THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1510 Calculus for Engineers (Fall 2021)
Homework 3
Deadline: Nov. 6 at 23:00

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Class: MATH1510G

I acknowledge that I am aware of University policy and regulations on honesty in academic work, and of the disciplinary guidelines and procedures applicable to breaches of such policy and regulations, as contained in the website http://www.cuhk.edu.hk/policy/academichonesty/

David — Signature

22-10-2021 — Date

General Guidelines for Homework Submission.

- Please submit your answer to Gradescope through the centralized course MATH1510A-I in Blackboard.
- In Gradescope, for each question, please indicate exactly which page(s) its answer locates. **Answers of incorrectly matched questions will not be graded**.
- **Late submission will NOT be graded and result in zero score.** Any answers showing evidence of plagiarism will also score zero; stronger disciplinary action may also be taken.
- Points will only be awarded for answers with sufficient justifications.
- All questions in **Part A** along with some selected questions in **Part B** will be graded. Question(s) labeled with * are more challenging.

Part A:

1. **Procedure for graphing functions using Calculus**

Step 1: Pre-calculus analysis:

(a) Find the domain of the function.
(b) Find the $x-$ and $y-$ intercepts.
(c) Test for symmetry with respect to the $y-$axis and the origin. (Verify whether the function is even or odd or neither or both).

Step 2: Calculus analysis:

(a) Use the first derivative to find the critical points and to find out where the graph is increasing and decreasing.
(b) Test the critical points for local maxima and minima.
(c) Use the second derivative to find out where the graph is concave upward and concave downward, and to locate inflection points.
(d) Find all asymptotes (horizontal, vertical), if any.

Step 3: Plot all critical points, inflection points, and $x-$ and $y-$ intercepts.

Step 4: Sketch the graph.

Sketch the graph of

$$f(x) = \frac{x}{(x-1)^2}$$

following the above procedure.

Domain of $f(x)$: $x \neq 1$

$= (-\infty, 1) \cup (1, \infty)$

Sub $f(x) = 0$: $x = 0$:

x-intercept: $(0, 0)$ = y-intercept

$\because f(-x) = \dfrac{-x}{(-x-1)^2}$

$= -\dfrac{x}{(x+1)^2}$

$\neq f(x)$ & $-f(x)$

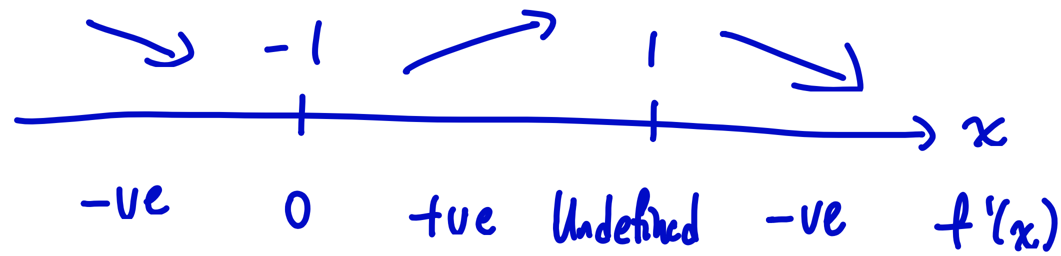
$\therefore$ The function is not a even or odd function. //

$$f'(x) = \frac{(x-1)^2 - x(2)(x-1)}{(x-1)^4}$$

$$= \frac{x-1-2x}{(x-1)^3}$$

$$= -\frac{x+1}{(x-1)^3}$$

$\therefore$ We have:



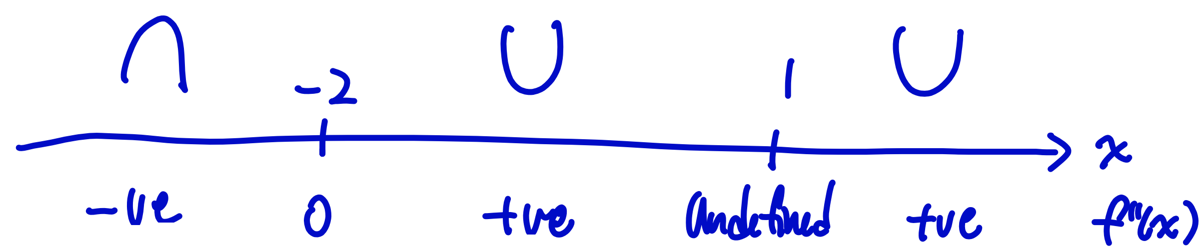
$$f''(x) = -\frac{(x-1)^3 - (x+1)(3)(x-1)^2}{(x-1)^6}$$

$$= -\frac{(x-1) - 3(x+1)}{(x-1)^4}$$

$$= \frac{3x+3-x+1}{(x-1)^4}$$

$$= \frac{2(x+2)}{(x-1)^4}$$

$\therefore$ We have:



$\because \lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} \frac{x}{(x-1)^2}$

$= \frac{1}{0}$

$= +\infty$

$\therefore$ $x = 1$ is the vertical asysptote of $f(x)$. //

$\because \lim\limits_{x = \infty} f(x) = \lim\limits_{x \to \infty} \frac{x}{(x-1)^2}$

$= \lim\limits_{x \to \infty} \frac{\frac{1}{x}}{1 - \frac{2}{x} + \frac{1}{x^2}}$

$= \frac{0}{1}$

$= 0$

$\therefore$ $y = 0$ is the horizontal asysptote of $f(x)$. //

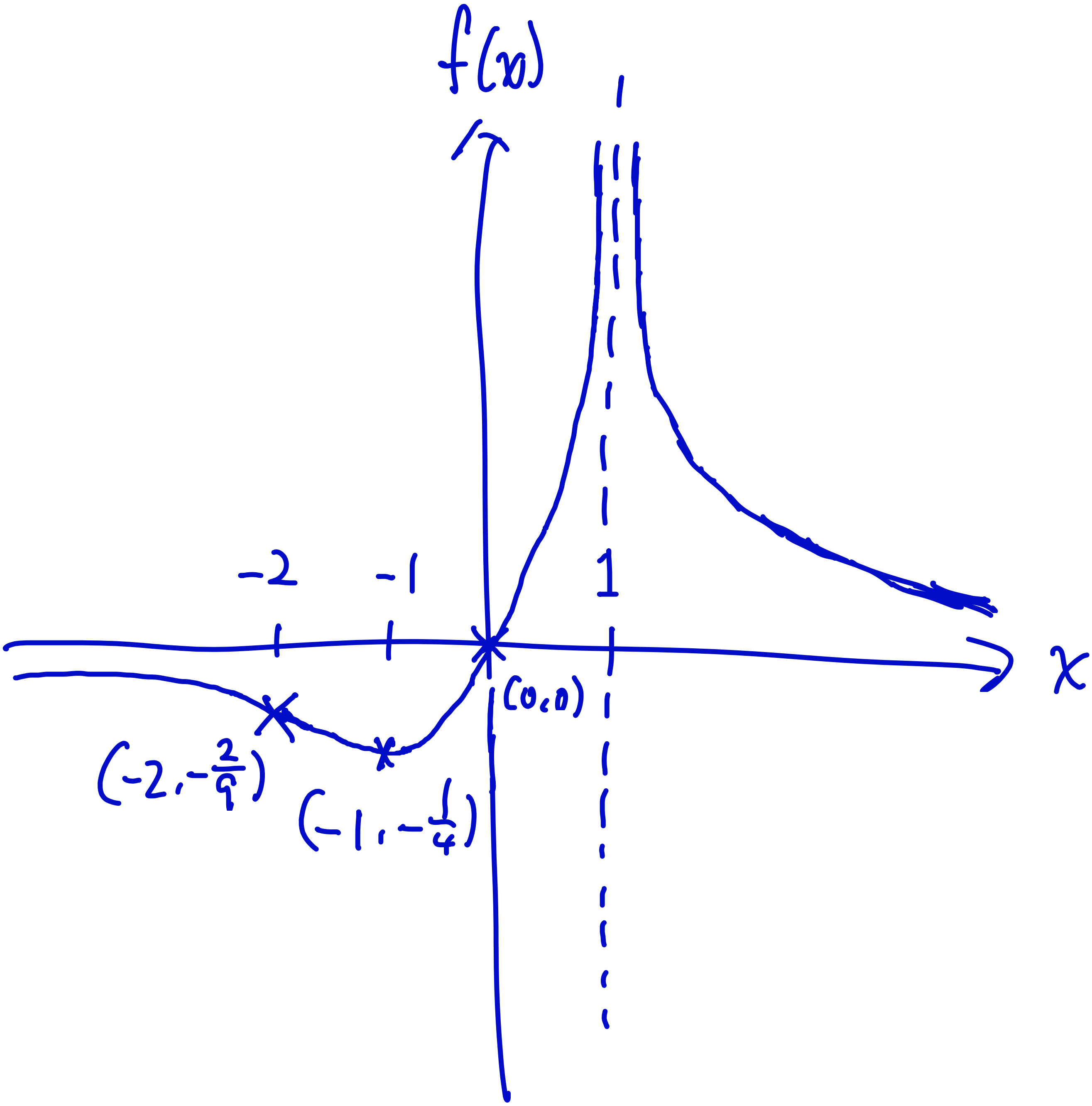
$\because f(-2) = -\frac{2}{9}$

$f(-1) = -\frac{1}{4}$

$\therefore$ The function passes through $(-2, -\frac{2}{9})$,

$(-1, -\frac{1}{4})$ //

$\therefore$ Curve-sketching :



$$\frac{dV}{dt} = 100\pi$$

Part B:

2. A spherical balloon is inflated with helium at the rate of $100\pi\text{ft}^3/\text{min}$. How fast is the balloon's radius increasing at the instant the radius is 5ft? How fast is the surface area increasing? Recall that

$$V(t) = \frac{4}{3}\pi r(t)^3, S(t) = 4\pi r(t)^2,$$

where $V(t)$ and $S(t)$ are the volume and surface of the sphere where the radius is given by $r(t)$.

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$$

$$100\pi = 4\pi r^2 \times \frac{dr}{dt}$$

$$\left.\frac{dr}{dt}\right|_{r=5} = \frac{\cancel{25}}{\cancel{25}}$$

$$= 1 \text{ ft/min}$$

$$\frac{dS}{dt} = \frac{dS}{dr} \times \frac{dr}{dt}$$

$$\frac{dS}{dt} = 8\pi r \times \frac{dr}{dt}$$

$$\left.\frac{dS}{dt}\right|_{r=5} = 40\pi \times 1$$

$$= 40\pi \text{ ft}^2/\text{min}$$

3. Show that
$$\frac{2}{\pi}x < \sin x < x, \quad x \in (0, \frac{\pi}{2}).$$

Let $f(x) = -\cos x$.

$\because$ The function of $f(x)$ is continuous & differentiable when $0 \leq x \leq \frac{\pi}{2}$.

$\therefore$ By the Mean Value Theorm, there exists

$$f'(c) = \frac{f(b)-f(a)}{b-a} \quad \text{for } c \in (a,b).$$

when $a < b$. //

Let $b = \frac{\pi}{2}$, $a = 0$:

$$\sin c = \frac{0+1}{\frac{\pi}{2}-0}$$

$$= \frac{2}{\pi}$$

$$c = \sin^{-1}\left(\frac{2}{\pi}\right) > \sin c$$

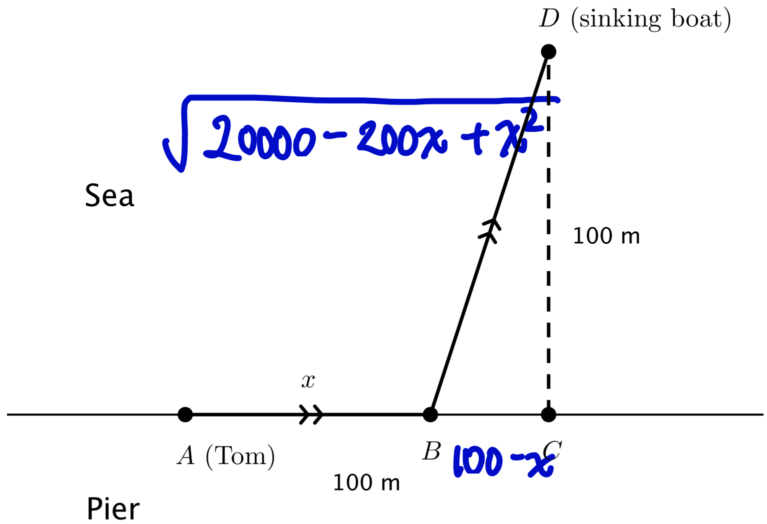
$\because$ c and sin c are smaller than 1,

$$c \sin c = \frac{2}{\pi} c$$

$$< \sin c$$

$\therefore$ We have $\frac{2}{\pi}x < \sin x < x$ //

4. Tom, a lifeguard stationed at point A, spotted a sinking boat at point D:



To get to point D as soon as possible, he decided to run from A to B and swim from B to D. Suppose his running and swimming speeds are 9 and 1.8 respectively, and $|AC| = |CD| = 100$. Denote $|AB| = x,\ x \in [0, 100]$.

(a) Express the total time taken of the trip T as a function of x.

(b) Find all the critical points of $T(x)$ over the interval $(0, 100)$.

(c) Find the value(s) of x that minimizes the total time taken and the corresponding T (correct to 2 d.p.).

