Calculus for Engineers

Jeff Chak-Fu WONG[1]

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Contents

Application of Derivatives

17.1 Motivation

In this chapter we shall study some general theorems which have very important roles in differential calculus. We begin with Rolle's theorem which is most fundamental. We shall show the nature of the function with some conditions defined in the close interval $[a, b]$. Before moving onto the topics of Rolle's theorem and Lagrange's mean value theorem, we recall that constant functions have derivatives of zero. Is it possible for a more complicated function to have derivative zero? In this section we shall answer this question and a related question: How are two functions with the same derivative related? Two important theorems in Section 17.2 and Section 17.3 are studied which give us more information on the behavior of a continuous function f on a closed interval $[a, b]$, when we add the extra assumption that f is differentiable on the open interval (a, b). In Section 17.4, by extending the Lagrange's mean value theorem, the so-called Cauchy mean value theorem is studied. In Section 17.5, the general mean value theorem is presented that explains how Rolle's, Lagrange's and Cauchy's theorems convert one form into another. In Section 17.6, we also presented the mean value theorem for second derivatives by adding the extra assumption that f' is continuous on a closed interval $[a, b]$. In Section 17.7, we study several types of the Taylor remainder theorem by means of generalized mean value theorem, where high-order derivatives of the remainder terms are used. In Section 17.8, we study several types of series expansion results by means of Maclaurin's theorem, where we shall see whether or not a single-valued function can be expressed on series form.

17.2 Rolle's Theorem

Theorem 1 If a function f defined on the closed interval $[a, b]$ exists such that

(i) f is continuous in the closed interval $[a, b]$,

(ii) $f'(x)$ exists for every point in the open interval[1] (a, b), and

(iii) $f(a) = f(b)$,

then there exists at least one real number c between a and b; that is, $c \in (a, b)$ such that

$$f'(c) = 0.$$

[1] $f(x)$ is differentiable in the open interval (a, b).

Proof:

A function f is continuous in $[a, b]$. By the property of continuity, f is bounded and attains its bounds in the closed interval $[a, b]$. That is, if M and m are the supremum and infimum of f in $[a, b]$, then there exists $c, d \in [a, b]$ such that

$$f(c) = m \quad \text{and} \quad f(d) = M.$$

Since $f(a) = f(b)$, therefore taking the value of x greater than a, either the function will be increasing or decreasing or constant.

Now two cases arise:

(i) $m = M$;

(ii) $m \neq M$

Case 1 When $m = M$, then f will be constant in $[a, b]$; this implies that

$$f(x) = M = m \ \forall\ x \in [a, b],$$

and

$$f'(x) = 0 \ \forall\ x \in [a, b].$$

Therefore for every value of x, $f'(x) = 0$, that is, the theorem is valid for any point $c \in (a, b)$.

Case 2 $M \neq m$, that is, $f(x)$ is not constant.

But using the condition $f(a) = f(b)$, by increasing the value of x from a, either the function will increase or decrease. Let $f(x)$ be decrease, then $m \neq f(a) = f(b)$ as shown in Figure 17.1(a). Now let $m = f(c) \neq f(a) = f(b)$. Clearly $c \in (a, b)$ or $a < c < b$ because $m = f(c)$ =infimum of f. This implies that

$$f(x) \geq f(c), \quad \forall x \in [a, b].$$

Therefore, $f(c-h) \geq f(c)$ and $f(c+h) \geq f(c), h > 0$ and $c \pm h \in [a, b]$. Hence, we have

$$\frac{f(c-h) - f(c)}{-h} \leq 0 \quad \text{and} \quad \frac{f(c+h) - f(c)}{h} \geq 0.$$

Now taking the limit when $h \to 0$

$$\lim_{h \to 0} \frac{f(c-h) - f(c)}{-h} \leq 0 \quad \text{implies} \quad Lf'(c) \leq 0$$

and

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \geq 0 \quad \text{implies} \quad Rf'(c) \geq 0,$$

that is,

$$Lf'(c) \leq 0, \quad Rf'(c) \geq 0. \tag{17.1}$$

The function is differentiable at $x = c \in (a, b)$, that is,

$$Rf'(c) = Lf'(c) = f'(c).$$

Therefore, from (17.8), we have

$$f'(c) = 0.$$

Similarly if $M \neq f(a) = f(b)$ as shown in Figure 17.1(b), the above theorem can be easily proved.

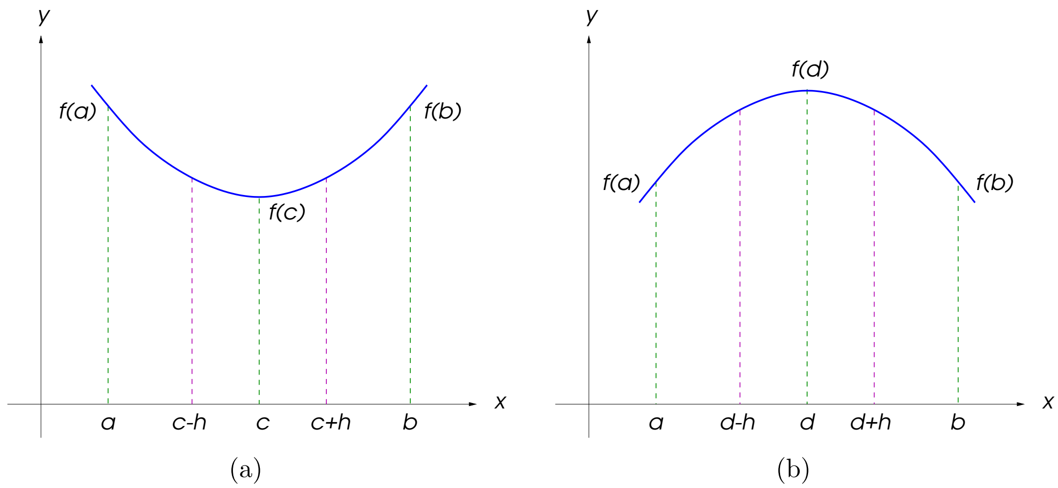


Figure 17.1: $y = f(x)$.

□

Note 1 Rolle's theorem is usually proved using the properties of a continuous function defined on a closed interval, say $[a, b]$.

Note 2 The reader should bear in mind the following observations:

- Rolle's theorem is not valid if $f'(x)$ does not exist for at least one point in the open interval (a, b).
- Rolle's theorem does not hold if f is continuous in the open interval (a, b) but not at the end points.
- Rolle's theorem is not valid if $f(a) \neq f(b)$.

17.2.1 Another Form of Rolle's Theorem

Theorem 2 If a function $f(x)$ defined on $[a, a + h]$ exists such that

(i) $f(x)$ is continuous in closed interval $[a, a + h]$,

(ii) $f(x)$ is differentiable in open interval $(a, a+h)$, and

(iii) $f(a) = f(a+h)$,

then there exists some θ that lies between 0 and 1 $(0 < \theta < 1)$ such that

$$f'(a + \theta h) = 0.$$

17.2.2 Geometrical Significance of Rolle's Theorem

If $f(a) = f(b)$, then the chord joining the points $P(a, f(a))$ and $Q(b, f(b))$ is parallel to the $x-$axis, and $f'(c) = 0$ implies that the tangent to the curve $y = f(x)$ is $R(c, f(c))$ which parallel to the $x-$axis.

From Figure 17.2, it is clear that if we move the chord PQ parallel to itself, then we touch at least one position where the line is a tangent to the curve $y = f(x)$ and this will certainly be so at a point which is farthest from the chord. It is worth mentioning that there can be more than one point where the tangent is parallel to the $x-$axis, as illustrated in Figure 17.3.

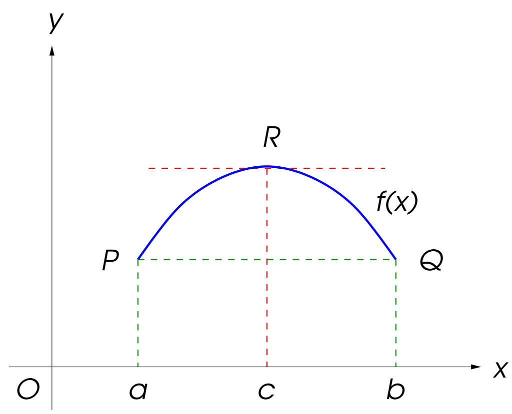
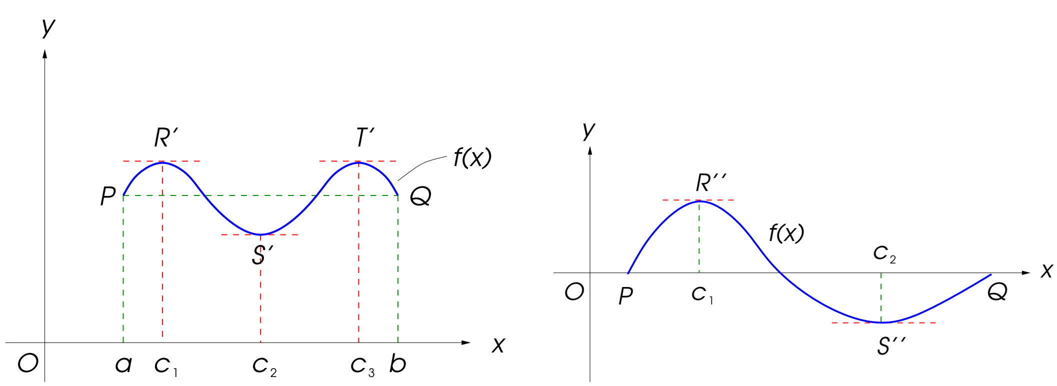


Figure 17.2: Slope of the tangent line $= f'(c) = 0$, where $c \in (a, b)$.



(a) Slope of the tangent line $= f'(c_1) = f'(c_2) = f'(c_3) = 0$, where $c_1, c_2, c_3 \in (a, b)$.

(b) Slope of the tangent line $= f'(c_1) = f'(c_2) = 0$, where $c_1, c_2 \in (a, b)$.

Figure 17.3: Note that the point c is not unique.

17.2.3 Algebraic Interpretation of Rolle's Theorem

Theorem 3 If $f(x)$ is a polynomial in x and the equation $f(x) = 0$ has two roots $x = a$ and $x = b$, then there lies at least one root of $f'(x) = 0$ between a and b.

Example 1 Let a polynomial $f(x)$ in x be

$$f(x) = \frac{a_0}{n+1}x^{n+1} + \frac{a_1}{n}x^n + \cdots + a_n x$$

where $a_0, a_1, \cdots, a_n$ are real numbers such that

$$\frac{a_0}{n+1} + \frac{a_1}{n} + \frac{a_2}{n-1} + \cdots + \frac{a_{n-1}}{2} + \frac{a_n}{1} = 0.$$

Clearly, $f(x)$ being a polynomial, it is continuous and differentiable.
Now, let us check the conditions **(i)** and **(ii)**. We have

$$f(0) = 0$$

and

$$f(1) = \frac{a_0}{n+1} + \frac{a_1}{n} + \cdots + a_n = 0,$$

that is,

$$f(0) = f(1) = 0.$$

The function f satisfies all the conditions in the interval $[0, 1]$, therefore there exists at least one real number c between 0 and 1 such that $f'(c) = 0$.
To see how it works, now, we have

$$f'(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n.$$

Therefore, this implies that at least one point exists in $(0, 1)$ such that

$$a_0 x^n + a_1 x^{n-1} + \cdots + a_n = 0.$$

□

Remark 1 Rolle's theorem holds only when all the conditions are satisfied. If even one of the conditions fails to hold, then Roll's theorem is not applicable.

Examples

Example 2 Verify Rolle's theorem for $f(x) = \sqrt{1 - x^2}$ in $[-1, 1]$.
Solution:
Let us check each of the criteria to see if this qualifies for evaluation under the Rolle theorem:

(i) The given function being an algebraic function of x is continuous in $[-1, 1]$.

(ii) Now, using the chain rule, the derivative of $f(x)$ is

$$f'(x) = \frac{d}{dx}\left(\sqrt{1-x^2}\right) = \frac{d\,(1-x^2)^{1/2}}{d\,(1-x^2)}\frac{(1-x^2)}{dx} = \frac{-x}{\sqrt{1-x^2}}.$$

for $-1 < x < 1$ so $f(x)$ is differentiable in $(-1,1)$.

(iii) Here

$$f(-1) = \sqrt{1-(-1)^2} = 0$$

and

$$f(1) = \sqrt{1-(1)^2} = 0.$$

Thus, $f(-1) = f(1) = 0$.

Hence all the three conditions of Rolle's theorem are satisfied. Therefore, $f'(x) = 0$ for at least one value of x in the open interval $(-1,1)$. □

Example 3 Example the applicability of Rolle's Theorem for f defined by: $f(x) = \ln\left|\frac{x^2+ab}{(a+b)x}\right|$ in $[a,b]$, $\quad 0 \notin [a,b]$.

Solution:

Let $f(x) = \ln\left(\frac{x^2+ab}{x(a+b)}\right)$. Using the properties of logarithms, one gets

$$f(x) = \ln(x^2+ab) - \ln x - \ln(a+b), \quad \forall x \in [a,b], \quad x \neq 0.$$

Let us check each of the criteria to see if this qualifies for evaluation under the Rolle theorem.

