THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1510 Calculus for Engineers (2020-2021)
Supplementary Exercise 10

Power Series

1. Simplify the following expression by using summation notation.

   (e.g) $x - x^3 + x^5 - \cdots - x^{15} = \sum_{r=1}^{8}(-1)^{r+1}x^{2r-1}$ or $\sum_{r=0}^{7}(-1)^{r}x^{2r+1}$

   (a) $x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{x^{2015}}{2015}$

   (b) $x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$

   (c) $\cos x + \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x + \cdots + \frac{1}{n^2}\cos nx$

   (d) $(\cos x - \sin x) + \frac{1}{2}(\cos 2x + \sin 2x) + \frac{1}{2^2}(\cos 3x - \sin 3x) + \cdots$

2. Let $P_k(x) = 1 + x + x^2 + \cdots + x^k = \sum_{n=0}^{k} x^n$, where $k \geq 0$.

   (a) Fix $x = 1/2$, note that $\{P_0(1/2), P_1(1/2), P_2(1/2), \cdots\}$ forms a sequence.

      (i) Write down the sequence explicitly:

      (ii) Does $\lim_{k\to\infty} P_k(\frac{1}{2})$ exist? Why?

   (b) Repeat the same procedure for $x = 2$. Does $\lim_{k\to\infty} P_k(2)$ exist?

   (c) Guess the range of x such that $\lim_{k\to\infty} P_k(x)$ exists. In this case, we say that the power series $\sum_{n=0}^{\infty} x^n$ converges.

   (Hint: $1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}$ if $x \neq 1$.)

3. Find the radius of convergence of each of the following power series.

   Recall: For a power series $\sum_{n=0}^{\infty} a_n(x-c)^n$, c is called the center. If the limit

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

   exists, the limit is said to be the **radius of convergence**. Given that the limit exists, the power series is convergent on the interval $(c - R, c + R)$ and divergent on $(-\infty, c - R) \cup (c + R, +\infty)$.

In particular, we allow R to be $+\infty$ here. If $R = +\infty$, it means that the power series converges for all real numbers x.

However, the fact does not tell the convergence of the power series at the boundary points $x = c - R$ and $c + R$. Also, it does not tell anything about the convergence if the limit does not exist.

(a) $\displaystyle\sum_{n=0}^{\infty} x^n$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n}$

(d) $\displaystyle\sum_{n=0}^{\infty} (x-3)^n$

(e) $\displaystyle\sum_{n=0}^{\infty} \frac{nx^n}{n+2}$

(f) $\displaystyle\sum_{n=0}^{\infty} \frac{3^n x^n}{n!}$

(g) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{\sqrt{n}+3}$

(h) $\displaystyle\sum_{n=0}^{\infty} \frac{n(x+3)^n}{5^n}$

(i) $\displaystyle\sum_{n=0}^{\infty} (-2)^n (n+1)(x-1)^n$

Taylor Polynomial and Taylor Series

4. Let $f(x)$ be a function with derivatives of order k for $k = 1, 2, \cdots, n$. Recall that the Taylor polynomial of order n generated by $f(x)$ at the point $x = c$ is the polynomial

$$\begin{aligned} T_n(x) &= f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n \\ &= \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k \end{aligned}$$

Prove that $T_n^{(k)}(c) = f^{(k)}(c)$ for all $k = 0, 1, 2, \cdots, n$.

5. Let $f(x) = \dfrac{1}{1-x}$, for $x \neq 1$.

(a) Find the Taylor polynomials $T_0(x)$, $T_1(x)$, $T_2(x)$ and $T_3(x)$ generated by $f(x)$ at $x = 0$ and plot the graphs of them by using MATLAB and compare with the graph of $f(x)$.

(b) Fix $x = 1/2$, note that $\{T_0(1/2), T_1(1/2), T_2(1/2), \cdots\}$ forms a sequence which is exactly the sequence in question 2(a).
Verify that $\lim_{k\to\infty} T_k(1/2) = f(1/2)$.

(c) Verify that $\lim_{k\to\infty} T_k(x) = f(x)$ for any $-1 < x < 1$.

6. Find the Taylor series generated by the following functions at given points.

   (a) $f(x) = \cos x$ at $x = \pi/2$;

   (b) $f(x) = \ln(1 + x)$ at $x = 0$;

   (c) $f(x) = e^x$ at $x = 1$.

7. (Harder Problem) Let $f(x)$ be a function defined by

$$f(x) = \begin{cases} 0 & \text{if } \ x = 0, \\ e^{-1/x^2} & \text{if } \ x \neq 0. \end{cases}$$

   (a) Show that $f(x)$ is differentiable at $x = 0$ and find $f'(0)$.
   (Hint: Show that $\lim_{x\to 0} \dfrac{f(x) - f(0)}{x}$ exists.)

   (b) Write down the function $f'(x)$ explicit as the following:

$$f'(x) = \begin{cases} \underline{\qquad\quad} & \text{if } \ x = 0, \\ \underline{\qquad\quad} & \text{if } \ x \neq 0. \end{cases}$$

   Show that $f'(x)$ is differentiable at $x = 0$ and find $f''(0)$.
   (Hint: Show that $\lim_{x\to 0} \dfrac{f'(x) - f'(0)}{x}$ exists.)

   (c) In general, is $f^{(n)}(0)$ defined for each positive integer n? If so, what is the value?

   (d) Find the Maclaurin series generated by $f(x)$, i.e. Taylor series generated by $f(x)$ at the point $x = 0$.

