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Calculus for Engineers
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Note on Fourier Series

If you have spotted any errors/typos, please email them to:
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Definition 1 Definition of Fourier series Under quite general conditions a function f defined on $[0, 2\pi]$ has the following (so-called) Fourier series expansion:

$$\begin{aligned} & a_0 + a_1 \cos\, x + a_2 \cos\, 2x + \cdots + a_n \cos\, nx + \cdots \\ & \qquad + b_1 \sin\, x + b_2 \sin\, 2x + \cdots + b_n \sin\, nx + \cdots \\ & = a_0 + \sum_{n=1}^{+\infty} \left(a_n \cos\, nx + b_n \sin\, nx \right), \end{aligned}$$

where the coefficients are determined as follows:

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_0^{2\pi} f(x) dx; \\ a_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \cos\, nx dx, \quad (n = 1, 2, 3, \cdots); \\ b_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \sin\, nx dx, \quad (n = 1, 2, 3, \cdots). \end{aligned}$$

Note that the Fourier series expansion is periodic with period 2π.

Example 1 Find the Fourier series that represents the square wave of period 2π and amplitude 3 shown in Figure 1.

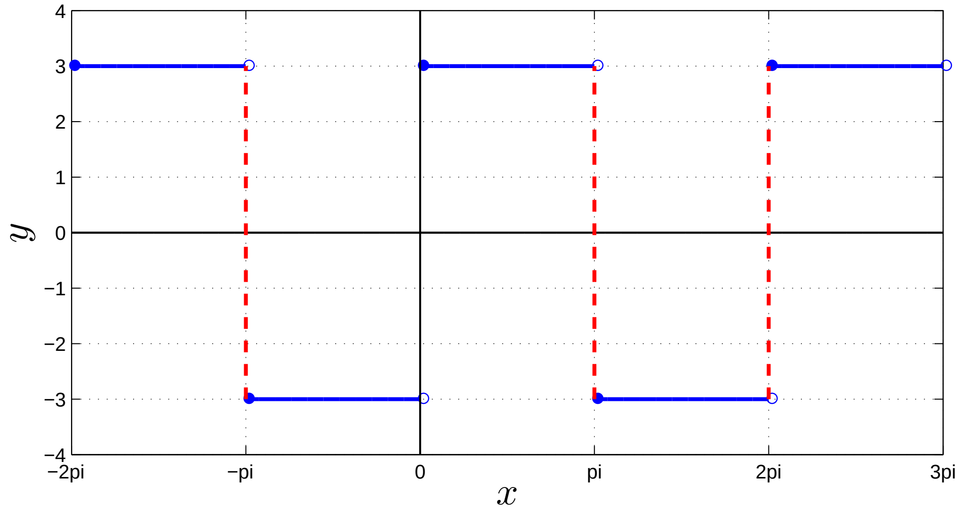

Figure 1:

Its formula is given by

$$f(x) = \begin{cases} 3, & 0 \le x < \pi \\ -3, & \pi \le x < 2\pi. \end{cases}$$

Solution: To calculate the coefficients, we will need to split up all of the integrals at $x = \pi$.

Finding a_0:

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_0^{2\pi} f(x)dx \\ &= \frac{1}{2\pi}\left(\int_0^{\pi} 3dx + \int_{\pi}^{2\pi} -3dx\right) \\ &= \frac{1}{2\pi}\left(3x|_0^{\pi} + (-3x)|_{\pi}^{2\pi}\right) \\ &= \frac{1}{2\pi}[3\pi - 0 + (-6\pi) - (-3\pi)] \\ &= 0. \end{aligned}$$

Find a_n:

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx, \quad n = 1, 2, 3, \ldots \\
&= \frac{1}{\pi}\left(\int_0^{\pi} 3\cos nx dx + \int_{\pi}^{2\pi} -3\cos nx dx\right) \\
&= \frac{1}{\pi}\left(\frac{3\sin nx}{n}\bigg|_0^{\pi} + \frac{-3\sin nx}{n}\bigg|_{\pi}^{2\pi}\right) \\
&= \frac{1}{\pi}\left(\frac{3\sin n\pi}{n} - \frac{3\sin 0}{n} + \frac{-3\sin 2n\pi}{n} - \frac{-3\sin n\pi}{n}\right) \\
&= \frac{1}{\pi}(0 - 0 + 0 - 0) \\
&= 0.
\end{aligned}$$

Therefore, the constant coefficient and all of the coefficients of the cosine terms are zero.

Find b_n:

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx, \quad n = 1, 2, 3, \ldots \\
&= \frac{1}{\pi}\left(\int_0^{\pi} 3\sin nx dx + \int_{\pi}^{2\pi} -3\sin nx dx\right) \\
&= \frac{1}{\pi}\left(\frac{-3\cos nx}{n}\bigg|_0^{\pi} + \frac{3\cos nx}{n}\bigg|_{\pi}^{2\pi}\right) \\
&= \frac{1}{\pi}\left(\frac{-3\cos n\pi}{n} - \frac{-3\cos 0}{n} + \frac{3\cos 2n\pi}{n} - \frac{3\cos n\pi}{n}\right) \\
&= \frac{1}{\pi}\left(\frac{6}{n} - \frac{6\cos n\pi}{n}\right) \qquad \text{since} \quad \begin{cases} \cos 0 = 1 \\ \cos(2n\pi) = 1 \end{cases} \\
&= \begin{cases} \frac{1}{\pi}\left(\frac{6}{n} - \frac{6}{n}\right) = 0 & \text{if } n \text{ is even} \\ \frac{1}{\pi}\left(\frac{6}{n} + \frac{6}{n}\right) = \frac{12}{n\pi} & \text{if } n \text{ is odd.} \end{cases}
\end{aligned}$$

Therefore, the Fourier series is

$$\begin{aligned} f(x) &= \frac{12}{(1)\pi}\sin 1x + \frac{12}{(3)\pi}\sin 3x + \frac{12}{(5)\pi}\sin 5x + \frac{12}{(7)\pi}\sin 7x + \cdots \\ &= \frac{12}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{\sin 7x}{7} + \cdots\right). \end{aligned}$$

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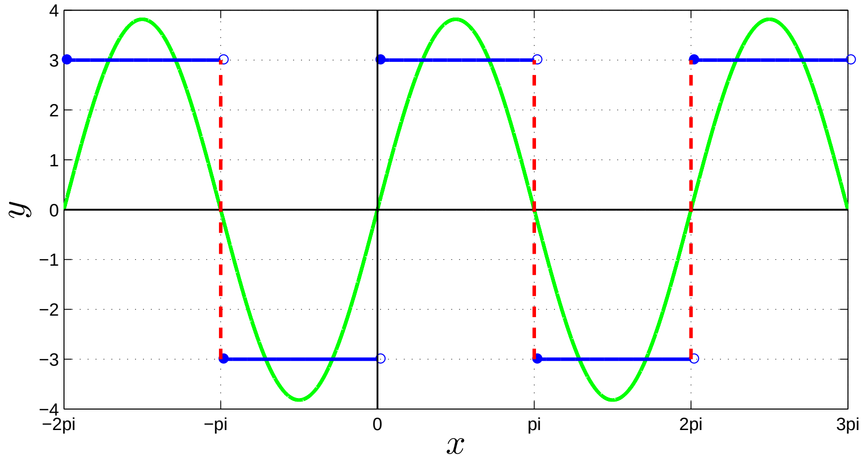

Figure 2: $g(x) = \dfrac{12}{\pi}(\sin x)$ in green.

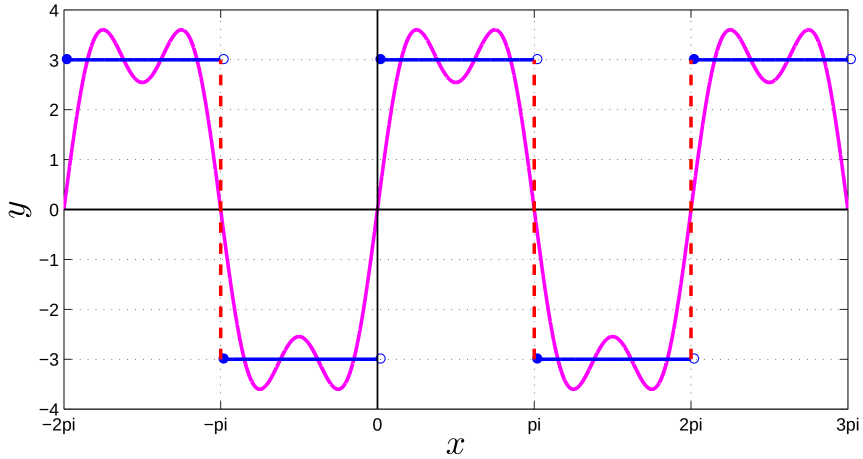

Figure 3: $g(x) = \dfrac{12}{\pi}\left(\sin x + \dfrac{\sin 3x}{3}\right)$ in pink.

