THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1510 Calculus for Engineers (Fall 2021)
Suggested solutions of homework 2

Part A:

1. Let $f(x) = \sin(2x + \pi)$. Use definition (first principle) to find $f'(x)$ for any $x \in \mathbb{R}$.

   Solution: For any $x \in \mathbb{R}$,

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(2(x+h)+\pi) - \sin(2x+\pi)}{h} \\
&= \lim_{h\to 0} \frac{2\cos\left(\frac{(2x+2h+\pi)+(2x+\pi)}{2}\right)\sin\left(\frac{(2x+2h+\pi)-(2x+\pi)}{2}\right)}{h} \\
&= \lim_{h\to 0} \frac{2\cos(2x+h+\pi)\sin(h)}{h} \\
&= \lim_{h\to 0} 2\cos(2x+h+\pi)\frac{\sin(h)}{h} \\
&= 2\cos(2x+\pi)\cdot 1 \\
&= -2\cos(2x).
\end{aligned}$$

2. Let $\mathcal{C}$ be the curve defined by the equation $xy = \ln x + y^3$. Given that $A = (1, 0)$ is a point on $\mathcal{C}$,

   (a) Find $\dfrac{dy}{dx}$ in terms of x and y.

   (b) Find $\left.\dfrac{d^2y}{dx^2}\right|_A$.

   Solution:

   (a) By implicit differentiation,

$$\begin{aligned}
\frac{d}{dx}(xy) &= \frac{d}{dx}\left(\ln x + y^3\right) \\
y + x\frac{dy}{dx} &= \frac{1}{x} + 3y^2\frac{dy}{dx} \\
(3y^2 - x)\frac{dy}{dx} &= y - \frac{1}{x} \qquad (*) \\
\frac{dy}{dx} &= \frac{y - \frac{1}{x}}{3y^2 - x} \\
\frac{dy}{dx} &= \frac{xy - 1}{3xy^2 - x^2}.
\end{aligned}$$

(b) Differentiating both sides of $(*)$ one more time, we have

$$(6y\frac{dy}{dx}-1)\frac{dy}{dx}+(3y^2-x)\frac{d^2y}{dx^2}=\frac{dy}{dx}+\frac{1}{x^2}.$$

At $A=(1,0)$, we have $\left.\frac{dy}{dx}\right|_A=\frac{0-\frac{1}{1}}{3(0)^2-1}=1$, and hence

$$\begin{aligned}(6(0)(1)-1)(1)+(3(0)^2-1)\left.\frac{d^2y}{dx^2}\right|_A&=1+\frac{1}{1^2}\\ -1-\left.\frac{d^2y}{dx^2}\right|_A&=2\\ \left.\frac{d^2y}{dx^2}\right|_A&=-3.\end{aligned}$$

Part B:

3. Determine the point(s) of discontinuity of the function:

$$f(x)=\begin{cases}x^2+3x-1, & \text{if } x\le 0,\\ \dfrac{\sin x}{x}, & \text{if } 0<x\le\pi,\\ \cos x+1, & \text{if } \pi<x.\end{cases}$$

Solution:

Note that $f(0)=0^3+3(0)-1=-1$ and

$$\begin{aligned}\lim_{x\to0^-}f(x)&=\lim_{x\to0^-}(x^2+3x-1)=-1,\\ \lim_{x\to0^+}f(x)&=\lim_{x\to0^+}\frac{\sin x}{x}=1.\end{aligned}$$

Since $\lim_{x\to0^-}f(x)\neq\lim_{x\to0^+}f(x)$, f is discontinuous at $x=0$.

Note that $f(\pi)=\frac{\sin\pi}{\pi}=0$ and

$$\begin{aligned}\lim_{x\to\pi^-}f(x)&=\lim_{x\to\pi^-}\frac{\sin x}{x}=\frac{\sin\pi}{\pi}=0,\\ \lim_{x\to\pi^+}f(x)&=\lim_{x\to\pi^+}(\cos x+1)=\cos\pi+1=0.\end{aligned}$$

Since $\lim_{x\to\pi^-}f(x)=\lim_{x\to\pi^+}f(x)=f(\pi)=0$, f is continuous at $x=\pi$.

Note that f is continuous on each of the intervals $(-\infty,0),(0,\pi),(\pi,\infty)$. Therefore the point of discontinuity of f is $x=0$ only.

4. Find the derivative of

$$f(x) = \begin{cases} x^2 + \cos x & \text{if } \ x < 0; \\ 1 & \text{if } \ x = 0; \\ 2x \sin x + 1 & \text{if } \ x > 0. \end{cases}$$

(Hint: You need to check the differentiability at 0.)

