Calculus for Engineers

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Contents

Partial Differentiation

28.1 Introduction

In Section 28.2, we will provide some information about multi-variable functions. In Section 28.3, we will give the definitions of partial derivatives for functions of two, three and nth variables. Due to the limitation of the graphical representation, we will present a geometrical interpretation of partial derivatives of the first order in Section 28.3. In Section 28.4, we will study higher order partial derivatives of multi-variable functions and the so-called Schwarz's theorem. In Section 28.5, we will study the differentiability of multi-variable functions. In Section 28.6, we will study the total differential coefficient of the function; first and second differential coefficients of an implicit functions are given. In Section 28.7, we will study the so-called Euler's theorem on homogeneous functions of degree n; homogenous functions arise naturally in many engineering and science models. In Section 28.8, Taylor's theorem for the functions of two variables is presented and the application of Taylor's theorem via approximation calculations is given. In Section 28.9, the definitions of the Jacobian for functions of two, three and nth variables are given and some theorems on Jacobians and related topics are also given (some background knowledge on linear algebra is required, for example, the determinant of a square matrix and solving a linear system of equations).

28.2 What are functions of multi-variables?

Up to now, we have considered functions of a single variable of the type $y = f(x)$, where y is the dependent variable and x is the independent variable. The derivative of y in this case is given by

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}. \tag{28.1}$$

Now let us consider the case when u is a function of two independent variables, say x and y given by the relation

$$u = f(x, y),$$

where u is a quantity which has one definite value for every pair of values of x and y.

Let us consider examples of functions of two variables:

Example 1 In geometry, the volume of a right circular cone with base radius r and altitude h is given by

$$V = \frac{1}{3}\pi r^2 h,$$

which clearly depends on the values of r and h, and so $V(r,h)$ is function of two variables.

Example 2 Consider the gas equation, that is,

$$PV = RT,$$

where V is the volume of a certain gas, P its pressure, T is absolute temperature and R is a constant. P can be regarded as a function of two variables V and T, that is,

$$P(V,T) = \frac{RT}{V}.$$

A similar definition can be given for functions of more than two variables.

28.3 Partial differential coefficient

28.3.1 Partial derivatives of functions of two variables

Let u be a function of two variables x and y given by the relation $u = f(x,y)$, then

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}, \tag{28.2}$$

provided this limit exists and is unique, is said to be partial differential coefficient of $f(x,y)$ or of u with respect to x and is denoted by $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial u}{\partial x}$ or f_x. Thus the partial differential coefficient of $f(x,y)$ with respect to x is the ordinary differential coefficient of $f(x,y)$ when y is treated as a constant.

Similarly, we have

$$\lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y} \tag{28.3}$$

if it exists and is unique, is called the partial differential coefficient of $f(x,y)$ or u with respect to y and is denoted as $\dfrac{\partial f}{\partial y}$ or $\dfrac{\partial u}{\partial y}$ or f_y. In the same way the partial differential coefficient of $f(x,y)$ with respect to y is the ordinary differential coefficient of $f(x,y)$ when x is treated as a constant.

Note 1 The partial derivatives at a particular point (a,b) are

$$f_x(a,b) = \left(\frac{\partial f}{\partial x}\right)_{(a,b)} = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

and

$$f_y(a,b) = \left(\frac{\partial f}{\partial y}\right)_{(a,b)} = \lim_{k \to 0} \frac{f(a,b+k) - f(a,b)}{k}.$$

28.3.2 Partial derivatives of functions of three variables

Let $u = f(x, y, z)$ be a function of three independent variables x, y, z and let $f(x, y, z)$ be defined in some neighbourhood of (x, y, z). Then the partial derivative of $f(x, y, z)$ with respect to x, y, z is defined as follows:

1. If $\dfrac{\partial f}{\partial x} = f_x = \lim_{\Delta x \to 0} \dfrac{f(x + \Delta x, y, z) - f(x, y, z)}{\Delta x}$ exists, then f_x is the partial differential coefficient of $f(x, y, z)$ with respect to x.

2. If $\dfrac{\partial f}{\partial y} = f_y = \lim_{\Delta y \to 0} \dfrac{f(x, y + \Delta y, z) - f(x, y, z)}{\Delta y}$ exists, then f_y is the partial differential coefficient of $f(x, y, z)$ with respect to y.

3. If $\dfrac{\partial f}{\partial z} = f_z = \lim_{\Delta z \to 0} \dfrac{f(x, y, z + \Delta z) - f(x, y, z)}{\Delta z}$ exists, then f_z is the partial differential coefficient of $f(x, y, z)$ with respect to z.

28.3.3 Partial derivatives of functions of nth variables

In general, if $f(x_1, x_2, \cdots, x_n)$ is a function of n variables $x_1, x_2, \cdots, x_n$, then the partial differential coefficient of f with respect to x_1 is the ordinary differential coefficient of f when all the other variables except x_1 are treated as constants:

$$\frac{\partial f}{\partial x_1} = \lim_{h \to 0} \frac{f(x_1 + h, x_2, \cdots, x_n) - f(x_1, x_2, \cdots, x_n)}{h}, \tag{28.4}$$

if this limit exists and is unique. Similarly, we can have the partial derivative with respect to other variables.

28.3.4 Geometrical interpretation of partial derivatives of the first order

The geometric interpretation of a partial derivative is the same as that for an ordinary derivative. It represents the slope of the tangent to that curve represented by the function at a particular point P. In the case of a function of two variables

$$u = f(x, y).$$

Figure 28.1 shows the interpretation of $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$, where

- $\dfrac{\partial f}{\partial x}$ corresponds to the slope of the tangent to the curve APB (in purple) at point P (where curve APB is the intersection of the surface with a plane through P perpendicular to the y axis).

- $\dfrac{\partial f}{\partial y}$ corresponds to the slope of the tangent to the curve CPD (in green) at point P (where curve CPD is the intersection of the surface with a plane through P perpendicular to the x axis).

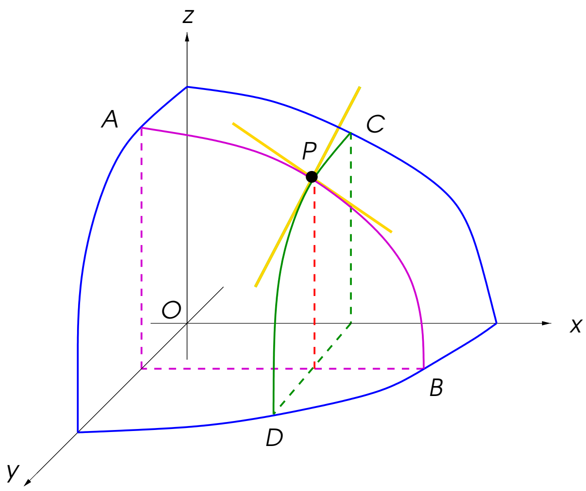

Figure 28.1: Geometrical interpretation of $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.

28.3.5 Worked examples

Example 3 If $f(x,y) = x^3 + y^2$, find $f_x(x,y)$ and $f_y(x,y)$ from the definition of partial derivatives.

Solution. We know

$$
\begin{aligned}
f_x = \frac{\partial u}{\partial x} &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{((x+\Delta x)^3 + y^2) - (x^3 + y^2)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{x^3 + 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3 + y^2 - x^3 - y^2}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left(3x^2 + 3x(\Delta x) + (\Delta x)^2\right) \\
&= 3x^2.
\end{aligned}
$$

and

$$\begin{aligned}
f_y = \frac{\partial u}{\partial y} &= \lim_{\Delta y \to 0} \frac{f(x, y+\Delta x) - f(x,y)}{\Delta y} \\
&= \lim_{\Delta y \to 0} \frac{(x^3 + (y+\Delta y)^2) - (x^3 + y^2)}{\Delta y} \\
&= \lim_{\Delta y \to 0} \frac{x^3 + y^2 + 2y\Delta y + (\Delta y)^2 - x^3 - y^2}{\Delta y} \\
&= \lim_{\Delta y \to 0} \frac{2y\Delta y + (\Delta y)^2}{\Delta y} \\
&= \lim_{\Delta y \to 0} (2y + (\Delta y)) \\
&= 2y.
\end{aligned}$$

□

Example 4 If $u = x^2y + y^2z + z^2x$, then show that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = (x+y+z)^2.$$

Solution. Given

$$u = x^2y + y^2z + z^2x. \tag{28.5}$$

Differentiating (28.5) partially with respect to x, we get

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial}{\partial x}\left(x^2y + y^2z + z^2x\right) \\
&= y\frac{\partial}{\partial x}\left(x^2\right) + y^2z\frac{\partial}{\partial x}(1) + z^2\frac{\partial}{\partial x}(x) \\
&= 2xy + z^2. \qquad (28.6)
\end{aligned}$$

Similarly, differentiating (28.5) partially with respect to y and z, we have

$$\begin{aligned}
\frac{\partial u}{\partial y} &= x^2\frac{\partial}{\partial y}(y) + z\frac{\partial}{\partial y}\left(y^2\right) + z^2x\frac{\partial}{\partial y}(1) \\
&= x^2 + 2yz \qquad (28.7)
\end{aligned}$$

and

$$\frac{\partial u}{\partial z} = x^2y\frac{\partial}{\partial z}(1) + y^2\frac{\partial}{\partial z}(z) + x\frac{\partial}{\partial z}\left(z^2\right)$$

$$= y^2 + 2zx. \tag{28.8}$$

Adding (28.6), (28.7) and (28.8), we get

$$\begin{aligned}
\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} &= x^2 + y^2 + z^2 + 2xy + 2yz + 2zx \\
&= (x+y+z)^2.
\end{aligned}$$

□

28.4 Partial derivatives of higher order

Let us consider second order partial derivatives. If f is a function of two variables x and y, then partial derivative $\dfrac{\partial f}{\partial x}$ or f_x will also be a function of x and y and so it can also possess partial derivatives. Thus for function $u = f(x, y)$, the partial derivative of $\dfrac{\partial f}{\partial x}$ or f_x and the partial derivative of $\dfrac{\partial f}{\partial x}$ or f_y with respect to x are

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

and

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y},$$

denoted by the symbol f_{xx} and f_{xy}, respectively. Similarly the partial derivative of $\dfrac{\partial f}{\partial y}$ or f_y and the partial derivative of $\dfrac{\partial f}{\partial x}$ or f_x with respect to y are

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

and

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x},$$

denoted by symbol f_{yy} and f_{xy} respectively and in all ordinary cases

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \qquad \text{or} \qquad f_{yx} = f_{xy}. \tag{28.9}$$

This equality is due to continuity. It can be proved that on every open set U on which f and its partial derivatives $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial^2 f}{\partial x \partial y}, \dfrac{\partial^2 f}{\partial y \partial x}$ exist, all are continuous.

28.4.1 Partial derivatives of second order

Partial derivatives of second order are illustrated as follows.

At a point (a, b), we have the following four results:

1.

$$\begin{aligned} f_{xx}(a,b) &= \left(\frac{\partial^2 f}{\partial x^2}\right)_{(a,b)} \\ &= \lim_{h \to 0} \frac{f_x(a+h,b) - f_x(a,b)}{h} \\ &= \lim_{h \to 0} \frac{f(a+2h,b) - 2f(a+h,b) - f(a,b)}{h^2}. \end{aligned}$$

2.

$$
\begin{aligned}
f_{yy}(a,b) &= \left(\frac{\partial^2 f}{\partial y^2}\right)_{(a,b)} \\
&= \lim_{k\to 0} \frac{f_y(a,b+k)-f_y(a,b)}{k} \\
&= \lim_{k\to 0} \frac{f(a,b+2k)-2f(a,b+k)-f(a,b)}{k^2}.
\end{aligned}
$$

3.

$$
\begin{aligned}
f_{yx}(a,b) &= \left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(a,b)} \\
&= \lim_{h\to 0} \frac{f_y(a+h,b)-f_y(a,b)}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left(\lim_{k\to 0} \frac{f(a+h,b+k)-f(a+h,b)}{k} - \lim_{k\to 0} \frac{f(a,b+k)-f(a,b)}{k}\right) \\
&= \lim_{h\to 0}\lim_{k\to 0} \frac{f(a+h,b+k)-f(a+h,b)-f(a,b+k)+f(a,b)}{hk}.
\end{aligned}
$$

4.

$$
\begin{aligned}
f_{xy}(a,b) &= \left(\frac{\partial^2 f}{\partial y \partial x}\right)_{(a,b)} \\
&= \lim_{k\to 0} \frac{f_x(a,b+k)-f_x(a,b)}{k} \\
&= \lim_{k\to 0} \frac{1}{k}\left(\lim_{h\to 0} \frac{f(a+h,b+k)-f(a,b+k)}{h} - \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}\right) \\
&= \lim_{k\to 0}\lim_{h\to 0} \frac{f(a+h,b+k)-f(a+h,b)-f(a,b+k)+f(a,b)}{hk}.
\end{aligned}
$$

Thus, from 3. and 4., we see that $f_{yx}(a,b)$ and $f_{xy}(a,b)$ are the repeated limits of the same expression taken in different orders. Hence $f_{yx}(a,b)$ and $f_{xy}(a,b)$ may or may not be equal.

In a similar fashion higher order partial derivatives, for example, $\dfrac{\partial^3 f}{\partial x \partial x \partial y}$, are defined.

Note 2 For most functions that one meets

$$f_{yx} = f_{xy}.$$

However, in some cases it is not true. Under what circumstances is it true? It is true if both functions f_{yx} and f_{xy} are continuous at the point 'where' the partials are being taken.

Theorem 1 (Schwarz's Theorem) If (a,b) is a point of the domain $D \subset \mathbb{R}$ of a real valued function $f(x,y)$ such that

1. f_x exists in a certain neighbourhood of (a,b),
2. f_{xy} is continuous at (a,b),

then $f_{yx}(a,b)$ exists and is equal to $f_{xy}(a,b)$, i.e., $f_{yx}(a,b) = f_{xy}(a,b)$.

Proof. Under the given conditions f_x, f_y and f_{xy} exist in a certain neighbourhood of (a,b). Let $(a+h, b+k)$ be any point of this neighbourhood.

Consider

$$\phi(h,k) = f(a+h,b+k) - f(a+h,b) - f(a,b+k) + f(a,b)$$

and

$$g(y) = f(a+h,b) - f(a,y)$$

so that

$$\phi(h,k) = g(a+h) - g(a). \tag{28.10}$$

Since f_x exists in a neighbourhood of (a,b), the function $g(y)$ is differentiable in open interval $(b, b+k)$ and therefore, by Lagrange's mean theorem, from (28.10), we get

$$\begin{aligned} \phi(h,k) &= kg(b+\theta k), \qquad\qquad 0<\theta<1 \\ &= k\left(f_y(a+h,b+\theta k) - f_y(a,b+\theta k)\right). \end{aligned} \tag{28.11}$$

Again, since f_{xy} exists in a neighbourhood of (a,b), the function f_y is differentiable with respect to x in $(a, a+h)$ and therefore, by Lagrange's mean theorem, from (28.11), we get

$$\phi(h,k) = hkf_{xy}(a+\hat{\theta}h, b+\theta k), \qquad\qquad 0<\hat{\theta}<1$$

or

$$\begin{aligned} &f(a+h,b+k) - f(a+h,b) - f(a,b+k) + f(a,b) \\ &\qquad = hkf_{xy}(a+\hat{\theta}h, b+\theta k), \quad 0<\theta<1 \text{ and } 0<\hat{\theta}<1. \end{aligned}$$

So

$$\frac{1}{k}\left(\frac{f(a+h,b+k) - f(a,b+k)}{h} - \frac{f(a+h,b) - f(a,b)}{h}\right) = f_{xy}(a+\hat{\theta}h, b+\theta k).$$

Since f_{xy} exists in a neighbourhood of (a,b), this gives when $h \to 0$, we get

$$\frac{f_x(a,b+k) - f_x(a,b)}{k} = \lim_{h\to 0} f_{xy}(a+\hat{\theta}h, b+\theta k).$$

Again, since f_{xy} is continuous at (a,b), taking the limit when $k \to 0$, we have

$$\lim_{k\to 0}\frac{f_x(a,b+k) - f_x(a,b)}{k} = \lim_{k\to 0}\lim_{h\to 0} f_{xy}(a+\hat{\theta}h, b+\theta k).$$

or

$$f_{yx}(a,b) = f_{xy}(a,b).$$

□

Note 3 The conditions of Theorem 1 imply that f_{xy} and f_{yx} must be defined in some neighbourhood of (a,b), and this of course also implies that f_x, f_y and f itself must also be defined at every point of some neighbourhood of (a,b). But we do not require f_x or f_y or even f to be continuous.

Note 4 Theorem 1 presents sufficient conditions for the equality of f_{xy} and f_{yx} at a point (a,b).

Corollary 2 If f_{xy} and f_{yx} are both continuous at (a,b), then

$$f_{yx}(a,b) = f_{xy}(a,b).$$

□

Example 5 If $f(x,y) = x^2 + xy - y^2$, find $f_x, f_y, f_{xx}, f_{xy}, f_{yy}$ and f_{yx} at $(1,2)$ by applying the definition of partial derivatives and show that $f_{xy} = f_{yx}$.