(i) A function f is continuous in $[a,b]$ because $f(x)$ is a composite function of a continuous function in the interval $[a,b]$.

(ii) The derivative of f is

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(\ln(x^2+ab) - \ln x - \ln(a+b)\right) \\ &= \frac{1}{x^2+ab}\cdot(-2x) - \frac{1}{x} \\ &= \frac{-2x}{x^2+ab} - \frac{1}{x} \end{aligned}$$

which exists finitely for each $a < x < b$. Thus, $f(x)$ is differentiable in (a,b).

(iii) Here

$$\begin{aligned} f(a) &= \ln(a^2+ab) - \ln a - \ln(a+b) \\ &= \ln a + \ln(a+b) - \ln a - \ln(a+b) = 0 \end{aligned}$$

and

$$f(b) = \ln(b^2+ab) - \ln b - \ln(a+b) = 0.$$

Therefore,

$$f(a) = f(b) = 0.$$

Hence, the given function satisfies all the conditions of Rolle's theorem, therefore there exists at least one c in (a, b) such that $f'(c) = 0$.

Now, setting $x = c$, one gets

$$\begin{aligned} f'(c) &= \frac{2c}{c^2 + ab} - \frac{1}{c} = 0 \\ &= \frac{c^2 - ab}{c(c^2 + ab)} = 0. \end{aligned}$$

Using the fact that $f'(c) = 0$, one obtains $c^2 = ab$ or $c = \pm\sqrt{ab}$.

Among these two values of c, one selects $c = +\sqrt{ab} \in (a, b)$. Since $\sqrt{ab}$ is the geometric mean of a and b, Rolle's theorem is verified. $\square$

Example 4 Verify Rolle's theorem for the following function:

$$f(x) = \begin{cases} x^2+1 & \text{when } 0 \le x \le 1; \\ 3-x & \text{when } 1 < x \le 2. \end{cases}$$

Solution:

Let us check each of the criteria carefully to see if this qualifies for evaluation under the Rolle theorem.

(i) $f(x)$ is continuous in $[0,2]$ except possibly at $x=1$. We check the continuity of f at $x=1$.

- $$f(1) = 1^2+1 = 2.$$
- $$\begin{aligned} f(1+0) = \lim_{x\to 1^+} f(x) &= \lim_{h\to 0} f(1+h) \\ &= \lim_{h\to 0} 3-(1+h) \\ &= 2. \end{aligned}$$
- $$\begin{aligned} f(1-0) = \lim_{x\to 1^-} f(x) &= \lim_{h\to 0} f(1-h) \\ &= \lim_{h\to 0} (1-h)^2+1 \\ &= 2. \end{aligned}$$

Here
$$f(1+0) = f(1-0) = f(1).$$
Therefore, $f(x)$ is continuous at $x=1$. Hence, $f(x)$ is continuous in $[0,2]$.

(ii) The derivative of f is

$$f'(x) = \begin{cases} 2x & \text{when } 0 \le x \le 1; \\ -1 & \text{when } 1 < x \le 2. \end{cases}$$

Clearly $f(x)$ is differentiable in $(0,2)$ except possibly at $x=1$.

We check the differentiability of $f(x)$ at $x=1$.

$$\begin{aligned} Rf'(1) = f'(1+0) &= \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} \\ &= \lim_{h\to 0} \frac{[3-(1+h)]-2}{h} \\ &= -1. \\ Lf'(1) = f'(1-0) &= \lim_{h\to 0} \frac{f(1-h)-f(1)}{-h} \\ &= \lim_{h\to 0} \frac{[(1-h)^2+1]-2}{-h} = \lim_{h\to 0} \frac{2h-h^2}{h} \\ &= 2. \end{aligned}$$

We observe here that $Rf'(1) \neq Lf'(f)$ which implies that $f(x)$ is not differentiable at $x = 1 \in (0, 2)$. Thus, $f(x)$ is not differentiable in $(0, 2)$.

(iii) Here

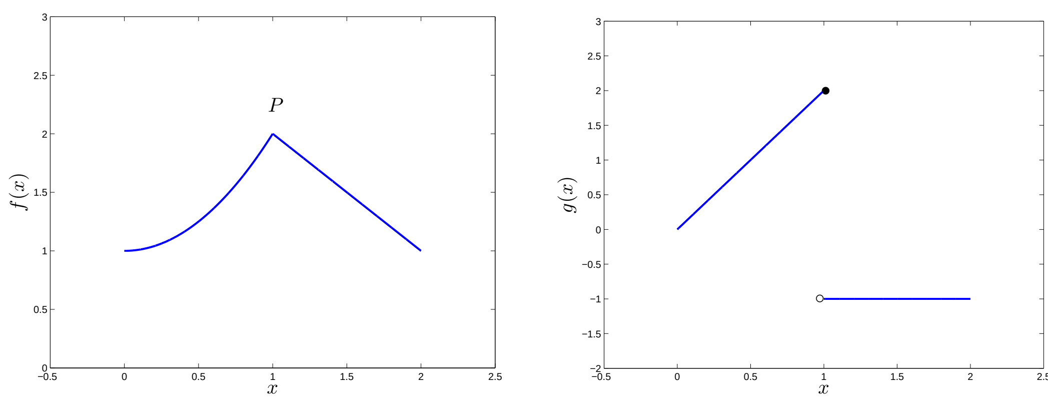
$$f(0) = 0^2 + 1 = 1,$$
$$f(2) = 3 - 2 = 1.$$

Thus,

$$f(0) = f(2) = 1.$$

Here, the function does not satisfy the second condition of Rolle's theorem in $[0, 2]$. Therefore, Rolle's theorem is not applicable. □

Note 3 In Figure 17.4, it is clear that $f(0) = f(2) = 1$, $f(x)$ is continuous at $x = 1$, but not differentiable at $x = 1$, and the tangents at P are not inclined at the same angle with x-axis.



(a) $f(x)$ (b) $g(x) = f'(x)$

Figure 17.4: Example 4.

Example 5 By considering the function $f(x) = (x - 4)\ln x$, prove that the equation

$$x \ln x = 4 - x$$

is satisfied by at least one value of $x \in (1, 4)$.

Solution:

(i) Since the given function is the product of two continuous function in $[1, 4]$, therefore it is continuous in $[1, 4]$.

(ii) The derivative of $f(x)$ is $f'(x) = \dfrac{x - 4}{x} + \ln x$ (using the product rule), which exists finitely in $[1, 4]$. Therefore, $f(x)$ is differentiable in $(0, 4)$.

(iii) Here

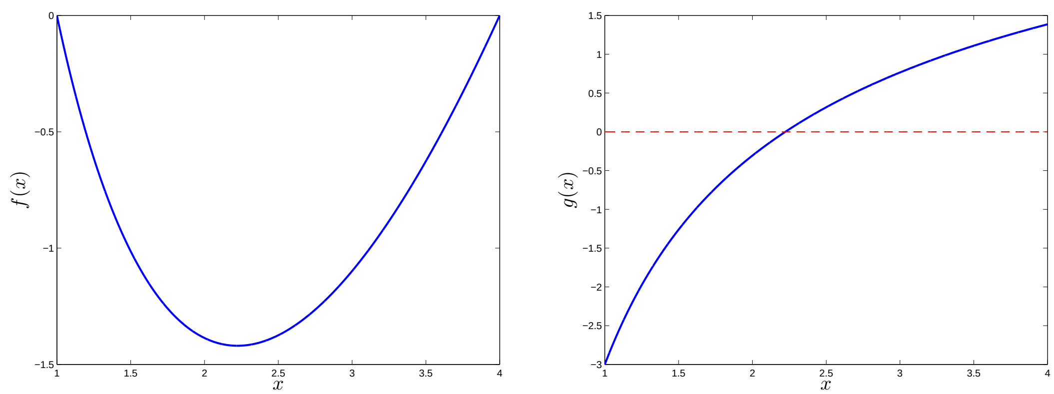
$$f(1) = (1-4)\ln 1 = 0,$$
$$f(4) = (4-4)\ln 4 = 0.$$

Thus, $f(1) = f(4) = 0$.

Hence, $f(x)$ satisfies all the conditions of Rolle's theorem. Thus for the at least one value in $(1, 4)$,

$$\begin{aligned} f'(x) &= 0, \\ \frac{x-4}{4} + \ln x &= 0, \\ x - 4 + 4\ln x &= 0, \\ 4\ln x &= 4 - x. \end{aligned}$$

That is, the above equation is satisfied for at least one value of x in $(1, 4)$, as shown in Figure 17.5. $\square$



(a) $f(x)$ (b) $g(x) = f'(x)$

Figure 17.5: Example 5.

Note 4 Using the Newton-Raphson method, one may find the root of a known function in the interval $(1, 4)$.

Example 6 Prove that there does not exist any real λ for which the equation

$$x^3 - 27x + \lambda = 0$$

has two different roots in $[0, 3]$.
Solution:
If possible, let $\lambda \in \mathbb{R}$. The equation

$$x^3 - 27x + \lambda = 0 \tag{17.2}$$

has two distinct roots α and β $(\alpha < \beta)$ in interval $[0, 3]$ and $0 \leq \alpha < \beta \leq 3$.

Since α and β are the root of equation

$$x^3 - 27x + \lambda = 0.$$

Therefore,

$$\left.\begin{array}{l}\alpha^3 - 27\alpha + \lambda = 0,\\ \beta^3 - 27\beta + \lambda = 0.\end{array}\right\} \tag{17.3}$$

Now we consider a function $f(x)$ in closed interval $[\alpha, \beta]$

$$f(x) = x^3 - 27x + \lambda \qquad \forall x \in [\alpha, \beta].$$

(i) Being a polynomial, $f(x)$ is continuous in $[\alpha, \beta]$.

(ii) $f'(x) = 3x^2 - 27$ exists finitely in (α, β). Therefore, $f(x)$ is differentiable in (α, β).

(iii) By further assuming

$$f(\alpha) = 0 = f(\beta),$$

the function $f(x)$ satisfies all the conditions of Rolle's theorem. There exists $c \in (\alpha, \beta)$ such that $f'(c) = 0$, that is,

$$\begin{aligned} f'(c) = 3c^2 - 27 &= 0,\\ c^2 &= 9,\\ c &= \pm 3. \end{aligned}$$

But $\pm 3 \notin (0, 3)$ implies that $c \notin (\alpha, \beta)$, which is contradictory.

Hence it is proved that there does not exist any real number λ for which the equation has two different roots in $[0, 3]$. $\square$

17.3 Lagrange's Mean Value Theorem

The following theorem gives an expression, $\frac{f(b)-f(a)}{b-a}$ for the mean of the values of the function at the end points in terms of the derivative of this function at an applicable intermediate point. It is stated as follows:

Theorem 4 If a function f defined on the closed interval $[a, b]$ exits such that it is

(i) continuous in the closed interval $[a, b]$ and

(ii) differentiable in the open interval (a, b),

then there exists at least one real number c, $a < c < b$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Proof:

To prove this theorem, we define a new function $\psi(x)$ in the closed interval $[a, b]$ involving the function $f(x)$ so that it satisfies all the conditions of Rolle's theorem.

Let

$$\psi(x) = f(x) + kx, \tag{17.4}$$

where k is a constant and to be determined such that $\psi(a) = \psi(b)$. That is,

$$\begin{aligned} f(a) + ka &= f(b) + kb \\ k &= -\frac{f(b) - f(a)}{b - a}, \quad a \neq b. \end{aligned} \tag{17.5}$$

We observe here that for the constant k, the function $f(x)$ and x are continuous in $[a, b]$ and differentiable in (a, b). Therefore, $\psi(x)$ being sum of two continuous functions, it is

(i) continuous in $[a, b]$,

(ii) differentiable in open interval (a, b), and

(iii) by our choice of k, we have $\psi(a) = \psi(b)$.

Thus, $\psi(x)$ satisfies all three conditions of Rolle's theorem. Hence, there exists $c \in (a, b)$ such that $\psi'(c) = 0$. Now, we have

$$\begin{aligned} \psi'(x) &= f'(x) + k && \text{Derivative of (17.4)} \\ f'(c) + k &= 0 && \text{Use } \psi'(c) = 0 \\ -k &= f'(c) && (17.6) \end{aligned}$$

Therefore, (17.6) = (17.5). This proves the theorem. □

17.3.1 Another Form of Lagrange's Theorem

Theorem 5 If a function $f(x)$ is defined in an interval $[a, a + h]$ in such a way that it is

(i) continuous in closed interval $[a, a + h]$, and

(ii) differentiable in open interval $(a, a + h)$,

then there lies at least one real number θ between 0 and 1, such that

$$f(a + h) - f(a) = hf'(a + \theta h).$$

Proof:

Let $a + h = b$. Then $h = b - a$ is the length of the interval.

Since c lies between a and $a + h$, therefore it is greater than a by fraction of h and may be written as $c = a + \theta h$, where $0 < \theta < 1$.

