Chapter 15
Derivatives and the Shapes of Graphs

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MATH1510

Calculus for Engineers

Table of Content

We shall especially look for the following aspects of a graph of function:

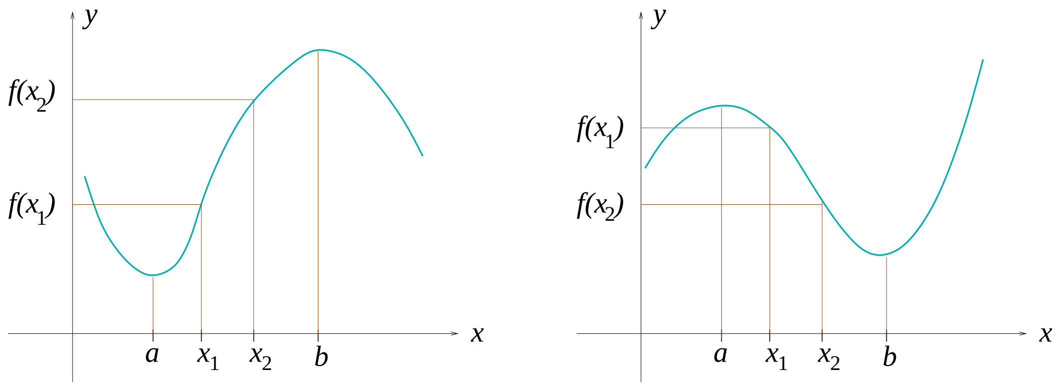
1. Behaviour as x becomes large positive and large negative, i.e., as $x \to \pm\infty$.
2. Regions of increase $\nearrow$ and decrease $\searrow$.
3. Regions of concave upward (concavity) $\nearrow\searrow$ and concave downward (convexity) $\nwarrow\swarrow$.
4. Extreme value points.
5. Inflection points.
6. Intersections with the coordinate axes.
7. Values of x near which y approaches infinity positive or negative, i.e., as $y \to \pm\infty$.

Maxima & Minima

Definition 1 **Definition of Increasing and Decreasing Functions.**

Let f be defined on an interval, let x_1 and x_2 denote numbers in the interval.

- A function f is increasing on an interval (a, b) if for any two numbers x_1 and x_2 in (a, b), $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ (Figure 1(a)).
- A function f is decreasing on an interval (a, b) if for any two numbers x_1 and x_2 in (a, b), $f(x_1) > f(x_2)$ whenever $x_1 < x_2$ (Figure 1(b)).
- A function f is constant on an interval (a, b) if $f(x_1) = f(x_2)$ for all x_1 and x_2.



(a) f is increasing on (a, b).

(b) f is decreasing on (a, b).

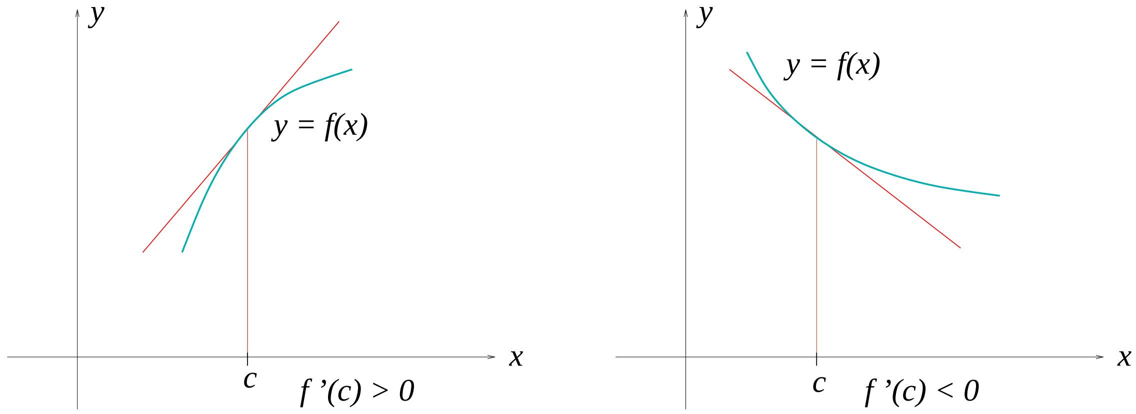
Figure 1: Increasing (or Rising $\nearrow$) and Decreasing (or Falling $\searrow$) Functions.

Theorem 1 Let f be a function that is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b).

(a) If $f'(x) > 0$ for each value of x in an interval (a, b), then f is increasing on $[a, b]$.

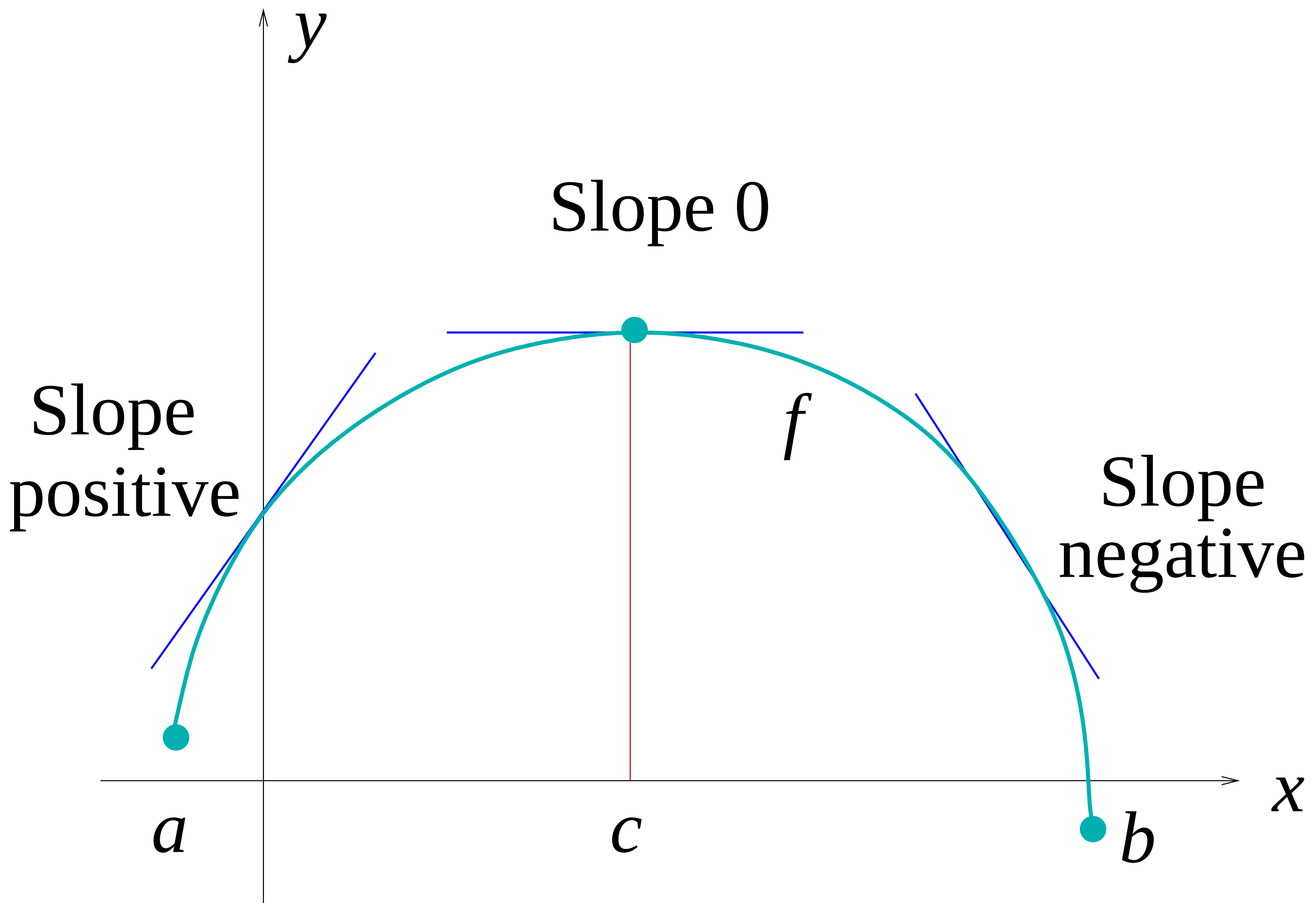
(b) If $f'(x) < 0$ for each value of x in an interval (a, b), then f is decreasing on $[a, b]$.

(c) If $f'(x) = 0$ for each value of x in an interval (a, b), then f is constant on $[a, b]$.



(a) f is increasing at $x = c$. Graph has positive slope.

(b) f is decreasing at $x = c$. Graph has negative slope.



Note that if $f'(c) = 0$, the graph of $y = f(x)$ will have a **horizontal tangent line** at $x = c$.

For the interval (a, b),

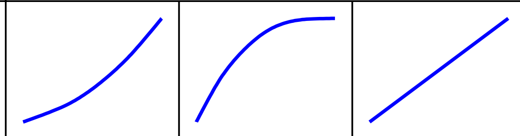
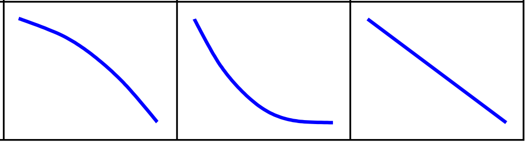
$f'(x)$	$f(x)$	Graph of f	Examples
+	Increases ↗	Rises ↗	
-	Decreases ↘	Falls ↘	

Figure 2: Increasing and Decreasing Functions

Remark 1 Theorem 1 states that it is only necessary to examine the derivative of f on the open interval (a, b) to determine whether f is increasing, deceasing or constant on the closed interval.

Besides the choice of the close interval $[a, b]$, Theorem 1 is also applicable to any interval I on which f is continuous and inside of which f is differentiable.

If f is continuous on $[a, \infty)$ and $f'(x) > 0$ for each x in the interval (a, ∞), then f is increasing on $[a, +\infty)$; if $f'(x) < 0$ on $(-\infty, +\infty)$, then f is decreasing on $(-\infty, +\infty)$ [the continuity on $(-\infty, +\infty)$ follows from the differentiability].

Definition of Partition Numbers

We begin a useful example

Example 1 Solve: $\dfrac{x+1}{x-2} > 0.$

Solution:

- Let $f(x) = \dfrac{x+1}{x-2}$.
- The rational function f is discontinuous at $x = 2$ and $f(x) = 0$ for $x = -1$ (a fraction is 0 when the numerator is 0 and the denominator is not 0).
- We use $x = 2$ and $x = -1$, which we call **partition numbers**, on a real number line.
- The partition numbers 2 and -1 determine three open intervals:

$$(-\infty, -1), (-1, 2), \text{ and } (2, \infty).$$

Remark 2

- Note that in general given a function f, we will call all values such that f is discontinuous at x or $f(x) = 0$ **partition numbers**.
- Partition numbers determine open intervals where $f(x)$ does not change sign, i.e., $+$ or $-$.
- By using a **test number** from each interval, we can construct a sign chart for $f(x)$ on the real number line.
- It is then an easy matter to determine where $f(x) < 0$ or $f(x) > 0$; that is to solve the inequality $f(x) < 0$ or $f(x) > 0$.

Definition 2 Critical points of f

The values of x in the domain of f, where

- $f'(x) = 0$ or
- $f'(x)$ does not exist

are called the critical points (or test points or critical values) of f.

Remark 3 We define a critical point for a function f to be a point in the domain of f at which either the graph of f has a horizontal tangent line or f is not differentiable.

Remark 4 To distinguish between the two types of critical points we call x **a stationary point** of f if $f'(x) = 0$.

Remark 5

- The critical points of f are always in the domain of f and are also partition numbers for f', but f' may have partition numbers that are not critical points.
- If f is a polynomial, then both the partition numbers for f' and the critical points of f are solutions of $f'(x) = 0$.

Partition Numbers and critical points

Let us explore the relationship between critical points and partition numbers.

Example 2 For each function,

1. find partition numbers for f', the critical points for f and
2. determine the intervals where f is increasing and those where f is decreasing.

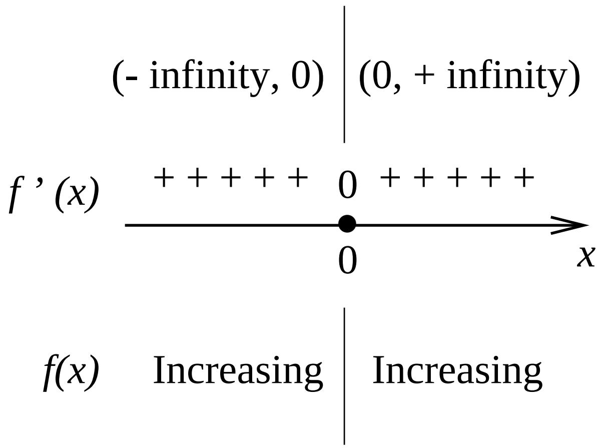
1. $f(x) = 1 + x^3$
2. $f(x) = (1 - x)^{1/3}$
3. $f(x) = \dfrac{1}{x - 2}$

Solution:

1. $f(x) = 1 + x^3$

   $f'(x) = 3x^2 = 0 \iff x = 0$

   The only partition number for f' is $x = 0$. Since 0 is the domain of f, $x = 0$ is also the only critical point for f.



Test Numbers	
x	$f'(x)$
- 1	3 (+)
+ 1	3 (+)

Figure 3: $f(x) = 1 + x^3$.

The sign chart indicates that $f(x)$ is increasing on $(-\infty, 0)$ and $(0, \infty)$. Since f is continuous at $x = 0$, it follows that $f(x)$ is increasing for all x.

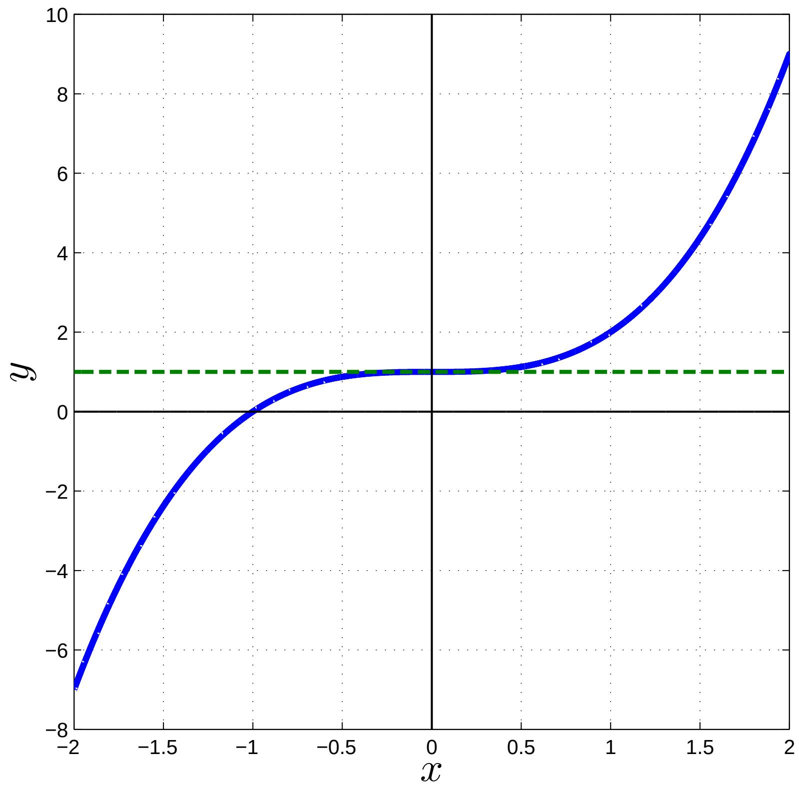


Figure 4: Graph of $f(x) = 1 + x^3$.

2.

$$f(x) = (1-x)^{1/3}$$

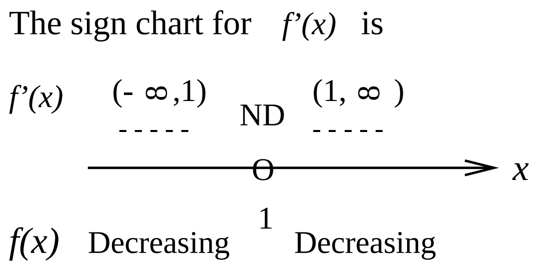
and

$$f'(x) = \frac{1}{3}(1-x)^{-2/3} = \frac{-1}{3(1-x)^{2/3}} \quad \text{by Chain rule.}$$

To find partition numbers for f', we note that f' is continuous for all x except for values of x for which the denominator is 0; that is, $f'(1)$ does not exist and f' is discontinuous at $x = 1$.

- Since the numerator is the constant -1, $f'(x) \neq 0$ for any value of x. Thus, $x = 1$ is *the only partition number* for f'.
- Since 1 is the domain of f, $x = 1$ is also *the only critical point* of f.
- When constructing the sign chart for f' we use the abbreviation **ND** note that the fact that $f'(x)$ is not defined at $x = 1$.

Sign chart for $f'(x) = \dfrac{-1}{3(1-x)^{2/3}}$ (partition number is 1):



Test Numbers	
x	*f'(x)*
0	-1/3 (-)
2	-1/3 (-)

Figure 5: $f(x) = (1-x)^{1/3}$ and $f'(x) = \dfrac{-1}{3(1-x)^{2/3}}$.

