THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1510 Calculus for Engineers (2020-2021)
Solution to Supplementary Exercise 10

Power Series

1. Simplify the following expression by using summation notation.

   (e.g) $x - x^3 + x^5 - \cdots - x^{15} = \sum_{r=1}^{8} (-1)^{r+1} x^{2r-1}$ or $\sum_{r=0}^{7} (-1)^r x^{2r+1}$

   (a) $x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{x^{2015}}{2015}$

   Ans: $\sum_{r=1}^{2015} \frac{(-1)^{r+1} x^r}{r}$

   (b) $x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$

   Ans: $\sum_{r=0}^{\infty} \frac{(-1)^r x^{2r+1}}{2r+1}$

   (c) $\cos x + \frac{1}{2^2} \cos 2x + \frac{1}{3^2} \cos 3x + \cdots + \frac{1}{n^2} \cos nx$

   Ans: $\sum_{r=1}^{n} \frac{\cos rx}{r^2}$

   (d) $(\cos x - \sin x) + \frac{1}{2}(\cos 2x + \sin 2x) + \frac{1}{2^2}(\cos 3x - \sin 3x) + \cdots$

   Ans: $\sum_{r=1}^{\infty} \frac{\cos rx + (-1)^r \sin rx}{2^{r-1}}$

2. Let $P_k(x) = 1 + x + x^2 + \cdots + x^k = \sum_{n=0}^{k} x^n$, where $k \geq 0$.

   (a) Fix $x = 1/2$, note that $\{P_0(1/2), P_1(1/2), P_2(1/2), \cdots\}$ forms a sequence.

      (i) Write down the sequence explicitly:

Ans:

$$
\begin{aligned}
P_0(\tfrac{1}{2}) &= 1\\
P_1(\tfrac{1}{2}) &= 1+\frac{1}{2} = \frac{3}{2}\\
P_2(\tfrac{1}{2}) &= \underline{1+\frac{1}{2}+(\frac{1}{2})^2} = \underline{\frac{7}{4}}\\
P_3(\tfrac{1}{2}) &= \underline{1+\frac{1}{2}+(\frac{1}{2})^2+(\frac{1}{2})^3} = \underline{\frac{15}{8}}\\
&\vdots\\
P_k(\tfrac{1}{2}) &= \underline{1+\frac{1}{2}+(\frac{1}{2})^2+(\frac{1}{2})^3+\cdots+(\frac{1}{2})^k} = \underline{\frac{1-(1/2)^{k+1}}{1-(1/2)}}\\
&= \underline{\frac{2^{k+1}-1}{2^k}} = \underline{2-\frac{1}{2^k}}
\end{aligned}
$$

(ii) Does $\lim_{k\to\infty} P_k(\frac{1}{2})$ exist? Why?

Ans: $\lim_{k\to\infty} P_k(\frac{1}{2}) = \lim_{k\to\infty}\left(2-\frac{1}{2^k}\right) = 2$

(b) Repeat the same procedure for $x = 2$. Does $\lim_{k\to\infty} P_k(2)$ exist?

Ans: $\lim_{k\to\infty} P_k(2) = \lim_{k\to\infty} 2^{k+1}-1$ which diverges to infinity. Therefore, the limit does not exist.

(c) Guess the range of x such that $\lim_{k\to\infty} P_k(x)$ exists. In this case, we say that the power series $\sum_{n=0}^{\infty} x^n$ converges.

(Hint: $1+x+x^2+\cdots+x^n = \dfrac{1-x^{n+1}}{1-x}$ if $x \neq 1$.)

Ans: By the hint, if $x \neq 1$, we have $\lim_{k\to\infty} P_k(x) = \lim_{k\to\infty} \dfrac{1-x^{n+1}}{1-x}$ which converges if and only if $|x| < 1$. In particular, when $x = 1$, $\lim_{k\to\infty} P_k(1) = k+1$ which diverges to infinity.

Therefore, when $|x| < 1$, $\lim_{k\to\infty} P_k(x)$ exists and we say that the power series $\sum_{n=0}^{\infty} x^n$ converges when $|x| < 1$.

