Calculus for Engineers

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Contents

Definite Integrals

21.1 Introduction

In Section 21.2, we will study the concept of the definite integral. In Section 21.3, some important properties of definite integrals are presented. In Section 21.4, we will give a geometrical interpretation of a definite integral. Some properties of definite integrals are discussed in Section 21.5.2. In Section 21.5, we will consider a definite integral as the limit of the sum of a certain number terms, when the number of terms tends to infinity and each term tends to zero. We will use a definite integral as the limit of the sum to derive some properties of definite integrals. We also derive some properties of definite integrals as the limit of the sum. In Section 21.6, we will see that the definition of the definite integral as the limit of the sum enables us to express the limits of sums of series of a certain type as definite integrals and thus evaluate them easily. In Chapter 22, we will study the relationship between antiderivatives and definite integrals. It is this special relationship that allows us to easily compute the exact values of many definite integrals without ever using Riemann sums. This connection is often referred to as Part II of Fundamental Theorem of Calculus.

21.2 Definite integral

In geometrical and other applications of Integral Calculus, it becomes necessary to find the difference in the values of an integral of a function $f(x)$ for two assigned values of the independent variable x say, a and b. This difference is called the *definite integral of* $f(x)$ over the interval $[a, b]$ and is denoted by

$$\int_a^b f(x)dx.$$

Thus

$$\int_a^b f(x)dx = F(b) - F(a).$$

where $F(x)$ is an integral of $f(x)$. The difference $(F(b) - F(a))$ is sometimes dented as

$$F(x)\big|_a^b .$$

Thus if $F(x)$ is an integral of $f(x)$, we write

$$\int_a^b f(x)dx = F(x)|_a^b = F(b) - F(a).$$

The number, a, is called the *lower limit* and the number, b, the *upper limit* of integration.

It should be noted that the *value of a definite integral is unique and is independent of the particular integral which we may employ to calculate it.* To see how it works, let us consider $F(x)+C$ instead of $F(x)$. We obtain

$$\begin{aligned}\int_a^b f(x)dx &= (F(x)+C)|_a^b \\ &= (F(b)+C) - (F(a)+C) \\ &= F(b) - F(a),\end{aligned}$$

so that the arbitrary constant, C, disappears in the process and we get the same value as when we considered only $F(x)$.

Example 1

$$\int_1^2 xdx = \left.\frac{x^2}{2}\right|_1^2 = \frac{2^2}{2} - \frac{1^2}{2} = \frac{3}{2}.$$

□

Example 2

$$\int_0^1 \frac{1}{1+x^2}dx = \tan^{-1} x\Big|_0^1 = \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4} - 0 = \frac{\pi}{4}.$$

□

Example 3

$$\int_0^{\pi/2} \sin xdx = -\cos x|_0^{\pi/2} = -\cos\frac{\pi}{2} + \cos 0 = 1.$$

□

21.3 Some important properties of definite integrals

We now give one definition and three theorems on the basic properties of definite integrals.

Since regions of zero width have zero area, integrals with equal limits of integration are defined to be zero.

Definition 1 (Zero Width Interval) For any function $f(x)$,

$$\int_a^a f(x)dx = 0. \tag{21.1}$$

□

Note 1 Definition 1 states that the area over a point is zero.

Example 4 What is the value of $\int_{10}^{10} x^2 dx$?

Solution. Definition 1 immediately gives $\int_{10}^{10} x^2 dx = 0$. □

Proposition 1 (Order of integration) If the integral of f from a to b is defined with $a < b$, then

$$\int_a^b f(x)dx = -\int_b^a f(x)dx.$$

Proof. Let $F(x)$ be an integral of $f(x)$, so that we have

$$\int_b^a f(x)dx = F(b) - F(a).$$

We have

$$\begin{aligned}\int_b^a f(x)dx &= F(x)|_b^a \\ &= F(a) - F(b) \\ &= -(F(b) - F(a)) \\ &= -\int_a^b f(x)dx.\end{aligned}$$

Hence the result is verified.

□

Proposition 2 (Integrals over adjacent intervals) If f is piecewise continuous on an interval containing the numbers a, b, and c, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx,$$

where c, is a point inside or outside the interval $[a, b]$.

Proof. Let $F(x)$ be an integral of $f(x)$, so that we have

$$\int_b^a f(x)dx = F(b) - F(a).$$

By considering the following results, illustrated in 21.1,

$$\begin{cases} \int_a^c f(x)dx = F(x)|_a^c = F(c) - F(a), \\ \int_c^b f(x)dx = F(x)|_c^b = F(b) - F(c), \end{cases}$$

we have

$$\begin{aligned} \int_a^c f(x)dx + \int_c^b f(x)dx &= (F(c) - F(a)) + (F(b) - F(c)) \\ &= F(b) - F(a) \\ &= \int_a^b f(x)dx. \end{aligned}$$

Hence the result is verified.

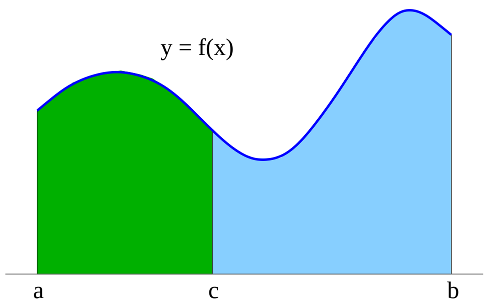


Figure 21.1: Integrals over adjacent intervals.

□

Note 2 Proposition 1 states that interchanging the limits of a definite integral does not change the absolute value but changes only the sign of the integral (or an integral from b to a with $a < b$ is defined to be the negative of the integral from a to b), while Proposition 2 states that it is possible to combine integrals of the same function over adjacent intervals. $\square$

Note 3 Proposition 2 also holds true even if the point c is exterior to the interval $[a, b]$, as shown in Figure 21.2.

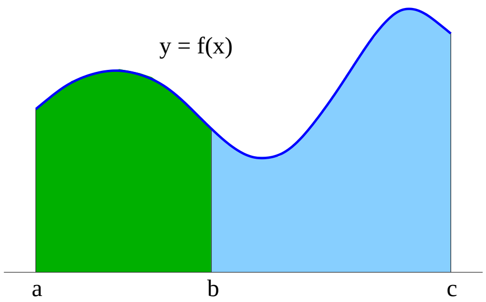


Figure 21.2: The point c is exterior to the interval $[a, b]$.

Example 5 Show that the following results are identical

$$\int_0^2 x^2 dx = \int_0^1 x^2 dx + \int_1^2 x^2 dx$$

and

$$\int_0^2 x^2 dx = \int_0^3 x^2 dx + \int_3^2 x^2 dx.$$

Solution. Exercise. We point out the fact that the function x^2 is integrable on any closed interval $[0, x]$. $\square$

Note 4 In place of one additional point c, we can take several points. Thus, an extension of Proposition 2 is

$$\int_a^b \phi(x)dx = \int_a^{c_1} \phi(x)dx + \int_{c_1}^{c_2} \phi(x)dx + \int_{c_2}^{c_3} \phi(x)dx + \cdots + \int_{c_{r-1}}^{c_r} \phi(x)dx + \cdots + \int_{c_n}^{b} \phi(x)dx.$$

Example 6 Given that $\int_0^1 x^2 dx = \frac{1}{3}$. What is the value of $\int_1^0 x^2 dx$?