Hence the results of Lagrange's mean value theorem can be written as

$$f(a + h) - f(a) = hf'(a + \theta h), \quad 0 < \theta < 1.$$

This proves the theorem. □

17.3.2 Geometrical Significance of Lagrange's Mean Value Theorem

In the graph of a continuous and differentiable function $f(x)$ there lies at least one point between any two points $P(a, f(a))$ and $Q(b, f(b))$, where the tangent line to the curve is parallel to chord joining P and Q.

In Figure 17.6, the slope of the chord is

$$\frac{f(b) - f(a)}{b - a}.$$

But $f'(c)$ represents the slope of the tangent at $R(c, f(c))$, therefore the tangent at the point $R(c, f(c))$ is parallel to a chord PQ. In Figure 17.7, there is a unique tangent at every point between P and Q, that is, $f'(c) = f'(d)$, where the points c and d are not the same position.

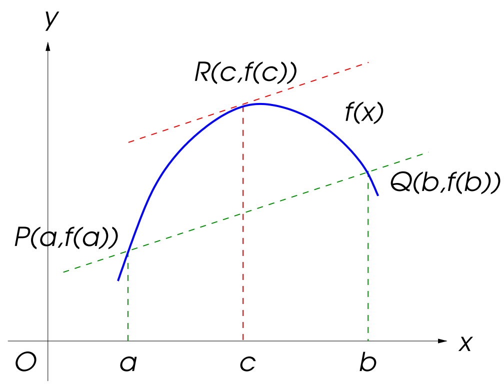


Figure 17.6: Slope of the tangent line $= f'(c)$.

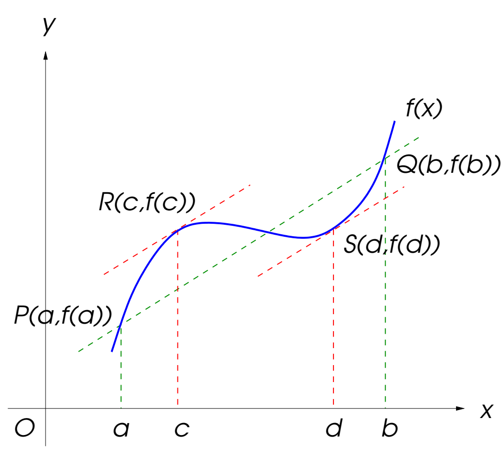


Figure 17.7: Slope of the tangent line $= f'(c) = f'(d)$, where $c \neq d$.

17.3.3 Some Important Deductions from Lagrange's Mean Value Theorem

Using Lagrange's mean value theorem, the following result defines relationship between the application of monotonicity and the sign of the derivative of f at a specific closed interval.

Theorem 6 If a function $f(x)$ is continuous in the interval $[a, b]$ and differentiable in (a, b), then for $\forall x \in (a, b)$

(a) If $f'(x) = 0$, then $f(x)$ is constant in $[a, b]$;

(b) If $f'(x) > 0$, then $f(x)$ is strictly increasing in $[a, b]$;

(c) If $f'(x) \geq 0$, then $f(x)$ is increasing in $[a, b]$;

(d) If $f'(x) < 0$, then $f(x)$ is strictly decreasing in $[a, b]$.

(e) If $f'(x) \leq 0$, then $f(x)$ is decreasing in $[a, b]$.

Proof: Let x_1 and x_2 be any two different points in $[a, b]$ such that if $a < x_1 < x_2 < b$, then $[x_1, x_2] \subset [a, b]$.

A function $f(x)$ satisfies all the conditions of Lagrange's mean value theorem in $[x_1, x_2]$. Therefore,

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c). \tag{17.7}$$

(a) If $f'(x) = 0$, $\forall x \in (a, b)$ which implies that when $x = c \in (a, b)$, then $f'(c) = 0$. From (17.7), $f(x_2) - f(x_1) = 0$ implies that

$$f(x_2) = f(x_1) \quad \forall x \in (a, b).$$

Hence $f(x)$ is constant in $[a, b]$.

(b) If $f'(x) > 0$, $\forall x \in (a, b)$ which implies that $x = c \in (a, b)$, then $f'(c) > 0$. From (17.7), $f(x_2) - f(x_1) > 0$ implies that $f(x_2) > f(x_1)$.

Hence $f(x)$ is monotonically increasing in $[a, b]$.

(c) Exercise.

(d) If $f'(x) < 0$, $\forall x \in (a, b)$ which implies that $x = c \in (a, b)$, then $f'(c) < 0$. From (17.7), $f(x_2) - f(x_1) < 0$ implies that $f(x_2) < f(x_1)$.

Hence $f(x)$ is monotonically decreasing in $[a, b]$.

(e) Exercise.

□

With the help of Lagrange's mean value theorem, we can now answer the questions we posed at the beginning of the section.

Theorem 7 If two functions f and ϕ exist such that they are

(i) continuous on $[a, b]$,

(ii) differentiable on (a, b), and

(iii) $f'(x) = \phi'(x) \quad \forall x \in (a, b)$,

then the difference of the functions $f - \phi$ is constant in $[a, b]$.

Proof: Let us define a new function $\psi(x)$ in $[a, b]$ as follows:

$$\psi(x) = f(x) - \phi(x) \quad \forall x \in [a, b].$$

The above function $\psi(x)$ is

(i) continuous in $[a, b]$, being the difference of two continuous functions in $[a, b]$;

(ii) differentiable in (a, b), being the difference of two differentiable functions in (a, b); and

(iii) $\psi'(x) = f'(x) - \phi'(x) = 0$.

Thus, according to Theorem 6 (a), $\psi(x)$ is a constant function in $[a, b]$. Hence, $f(x) - \phi(x)$ is constant in $[a, b]$. □

Examples

Example 7 Verify Lagrange's mean value theorem for the following function:

$$f(x) = x(x-1)(x-2) \qquad \forall x \in \left[0, \frac{1}{2}\right].$$

Solution:

(i) Being a polynomial, $f(x)$ is continuous in $\left[0, \frac{1}{2}\right]$.

(ii) The derivative of $f(x)$ is $f'(x) = 3x^2 - 6x + 2$, which exists finitely in $\left(0, \frac{1}{2}\right)$, therefore, $f(x)$ is differentiable in $\left(0, \frac{1}{2}\right)$.

Hence the function satisfies all the conditions of Lagrange's mean value theorem. There exists c in $\left(0, \frac{1}{2}\right)$, such that

$$\begin{aligned}
\frac{f\left(\frac{1}{2}\right) - f(0)}{\frac{1}{2} - 0} &= f'(c) && \text{By Theorem 17.3} \\
\left(\frac{3}{8} - 0\right) &= \frac{1}{2}(3c^2 - 6c + 2) && \text{By } \textbf{(ii)} \\
12c^2 - 24c + 5 = 0.
\end{aligned}$$

Solving for c, one gets

$$\begin{aligned} c &= \frac{24 \pm \sqrt{576-240}}{24} \\ &= 1 \pm \frac{\sqrt{21}}{6}. \end{aligned}$$

Clearly, $c = 1 + \frac{\sqrt{21}}{6} \notin \left(0, \frac{1}{2}\right)$, but $1 - \frac{\sqrt{21}}{6} \in \left(0, \frac{1}{2}\right)$. Thus, Lagrange's mean value theorem is verified. □

Example 8 Find the value of "c" in the Lagrange's mean value theorem,

$$f(b) - f(a) = (b-a)f'(c),$$

if

$$f(x) = \alpha x^2 + \beta x + \gamma$$

where α, β, γ are constants and $\alpha \neq 0$.
Solution:

(i) The given function is a polynomial in x therefore it is continuous in $[a, b]$.

(ii) The derivative of $f(x)$, $f'(x) = 2\alpha x + \beta$, exists finitely, hence the given function is differentiable in (a, b).

Therefore, $f(x)$ satisfies the required conditions of Lagrange's mean value theorem in (a, b).

Here, we have

$$\begin{aligned} f(b) &= \alpha b^2 + \beta b + \gamma, \\ f(a) &= \alpha a^2 + \beta a + \gamma, \\ f'(c) &= 2\alpha c + \beta. \end{aligned}$$

Thus, by $f(b) - f(a) = (b-a)f'(c)$, we have

$$\begin{aligned} \alpha(b^2 - a^2) + \beta(b-a) &= 2\alpha c(b-a) + \beta(b-a) \\ \alpha(b^2 - a^2) &= 2\alpha c(b-a) \\ \alpha(b-a)(b+a-2c) &= 0 && b \neq a \\ c &= \frac{a+b}{2} && \alpha \neq 0. \end{aligned}$$

□

The preceding results can be used to establish some important inequalities.

Example 9 Use Lagrange's mean value theorem to prove that

$$1 + x < e^x < 1 + xe^x \qquad \forall x > 0.$$

Solution:

We know that for positive real value of x, the function $f(x) = e^x$ is continuous in $[0, x]$ and obviously differentiable in $(0, x)$.

By Lagrange's mean value theorem, there exists $\theta \in (0, 1)$ such that

$$f(x) - f(0) = (x - 0)f'(0 + \theta x)$$

or

$$\begin{aligned} e^x - e^0 &= xf'(\theta x) && 0 < \theta < 1 \\ e^x - 1 &= xe^{\theta x} && \text{Because } f'(\theta x) = xe^{\theta x} \end{aligned}$$

or

$$e^x = 1 + xe^{\theta x}. \tag{17.8}$$

Since $0 < \theta < 1,\ x > 0$ and $e^x = 1 + x + \dfrac{x^2}{2!} + \cdots > 1 + x$, one gets

$$\begin{array}{ccc} 0 < \theta x < x & \text{and} & 1 + x < e^x \\ e^{\theta x} < e^x & \text{and} & 1 + x < e^x \end{array}$$

Thus, using $e^{\theta x} < e^x$ in (17.8), one gets

$$1 + x < e^x < 1 + xe^x.$$

□

Example 10 Examine the validity of the hypothesis and conclusion of Lagrange's mean value theorem for the following function in the given interval.

$$f(x) = |x|, \text{ where } x\ [-1, 2].$$

Solution:

Let $a \in [-1, 2]$. First the conditions of Lagrange's mean value theorem are to be checked.

To test for the continuity of f at $x = a$, let us examine the following cases:

- $$f(a) = |a|.$$

- $$\begin{aligned} f(a - 0) &= \lim_{h \to 0} |a - h| \\ &= |a|. \end{aligned}$$

- $$\begin{aligned} f(a+0) &= \lim_{h\to 0} |a+h| \\ &= |a|. \end{aligned}$$

Since $f(a) = f(a+0) = f(a-0)$, then $f(x)$ is continuous in $[-1, 2]$.
To test for the differentiability of f at $x = a$, let us examine the following cases:

At $x = 0 \in (-1, 2)$, we note that

$$\begin{aligned} Rf'(0) &= f'(0+0) \\ &= \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h\to 0} \frac{|0+h| - 0}{h} = 1. \end{aligned}$$

$$\begin{aligned} Lf'(0) &= f'(0-0) \\ &= \lim_{h\to 0} \frac{f(0-h) - f(0)}{-h} \\ &= \lim_{h\to 0} \frac{|0-h| - 0}{-h} = -1. \end{aligned}$$

Since

$$Rf'(0) \neq Lf'(0),$$

$f(x)$ is not differentiable in $(-1, 2)$. Hence Lagrange's theorem is not applicable.

Note that there would be no point on the graph at which the tangent is parallel to the $x-$axis.

□

Example 11 Show that under suitable conditions there exists at least one number $c \in (a, b)$ such that[2]

$$\begin{vmatrix} f(a) & f(b) \\ \phi(a) & \phi(b) \end{vmatrix} = (b-a) \begin{vmatrix} f(a) & f'(c) \\ \phi(a) & \phi'(c) \end{vmatrix}.$$

Solution:

Here both the function $f(x)$ and $\phi(x)$ are continuous in $[a, b]$ and differentiable in (a, b). Therefore, by Lagrange's mean value theorem, there exist at least one c in (a, b) such that

$$f(b) - f(a) = (b-a)f'(c) \tag{17.9}$$

[2]For the 2×2 matrix,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

where a, b, c and $d \in \mathbb{R}$. The determinant of A, denoted by $\det(A)$ or $|A|$, is the scalar

$$|A| = \left| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right| = ad - bc.$$

and

$$\phi(b) - \phi(a) = (b-a)\phi'(c). \tag{17.10}$$

Multiplying (17.9) by $\phi(a)$ and (17.10) by $f(a)$, that is,

$$\begin{aligned} \phi(a)f(b) - \phi(a)f(a) &= (b-a)\phi(a)f'(c), \\ f(a)\phi(b) - f(a)\phi(a) &= (b-a)f(a)\phi'(c), \end{aligned}$$

and then subtracting one from other, one gets

$$\phi(a)f(b) - \phi(b)f(a) = (b-a)[\phi(a)f'(c) - f(a)\phi'(c)]$$

or

$$\phi(b)f(a) - f(b)\phi(a) = (b-a)[f(a)\phi'(c) - \phi(a)f'(c)]$$

or

$$\begin{vmatrix} f(a) & f(b) \\ \phi(a) & \phi(b) \end{vmatrix} = (b-a)\begin{vmatrix} f(a) & f'(c) \\ \phi(a) & \phi'(c) \end{vmatrix}.$$

$\square$

Example 12 Separate the intervals in which the polynomial x^3+8x^2+5x-2 is increasing or decreasing.

