Calculus for Engineers

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Contents

SIGMA NOTATION AND RIEMANN SUMS

20.1 Introduction

Before studying the definite integral, we need to know the so-called sigma notation that is a method used to write out a long sum in a concise way. In Section 20.2, we look at ways of using sigma notation, and establish some useful rules. In Section 20.3, we will study some properties of sigma notation. In Section 20.4, we study the sums of the area under a curve using rectangles. To accomplish this, one divides the base interval into pieces (subintervals or mesh), where the partition of the base interval may be uniform or non-uniform. Then on each subinterval, build a rectangle that goes up to the curve. In Section 20.5, we study the concept of Riemann sums; one can estimate the sum of the rectangles for a monotone continuous piece of the function by choosing the height of kth rectangle corresponding to any value c_k of the function in the k-th interval: $x_{k-1} \leq c_k \leq x_k$. The estimate for the sum is $\sum\limits_{k=1}^{n} f(c_k)(x_k - x_{k-1})$, where n denotes the final number of rectangles (see Definition 3). Hence, in Section 20.6, we study two versions of Riemann sums: the upper sums and lower sums (see Definition 4 and Theorem 2). We will study the integral when the common limit of the upper and lower sums is considered in Chapter 21.

20.2 Sigma notation

Calculating the area of a region is simply done by cutting the region into simple shapes. Then one calculates the area of each simple shape, and then adds these smaller areas together to get the area of the whole region. Mathematically speaking, adding a lot of values together can be represented by a so-called sigma $\sum$ notation.

The variable (typically i, j, or k) often used in the summation is called the counter or index variable. The function to the right of the sigma is called the summand, and the numbers below and above the sigma are called the lower and upper limits of the summation, as illustrated in Figure 20.1.

Summation	Sigma notation	A way to read the sigma notation
$1^2+2^2+3^2+4^2+5^2$	$\sum_{k=1}^{5} k^2$	the sum of k squared for k equals 1 to k equals 5
$\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}$	$\sum_{k=3}^{7} \frac{1}{k}$	the sum of 1 divided by k for k equals 3 to k equals 7
$2^0+2^1+2^2+2^3+2^4+2^5$	$\sum_{j=0}^{5} 2^j$	the sum of 2 to the jth power for j equals 0 to j equals 5
$a_2+a_3+a_4+a_5+a_6+a_7$	$\sum_{i=2}^{7} a_i$	the sum of a sub i from i equals 2 to i equals 7

Table 20.1:

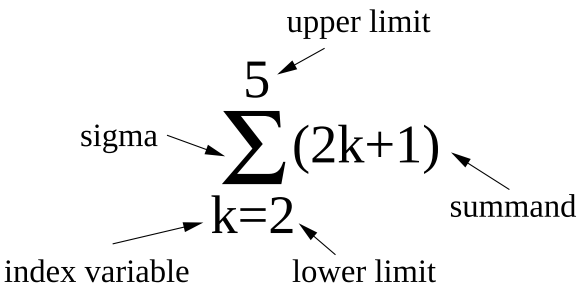

Figure 20.1: Sigma notation.

20.3 Definition of sigma notation and its properties

Definition 1 If $a_m, a_{m+1}, \cdots, a_n$ are real numbers such that $m \leq n$, then the summation of these numbers written in sigma notation is

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + \cdots + a_n$$

and also using functional notation,

$$\sum_{k=m}^{n} f(k) = f(m) + f(m+1) + \cdots + f(n-1) + f(n),$$

where $f(k) = a_k$. □

Since the sigma notation is simply a notation for addition, it has all of the familiar properties of addition.