Solution. We first evaluate $f(1,2) = 1^2 + 1 \cdot 2 - 2^2 = -1$.

- $$\begin{aligned} f_x(1,2) = \frac{\partial f(1,2)}{\partial x} &= \lim_{\Delta x \to 0} \frac{f(1+\Delta x, 2) - f(1,2)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{(1+\Delta x)^2 + (1+\Delta x)\cdot 2 - 2^2 - (-1)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{1 + 2\cdot \Delta x + (\Delta x)^2 + 2 + 2\cdot \Delta x - 4 + 1}{\Delta x} \\ &= \lim_{\Delta x \to 0} (4 + \Delta x) \\ &= 4. \end{aligned}$$

- $$\begin{aligned} f_y(1,2) = \frac{\partial f(1,2)}{\partial y} &= \lim_{\Delta y \to 0} \frac{f(1, 2+\Delta y) - f(1,2)}{\Delta y} \\ &= \lim_{\Delta y \to 0} \frac{1^2 + 1\cdot(2+\Delta y) - (2+\Delta y)^2 - (-1)}{\Delta y} \\ &= \lim_{\Delta y \to 0} \frac{1 + 2 + \Delta y - \left(4 + 4\cdot \Delta y + (\Delta y)^2\right) - (-1)}{\Delta y} \\ &= \lim_{\Delta y \to 0} \frac{-(\Delta y)^2 - 3\cdot \Delta y}{\Delta y} \\ &= \lim_{\Delta y \to 0} (-\Delta y - 3) \\ &= -3. \end{aligned}$$

- $$\begin{aligned}
f_{xx}(1,2) &= \frac{\partial^2 f(1,2)}{\partial x^2} \\
&= \lim_{\Delta x \to 0} \frac{f(1+2\Delta x, 2) - 2f(1+\Delta, 2) - f(1,2)}{(\Delta x)^2} \\
&= \lim_{\Delta x \to 0} \frac{\left((1+2\Delta x)^2 + (1+2\Delta x)\cdot 2 - 2^2\right) - 2\cdot\left((1+\Delta x)^2 + (1+\Delta x)\cdot 2 - 2^2\right) + (-1)}{(\Delta x)^2} \\
&= \lim_{\Delta x \to 0} \frac{-2\left(1 + (\Delta x)^2 + 2\cdot\Delta x + 2 + 2\cdot\Delta x - 4\right)}{(\Delta x)^2} \\
&= \lim_{\Delta x \to 0} \frac{2\,(\Delta x)^2}{(\Delta x)^2} \\
&= \lim_{\Delta x \to 0} 2 \\
&= 2.
\end{aligned}$$

- $$\begin{aligned}
f_{yy}(1,2) &= \frac{\partial^2 f(1,2)}{\partial y^2} \\
&= \lim_{\Delta y \to 0} \frac{f(1, 2+2\Delta y) - 2f(1, 2+\Delta y) - f(1,2)}{(\Delta y)^2} \\
&= \lim_{\Delta y \to 0} \frac{f(1, 2+2\Delta y) - 2f(1, 2+\Delta y) - f(1,2)}{(\Delta y)^2} \\
&= \lim_{\Delta y \to 0} \frac{1^2 + 1\cdot(2+2\Delta y) - (2+2\Delta y)^2 - 2\cdot\left(1^2 + 1\cdot(2+\Delta y) - (2+\Delta y)^2\right) + (-1)}{(\Delta y)^2} \\
&= \lim_{\Delta y \to 0} \frac{-6 - 2\Delta y + 2\cdot\left(4 + (\Delta y)^2 + 4\cdot\Delta y\right) - 1}{(\Delta y)^2} \\
&= \lim_{\Delta y \to 0} \frac{-2\,(\Delta y)^2}{(\Delta y)^2} \\
&= \lim_{\Delta y \to 0} -2 \\
&= -2.
\end{aligned}$$

- $$\begin{aligned}
f_{xy}(1,2) &= \frac{\partial^2 f(1,2)}{\partial x \partial y} \\
&= \lim_{\Delta y \to 0} \lim_{\Delta x \to 0} \frac{f(1+\Delta x, 2+\Delta y) - f(1+\Delta x, 2) - f(1, 2+\Delta y) + f(1,2)}{\Delta y \Delta x}.
\end{aligned}$$

Now,

$$
\begin{aligned}
&f(1+\Delta x, 2+\Delta y) - f(1+\Delta x, 2) - f(1, 2+\Delta y) + f(1,2) \\
&= (1+\Delta x)^2 + (1+\Delta x)\cdot(2+\Delta y) - (2+\Delta y)^2 - \big((1+\Delta x)^2 + (1+\Delta x)\cdot 2 - 2^2\big) \\
&\quad - \big(1^2 + 1\cdot(2+\Delta y) - (2+\Delta y)^2\big) + (-1) \\
&= 4\Delta x \Delta y.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
f_{xy}(1,2) = \frac{\partial^2 f(1,2)}{\partial x \partial y} &= \lim_{\Delta y\to 0}\lim_{\Delta x\to 0}\frac{4\Delta x\Delta y}{\Delta y \Delta x} \\
&= \lim_{\Delta y\to 0}\lim_{\Delta x\to 0} 4 \\
&= 4.
\end{aligned}
$$

•

$$
\begin{aligned}
f_{yx}(1,2) &= \frac{\partial^2 f(1,2)}{\partial y \partial x} \\
&= \lim_{\Delta x\to 0}\lim_{\Delta y\to 0}\frac{f(1+\Delta x, 2+\Delta y) - f(1+\Delta x, 2) - f(1, 2+\Delta y) + f(1,2)}{\Delta x\Delta y} \\
&= \lim_{\Delta x\to 0}\lim_{\Delta y\to 0}\frac{4\Delta y\Delta x}{\Delta y \Delta x} \\
&= \lim_{\Delta y\to 0}\lim_{\Delta x\to 0} 4 \\
&= 4.
\end{aligned}
$$

Hence, it is clear that

$$
\frac{\partial^2 f(1,2)}{\partial x \partial y} = \frac{\partial^2 f(1,2)}{\partial y \partial x}.
$$

□

28.4.2 Worked examples

Example 6 If $u = x^2 \tan^{-1}\dfrac{y}{x} - y^2 \tan^{-1}\dfrac{x}{y}$, prove that

$$
\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2 - y^2}{x^2 + y^2}.
$$

Solution. Given

$$
u = x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{x}{y}. \tag{28.12}
$$

Differentiating (28.12) partially with respect to y, we have

$$\begin{aligned}\frac{\partial u}{\partial y} &= x^2\frac{1}{1+\frac{y^2}{x^2}}\cdot\frac{1}{x} - y^2\frac{1}{1+\frac{x^2}{y^2}}\cdot\left(-\frac{x}{y^2}\right) - 2y\tan^{-1}\frac{x}{y}\\ &= \frac{x^3}{x^2+y^2} + \frac{xy^2}{x^2+y^2} - 2y\tan^{-1}\frac{x}{y}\\ &= \frac{x(x^2+y^2)}{x^2+y^2} - 2y\tan^{-1}\frac{x}{y}\\ &= x - 2y\tan^{-1}\frac{x}{y} \qquad (28.13)\end{aligned}$$

and differentiating (28.13) partially with respect to x, we have

$$\begin{aligned}\frac{\partial^2 u}{\partial x\partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(x - 2y\tan^{-1}\frac{x}{y}\right)\\ &= 1 - 2y\frac{1}{1+\frac{x^2}{y^2}}\cdot\frac{1}{y}\\ &= 1 - \frac{2y^2}{x^2+y^2}\\ &= \frac{x^2-y^2}{x^2+y^2}.\end{aligned}$$

□

Example 7 If $u = (x^2+y^2+z^2)^{-1/2}$, prove that

1. $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = -u;$$

2. $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

Solution. Given

$$u = \frac{1}{(x^2+y^2+z^2)^{1/2}}. \qquad (28.14)$$

1. Differentiating (28.14) partially with respect to x, we have

$$\begin{aligned}\frac{\partial u}{\partial x} &= -\frac{1}{2(x^2+y^2+z^2)^{3/2}}\cdot 2x\\ &= \frac{-x}{(x^2+y^2+z^2)^{3/2}}.\end{aligned}$$

Therefore, we have

$$x\frac{\partial u}{\partial x} = -\frac{x^2}{(x^2+y^2+z^2)^{3/2}}.$$

Similarly, we have

$$y\frac{\partial u}{\partial y} = -\frac{y^2}{(x^2+y^2+z^2)^{3/2}} \tag{28.15}$$

and

$$z\frac{\partial u}{\partial z} = -\frac{z^2}{(x^2+y^2+z^2)^{3/2}}. \tag{28.16}$$

Adding (28.14), (28.15) and (28.16), we get

$$\begin{aligned} x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} &= -\frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^{3/2}} \\ &= -\frac{1}{(x^2+y^2+z^2)^{1/2}} = -u. \end{aligned}$$

2. Now differentiating $\frac{\partial u}{\partial x}$ partially with respect to x, we have

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(-\frac{x}{(x^2+y^2+z^2)^{3/2}}\right) \\ &= -\left(\frac{1\cdot(x^2+y^2+z^2)^{3/2} - x\cdot\frac{3}{2}(x^2+y^2+z^2)^{1/2}\cdot 2x}{(x^2+y^2+z^2)^3}\right) \\ &= -\frac{(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3}(x^2+y^2+z^2-3x^2) \\ &= -\frac{1}{(x^2+y^2+z^2)^{5/2}}(y^2+z^2-2x^2). \end{aligned} \tag{28.17}$$

Similarly, we have

$$\frac{\partial^2 u}{\partial y^2} = -\frac{1}{(x^2+y^2+z^2)^{5/2}}(z^2+x^2-2y^2) \tag{28.18}$$

and

$$\frac{\partial^2 u}{\partial z^2} = -\frac{1}{(x^2+y^2+z^2)^{5/2}}(x^2+y^2-2z^2). \tag{28.19}$$

Adding (28.17), (28.18) and (28.19), we obtain

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} &= -\frac{1}{(x^2+y^2+z^2)^{5/2}}\big(y^2+z^2-2x^2+z^2+x^2 \\ &\quad -2y^2+x^2+y^2-2z^2\big) \\ &= 0. \end{aligned}$$

□

Example 8 If $\dfrac{x^2}{a^2+u}+\dfrac{y^2}{b^2+u}+\dfrac{z^2}{c^2+u}=1$, prove that

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=2\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right).$$

Solution. Differentiating both sides of the given relation partially with respect to x, we have

$$\frac{2x}{a^2+u}-\frac{x^2}{(a^2+u^2)}\frac{\partial u}{\partial x}-\frac{y^2}{(b^2+u)^2}\frac{\partial u}{\partial x}-\frac{z^2}{(c^2+u)^2}\frac{\partial u}{\partial x}=0.$$

Therefore, we have

$$\frac{2x}{a^2+u}-\left(\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u^2)}\right)\frac{\partial u}{\partial x}=0$$

or

$$\frac{\partial u}{\partial x}=\frac{\dfrac{2x}{a^2+u}}{\dfrac{x^2}{(a^2+u)^2}+\dfrac{y^2}{(b^2+u)^2}+\dfrac{z^2}{(c^2+u)^2}}. \tag{28.20}$$

Therefore, we have

$$\left(\frac{\partial u}{\partial x}\right)^2=\frac{\dfrac{4x^2}{(a^2+u)^2}}{\left(\dfrac{x^2}{(a^2+u)^2}+\dfrac{y^2}{(b^2+u)^2}+\dfrac{z^2}{(c^2+u)^2}\right)^2}.$$

So

$$\begin{aligned}\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2&=\frac{4\left(\dfrac{x^2}{(a^2+u)^2}+\dfrac{y^2}{(b^2+u)^2}+\dfrac{z^2}{(c^2+u)^2}\right)}{\left(\dfrac{x^2}{(a^2+u)^2}+\dfrac{y^2}{(b^2+u)^2}+\dfrac{z^2}{(c^2+u)^2}\right)^2}\\&=\frac{4}{\left(\dfrac{x^2}{(a^2+u)^2}+\dfrac{y^2}{(b^2+u)^2}+\dfrac{z^2}{(c^2+u)^2}\right)}. \end{aligned}\tag{28.21}$$

Now multiplying both sides of (28.20) by $2x$, we get

$$2x\frac{\partial u}{\partial x}=\frac{\dfrac{4x^2}{a^2+u}}{\left(\dfrac{x^2}{(a^2+u)^2}+\dfrac{y^2}{(b^2+u)^2}+\dfrac{z^2}{(c^2+u)^2}\right)}.$$

So

$$2\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right)=\frac{4\left(\dfrac{x^2}{(a^2+u)^2}+\dfrac{y^2}{(b^2+u)^2}+\dfrac{z^2}{(c^2+u)^2}\right)}{\left(\dfrac{x^2}{(a^2+u)^2}+\dfrac{y^2}{(b^2+u)^2}+\dfrac{z^2}{(c^2+u)^2}\right)^2}. \tag{28.22}$$

But from the given relation $\dfrac{x^2}{(a^2+u)^2}+\dfrac{y^2}{(b^2+u)^2}+\dfrac{z^2}{(c^2+u)^2}=1$. Therefore, from (28.22)

$$2\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right)=\frac{4}{\dfrac{x^2}{(a^2+u)^2}+\dfrac{y^2}{(b^2+u)^2}+\dfrac{z^2}{(c^2+u)^2}}. \tag{28.23}$$

Hence from (28.21) and (28.23), we immediately get

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=2\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right).$$

Example 9 If $\theta=t^ne^{-r^2/4t}$, what value of n will make

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)=\frac{\partial\theta}{\partial t}.$$

Solution. Differentiating θ partially with respect to t and r, we get

$$\begin{aligned}\frac{\partial\theta}{\partial t}&=nt^{n-1}e^{-r^2/4t}+t^ne^{-r^2/4t}\left(\frac{r^2}{4t^2}\right)\\&=e^{-r^2/4t}t^{n-2}\left(nt+\frac{r^2}{4}\right)\end{aligned} \tag{28.24}$$

and

$$\begin{aligned}\frac{\partial\theta}{\partial r}&=t^ne^{-r^2/4t}\left(-\frac{2r}{4t}\right)=t^ne^{-r^2/4t}\left(-\frac{r}{2t}\right)\\&=-\frac{r}{2}t^{n-1}e^{-r^2/4t}.\end{aligned} \tag{28.25}$$

Now

$$\begin{aligned}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)&=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\left(-\frac{r}{2}t^{n-1}e^{-r^2/4t}\right)\right)\\&=-\frac{t^{n-1}}{2r^2}\frac{\partial}{\partial r}(r^3e^{-r^2/4t}) && \text{using (28.25)}\\&=-\frac{t^{n-1}}{2r^2}\cdot\left(3r^2e^{-r^2/4t}-\frac{2r^4}{4t}e^{-r^2/4t}\right)\\&=\frac{1}{4}t^{n-2}e^{-r^2/4t}\cdot(-6t+r^2).\end{aligned} \tag{28.26}$$

Putting values from (28.24) and (28.26) in the given relation, we get

$$\frac{1}{4}t^{n-2}e^{-r^2/4t}(r^2-6t) = e^{-r^2/4t}t^{n-2}\left(nt+\frac{r^2}{4}\right).$$

Therefore,

$$\frac{r^2}{4}-\frac{3}{2}t = nt+\frac{r^2}{r}$$

or

$$n = -\frac{3}{2}.$$

□

Example 10 If $u = e^{xyz}$, show that

$$\frac{\partial^3 u}{\partial x\partial y\partial z} = (1+3xyz+^2y^2z^2)e^{xyz}$$

Solution. Given $u = e^{xyz}$. Differentiating u partially with respect to z, we have

$$\frac{\partial u}{\partial z} = xye^{xyz}. \tag{28.27}$$

Then

$$\begin{aligned}\frac{\partial^2 u}{\partial y\partial z} &= \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right)\\ &= \frac{\partial}{\partial y}\left(xye^{xyz}\right) && \text{using (28.27)}\\ &= x\frac{\partial}{\partial y}(ye^{xyz})\\ &= x\cdot(yzxe^{xyz}+e^{xyz})\\ &= e^{xyz}(x^2yz+x).\end{aligned} \tag{28.28}$$

Hence

$$\begin{aligned}\frac{\partial^3 u}{\partial x\partial y\partial z} &= \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial y\partial z}\right)\\ &= \frac{\partial}{\partial x}(e^{xyz}(x^2yz+x)) && \text{using (28.28)}\\ &= e^{xyz}(2xyz+1)+yze^{xyz}(x^2yz+x)\\ &= e^{xyz}(2xyz+1+x^2y^2z^2+xyz)\\ &= e^{xyz}(1+3xyz+x^2y^2z^2).\end{aligned}$$

□

28.5 Differentiability

Let $f(x, y)$ be a function of two independent variables x and y, defined in some neighbourhood of a point (a, b). Then the function $f(x, y)$ is said to be differentiable at (a, b), if there exists two constants α and β depending on f and (a, b) such that

$$\lim_{(h,k)\to(0,0)} \frac{f(a+h, b+k) - f(a, b) - \alpha h - \beta k}{\sqrt{h^2+k^2}} = 0. \tag{28.29}$$

From (28.29), when $k = 0$, we get

$$\lim_{h\to 0} \frac{f(a+h, b) - f(a, b) - \alpha h}{h} = 0$$

or

$$\lim_{h\to 0} \frac{f(a+h, b) - f(a, b)}{h} = \alpha$$

or

$$f_x(a, b) = \alpha.$$

Again, from (28.29), when $h = 0$, we get

$$\lim_{k\to 0} \frac{f(a, b+k) - f(a, b) - \beta k}{k} = 0$$

or

$$\lim_{k\to 0} \frac{f(a, b+k) - f(a, b)}{k} = \beta$$

or

$$f_y(a, b) = \beta.$$

Thus, the constants α and β are the partial derivatives of f with respect to x and y, respectively. Hence, a function which is differentiable at a point possesses the first order partial derivative.

