2021R1-MATH1510 HW 1

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TOTAL POINTS

18 / 20

QUESTION 1

1 Q1 **3 / 4**

(a), 2pts

✓ **- 0 pts** correct

(b), 2pts

✓ **- 0.5 pts** **should state that for n > 100, otherwise the inequality won't hold.**

✓ **- 0.5 pts** **computational error**

1 For n > 100

QUESTION 2

2 Q2 **3 / 3**

✓ **- 0 pts** **Correct**

QUESTION 3

3 Q3a **4 / 4**

✓ **- 0 pts** **Correct**

QUESTION 4

4 Q5a **2 / 3**

✓ **- 1 pts** **incorrect answer**

QUESTION 5

5 Q7 **3 / 3**

✓ **- 0 pts** **Correct**

QUESTION 6

6 Q8b **3 / 3**

\+ **0 pts** incorrect/no solution found on the selected page

\+ **2 pts** incorrect explanation

\+ **1 pts** incorrect example

✓ **+ 3 pts** **Correct**

\- **0.5 pts** No solution found (adjusted)

Part A:

1. Without using L'Hôpital's rule, evaluate the following limits of sequences. Furthermore, if the limit does not exist but diverges to $\pm\infty$, please indicate so and determine the correct sign.

   (a) $\lim_{n\to\infty}(\sqrt{n^2+n}-\sqrt{n^2-1})$

   (b) $\lim_{n\to\infty}\dfrac{\sin(n)+\cos(n^2)}{n-100}$

(a) $\lim_{n\to\infty}\left(\sqrt{n^2+n}-\sqrt{n^2-1} \times \dfrac{\sqrt{n^2+n}+\sqrt{n^2-1}}{\sqrt{n^2+n}+\sqrt{n^2-1}}\right)$

$= \lim_{n\to\infty}\left(\dfrac{n+1}{\sqrt{n^2+n}+\sqrt{n^2-1}}\right)$

$= \lim_{n\to\infty}\left(\dfrac{1+\frac{1}{n}}{\sqrt{1+\frac{1}{n}}+\sqrt{1-\frac{1}{n^2}}}\right)$

$= \dfrac{1}{1+1}$

$= \dfrac{1}{2}$ //

(b) $\because -2 \le \sin(n)+\cos(n^2) \le 2$

$-\dfrac{2}{n-100} \le \dfrac{\sin(n)+\cos(n^2)}{n-100} \le \dfrac{2}{n-100}$

$\because \lim_{n\to\infty}\left(-\dfrac{2}{n-100}\right) = -\dfrac{0}{1} = 0$

& $\lim_{n\to\infty}\left(\dfrac{2}{n-100}\right) = \dfrac{0}{1} = 0$

$\therefore$ By sandwich theorem, we have $\lim_{n\to\infty}\dfrac{\sin(n)+\cos(n^2)}{n-100} = 0$ //

1 Q1 3 / 4

(a), 2pts

✓ - 0 pts correct

(b), 2pts

✓ - 0.5 pts should state that for n > 100, otherwise the inequality won't hold.

✓ - 0.5 pts computational error

1 For n > 100

2. Let

$$f(x) = \begin{cases} \dfrac{1}{x}\tan\dfrac{x}{2} & \text{if } -1 < x < 0; \\ \dfrac{|x-1|}{2x-2} & \text{if } 0 < x < 1; \\ \dfrac{x^2-4x+3}{x^2+2x-3} & \text{if } x > 1. \end{cases}$$

Then find each of the following limits or state that it does not exist. Furthermore, if the limit does not exist but diverges to $\pm\infty$, please indicate so, and determine the correct sign.

(a) $\lim_{x\to 0^-} f(x)$;

(b) $\lim_{x\to 0^+} f(x)$;

(c) $\lim_{x\to 0} f(x)$.

(d) $\lim_{x\to 1} f(x)$;

(a) $$\lim_{x\to 0^-} f(x) = \lim_{x\to 0}\left(\frac{1}{x}\tan\frac{x}{2}\right)$$

$$= \lim_{x\to 0}\left(\frac{1}{x}\cdot\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right)$$

$$= \lim_{x\to 0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\cdot\frac{1}{2\cos\frac{x}{2}}\right)$$

$$= \lim_{x\to 0}\frac{1}{2\cos\frac{x}{2}}$$

$$= \frac{1}{2\times 1}$$

$$= \frac{1}{2} \quad //$$

(b) $$\lim_{x\to 0^+} f(x) = \lim_{x\to 0}\frac{-\cancel{(x-1)}}{2\cancel{(x-1)}}$$

$$= -\frac{1}{2}$$

(c) $\because \lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$

$\therefore \lim_{x \to 0} f(x)$ does not exist. //

(d) $\lim_{x \to 1^-} f(x) = \lim_{x \to 1} \frac{-(x-1)}{2x-2}$

$= -\frac{1}{2}$

$\therefore \lim_{x \to 1^-} f(x) = -\frac{1}{2}$ //

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1} \frac{x^2-4x+3}{x^2+2x-3}$$

$$= \lim_{x \to 1} \frac{(x-3)\cancel{(x-1)}}{\cancel{(x-1)}(x+3)}$$

$$= \lim_{x \to 1} \frac{x-3}{x+3}$$

$$= -\frac{1}{2}$$

$\because \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)$,

$\therefore \lim_{x \to 1} f(x) = -\frac{1}{2}$

2 Q2 3 / 3

✓ - 0 pts Correct

Part B:

3. (a) Let $f(x) = \dfrac{1}{\sqrt{5-4x-x^2}}$. Express the domain and range of f in interval notation.

   (b) Let $f(x) = x^2 - 1,\ g(x) = \dfrac{1}{3}\log_2 x$. Express the domain and range of $g \circ f$ in interval notation.

