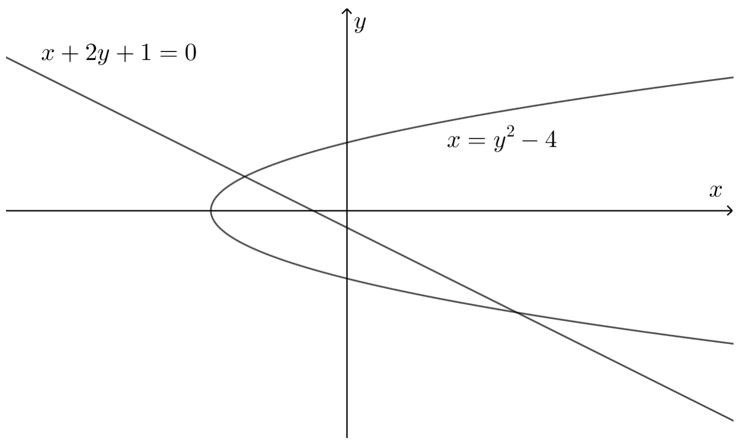
THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1510 Calculus for Engineers (Fall 2021)
Suggested solutions of coursework 11

Part A

1. Find the area of the region bounded by the curves:

$$x = y^2 - 4 \quad \text{and} \quad x + 2y + 1 = 0$$



Solution:

$$\begin{aligned} 0 &= x + 2y + 1 \\ &= (y^2 - 4) + 2y + 1 \\ &= y^2 + 2y - 3 \\ &= (y+3)(y-1) \end{aligned}$$

So, the intersections are $(5, -3)$ and $(-3, 1)$.

Thus, the area is

$$\begin{aligned} A &= \int_{-3}^{1} ((-2y - 1) - (y^2 - 4))\, dy \\ &= \int_{-3}^{1} (-y^2 - 2y + 3)\, dy \\ &= \left(-\frac{1}{3}y^3 - y^2 + 3y \right) \Bigg|_{-3}^{1} \\ &= \frac{32}{3}. \end{aligned}$$

2. Let R be the region bounded by the curves:

$$x = 0, \quad y = 0 \quad \text{and} \quad y = 2^{1-x} - 1$$

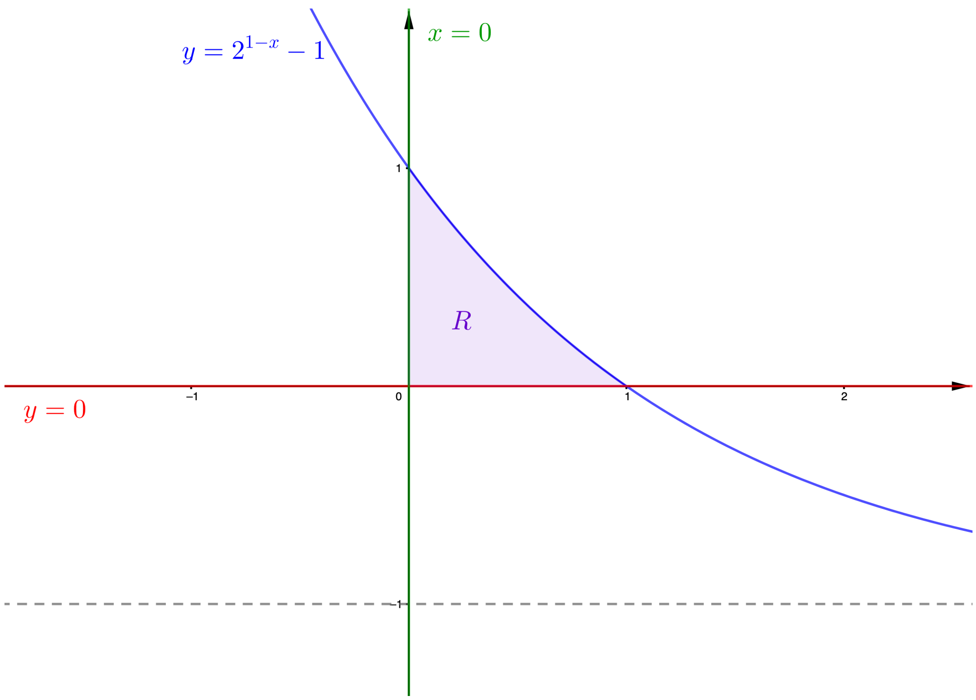
Write down a definite integral (do not evaluate) which computes the volume of the solid generated by rotating the region R about:

(a) the x-axis;

(b) the y-axis;

(c) the axis $x = 1$.

