THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1510 Calculus for Engineers (Fall 2021)
Suggested solutions of coursework 7

Part A

1. Evaluate the following indefinite integrals.

(a) $\displaystyle\int (x^{\sqrt{2}} + \sqrt{2}^x)\, dx$

(b) $\displaystyle\int \left(\sqrt[3]{x} + \frac{1}{\sqrt[3]{x}}\right) dx$

Solution:

(a)
$$\int (x^{\sqrt{2}} + \sqrt{2}^x)\, dx = \frac{1}{\sqrt{2}+1} x^{\sqrt{2}+1} + \frac{1}{\ln \sqrt{2}} \sqrt{2}^x + C.$$

(b)
$$\int \left(\sqrt[3]{x} + \frac{1}{\sqrt[3]{x}}\right) dx = \int \left(x^{\frac{1}{3}} + x^{-\frac{1}{3}}\right) dx = \frac{3}{4}x^{\frac{4}{3}} + \frac{3}{2}x^{\frac{2}{3}} + C.$$

2. Evaluate the following indefinite integrals.

(a) $\displaystyle\int (2x-3)^{1510}\, dx$

(b) $\displaystyle\int x\sqrt{x+1}\, dx$

Solution:

(a) Let $u = 2x - 3 \implies du = 2\, dx$. Hence
$$\begin{aligned}\int (2x-3)^{1510}\, dx &= \int u^{1510} \frac{1}{2}\, du \\ &= \frac{1}{2} \cdot \frac{1}{1511} u^{1511} + C \\ &= \frac{1}{3022}(2x-3)^{1511} + C.\end{aligned}$$

(b) Let $u = x + 1 \implies du = dx$. Hence
$$\begin{aligned}\int x\sqrt{x+1}\, dx &= \int (u-1)\sqrt{u}\, du \\ &= \int u^{\frac{3}{2}} - u^{\frac{1}{2}}\, du \\ &= \frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} + C \\ &= \frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}} + C.\end{aligned}$$

Part B

3. A particle is moving on the xy-plane and its position at time t is

$$\vec{r}(t) = (x(t), y(t)) = (e^{-t}\cos t, e^{-t}\sin t) \quad \text{for } t \geq 0.$$

(a) Find the velocity and acceleration of the particle at $t = \dfrac{\pi}{4}$.

(b) Find the position of the particle in the long run, i.e., $\lim\limits_{t\to\infty} \vec{r}(t)$.

Solution:

(a)

$$\begin{aligned} \text{velocity} &= \vec{r}\,'(t) \\ &= (x'(t), y'(t)) \\ &= (-e^{-t}\cos t - e^{-t}\sin t, -e^{-t}\sin t + e^{-t}\cos t), \end{aligned}$$

$$\begin{aligned} \text{acceleration} &= \vec{r}\,''(t) \\ &= (x''(t), y''(t)) \\ &= \big(e^{-t}\cos t + 2(-e^{-t})(-\sin t) + e^{-t}(-\cos t), \\ &\qquad e^{-t}\sin t + 2(-e^{-t})(\cos t) + e^{-t}(-\sin t)\big) \\ &= \big(2e^{-t}\sin t, -2e^{-t}\cos t\big)\,. \end{aligned}$$

Hence, at $t = \dfrac{\pi}{4}$,

$$\begin{aligned} \text{velocity} = \vec{r}\,'(4) &= \left(-\frac{2}{\sqrt{2}}e^{-\frac{\pi}{4}}, 0\right) \\ \text{acceleration} = \vec{r}\,''(4) &= \left(\frac{2}{\sqrt{2}}e^{-\frac{\pi}{4}}, -\frac{2}{\sqrt{2}}e^{-\frac{\pi}{4}}\right). \end{aligned}$$

(b) Since

$$-e^{-t} \leq e^{-t}\cos t \leq e^{-t}, \quad -e^{-t} \leq e^{-t}\sin t \leq e^{-t}, \qquad \text{for } t \in \mathbb{R},$$

