfrac{3}{6+4\cos\theta}$.

$$\begin{array}{ccccc} -1 & \leq & \cos\theta & \leq & 1 \\ \_\_\_\_\_\_ & \leq & 4\cos\theta & \leq & 4 \\ \_\_\_\_\_\_ & \leq & 6+4\cos\theta & \leq & 10 \\ \dfrac{1}{10} & \leq & \dfrac{1}{6+4\cos\theta} & \leq & \_\_\_\_\_\_ \\ \dfrac{3}{10} & \leq & \dfrac{3}{6+4\cos\theta} & \leq & \_\_\_\_\_\_ \end{array}$$

Ans:

$$\begin{array}{ccccc} -1 & \leq & \cos\theta & \leq & 1 \\ \underline{\quad -4 \quad} & \leq & 4\cos\theta & \leq & 4 \\ \underline{\quad 2 \quad} & \leq & 6+4\cos\theta & \leq & 10 \\ \dfrac{1}{10} & \leq & \dfrac{1}{6+4\cos\theta} & \leq & \underline{\quad \dfrac{1}{2} \quad} \\ \dfrac{3}{10} & \leq & \dfrac{3}{6+4\cos\theta} & \leq & \underline{\quad \dfrac{3}{2} \quad} \end{array}$$

2. (a) Prove that $\dfrac{1}{3} \leq \dfrac{2+\sin 3x}{3} \leq 1$.

   (b) By using (a) and the sandwich theorem, prove that $\displaystyle\lim_{x\to+\infty} \frac{2+\sin 3x}{3e^x} = 0$.

Ans:

(a)

$$\begin{array}{ccccc} -1 & \leq & \sin 3x & \leq & 1 \\ 1 & \leq & 2+\sin 3x & \leq & 3 \\ \dfrac{1}{3} & \leq & \dfrac{2+\sin 3x}{3} & \leq & 1 \end{array}$$

(b) By (a), $\dfrac{1}{3e^x} \leq \dfrac{2+\sin 3x}{3e^x} \leq \dfrac{1}{e^x}$.

Note that $\lim\limits_{x\to+\infty} \dfrac{1}{3e^x} = \lim\limits_{x\to+\infty} \dfrac{1}{e^x} = 0$, by the sandwich theorem, $\lim\limits_{x\to+\infty} \dfrac{2+\sin 3x}{3e^x} = 0$.

3. Show that $\lim\limits_{x\to 0} x \sin\left(\dfrac{1}{e^x - e^{-x}}\right) = 0$.

Ans: For $x \neq 0$, we have $-1 \leq \sin\left(\dfrac{1}{e^x - e^{-x}}\right) \leq 1$ and $-|x| \leq x \leq |x|$. Therefore, $-|x| \leq x \sin\left(\dfrac{1}{e^x - e^{-x}}\right) \leq |x|$.

Note that $\lim\limits_{x\to 0} -|x| = \lim\limits_{x\to 0} |x| = 0$ and then by the sandwich theorem, $\lim\limits_{x\to 0} x \sin\left(\dfrac{1}{e^x - e^{-x}}\right) = 0$.

(**Remark**: When we consider the limit at $x = 0$, we have to consider those x which is in a neighborhood of the point $x = 0$. That means x may be negative and so it is incorrect to say that $-x \leq x \sin\left(\dfrac{1}{e^x - e^{-x}}\right) \leq x$.)

4. Show that $\lim\limits_{x\to+\infty} \dfrac{e^{\cos x}}{x} = 0$.

Ans: For $x > 0$, we have $-1 \leq \cos x \leq 1$, then $e^{-1} \leq e^{\cos x} \leq e$, and so

$$\frac{e^{-1}}{x} \leq \frac{e^{\cos x}}{x} \leq \frac{e}{x}.$$

Note that $\lim\limits_{x\to+\infty} \dfrac{e^{-1}}{x} = \lim\limits_{x\to+\infty} = \dfrac{e}{x} = 0$ and then by the sandwich theorem, $\lim\limits_{x\to+\infty} \dfrac{e^{\cos x}}{x} = 0$.