3. Find the radius of convergence of each of the following power series.

Recall: For a power series $\sum_{n=0}^{\infty} a_n(x-c)^n$, c is called the center. If the limit

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

exists, the limit is said to be the **radius of convergence**. Given that the limit exists, the power series is convergent on the interval $(c-R, c+R)$ and divergent on $(-\infty, c-R) \cup (c+R, +\infty)$.

In particular, we allow R to be $+\infty$ here. If $R = +\infty$, it means that the power series converges for all real numbers x.

However, the fact does not tell the convergence of the power series at the boundary points $x = c - R$ and $c + R$. Also, it does not tell anything about the convergence if the limit does not exist.

(a) $\displaystyle\sum_{n=0}^{\infty} x^n$

Ans: For this power series, we have $c = 0$ and $a_n = 1$. Then

$$\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{1}{1}\right| = 1$$

Therefore, the radius of convergence $R = 1$, i.e the power series converges on the interval $(-1, 1)$.

See the discussion of convergence in question 2.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$

Ans: 1

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n}$

Ans: 1

(d) $\displaystyle\sum_{n=0}^{\infty} (x-3)^n$

Ans: 1

(e) $\displaystyle\sum_{n=0}^{\infty} \frac{nx^n}{n+2}$

Ans: 1

(f) $\displaystyle\sum_{n=0}^{\infty} \frac{3^n x^n}{n!}$

Ans: $+\infty$

(g) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{\sqrt{n}+3}$

Ans: 1

(h) $\displaystyle\sum_{n=0}^{\infty} \frac{n(x+3)^n}{5^n}$

Ans: 5

(i) $\displaystyle\sum_{n=0}^{\infty} (-2)^n (n+1)(x-1)^n$

Ans: $\dfrac{1}{2}$

Taylor Polynomial and Taylor Series

4. Let $f(x)$ be a function with derivatives of order k for $k = 1, 2, \cdots, n$. Recall that the Taylor polynomial of order n generated by $f(x)$ at the point $x = c$ is the polynomial

$$\begin{aligned} T_n(x) &= f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n \\ &= \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k \end{aligned}$$

Prove that $T_n^{(k)}(c) = f^{(k)}(c)$ for all $k = 0, 1, 2, \cdots, n$.

Ans: We have

$$\begin{aligned} T_n(x) &= f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n \\ T_n'(x) &= f'(c) + f''(c)(x-c) + \frac{f'''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{(n-1)!}(x-c)^{n-1} \\ T_n''(x) &= f''(c) + f'''(c)(x-c) + \cdots + \frac{f^{(n)}(c)}{(n-2)!}(x-c)^{n-2} \\ &\vdots \\ T_n^{(k)}(x) &= f^{(k)}(c) + f^{(k+1)}(c)(x-c) + \cdots + \frac{f^{(n)}(c)}{(n-k)!}(x-c)^{n-k} \end{aligned}$$

If we put $x = a$ into the equations, all terms on the right hand side vanish except the first one and the result follows.

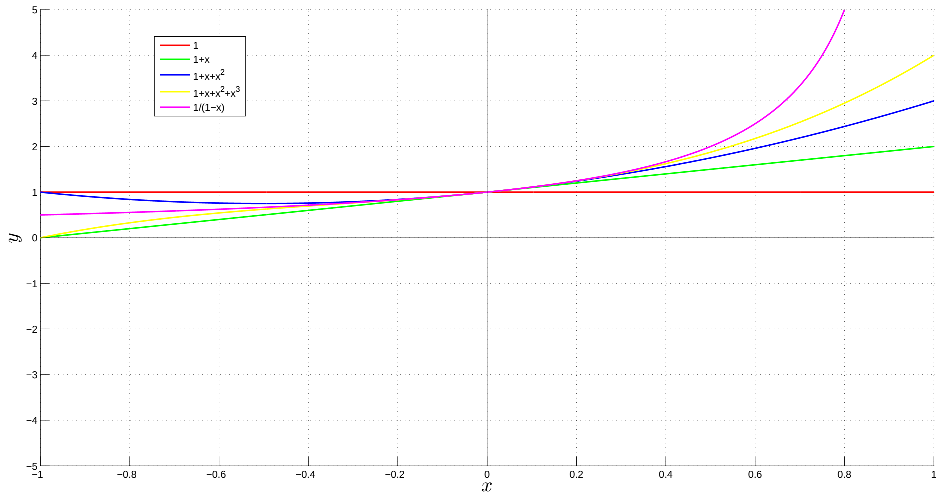
(Remark: The Taylor polynomial $T_n(x)$ generated by $f(x)$ at $x = c$ is a polynomial of degree n which approximates $f(x)$ in a sense that the k-th derivatives of $T_n(x)$ and $f(x)$ are the same at $x = c$, i.e. $T_n^{(k)}(c) = f^{(k)}(c)$, for all $k = 0, 1, 2, \cdots, n$.)