Solution. Proposition 1 gives $\int_1^0 x^2 dx = -\int_0^1 x^2 dx = -\frac{1}{3}$. $\square$

Example 7 What is the integral of g from 1 to 8 if its integral from 1 to 5 is -7 and its integral from 5 to 8 is 9?

Solution. Proposition 2 gives

$$\int_1^8 g(x)dx = \int_1^5 g(x)dx + \int_5^8 g(x)dx = -7 + 9 = 2.$$

□

Proposition 3 (Integrals of linear combinations) If f and g are piecewise continuous on an interval containing a to b, then for any constants α and β,

$$\int_a^b (\alpha f(x) + \beta g(x))\, dx = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx.$$

Proof. Exercise. □

Example 8 What is $\int_{-35}^{35} (2p(x) - 2q(x))\, dx$ if $\int_{-35}^{35} p(x)dx = 10$ and $\int_{-35}^{35} q(x)dx = 20$?

Solution. Proposition 3 gives

$$\int_{-35}^{35} (2p(x) - 2q(x))\, dx = 2\int_{-35}^{35} p(x)dx - 2\int_{-35}^{35} q(x)dx = 2 \cdot 10 - 2 \cdot 20 = -20.$$

□

21.4 Geometrical interpretation of a definite integral

Example 9 Show that the definite integral

$$\int_a^b f(x)dx$$

denotes the area bounded by the curve $y = f(x)$, the axis of x, and the two ordinates $x = a$ and $x = b$.

Solution.

Let $y = f(x)$ be the equation of a curve referred to by two rectangular axes.

Let, $\mathcal{A}$, denote the area bounded by the curve, the axis of x, a fixed ordinate AG, $(OA = a)$, and a variable ordinate MP.

Let $OM = x$ so that

$$MP = y = f(x).$$

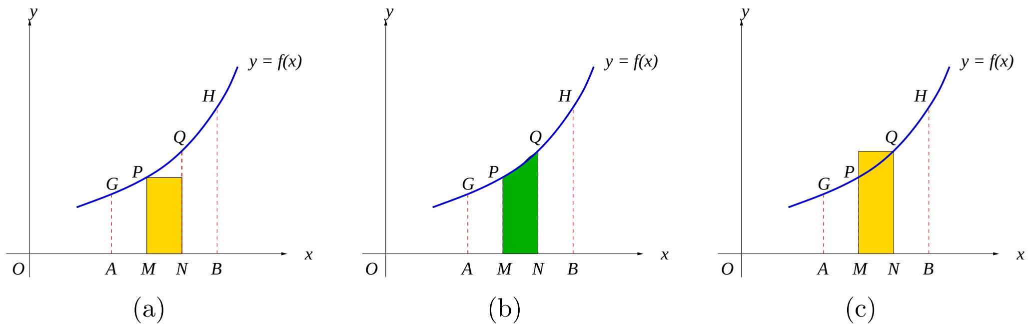


Figure 21.3: Example 9.

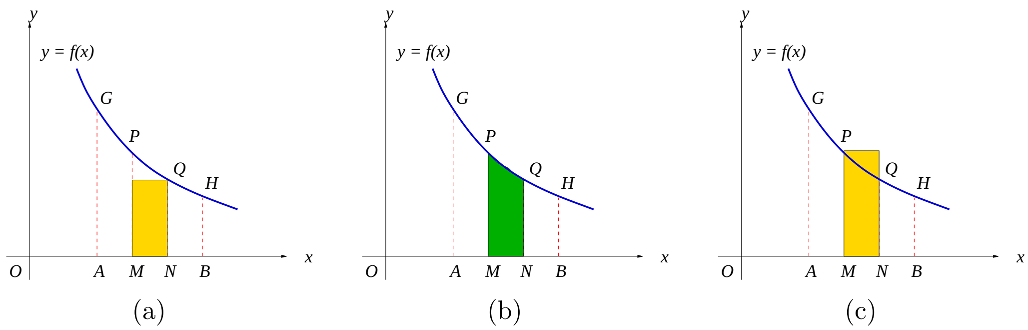


Figure 21.4: Example 9.

The area $\mathcal{A}$, depends on the position of the ordinate MP whose abscissa is x, and is, therefore, a function of x.

We take a point $Q(x + \Delta x, y + \Delta y)$ on the curve which lies so near P that, as a point moves along the curve from P to Q, its ordinate either *constantly* increases (see Figure 21.3) or constantly decreases (see Figure 21.4).

We have

$$\begin{aligned} ON &= x + \Delta x, \\ NQ &= y + \Delta y, \\ MN &= \Delta x. \end{aligned}$$

The increment $\Delta\mathcal{A}$ in $\mathcal{A}$, a consequence of the change Δx in x, is the area of the region $MNQPM$.

The area $\Delta\mathcal{A}$ of the figure $MNQPM$ (green region) lies between the area $y\Delta x$ and $(y + \Delta y)\Delta x$ of the two rectangles (see Figures 21.3 (a) and (c) or Figures 21.4 (a) and (c)).

For Figure 21.3, we have

$$(y + \Delta y)\Delta x > \Delta\mathcal{A} > y\Delta x.$$

Then

$$(y + \Delta y) > \frac{\Delta\mathcal{A}}{\Delta x} > y. \tag{21.2}$$

Let $Q \to P$ so that $\Delta x \to 0$. Then from (21.2), we obtain

$$\frac{d\mathcal{A}}{dx} = y = f(x).$$

For Figure 21.4, we have

$$y\Delta x > \Delta\mathcal{A} > (y + \Delta y)\Delta x.$$

Then

$$y > \frac{\Delta\mathcal{A}}{\Delta x} > (y + \Delta y),$$

so that, for this case also, we obtain the limit

$$\frac{d\mathcal{A}}{dx} = y = f(x).$$

Let BH be the ordinate $x = b$. We have

$$\begin{aligned}
\int_a^b f(x)dx &= \int_a^b \frac{d\mathcal{A}}{dx}dx \\
&= \mathcal{A}|_a^b \\
&= \text{the value of } \mathcal{A} \text{ for } x \text{ equal to } b - \text{the value of } \mathcal{A} \text{ for } x \text{ equal to } a \\
&= \text{area of the region } GABHGA - 0 \\
&= \text{area of the region } GABHGA,
\end{aligned}$$

which is the area bounded by the curve $y = f(x)$, the $x-$axis and the two ordinates $x = a$ and $x = b$.

□

Note 5 The definition of the area-function, $\mathcal{A}$, as given above, is not complete. To adequately define $\mathcal{A}$ in order to cover all possible cases, we agree to define $\mathcal{A}$ as the algebraic sum of the areas of all the portions enclosed by the curve, the axis of x and the two ordinates; each portion being equipped with a proper sign $+$ (in blue) or $-$, (in red) according to the following convention:

1. the areas of the portions to the right of the fixed ordinate GA lying above x-axis and also the areas of the portions to the left of GA lying below x-axis are positive, as shown in Figure 21.5 (a);

2. the areas of the portions to the left of GA lying above x-axis and the areas of the portions to the right of GA lying below x-axis are negative, as shown in Figure 21.5 (b).

It is easy to show that by applying these conventions to the meaning of the areas $\mathcal{A}$, the result of Section 21.4 holds no matter what the portion of the variable ordinate MP is relative to GA.