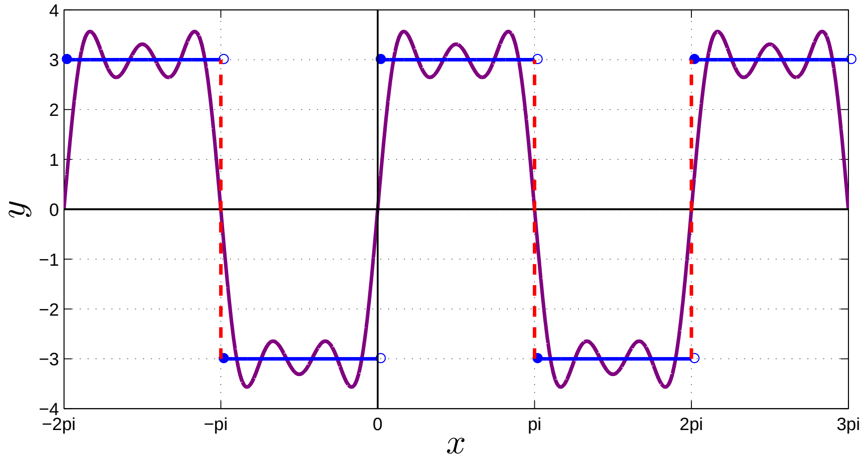

Figure 4: $g(x) = \dfrac{12}{\pi}\left(\sin x + \dfrac{\sin 3x}{3} + \dfrac{\sin 5x}{5}\right)$ in purple.

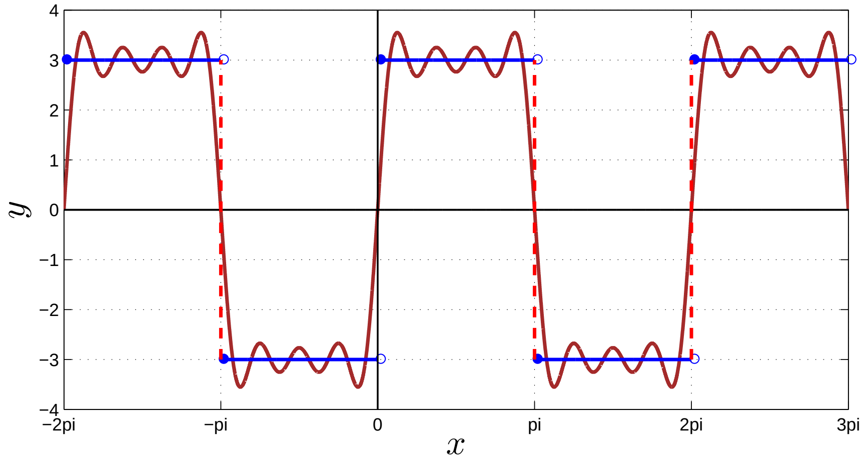

Figure 5: $g(x) = \dfrac{12}{\pi}\left(\sin x + \dfrac{\sin 3x}{3} + \dfrac{\sin 5x}{5} + \dfrac{\sin 7x}{7}\right)$ in brown.

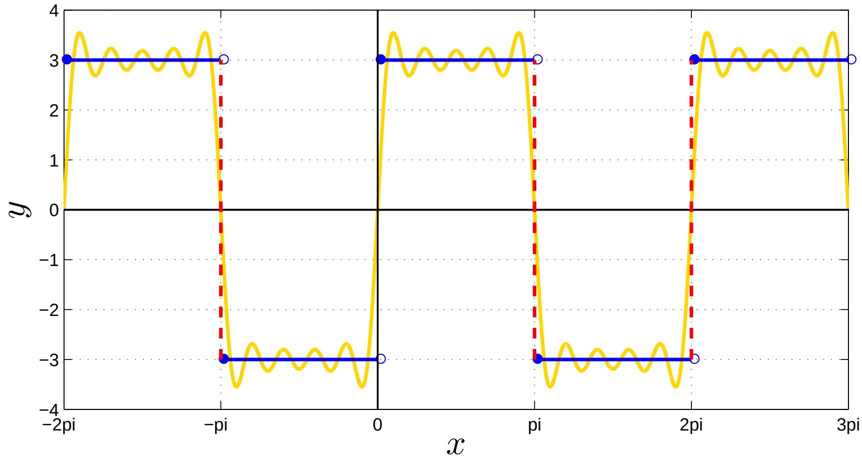

Figure 6: $g(x) = \dfrac{12}{\pi}\left(\sin x + \dfrac{\sin 3x}{3} + \dfrac{\sin 5x}{5} + \dfrac{\sin 7x}{7} + \dfrac{\sin 9x}{9}\right)$ in gold.

Exercise 1 Verify that

$$\int u \cos u du = \cos u + u \sin u + C \tag{1}$$

and

$$\int u \sin u du = \sin u - u \cos u + C, \tag{2}$$

where C is an arbitrary constant.

Example 2 Find the Fourier series which represents the wave function

$$f(x) = x, \; 0 \leq x < 2\pi$$

with period 2π shown in Figure 7.

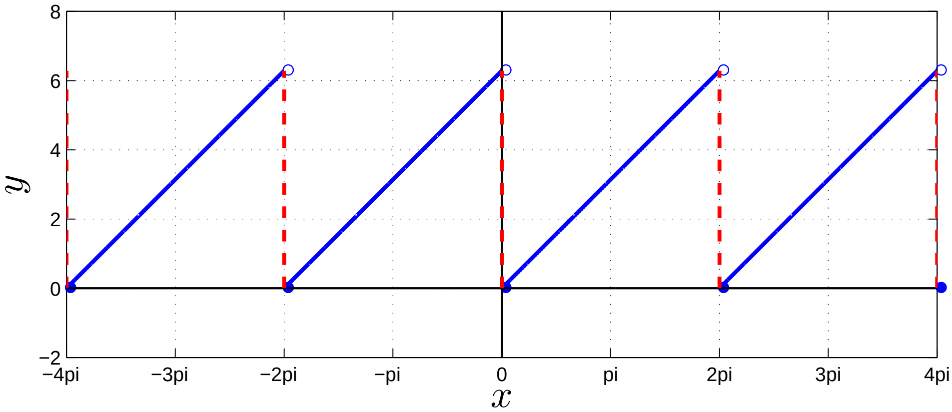

Figure 7:

Solution:

Finding a_0:

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} x dx = \frac{1}{2\pi}\left.\frac{x^2}{2}\right|_0^{2\pi} = \pi.$$

Find a_n:

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x \cos nx dx, \quad n = 1, 2, 3, \ldots$$

From Exercise 1, we have

$$\begin{aligned} \int x \cos nx &= \int \frac{u}{n} \cos u \frac{du}{n} \qquad \begin{cases} u = nx \\ du = u^{'} dx = n dx \end{cases} \\ &= \frac{1}{n^2}\int u \cos u du \\ &= \frac{1}{n^2}(\cos u + u \sin u) + C \\ &= \frac{1}{n^2}(\cos nx + nx \sin nx) + C. \end{aligned}$$

Thus,

$$\begin{aligned}
a_n &= \frac{1}{n^2}\left[\cos nx + nx\sin nx\right]\Big|_0^{2\pi} \\
&= \frac{1}{n^2\pi}\left[(\cos 2n\pi + 2n\pi\cos 2n\pi) - (\cos 0 + n(0)\sin 0)\right] \\
&= \frac{1}{n^2\pi}(1+0-1-0) = 0 \quad \text{(Recall that } n \text{ is a positive integer.)}
\end{aligned}$$

Find b_n:

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x\sin nx dx.$$

From Exercise 1, we have

$$\begin{aligned}
\int x\sin nx &= \int \frac{u}{n}\sin u\frac{du}{n} \qquad \left\{ \begin{array}{l} u = nx \\ du = u^{'}dx = ndx \end{array} \right. \\
&= \frac{1}{n^2}\int u\sin u du \\
&= \frac{1}{n^2}(\sin u - u\cos u) + C \\
&= \frac{1}{n^2}(\sin nx - nx\cos nx) + C.
\end{aligned}$$

Thus,

$$\begin{aligned}
b_n &= \frac{1}{n^2}\left(\sin nx - nx\cos nx\right)\Big|_0^{2\pi} \\
&= \frac{1}{n^2}\left[(\sin 2n\pi - 2n\pi\cos 2n\pi) - (\sin 0 - n(0)\cos 0)\right] \\
&= \frac{1}{n^2\pi}(0 - 2n\pi - 0 + 0) \\
&= -\frac{2}{n}.
\end{aligned}$$

That is $b_1 = -\frac{2}{1} = -2, b_2 = -\frac{2}{2} = -1, b_3 = -\frac{2}{3}, \ldots$ and the Fourier series is

$$f(x) = \pi - 2\sin x - 2\sin 2x - \frac{2}{3}\sin 3x - \cdots$$

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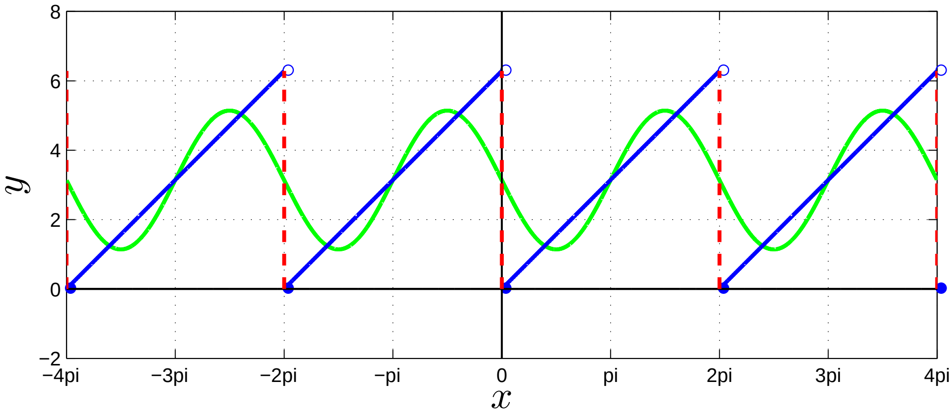

Figure 8: $g(x) = \pi - 2\sin x$ in green.