(a) $$|BD| = \sqrt{100^2 + (100-x)^2}$$

$$= \sqrt{20000 - 200x + x^2}$$

$\therefore$ Total time taken: $$\frac{x}{9} + \frac{\sqrt{20000-200x+x^2}}{1.8}$$

$$= T(x)$$

(b) $$T'(x) = \frac{1}{9} + \frac{5}{9} \times \frac{1}{2}(20000-200x+x^2)^{-\frac{1}{2}}(2x-200)$$

$$= \frac{1}{9} + \frac{5(x-100)}{9\sqrt{20000-200x+x^2}}$$

$T'(x) = 0$:

$$-\frac{1}{9} = \frac{5(x-100)}{9\sqrt{20000-200x+x^2}}$$

$$\sqrt{20000-200x+x^2} = 5(100-x)$$

$$20000-200x+x^2 = 25(10000-200x+x^2)$$

$$24x^2-4800x+230000 = 0$$

$$x = 79.58758548\ldots \text{ or } 120.4124145\ldots \text{ (rejected.)}$$

$\therefore$ The critical point $(79.58758548\ldots, 65.5442165\,1\ldots)$

$$\approx (79.59, 65.54) \quad (2.\text{dp.})$$

(c) We have $x_1 = 79.58758548\ldots$

x	$0 \leq x < x_1$	$x = x_1$	$x_1 < x \leq 100$
$T(x)$	$\searrow$		$\nearrow$
$T'(x)$	$-ve$	0	$+ve$

$\therefore$ The value of x: 79.59 (2.dp.) ;

value of T: 65.54 (2.dp.)

5. In physics, if the displacement of an object is described by a function $x(t)$, then its velocity, denoted by $v(t)$, and its acceleration, denoted by $a(t)$, are given by $x'(t) = \dfrac{dx}{dt}$ and $x''(t) = \dfrac{d^2x}{dt^2}$ respectively.

Ideally, an object attached to a spring oscillates in simple harmonic motion. Its displacement from the equilibrium position would then be a function of time t, given by

$$x(t) = A\cos(\omega t - \varphi),$$

where m is the mass of the object, k is the spring constant, $\omega = \sqrt{\dfrac{k}{m}}$ and A, φ are two constants determined by the initial situation.

(a) Find the velocity $v(t)$ and acceleration $a(t)$ of the object as a function of time.

(b) Find the maximum velocity and acceleration in magnitude and the value(s) of t achieving them.

(c) The kinetic and potential energy of the object are given by

$$K(t) = \frac{1}{2}m(v(t))^2 \quad \text{and} \quad U(t) = \frac{1}{2}k(x(t))^2$$

respectively. Show that the total mechanical energy, i.e. the sum of kinetic energy and potential energy, is independent of time t.

(a) $v(t) = -A\sin(wt - \varphi)(w)$

$= -Aw\sin(wt - \varphi)$ //

$a(t) = -Aw^2\cos(wt - \varphi)$ //

(b) $\because -1 \leq \sin(wt - \varphi) \leq 1$

To attain the maximum of $v(t)$,

$\sin(wt - \varphi) = -1$

$wt - \varphi = \dfrac{3\pi}{2}$, for $[0, 2\pi]$

$t = \dfrac{3\pi - 2\varphi}{2w}$

$\because \quad -1 \leq \cos(wt - \varphi) \leq 1$

To attain the maximum of $a(t)$,

$$\cos(wt-\varphi) = -1$$

$$wt - \varphi = \pi \qquad , \text{for } [0, 2\pi]$$

$$t = \frac{\pi + \varphi}{w}$$

(c) Total mechanic energy: $\underline{\underline{K(t) + U(T)}}$:

$$\frac{1}{2} m \left(-Aw \sin(wt - \varphi) \right)^2 + \frac{1}{2} k \left(A \cos(wt - \varphi) \right)^2$$

$$= \frac{1}{2} m (A^2) \left(\frac{k}{m} \right) \sin^2(wt - \varphi)$$

$$+ \frac{1}{2} k (A^2) \cos^2(wt - \varphi)$$

$$= \frac{1}{2} k (A^2) \left[\sin^2(wt - \varphi) + \cos^2(wt - \varphi) \right]$$

$$= \frac{1}{2} k A^2$$

$\because$ The variable of t is absent in the answer

$\therefore$ The total mechanic energy is independent of time t. //

6. Find the indicated limit, if it exists. Furthermore, if the limit does not exist but diverges to plus or minus infinity, please indicate so, and determine the correct sign. (Make sure that you have an indeterminate form of the right type before you apply L'Hôpital's rule.)

   (a) $\lim_{x\to 0^+}\left(\frac{1}{x}-\csc x\right)$;

   (b) $\lim_{x\to 0}(\cos x)^{1/x}$;

   (c) $\lim_{x\to +\infty}\left(\frac{e^x+\sin x}{e^x-\sin x}\right)$.

