Calculus for Engineers

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Curve sketching technique

16.1 Motivation

Graphing Strategy for $y = f(x)$

Step 1 Analyze $f(x)$.

1. **Domain:** Find the domain of f.
   - The domain of f is the set of all real numbers x that produce real values for $f(x)$
   - This will be useful when finding vertical asymptotes and determining critical numbers.
2. **Intercepts:** Find the $x-$and $y-$intercepts of the function, if possible.
   - The y intercept of $f(0)$, if it exists; the x intercepts are the solutions to $f(x)$, if they exist.
3. **Symmetry:** Determine whether the function is an odd function, an even function or neither odd nor even.
   - If $f(-x) = f(x)$ for all x in the domain, then f is even and symmetric about the $y-$axis.
   - If $f(-x) = -f(x)$ for all x in the domain, then f is odd and symmetric about the origin.
4. **Asymptotes:** Find vertical, horizontal and slant asymptotes.
   - use Chapter 5, if they apply; otherwise, calculate limits at points of discontinuity and as x increases and decreases without bound.

Step 2 Analyze $f'(x)$.

1. **Critical points:** Find any critical points for $f(x)$ and any partition numbers for $f'(x)$.
   - Remember, every critical point for $f(x)$ is also a partition number for $f'(x)$, but some partition numbers for $f'(x)$ may not be critical points for $f(x)$.
2. **Intervals of Increase and Decrease:** Construct a sign chart for $f'(x)$, determine the intervals where $f(x)$ is increasing and decreasing, and find local maxima and minima.

3. **Local Maximum/Minimum:** Find the critical numbers of the function. Remember that the number c in the domain is a critical number if $f'(c) = 0$ or $f'(c)$ does not exist. Use the first derivative test to find the local maximums and minimums of the function.

Step 3 Analyze $f''(x)$. **Concavity and Points of Inflection:**

1. Construct a sign chart for $f''(x)$,
2. Determine where the graph of f is concave upward and concave downward, and
3. Find any inflection points.

Step 4 Sketch the graph of f.

1. Draw asymptotes and locates intercepts, local maxima and minima, and inflection points.
2. Sketch in what you know from Steps 1 to 3.
3. In regions of uncertainty, use point-by-point plotting to complete the graph.

16.2 Worked Examples

Example 1 Use the steps from the curve sketching technique to sketch the graph of

$$f(x) = \frac{x^2 + 5x + 4}{x^2}.$$

1. Find the domain.
2. Find the $x-$ and $y-$intercept(s).
3. Determine if the function is symmetrical.
4. Find the vertical/horizontal asymptote(s), if any.
5. Find the slant asymptote(s), if any.
6. Find the intervals of increase and decrease.
7. Find the local maximum(s) and minimum(s).
8. Find the intervals of concavity.
9. Find the point(s) of inflection.
10. Sketch the curve.

Solution:

1. Take
$$f(x) = \frac{x^2 + 5x + 4}{x^2}.$$
The domain is given by
$$\{x|\, x^2 \neq 0\} = \{x|\, x \neq 0\} = (-\infty, 0) \cup (0, +\infty).$$

2. To find the $x-$intercept, we let $y = 0$ and solve the equation for x :
$$0 = \frac{x^2 + 5x + 4}{x^2}$$
$$0 = x^2 + 5x + 4$$
$$0 = (x+1)(x+4)$$
$$x = -1, -4.$$
So, the function crosses the $x-$axis at $(-1, 0)$ and $(-4, 0)$. Since the function is not defined at $x = 0$, there is no $y-$intercept.

3. Take
$$f(x) = \frac{x^2 + 5x + 4}{x^2}.$$
To check for symmetry, we must find the equation for $f(-x)$.
$$f(-x) = \frac{(-x)^2 + 5(-x) + 4}{(-x)^2} = \frac{x^2 - 5x + 4}{x^2} \neq -f(x) \neq f(x).$$
Since $f(-x) \neq -f(x)$ and $f(-x) \neq f(x)$, the function is neither odd nor even. Therefore, the function is not symmetrical.

4. Since the function is undefined when $x = 0$, we must check the limit as x approaches 0, to see if there is a vertical asymptote.