(a) Domain: $5 - 4x - x^2 > 0$

$$-5 < x < 1$$

$$\therefore (-5, 1) \ //$$

For $\sqrt{5-4x-x^2}$, the maximum value exists where $x = -\dfrac{-4}{2(-1)} = -2$.

where the value of $f(x)$ will attain the minimum which is $\dfrac{1}{\sqrt{9}} = \dfrac{1}{3}$.

$\therefore$ Range of f: $[\frac{1}{3}, \infty)$ //

(b) $g \circ f = g(f(x))$

$$= \frac{1}{3}\log_2(x^2 - 1)$$

Domain: $x^2 - 1 > 0$

$$x^2 > 1$$

$$x < -1 \quad \text{or} \quad x > 1$$

$$\therefore (\infty, -1) \cup (1, \infty) \ //$$

3 Q3a 4 / 4

✓ - 0 pts Correct

5. Without using L'Hôpital's rule, evaluate the following limits. Furthermore, if the limit does not exist but diverges to $\pm\infty$, please indicate so and determine the correct sign.

   (a) $\lim_{x\to 3} \frac{\sqrt{x+1}-2}{4-\sqrt{5x+1}}$;

   (b) $^*\lim_{x\to 8} \frac{x^2-7x-8}{\sqrt[3]{x}-2}$;

(a) $$\lim_{x\to 3} \frac{\sqrt{x+1}-2}{4-\sqrt{5x+1}} \times \frac{\sqrt{x+1}+2}{\sqrt{x+1}+2} \times \frac{4+\sqrt{5x+1}}{4+\sqrt{5x+1}}$$

$$= \lim_{x\to 3} \frac{\overset{1}{\cancel{(x+1-4)}}(4+\sqrt{5x+1})}{\underset{-5}{\cancel{(16-5x-1)}}(\sqrt{x+1}+2)}$$

$$= -5 \times \lim_{x\to 3} \frac{4+\sqrt{5x+1}}{\sqrt{x+1}+2}$$

$$= -5 \times \frac{8}{4}$$

$$= -10 \ //$$

(6) $\lim_{x \to 8} \dfrac{x^2 - 7x - 8}{\sqrt[3]{x} - 2}$

$= \lim_{x \to 8} \dfrac{(x^2 - 7x - 8)(x^{\frac{2}{3}} + 2x^{\frac{1}{3}} + 4)}{(x^{\frac{1}{3}} - 2)(x^{\frac{2}{3}} + 2x^{\frac{1}{3}} + 4)}$

$= \lim_{x \to 8} \dfrac{\cancel{(x-8)}(x+1)(x^{\frac{2}{3}} + 2x^{\frac{1}{3}} + 4)}{\cancel{x - 8}}$

$= 9 \times (4 + 4 + 4)$

$= 108$ //

4 Q5a **2 / 3**

✓ **- 1 pts** incorrect answer

$$\sqrt{6+\sqrt{6+\sqrt{6+a_n}}}$$

7. * Let $\{a_n\}$ be the sequence defined by the recursive relation

$$a_1 = \sqrt{6} \quad \text{and} \quad a_{n+1} = \sqrt{6 + a_n} \text{ for all postive integer } n.$$

Given that $\lim\limits_{n\to\infty} a_n$ exists and equals c. Find the value of c.

$$c = \lim_{n\to\infty} \underbrace{\sqrt{6+\sqrt{6+\sqrt{\cdots\sqrt{6+\sqrt{6}}}}}}_{(n-1) \text{ times}}$$

$$c^2 = \lim_{n\to\infty} \underbrace{6+\sqrt{6+\sqrt{\cdots\sqrt{6+\sqrt{6}}}}}_{(n-1) \text{ times}}$$

$$c^2 - 6 = \lim_{n\to\infty} \underbrace{\sqrt{6+\sqrt{6+\sqrt{\cdots\sqrt{6+\sqrt{6}}}}}}_{(n-1) \text{ times}} \quad (\text{ie. } c)$$

$$c^2 - c - 6 = 0$$

$$c = 3 \;/\!/ \text{ or } -2 \text{ (rejected)}$$

5 Q7 **3 / 3**

✓ **- 0 pts** Correct

8. * The following statements are both false. Give one counterexample for each of them.

(a) If $\lim_{n\to+\infty} a_n = 0$, $\lim_{n\to+\infty} b_n = +\infty$, then $\lim_{n\to+\infty} a_n b_n = 0$.

(b) If $f(x) > 0$ for all $x \in \mathbb{R}$ and $\lim_{x\to 0} f(x)$ exists,
then $\lim_{x\to 0} f(x) > 0$.

$$\frac{e^{2x}}{e^{x^2}}$$

(a) Let $a_n = \frac{1}{n}$ and $b_n = n$,

$$\lim_{n\to\infty} a_n b_n = \lim_{n\to\infty}\left(\frac{1}{n}\right) \times \lim_{n\to\infty}(n)$$

$$= 0 \cdot \infty$$, which is undefined.

$\therefore \lim_{n\to\infty} a_n b_n$ does not exist. //

(b) Let $f(x) = \begin{cases} 4x^2 & \text{for } x \neq 0 \\ 50 & \text{for } x = 0 \end{cases}$

$f(x) > 0$ for all $x \in \mathbb{R}$.

$$\because \lim_{x\to 0} f(x) = 4\left(\lim_{x\to 0} x\right)^2$$
$$= 4 \cdot 0$$
$$= 0$$

$\therefore$ The statement is incorrect. //

6 Q8b 3 / 3

+ **0 pts** incorrect/no solution found on the selected page
+ **2 pts** incorrect explanation
+ **1 pts** incorrect example

✓ + **3 pts** Correct

- **0.5 pts** No solution found (adjusted)