THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1510 Calculus for Engineers (Fall 2021)
Suggested solutions of homework 3
Deadline: Nov. 6 at 23:00

Part A:

1. **Procedure for graphing functions using Calculus**

Step 1: Pre-calculus analysis:

(a) Find the domain of the function.
(b) Find the $x-$ and $y-$ intercepts.
(c) Test for symmetry with respect to the $y-$axis and the origin. (Verify whether the function is even or odd or neither or both).

Step 2: Calculus analysis:

(a) Use the first derivative to find the critical points and to find out where the graph is increasing and decreasing.
(b) Test the critical points for local maxima and minima.
(c) Use the second derivative to find out where the graph is concave upward and concave downward, and to locate inflection points.
(d) Find all asymptotes (horizontal, vertical), if any.

Step 3: Plot all critical points, inflection points, and $x-$ and $y-$ intercepts.

Step 4: Sketch the graph.

Sketch the graph of

$$f(x) = \frac{x}{(x-1)^2}$$

following the above procedure.

Solution:

Step 1:
(a) The domain is $(-\infty, 1) \cup (1, +\infty)$, since f is not defined at $x = 1$.
(b) $f(0) = 0$ gives the y-intercept $(0, 0)$ which is also the only x-intercept.
(c) $f(-x) = \dfrac{(-x)}{(-x-1)^2} \neq \pm f(x)$, so it is neither even nor odd.

Step 2:
(a)

$$\begin{aligned} f'(x) &= \frac{(x-1)^2 - 2x(x-1)}{(x-1)^4} \\ &= \frac{(x-1)(x-1-2x)}{(x-1)^4} \\ &= -\frac{(x+1)}{(x-1)^3} \end{aligned}$$

x	$x < -1$	$x = -1$	$-1 < x < 1$	$x = 1$	$x > 1$
$f'(x)$	$-$	0	$+$	undefined	$-$

On $(-\infty, -1)$ and $(1, +\infty)$, $f'(x) < 0$; on $(-1, 1)$, $f'(x) > 0$. So f is decreasing on $(-\infty, -1)$ and $(1, +\infty)$; increasing on $(-1, 1)$. The only critical point is $x = -1$.

(b) By the First Derivative Test, $x = -1$ is a local minimum.

(c)

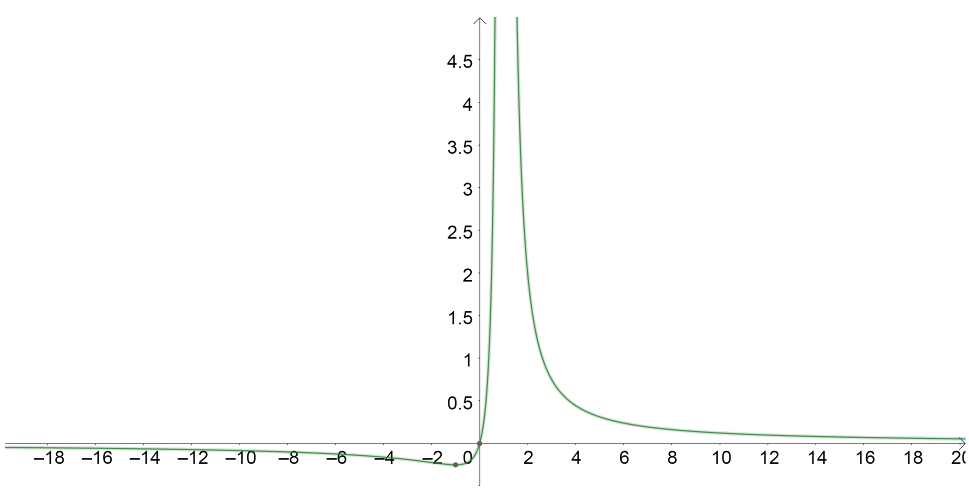
$$\begin{aligned} f''(x) &= -\frac{(x-1)^3 - 3(x-1)^2(x+1)}{(x-1)^6} \\ &= -\frac{(x-1)^2(-2x-4)}{(x-1)^6} \\ &= \frac{2(x+2)}{(x-1)^4} \end{aligned}$$

x	$x < -2$	$x = -2$	$-2 < x < 1$	$x = 1$	$x > 1$
$f''(x)$	$-$	0	$+$	undefined	$+$

On $(-\infty, -2)$, $f''(x) < 0$ and so f is concave down. On $(-2, 1)$ and $(1, \infty)$, $f''(x) > 0$, so f is concave up. $x = -2$ is an inflection point.