Definition 2 If α and β are real numbers that do not depend on integers m and n, then

- Constant Term Rule:

$$\sum_{k=1}^{n} \alpha = \alpha + \alpha + \cdots + \alpha \ (n \text{ terms}) = n \cdot \alpha.$$

- Addition and Subtraction Rules:

$$\sum_{k=1}^{n} (a_k \pm b_k) = \sum_{k=1}^{n} a_k \pm \sum_{k=1}^{n} b_k.$$

- Linearity Rule::

$$\sum_{k=1}^{n} (\alpha a_k + \beta b_k) = \alpha \sum_{k=1}^{n} a_k + \beta \sum_{k=1}^{n} b_k.$$

- Scalar Multiple Rule:

$$\sum_{k=1}^{n} \alpha \cdot a_k = \alpha \cdot \sum_{k=1}^{n} a_k.$$

- Subtotal Rule: If $1 < m < n$, then

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{m} a_k + \sum_{k=m+1}^{n} a_k.$$

- Dominance Rule: If $a_k \leq b_k$ for all k, then

$$\sum_{k=1}^{n} a_k \leq \sum_{k=1}^{n} b_k.$$

□

Example 1 Here are several worked examples:

1. $$\sum_{k=1}^{4} a_k = a_1 + a_2 + a_3 + a_4$$

2. $$\sum_{k=3}^{7} b_k = b_3 + b_4 + b_5 + b_6 + b_7$$

3. $$\sum_{k=-3}^{7} b_k = b_{-3} + b_{-2} + b_{-1} + b_0 + b_1 + b_2 + b_3 + b_4 + b_5 + b_6 + b_7$$

4. $$\sum_{k=1}^{4} k = 1 + 2 + 3 + 4$$

5.
$$\sum_{k=1}^{3}(k-1)^2 = 0^2+1^2+2^2 = 0+1+4 = 5$$

6.
$$\sum_{k=1}^{5} 2 = 2+2+2+2+2 = 10$$

7.
$$\sum_{k=3}^{5} -2 = (-2)+(-2)+(-2) = -6$$

8.
$$\sum_{k=3}^{7}(k+1) = 4+5+6+7+8 = 30$$

9.
$$\sum_{k=3}^{7}(-1)^k(k+1) = -4+5-6+7-8 = -6$$

10.
$$\sum_{k=2}^{4}\frac{1}{2^k} = \frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4} = \frac{1}{4}+\frac{1}{8}+\frac{1}{16} = \frac{7}{16}$$

□

Example 2 Write
$$\frac{2}{3}+\frac{3}{4}+\frac{4}{5}+\frac{5}{6}$$
using summation notation.

Solution. We have
$$\frac{2}{3}+\frac{3}{4}+\frac{4}{5}+\frac{5}{6} = \sum_{k=1}^{4}\frac{k+1}{k+2} = \sum_{k=2}^{5}\frac{k}{k+1} = \sum_{k=0}^{3}\frac{k+2}{k+3}.$$

□

Note 1 In Example 2, we get the same result even though the index variable and the summand exist in various forms.

Example 3 Write
$$3^4+4^3+5^4+6^4+7^3+8^4$$
using summation notation.

Solution. We have
$$3^4+4^3+5^4+6^4+7^3+8^4 = \sum_{k=3}^{8}k^4 = \sum_{k=1}^{6}(k+2)^4.$$

□

Example 4 Write

$$3^5 - 4^5 + 5^5 - 6^5 + 7^5 - 8^5$$

using summation notation.

Solution. We have

$$3^5 - 4^5 + 5^5 - 6^5 + 7^5 - 8^5 = \sum_{k=3}^{8}(-1)^{k+1}k^5 = \sum_{k=1}^{6}(-1)^{k+1}(k+2)^5 = \sum_{k=2}^{7}(-1)^k(k+1)^5.$$

$\square$

Example 5 Write

$$a + a^2 + a^3 + a^4 + a^5 + a^6$$

using summation notation.

Solution. We have

$$a + a^2 + a^3 + a^4 + a^5 + a^6 = \sum_{k=1}^{6} a^k = \sum_{k=0}^{5} a^{k+1} = \sum_{k=-2}^{3} a^{k+3}.$$

$\square$

Example 6 Write

$$a - a^2 + a^3 - a^4 + a^5 - a^6$$

using summation notation.

Solution. We have

$$a - a^2 + a^3 - a^4 + a^5 - a^6 = \sum_{k=1}^{6} a^k = \sum_{k=1}^{6}(-1)^{k+1}a^k = \sum_{k=4}^{9}(-1)^k a^{k-3}.$$

$\square$

Example 7 Write

$$1 - a + a^2 - a^3 + a^4 - a^5$$

using summation notation.