5. Let $f(x) = \dfrac{1}{1-x}$, for $x \neq 1$.

   (a) Find the Taylor polynomials $T_0(x)$, $T_1(x)$, $T_2(x)$ and $T_3(x)$ generated by $f(x)$ at $x = 0$ and plot the graphs of them by using MATLAB and compare with the graph of $f(x)$.

   Ans:

$$\begin{aligned} T_0(x) &= 1 \\ T_1(x) &= 1 + x \\ T_2(x) &= 1 + x + x^2 \\ T_3(x) &= 1 + x + x^2 + x^3 \end{aligned}$$



(b) Fix $x = 1/2$, note that $\{T_0(1/2), T_1(1/2), T_2(1/2), \cdots\}$ forms a sequence which is exactly the sequence in question 2(a).
Verify that $\lim_{k\to\infty} T_k(1/2) = f(1/2)$.

Ans: $\lim_{k\to\infty} T_k(1/2) = \lim_{k\to\infty} 1 + \frac{1}{2} + (\frac{1}{2})^2 + \cdots + (\frac{1}{2})^k = \lim_{k\to\infty} \frac{1-(1/2)^{k+1}}{1-(1/2)} = 2$

(c) Verify that $\lim_{k\to\infty} T_k(x) = f(x)$ for any $-1 < x < 1$.

Ans: For $-1 < x < 1$,

$$\lim_{k\to\infty} T_k(x) = \lim_{k\to\infty} 1 + x + x^2 + \cdots + x^k = \lim_{k\to\infty} \frac{1-x^{k+1}}{1-x} = \frac{1}{1-x} = f(x)$$

6. Find the Taylor series generated by the following functions at given points.

(a) $f(x) = \cos x$ at $x = \pi/2$;

Ans: $\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)!}(x - \frac{\pi}{2})^{2n-1}$

(b) $f(x) = \ln(1+x)$ at $x = 0$;

Ans: $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$

(c) $f(x) = e^x$ at $x = 1$.

Ans: $\sum_{n=0}^{\infty} \frac{e}{n!}(x-1)^n$

7. (Harder Problem) Let $f(x)$ be a function defined by

$$f(x) = \begin{cases} 0 & \text{if} \quad x = 0, \\ e^{-1/x^2} & \text{if} \quad x \neq 0. \end{cases}$$

(a) Show that $f(x)$ is differentiable at $x = 0$ and find $f'(0)$.

(Hint: Show that $\lim_{x\to 0} \frac{f(x) - f(0)}{x}$ exists.)

Ans:

$$
\begin{aligned}
\lim_{x\to 0} \frac{f(x) - f(0)}{x} &= \lim_{x\to 0} \frac{e^{-1/x^2}}{x} \\
&= \lim_{x\to 0} \frac{(\frac{1}{x})}{e^{\frac{1}{x^2}}} \qquad (\frac{\infty}{\infty}) \\
&= \lim_{x\to 0} \frac{(-\frac{1}{x^2})}{(-\frac{2}{x^3}e^{\frac{1}{x^2}})} \\
&= \lim_{x\to 0} \frac{x}{2e^{\frac{1}{x^2}}} \\
&= 0
\end{aligned}
$$

Therefore, $f'(0) = 0$.

(b) Write down the function $f'(x)$ explicit as the following:

$$
f'(x) = \begin{cases} \_\_\_\_\_\_ & \text{if} \quad x = 0, \\ \_\_\_\_\_\_ & \text{if} \quad x \neq 0. \end{cases}
$$

Show that $f'(x)$ is differentiable at $x = 0$ and find $f''(0)$.

(Hint: Show that $\lim_{x\to 0} \frac{f'(x) - f'(0)}{x}$ exists.)