(d) $\lim_{x \to +\infty} \frac{x}{(x-1)^2} = 0$ and $\lim_{x \to -\infty} \frac{x}{(x-1)^2} = 0$, so $y = 0$ is a horizontal asymptote. As $\lim_{x \to 1^+} \frac{x}{(x-1)^2} = \infty$ and $\lim_{x \to 1^-} \frac{x}{(x-1)^2} = \infty$, so $x = 1$ is the only vertical asymptote.

Step 3 and **Step 4**. See the graph below.



Part B:

2. A spherical balloon is inflated with helium at the rate of $100\pi\text{ft}^3/\text{min}$. How fast is the balloon's radius increasing at the instant the radius is 5ft? How fast is the surface area increasing? Recall that

$$V(t) = \frac{4}{3}\pi r(t)^3, S(t) = 4\pi r(t)^2,$$

where $V(t)$ and $S(t)$ are the volume and surface of the sphere where the radius is given by $r(t)$.

Solution:

Since the balloon is inflated at the rate of $100\pi\text{ft}^3/\text{min}$, $V'(t) = 100\pi$. Hence

$$100\pi = V'(t) = \frac{4}{3}\pi \cdot 3r(t)^2 r'(t) = 4\pi r(t)^2 r'(t).$$

Now, when $r(t) = 5$, we have

$$\begin{aligned} 100\pi &= 4\pi(5)^2 r'(t) \\ r'(t) &= 1. \end{aligned}$$

Thus the balloon's radius is increasing at the rate of 1 ft/min at the instant the radius is 5 ft.

For the surface area,

$$S'(t) = 4\pi \cdot 2r(t)r'(t) = 8\pi r(t)r'(t).$$

When $r(t) = 5$, $r'(t) = 1$, and hence

$$S'(t) = 8\pi(5)(1) = 40\pi.$$

Thus the balloon's surface area is increasing at the rate of 40π ft^2/min at the instant the radius is 5 ft.

3. Show that

$$\frac{2}{\pi}x < \sin x < x, \quad x \in (0, \frac{\pi}{2}).$$

Solution:

(i) Consider $f(x) = x - \sin x$, $x \in [0, \frac{\pi}{2}]$. $f(x)$ is continuous on $[0, \frac{\pi}{2}]$.

$f'(x) = 1 - \cos x > 0$ on $(0, \frac{\pi}{2})$, so $f(x)$ is strictly increasing on $[0, \frac{\pi}{2}]$.

Therefore, for $x \in (0, \frac{\pi}{2})$, $f(x) > f(0) = 0$ which implies $x > \sin x$.

(ii) Consider $g(x) = \sin x - \frac{2}{\pi}x$, $x \in [0, \frac{\pi}{2}]$. $g(x)$ is continuous on $[0, \frac{\pi}{2}]$.

$g'(x) = \cos x - \frac{2}{\pi} > 0$ on $(0, \arccos\frac{2}{\pi})$, $g'(x) < 0$ on $(\arccos\frac{2}{\pi}, \frac{\pi}{2})$,

so $g(x)$ is strictly increasing on $[0, \arccos\frac{2}{\pi}]$, and strictly decreasing on $[\arccos\frac{2}{\pi}, \frac{\pi}{2}]$.

