THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1510 Calculus for Engineers (2020-2021)
Solution to Supplementary Exercise 9

## Definite Integration

1. Evaluate the following integrals.

(a) $\displaystyle\int_0^2 x^2 - 3x + 4\,dx$

**Ans:** $\dfrac{14}{3}$

(b) $\displaystyle\int_{-2}^5 |x^2 - 3x + 2|\,dx$

**Ans:** $\dfrac{163}{6}$

(c) $\displaystyle\int_0^4 xe^{|2-x|}\,dx$

**Ans:** $4(e^2 - 1)$

(d) $\displaystyle\int_0^{\pi/6} (\sec x + \tan x)^2\,dx$

**Ans:** $2\sqrt{3} - \dfrac{\pi}{6} - 2$

(e) $\displaystyle\int_{\pi/2}^{\pi} \frac{\sin 2x}{2\sin x}\,dx$

**Ans:** $-1$

(f) $\displaystyle\int_{\ln(3/4)}^{\ln(4/3)} \frac{e^x}{(1+e^{2x})}\,dx$

**Ans:** $\tan^{-1}\dfrac{4}{3} - \tan^{-1}\dfrac{3}{4}$

(g) $\displaystyle\int_1^e \frac{1}{x\sqrt{1+(\ln x)^2}}\,dx$

**Ans:** $\ln(1+\sqrt{2})$

(h) $\displaystyle\int_0^{\ln 2} e^{-x}\ln(1+e^x)\,dx$

**Ans:** $3\ln 2 - \dfrac{3}{2}\ln 3$

(i) $\displaystyle\int_0^{\pi/2} e^{3x}\sin x\,dx$

**Ans:** $\dfrac{1}{10}(e^{3\pi/2} + 1)$

(j) $\displaystyle\int_1^3 \ln x\,dx$

**Ans:** $3\ln 3 - 2$

(k) $\displaystyle\int_1^4 \frac{1}{\sqrt{x}}(1+\sqrt{x})^4\,dx$

**Ans:** $\dfrac{422}{5}$

(l) $\displaystyle\int_0^1 \frac{x^2+4x}{\sqrt[3]{x^3+6x^2+1}}\,dx$

**Ans:** $\dfrac{3}{2}$

(m) $\displaystyle\int_{-2}^1 (x+1)\sqrt{x+3}\,dx$

**Ans:** $\dfrac{46}{15}$

(n) $\displaystyle\int_1^2 \frac{\ln x}{x}\,dx$

**Ans:** $\dfrac{1}{2}(\ln 2)^2$

(o) $\displaystyle\int_1^2 \frac{e^{2x}}{e^x - 1}\,dx$

**Ans:** $e^2 - e + \ln(e+1)$

(p) $\displaystyle\int_{-1}^{\sqrt{3}} \frac{1}{(1+x^2)^{3/2}}\,dx$

**Ans:** $\dfrac{\sqrt{3}+\sqrt{2}}{2}$

(q) $\displaystyle\int_{-1}^2 x^2\sqrt{4-x^2}\,dx$

**Ans:** $\dfrac{4\pi}{3} - \dfrac{\sqrt{3}}{4}$

2. Evaluate the following improper integrals.

(a) $\displaystyle\int_0^\infty e^{-x}\,dx$

**Ans:** 1

(b) $\displaystyle\int_1^\infty \frac{1}{x^3}\,dx$

**Ans:** $\frac{1}{2}$

(c) $\displaystyle\int_3^\infty \frac{1}{9+x^2}\,dx$

**Ans:** $\frac{\pi}{12}$

(d) $\displaystyle\int_{-\infty}^0 xe^x\,dx$

**Ans:** $-1$

3. (a) By using integration by parts, find $\displaystyle\int \sin(\ln x)\,dx$.

(b) Hence, evaluate $\displaystyle\int_1^{e^\pi} \sin(\ln x)\,dx$.

**Ans:**

(a) $\frac{1}{2}(x\sin(\ln x) - x\cos(\ln x)) + C$

(b) $\dfrac{e^\pi + 1}{2}$

4. Given that $I_n = \displaystyle\int_0^1 (1-x^3)^n\,dx$, where $n$ is an nonnegative integer. Show that for $n \geq 1$,

$$(3n+1)I_n = 3nI_{n-1}.$$

Hence, find $I_5$.