Ans:

$$
f'(x) = \begin{cases} \underline{\qquad 0 \qquad} & \text{if} \quad x = 0, \\ \underline{\quad 2x^{-3}e^{-1/x^2} \quad} & \text{if} \quad x \neq 0. \end{cases}
$$

Furthermore,

$$\begin{aligned}
\lim_{x\to 0}\frac{f'(x)-f'(0)}{x} &= \lim_{x\to 0}\frac{2x^{-3}e^{-1/x^2}}{x} \\
&= \lim_{x\to 0}\frac{(\frac{2}{x^4})}{e^{\frac{1}{x^2}}} \qquad (\frac{\infty}{\infty}) \\
&= \lim_{x\to 0}\frac{(-\frac{8}{x^5})}{(-\frac{2}{x^3}e^{\frac{1}{x^2}})} \\
&= \lim_{x\to 0}\frac{(\frac{4}{x^2})}{e^{\frac{1}{x^2}}} \qquad (\frac{\infty}{\infty}) \\
&= \lim_{x\to 0}\frac{(-\frac{8}{x^3})}{(-\frac{2}{x^3}e^{\frac{1}{x^2}})} \\
&= \lim_{x\to 0}\frac{4}{e^{\frac{1}{x^2}}} \\
&= 0
\end{aligned}$$

Therefore, $f''(0) = 0$.

(c) In general, is $f^{(n)}(0)$ defined for each positive integer n? If so, what is the value?

Ans: In general, $f^{(n)}(0) = 0$ for all positive integers n.

(d) Find the Maclaurin series generated by $f(x)$, i.e. Taylor series generated by $f(x)$ at the point $x = 0$.

Ans: Maclaurin series generated by $f(x)$ is $\displaystyle\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = 0$.

(Remark: The Maclaurin series is only a zero function which means that it converges for all real number x, however it does not converge to $f(x)$ except $x = 0$, i.e. $\displaystyle\lim_{k\to\infty} P_k(x) = 0 \neq f(x)$ for any $x \neq 0$.

Therefore, $P_k(x)$ **cannot** be used to approximate $f(x)$ if $x \neq 0$.)

8. By considering the Taylor series generated by e^x and $\cos x$ at $x = 0$, find the Taylor polynomials of degree 3 generated by the following functions at $x = 0$.

(a) $e^x \cos x$;

Ans:

$$\begin{aligned}
T(x) &= \left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\cdots\right)\left(1-\frac{x^2}{2}+\cdots\right) \\
&= 1+x+\left(\frac{1}{2}-\frac{1}{2}\right)x^2+\left(\frac{1}{6}-\frac{1}{2}\right)x^3+\cdots \\
\therefore \quad T_3(x) &= 1+x-\frac{x^3}{3}.
\end{aligned}$$

(b) $e^{\cos x}$;

Ans:

$$\begin{aligned}
T(x) &= 1+\cos x+\frac{\cos^2 x}{2!}+\frac{\cos^3 x}{3!}+\cdots \\
&= 1+\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right)+\frac{1}{2!}\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right)^2 \\
&\quad +\frac{1}{3!}\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right)^3+\cdots \\
\therefore\quad T_3(x) &= \left(1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots\right)+\left[-\frac{x^2}{2}+\frac{1}{2!}\cdot 2\cdot(-\frac{x^2}{2})+\frac{1}{3!}\cdot 3\cdot(-\frac{x^2}{2})+\cdots\right] \\
&= e-\frac{x^2}{2}\left(1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots\right) \\
&= e-\frac{e}{2}x^2.
\end{aligned}$$

(c) $\dfrac{e^x}{\cos x}$.

Ans: Suppose $T(x)=c_0+c_1x+c_2x^2+c_3x^3+\cdots$, then

$$\begin{aligned}
e^x &= (c_0+c_1x+c_2x^2+c_3x^3+\cdots)\cos x \\
1+x+\frac{x^2}{2}+\frac{x^3}{6}+\cdots &= \left(1-\frac{x^2}{2}+\cdots\right)(c_0+c_1x+c_2x^2+c_3x^3+\cdots) \\
\therefore\quad & \begin{cases} 1\cdot c_0=1 \\ 1\cdot c_1=1 \\ c_2-\frac{c_0}{2}=\frac{1}{2} \\ c_3-\frac{c_2}{2}=\frac{1}{6} \end{cases} \\
& c_0=1,\ c_1=1,\ c_2=1,\ c_3=\frac{2}{3}.
\end{aligned}$$

Therefore, $T_3(x)=1+x+x^2+\dfrac{2}{3}x^3$.

