THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1510 Calculus for Engineers (Fall 2021)
Coursework 6

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Class: MATH1510 G

I acknowledge that I am aware of University policy and regulations on honesty in academic work, and of the disciplinary guidelines and procedures applicable to breaches of such policy and regulations, as contained in the website http://www.cuhk.edu.hk/policy/academichonesty/

David — Signature

18/10/2021 — Date

General Guidelines for Coursework Submission.

- Please go to the class indicated by your registered course code via the CUSIS system. Failure to comply will result in a **2-point deduction** of the final score.
- Please write your answers using a black or blue pen, NOT any other color or a pencil.
- Points will only be awarded for answers with sufficient justifications.
- All questions in **Part A** along with some selected questions in **Part B** will be graded. Question(s) labeled with * are more challenging.

For internal use only:

1	3						
2	3						
3	3						
4	1						
					Total	10	/ 10

Part A

1. Let $f(x) = \sqrt[3]{x}$. $= x^{\frac{1}{3}}$

   (a) Find the equation of the tangent of $f(x)$ at $x = 1000$. Express your answer in form of $y = mx + c$.

   (b) Using the fact that
   $$y = L(x) = mx + c$$
   is close to $f(x)$ around the point $x = 1000$, give an approximation of $\sqrt[3]{999}$.

(a) $f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$

$f'(1000) = \frac{1}{3}(1000)^{-\frac{2}{3}} = \frac{1}{300}$

$\because f(1000) = \sqrt[3]{1000}$

$= 10$

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$\therefore$ Equation of tangent: $y - 10 = \frac{1}{300}(x - 1000)$

$y = \frac{1}{300}x + \frac{20}{3}$ //

(b) $\sqrt[3]{999} = f(999) \approx L(999)$

$\approx \frac{1}{300}(999) + \frac{20}{3}$

$\approx \frac{2999}{300}$ //

(b) Let $a = 20$, $b = 21$,

we have: $\dfrac{21-20}{21} < \ln 21 - \ln 20 < \dfrac{21-20}{20}$

$\dfrac{1}{21} < \ln 1.05 < \dfrac{1}{20}$ 3

2. (a) Let $0 < a < b$. Show that

$$\frac{b-a}{b} < \ln b - \ln a < \frac{b-a}{a}.$$

(b) Using the result obtained in part (a), show that

$$\frac{1}{21} < \ln 1.05 < \frac{1}{20}.$$

(a) Let $f : [a,b] \to \mathbb{R}$ be a function s.t.

1) f is continuous on $[a,b]$

2) f is differentiable on (a,b).

then $c \in (a,b)$ s.t. $f'(c) = \dfrac{f(b)-f(a)}{b-a}$.

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Consider $f(x) = \ln x$ on $[a,b]$

& $f'(x) = \dfrac{1}{x}$ on $[a,b]$ where $0 < a < b$.

We have the value c where: $\dfrac{1}{c} = \dfrac{\ln b - \ln a}{b-a}$.

$c = \dfrac{b-a}{\ln b - \ln a}$

$\because$ $a < c < b$

$a < \dfrac{b-a}{\ln a - \ln b} < b$

$\dfrac{a}{b-a} < \dfrac{1}{\ln a - \ln b} < \dfrac{b}{b-a}$

$\dfrac{b-a}{b} < \ln a - \ln b < \dfrac{b-a}{a}$

(b) Let $b = 1$, $a = 0$: We have

$$\frac{1}{2} < \frac{\tan^{-1} 1 - \tan^{-1} 0}{1} < \cancel{1}\ 4$$

$$2 < 4(\tan^{-1}(1) - \tan^{-1}(0)) < 4$$

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Part B

3. (a) Let $0 \le a < b$. Show that $2 < \pi < 4$ //

$$\frac{1}{1+b^2} < \frac{\tan^{-1} b - \tan^{-1} a}{b-a} < \frac{1}{1+a^2}.$$

(b) Using the result obtained in part (a), show that $2 < \pi < 4$.

(a) Let $f : [a,b] \to \mathbb{R}$ be a function s.t.

1) f is continuous on $[a,b]$

2) f is differentiable on (a,b)

then we have $c \in (a,b)$ s.t. $f'(c) = \dfrac{f(b) - f(a)}{b-a}$

Consider $f(x) = \tan^{-1}(x)$ on $[a,b]$

& $f'(x) = \dfrac{1}{1+x^2}$ on $[a,b]$ where $0 \le a < b$.

$\therefore$ We have the value c : $\dfrac{1}{1+c^2} = \dfrac{\tan^{-1} b - \tan^{-1} a}{b-a}$.

$\because$ $a < c < b$,

$$\frac{1}{b} < \frac{1}{c} < \frac{1}{a}$$

$$\frac{1}{b^2} < \frac{1}{c^2} < \frac{1}{a^2}$$

$$\frac{1}{1+b^2} < \frac{1}{1+c^2} < \frac{1}{1+a^2}$$

$$\frac{1}{1+b^2} < \frac{\tan^{-1} b - \tan^{-1} a}{b-a} < \frac{1}{1+a^2} \quad //$$

$$f'(x) = \frac{x-2}{(4x-x^2)^{-\frac{3}{2}}} \qquad f''(x) = \frac{(4x-x^2) - \frac{3}{2}(4x-x^2)^{\frac{1}{2}}(4-2x)}{(4x-x^2)^{-3}}$$

$=$

4. Let $f(x) = \dfrac{1}{\sqrt{4x-x^2}}$.

(a) Prove that

$$(4x-x^2)f'(x) = (x-2)f(x)$$

(b) Prove that for any positive integer n,

$$(4x-x^2)f^{(n+1)}(x) = (2n+1)(x-2)f^{(n)}(x) + n^2 f^{(n-1)}(x),$$

where $f^{(0)}(x) = f(x)$.

(a) $$f'(x) = \frac{-\frac{1}{2}(4x-x^2)^{-\frac{1}{2}}(4-2x)}{4x-x^2}$$

$$= \frac{x-2}{(4x-x^2)^{-\frac{3}{2}}}$$

$$(4x-x^2)f'(x) = (x-2)(4x-x^2)^{1-\frac{3}{2}}$$

$$= \frac{x-2}{\sqrt{4x-x^2}}$$

$$= (x-2) \cdot \frac{1}{\sqrt{4x-x^2}}$$

$$= (x-2)f(x) \quad //$$

(b) By General Leibniz Rule:

$$(fg)^n = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)}$$

Let $g(x) = 4x-x^2$ & $f(x) = f'(x)$

$g'(x) = 4-2x$

$g''(x) = -2$

$g'''(x) = 0$

$\vdots$

$g^{(n)}(x) = 0$

$$(fg)^n = \sum_{k=0}^{n} \binom{n}{n} \cancel{} f^{(n-k)} g^{(n)}$$

$$(fg)^n = f^{(n)} + n f^{(n-1)} g^{(1)} + \binom{n}{2} f^{(n-2)} g^{(2)}$$
$$+ \binom{n}{3} f^{(n-3)} g^{(3)}$$
$$+ 0 \quad (\text{when } n \geq 3)$$

$$= f^{(n)} + n f^{(n-1)} (4-2x)$$
$$+ (-2) f^{(n-2)} \frac{n(n-1)}{2}.$$
$$+ 0 \ldots$$

$$=$$