- The sign chart indicates that f is decreasing on $(-\infty, 1)$ and $(1, \infty)$.
- Since f is continuous at $x = 1$, it follows that $f(x)$ is decreasing for all x.
- Thus, a continuous function can be decreasing (or increasing) on an interval containing values of x where $f'(x)$ does <u>not</u> exist.

The graph of f is shown in Figure 6.

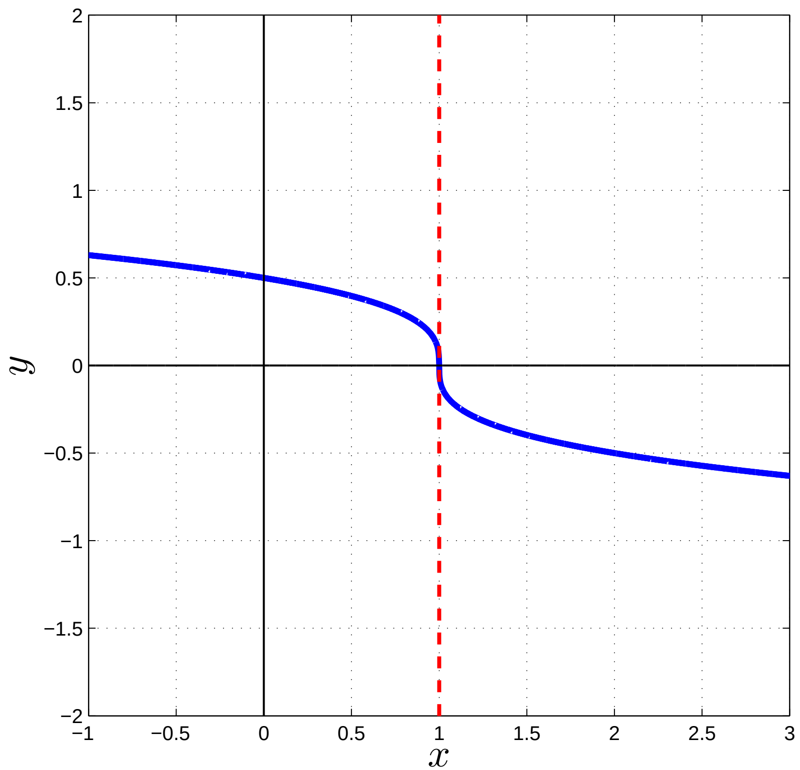


Figure 6: $f(x) = (1-x)^{1/3}$.

Note that the undefined derivative at $x = 1$ results in a vertical tangent line at $x = 1$. In general, a vertical tangent will occur at $x = c$ if f is continuous at $x = c$ and $|f'(x)|$ becomes larger and larger as x approaches c.

Compute

$$\lim_{x \to 1^+} f'(x)$$

and

$$\lim_{x \to 1^-} f'(x).$$

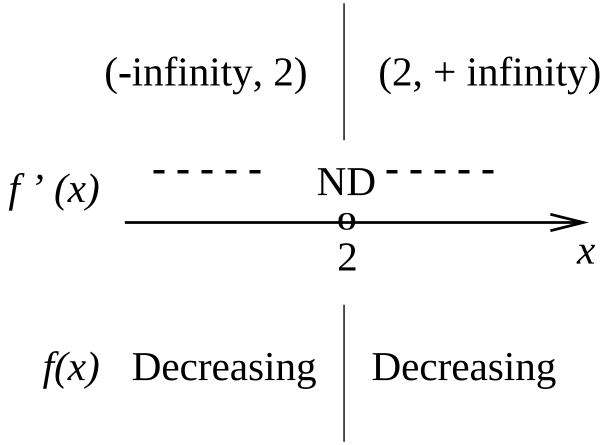
3.

$$f(x) = \frac{1}{x-2}$$

and

$$f'(x) = \frac{-1}{(x-2)^2} \quad \text{By Chain Rule and Product Rule.}$$

The partition numbers for f' is $x = 2$ since $f'(x) \neq 0$ for any x and f' is not defined at $x = 2$. However, $x = 2$ is not in the domain of f. Therefore, $x = 2$ is not a critical point of f. This function has no critical points.



Test Numbers	
x	$f'(x)$
1	-1 (-)
3	-1 (-)

Figure 7: $f(x) = \dfrac{1}{x-2}$ & $f'(x) = \dfrac{-1}{(x-2)^2}$.

Thus, f is decreasing on $(-\infty, 2)$ and $(2, \infty)$.

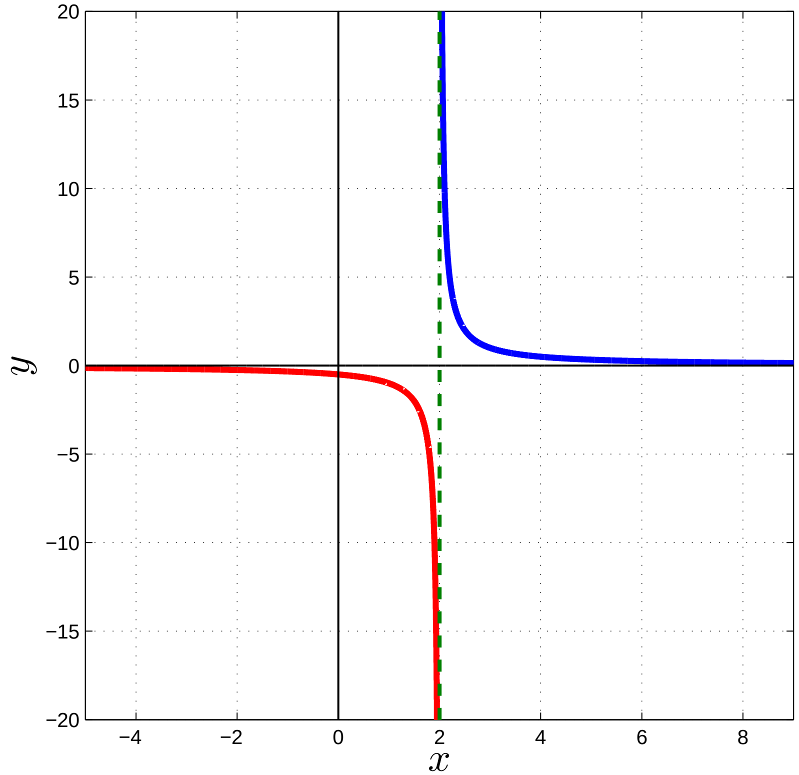


Figure 8: $f(x) = \dfrac{1}{x-2}$.

Remark 6

1. Do not assume that all partition numbers for the derivative f' are critical points of the function f.

   To be a critical point, a partition number must also be in the domain of f.

2. The values where a function is increasing or decreasing must always be expressed in terms of open intervals that are subsets of the domain of the function.

Determining the intervals where a function is increasing or decreasing

Step 1 Find *all values* of x (or partition numbers) for $f'(x) = 0$ or f' is discontinuous and identify the open intervals determined by these points.

Step 2 Select a critical point c in each interval found in Step 1 and determine the sign of $f'(c)$ in that interval.

1. If $f'(c) > 0$, f is increasing on that interval.
2. If $f'(c) < 0$, f is decreasing on that interval.

Note that the partition numbers for f' include the numbers c, where $f'(c)$ does not exist.

1. $f(c)$ does not exist or
2. $f(c)$ exists, but the slope of the tangent line at $x = c$ is undefined.

Local Extrema

- When the graph of a continuous function *changes from raising* $\nearrow$ *to falling* $\searrow$, a high point or local maximum, occurs;
- When the graph *changes from falling* $\searrow$ *to raising* $\nearrow$, a low point or local minimum, occurs.

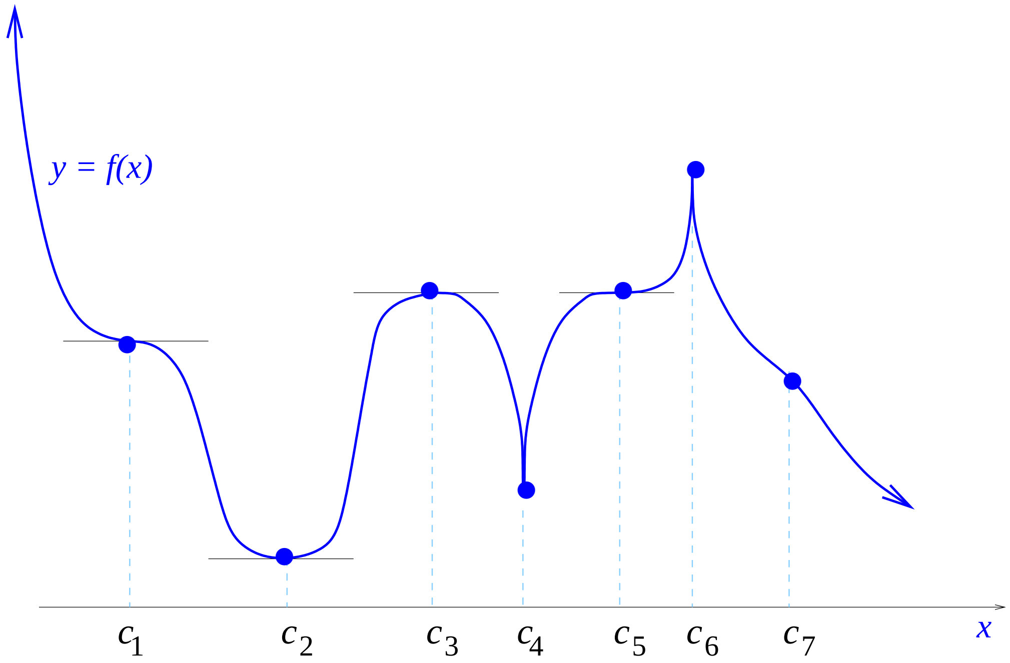


Figure 9: Looking for local extrema.

Definition 3 Local (or Relative) Maximum

A function f has a **local (or relative) maximum** at $x = c$ **if** there exists an open interval (a, b) containing c such that $f(x) \leq f(c)$ for all x in (a, b).

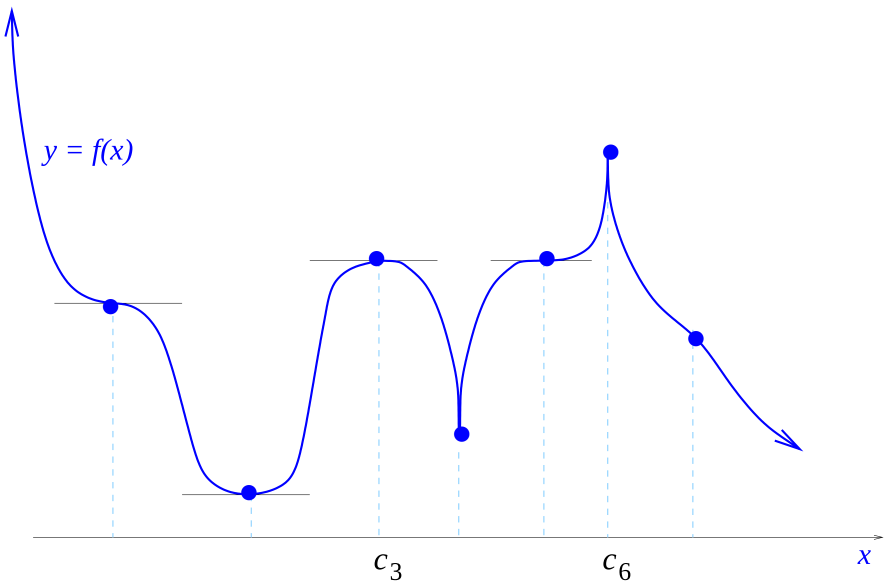


Figure 10: Local maxima occur at c_3 and c_6.

Note that this inequality need only hold for values of x near c, hence the use of term *local*.

Definition 4 Local (or Relative) Minimum

A function f has a **local or (relative) minimum** at $x = c$
if there exists an open interval (a, b) containing c such that $f(x) \geq f(c)$ for all x in (a, b).

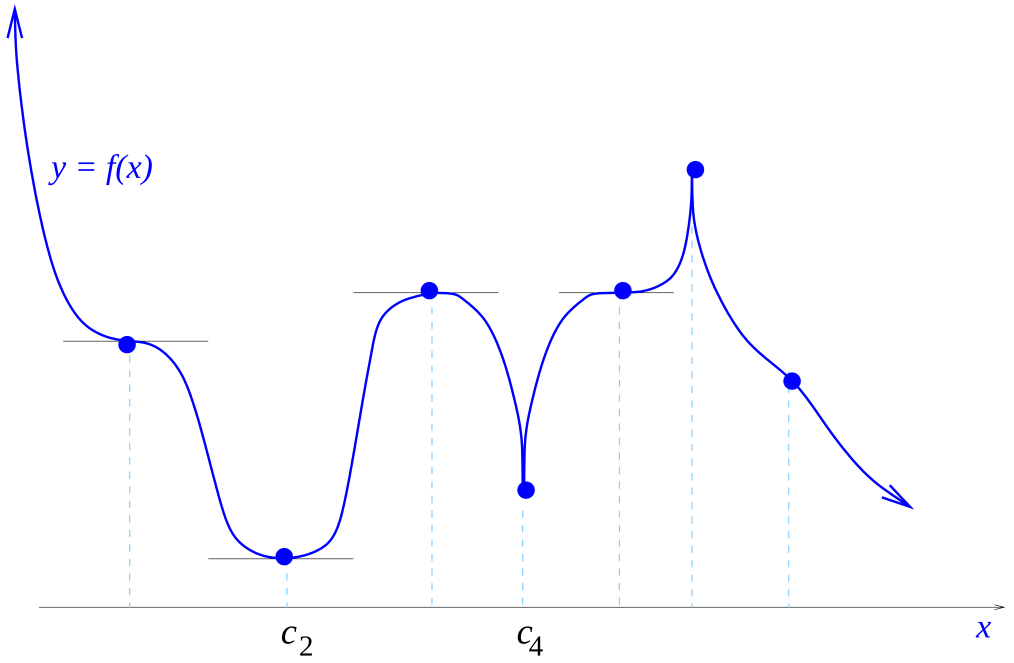


Figure 11: Local minima occur at c_2 and c_4.

Definition 5 Local Extremum

The quantity $f(c)$ is called a local extremum if it is either a local maximum or a local minimum. A point on graph where a local extremum occurs is also called *a turning point.*

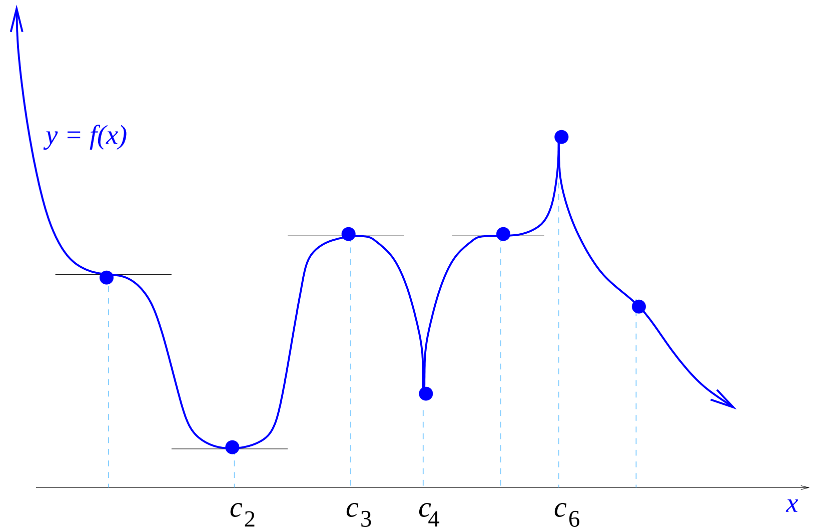


Figure 12: Local extrema occur at c_3 and c_6, and c_2 and c_4.

How can we locate local maxima and minima if we are given the equation for a function and not its graph?

To examine the critical points of the function

The local extrema of function f occur either at points where the derivative is 0, or

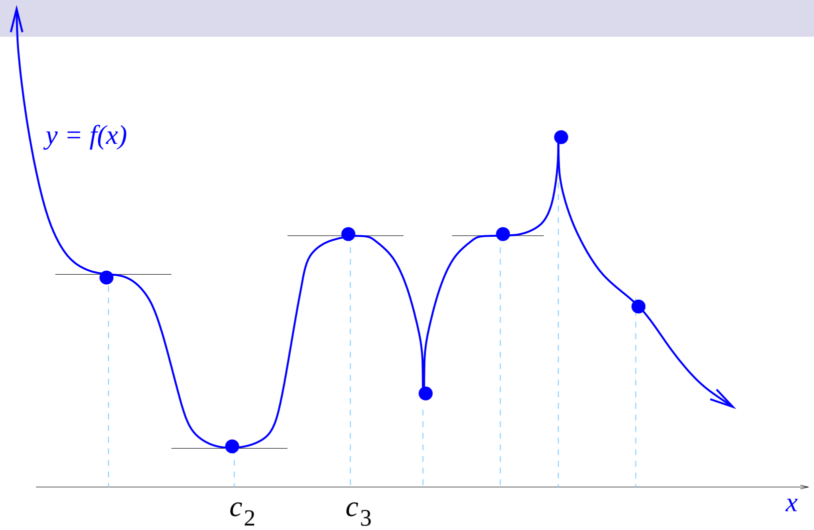


Figure 13: Derivative is 0; c_2 and c_3

at points where the derivative does not exists.

