Calculus for Engineers

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Contents

Integration of Transcendental Functions

25.1 Introduction

In this chapter, the method of evaluating a transcendental integral by reducing it via the trigonometric formulae to the standard form by some suitable substitution is accomplished simply by integration by substitution and term-by-term integration. In Section 25.2, we will study the integration of $\sin^m x$ and $\cos^m x$ when m is either an odd positive integer or an even positive integer. In Section 25.3, we will study the integration of $\sin^m x$ and $\cos^n x$ when m and n are chosen by a different integer index. In Section 25.4, we will study the integration of $\tan^m x$ and $\sec^n x$. In Section 25.5, we will study the integration of $\sin mx \sin nx$, $\sin mx \cos nx$ or $\cos mx \cos nx$.

25.2 Integration of $\sin^m x$ and $\cos^m x$

25.2.1 When m is an odd positive integer

In order to evaluate

$$\int \sin^m x dx$$

or

$$\int \cos^m x dx$$

when m is an odd positive integer, we may use the following methods:

1. when the index of $\sin x$ is an odd positive integer, then the substitution $\cos x = t$ is used.

2. when the index of $\cos x$ is an odd positive integer, then the substitution $\sin x = t$ is used.

Using the substitution method and the trigonometric identity, that is, $\sin^2 x + \cos^2 x = 1$, the integration can be done by transforming the given transcendental integrand into a sum of algebraic functions.

Example 1 Evaluate

$$\int \sin^3 x dx.$$

Solution. Let $m = 2n + 1$ and $n = 1$. Then the integrand becomes

$$\sin^3 x = \sin^{2+1} x = \sin^2 x \sin x.$$

We have

$$\begin{aligned} I &= \int \sin^3 x dx \\ &= \int \sin^2 x \sin x dx \\ &= \int \left(1 - \cos^2 x\right) \sin x dx. \end{aligned}$$

If we put $\cos x = t$, then $-\sin x dx = dt$. Therefore

$$\begin{aligned} I &= \int (1 - t^2) \sin x \frac{dt}{-\sin x} \\ &= -\int (1 - t^2) dt \\ &= \frac{t^3}{3} - t \\ &= \frac{\cos^3 x}{3} - \cos x + C, \end{aligned}$$

where C is a constant.

□

Example 2 Evaluate

$$\int \sin^m x dx$$

Solution. Let $m = 2n + 1$. Then

$$\begin{aligned} I &= \int \sin^m x dx \\ &= \int \sin^{2n+1} x dx \\ &= \int \sin^{2n} x \sin x dx \\ &= \int \left(1 - \cos^2 x\right)^n \sin x dx. \end{aligned}$$

If we put $\cos x = t$, then $-\sin x dx = dt$. Then

$$\begin{aligned} I &= \int \left(1 - t^2\right)^n \sin x \frac{dt}{-\sin x} \\ &= -\int \left(1 - t^2\right)^n dt. \end{aligned}$$

Now $(1-t^2)^n$ can be expanded in powers of t by the Newton binomial theorem and then term-by-term integration will be performed.

$$\begin{aligned} I &= -\int \left(1 - nt^2 + \frac{n(n-1)}{2!}t^4 - \cdots + (-1)^n t^{2n}\right) dt \\ &= -\left(t - n\frac{t^3}{3} + \frac{n(n-1)}{2!}\frac{t^5}{5} - \cdots + (-1)^n \frac{t^{2n+1}}{2n+1}\right) \\ &= -\left(\cos x - n\frac{\cos^3 x}{3} + \frac{n(n-1)}{2!}\frac{\cos^5 x}{5} - \cdots + (-1)^n \frac{\cos^{2n+1} x}{2n+1}\right). \end{aligned}$$

□

25.2.2 When m is an even positive integer

In order to evaluate

$$\int \sin^m x dx$$

or

$$\int \cos^m x dx$$

when m is an even positive integer, we simply express $\sin^m x$ or $\cos^m x$ in a series of cosine functions of multiple angles of x by using the trigonometric formulae[1]:

$$\sin^2 x = \frac{1-\cos(2x)}{2}$$

or

$$\cos^2 x = \frac{1+\cos(2x)}{2}$$

and then perform the term-by-term integration.

