Lecture Note 12

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MATH1020
General Mathematics

RECTANGULAR COORDINATES IN SPACE

In the two-dimensional plane, each point is associated with an ordered pair of real numbers.

In the three-dimensional space, each point is associate with an ordered triple of real numbers. Through a fixed point, called the **origin** O, draw three mutually perpendicular lines, the x-axis, the y-axis, and the z-axis. On each of these axes, select an appropriate scale and the positive direction. See Figure 1.

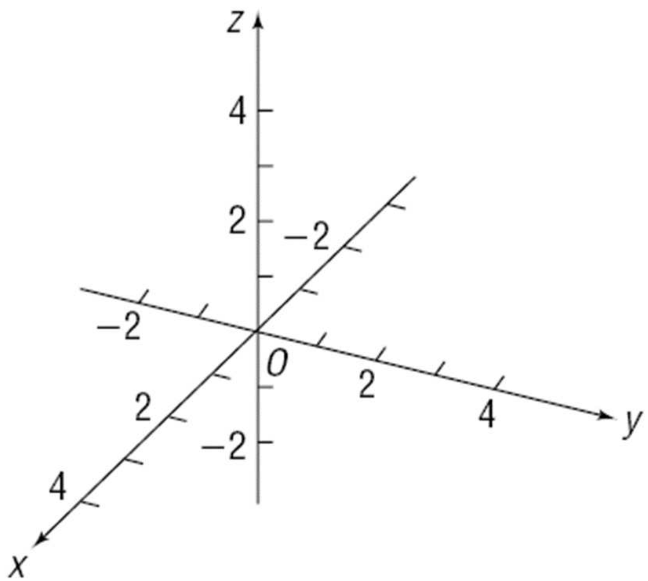

Figure 1:

The direction chosen for the positive z-axis in Figure 1 makes the system *right-handed*. This conforms to the *right-hand rule*, which states that, if the index finger of the right-hand points in the direction of the positive x-axis and the middle finger points in the direction of the positive y-axis, then the thumb will point in the direction of the positive z-axis. See Figure 2.

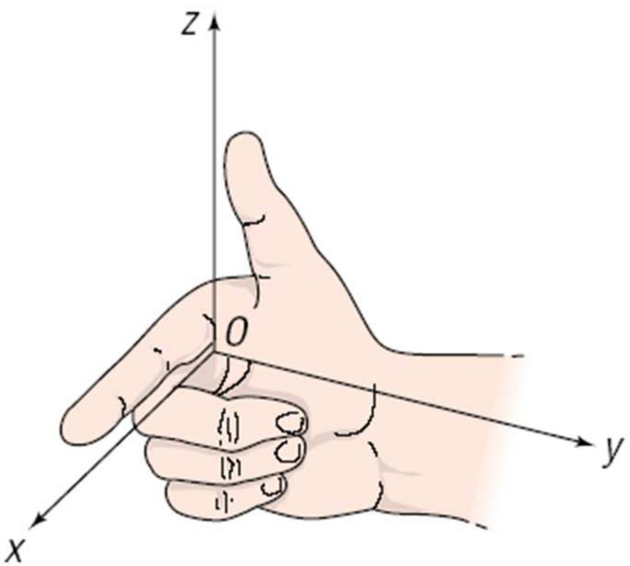

Figure 2:

We associate with each point P an ordered triple (x, y, z) of real numbers, the **coordinates of** P. For example, the point $(2, 3, 4)$ is located by starting at the origin and moving 2 units along the positive x-axis, 3 units in the direction of the positive z-axis. See Figure 3.

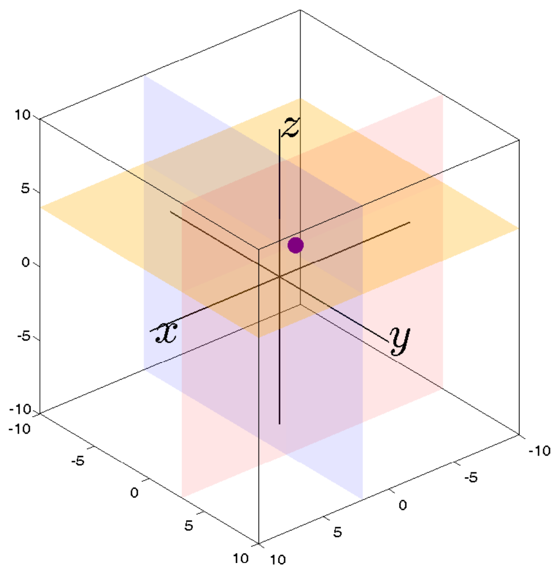

Figure 3:

Points of the form $(x, 0, 0)$ lie on the x-axis, while points of the form $(0, y, 0)$ and $(0, z, 0)$ lie on the y-axis and z-axis, respectively. Points of the form $(x, y, 0)$ lie in a plane, called the xy**-plane.** Its equation is $z = 0$. See Figure 4.