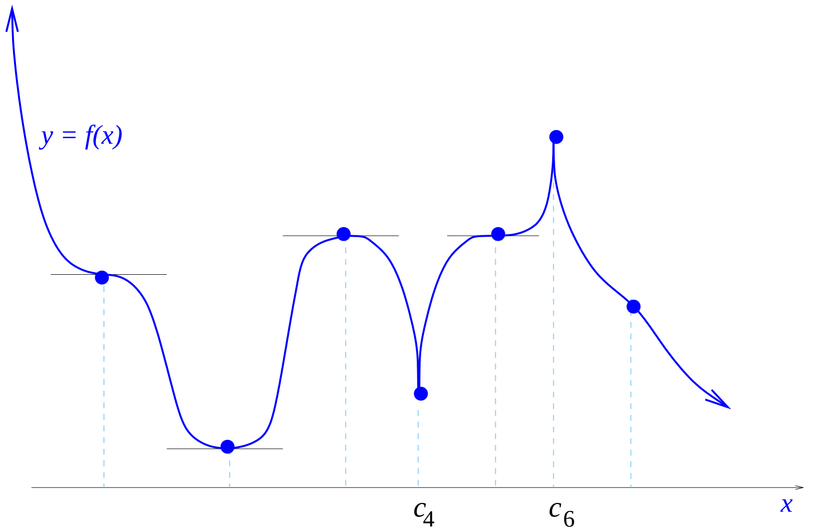


Figure 14: Derivative does not exists; c_4 and c_6

In other words, local extrema occur only at critical points of f.

Theorem 2 shows that this is true in general.

Theorem 2 **Existence of Local Extrema**

If f is continuous on the interval (a, b), c is a number in (a, b) and $f(c)$ is a local extremum, then either

- $f'(c) = 0$

or

- $f'(c)$ does not exist (or is not defined.)

Theorem 2 states that a local extremum can occur only at a critical point, but it does not imply that every critical point produces a local extremum.

Here, c_1 and c_5 are critical points (the slope is 0), but the function does not have *a local maximum* or *local minimum* at either of these values.

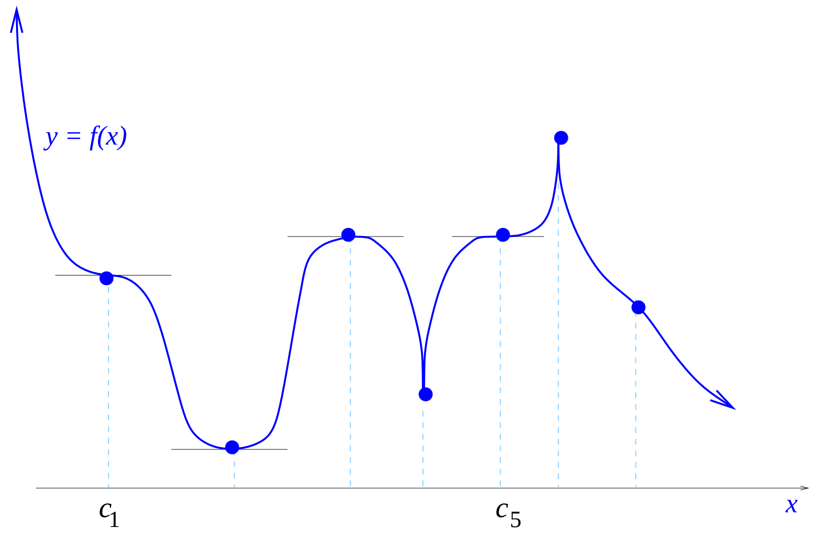


Figure 15:

What is the First Derivative Test?

If $f'(x)$ exists on both sides of a critical point c,
then the sign of $f'(x)$ can be used to determine whether the point $(c, f(c))$ is a local maximum, a local minimum, or neither.

Theorem 3 First Derivative Test

Suppose that f is continuous at a critical point c and that f is differentiable at every point in some interval containing c expect possibly at c itself. Moving across this interval from left to right,

1. If $f'(x) > 0$ on an open interval extending left from c and $f'(x) < 0$ on an open interval extending right from c, then f has a relative maximum at $x = c$.
2. If $f'(x) < 0$ on an open interval extending left from c and $f'(x) > 0$ on an open interval extending right from c, then f has a relative minimum $x = c$.
3. If $f'(x)$ has the same sign on an open interval extending left from c as it does on an open interval extending right from c, then f does not have a relative extremum at $x = c$.

Finding the local (or relative) extrema

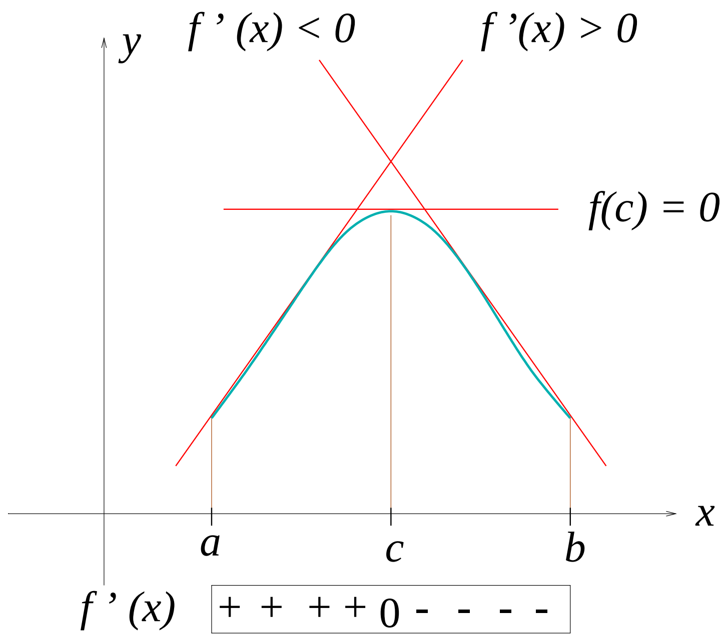
First-derivative test for local extrema

Let c be a critical point of f;

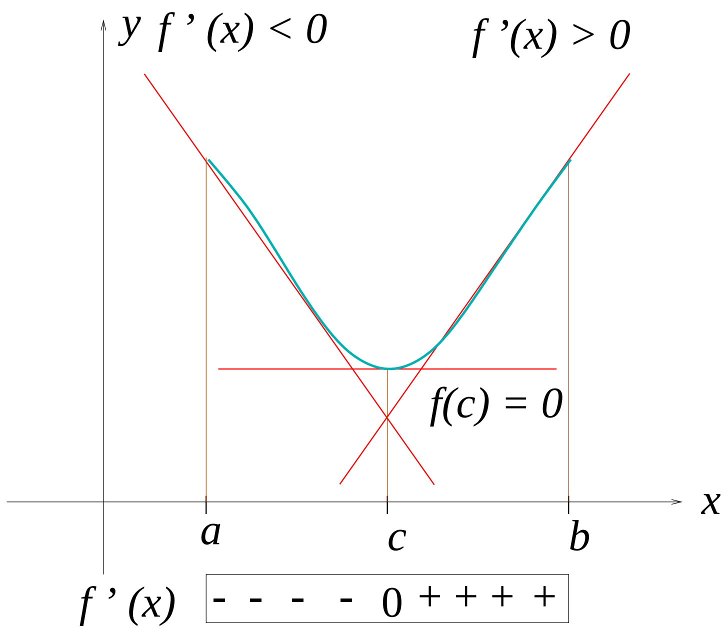
1. $f(c)$ defined and $f'(c) = 0$ (horizontal tangent)
2. $f'(c)$ is not defined but $f(c)$ is defined.

Construct a sign chart for $f'(x)$ close to and on either side of c.

An important characteristic of the local extrema of a differentiable function is: Any point c where f has a local extremum $f'(c) = 0$.

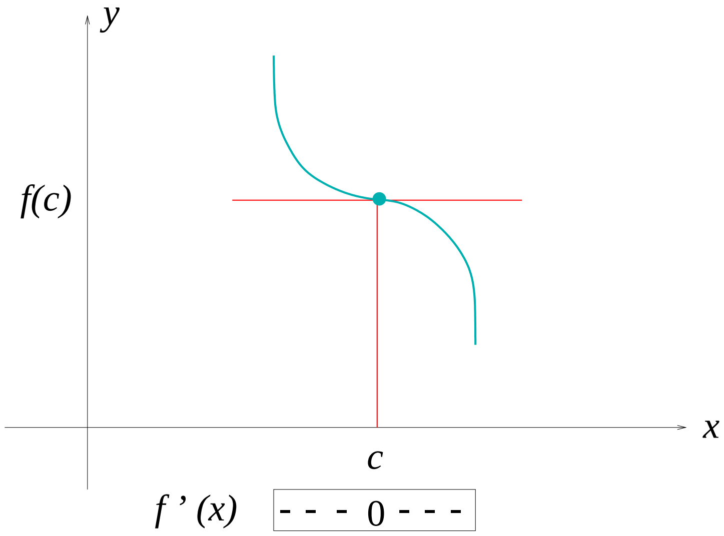


(a) f is a local maximum at $x = c$. c is a critical point and a stationary point.

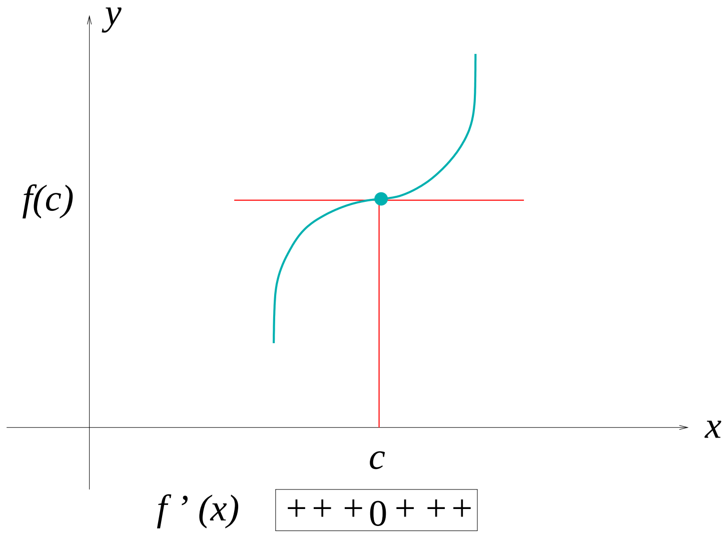


(b) f is a local minimum at $x = c$. c is a critical point and a stationary point.

Finding the local (or relative) extrema



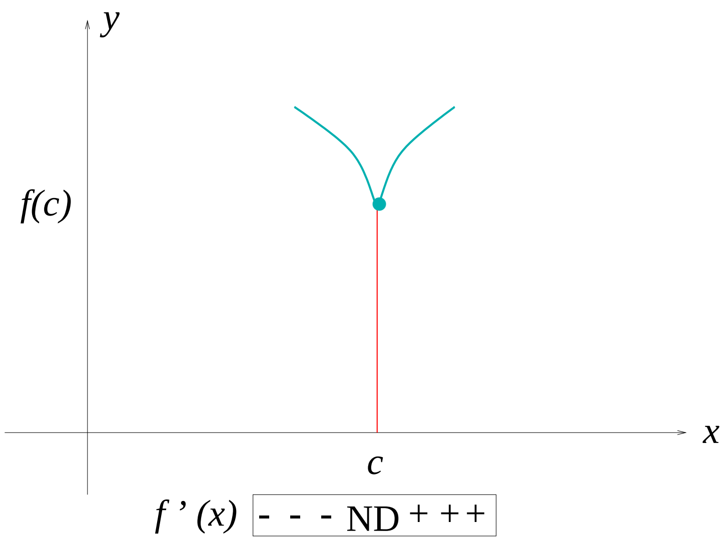
(c) f neither a local maximum nor a local minimum at $x = c$. c is a critical point, a stationary point and an inflection point.



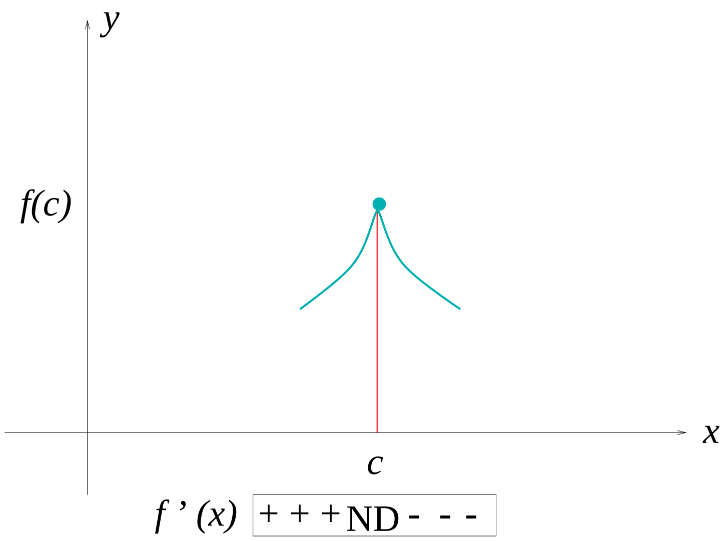
(d) f is neither a local maximum nor a local minimum at $x = c$. c is a critical point, a stationary point and an inflection point.

Figure 16: $f(c)$ is not a local extremum. If $f'(c)$ does not change sign at c, then $f(c)$ is neither a local maximum nor a local minimum.

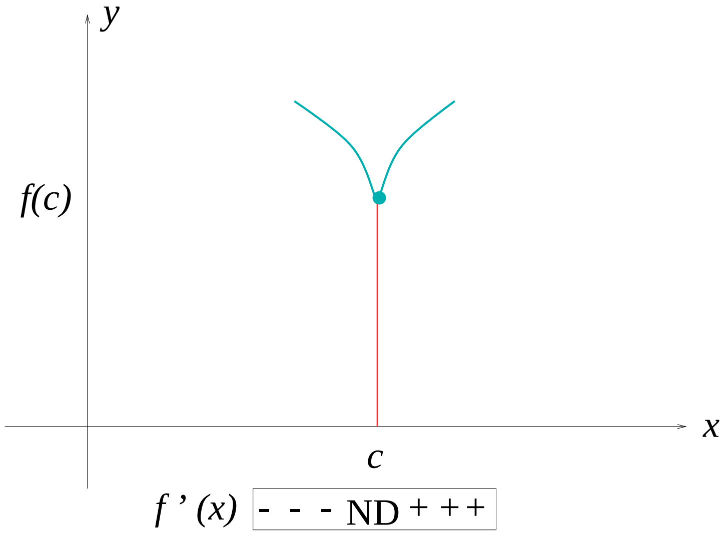
$f'(c)$ is not defined but $f(c)$ is defined.



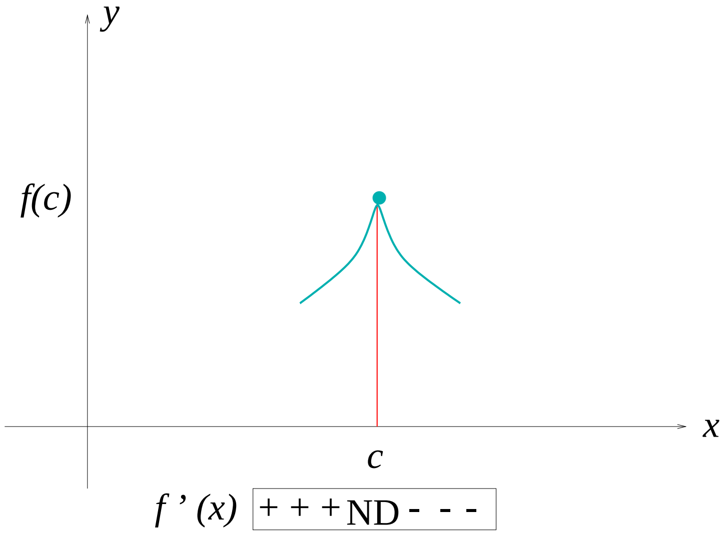
(a) f has a local minimum at $x = c$. c is a critical point and not a stationary point.



(b) f is a local maximum at $x = c$. c is a critical point and not a stationary point.