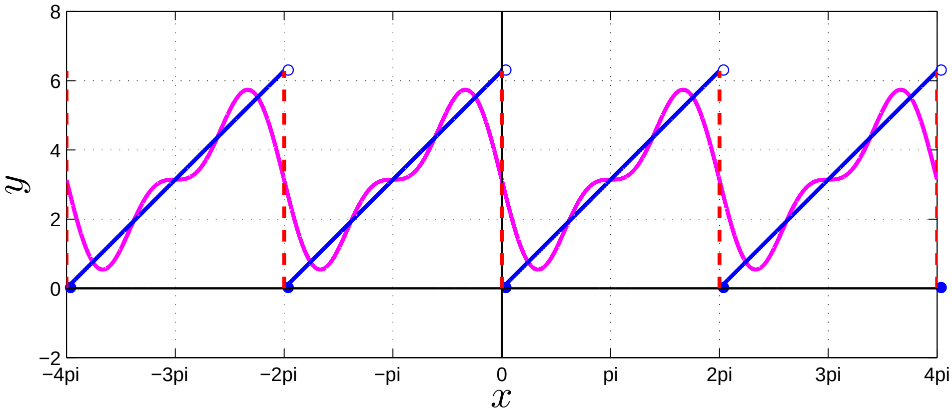

Figure 9: $g(x) = \pi - 2\sin x - 2\sin 2x$ in pink.

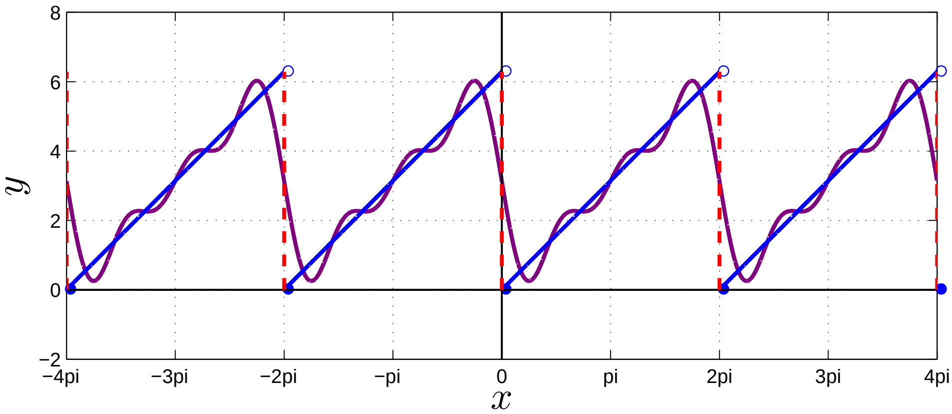

Figure 10: $g(x) = \pi - 2\sin x - 2\sin 2x - \dfrac{2}{3}\sin 3x$ in purple.

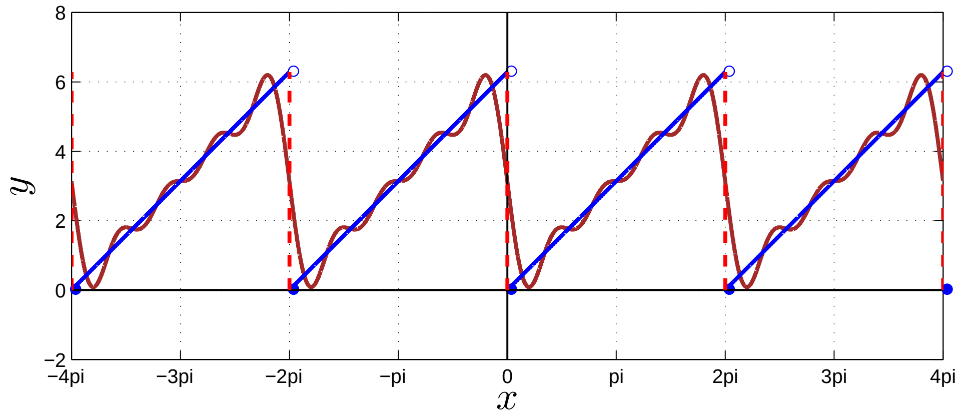

Figure 11: $g(x) = \pi - 2\sin x - 2\sin 2x - \dfrac{2}{3}\sin 3x - \dfrac{1}{2}\sin 4x$ in brown.

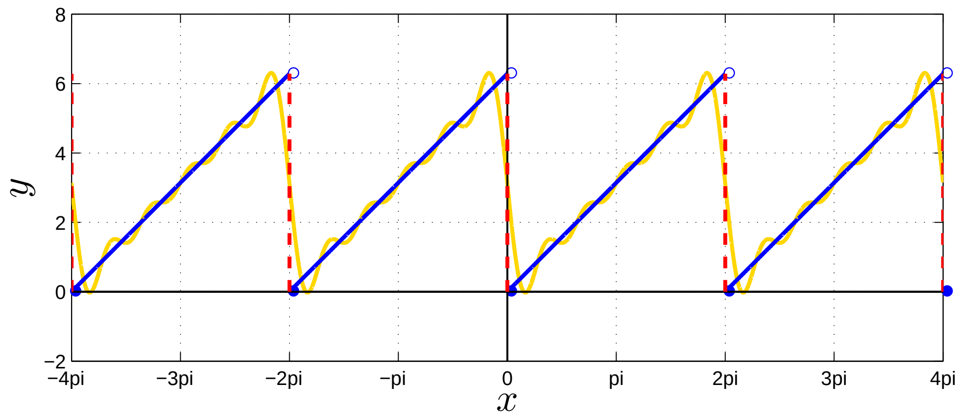

Figure 12: $g(x) = \pi - 2\sin x - 2\sin 2x - \dfrac{2}{3}\sin 3x - \dfrac{1}{2}\sin 4x - \dfrac{2}{5}\sin 5x$ in gold.

Example 3 Find the Fourier series for the wave function given by

$$f(x) = \begin{cases} \pi, & 0 \leq x < \pi\ ; \\ 2\pi - x, & \pi \leq x < 2\pi. \end{cases}$$

Figure 13 illustrates several periods of $f(x)$.

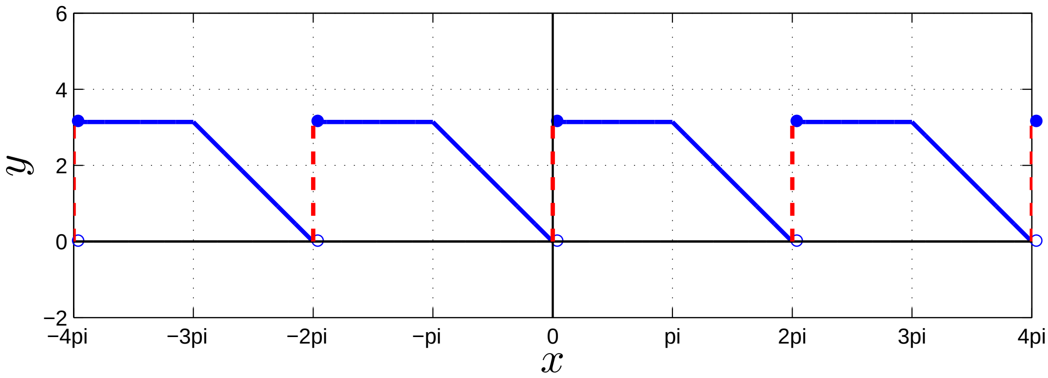

Figure 13:

Solution:

Finding a_0:

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_0^{\pi} \pi dx + \frac{1}{2\pi}\int_{\pi}^{2\pi} (2\pi - x) dx \\ &= \frac{1}{2\pi}(\pi x)|_0^{\pi} + \frac{1}{2\pi}\left(2\pi x - \frac{x^2}{2}\right)\Bigg|_{\pi}^{2\pi} \\ &= \frac{\pi}{2} + \frac{\pi}{4} \\ &= \frac{3\pi}{4}. \end{aligned}$$

Note that two separate integrals must be used to determine the coefficients. This is because the function is defined differently on the two intervals $0 \leq x < \pi$ and $\pi \leq x < 2\pi$.

