THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1510 Calculus for Engineers (Fall 2021)
Suggested solutions of homework 4

Part A:

1. Evaluate the following indefinite integrals by substitutions.

   (a) $\displaystyle\int (2021x+1)(x-1)^{1510}dx$;

   (b) $\displaystyle\int \frac{(\ln x)^3}{x}dx$.

Solution:

(a) Let $u = x - 1 \implies du = dx$. Hence,

$$\begin{aligned}\int (2021x+1)(x-1)^{1510}dx &= \int (2021(u+1)+1)u^{1510}\,du \\ &= 2021\int u^{1511} + 2022\int u^{1510}\,du \\ &= \frac{2021}{1512}u^{1512} + \frac{2022}{1511}u^{1511} + C \\ &= \frac{2021}{1512}(x-1)^{1512} + \frac{2022}{1511}(x-1)^{1511} + C.\end{aligned}$$

(b) Let $u = \ln x \implies du = \frac{1}{x}\,dx$. Hence,

$$\begin{aligned}\int \frac{(\ln x)^3}{x}dx &= \int u^3\,du \\ &= \frac{1}{4}u^4 + C \\ &= \frac{1}{4}(\ln x)^4 + C.\end{aligned}$$

2. Evaluate the following indefinite integrals by integration by parts.

   (a) $\displaystyle\int x^2 \sin x \, dx$;

   (b) $\displaystyle\int \ln(x + x^2) \, dx$.

   Solution

   (a) Using integration by parts twice,

$$
\begin{aligned}
\int x^2 \sin x \, dx &= -\int x^2 \, d(\cos x) \\
&= -x^2 \cos x + \int 2x \cos x \, dx \\
&= -x^2 \cos x + \int 2x \, d(\sin x) \\
&= -x^2 \cos x + 2x \sin x - \int 2 \sin x \, dx \\
&= -x^2 \cos x + 2x \sin x + 2 \cos x + C.
\end{aligned}
$$

   (b) Using integration by parts,

$$
\begin{aligned}
\int \ln(x + x^2) \, dx &= x \ln(x + x^2) - \int x \cdot \frac{1 + 2x}{x + x^2} \, dx \\
&= x \ln(x + x^2) - \int \frac{1 + 2x}{1 + x} \, dx \\
&= x \ln(x + x^2) - \int \left(\frac{2 + 2x}{1 + x} - \frac{1}{1 + x} \right) dx \\
&= x \ln(x + x^2) - \int 2 \, dx + \int \frac{1}{1 + x} \, dx \\
&= x \ln(x + x^2) - 2x + \ln|1 + x| + C.
\end{aligned}
$$

Part B:

3. Evaluate the following indefinite integrals by trigonometric substitutions.

   (a) $\displaystyle\int \frac{1}{x^2\sqrt{x^2-1}}\,dx$ where $x > 1$;

   (b) * $\displaystyle\int \frac{x^3}{\sqrt{4-x^2}}\,dx$ where $0 < x < 2$.

Solution:

(a) Let $x = \sec\theta$, where $\theta \in (0, \frac{\pi}{2})$. Then $dx = \sec\theta\tan\theta\,d\theta$, and

$$\sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = \sqrt{\tan^2\theta} = \tan\theta.$$

Hence,

$$\begin{aligned}
\int \frac{1}{x^2\sqrt{x^2-1}}\,dx &= \int \frac{1}{\sec^2\theta\tan\theta}\cdot\sec\theta\tan\theta\,d\theta \\
&= \int \frac{1}{\sec\theta}\,d\theta \\
&= \int \cos\theta\,d\theta \\
&= \sin\theta + C \\
&= \frac{\sqrt{x^2-1}}{x} + C.
\end{aligned}$$

(b) Let $x = 2\sin\theta$, where $\theta \in (0, \frac{\pi}{2})$. Then $dx = 2\cos\theta\,d\theta$, and

$$\sqrt{4-x^2} = \sqrt{4(1-\sin^2\theta)} = 2\sqrt{\cos^2\theta} = 2\cos\theta.$$

Hence,

$$\begin{aligned}
\int \frac{x^3}{\sqrt{4-x^2}}\,dx &= \int \frac{(2\sin\theta)^3}{2\cos\theta}\cdot 2\cos\theta\,d\theta \\
&= 8\int \sin^3\theta\,d\theta \\
&= -8\int (1-\cos^2\theta)(-\sin\theta\,d\theta) \\
&= 8\int (\cos^2\theta - 1)\,d(\cos\theta) \\
&= 8\left(\frac{1}{3}\cos^3\theta - \cos\theta\right) + C \\
&= 8\left(\frac{1}{3}\left(\frac{\sqrt{4-x^2}}{2}\right)^3 - \frac{\sqrt{4-x^2}}{2}\right) + C \\
&= -\frac{1}{3}\sqrt{4-x^2}(x^2+8) + C.
\end{aligned}$$

4. Evaluate the following indefinite integrals by partial fraction decomposition.

   (a) $\displaystyle\int \frac{8}{(x-1)(x+1)(x+3)}\,dx$;

   (b) $\displaystyle\int \frac{3x^2+7x}{(x-1)(x^2+4x+5)}\,dx$.