Solution:

Let

$$\begin{aligned} f(x) &= x^3+8x^2+5x-2 \\ f'(x) &= 3x^2+16x+5 \\ &= (x+5)(3x+1) \end{aligned}$$

If $f'(x)>0$, then either $(x+5)$ and $(3x+1)$ are both positive or both are negative.

Case 1 When $(3x+1)$ and $(x+5)$ are both positive

$$x+5>0,\ 3x+1>0$$
$$x>-5 \text{ and } x>-\frac{1}{3}.$$

Therefore, $x>-\frac{1}{3}$ implies $f'(x)>0$.

Case 2 When $(3x+1)<0$ and $(x+5)<0$

$$x<-\frac{1}{3} \text{ and } x<-5.$$

Therefore, $x<-5$ implies $f'(x)>0$ and $f(x)$ is an increasing function in $\left(-\frac{1}{3},\infty\right)$ and $(-\infty,-5)$.

If $f'(x)<0$, then $(x+5)$ and $(3x+1)$ have opposite signs.

Case 3 When $x+5<0$ and $3x+1>0$, it implies that $x<-5$ and $x>-\frac{1}{3}$, which is not possible.

Case 4 When $x+5>0$ and $3x+1<0$, it implies that $x>-5$ and $x<-\frac{1}{3}$, and $f(x)$ is a decreasing function in the interval $(-5,-1/3)$.

Figure 17.8 shows the graphs of f (blue) and f' (green). □

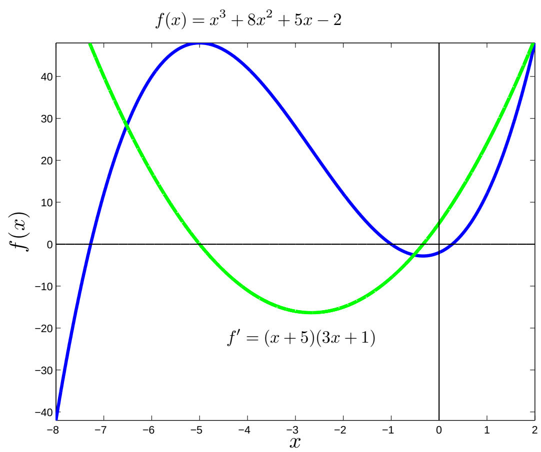


Figure 17.8: The graphs of f and f'.

Example 13 If $f'(x)$ exists in $[a,b]$ and

$$\frac{f(c)-f(a)}{c-a}=\frac{f(b)-f(c)}{b-c},$$

where $c\in(a,b)$, then there exists $\alpha\in(a,b)$ such that $f'(\alpha)=0$.

Solution:

It is given that f' exists in $[a,b]$. Therefore, f and f' are both continuous in $[a,b]$ since $c\in(a,b)$.

Applying Lagrange's theorem in $[a,c]$ and $[c,b]$, we have

$$\frac{f(c)-f(a)}{c-a}=f'(\alpha_1) \qquad \text{for } a<\alpha_1<c$$

and

$$\frac{f(b)-f(c)}{b-c}=f'(\alpha_2) \qquad \text{for } c<\alpha_2<b$$

where $[\alpha_1,\alpha_2]\subset[a,b]$, because

$$\frac{f(c)-f(a)}{c-a}=\frac{f(b)-f(c)}{b-c}.$$

Thus,

$$f'(\alpha_1)=f'(\alpha_2).$$

The function $f'(x)$ is continuous in $[\alpha_1,\alpha_2]$ and $f'(x)$ exists in (α_1,α_2), thus $f'(x)$ is differentiable.

Also $f'(\alpha_1)=f'(\alpha_2)$ implies that $f'(x)$ satisfies all the conditions of Rolle's theorem. Therefore, there exists $\alpha\in(\alpha_1,\alpha_2)\subset[a,b]$, such that

$$f'(\alpha)=0.$$

17.4 Cauchy's Mean Value Theorem

As we shall see in the next chapter, there is an interesting relationship between Cauchy's mean value theorem and L'Hôpital's rule. By adding condition **(iii)** to Theorem 8, the ratio of the values in (17.11) exists and is computable.

Theorem 8 If two functions f and ϕ defined on $[a, b]$ exist such that both are

(i) continuous on closed interval $[a, b]$,

(ii) differentiable on open interval (a, b), and

(iii) $\phi'(x) \neq 0$ for $\forall x \in (a, b)$,

then there exists at least one point c in the open interval (a, b) such that

$$\frac{f'(c)}{\phi'(c)} = \frac{f(b) - f(a)}{\phi(b) - \phi(a)}. \tag{17.11}$$

Proof:

Firstly, we notice that $\phi(b) = \phi(a)$. If $\phi(a) = \phi(b)$, then $\phi(x)$ satisfies all the conditions of Rolle's theorem and there exists $c \in (a, b)$ such that $\phi'(c) = 0$, which contradicts **(iii)**. Therefore,

$$\phi(a) \neq \phi(b).$$

Now let us define a new function $\psi(x)$ in the closed interval $[a, b]$ involving the given functions ϕ and f as follows:

$$\psi(x) = f(x) + \mathcal{A}\phi(x), \qquad \forall x \in [a, b]$$

where $\mathcal{A}$ is constant to be determined such that $\psi(a) = \psi(b)$. That is,

$$\begin{aligned} f(a) + \mathcal{A}\phi(a) &= f(b) + \mathcal{A}\phi(b), \\ \mathcal{A} &= -\frac{f(b) - f(a)}{\phi(b) - \phi(a)}. \end{aligned} \tag{17.12}$$

Here $\mathcal{A}$ exists because $\phi(b) \neq \phi(a)$.

Since $\phi(x)$ is the sum of two continuous functions f and ϕ, therefore

(i) ψ is continuous in $[a, b]$, and

(ii) ψ is differentiable in (a, b) such that

$$\psi'(c) = 0.$$

Therefore,

$$\begin{aligned} \psi'(c) &= f'(c) + \mathcal{A}\phi'(c) = 0 \\ \mathcal{A} &= -\frac{f'(c)}{\phi'(c)}. \end{aligned} \tag{17.13}$$

From (17.12) and (17.13), we have

$$\frac{f(b)-f(a)}{\phi(b)-\phi(a)}=\frac{f'(c)}{\phi'(c)}.$$

□

Note 5 Lagrange's mean value theorem is a particular case of Cauchy's mean value theorem. When $\phi(x)=x$, Cauchy's theorem reduces to Lagrange's mean value theorem. The way it works is as follows: $\phi(x)=x$ means $\phi(b)=b, \phi(a)=a, \phi'(x)=1$ and so $\phi'(c)=1$. Putting these values in Cauchy's mean value theorem, we immediately get the results of Lagrange's mean value theorem, that is,

$$f'(c)=\frac{f(b)-f(a)}{b-a}, \quad a<c<b.$$

17.4.1 Geometrical Significance of Cauchy's Mean Value Theorem

Let us consider a smooth curve in parametric representation

$$x=g(t), \quad y=f(t), \quad a\le t\le b.$$

In Figure 17.9, the slope of the chord joining the end point of the curve is

$$\frac{f(b)-f(a)}{g(b)-g(b)}.$$

But the slope of the tangent at $t=c$ with $R(c, f(c))$ is

$$\frac{f'(c)}{g'(c)}.$$

Cauchy's mean value theorem states that there will always be a value c in (a, b) for which the slope of the tangent at c is equal to the slope of the chord joining the end points, as shown in Figures 17.9 and 17.10.

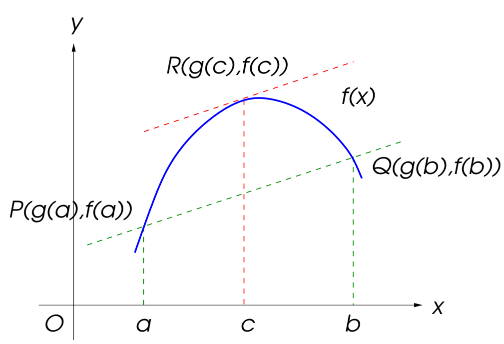


Figure 17.9: Slope of the tangent line $=\dfrac{f'(c)}{g'(c)}$.

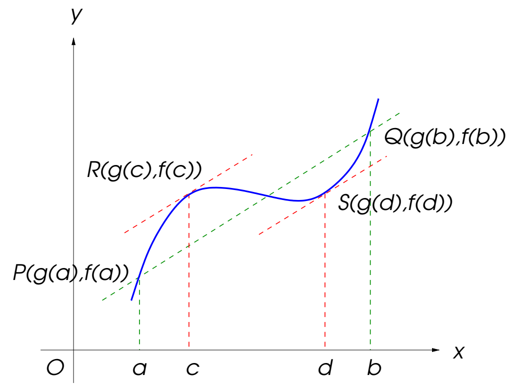


Figure 17.10: Slope of the tangent line $= \dfrac{f'(c)}{g'(c)} = \dfrac{f'(d)}{g'(d)}$, where $c \neq d$.

17.4.2 Another Form of Cauchy's Mean Value Theorem

Theorem 9 If two function $f(x)$ and $\phi(x)$ defined in $[a, a+h]$ exist such that both are

(i) continuous in $[a, a+h]$,

(ii) differentiable in $(a, a+h)$, and

(iii) $\phi'(x) \neq 0 \quad \forall x \in (a, a+h)$,

then there exists $\theta \in (0, 1)$ such that

$$\frac{f(a+h) - f(a)}{\phi(a+h) - \phi(a)} = \frac{f'(a+\theta h)}{\phi'(a+\theta h)}.$$

Note 6 In Cauchy's mean value theorem if we put $b = a + h$, then the interval $[a, b]$ becomes $[a, a+h]$, therefore a point c in $[a, a+h]$ will be taken as $c = a = \theta h$ where $0 < \theta < 1$. Hence Cauchy's mean value theorem takes the form shown above.

Examples

Example 14 Verify Cauchy's mean value theorem for the functions x^2 and x^3 in the interval $[1, 2]$.
Solution:
Let $f(x) = x^2$ and $\phi(x) = x^3$. We conclude that

(i) Clearly both the functions are continuous in $[1, 2]$.

(ii) Both $f'(x)$ and $\phi'(x)$ have unique and finite value in $(1, 2)$ therefore they are differentiable in $(1, 2)$.

(iii) Also,

$$\phi'(x) = 3x^2 \neq 0, \ \forall x \in (1, 2).$$

Therefore, f and ϕ satisfy all the conditions of Cauchy's theorem. Then there exists $c \in (1, 2)$ such that

$$\frac{f(2) - f(1)}{\phi(2) - \phi(1)} = \frac{f'(c)}{\phi'(c)},$$

$$\begin{aligned} \frac{4-1}{8-1} &= \frac{2c}{3c^2} && \text{Substitution} \\ 9c^2 - 14c &= 0 && \text{Simplification} \\ c = 0 \text{ or } c &= \frac{14}{9}. && \text{Solve for } c \end{aligned}$$

Here $0 \notin (1, 2)$, but $\frac{14}{9} \in (1, 2)$. Hence, Cauchy's theorem is verified. $\square$

Example 15 Find the value of θ for Cauchy's mean value theorem for the functions f and ϕ, defined on $\left[0, \frac{\pi}{2}\right]$ as follows:

$$\left.\begin{aligned} f(x) &= \sin x \\ \phi(x) &= \cos x \end{aligned}\right\} \qquad \forall x \in \left[0, \frac{\pi}{2}\right]$$

Solution:

By Cauchy's theorem, we have

$$\frac{f(a+h) - f(a)}{\phi(a+h) - \phi(a)} = \frac{f'(a+\theta h)}{\phi'(a+\theta h)}. \tag{17.14}$$

The left-hand side (LHS) of (17.14) is

$$\begin{aligned} \frac{\sin(a+h) - \sin a}{\cos(a+h) - \cos a} &= \frac{\cos\left(a + \frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}{-\sin\left(a + \frac{h}{2}\right)\sin\left(\frac{h}{2}\right)} \\ &= -\cot\left(a + \frac{h}{2}\right) \qquad\qquad h \neq 0 \text{ implies that } \frac{h}{2} \neq 0. \end{aligned}$$

The right-hand side (RHS) of (17.14) is

$$\begin{aligned} \frac{f'(a+\theta h)}{\phi'(a+\theta h)} &= -\frac{\cos(a+\theta h)}{\sin(a+\theta h)} \\ &= -\cot(a+\theta h) \qquad\qquad 0 < \theta < 1. \end{aligned}$$

Since LHS = RHS, we have

$$\begin{aligned} -\cot\left(a + \frac{h}{2}\right) &= -\cot(a+\theta h) \\ \theta &= \frac{1}{2} \in (0,1). \end{aligned}$$

□

Example 16 Use Cauchy's mean value theorem to evaluate:

$$\lim_{x\to 0} \frac{\cos \frac{1}{2}\pi x}{\ln\left(\frac{1}{x}\right)}.$$

Solution:

By Cauchy's mean value theorem, we have

$$\frac{f(b)-f(a)}{\phi(b)-\phi(a)} = \frac{f'(c)}{\phi'(c)}, \quad \text{where } a<c<b \tag{17.15}$$

Let $f(x)=\cos\left(\frac{1}{2}\pi x\right)$ and $\phi(x)=\ln x$. Also, let $a=x,\ b=1$.
From (17.15), we have

$$\frac{\cos\left(\frac{\pi}{2}\right)-\cos\left(\frac{\pi}{2}x\right)}{\ln 1-\ln x} = \frac{-\frac{1}{2}\pi\sin\left(\frac{\pi}{2}c\right)}{1/c}, \qquad a<c<b \text{ i.e., } x<c<1$$

$$\frac{\cos\left(\frac{1}{2}\pi x\right)}{\ln\left(\frac{1}{x}\right)} = \frac{\pi c}{2}\sin\left(\frac{\pi c}{2}\right). \tag{17.16}$$

If $x\to 1$, then $c\to 1$, from (17.16), and we have

$$\lim_{x\to 1}\frac{\cos\frac{\pi}{2}x}{\ln\frac{1}{x}} = \lim_{c\to 1}\frac{\pi c}{2}\sin\left(\frac{\pi c}{2}\right) = \lim_{c\to 1}\frac{\pi c}{2}\cdot\lim_{c\to 1}\sin\left(\frac{\pi c}{2}\right) = \frac{\pi}{2}.$$

□

Example 17 Show that the value of "c" satisfying Cauchy's mean value theorem in $[a,b]$ for the following pairs of functions is the arithmetic mean, the geometric mean and the harmonic mean, respectively, of a and b:

(a)	$f(x)=e^x$	$\phi(x)=e^{-x}$
(b)	$f(x)=\sqrt{x}$	$\phi(x)=\frac{1}{\sqrt{x}}$
(c)	$f(x)=\frac{1}{x^2}$	$\phi(x)=\frac{1}{x}$

Solution:

(a) Consider

$$\begin{array}{l|llll} f(x)=e^x & f(a)=e^a & f(b)=e^b & f'(c)=e^c; \\ \phi(x)=e^{-x} & \phi(a)=e^{-a} & \phi(b)=e^{-b} & \phi'(c)=-e^{-c}. \end{array}$$

By Cauchy's mean value theorem, we have

$$\frac{f(b)-f(a)}{\phi(b)-\phi(a)} = \frac{f'(c)}{\phi'(c)}. \tag{17.17}$$

Putting these results into (17.17) and using the fact that $\ln e^k=k$, one gets

$$\begin{aligned} \frac{e^b-e^a}{e^{-b}-e^{-a}} &= \frac{e^c}{e^{-c}} \\ -e^{a+b} &= -e^{2c} \\ 2c &= a+b \\ c &= \frac{a+b}{2}. \end{aligned}$$

Thus, c is the arithmetic mean of a and b; that is, $c\in(a,b)$.

(b) Consider

$$\begin{array}{l|lll} f(x)=\sqrt{x} & f(a)=\sqrt{a} & f(b)=\sqrt{b} & f(c)=\frac{1}{2}c^{-1/2}; \\ \phi(x)=\frac{1}{\sqrt{x}} & \phi(a)=\frac{1}{\sqrt{a}} & \phi(b)=\frac{1}{\sqrt{b}} & \phi'(c)=-\frac{1}{2}c^{-3/2}. \end{array}$$

Putting these results into (17.17) and using the fact that $\sqrt{b}\sqrt{a}=\sqrt{ab}$, one gets

$$\frac{\sqrt{b}-\sqrt{a}}{\frac{1}{\sqrt{b}}-\frac{1}{\sqrt{a}}}=\frac{\frac{1}{2}c^{-1/2}}{-\frac{1}{2}c^{-3/2}}$$
$$\sqrt{ab}=c.$$

Thus, $c=\sqrt{ab}$ is the geometric mean of a and b; that is, $c\in(a,b)$.

(c) Consider

$$\begin{array}{l|lll} f(x)=\frac{1}{x^2} & f(a)=\frac{1}{a^2} & f(b)=\frac{1}{b^2} & f'(c)=-2c^{-3}; \\ \phi(x)=\frac{1}{x} & \phi(a)=\frac{1}{a} & \phi(b)=\frac{1}{b} & \phi'(c)=-c^{2}. \end{array}$$

Putting these results into (17.17) and using the fact that $a^2-b^2=(a-b)(a+b)$, one gets

$$\frac{\frac{1}{b^2}-\frac{1}{a^2}}{\frac{1}{b}-\frac{1}{a}}=\frac{-2c^{-3}}{-c^{-2}}$$
$$\frac{1}{ab}\cdot\frac{a^2-b^2}{a-b}=\frac{2}{c}$$
$$c=\frac{2ab}{a+b}.$$

Thus, $c=\frac{2ab}{a+b}$ is the harmonic mean of a and b; that is, $c\in(a,b)$.

□

Example 18 Deduce from Cauchy's mean value theorem that

$$f(b)-f(a)=cf'(c)\ln\frac{b}{a},$$

where $f(x)$ is continuous and differentiable in $[a,b]$ and $c\in(a,b)$.

Solution:

By Cauchy's Mean Value Theorem, we have

$$\frac{f(b)-f(a)}{\phi(b)-\phi(a)}=\frac{f'(c)}{\phi'(c)}. \tag{17.18}$$

If

$$\phi(x)=\ln x \quad \text{then} \quad \phi'(c)=\frac{1}{c}. \tag{17.19}$$

From (17.18) and (17.19), we have

$$\frac{f(b)-f(a)}{\ln b-\ln a}=\frac{f'(c)}{1/c}.$$

Now

$$\frac{f(b)-f(a)}{\ln \frac{b}{a}} = cf'(c)$$

$$f(b)-f(a) = cf'(c)\ln\frac{b}{a}.$$

□

17.5 General Mean Value Theorem

Theorem 10 If the functions $f(x), g(x)$ and $h(x)$ are defined in the interval $[a, b]$ such that they are

(i) continuous in interval $[a, b]$, and

(ii) differentiable in the interval (a, b),

then there exists a number c in the interval (a, b), where[3]

$$\begin{vmatrix} f'(c) & g'(c) & h'(c) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0.$$

This theorem deduces both Cauchy's mean value theorem and Lagrange's mean value theorem.

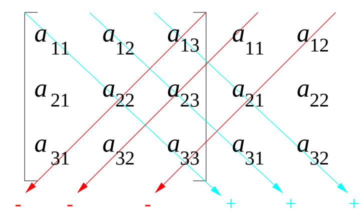


Figure 17.11: $\det(A)$

Proof: Let us define a new function ϕ in the interval $[a, b]$ as follows:

$$\phi(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix}.$$

Then ϕ is a linear function

$$\phi(x) = \alpha f(x) + \beta g(x) + \gamma h(x).$$

[3] For the 3×3 matrix,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

where $a_{ij} \in \mathbb{R}$, $1 \leq i, j \leq 3$. One way to compute the determinant of A, denoted by $\det(A)$ or $|A|$ which is the scalar is

$$\begin{aligned} \det(A) = {} & a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ & - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}. \end{aligned}$$

By repeating the first and second columns of A as illustrated in Figure 17.11, we can also obtain $\det(A)$. Sum of the products of the entries of the lines from left to right, and then subtract from this number the products of the entries on the lines from right to left.

The function ϕ is continuous in $[a, b]$, because it is the sum of the continuous functions f, g and h. Therefore it is differentiable in (a, b).

Also $\phi(a) = \phi(b) = 0$. Thus ϕ satisfies all the conditions of Rolle's theorem. There exists at least one $c \in [a, b]$ such that

$$\phi'(c) = 0;$$

that is,

$$\begin{vmatrix} f'(c) & g'(c) & h'(c) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0.$$

Deductions:

(i) When

- $h(x) = k$, (constant)
- $h(a) = h(b) = k$ implies that $h'(c) = 0$, where $c \in (a, b)$.

Applying this in the theorem

$$\begin{vmatrix} f'(c) & g'(c) & 0 \\ f(a) & g(a) & k \\ f(b) & g(b) & k \end{vmatrix} = 0, \tag{17.20}$$

and expanding (17.20), we get

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)},$$

which is Cauchy's mean value theorem.

(ii) When $g(x) = x$ and $h(x) = k$, then

$$g'(x) = 1 \quad \text{and} \quad h'(x) = 0.$$

Applying it in the theorem

$$\begin{vmatrix} f'(c) & 1 & 0 \\ f(a) & a & k \\ f(b) & b & k \end{vmatrix} = 0, \tag{17.21}$$

and expanding (17.21), we get

$$f(b) - f(a) = (b - a)f'(c),$$

which is Lagrange's theorem.

□

17.6 Mean Value Theorem for Second Derivatives

Theorem 11 If in the closed interval $[a, b]$, a function $f(x)$ is defined in such a way that,

(i) $f'(x)$ exists and is continuous in the closed interval $[a, b]$, and

(ii) $f(x)$ is differentiable in (a, b),

then there exists $c \in [a, b]$, such that,

$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!}f'(c).$$

Proof: Let us consider an auxiliary function ϕ as follows:

$$\phi(x) = f(x) + (b-x)f'(x) + \frac{(b-x)^2}{2!}\alpha, \tag{17.22}$$

where α is a constant to be determined such that

$$\phi(a) = \phi(b). \tag{17.23}$$

Now

$$x = a: \qquad \phi(a) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!}\alpha, \tag{17.24}$$

$$x = b: \qquad \phi(b) = f(b). \tag{17.25}$$

From (17.23), (17.24) and (17.25), we have

$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!}\alpha. \tag{17.26}$$

The function $\phi(x)$ satisfies all the conditions of Rolle's theorem in $[a, b]$; there exists $c \in [a, b]$ such that

$$\phi'(c) = 0. \tag{17.27}$$

But

$$\begin{aligned}\phi'(x) &= f'(x) + (b-x)f'(x) - f'(x) - \frac{2(b-x)}{2!}\alpha \\ &= (b-x)f'(x) - (b-x)\alpha\end{aligned}$$

Therefore, we have

$$x = c: \qquad \phi'(c) = (b-c)f'(c) - (b-c)\alpha = 0.$$

Since $b - c \neq 0$, we have

$$\alpha = f''(c).$$

Substituting in (17.26) gives

$$f(b) = f(a) + (b-a) + \frac{(b-a)^2}{2!}f''(c).$$

$\square$

17.6.1 Another Form of the Mean Value Theorem for Second Derivatives

If the function $f(x)$ is defined in the interval $[a, a+h]$, then there exists $0 < \theta < 1$, that is, $\theta \in (0,1)$ and by the second mean value theorem, we get

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a+\theta h).$$

Example 19 Find θ, if

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}\, f''(x+\theta h), \quad 0 < \theta < 1$$

when

$$f(x) = x^3 + x.$$

Solution:

Here

$$\begin{aligned} f(x) &= x^3 + x, \\ f'(x) &= 3x^2 + 1, \\ f''(x) &= 6x, \end{aligned}$$

and

$$\begin{aligned} f'(x+h) = (x+h)^3 + (x+h), \\ f''(x+\theta h) = 6(x+\theta h). \end{aligned}$$

Substituting these values in the above expression, that is,

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f'(x+\theta h)$$

gives

$$(x+h)^3 + (x+h) = x^3 + x + h(3x^2+1) + \frac{h^2}{2!}\,(6(x+\theta h))$$

or

$$h^3 = 3\theta h^3$$

or

$$\theta = \frac{1}{3} \in (0,1).$$

□

Example 20 If $f(x)$ is continuous in $[a, b]$ and possesses finite derivatives for $x = c \in (a, b)$, then prove that

$$f''(c) = \lim_{h \to 0} \frac{f(c+h) + f(c-h) - 2f(c)}{h^2}.$$

Solution:

Since the function is differentiable at $x = c$, $f'(x)$ and $f''(x)$ exist in the neighbourhood $(c-h, c+h)$ of $x = c$.

