Calculus for Engineers

Jeff Chak-Fu WONG[1]

August 2015

Contents

Methods of Integration: integration by substitution

23.1 Introduction

There are four principal methods of integration:

(I) Decomposition of the given integrand as a sum of integrands well known integrals;

(II) Integration by substitution;

(III) Integration by parts;

(IV) Integration by successive reduction.

The first method of integration which depends upon the two theorem proved in Chapter 19, has already been illustrated in the preceding chapter.

In Chapter 24 we will discuss the method of *integrating by parts*. Integration by parts is to integrals what the product rule is to derivatives. The method of integration by successive reduction is thus also on a development of the method of integration by parts.

In the present chapter, we shall be mainly emphasizing the different *methods* of integration by substitution. In Section 23.2, the formal definition of integration by substitution is given. In Section 23.3, by making use of trigonometric identities and algebraic manipulations, we shall study three important forms of integrals. In Section 23.4, by making use of trigonometric and inverse hyperbolic identities, we shall study six important forms of integrals.

23.2 Integration by substitution

This method consists of expressing the integral

$$\int f(x)dx,$$

where x is the independent variable, in terms of another integral where some other variable, say t, is the independent variable; x and t being connected by some suitable relation $x = \phi(t)$.

It leads to the result

$$\int f(x)dx = \int f(\phi(t))\phi'(t)dx,$$

which is proved as follows:

$v = \int f(x)dx$ implies that $\frac{dv}{dx} = f(x)$. We have

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = f(x)\frac{dx}{dt}.$$

This implies that

$$v = \int f(x)\frac{dx}{dt}dt = \int f[\phi(t)]\phi'(t)dt \quad \text{for} \quad x = \phi(t).$$

Thus we have shown that *the integral of a function $f(x)$ with respect to x is equal to the integral of $f(x)\frac{dx}{dt}$ with respect to t.*

Here x is to be replaced by $\phi(t)$.

This method proves useful only when a relation $x = \phi(t)$ can be so selected that the new integrand $f(x)\frac{dx}{dt}$ is of a form whose integral is known.

Note 1 It is worth mentioning that in the result

$$\int f(x)dx = \int f[\phi(t)]\phi'(t)dt$$

dx has been replaced by $\phi'(t)dt$ and this equality can be obtained from $\frac{dx}{dt} = \phi'(t)$, by *supposing* that dx and dt are separate quantities. This supposition greatly simplifies the presentation of the process of integration by substitution (See Example 3).

The logical justification for this supposition is not required here, for it has only been *formally* introduced for the sake of convenience. □

23.2.1 Worked examples

Example 1 Evaluate

$$\int e^x \sin e^x dx.$$

Solution. We put $e^x = t$. Differentiating on both sides of $e^x = t$ with respect to t, we obtain $e^x\frac{dx}{dt} = 1$. Hence

$$\frac{dx}{dt} = \frac{1}{e^x}.$$

Thus

$$\begin{aligned}\int e^x \sin e^x dx &= \int (e^x \sin e^x)\frac{dx}{dt}dt \\ &= \int (e^x \sin e^x)\frac{1}{e^x}dt \\ &= \int \sin t dt \\ &= -\cos t + C \\ &= -\cos e^x + C.\end{aligned}$$

□

Example 2 Evaluate
$$\int \cos^3 x \sin x dx.$$

Solution. We put $\cos x = t$. Differentiating on both sides of $\cos x = t$ with respect to t, we obtain $-\sin x \dfrac{dx}{dt} = 1$. Hence
$$\frac{dx}{dt} = \frac{-1}{\sin x}.$$
Therefore
$$\begin{aligned}\int \cos^3 x \sin x dx &= \int \cos^3 x \sin x \frac{dx}{dt} dt \\ &= \int \cos^3 x \sin x \frac{-1}{\sin x} dt \\ &= \int -t^3 dt \\ &= -\frac{t^4}{4} \\ &= -\frac{1}{4}\cos^4 x + C.\end{aligned}$$

□

Example 3 Evaluate
$$\int \frac{x^5}{1+x^{12}} dx.$$

Solution. We put $x^6 = t$. Differentiating on both sides of $x^6 = t$ with respect to t, we obtain $6x^5 \dfrac{dx}{dt} = 1$. Hence
$$6x^5 dx = dt.$$
Therefore
$$\begin{aligned}\int \frac{x^5}{1+x^{12}} dx &= \int \frac{dt}{6(1+t^2)} \\ &= \frac{1}{6}\tan^{-1} t + C \\ &= \frac{1}{6}\tan^{-1} x^6 + C.\end{aligned}$$

□

Example 4 Evaluate
$$\int \frac{dx}{x^{1/2}+x^{1/3}} dx.$$

Solution. By factoring out $x^{1/3}$ from $x^{1/2} + x^{1/3}$, we obtain
$$\int \frac{dx}{x^{1/2}+x^{1/3}} = \int \frac{dx}{x^{1/3}(1+x^{1/6})}.$$

If we put $x = y^6$. then $dx = 6y^5dy$. Therefore

$$\begin{aligned}\int \frac{6y^5dy}{y^2(1+y)}dx &= 6\int\left(y^2 - y + 1 - \frac{1}{1+y}\right)dy \\ &= 6\left(\frac{1}{3}y^3 - \frac{1}{2}y^2 + y - \log_e|1+y|\right) + C \\ &= 2\sqrt{x} - 3x^{1/3} + 6x^{1/6} - 6\log_e|1+x^{1/6}| + C.\end{aligned}$$