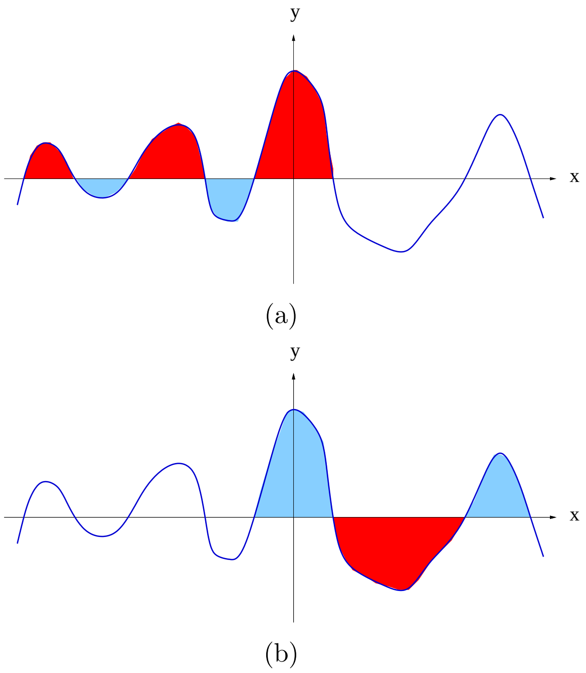


Figure 21.5: Graphs of Note 5.

21.5 Definite integral as the limit of a sum.

Definition 2 If $f(x)$ is defined on the closed interval $[a, b]$ and the limit

$$\lim_{\text{mesh}\to 0} \sum_{k=1}^{n} f(c_k) \cdot \Delta x_k$$

exists (as described in Chapter 20), then $f(x)$ is integrable on $[a, b]$ and the limit is denoted by

$$\lim_{\text{mesh}\to 0} \sum_{k=1}^{n} f(c_k) \cdot \Delta x_k = \int_a^b f(x)dx.$$

The limit is called the definite integral of f from a to b. The number a is the lower limit of integration, and the number b is the upper limit of integration. □

Note 6 Definition 2 also means that if the function $f(x)$ monotonically increases from a to b, the interval $[a, b]$ is divided into n equal parts and the length of each part is h so that $nh = b - a$, then

$$\int_a^b f(x)dx = \lim h[f(a + h) + f(a + 2h) + \cdots + f(a + nh)],$$

when $n \to \infty, h \to 0$ and $nh = b - a$ (one proceeds to limit as $h \to 0$ only after putting $nh = b - a$). Here

$$a, a + h, a + 2h, \cdots, a + (k - 1)h, \cdots, a + (n - 1)h, a + nh$$

are the points of division obtained when the interval $[a, b]$ is divided into n equal parts; h being the length of each part. More precisely, we have

$$\int_a^b f(x)dx = \lim \sum_{k=1}^{n} hf(a + kh),$$

when $h \to 0, n \to \infty$ and $nh = b - a$.

Note 7 We state that the result remains true in the case when $f(x)$ monotonically *decreases* as x increases from a to b.

Note 8 It should be noted that, since each term in $\sum hf(a+kh)$ tends to 0, the addition or omission of a certain finite number of terms of a similar type will not alter the limit.

Example 10 Evaluate $\int_a^b x^2 dx$ as the limit of a sum.

Solution. Here $f(x) = x^2$. So

$$\begin{aligned}
\int_a^b x^2 dx &= \lim h[f(a+h) + f(a+2h) + \cdots + f(a+nh)] \\
&\quad \text{when } h \to 0,\ n \to \infty \text{ and } nh = b-a, \\
&= \lim h[(a+h)^2 + (a+2h)^2 + \cdots + (a+nh)^2] \\
&= \lim h\left[na^2 + 2ah\underbrace{(1+2+\cdots+n)}_{?} + h^2\underbrace{(1^2+2^2+\cdots+n^2)}_{?}\right] \\
&= \lim h\left[na^2 + n(n+1)ah + \frac{1}{6}n(n+1)(2n+1)h^2\right] \\
&= \lim\left[nha^2 + nh(nh+h)a + \frac{1}{6}nh(nh+h)(2nh+h)\right] \\
&= \lim\left[(b-a)a^2 + (b-a)(b-a+h)a + \frac{1}{6}(b-a)(b-a+h)(2(b-a+h))\right] \\
&= (b-a)a^2 + (b-a)^2a + \frac{1}{3}(b-a)^3 \\
&= (b-a)\left(a^2 + (b-a)a + \frac{1}{3}(b-a)^2\right) \\
&= (b-a)\left(a^2 + (b-a)\left(a + \frac{1}{3}(b-a)\right)\right) \\
&= (b-a)\left(a^2 + (b-a)\frac{(b+2a)}{3}\right) \\
&= (b-a)\left(\frac{3a^2 + b^2 + 2ab - ab - 2a^2}{3}\right) \\
&= \frac{(b-a)(a^2+b^2+ab)}{3} \\
&= \frac{1}{3}(b^3 - a^3).
\end{aligned}$$

□

Example 11 Evaluate $\int_a^b e^x dx$ as the limit of a sum.

Solution. Here $f(x) = e^x$. So

$$\begin{aligned}\int_a^b e^x dx &= \lim h[e^{a+h} + e^{a+2h} + \cdots + e^{a+nh}],\\ &\qquad \text{when } h \to 0,\ n \to \infty \text{ and } nh = b - a,\\ &= \lim \frac{he^{a+h}(1 - e^{nh})}{1 - e^h}\\ &= \lim \left[e^a \cdot (1 - e^{b-a})e^h \cdot \frac{h}{1 - e^h}\right]\\ &= (e^a - e^b) \cdot \lim e^h \cdot \lim \frac{h}{1 - e^h}\\ &= (e^a - e^b)(-1)\\ &= e^b - e^a,\end{aligned}$$

for, $\lim \dfrac{h}{1 - e^h} = -1$, as $h \to 0$. □

Example 12 Evaluate $\int_a^b \cos x dx$ as the limit of a sum.

Solution. Here $f(x) = \cos x$. Therefore

$$\int_a^b \cos x dx = \lim h[\cos(a + h) + \cos(a + 2h) + \cdots + \cos(a + nh)].$$

Let

$$S = \cos(a + h) + \cos(a + 2h) + \cdots + \cos(a + nh). \tag{21.3}$$

Multiplying both sides of (21.3) by $2 \sin \left(\frac{1}{2}h\right)$, we get

$$\begin{aligned}2 \sin \left(\frac{1}{2}h\right) \cdot S &= 2 \sin \left(\frac{1}{2}h\right) \cos(a + h) + 2 \sin \frac{1}{2}h \cos(a + 2h)\\ &\qquad + \cdots + 2 \sin \left(\frac{1}{2}h\right) \cos(a + nh)\\ &= \sin \left(a + \frac{3}{2}h\right) - \sin \left(a + \frac{1}{2}h\right) + \sin \left(a + \frac{5}{2}h\right)\\ &\quad - \sin \left(a + \frac{3}{2}h\right) + \cdots + \sin \left(a + \frac{1}{2}(2n + 1)h\right) - \sin \left(a + \frac{1}{2}(2n - 1)h\right)\\ &= \sin \left(a + \frac{1}{2}(2n + 1)h\right) - \sin \left(a + \frac{1}{2}h\right)\\ &= \sin \left(b + \frac{1}{2}h\right) - \sin \left(a + \frac{1}{2}h\right), \quad \text{for } nh = b - a.\end{aligned}$$

Thus

$$\begin{aligned}\int_a^b \cos x dx &= \lim \frac{h\left(\sin\left(b+\frac{1}{2}h\right) - \sin\left(a+\frac{1}{2}h\right)\right)}{2\sin\frac{1}{2}h}\\ &= \lim \frac{\frac{1}{2}h}{\sin\left(\frac{1}{2}h\right)}\left(\sin\left(b+\frac{1}{2}h\right) - \sin\left(a+\frac{1}{2}h\right)\right)\\ &= 1\cdot(\sin b - \sin a)\\ &= \sin b - \sin a.\end{aligned}$$

□

21.5.1 Properties of definite integrals: revisited

Being familiar with the basic properties of finite sums and limits many of the properties listed below are easy to understand.

