THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1510 Calculus for Engineers (Fall 2021)
Homework 2
Deadline: October 16 at 23:00

Name: Chan Cho Kit, David Student No.: 1155175546

Class: MATH 1510 G

I acknowledge that I am aware of University policy and regulations on honesty in academic work, and of the disciplinary guidelines and procedures applicable to breaches of such policy and regulations, as contained in the website http://www.cuhk.edu.hk/policy/academichonesty/
David — Signature   1/10/2021 — Date

General Guidelines for Homework Submission.

- Please submit your answer to Gradescope through the centralized course MATH1510A-I in Blackboard.
- In Gradescope, for each question, please indicate exactly which page(s) its answer locates. **Answers of incorrectly matched questions will not be graded**.
- **Late submission will NOT be graded and result in zero score.** Any answers showing evidence of plagiarism will also score zero; stronger disciplinary action may also be taken.
- Points will only be awarded for answers with sufficient justifications.
- All questions in **Part A** along with some selected questions in **Part B** will be graded. Question(s) labeled with * are more challenging.

Part A:

1. Let $f(x) = \sin(2x + \pi)$. Use definition (first principle) to find $f'(x)$ for any $x \in \mathbb{R}$.

By definition of first principle, we have:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Let $f(x) = \sin(2x+\pi)$, we have:

$$f'(x) = \lim_{\Delta x \to 0} \frac{\sin(2x + 2\Delta x + \pi) - \sin(2x+\pi)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2\cos(2x+\Delta x+\pi)\sin(\Delta x)}{\Delta x}$$

$$= 1 \times \lim_{\Delta x \to 0} 2\cos(2x + \Delta x + \pi)$$

$$= 1 \times 2\cos(2x+\pi)$$

$$= 2\cos(2x+\pi) \ //$$

2. Let $\mathcal{C}$ be the curve defined by the equation $xy = \ln x + y^3$. Given that $A = (1, 0)$ is a point on $\mathcal{C}$,

   (a) Find $\dfrac{dy}{dx}$ in terms of x and y.

   (b) Find $\left.\dfrac{d^2y}{dx^2}\right|_A$.

(a)

$$xy = \ln x + y^3$$

$$xy - y^3 = \ln x$$

$$y + \frac{dy}{dx}(x - 3y^2) = \frac{1}{x}$$

$$\frac{dy}{dx}(x - 3y^2) = \frac{1 - xy}{x}$$

$$\frac{dy}{dx} = \frac{1 - xy}{x(x - 3y^2)} \quad //$$

(b)

$$\frac{dy}{dx} = \frac{1}{x(x-3y^2)} - \frac{y}{x-3y^2}$$

$$\left.\frac{dy}{dx}\right|_A = \frac{1}{(1-0)} - \frac{0}{1-0}$$

$$= 1$$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{-(x-3y^2) - x\left[1 - \frac{dy}{dx}(6y)\right]}{x^2(x-3y^2)^2}$$

$$- \frac{\frac{dy}{dx}(1)(x-3y^2) - y\left[1 - \frac{dy}{dx}(6y)\right]}{(x-3y^2)^2}$$

$$\left.\frac{d^2y}{dx^2}\right|_A = \frac{-(1) - (1-0)}{1^2(1-0)^2} - \frac{1-0}{(1-0)^2}$$

$$= -2 - 1$$

$$= -3 \quad //$$

Part B:

3. Determine the point(s) of discontinuity of the function:

$$f(x) = \begin{cases} x^2 + 3x - 1, & \text{if } x \leq 0, \\ \dfrac{\sin x}{x}, & \text{if } 0 < x \leq \pi, \\ \cos x + 1, & \text{if } \pi < x. \end{cases}$$

$\because \lim_{x \to 0^-} f(x) = 0^2 + 3(0) - 1$

$= -1$

$f(0) = -1$

$\lim_{x \to 0^+} f(x) = 1 \neq \lim_{x \to 0^-} f(x)$ & $f(0)$.

$\therefore$ The function is discontinuous when $x = 0$.

$\because \lim_{x \to \pi^-} f(x) = \frac{\sin \pi}{\pi}$

$= 0$

$f(\pi) = 0$

$\lim_{x \to \pi^+} f(x) = \cos \pi + 1$

$= 0$

$\because \lim_{x \to \pi^-} f(x) = f(\pi) = \lim_{x \to \pi^+} f(x)$,

$\therefore$ The function is continuous when $x = \pi$.

$\therefore$ There is one point of discontinuity when $x = 0$.