(c)



(d)

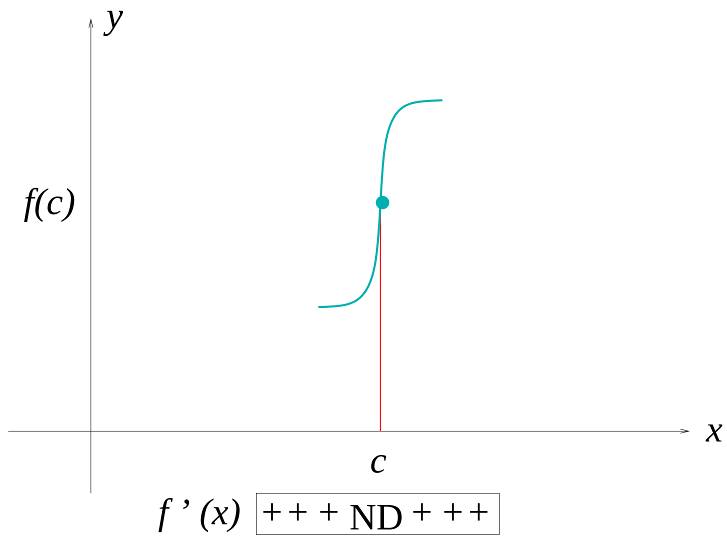
$$\lim_{x \to c^+} f'(x) = +\infty$$

$$\lim_{x \to c^-} f'(x) = -\infty$$

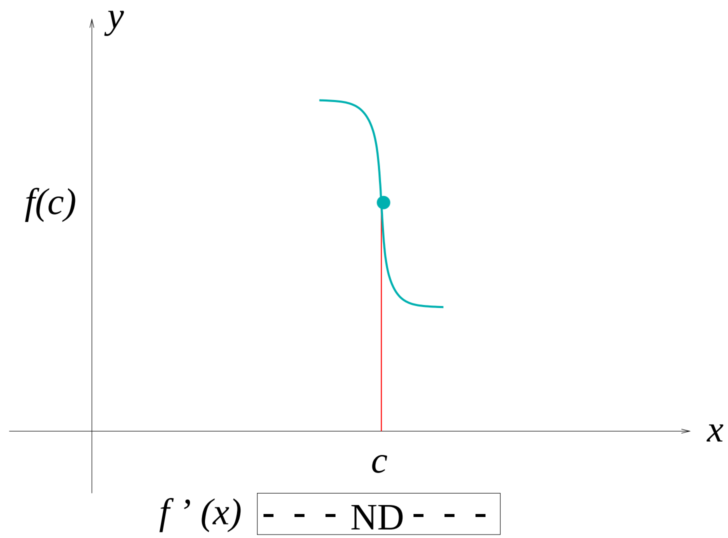
$$\lim_{x \to c^+} f'(x) = -\infty$$

$$\lim_{x \to c^-} f'(x) = +\infty$$

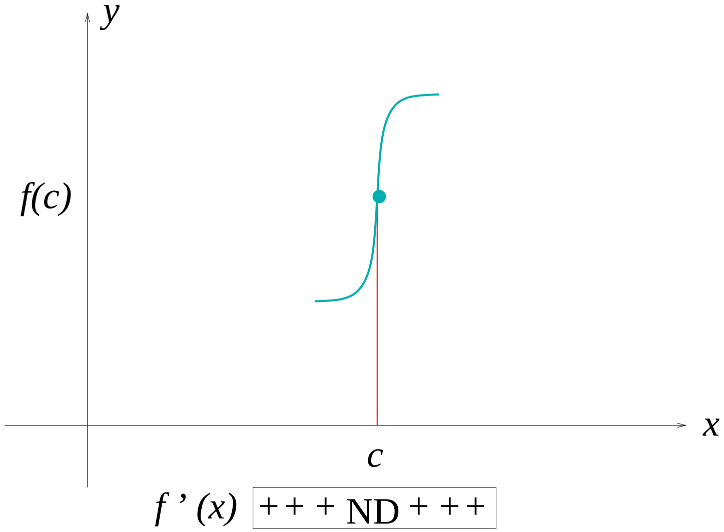
$f'(c)$ is not defined but $f(c)$ is defined.



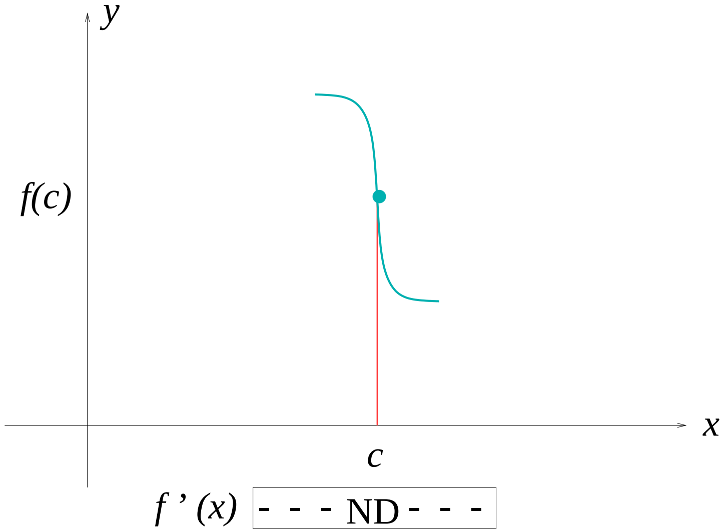
(e) f neither a local maximum nor a local minimum at $x = c$. c is a critical point, not a stationary point and an inflection point.



(f) f is neither a local maximum nor a local minimum at $x = c$. c is a critical point, not a stationary point and an inflection point.



(g)



(h)

$$\lim_{x \to c^+} f'(x) = +\infty \qquad \lim_{x \to c^+} f'(x) = -\infty$$

$$\lim_{x \to c^-} f'(x) = +\infty \qquad \lim_{x \to c^-} f'(x) = -\infty$$

Critical points of f

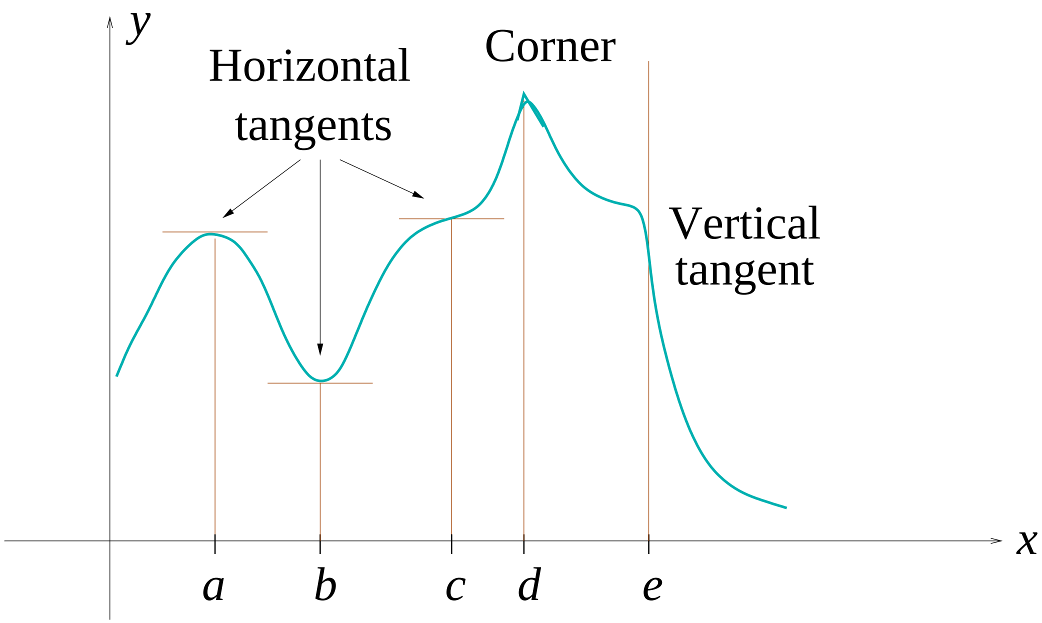


Figure 17: Critical points of f.

A critical point of a function f is any point x in the domain of f such that $f'(x) = 0$ or $f'(x)$ does not exist.

The point d is a critical point and not a stationary point, but f is a local maximum at $x = d$.

Example 3 Apply the first derivative text to classify the critical point of

$$f(x) = 2x^3 + 3x^2 - 36x + 17.$$

Solution:

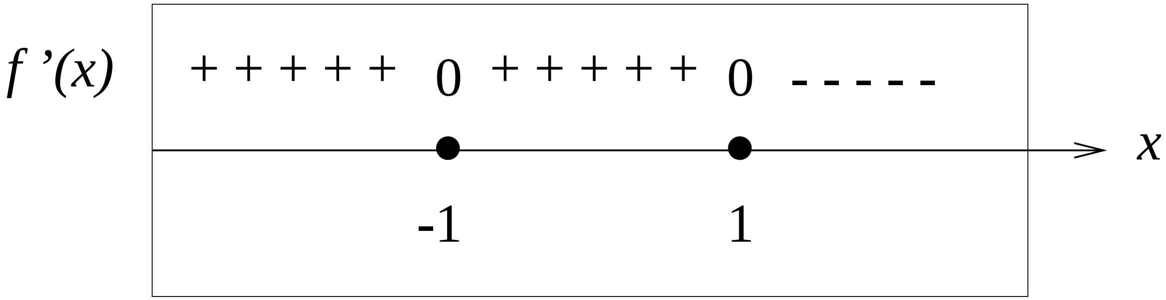
- Since $f(x)$ is a polynomial, $f(x)$ is differentiable (and so continuous) in its domain $(\infty, -\infty)$.
- Compute

  $$f'(x) = 6x^2 + 6x - 36 = 6(x^2 + x - 6) = 6(x - 2)(x + 3).$$

  Thus the critical points of $f(x)$ are $x = 2$ and $x = -3$. These points divide the domain of $f(x)$ into three intervals: $(-\infty, -3)$, $(-3, 2)$, $(2, \infty)$.
- Since $f'(-4) > 0$, $f'(0) < 0$ and $f'(3) > 0$, we conclude that $f'(x) > 0$ in both $(-\infty, 3)$ and $(2, \infty)$, and that $f'(x) < 0$ in $(3, 2)$. Therefore, $f(x)$ is increasing in both $(-\infty, -3)$ and $(2, \infty)$, and $f(x)$ is decreasing in $(-3, 2)$.
- By The First Derivative Test, $f(-3)$ is a local minimum value and $f(2)$ is a local maximum value of $f(x)$ in its domain.

		CP		CP	
		-3		2	
Interval	$(-\infty, -3)$	-3	$(-3, 2)$	2	$(2, \infty)$
Test Value	-4		0		3
f'	$f'(-4) < 0$		$f'(0) > 0$		$f'(3) < 0$
	$-$		$+$		$-$
f	decreasing		increasing		decreasing
	$\searrow$		$\nearrow$		$\searrow$
The First Derivative Test		a local min.		a local max.	

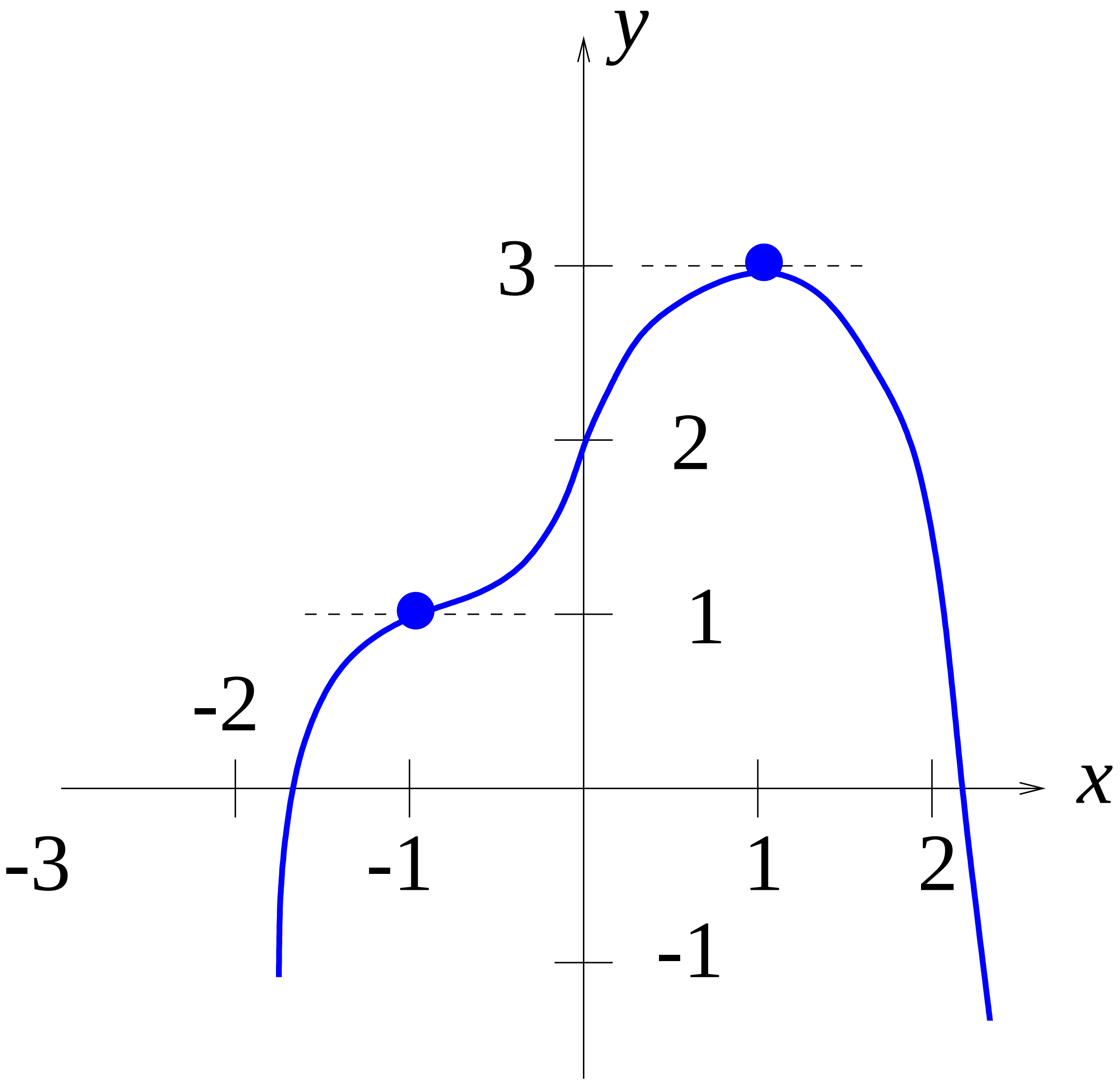
Example 4 $f(x)$ is continuous on $(-\infty, +\infty)$. Use the given information to sketch the graph of f.



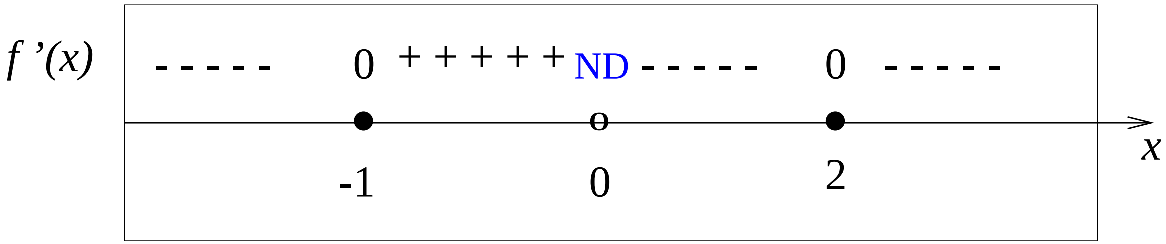
x	-2	-1	0	1	2
$f(x)$	-1	1	2	3	1

x	$f'(x)$	$f(x)$	Graph of f
$(-\infty, -1)$	$+$	Increasing	Rising
$x = -1$	0	Neither local max. nor local min	Horizontal Tangent
$(-1, 1)$	$+$	Increasing	Rising
$x = 1$	0	Local max.	Horizontal Tangent
$(1, \infty)$	$-$	Decreasing	Falling

Using this information together with the points $(-2, -1)$, $(-1, 1)$, $(0, 2)$, $(1, 3)$, $(2, 1)$ on the graph, we have


y
3
2
1
-2
x
-3
-1
1
2
-1

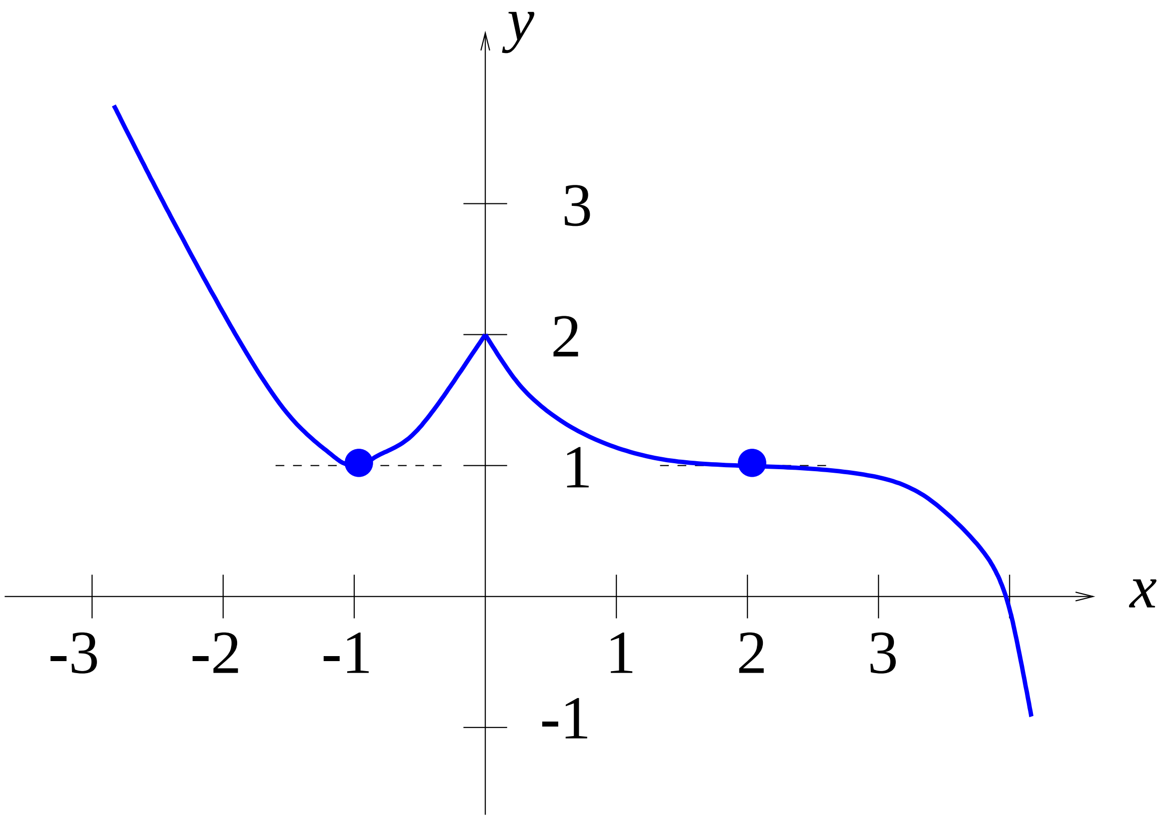
Example 5 $f(x)$ is continuous on $(-\infty, +\infty)$. Use the given information to sketch the graph of f.



x	-2	-1	0	2	4
f(x)	2	1	2	1	0

x	$f'(x)$	$f(x)$	Graph of f
$(-\infty, -1)$	$-$	Decreasing	Falling
$x = -1$	0	Local min.	Horizontal Tangent
$(-1, 0)$	$+$	Increasing	Rising
$x = 0$	Not defined	Local max.	Corner
$(0, 2)$	$-$	Decreasing	Falling
$x = 2$	0	Neither local max. nor local min	Horizontal Tangent
$(2, \infty)$	$-$	Decreasing	Falling

Using this information together with the points $(-2, 2)$, $(-1, 1)$, $(0, 2)$, $(2, 1)$, $(4, 0)$ on the graph, we have


y
3
2
1
-1
x
-3
-2
-1
1
2
3

Example 6 Find the critical points, the intervals where $f(x)$ is increasing, the intervals where $f(x)$ is decreasing, and the local extrema. Do not graph

1. $f(x) = 1 + \dfrac{1}{x} + \dfrac{1}{x^2}$;
2. $f(x) = x^4(x-6)^2$.