□

Example 5 Evaluate

$$\int \sin x\cos x(2\sin x + 3\cos x)dx.$$

Solution. By expanding the integrand into two integrals, we obtain

$$\int \sin x\cos x(2\sin x + 3\cos x)dx = 2\int \sin^2 x\cos x dx + 3\int \sin x\cos^2 x dx.$$

By substituting $\sin x = y$ in the first integral and $\cos x = t$ in the second integral, we get

$$\begin{aligned}2\int y^2dy - 3\int t^2dt &= \frac{2}{3}y^3 - t^3 + C \\ &= \frac{2}{3}\sin^3 x - \cos^3 x + C.\end{aligned}$$

□

Example 6 Evaluate

$$\int \frac{x\sin^{-1}(x^2)}{\sqrt{1-x^4}}dx.$$

Solution. If we let $\sin^{-1}(x^2) = y$, then $\dfrac{2xdx}{\sqrt{1-x^4}} = dy$. Then, we have

$$\begin{aligned}\int \frac{1}{2}ydy &= \frac{1}{4}y^2 + C \\ &= \frac{1}{4}\left(\sin^{-1}(x^2)\right)^2 + C.\end{aligned}$$

□

Example 7 Evaluate

$$\int \frac{\cos x - \sin x}{\cos x + \sin x}dx.$$

Solution. If we let $\cos x + \sin x = y$, then $(-\sin x + \cos x)dx = dy$. Then, we have

$$\begin{aligned}\int \frac{dy}{y} &= \log_e|y| + C \\ &= \log_e|\cos x + \sin x| + C.\end{aligned}$$

□

Example 8 Evaluate

$$\int \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x} dx.$$

Solution. Let us write the integral as:

$$\int \frac{dx}{\cos^2 x (b^2 + a^2 \tan^2 x)} = \int \frac{\sec^2 x dx}{b^2 + a^2 \tan^2 x}.$$

If we put $a \tan x = by$, then $a \sec^2 x dx = bdy$. Then, the given integral becomes

$$\begin{aligned} \int \frac{(b/a)dy}{b^2(1+y^2)} &= \frac{1}{ab} \int \frac{dy}{1+y^2} \\ &= \frac{1}{ab} \tan^{-1} y + C \\ &= \frac{1}{ab} \tan^{-1} \left(\frac{a}{b} \tan x \right) + C. \end{aligned}$$

□

Example 9 Evaluate

$$\int \frac{e^{\sqrt{x}} \cos(e^{\sqrt{x}}) dx}{\sqrt{x}} dx.$$

Solution. If we put $e^{\sqrt{x}} = y$, then we have $\frac{1}{2\sqrt{x}} e^{\sqrt{x}} dx = dy$. Therefore, we obtain

$$\begin{aligned} \int 2 \cos y dy &= 2 \sin y + C \\ &= 2 \sin(e^{\sqrt{x}}) + C. \end{aligned}$$

□

Example 10 Evaluate

$$\int \sqrt{e^x - 1} dx.$$

Solution. If we let $e^x - 1 = t^2$, then we have $e^x dt = 2tdt$ or $(1+t^2)dx = 2tdt$. So

$$dx = \frac{2t}{1+t^2} dt.$$

Therefore, we have

$$\begin{aligned} \int \sqrt{t^2} \left(\frac{2t}{1+t^2} \right) dt &= 2 \int \frac{t^2}{1+t^2} dt \\ &= 2 \int \left(1 - \frac{1}{1+t^2} \right) dt \\ &= 2(t - \tan^{-1} t) + C \\ &= 2 \left(\sqrt{e^x - 1} - \tan^{-1}(\sqrt{e^x - 1}) \right) + C. \end{aligned}$$

□

Example 11 Evaluate

$$\int \frac{\sin 2x}{p\cos^2 x + q\sin^2 x} dx.$$

Solution. Let us simplify the above integral:

$$\int \frac{\sin 2x}{p\cos^2 x + q\sin^2 x} dx = \int \frac{2\sin x \cos x dx}{p + (q-p)\sin^2 x}.$$

If we put $\sin^2 x = y$, then we have $2\sin x \cos x dx = dy$. The integral simplifies to

$$\begin{aligned}\int \frac{dy}{p + (q-p)y} &= \frac{1}{(q-p)} \log_e |p + (q-p)y| + C \\ &= \frac{1}{(q-p)} \log_e |p + (q-p)\sin^2 x| + C.\end{aligned}$$

$\square$

23.3 Some important forms of integrals

In this section, we shall study three important forms of integrals. The technique for integration by substitution plays an important role for these integrals. Using certain trigonometric identities and applying algebraic manipulations, we convert the following forms of integrals into standard forms. Hence, the results are easily obtained.

23.3.1

$$\int \frac{f'(x)}{f(x)}dx = \log_e|f(x)| + C.$$

Let us verify the following integral:

$$\int \frac{f'(x)}{f(x)}dx = \log_e|f(x)| + C.$$

If we put $f(x) = t$, then $f'(x)dx = dt$. Therefore, we have

$$\begin{aligned}\int \frac{f'(x)}{f(x)}dx &= \int \frac{f'(x)}{f(x)} \cdot \frac{dt}{f'(x)} \\ &= \int \frac{dt}{t} \\ &= \log_e|t| + C \\ &= \log_e|f(x)| + C.\end{aligned}$$

Thus we see that the integral of a fraction whose numerator is the derivative of its denominator is equal to the logarithm of the denominator.