Theorem 3 If $f(x, y)$ is differentiable at a point (a, b), then it is continuous at (a, b) but the converse need not be true.

Proof. Since f(x,y) is differentiable at (a,b), then we have

$$\begin{aligned}
\lim_{(h,k)\to(0,0)} \frac{f(a+h, b+k) - f(a, b) - hf_x(a, b) - kf_y(a, b)}{\sqrt{h^2+k^2}} &= 0,\\
\lim_{(h,k)\to(0,0)} f(a+h, b+k) - f(a, b) - hf_x(a, b) - kf_y(a, b) &= 0,\\
\lim_{(h,k)\to(a,b)} f(a+h, b+k) &= f(a, b).
\end{aligned}$$

Hence, $f(x, y)$ is continuous at (a, b). $\square$

Example 12 illustrates the converse part of Theorem 3. Before examining the claim above, we first study Example 11.

Example 11 Let

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & (x,y) \neq (0,0), \\ 0 & (x,y) = (0,0). \end{cases}$$

Show that both partial derivatives $f_x(0,0)$ and $f_y(0,0)$ exist but the function is not continuous at the origin.

Solution. Using the definition of partial derivatives, we have

$$\lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$

and

$$\lim_{k\to 0} \frac{f(0,0+k)-f(0,0)}{h} = \lim_{k\to 0} \frac{0}{k} = 0.$$

Thus, partial derivatives exist at the origin.

Now, for testing the continuity at the origin, if we let the path of approach along the line $y = mx$, we get

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{x\cdot(mx)}{x^2+m^2x^2} = \frac{m}{1+m^2}$$

so that the limit depends on the value of m. Therefore, the limit is not unique and is different for the different paths. It follows that the limit does not exist. Hence, the function $f(x,y)$ is not continuous at the origin.

□

Example 12 Let

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & (x,y) \neq (0,0), \\ 0 & (x,y) = (0,0). \end{cases}$$

Show that the function $f(x,y)$ is continuous and partial derivatives $f_x(0,0)$ and $f_y(0,0)$ exist but are not differentiable at $(0,0)$.

Solution. Firstly, Example 11 showed the continuity of the function f. Now, by the definition of partial derivatives, we have

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$

and

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,0+k)-f(0,0)}{h} = \lim_{k\to 0} \frac{0}{k} = 0.$$

Therefore, we have

$$\begin{aligned}
&\lim_{(h,k)\to(0,0)} \frac{f(0+h,0+k)-f(0,0)-hf_x(0,0)-kf_y(0,0)}{\sqrt{h^2+k^2}} \\
&= \lim_{(h,k)\to(0,0)} \frac{f(h,k)}{\sqrt{h^2+k^2}} \\
&= \lim_{(h,k)\to(0,0)} \frac{1}{\sqrt{h^2+k^2}} \frac{hk}{\sqrt{h^2+k^2}} \\
&= \lim_{(h,k)\to(0,0)} \frac{hk}{h^2+k^2}.
\end{aligned}$$

By letting $(h,k)\to(0,0)$ along the straight line $k=mh$, we have

$$\begin{aligned}
&\lim_{(h,k)\to(0,0)} \frac{f(0+h,0+k)-f(0,0)-hf_x(0,0)-kf_y(0,0)}{\sqrt{h^2+k^2}} \\
&= \lim_{(h,k)\to(0,0)} \frac{h\cdot(mh)}{h^2+(mh)^2} \\
&= \lim_{(h,k)\to(0,0)} \frac{h^2}{h^2}\cdot\frac{m}{1+m^2} \\
&= \frac{m}{(1+m^2)} \neq 0.
\end{aligned}$$

Hence, the function $f(x,y)$ is not differentiable at $(0,0)$. $\square$

28.6 Total differential coefficient

Let us consider the function $u = f(x,y)$, where x and y are functions of a third variable t given by the relations

$$x = \phi(t) \qquad \text{and} \qquad y = \psi(t)$$

characterizing u as a composite function of t.

At first, let us make a clear distinction between the total differential coefficient and the total differential coefficient of u:

- The function u can be expressed in terms of t alone by the substitution of the values of x and y in $u(x,y)$. Thus we can regard u as a function of the single variable t and find the ordinary differential coefficient $\dfrac{du}{dt}$, which is known as the total differential coefficient.
- To distinguish $\dfrac{du}{dt}$ from the partial derivatives $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial u}{\partial y}$, it is called the total differential coefficient of u.

Our aim here is to find $\dfrac{du}{dt}$ without actually substituting the values of x and y in terms of t by $u = f(x,y)$ that consists of the following steps:

- Let Δx, Δy and Δu be the small increments in the values of x, y and u respectively, corresponding to a small increment Δt in t.

  Therefore,
  $$u + \Delta u = f(x + \Delta x, y + \Delta y)$$
  and
  $$\Delta u = f(x + \Delta x, y + \Delta y) - f(x,y).$$
  By adding and subtracting $f(x, y + \Delta y)$, Δu can also be written as
  $$\Delta u = (f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y)) + (f(x, y + \Delta y) - f(x,y)).$$
  Hence
  $$\begin{aligned}\frac{\Delta u}{\Delta t} &= \frac{(f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y))}{\Delta t} + \frac{(f(x, y + \Delta y) - f(x,y))}{\Delta t} \\ &= \left(\frac{f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y)}{\Delta x}\right) \cdot \frac{\Delta x}{\Delta t} + \left(\frac{f(x, y + \Delta y) - f(x,y)}{\Delta y}\right) \cdot \frac{\Delta y}{\Delta t}.\end{aligned}$$
- Now, let Δt and consequently $\Delta x, \Delta y$, and Δu tend to zero, then taking limits accordingly
  $$\begin{aligned}\lim_{\Delta t \to 0} \frac{\Delta u}{\Delta t} = &\lim_{\Delta x \to 0} \left(\frac{f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y)}{\Delta x}\right) \cdot \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} \\ &+ \lim_{\Delta y \to 0} \left(\frac{f(x, y + \Delta y) - f(x,y)}{\Delta y}\right) \cdot \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t}. \qquad (28.30)\end{aligned}$$

- Now by the definition of partial derivatives

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y)}{\Delta x} = \frac{\partial f(x, y+\Delta y)}{\partial x}$$

but Δy tends to zero with Δx and it is under supposition that x is the variable that varies whereas the variable y remains constant, therefore, the above limit is the differential coefficient of $f(x, y)$ or u with respect to x. Therefore,

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y)}{\Delta x} = \frac{\partial f(x, y)}{\partial x} = \frac{\partial u}{\partial x}.$$

Also

$$\lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y} = \frac{\partial f(x, y)}{\partial y} = \frac{\partial u}{\partial y}.$$

Hence (28.30) reduces to

$$\frac{du}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

or

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} \tag{28.31}$$

In general, if $u = f(x_1, x_2, \cdots, x_n)$ and $x_1, x_2, \cdots, x_n$ are all functions of t, we can prove that

$$\frac{du}{dt} = \frac{\partial u}{\partial x_1} \cdot \frac{dx_1}{dt} + \frac{\partial u}{\partial x_2} \cdot \frac{dx_2}{dt} + \cdots \frac{\partial u}{\partial x_n} \cdot \frac{dx_n}{dt}. \tag{28.32}$$

This is called the chain rule of partial differentiation. Equation (28.32) says that the total differential is the sum of the partial differentials.

Corollary 4 If $u = (x, y)$, where x and y are functions of two other variables, say t_1, t_2 given by the relations

$$x = \phi(t_1, t_2) \qquad \text{and} \qquad y = \psi(t_1, t_2),$$

then

$$\frac{\partial u}{\partial t_1} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t_1} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial t_1}$$

and

$$\frac{\partial u}{\partial t_2} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t_2} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial t_2}.$$

Proof. To obtain $\dfrac{\partial u}{\partial t_1}$, we regard t_2 as a constant so that x and y may be supposed to be functions of t_1 alone. Hence by replacing the total differential coefficients by partial differential coefficients and t by t_1 in (28.31), we get

$$\frac{\partial u}{\partial t_1} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t_1} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial t_1}.$$

Similarly on replacing t by t_2, we have

$$\frac{\partial u}{\partial t_2} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t_2} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial t_2}.$$

□

Note 5 If values of t_1 and t_2 are given by the relations

$$t_1 = f_1(x, y) \qquad \text{and} \qquad t_2 = f_2(x, y),$$

then we have

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t_1} \cdot \frac{\partial t_1}{\partial x} + \frac{\partial u}{\partial t_2} \cdot \frac{\partial t_2}{\partial x}$$

and

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial t_1} \cdot \frac{\partial t_1}{\partial y} + \frac{\partial u}{\partial t_2} \cdot \frac{\partial t_2}{\partial y}.$$

These results are useful for change of variables.

Corollary 5 If $u = f(x, y)$, where y is itself a function of x, then

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}.$$

Proof. Since u is a function of x alone, replacing t by x in above relation (28.31) produces

$$\frac{du}{dx} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}.$$

Therefore,

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} \tag{28.33}$$

□

Example 13 Find the total differential of a rectangle.

Solution. Let $\mathcal{A}$ be the area of the rectangle given by $\mathcal{A} = xy$.

An increase in the dimensions of the rectangle to $x+\Delta x$ and $y+\Delta y$ produces a change in the area

$$\begin{aligned} \Delta A &= (x + \Delta x) \cdot (y + \Delta y) - xy \\ &= x\Delta y + y\Delta x + \Delta x \Delta y. \end{aligned}$$

The differential estimate for this change in area is

$$d\mathcal{A} = \frac{\partial \mathcal{A}}{\partial x} \cdot \Delta x + \frac{\partial \mathcal{A}}{\partial y} \cdot \Delta y = y\Delta x + x\Delta y.$$

Hence, the error of our estimate, the difference between the actual change and the estimated change, is the difference

$$\Delta\mathcal{A} - d\mathcal{A} = \Delta x \Delta y,$$

as illustrated in Figure 28.2.

□

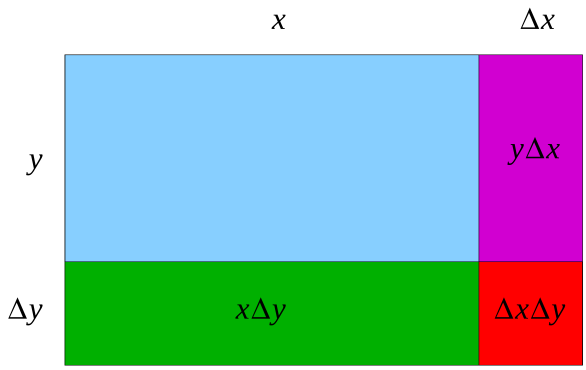

Figure 28.2: Graph of $\Delta\mathcal{A} - d\mathcal{A} = \Delta x \Delta y$ (in red).

Example 14 Find the differential of $f(x, y) = x^3 y - x^2 y^2$.

Solution. We have

$$df = \frac{\partial f}{\partial x} \cdot \Delta x + \frac{\partial f}{\partial y} \cdot \Delta y = \left(3x^2 y - 2xy^2\right) \Delta x + \left(x^3 - 2x^2 y\right) \Delta y.$$

□

Example 15 Compute Δu and du for

$$u(x, y) = x^2 - 3xy + 2y^2$$

at $x = 1, y = -3, \Delta x = -0.3$ and $\Delta y = 0.2$.

Solution. We have

$$\begin{aligned}
\Delta u &= f(x + \Delta x, y + \Delta y) - f(x, y) \\
&= (x + \Delta x)^2 - 3(x + \Delta x)(y + \Delta y) + 2(y + \Delta y)^2 - x^2 + 3xy - 2y^2 \\
&= x^2 + 2x\Delta x + (\Delta x)^2 - 3(xy + x\Delta y + y\Delta x + \Delta x \Delta y) \\
&\qquad + 2(y^2 + 2y\Delta y + (\Delta y)^2) - x^2 + 3xy - 2y^2 \\
&= 2x\Delta x + (\Delta x)^2 - 3x\Delta y - 3\Delta x \Delta y + 4y\Delta y + 2\left(\Delta y\right)^2.
\end{aligned}$$

Substituting $x = 1, y = -3, \Delta x = -0.3$ and $\Delta y = 0.2.$ into the above expression gives $\Delta u = -7.15$.

Now, in turn, we have

$$\begin{aligned}
du &= \frac{\partial u}{\partial x} \cdot \Delta x + \frac{\partial u}{\partial y} \cdot \Delta y \\
&= (2x - 3y)\,\Delta x + (-3x + 4y)\,\Delta y \\
&= (4 + 9) \cdot (-0.3) + (-6 - 12) \cdot (0.2) \\
&= -3.9 - 3.6 \\
&= -7.5.
\end{aligned}$$

□

28.6.1 First differential coefficient of an implicit function

If $u = f(x, y)$, where x and y are functions of t, then we have

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}. \tag{28.34}$$

If $t = x$, that is, y is a function of x, then from (28.34), we have

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}. \tag{28.35}$$

But if an implicit function of x and y is of the form $f(x, y) = c$ or 0, then

$$\frac{du}{dx} = 0.$$

Hence from (28.35), we have

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} = 0$$

or

$$\frac{dy}{dx} = -\frac{\dfrac{\partial u}{\partial x}}{\dfrac{\partial u}{\partial y}} \qquad \text{or} \qquad -\frac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial y}}. \tag{28.36}$$

28.6.2 Second differential coefficient of an implicit function

Let $f(x, y) = 0$ be an implicit function of x and y, then from (28.36), we have

$$\frac{dy}{dx} = -\frac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial y}}. \tag{28.37}$$

Here, we are concerned with the second differential coefficient, therefore let us denote $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial^2 f}{\partial x^2}, \dfrac{\partial^2 f}{\partial x \partial y}$ and $\dfrac{\partial^2 f}{\partial y^2}$ by p, q, r, s and t, respectively.

Then (28.37) becomes

$$\frac{dy}{dx} = -\frac{p}{q}$$

and

$$\frac{d^2y}{dx^2} = -\frac{q\dfrac{dp}{dx} - p\dfrac{dq}{dx}}{q^2}. \tag{28.38}$$

But

$$\begin{aligned}\frac{dp}{dx} = \frac{\partial p}{\partial x} + \frac{\partial p}{\partial y}\frac{dy}{dx} &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y \partial x}\frac{dy}{dx} \\ &= r + s\left(-\frac{p}{q}\right)\end{aligned}$$

or

$$\frac{dp}{dx} = \frac{qr - ps}{q}.$$

Also

$$\begin{aligned}\frac{dq}{dx} = \frac{\partial q}{\partial x} + \frac{\partial q}{\partial y}\frac{dy}{dx} &= \frac{\partial^2 f}{\partial x \partial} + \frac{\partial^2 y}{\partial y^2}\frac{dy}{dx} \\ &= s + t\left(-\frac{p}{q}\right)\end{aligned}$$

or

$$\frac{dq}{dx} = \frac{sq - pt}{q}.$$

Substituting these values in relation (28.38), we have

$$\frac{d^2y}{dx^2} = -\frac{q^2r - 2pqs + p^2t}{q^3}. \tag{28.39}$$

28.6.3 Worked examples

Example 16 If $u = x^2 + y^2$, where $x = at^2$ and $y = 2at$, find $\frac{du}{dt}$.

Solution. We recognize that,

$$\begin{aligned}\frac{du}{dt} &= \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} \\ &= 2x \cdot 2at + 2y \cdot 2a \\ &= 4xat + 4ay.\end{aligned}$$

Substituting the values of x and y in terms of t, we have

$$\begin{aligned}\frac{du}{dt} &= 4(at^2)at + 4a(2at) \\ &= 4a^2t^3 + 8a^2t.\end{aligned}$$

□

Example 17 If $u = \sin(x^2 + y^2)$, where $a^2x^2 + b^2y^2 = c^2$, find $\frac{du}{dx}$.