Solution. We have

$$1 - a + a^2 - a^3 + a^4 - a^5 = \sum_{k=0}^{5}(-1)^k a^k = \sum_{k=1}^{6}(-1)^{k+1}a^{k-1}.$$

$\square$

Example 8 Write

$$\frac{7}{10\cdot 12}+\frac{8}{11\cdot 13}+\frac{9}{12\cdot 14}+\frac{10}{13\cdot 15}$$

using summation notation.

Solution. We have

$$\frac{7}{10\cdot 12}+\frac{8}{11\cdot 13}+\frac{9}{12\cdot 14}+\frac{10}{13\cdot 15}=\sum_{k=7}^{10}\frac{k}{(k+3)(k+5)}=\sum_{k=1}^{4}\frac{k+6}{(k+9)(k+11)}.$$

□

Proposition 1 The summation formulas for $\sum_{k=1}^{n} k_i$ when $i=1,2,\cdots,7,$ are

1. $$\sum_{k=1}^{n} k=\frac{1}{2}n(n+1).$$

2. $$\sum_{k=1}^{n} k^2=\frac{1}{6}n(n+1)(2n+1).$$

3. $$\sum_{k=1}^{n} k^3=\frac{1}{4}n^2(n+1)^2.$$

4. $$\sum_{k=1}^{n} k^4=\frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1)$$

5. $$\sum_{k=1}^{n} k^5=\frac{1}{12}n^2(n+1)^2(2n^2+2n-1)$$

6. $$\sum_{k=1}^{n} k^6=\frac{1}{42}n(n+1)(2n+1)(3n^4+6n^3-3n+1).$$

7. $$\sum_{k=1}^{n} k^7=\frac{1}{24}n^2(n+1)^2(3n^4+6n^3-n^2-4n+2).$$

□

Example 9 Find the value of the sum $\sum_{j=1}^{100} j(j^2+1)$.

Solution. The value of the sum is

$$\begin{aligned}\sum_{j=1}^{100} j(j^2+1) &= \sum_{j=1}^{100}(j^3+j)\\ &= \sum_{j=1}^{100} j^3 + \sum_{j=1}^{100} j\\ &= \frac{1}{4}100^2(100+1)^2 + \frac{1}{2}100(100+1)\\ &= 25502500 + 5050 = 25507550.\end{aligned}$$

□

20.4 Sums of areas of rectangles

Example 10 Evaluate the sum of the rectangular areas in Figure 20.2, and write the sum using sigma notation.

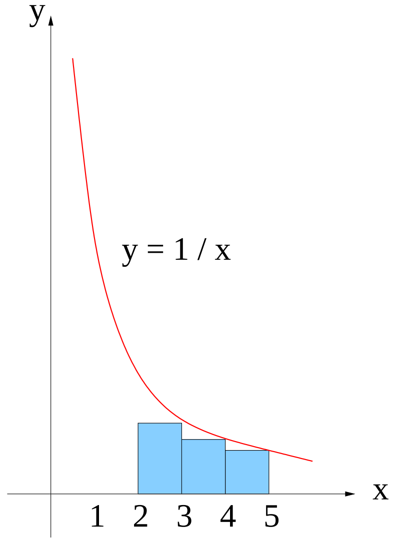

Figure 20.2: Example 10.

Solution. We have the following formula:

$$\begin{aligned}\text{Sum of the rectangular areas } &= \text{sum of (base} \times \text{height) for each rectangle}\\ &= 1\cdot\frac{1}{3} + 1\cdot\frac{1}{4} + 1\cdot\frac{1}{5} = \frac{47}{50}.\end{aligned}$$

Using sigma notation, we have

$$1\cdot\frac{1}{3} + 1\cdot\frac{1}{4} + 1\cdot\frac{1}{5} = \sum_{k=3}^{5}\frac{1}{k}.$$

□

Example 11 Let us examine a case where the bases of the rectangles do not have to be equal. For the rectangular areas in Figure 20.3, we have

