THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1510 Calculus for Engineers (Fall 2021)
Homework 1
Deadline: September 25 at 23:00

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Class: MATH 1510 G

I acknowledge that I am aware of University policy and regulations on honesty in academic work, and of the disciplinary guidelines and procedures applicable to breaches of such policy and regulations, as contained in the website http://www.cuhk.edu.hk/policy/academichonesty/

David — 16-9-2021

Signature Date

General Guidelines for Homework Submission.

- Please submit your answer to Gradescope through the centralized course MATH1510A-I in Blackboard.
- In Gradescope, for each question, please indicate exactly which page(s) its answer locates. **Answers of incorrectly matched questions will not be graded**.
- **Late submission will NOT be graded and result in zero score.** Any answers showing evidence of plagiarism will also score zero; stronger disciplinary action may also be taken.
- Points will only be awarded for answers with sufficient justifications.
- All questions in **Part A** along with some selected questions in **Part B** will be graded. Question(s) labeled with * are more challenging.

Part A:

1. Without using L'Hôpital's rule, evaluate the following limits of sequences. Furthermore, if the limit does not exist but diverges to $\pm\infty$, please indicate so and determine the correct sign.

   (a) $\lim_{n\to\infty}(\sqrt{n^2+n}-\sqrt{n^2-1})$

   (b) $\lim_{n\to\infty}\frac{\sin(n)+\cos(n^2)}{n-100}$

(a) $\lim_{n\to\infty}\left(\sqrt{n^2+n}-\sqrt{n^2-1} \times \frac{\sqrt{n^2+n}+\sqrt{n^2-1}}{\sqrt{n^2+n}+\sqrt{n^2-1}}\right)$

$= \lim_{n\to\infty}\left(\frac{n+1}{\sqrt{n^2+n}+\sqrt{n^2-1}}\right)$

$= \lim_{n\to\infty}\left(\frac{1+\frac{1}{n}}{\sqrt{1+\frac{1}{n}}+\sqrt{1-\frac{1}{n^2}}}\right)$

$= \frac{1}{1+1}$

$= \frac{1}{2}$ //

(b) $\because -2 \leq \sin(n)+\cos(n^2) \leq 2$

$-\frac{2}{n-100} \leq \frac{\sin(n)+\cos(n^2)}{n-100} \leq \frac{2}{n-100}$

$\because \lim_{n\to\infty}\left(-\frac{2}{n-100}\right) = -\frac{0}{1} = 0$

& $\lim_{n\to\infty}\left(\frac{2}{n-100}\right) = \frac{0}{1} = 0$

$\therefore$ By sandwich theorem, we have $\lim_{n\to\infty}\frac{\sin(n)+\cos(n^2)}{n-100} = 0$ //

2. Let

$$f(x) = \begin{cases} \frac{1}{x}\tan\frac{x}{2} & \text{if } -1 < x < 0; \\ \frac{|x-1|}{2x-2} & \text{if } 0 < x < 1; \\ \frac{x^2-4x+3}{x^2+2x-3} & \text{if } x > 1. \end{cases}$$

Then find each of the following limits or state that it does not exist. Furthermore, if the limit does not exist but diverges to $\pm\infty$, please indicate so, and determine the correct sign.

(a) $\lim_{x\to 0^-} f(x)$;

(b) $\lim_{x\to 0^+} f(x)$;

(c) $\lim_{x\to 0} f(x)$.

(d) $\lim_{x\to 1} f(x)$;

(a)
$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0}\left(\frac{1}{x}\tan\frac{x}{2}\right)$$
$$= \lim_{x\to 0}\left(\frac{1}{x}\cdot\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right)$$
$$= \lim_{x\to 0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\cdot\frac{1}{2\cos\frac{x}{2}}\right)$$
$$= \lim_{x\to 0}\frac{1}{2\cos\frac{x}{2}}$$
$$= \frac{1}{2\times 1}$$
$$= \frac{1}{2} \quad //$$

(b)
$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0}\frac{-\cancel{(x-1)}}{2\cancel{(x-1)}}$$
$$= -\frac{1}{2}$$

(c) $\because \lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$

$\therefore \lim_{x \to 0} f(x)$ does not exist. //

(d) $\lim_{x \to 1^-} f(x) = \lim_{x \to 1} \frac{-(x-1)}{2x-2}$

$= -\frac{1}{2}$

$\therefore \lim_{x \to 1^-} f(x) = -\frac{1}{2}$ //

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1} \frac{x^2-4x+3}{x^2+2x-3}$

$= \lim_{x \to 1} \frac{(x-3)\cancel{(x-1)}}{\cancel{(x-1)}(x+3)}$

$= \lim_{x \to 1} \frac{x-3}{x+3}$

$= -\frac{1}{2}$

$\because \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)$,

$\therefore \lim_{x \to 1} f(x) = -\frac{1}{2}$

Part B:

3. (a) Let $f(x) = \dfrac{1}{\sqrt{5-4x-x^2}}$. Express the domain and range of f in interval notation.