Proposition 4 Suppose $\phi(x) = \alpha$ is a constant function on $[a, b]$, where $\alpha \in \mathbb{R}$. Then

$$\int_a^b \alpha dx = \alpha \cdot (b - a).$$

Proof. We provide two ways of getting the result of Proposition 4.

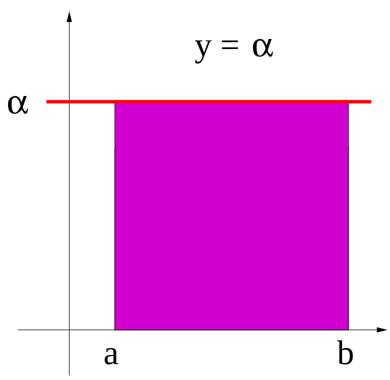


Figure 21.6: Proposition 4.

1. Using an area approach, if $k > 0$, then

$$\int_a^b \alpha dx$$

represents the area of the rectangle with base $= b - a$ and height k, thus

$$\int_a^b \alpha dx = (\text{height}) \cdot (\text{base}) = \alpha \cdot (b - a).$$

2. Using Riemann sums, for every partition

$$\{x_0 = a, x_1, x_2, x_3, \cdots, x_{n-1}, x_n = b\}$$

of the interval $[a, b]$, and every choice of c_k, the Riemann sum is

$$\begin{aligned}
\sum_{k=1}^{n} f(c_k) \cdot \Delta x_k = \sum_{k=1}^{n} \alpha \cdot \Delta x_k = \alpha \cdot \sum_{k=1}^{n} \Delta x_k \\
&= \alpha \cdot \sum_{k=1}^{n} (x_k - x_{k-1}) \\
&= \alpha \cdot ((x_1 - x_0) + (x_2 - x_1) + (x_3 - x_2) + \cdots + (x_{n-1} - x_n)) \\
&= \alpha \cdot (x_n - x_0) \\
&= \alpha \cdot (b - a).
\end{aligned}$$

□

Note 9 Proposition 4 states that the integral of a constant function $f(x) = \alpha$ is the constant α multiplied by the length of the interval.

Proposition 5 Suppose $\phi(x)$ is an integrable function on $[a, b]$, and $\beta \in \mathbb{R}$ is a constant. Then

$$\int_a^b \beta \cdot \phi(x) dx = \beta \cdot \int_a^b \phi(x) dx.$$

Proof. Using Riemann sums, we have

$$\begin{aligned}
\int_a^b \beta \cdot \phi(x) dx &= \lim_{\text{mesh}\emptyset 0} \sum_{k=1}^{n} \beta \cdot f(c_k) \cdot \Delta x_k \\
&= \beta \cdot \left(\lim_{\text{mesh}\emptyset 0} \sum_{k=1}^{n} f(c_k) \cdot \Delta x_k \right) \\
&= \beta \cdot \int_a^b \phi(x) dx.
\end{aligned}$$

□

Note 10 Proposition 5 states that the integral of a constant β times a function is the constant times the integral of that function. This means that a constant (and a constant alone) can be moved in front of the integral sign before integrating.

Proposition 6 Suppose f and g are integrable functions on $[a,b]$. Then so are their sum and difference, and

$$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx.$$

Proof. Using Riemann sums, we have

$$\begin{aligned}
\int_a^b (f(x) \pm g(x))\,dx &= \lim_{\text{mesh}\to 0} \sum_{k=1}^{n} (f(c_k) \pm g(c_k)) \cdot \Delta x_k \\
&\lim_{\text{mesh}\to 0} \left(\sum_{k=1}^{n} f(c_k) \pm \sum_{k=1}^{n} g(c_k) \right) \cdot \Delta x_k \\
&= \lim_{\text{mesh}\to 0} \sum_{k=1}^{n} f(c_k) \cdot \Delta x_k \pm \lim_{\text{mesh}\to 0} \sum_{k=1}^{n} g(c_k) \cdot \Delta x_k \\
&= \int_a^b f(x)dx \pm \int_a^b g(x)dx.
\end{aligned}$$

□

Note 11 Proposition 6 states that the integral of a sum is the sum of those integrals and the integral of a difference is the difference of those integrals. In other words the area under $f+g$ is the area under f plus the area under g just as the area under $f-g$ is the area under f minus the area under g. Figure 21.7 shows that the values of integrals of sums of functions are equal to the sums of integrals of the individual functions.

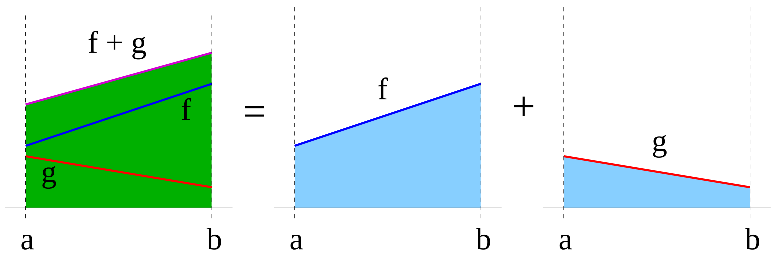


Figure 21.7: Proposition 6.

Proposition 7 (Domination) Suppose f and g are integrable functions on $[a,b]$. If $f(x) \leq g(x)$ for all x in $[a,b]$, then

$$\int_a^b f(x)(x)dx \leq \int_a^b g(x)(x)dx.$$

Proof. Using Riemann Sums, if the same partition and sampling points c_k are used to get Riemann sums for f and g, then $f(c_k) \le g(c_k)$ for each k and

$$\sum_{k=1}^{n} f(c_k) \cdot \Delta x_k \le \sum_{k=1}^{n} g(c_k) \cdot \Delta x_k.$$

Thus

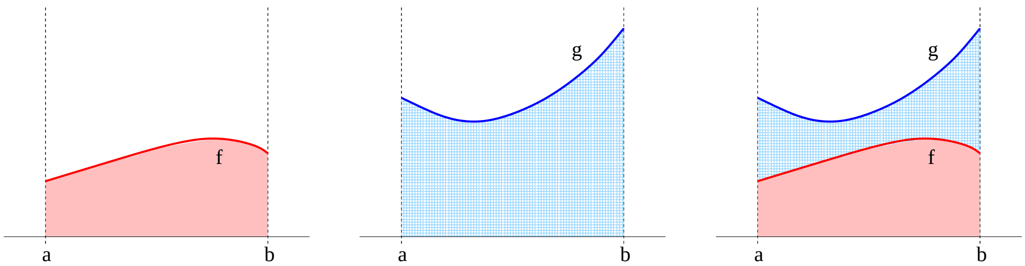
$$\lim_{\text{mesh}\to 0} \left(\sum_{k=1}^{n} f(c_k) \cdot \Delta x_k \right) \le \lim_{\text{mesh}\to 0} \left(\sum_{k=1}^{n} g(c_k) \cdot \Delta x_k \right).$$

□

Note 12 Proposition 7 says that if one function is larger than another function on an interval, then the definite integral of the larger function on that interval is bigger than the definite integral of the smaller function.