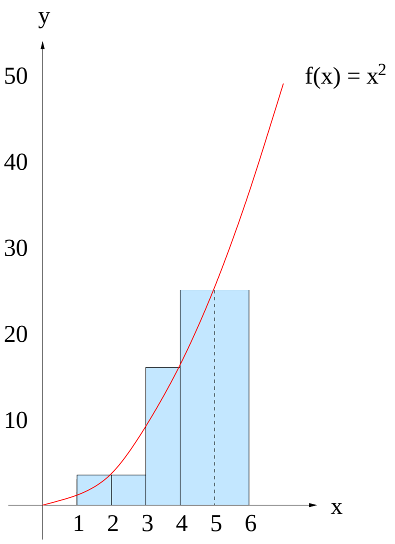

Figure 20.3: Example 11.

Rectangle	Base	Height	Area
1	$3-1=2$	$f(2)=4$	$2\cdot 4=8$
2	$4-3=1$	$f(4)=16$	$1\cdot 16=16$
3	$6-4=2$	$f(5)=25$	$2\cdot 25=50$

Table 20.2:

Thus, the sum of the rectangular areas is $8+16+50=74$. □

Example 12 Write the sum of the areas of the rectangles in Figure 20.4 using sigma notation.

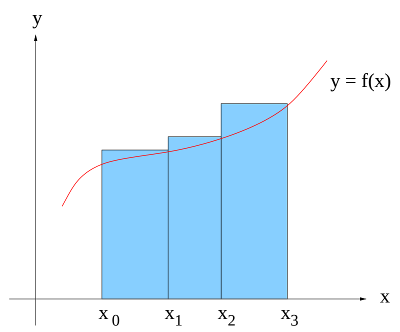

Figure 20.4: Example 12.

Solution. The area of each rectangle is (base $\cdot$ height).

Rectangle	Base	Height	Area
1	$x_1 - x_0$	$f(x_1)$	$(x_1 - x_0) \cdot f(x_1)$
2	$x_2 - x_1$	$f(x_2)$	$(x_2 - x_1) \cdot f(x_2)$
3	$x_3 - x_2$	$f(x_3)$	$(x_3 - x_2) \cdot f(x_3)$

Table 20.3:

In Table 20.3, we note that the area of the kth rectangle is

$$(x_k - x_{k-1}) \cdot f(x_k),$$

and the total area of the rectangles is the sum

$$\sum_{k=1}^{3} (x_k - x_{k-1}) \cdot f(x_k).$$

□

20.5 Area under a curve - Riemann sums

Suppose we want to calculate the area between the graph of a positive function $f(x)$ and the interval $[a, b]$ on the $x-$axis, as shown in Figure 20.5.

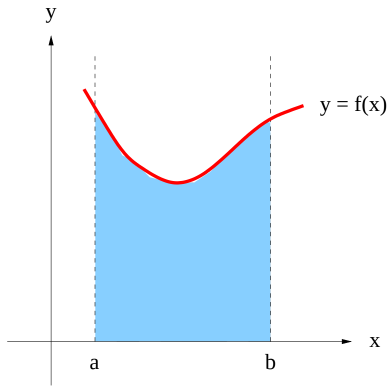

Figure 20.5:

With the Riemann sum method, several rectangles are built with bases on the interval $[a, b]$ and sides that reach up to the graph of $f(x)$, as shown in Figure 20.6.

Then the areas of the rectangles can be calculated and summed together to get a number called the Riemann sum of $f(x)$ on $[a, b]$. The area of the region formed by the rectangles is an approximation of the area by taking as many rectangles as we please.

Example 13 Approximate the area in Figure 20.7(a) between the graph of $f(x)$ and the interval $[2, 5]$ on the $x-$axis by summing the areas of the rectangles in Figure 20.7(b).

Solution. The total area of rectangles is $(2) \cdot (3) + (1) \cdot (5) = 11$ square units.