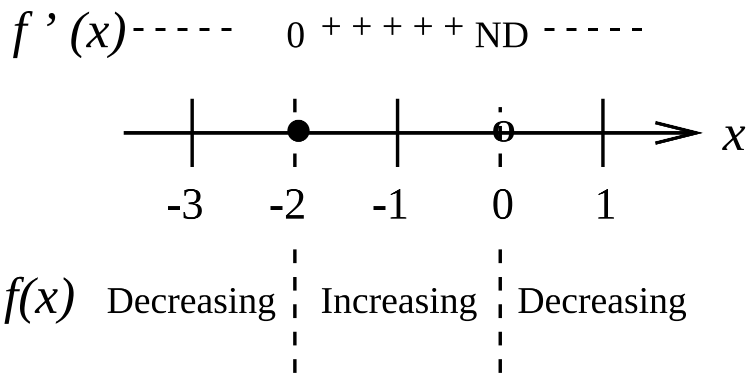
Solution:

1. $f(x) = 1 + \dfrac{1}{x} + \dfrac{1}{x^2}$. Note that f is not defined at $x = 0$. $f'(x) = -\dfrac{1}{x^2} - \dfrac{2}{x^3}$.
   Critical points: $x = 0$ is not a critical point of f since 0 is not in the domain of f; $x = 0$ is a partition number for f'.

$$\begin{aligned} f'(x) = -\frac{1}{x^2} - \frac{2}{x^3} &= 0 \\ -x - 2 &= 0 \\ x &= -2. \end{aligned}$$

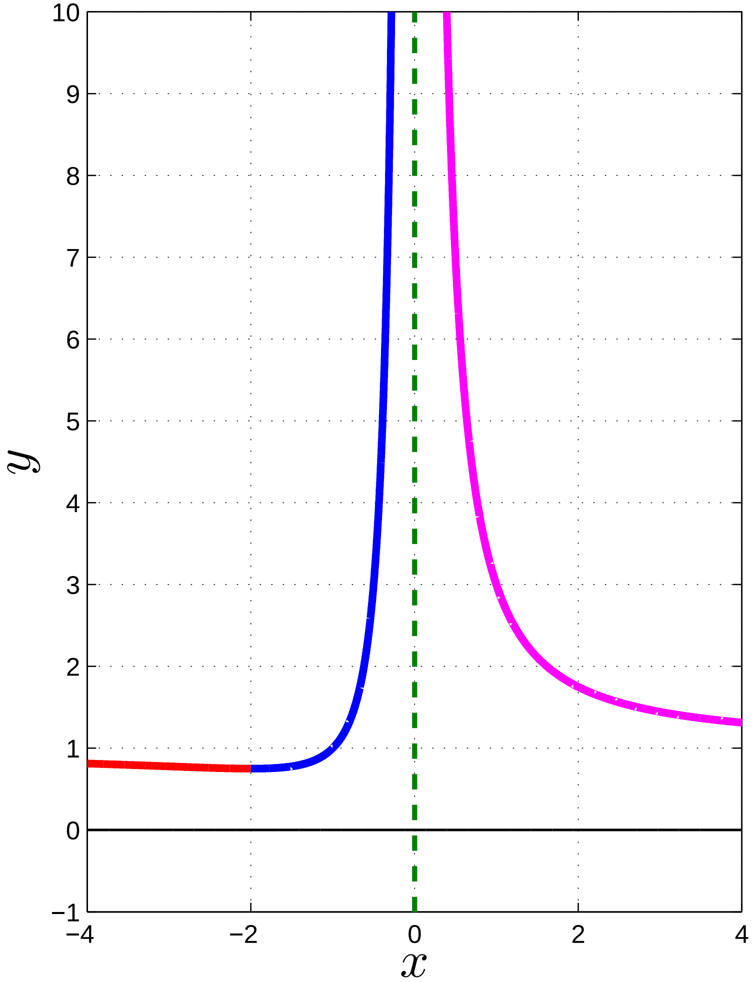
   Thus, the critical point is $x = -2$; -2 is also a partition number for f'.

The sign chart for *f ' (x)* is



Test Numbers	
x	*f ' (x)*
-3	- 1/27 (-)
-1	1 (+)
1	-3 (-)

Therefore, f is increasing on $(-2, 0)$ and f is decreasing on $(-\infty, -2)$ and on $(0, +\infty)$; f has a local minimum at $x = -2$.



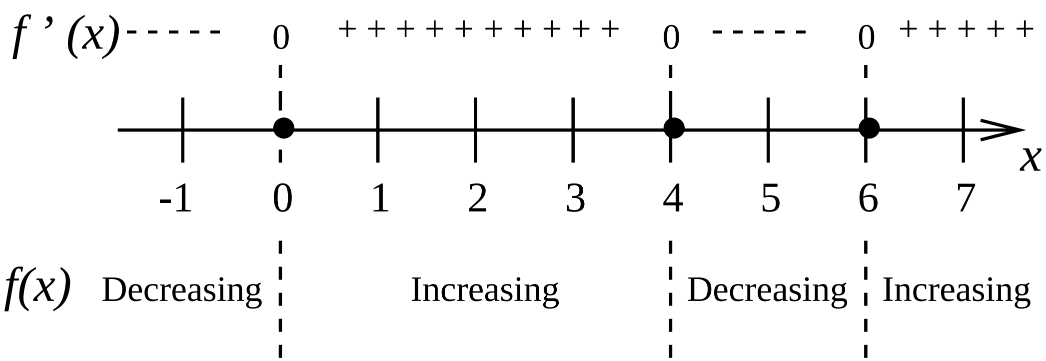
2. $f(x) = x^4(x-6)^2$

$$\begin{aligned} f'(x) &= x^4(2)(x-6)(1) + (x-6)^2(4x^3) \\ &= 2x^3(x-6)[x+2(x-6)] \\ &= 2x^3(x-6)(3x-12) \\ &= 6x^3(x-4)(x-6). \end{aligned}$$

Thus, the critical points of f are $x = 0$, $x = 4$, and $x = 6$.

Now we construct the sign chart for f' ($x = 0$, $x = 4$, $x = 6$ are partition numbers).

The sign chart for $f'(x)$ is



Test Numbers	
x	$f'(x)$
-1	-210 (-)
1	90 (+)
5	-750 (-)
7	+

Therefore, f is increasing on $(0, 4)$ and on $(6, \infty)$, f is decreasing on $(-\infty, 0)$ and on $(4, 6)$; f has a local maximum at $x = 4$ and local minima at $x = 0$ and $x = 6$.

SECOND DERIVATIVE AND GRAPHS

- Concavity
- Inflection Points
- Second Derivative Test
- Analyzing Graphs

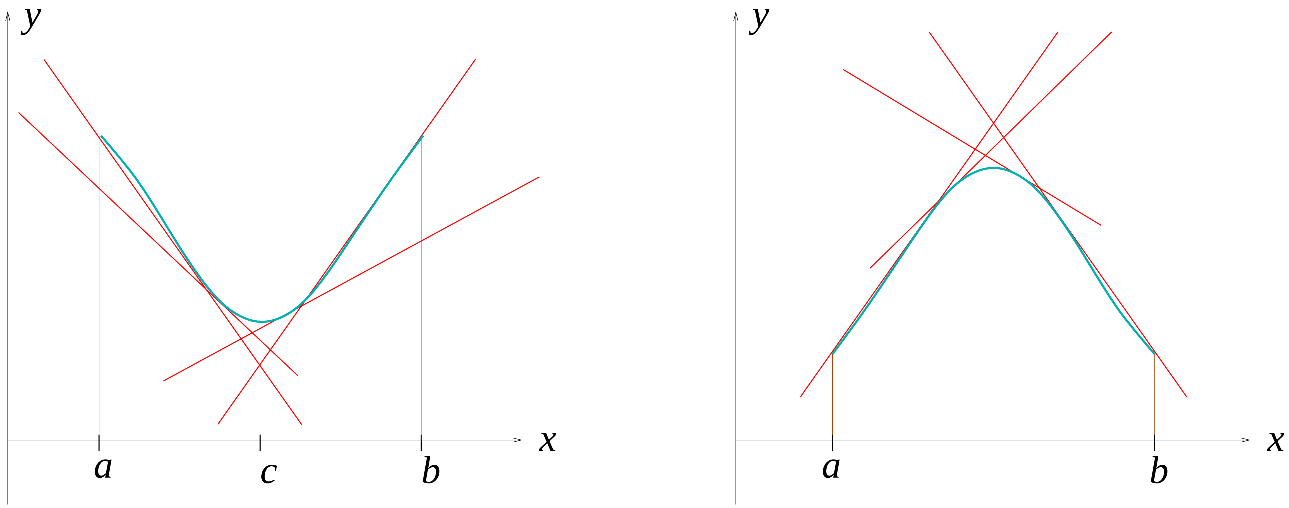
We saw that the derivative can be used to determine when a graph is rising and falling.

Now we want to see what the second derivative (the derivative of the derivative) can tell us about the shape of a graph.

Definition 6 **Concavity of a differentiable function f**

Let the function f be differentiable on an interval (a, b). Then,

1. f is **concave upward** on (a, b) if f' is increasing $\nearrow$ on (a, b).
2. f is **concave downward** on (a, b) if f' is decreasing $\searrow$ on (a, b).



(a) f is concave upward on (a, b). Increasing slopes.

(b) f is concave downward on (a, b). Decreasing slopes.

Figure 18:

How can we determine when $f'(x)$ is increasing $\nearrow$ or decreasing $\searrow$?

As we mentioned earlier, we used the derivative of a function to determine when the function is increasing $\nearrow$ or decreasing $\searrow$.

Thus, to determine when the function $f'(x)$ is increasing and decreasing, we use the derivative of $f'(x)$.

- The derivative of the derivative of a function is called *the second derivative of the function.*

Second Derivative

For $y = f(x)$, the Second Derivative of f, provided it exists, is:

$$f''(x) = \frac{d}{dx} f'(x)$$

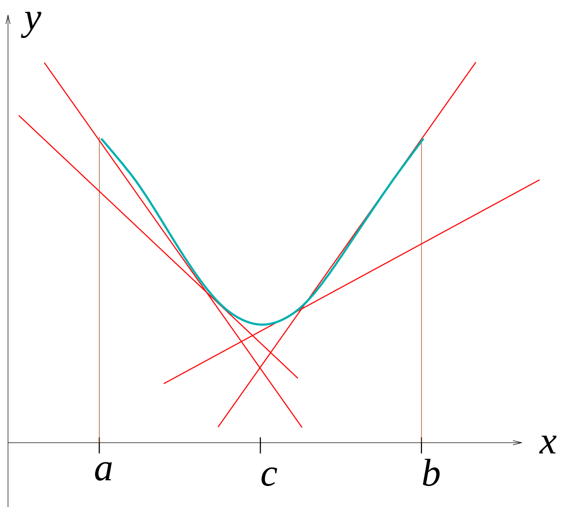
Other notations for $f''(x)$ are:

$$\frac{d^2y}{dx^2} \quad \text{and} \quad y''.$$

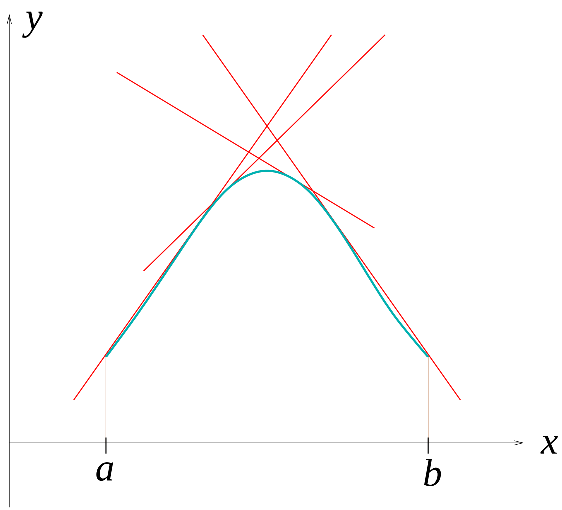
Theorem 4 Let f be twice differentiable on an open interval (a, b).

(a) If $f''(x) > 0$ for each value of x in an interval (a, b), then f is concave upward on (a, b).

(b) If $f''(x) < 0$ for each value of x in an interval (a, b), then f is concave downward on (a, b).



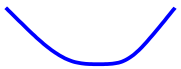
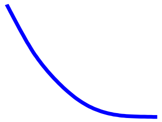
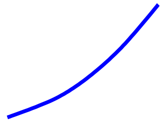
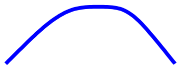
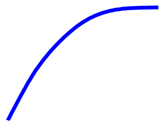
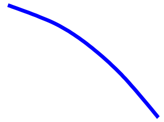
(a) f is concave upward on (a, b).



(b) f is concave downward on (a, b).

Concavity

Find the interval (a, b)

$f''(x)$	$f'(x)$	Graph of f	Examples		
+	Increasing	Concave upward			
-	Decreasing	Concave downward			

- Be careful not to confuse concavity with falling and rising.
- As Figures 19 and 20 illustrate, a graph that is concave upward on an interval may be falling, rising, or both falling and rising on that interval.