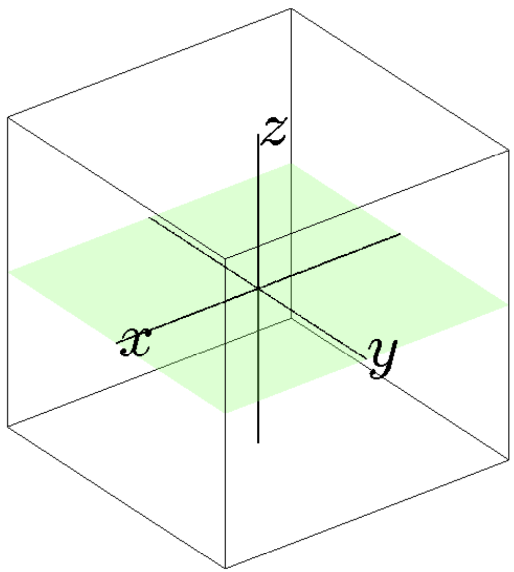

Figure 4:

Similarly, points of the form $(x, 0, z)$ lie in the xz-**plane** (equation $y = 0$). See Figure 5.

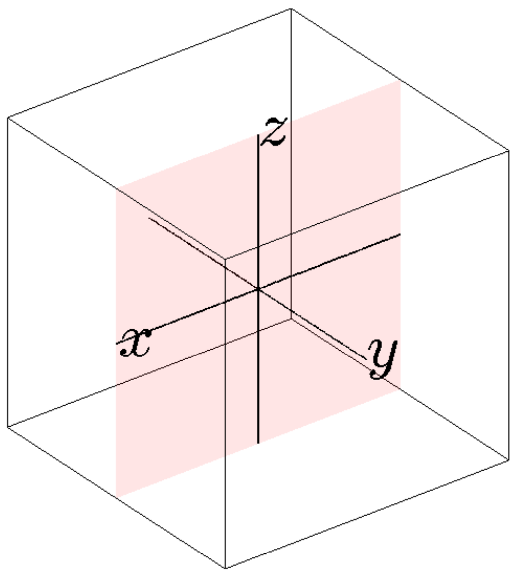

Figure 5:

Similarly, points of the form $(0, y, z)$ lie in the yz-plane (equation $x = 0$). See Figure 6.

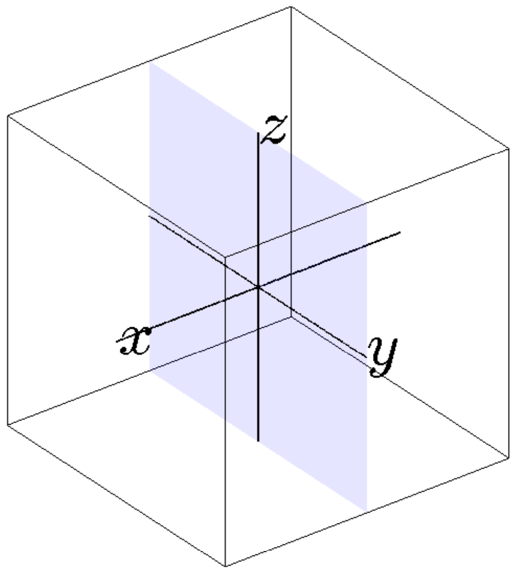

Figure 6:

By extension of these ideas, all points obeying the equation $z = 3$ will lie in a plane parallel to and 3 units above the xy-plane. See Figure 7.

