THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1510 Calculus for Engineers (2020-2021)
Supplementary Exercise 5

Differentiability of Functions (First Principle)

1. A function f is said to be differentiable at a point $x = c$ if the limit

$$\lim_{h\to 0}\frac{f(c+h)-f(c)}{h} \tag{1}$$

exists. The value of the limit is said to be the derivative of the function f at the point $x = c$, which is denoted by $f'(c)$. The above definition is also called the first principle.

Compute the derivative of each of the following function at the given point by using the definition (i.e. the first principle).

(a) $f(x) = x^2 + 1$ at the point $x = 2$;

(b) $f(x) = \dfrac{1}{x}$ at the point $x = 3$;

(c) $f(x) = \cos x$ at the point $x = \dfrac{\pi}{2}$;

(d) (Harder Problem) $f(x) = x^n$, where n is a natural number, at the point $x = 2$.

2. The **left derivative** of a function $f(x)$ at $x = c$ is by definition:

$$Lf'(c) = \lim_{h\to 0^-}\frac{f(c+h)-f(c)}{h};$$

and the **right derivative** at $x = c$ is:

$$Rf'(c) = \lim_{h\to 0^+}\frac{f(c+h)-f(c)}{h}.$$

From the definition (see (1)), the function $f(x)$ is differentiable at $x = c$ if and only if

$$\lim_{h\to 0^-}\frac{f(c+h)-f(c)}{h} = \lim_{h\to 0^+}\frac{f(c+h)-f(c)}{h},$$

i.e. $Lf'(c) = Rf'(c)$. In this case, $f'(c)$ equals to their common value.

Suppose

$$f(x) = \begin{cases} 3 - \sin x & \text{if } \; x < 0, \\ a & \text{if } \; x = 0, \\ bx + c & \text{if } \; x > 0, \end{cases}$$

where a, b are some real numbers. Given that $f(x)$ is continuous at $x = 0$.

(a) What is the values of a and c?

(b) Find $Lf'(0)$.

(c) Find $Rf'(0)$ (in terms of b).

(d) For what value of b is the function $f(x)$ differentiable at 0?

3. Let us study the derivative as a function. The function $f'(x)$ is still defined as a limit, but the fixed number c in the definition (see (1)) is replaced by the variable x:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}. \tag{2}$$

If $y = f(x)$, we also write y' or $y'(x)$ for $f'(x)$. The domain of $f'(x)$ consists of all values of x in the domain of $f(x)$ for which the limit in equation (2) exists. If f is differentiable at every point in the domain, then f is said to be a differentiable function.

Using Equation (2), determine the domain of f', then give a formula describing $f'(x)$ where

$$f(x) = \sqrt{2-x} \quad \text{with domain } D_f = (-\infty, 2].$$

4. Compute the derivative function of each of the following functions by using the definition (i.e. the first principle).

(a) $f(x) = x^2 + 1$;

(b) $f(x) = \dfrac{1}{x}$, for $x \neq 0$;

(c) $f(x) = \cos x$;

(d) (Harder Problem) $f(x) = x^n$, where n is a natural number.

5. Let $f(x) = |x|$. $f(x)$ can be described as the following

$$f(x) = \begin{cases} x & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -x & \text{if } x < 0. \end{cases}$$

(a) Write down $\dfrac{f(0+h) - f(0)}{h}$ explicitly for the cases $h > 0$ and $h < 0$.

(b) By using the result in (a), find left derivative, i.e.

$$Lf'(0) = \lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h},$$

and the right derivative, i.e.

$$Rf'(0) = \lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h},$$

at the point $x = 0$.

(Hint: You have to use the first expression in (a) to find the left hand limit and the second expression to compute the right hand limit.)

(c) Does $f'(0)$ exist?

(d) Find $f'(x)$ by using the first principle for the cases $x > 0$ and $x < 0$. Hence, write down the domain of $f'(x)$.

6. Let $f(x)$ be a function defined by

$$f(x) = \begin{cases} x^2 & \text{if } \; x \geq 0, \\ 0 & \text{if } \; x < 0. \end{cases}$$

(a) Is $f(x)$ differentiable at $x = 0$? If yes, find $f'(0)$.

(b) Compute $f'(x)$ for the cases $x > 0$ and $x < 0$.

(c) Is $f'(x)$ differentiable at $x = 0$?