[1]Verify the following trigonometric identity

$$\sin^2 x = \frac{1}{2}(1-\cos(2x)).$$

Solution. Let us recall that

$$\cos(A+B) = \cos A \cos B - \sin A \sin B.$$

If $B = A$, then

$$\begin{aligned} \cos(A+A) &= \cos A \cos A - \sin A \sin A \\ \cos(2A) &= \cos^2 A - \sin^2 A. \end{aligned}$$

Putting the identity $\cos^2 A + \sin^2 A = 1$ into the above formula, we have

$$\begin{aligned} \cos(2A) &= \left(1 - \sin^2 A\right) - \sin^2 A \\ &= 1 - 2\sin^2 A. \end{aligned}$$

Hence

$$\sin^2 A = \frac{1}{2}(1-\cos(2A)).$$

The verification was done. □

Example 3 Evaluate
$$\int \sin^4 x dx$$

Solution. We have
$$\begin{aligned}
I &= \int \sin^4 x dx \\
&= \int \left(\sin^2 x\right)^2 dx && \text{Exponentiation} \\
&= \int \left(\frac{1-\cos(2x)}{2}\right)^2 dx && \text{Trigonometric formula} \\
&= \int \frac{1}{4}\left(1 - 2\cos(2x) + \cos^2(2x)\right) dx && \text{Quadratic expansion} \\
&= \frac{1}{4}\int \left(1 - 2\cos(2x) + \left(\frac{1+\cos(4x)}{2}\right)\right) dx && \text{Trigonometric formula} \\
&= \frac{1}{8}\int (3 - 4\cos(2x) + \cos(4x))\, dx && \text{Simplification} \\
&= \frac{1}{8}\left(3 - 4\sin(2x) + \frac{\sin(4x)}{4}\right) + C, && \text{Substitution method}
\end{aligned}$$
where C is a constant. □

25.3 Integration of $\sin^m x$ and $\cos^n x$

In order to evaluate
$$\int \sin^m x \cos^n x dx,$$
we may use the following rules:

1. If m is an odd positive integer, then the substitution $\cos x = t$ is used.
2. If n is an odd positive integer, then the substitution $\sin x = t$ is used.
3. If $(m+n)$ is an even negative integer, then one puts $\tan x = t$.
   - One converts the given integral in terms of $\tan x$ and $\sec x$ and puts $\tan x = t$. Then one expands using the Newton binomial theorem, if necessary, and performs term-by-by integration.
4. If both m an n are odd integers, then the substitution either $\sin x = t$ or $\cos x = t$ is used. It is advisable to use $\sin x = t$ if $m \geq n$ and $\cos x = t$ if $n \geq m$.
5. If both m and n are even integers, then one converts $\sin^m x \cos^n x$ in terms of multiple angles of x using the following formulae:
$$\cos^2 x = \frac{1}{2}(1 + \cos(2x)),$$

$$\sin^2 x = \frac{1}{2}(1 - \cos(2x)),$$
$$\sin x \cos x = \frac{1}{2}\sin(2x),$$

and

$$2\cos x \cos y = \cos(x + y) + \cos(x - y).$$

Let us try to understand the above mentioned rules with the help of the following worked examples.

Example 4 Evaluate

$$\int \sin^3 x \cos^2 x dx.$$

Solution. We have

$$\begin{aligned} I &= \int \sin^3 x \cos^2 x dx \\ &= \int \sin^2 x \cos^2 x \sin x dx \\ &= \int \left(1 - \cos^2 x\right) \cos^2 x \sin x dx. \end{aligned}$$

If we put $\cos x = t$, then $-\sin x dx = dt$. Therefore

$$\begin{aligned} I &= \int (1 - t^2)t^2 \sin x \frac{dt}{-\sin x} \\ &= -\int (t^2 - t^4)dt \\ &= \int (t^4 - t^2)dt \\ &= \frac{t^5}{5} - \frac{t^3}{3} \\ &= \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C, \end{aligned}$$

where C is a constant.

□

Example 5 Evaluate

$$\int \frac{\cos^5 x}{\sin^2 x} dx.$$

Solution. We have

$$\begin{aligned} I &= \int \frac{\cos^5 x}{\sin^2 x} dx \\ &= \int \frac{\cos^4 x}{\sin^2 x} \cos x dx \\ &= \int \frac{\left(1 - \sin^2 x\right)^2}{\sin^2 x} \cos x dx. \end{aligned}$$

If we put $\sin x = t$, then $\cos x dx = dt$. Therefore

$$\begin{aligned} I &= \int \frac{(1-t^2x)^2}{t^2} \cos x \frac{dt}{\cos x} \\ &= \int \frac{(1-2t^2+t^4)}{t^2} dt \\ &= \int \left(\frac{1}{t^2} - 2 + t^2 \right) dt \\ &= -\frac{1}{t} - 2t + \frac{t^3}{3} \\ &= -\frac{1}{\sin x} - 2\sin x + \frac{\sin^3 x}{3} \\ &= -\operatorname{cosec} x - 2\sin x + \frac{\sin^3 x}{3} + C, \end{aligned}$$

where C is a constant.