Note 13 Figure 21.8 illustrates that if f and g are both positive and $f(x) \le g(x)$ for all x in $[a, b]$, then the area of region F is smaller than the area of region G and

$$\int_a^b f(x)(x)dx \le \int_a^b g(x)(x)dx.$$



(a) the area of region F. (b) the area of region G. (c) $\int_a^b f(x)(x)dx \le \int_a^b g(x)(x)dx$.

Figure 21.8: Proposition 7.

Proposition 8 (Max-Min Inequality) Suppose f and g are integrable functions on $[a, b]$. If f has a maximum value M and a minimum value m, then

$$(b-a) \cdot m \le \int_a^b f(x)(x)dx \le (b-a) \cdot M.$$

Proof. Proposition 8 follows easily from Proposition 7.

Let $g(x) = M =$ (max of f on $[a, b]$). Then $f(x) \leq M = g(x)$ for all x in $[a, b]$. Then

$$\int_a^b f(x)(x)dx \leq \int_a^b g(x)(x)dx = \int_a^b M dx = (b-a) \cdot M.$$

Similarly, let $h(x) = m =$ (min of f on $[a, b]$). Then $h(x) = m \leq f(x)$ for all x in $[a, b]$. Then

$$(b-a) \cdot m = \int_a^b m(x)dx = \int_a^b h(x)(x)dx \leq \int_a^b f(x)(x)dx.$$

$\square$

Note 14 Proposition 7 leads to Proposition 8 which provides a quick method for determining the size of the bounds.

Note 15 Figure 21.9 illustrates Proposition 8.

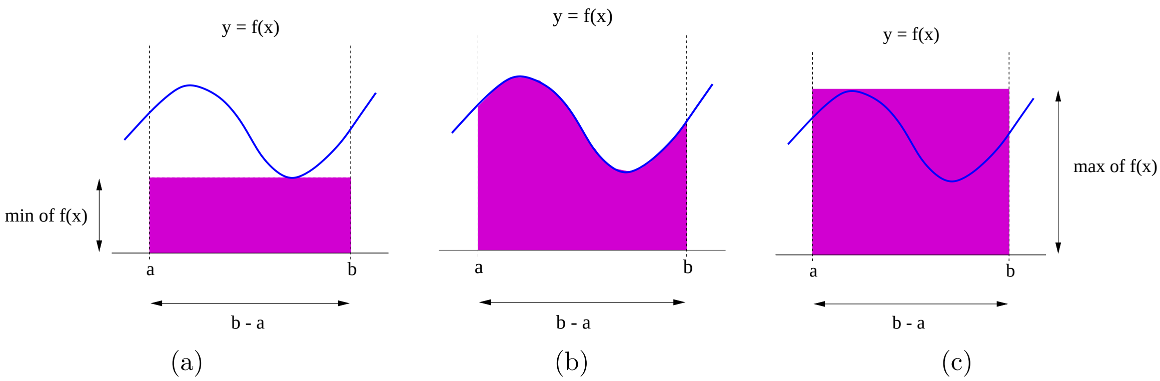


Figure 21.9: $(b-a) \cdot (\text{min of } f) \leq \int_a^b f(x)(x)dx \leq (b-a) \cdot (\text{max of } f)$.

Note 16 Similar sketches for the situations when f or g are either sometimes or always negative illustrate that Proposition 8 is always true, but we can avoid all of the different cases by using Riemann sums.

Example 13 Determine the lower and upper bounds for the value of $\int_1^5 f(x)dx$ in Figure 21.10.

Solution. If $1 \leq x \leq 5$, then $2 \leq f(x) \leq 9$. Thus a lower bound is

$$\text{min of } f \text{ on } [a, b] = 4 \cdot 2 = 8$$

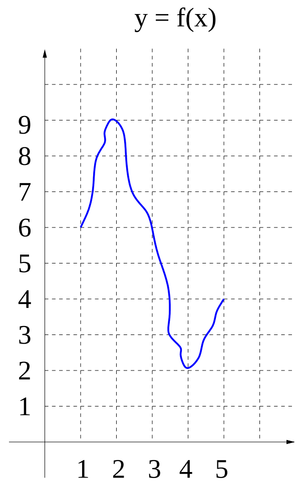


Figure 21.10: Example 13.

and an upper bound is

$$\max \text{ of } f \text{ on } [a,b] = 4 \cdot 9 = 36.$$

Hence,

$$8 \leq \int_a^b f(x)dx \leq 36.$$

This range, from 8 to 36 is rather wide. Proposition 8 is not useful for finding the exact value of the integral, but it is very straight-forward to use and it can help us avoid an unreasonable value for an integral.

□

21.5.2 Further properties of definite integrals

Proposition 9

$$\int_a^b \phi(x)dx = \int_a^b \phi(t)dt.$$

Proof. Let

$$\int \phi(x)dx = F(x) + C_1$$

and

$$\int \phi(t)dt = F(t) + C_2.$$

Then

$$\int_a^b \phi(x)dx = F(x) + C_1|_a^b = F(b) - F(a). \tag{21.4}$$

Similarly,

$$\int_a^b \phi(t)dt = F(t) + C_2|_a^b = F(b) - F(a). \tag{21.5}$$

From (21.4) and (21.5), we have

$$\int_a^b \phi(x)dx = \int_a^b \phi(t)dt.$$

□

Note 17 Proposition 9 states that the value of a definite integral does not change with the change of the variable of integration (also called the 'argument') provided the limits of integration remain the same.

Proposition 10

$$\int_0^a \phi(x)dx = \int_0^a \phi(a-x)dx.$$

Proof. Let

$$\text{RHS} = \int_0^a \phi(a-x)dx.$$

Putting $a - x = t$ so that $-dx = dt$, we have

- the upper limit for $x = 0$ and $t = 0$, and
- the lower limit for $x = 0$ and $t = a$.

Therefore

$$\begin{aligned}
\text{RHS} &= -\int_a^0 \phi(t)dt \\
&= \int_0^a \phi(t)dt && \text{Proposition 1} \\
&= \int_0^a \phi(x)dx && \text{Proposition 9} \\
&= \text{LHS}.
\end{aligned}$$

□

Proposition 11

$$\int_0^{2a} \phi(x)dx = \begin{cases} 2\int_0^{a} \phi(x)dx, & \text{if} \quad \phi(2a-x)=\phi(x) \\ 0 & \text{if} \quad \phi(2a-x)=-\phi(x). \end{cases}$$

Proof. We have

$$\text{LHS} = \int_0^{2a} \phi(x)dx = \int_0^{a} \phi(x)dx + \int_a^{2a} \phi(x)dx.$$

Now consider

$$\int_a^{2a} \phi(x)dx.$$

If we let $2a - x = t$ or $2a - t = x$, then $-dx = dt$. We have

- the upper limit for $x = 2a$ and $t = 0$, and
- the lower limit for $x = a$ and $t = a$.