23.3.2 Worked examples

Example 12 Evaluate

$$\int \frac{3x^2}{1+x^3}dx.$$

Solution. The numerator $3x^2$ is the derivative of the denominator $1 + x^3$. Therefore,

$$\int \frac{3x^2}{1+x^3}dx = \log_e|1 + x^3| + C.$$

□

Example 13 Evaluate

$$\int \frac{e^x}{1+e^x}dx.$$

Solution. The numerator e^x is the derivative of the denominator $1 + e^x$. Therefore,

$$\int \frac{e^x}{1+e^x}dx = \log_e|1 + e^x| + C.$$

□

23.3.3 Integrals of $\tan x$, $\cot x$, $\sec x$, $\mathrm{cosec} x$.

The result given above enables us to obtain the integrals of $\tan x$, $\cot x$, $\sec x$ and $\mathrm{cosec} x$ as shown below:

(I)

$$\int \tan x dx = \int \frac{\sec x \tan x}{\sec x} dx = \log_e |\sec x| + C.$$

(II)

$$\int \cot x dx = \int \frac{\cos x}{\sin x} dx = \log_e |\sin x| + C.$$

(III) We have two results for $\displaystyle\int \cot x dx$:

1.

$$\int \sec x dx = \int \frac{\sec x (\sec x + \tan x)}{\sec x + \tan x} dx = \log_e |\sec x + \tan x| + C,$$

since the numerator, $\sec^2 x + \sec x \tan x$, is the derivative of the denominator $\sec x + \tan x$.

2. To put the result in another form, we write

$$\sec x + \tan x = \frac{1 + \sin x}{\cos x}$$

and employ the results

$$\sin x = \frac{2\tan(x/2)}{1 + \tan^2(x/2)}, \quad \text{and} \quad \cos x = \frac{1 - \tan^2(x/2)}{1 + \tan^2(x/2)},$$

so that we obtain

$$\sec x + \tan x = \frac{1 + \tan(x/2)}{1 - \tan(x/2)} = \tan\left(\frac{\pi}{4} + \frac{x}{2}\right).$$

Thus

$$\int \sec x dx = \log_e \left| \tan\left(\frac{\pi}{4} + \frac{x}{2}\right) \right| + C.$$

(IV)

$$\int \mathrm{cosec} x dx = \int \frac{\mathrm{cosec} x (\mathrm{cosec} x - \cot x)}{\mathrm{cosec} x - \cot x} dx$$
$$= \log_e |\mathrm{cosec} x - \cot x|.$$

As in (III) above, we have

$$\mathrm{cosec} x - \cot x = \frac{1 - \cos x}{\sin x} = \frac{1 - \frac{1 - \tan^2(x/2)}{1 + \tan^2(x/2)}}{\frac{2\tan^2(x/2)}{1 + \tan^2(x/2)}} = \tan\frac{x}{2}.$$

Thus

$$\int \mathrm{cosec} x dx = \log_e \left| \tan\frac{x}{2} \right| + C.$$

23.3.4 Worked examples

Example 14 Evaluate
$$\int \frac{\sin x}{\sin(x+a)}dx.$$

Solution. If we put $x + a = y$, then we have $dx = dy$. By substituting in the given integral, we get
$$\begin{aligned}\int \frac{\sin(y-a)dy}{\sin y} &= \int \frac{\sin y\cos a - \cos y \sin a}{\sin y}dy\\ &= \cos a \int dy - \sin a \int \cot y dy\\ &= (\cos a)y - (\sin a)\log_e|\sin y| + C\\ &= (\cos a)(x+a) - (\sin a)\log_e|\sin(x+a)| + C.\end{aligned}$$

□

Example 15 Evaluate
$$\int \frac{dx}{\cos(x+a)\cos(x+b)}.$$

Solution.

$$\begin{aligned}\int \frac{dx}{\cos(x+a)\cos(x+b)} &= \frac{1}{\sin(a-b)}\int \frac{\sin(a-b)}{\cos(x+a)\cos(x+b)}dx\\ &= \frac{1}{\sin(a-b)}\int \frac{\sin((x+a)-(x+b))}{\cos(x+a)\cos(x+b)}dx\\ &= \frac{1}{\sin(a-b)}\int \frac{\sin(x+a)\cos(x+b)-\cos(x+a)\sin(x+b)}{\cos(x+a)\cos(x+b)}dx\\ &= \frac{1}{\sin(a-b)}\int (\tan(x+a)-\tan(x+b))dx\\ &= \frac{1}{\sin(a-b)}(\log_e|\sec(x+a)| - \log_e|\sec(x+b)| + C\\ &= \frac{1}{\sin(a-b)}\log_e\left|\frac{\sec(x+a)}{\sec(x+b)}\right| + C.\end{aligned}$$

□

Example 16 Evaluate
$$\int \frac{dx}{x\cos^2(1+\log_e x)}.$$

Solution. If we let $1 + \log_e x = y$, then we get $\dfrac{dx}{x} = dy$. Therefore, we have
$$\begin{aligned}\int \frac{dy}{\cos^2 y} &= \int \sec^2 y dy\\ &= \tan y + C\\ &= \tan(1+\log_e|x|) + C.\end{aligned}$$

□

Example 17 Evaluate
$$\int e^x \operatorname{cosec} x(1-\cot x)dx.$$

Solution. If we let $e^x \operatorname{cosec} x = y$, then we have
$$(e^x \operatorname{cosec} x - e^x \operatorname{cosec} x \cot x)dx = dy.$$
Thus
$$e^x \operatorname{cosec} x(1-\cot x)dx = dy.$$
Therefore, we have
$$\begin{aligned}\int dy &= y + C\\ &= e^x \operatorname{cosec} x + C.\end{aligned}$$