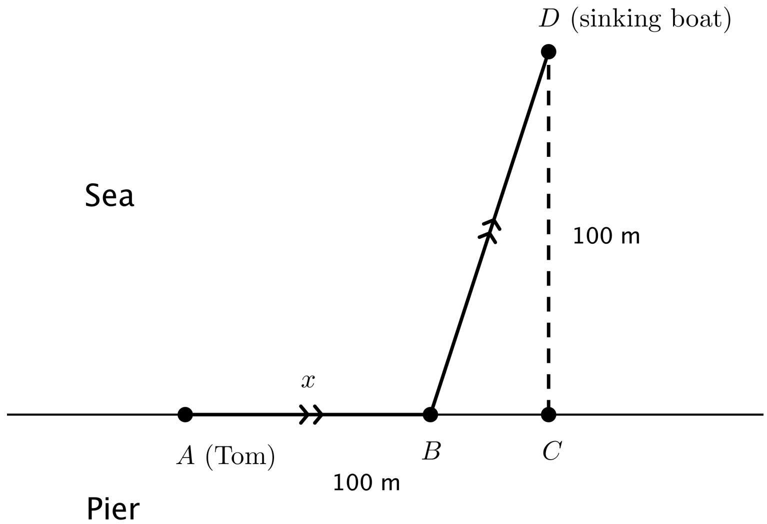
Since $g(0) = g(\frac{\pi}{2}) = 0$, it follows from the monotonicity that $g(x) > 0$ for $x \in (0, \frac{\pi}{2})$,

which implies $\sin x > \frac{2}{\pi}x$.

Combining part (i) and (ii), we obtain

$$\frac{2}{\pi}x < \sin x < x, \quad x \in (0, \frac{\pi}{2}).$$

4. Tom, a lifeguard stationed at point A, spotted a sinking boat at point D:



To get to point D as soon as possible, he decided to run from A to B and swim from B to D. Suppose his running and swimming speeds are 9 and 1.8 respectively, and $|AC| = |CD| = 100$. Denote $|AB| = x,\ x \in [0, 100]$.

(a) Express the total time taken of the trip T as a function of x.

(b) Find all the critical points of $T(x)$ over the interval $(0, 100)$.

(c) Find the value(s) of x that minimizes the total time taken and the corresponding T (correct to 2 d.p.).

Solution:

(a) $AB = x, BC = 100 - x, CD = 100$, So, $BD = \sqrt{(100-x)^2 + 100^2}$.
Hence, the total time taken is

$$T(x) = \frac{AB}{9} + \frac{BD}{1.8} = \frac{x}{9} + \frac{\sqrt{(100-x)^2+100^2}}{1.8},$$

where $x \in [0, 100]$.

(b)

$$\begin{aligned} T'(x) &= \frac{1}{9} + \frac{-2(100-x)}{2(1.8)\sqrt{(100-x)^2+100^2}} \\ &= \frac{1}{9} + \frac{x-100}{(1.8)\sqrt{(100-x)^2+100^2}} \\ &= \frac{\sqrt{(100-x)^2+100^2} + 5(x-100)}{9\sqrt{(100-x)^2+100^2}} \end{aligned}$$

$T'(x)$ exists everywhere over $[0, 100]$.

When $T'(x) = 0$, we have

$$\begin{aligned}
\sqrt{(100-x)^2 + 100^2} &= 5(100-x) \\
[(100-x)^2 + 100^2] &= 25(100-x)^2 \\
(24)(100-x)^2 &= 100^2 \\
x &= \frac{25}{3}(12-\sqrt{6}) \qquad \left(\frac{25}{3}(12+\sqrt{6}) \text{ is rejected}\right)
\end{aligned}$$

So $\frac{25}{3}(12-\sqrt{6})$ is the only critical point of $T(x)$ over $(0, 100)$.

(c) Note that

$$T(0) \approx 78.57, \quad T(\frac{25}{3}(12-\sqrt{6})) \approx 65.54, \quad T(100) \approx 66.67.$$

Thus the minimum total time taken is 65.54 at $x = \frac{25}{3}(12-\sqrt{6})$.

5. In physics, if the displacement of an object is described by a function $x(t)$, then its velocity, denoted by $v(t)$, and its acceleration, denoted by $a(t)$, are given by $x'(t) = \frac{dx}{dt}$ and $x''(t) = \frac{d^2x}{dt^2}$ respectively.
   Ideally, an object attached to a spring oscillates in simple harmonic motion. Its displacement from the equilibrium position would then be a function of time t, given by
   $$x(t) = A\cos(\omega t - \varphi),$$
   where m is the mass of the object, k is the spring constant, $\omega = \sqrt{\frac{k}{m}}$ and A, φ are two constants determined by the initial situation.