(b) Let $f(x) = x^2 - 1,\ g(x) = \dfrac{1}{3}\log_2 x$. Express the domain and range of $g \circ f$ in interval notation.

(a) Domain: $5 - 4x - x^2 > 0$

$$-5 < x < 1$$

$$\therefore (-5, 1) \ //$$

For $\sqrt{5-4x-x^2}$, the maximum value exists where $x = -\frac{-4}{2(-1)} = -2$.

where the value of $f(x)$ will attain the minimum which is $\frac{1}{\sqrt{9}} = \frac{1}{3}$.

$\therefore$ Range of f: $[\frac{1}{3}, \infty) \ //$

(b) $g \circ f = g(f(x))$

$$= \frac{1}{3}\log_2(x^2-1)$$

Domain: $x^2 - 1 > 0$

$$x^2 > 1$$

$$x < -1 \quad \text{or} \quad x > 1$$

$$\therefore (\infty, -1) \cup (1, \infty) \ //$$

4. Let $\gamma(t) = (x(t), y(t)) = (3\cos 2t - 1, 3\sin 2t + 2)$, $t \in \mathbb{R}$ be a curve.

(a) Write down an equation of the curve in terms of x and y without t.

(b) Sketch the curve in xy-plane, and indicate the direction for t increasing with an arrow.

(a) $y = 3\sin 2t + 2$ & $x = 3\cos 2t - 1$

$\frac{y-2}{3} = \sin 2t$ & $\frac{x+1}{3} = \cos 2t$

$\frac{y-2}{6} = \sin t \cos t$ ... ③ $\qquad = 1 - 2\sin^2 t$

$\sin t = \sqrt{\frac{2-x}{6}}$ ... ①

$\frac{x+1}{3} = 2\cos^2 t - 1$

$\cos t = \sqrt{\frac{x+4}{6}}$ ... ②

∴ Sub ① & ② into ③:

