**THE CHINESE UNIVERSITY OF HONG KONG**
**Department of Mathematics**
**MATH1510 Calculus for Engineers (Fall 2021)**
**Suggested solutions of coursework 10**

## Part A

1. Evaluate each of the following definite integrals.

   (a) $\displaystyle\int_0^2 x\ln(x^2+1)\,dx$

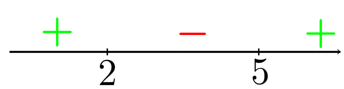
   (b) $\displaystyle\int_0^5 |-x^2+7x-10|\,dx$

**Solution:**

(a) Let $u = x^2+1 \implies du = 2x\,dx$. Hence

$$\begin{aligned}\int_0^2 x\ln(x^2+1)\,dx &= \frac{1}{2}\int_0^2 \ln(x^2+1)(2x\,dx)\\ &= \frac{1}{2}\int_1^5 \ln u\,du\\ &= \frac{1}{2}\left[u\ln u\Big|_1^5 - \int_1^5 u\cdot\frac{1}{u}\,du\right]\\ &= \frac{1}{2}\left[5\ln 5 - 0 - \left(u\Big|_1^5\right)\right]\\ &= \frac{1}{2}\left(5\ln 5 - 4\right).\end{aligned}$$

(b) Note that $x^2-7x+10 = (x-2)(x-5)$, and




Hence,

$$\begin{aligned}\int_0^5 |-x^2+7x-10|\,dx &= \int_0^5 |x^2-7x+10|\,dx\\ &= \int_0^2 (x^2-7x+10)\,dx + (-1)\int_2^5 (x^2-7x+10)\,dx\\ &= \left(\frac{1}{3}x^3 - \frac{7}{2}x^2 + 10x\right)\Bigg|_0^2 - \left(\frac{1}{3}x^3 - \frac{7}{2}x^2 + 10x\right)\Bigg|_2^5\\ &= \frac{79}{6}.\end{aligned}$$

2. Let
$$f(x) = \frac{1}{(1+x)\sqrt{x}}.$$
Evaluate each of the following improper integrals.

(a) $\displaystyle\int_1^\infty f(x)\,dx$

(b) $\displaystyle\int_0^1 f(x)\,dx$

**Solution:**

(a) Let $x = u^2 \implies dx = 2u\,du$. Then,
$$\begin{aligned}\int \frac{1}{(1+x)\sqrt{x}}\,dx &= \int \frac{2u}{(1+u^2)u}\,du \\ &= 2\int \frac{1}{1+u^2}\,du \\ &= 2\arctan u + C \\ &= 2\arctan\sqrt{x} + C.\end{aligned}$$

So,
$$\begin{aligned}\int_1^\infty f(x)\,dx &= \lim_{b\to\infty}\int_1^b f(x)\,dx \\ &= \lim_{b\to\infty}\left(2\arctan\sqrt{x}\Big|_1^b\right) \\ &= \lim_{b\to\infty}\left(2\arctan\sqrt{b} - \frac{\pi}{2}\right) \\ &= \frac{\pi}{2}.\end{aligned}$$

(b) Note that $0 \notin D_f$. So,
$$\begin{aligned}\int_0^1 f(x)\,dx &= \lim_{a\to 0^+}\int_a^1 f(x)\,dx \\ &= \lim_{a\to 0^+}\left(2\arctan\sqrt{x}\Big|_a^1\right) \\ &= \lim_{a\to 0^+}\left(\frac{\pi}{2} - 2\arctan\sqrt{a}\right) \\ &= \frac{\pi}{2}.\end{aligned}$$

## Part B

3. (a) Find $\dfrac{d}{dx}\displaystyle\int_0^x e^{(t^2)}\,dt$.

(b) Find $\dfrac{d}{dx}\displaystyle\int_0^{\sin 2x} e^{\sin t}\,dt$.

(c) By L'Hôpital's rule and parts (a),(b), evaluate

$$\lim_{x\to 0}\frac{\displaystyle\int_0^x e^{(t^2)}\,dt}{\displaystyle\int_0^{\sin 2x} e^{\sin t}\,dt}$$

**Solution:**

(a) By the Fundamental Theorem of Calculus,

$$\frac{d}{dx}\int_0^x e^{(t^2)}\,dt = e^{(x^2)}.$$

(b) By the Fundamental Theorem of Calculus,

$$\frac{d}{dx}\int_0^{\sin 2x} e^{\sin t}\,dt = e^{\sin(\sin 2x)}\cdot(\cos 2x)(2).$$

(c)

$$\begin{aligned}
&\lim_{x\to 0}\frac{\displaystyle\int_0^x e^{(t^2)}\,dt}{\displaystyle\int_0^{\sin 2x} e^{\sin t}\,dt} \qquad \left(\frac{0}{0}\right)\\
&= \lim_{x\to 0}\frac{e^{(x^2)}}{e^{\sin(\sin 2x)}(2\cos 2x)}\\
&= \frac{1}{2}.
\end{aligned}$$