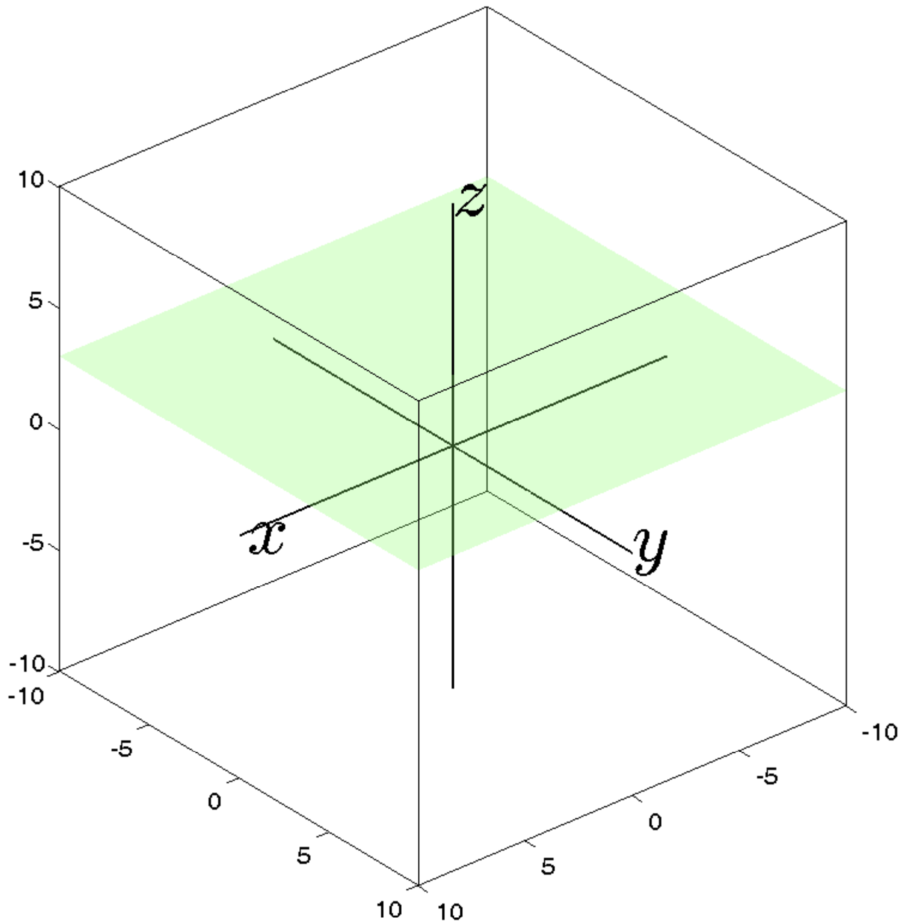

Figure 7:

Similarly, the equation $y = 3$ represents a plane parallel to the xz-plane and 3 units to the right of the plane $y = 0$. See Figure 8.

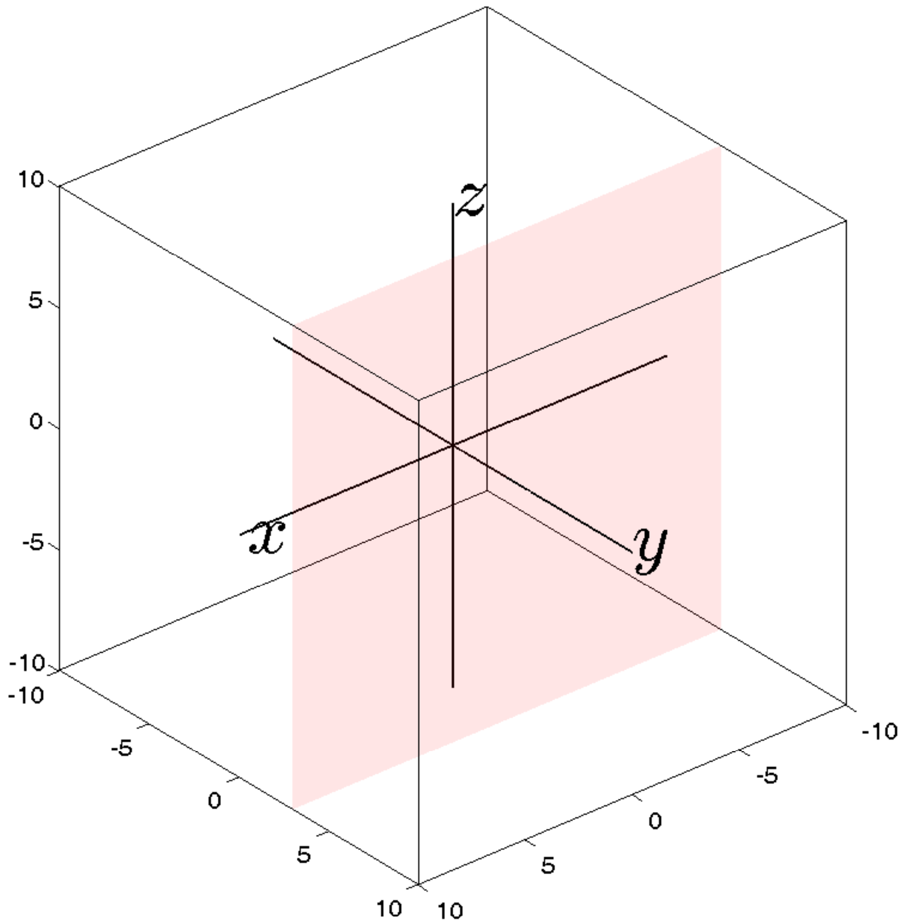

Figure 8:

Similarly, the equation $x = 3$ represents a plane parallel to the yz-plane and 3 units to the right of the plane $x = 0$. See Figure 9.