$$\frac{y-2}{6} = \sqrt{\frac{2-x}{6}} \times \sqrt{\frac{x+4}{6}}$$

$$\frac{y-2}{6} = \frac{\sqrt{(2-x)(x+4)}}{6}$$

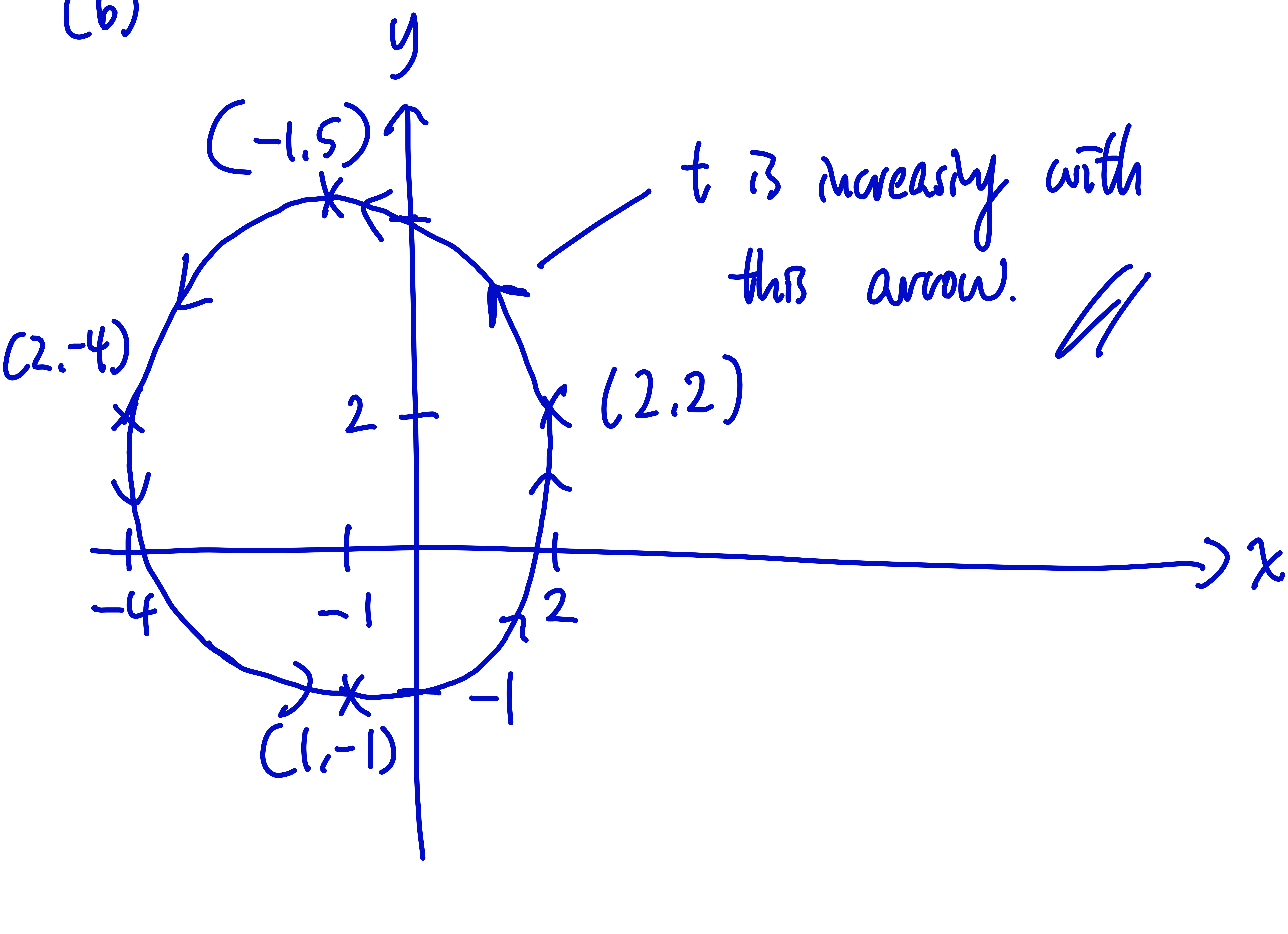
$$y-2 = \sqrt{(2-x)(x+4)}$$

$$y^2-4y+4 = (2-x)(x+4)$$

$$y^2-4y+4 = -x^2-2x+8$$

$$x^2+y^2+2x-4y-4 = 0$$ //

(b)



t is increasing with this arrow. //

5. Without using L'Hôpital's rule, evaluate the following limits. Furthermore, if the limit does not exist but diverges to $\pm\infty$, please indicate so and determine the correct sign.

   (a) $\lim_{x\to 3} \dfrac{\sqrt{x+1}-2}{4-\sqrt{5x+1}}$;

   (b) $^*\lim_{x\to 8} \dfrac{x^2-7x-8}{\sqrt[3]{x}-2}$;

(a) $$\lim_{x\to 3} \frac{\sqrt{x+1}-2}{4-\sqrt{5x+1}} \times \frac{\sqrt{x+1}+2}{\sqrt{x+1}+2}$$

$$\times \frac{4+\sqrt{5x+1}}{4+\sqrt{5x+1}}$$

$$= \lim_{x\to 3} \frac{\overset{1}{\cancel{(x+1-4)}}(4+\sqrt{5x+1})}{\underset{-5}{\cancel{(16-5x-1)}}(\sqrt{x+1}+2)}$$

$$= -5 \times \lim_{x\to 3} \frac{4+\sqrt{5x+1}}{\sqrt{x+1}+2}$$

$$= -5 \times \frac{8}{4}$$

$$= -10 \,//$$

(6) $\displaystyle \lim_{x \to 8} \frac{x^2 - 7x - 8}{\sqrt[3]{x} - 2}$

$$= \lim_{x \to 8} \frac{(x^2 - 7x - 8)(x^{\frac{2}{3}} + 2x^{\frac{1}{3}} + 4)}{(x^{\frac{1}{3}} - 2)(x^{\frac{2}{3}} + 2x^{\frac{1}{3}} + 4)}$$

$$= \lim_{x \to 8} \frac{\cancel{(x-8)}(x+1)(x^{\frac{2}{3}} + 2x^{\frac{1}{3}} + 4)}{\cancel{x - 8}}$$

$$= 9 \times (4 + 4 + 4)$$

$$= 108 \;//$$

6. Without using L'Hôpital's rule, evaluate the following limits. Furthermore, if the limit does not exist but diverges to $\pm\infty$, please indicate so and determine the correct sign.