   (a) Find the velocity $v(t)$ and acceleration $a(t)$ of the object as a function of time.
   (b) Find the maximum velocity and acceleration in magnitude and the value(s) of t achieving them.
   (c) The kinetic and potential energy of the object are given by
   $$K(t) = \frac{1}{2}m(v(t))^2 \quad \text{and} \quad U(t) = \frac{1}{2}k(x(t))^2$$
   respectively. Show that the total mechanical energy, i.e. the sum of kinetic energy and potential energy, is independent of time t.

Solution:

(a)
$$\begin{aligned} v(t) = x'(t) &= \frac{dx}{dt} = -A\omega\sin(\omega t - \varphi) \\ a(t) = x''(t) &= \frac{d^2x}{dt^2} = -A\omega^2\cos(\omega t - \varphi) \end{aligned}$$

(b) $\max|v(t)| = A\omega$, the maximum is attained at $\omega t - \varphi = \frac{\pi}{2} + n\pi, n \in \mathbb{Z}$, i.e. $t = \frac{1}{\omega}\left(\frac{\pi}{2} + n\pi + \varphi\right)$, $n \in \mathbb{Z}$.
$\max|a(t)| = A\omega^2$, the maximum is attained at $\omega t - \varphi = n\pi, n \in \mathbb{Z}$, i.e. $t = \frac{1}{\omega}(n\pi + \varphi), n \in \mathbb{Z}$.

(c) The total mechanical energy
$$\begin{aligned} E &= K(t) + U(t) \\ &= \frac{1}{2}m(v(t))^2 + \frac{1}{2}k(x(t))^2 \\ &= \frac{1}{2}m(-A\omega\sin(\omega t - \varphi))^2 + \frac{1}{2}k(A\cos(\omega t - \varphi))^2 \\ &= \frac{1}{2}mA^2\omega^2\sin^2(\omega t - \varphi) + \frac{1}{2}kA^2\cos^2(\omega t - \varphi) \end{aligned}$$

Since $\omega = \sqrt{\dfrac{k}{m}}$, so $k = m\omega^2$.
And $\sin^2(\omega t - \varphi) + \cos^2(\omega t - \varphi) \equiv 1$, so we have

$$\begin{aligned}
E &= \frac{1}{2}mA^2\omega^2 \sin^2(\omega t - \varphi) + \frac{1}{2}kA^2 \cos^2(\omega t - \varphi) \\
&= \frac{1}{2}mA^2\omega^2 \sin^2(\omega t - \varphi) + \frac{1}{2}mA^2\omega^2 \cos^2(\omega t - \varphi) \\
&= \frac{1}{2}mA^2\omega^2.
\end{aligned}$$

So the total mechanical energy is independent of time t.

6. Find the indicated limit, if it exists. Furthermore, if the limit does not exist but diverges to plus or minus infinity, please indicate so, and determine the correct sign. (Make sure that you have an indeterminate form of the right type before you apply L'Hôpital's rule.)

   (a) $\displaystyle\lim_{x\to 0^+}\left(\frac{1}{x}-\csc x\right)$;

   (b) $\displaystyle\lim_{x\to 0}(\cos\, x)^{1/x}$;

   (c) $\displaystyle\lim_{x\to +\infty}\left(\frac{e^x+\sin x}{e^x-\sin x}\right)$.

Solution:

(a)

$$\begin{aligned}\lim_{x\to 0^+}\left(\frac{1}{x}-\csc x\right) &= \lim_{x\to 0^+}\left(\frac{\sin x - x}{x\sin x}\right) \qquad \left(\frac{0}{0}\text{ form}\right)\\ &= \lim_{x\to 0^+}\left(\frac{\cos x - 1}{\sin x + x\cos x}\right) \qquad \left(\frac{0}{0}\text{ form}\right)\\ &= \lim_{x\to 0^+}\frac{-\sin x}{2\cos x - x\sin x}\\ &= \frac{0}{2\cdot 1 - 0\cdot 0}\\ &= 0;\end{aligned}$$