□

Example 6 Evaluate

$$\int \frac{1}{\sin^3 x \cos^5 x} dx.$$

Solution. We have

$$I = \int \frac{1}{\sin^3 x \cos^5 x} dx.$$

Here if $m = -3$ and $n = -5$, then $m + n = -8$ is an even negative integer. By dividing the numerator and the denominator of the integrand by $\cos^8 x$, we have

$$\begin{aligned} I &= \int \frac{1/\cos^8 x}{\sin^3 x \cos^5 x / \cos^8 x} dx \\ &= \int \frac{\sec^8 x}{\tan^3 x} dx \\ &= \int \frac{\sec^6 x \sec^2 x}{\tan^3 x} dx \\ &= \int \frac{(1+\tan^2 x)^3 \sec^2 x}{\tan^3 x} dx. \end{aligned}$$

If we put $\tan x = t$, then $\sec^2 x dx = dt$. Therefore

$$\begin{aligned} I &= \int \frac{(1+t^2)^3 \sec^2 x}{t^3} \frac{dt}{\sec^2 x} \\ &= \int \frac{(1+t^2)^3}{t^3} dt \\ &= \int \frac{1+t^6+3t^2+3t^4}{t} dt \\ &= \int \left(\frac{1}{t^3} + t^3 + \frac{3}{t} + 3 \right) dt \\ &= -\frac{1}{2t^2} + \frac{t^4}{4} + 3\ln|t| + \frac{3t^2}{2} \\ &= -\frac{1}{2\tan^2 x} + \frac{\tan^4 x}{4} + 3\ln|\tan x| + \frac{3\tan^2 x}{2} + C, \end{aligned}$$

where C is a constant.

□

Example 7 Evaluate

$$\int \sin^2 x \cos^4 x dx.$$

Solution. We have

$$\begin{aligned} \sin^2 x \cos^4 x &= \left(\sin^2 x \cos^2 x\right) \cos^2 x \\ &= (\sin x \cos x)^2 \cos^2 x \\ &= \left(\frac{1}{2}\sin(2x)\right)^2 \left(\frac{1}{2}(1+\cos(2x))\right) \\ &= \frac{1}{8}\left(\frac{1}{2}(1-\cos(4x))\right)(1+\cos(2x)) \\ &= \frac{1}{16}\left(1+\cos(2x)-\cos(4x)-\cos(4x)\cdot\cos(2x)\right) \\ &= \frac{1}{16}\left(1+\cos(2x)-\cos(4x)-\frac{1}{2}\left(\cos(6x)+\cos(2x)\right)\right) \\ &= \frac{1}{16}\left(1+\frac{1}{2}\cos(2x)-\cos(4x)-\frac{1}{2}\cos(6x)\right) \\ &= \frac{1}{32}\left(2+\cos(2x)-2\cos(4x)-\cos(6x)\right). \end{aligned}$$

Applying integration by substitution and term-by-term integration give

$$\begin{aligned} \int \sin^2 x \cos^4 x dx &= \frac{1}{32}\int \left(2+\cos(2x)-2\cos(4x)-\cos(6x)\right) dx \\ &= \frac{1}{32}\left(2x+\frac{1}{2}\sin 2x-\frac{1}{2}\sin 4x-\frac{1}{6}\sin 6x\right)+C, \end{aligned}$$

where C is a constant.