4. Find the derivative of

$$f(x) = \begin{cases} x^2 + \cos x & \text{if} \quad x < 0; \\ 1 & \text{if} \quad x = 0; \\ 2x \sin x + 1 & \text{if} \quad x > 0. \end{cases}$$

(Hint: You need to check the differentiability at 0.)

Let $y = f(x)$.

$\therefore \dfrac{dy}{dx} = 2x - \sin x \quad \text{if } x < 0,$

$\left.\dfrac{dy}{dx}\right|_{x=0^-} = -\sin(0) = 0$

& $\dfrac{dy}{dx} = 2\sin x + 2x\cos x \quad \text{if } x > 0,$

$\left.\dfrac{dy}{dx}\right|_{x=0^+} = 0 + 0 = 0$

$= \left.\dfrac{dy}{dx}\right|_{x=0^-}$

& $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^+} f(x) = f(0) = 1$

$$\therefore f'(x) = \begin{cases} 2x - \sin x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 2\sin x + 2x\cos x & \text{if } x > 0 \end{cases}$$ //

5. Find $\frac{dy}{dx}$ by logarithmic differentiation if

(a) $y = \frac{(x^2+5)^4}{(e^{-x}+2)\sqrt{x^4+1}}$;

(b) $y = x^{x+1}$, for $x > 0$.

(a) $\ln y = 4\ln(x^2+5) - \ln(e^{-x}+2)$
$\quad - \frac{1}{2}\ln(x^4+1)$

$\frac{d}{dx}(\ln y) = 4 \times \frac{1}{x^2+5}(2x) - \frac{1}{e^{-x}+2}(-e^{-x})$
$\quad - \frac{1}{2} \times \frac{1}{x^4+1}(4x^3)$

$\frac{dy}{dx}\left(\frac{1}{y}\right) = \frac{8x}{x^2+5} + \frac{e^{-x}}{e^{-x}+2} - \frac{2x^3}{x^4+1}$

$\frac{dy}{dx} = \left(\frac{8x}{x^2+5} + \frac{e^{-x}}{e^{-x}+2} - \frac{2x^3}{x^4+1}\right)\left[\frac{(x^2+5)^4}{(e^{-x}+2)\sqrt{x^4+1}}\right]$ //

(b) $y = x^{x+1}$

$\ln y = (x+1)\ln x$

$\frac{dy}{dx}\left(\frac{1}{y}\right) = \ln x + (x+1)\frac{1}{x}$

$\frac{dy}{dx} = (x\ln x + x + 1)\,x^{x}$ //

a, b

6. * Let a and b be real numbers with $a < b$. Show that the function

$$F(x) = (x-a)(x-b)^2 + x$$

takes on the value $\frac{a+b}{2}$ for some value of x.

$\because$ We have $a < \frac{a+b}{2} < b$,

$$F(a) = (a-a)(a-b)^2 + a$$
$$= a$$
$$F(b) = (b-a)(b-b)^2 + b$$
$$= b$$
$$> a$$

& $F(x)$ is continuous for all $x \in \mathbb{R}$. (including $[a,b]$) //

$\therefore$ By Intermediate Value Theorm, there exists at least one $c \in (a,b)$ such that

$$f(c) = (c-a)(c-b)^2 + c = \frac{a+b}{2}. \quad //$$

7. * Let u, v be functions of x. The first order derivative of uv can be obtained by the product rule:

$$(uv)' = u'v + uv'.$$

The general formula for n-th order derivative of uv was derived by the German mathematician Gottfried Wilhelm Leibniz:

$$(uv)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} \underline{u^{(k)}}\, \underline{v^{(n-k)}},$$

where $\binom{n}{k} = C_k^n = \frac{n!}{k!(n-k)!}$, the symbol $u^{(k)} = \frac{d^k u}{dx^k}$ means the k-th order derivative of u and $u^{(0)} = u$.

By Leibniz's formula, compute $f^{(100)}(x)$ if

$$f(x) = (2x^3 + 5x^2 - x + 3)\cos x.$$

$$f^{(100)}(x) = \sum_{k=0}^{100} C_k^{100} (2x^3+5x^2-x+3)^{(k)} (\cos x)^{(100-k)}$$

$$\begin{aligned} = \; & (2x^3+5x^2-x+3)(\cos x) \\ & + 100(6x^2+10x-1)(\sin x) \\ & + 4950(12x+10)(-\cos x) \\ & + 161700(12)(-\sin x) \\ & + 3921225(0)(\cos x) \\ & + \ldots \end{aligned}$$

$$\begin{aligned} = \; & \cos x (2x^3+5x^2-59401x-49497) \\ & + \sin x (600x^2+1000x-1940500) \end{aligned}$$

//