Therefore,

$$\begin{aligned} \int_a^{2a} \phi(x)dx &= -\int_a^{0} \phi(2a-t)dt \\ &= \int_0^{a} \phi(2a-t)dt \\ &= \int_0^{a} \phi(2a-x)dx \end{aligned}$$

Then

$$\text{LHS} = \int_0^{a} \phi(2a-x)dx + \int_0^{a} \phi(x)dx.$$

Case I: Now, if $\phi(2a-x) = \phi(x)$. Then

$$\begin{aligned} \int_0^{2a} \phi(x)dx &= \int_0^{a} \phi(x)dx + \int_0^{a} \phi(x)dx \\ &= 2\int_0^{a} \phi(x)dx. \end{aligned}$$

Case II: Again, if $\phi(2a-x) = -\phi(x)$. Then

$$\begin{aligned}\int_0^{2a} \phi(x)dx &= \int_0^{a} \phi(x)dx - \int_0^{a} \phi(x)dx \\ &= 0.\end{aligned}$$

□

Note 18 Before going on to study Proposition 12, let us recall the definitions for odd and even functions: A function $f(x)$ is said to be

1. an odd function of x if $f(-x) = -f(x)$,
2. an even function of x if $f(-x) = f(x)$.

Proposition 12

$$\begin{cases} \int_{-a}^{a} \phi(x)dx = 0, & \text{if} \quad \phi(-x) = -\phi(x) \\ 2\int_0^{a} \phi(x)dx, & \text{if} \quad \phi(-x) = \phi(x) \end{cases}$$

Proof. By Proposition 2, we have

$$\int_{-a}^{a} \phi(x)dx = \int_{-a}^{0} \phi(x)dx + \int_0^{a} \phi(x)dx.$$

Let us evaluate the following integral

$$\int_{-a}^{0} \phi(x)dx.$$

Put $x = -t$ so that $dx = -dt$. We have

- the upper limit for $x = 0$ and $t = 0$, and
- the lower limit for $x = -a$ and $t = a$.

Therefore

$$\begin{aligned}\int_{-a}^{0} \phi(x)dx = -\int_{-a}^{0} \phi(-t)dt &= \int_0^{a} \phi(-t)dt \\ &= \int_0^{a} \phi(-x)dx\end{aligned}$$

and

$$\int_{-a}^{a} \phi(x)dx = \int_{0}^{a} \phi(-x)dx + \int_{0}^{a} \phi(x)dx.$$

Two cases are:

Case I: If $\phi(-x) = -\phi(x)$, then

$$\begin{aligned}\int_{-a}^{a} \phi(x)dx &= -\int_{0}^{a} \phi(x)dx + \int_{0}^{a} \phi(x)dx \\ &= 0.\end{aligned}$$

Case II: Again, if $\phi(-x) = \phi(x)$, then

$$\begin{aligned}\int_{a}^{a} \phi(x)dx &= \int_{0}^{a} \phi(x)dx + \int_{0}^{a} \phi(x)dx \\ &= 2\int_{0}^{a} \phi(x)dx.\end{aligned}$$

□

Note 19 The following results should be committed to memory.

1.

$$\begin{aligned}\int_{0}^{\pi/2} f(\sin x)dx &= \int_{0}^{\pi/2} f\left(\sin\left(\frac{\pi}{2} - x\right)\right) dx \\ &= \int_{0}^{\pi/2} f(\cos x)dx.\end{aligned}$$

2.

$$\begin{aligned}\int_{0}^{\pi/2} f(\sin 2x)\cos x dx &= \int_{0}^{\pi/2} f\left(\sin 2\left(\frac{\pi}{2} - x\right)\right) \cos\left(\frac{\pi}{2} - x\right) dx \\ &= \int_{0}^{\pi/2} f(\sin 2x)\sin x dx.\end{aligned}$$

3.

$$\int_0^{\pi/2} \sin^n x dx = \int_0^{\pi/2} \sin^n \left(\frac{\pi}{2} - x\right) dx$$
$$= \int_0^{\pi/2} \cos^n x dx.$$

These propositions are applied when solving problems with definite integrals. Some equations are being solved here to illustrate the procedure.

Example 14 Show that

$$\int_0^{\pi/2} \log_e \sin x dx = \frac{\pi}{2} \log_e \frac{1}{x}.$$

Solution. Let the value of the given integral be I, so that

$$I = \int_0^{\pi/2} \log_e \sin x dx$$
$$= v \int_0^{\pi/2} \log_e \sin \left(\frac{\pi}{2} - x\right) dx \qquad \text{Proposition 10}$$
$$= \int_0^{\pi/2} \log_e \cos x dx$$

so that

$$2I = \int_0^{\pi/2} \log_e \sin x dx + \int_0^{\pi/2} \log_e \cos x dx$$
$$= \int_0^{\pi/2} \log_e (\sin x \cos x) dx$$
$$= \int_0^{\pi/2} \log_e \left(\frac{\sin 2x}{2}\right) dx$$
$$= \int_0^{\pi/2} \log_e \frac{\sin 2x}{d} x - \int_0^{\pi/2} \log_e 2 dx.$$

Regarding the first integral, let us put $2x = t$ so that $dx = \frac{1}{2}dt$ and the new limits shall be 0 and π, so

$$\begin{aligned}
\int_0^{\pi/2} \log_e \frac{\sin 2x}{d} x &= \int_0^{\pi} \log_e \frac{1}{2} \sin t dt \\
&= \frac{1}{2} \int_0^{\pi} \log_e \sin t dt \qquad \text{Proposition } 5 \\
&= \int_0^{\pi/2} \log_e \sin t dt
\end{aligned}$$

The last expression is taken from the fact that

$$\int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx \quad \text{if } f(2a-x) = f(x).$$

Hence

$$\int_0^{\pi/2} \log_e \frac{\sin 2x}{d} x = I.$$

So

$$2I = I - \log_e 2 \cdot \int_0^{\pi/2} dx$$

and

$$I = -\frac{\pi}{2} \log_e 2 \quad \text{or} \quad \frac{\pi}{2} \log_e \frac{1}{2}.$$

□

21.5.3 Functions defined by integrals

If one of the endpoints a or b of the interval $[a, b]$ changes, then the value of the integral $\int_a^b f(t)dt$ typically changes. A definite integral of the form $\int_a^b f(t)dt$ defines a function of x, and functions defined by definite integrals in this way have interesting and useful properties. The next examples illustrate one of them: the derivative of a function defined by an integral is closely related to the integrand, the function "inside" the integral.

Example 15 For the function $f(t) = 2$, define $F(x)$ to be the area of the region bounded by f, the $t-$axis, and vertical lines at $t = 1$ and $t = x$, as shown in Figure 21.11.