   If there is a vertical asymptote, we want to check if the function approaches positive or negative infinity on either side of the asymptote:
$$\begin{aligned}
&\lim_{x \to 0^-} \frac{x^2 + 5x + 4}{x^2} \\
&= \frac{(-0.000\cdots01)^2 + 5(-0.000\cdots01) + 4}{(-0.000\cdots01)^2} \\
&= \frac{[\text{very small positive number}] + 5[\text{very small negative number}] + 4}{[\text{very small positive number}]} \\
&= \frac{[\text{positive number}]}{[\text{very small positive number}]} \\
&= +\infty.
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{x\to 0^+} \frac{x^2+5x+4}{x^2} \\
&= \frac{(0.000\cdots 01)^2 + 5(0.000\cdots 01) + 4}{(0.000\cdots 01)^2} \\
&= \frac{[\text{very small positive number}] + 5[\text{very small positive number}] + 4}{[\text{very small positive number}]} \\
&= \frac{[\text{positive number}]}{[\text{very small positive number}]} \\
&= +\infty.
\end{aligned}$$

Since $\lim_{x\to 0^-} f(x) = +\infty$, the line $x = 0$ is a vertical asymptote. The function approaches infinity on either side of the asymptote.

To find the horizontal asymptotes of the function, we must find the limit as x approaches positive and negative infinity.

$$\begin{aligned}
\lim_{x\to -\infty} \frac{x^2+5x+4}{x^2} &= \lim_{x\to -\infty} \frac{x^2+5x+4\left(\frac{1}{x^2}\right)}{x^2\left(\frac{1}{x^2}\right)} \\
&= \lim_{x\to -\infty} \frac{\frac{x^2}{x^2}+\frac{5x}{x^2}+\frac{4}{x^2}}{\frac{x^2}{x^2}} \\
&= \lim_{x\to -\infty} \frac{1+\frac{5}{x}+\frac{4}{x^2}}{1} \\
&= \frac{1+0+0}{1} \\
&= 1.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\lim_{x\to+\infty}\frac{x^2+5x+4}{x^2} &= \lim_{x\to+\infty}\frac{x^2+5x+4\left(\dfrac{1}{x^2}\right)}{x^2\left(\dfrac{1}{x^2}\right)}\\
&= \lim_{x\to+\infty}\frac{\dfrac{x^2}{x^2}+\dfrac{5x}{x^2}+\dfrac{4}{x^2}}{\dfrac{x^2}{x^2}}\\
&= \lim_{x\to+\infty}\frac{1+\dfrac{5}{x}+\dfrac{4}{x^2}}{1}\\
&= \frac{1+0+0}{1}\\
&= 1.
\end{aligned}$$

Therefore, the function has a horizontal asymptotes of $y = 1$. The function approaches the asymptote as x approaches positive and negative infinity.

5. To find the intervals of increase and decrease, we must take the first derivative of the function:

$$f(x) = \frac{x^2+5x+4}{x^2}.$$

$$\begin{aligned}
f'(x) &= \frac{(x^2)\dfrac{d}{dx}(x^2+5x+4)-(x^2+5x+4)\dfrac{d}{dx}(x^2)}{x^4}\\
&= \frac{-(5x+8)}{x^3}.
\end{aligned}$$

We must solve the equation $f'(x) = 0$ to find the endpoints of the interval

$$\begin{aligned}
0 &= \frac{-(5x+8)}{x^3}\\
0 &= -5x-8\\
x &= -\frac{8}{5}
\end{aligned}$$

The derivative $f'(x) = 0$ when $x = -\frac{8}{5}$. The derivative is undefined when $x = 0$. These two values give us the intervals in which the function will be constantly increasing or constantly decreasing. The intervals are calculated in the chart below:

Interval	$-(5x+8)$	x^3	$f'(x)$	f
$(-\infty, -8/5)$	$+$	$-$	$-$	decreasing $\searrow$
$(-8/5, 0)$	$-$	$-$	$+$	increasing $\nearrow$
$(0, +\infty)$	$-$	$+$	$-$	decreasing $\searrow$

So, the function is increasing on $(-\infty, -8/5)$ and decreasing $(-8/5, 0)$ and $(0, +\infty)$.