8. By considering the Taylor series generated by e^x and $\cos x$ at $x = 0$, find the Taylor polynomials of degree 3 generated by the following functions at $x = 0$.

   (a) $e^x \cos x$;

   (b) $e^{\cos x}$;

   (c) $\dfrac{e^x}{\cos x}$.

9. By considering $\cos^2 x = \dfrac{1}{2}(1 + \cos 2x)$, find the Taylor polynomial of degree 4 generated by $\cos^2 x$.
(Remark: You may compare the one obtained by considering $\cos^2 x = (\cos x)(\cos x)$.)

10. Let $f(x) = \sin x$.
    (a) Find the Maclaurin series generated by $f(x)$.
    (b) By considering $f'(x)$ and term-by-term differentiation, find the Maclaurin series generated by $\cos x$. Do they match with each other?

11. Let $f(x) = \dfrac{1}{1-x}$. By considering $f'(x)$, $f''(x)$ and term-by-term differentiation, find the Maclaurin series generated by $\dfrac{1}{(1-x)^2}$ and $\dfrac{1}{(1-x)^3}$.

12. By using the fact that $\displaystyle\int -\sin x\,dx = \cos x + C$, find the Taylor series generated by $\cos x$ at $x = 0$.

13. (a) By considering $\dfrac{2x}{1-x^2} = \dfrac{1}{1-x} - \dfrac{1}{1+x}$, find the Taylor series generated by $\dfrac{2x}{1-x^2}$ at $x = 0$.
    (b) By using the fact that $\displaystyle\int -\frac{2x}{1-x^2}\,dx = \ln(1-x^2) + C$, find the Talyor series generated by $\ln(1-x^2)$ at $x = 0$.

14. Let $f(x) = \ln(1-x)$ for $x < 1$.
    (a) Find the Taylor series generated by $f(x)$ at $x = 0$ and find the radius of convergence.
    (b) Write down the Taylor polynomial $T_3(x)$ of degree 3 generated by $f(x)$ at $x = 0$ and the Lagrange remainder $R_3(x)$.
    (c) Hence, approximate $\ln 0.9$ and show that the error of this approximation is less than $\dfrac{1}{4 \times 9^4}$.

Fourier Series

(Periodic Function) A function $f : \mathbb{R} \to \mathbb{R}$ is said to be a periodic function if there is a constant $T > 0$ such that

$$f(x + T) = f(x)$$

for all real numbers x. Furthermore, if T is the least positive real number with the above property, then T is said to be the period of the function f. For example, $\sin x$, $\cos x$ and $\tan x$ are periodic function but the periods of $\sin x$ and $\cos x$ are 2π while the period of $\tan x$ is π.

15. Suppose that $f : [-1, 1) \to \mathbb{R}$ is a function defined by $f(x) = x$. If f is extended to be a periodic function with period 2, try to sketch the graph of the extended function.

Let $L > 0$ and $f : \mathbb{R} \to \mathbb{R}$ be a periodic function with period $2L$. The Fourier Series generated by f is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{L}) + \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{L}),$$

where the Fourier coefficients a_n's and b_n's are given by

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{n\pi x}{L})\,dx,\ n \geq 0; \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{n\pi x}{L})\,dx,\ n \geq 1. \end{aligned}$$

The idea of Fourier Series is experssing a period function $f(x)$ as a sum of sines and cosines. With suitable assumptions (beyond the scope of this course), we have the point-wise convergence, that is

$$f(x_0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x_0}{L}) + \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x_0}{L})$$

if $f(x)$ is continuous at $x = x_0$.

16. Let $f(x)$ be a periodic function with period 2 (i.e $L = 1$) which is defined by

$$f(x) = \begin{cases} x+1 & \text{if} \quad -1 \leq x \leq 0, \\ 1-x & \text{if} \quad 0 < x < 1. \end{cases}$$

   (a) Find the Fourier series generated by $f(x)$.
   (Hint: $f(x)\sin(n\pi x)$ is an odd function for any $n = 1, 2, \cdots$.)

   (b) For any natural number N, define

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos(n\pi x) + \sum_{n=1}^{N} b_n \sin(n\pi x),$$

   where a_n's and b_n's are Fourier coefficients found in (a).
   Plot the graphs of $S_N(x)$ for $N = 1, 3, 5$ by using MATLAB or other softwares.

17. Let $f(x)$ be a periodic function with period 2π such that

$$f(x) = \begin{cases} 1 & \text{if} \quad -\pi < x < 0, \\ 0 & \text{if} \quad x = -\pi \text{ or } 0, \\ -1 & \text{if} \quad 0 < x < \pi. \end{cases}$$

   Find the Fourier Series generated by $f(x)$.

18. Let $f(x)$ be a periodic function with period 2π such that $f(x) = x^2$ for $-\pi < x \leq \pi$.

    (a) Show that the Fourier series of $f(x)$ is

$$\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx.$$

    (b) By considering a suitable value of x, show that

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots .$$

19. Let $f(x)$ be a function defined on $[-\pi, \pi]$ such that

$$f(x) = \begin{cases} x & \text{if} \quad 0 < x \leq \pi, \\ 0 & \text{if} \quad -\pi \leq x \leq 0. \end{cases}$$

    (a) Find the Fourier series of $f(x)$.

    (b) Show that

        (i) $\dfrac{\pi}{4} = 1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \cdots;$

        (ii) $\dfrac{\pi^2}{8} = 1 + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \dfrac{1}{7^2} + \cdots.$