香港中文大學

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The Chinese University of Hong Kong

二〇一七至一八年度上學期科目考試 Course Examination 1st Term, 2017-18

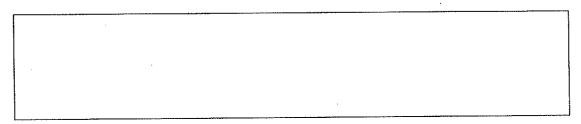
科目編號及名稱 Course Code & Title 時間 Time allowed 學號 Student I.D. No	: MATH1510A/B/C/D	小時 hours 0 座號	分鐘 0 minutes		
	all questions. Pleable for every step.	se show the	work with as mu	ıch detail	
• There are a total of 100 points and 11 questions.					
• Answer	• Answer Questions 1 and 10 in the question paper.				
	• Answer Questions 2 - 9, and 11 in the examination answer book. If you need extra room to answer questions, raise your hands.				
	• You must return your question paper and examination answer book(s) with your scratch paper at the end of the examination.				
Please answer	all questions:				
1. Write yo	ur answer only inside	the box. No	justification is need	ded.	
(a) (1 pc	oint) The derivative o	of $f(x)$ with re	espect to x , $f'(x)$, is	defined as	
	$f'(x) = \lim_{x \to a} f'(x)$			•	
				•	

(b) (1 point) (Chain Rule) If g is differentiable at x and f is differentiable at y = g(x), then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and the derivative is given by

$$\frac{d}{dx}\Big(\left(f\circ g\right)(x)\Big) = \underline{\hspace{1cm}}$$



(c) (1 point) The derivative of $\cot x$ is



(d) (1 point) The derivative of $\sin^{-1}(x)$, -1 < x < 1, is

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- (e) (1 point) (Mean Value Theorem) Suppose f(x) satisfies:
 - i. f(x) is continuous on the closed interval [a, b].
 - ii. f(x) is differentiable on the open interval (a, b).

Then there exists c satisfying a < c < b such that

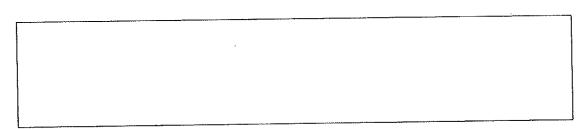
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(f) (1 point) Let a > 1. Then $\int a^x dx$ is



(g) (1 point) Let b > 0. Then $\int \sin(bx) dx$ is



(h) (1 point) The Maclaurin series (or Taylor series at x=0) of e^x is

$$e^x = \sum$$

(i) (1 point) Suppose that f is a function of two independent variables x and y. The partial derivative of f with respect to x is defined as

$$\frac{\partial f(x,y)}{\partial x} = \lim$$

Question 1 is worth 9 points. You must return your question paper and examination answer book(s) with your scratch paper at the end of the examination.

2. Let

$$f(x) = \begin{cases} \frac{x^2 - x - 6}{x - 3} & \text{if } x \neq 3; \\ c & \text{if } x = 3. \end{cases}$$

- (a) (3 points) Find $\lim_{x\to 3} f(x)$;
- (b) (2 points) Find the value of c for which f(x) is continuous at x = 3.
- 3. (3 points) Let

$$f(x) = \begin{cases} 4 & \text{if } x < 2; \\ 2x & \text{if } x \ge 2. \end{cases}$$

Use the definition of derivative to determine whether f(x) is differentiable at x = 2 or not.

4. (a) (2 points) Find $\frac{dy}{dx}$ if

$$y = \left(\frac{1}{x} + \frac{1}{x^2}\right)^{-2};$$

(b) (2 points) Find $\frac{dy}{dx}$ if

$$y = \cos (1 + \sin x);$$

(c) (2 points) Find $\frac{dy}{dx}$ if

$$y = x^{\sqrt{x}}$$
, for $x > 0$;

(d) (2 points) Evaluate $\frac{dy}{dx}\Big|_{(x,y)=(1,1)}$ if

$$y^3x^2 - yx + 2y^2 = 2x;$$

(e) (2 point) Find

$$\frac{d}{dx} \left\{ \int_{x^2}^{x^3} \sin^2 t \, dt \right\}.$$

5. Evaluate the following integrals:

(a) (2 points)
$$\int_0^3 2x \, e^{x^2} \, dx$$
;

(b) (2 points)
$$\int_4^9 \frac{2}{x-3} dx$$
;

(c) (2 points)
$$\int (1 - \cos x)^5 \sin x \, dx$$
;

(d) (2 points)
$$\int x \cos x \, dx$$
;

(e) (2 points)
$$\int \frac{1}{\sqrt{x^2-1}} dx$$
, where $x>1$.

- 6. Solve the following problems separately.
 - (a) Let

$$f(x) = x^3 + 3x^2 - 24x + 7.$$

- i. (4 points) Find all critical point(s)(or stationary point(s)) of f. Then, find the interval(s) on which f is increasing, and those on which f is decreasing.
- ii. (2 points) Determine whether each critical point is a local minimum or maximum, or neither.
- (b) (4 points) Let

$$g(x) = 2\sin^2 x - x, \ 0 < x < \pi.$$

Find all the critical point(s) of g(x) on the interval $(0, \pi)$ and apply the Second Derivative Test for them.

