2021R1-MATH1510 Midterm exam

Cho Kit CHAN

TOTAL POINTS

100 / 150

QUESTION 1

1 Q1-4 **9 / 12**

Q1

✓ **- 0 pts** Correct

Q2

✓ **- 0 pts** Correct

Q3

✓ **- 0 pts** Correct

Q4

✓ **- 3 pts** Incorrect "NONE"

QUESTION 2

2 Q5-9 **15 / 15**

Q5

✓ **- 0 pts** Correct

Q6

✓ **- 0 pts** Correct

Q7

✓ **- 0 pts** Correct

Q8

✓ **- 0 pts** Correct

Q9

✓ **- 0 pts** Correct

QUESTION 3

3 Q10-15 **15 / 18**

Q10

✓ **- 0 pts** Correct

Q11

✓ **- 0 pts** Correct

Q12

✓ **- 0 pts** Correct

Q13

✓ **- 0 pts** Correct

Q14

✓ **- 0 pts** Correct

Q15

✓ **- 3 pts** Incorrect

QUESTION 4

4 Q16-18 **6 / 9**

Q16

✓ **- 0 pts** Correct

Q17

✓ **- 3 pts** Incorrect

Q18

✓ **- 0 pts** Correct

QUESTION 5

5 Q19 **6 / 12**

Value of b

✓ **- 0 pts** Correct

Value of a

✓ **- 6 pts** Mistake: Take limit of f'.

QUESTION 6

6 Q20 **14 / 14**

Q20(a)

✓ **- 0 pts** Correct

Q20(b)

✓ **- 0 pts** Correct

QUESTION 7

7 Q21 **10 / 10**

✓ - 0 pts Correct

QUESTION 8

8 Q22 12 / 16

Part (a)

✓ - 0 pts Correct

Part (b)

✓ - 2 pts **Incorrect formula regarding $$f'(t)$$**

✓ - 2 pts **Incorrect final answer**

QUESTION 9

9 Q23 4 / 14

✓ - 2 pts **incorrect explanation for domain of f'(x)**

✓ - 8 pts **incorrect argument for f'(0)**

QUESTION 10

10 Q24 7 / 20

Part (a)

✓ - 1 pts **Should state that you are applying MVT on the interval $$[k,k+1]$$ explicitly**

Part (b)

✓ - 6 pts **Incorrect**

Part (c)

✓ - 6 pts **Incorrect**

QUESTION 11

11 Q25 2 / 10

+ **10 pts** Correct
+ **2 pts** Essentially expresses lim sin(x^3)f(ax) interms of f(x)
+ **2 pts** Essentially expresses lim sin^3(x)g(bx) interms of g(x)
- **2 pts** Misc. mistakes in re-expressing sin(x^3)f(ax) and sin^3(x)g(bx)
+ **4 pts** Correctly expresses lim f(x)/g(x) in way which takes advantage of known identities.

✓ + 2 pts **Expresses lim f(x)/g(x) in way which takes advantage of known identities, with at least one significant mistake or omission in the reasoning.**

+ **2 pts** Correct final answer.
+ **0 pts** Assumes without justification the limit of individual term(s) exists.
+ **0 pts** Incorrect.
+ **0 pts** No answer found on the first selected page

$x > \frac{7}{2}$ $x > -\frac{5}{3}$

$2x-7>0$ & $3x+5>0$

or $2x-7<0$ & $3x+5<0$

$x < \frac{7}{2}$ $x < -\frac{5}{3}$

Short Questions

Each of question 1-18 is worth 3 points.

1. Find the domain of the function

$$f(x) = \ln((2x-7)(3x+5))$$

Answer: $(-\infty, -\frac{5}{3}) \cup (\frac{7}{2}, \infty)$

2. Find the range of the function

$$f(x) = x^2 - 1$$

with domain $[-2, 6]$. $= (x-1)(x+1)$

Answer: $[-1, 35]$

3. Which of the following functions have minimum value on the specified intervals? If none of them has minimum value, write NONE.