Solution. Here u is a function of x and y, where y is a function of x, so we have

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}. \tag{28.40}$$

Given $u = \sin(x^2 + y^2)$. Then

$$\frac{\partial u}{\partial x} = \cos(x^2 + y^2) \cdot 2x \tag{28.41}$$

and

$$\frac{\partial u}{\partial y} = \cos(x^2 + y^2) \cdot 2y. \tag{28.42}$$

Also differentiating $a^2x^2 + b^2y^2 = c^2$ with respect to x, we have

$$2a^2x + 2b^2y\frac{dy}{dx} = 0$$

So

$$\frac{dy}{dx} = -\frac{a^2x}{b^2y}. \tag{28.43}$$

Substituting the values of (28.41), (28.42) and (28.43) in (28.40), we have

$$\frac{du}{dx} = 2x\cos(x^2 + y^2) + 2y\cos(x^2 + y^2)\left(-\frac{a^2x}{b^2y}\right)$$

or

$$\frac{du}{dx} = \frac{2x}{b^2}\cos(x^2 + y^2) \cdot (b^2 - a^2).$$

□

Example 18 If $u = x\log_e xy$, where $x^3 + y^3 + 3xy = 1$, find $\dfrac{du}{dx}$.

Solution. Given

$$u = x\log_e xy. \tag{28.44}$$

We know

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}. \tag{28.45}$$

Now from (28.44), we have

$$\frac{\partial u}{\partial x} = x \cdot \frac{1}{xy} \cdot y + \log_e xy = 1 + \log_e xy \tag{28.46}$$

and

$$\frac{\partial u}{\partial y} = x\frac{1}{xy} \cdot x = \frac{x}{y}. \tag{28.47}$$

Again it is given that $x^3 + y^3 + 3xy = 1$.

Differentiating it with respect to x, we have

$$3x^2 + 3y^2\frac{dy}{dx} + 3\left(x\frac{dy}{dx} + y \cdot 1\right) = 0$$

or

$$3(y^2+x)\frac{dy}{dx}+3(x^2+y)=0$$

or

$$\frac{dy}{dx}=-\frac{x^2+y}{y^2+x}. \tag{28.48}$$

Substituting the values of (28.46), (28.47) and (28.48) in (28.45), we get

$$\begin{aligned}\frac{du}{dx}&=(1+\log_e xy)+\frac{x}{y}\left(-\left(\frac{x^2+y}{y^2+x}\right)\right)\\&=1+\log_e xy-\frac{x\left(x^2+y\right)}{y\left(y^2+x\right)}.\end{aligned}$$

□

Example 19 If $u=f(x,y)$, where $x=r\cos\theta$ and $y=r\sin\theta$, prove that the equation $\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}=0$ transforms into $\frac{\partial^2u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2u}{\partial\theta^2}=0$.

Solution. Given $x=r\cos\theta$ and $y=r\sin\theta$.

We have $r=\sqrt{x^2+y^2}$ and $\theta=\tan^{-1}\left(\frac{y}{x}\right)$.

Hence

$$\frac{\partial r}{\partial x}=\frac{2x}{2\sqrt{x^2+y^2}}=\frac{x}{\sqrt{x^2+y^2}}=\frac{x}{r}=\cos\theta. \tag{28.49}$$

Similarly,

$$\frac{\partial r}{\partial y}=\frac{y}{\sqrt{x^2+y^2}}=\sin\theta, \tag{28.50}$$

$$\frac{\partial\theta}{\partial x}=-\frac{y}{x^2+y^2}=-\frac{\sin\theta}{r}, \tag{28.51}$$

$$\frac{\partial\theta}{\partial y}=\frac{x}{x^2+y^2}=\frac{\cos\theta}{r}. \tag{28.52}$$

Now, substituting the values of (28.49) and (28.51) in $\frac{\partial u}{\partial x}$, we have

$$\begin{aligned}\frac{\partial u}{\partial x}&=\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial u}{\partial\theta}\frac{\partial\theta}{\partial x}\\&=\frac{\partial u}{\partial r}\cdot\cos\theta+\frac{\partial u}{\partial\theta}\left(-\frac{\sin\theta}{r}\right).\end{aligned}$$

Thus

$$\frac{\partial}{\partial x}\equiv\left(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right). \tag{28.53}$$

Hence

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)\\
&= \frac{\partial}{\partial x}\left[\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)u\right]\\
&= \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial\theta}\right) \qquad \text{From (28.53)}\\
&= \cos\theta\frac{\partial}{\partial r}\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial\theta}\right)\\
&\quad - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial\theta}\right)\\
&= \cos^2\theta\frac{\partial^2 u}{\partial r^2} - \cos\theta\sin\theta\left(\frac{1}{r}\frac{\partial^2 u}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial u}{\partial\theta}\right)\\
&\quad - \frac{\sin\theta}{r}\left(cos\theta\frac{\partial^2 u}{\partial r\partial\theta} - \sin\theta\frac{\partial u}{\partial r}\right)\\
&\quad + \frac{\sin\theta}{r}\left(\frac{\sin\theta}{r}\frac{\partial^2 u}{\partial\theta^2} + \frac{\cos\theta}{r}\frac{\partial u}{\partial\theta}\right)
\end{aligned}$$

or

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \cos^2\theta\frac{\partial^2 u}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta} + \frac{\sin^2\theta}{r}\frac{\partial u}{\partial r}\\
&\quad + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial\theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2}.
\end{aligned} \tag{28.54}$$

or

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \cos^2\theta\frac{\partial^2 u}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta} + \frac{\sin^2\theta}{r}\frac{\partial u}{\partial r}\\
&\quad + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial\theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2}.
\end{aligned} \tag{28.55}$$

Similarly, we can get

$$\begin{aligned}
\frac{\partial^2 u}{\partial y^2} &= \sin^2\theta\frac{\partial^2 u}{\partial r^2} + \frac{2\sin\theta\cos\theta}{r}\frac{\partial^u}{\partial r\partial\theta} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2}\\
&\quad + \frac{\cos^2\theta}{r}\cdot\frac{\partial u}{\partial r} - \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial\theta}.
\end{aligned} \tag{28.56}$$

Adding (28.55) and (28.56), we get

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= (\cos^2\theta + \sin^2\theta)\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}(\sin^2\theta + \cos^2\theta)\frac{\partial^2\theta}{\partial\theta^2} + \frac{1}{r}(\sin^2\theta + \cos^2\theta)\frac{\partial u}{\partial r}\\
&= \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{1}{r}\frac{\partial u}{\partial r} = 0.
\end{aligned}$$

Hence the transformed equation is:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} = 0.$$

□

Example 20 If $u = f(y-z, z-x, x-y)$, prove that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$$

Solution. Let $y - z = t_1, z - x = t_2, x - y = t_3$. Then $u = f(t_1, t_2, t_3)$, that is, u is a function of t_1, t_2 and t_3, where t_1, t_2, and t_3 are each a function of x, y and z.

Now

$$\frac{\partial t_1}{\partial x} = 0, \frac{\partial t_1}{\partial y} = 1, \frac{\partial t_1}{\partial z} = -1,$$

$$\frac{\partial t_2}{\partial x} = -1, \frac{\partial t_2}{\partial y} = 0, \frac{\partial t_2}{\partial z} = 1$$

and

$$\frac{\partial t_3}{\partial x} = 1, \frac{\partial t_3}{\partial y} = -1, \frac{\partial t_3}{\partial z} = 0.$$

Now applying the chain rule gives

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t_1}\frac{\partial t_1}{\partial x} + \frac{\partial u}{\partial t_2}\frac{\partial t_2}{\partial x} + \frac{\partial u}{\partial t_3}\frac{\partial t_3}{\partial x}.$$

Therefore

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial t_1}\cdot 0 + \frac{\partial u}{\partial t_2}\cdot(-1) + \frac{\partial u}{\partial t_3}(1)\\ &= \frac{\partial u}{\partial t_3} - \frac{\partial u}{\partial t_2} \qquad (28.57)\end{aligned}$$

$$\begin{aligned}\frac{\partial u}{\partial y} &= \frac{\partial u}{\partial t_1}\frac{\partial t_1}{\partial y} + \frac{\partial u}{\partial t_2}\frac{\partial t_2}{\partial y} + \frac{\partial u}{\partial t_3}\frac{\partial t_3}{\partial z}\\ &= \frac{\partial u}{\partial t_1}(1) + \frac{\partial u}{\partial t_2}(0) + \frac{\partial u}{\partial t_3}(-1)\\ &= \frac{\partial u}{\partial t_1} - \frac{\partial u}{\partial t_3} \qquad (28.58)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial u}{\partial z} &= \frac{\partial u}{\partial t_1}\frac{\partial t_1}{\partial z} + \frac{\partial u}{\partial t_2}\frac{\partial t_2}{\partial z} + \frac{\partial u}{\partial t_3}\frac{\partial t_3}{\partial z}\\ &= \frac{\partial u}{\partial t_1}(-1) + \frac{\partial u}{\partial t_2}(1) + \frac{\partial u}{\partial t_3}(0)\\ &= \frac{\partial u}{\partial t_2} - \frac{\partial u}{\partial t_1}. \qquad (28.59)\end{aligned}$$

Adding (28.57), (28.58) and (28.59), we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$$

□

Example 21 If $x^y + y^x = c$, find the value of $\dfrac{dy}{dx}$.

Solution. Let $f(x,y) = x^y + y^x$, then we have

$$\frac{\partial f}{\partial x} = yx^{y-1} + y^x \log_e y$$

and

$$\frac{\partial f}{\partial y} = x^y \log_e x + xy^{x-1}.$$

Then

$$\frac{dy}{dx} = \frac{-\partial f/\partial x}{\partial f/\partial y} = -\frac{y^x \log_e y + yx^{y-1}}{x^y \log_e x + xy^{x-1}}.$$

□

Example 22 If $f(x,y) = 0, \phi(y,z) = 0$, show that

$$\frac{\partial f}{\partial y} \cdot \frac{\partial \phi}{\partial z} \cdot \frac{dz}{dx} = \frac{\partial f}{\partial x} \cdot \frac{\partial \phi}{\partial y}.$$

Solution. Given $f(x,y) = 0$. Then

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}. \tag{28.60}$$

Also $\phi(y,z) = 0$. Then

$$\frac{dy}{dz} = -\frac{\partial \phi/\partial z}{\partial \phi/\partial y}. \tag{28.61}$$

Dividing (28.60) by (28.61), we have

$$\frac{\frac{dy}{dx}}{\frac{dy}{dz}} = \frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \cdot \frac{\frac{\partial \phi}{\partial y}}{\frac{\partial \phi}{\partial z}}$$

or

$$\frac{dz}{dx} = \frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \cdot \frac{\frac{\partial \phi}{\partial y}}{\frac{\partial \phi}{\partial z}}$$

or

$$\frac{\partial f}{\partial y} \cdot \frac{\partial \phi}{\partial z} \cdot \frac{dz}{dx} = \frac{\partial f}{\partial x} \cdot \frac{\partial \phi}{\partial y}.$$

□

Example 23 If $ax^2 + 2hxy + by^2 = 1$, find $\dfrac{d^2y}{dx^2}$.

Solution. Let $f(x, y) = ax^2 + 2hxy + by^2$. Hence

$$\begin{aligned}
p &= \frac{\partial f}{\partial x} = 2(ax + hy) \\
q &= \frac{\partial f}{\partial y} = 2(hx + by) \\
r &= \frac{\partial^2 f}{\partial x^2} = 2a \\
s &= \frac{\partial^2 f}{\partial x \partial y} = 2h \\
t &= \frac{\partial^2 f}{\partial x^2} = 2b
\end{aligned}$$

Now

$$\begin{aligned}
\frac{d^2y}{dx^2} &= -\frac{q^2r - 2qps + p^2t}{q^3} \\
&= -\frac{2^2(hx + by)^2 2a - 2 \cdot 2(h + by) \cdot 2(ax + hy)2h + 2^2(ax + hy)^2 \cdot 2b}{2^3(hx + by)^3} \\
&= -\frac{(hx + by)^2 a - 2(hx + by)(ax + hy)h + (ax + hy)^2 b}{(hx + by)^3}.
\end{aligned}$$

□

28.7 Homogeneous functions

A function $f(x, y)$ of two variables x and y is said to be a homogeneous function of degree n if the sum of the indices of x and y in every term is same and is equal to n.

While judging the degree of the terms, we have to add the indices of x and y. For example the term x^2y^4 is of sixth degree and $\dfrac{x^{10}}{y^4}$ is also of sixth degree.

Let us consider the homogeneous function of nth degree as

$$f(x, y) = a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \cdots + a_{n-1}xy^{n-1} + a_ny^n \tag{28.62}$$

where $a_0, a_1, \cdots, a_n$ are constants. The function $f(x, y)$ can also be expressed as:

$$f(x, y) = x^n \left(a_0 + a_1 \left(\frac{y}{x}\right) + a_2 \left(\frac{y}{x}\right)^2 + \cdots + a_n \left(\frac{y}{x}\right)^n \right) \tag{28.63}$$

and is denoted by

$$x^n f\left(\frac{y}{x}\right).$$

Hence a homogeneous function of degree n can also be expressed as

$$x^n f\left(\frac{y}{x}\right) \qquad \text{or} \qquad y^n f\left(\frac{x}{y}\right).$$

Thus $x^3 \sin\left(\frac{y}{x}\right)$ is a homogeneous function of x and y of degree 3.

In general, the function $f(x_1, x_2, \cdots, x_m)$ of the m variables $x_1, x_2, \cdots, x_m$ can be written as

$$x_i^n F\left(\frac{x_1}{x_i}, \frac{x_2}{x_i}, \cdots, \frac{x_m}{x_i}\right),$$

where $i = 1, 2, \cdots, m$.

28.7.1 Euler's theorem on homogeneous functions

Theorem 6 If $f(x, y)$ is a homogeneous function of x and y of degree n, then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf. \tag{28.64}$$

Proof. Since $f(x, y)$ is a homogeneous function of degree n, it can be expressed as

$$f(x, y) = x^n f\left(\frac{y}{x}\right). \tag{28.65}$$

Now differentiating (28.65) partially with respect to x, we have

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\left(x^n f\left(\frac{y}{x}\right)\right) \\
&= nx^{n-1} f\left(\frac{y}{x}\right) + x^n f\left(\frac{y}{n}\right)\left(\frac{-y}{x^2}\right) && \text{Denoting } \frac{df}{dx} \text{ by } f' \\
&= nx^{n-1} f\left(\frac{y}{x}\right) - x^{n-2} y f'\left(\frac{y}{x}\right).
\end{aligned}$$

Therefore,

$$x\frac{\partial f}{\partial x} = nx^n f\left(\frac{y}{x}\right) - x^{n-1} y f'\left(\frac{y}{x}\right). \tag{28.66}$$

Again differentiating (28.65) partially with respect to y, we get

$$\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(x^n f\left(\frac{y}{x}\right)\right) \\
&= x^n f'\left(\frac{y}{x}\right) \cdot \frac{1}{x} \\
&= x^{n-1} f'\left(\frac{y}{x}\right).
\end{aligned} \tag{28.67}$$

Therefore,

$$y\frac{\partial f}{\partial y} = x^{n-1} y f'\left(\frac{y}{x}\right). \tag{28.68}$$

Adding (28.66) and (28.68), we get

$$\begin{aligned}
x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} &= nx^n f\left(\frac{y}{x}\right) \\
&= nf(x, y). && \text{By (28.65)}
\end{aligned}$$

Hence

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf.$$

In general if $f(x_1, x_2, \cdots, x_m)$ is a homogeneous function of degree n, then

$$x_1\frac{\partial f}{\partial x_1} + x_2\frac{\partial f}{\partial x_2} + \cdots + x_m\frac{\partial f}{\partial x_m} = nf \tag{28.69}$$

□

The proof is similar to that for two variables given above.

Corollary 7 If u is a homogeneous function of degree n, then

1. $x\dfrac{\partial^2 u}{\partial x^2} + y\dfrac{\partial^2 u}{\partial x \partial y} = (n-1)\dfrac{\partial u}{\partial x}$.
2. $x\dfrac{\partial^2 u}{\partial x \partial y} + y\dfrac{\partial^2 u}{\partial y^2} = (n-1)\dfrac{\partial u}{\partial y}$.
3. $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = n(n-1)u$

Proof. Since u is a homogeneous function of degree n, by Euler's theorem, we have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu. \tag{28.70}$$

Differentiating (28.70) partially with respect to x, we have

$$x\frac{\partial^2 u}{\partial x^2} + 1 \cdot \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x \partial y} = n\frac{\partial u}{\partial x}$$

or

$$x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = (n-1)\frac{\partial u}{\partial x}. \tag{28.71}$$

Again differentiating (28.70) partially with respect to y, we have

$$x\frac{\partial^2 u}{\partial y \partial x} + 1 \cdot \frac{\partial u}{\partial y} + y\frac{\partial^2 u}{\partial y^2} = n\frac{\partial u}{\partial y}$$

or

$$x\frac{\partial^2 u}{\partial y \partial x} + y\frac{\partial^2 u}{\partial y^2} = (n-1)\frac{\partial u}{\partial y}. \tag{28.72}$$

Now multiplying (28.71) by x and (28.72) by y and adding, we get

$$x^2\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial x \partial y} + xy\frac{\partial^2 u}{\partial y \partial x} + y^2\frac{\partial^2 u}{\partial y^2} = (n-1) \left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right]$$

or

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=(n-1)\cdot nu \qquad \left[\text{because } \frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 u}{\partial y\partial x}\right]$$

or

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u.$$

□

Theorem 8 (Euler's Theorem) If $u=\phi(H_n)$; where H_n is a homogeneous function of degree n and supposeing that from the above relation we get $H_n=f(u)$, then

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\frac{f(u)}{f'(u)}.$$

Proof. Since H_n is a homogeneous function of degree n, therefore by Euler's theorem, we get

$$x\frac{\partial}{\partial x}f(u)+y\frac{\partial}{\partial y}f(u)=nf(u)$$

or

$$xf'(u)\frac{\partial u}{\partial x}+yf'(u)\frac{\partial u}{\partial y}=nf(u)$$

or

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{nf(u)}{f'(u)}.$$

□

28.7.2 Worked examples

Example 24 If $u=\dfrac{x+y}{x^2+y^2}$, prove that the order of u is -1.