Derivatives

7. Find the first derivatives of the following functions.

(a) $y = 2x^3 - 4x + 2$

(b) $y = 5x^3 - 4x^2 + 7$

(c) $y = e^{3x}$

(d) $y = \cos 2x$

(e) $y = \sin 5x$

(f) $y = -\tan 3x$

(g) $y = \sqrt{x}$

(h) $y = \ln(1 + x^2)$

8. Find the first derivatives of the following functions.

(a) $y = 4\sqrt{x} + \dfrac{2}{\sqrt{x}}$

(b) $y = x^3 e^{-2x}$

(c) $y = \sin x \ln x$

(d) $y = \sec x - 3 \tan x$

(e) $y = x \csc x$

(f) $y = \dfrac{3x - 4}{x + 2}$

(g) $y = \dfrac{x^2 + 1}{x + 1}$

(h) $y = \dfrac{\sin x}{x}$

(i) $y = (3x^2 - 4)^{10}$

(j) $y = \sqrt{x^3 + 1}$

(k) $y = \ln(\ln x)$

(l) $y = e^{\cot x}$

(m) $y = \ln(x + \sqrt{x})$

9. Let $\mathcal{C}$ be the graph of the function $y = 4e^x(x + 1)$.

(a) Show that $A = (0, 4)$ is a point lies on $\mathcal{C}$.

(b) Find the equations of tangent and normal of $\mathcal{C}$ at the point A.

10. Let $\mathcal{C}$ be the graph of the function $y = \sin x + \cos x$.

(a) Show that $A = (\frac{\pi}{2}, 1)$ is a point lies on $\mathcal{C}$.

(b) Find the equations of tangent and normal of $\mathcal{C}$ at the point A.

11. By using the logarithmic differentiation, find the first derivative of the following functions.

    (a) $y = (2x+1)^3(x-1)^4\sqrt{(3x+2)^5}$

    (b) $y = \dfrac{e^{2x}}{(x-1)^4}$

    (c) $y = x^x$

    (d) $y = (\sin x)^{(\cos x)}$

12. Find $\dfrac{dy}{dx}$ in terms of x and y for the following implicit functions.

    (a) $x^2 + y^2 = 9$

    (b) $x^3y + xy^2 = 1$

    (c) $x^3 + y^3 = 2xy$

    (d) $ye^{xy} = 1$

13. Let $\mathcal{C}$ be the curve given by the equation $x^3 + xy + y^3 = 11$.

    (a) Show that $A = (1, 2)$ is a point lies on $\mathcal{C}$.

    (b) Find the equation of tangent of $\mathcal{C}$ at the point A.

14. If $y = x^2e^x$, show that $\dfrac{d^2y}{dx^2} = 2\dfrac{dy}{dx} - y + 2e^x$.

15. Let $\mathcal{C} : (x(t), y(t)) = (\sqrt{2}\cos t, \sqrt{2}\sin t)$, for $t \in \mathbb{R}$, be a curve defined on $\mathbb{R}^2$.

    (a) Find $\dfrac{dy}{dx}$ in terms of t.

    (b) Find the equation of tangent of $\mathcal{C}$ at the point $(x(\frac{\pi}{4}), y(\frac{\pi}{4})) = (1, 1)$.

16. Let $\mathcal{C} : (x(t), y(t)) = (t^2, t^3)$, for $t \in \mathbb{R}$, be a curve defined on $\mathbb{R}^2$.

    (a) Find $\dfrac{dy}{dx}$ in terms of t.

    (b) Find the equation of tangent of $\mathcal{C}$ at the point $(x(1), y(1) = (1, 1)$.

Linearization

17. Let $f(x)$ be a function which is differentiable at $x = a$.

    (a) Show that the equation of tangent at the point $x = a$ is

$$y = L(x) = f'(a)(x - a) + f(a).$$

(b) The function $L(x)$ obtained in (a) is a linear function (i.e. a polynomial function of degree 1) which is called the linearization of $f(x)$ at the point $x = a$ (also called the Talyor polynomial of degree 1 generated by $f(x)$ at the point $x = a$, which is commonly denoted by $T_1(x)$).

Now, suppose that $f(x) = \sqrt{x}$. By stekching the graphs of $f(x)$ and $L(x)$, one can observe that when x is close to 9, $L(x)$ approximately equals $f(x)$.
By using the result in (a), approximate the value of $\sqrt{9.1}$.
(Remark: Compare your approximated value obtained and the value obtained by using calculator.)

18. Approximate the value of $e^{0.1}$ by linearizing an appropriately chosen function at an appropriately chosen point.