and

$$\lim_{t\to\infty} e^{-t} = \lim_{t\to\infty} -e^{-t} = 0,$$

it follows from Squeeze Theorem that

$$\lim_{t\to\infty} e^{-t}\cos t = \lim_{t\to\infty} e^{-t}\sin t = 0.$$

Hence,

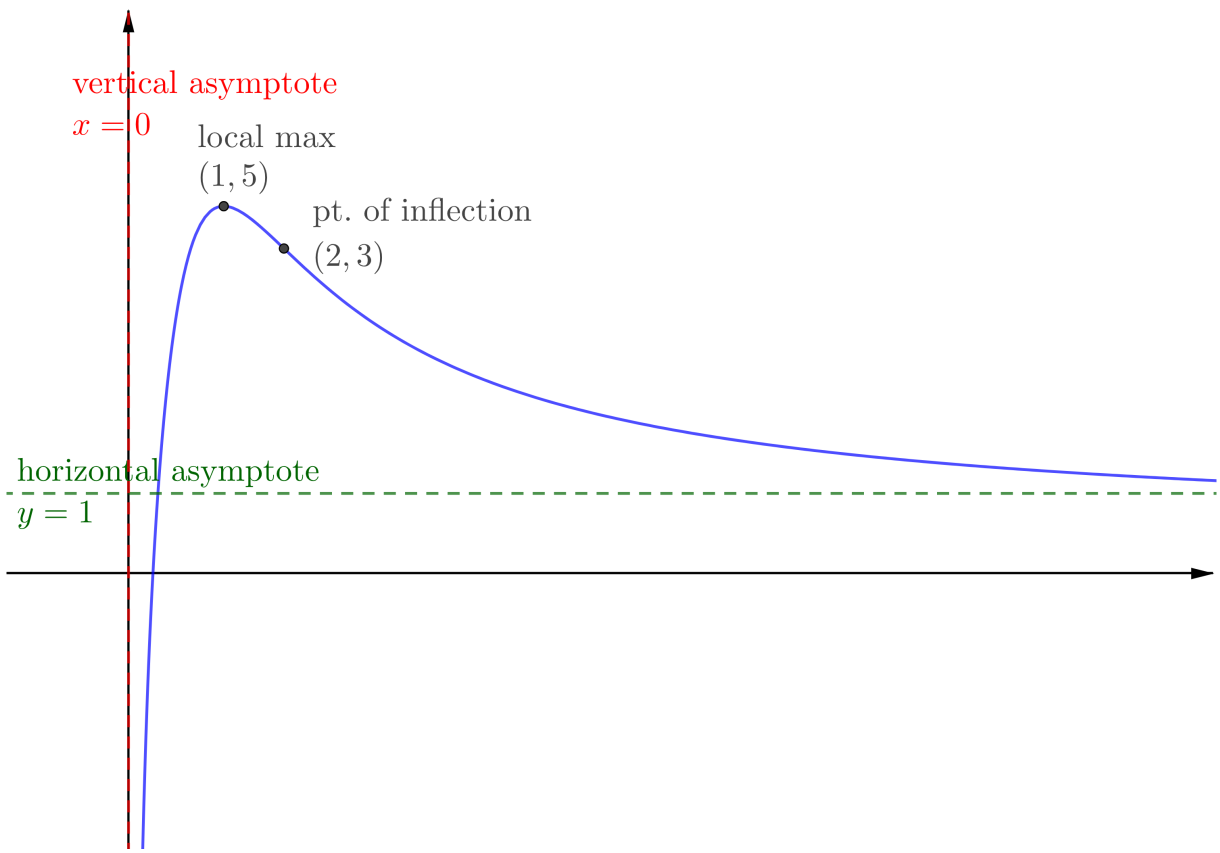
$$\lim_{t\to\infty} \vec{r}(t) = \lim_{t\to\infty} (e^{-t}\cos t, e^{-t}\sin t) = (0, 0) = \vec{0}.$$

4. Sketch a graph of a twice-differentiable function $f : (0, \infty) \to \mathbb{R}$ which satisfies the followings:

   - $f(1) = 5$ and $f(2) = 3$
   - $\lim_{x \to 0^+} f(x) = -\infty$ (DNE) and $\lim_{x \to \infty} f(x) = 1$
   - $f'(x) > 0$ over $(0, 1)$ and $f'(x) < 0$ over $(1, \infty)$
   - $f''(x) < 0$ over $(0, 2)$ and $f''(x) > 0$ over $(2, \infty)$

   On your graph, label any local maximum(s), local maximum(s), point of inflection(s) and asymptote(s) (if any).

   Solution:



5. A new cylindrical container will be built in a factory to store hazardous chemicals. The capacity of the container should be 3200 m^3. Due to the safety regulation, the radius of the base of the container must be at least 6 m and at most 10 m. It is known that the building cost of the container is directly proportional to the total surface area (including both the top and bottom).

   Find the base radius of the container that minimizes the building cost (correct to 4 decimal places).

   Solution: Let r be the base radius and h be the height of the container.

   By considering the volume of the container, we have

$$3200 = \text{volume} = \pi r^2 h \implies h = \frac{3200}{\pi r^2}.$$

   The total surface area of the container is

$$\begin{aligned} S(r) &= 2\pi r^2 + 2\pi r h \\ &= 2\pi r^2 + 2\pi r \cdot \frac{3200}{\pi r^2} \\ &= 2\pi r^2 + \frac{6400}{r}, \qquad r \in [6, 10]. \end{aligned}$$

   So S is differentiable over $(0, \infty)$ and $S'(r) = 4\pi r - \dfrac{6400}{r^2}$.

   Setting $S'(r_0) = 0$, we have

$$\begin{aligned} 4\pi r_0^3 &= 6400 \\ r_0 &= \sqrt[3]{\frac{1600}{\pi}} \approx 7.9859, \end{aligned}$$

   which is the only critical point.

   By comparing the values:

$$\begin{aligned} S(6) &\approx 1292.86 \\ S(r_0) &\approx 1202.12 \\ S(10) &\approx 1268.32, \end{aligned}$$

   we can conclude that the surface area (hence, building cost) is minimized when $r = r_0 \approx 7.9859$.