Solution. Differentiating u partially with respect to x, we have

$$\frac{\partial u}{\partial x}=\frac{(x^2+y^2)\cdot 1-(x+y)\cdot 2x}{(x^2+y^2)^2}=\frac{y^2-2xy-x^2}{(x^2+y^2)^2}.$$

Therefore

$$x\frac{\partial u}{\partial x}=\frac{xy^2-2x^2y-x^3}{(x^2+y^2)^2}.$$

Similarly

$$y\frac{\partial u}{\partial y}=\frac{yx^2-2xy^2-y^3}{(x^2+y^2)^2}.$$

Therefore

$$\begin{aligned}x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}&=-\frac{x^3+y^3+x^2y+xy^2}{(x^2+y^2)^2}=-\frac{(x+y)(x^2+y^2)}{(x^2+y^2)^2}\\&=-\frac{(x+y)}{x^2+y^2}=-u.\end{aligned}$$

But by Euler's theorem, we have

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=mu=-1\cdot u.$$

Hence $n=-1$.

□

Example 25 If $u=\tan^{-1}\left(\dfrac{x^3+y^3}{x-y}\right)$, prove that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\sin 2u.$$

Solution. We have

$$u=\tan^{-1}\left(\frac{x^3+y^3}{x-y}\right).$$

Therefore $\tan u=\dfrac{x^3+y^3}{x-y}=f$ (say),
is a homogeneous function of degree 2, hence by Euler's theorem

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nf$$

or

$$x\frac{\partial}{\partial x}(\tan u)+y\frac{\partial}{\partial y}(\tan u)=2\tan u$$

or

$$x\left(\sec^2 u\frac{\partial u}{\partial x}\right)+y\left(\sec^2 u\frac{\partial u}{\partial y}\right)=2\tan u$$

or

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{2\tan u}{\sec^2 u}=2\sin u\cos u$$

or

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\sin 2u.$$

□

Example 26 If $u=\sin^{-1}\left(\dfrac{x}{y}\right)+\tan^{-1}\left(\dfrac{y}{x}\right)$ prove that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0.$$

Solution. Given

$$u=\sin^{-1}\left(\frac{x}{y}\right)+\tan^{-1}\left(\frac{y}{x}\right).$$

Differentiating partially with respect to x, we get

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \cdot \frac{1}{y} + \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \left(-\frac{y}{x^2}\right)$$

or

$$x\frac{\partial u}{\partial x} = \frac{x}{\sqrt{y^2-x^2}} - \frac{xy}{x^2+y^2}. \tag{28.73}$$

Again differentiating u partially with respect to y, we get

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \cdot \left(-\frac{x}{y^2}\right) + \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x}$$

or

$$y\frac{\partial u}{\partial y} = -\frac{x}{\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2}. \tag{28.74}$$

Adding (28.73) and (28.74), we get

$$x\frac{\partial u}{\partial y} + y\frac{\partial u}{\partial y} = 0.$$

□

Example 27 If $u = \sin^{-1}\left(\frac{y}{x}\right)$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0.$$

Solution. Given

$$u = x\sin^{-1}\left(\frac{y}{x}\right) = xf\left(\frac{y}{x}\right).$$

Therefore, u is a homogeneous function of x and y of degree 1, hence by Euler's theorem, we have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1 \cdot u = u. \tag{28.75}$$

Differentiating (28.75) partially with respect to x, we get

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial u}{\partial x}$$

or

$$x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = 0$$

or

$$x^2\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial x \partial y} = 0. \tag{28.76}$$

Again differentiating (28.75) partially with respect to y, we get

$$x\frac{\partial^2 u}{\partial y \partial x} + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = \frac{\partial u}{\partial y}$$

or

$$x\frac{\partial^2 u}{\partial y \partial x} + y\frac{\partial^2 u}{\partial y^2} = 0$$

or

$$xy\frac{\partial^2 u}{\partial y \partial x} + y^2\frac{\partial^2 u}{\partial y^2}. \tag{28.77}$$

Adding (28.76) and (28.77), we have

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0.$$

□

Example 28 If $V = \log_e \sin\left(\frac{\pi}{2}\frac{(2x^2+y^2+zx)^{1/2}}{(x^2+xy+2yz+z^2)^{1/3}}\right)$, find the value of

$$x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} + z\frac{\partial V}{\partial z},$$

when $x = 0, y = 1$ and $z = 2$.

Solution. Let

$$u = \frac{\pi}{2}\frac{(2x^2+y^2+zx)^{1/2}}{(x^2+xy+2yz+z^2)^{1/3}},$$

then

$$V = \log\sin u.$$

Therefore,

$$\frac{\partial V}{\partial x} = \cot u\frac{\partial u}{\partial x}.$$

Similarly

$$\frac{\partial V}{\partial y} = \cot u\frac{\partial u}{\partial y}$$

and

$$\frac{\partial V}{\partial z} = \cot u\frac{\partial u}{\partial z}.$$

Therefore,

$$x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} + z\frac{\partial V}{\partial z} = \cot u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right). \tag{28.78}$$

But u is a homogeneous function of degree $\frac{1}{3}$.

Therefore,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=\frac{1}{3}u.$$

Hence from (28.78), we have

$$x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}+z\frac{\partial V}{\partial z}=\frac{1}{3}u\cot u. \tag{28.79}$$

Now when $x=0, y=1, z=2$, we get $u=\frac{\pi}{4}$.

Therefore from (28.79), we have

$$\begin{aligned}x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}+z\frac{\partial V}{\partial z}&=\frac{1}{3}\cdot\frac{\pi}{4}\cdot\cot\frac{\pi}{4}.\\&=\frac{\pi}{12}.\end{aligned}$$

□

28.8 Taylor's theorem for functions of two variables

Theorem 9 If a function $f(x,y)$ of two independent variables x and y, possesses continuous partial derivatives of order n in any domain of a point (a,b) and $f(a+th,b+tk)$ is any point of D, then there exists a real number θ such that

$$\begin{aligned}f(x+h,y+k)=f(x,y)&+\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f+\frac{1}{2!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f\\&+\cdots+\frac{1}{(n-1)!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n-1} f+R\end{aligned}$$

where

$$R_n=\frac{1}{n!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f(a+\theta h,b+\theta k),\quad 0<\theta<1$$

is called the remainder after $n-$terms.

Proof. Let (x,y) be a point in the neighbourhood of (a,b) such that

$$x=a+th \qquad \text{and} \qquad y=b+tk$$

where $0\le t\le 1$ is a parameter.

Let us define a composite function such that

$$F(t)=f(x,y)=f(a+th,b+tk).$$

The partial derivatives of $f(x,y)$ of order n are continuous in the domain D. Therefore, the nth derivative $F^n(t)$ of the composite function $F(t)$ is continuous in $[0,1]$.

Hence, by Maclaurin's theorem for the function $F(t)$ on $[0,1]$, we get

$$F(t) = F(0) + tF'(0) + \frac{t^2}{2!}F^2(0) + \cdots + \frac{t^{n-1}}{(n-1)!}F^{n-1}(0) + \frac{t^n}{n!}F^n(\theta t), \tag{28.80}$$

where $0 < \theta < 1$.

By putting $t = 1$, we obtain

$$F(1) = F(0) + F'(0) + \frac{1}{2!}F^2(0) + \cdots + \frac{1}{(n-1)!}F^{n-1}(0) + \frac{1}{n!}F^n(0). \tag{28.81}$$

But

$$\begin{aligned} F'(t) = \frac{dF}{dt} = \frac{df}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \\ &= h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} \\ &= \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f. \end{aligned}$$

Again,

$$\begin{aligned} F''(t) &= \frac{d}{dt}F' = \frac{d}{dt}\left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right) \\ &= \frac{\partial}{\partial x}\left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right)\frac{dx}{dt} + \frac{\partial}{\partial y}\left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right)\frac{dy}{dt} \\ &= \left(h\frac{\partial^2 f}{\partial x^2} + k\frac{\partial^2 f}{\partial x \partial y}\right)\frac{dx}{dt} + \left(h\frac{\partial^2 f}{\partial x \partial y} + k\frac{\partial^2 f}{\partial y^2}\right)\frac{dy}{dt} \\ &= \left(h\frac{\partial^2 f}{\partial x^2} + k\frac{\partial^2 f}{\partial x \partial y}\right)\cdot h + \left(h\frac{\partial^2 f}{\partial x \partial y} + k\frac{\partial^2 f}{\partial y^2}\right)\cdot k \\ &= h^2 \cdot \frac{\partial^2 f}{\partial x^2} + 2hk \cdot \frac{\partial^2 f}{\partial x \partial y} + k^2 \cdot \frac{\partial^2 f}{\partial y^2} \\ &= \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f. \end{aligned}$$

Similarly,

$$F^3(t) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f.$$

In general, we have

$$F^n(t) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f.$$

Now, by letting $t = 0$, we have

$$
\begin{aligned}
F(0) &= f(a,b)\\
F'(0) &= \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(a,b)\\
F''(0) &= \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(a,b)\\
&\vdots\\
F^{n-1}(0) &= \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n-1} f(a,b)\\
F^{n}(\theta) &= \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n} f(a+\theta h, b+\theta k)
\end{aligned}
$$

Substituting these values in (28.81) gives

$$
\begin{aligned}
f(a+\theta h, b+\theta k) = f(a,b) &+ \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(a,b) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(a,b)\\
&+ \cdots + \frac{1}{(n-1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n-1} f(a,b) + R_n, \qquad (28.82)
\end{aligned}
$$

where

$$R_n = \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n} f(a+\theta h, b+\theta k), \quad 0 < \theta < 1$$

is Taylor's theorem with a remainder for the function of two variables. □

28.8.1 Maclaurin's theorem

If we put $a = b = 0, h = x$, and $k = y$ in Taylor's theorem, we get

$$
\begin{aligned}
f(x,y) = f(0,0) &+ \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(0,0) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(0,0)\\
&+ \cdots + \frac{1}{(n-1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n-1} f(0,0) + R_n, \qquad (28.83)
\end{aligned}
$$

where

$$R_n = \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n} f(\theta x, \theta y), \quad 0 < \theta < 1.$$

This form of expansion is called Maclaurin's theorem or Maclaurin's expansion in finite form.

28.8.2 Another form of Taylor's theorem

If we put $a+h=x$, or $h=x-a$, and $b+k=y$, or $k=y-b$ in Taylor's theorem, we get

$$f(x,y)=f(a,b)+\left((x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right)f(a,b)+\frac{1}{2!}\left((x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right)^2 f(a,b)$$
$$+\cdots+\frac{1}{(n-1)!}\left((x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right)^{n-1} f(a,b)+R_n, \qquad (28.84)$$

where

$$R_n=\frac{1}{n!}\left((x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right)^n f(a+\theta(x-a),b+\theta(y-b)), \quad 0<\theta<1.$$

This form of expansion is called Taylor's expansion in finite form for the function $f(x,y)$ about the point (a,b) in the powers of $(x-a)$ and $(y-b)$.

28.8.3 Taylor's series

If partial derivatives of all orders of $f(x,y)$ exist and are continuous in domain D and $R_n\to 0$ as $n\to\infty$, where

$$R_n=\frac{1}{n!}\left((x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right)^n f(a+\theta(x-a),b+\theta(y-b)), \quad 0<\theta<1$$

is Taylor's remainder after n terms, then we obtain Taylor's series,

$$f(x,y)=f(a,b)+\left((x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right)f(a,b)+\frac{1}{2!}\left((x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right)^2 f(a,b)$$
$$+\cdots+\frac{1}{(n-1)!}\left((x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right)^{n-1} f(a,b)+\cdots. \qquad (28.85)$$

28.8.4 Application of Taylor's theorem: approximate calculations

Let $f(x,y)$ be a function of two variables x and y. If Δx and Δy are small increments in the values of x and y, then the corresponding value of the function becomes $f(x+\Delta x,y+\Delta y)$.

Hence the corresponding increase $\Delta f(x)$ in $f(x,y)$ will be given by

$$\Delta f(x,y)=f(x+\Delta x,y+\Delta y)-f(x,y).$$

Now expanding $f(x+\Delta x,y+\Delta y)$ by Taylor's theorem, and assuming that Δx and Δy are so small that their squares and higher power terms can be neglected, we get

$$\Delta f=\left(f(x,y)+\frac{\partial f}{\partial x}\Delta x+\frac{\partial f}{\partial y}\Delta y\right)-f(x,y) \qquad \text{(approximately)}.$$

Hence

$$\Delta f=\frac{\partial f}{\partial x}\Delta x+\frac{\partial f}{\partial y}\Delta y,$$

approximately. This formula gives a small error in f due to the small errors Δx and Δy in x and y, respectively.

In general, if f is a function of n variables $x_1, x_2, \cdots, x_n$ then we have

$$\Delta f = \frac{\partial f}{\partial x_1}\Delta x_1 + \frac{\partial f}{\partial x_2}\Delta x_2 + \cdots \frac{\partial f}{\partial x_n}\Delta x_n \qquad \text{(approximately)}.$$

The above result is useful in computing the effect of small errors in measured quantities.

Note 6 By Newtons binomial formula, we obtain

$$\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f = h^n\frac{\partial^n f}{\partial x^n} + nh^{n-1}k\frac{\partial^n f}{\partial x^{n-1}\partial y} + \frac{n(n-1)}{2!}h^{n-2}k^2\frac{\partial^n f}{\partial x^{n-2}\partial y^2}$$
$$+ \cdots + nhk^{n-1}\frac{\partial^n f}{\partial x\partial y^{n-1}} + k^n\frac{\partial^n f}{\partial y^n}.$$

28.8.5 Worked examples

Example 29 If $f(x,y) = e^{x+y}$, then find Taylor's expansion at the point $(0,0)$ for $n = 3$.

Solution. For $n = 3$ at the point $(0,0)$, Taylor's expansion is given as

$$f(x,y) = f(0,0) + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) f(0,0) + \frac{1}{2!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 + R_3,$$

where

$$R_n = \frac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f(\theta x, \theta y), \quad 0 < \theta < 1$$
$$= \frac{1}{3!}\left(x^3\frac{\partial^3}{\partial x^3} + 3x^2y\frac{\partial^3}{\partial x^2\partial y} + 3xy^2\frac{\partial^3}{\partial x\partial y^2} + y^3\frac{\partial^3}{\partial y^3}\right) f(\theta x, \theta y).$$

We are given $f(x,y) = e^{x+y}$, thus, $f(0,0) = 1$.

Now, we have

- $$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = e^{x+y};$$
- $$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial x\partial y} = \frac{\partial^2 f}{\partial y^2} = e^{x+y};$$
- $$\frac{\partial^3 f}{\partial x^3} = \frac{\partial^3 f}{\partial x^2\partial y} = \frac{\partial^3 f}{\partial x\partial y^2} = \frac{\partial^3 f}{\partial y^3} = e^{x+y}.$$

Therefore, at the point $(0,0)$, we have

$$f_x = f_y = 1$$

and

$$f_{xx} = f_{xy} = f_{yy} = 1,$$

and at the point $(\theta x, \theta y)$, we have

$$\frac{\partial^3 f}{\partial x^3} = \frac{\partial^3 f}{\partial x^2 \partial y} = \frac{\partial^3 f}{\partial x \partial y^2} = \frac{\partial^3 f}{\partial y^3} = e^{\theta(x+y)}.$$

Thus, we have

$$e^{x+y} = 1 + (x+y) + \frac{1}{2!}\left(x^2 + 2xy + y^2\right) + \frac{1}{3!}\left(x^3 + 3x^2y + 3xy^2 + y^3\right) e^{\theta(x+y)}.$$

□

Example 30 Suppose $f(x, y) = e^{xy}$. Expand $f(x, y)$ by Taylor's series about the point $(1, 1)$.

Solution. We are given $f(x, y) = e^{xy}$. Then $f(1, 1) = e$.