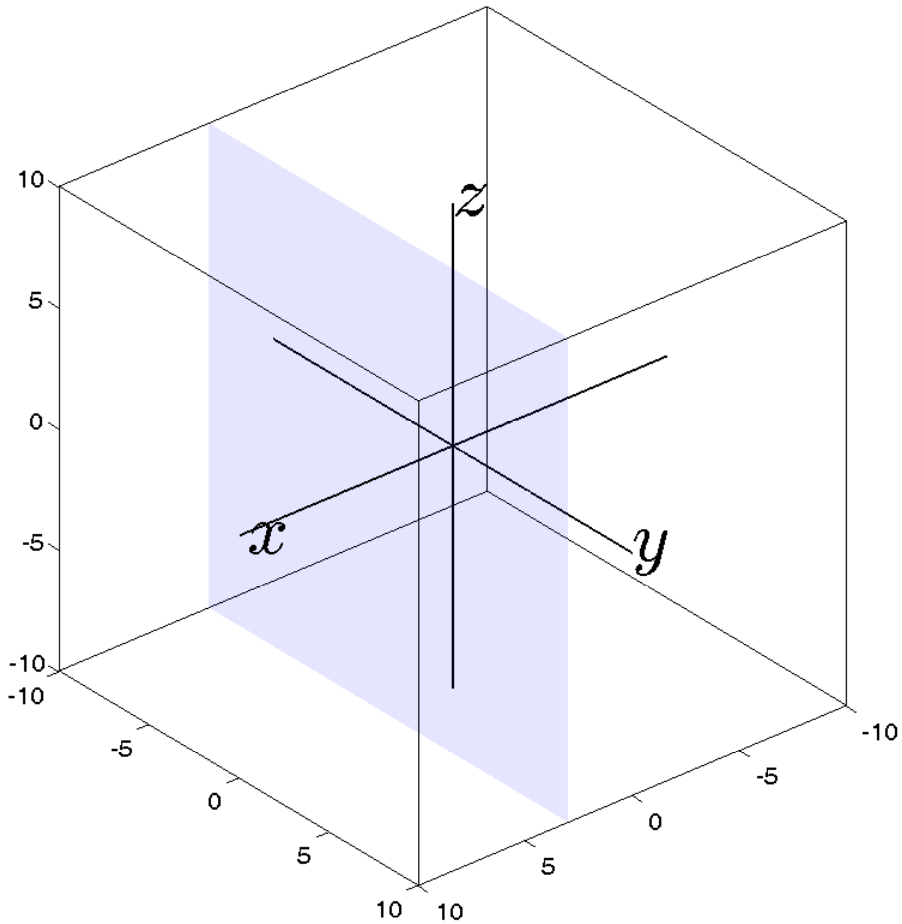

Figure 9:

The formula for the distance between two points in space is an extension of the Distance Formula for points in the plane revisited as follows.

Theorem 1 **Distance Formula in Space** If $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ are two points in space, the distance d from P_1 to P_2 is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_1 - z_2)^2}. \tag{1}$$

Example 1 Using the Distance Formula

Find the distance from $P_1 = (-1, 3, 2)$ to $P_2 = (5, -2, 4)$.

Solution:

$$d = \sqrt{[5-(-1)]^2 + [-2-3]^2 + [4-2]^2} = \sqrt{36+25+4} = \sqrt{65}.$$

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Find Position Vectors in Space

To represent vectors in space, we introduce the unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ whose direction are along the positive x-axis, positive y-axis and positive z-axis, respectively.

If $\mathbf{v}$ is a vector with initial point at the origin O and terminal point at $P = (a, b, c)$, we can represent $\mathbf{v}$ in terms of the vectors $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ as

$$\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}.$$

See Figure 10.

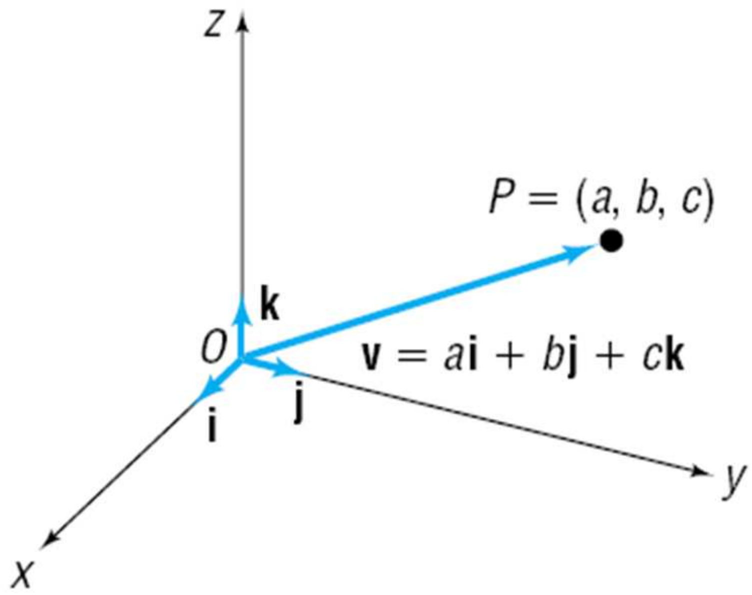

Figure 10:

The scalars $a,\ b$, and c are called the **components** of the vector $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, and c the component in the direction $\mathbf{k}$.