(c) (3 points) Let

$$h(x) = e^{2x} + 16e^x - 10x^2.$$

Find all the inflection point(s) of h(x).

- 7. Solve the following problems separately.
 - (a) (4 points) Find the area of the region in the xy-plane bounded by the curves C_1 and C_2 :

$$\begin{cases} C_1: & y=x, \\ C_2: & x^2-y=2. \end{cases}$$

(b) Let \mathcal{R} be the region in the xy-plane bounded by

the curves $y = x^3$, x = 0, and y = 1.

Express the volumes of the following solids as integrals (You do not need to evaluate the integrals):

- i. (2 points) The solid obtained by revolving \mathcal{R} about the x-axis.
- ii. (2 points) The solid obtained by revolving \mathcal{R} about the y-axis.

- 8. Solve the following problems separately.
 - (a) Given that

$$u(x,y) = x \ln y$$
.

- i. (2 points) Find $u_x = \frac{\partial u}{\partial x}$ and $u_y = \frac{\partial u}{\partial y}$.
- ii. (2 points) Let $u_{yx} = \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial y} \right)$ and $u_{xy} = \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial x} \right)$. Show that $u_{yx} = u_{xy}$.
- iii. (1 point) Find $u_{xx} u_{yy} u_{yx} u_{xy}$.
- (b) (3 points) Let

$$z = f(r, s),$$

and

$$r = e^x + e^{-y}, \quad s = e^{-x} - e^y$$

where f, r and s are assumed differentiable, and x and y are independent variables of real numbers.

Show that

$$\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = r \frac{\partial z}{\partial r} - s \frac{\partial z}{\partial s}.$$

(c) (3 points) Compute

$$\int_0^1 \int_0^1 \cos(2x + y) \, dy \, dx.$$

(d) (3 points) Find the volume of the solid under the graph of the function

$$w(x,y) = xe^{y^2}$$

over the region

$$\mathcal{D} = \{(x, y) \mid 0 \le x \le 1, x^2 \le y \le 1\}.$$

- 9. Solve the following problems.
 - (a) (3 points) Recall that the Taylor polynomial of degree n about x = a for the function f(x) is given by

$$T_n(x) = \sum_{i=0}^n \frac{f^{(i)}(x)}{n!} \Big|_{x=a} (x-a)^i.$$

Find the Taylor polynomial of degree 3 for $f(x) = \sqrt{x}$ at x = 1, i.e., $T_3(x)$.

(b) i. (2 points) Find constants A and B such that

$$\frac{-2x+7}{x^2-7x+12} = \frac{A}{x-3} + \frac{B}{x-4}.$$

ii. (2 points) Given that

$$\frac{1}{1-x} = \sum_{k=0}^{+\infty} x^k = 1 + x + x^2 + x^3 + \cdots, -1 < x < 1.$$

Find the Taylor series at x = 0 for

$$f(x) = \frac{-2x+7}{x^2 - 7x + 12}.$$

- iii. (2 points) Find the radius of convergence for the above Taylor series for f(x).
- (c) (2 points) Given that

$$f(x) = x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} \cdots,$$

$$g(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

Find the first 3 nonzero terms in the Taylor series for f(x)g(x).

10. True or False Questions Not to be provided P.10 - P.11 (Question 10)

- 11. Solve the following problems separately.
 - (a) (2 point) Suppose

$$x^x y^y z^z = 7$$

determines a function z = z(x, y). Use the technique of logarithmic differentiation to show that at x = y = z,

$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{1}{x(1 + \ln x)}.$$

- (b) i. (2 point) Let $f(x) = x^{\frac{1}{3}} \frac{1}{3}x \frac{2}{3}$ for x > 0. Show that $f(x) \le 0$ for all x > 0.
 - ii. (1 point) By using the previous result in i, or otherwise, show that for any a,b>0,

 $a^{\frac{1}{3}}b^{\frac{2}{3}} \le \frac{1}{3}a + \frac{2}{3}b.$

(c) (2 point) Define $I_n = \int (\ln x)^n dx$ for any nonnegative integer n. Show that for any positive integer n,

$$I_n = x \left(\ln x\right)^n - nI_{n-1}.$$

Hence, evaluate I_3 .

(d) (2 point) Let $a \in \mathbb{R}$ and let $f : [0, a] \to \mathbb{R}$ be a continuous function. Show that

$$\int_0^a f(x) \, dx = \int_0^a f(a - x) \, dx.$$

Hence, find

$$\int_0^{\pi/2} \frac{\cos^3 x}{\sin x + \cos x} \, dx.$$

(e) (2 point) Write the first 2 non-zero terms in the Taylor series of $\sin(x^2)$, and compute

$$\lim_{x\to 0} \frac{\sin\left(x^2\right) - x^2}{x^6}.$$

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