Nested Square roots

Yue Kwok Choy

Nested square roots problems are very interesting. In this article, we investigate some mathematical techniques applied to this topic that most senior secondary school students can understand.

1. $\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}$

(a) We put $x=\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}$

Then $x^2=1+\sqrt{1+\sqrt{1+\cdots}}$

$x^2-1=\sqrt{1+\sqrt{1+\cdots}}=x$

We get a quadratic equation : $x^2-x-1=0$

Solving, we have $x=\frac{1+\sqrt{5}}{2}\approx 1.6180339887\ \ldots$

Note that the negative root is rejected since $x>0$.

You may also notice that $\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}=\frac{1+\sqrt{5}}{2}=\varphi$, the famous ***Golden Ratio***.

(b) It seems that our job is done, but in fact we still need to show the **convergence** of $\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}$.

We write $x_1=\sqrt{1}=1$

$$x_2=\sqrt{1+\sqrt{1}}=\sqrt{2}\approx 1.4142$$

$$x_3=\sqrt{1+\sqrt{1+\sqrt{1}}}=\sqrt{1+\sqrt{2}}\approx 1.5538$$

$$x_4=\sqrt{1+\sqrt{1+\sqrt{2}}}\approx 1.5981$$

…. $x_n=\sqrt{1+\sqrt{1+\sqrt{1+\cdots+\sqrt{1}}}}$ (n square roots)

We note that: **(1)** The sequence x_n 'may be' increasing.

(2) $x_n=\sqrt{1+x_{n-1}}$

We apply the ***Monotone Convergence Theorem***, which states that every monotonic increasing (or decreasing) sequence bounded above (below) has a limit.

(i) To prove x_n is **increasing**, we use ***Mathematical Induction***.

Let P(n) : $x_n < x_{n+1}$

P(1) is true since $x_1 = 1 < \sqrt{2} = x_2$

Assume P(k) is true for some k∈**N** , that is $x_k < x_{k+1}$ …(1)

For P(k + 1), From (1) $1 + x_k < 1 + x_{k+1}$

$$x_{k+1} = \sqrt{1 + x_k} < \sqrt{1 + x_{k+1}} = x_{k+2}$$

∴ P(k + 1) is true.

By the Principle of Mathematical Induction, P(n) is true ∀n∈**N** .

(ii) To prove x_n is **bounded**, we also use ***Mathematical Induction***.

Let P(n) : $x_n < 2$ (We use 2 here instead of $\frac{1+\sqrt{5}}{2}$ to simplify our writing.)

P(1) is true since $x_1 = 1 < 2$

Assume P(k) is true for some k∈**N** , that is $x_k < 2$ …(2)

For P(k + 1), From (2) $1 + x_k < 3$

$$x_{k+1} = \sqrt{1 + x_k} < \sqrt{3} < 2$$

∴ P(k + 1) is true.

By the Principle of Mathematical Induction, P(n) is true ∀n∈**N** .

Finally we use the ***Monotone Convergence Theorem***, x_n has a limit, and $\lim_{n\to\infty} x_n = \frac{1+\sqrt{5}}{2}$.

<u>Quiz</u> In **(1)** – **(3)** below, you may omit the proof of convergence.

(1) Show that $\sqrt{a + \sqrt{a + \sqrt{a + \cdots}}} = \frac{1+\sqrt{1+4a}}{2}$ (a > 0)

(2) Show that $\sqrt{a + b\sqrt{a + b\sqrt{a + \cdots}}} = \frac{b+\sqrt{b^2+4a}}{2}$ (a, b > 0)

(3) Show that $\sqrt{2 - \sqrt{2 + \sqrt{2 - \sqrt{2 + \cdots}}}} = \frac{\sqrt{5}-1}{2}$

(4) Find the mistake to prove that 1 = 0 :

In **(1)**, take limit a → 0, $\lim_{a\to 0} \sqrt{a + \sqrt{a + \sqrt{a + \cdots}}} = \lim_{a\to 0} \frac{1+\sqrt{1+4a}}{2} = 1$

But obviously, $\sqrt{0 + \sqrt{0 + \sqrt{0 + \cdots}}} = 0$

∴ 1 = 0 .

2. $\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}$

We change our direction to employ trigonometry to tackle this.