(a) $f(x) = \frac{1}{x}$ on $[1, \infty)$. X

(b) $g(x) = (x-3)e^x \sin x$ on $[-3, 7]$. ✓

(c) $h(x) = x$ on $(0, 4]$. X

Answer: (b),

4. Let

$$f(x) = \begin{cases} |x| & \text{if } |x| \geq 1; \\ 2x-3 & \text{if } 0 \leq x < 1; \\ x^2 & \text{if } -1 < x < 0. \end{cases}$$

Write down all the point(s) on $\mathbb{R}$ where $f(x)$ is not continuous. If there is no such point, write NONE.

Answer: NONE

$\cancel{\lim_{x\to 1^-}} = 2-3 = -1$ $\cancel{\lim_{x\to 1^+}} = 1$ $\lim_{x\to 0^-} = 0$ $\lim_{x\to 0^+} 0-3 = -3$

1 Q1-4 **9 / 12**

Q1

✓ **- 0 pts** Correct

Q2

✓ **- 0 pts** Correct

Q3

✓ **- 0 pts** Correct

Q4

✓ **- 3 pts** Incorrect "NONE"

$\lim_{x\to 1^-} = 1-a$ $\lim_{x\to 1^+} = a+2$

5. Let

$$f(x) = \begin{cases} ax+2 & \text{if } x \geq 1; \\ x-a & \text{if } x < 1. \end{cases}$$

Find all the value(s) of a so that $f(x)$ is continuous on $\mathbb{R}$.

$1-a = a+2$

$-1 = 2a$

Answer: $a = -\frac{1}{2}$

6. Find $\frac{dy}{dx}$ if

$$y = \frac{x \sin x}{2 - \cos x}$$

$\frac{(\sin x + x\cos x)(2-\cos x) - x\sin^2 x}{(2-\cos x)^2}$

Answer: $\frac{2\sin x - \sin x\cos x + 2x\cos x - x}{(2-\cos x)^2}$

7. Find $\frac{dy}{dx}$ if

$$y = \sin(\cos(\sin x))$$

$\frac{dy}{dx} = \cos(\cos(\sin x)) \times -\sin(\sin x) \times \cos x$

Answer: $-(\cos x)(\sin(\sin x))[\cos(\cos(\sin x))]$

8. Find $\frac{dy}{dx}$ if

$$y = \arctan x + e^x \arcsin x, \quad \text{where } x \in (-1, 1)$$

Answer: $\frac{1}{1+x^2} + e^x\left(\arcsin x + \frac{1}{\sqrt{1-x^2}}\right)$

9. Find $\frac{dy}{dx}$ if

$$y = e^{\sin x} \ln x$$

Answer: $\frac{e^{\sin x}}{x} + \ln x \cdot (\cos x) e^{\sin x}$

$\sin^{-1} x = y$

$x = \sin y$

$1 = \cos y \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

2 Q5-9 **15** / **15**

Q5

✓ **- 0 pts** Correct

Q6

✓ **- 0 pts** Correct

Q7

✓ **- 0 pts** Correct

Q8

✓ **- 0 pts** Correct

Q9

✓ **- 0 pts** Correct

$f'(x) = -\sin x$
$f''(x) = -\cos x$
$f'''(x) = \sin x$ $\quad f^{(4)}(x) = \cos x$

10. Let $f(x) = \cos x$. Find $f^{(1510)}\left(\frac{\pi}{6}\right)$. $\rightarrow -\cos\left(\frac{\pi}{6}\right)$

Answer: $-\frac{\sqrt{3}}{2}$

11. Let $f(x) = (x^2 + x + 1)e^{ax}$ for some $a > 0$. Find $f^{(8)}(x)$.

Answer: $(x^2+x+1)a^8e^{ax} + (16x+8)a^7e^{ax} + 56a^6a^{ax}$

12. Evaluate

$$\lim_{t\to\infty} \frac{\log_3(t+1)}{\ln(3t)}$$

$$\frac{\lim_{t\to\infty} \frac{1}{(t+1)\ln 3}}{\lim_{t\to\infty}\left(\frac{1}{t}\right)} = \frac{t}{(t+1)\ln 3} = \frac{1}{\ln 3\,(1)}$$