Solution:



(a)

$$V = \int_0^1 \pi(2^{1-x} - 1)^2 \, dx$$

(b)

$$\begin{aligned} y = 2^{1-x} - 1 &\implies \log_2(y+1) = 1 - x \\ &\implies x = 1 - \log_2(y+1) \end{aligned}$$

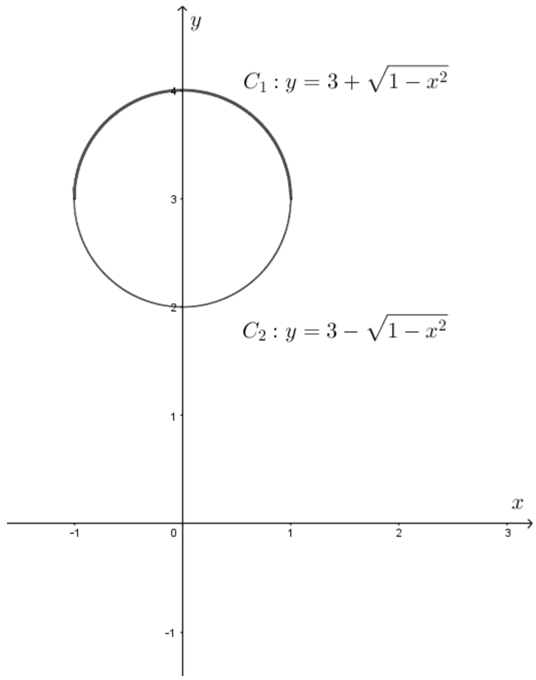
$$V = \int_0^1 \pi(1 - \log_2(y+1))^2 \, dy$$

(c)

$$V = \int_0^1 \left(\pi 1^2 - \pi(\log_2(y+1))^2\right) \, dy$$

Part B

3.



In the above diagram, the upper half and lower half of the circles are given by

$$C_1 : y = 3 + \sqrt{1-x^2} \quad \text{and} \quad C_2 : y = 3 - \sqrt{1-x^2}$$

respectively. Suppose the region enclosed by the circle is rotated about the x-axis, find the volume of the solid generated, which is called a **solid torus**.

Solution:

$$\begin{aligned} V &= \int_{-1}^{1} \left(\pi(3+\sqrt{1-x^2})^2 - \pi(3-\sqrt{1-x^2})^2 \right) dx \\ &= \pi \int_{-1}^{1} \left(9 + 6\sqrt{1-x^2} + (1-x^2) - 9 + 6\sqrt{1-x^2} - (1-x^2) \right) dx \\ &= 12\pi \int_{-1}^{1} \sqrt{1-x^2}\, dx \end{aligned}$$

Let $x = \sin\theta \implies dx = \cos\theta\, d\theta$. Then

$$\begin{aligned} V &= 12\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1-\sin^2\theta}\cos\theta\, d\theta \\ &= 12\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\, d\theta & (\cos\theta \geq 0) \\ &= 6\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1+\cos 2\theta)\, d\theta \\ &= 6\pi \left(\theta + \frac{1}{2}\sin 2\theta \right) \Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\ &= 6\pi \left(\frac{\pi}{2} + \frac{\pi}{2} \right) = 6\pi^2. \end{aligned}$$

4. Find the center and radius of convergence of each of the following power series.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{3^n}(x+1)^n$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}(2x+1)^n$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n!}x^n$

Solution:

(a) Center $= -1$, $c_n = \dfrac{(-1)^n n}{3^n}$.

$$R = \lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-1)^n n}{3^n}}{\frac{(-1)^{n+1}(n+1)}{3^{n+1}}}\right| = \lim_{n\to\infty}\frac{n}{n+1}\cdot 3 = 3.$$

So, radius of convergence is 3.

(b)

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}(2x+1)^n = \sum_{n=1}^{\infty}\frac{(-1)^n 2^n}{n!}(x+\frac{1}{2})^n$$

So, center $= -\dfrac{1}{2}$, $c_n = \dfrac{(-1)^n 2^n}{n!}$.

$$R = \lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-1)^n 2^n}{n!}}{\frac{(-1)^{n+1}2^{n+1}}{(n+1)!}}\right| = \lim_{n\to\infty}\frac{n+1}{2} = \infty \quad \text{(DNE)}.$$

So, radius of convergence is ∞.

(c) Center $= 0$, $c_n = \dfrac{\ln n}{n!}$.

$$R = \lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{\frac{\ln n}{n!}}{\frac{\ln(n+1)}{(n+1)!}}\right| = \lim_{n\to\infty}(n+1)\frac{\ln n}{\ln(n+1)} = \infty \quad \text{(DNE)},$$

because

$$\begin{aligned}
&\lim_{x\to\infty}\frac{\ln x}{\ln(x+1)} && \left(\frac{\infty}{\infty}\right)\\
&= \lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{x+1}} = \lim_{x\to\infty}\frac{x+1}{x} = 1\\
\Longrightarrow\ &\lim_{n\to\infty}\frac{\ln n}{\ln(n+1)} = 1.
\end{aligned}$$

So, radius of convergence is ∞.