4. Let
$$F(x) = \int_0^x |t|\,dt.$$

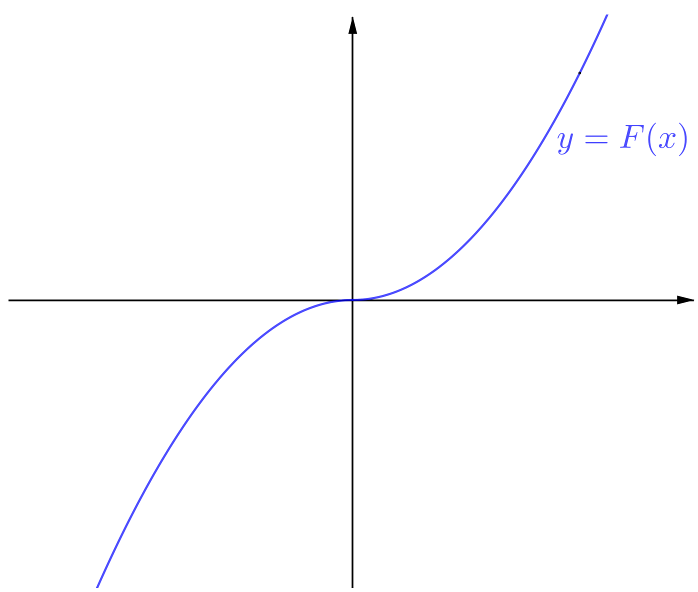
(a) $F(x)$ can be stated explicitly in the form
$$F(x) = \begin{cases} g(x) & \text{if } x \geq 0 \\ h(x) & \text{if } x < 0, \end{cases}$$
where $g, h$ are polynomials. Find $g(x), h(x)$.

(b) Sketch the graph of $F(x)$.

**Solution:**

(a) When $x \geq 0$,
$$F(x) = \int_0^x |t|\,dt = \int_0^x t\,dt = \frac{1}{2}t^2\Big|_0^x = \frac{1}{2}x^2,$$
so
$$g(x) = \frac{1}{2}x^2.$$
When $x < 0$,
$$F(x) = \int_0^x |t|\,dt = -\int_x^0 |t|\,dt = \int_x^0 t\,dt = \frac{1}{2}t^2\Big|_x^0 = -\frac{1}{2}x^2,$$
so
$$h(x) = -\frac{1}{2}x^2.$$

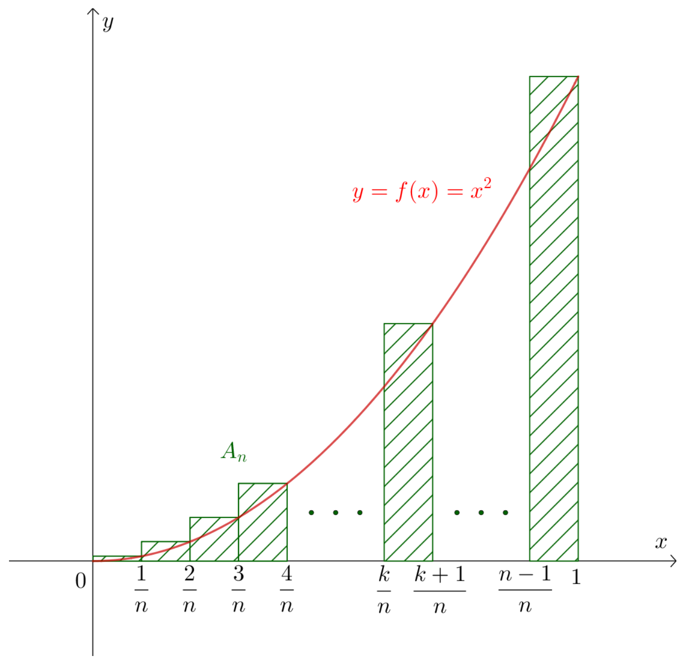
(b)



5. Let $f(x) = x^2$.

   (a) Evaluate $\displaystyle\int_0^1 f(x)\,dx$.

   (b) Suppose that the interval $[0, 1]$ is subdivided into $n$ equal subintervals. Define $A_n$ to be the Riemann sum of $f(x)$ as shown below.




   Find $A_n$ in terms of $n$.

   (Hint: $\displaystyle\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$ and $\displaystyle\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$)

   (c) By parts (a), (b), verify that

$$\lim_{n\to\infty} A_n = \int_0^1 f(x)\,dx$$

**Solution:**

(a) $\displaystyle\int_0^1 x^2\,dx = \frac{1}{3}x^3\Big|_0^1 = \frac{1}{3}$.

(b)

$$\begin{aligned}
A_n &= \sum_{k=1}^{n} f(\frac{k}{n})\frac{1}{n} \\
&= \frac{1}{n}\left[f(\frac{1}{n}) + f(\frac{2}{n}) + \cdots + f(\frac{n}{n})\right] \\
&= \frac{1}{n}\left[\frac{1^2}{n^2} + \frac{2^2}{n^2} + \cdots + \frac{n^2}{n^2}\right] \\
&= \frac{1}{n^3}\left(1^2 + 2^2 + \cdots + n^2\right) \\
&= \frac{1}{n^3}\cdot\frac{1}{6}n(n+1)(2n+1) \\
&= \frac{(n+1)(2n+1)}{6n^2}.
\end{aligned}$$

(c)

$$\begin{aligned}
\lim_{n\to\infty} A_n &= \lim_{n\to\infty}\frac{(n+1)(2n+1)}{6n^2} \\
&= \lim_{n\to\infty}\frac{(1+\frac{1}{n})(2+\frac{1}{n})}{6} \\
&= \frac{2}{6} = \frac{1}{3} = \int_0^1 x^2\,dx.
\end{aligned}$$