□

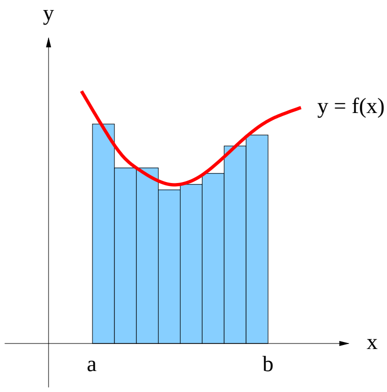

Figure 20.6:

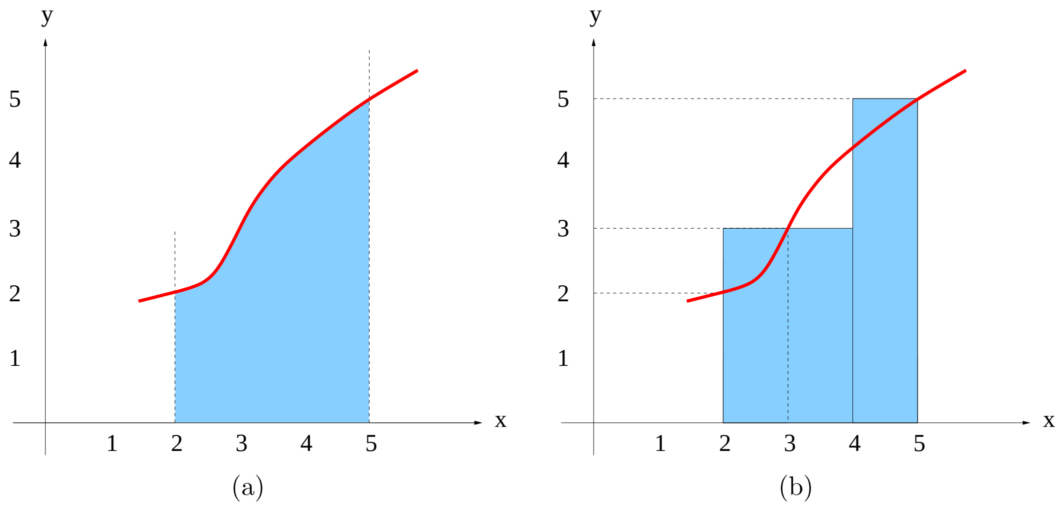

Figure 20.7: Example 13.

In order to effectively describe this process, some new vocabulary is helpful: a partition of an interval and the mesh of a partition.

A partition P of a closed interval $[a, b]$ divided into n subintervals is a set of $n + 1$ points

$$\{x_0 = a, x_1, x_2, x_3, \cdots, x_{n-1}, x_n = b\}$$

with increasing order,

$$a = x_0 < x_1 < x_2 < x_3 < \cdots < x_{n-1} < x_n = b.$$

(A partition is a collection of points on the axis and it does not depend on the function in any way.)

The points of the partition P divide the interval into n subintervals, as shown in Figure 20.8:

$$[x_0, x_1],\ [x_1, x_2],\ [x_2, x_3],\ \cdots,\ \text{and}\ [x_{n-1}, x_n]$$

with lengths

$$\Delta x_1 = x_1 - x_0,\ \Delta x_2 = x_2 - x_1,\ \Delta x_3 = x_3 - x_2,\ \cdots,\ \text{and}\ \Delta x_n = x_n - x_{n-1}.$$

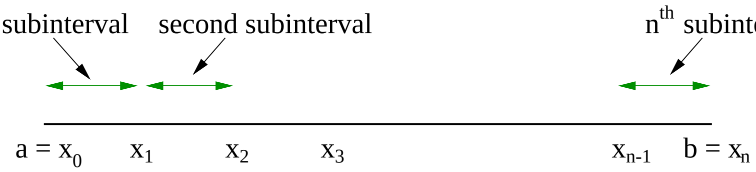

Figure 20.8:

The points x_k of the partition P are the locations of the vertical lines for the sides of the rectangles, and the bases of the rectangles have lengths Δx_k for $k = 1, 2, 3, \cdots, n$.