□

25.4 Integration of $\tan^m x$ and $\sec^n x$

In order to evaluate

$$\int \tan^m x \sec^n x dx,$$

we may use the following rules:

1. If m is an odd positive integer, then the substitution $\sec x = t$ is used.

2. If n is an even positive integer, then the substitution $\tan x = t$ is used.

3. In other cases the guidelines are not as clear-cut.

Example 8 Evaluate

$$\int \tan^8 x \sec^4 x dx.$$

Solution. We have

$$\begin{aligned} I &= \int \tan^8 x \sec^4 x dx \\ &= \int \tan^8 x \sec^2 x \sec^2 x dx \\ &= \int \tan^8 x \left(1 + \tan^2 x\right) \sec^2 x dx. \end{aligned}$$

If we put $\tan x = t$, then $\sec^2 x dx = dt$. Therefore

$$\begin{aligned} I &= \int t^8 \left(1 + t^2\right) \sec^2 x \frac{dt}{\sec^2 x} \\ &= \int t^8 \left(1 + t^2\right) dt \\ &= \int \left(t^8 + t^{10}\right) dt \\ &= \frac{1}{9} t^9 + \frac{1}{11} t^{11} \\ &= \frac{1}{9} \tan^9 x + \frac{1}{11} \tan^{11} x + C, \end{aligned}$$

where C is a constant.

□

Example 9 Evaluate

$$\int \tan^3 x dx.$$

Solution. We have

$$\begin{aligned}
I &= \int \tan^3 x dx \\
&= \int \tan x \cdot \tan^2 x dx \\
&= \int \tan x \cdot \left(\sec^2 x - 1\right) dx \\
&= \int \left(\tan x \sec^2 x - \tan x\right) dx \\
&= \int \tan x \sec^2 x dx - \int \tan x dx.
\end{aligned}$$

In the first integral, if we put $\tan x = t$, then $\sec^2 x dx = dt$. Therefore

$$\begin{aligned}
I &= \int t \sec^2 x \frac{dt}{\sec^2 x} - \int \tan x dx \\
&= \int t dt - \ln|\sec x| \\
&= \frac{1}{2} t^2 - \ln|\sec x| \\
&= \frac{1}{2} \tan^2 x - \ln|\sec x| + C,
\end{aligned}$$

where C is a constant.

□

Example 10 Evaluate

$$\int \sec^3 x dx.$$

Solution. We have

$$\begin{aligned}
I &= \int \sec^3 x dx \\
&= \int \sec^2 x \cdot \sec x dx.
\end{aligned}$$

Using integration by parts, we have

$$\begin{aligned}
I &= \sec x \tan x - \int \sec x \tan^2 x dx \\
&= \sec x \tan x - \int \sec x \left(\sec^2 x - 1\right) dx \\
&= \sec x \tan x - \int \left(\sec^3 x - \sec x\right) dx \\
&= \sec x \tan x - \int \sec^3 x dx + \int \sec x dx.
\end{aligned}$$

It immediately follows that

$$2I = \sec x \tan x + \int \sec x dx.$$

Therefore, we have

$$\begin{aligned} I &= \frac{1}{2}\left(\sec x \tan x + \int \sec x dx\right) \\ &= \frac{1}{2}\left(\sec x \tan x + \ln|\sec x + \tan x|\right) + C, \end{aligned}$$

where C is a constant.

□

25.5 Integration of $\sin mx \sin nx$, $\sin mx \cos nx$ or $\cos mx \cos nx$

In order to evaluate

$$\int \sin mx \sin nx dx,$$

$$\int \sin mx \cos nx dx,$$

or

$$\int \cos mx \cos nx dx,$$

we may use the following trigonometric formulae:

1. $$\sin A \sin B = \frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right).$$

2. $$\cos A \cos B = \frac{1}{2}\left(\cos(A+B) + \cos(A-B)\right).$$

3. $$\sin A \cos B = \frac{1}{2}\left(\sin(A+B) + \sin(A-B)\right).$$

Example 11 Evaluate

$$\int \cos 3x \cos 2x dx.$$

Solution. We have

$$\begin{aligned} I &= \int \cos 3x \cos 2x dx \\ &= \int \frac{1}{2}\left(\cos(3x+2x) + \cos(3x-2x)\right) dx \\ &= \frac{1}{2}\int \cos(5x) dx + \frac{1}{2}\int \cos x dx \\ &= \frac{1}{10}\sin(5x) + \frac{1}{2}\sin x + C, \end{aligned}$$

where C is a constant.

□

Example 12 Evaluate

$$\int \sin 2x \cos 3x dx.$$

Solution. We have

$$\begin{aligned}
I &= \int \sin 2x \cos 3x dx \\
dx &= \int \frac{1}{2} \left(\sin(3x + 2x) - \cos(3x - 2x) \right) \\
&= \frac{1}{2} \int \sin(5x) dx - \frac{1}{2} \int \sin x dx \\
&= -\frac{1}{10} \cos(5x) + \frac{1}{2} \cos x + C,
\end{aligned}$$

where C is a constant.

□