Interval	$-(5x+8)$	x^3	$f'(x)$	f
$(-\infty, -8/5)$	$+$	$-$	$-$	decreasing $\searrow$
$(-8/5, 0)$	$-$	$-$	$+$	increasing $\nearrow$
$(0, +\infty)$	$-$	$+$	$-$	decreasing $\searrow$

6. Since $f'(x) = 0$ when $x = -\frac{8}{5}$, it is a critical value of the function. Since $x = 0$ is not in the domain of the function, it is not a critical value.

   The chart above shows that at $x = -\frac{8}{5}$, the derivative changes from negative to positive. By the first-derivative test, there is a local minimum at $x = -\frac{8}{5}$.

$$f\left(-\frac{8}{5}\right) = \frac{\left(-\frac{8}{5}\right)^2 + 5\left(-\frac{8}{5}\right) + 4}{\left(-\frac{8}{5}\right)^2} = -\frac{9}{16}.$$

   So the function has a local minimum at the point $\left(-\frac{8}{5}, -\frac{9}{16}\right)$.

7. To find the intervals of concavity, we must take the second derivative of the function. By the quotient rule, we have

$$f''(x) = \frac{10x + 24}{x^4}.$$

   We must find the values of x for which $f''(x) = 0$ or $f''(x)$ is undefined, in order to find the endpoints of the interval of concavity. By inspection, $f''(x)$ is undefined when $x = 0$. We must determine when $f''(x) = 0$.

$$0 = \frac{10x + 24}{x^4}$$

$$0 = 10x + 24$$

$$x = -\frac{12}{5}$$

   These two values of x give us three intervals in which the function will have a constant concavity. The intervals are calculated in the chart below:

Interval	$10x + 24$	x^4	$f''(x)$	f
$(-\infty, -12/5)$	$-$	$+$	$-$	Concave downward
$(-12/5, 0)$	$+$	$+$	$+$	Concave upward
$(0, +\infty)$	$+$	$+$	$+$	Concave upward

Thus, the function is concave downward on $(-\infty, -12/5)$ and concave upward on $(-12/5, 0)$ and $(0, +\infty)$. Since the function changes concavity when $x = -\dfrac{12}{5}$, there is a point of inflection at that value of x.

Substituting $x = -\dfrac{12}{5}$ into $f(x)$, we have

$$f\left(-\frac{12}{5}\right) = \frac{\left(-\frac{12}{5}\right)^2 + 5\left(-\frac{12}{5}\right) + 4}{\left(-\frac{12}{5}\right)^2} = -\frac{7}{18}.$$

So the function has a point of inflection at $\left(-\dfrac{12}{5}, -\dfrac{7}{18}\right)$.

A sketch of the curve is shown below:

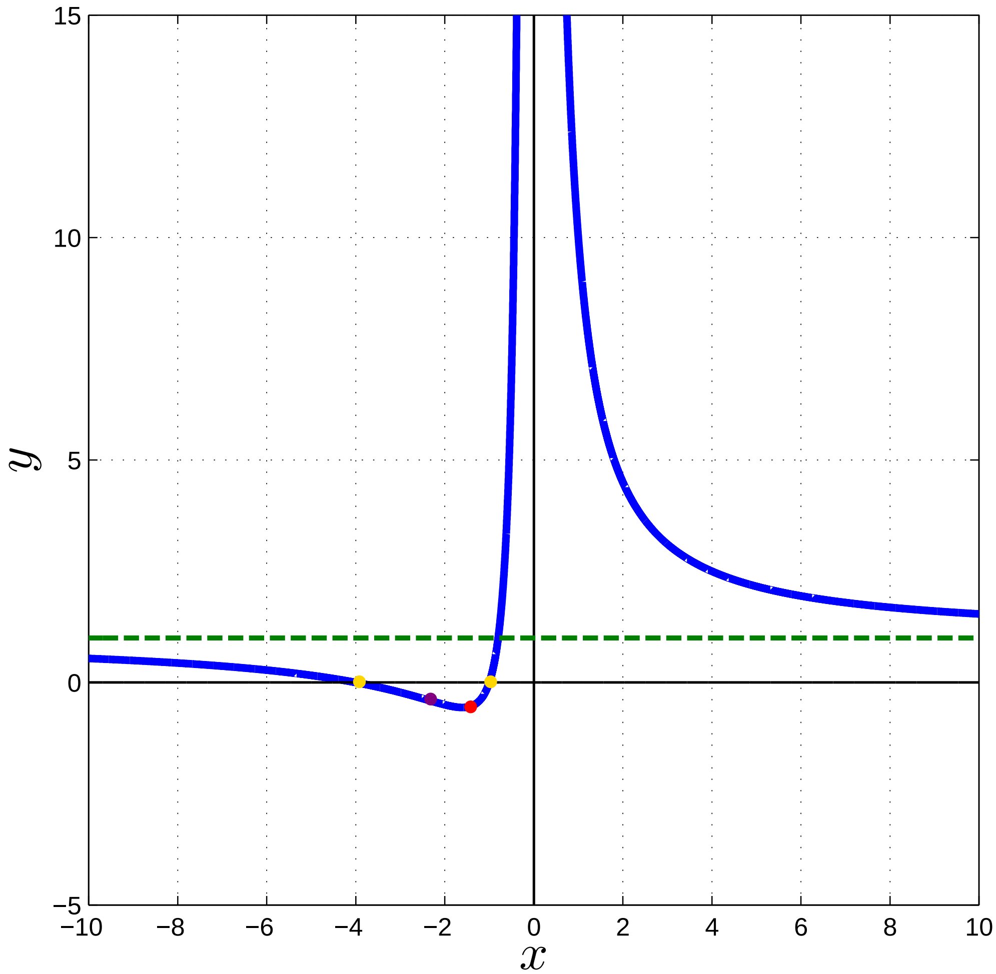

Figure 16.1: What do the dots mean?

□

Example 2 Sketch a graph of the following functions:

1. $f(x) = \dfrac{x^2+12}{2x+1}$;
2. $f(x) = 2 - x^{2/3} + x^{4/3}$.

If an interval is not specified, graph the function on its domain. Use a graphing utility to check your work. Answer the following questions, if any:

1. Identify the domain or interval of interest.
2. Exploit symmetry.
3. Find the first and second derivatives.

4. Find critical points and possible inflection points.

5. Find intervals on which the function is increasing/descreasing and concave up/down.

6. Identify extreme values and inflection points.

7. Locate vertical/horizontal/slant asymptotes and determine end behaviour.

8. Find the intercepts.

9. Choose an appropriate graphing window and make a graph.

Solutions:

1. Take $f(x) = \dfrac{x^2+12}{2x+1}$.

1. Domain of $f = \{x \in \mathbb{R} | x \neq -0.5\} = (-\infty, -0.5) \cup (-0.5, +\infty)$.

2. Note that $f(-x) = \dfrac{x^2+12}{-2x+1}$, then $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$ for some x. Let us take $x = 1$, then $f(-1) = -13$ and $f(1) = \frac{13}{3}$, so $f(-1) \neq f(1)$ and $f(-1) \neq -f(1)$. Thus, f is neither odd nor even.

3. Compute the first and the second derivatives of f:

$$f' = \frac{2x(2x+1) - 2(x^2+12)}{(2x+1)^2} = 2\frac{(x+4)(x-3)}{(2x+1)^2}$$

and

$$f'' = 2\left(\frac{x^2+x-12}{(2x+1)^2}\right)' = 2\frac{(2x+1)(2x+1)^2 - 2\cdot 2(2x+1)(x^2+x-12)}{(2x+1)^4} = \frac{98}{(2x+1)^3}.$$

4. Solve $f'(x) = 0$, then we get $x = -4$ and $x = 3$ are the critical points of f; Since $f'' = \frac{98}{(2x+1)^3}$, then $x = -0.5$ is the only one point that can change concavity sign from plus to minus or from minus to plus. Thus, $x = -0.5$ is the only one possible inflection point of f, but it's not in the domain. Thus, there exists no inflection point of f.

5. Here are the results

$x \leq -4$	$-4 < x < -0.5$	-0.5	$-0.5 < x < 3$	$x \geq 3$
Increasing	Decreasing	Not defined	Decreasing	Increasing
Concave downward	Concave downward	Not defined	Concave upward	Concave upward

6. Local maximum point: $(-4, -4)$, since $f''(-4) < 0$;

   Local minimum point: $(3, 3)$, since $f''(3) > 0$.