□

Example 18 Evaluate
$$\int \sqrt{(1+2\tan x(\tan x+\sec x))}dx.$$

Solution. Let
$$\tan x + \sec x = \frac{1+\sin x}{\cos x}.$$
Then
$$\begin{aligned}1+2\tan x(\tan x+\sec x) &= 1+\frac{2\sin x(1+\sin x)}{\cos^2 x}\\ &= \frac{\cos^2 x+2\sin x+2\sin^2 x}{\cos^2 x}\\ &= \frac{1+2\sin x+\sin^2 x}{\cos^2 x}\\ &= \frac{(1+\sin x)^2}{\cos^2 x}.\end{aligned}$$
Therefore, we have
$$\begin{aligned}\int \frac{1+\sin x}{\cos x}dx &= \int (\sec x+\tan x)dx\\ &= \log_e|\sec x+\tan x| + \log_e|\sec x| + C\\ &= \log_e|\sec x(\sec x+\tan x)| + C\\ &= \log_e\left|\frac{1+\sin x}{\cos^2 x}\right| + C\\ &= \log_e\left|\frac{1+\sin x}{\cos^2 x}\cdot\frac{1-\sin x}{1-\sin x}\right| + C\\ &= \log_e\left|\frac{1}{1-\sin x}\right| + C\\ &= -\log_e|1-\sin x| + C.\end{aligned}$$

□

23.3.5

$$\int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1} + C, \text{ when } n \neq -1.$$

If we put $f(x) = t$, then $f'(x)dx = dt$. Therefore, we have

$$\begin{aligned}\int (f(x))^n f'(x)dx = \int t^n dt &= \frac{t^{n+1}}{n+1} + C, \quad \text{for} \quad n \neq -1 \\ &= \frac{(f(x))^{n+1}}{n+1} + C.\end{aligned}$$

Note that the integrand consists of the product of a power of a function $f(x)$ and the derivative $f'(x)$ of $f(x)$. The integral then is obtained by increasing the index by unity and dividing by the increased index.

The case of $n = -1$ corresponds to that of the preceding subsection 23.3.1.

23.3.6 Worked examples

Example 19 Evaluate

$$\int \sin^4 x \cos x dx.$$

Solution. By taking $f(x) = \sin x$, we see that the given integral is of the form

$$\int (f(x))^4 f'(x)dx$$

so that

$$\begin{aligned}\int \sin^4 x \cos x dx &= \frac{\sin^{4+1} x}{4+1} + C \\ &= \frac{1}{5} \sin^5 x + C.\end{aligned}$$

□

23.3.7

$$\int f'(ax+b)dx = \frac{f(ax+b)}{a} + C.$$

Let us verify the following integral:

$$\int f'(ax+b)dx = \frac{f(ax+b)}{a} + C.$$

If we put $ax+b=t$, then $adx=dt$ or $dx=\dfrac{dt}{a}$. Therefore, we have

$$\begin{aligned}\int f'(ax+b)dx &= \int f'(t)\frac{dt}{a} \\ &= \frac{1}{a}\int f'(t)dt \\ &= \frac{1}{a}f(t)+C \\ &= \frac{1}{a}f(ax+b)+C.\end{aligned}$$

Thus the integral of a function of $(ax+b)$ is of the same form as the integral of the same function of x divided by a, which is the coefficient of x.

23.3.8 Worked examples

Example 20 Evaluate

$$\int \cos(ax+b)dx.$$

Solution. We recall that the integral of $\cos x$ is $\sin x$. Therefore, we have

$$\int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b)+C.$$

□

Example 21 Evaluate

$$\int (2x+3)^3dx.$$

Solution. We recall that the integral of x^3 is $\dfrac{x^4}{4}$. Therefore, we have

$$\begin{aligned}\int (2x+3)^3dx &= \frac{(2x+3)^4}{(4)(2)}+C \\ &= \frac{(2x+3)^4}{8}+C.\end{aligned}$$

□

Example 22 Evaluate

$$\int \sin^2 xdx.$$

Solution. By the trigonometric identity, we have

$$\cos 2x = 1-2\sin^2 x.$$

Hence

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x).$$

Therefore, we have

$$\begin{aligned}
\int \sin^2 x dx &= \frac{1}{2}\int (1 - \cos 2x)dx + C \\
&= \frac{1}{2}\left(\int 1dx - \int \cos 2x dx\right) + C \\
&= \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right) + C \\
&= \frac{1}{2}(x - \sin x \cos x) + C.
\end{aligned}$$

$\square$

Example 23 Evaluate

$$\int \sin^3 x dx.$$

Solution. By the trigonometric identity, we have

$$\sin 3x = 3\sin x - 4\sin^3 x.$$

Hence

$$\sin^3 x = \frac{1}{4}(3\sin x - \sin^3 x).$$

Therefore, we have

$$\begin{aligned}
\int \sin^3 x dx &= \frac{1}{4}\left(\int 3\sin x dx - \int \sin 3x dx\right) + C \\
&= \frac{1}{4}\left(-3\cos x + \frac{\cos 3x}{3}\right) + C \\
&= \frac{1}{12}(\cos 3x - 9\cos x) + C.
\end{aligned}$$

$\square$

Example 24 Evaluate

$$\int \cos x \cos 2x dx.$$

Solution. Applying the trigonometric identity

$$\cos x \cos 2x = \frac{1}{2}(\cos 3x + \cos x),$$

we have

$$\begin{aligned}\int \cos x \cos 2x dx &= \frac{1}{2}\left(\int \cos 3x dx + \int \cos x dx\right) + C\\ &= \frac{1}{2}\left(\frac{\sin 3x}{3} + \sin x\right) + C\\ &= \frac{\sin 3x + 3\sin x}{6} + C.\end{aligned}$$