A vector whose initial point is at the origin is called a **position vector.** The next result states that any vector whose initial point is not at the origin is equal to a unique position vector.

Theorem 2 Suppose that $\mathbf{v}$ is a vector with initial point $P_1 = (x_1, x_1, z_1)$, not necessarily the origin, and terminal point $P_1 = (x_2, y_2, z_2)$. If $\mathbf{v} = \overrightarrow{P_1P_2}$, then $\mathbf{v}$ is equal to the position vector

$$\mathbf{v} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}. \qquad (2)$$

Figure 11 illustrates this result.

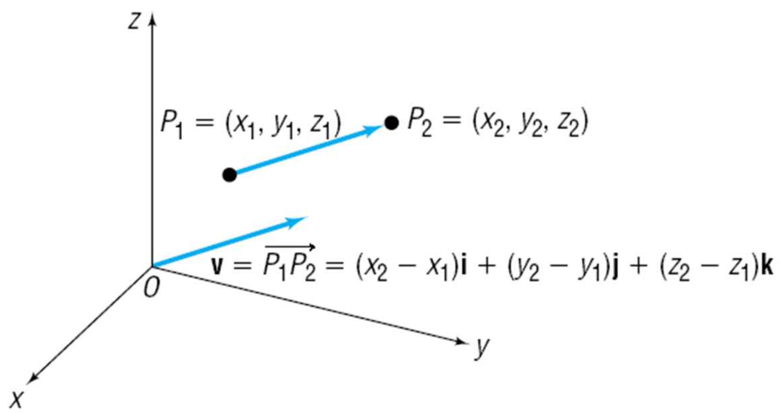

Figure 11:

Example 2 Finding a Position Vector

Find the position of the vector $\mathbf{v} = \overrightarrow{P_1P_2}$ if $P_1 = (-1, 6, 1)$ and $P_2 = (7, 2, 3)$.

Solution:

By equation (2), the position vector equal to $\mathbf{v}$ is

$$\begin{aligned} \mathbf{v} &= [7 - (-1)]\mathbf{i} + (2 - 6)\mathbf{j} + (3 - 1)\mathbf{k} \\ &= 8\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}. \end{aligned}$$

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Perform Operations on Vectors

Let us define equality, addition, subtraction, scalar product, an magnitude in terms of the components of a vector.

Definition 1 Let $\mathbf{v} = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$ and $\mathbf{w} = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$ be two vectors, and let α be a scalar. Then

$$\mathbf{v} = \mathbf{w} \quad \text{if and only if} \quad a_1 = a_2, b_1 = b_2, \quad \text{and} \quad c_1 = c_2;$$

$$\mathbf{v} + \mathbf{w} = (a_1 + a_2)\mathbf{i} + (b_1 + b_2)\mathbf{j} + (c_1 + c_2)\mathbf{k};$$

$$\mathbf{v} - \mathbf{w} = (a_1 - a_2)\mathbf{i} + (b_1 - b_2)\mathbf{j} + (c_1 - c_2)\mathbf{k};$$

$$\alpha\mathbf{v} = (\alpha a_1)\mathbf{i} + (\alpha b_1)\mathbf{j} + (\alpha c_1)\mathbf{k};$$

$$||\mathbf{v}|| = \sqrt{a_1^2 + b_1^2 + c_1^2}.$$

Example 3 Adding and Subtracting Vectors

If $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ and $\mathbf{w} = 3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$, find:

(a) $\mathbf{v} + \mathbf{w}$ (b)$\mathbf{v} - \mathbf{w}$

Solution:

(a)

$$\begin{aligned} \mathbf{v} + \mathbf{w} &= (2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}) + (3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}) \\ &= (2+3)\mathbf{i} + (3-4)\mathbf{j} + (-2+5)\mathbf{k} \\ &= 5\mathbf{i} - \mathbf{j} + 3\mathbf{k}. \end{aligned}$$

(b)

$$\begin{aligned} \mathbf{v} - \mathbf{w} &= (2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}) - (3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}) \\ &= (2-3)\mathbf{i} + [3-(-4)]\mathbf{j} + [-2-5]\mathbf{k} \\ &= -\mathbf{i} + 7\mathbf{j} - 7\mathbf{k}. \end{aligned}$$