Answer: $\frac{1}{\ln 3}$

13. Evaluate

$$\lim_{x\to 1} \left(\tan\left(\frac{\pi x}{2}\right)\ln x\right)$$

$$\frac{\ln x}{\cot\left(\frac{\pi x}{2}\right)} = \frac{\frac{1}{x}}{-\csc^2\left(\frac{\pi x}{2}\right)\left(\frac{\pi}{2}\right)} = \frac{2\sin^2\left(\frac{\pi x}{2}\right)}{-\pi x} = \frac{2}{\pi}$$

Answer: $-\frac{2}{\pi}$

14. Evaluate

$$\lim_{x\to\infty} \frac{ae^x + e^{-ax}}{be^x + e^{-bx}},$$

where $a, b > 0$.

$$\frac{a + e^{-2ax}}{b + e^{-2bx}}$$

Answer: $\frac{a}{b}$ //

15. Evaluate

$$\lim_{x\to\infty} \frac{(ax)^2 + x\sin(\pi x)}{bx^2 + c},$$

where $a, b, c \neq 0$.

$$\frac{a^2x^2 + x\sin(\pi x)}{bx^2 + c}$$

Answer:

$$= \frac{a^2 + \pi\cos(\pi x) + \frac{x}{2}\pi^2\sin(\pi x)}{b} = \frac{2a^2x + \sin(\pi x) + x\pi\cos(\pi x)}{2bx}$$

3 Q10-15 15 / 18

Q10

✓ - 0 pts Correct

Q11

✓ - 0 pts Correct

Q12

✓ - 0 pts Correct

Q13

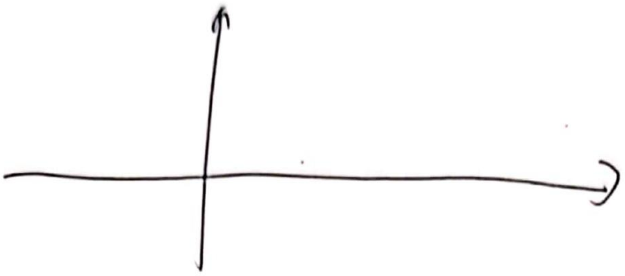
✓ - 0 pts Correct

Q14

✓ - 0 pts Correct

Q15

✓ - 3 pts Incorrect



16. Given that

$$\lim_{x\to\infty} f(x) = a, \quad \lim_{x\to-\infty} f(x) = b, \quad \lim_{x\to 0^+} f(x) = c, \quad \lim_{x\to 0^-} f(x) = d$$

for some $a, b, c, d > 0$, evaluate

$$\lim_{x\to 0} f\left(\left|\frac{1}{x}\right|\right) = \lim_{x\to\infty} f(|x|)$$

Answer: a

17. Apply linearization of the function

$$f(x) = \sqrt{x}$$

$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$

$= \frac{1}{2\sqrt{x}}$

at $x = 16$ to approximate $\sqrt{17}$.

$x=16,\ y=4$

Answer: $\frac{\sqrt{17}}{8} + 2 \approx 2.5154$ (4 d.p.) //

$y = \frac{1}{8}x + 2$.

18. Let

$$C : x^3 + 5xy - 3y^2 = 3$$

be a curve. Find the equation of the tangent of C at the point $(1, 1)$.

