THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1510 Calculus for Engineers (Fall 2021)
Suggested solutions of coursework 8

Part A

1. Evaluate $\displaystyle\int \frac{x^3}{(x^2+1)^3}\,dx$ by using the given substitution.

   (a) $x = \tan\theta$

   (b) $u = 1 + x^2$

Solution:

(a) Let $x = \tan\theta \implies dx = \sec^2\theta\,d\theta$. Hence,

$$\begin{aligned}
\int \frac{x^3}{(x^2+1)^3}\,dx &= \int \frac{\tan^3\theta}{(\tan^2\theta+1)^3}\sec^2\theta\,d\theta \\
&= \int \frac{\tan^3\theta}{(\sec^2\theta)^3}\sec^2\theta\,d\theta \\
&= \int \frac{\tan^3\theta}{\sec^4\theta}\,d\theta \\
&= \int \sin^3\theta\cos\theta\,d\theta \\
&= \int \sin^3\theta\,d(\sin\theta) \\
&= \frac{1}{4}\sin^4\theta + C \\
&= \frac{1}{4}\sin^4(\arctan x) + C.
\end{aligned}$$

(b) Let $u = 1 + x^2 \implies du = 2x\,dx$. Hence,

$$\begin{aligned}
\int \frac{x^3}{(x^2+1)^3}\,dx &= \frac{1}{2}\int \frac{x^2}{(1+x^2)^3}(2x\,dx) \\
&= \frac{1}{2}\int \frac{u-1}{u^3}\,du \\
&= \frac{1}{2}\int (u^{-2} - u^{-3})\,du \\
&= \frac{1}{2}\left(-u^{-1} + \frac{1}{2}u^{-2} + C\right) \\
&= \frac{1}{2}\left(-\frac{1}{1+x^2} + \frac{1}{2}\frac{1}{(1+x^2)^2} + C\right).
\end{aligned}$$

Remark: In (a), note that $x = \tan\theta$, so $\sin\theta = \dfrac{x}{\sqrt{1+x^2}}$. We can rewrite the answer as following:

$$\frac{1}{4}\sin^4\theta + C = \frac{1}{4}\left(\frac{x}{\sqrt{1+x^2}}\right)^4 + C = \frac{1}{4}\frac{x^4}{(1+x^2)^2} + C$$

Consider the difference of the answers in (a) and (b), we have

$$\frac{1}{4}\frac{x^4}{(1+x^2)^2} - \frac{1}{2}\left(-\frac{1}{1+x^2} + \frac{1}{2}\frac{1}{(1+x^2)^2}\right) = \frac{1}{4(1+x^2)^2}\left[x^4 + 2(1+x^2) - 1\right] = \frac{1}{4}$$

which is a constant.

2. Evaluate the following indefinite integrals.

   (a) $\displaystyle\int \frac{x^2+4x+1}{x-2}\,dx$

   (b) $\displaystyle\int \frac{9x-1}{3x^2+x-2}\,dx$

   (c) $\displaystyle\int \frac{6x+17}{x^2+4x+20}\,dx$

   Solution:

   (a) By long division,

$$\begin{array}{r@{}l} & x + 6 \\ \cline{2-2} x-2 & \big) x^2 + 4x + 1 \\ & x^2 - 2x \\ \cline{2-2} & \phantom{)x^2 - {}}6x + 1 \\ & \phantom{)x^2 - {}}6x - 12 \\ \cline{2-2} & \phantom{)x^2 - 6x - {}}13 \end{array}$$

   Hence,

$$\begin{aligned}\int \frac{x^2+4x+1}{x-2}\,dx &= \int \left(x + 6 + \frac{13}{x-2}\right)dx \\ &= \frac{1}{2}x^2 + 6x + 13\ln|x-2| + C.\end{aligned}$$

   (b) By partial fractions decomposition,

$$\frac{9x-1}{3x^2+x-2} = \frac{9x-1}{(3x-2)(x+1)} = \frac{A}{3x-2} + \frac{B}{x+1}$$

   for some real constants A, B.
   Multiplying both sides by $(3x-2)(x+1)$, we get

$$9x - 1 = A(x+1) + B(3x-2).$$

$$x \to -1: \qquad -10 = 0 + B(-5) \implies B = 2$$
$$x \to \frac{2}{3}: \qquad 5 = A(\frac{5}{3}) + 0 \implies A = 3.$$

Thus,

$$\begin{aligned}\int \frac{9x-1}{3x^2+x-2}\,dx &= \int \left(\frac{3}{3x-2} + \frac{2}{x+1}\right) dx \\ &= \ln|3x-2| + 2\ln|x+1| + C.\end{aligned}$$