□

Example 25 Evaluate

$$\int \frac{1}{a\sin x + b\cos x} dx.$$

Solution. We find two numbers r and α such that

$$a = r\cos\alpha, \quad \text{and} \quad b = r\sin\alpha.$$

From these, squaring and adding, we obtain

$$r = \sqrt{a^2+b^2},$$

and on dividing one by the other, we obtain

$$\tan\alpha = \left(\frac{b}{a}\right) \quad \text{or} \quad \alpha = \tan^{-1}\left(\frac{b}{a}\right).$$

Hence, we have

$$a\sin x + b\cos x = r(\sin x\cos\alpha + \cos x\sin\alpha).$$

Therefore, we have

$$\begin{aligned}\int \frac{dx}{a\sin x + b\cos x} &= \frac{1}{r}\int \frac{dx}{\sin(x+\alpha)}\\ &= \frac{1}{r}\int \operatorname{cosec}(x+\alpha)dx\\ &= \frac{1}{r}\log_e\left|\tan\left(\frac{x+\alpha}{2}\right)\right| + C,\end{aligned}$$

where $r = \sqrt{a^2+b^2}$, and $\alpha = \tan^{-1}\left(\frac{b}{a}\right)$. □

23.4 Six important integrals

We now obtain the integrals of

$$\begin{array}{llll}\displaystyle\int \frac{1}{\sqrt{a^2-x^2}}dx & \text{Example } 26 & \displaystyle\int \sqrt{a^2-x^2}dx & \text{Example } 29\\ \displaystyle\int \frac{1}{\sqrt{a^2+x^2}}dx & \text{Example } 27 & \displaystyle\int \sqrt{a^2+x^2}dx & \text{Example } 30\\ \displaystyle\int \frac{1}{\sqrt{x^2-a^2}}dx & \text{Example } 28 & \displaystyle\int \sqrt{x^2-a^2}dx & \text{Example } 31\end{array}$$

The *minus* sign in Example 31 should be specially noted.

Example 26 Evaluate
$$\int \frac{1}{\sqrt{a^2 - x^2}} dx.$$

Solution. We put $x = a \sin\theta$, then $dx = a\cos\theta d\theta$. Also
$$a^2 - x^2 = a^2(1 - \sin^2\theta) = a^2\cos^2\theta.$$

Therefore, we have
$$\begin{aligned} \int \frac{1}{\sqrt{a^2 - x^2}} dx &= \int \frac{1}{a\cos\theta} a\cos\theta d\theta \\ &= \int 1\, d\theta \\ &= \theta + C \\ &= \sin^{-1}\frac{x}{a} + C. \end{aligned}$$

□

Example 27 Evaluate
$$\int \frac{1}{\sqrt{a^2 + x^2}} dx.$$

Solution. We put $x = a\text{sinh}\theta$, then $dx = a\text{cosh}\theta d\theta$. Also
$$a^2 + x^2 = a^2(1 + \text{sinh}^2\theta) = a^2\text{cosh}^2\theta.$$

Therefore, we have
$$\begin{aligned} \int \frac{1}{\sqrt{a^2 + x^2}} dx &= \int \frac{1}{a\text{cosh}\theta} a\text{cosh}\theta d\theta \\ &= \int 1\, d\theta \\ &= \theta + C \\ &= \text{sinh}^{-1}\frac{x}{a} + C \\ &= \log_e\left|\frac{x + \sqrt{x^2 + a^2}}{a}\right| + C. \end{aligned}$$

□

Example 28 Evaluate
$$\int \frac{1}{\sqrt{x^2 - a^2}} dx.$$

Solution. We put $x = a\text{cosh}\theta$, then $dx = a\text{sinh}\theta d\theta$. Also
$$x^2 - a^2 = a^2(\text{cosh}^2\theta - 1) = a^2\text{sinh}^2\theta.$$

Therefore, we have

$$\begin{aligned}\int \frac{1}{\sqrt{x^2-a^2}}dx &= \int \frac{1}{a\mathrm{sinh}\theta} a\mathrm{sinh}\theta d\theta \\ &= \int 1 \cdot d\theta \\ &= \theta + C \\ &= \cosh^{-1}\frac{x}{a} + C \\ &= \log_e\left|\frac{x+\sqrt{x^2-a^2}}{a}\right| + C.\end{aligned}$$

□

Example 29 Evaluate

$$\int \sqrt{(a^2-x^2)}dx.$$

Solution. Putting $x = a\sin\theta$, we get

$$\begin{aligned}\int \sqrt{a^2-x^2}dx &= \int a^2\cos^2\theta d\theta \\ &= \frac{a^2}{2}\int(\cos 2\theta + 1)d\theta + C \\ &= \frac{a^2}{2}\left(\frac{\sin 2\theta}{2} + \theta\right) + C \\ &= \frac{a^2}{2}(\sin\theta\cos\theta + \theta) + C \\ &= \frac{a^2}{2}\left(\sin\theta\sqrt{1-\sin^2\theta} + \theta\right) + C \\ &= \frac{a^2}{2}\left(\frac{x}{a}\sqrt{1-\frac{x^2}{a^2}} + \sin^{-1}\frac{x}{a}\right) + C \\ &= \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C.\end{aligned}$$