A function, a partition, and a point in each subinterval determine a Riemann sum. Suppose

- $f(x)$ is a positive function on the interval $[a, b]$,
- $$P = \{x_0 = a, x_1, x_2, x_3, \cdots, x_{n-1}, x_n = b\}$$
  is a partition of $[a, b]$
- c_k is an $x-$value in the kth subinterval $[x_{k-1}, x_k]$ such that $x_{k-1} \leq c_k \leq x_k$.

Then the area of the kth rectangle is

$$f(c_k) \cdot (x_k - x_{k-1}) = f(c_k) \cdot \Delta x_k,$$

as shown in Figure 20.9.

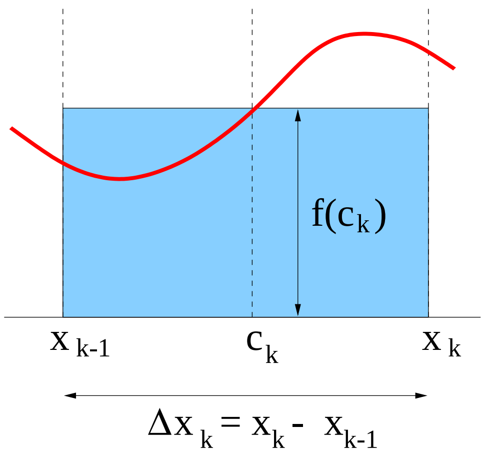

Figure 20.9:

Definition 3 A summation of the form

$$\sum_{k=1}^{n} f(c_k) \cdot \Delta x_k$$

is called the Riemann sum of $f(x)$ for the partition P. □

This Riemann sum is the total of the areas of the rectangular regions and is an approximation of the area between the graph of $f(x)$ and the $x-$axis.

Example 14 Find the Riemann sum for $f(x) = \dfrac{1}{x}$ and the partition $\{1, 4, 5\}$ using the values $c_1 = 2$ and $c_2 = 5$, as shown in Figure 20.10.

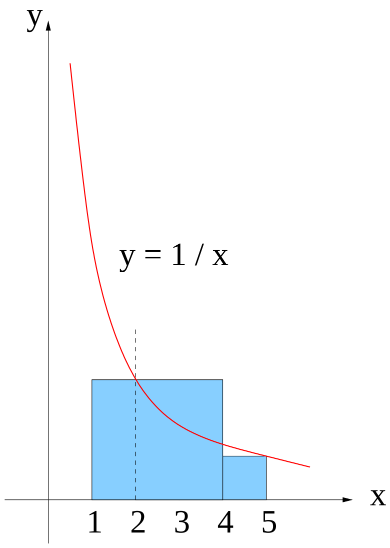

Figure 20.10: Example 14.

Solution. The two subintervals are $[1, 4]$ and $[4, 5]$ so $\Delta x_1 = 3$ and $\Delta x_2 = 1$. Then the Riemann sum for this partition is

$$\begin{aligned}\sum_{k=1}^{n} f(c_k)\cdot\Delta x_k &= \sum_{k=1}^{2} f(c_k)\cdot\Delta x_k \\ &= f(c_1)\cdot\Delta x_1 + f(c_2)\cdot\Delta x_2 \\ &= f(2)\cdot 3 + f(5)\cdot 1 \\ &= \frac{1}{2}\cdot 3 + \frac{1}{5}\cdot 1 \\ &= 1.7.\end{aligned}$$

□

Example 15 Table 20.4 shows the computational results of the calculated Riemann sums for the function $f(x) = \dfrac{1}{x}$ with different numbers of subintervals and different ways of selecting the points c_i in each subinterval. When the mesh of the partition is small (and the number of subintervals large), all of the ways of selecting the point c_i lead to approximately the same number for the Riemann sums. For this decreasing function, using the left endpoint of the subinterval always resulted in a sum that was larger than the area. What can you say about the following example when choosing the right end point gave a value smaller than the area?

As the mesh gets smaller, all of the Riemann sums seem to be approaching the same value, approximately 1.609. ($\ln 5 = 1.609437912$). □

number of subintervals	mesh (or length)	c_k= left edge = x_{k-1}	c_k= random point in $[x_{k-1}, x_k]$	c_k= right edge = x_k
4	1	2.083333	1.473523	1.283333
8	.5	1.828968	1.633204	1.428968
16	.25	1.714406	1.577806	1.514406
40	.1	1.650237	1.606364	1.570237
400	.01	1.613446	1.609221	1.605446
4000	.001	1.609838	1.609436	1.609038

Table 20.4: Riemann sums for $f(x) = \dfrac{1}{x}$ on the interval $[1, 5]$.