   Solution:

   (a) By partial fractions decomposition,
$$\frac{8}{(x-1)(x+1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+3},$$
for some real constants A, B and C.
Multiplying both sides by $(x-1)(x+1)(x+3)$, we get
$$8 = A(x+1)(x+3) + B(x-1)(x+3) + C(x-1)(x+1).$$
$$\begin{aligned} x\to 1: &\qquad 8 = A(2)(4)+0+0 \implies A = 1\\ x\to -1: &\qquad 8 = 0+B(-2)(2)+0 \implies B=-2\\ x\to -3: &\qquad 8 = 0+0+C(-4)(-2) \implies C = 1.\end{aligned}$$
Thus,
$$\begin{aligned}\int \frac{8}{(x-1)(x+1)(x+3)}\,dx &= \int\left(\frac{1}{x-1}+\frac{-2}{x+1}+\frac{1}{x+3}\right)dx\\ &= \ln|x-1| - 2\ln|x+1| + \ln|x+3| + C'.\end{aligned}$$

   (b) By partial fractions decomposition,
$$\frac{3x^2+7x}{(x-1)(x^2+4x+5)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4x+5},$$
for some real constants A, B and C.
Multiplying both sides by $(x-1)(x^2+4x+5)$, we get
$$3x^2+7x = A(x^2+4x+5) + (Bx+C)(x-1).$$
$$\begin{aligned} x\to 1: &\qquad 10 = A(10) \implies A = 1\\ \text{coefficient of } x^2: &\qquad 3 = A+B \implies B = 2\\ \text{constant term}: &\qquad 0 = 5A - C \implies C = 5.\end{aligned}$$
Thus,
$$\begin{aligned}\int \frac{3x^2+7x}{(x-1)(x^2+4x+5)}\,dx &= \int\left(\frac{1}{x-1}+\frac{2x+5}{x^2+4x+5}\right)dx\\ &= \int\frac{1}{x-1}\,dx + \int\frac{2x+4}{x^2+4x+5}\,dx + \int\frac{1}{x^2+4x+5}\,dx\\ &= \ln|x-1| + \int\frac{d(x^2+4x+5)}{x^2+4x+5} + \int\frac{1}{(x+2)^2+1}\,d(x+2)\\ &= \ln|x-1| + \ln|x^2+4x+5| + \arctan(x+2) + C'.\end{aligned}$$

5. Evaluate the following indefinite integrals by t-substitution.

   (a) $\displaystyle\int \frac{1}{2\sin x + \cos x + 1}\, dx$;

   (b) $\displaystyle\int \frac{1}{2 + \cos x}\, dx$.

   Solution:

   (a) By t-substitution, we have

$$\begin{aligned}
\int \frac{1}{2\sin x + \cos x + 1}\, dx &= \int \frac{1}{2 \cdot \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2} + 1} \cdot \frac{2}{1+t^2} dt \\
&= \int \frac{2}{4t + 1 - t^2 + 1 + t^2}\, dt \\
&= \int \frac{1}{2t+1}\, dt \\
&= \frac{1}{2} \ln|2t+1| + C \\
&= \frac{1}{2} \ln|2\tan\frac{x}{2} + 1| + C.
\end{aligned}$$

   (b) By t-substitution, we have

$$\begin{aligned}
\int \frac{1}{2 + \cos x}\, dx &= \int \frac{1}{2 + \frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2} dt \\
&= \int \frac{2}{2 + 2t^2 + 1 - t^2}\, dt \\
&= \int \frac{2}{t^2+3}\, dt \\
&= \frac{2}{\sqrt{3}} \int \frac{1}{(\frac{t}{\sqrt{3}})^2 + 1}\, d(\frac{t}{\sqrt{3}}) \\
&= \frac{2}{\sqrt{3}} \arctan(\frac{t}{\sqrt{3}}) + C \\
&= \frac{2}{\sqrt{3}} \arctan(\frac{\tan\frac{x}{2}}{\sqrt{3}}) + C.
\end{aligned}$$

6. Derive a reduction formula for

$$I_n = \int x^n \sin x \, dx$$

where n is an integer, $n \geq 2$. Hence, compute I_4.