Answer: $y = 8x - 7$

$3x^2 + 5y + 5x\left(\frac{dy}{dx}\right) - 6y\left(\frac{dy}{dx}\right) = 0$

$\frac{dy}{dx}(5x - 6y) = -3x^2 - 5y$

slope $= 8$ $\quad \frac{-3x^2 - 5y}{5x - 6y} = \frac{dy}{dx}$

$y = 8x + -7$

4 Q16-18 6 / 9

Q16

✓ - 0 pts Correct

Q17

✓ - 3 pts Incorrect

Q18

✓ - 0 pts Correct

Long Questions

19. (12 points) Suppose that

$$f(x) = \begin{cases} \tan x - 1 & \text{if } x \in (-\frac{\pi}{2}, 0] \\ x^2 + ax + \cancel{b} & \text{if } x > 0, \end{cases}$$

-1

where a and b are real numbers.

Given that f is differentiable at $x = 0$, without using L'Hôpital's rule, find the values of a and b.

$\because f$ is differentiable at $x = 0$,

$\therefore f$ is continuous at $x = 0$:

$\therefore$ We have $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$

$$0 - 1 = b$$

$$b = -1 \;//$$

Also, Let $y(x) = \tan x - 1$ & $z(x) = x^2 + ax - 1$.

We have $y'(0) = z'(0)$

$\frac{1}{\cos^2(0)} \to$ $\quad \sec^2(0) = 2(0) + a$

$$a = 1 \;//$$

Scanned with CamScanner

5 Q19 **6 / 12**

Value of b

✓ **- 0 pts** Correct

Value of a

✓ **- 6 pts** **Mistake: Take limit of f'.**

20. (14 points) Let C be the curve defined by the equation

$$y^4 - x = xy + \cos x$$

Given that $A = (0, 1)$ is a point on C,

(a) Find $\left.\frac{dy}{dx}\right|_A$

(b) Find $\left.\frac{d^2y}{dx^2}\right|_A$

(a) By implicit differentiation,

$$4y^3\left(\frac{dy}{dx}\right) - 1 = y + x\left(\frac{dy}{dx}\right) - \sin x$$

$$\left(\frac{dy}{dx}\right)(4y^3 - x) = 1 + y - \sin x$$

$$\frac{dy}{dx} = \frac{1 + y - \sin x}{4y^3 - x}$$

$$\left.\frac{dy}{dx}\right|_A = \frac{1 + 1 - 0}{4 - 0}$$

$$= \frac{1}{2} \quad //$$

(b)

$$\frac{d^2y}{dx^2} = \frac{\left(\frac{dy}{dx} - \cos x\right)(4y^3 - x) - (1 + y - \sin x)\left(12y^2\frac{dy}{dx} - 1\right)}{(4y^3 - x)^2}$$

$$\left.\frac{d^2y}{dx^2}\right|_A = \frac{\left(\frac{1}{2} - 1\right)(4 - 0) - (1 + 1 - 0)\left(12 \cdot \frac{1}{2} - 1\right)}{(4 - 0)^2}$$

$$= \frac{-2 - 2 \times 5}{16}$$

$$= -\frac{3}{4} \quad //$$

6 Q20 **14 / 14**

Q20(a)

✓ **- 0 pts** Correct

Q20(b)

✓ **- 0 pts** Correct

21. (10 points) Show that the function

$$f(x) = x^4 - 5x + 1$$

has at least two real roots.

$\because$ $f(x)$ is a continuous function for a $\forall x \to \mathbb{R}$.

① $f(1) = -3$
$f(2) = 7$

By Intermediate value Theorem,
there exists a real value $c \in (1, 2)$ where
$f(c) = 0$.

② $f(0) = 1$
$f(1) = -3$

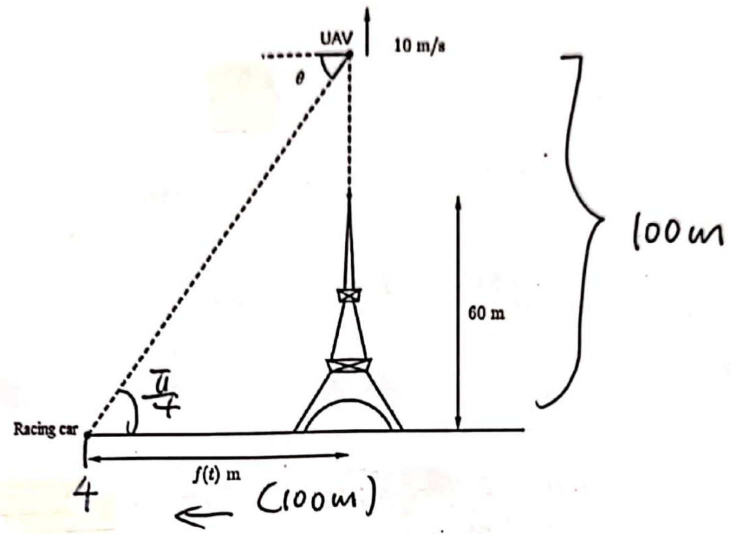
By Intermediate value Theorem,
there exists a real value $d \in (0, 1)$
where $f(d) = 0$.

From the above, the function has at least two real roots which are c and d. //

7 Q21 10 / 10

✓ - 0 pts Correct

22. (16 points) A racing car moves away from a tower, and the horizontal distance between the racing car and the tower after $t > 0$ seconds is given by a differentiable function $f(t)$ meter. To monitor the racing car, when it starts to move, an unmanned aerial vehicle (UAV) flies vertically upwards with constant speed 10 m/s from the top of a tower 60 meters in height.



Given that at $t = 4$, the angle of depression θ from the UAV to the racing car is $\frac{\pi}{4}$ radian.

(a) Find $f(4)$.

(b) At $t = 4$, the angle of depression θ decreases at a rate of 0.15 radian/s. Find the speed of the racing car at that moment.

(a) $\tan\frac{\pi}{4} = \dfrac{60 + 10\times 4}{f(4)}$

$f(4) = 100$ //

(b) $\dfrac{d}{d\theta}\tan\theta = \dfrac{10f(t) - (60+10t)f'(t)}{f'(t)}$

Sub $\theta = -0.15$, $t=4$ and $f(4) = 100$:

$\sec^2(-0.15) = \dfrac{10\times 100 - 100\times f'(4)}{f'(4)}$

$f'(4)\sec^2(-0.15) = 1000 - 100f'(4)$

$$f'(4)\left(100 + \sec^2(-0.15)\right) = 1000$$

$$f'(4) = 9.8988 \text{ m/s}$$

(4 d.p.)

8 Q22 **12 / 16**

Part (a)

✓ - **0 pts** Correct

Part (b)

✓ - **2 pts** Incorrect formula regarding $$f'(t)$$

✓ - **2 pts** Incorrect final answer

$$\frac{-2c(\tfrac{1}{x})}{x^2}$$

23. (14 points) Suppose

$$f(x) = \begin{cases} 2x - \sin x & \text{if } x \le 0 \\ x^2 \cos\left(\dfrac{2}{x}\right) & \text{if } x > 0 \end{cases}$$

Find $D_{f'}$ (the domain of f') and $f'(x)$ for any $x \in D_{f'}$.

$$f'(x) = \begin{cases} 2 - \cos x & \text{if } x < 0 \\ \text{Undefined} & \text{if } x = 0 \\ 2x \cos\left(\frac{2}{x}\right) + x^2 \sin\left(\frac{2}{x}\right) \cdot \frac{2}{x^2} & \text{if } x > 0. \end{cases}$$

$$= \begin{cases} 2 - \cos x & \text{if } x < 0 \\ \text{Undefined} & \text{if } x = 0 \\ 2x \cos\left(\frac{2}{x}\right) + 2 \sin\left(\frac{2}{x}\right) & \text{if } x > 0 \end{cases}$$