Example 4 Finding Scalar Products and Magnitudes

If $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ and $\mathbf{w} = 3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$, find:

(a) $3\mathbf{v}$ (b)$2\mathbf{v} - 3\mathbf{w}$ (c)$||\mathbf{v}||$

Solution:

(a)

$$3\mathbf{v} = 3(2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}) = 6\mathbf{i} + 9\mathbf{j} - 6\mathbf{k}.$$

(b)

$$\begin{aligned} 2\mathbf{v} - 3\mathbf{w} &= 2(2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}) - 3(3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}) \\ &= 4\mathbf{i} + 6\mathbf{j} - 4\mathbf{k} - 9\mathbf{i} + 12\mathbf{j} - 15\mathbf{k} \\ &= -5\mathbf{i} + 18\mathbf{j} - 19\mathbf{k}. \end{aligned}$$

(c)

$$||\mathbf{v}|| = ||2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}|| = \sqrt{2^2 + 3^2 + (-2)^2} = \sqrt{17}.$$

Recall that a unit vector $\mathbf{u}$ is one for which $||\mathbf{u}|| = 1$. In many applications, it is useful to be able to find a unit vector $\mathbf{u}$ that has the same direction as a given vector $\mathbf{v}$.

Theorem 3 Unit Vector in the Direction of $\mathbf{v}$

For any nonzero vector $\mathbf{v}$, the vector

$$\mathbf{u} = \frac{\mathbf{v}}{||\mathbf{v}||}$$

is a unit vector that has the same direction as $\mathbf{v}$.

As a consequence of this theorem, if $\mathbf{u}$ is a unit vector in the same direction as a vector $\mathbf{v}$, then $\mathbf{v}$ may be expressed as

$$\mathbf{v} = ||\mathbf{v}||\mathbf{u}.$$

Example 5 Finding a Unit Vector

Find the unit vector in the same direction as $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j} - 6\mathbf{k}$.

Solution:

We find $||\mathbf{v}||$ first,

$$||\mathbf{v}|| = ||2\mathbf{i} - 3\mathbf{j} - 6\mathbf{k}|| = \sqrt{4 + 9 + 25} = \sqrt{38}.$$

Now we multiply $\mathbf{v}$ by the scalar $\frac{1}{||\mathbf{v}||} = \frac{1}{\sqrt{38}}$. The result is the unit vector

$$\mathbf{u} = \frac{\mathbf{v}}{||\mathbf{v}||} = \frac{2\mathbf{i} - 3\mathbf{j} - 5\mathbf{k}}{\sqrt{38}} = \frac{2}{\sqrt{38}}\mathbf{i} - \frac{3}{\sqrt{38}}\mathbf{j} - \frac{6}{\sqrt{38}}\mathbf{k}.$$

Check:

$$\left(\frac{2}{\sqrt{38}}\right)^2 + \left(-\frac{3}{\sqrt{38}}\right)^2 + \left(-\frac{6}{\sqrt{38}}\right)^2 = 1.$$

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Find the Dot Product

The definition of dot product is an extension of the definition given for vectors in the plane.

Definition 2 If $\mathbf{v} = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$ and $\mathbf{w} = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$ are two vectors, the **dot product** $\mathbf{v} \cdot \mathbf{w}$ is defined as

$$\mathbf{v} \cdot \mathbf{w} = a_1 a_2 + b_1 b_2 + c_1 c_2. \tag{3}$$

Example 6 Finding Dot Products

If $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ and $\mathbf{w} = 3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$, find:

(a) $\mathbf{v} \cdot \mathbf{w}$ (b) $\mathbf{w} \cdot \mathbf{v}$ (c) $\mathbf{v} \cdot \mathbf{v}$

(a) $\mathbf{w} \cdot \mathbf{w}$ (e) $||\mathbf{v}||$ (f) $||\mathbf{w}||$

Solution:

$$
\begin{array}{ll}
\text{(a)} & \mathbf{v} \cdot \mathbf{w} = 2(5) + (-3)3 + 6(-1) = -5. \\
\text{(b)} & \mathbf{w} \cdot \mathbf{v} = 5(2) + 3(-3) + (-1)6 = -5. \\
\text{(c)} & \mathbf{v} \cdot \mathbf{v} = 2(2) + (-3)(-3) + 6(6) = 49. \\
\text{(d)} & \mathbf{w} \cdot \mathbf{w} = 5(5) + 3(3) + (-1)(-1) = 35. \\
\text{(e)} & ||\mathbf{v}|| = \sqrt{2^2 + (-3)^2 + 6^2} = \sqrt{49} = 7. \\
\text{(f)} & ||\mathbf{w}|| = \sqrt{5^2 + 3^2 + (-1)^2} = \sqrt{35}.
\end{array}
$$

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Theorem 4 **Properties of the Product** If $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ are vectors, then

Commutative Property

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}. \tag{4}$$

Distributive Property

$$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}. \tag{5}$$

$$\mathbf{v} \cdot \mathbf{v} = ||\mathbf{v}||. \tag{6}$$

$$\mathbf{0} \cdot \mathbf{v} = 0. \tag{7}$$

Find the Angle between Two Vectors

The angle θ between two vectors in space follows the same formula as for two vectors in the plane.