Now, applying the mean value theorem for second derivatives for the intervals $(c-h, c)$ and $(c, c+h)$, we have

$$f(c-h) = f(c) - hf'(c) + \frac{h^2}{2!}f''(c-\theta_1 h) \quad 0 < \theta_1 < 1, \tag{17.28}$$

and

$$f(c+h) = f(c) + hf'(c) + \frac{h^2}{2!}f''(c+\theta_2 h) \quad 0 < \theta_2 < 1. \tag{17.29}$$

Adding (17.28) and (17.29), we obtain

$$f(c-h) + f(c+h) = 2f(c) + \frac{h^2}{2!}[f''(c-\theta_1 h) + f''(c+\theta_2 h)],$$

or

$$f(c-h) + f(c+h) - 2f(c) = \frac{h^2}{2!}[f''(c-\theta_1 h) + f''(c+\theta_2 h)].$$

Taking the limit of the above result, we have

$$\lim_{h\to 0}\frac{f(c-h) + f(c+h) - 2f(c)}{h^2} = \lim_{h\to 0}\frac{1}{2!}[f''(c-\theta_1 h) + f''(c+\theta_2 h)],$$

or

$$\lim_{h\to 0}\frac{f(c-h) + f(c+h) - 2f(c)}{h^2} = f''(c).$$

$\square$

17.7 Generalised Mean Value Theorem

17.7.1 Taylor's Theorem with Lagrange's Form of Remainder

Theorem 12 If in the interval $[a, a+h]$, a function f is defined in such a way that the differentials $f'(x), f''(x), \cdots, f^{n-1}(x)$ up to the order $(n-1)$ are

(i) continuous in the interval $[a, a+h]$, and

(ii) the n^{th} derivative of $f(x)$ exists in the interval $(a, a+h)$,

then there exists at least one number θ between 0 and 1, such that

$$\begin{aligned} f(a+h) &= f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots \\ &+ \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{h^n}{n!}f^{(n)}(a+\theta h). \end{aligned}$$

Proof: Let us define a new function $\phi(x)$ involving the function f and the derivatives of f i.e. $f', f'', f'', \cdots$ in the interval $[a, a+h]$ as follows:

$$\begin{aligned} \phi(x) &= f(x) + (a+h-x)f'(x) + \frac{(a+h-x)^2}{2!}f''(x) + \cdots \\ &+ \frac{(a+h-x)^{n-1}}{(n-1)!}f^{n-1}(x) - \frac{(a+h-x)^n}{n!}\mathcal{A}, \end{aligned} \tag{17.30}$$

where $\mathcal{A}$ is a constant to be taken as

$$\phi(a) = \phi(a+h),$$

where

$$f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{h^n}{n!}\mathcal{A} = f(a+h).$$

We know $f(x), f'(x), \cdots, f^{n-1}(x)$ are continuous in $[a, a+h]$ and also $f'(x), f''(x) \cdots f^n(x)$ exist finitely in $(a, a+h)$. Also $(a+h-x), \frac{(a+h-x)^2}{2!}, \cdots, \frac{(a+h-x)^{n-1}}{(n-1)!}$ are polynomials in x. Thus $\phi(x)$ being the linear combination of continuous functions is

(i) continuous in $[a, a+h]$,

(ii) differentiable in $(a, a+h)$, and

(iii) $\phi(a) = \phi(a+h)$.

Therefore, $\phi(x)$ satisfies all the conditions of Rolle's theorem. This implies that there exists θ $(0 < \theta < 1)$ such that

$$\phi'(a+\theta h) = 0.$$

Differentiating (17.30) with respect to x, we obtain

$$\begin{aligned}\phi'(x) &= f'(x) - f'(x) + (a+h-x)f''(x) - (a+h-x)f''(x) \\ &\quad + \frac{(a+h-x)^2}{2!}f'''(x) - \frac{(a+h-x)^2}{2!}f'''(x) + \cdots + \frac{(a+h-x)^{n-1}}{(n-1)!}f^n(x) \\ &\quad - \frac{(a+h-x)^{n-1}}{(n-1)!}\mathcal{A}.\end{aligned}$$

Hence, we have

$$\phi'(x) = \frac{(a+h-x)^{n-1}}{(n-1)}\left(f^n(x) - \mathcal{A}\right).$$

Now taking $\phi'(a+\theta h) = 0$, we have

$$\frac{(a+h-a-\theta h)^{n-1}}{(n-1)!}\left(f^n(a+\theta h) - \mathcal{A}\right) = 0,$$

or

$$\frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!}\left(f^n(a+\theta h) - \mathcal{A}\right) = 0.$$

Since $1-\theta \neq 0$, we have

$$f^n(a+\theta h) = \mathcal{A}.$$

Substituting this value of $\mathcal{A}$ in (17.30) gives

$$\begin{aligned}f(a+h) = f(a) + nf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) \\ + \frac{h^n}{n!}f^n(a+\theta h),\ 0 < \theta < 1.\end{aligned}$$

The $(n+1)$th term of the RHS of (17.30) is simply $\frac{h^n}{n!}f^n(a+\theta h) = R_n$, which is known as Lagrange's form of remainder and is denoted by R_n. Here R_n stands for Taylor's expansion $f(a+h)$ (in the ascending integral powers of h) after n terms. □

Note 7

1. In Theorem 12, by putting $n = 1$, Lagrange's Mean Value theorem is immediately obtained.

2. Substituting $h = b - a$, we get the theorem as follows:

$$\begin{aligned}f(b) = f(a) + (b-a)f'(a) + \cdots + \frac{(b-a)^{n-1}}{(n-1)!}f^{n-1}(a) \\ + \frac{(b-a)^n}{n!}f^n(a+\theta(b-a)), \quad 0 < \theta < 1.\end{aligned}$$

3. Substituting $h = x - a$ in above theorem, we get the following useful form:

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}f^{n-1}(a)$$
$$+ \frac{(x-a)^n}{n!}f^n(c), \quad a < c < x.$$

17.7.2 Taylor's Theorem with Cauchy's Form of Remainder

Theorem 13 If in the interval $[a, a+h]$, a function $f(x)$ is defined in such a way that,

(i) all derivatives of $f(x)$ up to the order $(n-1)$ are continuous in the interval $[a, a+h]$, and

(ii) all derivatives of $f(x)$ up to the order n exist in the interval $(a, a+h)$,

then there exists at least one number θ between 0 and 1, such that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}}{n-1}f^{n-1}(a) + \frac{h^n}{(n-1)!}(1-\theta)^{n-1}f^n(a+\theta h).$$

Proof: We define the new function $\phi(x)$ as follows:

$$\phi(x) = f(x) + (a+h-x)f'(x) + \frac{(a+h-x)^2}{2!}f''(x) + \cdots + \frac{(a-h-x)^{n-1}}{(n-1)!}f^{n-1}(x) + (a-h-x)\mathcal{A}, \tag{17.31}$$

where we choose the constant $\mathcal{A}$ in such a way, so that

$$\phi(a) = \phi(a+h).$$

This implies that

$$f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + h\mathcal{A} = f(a+h). \tag{17.32}$$

We know that $f(x), f'(x), \cdots, f^{n-1}(x)$ are continuous in $[a, a+h]$ and $f'(x), f''(x), \cdots, f^n(x)$ are differentiable in $(a, a+h)$. Also $(a+h-x)$, $\frac{(a+h-x)^2}{2!}, \cdots, \frac{(a+h-x)^{n-1}}{(n-1)!}$ being the polynomials in x are continuous in $[a, a+h]$ and differentiable in $(a, a+h)$.

Thus the function $\phi(x)$

(i) is continuous in $[a, a+h]$,

(ii) differentiable in $(a, a+h)$, and

(iii) $\phi(a) = \phi(a+h)$.

Thus $\phi(x)$ satisfies all the conditions of Rolle's theorem; there exists θ $(0 < \theta < 1)$, such that

$$\phi'(a+\theta h) = 0.$$

Differentiating (17.31) with respect to x, we have

$$\begin{aligned}\phi'(x) &= f'(x) - f'(x) + (a+h-x)f''(x) - (a+h-x)f''(x) \\ &\quad + \frac{(a+h-x)^2}{2!}f'''(x) - \frac{(a+h-x)^2}{2!}f'''(x) + \cdots \\ &\quad + \frac{(a+h-x)^{(n-2)}}{(n-2)!}f^{n-1}(x) + \frac{(a+h-x)^{n-1}}{(n-1)!}f^n(x) - \mathcal{A}.\end{aligned}$$

Hence

$$\phi'(x) = \frac{(a+h-x)^{n-1}}{(n-1)!} f^n(x) - \mathcal{A}.$$

After cancelling the other terms, we have

$$\phi'(a+\theta h) = 0$$

which implies that

$$\frac{(a+h-a-\theta h)^{n-1}}{(n-1)!} f^n(a+\theta h) - \mathcal{A} = 0$$

or

$$\mathcal{A} = \frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!} f^n(a+\theta h).$$

Substituting the value of $\mathcal{A}$ in (17.32) gives

$$\begin{aligned} f(a+h) &= f(a) + hf'(a) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) \\ &\qquad + \frac{h^n}{(n-1)!}(1-\theta)^{n-1} f^n(a+\theta h). \end{aligned}$$

The $(n+1)$th term of the RHS of (17.31) is

$$\frac{h^n}{(n-1)!}(1-\theta)^{n-1} f^n(a+\theta h)$$

which is known as Cauchy's form of remainder and is denoted by R_n. $\square$

17.7.3 Taylor's Theorem with the Schlömitch and Röche Form of Remainder

Theorem 14 If the function f in $[a, a+h]$ is defined as follows:

(i) $f'(x), f''(x), \cdots, f^n(x)$ are continuous in $[a, a+h]$,

(ii) $f'(x), f''(x), \cdots, f^n(x)$ are differentiable in $(a, a+h)$, and

(iii) $p \in \mathbb{N}$,

then there exists $\theta \in (0,1)$ such that

$$\begin{aligned} f(a+h) &= f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) \\ &\quad + \frac{h^n(1-\theta)^{n-p}}{p(n-1)!}f^n(a+\theta h), \qquad 0<\theta<1. \end{aligned}$$

Proof: We define a new function $\phi(x)$ involving the derivatives of f; that is, $f', f'', \cdots$ as follows:

$$\begin{aligned} \phi(x) &= f(x) + (a+h-x)f'(x) + \frac{(a+h-x)^2}{2!}f''(x) + \cdots \\ &\quad + \frac{(a+h-x)^{n-1}}{(n-1)!}f^{n-1}(x) + (a+h-x)^p\mathcal{A} \end{aligned} \tag{17.33}$$

where $\mathcal{A}$ is a constant to be determined such that

$$f(a+h) = \phi(a).$$

This implies that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + h^p\mathcal{A}. \tag{17.34}$$

Since $f(x), f'(x), \cdots, f^{n-1}(x)$ are continuous in $[a, a+h]$ and $f'(x), f''(x) \cdots f^n(x)$ exists finitely in $(a, a+h)$. Also, $a+h-x, \frac{(a+h-x)^2}{2!}, \cdots, \frac{(a+h-x)^{(n-1)}}{(n-1)!}$ are polynomials in x, and therefor continuous in $[a, a+h]$ and are differentiable in $(a, a+h)$.

Hence the function ϕ is continuous in $[a, a+h]$, and differentiable in $(a, a+h)$. Also $\phi(a) = \phi(a+h)$.

Thus, the function ϕ satisfies all the conditions of Rolle's theorem; there exists $\theta \in (0,1)$ such that

$$\phi'(a+\theta h) = 0.$$

Differentiating (17.33) with respect to x, one gets

$$\begin{aligned} \phi'(x) &= f'(x) - f'(x) + (a+h-x)f''(x) - (a+h-x)f''(x) + \cdots \\ &\quad - \frac{(a+h-x)^{n-2}}{(n-2)!}f^{n-1}(x) + \frac{(a+h-x)^{(n-1)}}{(n-1)!}f^n(x) \\ &\quad - p(a+h-x)^{p-1}\mathcal{A}. \end{aligned}$$

Now,

$$\phi'(a+\theta h) = 0$$

which implies that

$$\frac{(a+h-a-\theta h)^{n-1}}{(n-1)!} f^n(a+\theta h) - p(h-\theta h)^{p-1}\mathcal{A} = 0,$$

or

$$\frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!} f^n(a+\theta h) = \mathcal{A}ph^{p-1}(1-\theta)^{p-1}$$

implies that

$$\mathcal{A} = \frac{h^{n-p}(1-\theta)^{n-p}}{p(n-1)!} f^n(a+\theta h)$$

Substituting this value of $\mathcal{A}$ in (17.34) gives

$$\begin{aligned} f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) \\ + \frac{h^n(1-\theta)^{n-p}}{p(n-1)!}f^n(a+\theta h), \end{aligned}$$

where $R_n = \frac{h^n(1-\theta)^{n-p}}{p(n-1)!}f^n(a+\theta h)$ is known as the Schlömitch and Röche form of remainder.

Special Cases:

1. When $p = n$, then $R_n = \frac{h^n}{n!}f^n(a+\theta h)$ is Lagrange's form of remainder since $n! = n(n-1)!$ and $(1-\theta)^{p-p} = (1-\theta)^0 = 1$.

2. When $p = 1$, then $R_n(x) = \frac{h^n(1-\theta)^{n-1}}{(n-1)!}f^n(a+\theta h)$ which is Cauchy's form of remainder.

□

17.8 Maclaurin's Theorem

Theorem 15 If a function f in $[0, x]$ exists such that

(i) $f, f' \cdots f^{n-1}(x)$ is continuous in $[0, x]$,

(ii) f^n exists in $(0, x)$, and

(iii) $p \in \mathbb{N}$,

then there exists $\theta \in (0, 1)$ such that

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!}f^{n-1}(0) + R_n, \tag{17.35}$$

where

$$R_n = \frac{x^n(1-\theta)^{n-p}}{p(n-1)!}f^n(\theta x), \quad 0 < \theta < 1$$

is called Maclaurin's remainder due to Schlömitch and Röche.

We obtain the above theorem by putting $a = 0$ and $h = x$ in Taylor's remainder due to Schlömitch and Röche (see Theorem 14).