(a) $\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{\sin x}\right)$ $(\infty-\infty)$

$= \lim_{x\to 0^+}\left(\frac{\sin x - x}{x\sin x}\right)$ $\left(\frac{0}{0}\right)$

$= \lim_{x\to 0^+}\left(\frac{\cos x - 1}{\sin x + x\cos x}\right)$ $\left(\frac{0}{0}\right)$

$= \lim_{x\to 0^+}\frac{-\sin x}{\cos x+\cos x - \sin x}$

$= \frac{0}{2}$

$= 0$ //

(b) $\lim_{x\to 0}(\cos x)^{\frac{1}{x}}$ (1^∞)

Let $y = (\cos x)^{\frac{1}{x}}$, we have :

$$\ln y = \frac{1}{x}\ln(\cos x)$$

$$\lim_{x\to 0}\ln y = \lim_{x\to 0}\frac{\ln(\cos x)}{x} \qquad \left(\frac{0}{0}\right)$$

$$\ln \lim_{x\to 0} y = \lim_{x\to 0}\frac{\frac{1}{\cos x}(-\sin x)}{1}$$

$$\ln \lim_{x\to 0}(\cos x)^{\frac{1}{x}} = \frac{0}{1}$$

$$\lim_{x\to 0}(\cos x)^{\frac{1}{x}} = e^{0} = 1 \quad //$$

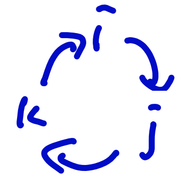
(c) $$\lim_{x\to +\infty}\left(\frac{e^{x}+\sin x}{e^{x}-\sin x}\right) \qquad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to +\infty}\left[\frac{1+(\sin x)e^{-x}}{1-(\sin x)e^{-x}}\right]$$

$$= \frac{1}{1}$$

$$= 1 \quad //$$

7. Assume that $\mathbf{r}_1(t) = \langle a(t), b(t), c(t) \rangle$ and $\mathbf{r}_2(t) = \langle x(t), y(t), z(t) \rangle$ are differentiable. Show that



(a)

$$\frac{d}{dt}(\mathbf{r}_1(t) \cdot \mathbf{r}_2(t)) = \mathbf{r}_1(t) \cdot \mathbf{r}_2'(t) + \mathbf{r}_1'(t) \cdot \mathbf{r}_2(t).$$

(b)

$$\frac{d}{dt}(\mathbf{r}_1(t) \times \mathbf{r}_2(t)) = \mathbf{r}_1(t) \times \mathbf{r}_2'(t) + \mathbf{r}_1'(t) \times \mathbf{r}_2(t).$$

(a) $r_1(t) \cdot r_2(t) = a(t) \times x(t) + b(t) \times y(t) + c(t) \times z(t)$

$$\frac{d}{dt}(r_1(t) \cdot r_2(t)) = a'(t) \times x(t) + a(t) \times x'(t)$$
$$+ b'(t) \times y(t) + b(t) \times y'(t)$$
$$+ c'(t) \times z(t) + c(t) \times z'(t)$$

$\because$ $r_1'(t) = \langle a'(t), b'(t), c'(t) \rangle$

$r_2'(t) = \langle x'(t), y'(t), z'(t) \rangle$

$\therefore \frac{d}{dt}(r_1(t) \cdot r_2(t)) = r_1(t) \cdot r_2'(t) + r_1'(t) \cdot r_2(t)$ //

(b) $r_1(t) \times r_2(t) = \langle b(t) \times z(t) - c(t) \times y(t),$

$a(t) \times z(t) - c(t) \times x(t),$

$a(t) \times y(t) - b(t) \times x(t) \rangle$

$$\frac{d}{dt}\left(r_1(t) \times r_2(t)\right) = \langle b'(t) \times z(t) + b(t) \times z'(t) - c'(t) \times y(t) - c(t) \times y'(t),$$
$$a'(t) \times z(t) + a(t) \times z'(t) - c'(t) \times x(t) - c(t) \times x'(t),$$
$$a'(t) \times y(t) + a(t) \times y'(t) - b'(t) \times x(t) - b(t) \times x'(t) \rangle$$

$$\because \; r_1'(t) = \langle a'(t), b'(t), c'(t) \rangle$$
$$r_2'(t) = \langle x'(t), y'(t), z'(t) \rangle$$

$$\therefore \frac{d}{dt}\left(r_1(t) \times r_2(t)\right) = r_1(t) \times r_2'(t) + r_1'(t) \times r_2(t) \quad //$$