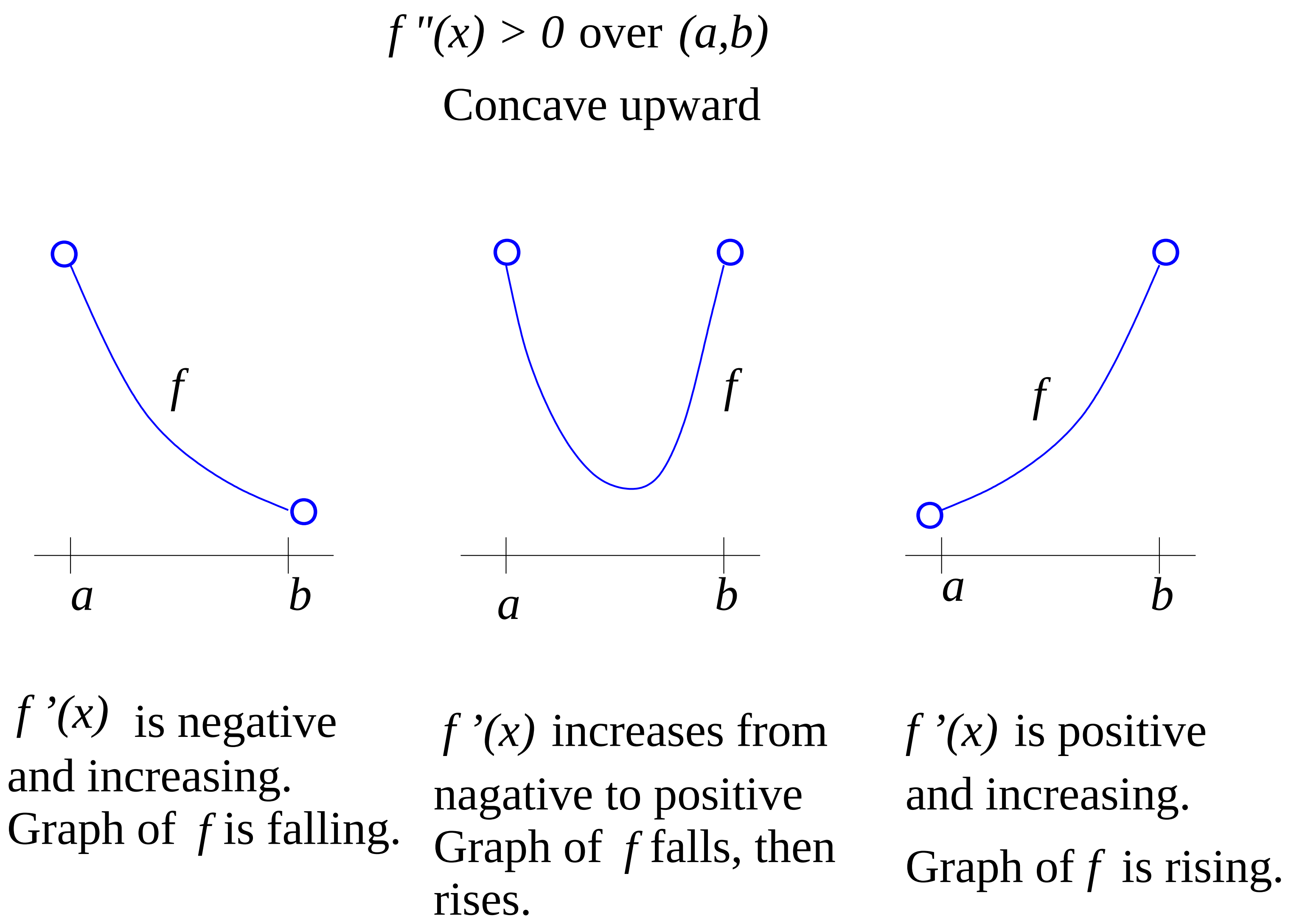
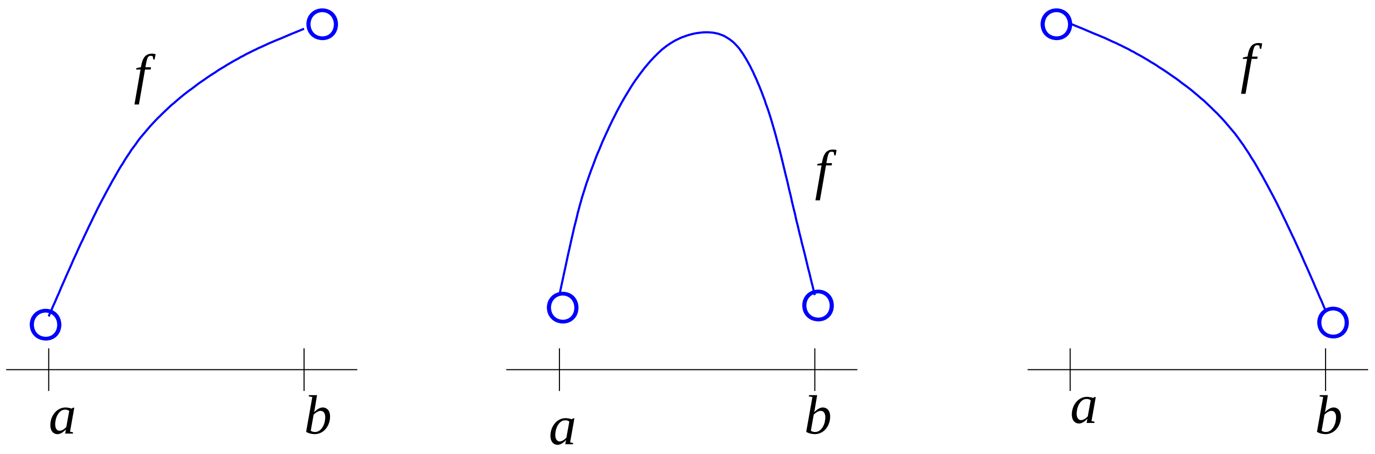


Figure 19: Concave upward

$f''(x) < 0$ over (a,b)

Concave downward



$f'(x)$ is positive and decreasing.
Graph of f is rising.

$f'(x)$ decreases from positive to negative
Graph of f rises, the falls.

$f'(x)$ is negative and decreasing.
Graph of f is falling.

Figure 20: Concave downward

Determining the intervals of concavity of f

1. Determine the values of x for which f'' is not defined, and identify the open intervals determined by these points.

2.
   - Determine the sign of f'' in each interval found in Step 1.
   - To do this, compute $f''(c)$, where c is any conveniently chosen critical point in the interval.

   (a) If $f''(c) > 0$, f is concave upward on that interval.
   (b) If $f''(c) < 0$, f is concave downward on that interval.

Example 7 Test for Concavity

Determine the intervals on which the graph of $f(x) = x^3 + \frac{9}{2}x^2$ is concave up and the intervals on which the graph is concave down.

Solution:

From $f''(x) = 3x^2 + 9x$ we obtain

$$f''(x) = 6x + 9 = 6\left(x + \frac{3}{2}\right).$$

We see that $f''(x) < 0$ when $6(x + \frac{3}{2})$ or $x < -\frac{3}{2}$ and that $f''(x) > 0$ when $6(x + \frac{3}{2}) > 0$ or $x > -\frac{3}{2}$. It follows from Theorem 4 that the graph of f is concave downward on the interval $(-\infty, -\frac{3}{2})$ and concave upward on $(-\frac{3}{2}, \infty)$. See Figure 21.

		CP $-\frac{3}{2}$	
Interval	$(-\infty, -\frac{3}{2})$		$(-\frac{3}{2}, \infty)$
Test Value	-2		1
f''	$f''(-2) < 0$		$f''(0) > 0$
	$-$		$+$
f	Concave downward		Concave upward
	$\frown$		$\smile$

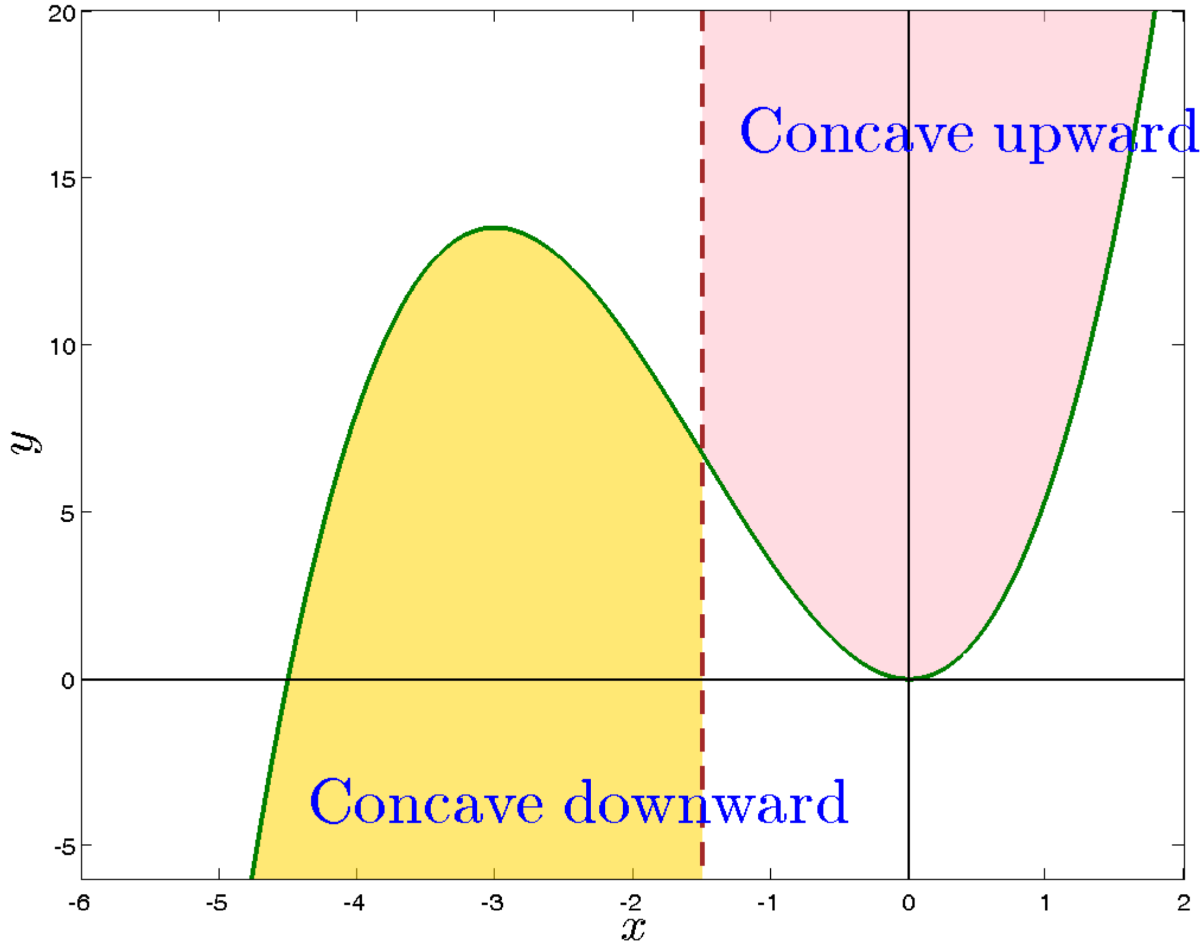


Figure 21: $f''(-\frac{3}{2}) = 0$ and $(-\frac{3}{2}, -\frac{27}{4})$ is a point of inflection.

What is an Inflection Point?

A point on the graph of a differentiable function f at which the concavity changes is called an inflection point.

Theorem 5 If $y = f(x)$ is continuous on (a, b) and has an inflection point at $x = c$, then either $f^{''}(x) = 0$ or $f^{''}(c)$ does not exist.

Remark 7 Theorem 5 states that if f is continuous on an open interval containing a value c, and if f changes the direction of its concavity at the point $(c, f(c))$, then we say that f has an inflection point at $x = c$, and we call the point $(c, f(c))$on the graph of f an inflection point of f.

Remark 8 A partition number c for $f^{''}$ produces an inflection point for the graph of f only if

1. $f^{''}(x)$ changes sign at c, and
2. c is in the domain of f.

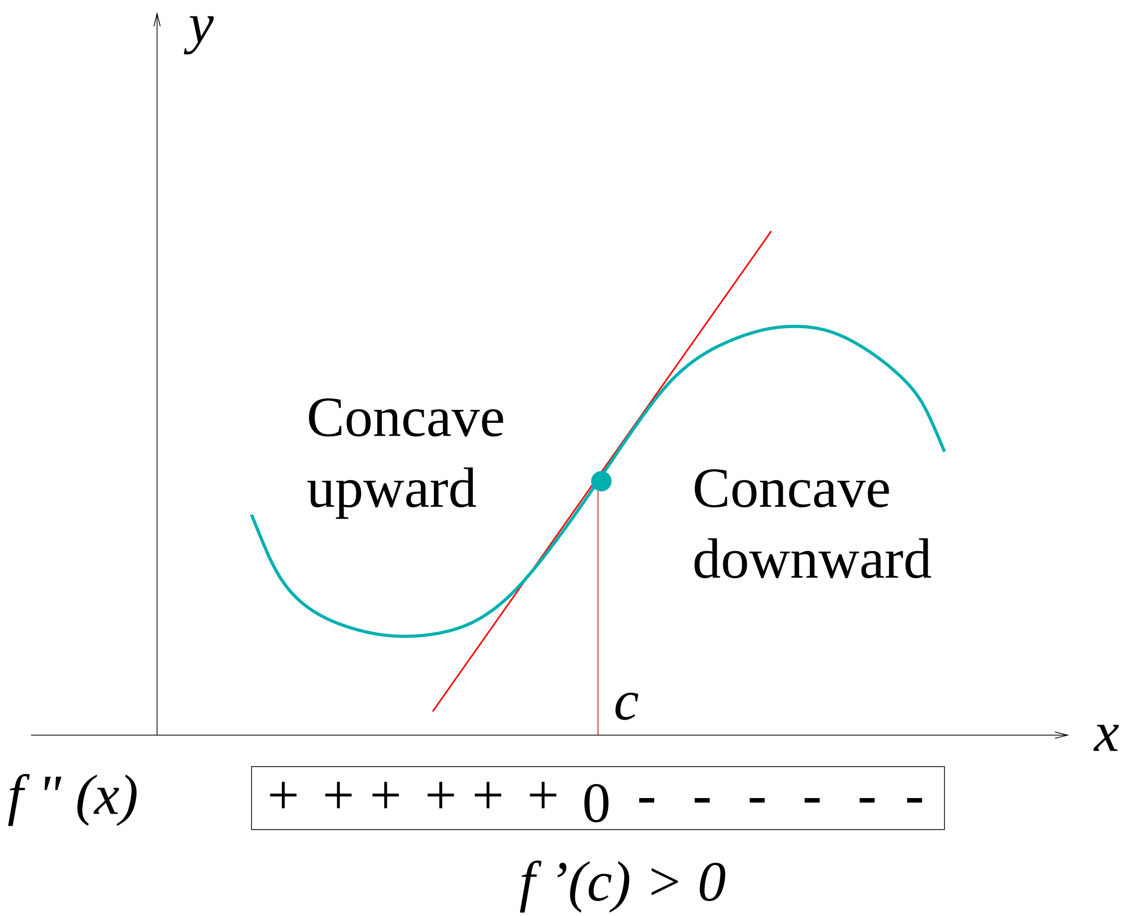


Figure 22: At each point of inflection, the graph of a function crosses its tangent line.

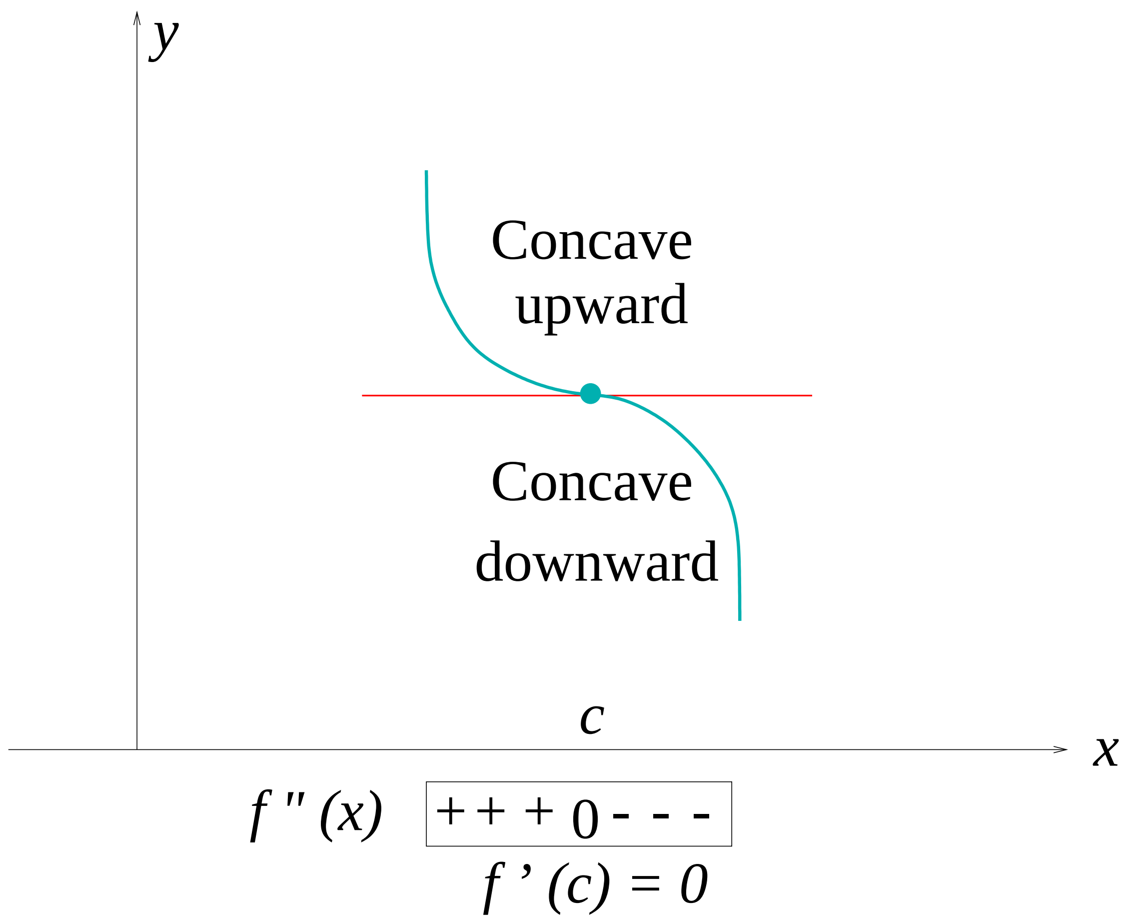


Figure 23: At each point of inflection, the graph of a function crosses its tangent line.