   There exists no inflection point. (See (4)).

7. (a) Vertical asymptote : $x = -0.5$.
Furthermore, we also compute

$$\lim_{x\to -0.5^+} f(x) = \lim_{x\to -0.5^+} \frac{x^2+12}{2x+1} = +\infty$$

and

$$\lim_{x\to -0.5^-} f(x) = \lim_{x\to -0.5^-} \frac{x^2+12}{2x+1} = -\infty.$$

This leads to the following geometrical observation:

- As $x \to -0.5^+$, $f(x) \to +\infty$ which means that the graph shoots upward as it approaches its vertical asymptote $x = -0.5$ from the right.
- As $x \to -0.5^-$, $f(x) \to -\infty$ which means that the graph shoots downward as it approaches its vertical asymptote $x = -0.5$ from the left.

(b) Slant asymptote: $y = \frac{1}{2}x - \frac{1}{4}$.
We investigate whether the function stabilizes toward a linear function as $x \to \pm\infty$.
By the long division, we have

$$\begin{array}{r|llll}
 & 2x & & -\frac{1}{4} & \\
\hline
2x+1 & x^2 & +2 & & \\
 & x^2 & +\frac{1}{2}x & & \\
\hline
 & & -\frac{1}{2}x & +12 & \\
 & & -\frac{1}{2}x & -\frac{1}{4} & \\
\hline
 & & & \frac{49}{4} &
\end{array}$$

So we have

$$f(x) = \frac{1}{2}x - \frac{1}{4} + \frac{49}{4(2x+1)}.$$

Then

$$\lim_{x\to +\infty} f(x) - \frac{1}{2}x - \frac{1}{4} = \lim_{x\to +\infty} \frac{49}{4(2x+1)} = 0$$

and

$$\lim_{x\to -\infty} f(x) - \frac{1}{2}x - \frac{1}{4} = \lim_{x\to -\infty} \frac{49}{4(2x+1)} = 0.$$

That is, $f(x)$ will behave like the line $y = \frac{1}{2}x - \frac{1}{4}$ as $x \to \pm\infty$.
Therefore, the slant asymptote is $y = \frac{1}{2}x - \frac{1}{4}$.

(c) Horizontal asymptote: There exists no horizontal asymptotes.
We investigate whether the function stabilizes toward a constant as $x \to \pm\infty$.
That is,

$$\lim_{x\to\pm\infty} \frac{x^2+12}{2x+1} = \lim_{x\to\pm\infty} \frac{x+\frac{12}{x}}{2+\frac{1}{x}} = \pm\infty.$$

That is, $f(x)$ goes to $+\infty$ as x goes to $+\infty$; $f(x)$ goes to $-\infty$ as x goes to $-\infty$. We know that there is no horizontal asymptotes.

8. There exists no $x-$intercept, since there is no solution for $f(x) = 0$. And $y-$intercept is $y = 12$, since $f(0) = 12$.