- $\frac{\partial f}{\partial x} = ye^{xy}$ and $\frac{\partial f}{\partial y} = xe^{xy}$;
- $\frac{\partial^2 f}{\partial x^2} = y^2e^{xy}$, $\frac{\partial^2 f}{\partial x \partial y} = e^{xy} + e^{xy} \cdot xy = e^{xy}(1 + \cdot xy)$ and $\frac{\partial^2 f}{\partial y^2} = x^2e^{xy}$;
- $\frac{\partial^3 f}{\partial x^3} = y^3e^{xy}$, $\frac{\partial^3 f}{\partial x^2 \partial y} = (2x + yx^2)\, e^{xy}$, $\frac{\partial^3 f}{\partial x \partial y^2} = (2x + yx^2)\, e^{xy}$ and $\frac{\partial^3 f}{\partial y^3} = x^3e^{xy}$.

At the point $(1, 1)$, we have

$$\begin{aligned}
&f_x(1,1) = f_y(1,1) = e;\\
&f_{xx}(1,1) = f_{yy}(1,1) = e;\\
&f_{xy}(1,1) = (1+1)e = 2e;\\
&f_{xxx}(1,1) = f_{yyy}(1,1) = e;\\
&f_{xxy}(1,1) = f_{xyy}(1,1) = (2+1)e = 3e.
\end{aligned}$$

Therefore, by Taylor's series expansion about $(1, 1)$, we have

$$f(x,y) = f(1,1) + \left((x-1)\frac{\partial}{\partial x} + (y-1)\frac{\partial}{\partial y}\right) f(1,1) + \frac{1}{2!}\left((x-1)\frac{\partial}{\partial x} + (y-1)\frac{\partial}{\partial y}\right)^2 f(1,1) + \cdots.$$

Hence,

$$\begin{aligned}
e^{xy} &= e + ((x-1)e + (y-1)e) + \frac{1}{2!}\left((x-1)^2e + 2(x-1)(y-1)e + (y-1)^2e\right)^2 f(1,1) + \cdots\\
&= e\left(1 + (x-1) + (y-1) + \frac{1}{2}(x-1)^2 + (x-1)(y-1) + \frac{1}{2}(y-1)^2 + \cdots\right).
\end{aligned}$$

□

Example 31 The side a and the opposite angle A of a triangle ABC remain constant. Show that when the other sides and angles are slightly varied:

$$\frac{\Delta b}{\cos B} + \frac{\Delta c}{\cos C} = 0.$$

Solution. We know that in a triangle,

$$a = b\cos C + c\cos B.$$

Differentiating partially, we have

$$0 = \Delta b\cos C + \Delta c\cos B - b\sin C\Delta C - c\sin B\Delta B,$$

where a is constant, or

$$\begin{aligned}\Delta b\cos C + \Delta c\cos B &= b\sin C\cdot(\Delta B + \Delta C)\\ &= b\sin C\Delta(\pi - A) && \text{because} \quad c\sin B = b\sin C\\ &= 0. && \text{because} \quad A \text{ is constant}\end{aligned}$$

Hence, we have

$$\frac{\Delta b}{\cos B} + \frac{\Delta c}{\cos C} = 0.$$

□

Example 32 If the sides of a plane triangle ABC vary in such a way that its circum-radius remains constant, prove that

$$\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 0.$$

Solution. From trigonometry, we know that the triangle's circum-radius is

$$R = \frac{a}{2\sin A} = \text{Constant}$$

or

$$\frac{a}{\sin A} = K.$$

Taking differentials, we get

$$\frac{da\sin A - a\cos A dA}{\sin^2 A} = 0$$

or

$$\frac{da}{\cos A} = \frac{a}{\sin A}dA. \tag{28.86}$$

Similarly, we can get

$$\frac{db}{\cos B} = \frac{b}{\sin B}dB \tag{28.87}$$

and

$$\frac{dc}{\cos C} = \frac{c}{\sin C} dC. \tag{28.88}$$

Adding (28.86), (28.87) and (28.88), we get

$$\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = \left(\frac{adA}{\sin A} + \frac{bdB}{\sin B} + \frac{cdC}{\sin C} \right). \tag{28.89}$$

But from trigonometry, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Therefore, on the RHS of (28.89), we have

$$= \frac{a}{\sin A}(dA + dB + dC).$$

Now in the triangle

$$A + B + C = \pi.$$

Then

$$dA + dB + dC = 0.$$

Therefore (28.89) reduces to

$$\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 0.$$

□

Example 33 The height h and the semi-vertical angle α of a cone are measured, and from them $\mathcal{A}$, the total area of a cone, including the base, is calculated. If h and α are in error by the small quantities Δh and $\Delta \alpha$ respectively, find the corresponding error in the area. Show further that, if $\alpha = \frac{\pi}{6}$, an error of $+1$ in h will be approximately compensated by an error of -19.8% in α.

Solution. We have

$$\text{Area of the base of the cone} = \pi h^2 \tan^2 \alpha$$

and

$$\begin{aligned} \text{Area of the curved surface of the cone} &= \pi(h \tan \alpha) \cdot (h \sec \alpha) \\ &= \pi h^2 \tan \alpha \sec \alpha. \end{aligned}$$

Hence

$$\begin{aligned} \mathcal{A} &= \text{Total area of the cone including the base} \\ &= \pi h^2 (\tan^2 \alpha + \tan \alpha \sec \alpha). \end{aligned}$$

Now by Taylor's theorem, we have

$$\begin{aligned}\Delta\mathcal{A} &= \frac{\partial\mathcal{A}}{\partial h}\Delta h + \frac{\partial\mathcal{A}}{\partial\alpha}\Delta\alpha, \quad \text{approximately}\\ &= 2\pi h\tan\alpha(\tan\alpha+\sec\alpha)\Delta h + \pi h^2(2\tan\alpha\sec^2\alpha+\sec^3\alpha+\tan^2\alpha\sec\alpha)\Delta\alpha\\ &= 2\pi h\tan\alpha(\tan\alpha+\sec\alpha)\Delta h + \pi h^2\sec\alpha\cdot\left(2\tan\alpha\sec\alpha+\sec^2\alpha+\tan^2\alpha\right)\Delta\alpha.\end{aligned} \tag{28.90}$$

This is the relation which gives the error $\Delta\mathcal{A}$ in $\mathcal{A}$ corresponding to errors Δh and $\Delta\alpha$, in h and α respectively.

If $\alpha = \frac{\pi}{6}$ and $\Delta h = \frac{h}{100}$, then (28.90) gives

$$\begin{aligned}\Delta A &= 2\pi h^2\frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}+\frac{2}{\sqrt{3}}\right)\frac{1}{100} + \pi h^2\cdot\frac{2}{\sqrt{3}}\left(2\frac{1}{\sqrt{3}}\cdot\frac{2}{\sqrt{3}}+\frac{4}{3}+\frac{1}{3}\right)\Delta\alpha\\ &= 2\pi h^2\times 0.01 + \pi h^2\cdot 2\sqrt{3}\Delta\alpha\\ &= 2\pi h^2\times 0.01 + \pi h^2 3.4646\Delta\alpha.\end{aligned}$$

Hence if $\Delta\mathcal{A} = 0$, then

$$\begin{aligned}\Delta\alpha &= -\frac{2\pi h^2\times 0.1}{\pi h^2\times 3.4646} = -\frac{2\times 0.01}{3.4646} && \text{radians}\\ &= -\frac{2\times 0.01\times 57.3}{3.4646} && \text{degree [1 radian} = 57.3^\circ]\\ &= -19.8' && \text{(approximately).}\end{aligned}$$

□

Example 34 Find the percentage error in the area of an ellipse when an error of 1% is made in measuring its major and minor axes.

Solution. Let a and b be the major and minor axes of an ellipse, respectively. If $\mathcal{A}$ is the area of the ellipse, then

$$A = \frac{\pi ab}{4}.$$

Now, we have

$$\begin{aligned}\Delta\mathcal{A} &= \frac{\partial\mathcal{A}}{\partial a}\cdot\Delta a + \frac{\partial\mathcal{A}}{\partial b}\Delta b\\ &= \frac{\pi b}{4}\Delta a + \frac{\pi a}{4}\Delta b.\end{aligned} \tag{28.91}$$

Given that

$$\frac{\Delta a}{a} = \frac{1}{100} \quad\text{and}\quad \frac{\Delta b}{b} = \frac{1}{100}$$

or

$$\Delta a = \frac{a}{100} \quad\text{and}\quad \Delta b = \frac{b}{100}.$$

Substituting these values in (28.91), we get

$$\Delta\mathcal{A} = \frac{\pi b}{4}\cdot\frac{a}{100} + \frac{\pi a}{4}\cdot\frac{b}{100} = \frac{\pi ab}{200}.$$

Therefore

$$\frac{\Delta\mathcal{A}}{\mathcal{A}} = \frac{\pi ab}{200}\cdot\frac{4}{\pi ab} = \frac{1}{50}$$

or

$$\frac{\Delta\mathcal{A}}{\mathcal{A}}\times 100 = \frac{1}{50}\times 100 = 2.$$

Hence the percentage error in the area = 2%.

□

28.9 Jacobians

28.9.1 Jacobians of functions of two independent variables

Definition 1 Let u and v be the differentiable functions of two independent variables x and y. Then the determinant

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

is called the Jacobian of u and v with respect to x and y. It is denoted by $\frac{\partial(u,v)}{\partial(x,y)}$ or $J(u,v)$.

□

28.9.2 Jacobians of functions of three independent variables

Definition 2 Let u, v and w be the differentiable functions of three independent variables x, y and z. Then the determinant

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

is called the Jacobian of u, v and w with respect to x, y and z. It is denoted by $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ or $J(u,v,w)$. □

28.9.3 Jacobians of functions of n independent variables

Definition 3 Let $u_1, u_2, u_3, \cdots, u_n$ be the differentiable functions of nth independent variables $x_1, x_2, x_3, \cdots, x_n$. Then the determinant

$$\begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_2}{\partial x_1} & \frac{\partial u_3}{\partial x_1} & \cdots & \frac{\partial u_n}{\partial x_1} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_3}{\partial x_2} & \cdots & \frac{\partial u_n}{\partial x_2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \frac{\partial u_n}{\partial x_2} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

is called the Jacobian of $u_1, u_2, u_3, \cdots, u_n$ with respect to $x_1, x_2, x_3, \cdots, x_n$. It is denoted by $\frac{\partial(u_1, u_2, u_3, \cdots, u_n)}{\partial(x_1, x_2, x_3, \cdots, x_n)}$ or $J(u_1, u_2, u_3, \cdots, u_n)$. □

28.9.4 Some theorems on Jacobians

Theorem 10 If $u_1, u_2, u_3, \cdots, u_n$ are functions of $x_1, x_2, x_3, \cdots, x_n$ of the form:

$$\begin{aligned} u_1 &= f_1(x_1) \\ u_2 &= f_2(x_1, x_2) \\ u_3 &= f_3(x_1, x_2, x_3) \\ &\vdots \\ u_n &= f_n(x_1, x_2, x_3, \cdots, x_n), \end{aligned}$$

then

$$\frac{\partial(u_1, u_2, u_3, \cdots, u_n)}{\partial(x_1, x_2, x_3, \cdots, x_n)} = \frac{\partial u_1}{\partial x_1} \cdot \frac{\partial u_2}{\partial x_2} \cdot \frac{\partial u_3}{\partial x_3} \cdots \frac{\partial u_n}{\partial x_n}.$$

Proof. We know that

$$\frac{\partial(u_1, u_2, u_3, \cdots, u_n)}{\partial(x_1, x_2, x_3, \cdots, x_n)} = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_2}{\partial x_1} & \frac{\partial u_3}{\partial x_1} & \cdots & \frac{\partial u_n}{\partial x_1} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_3}{\partial x_2} & \cdots & \frac{\partial u_n}{\partial x_2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \frac{\partial u_n}{\partial x_2} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}. \tag{28.92}$$

Since u_1 is a function of x_1 only, then,

$$\frac{\partial u_1}{\partial x_2} = 0, \frac{\partial u_1}{\partial x_3} = 0, \cdots, \frac{\partial u_1}{\partial x_n} = 0.$$

Also since u_2 is a function of x_1 and x_2, only $\frac{\partial u_2}{\partial x_1}$ and $\frac{\partial u_2}{\partial x_2}$ will exist, therefore,

$$\frac{\partial u_2}{\partial x_3} = 0, \frac{\partial u_2}{\partial x_4} = 0, \cdots, \frac{\partial u_2}{\partial x_n} = 0.$$

Also since u_3 is a function of x_1, x_2 and x_3, therefore $\dfrac{\partial u_3}{\partial x_1}$, $\dfrac{\partial u_3}{\partial x_2}$ and $\dfrac{\partial u_3}{\partial x_3}$ will exist and the rest will be zero.

Similarly, since u_n is a function of $x_1, x_2, \cdots, x_3$, then $\dfrac{\partial u_n}{\partial x_1}$, $\dfrac{\partial u_n}{\partial x_2}, \cdots, \dfrac{\partial u_n}{\partial x_n}$ will exist.

Putting these values in (28.92), we get

$$\frac{\partial(u_1, u_2, u_3, \cdots, u_n)}{\partial(x_1, x_2, x_3, \cdots, x_n)} = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & 0 & 0 & \cdots & 0 \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & 0 & \cdots & 0 \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \frac{\partial u_n}{\partial x_2} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}.$$

Expanding the determinant in terms of the first row, we immediately get

$$\frac{\partial(u_1, u_2, u_3, \cdots, u_n)}{\partial(x_1, x_2, x_3, \cdots, x_n)} = \frac{\partial u_1}{\partial x_1} \cdot \frac{\partial u_2}{\partial x_2} \cdot \frac{\partial u_3}{\partial x_3} \cdots \frac{\partial u_n}{\partial x_n}.$$

□

28.9.5 Jacobians of composite functions

Theorem 11 If u_1, u_2, u_3 are the functions of y_1, y_2, y_3 and y_1, y_2, y_3 are the functions of x_1, x_2, x_3, then

$$\frac{\partial(u_1, u_2, u_3)}{\partial(x_1, x_2, x_3)} = \frac{\partial(u_1, u_2, u_3)}{\partial(y_1, y_2, y_3)} \frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)}.$$

Proof. Given that u_1, u_2, u_3 are functions of y_1, y_2, y_3 and y_1, y_2, y_3 are functions of x_1, x_2, x_3, we get

$$\frac{\partial u_1}{\partial x_1} = \frac{\partial u_1}{\partial y_1}\frac{\partial y_1}{\partial x_1} + \frac{\partial u_1}{\partial y_2}\frac{\partial y_2}{\partial x_1} + \frac{\partial u_1}{\partial y_3}\frac{\partial y_3}{\partial x_1} = \sum_{i=1}^{3} \frac{\partial u_1}{\partial y_i}\frac{\partial y_i}{\partial x_1},$$

$$\frac{\partial u_1}{\partial x_2} = \frac{\partial u_1}{\partial y_1}\frac{\partial y_1}{\partial x_2} + \frac{\partial u_1}{\partial y_2}\frac{\partial y_2}{\partial x_2} + \frac{\partial u_1}{\partial y_3}\frac{\partial y_3}{\partial x_1} = \sum_{i=1}^{3} \frac{\partial u_1}{\partial y_i}\frac{\partial y_i}{\partial x_2},$$

and

$$\frac{\partial u_1}{\partial x_3} = \frac{\partial u_1}{\partial y_1}\frac{\partial y_1}{\partial x_3} + \frac{\partial u_1}{\partial y_2}\frac{\partial y_2}{\partial x_3} + \frac{\partial u_1}{\partial y_3}\frac{\partial y_3}{\partial x_1} = \sum_{i=1}^{3} \frac{\partial u_1}{\partial y_i}\frac{\partial y_i}{\partial x_3}.$$

Similarly,

$$\frac{\partial u_2}{\partial x_1} = \sum_{i=1}^{3} \frac{\partial u_2}{\partial y_i}\frac{\partial y_i}{\partial x_1}, \frac{\partial u_2}{\partial x_2} = \sum_{i=1}^{3} \frac{\partial u_2}{\partial y_i}\frac{\partial y_i}{\partial x_2}, \frac{\partial u_2}{\partial x_3} = \sum_{i=1}^{3} \frac{\partial u_2}{\partial y_i}\frac{\partial y_i}{\partial x_3},$$

and

$$\frac{\partial u_3}{\partial x_1} = \sum_{i=1}^{3} \frac{\partial u_3}{\partial y_i}\frac{\partial y_i}{\partial x_1}, \frac{\partial u_3}{\partial x_2} = \sum_{i=1}^{3} \frac{\partial u_3}{\partial y_i}\frac{\partial y_i}{\partial x_2}, \frac{\partial u_3}{\partial x_3} = \sum_{i=1}^{3} \frac{\partial u_3}{\partial y_i}\frac{\partial y_i}{\partial x_3}$$