□

Example 30 Evaluate

$$\int \sqrt{a^2+x^2}dx.$$

Solution. We put $x = a\text{sinh}\theta$, then $dx = a\text{cosh}\theta d\theta$. We know that

$$\int \sqrt{a^2+x^2}dx = \int a^2\text{cosh}^2\theta d\theta$$

and

$$\begin{cases} \text{cosh}2\theta = \text{cosh}^2\theta + \text{sinh}^2\theta \\ 1 = \text{cosh}^2\theta - \text{sinh}^2\theta. \end{cases}$$

Hence

$$\frac{1+\text{cosh}2\theta}{2} = \text{cosh}^2\theta.$$

Therefore, we have

$$\begin{aligned}
\int \sqrt{a^2+x^2}dx &= \frac{a^2}{2}\int(\text{cosh}2\theta+1)d\theta \\
&= \frac{a^2}{2}\left(\frac{\text{sinh}2\theta}{2}+\theta\right)+C \\
&= \frac{a^2}{2}(\text{sinh}\theta\text{cosh}\theta+\theta)+C \\
&= \frac{a^2}{2}\left(\text{sinh}\theta\sqrt{1+\text{sinh}^2\theta}+\theta\right)+C \\
&= \frac{a^2}{2}\left(\frac{x}{2}\sqrt{1+\frac{x^2}{a^2}}+\text{sinh}^{-1}\frac{x}{a}\right)+C \\
&= \frac{x\sqrt{a^2+x^2}}{2}+\frac{a^2}{2}\text{sinh}^{-1}\frac{x}{a}+C \\
&= \frac{x\sqrt{a^2+x^2}}{2}+\frac{a^2}{2}\log_e\left|\frac{x+\sqrt{x^2+a^2}}{a}\right|+C.
\end{aligned}$$

□

Example 31 Evaluate

$$\int \sqrt{x^2-a^2}dx.$$

Solution. We put $x = a\text{cosh}\theta$, then $dx = a\text{sinh}\theta d\theta$. We know that

$$\int \sqrt{x^2-a^2}dx = \int a^2\text{sinh}^2\theta d\theta.$$

and

$$\begin{cases} \text{cosh}2\theta = \text{cosh}^2\theta + \text{sinh}^2\theta \\ 1 = \text{cosh}^2\theta - \text{sinh}^2\theta. \end{cases}$$

Hence
$$\frac{\cosh 2\theta - 1}{2} = \sinh^2\theta.$$
Therefore, we have
$$\begin{aligned}
\int \sqrt{x^2-a^2}dx &= \frac{a^2}{2}\int (\cosh 2\theta - 1)d\theta \\
&= \frac{a^2}{2}\left(\frac{\sinh 2\theta}{2} - \theta\right) + C \\
&= \frac{a^2}{2}(\sinh\theta\cosh\theta - \theta) + C \\
&= \frac{a^2}{2}\left(\sqrt{\cosh^2\theta - 1}\cosh\theta - \theta\right) + C \\
&= \frac{a^2}{2}\left(\frac{x}{a}\sqrt{\frac{x^2}{a^2} - 1} - \cosh^{-1}\frac{x}{a}\right) + C \\
&= \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a} + C \\
&= \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\log_e\left|\frac{x+\sqrt{x^2-a^2}}{a}\right| + C.
\end{aligned}$$

□

23.4.1 Worked examples

Example 32 Evaluate

1. $$\int \frac{1}{\sqrt{a^2-(bx+c)^2}},$$
2. $$\int \frac{1}{\sqrt{a^2+(bx+c)^2}},$$
3. $$\int \frac{1}{\sqrt{(bx+c)^2-a^2}}$$

Solution.

1. $$\int \frac{dx}{\sqrt{a^2-(bx+c)^2}} = \frac{1}{b}\sin^{-1}\left(\frac{bx+c}{a}\right) + C.$$
2. $$\int \frac{dx}{\sqrt{a^2+(bx+c)^2}} = \frac{1}{b}\sinh^{-1}\left(\frac{bx+c}{a}\right) + C.$$

3.

$$\int \frac{dx}{\sqrt{(bx+c)^2 - a^2}} = \frac{1}{b}\cosh^{-1}\left(\frac{bx+c}{a}\right) + C.$$

$\square$

Example 33 Evaluate

1. $$\int \frac{1}{\sqrt{x^2+2x+2}},$$

2. $$\int \frac{1}{\sqrt{x^2+4x+2}},$$

3. $$\int \frac{1}{\sqrt{-2x^2+3x+4}}$$

Solution.

1. We have
$$(x^2+2x+2) = (x+1)^2+1^2.$$
Therefore, we have
$$\begin{aligned}\int \frac{1}{\sqrt{x^2+2x+2}}dx &= \int \frac{dx}{\sqrt{(x+1)^2+1^2}}\\ &= \sinh^{-1}(x+1)+C.\end{aligned}$$

2. We have
$$(x^2+4x+2) = (x+2)^2-(\sqrt{2})^2.$$
Therefore, we have
$$\begin{aligned}\int \frac{1}{\sqrt{x^2+4x+2}}dx &= \int \frac{dx}{\sqrt{(x+2)^2-(\sqrt{2})^2}}\\ &= \cosh^{-1}\left(\frac{x+2}{\sqrt{2}}\right)+C.\end{aligned}$$

3. We have
$$\begin{aligned}-2x^2+3x+4 &= -2\left[x^2-\frac{3}{2}x-2\right]\\ &= -2\left[\left(x-\frac{3}{4}\right)^2-\left(\frac{\sqrt{41}}{4}\right)^2\right]\\ &= 2\left[\left(\frac{\sqrt{41}}{4}\right)^2-\left(x-\frac{3}{4}\right)^2\right].\end{aligned}$$

Therefore, we have

$$
\begin{aligned}
\int \frac{1}{\sqrt{-2x^2+3x+4}}dx &= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\left(\frac{\sqrt{41}}{4}\right)^2 - \left(x-\frac{3}{4}\right)^2}} \\
&= \frac{1}{\sqrt{2}} \sin^{-1}\left(\frac{x-\frac{3}{4}}{\sqrt{41}}\right) + C \\
&= \frac{1}{\sqrt{2}} \sin^{-1}\left(\frac{4x-3}{\sqrt{41}}\right) + C.
\end{aligned}
$$

$\square$

Example 34 Evaluate

1. $$\int \frac{1}{x^4+a^2}dx,$$

2. $$\int \frac{2x^2}{x^4+a^2}dx.$$

Solution.