Finding a_n:

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^{\pi} \pi \cos nx dx + \frac{1}{\pi}\int_{\pi}^{2\pi} (2\pi - x)\cos nx dx \\
&= \int_0^{\pi} \cos nx dx + 2\int_{\pi}^{2\pi} \cos nx dx - \frac{1}{\pi}\int_{\pi}^{2\pi} x\cos nx dx \\
&= \left.\frac{\sin nx}{n}\right|_0^{\pi} + \left.\frac{2\sin nx}{n}\right|_{\pi}^{2\pi} - \left.\frac{\cos nx + nx\sin nx}{n^2\pi}\right|_{\pi}^{2\pi} \qquad \text{(from Example 2)} \\
&= \begin{cases} 0 + 0 - \dfrac{2}{n^2\pi} = -\dfrac{2}{n^2\pi}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even.} \end{cases}
\end{aligned}
$$

Finding b_n:

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_0^{\pi} \pi \sin nx dx + \frac{1}{\pi}\int_{\pi}^{2\pi} (2\pi - x)\sin nx dx \\
&= \int_0^{\pi} \sin nx dx + 2\int_{\pi}^{2\pi} \sin nx dx - \frac{1}{\pi}\int_{\pi}^{2\pi} x\sin nx dx \\
&= -\left.\frac{\cos nx}{n}\right|_0^{\pi} + \left.\frac{-2\cos nx}{n}\right|_{\pi}^{2\pi} - \left.\frac{\sin nx - nx\cos nx}{n^2\pi}\right|_{\pi}^{2\pi} \qquad \text{(from Example 2)} \\
&= \begin{cases} \dfrac{2}{n} - \dfrac{4}{n} + \dfrac{3}{n} = \dfrac{1}{n}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even.} \end{cases}
\end{aligned}
$$

We thus obtain the Fourier series

$$
\begin{aligned}
f(x) = \frac{3\pi}{4} - \frac{2}{\pi}&\left(\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \cdots\right) \\
&+ \left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right).
\end{aligned}
$$

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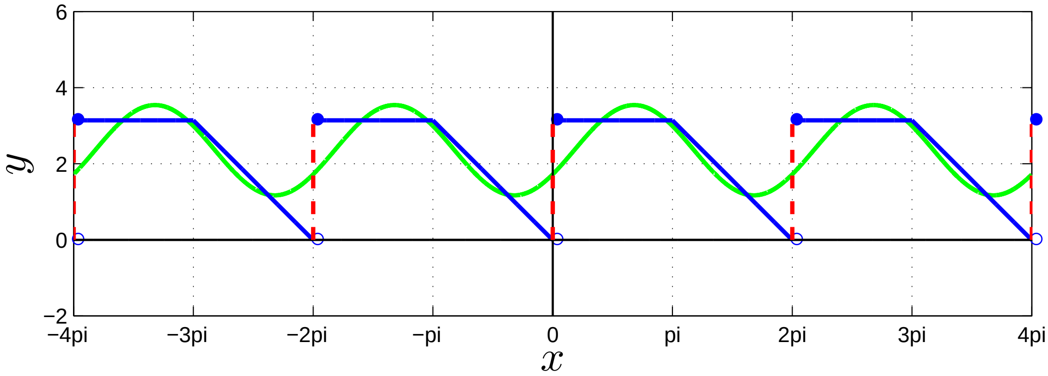

Figure 14: $g(x) = \frac{3\pi}{4} - \frac{2}{\pi}\cos x + \sin x$ in green.

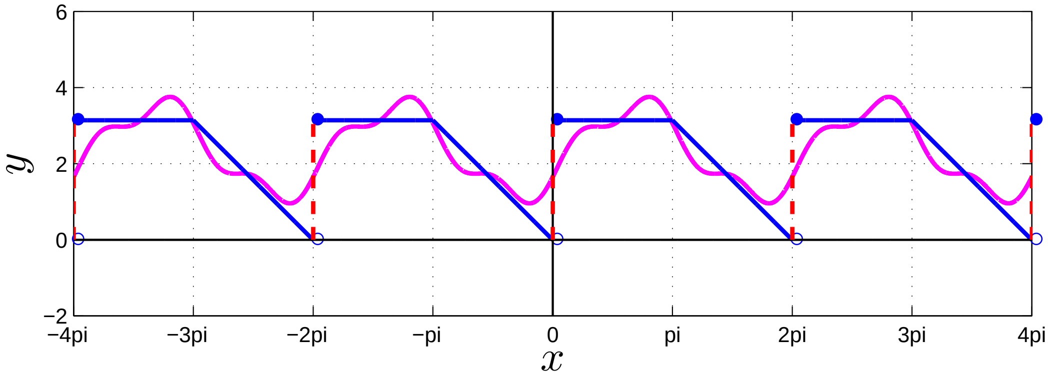

Figure 15: $g(x) = \frac{3\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{1}{9}\cos 3x\right) + \left(\sin x + \frac{1}{3}\sin 3x\right)$ in pink.

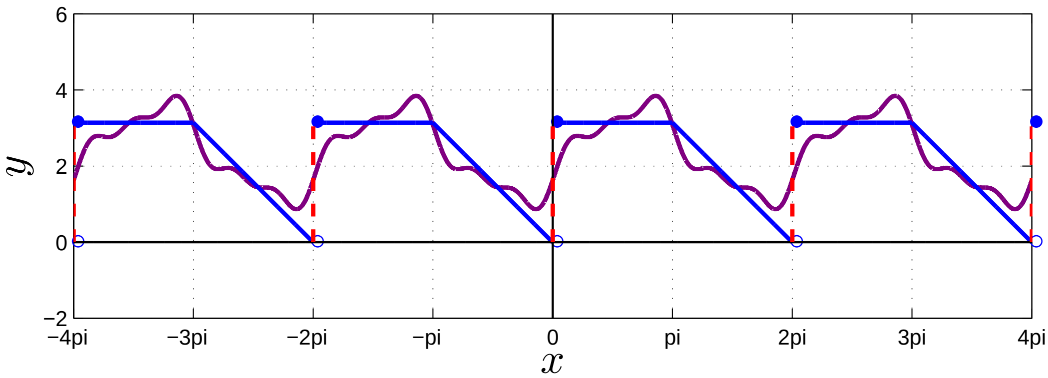

Figure 16: $g(x) = \frac{3\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x\right) + \left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x\right)$ in purple.

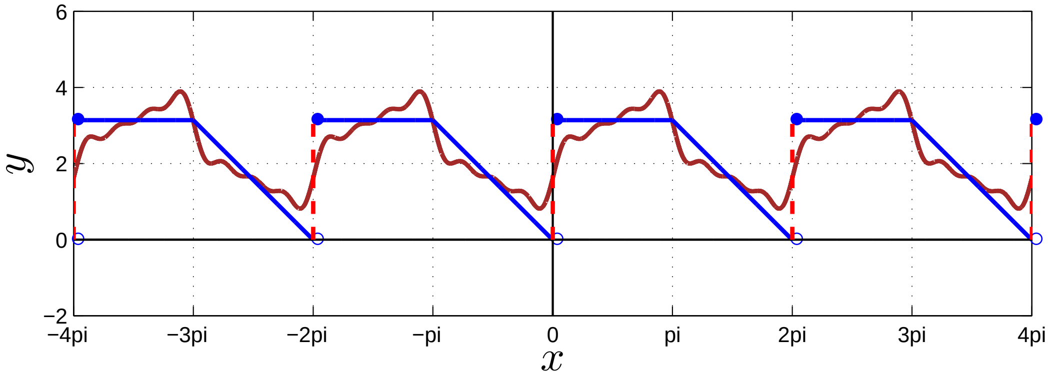

Figure 17: $g(x) = \dfrac{3\pi}{4} - \dfrac{2}{\pi}\left(\cos x + \dfrac{1}{9}\cos 3x + \dfrac{1}{25}\cos 5x + \dfrac{1}{49}\cos 7x\right) + \left(\sin x + \dfrac{1}{3}\sin 3x + \dfrac{1}{5}\sin 5x + \dfrac{1}{7}\sin 7x\right)$ in brown.

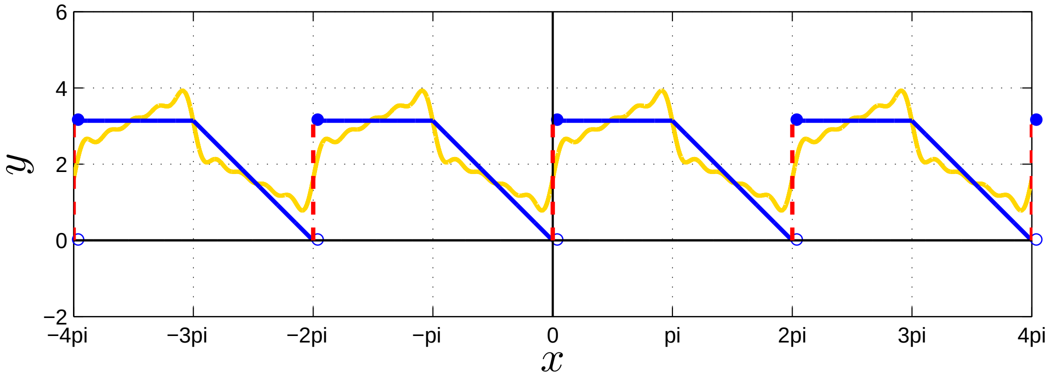

Figure 18: $g(x) = \dfrac{3\pi}{4} - \dfrac{2}{\pi}\left(\cos x + \dfrac{1}{9}\cos 3x + \dfrac{1}{25}\cos 5x + \dfrac{1}{49}\cos 7x + \dfrac{1}{81}\cos 9x\right) + \left(\sin x + \dfrac{1}{3}\sin 3x + \dfrac{1}{5}\sin 5x + \dfrac{1}{7}\sin 7x + \dfrac{1}{9}\sin 9x\right)$ in gold.