9. See Figure 16.2.

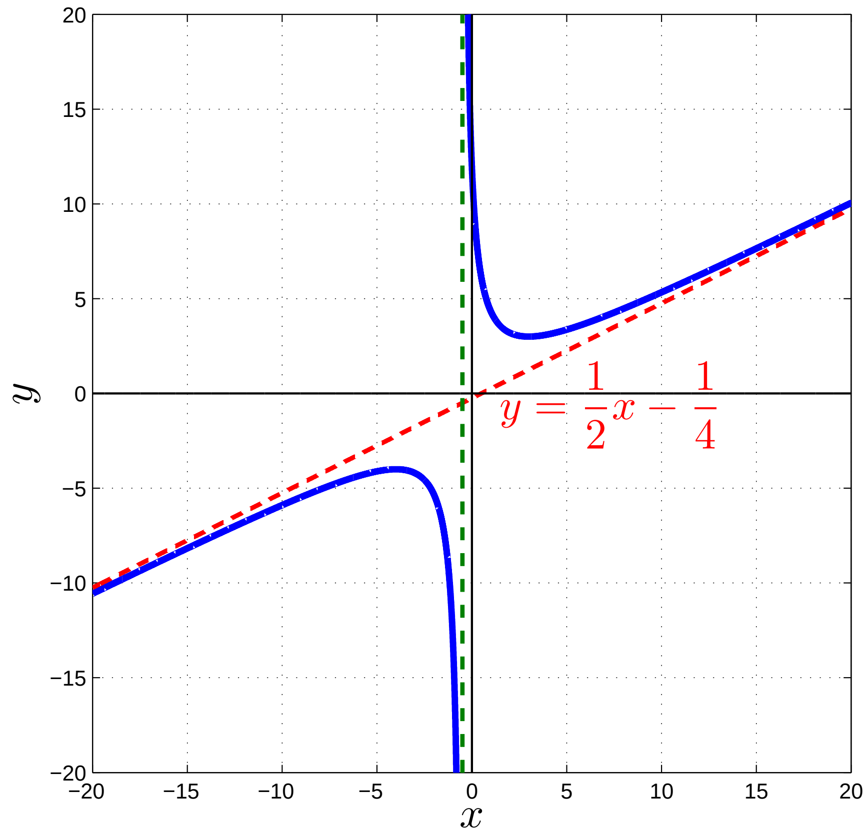

Figure 16.2:

2. Take $f(x) = 2 - x^{2/3} + x^{4/3}$.
Solutions:

1. Domain of $f = \mathbb{R} = (-\infty, +\infty)$.
2. Even. Since $f(x) = 2 - x^{2/3} + x^{4/3} = 2 - (-x)^{2/3} + (-x)^{4/3} = f(-x)$.
3. Compute the first and the second derivatives of f :
$$f' = -\frac{2}{3}x^{-\frac{1}{3}} + \frac{4}{3}x^{\frac{1}{3}}$$
and
$$f'' = \frac{2}{9}x^{-\frac{4}{3}} + \frac{4}{9}x^{-\frac{2}{3}}.$$
4. Critical points: $(-\frac{\sqrt{2}}{4}, \frac{7}{4})$ and $(\frac{\sqrt{2}}{4}, \frac{7}{4})$. Because solve $f'(x) = 0$, we get $x = -\frac{\sqrt{2}}{4}$ and $x = \frac{\sqrt{2}}{4}$.

   No inflection point. Because there is no solutions of $f''(x) = 0$.
5. Here are the results:

$x < -\frac{\sqrt{2}}{4}$	$-\frac{\sqrt{2}}{4}$	$-\frac{\sqrt{2}}{4} < x < 0$	0	$0 < x < \frac{\sqrt{2}}{4}$	$\frac{\sqrt{2}}{4}$	$x > \frac{\sqrt{2}}{4}$
Decreasing		Increasing		Decreasing		Increasing
Concave upward		Concave upward		Concave upward		Concave upward

6. Local minimum points: $(-\frac{\sqrt{2}}{4}, \frac{7}{4})$ and $(\frac{\sqrt{2}}{4}, \frac{7}{4})$, since $f''(-\frac{\sqrt{2}}{4})$ and $f''(\frac{\sqrt{2}}{4})$ are positive.

   Local maximum point: $(0, 2)$. Because when $x_0 = 0$, $f(0) > f(x), \forall x$ very close to $x_0 = 0$.

   No inflection point. (See (4)).
7. No any asymptotes. Do you know why?
   (a) Vertical asymptote: There is no such a point x_0 that $\lim\limits_{x\to x_0^{\pm}} f(x) = \pm\infty$.

   (b) Slant asymptote: No.
   We investigate whether the function stabilizes toward a linear function as $x \to \pm\infty$.
   Thm: $y = kx + b$ is the slant or horizontal asymptote (line) of f as $x \to +\infty$if and only if the following two limits exist:
$$k = \lim_{x\to+\infty} \frac{f(x)}{x}$$
   and
$$b = \lim_{x\to+\infty} (f(x) - kx).$$
   We compute $\lim\limits_{x\to-\infty} \frac{2-x^{2/3}+x^{4/3}}{x}$ and $\lim\limits_{x\to+\infty} \frac{2-x^{2/3}+x^{4/3}}{x}$, we get $\frac{2-x^{2/3}+x^{4/3}}{x} \to +\infty$ as $x \to \pm\infty$. Thus, there is no slant asymptote or horizontal one.