1. Let us split the integral into sub-integrals:

$$\begin{aligned}
I &= \int \frac{dx}{x^4+a^2} \\
&= \frac{1}{2a}\int \frac{(x^2+a)-(x^2-a)}{x^4+a^2}dx \\
&= \frac{1}{2a}\int \frac{x^2+a}{x^4+a^2}dx - \frac{1}{2a}\int \frac{x^2-a}{x^4+a^2}dx \\
&= \frac{1}{2a}(I_1 - I_2).
\end{aligned}$$

We have

$$\begin{aligned}
I_1 &= \int \frac{x^2+a}{x^4+a^2}dx \\
&= \int \frac{1+\frac{a}{x^2}}{x^2+\frac{a^2}{x^2}}dx \\
&= \int \frac{1+\frac{a}{x^2}}{\left(x-\frac{a}{x}\right)^2+2a}dx.
\end{aligned}$$

If $x-\frac{a}{x}=y$, then, $\left(1+\frac{a}{x^2}\right)dx = dy$.
Therefore, we get

$$\begin{aligned}
I_1 &= \int \frac{dy}{y^2+2a} = \frac{1}{\sqrt{2a}}\tan^{-1}\left(\frac{y}{\sqrt{2a}}\right)+C_1 \\
&= \frac{1}{\sqrt{2a}}\tan^{-1}\left(\frac{x^2-a}{x\sqrt{2a}}\right)+C_1.
\end{aligned}$$

Similarly, we write

$$\begin{aligned}
I_2 &= \int \frac{x^2-a}{x^4+a^2}dx \\
&= \int \frac{1-\frac{a}{x^2}}{x^2+\frac{a^2}{x^2}}dx \\
&= \int \frac{1-\frac{a}{x^2}}{\left(x+\frac{a}{x}\right)^2-2a}
\end{aligned}$$

If $x + \frac{a}{x} = y$, then $\left(1 - \frac{a}{x^2}\right) dx = dy$. The integral reduces to

$$\begin{aligned} I_2 &= \int \frac{dy}{y^2 - 2a} \\ &= \frac{1}{2\sqrt{2a}} \log_e \left| \frac{y - \sqrt{2a}}{y + \sqrt{2a}} \right| + C_2 \\ &= \frac{1}{2\sqrt{2a}} \log_e \left| \frac{x^2 - \sqrt{2a}x + a}{x^2 + \sqrt{2a}x + a} \right| + C_2. \end{aligned}$$

Hence, we obtain

$$I = \frac{1}{2a\sqrt{2a}} \left(\tan^{-1} \left(\frac{x^2 - a}{x\sqrt{2a}} \right) - \frac{1}{2} \log_e \left| \frac{x^2 - \sqrt{2a}x + a}{x^2 + \sqrt{2a}x + a} \right| \right) + C$$

where $C = \dfrac{C_1 - C_2}{2}$ is another arbitrary constant.

2. Let us decompose the integral into two sub integrals:

$$\begin{aligned} I &= \int \frac{2x^2}{x^4 + a^2} dx \\ &= \int \frac{(x^2 + a) + (x^2 - a)}{x^4 + a^2} dx \\ &= \int \frac{x^2 + a}{x^4 + a^2} dx + \int \frac{x^2 - a}{x^4 + a^2} dx \\ &= I_1 + I_2. \end{aligned}$$

Using the results from 1, we obtain

$$I = \frac{1}{\sqrt{2a}} \left(\tan^{-1} \left(\frac{x^2 - a}{x\sqrt{2a}} \right) + \frac{1}{2} \log_e \left| \frac{x^2 - \sqrt{2a}x + a}{x^2 + \sqrt{2a}x + a} \right| \right) + C.$$

□

Example 35 Evaluate

1. $$\int \frac{xdx}{(9-x^2)^{3/2}}dx,$$

2. $$\int \frac{\sin(2x)\cos(2x)dx}{\sqrt{9-\cos^4(2x)}}dx.$$

Solution.

1. If we put $x = 3\sin\theta$, then $dx = 3\cos\theta d\theta$.

   Therefore, we get

$$\begin{aligned}\int \frac{9\sin\theta\cos\theta d\theta}{27\cos^3\theta} &= 7\frac{1}{3}\int \sec\theta\tan\theta d\theta \\ &= \frac{1}{3}\sec\theta + C \\ &= \frac{1}{3\sqrt{1-\sin^2\theta}} + C \\ &= \frac{1}{\sqrt{(9-x^2)}} + C.\end{aligned}$$

2. If we put $\cos^2(2x) = y$, then we get $4\sin 2x\cos 2xdx = -dy$.

   Therefore, we get

$$\begin{aligned}-\frac{1}{4}\frac{dy}{\sqrt{9-y^2}} &= -\frac{1}{4}\sin^{-1}\left(\frac{y}{3}\right) + C \\ &= -\frac{1}{4}\sin^{-1}\left(\frac{1}{3}\cos^2(2x)\right) + C.\end{aligned}$$

□

Example 36 Evaluate

$$\int \frac{dx}{\sqrt{2x+x^2}}dx.$$

Solution.