Continuity of Functions

5. Let $f(x)$ be a function defined by

$$f(x) = \begin{cases} \dfrac{\sin x}{2x} & \text{if} \quad x \neq 0, \\ a & \text{if} \quad x = 0. \end{cases}$$

If $f(x)$ is a continuous at $x = 0$, then what is the value of a?

Ans: Since $f(x)$ is a continuous at $x = 0$, we have

$$\begin{aligned} \lim_{x\to 0} f(x) &= f(0) \\ \lim_{x\to 0} \frac{\sin x}{2x} &= a \\ a &= \left(\lim_{x\to 0} \frac{1}{2}\right)\left(\lim_{x\to 0} \frac{\sin x}{x}\right) \\ a &= \frac{1}{2} \times 1 \\ &= \frac{1}{2} \end{aligned}$$

6. Let $f(x)$ be a function defined by

$$f(x) = \begin{cases} \log_{10} x & \text{if} \quad x > 10, \\ a & \text{if} \quad x = 10 \\ mx - 1 & \text{if} \quad x \leq 10, \end{cases}$$

where m, a are real numbers.

(a) Find $\lim\limits_{x \to 10^+} f(x)$.

(b) Find $\lim\limits_{x \to 10^-} f(x)$ in terms of m.

(c) If $f(x)$ is continuous at $x = 10$, what is the values of a and m.

Ans:

(a) $\lim\limits_{x \to 10^+} f(x) = \lim\limits_{x \to 10^+} \log_{10} x = 1$.

(b) $\lim\limits_{x \to 10^-} f(x) = \lim\limits_{x \to 10^-} (mx - 1) = 10m - 1$.

(c) Since $f(x)$ is continuous at $x = 10$, $\lim\limits_{x \to 10^+} f(x) = \lim\limits_{x \to 10^-} f(x) = f(10)$. Then $1 = 10m - 1 = a$, so $m = \dfrac{1}{5}$ and $a = 1$.

7. Let a be a real number and let $f(x)$ be a function defined by

$$f(x) = \begin{cases} e^x & \text{if} \quad x > 0, \\ 1 & \text{if} \quad x = 0, \\ \cos x & \text{if} \quad x < 0. \end{cases}$$

(a) Find $\lim\limits_{x \to 0^-} f(x)$ and $\lim\limits_{x \to 0^+} f(x)$.

(b) Is $f(x)$ is continuous at $x = 0$?

Ans:

(a) $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} \cos x = 1$ and $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} e^x = 1$.

(b) Since $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^-} f(x) = f(0) = 1$, $f(x)$ is continuous at $x = 0$.

8. Let a and b are real numbers and let $f(x)$ be a function defined by

$$f(x) = \begin{cases} x^2 & \text{if} \quad x > 3, \\ b & \text{if} \quad x = 3, \\ 2x + a & \text{if} \quad x < 3. \end{cases}$$

Given that $f(x)$ is continuous at $x = 3$. What are the values of a and b?

Ans:

$$\begin{aligned}
&\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = f(3) \\
&2\times 3 + a = 3^2 = b \\
&a = 3, b = 9.
\end{aligned}$$

9. Let $f(x)$ be a function defined by

$$f(x) = \begin{cases} x^2 & \text{if} \quad x > 1, \\ ax+b & \text{if} \quad -1 \le x \le 1, \\ \sin \pi x & \text{if} \quad x < -1. \end{cases}$$

If $f(x)$ is a continuous function, find the value of a and b.

(Hint: In particular, $f(x)$ is continuous at $x = -1$ and $x = 1$.)