(b) Since

$$\begin{aligned}\lim_{x\to 0}\ln(\cos\, x)^{1/x} &= \lim_{x\to 0}\frac{\ln(\cos\, x)}{x} \qquad \left(\frac{0}{0}\text{ form}\right)\\ &= \lim_{x\to 0}\frac{\frac{1}{\cos x}(-\sin x)}{1}\\ &= \lim_{x\to 0} -\tan x\\ &= 0,\end{aligned}$$

we have $\displaystyle\lim_{x\to 0}(\cos\, x)^{1/x} = \lim_{x\to 0} e^{\ln(\cos\, x)^{1/x}} = e^0 = 1$;

(c) Since $-e^{-x}\le e^{-x}\sin x\le e^{-x}$ for any $x\in\mathbb{R}$, and $\displaystyle\lim_{x\to+\infty} e^{-x}=0$, it follows from the Sandwich theorem that $\displaystyle\lim_{x\to+\infty} e^{-x}\sin x = 0$. Hence

$$\begin{aligned}\lim_{x\to +\infty}\left(\frac{e^x+\sin x}{e^x-\sin x}\right) &= \lim_{x\to +\infty}\left(\frac{1+e^{-x}\sin x}{1-e^{-x}\sin x}\right)\\ &= \frac{1+0}{1-0}\\ &= 1.\end{aligned}$$

7. Assume that $\mathbf{r}_1(t) = \langle a(t), b(t), c(t)\rangle$ and $\mathbf{r}_2(t) = \langle x(t), y(t), z(t)\rangle$ are differentiable. Show that

   (a)
   $$\frac{d}{dt}\left(\mathbf{r}_1(t)\cdot\mathbf{r}_2(t)\right) = \mathbf{r}_1(t)\cdot\mathbf{r}_2'(t) + \mathbf{r}_1'(t)\cdot\mathbf{r}_2(t).$$

   (b)
   $$\frac{d}{dt}\left(\mathbf{r}_1(t)\times\mathbf{r}_2(t)\right) = \mathbf{r}_1(t)\times\mathbf{r}_2'(t) + \mathbf{r}_1'(t)\times\mathbf{r}_2(t).$$

   Solution:

   (a) Note that
   $$\mathbf{r}_1(t)\cdot\mathbf{r}_2(t) = ax + by + cz.$$
   Hence
   $$\begin{aligned}\frac{d}{dt}\left(\mathbf{r}_1(t)\cdot\mathbf{r}_2(t)\right) &= \frac{d}{dt}(ax+by+cz) = a'x + ax' + b'y + by' + c'z + cz' \\ &= (ax' + by' + cz') + (a'x + b'y + c'z) \\ &= \langle a, b, c\rangle\cdot\langle x', y', z'\rangle + \langle a', b', c'\rangle\cdot\langle x, y, z\rangle \\ &= \mathbf{r}_1(t)\cdot\mathbf{r}_2'(t) + \mathbf{r}_1'(t)\cdot\mathbf{r}_2(t).\end{aligned}$$

   (b) Note that
   $$\mathbf{r}_1(t)\times\mathbf{r}_2(t) = \langle bz - cy, cx - az, ay - bx\rangle.$$
   Hence
   $$\begin{aligned}&\frac{d}{dt}\left(\mathbf{r}_1(t)\times\mathbf{r}_2(t)\right) = \frac{d}{dt}\langle bz - cy, cx - az, ay - bx\rangle \\ &= \langle bz' + b'z - cy' - c'y, cx' + c'x - az' - a'z, ay' + a'y - bx' - b'x\rangle \\ &= \langle bz' - cy', cx' - az', ay' - bx'\rangle + \langle b'z - c'y, c'x - a'z, a'y - b'x\rangle \\ &= \langle a, b, c\rangle\times\langle x', y', z'\rangle + \langle a', b', c'\rangle\times\langle x, y, z\rangle \\ &= \mathbf{r}_1(t)\times\mathbf{r}_2'(t) + \mathbf{r}_1'(t)\times\mathbf{r}_2(t).\end{aligned}$$