Solution: By formulas,

$$f'(x) = \begin{cases} 2x - \sin x & \text{if } \ x < 0; \\ 2 \sin x + 2x \cos x & \text{if } \ x > 0. \end{cases}$$

At $x = 0$, note that

$$\begin{aligned} Lf'(0) &= \lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h \to 0^-} \frac{h^2 + \cos h - 1}{h} \\ &= \lim_{h \to 0^-} \left(\frac{h^2}{h} + \frac{\cos h - 1}{h} \right) \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} Rf'(0) &= \lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h \to 0^+} \frac{2h \sin h + 1 - 1}{h} \\ &= \lim_{h \to 0^+} 2 \sin h \\ &= 0. \end{aligned}$$

Hence, f is differentiable at $x = 0$ and $f'(0) = 0$. Therefore,

$$f'(x) = \begin{cases} 2x - \sin x & \text{if } \ x < 0; \\ 0 & \text{if } \ x = 0; \\ 2 \sin x + 2x \cos x & \text{if } \ x > 0. \end{cases}$$

5. Find $\dfrac{dy}{dx}$ by logarithmic differentiation if

   (a) $y = \dfrac{(x^2+5)^4}{(e^{-x}+2)\sqrt{x^4+1}}$;

   (b) $y = x^{x+1}$, for $x > 0$.

   Solution:

   (a) By logarithmic differentiation,

$$\begin{aligned}
\ln y &= 4\ln(x^2+5) - \ln(e^{-x}+2) - \frac{1}{2}\ln(x^4+1) \\
\frac{1}{y}\frac{dy}{dx} &= \frac{4(2x)}{x^2+5} - \frac{-e^{-x}}{e^{-x}+2} - \frac{4x^3}{2(x^4+1)} \\
\frac{dy}{dx} &= \frac{(x^2+5)^4}{(e^{-x}+2)\sqrt{x^4+1}}\left(\frac{8x}{x^2+5} + \frac{e^{-x}}{e^{-x}+2} - \frac{2x^3}{x^4+1}\right).
\end{aligned}$$

   (b) By logarithmic differentiation,

$$\begin{aligned}
\ln y &= (x+1)\ln x \\
\frac{1}{y}\frac{dy}{dx} &= \ln x + \frac{x+1}{x} \\
\frac{dy}{dx} &= x^{x+1}(\ln x + \frac{x+1}{x}) \\
&= x^x(x\ln x + x + 1).
\end{aligned}$$

6. * Let a and b be real numbers with $a < b$. Show that the function

$$F(x) = (x-a)(x-b)^2 + x$$

   takes on the value $\dfrac{a+b}{2}$ for some value of x.

   Solution:

   Note that

$$F(a) = (a-a)^2(a-b)^2 + a = a,$$

   and

$$F(b) = (b-a)^2(b-b)^2 + b = b.$$

   Then

$$F(a) = a < \frac{a+b}{2} < b = F(b).$$

   As a polynomial function, F is continuous on $[a, b]$.

   By Intermediate Value Theorem, $F(c) = \dfrac{a+b}{2}$ for some number $c \in [a, b]$.

7. * Let u, v be functions of x. The first order derivative of uv can be obtained by the product rule:

$$(uv)' = u'v + uv'.$$

The general formula for n-th order derivative of uv was derived by the German mathematician Gottfried Wilhelm Leibniz:

$$(uv)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} u^{(k)} v^{(n-k)},$$

where $\binom{n}{k} = C_k^n = \dfrac{n!}{k!(n-k)!}$, the symbol $u^{(k)} = \dfrac{d^k u}{dx^k}$ means the k-th order derivative of u and $u^{(0)} = u$.

By Leibniz's formula, compute $f^{(100)}(x)$ if

$$f(x) = (2x^3 + 5x^2 - x + 3)\cos x.$$

Solution:

Let $u = 2x^3 + 5x^2 - x + 3$ and $v = \cos x$. Then $f(x) = uv$.

Note that

$$\begin{aligned}
u^{(0)} &= 2x^3 + 5x^2 - x + 3 \\
u^{(1)} &= 6x^2 + 10x - 1 \\
u^{(2)} &= 12x + 10 \\
u^{(3)} &= 12 \\
u^{(k)} &= 0 \qquad \text{for} \quad k \geq 4.
\end{aligned}$$

Also, for $l \geq 0$,

$$\begin{aligned}
v^{(4l)} &= \cos x \\
v^{(4l+1)} &= -\sin x \\
v^{(4l+2)} &= -\cos x \\
v^{(4l+3)} &= \sin x.
\end{aligned}$$

By Leibniz's formula,

$$\begin{aligned}
&f^{(100)}(x) \\
={}& \binom{100}{0} u^{(0)} v^{(100)} + \binom{100}{1} u^{(1)} v^{(99)} + \binom{100}{2} u^{(2)} v^{(98)} + \binom{100}{3} u^{(3)} v^{(97)} \\
={}& (2x^3 + 5x^2 - x + 3)\cos x + 100(6x^2 + 10x - 1)(\sin x) \\
&+ \frac{100 \times 99}{1 \times 2}(12x + 10)(-\cos x) + \frac{100 \times 99 \times 98}{1 \times 2 \times 3}(12)(-\sin x) \\
={}& (2x^3 + 5x^2 - 59401x - 49497)\cos x + 100(6x^2 + 10x - 19405)\sin x.
\end{aligned}$$