Ans: It suffices to show that $f(x)$ is continuous at $x = -1$ and $x = 1$, where the former implies that

$$\begin{aligned}
&\lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x) = f(-1) \\
&\sin(-\pi) = -a + b \\
&a = b.
\end{aligned}$$

Similarly, $f(x)$ is continuous at $x = 1 \Rightarrow a + b = 1$.
Therefore, $a = b = \dfrac{1}{2}$.

10. Let $f(x)$ be a function defined by

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & \text{if} \quad x \neq 0, \\ 0 & \text{if} \quad x = 0. \end{cases}$$

Prove that $f(x)$ is continuous at $x = 0$.

Ans: Note that for $x \neq 0$, we have $-x^2 \le x^2 \sin \dfrac{1}{x} \le x^2$.

Also, $\lim_{x\to 0}(-x^2) = \lim_{x\to 0} x^2 = 0$.

By the sandwich theorem, $\lim_{x\to 0} f(x) = \lim_{x\to 0} x^2 \sin \dfrac{1}{x} = 0$ which equals to $f(0)$. Therefore, $f(x)$ is continuous at $x = 0$.

11. Let $f(x) = |x|$. Prove that $f(x)$ is a continuous function, i.e. $f(x)$ is continuous at every point a.

    (Hint: Consider three cases, $a > 0$, $a = 0$ and $a < 0$.)

    Ans: We can rewrite the function $f(x)$ as a piecewise defined function:

    $$f(x) = \begin{cases} x & \text{if} \quad x \geq 0, \\ \\ -x & \text{if} \quad x < 0. \end{cases}$$

    (a) If $a < 0$, $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a}(-x) = -a = f(a)$.

    (b) If $a > 0$, $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} x = a = f(a)$.

    (c) If $a = 0$, $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} x = 0$, $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-}(-x) = 0$ and $f(0) = 0$. Therefore $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^-} f(x) = f(0) = 0$ and $f(x)$ is continuous at $x = 0$.

    (**Remark**: When we consider the limit of $f(x)$ at $x = 0$, we have to consider both the left hand side and right hand side of 0, but the function has different definition on different side. Therefore, we have to consider the right hand limit $\lim\limits_{x \to 0^+} f(x)$ and the left hand limit $\lim\limits_{x \to 0^-} f(x)$ seperately.)

Intermediate Value Theorem

12. By using the intermediate-value theorem, show that the equation $2^x = 10 - x$ has at least one solution on $[2, 3]$, i.e. there exists $c \in [2, 3]$ such that $2^c = 10 - c$.

    (Hint: Consider the function $f(x) = 2^x + x - 10$.)

    Ans: Let $f(x) = 2^x + x - 10$. Since the exponential function 2^x and the polynomial $x - 10$ are continuous functions, in particular, they are continuous on $[2, 3]$. Therefore, $f(x) = 2^x + x - 10$ is also continuous on $[2, 3]$.

    Note that $f(2) = 2^2 + 2 - 10 = -4 < 0$ and $f(3) = 2^3 + 3 - 10 = 1 > 0$. By intermediate value theorem, there exists $c \in (2, 3) \subset [2, 3]$ such that $f(c) = 2^c + c - 10 = 0$, i.e. $2^c = 10 - c$.

13. Show that the equation $4^x = 3^x + 2^x$ has at least one solution.

    Ans: Let $f(x) = 4^x - 3^x - 2^x$ which is a continuous function. In particular, $f(x)$ is also continuous on $[1, 2]$.

    Note that $f(1) = 4^1 - 3^1 - 2^1 = -1 < 0$ and $f(2) = 4^2 - 3^2 - 2^1 = 3 > 0$. By intermediate value theorem, there exists $c \in (1, 2)$ such that $f(c) = 4^c - 3^c - 2^c = 0$, i.e. $4^c = 3^c + 2^c$.

    Therefore, the original equation has at least one solution.

    (**Remark**: You may use another pair of points a and b other than 1 and 2, but you have to make sure that $f(a)$ and $f(b)$ are in opposite sides.)