Exercise 2 Verify that

$$\int_0^{2\pi} \cos nx dx = 0, \quad n = 1, 2, 3, \cdots \tag{3}$$

and

$$\int_0^{2\pi} \sin nx dx = 0, \quad n = 1, 2, 3, \cdots . \tag{4}$$

Exercise 3 Verify that

$$\int_0^{2\pi} \cos mx \, \cos nx dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases} \quad n = 1, 2, 3, \cdots \tag{5}$$

and

$$\int_0^{2\pi} \sin mx \, \sin nx dx \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases} \quad n = 1, 2, 3, \cdots \tag{6}$$

Exercise 4 Verify that

$$\int_0^{2\pi} \cos mx \, \sin nx dx = 0, \quad n = 1, 2, 3, \cdots \tag{7}$$

and

$$\int_0^{2\pi} \sin mx \, \cos nx dx = 0, \quad n = 1, 2, 3, \cdots . \tag{8}$$

We will now show how the formulas for the coefficients a_0, a_n and b_n are obtained. Note that if we integrate each side of the equation

$$\begin{aligned} f(x) = a_0 &+ a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \cdots \\ &+ b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots . \end{aligned} \tag{9}$$

from 0 to 2π, then the integrals should be equal. That is,

$$\begin{aligned} \int_0^{2\pi} f(x)dx &= \int_0^{2\pi} a_0 dx \\ &\quad + \int_0^{2\pi} a_1 \cos x dx + \int_0^{2\pi} a_2 \cos 2x dx + \int_0^{2\pi} a_3 \cos 3x dx + \cdots \\ &\quad\quad + \int_0^{2\pi} b_1 \sin x dx + \int_0^{2\pi} b_2 \sin 2x dx + \int_0^{2\pi} b_3 \sin 3x dx + \cdots \\ &= a_0 \int_0^{2\pi} dx \\ &\quad + a_1 \cancelto{0}{\int_0^{2\pi} \cos x dx} + a_2 \cancelto{0}{\int_0^{2\pi} \cos 2x dx} + a_3 \cancelto{0}{\int_0^{2\pi} \cos 3x dx} + \cdots \\ &\quad\quad + b_1 \cancelto{0}{\int_0^{2\pi} \sin x dx} + b_2 \cancelto{0}{\int_0^{2\pi} \sin 2x dx} + b_3 \cancelto{0}{\int_0^{2\pi} \sin 3x dx} + \cdots . \end{aligned}$$

From Exercise 2, all terms on the right-hand side are zero except for

$$\int_0^{2\pi} a_0 dx = 2\pi a_0,$$

so

$$\int_0^{2\pi} f(x)dx = 2\pi a_0.$$

Then

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x)dx.$$

Multiplying each side of Equation (9) by $\cos nx$, and integrating each term, we obtain

$$
\begin{aligned}
\int_0^{2\pi} f(x)\cos nx dx = {} & \int_0^{2\pi} a_0 \cos nx dx + \int_0^{2\pi} a_1 \cos x \cos nx dx \\
& + \int_0^{2\pi} a_2 \cos 2x \cos nx dx + \int_0^{2\pi} a_3 \cos 3x \cos nx dx + \cdots \\
& + \int_0^{2\pi} a_m \cos mx \cos nx dx + + \int_0^{2\pi} a_m \cos nx \cos nx dx + \cdots \\
& + \int_0^{2\pi} b_1 \sin x \cos nx dx + \int_0^{2\pi} b_2 \sin 2x \cos nx dx \\
& + \int_0^{2\pi} b_3 \sin 3x \cos nx dx + \cdots \\
& + \int_0^{2\pi} b_m \sin mx \cos nx dx + \int_0^{2\pi} b_n \sin nx \cos nx dx + \cdots \\
= {} & a_0 \cancelto{0}{\int_0^{2\pi} \cos nx dx} + a_1 \cancelto{0}{\int_0^{2\pi} \cos x \cos nx dx} \\
& + a_2 \cancelto{0}{\int_0^{2\pi} \cos 2x \cos nx dx} + a_3 \cancelto{0}{\int_0^{2\pi} \cos 3x \cos nx dx} + \cdots \\
& + a_m \cancelto{0}{\int_0^{2\pi} \cos mx \cos nx dx} + a_n \int_0^{2\pi} \cos nx \cos nx dx + \cdots \\
& + b_1 \cancelto{0}{\int_0^{2\pi} \sin x \cos nx dx} + b_2 \cancelto{0}{\int_0^{2\pi} \sin 2x \cos nx dx} \\
& + b_3 \cancelto{0}{\int_0^{2\pi} \sin 3x \cos nx dx} + \cdots \\
& + b_m \cancelto{0}{\int_0^{2\pi} \sin mx \cos nx dx} + b_n \cancelto{0}{\int_0^{2\pi} \sin nx \cos nx dx} + \cdots .
\end{aligned}
$$

From Exercises 3 and 4, all terms on the right-hand side are zero except the term

$$\int_0^{2\pi} a_n (\cos nx)(\cos nx) dx = \pi a_n.$$

So,

$$\int_0^{2\pi} f(x)\cos nx dx = \pi a_n.$$

Then

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx.$$

Multiplying each side of Equation (9) by $\sin nx$, and integrating each term, we obtain

$$\begin{aligned}
\int_0^{2\pi} f(x)\sin nx dx &= \int_0^{2\pi} a_0 \sin nx dx + \int_0^{2\pi} a_1 \cos x \sin nx dx \\
&\quad + \int_0^{2\pi} a_2 \cos 2x \sin nx dx + \int_0^{2\pi} a_3 \cos 3x \sin nx dx + \cdots \\
&\quad + \int_0^{2\pi} a_m \cos mx \sin nx dx + \cdots \\
&\quad + \int_0^{2\pi} b_1 \sin x \sin nx dx + \int_0^{2\pi} b_2 \sin 2x \sin nx dx \\
&\quad + \int_0^{2\pi} b_3 \sin 3x \sin nx dx + \cdots \\
&\quad + \int_0^{2\pi} b_m \sin mx \sin nx dx + \cdots \\
&= a_0 \cancelto{0}{\int_0^{2\pi} \sin nx dx} + a_1 \cancelto{0}{\int_0^{2\pi} \cos x \sin nx dx} \\
&\quad + a_2 \cancelto{0}{\int_0^{2\pi} \cos 2x \sin nx dx} + a_3 \cancelto{0}{\int_0^{2\pi} \cos 3x \sin nx dx} + \cdots \\
&\quad + a_m \cancelto{0}{\int_0^{2\pi} \cos mx \sin nx dx} + a_n \cancelto{0}{\int_0^{2\pi} \cos nx \sin nx dx} + \cdots \\
&\quad + b_1 \cancelto{0}{\int_0^{2\pi} \sin x \sin nx dx} + b_2 \cancelto{0}{\int_0^{2\pi} \sin 2x \sin nx dx} \\
&\quad + b_3 \cancelto{0}{\int_0^{2\pi} \sin 3x \sin nx dx} + \cdots \\
&\quad + b_m \cancelto{0}{\int_0^{2\pi} \sin mx \sin nx dx} + b_n \int_0^{2\pi} \sin nx \sin nx dx + \cdots .
\end{aligned}$$

From Exercises 3 and 4, all terms on the right-hand side are zero except the term

$$\int_0^{2\pi} b_n(\sin nx)(\sin nx)dx = \pi b_n.$$

So,

$$\int_0^{2\pi} f(x)\sin nx dx = \pi b_n.$$

Then

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx.$$

Note that if the function to be analyzed ranges periodically from $-\pi$ to π, then the coefficients become

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx;\\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx;\\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx.
\end{aligned}$$

Example 4 Find the Fourier series for the square wave function

$$f(x) = \begin{cases} -1, & -\pi \leq x < 0 \\ 1, & 0 \leq x < \pi. \end{cases}$$

Solution:

Since $f(x)$ is defined differently for the intervals of x indicated, it requires two integrals for each coefficients:

Finding a_0:

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{0}(-1)dx + \frac{1}{2\pi}\int_{0}^{\pi}(1)dx\\
&= -\left.\frac{x}{2\pi}\right|_{-\pi}^{0} + \left.\frac{x}{2\pi}\right|_{0}^{\pi}\\
&= -\frac{1}{2} + \frac{1}{2}\\
&= 0.
\end{aligned}$$

Find a_n:

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{0}(-1)\cos(nx)\, dx + \frac{1}{\pi}\int_{0}^{\pi}(1)\cos(nx)dx\\
&= -\frac{1}{n\pi}\sin(nx)|_{-\pi}^{0} + \frac{1}{n\pi}\sin(nx)|_{0}^{\pi}\\
&= 0 + 0\\
&= 0
\end{aligned}$$

for all values of n, since $\sin(n\pi) = 0$;

Find b_n:

$$
\begin{aligned}
b_1 &= \frac{1}{\pi}\int_{-\pi}^{0}(-1)\sin\, x\, dx + \frac{1}{\pi}\int_{0}^{\pi}(1)\sin\, x dx \\
&= \frac{1}{\pi}\cos\, x|_{-\pi}^{0} - \frac{1}{\pi}\cos\, x|_{0}^{\pi} \\
&= \frac{1}{\pi}(1+1) - \frac{1}{\pi}(-1-1) \\
&= \frac{4}{\pi}.
\end{aligned}
$$

$$
\begin{aligned}
b_2 &= \frac{1}{\pi}\int_{-\pi}^{0}(-1)\sin(2x)\, dx + \frac{1}{\pi}\int_{0}^{\pi}(1)\sin(2x) dx \\
&= \frac{1}{\pi}\cos(2x)|_{-\pi}^{0} - \frac{1}{\pi}\cos(2x)|_{0}^{\pi} \\
&= \frac{1}{2\pi}(1-1) - \frac{1}{2\pi}(1-1) \\
&= 0.
\end{aligned}
$$

$$
\begin{aligned}
b_3 &= \frac{1}{\pi}\int_{-\pi}^{0}(-1)\sin(3x)\, dx + \frac{1}{\pi}\int_{0}^{\pi}(1)\sin(3x) dx \\
&= \frac{1}{\pi}\cos(3x)|_{-\pi}^{0} - \frac{1}{\pi}\cos(3x)|_{0}^{\pi} \\
&= \frac{1}{3\pi}(1+1) - \frac{1}{3\pi}(-1-1) \\
&= \frac{4}{3\pi}.
\end{aligned}
$$

$$
\begin{aligned}
b_4 &= \frac{1}{\pi}\int_{-\pi}^{0}(-1)\sin(4x)\, dx + \frac{1}{\pi}\int_{0}^{\pi}(1)\sin(4x) dx \\
&= \frac{1}{\pi}\cos(4x)|_{-\pi}^{0} - \frac{1}{\pi}\cos(4x)|_{0}^{\pi} \\
&= \frac{1}{4\pi}(1-1) - \frac{1}{4\pi}(1-1) \\
&= 0.
\end{aligned}
$$

In general, if n is even, $b_n = 0$, and if n is odd, then $b_n = \dfrac{4}{n\pi}$. Therefore, the Fourier series is

$$\begin{aligned} f(x) &= \frac{4}{\pi}\sin x + \frac{4}{3\pi}\sin(3x) + \frac{4}{5\pi}\sin(5x) + \cdots \\ &= \frac{4}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{\sin 7x}{7} + \cdots\right). \end{aligned}$$

■

Even Functions and Odd Functions

Example 5 The function $f(x) = \cos x$ is an even function by using the Taylor expansions for $\cos x$ and $\cos(-x)$. These are

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots$$

and

$$\cos(-x) = 1 - \frac{(-x)^2}{2} + \frac{(-x)^4}{24} - \cdots .$$

Since the expansions are the same, $\cos x$ an even function, i.e., $(-1)^{2n}$, $n = 1, 2, \cdots$.

Remark 1 Since $\cos x$ is an even function and all of its terms are even functions, it follows that an even function will have a Fourier series that contains only cosine terms (and possibly a constant term).

Example 6 The Fourier series for the function

$$f(x) = \begin{cases} 0, & -\pi \leq x < -\pi/2; \\ 1, & -\pi/2 \leq x < \pi/2; \\ 0, & \pi/2 \leq x < \pi. \end{cases} \tag{10}$$

is

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\left(\cos\, x - \frac{1}{3}\cos\, 3x + \frac{1}{5}\cos\, 5x - \cdots\right).$$

We see that $f(x) = f(-x)$, which means it is an even function. We also see that its Fourier series expansion contains only cosine terms (and a constant). Thus, *when finding the Fourier series, we do not have to find any sine terms.* The graph of $f(x)$ in Figure 19 shows its symmetry to the $y-$axis. ■

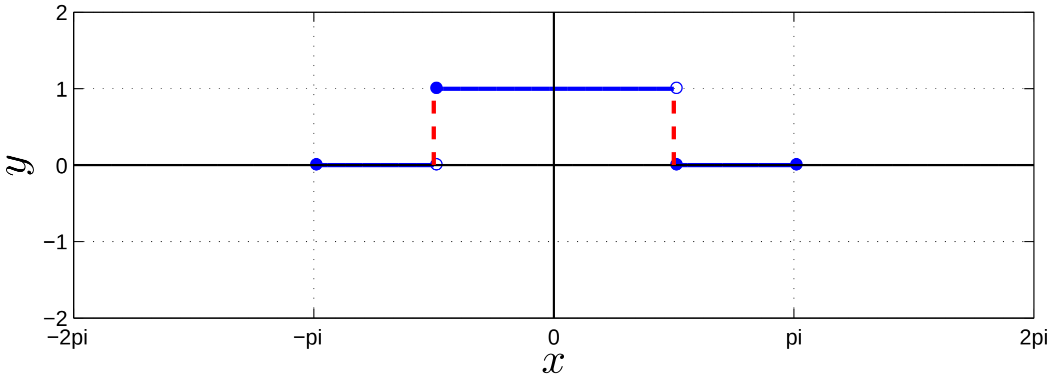

Figure 19: Example 6.

Example 7 The function $y = \sin\, x$ is an odd function by using the Maclaurin expansions for $\sin\, x$ and $-\sin\,(-x)$. Note that the negative sign, $-$ve, before $\sin\,(-x)$ is equivalent to making y negative.

These are

$$\sin\, x = x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots$$

and

$$-\sin\,(-x) = -\left[(-x) - \frac{(-x)^3}{6} + \frac{(-x)5}{120} - \cdots\right].$$

Since $\sin\, x = -\sin\,(-x)$, $\sin\, x$ is an odd function. ■

Remark 2 Since cos x is an odd function and all of its terms are odd functions, it follows that an odd function will have a Fourier series that contains only sine terms (and possibly **no** constant term).

Example 8 The Fourier series for the function

$$f(x) = \begin{cases} -1, & -\pi \leq x < 0; \\ 1, & 0 \leq x < \pi. \end{cases} \tag{11}$$

is

$$f(x) = \frac{4}{\pi}\left(\sin\, x + \frac{1}{3}\sin\, 3x + \frac{1}{5}\sin\, 5x + \cdots\right).$$

We see that $f(-x) = -f(-x)$, which means it is an odd function. We also see that its Fourier series expansion contains only sine terms. Thus, *when finding the Fourier series, we do not have to find any cosine terms.* The graph of $f(x)$ in Figure 20 shows its symmetry to the origin. ■

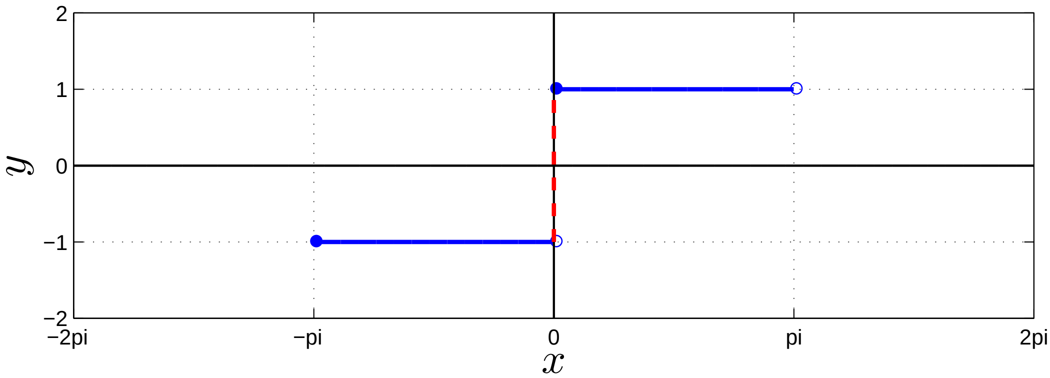

Figure 20: Example 8.

Remark 3 If a constant k is added to a function k is added to a function $f_1(x)$, the resulting function $f(x)$ is

$$f(x) = k + f_1(x).$$

Therefore, if we know the Fourier series expansion for $f_1(x)$, the Fourier series expansion of$f(x)$ is found by adding k to the Fourier series expansion of $f_1(x)$.