(c) (More reason) Horizontal asymptote: There are no horizontal asymptotes.
We investigate whether the function stabilizes toward a constant as $x \to \pm\infty$. That is,

$$\lim_{x\to\pm\infty} 2 - x^{2/3} + x^{4/3} = +\infty,$$

since the power of the positive term is bigger than the negative term's.
That is, $f(x)$ goes to $+\infty$ as x goes to $+\infty$; $f(x)$ goes to $+\infty$ as x goes to $-\infty$.We know that there is no horizontal asymptotes.

8. $y = 2$ is the $y-$intercept by computing $f(0)$. No $x-$intercepts, since there is no solution of $f(x) = 0$.

9. See Figures 16.3 and 16.4.

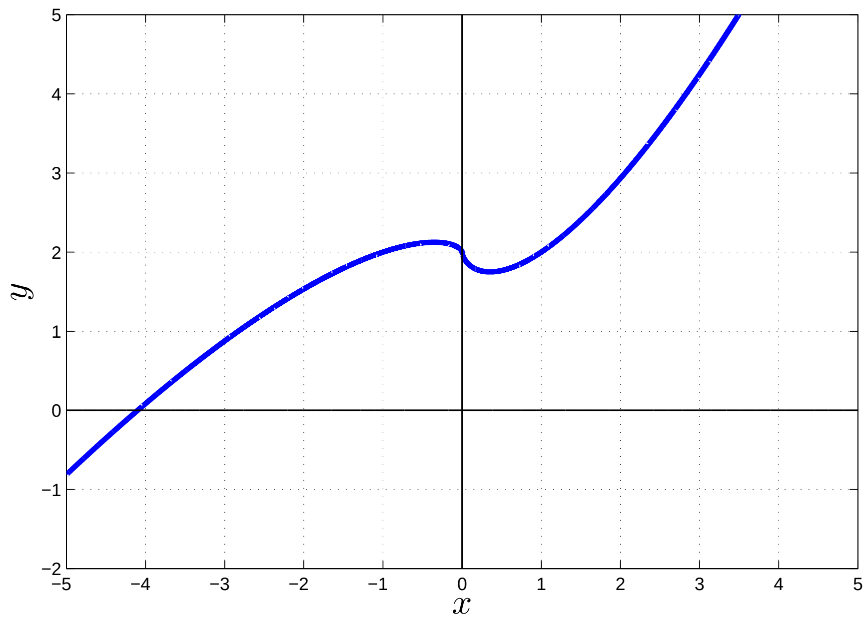

Figure 16.3: Graph of f, where $x \in [-5, 5]$.

$\square$

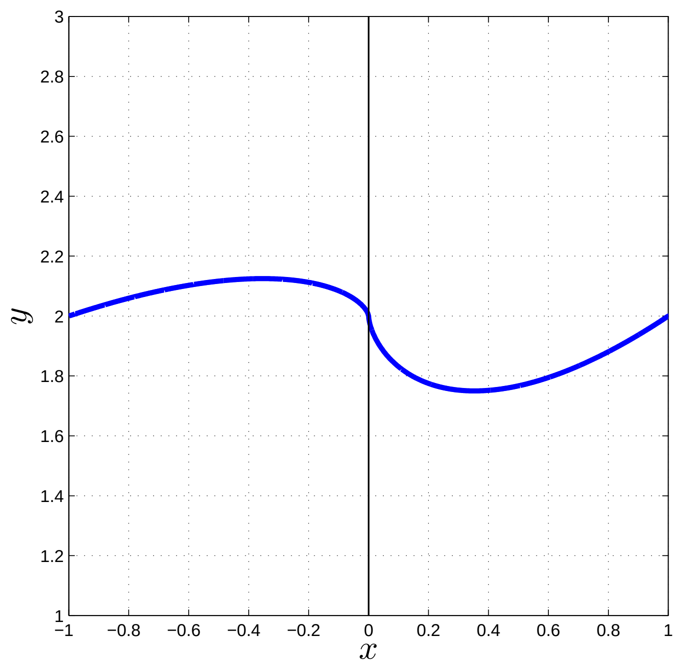

Figure 16.4: Graph of f, where $x \in [-1, 1]$.