1. Evaluate $F(1), F(2), F(3)$, and $F(4)$.

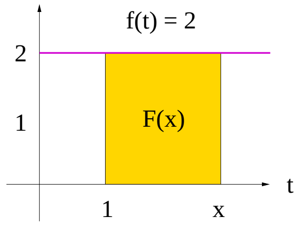


Figure 21.11: Example 15.

2. Find an algebraic formula for $F(x)$ for $x \geq 1$.

3. Calculate $\dfrac{d}{dx}F(x)$.

4. Describe $F(x)$ as a definite integral.

Solution.

1. $F(1) = 0, F(2) = 2, F(3) = 4$, and $F(4) = 6$.

2. $F(x) =$ area of a rectangle $=$ (base) $\cdot$ (height) $= (x?1) \cdot (2) = 2x?2$.

3. $\dfrac{d}{dx}F(x) = \frac{d}{dx}(2x - 2) = 2$.

4. $F(x) = \displaystyle\int_1^x 2dt$.

□

Example 16 For the function $f(t) = 1 + t$, define $G(x)$ to be the area of the region bounded by the graph of f, the $t-$axis, and vertical lines at $t = 0$ and $t = x$, as shown in Figure 21.12.

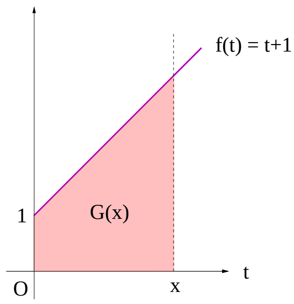


Figure 21.12: Example 16.

1. Evaluate $G(0), G(1), G(2)$, and $G(3)$.

2. Find an algebraic formula for $G(x)$ for $x \geq 0$.

3. Calculate $\frac{d}{dx}G(x)$.

4. Describe $G(x)$ as a definite integral.

Solution.

1. $G(0) = 0, G(1) = 1.5, G(2) = 4$, and $G(3) = 7.5$

2. $G(x) =$ area of trapezoid $=$ (base)$\cdot$(average height) $= (x) \cdot \left(\frac{1+(1+x)}{2}\right) = x + \frac{x^2}{2}$.

3. $\frac{d}{dx}G(x) = \frac{d}{dx}\left(x + \frac{x^2}{2}\right) = 1 + x$.

4. $G(x) = \int\limits_0^x (1+t)dt$.

$\square$

21.5.4 Which functions are integrable?

It was Riemann in 1850 who proved that a function must be badly discontinuous to not be integrable. In turn, we have the following Theorem 13 (given without proof) as follows.

Theorem 13 (Every continuous function is integrable) If f is continuous on the interval $[a, b]$, then

$$\lim_{\text{mesh}\to 0} \sum_{k=1}^{n} f(c_k) \cdot \Delta x_k$$

is always the same finite number, $\int\limits_a^b f(x)dx$, so f is integrable on $[a, b]$. $\square$

In fact, a function can even have any finite number of breaks and still be integrable.

Theorem 14 (Every bounded, piecewise continuous function is integrable) If f is defined and bounded ($K \leq f(x) \leq K$ for some K) for all x in $[a, b]$ and continuous except at a finite number of points in $[a, b]$, then

$$\lim_{\text{mesh}\to 0} \sum_{k=1}^{n} f(c_k) \cdot \Delta x_k$$

is always the same finite number, $\int\limits_a^b f(x)dx$, so f is integrable on $[a, b]$. $\square$

Example 17 In Figure 21.13, the function f is always in ? and 3, that is, $f \in [-3, 3]$(in fact, always between ? and 3). Then it is bounded, and it is continuous except at 2 and 3. As long as the values of $f(2)$ and $f(3)$ are finite numbers, their actual values will not effect the value of the definite integral, and

$$\int_0^5 f(x)dx = 0 + 3 + 2 = 5.$$

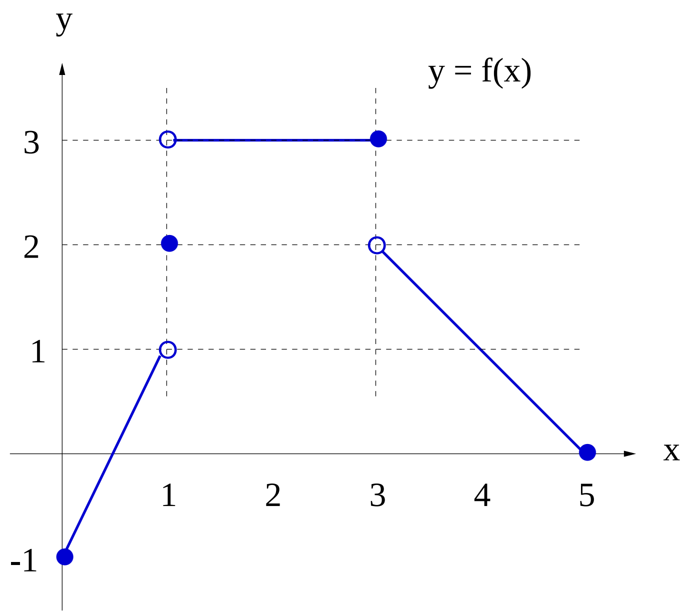


Figure 21.13: Example 17.

□

In Figure 21.14, the relationships among differentiable, continuous, and integrable functions are summarized as follows:

- Every differentiable function is continuous, but there are continuous functions which are not differentiable. (consider $f(x) = |x|$ is which continuous but not differentiable at $x = 0$.)

- Every continuous function is integrable, but there are integrable functions which are not continuous. (see Example 17: the function in Figure 21.13 is integrable on $[0, 5]$ but is not continuous at 2 or 3.)

- Finally, as shown in Example 18, there are functions which are not integrable.

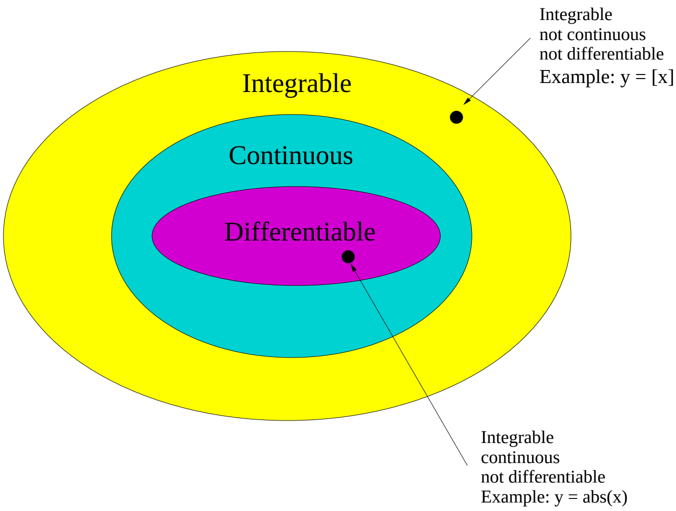


Figure 21.14: The relationships among differentiable, continuous, and integrable functions.

21.5.5 A non-integrable function

If f is continuous or piecewise continuous on $[a, b]$, then f is integrable on $[a, b]$. Fortunately, the functions we have discussed and will use in later chapters are all integrable. However, there are functions for which the limit of the Riemann sums does not exist, and those functions are not integrable.

Example 18 Show that the function

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is a rational number,} \\ 2 & \text{if } x \text{ is an irrational number,} \end{cases}$$

is not integrable on $[0, 3]$. Figure 21.15 shows the graph of f.