Example 9 The value of the function

$$f(x) = \begin{cases} 1, & -\pi \le x < -\pi/2; \\ 2, & -\pi/2 \le x < \pi/2; \\ 1, & \pi/2 \le x < \pi; \end{cases} \tag{12}$$

are all 1 greater than those of the function of Example 6. Therefore, denoting the function of Example 6 as $f_1(x)$, we have $f(x) = 1 + f_1(x)$. This means that the Fourier series for $f(x)$ is

$$\begin{aligned} f(x) &= 1 + \left[\frac{1}{2} + \frac{2}{\pi}\left(\cos x - \frac{1}{3}\cos(3x) + \frac{1}{5}\cos(5x) - \cdots\right)\right] \\ &= \frac{3}{2} + \frac{2}{\pi}\left(\cos x - \frac{1}{3}\cos(3x) + \frac{1}{5}\cos(5x) - \cdots\right) \end{aligned}$$

In Figure 21, we see that the graph of $f(x)$ is shifted up vertically by 1 unit from the graph of $f_1(x)$ in Figure 19. This is equivalent to a vertical translation of axes. We also note that $f(x)$ is an even function. ■

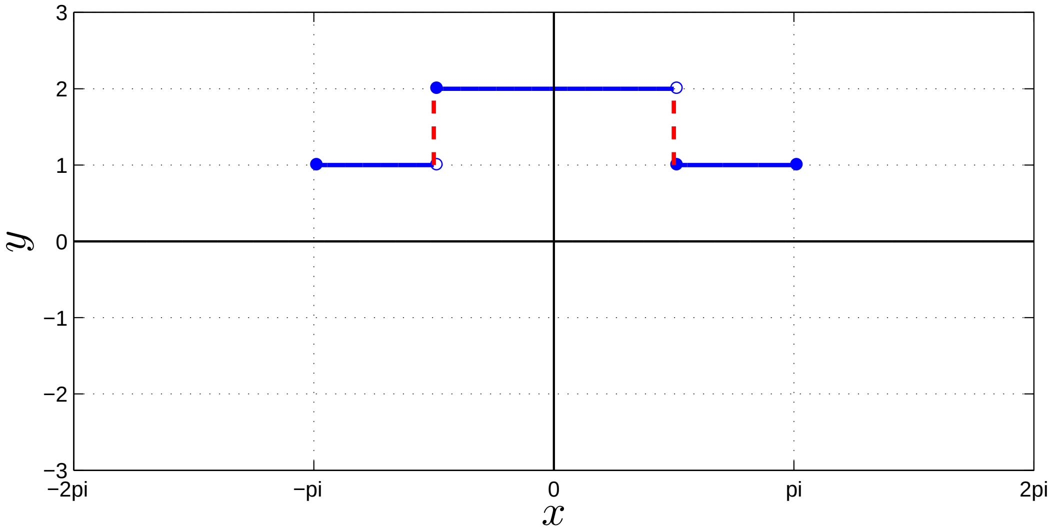

Figure 21: Example 9.

Example 10 The value of the function

$$f(x) = \begin{cases} -3/2 & -\pi \le x < 0; \\ 1/2 & 0 \le x < \pi. \end{cases} \tag{13}$$

are all $1/2$ less than those of the function of Example 8. Therefore, denoting the function of Example 8 as $f_1(x)$, we have $f(x) = -\dfrac{1}{2} + f_1(x)$. This means that the Fourier series for $f(x)$ is

$$f(x) = -\frac{1}{2} + \frac{4}{\pi}\left(\sin x - \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) - \cdots\right).$$

In Figure 22, we see that the graph of $f(x)$ is shifted vertically down by $1/2$ unit from the graph of $f_1(x)$ in Figure 20. Although $f(x)$ is not an odd function, it would be an odd function if its origin were translated to $\left(0, -\dfrac{1}{2}\right)$.

■

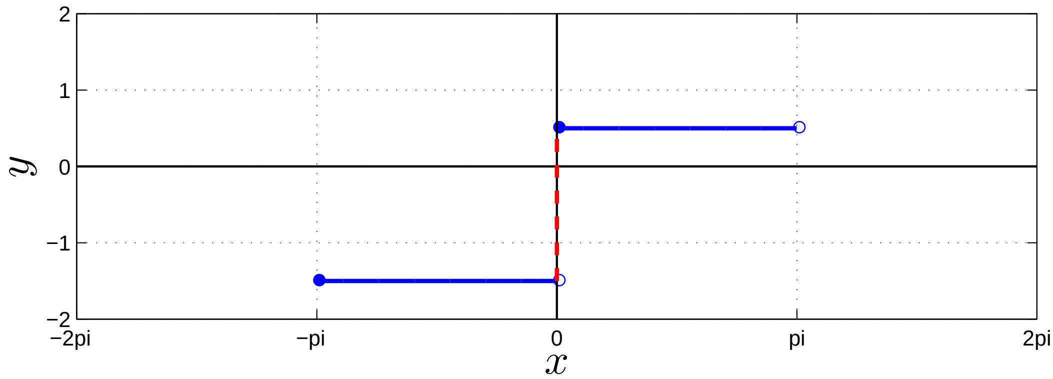

Figure 22: Example 10.

Fourier Series with period $2L$

The standard form of a Fourier series we have considered to this point is defined over the interval from $x = -\pi$ to $x = \pi$. At times, it is preferable to have a series that is defined over a different interval.

Note that

$$\sin \frac{n\pi}{L}(x + 2L) = \sin n\left(\frac{\pi x}{L} + 2\pi\right) = \sin\left(\frac{n\pi x}{L}\right)$$

we see that

$$\sin\left(\frac{n\pi x}{L}\right)$$

has a period of $2L$. Thus, using

$$\sin\left(\frac{n\pi x}{L}\right) \quad \text{and} \quad \cos\left(\frac{n\pi x}{L}\right)$$

and the same method of derivation, the following equations are found for the coefficients for the Fourier series for the interval from $x = -L$ and $x = L$.

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx \tag{14}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \tag{15}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \tag{16}$$

Then the Fourier series is

$$\begin{aligned} f(x) = a_0 &+ a_1\cos\frac{\pi x}{L} + a_2\cos\frac{2\pi x}{L} + a_3\cos\frac{3\pi x}{L} + \cdots + a_n\cos\frac{n\pi x}{L} + \cdots \\ &+ b_1\sin\frac{\pi x}{L} + b_2\sin\frac{2\pi x}{L} + b_3\sin\frac{3\pi x}{L} + \cdots + b_n\sin\frac{n\pi x}{L} + \cdots . \end{aligned}$$

Example 11 The square wave function is given by

$$f(t) = \begin{cases} 0, & -4 \le t < 0; \\ 2, & 0 \le t < 4. \end{cases}$$

with the period of 8. See Figure 23.

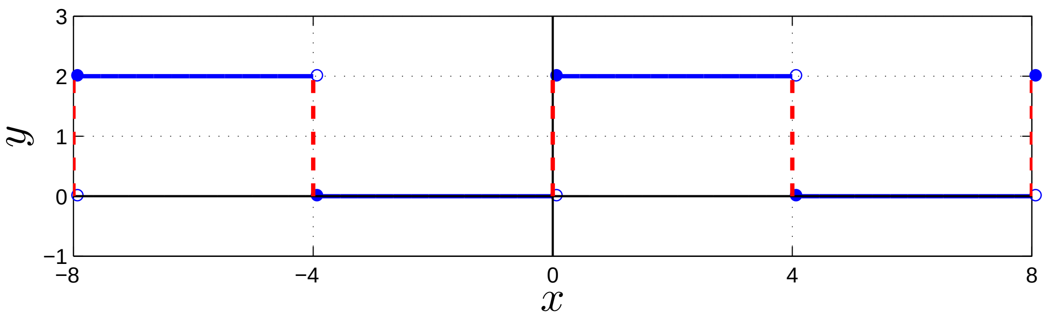

Figure 23: Example 11.

Since the period of f is 8 with $L = 4$. Next, we note that

$$f(t) = 1 + f_1(t),$$

where $f_1(t)$ is an odd function from the definition of $f(t)$, and from we can see the symmetry to the point $(0, 1)$. Therefore, the constant is 1 and there are no cosine terms in the Fourier series for $f(t)$. Now, finding the sine terms, we have

$$\begin{aligned}
b_n &= \cancelto{0}{\frac{1}{4}\int_{-4}^{0} (0)\,\sin\left(\frac{n\pi t}{4}\right) dt} + \frac{1}{4}\int_{0}^{4} (2)\,\sin\left(\frac{n\pi t}{4}\right) dt \\
&= \frac{1}{2}\left(\frac{4}{n\pi}\right)\int_{0}^{4} \sin\left(\frac{n\pi t}{4}\right)\left(\frac{n\pi}{4} dt\right) \\
&= -\frac{2}{n\pi}\cos\left(\frac{n\pi t}{4}\right)\Big|_{0}^{4} \\
&= -\frac{2}{n\pi}\left(\cos\left(n\pi\right) - \cos\,0\right) \\
&= \frac{2}{n\pi}\left(1 - \cos\left(n\pi\right)\right).
\end{aligned}$$

Let us calculate:

$$\begin{aligned} b_1 &= \frac{2}{\pi}(1-(-1)) = \frac{4}{\pi}; \\ b_2 &= \frac{2}{2\pi}(1-1) = 0; \\ b_3 &= \frac{2}{3\pi}(1-(-1)) = \frac{4}{3\pi}; \\ b_4 &= \frac{2}{4\pi}(1-1) = 0. \end{aligned}$$

Therefore, the Fourier series is

$$f(t) = 1 + \frac{4}{\pi}\sin\left(\frac{\pi t}{4}\right) + \frac{4}{3\pi}\sin\left(\frac{3\pi t}{4}\right) + \cdots .$$

■

Example 12 Find the Fourier series for the function

$$f(x) = x^2, \quad -1 \leq x < 1. \tag{17}$$

for which the period is 2. See Figure 24.