**Ans:** By using integration by parts,

$$\begin{aligned}
I_n &= \int_0^1 (1-x^3)^n\,dx \\
&= [x(1-x^3)^n]_0^1 - \int_0^1 x\,d(1-x^3)^n \\
&= -\int_0^1 x[-3nx^2(1-x^3)^{n-1}]\,dx \\
&= \int_0^1 3nx^3(1-x^3)^{n-1}\,dx \\
&= \int_0^1 -3n(1-x^3)^n + 3n(1-x^3)^{n-1} \\
&= -3nI_n - 3nI_{n-1} \\
(3n+1)I_n &= 3nI_{n-1}
\end{aligned}$$

By using the above,

$$I_5 = \frac{15}{16}I_4 = \frac{15}{16}\cdot\frac{12}{13}I_3 = \cdots = \frac{15}{16}\cdot\frac{12}{13}\cdot\frac{9}{10}\cdot\frac{6}{7}\cdot\frac{3}{4}I_0 = \frac{729}{1456}$$

Note that $I_0 = \displaystyle\int_0^1 (1-x^3)^0\,dx = \int_0^1 1\,dx = 1$.

5. Let $p$ and $q$ be positive integers. Show that

$$\int_0^1 x^p(1-x)^q\,dx = \frac{q}{p+1}\int_0^1 x^{p+1}(1-x)^{q-1}\,dx.$$

Hence, find $\displaystyle\int_0^1 x^4(1-x)^3\,dx$.

**Ans:** By using integration by parts,

$$\begin{aligned}
\int_0^1 x^p(1-x)^q\,dx &= \int_0^1 (1-x)^q\,d\frac{x^{p+1}}{p+1} \\
&= \left[\frac{x^{p+1}}{p+1}\cdot(1-x)^q\right]_0^1 - \int_0^1 \frac{x^{p+1}}{p+1}\,d(1-x)^q \\
&= -\int_0^1 \frac{x^{p+1}}{p+1}\cdot[-q(1-x)^{q-1}\,dx \\
&= \frac{q}{p+1}\int_0^1 x^{p+1}(1-x)^{q-1}\,dx
\end{aligned}$$

By using the above,

$$\begin{aligned}
\int_0^1 x^4(1-x)^3\,dx &= \frac{3}{5}\int_0^1 x^5(1-x)^2\,dx \\
&= \frac{3}{5}\cdot\frac{2}{6}\int_0^1 x^6(1-x)\,dx \\
&= \frac{3}{5}\cdot\frac{2}{6}\cdot\frac{1}{7}\int_0^1 x^7\,dx \\
&= \frac{3}{5}\cdot\frac{2}{6}\cdot\frac{1}{7}\cdot\frac{1}{8} \\
&= \frac{1}{280}
\end{aligned}$$

6. Let $f:\mathbb{R}\to\mathbb{R}$ be a function,

- if $f(x)=f(-x)$ for all $x\in\mathbb{R}$, $f(x)$ is called an even function;
- if $-f(x)=f(-x)$ for all $x\in\mathbb{R}$, $f(x)$ is called an odd function.

(a) Show that $x^2$ and $\cos x$ are even functions.

**Ans:** Let $f(x)=\cos x$. Then $f(-x)=\cos(-x)=\cos x=f(x)$, so $f(x)=\cos x$ is an even function. Similar for $x^2$.

(b) Show that $x^3$ and $\sin x$ are odd functions.

**Ans:** Let $f(x)=\sin x$. Then $f(-x)=\sin(-x)=-\sin x=-f(x)$, so $f(x)=\sin x$ is an odd function. Similar for $x^3$.

(Remark: The graph of an even function must be symmetric along the $y$-axis and the graph of an odd function must be symmetric about the origin.)

7. (Harder Problem) Let $a > 0$ and let $f(x)$ be an even function. Show that

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$$

Hence, evaluate $\displaystyle\int_{-\pi}^{\pi} |x|\sin|x|\,dx$.

**Ans:** Note that $\displaystyle\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$, so if we can show that $\displaystyle\int_{-a}^{0} f(x)\,dx = \int_{0}^{a} f(x)\,dx$, then the result follows.

Let $y = -x$, then $dx = -dy$. When $x = -a$, $y = a$; when $x = 0$, $y = 0$.

$$\begin{aligned}
\int_{-a}^{0} f(x)\,dx &= \int_{a}^{0} -f(-y)\,dy \\
&= \int_{0}^{a} f(-y)\,dy \\
&= \int_{0}^{a} f(y)\,dy \qquad (\because\ f(x) \text{ is an even function}) \\
&= \int_{0}^{a} f(x)\,dx \qquad (\text{dummy variable})
\end{aligned}$$

(Remark: The last equality just says that they are computing the area under the graph of $f(x)$ over the interval $[0, a]$, which is independent from the variable we are using.)