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}||||\mathbf{v}||}. \tag{8}$$

Example 7 Finding the Angle θ between Two Vectors

Find the angle θ between $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}$.

Solution:

We compute the quantities $\mathbf{u} \cdot \mathbf{v}$, $||\mathbf{u}||$, and $||\mathbf{v}||$.

$$\begin{aligned}
\mathbf{u} \cdot \mathbf{v} &= 2(2) + (-3)(5) + 6(-2) = -23; \\
||\mathbf{u}|| &= \sqrt{2^2 + (-3)^2 + 6^2} = \sqrt{49} = 7; \\
||\mathbf{v}|| &= \sqrt{2^2 + 5^2 + (-2)^2} = \sqrt{33}.
\end{aligned}$$

By formula (8), if θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}||||\mathbf{v}||} = \frac{-23}{7\sqrt{33}} \approx -0.5720.$$

We find that $\theta \approx 124.8°$.

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Find the Direction Angles of a Vector

A nonzero vector **v** in space can be described by specifying its magnitude and its three **direction angles** α, β, and γ. These direction angles are defined as

$$\begin{cases} \alpha = \text{the angle between } \mathbf{v} \text{ and } \mathbf{i}, \text{ the positive } x\text{-axis}, 0 \le \alpha \le \pi; \\ \beta = \text{the angle between } \mathbf{v} \text{ and } \mathbf{j}, \text{ the positive } y\text{-axis}, 0 \le \beta \le \pi; \\ \gamma = \text{the angle between } \mathbf{v} \text{ and } \mathbf{k}, \text{ the positive } z\text{-axis}, 0 \le \gamma \le \pi. \end{cases}$$

See Figure 12.