   (a) $\displaystyle\lim_{x\to+\infty}\left(1+\frac{2}{4x-1}\right)^{x}$

   (b) $\displaystyle\lim_{x\to+\infty}\left(\frac{2x-1}{2x+1}\right)^{x}$

(a) $\displaystyle\lim_{x\to+\infty}\left(1+\frac{2}{4x-1}\right)^{x}$

$$= \lim_{x\to+\infty}\left(1+\frac{1}{2x-\frac{1}{2}}\right)^{(2x-\frac{1}{2})\frac{1}{2}+\frac{1}{4}}$$

$$= \lim_{x\to+\infty}\left[\left(1+\frac{1}{2x-\frac{1}{2}}\right)^{2x-\frac{1}{2}}\right]^{\frac{1}{2}}$$

$$\times\left(1+\frac{1}{2x-\frac{1}{2}}\right)^{\frac{1}{4}}$$

$$= e^{\frac{1}{2}} \times 1$$

$$= e^{\frac{1}{2}} \,/\!/$$

(b) $\lim\limits_{x \to +\infty} \left(\dfrac{2x-1}{2x+1} \right)^x$

$$= \lim_{x \to +\infty} \left(\frac{2x+1}{2x+1} - \frac{2}{2x+1} \right)^x$$

$$= \lim_{x \to +\infty} \left(1 - \frac{1}{x+\frac{1}{2}} \right)^x$$

$$= \lim_{x \to +\infty} \left(1 - \frac{1}{x+\frac{1}{2}} \right)^{x+\frac{1}{2}-\frac{1}{2}}$$

$$= e^{-1} \times \lim_{x \to +\infty} \left(1 - \frac{1}{x-\frac{1}{2}} \right)^{-\frac{1}{2}}$$

$$= e^{-1} \times 1$$

$$= e^{-1} \;//$$

$$\sqrt{6+\sqrt{6+\sqrt{6+a_n}}}$$

7. * Let $\{a_n\}$ be the sequence defined by the recursive relation

$$a_1 = \sqrt{6} \quad \text{and} \quad a_{n+1} = \sqrt{6 + a_n} \text{ for all postive integer } n.$$

Given that $\lim_{n\to\infty} a_n$ exists and equals c. Find the value of c.

$$c = \lim_{n\to\infty} \underbrace{\sqrt{6+\sqrt{6+\sqrt{\cdots\sqrt{}}}}}_{(n-1)\text{ times}} \sqrt{6+\sqrt{6}}$$

$$c^2 = \lim_{n\to\infty} 6+\underbrace{\sqrt{6+\sqrt{\cdots\sqrt{6+\sqrt{6}}}}}_{(n-1)\text{ times}}$$

$$c^2 - 6 = \lim_{n\to\infty} \underbrace{\sqrt{6+\sqrt{6+\sqrt{\cdots}}}}_{(n-1)\text{ times}} \sqrt{6+\sqrt{6}} \quad (\text{i.e. } c)$$

$$c^2 - c - 6 = 0$$

$$c = 3 \;/\!/ \text{ or } -2 \text{ (rejected)}$$

8. * The following statements are both false. Give one counterexample for each of them.

   (a) If $\lim_{n\to+\infty} a_n = 0$, $\lim_{n\to+\infty} b_n = +\infty$, then $\lim_{n\to+\infty} a_n b_n = 0$.

   (b) If $f(x) > 0$ for all $x \in \mathbb{R}$ and $\lim_{x\to 0} f(x)$ exists, then $\lim_{x\to 0} f(x) > 0$.

$$\frac{e^{2x}}{e^{x^2}}$$

(a) Let $a_n = \frac{1}{n}$ and $b_n = n$,

$$\lim_{n\to\infty} a_n b_n = \lim_{n\to\infty}\left(\frac{1}{n}\right) \times \lim_{n\to\infty}(n)$$

$$= 0 \cdot \infty \text{, which is undefined.}$$

$\therefore \lim_{n\to\infty} a_n b_n$ does not exist. //

(b) Let $f(x) = \begin{cases} 4x^2 & \text{for } x \neq 0 \\ 50 & \text{for } x = 0 \end{cases}$

$f(x) > 0$ for all $x \in \mathbb{R}$.

$$\because \lim_{x\to 0} f(x) = 4\left(\lim_{x\to 0} x\right)^2$$
$$= 4 \cdot 0$$
$$= 0$$

$\therefore$ The statement is incorrect. //