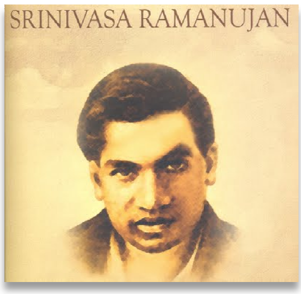
$\cos\frac{\pi}{4}=\frac{\sqrt{2}}{2}$ (for those who don't know radian, take π radian $=180^{\circ}$)

$\cos\frac{\pi}{8}=\frac{\sqrt{2(1+\sin\frac{\pi}{4})}}{2}=\frac{\sqrt{2+\sqrt{2}}}{2}$ (Here we use half-angle formula. Please fill in the calculations.)

$\cos\frac{\pi}{16}=\frac{\sqrt{2(1+\sin\frac{\pi}{8})}}{2}=\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}$ ….

Hence, $\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}=2\lim_{n\to\infty}\cos\frac{\pi}{2^n}=2$

3. $\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+\cdots}}}}$



The Indian mathematics genius Ramanujan once published this problem in the Indian Mathematical journal. He waited for more than 6 months, but no one came forward with a solution.

He then discovered that:

$$x+n+a=$$

$$\sqrt{ax+(n+a)^2+x\sqrt{a(x+n)+(n+a)^2+(x+n)\sqrt{a(x+2n)+(n+a)^2+(x+2n)\sqrt{\ldots}}}} \quad \ldots (3)$$

This problem is then a special case where a = 0, n = 1, x = 2.

We don't discuss the story here as it is a bit involved. Readers interested may study:

http://en.wikipedia.org/wiki/Denesting_radicals#Infinitely_nested_radicals

However, we can still carry out our informal investigation on $\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+\cdots}}}}$

Firstly study that,

$$x_1=\sqrt{1}=1$$

$$x_2=\sqrt{1+2\sqrt{1}}=\sqrt{3}\approx 1.7321$$

$$x_3=\sqrt{1+2\sqrt{1+3\sqrt{1}}}\approx 2.2361$$

$$x_4 = \sqrt{1 + 2\sqrt{1 + +3\sqrt{1 + 4\sqrt{1}}}} \approx 2.5598$$

We therefore **guess** that :

$$x_n = \sqrt{1 + 2\sqrt{1 + +3\sqrt{1 + 4\sqrt{1 + \cdots n\sqrt{1}}}}} \to 3$$

We then apply the following identity: $n = \sqrt{1 + (n-1)(n+1)}$, for $n \geq 1$ repeatedly:

$$3 = \sqrt{1 + 2 \times 4} = \sqrt{1 + 2\sqrt{3 \times 5}} = \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4 \times 6}}}$$

$$= \cdots = \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + \cdots + (k-1)\sqrt{1 + k(k+2)}}}}}$$

$$\overset{??}{=} \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + \cdots}}}}$$

The proof above may already look for most readers, but I still put "??" in the last step, as it seems to be a not so mathematical "**deduction**".

Studying the last expression above more thoroughly, we may **guess** a more general formula :

$$4 = \sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + \cdots}}}$$

$$5 = \sqrt{1 + 4\sqrt{1 + 5\sqrt{1 + \cdots}}}$$

…

$$n = \sqrt{1 + (n-1)\sqrt{1 + n\sqrt{1 + (n+1)\sqrt{1 + \cdots}}}}$$

If we **assume** $3 = \sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+\cdots}}}}$ is proved, we can begin our induction as follows:

Let P(n) : $n = \sqrt{1+(n-1)\sqrt{1+n\sqrt{1+(n+1)\sqrt{1+\cdots}}}}$

Assume P(k) is true for some $k\in\mathbf{N}$, that is

$$k = \sqrt{1+(k-1)\sqrt{1+k\sqrt{1+(k+1)\sqrt{1+\cdots}}}} \qquad \ldots(4)$$

For P(k + 1), From (4), $k^2 = 1+(k-1)\sqrt{1+k\sqrt{1+(k+1)\sqrt{1+\cdots}}}$

$$k^2 - 1 = (k-1)\sqrt{1+k\sqrt{1+(k+1)\sqrt{1+\cdots}}}$$

$$(k-1)(k+1) = (k-1)\sqrt{1+k\sqrt{1+(k+1)\sqrt{1+\cdots}}}$$

$$k+1 = \sqrt{1+k\sqrt{1+(k+1)\sqrt{1+\cdots}}}$$

$\therefore$ P(k + 1) is true.

By the Principle of Mathematical Induction, P(n) is true $\forall n\in\mathbf{N}$.

So we have proved $n = \sqrt{1+(n-1)\sqrt{1+n\sqrt{1+(n+1)\sqrt{1+\cdots}}}}$ which is a step closer to the Ramanujan formula given by (3).

Quiz

Find $\sqrt[3]{1+\sqrt[3]{1+\sqrt[3]{1+\cdots}}}$