Particular Cases:

1. When $p = 1$, then
$$R_n = \frac{x^n(1-\theta)^{n-1}}{(n-1)!}\ f^n(\theta x),$$
which is Cauchy's form of remainder. In the above theorem, substituting $p = 1$, we get Taylor's theorem due to Cauchy's form of remainder.

2. When $p = n$, then
$$R_n = \frac{x^n}{n!}f^n(\theta x)$$
which is Lagrange's form of remainder. Thus putting $p = n$, we get Taylor's theorem due to Lagrange's form of remainder.

Examples

Example 21 Show that the number θ which occurs in the Taylor's theorem with Lagrange's form of remainder after n terms approaches the limit $\frac{1}{n+1}$ as h approaches zero provided that $f^{n+1}(x)$ is continuous and different from zero at $x = a$.

Solution:

Taylor's theorem after n and $n+1$ terms of Lagrange's form of remainder, respectively, are

$$f(a+h) = f(a) + hf'(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{h^n}{n!}f^n(a+\theta h), \qquad 0 < \theta < 1 \quad (17.36)$$

and

$$f(a+h) = f(a) + hf'(a) + \cdots + \frac{h^n}{n!}f^n(a) + \frac{h^{n+1}}{(n+1)!}f^{n+1}(a+\theta_1 h), \quad 0 < \theta_1 < 1 \quad (17.37)$$

Subtracting (17.37) from (17.36), we have

$$\frac{h^n}{n!}\left(f^n(a+\theta h) - f^n(a)\right) - \frac{h^{n+1}}{(n+1)!}f^{n+1}(a+\theta_1 h) = 0$$

or

$$f^n(a+\theta h) - f^n(a) = \frac{h}{n+1}f^{n+1}(a+\theta_1 h) \qquad (17.38)$$

since $\dfrac{n!}{(n+1)!} = \dfrac{n!}{(n+1)n!} = \dfrac{1}{n+1}$.

In the LHS of (17.38), applying Lagrange's mean value theorem, one gets

$$\theta h f^{n+1}(a+\theta\theta_2 h) = \frac{h}{n+1}f^{n+1}(a+\theta_1 h),$$

or

$$\theta = \frac{f^{n+1}(a+\theta_1 h)}{(n+1)f^{n+1}(a+\theta\theta_2 h)}, \qquad h \neq 0.$$

This implies that

$$\begin{aligned}\lim_{\theta\to 0}\theta &= \lim_{h\to 0}\frac{1}{n+1}\cdot\frac{f^{n+1}(a+\theta_1 h)}{f^{n+1}(a+\theta\theta_2 h)} \\ &= \frac{1}{n+1}\cdot\lim_{h\to 0}\frac{f^{n+1}(a+\theta_1 h)}{f^{n+1}(a+\theta\theta_2 h)} \\ &= \frac{1}{n+1}\end{aligned}$$

because $f^{n+1}(a) \neq 0$. $\square$

Example 22 Expand sin x by Maclaurin's theorem.
Solution: Let $f(x) = \sin\ x$. Then $f(0) = \sin\ 0 = 0$.

$$\begin{aligned} f'(x) &= \cos\ x & f'(0) &= \cos\ 0 = 1 \\ f''(x) &= -\sin\ x & f''(0) &= 0 \\ f^{(3)}(x) &= -\cos\ x & f^{(3)}(0) &= -1 \\ f^{(4)}(x) &= \sin\ x & f^{(4)}(0) &= 0 \\ f^{(5)}(x) &= \cos\ x & f^{(5)}(0) &= 1 \\ &\vdots & &\vdots \\ f^{(n)}(x) &= \sin\left(x + \frac{n\pi}{2}\right) & f^{(n)}(0) &= \sin\left(\frac{n\pi}{2}\right). \end{aligned}$$

Now two cases are considered.

Case I When n is even, e.g., $n = 2p$, then

$$f^{(n)}(0) = \sin\left(\frac{2p\pi}{2}\right) = \sin\ p\pi = 0.$$

Case II When n is odd, e.g., $n = 2p + 1$, then

$$f^{(n)}(0) = \sin\left(\frac{(2p+1)\pi}{2}\right) = \sin\left(\frac{\pi}{2} + p\pi\right) = \cos\ p\pi = (-1)^p.$$

Substituting these values into (17.35) gives

$$\begin{aligned} \sin\ x &= f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!}f^{n-1}(0) + \frac{x^n(1-\theta)^{n-p}}{p(n-1)!}f^n(\theta x) \\ &= 0 + x(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-1) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(1) + \cdots + \frac{x^{2p+1}}{(2p+1)!}(-1)^p + \cdots \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^p\frac{x^{2p+1}}{(2p+1)!} + \cdots. \end{aligned}$$

$\square$

Example 23 Expand e^x by Maclaurin's theorem.

Solution: Let $f(x) = e^x$. Then $f(0) = e^0 = 1$.

$$\begin{aligned} f'(x) &= e^x & f'(0) &= \cos\ 0 = 1 \\ f''(x) &= e^x & f''(0) &= 1 \\ f^{(3)}(x) &= e^x & f^{(3)}(0) &= 1 \\ f^{(4)}(x) &= e^x & f^{(4)}(0) &= 1 \\ f^{(5)}(x) &= e^x & f^{(5)}(0) &= 1 \\ &\vdots & &\vdots \\ f^{(n)}(x) &= e^x & f^{(n)}(0) &= 1. \end{aligned}$$

Substituting these values into (17.35) gives

$$\begin{aligned} e^x &= f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots \\ &= 1 + x(1) + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(1) + \frac{x^4}{4!}(1) + \frac{x^5}{5!}(1) + \cdots + \frac{x^{2n+1}}{(2n+1)!}(-1)^n + \cdots \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}(1) + \cdots + \frac{x^n}{n!} + \cdots. \end{aligned}$$

□

Example 24 Expand $\ln x$ by Maclaurin's theorem.

Solution: Let $f(x) = \ln\ x$. Then $f(0) = \ln\ 0$ (undefined).

$$\begin{aligned} f'(x) &= \frac{1}{x} & f'(0) &= \frac{1}{0} \text{ (undefined)} \\ f''(x) &= -\frac{1}{x^2} & f''(0) &= -\frac{1}{0^2} \text{ (undefined).} \end{aligned}$$

Neither the function nor any of its derivatives exist at $x = 0$, so there is no polynomial Maclaurin expansion of the natural logarithm function $\ln x$. □

Example 25 Expand $\ln(1+x)$ by Maclaurin's theorem.

Solution: Let $f(x) = \ln(1+x)$. Then $f(0) = \ln(1+0) = \ln 1 = 0$.

$$\begin{aligned}
f'(x) &= \frac{1}{1+x} & f'(0) &= \frac{1}{1+0} = 1 \\
f''(x) &= -\frac{1}{(1+x)^2} & f''(0) &= -\frac{1}{(1+0)^2} = -1 \\
f^{(3)}(x) &= \frac{2}{(1+x)^3} & f^{(3)}(0) &= \frac{2}{(1+0)^3} = 2 \\
f^{(4)}(x) &= -\frac{(3)(2)}{(1+x)^4} & f^{(4)}(0) &= -\frac{(3)(2)}{(1+0)^4} = -(3)(2) \\
f^{(5)}(x) &= \frac{(4)(3)(2)}{(1+x)^5} & f^{(5)}(0) &= \frac{(4)(3)(2)}{(1+0)^5} = (4)(3)(2) \\
f^{(6)}(x) &= -\frac{(5)(4)(3)(2)}{(1+x)^6} & f^{(6)}(0) &= -\frac{(5)(4)(3)(2)}{(1+0)^6} = -(5)(4)(3)(2) \\
&\vdots & &\vdots \\
f^{(n)}(x) &= \frac{n\cdots 2}{(1+x)^{n+1}} & f^{(n)}(0) &= \frac{n\cdots 2)}{(1+0)^{n+1}} = n\cdots 2 \\
f^{(n+1)}(x) &= -\frac{(n+1)n\cdots 2}{(1+x)^{n+2}} & f^{(n+1)}(0) &= -\frac{(n+1)n\cdots 2}{(1+x)^{n+2}} = -(n+1)n\cdots 2.
\end{aligned}$$

Substituting these values into (17.35) gives

$$\begin{aligned}
\ln(1+x) &= f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!}f^{n-1}(0) + \cdots \\
&= 0 + x(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-(3)(2)) + \frac{x^5}{5!}((4)(3)(2)) + \cdots + \frac{x^{p+1}}{(p+1)!}(p\cdots 2) + \cdots \\
&= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots - \frac{x^p}{p} + \frac{x^{p+1}}{p+1} + \cdots .
\end{aligned}$$

Note 8 Although the series for $\ln(1+x)$ exists (since the function and its derivatives existing and the function and its derivatives being defined at $x = 0$). But the function itself does not exist for $x \leq -1$, in which case the function expansion does not exist for either. In fact, the series approximation is only valid within the narrow range $-1 < x \leq 1$. □

Example 26 Show that θ which occurs in the Lagrange's Mean Value theorem approaches the limit $\frac{1}{2}$ as h approaches zero provided that $f''(x)$ is continuous and $f''(x) \neq 0$.

Solution:

In Example 21, substitute $n = 1$ to obtain the solution. □

Example 27 Find Lagrange's and Cauchy's remainder after n terms in the expansion of the following functions:

1. e^{kx};
2. $\ln(1+x)$.

Solution:

By Maclaurin's expansion for $f(x)$, we find that

$$\text{Lagrange's remainder} = \frac{x^n}{n!} f^n(\theta x)$$

and

$$\text{Cauchy's remainder} = \frac{x^n(1-\theta)^{n-1}}{(n-1)!} f^n(\theta).$$

1. Here we have $f(x) = e^{kx}$ and $f^n(x) = k^n e^{kx}$.

   Thus we have

   $$\text{Lagrange's remainder} = \frac{x^n}{n!} k^n e^{\theta k x}$$

   and

   $$\text{Cauchy's remainder} = \frac{x^n(1-\theta)^{n-1}}{(n-1)!} k^n e^{k\theta x}.$$

2. Here $f(x) = \ln(1+x)$ and $f^n(x) = \dfrac{(-1)^{n-1}(n-1)!}{(1+n)^n}$.

   Thus we have

   $$\begin{aligned} \text{Lagrange's remainder} &= \frac{x^n(-1)^{n-1}(n-1)!}{n!(1+\theta x)^n} \\ &= \frac{(-1)^{n-1}}{n} \left(\frac{x}{1+\theta x} \right)^n. \end{aligned}$$

   and

   $$\begin{aligned} \text{Cauchy's remainder} &= \frac{x^n(1-0)^{n-1}}{(n-1)!} \frac{(-1)^{n-1}(n-1)!}{(1+\theta x)^n} \\ &= (-1)^{n-1} \frac{x^n}{1+\theta x} \left(\frac{1-\theta}{1+\theta x} \right)^{n-1}. \end{aligned}$$

□

17.8.1 Power Series

Definition 1

If x is a continuous real variable and "a" is any constant, then the series

$$\sum_{n=0}^{\infty} A_n (x-a)^n,$$

where A_n is constant, is called a power series about the point $x = a$.

If $a = 0$, then $\sum\limits_{n=0}^{\infty} A_n x^n$ is called *the standard power series.*

17.8.2 Taylor Series

If a function $f(x)$ defined in $[a, a+h]$ satisfies all the conditions of Taylor's theorem, then by Taylor's theorem:

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + R_n.$$

We have the following results:

- If
$$S_n = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a),$$
then
$$f(a+h) = S_n + R_n.$$
Here R_n is called Taylor's remainder after n terms.

- If
$$n \to \infty \ \forall x \in [a, a+h],$$
then
$$\lim_{n\to\infty} f(a+h) = \lim_{n\to\infty} S_n + \lim_{n\to\infty} R_n$$

- If
$$\lim_{n\to\infty} R_n = 0,$$
then
$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots$$
which is called Taylor's series.

17.8.3 Maclaurin Series

If a function f possesses derivatives of every order in $[0, h]$ and Maclaurin's remainder $R_n \to 0$ as $n \to \infty$, then $\forall x \in [0, h]$. This implies that

$$f(x) = f(0) + xf'(0) + \cdots + \frac{x^n}{n!}f^n(0) + \cdots.$$

Note 9 Taylor's remainder R_n can be of any form.

17.8.4 Power Series Expansion of Some Basic Functions

With the help of Maclaurin's series expansion we shall find the power series expansion of the following elementary functions.