9. By considering $\cos^2 x=\dfrac{1}{2}(1+\cos 2x)$, find the Taylor polynomial of degree 4 generated by $\cos^2 x$.
(Remark: You may compare the one obtained by considering $\cos^2 x=(\cos x)(\cos x)$.)

Ans: $T_4(x)=1-x^2+\dfrac{x^4}{3}$.

10. Let $f(x)=\sin x$.

(a) Find the Maclaurin series generated by $f(x)$.

Ans: $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots$

(b) By considering $f'(x)$ and term-by-term differentiation, find the Maclaurin series generated by $\cos x$. Do they match with each other?

Ans:

$$\begin{aligned}
\frac{d}{dx}\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}\right) &= \sum_{n=0}^{\infty}\frac{d}{dx}\left(\frac{(-1)^n}{(2n+1)!}x^{2n+1}\right)\\
&= \sum_{n=0}^{\infty}\left(\frac{(-1)^n}{(2n+1)!}\cdot(2n+1)x^{2n}\right)\\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}
\end{aligned}$$

which is the Maclaurin series generated by $\cos x$.

(Remark: Assume the convergence, if we term-by-term differentiate the Maclaurin series generated by $f(x)$, then we can get the Maclaurin series of $f'(x)$.)

11. Let $f(x) = \dfrac{1}{1-x}$. By considering $f'(x)$, $f''(x)$ and term-by-term differentiation, find the Maclaurin series generated by $\dfrac{1}{(1-x)^2}$ and $\dfrac{1}{(1-x)^3}$.

Ans: Maclaurin series of generated by $\dfrac{1}{(1-x)^2}$ is

$$\sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \cdots .$$

Maclaurin series of generated by $\dfrac{1}{(1-x)^3}$ is

$$\sum_{n=2}^{\infty}\frac{n(n-1)}{2}x^{n-2} = x + 3x + 6x^2 + \cdots .$$

(Note: $\dfrac{d}{dx}\dfrac{1}{(1-x)^2} = \dfrac{2}{(1-x)^3}$.)

12. By using the fact that $\displaystyle\int -\sin x\, dx = \cos x + C$, find the Taylor series generated by $\cos x$ at $x = 0$.

Ans:

$$\begin{aligned}
\int -\sin x\, dx &= \cos x + C\\
\cos x &= C - \int \sin x\, dx\\
&= C - \int (x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots)\, dx\\
&= C - \underline{\ \frac{1}{2!}\ }\ x^2 + \underline{\ \frac{1}{4!}\ }\ x^4 - \underline{\ \frac{1}{6!}\ }\ x^6 + \cdots
\end{aligned}$$

By putting $x = 0$ on both sides, $C = 1$.

13. (a) By considering $\dfrac{2x}{1-x^2} = \dfrac{1}{1-x} - \dfrac{1}{1+x}$, find the Taylor series generated by $\dfrac{2x}{1-x^2}$ at $x=0$.

Ans: $\displaystyle\sum_{n=0}^{\infty} 2x^{2n+1} = 2x + 2x^3 + 2x^5 + \cdots$

(b) By using the fact that $\displaystyle\int -\frac{2x}{1-x^2}\,dx = \ln(1-x^2) + C$, find the Talyor series generated by $\ln(1-x^2)$ at $x=0$.

Ans: $\displaystyle\sum_{n=1}^{\infty} -\frac{1}{n}x^{2n} = -x^2 - \frac{x^4}{2} - \frac{x^6}{3} - \cdots$

14. Let $f(x) = \ln(1-x)$ for $x < 1$.

(a) Find the Taylor series generated by $f(x)$ at $x=0$ and find the radius of convergence.

Ans: Taylor series generated by $f(x)$ at $x=0$ is

$$\sum_{n=1}^{\infty} -\frac{1}{n}x^n = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots$$

and its radius of convergence is $R=1$.

(b) Write down the Taylor polynomial $T_3(x)$ of degree 3 generated by $f(x)$ at $x=0$ and the Lagrange remainder $R_3(x)$.