Now, putting the values of each element of the determinant[1] from the above relations, we get

$$\begin{aligned}
\frac{\partial(u_1,u_2,u_3)}{\partial(y_1,y_2,y_3)}\frac{\partial(y_1,y_2,y_3)}{\partial(x_1,x_2,x_3)} &= \begin{vmatrix} \frac{\partial u_1}{\partial y_1} & \frac{\partial u_1}{\partial y_2} & \frac{\partial u_1}{\partial y_3} \\ \frac{\partial u_2}{\partial y_1} & \frac{\partial u_2}{\partial y_2} & \frac{\partial u_2}{\partial y_3} \\ \frac{\partial u_3}{\partial y_1} & \frac{\partial u_3}{\partial y_2} & \frac{\partial u_3}{\partial y_3} \end{vmatrix} \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix} \\
&= \left| \begin{bmatrix} \frac{\partial u_1}{\partial y_1} & \frac{\partial u_1}{\partial y_2} & \frac{\partial u_1}{\partial y_3} \\ \frac{\partial u_2}{\partial y_1} & \frac{\partial u_2}{\partial y_2} & \frac{\partial u_2}{\partial y_3} \\ \frac{\partial u_3}{\partial y_1} & \frac{\partial u_3}{\partial y_2} & \frac{\partial u_3}{\partial y_3} \end{bmatrix} \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{bmatrix} \right| \\
&= \begin{vmatrix} \sum_{i=1}^{3} \frac{\partial u_1}{\partial y_i}\frac{\partial y_i}{\partial x_1} & \sum_{i=1}^{3} \frac{\partial u_1}{\partial y_i}\frac{\partial y_i}{\partial x_2} & \sum_{i=1}^{3} \frac{\partial u_1}{\partial y_i}\frac{\partial y_i}{\partial x_3} \\ \sum_{i=1}^{3} \frac{\partial u_2}{\partial y_i}\frac{\partial y_i}{\partial x_1} & \sum_{i=1}^{3} \frac{\partial u_2}{\partial y_i}\frac{\partial y_i}{\partial x_2} & \sum_{i=1}^{3} \frac{\partial u_2}{\partial y_i}\frac{\partial y_i}{\partial x_3} \\ \sum_{i=1}^{3} \frac{\partial u_3}{\partial y_i}\frac{\partial y_i}{\partial x_1} & \sum_{i=1}^{3} \frac{\partial u_3}{\partial y_i}\frac{\partial y_i}{\partial x_2} & \sum_{i=1}^{3} \frac{\partial u_3}{\partial y_i}\frac{\partial y_i}{\partial x_3}. \end{vmatrix} \\
&= \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{vmatrix}.
\end{aligned}$$

Therefore,

$$\frac{\partial(u_1,u_2,u_3)}{\partial(y_1,y_2,y_3)}\frac{\partial(y_1,y_2,y_3)}{\partial(x_1,x_2,x_3)} = \frac{\partial(u_1,u_2,u_3)}{\partial(x_1,x_2,x_3)}.$$

□

[1] Let A and B be square matrices. Then the determinant of a product of two matrices is the product of their determinants, that is, $|AB| = |A||B|$.

Theorem 12 If $u_1, u_2, u_3, \cdots, u_n$ are functions of $y_1, y_2, y_3, \cdots, y_n$ and $y_1, y_2, y_3, \cdots, y_n$ are functions of $x_1, x_2, x_3, \cdots, x_n$, then

$$\frac{\partial(u_1, u_2, u_3, \cdots, u_n)}{\partial(x_1, x_2, x_3, \cdots, x_n)} = \frac{\partial(u_1, u_2, u_3, \cdots, u_n)}{\partial(y_1, y_2, y_3, \cdots, y_n)} \frac{\partial(y_1, y_2, y_3, \cdots, y_n)}{\partial(x_1, x_2, x_3, \cdots, x_n)}.$$

Proof. Given that $u_1, u_2, u_3, \cdots, u_n$ are functions of $y_1, y_2, y_3, \cdots, y_n$ and $y_1, y_2, y_3, \cdots, y_n$ are functions of $x_1, x_2, x_3, \cdots, x_n$, we get

$$\begin{aligned} \frac{\partial u_j}{\partial x_i} &= \frac{\partial u_j}{\partial y_1}\frac{\partial y_1}{\partial x_i} + \frac{\partial u_j}{\partial y_2}\frac{\partial y_2}{\partial x_i} + \cdots + \frac{\partial u_j}{\partial y_n}\frac{\partial y_n}{\partial x_i} \\ &= \sum_{r=1}^{n} \frac{\partial u_j}{\partial y_r}\frac{\partial y_r}{\partial x_i} \end{aligned}$$

Thus,

$$\frac{\partial u_1}{\partial x_i} = \sum_{r=1}^{n} \frac{\partial u_1}{\partial y_r}\frac{\partial y_r}{\partial x_i},$$

$$\frac{\partial u_2}{\partial x_i} = \sum_{r=1}^{n} \frac{\partial u_2}{\partial y_r}\frac{\partial y_r}{\partial x_i},$$

$\cdots$ and so on.

Now, by the row by column rule for matrix multiplication of determinants, we have

$$\frac{\partial(u_1,u_2,u_3,\cdots,u_n)}{\partial(y_1,y_2,y_3,\cdots,y_n)}\frac{\partial(y_1,y_2,y_3,\cdots,y_n)}{\partial(x_1,x_2,x_3,\cdots,x_n)} = \begin{vmatrix} \frac{\partial u_1}{\partial y_1} & \frac{\partial u_1}{\partial y_2} & \cdots & \frac{\partial u_1}{\partial y_n} \\ \frac{\partial u_2}{\partial y_1} & \frac{\partial u_2}{\partial y_2} & \cdots & \frac{\partial u_2}{\partial y_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial u_n}{\partial y_1} & \frac{\partial u_n}{\partial y_2} & \cdots & \frac{\partial u_n}{\partial y_n} \end{vmatrix} \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & \cdots & \frac{\partial y_n}{\partial x_n} \end{vmatrix}$$

$$= \left| \begin{bmatrix} \frac{\partial u_1}{\partial y_1} & \frac{\partial u_1}{\partial y_2} & \cdots & \frac{\partial u_1}{\partial y_n} \\ \frac{\partial u_2}{\partial y_1} & \frac{\partial u_2}{\partial y_2} & \cdots & \frac{\partial u_2}{\partial y_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial u_n}{\partial y_1} & \frac{\partial u_n}{\partial y_2} & \cdots & \frac{\partial u_n}{\partial y_n} \end{bmatrix} \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & \cdots & \frac{\partial y_n}{\partial x_n} \end{bmatrix} \right|$$

$$= \begin{vmatrix} \frac{\partial u_1}{\partial y_i}\frac{\partial y_i}{\partial x_1} & \frac{\partial u_1}{\partial y_i}\frac{\partial y_i}{\partial x_2} & \cdots & \frac{\partial u_1}{\partial y_i}\frac{\partial y_i}{\partial x_n} \\ \frac{\partial u_2}{\partial y_i}\frac{\partial y_i}{\partial x_1} & \frac{\partial u_2}{\partial y_i}\frac{\partial y_i}{\partial x_2} & \cdots & \frac{\partial u_2}{\partial y_i}\frac{\partial y_i}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial u_n}{\partial y_i}\frac{\partial y_i}{\partial x_1} & \frac{\partial u_n}{\partial y_i}\frac{\partial y_i}{\partial x_2} & \cdots & \frac{\partial u_n}{\partial y_i}\frac{\partial y_i}{\partial x_n} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \cdots & \frac{\partial u_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}.$$

Therefore,

$$\frac{\partial(u_1,u_2,u_3,\cdots,u_n)}{\partial(y_1,y_2,y_3,\cdots,y_n)}\frac{\partial(y_1,y_2,y_3,\cdots,y_n)}{\partial(x_1,x_2,x_3,\cdots,x_n)} = \frac{\partial(u_1,u_2,u_3,\cdots,u_n)}{\partial(x_1,x_2,x_3,\cdots,x_n)}.$$

□

28.9.6 Jacobians of implicit functions

Theorem 13 If u_1, u_2, and u_3 are the functions of independent variables, x_1, x_2, and x_3 given by the implicit relations

$$\begin{aligned} F_1(u_1,u_2,u_3,x_1,x_2,x_3) &= 0, \\ F_2(u_1,u_2,u_3,x_1,x_2,x_3) &= 0, \\ F_3(u_1,u_2,u_3,x_1,x_2,x_3) &= 0, \end{aligned}$$

then

$$\frac{\partial(u_1,u_2,u_3)}{\partial(x_1,x_2,x_3)} = (-1)^3 \frac{\dfrac{\partial(F_1,F_2,F_3)}{\partial(x_1,x_2,x_3)}}{\dfrac{\partial(F_1,F_2,F_3)}{\partial(u_1,u_2,u_3)}}.$$

Proof. Differentiating F_1, F_2 and F_3 with respect to x_1, x_2, and x_3, we get

$$\frac{\partial F_1}{\partial x_1} + \frac{\partial F_1}{\partial u_1}\cdot\frac{\partial u_1}{\partial x_1} + \frac{\partial F_1}{\partial u_2}\cdot\frac{\partial u_2}{\partial x_1} + \frac{\partial F_1}{\partial 3_1}\cdot\frac{\partial u_3}{\partial x_1} = 0.$$

Then

$$\sum_{r=1}^{3}\frac{\partial F_1}{\partial u_r}\cdot\frac{\partial u_r}{\partial x_1} = -\frac{\partial F_1}{\partial x_1}.$$

Similarly, we have

$$\sum_{r=1}^{3}\frac{\partial F_1}{\partial u_r}\cdot\frac{\partial u_r}{\partial x_2} = -\frac{\partial F_1}{\partial x_2}$$

and

$$\sum_{r=1}^{3}\frac{\partial F_1}{\partial u_r}\cdot\frac{\partial u_r}{\partial x_3} = -\frac{\partial F_1}{\partial x_3}.$$

Likewise, we have

$$\sum_{r=1}^{3}\frac{\partial F_2}{\partial u_r}\cdot\frac{\partial u_r}{\partial x_1} = -\frac{\partial F_2}{\partial x_1}.$$

$$\sum_{r=1}^{3}\frac{\partial F_2}{\partial u_r}\cdot\frac{\partial u_r}{\partial x_2} = -\frac{\partial F_2}{\partial x_2},$$

$$\sum_{r=1}^{3}\frac{\partial F_2}{\partial u_r}\cdot\frac{\partial u_r}{\partial x_3} = -\frac{\partial F_2}{\partial x_3},$$

$$\sum_{r=1}^{3}\frac{\partial F_3}{\partial u_r}\cdot\frac{\partial u_r}{\partial x_1} = -\frac{\partial F_3}{\partial x_1}.$$

$$\sum_{r=1}^{3}\frac{\partial F_3}{\partial u_r}\cdot\frac{\partial u_r}{\partial x_2} = -\frac{\partial F_3}{\partial x_2},$$

and

$$\sum_{r=1}^{3}\frac{\partial F_3}{\partial u_r}\cdot\frac{\partial u_r}{\partial x_3} = -\frac{\partial F_3}{\partial x_3}.$$

Substituting the values of each summation, we immediately get

$$\begin{aligned}
\frac{\partial(F_1,F_2,F_3)}{\partial(u_1,u_2,u_3)}\frac{\partial(u_1,u_2,u_3)}{\partial(x_1,x_2,x_3)} &= \begin{vmatrix} \frac{\partial F_1}{\partial u_1} & \frac{\partial F_1}{\partial u_2} & \frac{\partial F_1}{\partial u_3} \\ \frac{\partial F_2}{\partial u_1} & \frac{\partial F_2}{\partial u_2} & \frac{\partial F_2}{\partial u_3} \\ \frac{\partial F_3}{\partial u_1} & \frac{\partial F_3}{\partial u_2} & \frac{\partial F_3}{\partial u_3} \end{vmatrix} \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{vmatrix} \\
&= \begin{vmatrix} \sum_{r=1}^{3}\frac{\partial F_1}{\partial u_r}\cdot\frac{\partial u_r}{\partial x_1} & \sum_{r=1}^{3}\frac{\partial F_1}{\partial u_r}\cdot\frac{\partial u_r}{\partial x_2} & \sum_{r=1}^{3}\frac{\partial F_1}{\partial u_r}\cdot\frac{\partial u_r}{\partial x_3} \\ \sum_{r=1}^{3}\frac{\partial F_2}{\partial u_r}\cdot\frac{\partial u_r}{\partial x_1} & \sum_{r=1}^{3}\frac{\partial F_2}{\partial u_r}\cdot\frac{\partial u_r}{\partial x_2} & \sum_{r=1}^{3}\frac{\partial F_2}{\partial u_r}\cdot\frac{\partial u_r}{\partial x_3} \\ \sum_{r=1}^{3}\frac{\partial F_3}{\partial u_r}\cdot\frac{\partial u_r}{\partial x_1} & \sum_{r=1}^{3}\frac{\partial F_3}{\partial u_r}\cdot\frac{\partial u_r}{\partial x_2} & \sum_{r=1}^{3}\frac{\partial F_3}{\partial u_r}\cdot\frac{\partial u_r}{\partial x_3} \end{vmatrix} \\
&= \begin{vmatrix} -\frac{\partial F_1}{\partial x_1} & -\frac{\partial F_1}{\partial x_2} & -\frac{\partial F_1}{\partial x_3} \\ -\frac{\partial F_2}{\partial x_1} & -\frac{\partial F_2}{\partial x_2} & -\frac{\partial F_2}{\partial x_3} \\ -\frac{\partial F_3}{\partial x_1} & -\frac{\partial F_3}{\partial x_2} & -\frac{\partial F_3}{\partial x_3} \end{vmatrix} \\
&= (-1)^3 \begin{vmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \frac{\partial F_1}{\partial x_3} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \frac{\partial F_2}{\partial x_3} \\ \frac{\partial F_3}{\partial x_1} & \frac{\partial F_3}{\partial x_2} & \frac{\partial F_3}{\partial x_3} \end{vmatrix} \\
&= (-1)^3 \frac{\partial(F_1,F_2,F_3)}{\partial(x_1,x_2,x_3)}.
\end{aligned}$$

Therefore, we get

$$\frac{\partial(u_1,u_2,u_3)}{\partial(x_1,x_2,x_3)} = (-1)^3 \frac{\frac{\partial(F_1,F_2,F_3)}{\partial(x_1,x_2,x_3)}}{\frac{\partial(F_1,F_2,F_3)}{\partial(u_1,u_2,u_3)}}.$$

□

28.9.7 Generalization of the Jacobians of implicit functions

Theorem 14 If $u_1, u_2, u_3, \cdots, u_n$ are the functions of independent variables, $x_1, x_2, x_3, \cdots, x_n$ given by the implicit relations

$$\begin{aligned}
F_1(u_1,u_2,u_3,\cdots,u_n,x_1,x_2,x_3,\cdots,x_n) &= 0, \\
F_2(u_1,u_2,u_3,\cdots,u_n,x_1,x_2,x_3,\cdots,x_n) &= 0, \\
F_3(u_1,u_2,u_3,\cdots,u_n,x_1,x_2,x_3,\cdots,x_n) &= 0, \\
&\vdots \\
F_n(u_1,u_2,u_3,\cdots,u_n,x_1,x_2,x_3,\cdots,x_n) &= 0,
\end{aligned}$$

then

$$\frac{\partial(u_1, u_2, u_3, \cdots, u_n)}{\partial(x_1, x_2, x_3, \cdots, x_n)} = (-1)^n \frac{\dfrac{\partial(F_1, F_2, F_3, \cdots, F_n)}{\partial(x_1, x_2, x_3, \cdots, x_n)}}{\dfrac{\partial(F_1, F_2, F_3, \cdots, F_n)}{\partial(u_1, u_2, u_3, \cdots, u_n)}}.$$

□

Corollary 15 If $u_1, u_2, u_3, \cdots, u_n$ are the functions of independent variables, $x_1, x_2, x_3, \cdots, x_n$ given by the implicit relations of the form:

$$\begin{aligned}
F_1(u_1, x_1, x_2, x_3, \cdots, x_n) &= 0, \\
F_2(u_1, u_2, x_1, x_2, x_3, \cdots, x_n) &= 0, \\
F_3(u_1, u_2, u_3, x_1, x_2, x_3, \cdots, x_n) &= 0, \\
&\vdots \\
F_n(u_1, u_2, u_3, \cdots, u_n, x_1, x_2, x_3, \cdots, x_n) &= 0,
\end{aligned}$$

then

$$\frac{\partial(u_1, u_2, u_3, \cdots, u_n)}{\partial(x_1, x_2, x_3, \cdots, x_n)} = (-1)^n \frac{\dfrac{\partial F_1}{\partial x_1} \cdot \dfrac{\partial F_2}{\partial x_2} \cdot \dfrac{\partial F_3}{\partial x_3} \cdots \dfrac{\partial F_n}{\partial x_n}}{\dfrac{\partial F_1}{\partial u_1} \cdot \dfrac{\partial F_2}{\partial u_2} \cdot \dfrac{\partial F_3}{\partial u_3} \cdots \dfrac{\partial F_n}{\partial u_n}}.$$

Proof. By hypothesis (or by assumption), we have

$$\frac{\partial(F_1, F_2, F_3, \cdots, F_n)}{\partial(x_1, x_2, x_3, \cdots, x_n)} = \frac{\partial F_1}{\partial x_1} \cdot \frac{\partial F_2}{\partial x_2} \cdot \frac{\partial F_3}{\partial x_3} \cdots \frac{\partial F_n}{\partial x_n}$$

and

$$\frac{\partial(F_1, F_2, F_3, \cdots, F_n)}{\partial(u_1, u_2, u_3, \cdots, u_n)} = \frac{\partial F_1}{\partial u_1} \cdot \frac{\partial F_2}{\partial u_2} \cdot \frac{\partial F_3}{\partial u_3} \cdots \frac{\partial F_n}{\partial u_n}.$$

Therefore, we have

$$\begin{aligned}
\frac{\partial(u_1, u_2, u_3, \cdots, u_n)}{\partial(x_1, x_2, x_3, \cdots, x_n)} &= (-1)^n \frac{\dfrac{\partial(F_1, F_2, F_3, \cdots, F_n)}{\partial(x_1, x_2, x_3, \cdots, x_n)}}{\dfrac{\partial(F_1, F_2, F_3, \cdots, F_n)}{\partial(u_1, u_2, u_3, \cdots, u_n)}} \\
&= (-1)^n \frac{\dfrac{\partial F_1}{\partial x_1} \cdot \dfrac{\partial F_2}{\partial x_2} \cdot \dfrac{\partial F_3}{\partial x_3} \cdots \dfrac{\partial F_n}{\partial x_n}}{\dfrac{\partial F_1}{\partial u_1} \cdot \dfrac{\partial F_2}{\partial u_2} \cdot \dfrac{\partial F_3}{\partial u_3} \cdots \dfrac{\partial F_n}{\partial u_n}}.
\end{aligned}$$

□

28.9.8 Functional relationship

Theorem 16 If the functions u, v, and w of three independent variables, x, y, and z are not independent then the Jacobian of u, v, and w with respect to x, y and z vanishes.