Example 28 Expand $\sin x$.
Solution:
Let $f(x) = \sin x$

$$f^n(x) = \sin\left(x + n\frac{\pi}{2}\right), \quad n \in \mathbb{N}, \quad x \in \mathbb{R}. \tag{17.39}$$

Thus, in the interval $(0, x)$, $f, f', \cdots, f^n(n)$ exist finitely $\forall n \in \mathbb{N}$.
Putting $x = 0$ in (17.39), we have

$$f^n(0) = \sin\left(\frac{n\pi}{2}\right).$$

This implies that $f^n(0) = \begin{cases} 0 & \text{if } n \text{ is even} \\ (-1)^{\frac{n-1}{2}} & \text{if } n \text{ is odd} \end{cases}$

In the expansion of the power series we will prove that when $n \to \infty$ then $R_n \to 0$. Now we consider Lagrange's form of remainder in Maclaurin's expansion, that is,

$$R_n = \frac{x^n}{n!} f^n(\theta x), \quad 0 < \theta < 1.$$

Here

$$R_n = \frac{x^n}{n!} \sin\left(\theta x + n\frac{\pi}{2}\right) \qquad \theta \in (0, 1).$$

Now

$$\begin{aligned} |R_n| &= \left|\frac{x^n}{n!} \sin\left(\theta x + \frac{n\pi}{2}\right)\right| \\ &= \left|\frac{x^n}{n!}\right| \left|\sin\left(\theta x + \frac{n\pi}{2}\right)\right| \\ &\le \left|\frac{x^n}{n!}\right| \cdot 1 = \left|\frac{x^n}{n!}\right| \qquad \text{Because } |\sin\phi| \le 1. \end{aligned}$$

This implies that

$$\lim_{n\to\infty} |R_n| \le \lim_{n\to\infty} \left|\frac{x^n}{n!}\right| = 0$$

because $\sum \left|\frac{x^n}{n!}\right|$ is a convergent series or

$$\lim_{n\to\infty} |R_n| = 0 \text{ implies } \lim_{n\to\infty} R_n = 0.$$

Thus $\forall x \in \mathbb{R}, \sin x$ satisfies both the properties of Maclaurin's series. Thus,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots.$$

□

Example 29 Expand $\cos x$.
Solution:

Here $f(x) = \cos x$. Therefore,

$$f^n(x) = \cos\left(x + \frac{n\pi}{2}\right) \qquad n \in \mathbb{N}, x \in \mathbb{R}. \tag{17.40}$$

Thus every order of derivative for the function $f(x) = \cos x$ exists, $\forall n \in \mathbb{N}$.
From (17.40), $f^n(0) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ (-1)^{\frac{n}{2}} & \text{if } n \text{ is even} \end{cases}$
Now for the expansion of the power series we will prove that when $n \to \infty$ then $R_n \to 0$. For this we consider Lagrange's form of remainder in Maclaurin's expansion is:

$$R_n = \frac{x^n}{n!} f^n(\theta x) = \frac{x^n}{n!} \cos\left(\theta x + \frac{n\pi}{2}\right), \qquad \theta \in (0,1). \tag{17.41}$$

Taking the absolute value of (17.41), we obtained the bound, that is,

$$|R_n| = \left|\frac{x^n}{n!} \cos\left(\theta x + \frac{n\pi}{2}\right)\right| \le \left|\frac{x^n}{n!}\right| \theta \in (0,1) \qquad \text{Because } |\cos\phi| \le 1. \tag{17.42}$$

Passing the limit of (17.42) as $n \to \infty$, we have

$$\lim_{n\to\infty} |R_n| \le \lim_{n\to\infty} \left|\frac{x^n}{n!}\right| = 0 \tag{17.43}$$

because $\sum \left|\frac{x^n}{n!}\right|$ is a convergent series. Hence,

$$\lim_{n\to\infty} |R_n| = 0 \quad \text{implies} \quad \lim_{n\to\infty} R_n = 0.$$

Therefore, we have $\cos\theta$, $\forall x \in \mathbb{R}$ satisfies both the conditions of Maclaurin's series. Thus

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^5}{4!} - \frac{x^6}{6!} + \cdots.$$

$\square$

Example 30 Expand e^x:
Solution:

Let $f(x) = e^x$ and

$$f^n(x) = e^x, \quad n \in \mathbb{N},\ x \in \mathbb{R}. \tag{17.44}$$

Thus $\forall n \in \mathbb{N}$, e^x possesses every order of derivatives.

Putting $x = 0$ in (17.44), we have

$$f^n(0) = e^0 = 1, \quad n \in \mathbb{N}$$

In the expansion of the power series when $n \to \infty$, then $R_n \to 0$. Now Lagrange's form of remainder in Maclaurin's expansion is

$$R_n = \frac{x^n}{n!} f^n(\theta x) = \frac{x^n}{n!} e^{\theta x}, \quad 0 < \theta < 1.$$

Now, we have

$$\begin{aligned}\lim_{n\to\infty} \frac{R_n}{R_{n+1}} &= \lim_{n\to\infty} \left[\frac{x^n}{n!} e^{\theta x} \frac{(n+1)!}{x^{n+1} e^{\theta x}}\right] \\ &= \lim_{n\to\infty} \frac{n+1}{x} \to \infty, \quad \forall x \in \mathbb{R}.\end{aligned}$$

or

$$\lim_{n\to\infty} \frac{R_n}{R_{n+1}} = \infty, \quad \forall x \in \mathbb{R}.$$

Thus, by D'Alembert's Ratio Test, $\sum R_n$ is convergent hence $\lim\limits_{n\to\infty} R_n = 0$.

Now $f(x) = e^x$, $\forall x \in \mathbb{R}$ satisfies both the conditions of Maclaurin's series. Therefore, by Maclaurin's series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots, \ \forall x \in \mathbb{R}.$$

□

Example 31 Expand $\ln(1+x)$.

Solution:

Let

$$f(x) = \ln(1+x), \quad \forall x > -1.$$

If $1+x>0$, that is, $x>-1$, then $\ln(1+x)$ possesses derivatives of every order, that is,

$$f^n(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}, \quad \forall n \in \mathbb{N} \text{ and } x > -1. \tag{17.45}$$

From (17.45), we have

$$f^n(0) = (-1)^{n-1}(n-1)! = \begin{cases} -(n-1)! & \text{if } n \text{ is even} \\ (n-1)! & \text{if } n \text{ is odd} \end{cases}$$

Now for the power series expansion we will show that when $n \to \infty$, then $R_n \to 0$. For this Lagrange's form of remainder in Maclaurin's expansion is

$$R_n = \frac{x^n}{n!} f^n(\theta x) = \frac{x^n}{n!}(-1)^{n-1}\frac{(n-1)!}{(1+\theta x)^n} \quad 0 < \theta < 1$$

or

$$R_n = \frac{(-1)^{n-1}}{n}\left(\frac{x}{1+\theta x}\right)^n.$$

Two cases are considered:

Case 1

$$0 \le x \le 1 \quad \text{and} \quad 0 < \theta < 1$$

or

$$0 < x < 1 + \theta x$$

or

$$\frac{x}{1+\theta x} < 1 \quad \text{implies} \quad \left(\frac{x}{1+\theta x}\right)^n < 1$$

or

$$\lim_{n\to\infty}\left(\frac{x}{1+\theta x}\right)^n = 0.$$

Thus, we have

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty}\frac{(-1)^{n-1}}{n}\left(\frac{x}{1+\theta x}\right)^n = 0.$$

Case 2 $-1 \le x < 0$

In this case it is not necessary that $\dfrac{x}{1+\theta x} < 1$. Therefore taking Cauchy's form of remainder in Maclaurin's expansion:

$$\begin{aligned} R_n &= \frac{x^n}{(n-1)!}(1-\theta)^{n-1} f^n(\theta x), \qquad 0<\theta<1 \\ &= \frac{x^n}{(n-1)!}(1-\theta)^{n-1}\frac{(-1)^{n-1}(n-1)!}{(1+\theta x)^n} \\ &= (-1)^{n-1}\frac{x^n}{1+\theta x}\left(\frac{1-\theta}{1+\theta x}\right)^{n-1} \end{aligned}$$

because $0<\theta<1 \Rightarrow\ -1<-\theta<0$ and $-1<x<0 \Rightarrow |x|<1$

or

$$\Rightarrow\ -1<-\theta<\theta x\ \Rightarrow\ 0<1-\theta<1+\theta x$$

or

$$0<\frac{1-\theta}{1+\theta x}<1$$

or

$$\lim_{n\to\infty}\left(\frac{1-\theta}{1+\theta x}\right)^{n-1} \to 0. \tag{17.46}$$

Also because $|x|<1$ and $-1<x<0$, we have

$$\lim_{n\to\infty}|x|^n = 0. \tag{17.47}$$

So that

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty}(-1)^{n-1}\frac{x^n}{1+\theta x}\left(\frac{1-\theta}{1+\theta x}\right)^{n-1} = 0$$

[From (17.46) and (17.47)].

Therefore for power series expansion, the function $f(x)$ satisfies both the conditions of Maclaurin's series. Thus,

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots,$$

where $-1 \le x \le 1$. □

Example 32 Expand $(1+x)^m$, $x \in \mathbb{R}$.

Solution:
Let

$$f(x) = (1+x)^m, \quad x \in \mathbb{R}$$

then

$$f^n(x) = m(m-1)(m-2)\cdots(m-n+1)(1+x)^{m-n}. \tag{17.48}$$

Two cases are considered.

Case I When $m \in \mathbb{N}$ and $n > m$, $\forall x \in \mathbb{R}$, $f(x)$ possesses every order of derivatives by (17.48), that is,

$$f^n(x) = 0 \Rightarrow R_n = 0.$$

Therefore,

$$\lim_{n\to\infty} R_n = 0.$$

The expansion of $(1+x)^m$ will be

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}\frac{x^2}{2!} + \cdots + x^m,$$

which contains a finite number of terms.

Case 2 When $n \notin \mathbb{N}$, that is, $m \in \mathbb{R}\backslash\mathbb{N}$, then the derivative of any order does not vanish. Taking Cauchy' form of remainder, we have

$$R_n = \frac{x^n(1-\theta)^{n-1}}{(n-1)!} f^n(\theta x) \quad 0 < \theta < 1$$

or

$$\begin{aligned} R_n &= \frac{x^n(1-\theta)^{n-1}}{(n-1)!} \cdot m(m-1)\cdots(m-n+1)(1+\theta x)^{m-n} \\ &= \frac{m(m-1)\cdots(m-n+1)}{(n-1)!} \cdot x^n(1+\theta x)^{m-1}\left(\frac{1-\theta}{1+\theta x}\right)^{n-1}. \end{aligned}$$

Now, we have

$$\frac{R_n}{R_{n+1}} = \frac{n}{m-n} \cdot \frac{1+\theta x}{x(1-\theta)}.$$

This implies that

$$\begin{aligned} \lim_{n\to\infty}\left|\frac{R_n}{R_{n+1}}\right| &= \lim_{n\to\infty}\left|\frac{n}{m-n}\frac{1+\theta x}{x(1-\theta)}\right| \quad 0 < \theta < 1 \\ &= \left|\frac{1+\theta x}{x(1-\theta)}\right| \end{aligned}$$

because $0 < \theta < 1$ implies $-1 < -\theta < 0$ and $-1 < x < 1 \Rightarrow |x| < 1$ or

$$-1 < -\theta < \theta x$$

or

$$0 < 1 - \theta < 1 + \theta x \Rightarrow \frac{1 + \theta x}{1 - \theta} > 1.$$

Also $\frac{1}{|x|} > 1$. Therefore,

$$\left| \frac{1}{x} \frac{(1 + \theta x)}{1 - \theta} \right| > 1$$

or

$$\lim_{n \to \infty} \left| \frac{R_n}{R_{n+1}} \right| = \frac{1}{|x|} \left| \frac{1 + \theta x}{1 - \theta} \right| > 1$$

or $\sum R_n$ is convergent, which implies that

$$\lim_{n \to \infty} R_n = 0.$$

Hence $f(x) = (1+x)^m$ satisfies both the conditions of Maclaurin's series for $-1 < x < 1$. Therefore, we have

$$(1 + x)^m = 1 + mx + \frac{m(m-1)}{2!} x^2 + \cdots .$$

where $-1 < x < 1$. □

17.8.5 Failure of Taylor's Series Expansion

In the following cases, Taylor's series expansion of the functions is not possible:

1. If $f(x)$ or its derivative exist infinitely for any value of x.
2. If $\lim\limits_{n \to \infty} R_n \neq 0$, that is, $\sum R_n$ is not convergent.

17.8.6 Failure of Maclaurin's series expansion

In the following cases, the Maclaurin's expansion of the functions is not possible:

1. $f(0), f'(0), f''(0), \cdots, f^n(0)$ exist infinitely.
2. $f(0)$ or its derivative are not continuous for $x \to 0$.
3. $\lim\limits_{n \to \infty} R_n \neq 0$.

Example 33 Prove that the following functions cannot be expanded by Maclaurin's series:

(a) $e^{\frac{1}{x}}$

(b) $\ln x$

Solution:

(a) Let $f(x) = e^{1/x}$. This implies that

$$f'(x) = e^{1/x}\left(-\frac{1}{x^2}\right), \; f''(x) = e^{1/x}\left(\frac{2}{x^2} + \frac{1}{x^4}\right), \cdots.$$

Clearly at $x = 0$ the function and its derivative are not finite. Therefore Maclaurin's expansion of $e^{1/x}$ is not valid.

(b) Let $f(x) = \ln x$. This implies that

$$f'(x) = \frac{1}{x}, \; f''(x) = -\frac{1}{x^2}, \cdots.$$

We find that the function and its derivatives are not finite at $x = 0$. Hence it is not possible to expand $\ln x$ by Maclaurin's expansion.