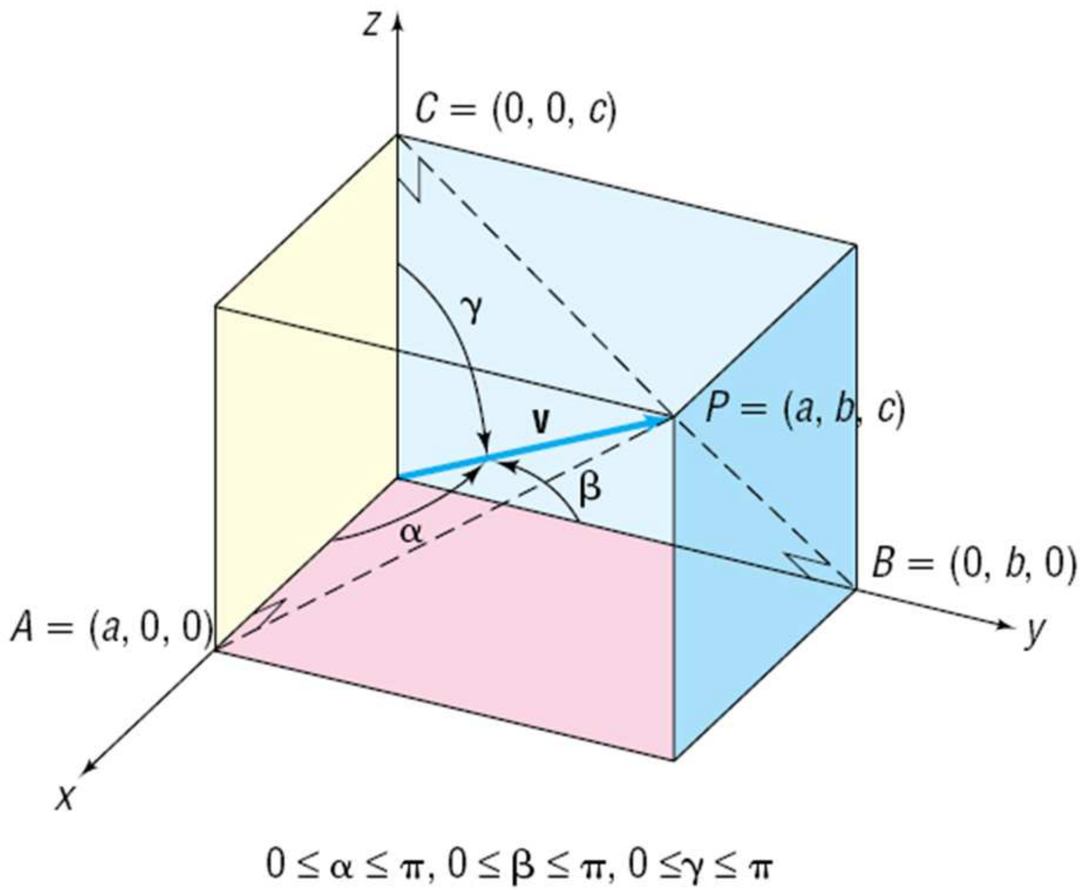

Figure 12:

Our first goal is to find expressions for α, β, and γ in terms of the components of a vector. Let $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ denote a nonzero vector. The angle α between $\mathbf{v}$ and $\mathbf{i}$, the positive x-axis obeys

$$\cos\alpha = \frac{\mathbf{u}\cdot\mathbf{i}}{||\mathbf{v}||\cdot||\mathbf{i}||} = \frac{a}{||\mathbf{v}||}.$$

Similarly,

$$\cos\beta = \frac{b}{||\mathbf{v}||}$$

and

$$\cos\gamma = \frac{c}{||\mathbf{v}||}.$$

Since $||\mathbf{v}|| = \sqrt{a^2+b^2+c^2}$, we have the following result:

Theorem 5 Direction Angles

If $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is a nonzero vector in space, the direction angles α, β, and γ obey

$$\begin{cases} \cos\alpha = \dfrac{a}{\sqrt{a^2+b^2+c^2}} = \dfrac{a}{||\mathbf{v}||}; \\ \cos\beta = \dfrac{b}{\sqrt{a^2+b^2+c^2}} = \dfrac{b}{||\mathbf{v}||}; \\ \cos\gamma = \dfrac{c}{\sqrt{a^2+b^2+c^2}} = \dfrac{c}{||\mathbf{v}||}. \end{cases} \tag{9}$$

The numbers $\cos\alpha, \cos\beta$, and $\cos\gamma$ are called the **direction cosines** of the vector $\mathbf{v}$.

Example 8 Finding the Direction Angles of a Vector

Find the direction angles of $\mathbf{v} = -3\mathbf{i} + 2\mathbf{j} - 6\mathbf{k}$.

Solution:

Using the formulas in equation (9), we have

$$\cos\alpha = \frac{-3}{7} \qquad \cos\beta = \frac{2}{7} \qquad \cos\gamma = \frac{-6}{7}$$
$$\alpha \approx 115.4^\circ \qquad \beta \approx 73.4^\circ \qquad \alpha \approx 149.0^\circ.$$

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Theorem 6 Property of the Direction Cosines

If $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are direction angles of a nnzero vector $\mathbf{v}$ in space, then

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1. \tag{10}$$

Based on equation (10), when two direction cosines are known, the third is determined up to its sign. Knowing two direction cosines is not sufficient to uniquely determine the direction of a vector in space.

Example 9 Finding the Direction Angle of a Vector

The vector **v** makes an angle of $\alpha = \frac{\pi}{3}$ with the positive x-axis, an angle of $\beta = \frac{\pi}{3}$ with the positive y-axis, and an acute angle γ with the positive z-axis. Find γ.

Solution:

By equation (10), we have

$$\begin{aligned}
\cos^2\left(\frac{\pi}{3}\right) + \cos^2\left(\frac{\pi}{3}\right) + \cos^2\left(\frac{\pi}{3}\right) &= 1 \qquad \left(0 < \gamma < \frac{\pi}{2}\right)\\
\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \cos^2\gamma &= 1\\
\cos^2\gamma &= \frac{1}{2}\\
\cos\gamma = \frac{\sqrt{2}}{2} \qquad \text{or} \qquad \cos\gamma &= -\frac{\sqrt{2}}{2}\\
\gamma = \frac{\pi}{4} \qquad \text{or} \qquad \gamma &= \frac{3\pi}{4}.
\end{aligned}$$

Since we are requiring that γ be acute, $\gamma = \frac{\pi}{4}$.

The direction cosines of a vector give information about only the direction of the vector; they provide no information about its magnitude. For example, any vector parallel to the xy-plane and making an angle of $\frac{\pi}{4}$ radian with the positive x-axis and y-axis has direction cosines

$$\cos\alpha = \frac{\sqrt{2}}{2} \qquad \cos\beta = \frac{\sqrt{2}}{2} \qquad \cos\gamma = 0.$$

However, if the direction angles and the magnitude of a vector are known, the vector is uniquely determined.

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