THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1510 Calculus for Engineers (Fall 2021)
Suggested solutions of coursework 5

Part A

1. Find $f'(x)$ if

   (a) $f(x) = ex^{\pi} + \sqrt{2}\pi^{x} + \pi^{\pi}$

   (b) $f(x) = \dfrac{\log x}{x+1}$

   (c) $f(x) = \sec(\tan x)$

   (d) $f(x) = \ln(\ln(e^{x} + x))$

   (e) $f(x) = \cos^{2}(2^{x})$

   Solution:

   (a) $f'(x) = e\pi x^{\pi-1} + \sqrt{2}(\ln \pi)\pi^{x}$

   (b) $f'(x) = \dfrac{\frac{1}{(\ln 10)x}(x+1) - \log x}{(x+1)^{2}} = \dfrac{x+1-x\ln x}{(\ln 10)x(x+1)^{2}}$

   (c) $f'(x) = \sec(\tan x)\tan(\tan x) \cdot \sec^{2} x$

   (d) $f'(x) = \dfrac{1}{\ln(e^{x}+x)} \cdot \dfrac{1}{e^{x}+x} \cdot (e^{x}+1)$

   (e) $f'(x) = 2\cos(2^{x})(-\sin(2^{x}))(\ln 2)2^{x} = -2^{x+1}(\ln 2)\sin(2^{x})\cos(2^{x})$

2. Find $\dfrac{dy}{dx}$ if

   (a) $y = \dfrac{3^{x}\sqrt[3]{x^{2}+4}}{\sqrt{e^{x}+1}}$

   (b) $y = \sin^{-1}(3x+1)$

   (c) $y = (\sin x)^{x}$

   (d) $x^{3} - 2xy + 2y^{2} = 5$
   Express your answer in terms of x, y.

Solution:

(a) By logarithmic differentiation,

$$\begin{aligned}\frac{d}{dx}\ln y &= \frac{d}{dx}\left[x\ln 3 + \frac{1}{3}\ln(x^2+4) - \frac{1}{2}\ln(e^x+1)\right]\\ \frac{1}{y}\cdot y' &= \ln 3 + \frac{1}{3}\cdot\frac{2x}{x^2+4} - \frac{1}{2}\cdot\frac{e^x}{e^x+1}\\ y' &= \left[\ln 3 + \frac{2x}{3(x^2+4)} - \frac{e^x}{2(e^x+1)}\right]\frac{3^x\sqrt[3]{x^2+4}}{\sqrt{e^x+1}}.\end{aligned}$$

(b) $y' = \dfrac{1}{\sqrt{1-(3x+1)^2}}\cdot(3) = \dfrac{3}{\sqrt{1-(3x+1)^2}}$

(c) By logarithmic differentiation,

$$\begin{aligned}\frac{d}{dx}\ln y &= \frac{d}{dx}\left(x\ln(\sin x)\right)\\ \frac{1}{y}\cdot y' &= \ln(\sin x) + x\cdot\frac{\cos x}{\sin x}\\ y' &= \left(\ln(\sin x) + \frac{x\cos x}{\sin x}\right)(\sin x)^x.\end{aligned}$$

(d) By implicit differentiation,

$$\begin{aligned}\frac{d}{dx}\left(x^3-2xy+2y^2\right) &= \frac{d}{dx}(5)\\ 3x^2-2y-2xy'+4yy' &= 0\\ (4y-2x)y' &= 2y-3x^2\\ y' &= \frac{2y-3x^2}{4y-2x}.\end{aligned}$$

Part B

3. Let $f(x) = \ln(2x+4)$ for $x > -2$.

(a) Find $f'(x)$, $f''(x)$ and $f'''(x)$.

(b) Let n be a positive integer. Write down $f^{(n)}(x)$.

Solution:

(a) $f'(x) = 2(2x+4)^{-1}$

$f''(x) = 2^2(-1)(2x+4)^{-2}$

$f'''(x) = 2^3(-1)(-2)(2x+4)^{-3}$

(b) $f^{(n)} = 2^n(-1)(-2)\cdots(-(n-1))(2x+4)^{-n} = (-1)^{n-1}2^n(n-1)!(2x+4)^{-n}$

4. Let $f(x) = \sin 2x$.

   (a) Find $f(0)$, $f'(0)$, $f''(0)$, $f'''(0)$ and $f^{(4)}(0)$.

   (b) Let n be a positive integer. Write down $f^{(2n-1)}(0)$ and $f^{(2n)}(0)$.

   Solution:

   (a) $f(0) = 0$
   $f'(x) = 2\cos 2x \implies f'(0) = 2$
   $f''(x) = -4\sin 2x \implies f''(0) = 0$
   $f'''(x) = -8\cos 2x \implies f'''(0) = -8$
   $f^{(4)}(x) = 16\sin 2x \implies f^{(4)}(0) = 0$

   (b) $f^{(2n-1)}(0) = (-1)^{n-1}2^{2n-1}$
   $f^{(2n)}(0) = 0$

5. Let $y = f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$.

   (a) Show that $\dfrac{dy}{dx} = 1 - y^2$.

   (b) Show that

   $$\frac{d^3y}{dx^3} + 2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = C$$

   for some constant C. Also, find the value of C.

   Solution:

   (a)

   $$\begin{aligned} \text{LHS} = \frac{dy}{dx} &= \frac{(e^x + e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2} \\ &= \frac{(e^{2x} + 2 + e^{-2x}) - (e^{2x} - 2 + e^{-2x})}{(e^x + e^{-x})^2} \\ &= \frac{4}{(e^x + e^{-x})^2} \end{aligned}$$

   $$\begin{aligned} \text{RHS} = 1 - y^2 = 1 &- \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)^2 \\ &= \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2} \\ &= \frac{4}{(e^x + e^{-x})^2} \\ &= \text{LHS} \end{aligned}$$

(b) By part (a),

$$
\begin{aligned}
\frac{dy}{dx} &= 1 - y^2 \\
\frac{d}{dx}\left(\frac{dy}{dx}\right) &= \frac{d}{dx}\left(1 - y^2\right) \\
\frac{d^2y}{dx^2} &= -2y\frac{dy}{dx} \\
\frac{d^3y}{dx^3} &= \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}\left(-2y\frac{dy}{dx}\right) \\
&= -2y\frac{d^2y}{dx^2} - 2\left(\frac{dy}{dx}\right)^2
\end{aligned}
$$

Hence

$$\frac{d^3y}{dx^3} + 2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = 0,$$

and $C = 0$.