(c)

$$\begin{aligned}I &= \int \frac{6x+17}{x^2+4x+20}\,dx \\ &= \int \frac{6x+17}{(x+2)^2+16}\,dx \\ &= \frac{1}{16}\int \frac{6x+17}{\left(\frac{x+2}{4}\right)^2+1}\,dx.\end{aligned}$$

Let $u = \dfrac{x+2}{4} \implies du = \dfrac{1}{4}\,dx$. Hence,

$$\begin{aligned}I &= \frac{1}{16}\int \frac{6(4u-2)+17}{u^2+1} \cdot 4\,du \\ &= \frac{1}{4}\int \left(\frac{24u}{u^2+1} + \frac{5}{u^2+1}\right) du \\ &= \frac{1}{4}\left(12\int \frac{1}{u^2+1}\,d(u^2+1) + 5\arctan u\right) \\ &= \frac{1}{4}\left(12\ln|u^2+1| + 5\arctan u + C\right) \\ &= 3\ln\left|\left(\frac{x+2}{4}\right)^2+1\right| + \frac{5}{4}\arctan\left(\frac{x+2}{4}\right) + C.\end{aligned}$$

Part B

3. Evaluate the following indefinite integrals.

   (a) $\displaystyle\int \sin^6 x \cos^3 x\,dx$

   (b) $\displaystyle\int \sin^4 x \cos^4 x\,dx$
   (Hint: Consider the double angle formula for sine.)

Solution:

(a) Let $u = \sin x \implies du = \cos x\, dx$. Hence,

$$\begin{aligned} \int \sin^6 x \cos^3 x\, dx &= \int \sin^6 x(1 - \sin^2 x)(\cos x\, dx) \\ &= \int u^6(1-u^2)\, du \\ &= \int (u^6 - u^8)\, du \\ &= \frac{1}{7}u^7 - \frac{1}{9}u^9 + C \\ &= \frac{1}{7}\sin^7 x - \frac{1}{9}\sin^9 x + C. \end{aligned}$$

(b)

$$\begin{aligned} \int \sin^4 x \cos^4 x\, dx &= \int \left(\frac{1}{2}\sin 2x\right)^4 dx \\ &= \frac{1}{16}\int \sin^4 2x\, dx \\ &= \frac{1}{16}\int \left(\frac{1}{2}(1-\cos 4x)\right)^2 dx \\ &= \frac{1}{64}\int (\cos^2 4x - 2\cos 4x + 1)\, dx \\ &= \frac{1}{64}\int \left(\frac{1}{2}(1+\cos 8x) - 2\cos 4x + 1\right) dx \\ &= \frac{1}{64}\int \left(\frac{1}{2}\cos 8x - 2\cos 4x + \frac{3}{2}\right) dx \\ &= \frac{1}{64}\left(\frac{1}{16}\sin 8x - \frac{1}{2}\sin 4x + \frac{3}{2}x + C\right). \end{aligned}$$

4. Evaluate the following indefinite integrals.

(a) $\displaystyle\int \frac{x}{\sqrt[3]{1+x^2}}\, dx$

(b) $\displaystyle\int \frac{1}{\sqrt{2x+3} - \sqrt{2x+1}}\, dx$

Solution:

(a) Let $u = x^2 + 1 \implies du = 2x\, dx$. Hence,

$$\begin{aligned} \int \frac{x}{\sqrt[3]{1+x^2}}\, dx &= \frac{1}{2}\int \frac{1}{\sqrt[3]{1+x^2}}\,(2x\, dx) \\ &= \frac{1}{2}\int \frac{1}{\sqrt[3]{u}}\, du \\ &= \frac{1}{2}\cdot\frac{3}{2}u^{\frac{2}{3}} + C \\ &= \frac{3}{4}(x^2+1)^{\frac{2}{3}} + C. \end{aligned}$$

(b)

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\begin{aligned}
\int \frac{1}{\sqrt{2x+3}-\sqrt{2x+1}}\,dx &= \int \frac{1}{\sqrt{2x+3}-\sqrt{2x+1}} \cdot \frac{\sqrt{2x+3}+\sqrt{2x+1}}{\sqrt{2x+3}+\sqrt{2x+1}}\,dx \\
&= \int \frac{\sqrt{2x+3}+\sqrt{2x+1}}{(2x+3)+(2x+1)}\,dx \\
&= \frac{1}{2}\int \left(\sqrt{2x+3}+\sqrt{2x+1}\right)\,dx \\
&= \frac{1}{2}\left(\frac{1}{2}\cdot\frac{2}{3}(2x+3)^{\frac{3}{2}}+\frac{1}{2}\cdot\frac{2}{3}(2x+1)^{\frac{3}{2}}+C\right) \\
&= \frac{1}{6}(2x+3)^{\frac{3}{2}}+\frac{1}{6}(2x+1)^{\frac{3}{2}}+\frac{C}{2}.
\end{aligned}
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