We have $2x - x^2 = 1 - (1 - 2x + x^2) = 1 - (1-x)^2$.

Therefore, we have

$$\begin{aligned}\int \frac{dx}{\sqrt{1-(1-x)^2}} &= -\int \frac{dy}{\sqrt{1-y^2}}, \text{ where } 1-x=y \\ &= -\sin^{-1} y + C \\ &= -\sin^{-1}(1-x) + C.\end{aligned}$$

□

Example 37 Evaluate

1.
$$\int \frac{dx}{\sqrt{7-6x-x^2}},$$

2.
$$\int \frac{x^4 dx}{x^2-x+1}dx,$$

3.
$$\int \frac{xdx}{x^4-x^2+1}dx.$$

Solution.

1. Let us write
$$\int \frac{dx}{\sqrt{7-6x-x^2}} = \int \frac{dx}{\sqrt{16-(x-3)^2}}.$$

   If we put $x+3=4y$, then, $dx = 4dy$.

   Therefore, we get
$$\begin{aligned}\int \frac{4dy}{4\sqrt{1-y^2}} &= \sin^{-1}(y) + C \\ &= \sin^{-1}\left(\frac{x+3}{4}\right) + C.\end{aligned}$$

2. Let us decompose the given integral into two integrals:
$$\begin{aligned}\int \frac{x^4 dx}{x^2-x+1} &= \int \left(x^2 + x - \frac{x}{x^2-x+1}\right) dx \\ &= \int (x^2+x)dx - \int \frac{xdx}{x^2-x+1} \\ &= I_1 - I_2.\end{aligned}$$

   We have
$$I_1 = \int (x^2+x)dx = \frac{1}{3}x^3 + \frac{1}{2}x^2 + C_1,$$

   and
$$I_2 = \int \frac{xdx}{x^2-x+1}.$$

   Let $x = A(2x-1)+B$. Then we find A and B.

Comparing coefficients on both sides, we get $2A = 1$ and $B - A = 0$. Hence, $A = B = \frac{1}{2}$.

Therefore, we get

$$\begin{aligned}
I_2 &= \frac{1}{2}\int \frac{(2x-1)dx}{x^2-x+1} + \frac{1}{2}\int \frac{dx}{x^2-x+1} \\
&= \frac{1}{2}\log_e\left|x^2-x+1\right| + \frac{1}{2}\int \frac{dx}{(x-1/2)^2+(\sqrt{3}/2)^2} \\
&= \frac{1}{2}\log_e\left|x^2-x+1\right| + \frac{1}{2}\left(\frac{2}{\sqrt{3}}\right)\tan^{-1}\left(\frac{x-1/2}{\sqrt{3}/2}\right) + C_2 \\
&= \frac{1}{2}\log_e\left|x^2-x+1\right| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C_2.
\end{aligned}$$

Hence, we get

$$I = \frac{1}{3}x^3 + \frac{1}{2}x^2 - \frac{1}{2}\log_e\left|x^2-x+1\right| - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C.$$

3. If we put $x^2 = y$, then $2xdx = dy$. The integral reduces to

$$\begin{aligned}
\frac{1}{2}\int \frac{dy}{y^2-y+1} &= \frac{1}{2}\int \frac{dy}{(y-1/2)^2+(\sqrt{3}/2)^2} \\
&= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2y-1}{\sqrt{3}}\right) + C \\
&= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x^2-1}{\sqrt{3}}\right) + C.
\end{aligned}$$

□

Example 38 Evaluate

1.
$$\int \sqrt{\frac{1+x}{1-x}}dx,$$

2.
$$\int \frac{dx}{\sqrt{x(1-2x)}}.$$

Solution.

1. We have

$$\begin{aligned}
\int \sqrt{\frac{1+x}{1-x}}dx &= \int \frac{1+x}{\sqrt{1-x^2}}dx \\
&= \int \frac{dx}{\sqrt{1-x^2}} + \int \frac{xdx}{\sqrt{1-x^2}} \\
&= \sin^{-1} x + I_1 + C_1.
\end{aligned}$$

In I_1, if we put $1-x^2 = t^2$, then we get $-2xdx = 2tdt$ or $xdx = -t$. Hence, we get

$$\begin{aligned}
-\int \frac{tdt}{\sqrt{t^2}} &= -\int dt \\
&= -t + C_2 \\
&= -\sqrt{1-x^2} + C_2.
\end{aligned}$$

Therefore, we have

$$\int \sqrt{\frac{1+x}{1-x}}dx = \sin^{-1} x - \sqrt{1-x^2} + C,$$

where $C = C_1 + C_2$.

2. We have

$$\begin{aligned}
I &= \int \frac{dx}{\sqrt{x(1-2x)}} \\
&= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{(x/2) - x^2}} \\
&= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{(1/16) - (x-(1/16))^2}}
\end{aligned}$$

If we put $x - (1/4) = y$, then we get $dx = dy$.

Therefore, we have

$$\begin{aligned}
\frac{1}{\sqrt{2}} \int \frac{dy}{\sqrt{(1/16) - y^2}} &= \frac{1}{\sqrt{2}} \sin^{-1}\left(\frac{y}{1/4}\right) + C \\
&= \frac{1}{\sqrt{2}} \sin^{-1}(4(x-(1/4))) + C \\
&= \frac{1}{\sqrt{2}} \sin^{-1}(4x-1) + C.
\end{aligned}$$

$\square$