THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1510 Calculus for Engineers (2020-2021)
Supplementary Exercise 6

Extrema, Inflection Points and Graphing

1. (a) Factor theorem states that if $P(x)$ is a polynomial and $P(a) = 0$, then $x - a$ is a factor of $P(x)$.
   By using factor theorem, factorize the following polynomials.
   (i) $x^3 + 2x^2 - 5x - 6$
   (ii) $2x^3 - 3x^2 + 1$
   (iii) $3x^3 - x^2 - x - 1$

   (b) State the domain of the function $f(x) = \dfrac{1}{x^3 + 2x^2 - 5x - 6}$.

2. Let $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 3$.

   (a) Find $f'(x)$. By using the factor theorem or otherwise, show that $f'(x) = 4(x - 1)(x - 2)(x - 3)$.

   (b) In the following table, fill in the signs of the factors in the corresponding intervals.

	$x < 1$	$x = 1$	$1 < x < 2$	$x = 2$	$2 < x < 3$	$x = 3$	$x > 3$
$x - 1$	$-$	0	$+$	$+$	$+$	$+$	$+$
$x - 2$							
$x - 3$							
$f'(x)$							

   (c) Solve $f'(x) > 0$ and $f'(x) < 0$.
   Hence, find the extreme points of the graph $y = f(x)$.

3. Let $f(x) = x^2 \ln x$ for $x > 0$.

   Find $f'(x)$ and $f''(x)$. Hence, determine the extreme point(s) of the function.

4. Find the greatest and least values of the following functions on the given closed interval:

   (a) $f(x) = x - 2\sqrt{x}$ on $[0, 9]$;
   (b) $f(x) = x^4 - 8x^2 + 2$ on $[-1, 3]$;
   (c) $f(x) = e^x \ln x$ on $[1, 2]$.

5. Let $f(x) = \dfrac{x^2 + 3x}{x - 1}$.

   (a) Find $f'(x)$.
   (b) Determine the values of x for each of the following cases:

(i) $f'(x) = 0$; (ii) $f'(x) > 0$; (iii) $f'(x) < 0$.

(c) Find all relative extrema of $f(x)$.

(d) Find all asymptotes of $f(x)$.

(e) Sketch the graph of $f(x)$.

6. Let $f(x) = xe^{-x^2}$.

(a) Find $f'(x)$ and $f''(x)$.

(b) Determine the values of x for each of the following cases:

(i) $f'(x) = 0$;	(iii) $f'(x) < 0$;	(v) $f''(x) > 0$;
(ii) $f'(x) > 0$;	(iv) $f''(x) = 0$;	(vi) $f''(x) < 0$.

(c) Find all relative extrema and points of inflexion of $f(x)$.

(d) Sketch the graph of $f(x)$.

7. Let $f(x) = \dfrac{e^x}{x^e}$, for $x > 0$.

(a) Solving $f'(x) > 0$ and $f'(x) < 0$. Hence, find the least value of $f(x)$.

(b) Show that $e^\pi > \pi^e$.

Mean Value Theorem

8. By considering the function $f(x) = \sin x$ on $[0, 1]$ and applying the mean value theorem, show that $\sin 0.1 \leq 0.1$.

9. By using the mean value theorem, prove that for all $x, y \in \mathbb{R}$,

$$|\cos x - \cos y| \leq |x - y|.$$

10. By using the mean value theorem, prove that for all $x > 0$,

$$1 + x < e^x < 1 + xe^x.$$

L'Hôpital Rule

11. By using L'Hôpital rule, find the following limits.

(a) $\displaystyle\lim_{x\to 0} \frac{\ln(\cos x)}{x^2}$

(b) $\displaystyle\lim_{x\to 0} \frac{e^x + e^{-x} - 2}{1 - \cos 2x}$

(c) $\displaystyle\lim_{x\to \pi^+} \frac{\sin x}{\sqrt{x - \pi}}$

(d) $\lim_{x\to 0^+} \frac{\ln(\cos 3x)}{\ln(\cos 2x)}$

12. By using L'Hôpital rule, find the following limits.

(a) $\lim_{x\to 0^+} \frac{\ln(\sin x)}{\ln(\tan x)}$

(b) $\lim_{x\to 0^+} \frac{\ln(\tan 2x)}{\ln(\tan 3x)}$

(c) $\lim_{x\to \frac{\pi}{2}^-} \frac{4\tan x}{1+\sec x}$

(d) $\lim_{x\to\infty} x^n e^{-ax}$, where n is a natural number and a is a positive real number.

13. By using L'Hôpital rule, find the following limits.

(a) $\lim_{x\to 0^+} x^2 \ln x$

(b) $\lim_{x\to \frac{\pi}{2}} (2x-\pi)\sec x$

(c) $\lim_{x\to 1^+} (x^2-1)\tan\frac{\pi x}{2}$

(d) $\lim_{x\to\infty} x(\frac{\pi}{2} - \tan^{-1} x)$

14. By using L'Hôpital rule, find the following limits.

(a) $\lim_{x\to 0} \left(\frac{1}{e^x-1} - \frac{1}{x}\right)$

(b) $\lim_{x\to 0} \left(\frac{1}{\sin^2 x} - \frac{1}{x^2}\right)$

15. By using L'Hôpital rule, find the following limits.

(a) $\lim_{x\to 0} x^x$

(b) $\lim_{x\to\infty} (e^{3x}-5x)^{1/x}$

(c) $\lim_{x\to 0} (\cos x)^{\frac{1}{x^2}}$

(d) $\lim_{x\to 0} \sin x \ln(\sin x)$