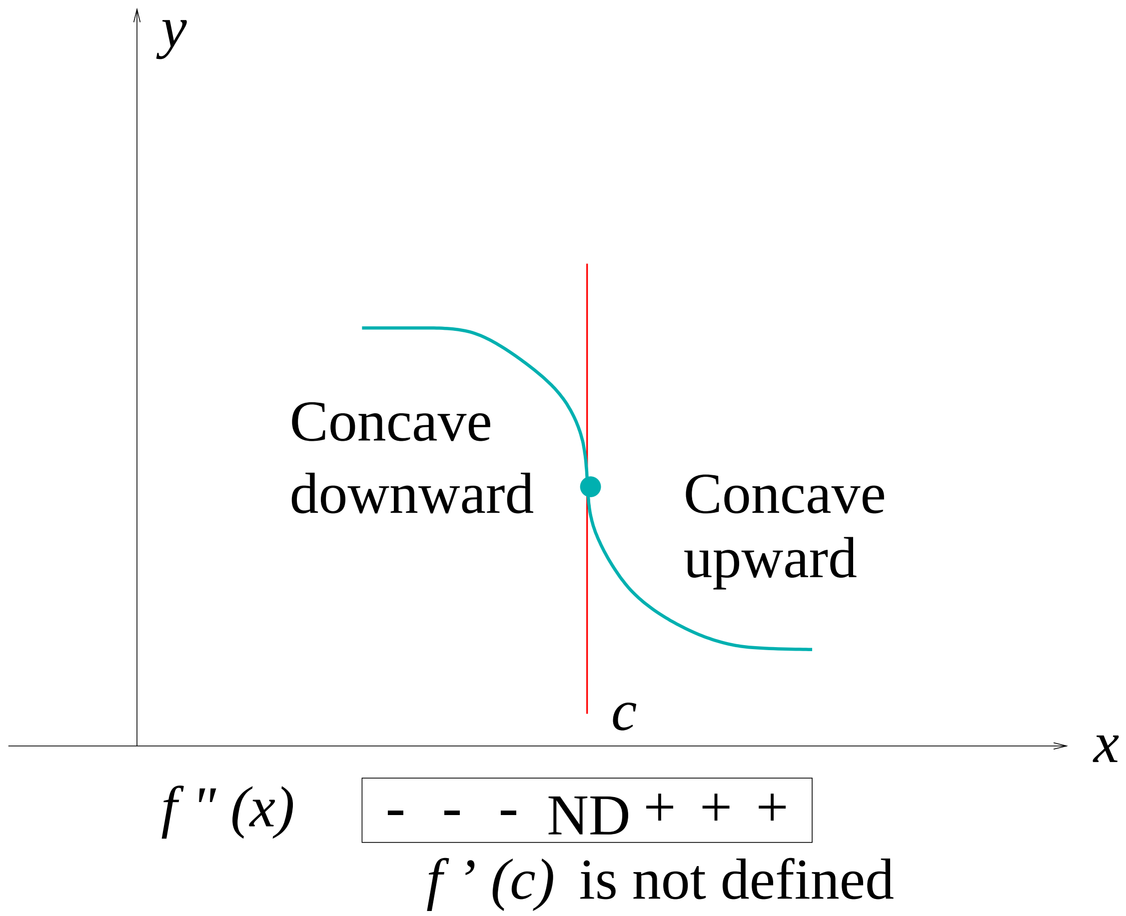


Figure 24: At each point of inflection, the graph of a function crosses its tangent line.

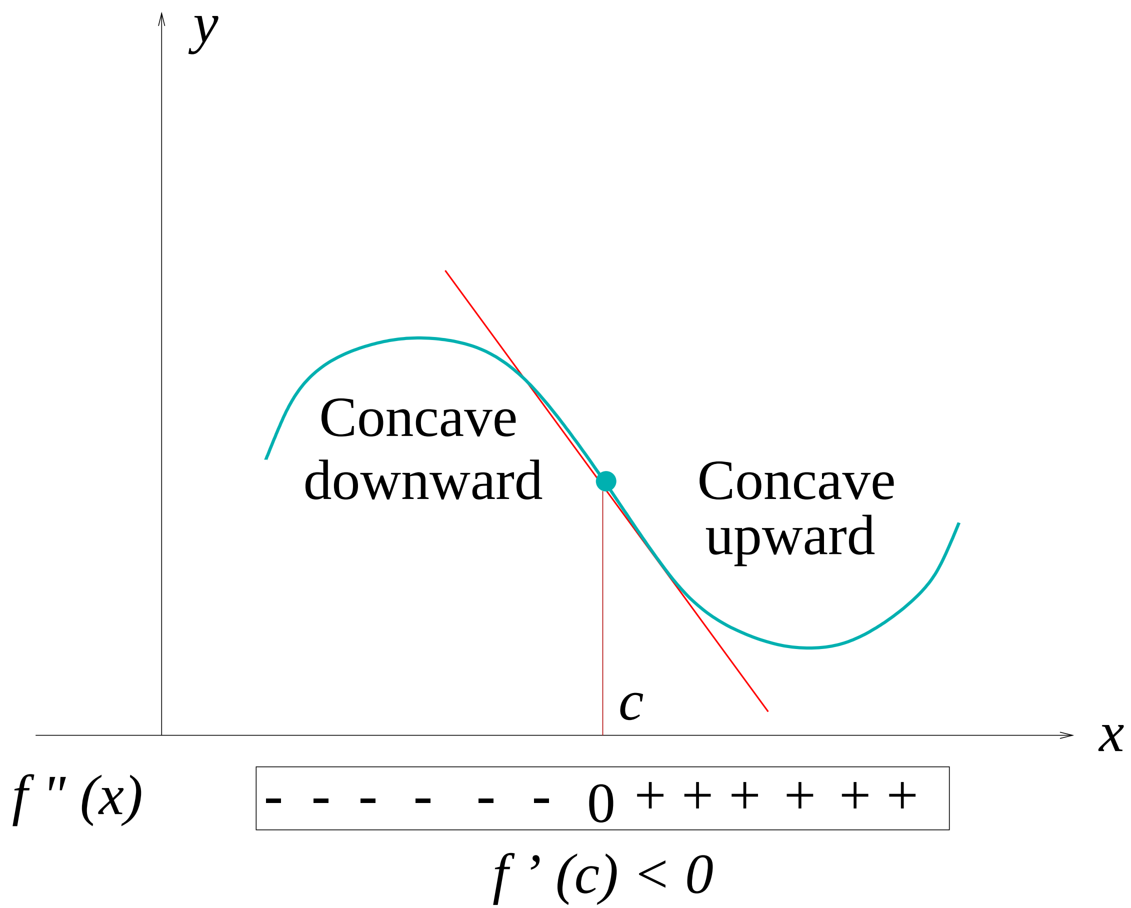


Figure 25: At each point of inflection, the graph of a function crosses its tangent line.

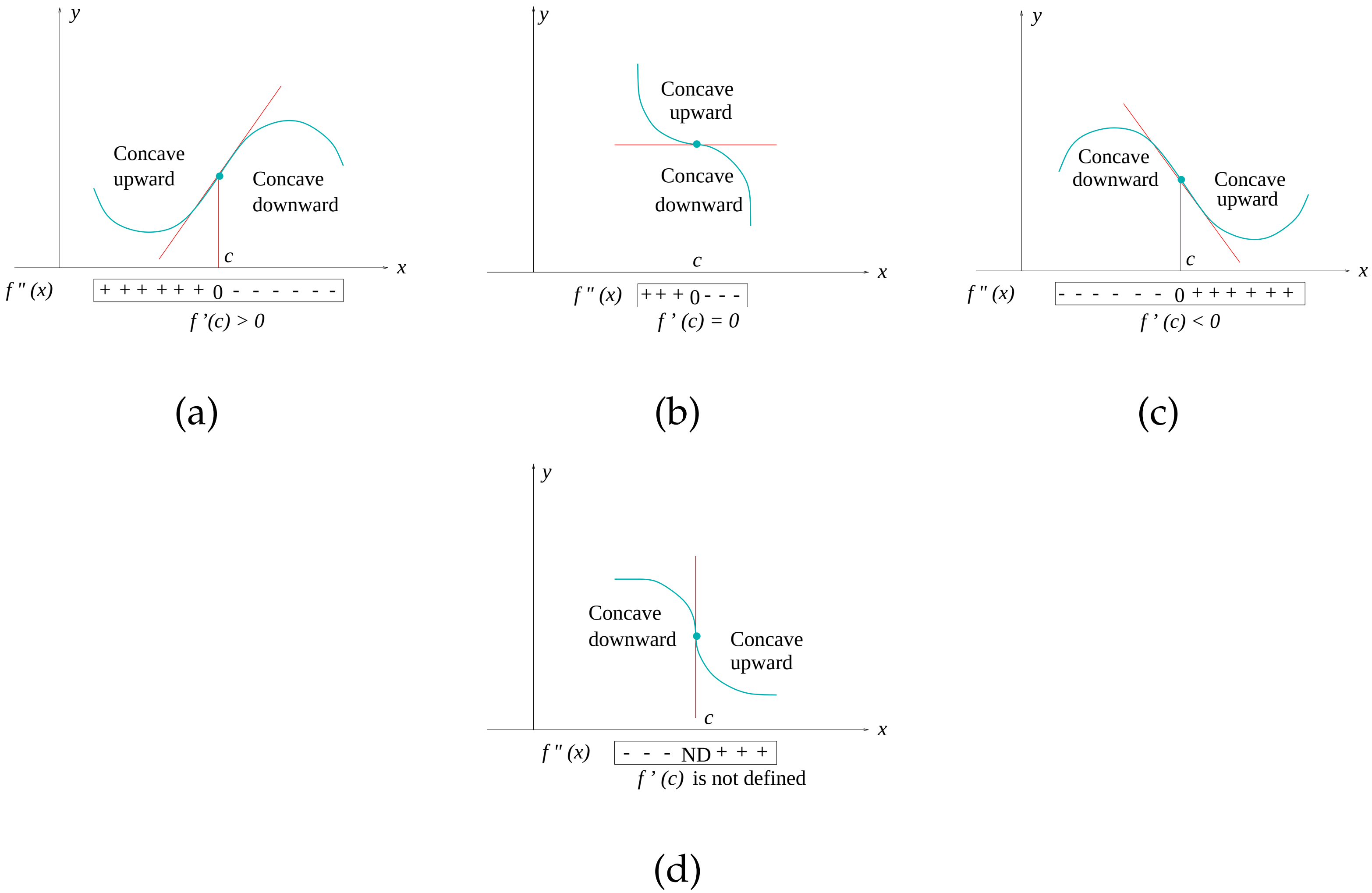


Figure 26: At each point of inflection, the graph of a function crosses its tangent line.

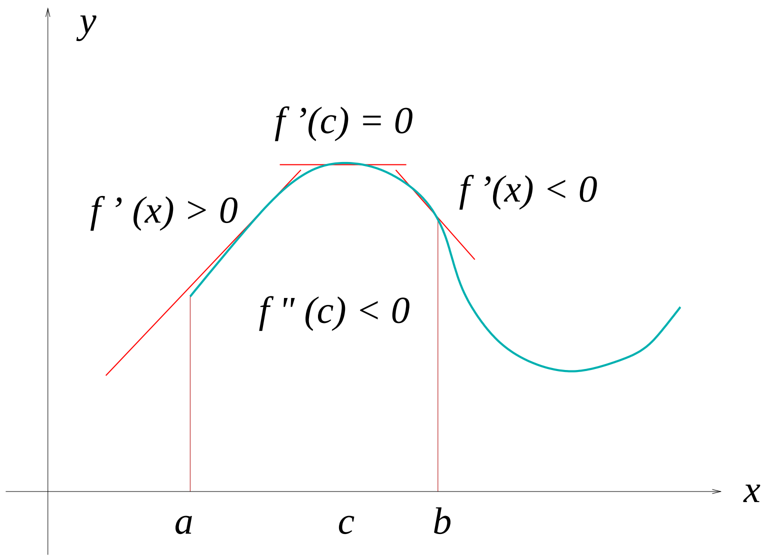
Finding inflection points

1. Compute $f^{''}(x)$.
2. Determine the points in the domain of f for which $f^{''}(x) = 0$ or $f^{''}(x)$ does not exist.
3. - Determine the sign of $f^{''}(x)$ to the left and right of each point $x = c$ found in Step 2.
   - If there is a change in the sign of $f^{''}(x)$ as we move across the point $x = c$, then $(c, f(c))$ is an inflection point of f.

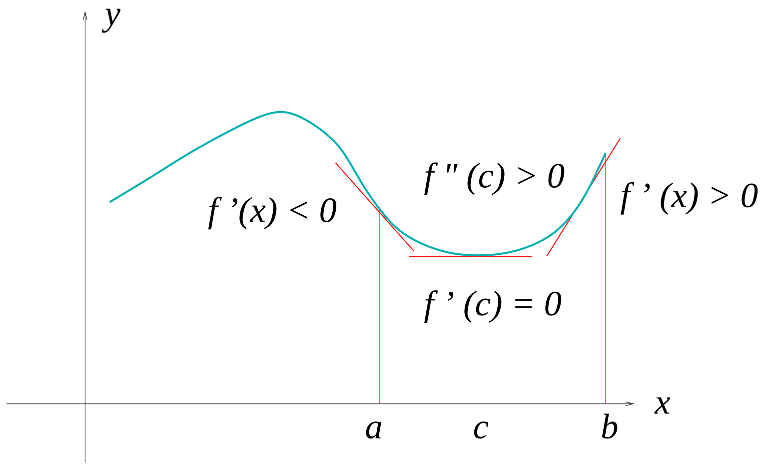
Theorem 6 Let f be a function for which $f^{''}$ exists on an interval (a, b) that contains the critical number c.

1. If $f^{''}(c) > 0$, then $f(c)$ is a relative (or local) minimum.
2. If $f^{''}(c) < 0$, then $f(c)$ is a relative (or local) maximum.
3. If $f^{''}(c) = 0$, the test fails and $f(c)$ may or may not be a relative extremum. In this case, use The First Derivative Test.

The Second Derivative Test



(a) $f'(c) = 0$ and $f^{''}(c) < 0$ implies $f(c)$ has a local at $x = c$.

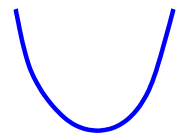
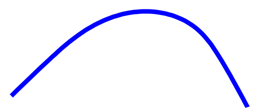


(b) $f'(c) = 0$ and $f^{''}(c) > 0$ implies $f(c)$ has a local minimum at $x = c$.

Figure 27: How the second derivative can be used to find local extrema.

Second-Derivative Test for Local Maxima and Minima

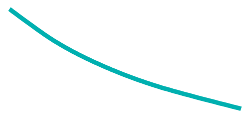
Let c be a critical value for $f(x)$

$f'(c)$	$f''(c)$	Graph of f is:	$f(c)$	Example
0	+	Concave upward	Local minimum	
0	-	Concave downward	Local maximum	
0	0	?	Test fails	

The first-derivative test must be used whenever $f''(c) = 0$ or $f''(c)$ does not exist

The Second Derivative Test

1. Compute $f^{'}(x)$ and $f^{''}(x)$.
2. Find all the critical points of f at which $f^{'}(x) = 0$.
3. Compute $f^{''}(c)$ for each such critical point c.
   (a) If $f^{''}(c) < 0$, then f has a local (or relative) maximum at c.
   (b) If $f^{''}(c) > 0$, then f has a local (or relative) minimum at c.
   (c) If $f^{''}(c) = 0$, the test fails; that is, it is inconclusive.

Signs of f' and f''	Properties of the Graph of f	General Shape of the Graph of f
$f'(x) > 0$ $f''(x) > 0$	f increasing f concave upward	
$f'(x) > 0$ $f''(x) < 0$	f increasing f concave downward	
$f'(x) < 0$ $f''(x) > 0$	f decreasing f concave upward	
$f'(x) < 0$ $f''(x) < 0$	f decreasing f concave downward	

Example 8 **Testing for local extrema**. Find the local maxima and minima for each function. Use the second-derivative test when it applies.

1. $f(x) = x^3 - 6x^2 + 9x + 1$
2. $f(x) = \frac{1}{6}x^6 - 4x^5 + 25x^4$

Solution:

1. $f(x) = x^3 - 6x^2 + 9x + 1.$

   Take first and second derivatives and find critical points

$$f(x) = x^3 - 6x^2 + 9x + 1$$

$$f'(x) = 3x^2 - 12x + 9 = 3(x-1)(x-3)$$

$$f''(x) = 6x - 12 = 6(x-2)$$

   Critical points are $x = 1$ and $x = 3$.

$$f''(1) = -6 < 0, \quad f \text{ has a local maximum at } x = 1$$

$$f''(3) = 6 > 0, \quad f \text{ has a local minimum at } x = 3$$

2. $f(x) = \frac{1}{6}x^6 - 4x^5 + 25x^4$

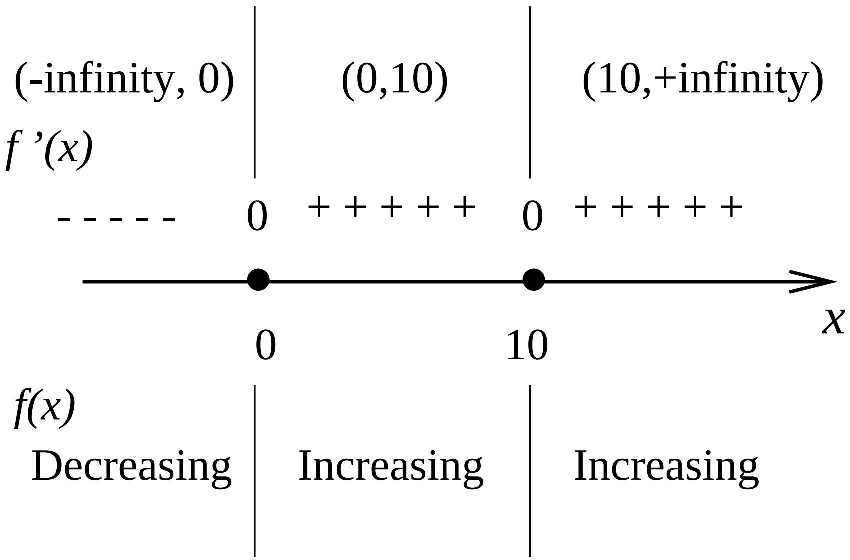
$$f(x) = \frac{1}{6}x^6 - 4x^5 + 25x^4$$
$$f'(x) = x^5 - 20x^4 + 100x^3 = x^3(x-10)^2$$
$$f''(x) = 5x^4 - 80x^3 + 300x^2$$

Critical points are $x = 0$ and $x = 10$.

$$f''(0) = 0$$
$$f''(10) = 0$$

The second-derivative test fails at both critical points, so the first derivative test must be used.