Ans: $T_3(x) = -x - \dfrac{x^2}{2} - \dfrac{x^3}{3}$ and

$R_3(x) = \dfrac{f^{(4)}(c)}{4!}x^4$ for some c lying between 0 and x (and so c depends on x). Therefore,

$$\begin{aligned} f(x) &= T_3(x) + R_3(x) \\ \ln(1-x) &= -x - \frac{x^2}{2} - \frac{x^3}{3} + \frac{f^{(4)}(c)}{4!}x^4 \end{aligned}$$

(c) Hence, approximate $\ln 0.9$ and show that the error of this approximation is less than $\dfrac{1}{4 \times 9^4}$.

Ans: By putting $x = 0.1$, we have

$$\begin{aligned}
\ln(1-0.1) &= -0.1 - \frac{0.1^2}{2} - \frac{0.1^3}{3} + \frac{f^{(4)}(c)}{4!}(0.1^4) \\
\ln 0.9 &= -\frac{79}{750} + \frac{(\frac{-3!}{(1-c)^4})}{4!}(0.1)^4 \\
\left|\ln 0.9 - (-\frac{79}{750})\right| &= \frac{1}{4(1-c)^4}(0.1)^4 \\
&< (\frac{1}{4})(\frac{10}{9})^4(\frac{1}{10})^4 \\
&= \frac{1}{4 \times 9^4}
\end{aligned}$$

Note that $0 < c < 0.1$, so $\dfrac{1}{1-c} < \dfrac{1}{0.9} = \dfrac{10}{9}$

$\ln 0.9$ can be approximated by $-\dfrac{79}{750} \approx -0.1053333$ with absolute error less than $\dfrac{1}{4 \times 9^4}$.

Fourier Series

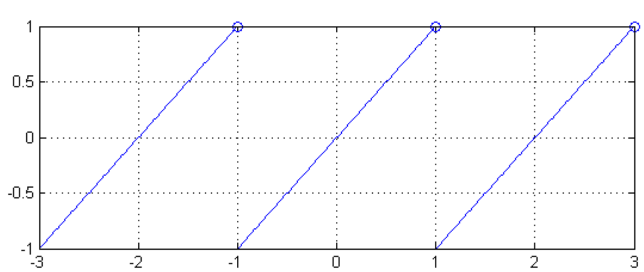
(Periodic Function) A function $f : \mathbb{R} \to \mathbb{R}$ is said to be a periodic function if there is a constant $T > 0$ such that

$$f(x+T) = f(x)$$

for all real numbers x. Furthermore, if T is the least positive real number with the above property, then T is said to be the period of the function f. For example, $\sin x$, $\cos x$ and $\tan x$ are periodic function but the periods of $\sin x$ and $\cos x$ are 2π while the period of $\tan x$ is π.

15. Suppose that $f : [-1, 1) \to \mathbb{R}$ is a function defined by $f(x) = x$. If f is extended to be a periodic function with period 2, try to sketch the graph of the extended function.

    Ans:



(Remark: In general, let $L > 0$, a function $f : [-L, L) \to \mathbb{R}$ (or $f : (-L, L] \to \mathbb{R}$) can be extended as a periodic function with period $2L$.)

Let $L > 0$ and $f : \mathbb{R} \to \mathbb{R}$ be a periodic function with period $2L$. The Fourier Series generated by f is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{L}) + \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{L}),$$

where the Fourier coefficients a_n's and b_n's are given by

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{n\pi x}{L})\,dx,\ n \geq 0; \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{n\pi x}{L})\,dx,\ n \geq 1. \end{aligned}$$

The idea of Fourier Series is experssing a period function $f(x)$ as a sum of sines and cosines. With suitable assumptions (beyond the scope of this course), we have the pointwise convergence, that is

$$f(x_0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x_0}{L}) + \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x_0}{L})$$

if $f(x)$ is continuous at $x = x_0$.