Proof. It is given that since the functions u, v, and w are not independent, then there will be a relation $F(u, v, w) = 0$, which will connect these independent variables.

Differentiating this relation with respect to x, y and z, we get

$$\frac{\partial F}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial F}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial F}{\partial w} \cdot \frac{\partial w}{\partial x} = 0, \tag{28.93}$$

$$\frac{\partial F}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial F}{\partial v} \cdot \frac{\partial v}{\partial y} + \frac{\partial F}{\partial w} \cdot \frac{\partial w}{\partial y} = 0, \tag{28.94}$$

$$\frac{\partial F}{\partial u} \cdot \frac{\partial u}{\partial z} + \frac{\partial F}{\partial v} \cdot \frac{\partial v}{\partial z} + \frac{\partial F}{\partial w} \cdot \frac{\partial w}{\partial z} = 0. \tag{28.95}$$

Eliminating $\frac{\partial F}{\partial u}, \frac{\partial F}{\partial v}$ and $\frac{\partial F}{\partial w}$ from (28.93), (28.94) and (28.95), we get

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial z} \\ \frac{\partial u}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial z} \\ \frac{\partial u}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

or

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = 0.$$

□

Theorem 17 Let $u_1, u_2, u_3, \cdots, u_n$ be the functions of n independent variables, $x_1, x_2, x_3, \cdots, x_n$. Then a necessary and sufficient condition for the existence of a relation of the form

$$F(u_1, u_2, u_3, \cdots, u_n) = 0$$

is that the Jacobian

$$\frac{\partial(u_1, u_2, u_3, \cdots, u_n)}{\partial(x_1, x_2, x_3, \cdots, x_n)}$$

should vanish identically.

Proof.

Necessary Condition: Assume there exists a relation such that

$$F(u_1, u_2, u_3, \cdots, u_n) = 0, \tag{28.96}$$

that is, $u_1, u_2, u_3, \cdots, u_n$ are not independent. Then we are to show that

$$\frac{\partial(u_1, u_2, u_3, \cdots, u_n)}{\partial(x_1, x_2, x_3, \cdots, x_n)} = 0.$$

Now, differentiating (28.96) partially with respect to $x_1, x_2, x_3, \cdots, x_n$, respectively, we get

$$
\begin{aligned}
\frac{\partial F}{\partial u_1}\cdot\frac{\partial u_1}{\partial x_1}+\frac{\partial F}{\partial u_2}\cdot\frac{\partial u_2}{\partial x_1}+\cdots+\frac{\partial F}{\partial u_n}\cdot\frac{\partial u_n}{\partial x_1}&=0,\\
\frac{\partial F}{\partial u_1}\cdot\frac{\partial u_1}{\partial x_2}+\frac{\partial F}{\partial u_2}\cdot\frac{\partial u_2}{\partial x_2}+\cdots+\frac{\partial F}{\partial u_n}\cdot\frac{\partial u_n}{\partial x_2}&=0,\\
&\vdots\\
\frac{\partial F}{\partial u_1}\cdot\frac{\partial u_1}{\partial x_n}+\frac{\partial F}{\partial u_2}\cdot\frac{\partial u_2}{\partial x_n}+\cdots+\frac{\partial F}{\partial u_n}\cdot\frac{\partial u_n}{\partial x_n}&=0.
\end{aligned}
$$

Since we cannot have the following result simultaneously

$$\frac{\partial F}{\partial u_1}=\frac{\partial F}{\partial u_2}=\frac{\partial F}{\partial u_3}=\cdots=\frac{\partial F}{\partial u_n}=0$$

otherwise (28.96) will reduce to a trivial identity, hence eliminating

$$\frac{\partial F}{\partial u_1},\frac{\partial F}{\partial u_2},\frac{\partial F}{\partial u_3},\cdots,\frac{\partial F}{\partial u_n}$$

from these above equations, we get

$$\begin{vmatrix}
\frac{\partial u_1}{\partial x_1} & \frac{\partial u_2}{\partial x_1} & \cdots & \frac{\partial u_n}{\partial x_1}\\
\frac{\partial u_1}{\partial x_2} & \frac{\partial u_2}{\partial x_2} & \cdots & \frac{\partial u_n}{\partial x_2}\\
\vdots & \vdots & \ddots & \vdots\\
\frac{\partial u_1}{\partial x_n} & \frac{\partial u_2}{\partial x_n} & \cdots & \frac{\partial u_n}{\partial x_n}
\end{vmatrix}=0.$$

Now, interchanging rows and columns[2], we get

$$\begin{vmatrix}
\frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \cdots & \frac{\partial u_1}{\partial x_3}\\
\frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \cdots & \frac{\partial u_2}{\partial x_3}\\
\vdots & \vdots & \ddots & \vdots\\
\frac{\partial u_n 1}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \cdots & \frac{\partial u_n}{\partial x_3}
\end{vmatrix}=0.$$

or

$$\frac{\partial(u_1,u_2,u_3,\cdots,u_n)}{\partial(x_1,x_2,x_3,\cdots,x_n)}=0.$$

[2]Let A be a square matrix. Then the determinants of a matrix and its transpose are equal, that is, $|A|=|A^T|$.

Sufficient Condition: If
$$\frac{\partial(u_1, u_2, u_3, \cdots, u_n)}{\partial(x_1, x_2, x_3, \cdots, x_n)} = 0,$$
then we have to prove that $u_1, u_2, u_3, \cdots, u_n$ are not independent, that is, there must exist a relation $u_1, u_2, u_3, \cdots, u_n$.

Let the function $u_1, u_2, u_3, \cdots, u_n$ and the independent variables $x_1, x_2, x_3, \cdots, x_n$ be connected by the following set of equations,

$$\begin{aligned}
F_1(u_1, x_1, x_2, x_3, \cdots, x_n) &= 0,\\
F_2(u_1, u_2, , x_2, x_3, \cdots, x_n) &= 0,\\
F_3(u_1, u_2, u_3, x_3, \cdots, x_n) &= 0,\\
&\vdots\\
F_r(u_1, u_2, u_3, u_r, x_r, x_{r+1}, \cdots, x_n) &= 0,\\
&\vdots\\
F_n(u_1, u_2, u_3, \cdots, u_n) &= 0.
\end{aligned}$$

Now,

$$\frac{\partial(u_1, u_2, u_3, \cdots, u_n)}{\partial(x_1, x_2, x_3, \cdots, x_n)} = (-1)^n \frac{\dfrac{\partial F_1}{\partial x_1} \cdot \dfrac{\partial F_2}{\partial x_2} \cdot \dfrac{\partial F_3}{\partial x_3} \cdots \dfrac{\partial F_n}{\partial x_n}}{\dfrac{\partial F_1}{\partial u_1} \cdot \dfrac{\partial F_2}{\partial u_2} \cdot \dfrac{\partial F_3}{\partial u_3} \cdots \dfrac{\partial F_n}{\partial u_n}},$$

since

$$\frac{\partial(u_1, u_2, u_3, \cdots, u_n)}{\partial(x_1, x_2, x_3, \cdots, x_n)} = 0. \tag{28.97}$$

Therefore, from (28.97), we have

$$\frac{\partial F_1}{\partial x_1} \cdot \frac{\partial F_2}{\partial x_2} \cdot \frac{\partial F_3}{\partial x_3} \cdots \frac{\partial F_n}{\partial x_n}.$$

Hence, $\dfrac{\partial F_r}{\partial x_r}$ =0, for some value of r between 1 and n. Thus for a particular value of r, F_r must be free from x_r. Consequently, F_r is of the form,

$$F_r(u_1, u_2, u_3, u_r, x_{r+1}, x_{r+2} \cdots, x_n) = 0$$

which contains only the independent variables $x_{r+1}, x_{r+2} \cdots, x_n$ in addition to the u's. We can now eliminate the $(n-r)$ independent variables $x_{r+1}, x_{r+2} \cdots, x_n$ from $(n-r+1)$ equations given by

$$F_r = F_{r+1} = \cdots = F_n = 0$$

and we can find a functional relation, that is

$$F_n(u_1, u_2, u_3, \cdots, u_n) = 0$$

between the $u-$variables.

□

Theorem 18

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} \cdot \frac{\partial(x,y,z)}{\partial(u,v,w)} = 1 \tag{28.98}$$

or

$$J \cdot J^T = 1$$

where T stands for the transpose.

Proof.
Let $u = f_1(x,y,z)$, $v = f_2(x,y,z)$ and $w = f_3(x,y,z)$ then we may write these equations as

$$\begin{aligned} x &= \phi_1(u,v,w); \\ y &= \phi_2(u,v,w); \\ z &= \phi_3(u,v,w). \end{aligned}$$

Differentiating $u = f_1(x,y,z)$ partially with respect to u, v and w, we get

$$\begin{aligned} \frac{\partial u}{\partial u} &= \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial u} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial u} \\ 1 &= \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial u} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial u}, \end{aligned} \tag{28.99}$$

$$\begin{aligned} \frac{\partial u}{\partial v} &= \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial v} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial v} \\ 0 &= \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial v} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial v}, \end{aligned} \tag{28.100}$$

and

$$\begin{aligned} \frac{\partial u}{\partial w} &= \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial w} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial w} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial w} \\ 0 &= \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial w} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial w} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial w}. \end{aligned} \tag{28.101}$$

Now differentiating $v = f_2(x,y,z)$ and $w = f_3(x,y,z)$ partially with respect to u, v and w, we get

$$0 = \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial u} + \frac{\partial v}{\partial z} \cdot \frac{\partial z}{\partial u}, \tag{28.102}$$

$$1 = \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial v} + \frac{\partial v}{\partial z} \cdot \frac{\partial z}{\partial v}, \tag{28.103}$$

$$0 = \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial w} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial w} + \frac{\partial v}{\partial z} \cdot \frac{\partial z}{\partial w}., \tag{28.104}$$

$$0 = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial u} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial u}, \tag{28.105}$$

$$0 = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial v} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial v}, \tag{28.106}$$

and

$$1 = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial w} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial w} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial w}.. \tag{28.107}$$

Putting these results into the (28.98), we have

$$\begin{aligned}
\frac{\partial(u,v,w)}{\partial(x,y,z)} \cdot \frac{\partial(x,y,z)}{\partial(u,v,w)} &= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial z} \\ \frac{\partial u}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial z} \\ \frac{\partial u}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} \cdot \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \\
&= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \\
&= 1.
\end{aligned}$$

□

28.9.9 Worked examples

Example 35 If $x = r\cos\theta$ and $y = r\sin\theta$ show that

$$\frac{\partial(x,y)}{\partial(r,\theta)} = r.$$

Solution. We have

$$\begin{aligned}
J &= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} \\
&= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos \end{vmatrix} \\
&= r\cos^2\theta + r^2\sin^2\theta \\
&= r.
\end{aligned}$$

□

Example 36 If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$ and $z = r\cos\theta$, show that

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta.$$

Solution. We have

$$\begin{aligned} J &= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} \\ &= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} \\ &= r^2\sin\theta. \end{aligned}$$

□

Example 37 The roots of the equation in λ :

$$(\lambda - x)^3 + (\lambda - y)^3 + (\lambda - z)^3 = 0$$

are u, v, and w. Prove that

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = -2\frac{(y-z)(z-x)(x-y)}{(u-v)(v-w)(w-u)}.$$

Solution. The given equation can be written as

$$3\lambda^3 - 3\lambda^2(x+y+z) + 3\lambda(x^2+y^2+z^2) - (x^3+y^3+z^3) = 0.$$

If u, v, and w are the roots of the above equation, then we shall have

$$\begin{aligned} \sum u &= x+y+z \\ \sum uv &= x^2+y^2+z^2 \\ uvw &= \frac{1}{3}(x^3+y^3+z^3). \end{aligned}$$

Now let us consider the functions

$$\begin{aligned} f_1 &= u+v+w-x-y-z = 0 \\ f_2 &= uv+vw+wu-x^2-y^2-z^2 \\ f_3 &= uvw - \frac{1}{3}(x^3+y^3+z^3). \end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)} &= \begin{vmatrix} -1 & -1 & -1 \\ 2x & -2y & -2z \\ -x^2 & -y^2 & -z^2 \end{vmatrix} \\ &= -2(x-y)(y-z)(z-x).\end{aligned}$$

Also

$$\begin{aligned}\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)} &= \begin{vmatrix} 1 & 1 & 1 \\ v+w & w+u & u+v \\ vw & wu & uv \end{vmatrix} \\ &= (u-v)(v-w)(w-u).\end{aligned}$$

Hence, applying Theorem 13 gives

$$\begin{aligned}\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)} &= \frac{\dfrac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}}{\dfrac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}} \\ &= -2\frac{(y-z)(z-x)(x-y)}{(u-v)(v-w)(w-u)}.\end{aligned}$$

□

Example 38 Show that the functions

$$\begin{aligned} u &= x+y-z, \\ v &= x-y+z, \\ w &= x^2+y^2+z^2-2yz \end{aligned}$$

are not independent of one another. Also find the relation between them.

Solution. We have

$$\begin{aligned}\frac{\partial(u,v,w)}{\partial(x,y,z)} &= \begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 2x & 2(y-z) & 2(z-y) \end{vmatrix} \\ &= \begin{vmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 2x & 2(y-z) & 0 \end{vmatrix}.\end{aligned}$$

Since the Jacobian has vanished, then from Theorem 13, it can be said that the functions u, v and w are not independent.

Now, we have

$$\begin{aligned} u+v &= 2x \\ u-v &= 2(y-z) \end{aligned}$$

so

$$\begin{aligned}(u+v)^2+(u-v)^2 &= 4x^2+4(y-z)^2\\ &= 4\left(x^2+y^2+z^2-2yz\right)\\ &= 4w.\end{aligned}$$

□

Example 39 If $u=\dfrac{x+y}{1-xy}$ and $v=\tan^{-1}x+\tan^{-1}y$, find $\dfrac{\partial(u,v)}{\partial(x,y)}$. Are u and v functionally related? If so, find the relationship between them.

Solution. We have

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{1\cdot(1-xy)-(-y)\cdot(x+y)}{(1-xy)^2}=\frac{1+y^2}{(1-xy)^2},\\ \frac{\partial u}{\partial y} &= \frac{1\cdot(1-xy)-(-x)\cdot(x+y)}{(1-xy)^2}=\frac{1+x^2}{(1-xy)^2},\\ \frac{\partial v}{\partial x} &= \frac{1}{1+x^2},\\ \frac{\partial v}{\partial y} &= \frac{1}{1+y^2}.\end{aligned}$$

Now, we have

$$\begin{aligned}\frac{\partial(u,v)}{\partial(x,y)} &= \begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}\\ &= \frac{1+y^2}{(1-xy)^2}\cdot\frac{1}{1+y^2}-\frac{1+x^2}{(1-xy)^2}\cdot\frac{1}{1+x^2}\\ &= 0.\end{aligned}$$

Since the Jacobian of the functions u and v is zero, these functions are not independent so and they must be functionally related. We have

$$v=\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)=\tan^{-1}u.$$

Thus $v=\tan^{-1}u$ or $\tan v=u$ is the required relation between u and v.

□