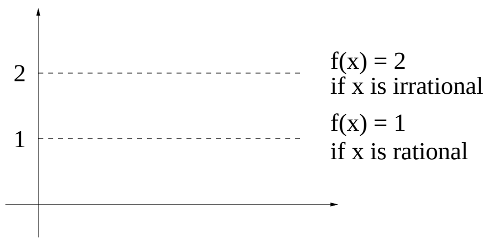


Figure 21.15: Example 18.

Proof. For any partition P, we consider the following two cases:

- Suppose one always selects values of c_k which are rational numbers. (Every subinterval contains rational numbers and irrational numbers, so one we can always pick c_k to be a rational number.) Then $f(ck) = 1$, and let RS be a Riemann sum of the case above:

$$RS_P = \sum_{k=1}^{n} f(c_k)\cdot\Delta x_k = \sum_{k=1}^{n} 1\cdot\Delta x_k = 1\cdot\sum_{k=1}^{n}\Delta x_k 3.$$

- Suppose however, one always selects values of c_k which are irrational numbers. Then $f(c_k) = 2$, and let IRS be a Riemann sum of the case above:

$$IRS_P = \sum_{k=1}^{n} f(c_k)\cdot\Delta x_k = \sum_{k=1}^{n} 2\cdot\Delta x_k = 2\cdot\sum_{k=1}^{n}\Delta x_k = 6.$$

Then

$$\lim_{\text{mesh}\to 0} RS_P = 3 \quad \text{and} \quad \lim_{\text{mesh}\to 0} IRS_P = 6.$$

So

$$\lim_{\text{mesh}\to 0}\sum_{k=1}^{n} f(c_k)\cdot\Delta x_k$$

does not exist, and this f is not integrable on $[0,3]$ or on any other interval either.

□

21.6 Summation of series

It is possible to express the limits of sums of certain types of series as definite integrals, and thus to evaluate them. We shall now consider the characteristics of the series as the limit of whose sum can be so expressed and also learn how to obtain the corresponding definite integrals. Our aim here is to transform such a series so that the lower and upper limits of the corresponding definite integral are 0 and 1 respectively.

We have seen that when $h\to 0, n\to\infty$ and $nh = b-a$,

$$\int_a^b f(x)dx = \lim h[f(a+h)+f(a+2h)+\cdots+f(a+kh)+\cdots+f(a+nh)].$$

Changing h to $\dfrac{(b-a)}{n}$, we see that when $n\to\infty$, we have

$$\begin{aligned}\int_a^b f(x)dx &= (b-a)\lim\frac{1}{n}\left[f\left(a+(b-a)\frac{1}{n}\right)+f\left(a+(b-a)\frac{2}{n}\right)+\cdots\right.\\ &\qquad\left.+f\left(a+(b-a)\frac{k}{n}\right)+\cdots+f\left(a+(b-a)\frac{n}{n}\right)\right]\\ &= (b-a)\lim\frac{1}{n}\sum_{k=1}^{k=n} f\left(a+(b-a)\frac{k}{n}\right).\end{aligned}$$

We can easily see that when $n \to \infty$, we have

$$\lim \frac{1}{n} \sum_{k=1}^{n} f\left(a + (b-a)\frac{k}{n}\right) = \int_0^1 f(a + (b-a)x)dx.$$

Thus

$$\int_a^b f(x)dx = (b-a) \int_0^1 f(a+(b-a)x)dx.$$

We now notice the following:

- The general term $\dfrac{1}{n} f\left(a + (b-a)\dfrac{k}{n}\right)$ of the sum

$$\frac{1}{n} \sum f\left(a + (b-a)\frac{k}{n}\right)$$

  is a function of $\dfrac{k}{n}$ such that the various terms in the series are obtained by giving values $1, 2, 3, \cdots, n$ to k.

- The new integrand $f(a + (b-a)x)$ is obtained by changing $\dfrac{k}{n}$ to x.

- $\dfrac{1}{n}$ is a factor of each term.

From the above we deduce that in the case of a series the limit of whose sum can be expressed as a definite integral, the general term is the product of $\dfrac{1}{n}$ and a function $\varphi\left(\dfrac{k}{n}\right)$ of $\dfrac{k}{n}n$, so that the various terms of the series can be obtained from it by changing k to $1, 2, 3, \cdots, n$, successively. The equivalent definite integral, then, is obtained by *changing* $\left(\dfrac{k}{n}\right)$, *in the general term, to* x *and taking* 0, 1 *as the two limits.*

A rule of thumb: Thus it is important to remember that

$$\lim_{n\to\infty} \left[\sum_{k=1}^{n} \frac{1}{n} \varphi\left(\frac{k}{n}\right)\right] = \int_0^1 \varphi(x)dx.$$

Note 20 Here the number of terms is n but, since each term tends to 0, the addition or omission of a *finite* number of terms will not affect the required limit.

Example 19 Find the limit when n tends to infinity of the following series.

$$\frac{n^2}{(n^2+1)^{3/2}} + \frac{n^2}{(n^2+2^2)^{3/2}} + \frac{n^2}{(n^2+3^2)^{3/2}} + \cdots + \frac{n^2}{(n^2+(n-1)^2)^{3/2}}.$$

Solution. Here

$$\text{the general } k\text{th term} = \frac{n^2}{(n^2+k^2)^{3/2}}$$
$$= \frac{1}{n}\left(\frac{1}{\left(1+\left(\frac{k}{n}\right)^2\right)^{3/2}}\right),$$

which is the product of $\dfrac{1}{n}$ and the function $\dfrac{1}{\left(1+\left(\frac{k}{n}\right)^2\right)^{3/2}}$ of $\dfrac{k}{n}$.

Thus, by the above rule, the required limit is equal to

$$\int_0^1 \frac{1}{(1+x^2)^{3/2}}dx.$$

Putting $x = \tan\theta$, it can be seen that this integral is equal to $1/\sqrt{2}$. □

Example 20 Find the limit, when n tends to infinity, of the following series.

$$\frac{\sqrt{n}}{\sqrt{n^3}} + \frac{\sqrt{n}}{\sqrt{(n+4)^3}} + \frac{\sqrt{n}}{\sqrt{(n+8)^3}} + \frac{\sqrt{n}}{\sqrt{(n+12)^3}} + \cdots + \frac{\sqrt{n}}{\sqrt{(n+4(n-1)]^3}}.$$

Solution. Here

$$\text{the general } k\text{th term} = \frac{\sqrt{n}}{\sqrt{(n+4(k-1))^3}}$$
$$= \frac{1}{n}\frac{1}{\sqrt{\left(1+\frac{4(k-1)}{n}\right)^3}}.$$

As the rth term contains $(r-1)$, we consider the $(k+1)$th term. Now

$$\text{The general } (k+1)\text{th term} = \frac{1}{n}\frac{1}{\sqrt{\left(1+\frac{4k}{n}\right)^3}}.$$

Changing $\frac{k}{n}$ in $1/\sqrt{\left(1+\frac{4k}{n}\right)^3}$ to x, we see that

$$\text{the required limit} = \int_0^1 \frac{dx}{\sqrt{(1+4x)^3}}$$
$$= \int_0^1 (1+4x)^{-3/2}dx$$
$$= \frac{1}{10}(5-\sqrt{5}).$$

□