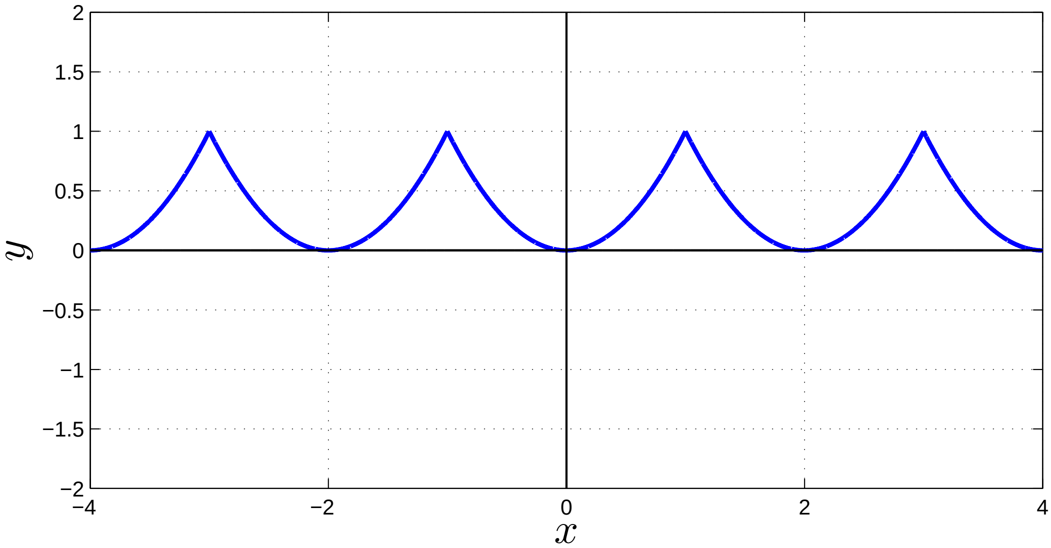

Figure 24: Example 12.

Since the period is 2, $L = 1$. Next, we note that $f(x) = f(-x)$, which means it is an even function, Therefore, there are no sine terms in the

Fourier series. Finding the constant term and the cosine terms, we have

$$
\begin{aligned}
a_0 &= \frac{1}{2(1)} \int_{-1}^{1} x^2 dx \\
&= \frac{1}{6} x^3 \Big|_{-1}^{1} \\
&= \frac{1}{6}(1+1) \\
&= \frac{1}{3}
\end{aligned}
$$

and

$$
\begin{aligned}
a_n &= \frac{1}{1} \int_{-1}^{1} x^2 \cos\left(\frac{n\pi x}{1}\right) dx \\
&= \int_{-1}^{1} x^2 \cos\,(n\pi x)\, dx \\
&= x^2 \left(\frac{1}{n\pi} \sin\,(n\pi x)\right)\Bigg|_{-1}^{1} - \frac{2}{n\pi} \int_{-1}^{1} x \sin\,(n\pi x)\, dx \\
&= \frac{1}{n\pi} \sin\, n\pi - \frac{1}{n\pi} \sin\,(-n\pi) \\
&\quad - \frac{2}{n\pi} \left[\left(-\frac{1}{n\pi} \cos(n\pi x) \right)\Bigg|_{-1}^{1} - \left(-\frac{1}{n\pi} \int_{-1}^{1} \cos\,(n\pi x)\, dx \right) \right] \\
&= \frac{2}{n^2\pi^2} \left(\cos\,(n\pi) + \cos\,(-n\pi)\right) + \frac{1}{n^2\pi^2} \sin\,(n\pi x)\big|_{-1}^{1} \\
&= \frac{4}{n^2\pi^2} \cos\,(n\pi).
\end{aligned}
$$

Let us calculate:

$$
\begin{aligned}
a_1 &= \frac{4}{\pi^2} \cos\,(\pi) = -\frac{4}{\pi^2}; \\
a_2 &= \frac{4}{4\pi^2} \cos\,(2\pi) = \frac{4}{4\pi^2}; \\
a_3 &= \frac{4}{9\pi^2} \cos\,(3\pi) = -\frac{4}{9\pi^2}.
\end{aligned}
$$

Therefore, the Fourier series is

$$
f(t) = \frac{1}{3} - \frac{4}{\pi^2} \left(\cos\,(\pi x) - \frac{1}{4} \cos\,(2\pi x) + \frac{1}{9} \cos\,(3\pi x) + \cdots \right).
$$

■

Example 13 Find $f(x) = x$ in a half-range cosine series for $0 \leq x < 2$.

Since we are to have a cosine series, we extend the function to be an even function with its graph as shown in Figure 25. The blue line between $x = 0$ and $x = L$, shows the given function as defined, green line portions show that the extension that makes it an even function.

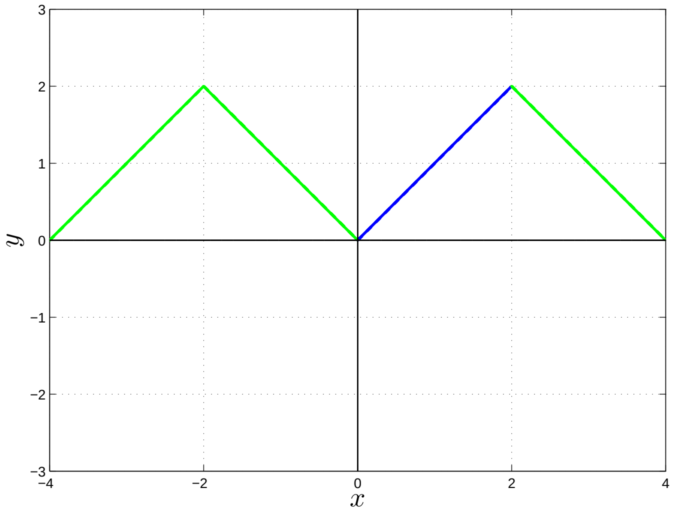

Figure 25: Example 13.

We find the Fourier expansion coefficients, with $L = 2$.

$$\begin{aligned} a_0 &= \frac{1}{2}\int_0^2 x dx \\ &= \frac{1}{4}x^2\Big|_0^2 \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} a_n &= \frac{2}{2}\int_0^2 x\cos\left(\frac{n\pi x}{2}\right)dx \\ &= \frac{4}{n^2\pi^2}\left(\cos(n\pi) - 1\right). \end{aligned}$$

If n is even, $\cos(n\pi) = -1 = 0$. Therefore, we evaluate a_n for the odd values of n, and find the expansion is

$$f(x) = 1 - \frac{8}{\pi^2}\left(\cos\frac{\pi x}{2} + \frac{1}{9}\cos\frac{3\pi x}{2} + \frac{1}{25}\cos\frac{5\pi x}{2} + \cdots\right).$$

■

Example 14 Find $f(x) = x$ in a half-range sine series for $0 \leq x < 2$.

Since we are to have a sine series, we extend the function to be an odd function with its graph as shown in Figure 26. The blue line between $x = 0$ and $x = L$, shows the given function as defined, green line portions show that the extension that makes it an odd function.

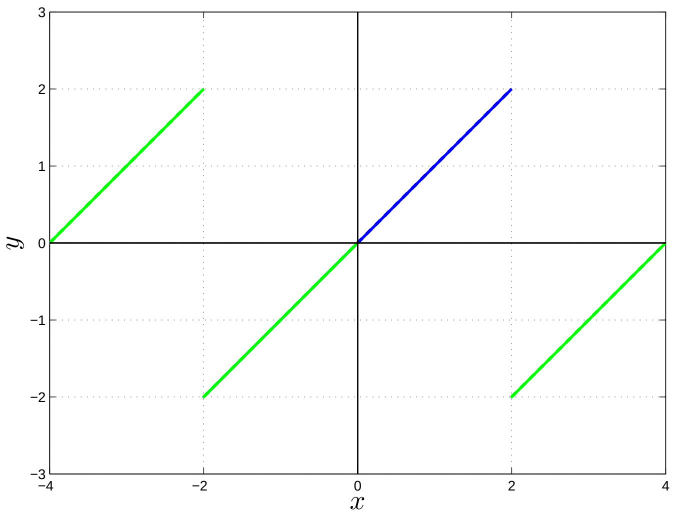

Figure 26: Example 14.

We find the Fourier expansion coefficients, with $L = 2$.

$$
\begin{aligned}
b_n &= \frac{2}{2}\int_0^2 x \sin\left(\frac{n\pi x}{2}\right) dx \\
&= -\frac{4}{n\pi}\cos(n\pi).
\end{aligned}
$$

the Fourier series is

$$
f(x) = \frac{4}{\pi}\left(\sin\frac{\pi x}{2} - \frac{1}{2}\sin \pi x + \frac{1}{3}\sin\frac{3\pi x}{2} + \cdots\right).
$$

■