Solution: For $n \geq 2$, using integration by parts twice,

$$\begin{aligned}
I_n &= \int x^n \sin x \, dx \\
&= -\int x^n \, d(\cos x) \\
&= -x^n \cos x + \int n x^{n-1} \cos x \, dx \\
&= -x^n \cos x + \int n x^{n-1} \, d(\sin x) \\
&= -x^n \cos x + n x^{n-1} \sin x - n \int (n-1) x^{n-2} \sin x \, dx \\
&= -x^n \cos x + n x^{n-1} \sin x - n(n-1) I_{n-2}.
\end{aligned}$$

By the reduction formula above,

$$\begin{aligned}
I_4 &= -x^4 \cos x + 4x^3 \sin x - 4 \cdot 3 \cdot I_2 \\
&= -x^4 \cos x + 4x^3 \sin x - 12 \left(-x^2 \cos x + 2x \sin x - 2 \cdot 1 \cdot I_0\right) \\
&= -x^4 \cos x + 4x^3 \sin x + 12x^2 \cos x - 24x \sin x - 24 \cos x + C.
\end{aligned}$$

7. * Evaluate the following indefinite integrals.

(a) $\displaystyle\int \frac{\sin\sqrt{x}}{\sqrt{x}\cos^3\sqrt{x}}dx$;

(b) $\displaystyle\int \frac{3\sin x}{2-\cos x-\cos^2 x}\,dx$;

(c) $\displaystyle\int \frac{2-\sqrt{x}}{x+1}dx$;

(d) $\displaystyle\int \frac{2}{x(x^{1/3}+2)}\,dx$;

(e) $\displaystyle\int \frac{\sqrt{x}}{e^{\sqrt{x}}}dx$.

Solution:

(a)

$$\begin{aligned}\int \frac{\sin\sqrt{x}}{\sqrt{x}\cos^3\sqrt{x}}dx &= \int \frac{-2}{\cos^3\sqrt{x}}\,d(\cos\sqrt{x})\\ &= \frac{1}{\cos^2\sqrt{x}}+C.\end{aligned}$$

(b)

$$\begin{aligned}\int \frac{3\sin x}{2-\cos x-\cos^2 x}\,dx &= \int \frac{-3}{2-\cos x-\cos^2 x}\,d(\cos x)\\ &= \int \frac{3}{(\cos x+2)(\cos x-1)}\,d(\cos x)\\ &= \int \left(\frac{1}{\cos x-1}-\frac{1}{\cos x+2}\right)\,d(\cos x)\\ &= \ln|\cos x-1|-\ln|\cos x+2|+C.\end{aligned}$$

(c)

$$\begin{aligned}\int \frac{2-\sqrt{x}}{x+1}dx &= \int \frac{2}{x+1}-\int \frac{\sqrt{x}}{x+1}\,dx\\ &= 2\ln|x+1|-\int \frac{\sqrt{x}}{x+1}\,2\sqrt{x}\,d(\sqrt{x})\\ &= 2\ln|x+1|-\int \frac{2x+2}{x+1}\,d(\sqrt{x})+\int \frac{2}{x+1}\,d(\sqrt{x})\\ &= 2\ln|x+1|-\int 2\,d(\sqrt{x})+2\int \frac{1}{(\sqrt{x})^2+1}\,d(\sqrt{x})\\ &= 2\ln|x+1|-2\sqrt{x}+2\arctan(\sqrt{x})+C.\end{aligned}$$

(d) Let $u = x^{1/3} \implies du = \frac{1}{3}x^{-2/3}\,dx$. Hence,

$$\begin{aligned}
\int \frac{2}{x(x^{1/3}+2)}\,dx &= 3\int \frac{2}{x^{1/3}(x^{1/3}+2)}\left(\frac{1}{3}x^{-2/3}\,dx\right) \\
&= 3\int \frac{2}{u(u+2)}\,du \\
&= 3\int \left(\frac{1}{u} - \frac{1}{u+2}\right)du \\
&= 3\left(\ln|u| - \ln|u+2|\right) + C \\
&= 3\left(\ln|x^{1/3}| - \ln|x^{1/3}+2|\right) + C.
\end{aligned}$$

(e) Let $u = \sqrt{x} \implies du = \frac{1}{2\sqrt{x}}\,dx$. Then,

$$\int \frac{\sqrt{x}}{e^{\sqrt{x}}}dx = \int \frac{2(\sqrt{x})^2}{e^{\sqrt{x}}}\left(\frac{1}{2\sqrt{x}}\,dx\right) = 2\int u^2 e^{-u}\,du.$$

Using integration by parts twice,

$$\begin{aligned}
\int u^2 e^{-u}\,du &= -\int u^2\,d(e^{-u}) \\
&= -u^2e^{-u} + \int 2ue^{-u}\,du \\
&= -u^2e^{-u} - \int 2u\,d(e^{-u}) \\
&= -u^2e^{-u} - 2ue^{-u} + \int 2e^{-u}\,du \\
&= -u^2e^{-u} - 2ue^{-u} - 2e^{-u} + C.
\end{aligned}$$

Hence,

$$\int \frac{\sqrt{x}}{e^{\sqrt{x}}}dx = -2xe^{-\sqrt{x}} - 4\sqrt{x}e^{-\sqrt{x}} - 4e^{-\sqrt{x}} + C.$$