16. Let $f(x)$ be a periodic function with period 2 (i.e $L = 1$) which is defined by

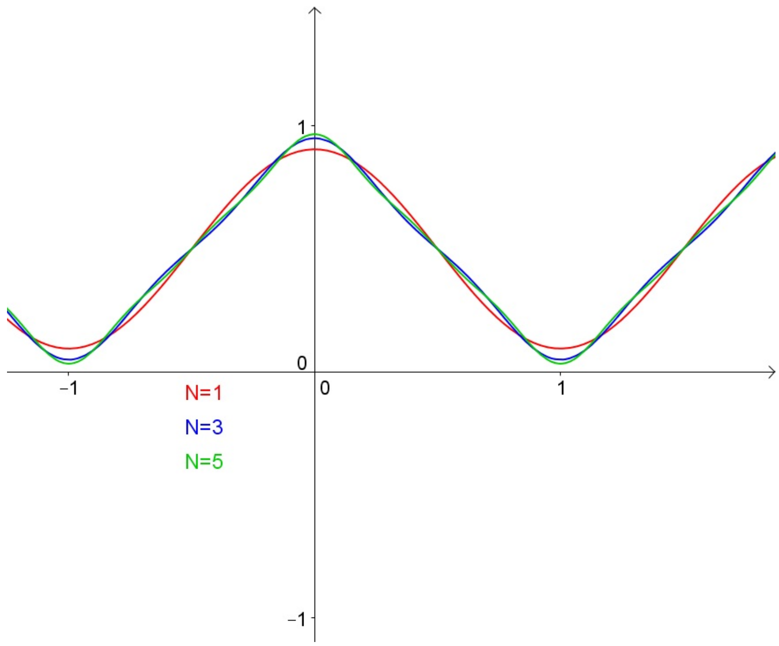
$$f(x) = \begin{cases} x+1 & \text{if} \quad -1 \leq x \leq 0, \\ 1-x & \text{if} \quad 0 < x < 1. \end{cases}$$

(a) Find the Fourier series generated by $f(x)$.
(Hint: $f(x)\sin(n\pi x)$ is an odd function for any $n = 1, 2, \cdots$.)
Ans: $\displaystyle \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2[1-(-1)^n]}{(n\pi)^2}\cos(n\pi x)$

(b) For any natural number N, define

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos(n\pi x) + \sum_{n=1}^{N} b_n \sin(n\pi x),$$

where a_n's and b_n's are Fourier coefficients found in (a).
Plot the graphs of $S_N(x)$ for $N = 1, 3, 5$ by using MATLAB or other softwares.
Ans:



17. Let $f(x)$ be a periodic function with period 2π such that

$$f(x) = \begin{cases} 1 & \text{if} \quad -\pi < x < 0, \\ 0 & \text{if} \quad x = -\pi \text{ or } 0, \\ -1 & \text{if} \quad 0 < x < \pi. \end{cases}$$

Find the Fourier Series generated by $f(x)$.

Ans: $\displaystyle\sum_{n=1}^{\infty} -\frac{2[1-(-1)^n]}{n\pi} \sin(nx)$

18. Let $f(x)$ be a periodic function with period 2π such that $f(x) = x^2$ for $-\pi < x \le \pi$.

(a) Show that the Fourier series of $f(x)$ is

$$\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx.$$

Ans: For $n \ge 0$,

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx\, dx \\ &= (-1)^n \frac{4}{n^2} \end{aligned}$$

For $n \geq 1$,

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin nx\,dx \\
&= = 0
\end{aligned}$$

Note that $x^2 \sin nx$ is an odd function.

(b) By considering a suitable value of x, show that

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots .$$

Ans: By putting $x = \pi$, we have

$$\begin{aligned}
f(\pi) &= \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\pi \\
\pi^2 &= \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}(-1)^n \\
&= \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{1}{n^2} \\
\sum_{n=1}^{\infty} \frac{1}{n^2} &= \frac{\pi^2 - \frac{\pi^2}{3}}{4} \\
&= \frac{\pi^2}{6}
\end{aligned}$$

19. Let $f(x)$ be a function defined on $[-\pi, \pi]$ such that

$$f(x) = \begin{cases} x & \text{if} \quad 0 < x \leq \pi, \\ \\ 0 & \text{if} \quad -\pi \leq x \leq 0. \end{cases}$$

(a) Find the Fourier series of $f(x)$.

Ans: $\displaystyle \frac{\pi}{4} - \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^2\pi} \cos nx + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$

(b) Show that

(i) $\displaystyle \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots;$
Ans: Put $x = \pi/2$, the result follows.

(ii) $\displaystyle \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots.$
Ans: Put $x = 0$, the result follows.