Therefore,

$$\begin{aligned}
\int_{-a}^{0} f(x)\,dx &= \int_{0}^{a} f(x)\,dx \\
\int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx &= \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx \\
\int_{-a}^{a} f(x)\,dx &= 2\int_{0}^{a} f(x)\,dx
\end{aligned}$$

Now, let $f(x) = |x|\sin|x|$, we have

$$f(-x) = |-x|\sin|-x| = |x|\sin|x| = f(x).$$

Therefore, $f(x) = |x|\sin|x|$ is an even function. By the above result,

$$\begin{aligned}
\int_{-\pi}^{\pi} |x|\sin|x|\,dx &= 2\int_{0}^{\pi} |x|\sin|x|\,dx \\
&= 2\left(\int_{0}^{\pi} x\sin x\,dx\right) \\
&= 2\left(\int_{0}^{\pi} x\,d(-\cos x)\right) \\
&= 2\left([-x\cos x]_0^{\pi} + \int_{0}^{\pi} \cos x\,dx\right) \\
&= 2\left([-x\cos x]_0^{\pi} + [\sin x]_0^{\pi}\right) \\
&= 2\pi
\end{aligned}$$

8. (Harder Problem) Let $a > 0$ and let $f(x)$ be an odd function. Show that

$$\int_{-a}^{a} f(x)\,dx = 0.$$

Hence, evaluate $\displaystyle\int_{-\pi}^{\pi} x^4 \tan 3x\,dx$.

**Ans:** Note that $\displaystyle\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$, so if we can show that $\displaystyle\int_{-a}^{0} f(x)\,dx = -\int_{0}^{a} f(x)\,dx$, then the result follows.

Let $y = -x$, then $dx = -dy$. When $x = -a$, $y = a$; when $x = 0$, $y = 0$.

$$\begin{aligned}
\int_{-a}^{0} f(x)\,dx &= \int_{a}^{0} -f(-y)\,dy \\
&= \int_{0}^{a} f(-y)\,dy \\
&= \int_{0}^{a} -f(y)\,dy \qquad (\because\ f(x) \text{ is an even function}) \\
&= -\int_{0}^{a} f(x)\,dx \qquad \text{(dummy variable)}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_{-a}^{0} f(x)\,dx &= -\int_{0}^{a} f(x)\,dx \\
\int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx &= -\int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx \\
\int_{-a}^{a} f(x)\,dx &= 0
\end{aligned}$$

Now, let $f(x) = x^4 \tan 3x$, we have

$$f(-x) = (-x)^4 \tan(-3x) = -x^4 \tan 3x = -f(x).$$

Therefore, $f(x) = x^4 \tan 3x$ is an odd function. By the above result,

$$\int_{-\pi}^{\pi} x^4 \tan 3x\,dx = 0.$$

**Volumes of Solids of Revolution**

9. Find the volume of the solid generated by revolving the area bounded by the graph of $y = \sin x$ and the $x$-axis between $x = 0$ and $x = \pi$ about the $x$-axis.

**Ans:** $\displaystyle\int_{0}^{\pi} \pi \sin^2 x\,dx = \frac{\pi^2}{2}$

10. Find the volume of the solid generated by revolving the area bounded by the graph of $y = x^2$ and the line $x + y - 6 = 0$ about the $x$-axis.

    **Ans:** $\displaystyle\int_{-3}^{2} \pi[(-x+6)^2 - (x^2)^2]\,dx = \frac{500\pi}{3}$

11. Find the volume of the solid generated by revolving the area bounded by the curves $y = x^2$ and $y^2 = x$ about

    (a) the $x$-axis.

    **Ans:** $\displaystyle\int_{0}^{1} \pi[(\sqrt{x})^2 - (x^2)^2]\,dx = \frac{3\pi}{10}$

    (b) the line $y = 1$.

    **Ans:** $\displaystyle\int_{0}^{1} \pi[(1-x^2)^2 - (1-\sqrt{x})^2]\,dx = \frac{11\pi}{30}$

12. Find the volume of the solid generated by revolving the area bounded by the curve $x - 2y + 4 = 0$ and $y^2 = x + 4$ about

    (a) the $x$-axis.

    **Ans:** $\displaystyle\int_{-4}^{0} \pi[(\sqrt{x+4})^2 - (\frac{x+4}{2})^2]\,dx = \frac{8\pi}{3}$

    (b) the $y$-axis.

    **Ans:** $\displaystyle\int_{0}^{2} \pi[(y^2-4)^2 - (2y-4)^2]\,dy = \frac{32\pi}{5}$

    (c) the line $x = -4$.

    **Ans:** $\displaystyle\int_{0}^{2} \pi[((2y-4)+4)^2 - ((y^2-4)+4)^2]\,dy = \frac{64\pi}{15}$