20.6 Two special Riemann sums: lower & upper sums

Two particular Riemann sums are of special interest because they represent the extreme possibilities for the Riemann sums for a given partition.

Definition 4 Suppose $f(x)$ is a positive function on $[a, b]$, and P is a partition of $[a, b]$. Let m_k be the $x-$value in the kth subinterval so that $f(m_k)$ is the minimum value of $f(x)$ in that interval, and let M_k be the $x-$value in the kth subinterval so that $f(M_k)$ is the maximum value of $f(x)$ in that interval.

$$\mathsf{LS}_P = \sum_{k=1}^{n} f(m_k) \cdot \Delta x_k \quad \text{is the lower sum of } f(x) \text{ for the partition } P,$$
$$\mathsf{US}_P = \sum_{k=1}^{n} f(M_k) \cdot \Delta x_k \quad \text{is the upper sum of } f(x) \text{ for the partition } P.$$

□

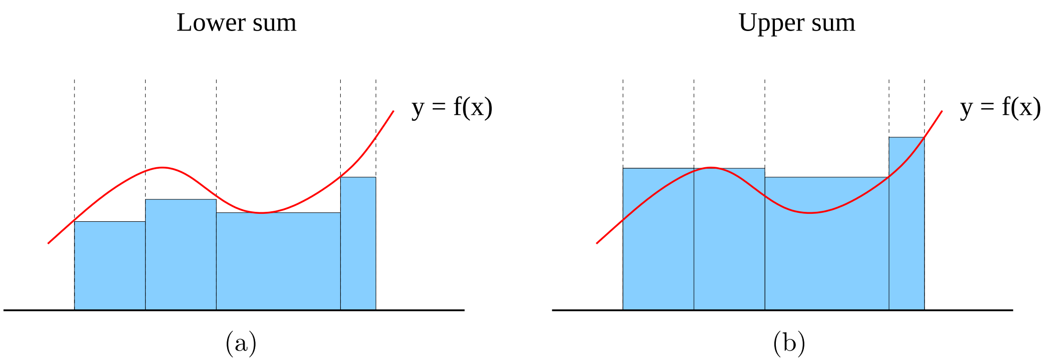

Figure 20.11: Lower sum LS_P and Upper sum US_P.

Geometrically speaking, in Figure 20.11, the lower sum stems from building rectangles under the graph of $f(x)$, and the lower sum (every lower sum) is less than or equal to the exact area $\mathcal{A}$ such that

$$\mathsf{LS}_P \leq \mathcal{A} \quad \text{for every partition } P.$$

The upper sum comes from building rectangles over the graph of $f(x)$, and the upper sum (every upper sum) is greater than or equal to the exact area $\mathcal{A}$ such that

$$\mathsf{US}_P \geq \mathcal{A} \quad \text{for every partition } P.$$

The lower and upper sums provide bounds on the size of the exact area:

$$\mathsf{LS}_P \leq \mathcal{A} \leq \mathsf{US}_P.$$

For any c_k value in the kth subinterval,

$$f(m_k) \leq f(c_k) \leq f(M_k);$$

so, for any choice of the c_k values, the Riemann sum

$$\mathsf{RS}_P = \sum_{k=1}^{n} f(c_k) \cdot \Delta x_k$$

satisfies

$$\sum_{k=1}^{n} f(m_k) \cdot \Delta x_k \leq \sum_{k=1}^{n} f(c_k) \cdot \Delta x_k \leq \sum_{k=1}^{n} f(M_k) \cdot \Delta x_k.$$

or, equivalently

$$\mathsf{LS}_P \leq \mathsf{RS}_P \leq \mathsf{US}_P.$$

Example 16 Approximate the area under the curve

$$f(x) = x^2 + 2, \quad -2 \leq x \leq 1$$

with a Riemann sum, using six regular sub-intervals and right endpoints.