Sign chart for $f'(x) = x^3(x - 10^2)$ (partition numbers are 0 and 10):



Test Numbers	
x	$f'(x)$
- 1	-121(-)
1	81 (+)
11	1,331 (+)

A common error is to assume that $f''(c) = 0$ implies that $f(c)$ is not a local extremum.

As Part 2 illustrates, if $f''(c) = 0$, then $f(c)$ may or may not be a local extremum.

The first-derivative test must be used whenever $f''(c) = 0$ or $f''(c)$ does not exist.

Example 9 Find all local maxima and minima using the second derivative test whenever it applies (do not graph). If the second-derivative test fails, use the first derivative test.

1. $f(x) = 2x^2 - 8x + 6$
2. $f(x) = 2x^3 - 3x^2 - 12x - 5$
3. $f(x) = 3 - x^3 + 3x^2 - 3x$

Solution:

1. $f(x) = 2x^2 - 8x + 6$

$$f(x) = 2x^2 - 8x + 6$$
$$f'(x) = 4x - 8 = 4(x-2)$$
$$f''(x) = 4$$

Critical point: $x = 2$

Now, $f''(2) = 4 > 0$. Therefore, $f(2) = 2(2)^2 - 8(2) + 6 = -2$ is a local minimum.

2. $f(x) = 2x^3 - 3x^2 - 12x - 5$

$$f(x) = 2x^3 - 3x^2 - 12x - 5$$
$$f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2) = 6(x-2)(x+1)$$
$$f''(x) = 12x - 6 = 6(2x - 1)$$

Critical points: $x = 2$ and $x = -1$.

Now,

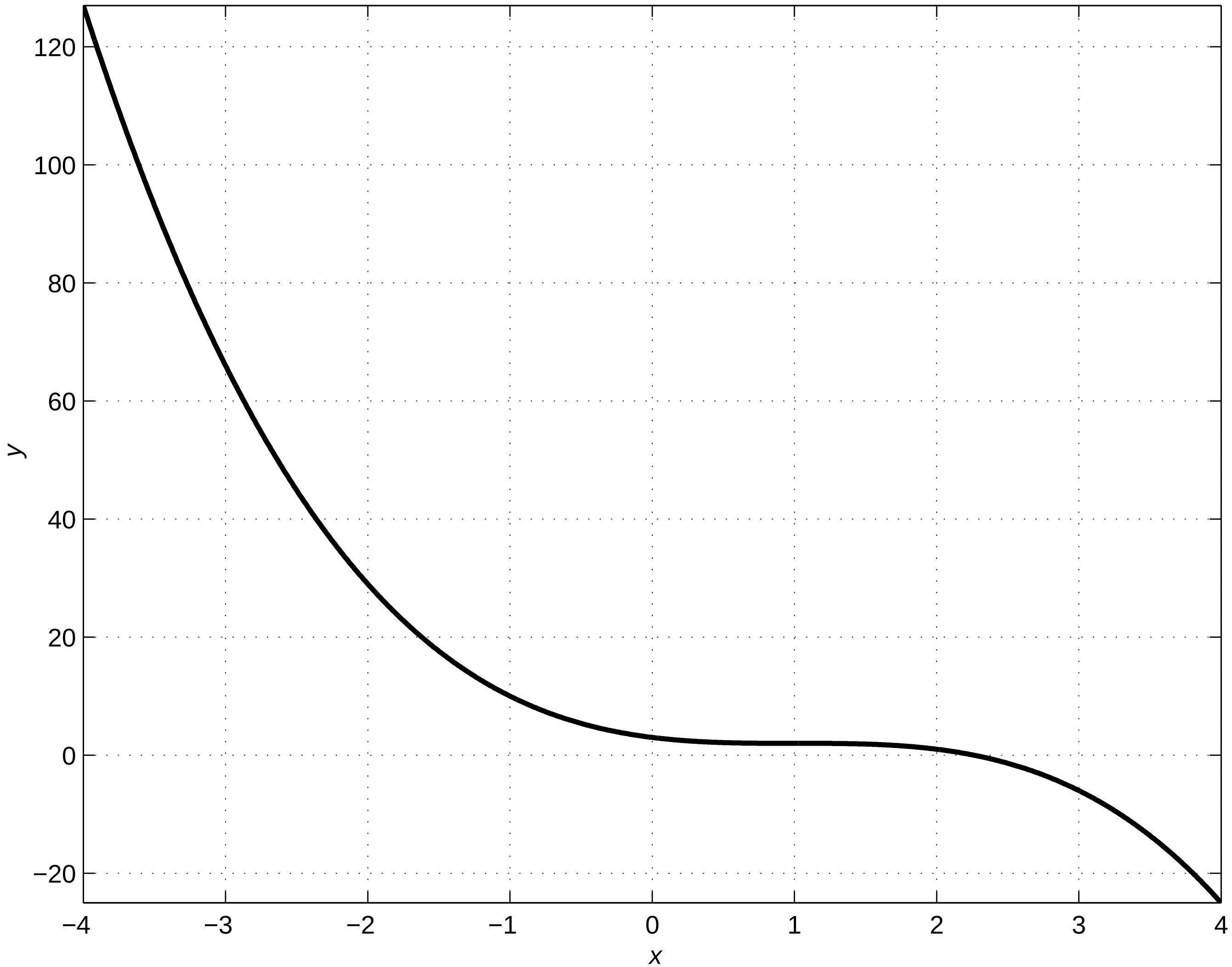
- $f''(2) = 6(2(2) - 1) = 18 > 0$. Therefore, $f(2) = 2(2)^3 - 3(2)^2 - 12(2) - 5 = -25$ is a local minimum.
- $f''(-1) = 6(2(-1) - 1) = 18 < 0$. Therefore, $f(-1) = 2(-1)^3 - 3(-1)^2 - 12(-1) - 5 = 2$ is a local maximum.

3. $f(x) = 3 - x^3 + 3x^2 - 3x$

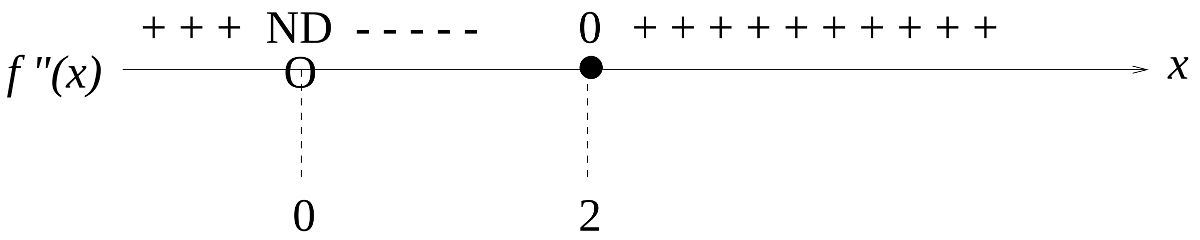
$$
\begin{aligned}
f(x) &= 3 - x^3 + 3x^2 - 3x \\
f'(x) &= -3x^2 + 6x - 3 = -3(x^2 - 2x + 1) = -3(x-1)^2 \\
f''(x) &= -6(x-1)
\end{aligned}
$$

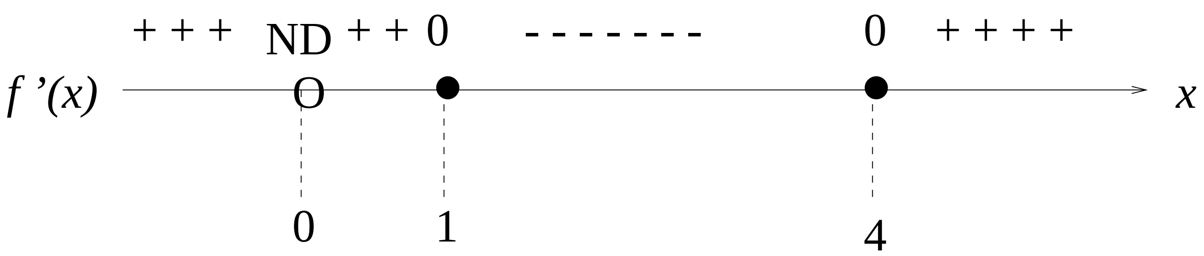
Critical points: $x = 1$

- Now, $f''(1) = -6(1-1) = 0$.
- Thus the second-derivative test fails.
- Since $f'(x) = -3(x-1)^2 < 0$ for all $x \neq 1$, $f(x)$ is decreasing on $(-\infty, +\infty)$.
- Therefore, $f(x)$ has no local extrema.


120
100
80
60
40
20
0
−20
y
−4
−3
−2
−1
0
1
2
3
4
x

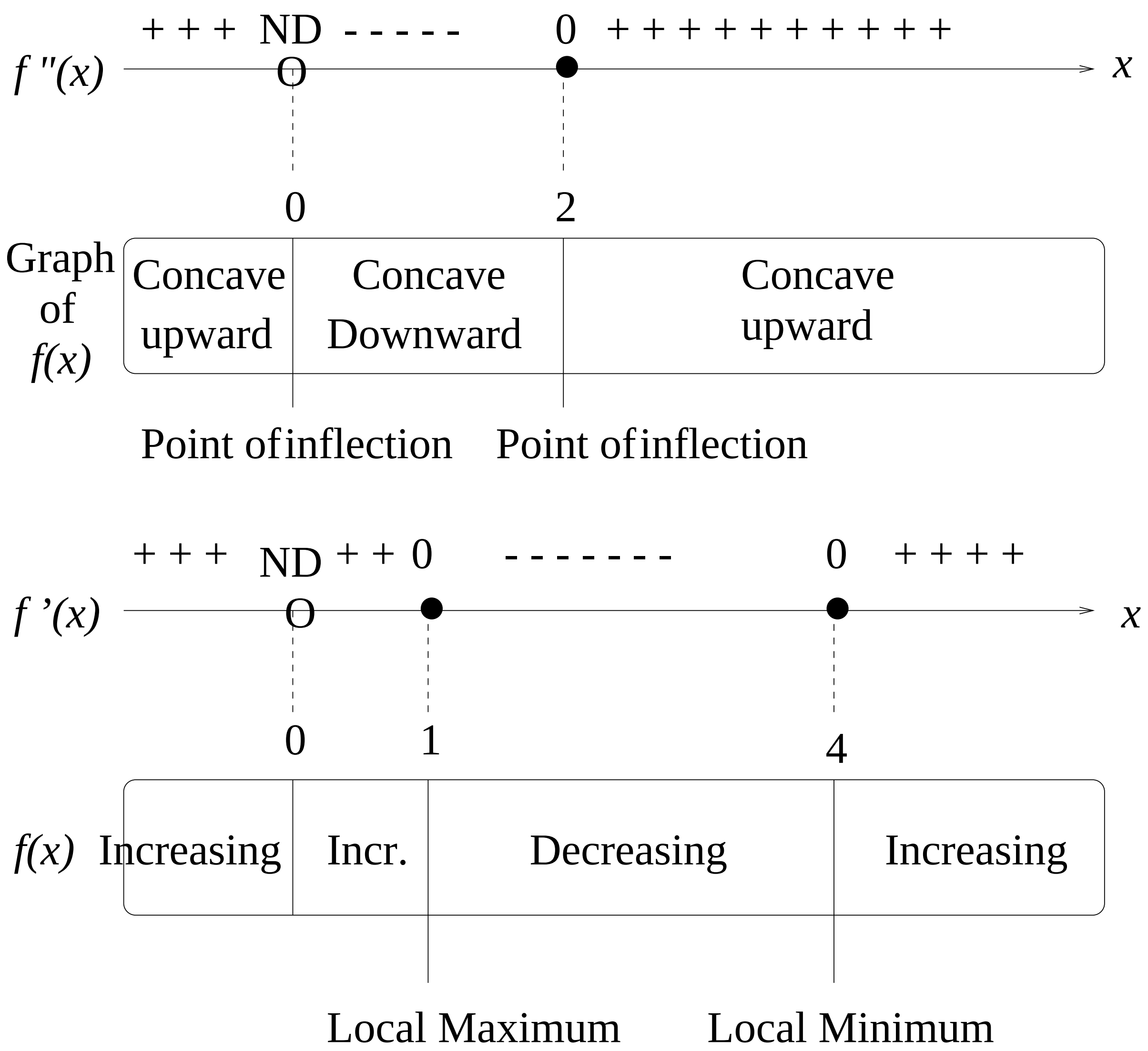
Example 10 $f(x)$ is continuous on $(-\infty, +\infty)$. Use the given information to sketch the graph of f.





x	-3	0	1	2	4	5
$f(x)$	-4	0	2	1	-1	0

Solution:


+ + + ND - - - - - 0 + + + + + + + + + + +
f ''(x)
x
0
2
Graph of f(x)
Concave upward
Concave Downward
Concave upward
Point of inflection
Point of inflection
+ + + ND + + 0 - - - - - - - 0 + + + +
f '(x)
x
0
1
4
f(x)
Increasing
Incr.
Decreasing
Increasing
Local Maximum
Local Minimum

Using this information together with points $(-3,-4)$, $(0,0)$, $(1,2)$, $(2,1)$, $(4,-1)$, $(5,0)$ on the graph, we have

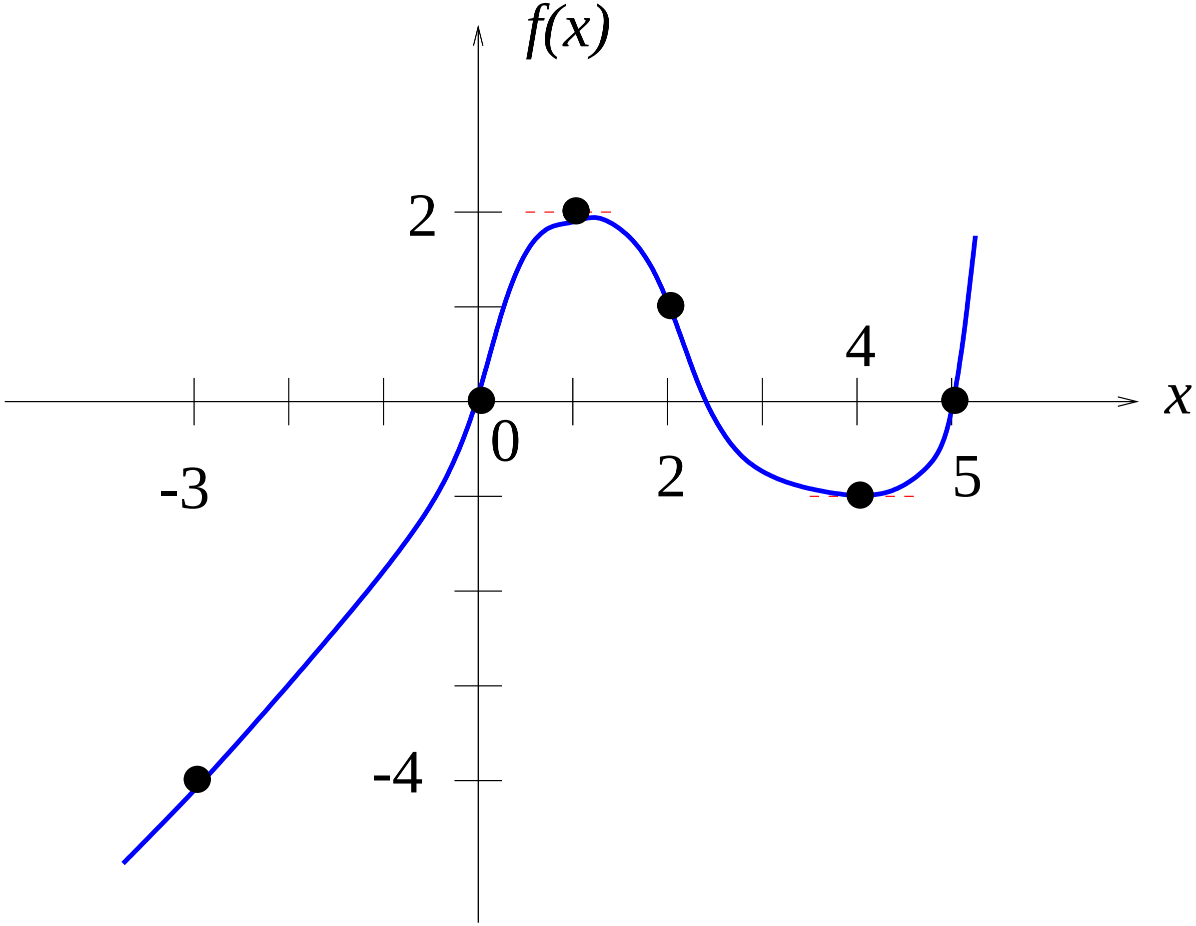


Figure 28: