香 港 中 文 大 學
The Chinese University of Hong Kong

二〇一九至二〇年度 下學期科目考試
Course Examination 2^{nd} Term, 2019-20

科目編號及名稱
Course Code & Title : **MATH1510J Calculus for Engineers**

時間 Time allowed : 2 小時 hours 00 分鐘 minutes

學號 Student I.D. No. : ________ 座號 Seat No. : ________

- Time allowed: **2 hours**. Total score: **200**.
- Write your name and student ID on the first page of your solution. Scan your solutions and combine them into **one single PDF file** named by your student ID for submission.
- Any violation of the regulations of CUHK academic honesty will be reported to the disciplinary committee.

Part A: Short Questions

Please write down your answers directly without explanation.

Each of question 1-17 is worth 3 points.

1. Find the domain of $f(x) = \sqrt{1-x^2}$.

2. Given the parametric equations $\begin{cases} x = 2\cos t, \\ y = 3\sin t. \end{cases}$

   Write down an equation relating x and y without t.

3. Evaluate $\lim_{x\to+\infty} \frac{\pi x^7 - 3x^6}{5x^7 + x + 14}$.

4. Evaluate $\lim_{\theta\to+\infty} \theta \sin\left(\frac{\pi}{\theta}\right)$.

5. Find $\frac{d^{100}}{dx^{100}}(xe^x)$.

6. Find the linearization of $f(x) = \sqrt{x}$ at $x = 4$.

7. Suppose the side length of a square is increasing at a constant rate of 2 unit/min. Find the rate of change of its area when its side length is 10 unit.

8. Let $f(x), g(x)$ be differentiable functions satisfying

$g(0) = 1$	$f(0) = 3$	$f(1) = 5$	$f(2) = 7$
$g'(0) = 2$	$f'(0) = 4$	$f'(1) = 6$	$f'(2) = 8$

   Find $(f \circ g)'(0)$.

9. Find all the inflection point(s) of $f(x) = \frac{1}{5}x^5 - \frac{1}{3}x^4$.

10. Find all the vertical and horizontal asymptotes of the graph $y = \frac{2x^2}{x^2-1}$.

11. Evaluate $\displaystyle\int \sin x \cos^3 x \, dx$.

12. Find the average value of $f(x) = \dfrac{1}{1+x^2}$ over the interval $[0, 3]$.

13. Evaluate the improper integral $\displaystyle\int_1^2 \frac{1}{\sqrt{x-1}} \, dx$.

14. Find $f'(x)$ if $f(x) = \displaystyle\int_1^{5x} e^{t^2} \, dt$.

15. Find the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{n}{6^n}(x+1)^n.$$

16. Let

$$f(x) = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + x^4 + \cdots$$

$$g(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$$

Find the third order Maclaurin polynomial of $f(x)g(x)$.

17. Let

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n+1} = 1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \cdots$$

$$g(x) = \sum_{n=1}^{\infty} 2^n x^n = 2x + 4x^2 + 8x^3 + \cdots$$

Find the second order Maclaurin polynomial of $(f \circ g)(x)$.

Part B: Long Questions

Please write down the detailed steps.

18. (20 points) Evaluate the following limits.

(a) $\lim_{x\to+\infty}(\sqrt{x^2+x}-\sqrt{x^2-x})$.

(b) $\lim_{x\to+\infty}\left(1-\frac{2}{3x}\right)^x$.

(c) $\lim_{x\to 2^+}(x-2)\sin\left(\frac{1}{\sqrt{x-2}}\right)$.

(d) $\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)$.

19. (16 points) Find $\frac{dy}{dx}$ if

(a) $y=\frac{\tan(2x)}{3x+1}$.

(b) $y=\arcsin(x^2+1)$.

(c) $y=e^{\sqrt{2+\cos x}}$.

(d) $y=(\sin x)^x$.

20. (30 points) Evaluate the following integrals.

(a) $\int \frac{1}{x(x^2+1)}\,dx$.

(b) $\int \frac{1}{1+\sin x+\cos x}\,dx$.

(c) $\int \sin(6x)\cos(4x)\,dx$.

(d) $\int (x^2+x)e^x\,dx$.

(e) $\int_0^\infty \frac{1}{x^2+4}\,dx$.

21. (10 points) Let

$$f(x)=\begin{cases} 3x+7 & \text{if } x<0, \\ x^2+ax+b & \text{if } x\geq 0 \end{cases}$$

where a, b are constants.

(a) Suppose f is continuous at $x=0$. Find b.

(b) Suppose f is differentiable at $x=0$. Find a.

22. (10 points) Suppose the equation

$$xy = \ln x + y^3$$

implicitly determines a function $y = y(x)$ near the point $(1, 0)$.

(a) Find $\dfrac{dy}{dx}$ at $(1, 0)$.

(b) Find $\dfrac{d^2y}{dx^2}$ at $(1, 0)$.

23. (10 points) Solve the following problems separately.

(a) Let

$$f(x) = x^3 - 3x^2 - 9x + 1.$$

Find all the critical point(s) of f, and use first derivative test to determine whether each critical point is a local minimum, maximum, or neither.

(b) Let

$$g(x) = (x - 3)e^x + e^2.$$

Find all the critical point(s) of g, and use second derivative test to determine whether each critical point is a local minimum, maximum, or neither.

24. (10 points) Let $\mathcal{R}$ be the region in xy-plane bounded by $y = 2^x - 1$, $x = 1$ and x-axis.

(a) Find the area of $\mathcal{R}$.

(b) Express the volumes of the following solids as definite integrals (**Do not evaluate**):

(i) The solid obtained by revolving $\mathcal{R}$ about the x-axis.

(ii) The solid obtained by revolving $\mathcal{R}$ about the y-axis.

25. (10 points)

(a) Find the Taylor series generated by the function $\dfrac{1}{1-3x}$ at $x = 0$, and express your answer in summation notation.

(b) By using (a), find the Taylor series generated by the function $\dfrac{1}{(1-3x)^2}$ at $x = 0$, and express your answer in summation notation. Find the radius of convergence of the series obtained.

26. (14 points)

(a) Show that

$$\arctan x - \arctan y < x - y$$

for any real numbers x, y such that $x > y \geq 0$.

(b) Use the result in part (a) to show that $\pi < 2\sqrt{3}$.

27. (19 points) Let $a \in \mathbb{R}$ and let $f : [0, a] \to \mathbb{R}$ be a continuous function.

(a) Show that
$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx.$$

(b) Find
$$\int_0^{\pi/2} \frac{\cos^3 x}{\sin x + \cos x}\,dx.$$

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