Solution. Take $a = -2, b = 1, \Delta x = \dfrac{b-a}{n} = \dfrac{1-(-2)}{6} = \dfrac{1}{2}$, and $f(x) = x^2 + 2$. The area under the curve is

$$\begin{aligned}
\mathcal{A} &\approx \mathsf{US}_P \\
&= \Delta x(f(-1.5) + f(-1) + f(-0.5) + f(0) + f(0.5) + f(1)) \\
&= \frac{1}{2}((-1.5)^2 + 2 + (-1)^2 + 2 + (-0.5)^2 + 2 + 0^2 + 2 + (0.5)^2 + 2) + 1^2 + 2) \\
&= 8.375.
\end{aligned}$$

□

Example 17 Approximate the area under the curve

$$f(x) = \sqrt{x+1}, \quad -1 \leq x \leq 0$$

with a Riemann sum, using four regular sub-intervals and left endpoints.

Solution. Take $a = -1, b = 0, \Delta x = \dfrac{b-a}{n} = \dfrac{0-(-1)}{4} = \dfrac{1}{4}$, and $f(x) = \sqrt{x+1}$. The area under the curve is

$$\begin{aligned}
\mathcal{A} &\approx \mathsf{LS}_P \\
&= \Delta x(f(-1) + f(-0.75) + f(-0.5) + f(-0.25)) \\
&= \frac{1}{4}(\sqrt{0} + \sqrt{0.25} + \sqrt{0.5} + \sqrt{0.75}) \\
&= 0.5183.
\end{aligned}$$

□

Note 2 Finding minimums and maximums can be a time-consuming task, and it is usually not practical to determine lower and upper sums for wiggly or wavy functions. If $f(x)$ is monotonic, however, then finding the values for m_k and M_k become easier, and sometimes we can explicitly calculate the limits of the lower and upper sums.

Theorem 2 states that for a monotonic bounded function we can guarantee that a Riemann sum is within a certain distance of the exact value of the area it is approximating.

Theorem 2 If $f(x)$ is a positive, monotonically increasing, bounded function on $[a, b]$, then for any partition P and any Riemann sum for P,

$$\left\{\begin{array}{c}\text{distance between}\\ \text{the Riemann sum and}\\ \text{the exact area}\end{array}\right\} \le \left\{\begin{array}{c}\text{distance between}\\ \text{the upper sumand}\\ \text{the lower sum}\end{array}\right\} \le (f(b) - f(a)) \cdot (\text{mesh of } P)$$

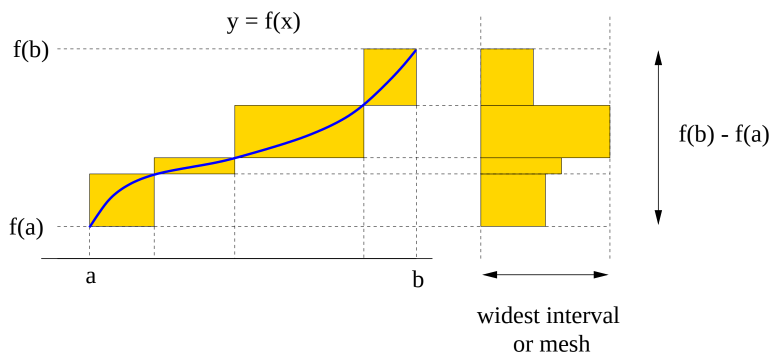

Figure 20.12:

Proof. The Riemann sum and the exact area are both between the upper and lower sums because the distance between the Riemann sum and the exact area is less than or equal to the distance between the upper and lower sums. Since $f(x)$ is monotonically increasing, the areas representing the difference between the upper and lower sums can be combined into a rectangle, as shown in Figure 20.12, whose height equals

$$f(b) - f(a)$$

and whose base equals the widest interval of P or the mesh of P. Thus the total difference between the upper and lower sums is less than or equal to the area of the rectangle,

$$(f(b) - f(a)) \cdot (\text{mesh of } P).$$

$\square$