$\therefore D_{f'} = (-\infty, 0) \cup (0, \infty)$ //

Scanned with CamScanner

9 Q23 4 / 14

✓ - 2 pts incorrect explanation for domain of f'(x)

✓ - 8 pts incorrect argument for f'(0)

24. (20 points)

(a) Let k be a positive integer. Show that

$$\frac{1}{k+1} < \ln(k+1) - \ln(k) < \frac{1}{k}$$

(b) Let n be a positive integer. Show that

$$\ln(n+1) < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < 1 + \ln n$$

(c) By part (b), evaluate

$$\lim_{n\to\infty} \frac{1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}}{\ln n}.$$

(a) Let $f(x) = \ln x$

$$f'(x) = \frac{1}{x}.$$

By the mean value theorem, there exists a real value c where $c \in (a, b)$ and

$$f'(c) = \frac{f(b) - f(a)}{b - a},$$

given that $f(x)$ is continuous & differentiable.

$\therefore$ Sub $b = k+1$ & $a = k$,

we have: $$\frac{1}{c} = \frac{\ln(k+1) - \ln k}{k+1-k}$$

$$\frac{1}{c} = \ln(k+1) - \ln k$$

$\therefore$ Also, we have: $$\frac{1}{b} < \frac{1}{c} < \frac{1}{a}$$

$$\therefore \frac{1}{k+1} < \ln(k+1) - \ln(k) < \frac{1}{k} \quad //$$

13

(b) Sub n into k for (a).

$$\frac{1}{n+1} < \ln(n+1) - \ln(n) < \frac{1}{n}$$

$$<\qquad\qquad <$$

$$<\qquad\qquad <$$

(c) $\lim_{n \to \infty}$

Scanned with CamScanner

10 Q24 **7 / 20**

Part (a)

✓ **- 1 pts** **Should state that you are applying MVT on the interval $$[k,k+1]$$ explicitly**

Part (b)

✓ **- 6 pts** Incorrect

Part (c)

✓ **- 6 pts** Incorrect

25. (10 points) Given that

$$\lim_{x\to 0}(\sin(x^3)f(ax)) = \lim_{x\to 0}(\sin^3(x)g(bx)) = 1$$

for some $a, b \neq 0$, evaluate

$$\lim_{x\to 0}\frac{f(x)}{g(x)}$$

$$\lim_{x\to 0} \frac{\sin(x^3)\ f(ax)}{x^3} \times x^3 = 1$$

$\lim_{x\to 0} \frac{\sin x}{x} = 1 \rightarrow$

$$\lim_{x\to 0} f(ax)\cdot x^3 = 1$$

$$f(ax) = 0$$

$$f(x) = 0.$$

$$\lim_{x\to 0} \frac{\sin^3(x)\ g(bx)\cdot x^3}{x^3} = 1$$

$$\lim_{x\to 0} g(bx)\cdot x^3 = 1$$

$$g(bx) = 0$$

$$g(x) = 0$$

$$\therefore \lim_{x\to 0}\frac{f(x)}{g(x)} = \frac{0}{0}$$

$= \cancel{}$ DONE //

11 Q25 2 / 10

+ **10 pts** Correct
+ **2 pts** Essentially expresses lim sin(x^3)f(ax) interms of f(x)
+ **2 pts** Essentially expresses lim sin^3(x)g(bx) interms of g(x)
- **2 pts** Misc. mistakes in re-expressing sin(x^3)f(ax) and sin^3(x)g(bx)
+ **4 pts** Correctly expresses lim f(x)/g(x) in way which takes advantage of known identities.

✓ + **2 pts** **Expresses lim f(x)/g(x) in way which takes advantage of known identities, with at least one significant mistake or omission in the reasoning.**

+ **2 pts** Correct final answer.
+ **0 pts** Assumes without justification the limit of individual term(s) exists.
+ **0 pts** Incorrect.
+ **0 pts** No answer found on the first selected page