THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1510 Calculus for Engineers (2020-2021)
Solution to Supplementary Exercise 2

Set Notations

1. Describe the elements in the following sets.

   (a) $\{2, 4\}$;
   Ans: The set only consists of two elements 2 and 4.

   (b) $(2, 4)$;
   Ans: The set of all real numbers x such that $2 < x < 4$.

   (c) $[2, 4]$.
   Ans: The set of all real numbers x such that $2 \leq x \leq 4$.

2. Describe the elements in the following sets.

   (a) $\mathbb{R}\backslash[2, 4]$;
   Ans: The set of all real numbers x such that $x < 2$ or $x > 4$.

   (b) $\mathbb{R}\backslash\{2, 4\}$;
   Ans: The set of all real numbers x except 2 and 4.

   (c) $(-\infty, 2) \cup (4, \infty)$;
   Ans: The set of all real numbers x such that $x < 2$ or $x > 4$.

   (d) $\mathbb{Z}^+ \cap (5, \infty)$;
   Ans: The set of all positive integers n such that $n > 5 = \{6, 7, 8, \ldots\}$.

   (e) $\mathbb{Z}^+ \cap [5, \infty)$.
   Ans: The set of all positive integers n such that $n \geq 5 = \{5, 6, 7, \ldots\}$.

   Remark: Here we use $\mathbb{Z}$ to denote the set of all integers and $\mathbb{Z}^+$ to denote the set of all positive integers.

3. Describe the elements in the following sets.

   (a) $\{x \in \mathbb{R} : x \geq 3\}$;
   Ans: The set of all real numbers x such that $x \geq 3 = [3, \infty)$.

   (b) $\{n \in \mathbb{Z}^+ : n \geq 3\}$;
   Ans: The set of all positive integers n such that $n \geq 3 = \{3, 4, 5, \ldots\}$.

   (c) $\{m \in \mathbb{Z} : -5 < m < 5\}$;
   Ans: The set of all integers m such that $-5 < m < 5 = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$.

   (d) $\{2m - 1 : m \in \mathbb{Z}^+\}$;
   Ans: The set of all positive odd integers $= \{1, 3, 5, \ldots\}$.

(e) $\{3n : n \in \mathbb{Z}^+\}$.
Ans: The set of all positive integers which are divisible by $3 = \{3, 6, 9, \ldots\}$.

4. Using set notations to describe the following sets.

   (a) the set of all real numbers except -1 and 1;
   Ans: $\mathbb{R}\backslash\{-1, 1\}$ or $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

   (b) the set of all positive real numbers x such that $x < 1$ or $x > 6$;
   Ans: $\{x \in \mathbb{R}^+ : x < 1 \,\text{or}\, x > 6\}$ or $(0, 1) \cup (6, \infty)$.

   (c) the set of all positive even integers;
   Ans: $\{2n : n \in \mathbb{Z}^+\}$ or $\{2, 4, 6, \ldots\}$.

   (d) the set of all integers which are divisible by 5.
   Ans: $\{5m : m \in \mathbb{Z}\}$ or $\{\ldots, -10, -5, 0, 5, 10, \ldots\}$.

   (Remark: The way of describing a set is not unique.)

Functions

5. Describe the domain and range of each of the following functions.

   (a) $f(x) = \sqrt{x - 1}$;
   Ans: Domain $= [1, \infty)$; Range $= [0, \infty)$.

   (b) $f(x) = \dfrac{1}{x^2}$;
   Ans: Domain $= (-\infty, 0) \cup (0, \infty) = \mathbb{R}\backslash\{0\}$; Range $= (0, \infty)$.

   (c) $g(x) = \sin x$;
   Ans: Domain $= (-\infty, \infty) = \mathbb{R}$; Range $= [-1, 1]$.

   (d) $g(x) = 2 + 3\cos x^2$;
   Ans: Domain $= (-\infty, \infty) = \mathbb{R}$; Range $= [-1, 5]$.

   (e) $h(x) = \log_2 x$;
   Ans: Domain $= (0, \infty)$; Range $= (-\infty, \infty) = \mathbb{R}$.

   (f) $h(x) = 3^x$.
   Ans: Domain $= (-\infty, \infty) = \mathbb{R}$; Range $= (0, \infty)$.

6. Describe the domain of each of the following functions.

   (a) $f(x) = \dfrac{1}{x^2 - 4x - 12}$;
   Ans: Domain $= \mathbb{R}\backslash\{-2, 6\}$.

   (b) $f(x) = \dfrac{1}{\sqrt{4 - x^2}}$;
   Ans: Domain $= (-2, 2)$.

7. Consider the following functions:

$$f(x) = \sqrt{x} \qquad \text{and} \qquad g(x) = x + 5.$$

Find the formulas explicitly describing $f+g$, fg, $f \circ g$ and $g \circ f$; and state the domains of the functions. Furthermore, state the range of $f \circ g$ and $g \circ f$.

Ans: $(f+g)(x) = x + \sqrt{x} + 5$, Domain $= [0, \infty)$;

$(fg)(x) = \sqrt{x}(x+5) = x^{3/2} + 5x^{1/2}$, Domain $= [0, \infty)$;

$(f \circ g)(x) = f(g(x)) = \sqrt{x+5}$, Domain $= [-5, \infty)$, Range $= [0, \infty)$;

$(g \circ f)(x) = g(f(x)) = \sqrt{x} + 5$, Domain $= [0, \infty)$, Range $=[5, \infty)$.

8. Consider the function $f(x)$ defined by

$$f(x) = \begin{cases} x+1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Find the value of $f(-1)$, $f(0)$ and $f(1)$.

Ans: $f(-1) = 0$, $f(0) = 1$ and $f(1) = 2$.

9. Consider the function $f(x)$ defined by

$$f(x) = \begin{cases} \sqrt{x} & \text{if } x \geq 4, \\ \dfrac{1}{x-4} & \text{if } x < 4. \end{cases}$$

Find the value of $f(0)$, $f(4)$ and $f(9)$.

Ans: $f(0) = -\frac{1}{4}$, $f(4) = 2$ and $f(9) = 3$.

10. Fill in the blanks.

Ans:

(a) Consider the function $f(x) = |x|$. The function can be described explicitly by

$$f(x) = \begin{cases} \underline{\quad x \quad} & \text{if } x \geq 0, \\ \underline{\quad -x \quad} & \text{if } x < 0. \end{cases}$$

(b) Consider the function $f(x) = |x^2 - 9|$. The function can be described explicitly by

$$f(x) = \begin{cases} \underline{\quad x^2 - 9 \quad} & \text{if } x \geq 3, \\ \underline{\quad 9 - x^2 \quad} & \text{if } -3 < x < 3, \\ \underline{\quad x^2 - 9 \quad} & \text{if } x \leq -3 \end{cases}$$

Graphs of Functions

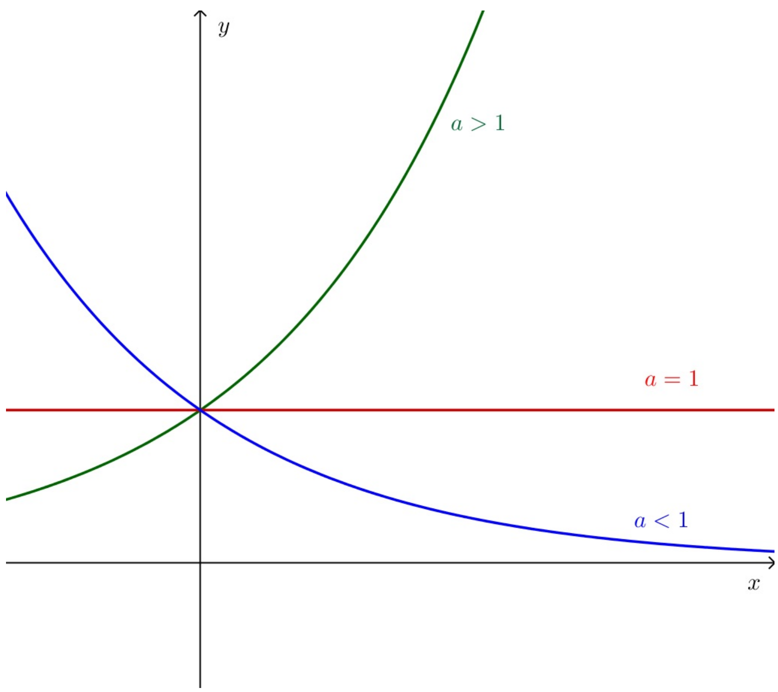
11. Sketch the graph of $y = f(x) = a^x$ if

    (a) $a > 1$;

    (b) $a = 1$;

    (c) $0 < a < 1$.

    Ans:



12. Let $f(x) = e^x$. Sketch the graphs of the following functions.

    (a) $y = 3f(x)$;

    (b) $y = -3f(x)$;

    (c) $y = f(x + 3)$;

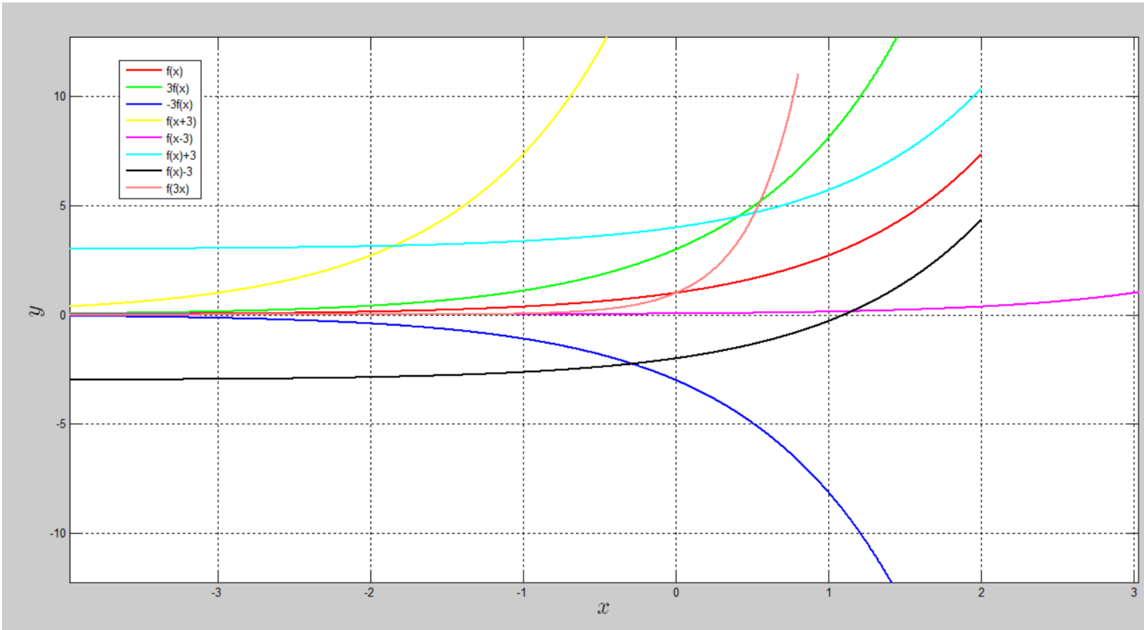
    (d) $y = f(x - 3)$;

    (e) $y = f(x) + 3$;

    (f) $y = f(x) - 3$;

    (g) $y = f(3x)$.

    (Remark: What is the relation between each of the graph and the graph of $f(x)$?)
    Ans:



Summation Notation

13. Write down the expansion of the following expressions.

(e.g.) $\displaystyle\sum_{i=1}^{5} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2;$

(a) $\displaystyle\sum_{i=1}^{4} (2i+3)^2;$

Ans: $5^2 + 7^2 + 9^2 + 11^2$.

(b) $\displaystyle\sum_{i=2}^{5} (i^2+3);$

Ans: $7 + 12 + 19 + 28$.

(c) $\displaystyle\sum_{r=0}^{5} 2^r;$

Ans: $1 + 2 + 4 + 8 + 16 + 32$.

(d) $\displaystyle\sum_{r=0}^{7} \left(-\frac{1}{2}\right)^r.$

Ans: $1 - \dfrac{1}{2} + \dfrac{1}{4} - \dfrac{1}{8} + \dfrac{1}{16} - \dfrac{1}{32} + \dfrac{1}{64} - \dfrac{1}{128}$.

14. Write down the expansion of the following expressions.

(e.g.) $\displaystyle\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots;$

(a) $\displaystyle\sum_{i=5}^{n}\left(\frac{1}{3}\right)^i$;

Ans: $\displaystyle\sum_{i=5}^{n}\left(\frac{1}{3}\right)^i = \frac{1}{3^5}+\frac{1}{3^6}+\frac{1}{3^7}+\cdots+\frac{1}{3^n}$;

(b) $\displaystyle\sum_{r=0}^{4}\frac{x^r}{r!}$;

Ans: $\displaystyle\sum_{r=0}^{4}\frac{x^r}{r!} = 1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}$.

(c) $\displaystyle\sum_{r=0}^{\infty}(-1)^r\frac{x^{2r+1}}{(2r+1)!}$;

(Recall: If n is a positive integer, $n! = 1\times 2\times 3\times\cdots\times n$ and we define $0! = 1$.)

Ans: $\displaystyle\sum_{r=0}^{\infty}(-1)^r\frac{x^{2r+1}}{(2r+1)!} = x-\frac{x^3}{6}+\frac{x^5}{120}-\cdots+(-1)^r\frac{x^{2r+1}}{(2r+1)!}+\cdots$.

(d) $\displaystyle\sum_{r=0}^{n}(-1)^r\frac{x^{2r}}{(2r)!}$;

Ans: $\displaystyle\sum_{r=0}^{n}(-1)^r\frac{x^{2r}}{(2r)!} = 1-\frac{x^2}{2}+\frac{x^4}{24}-\cdots+(-1)^n\frac{x^{2n}}{(2n)!}$.

(e) $\displaystyle\sum_{r=1}^{3}\frac{1}{r^2}\sin rx$;

Ans: $\displaystyle\sum_{r=1}^{3}\frac{1}{r^2}\sin rx = \sin x+\frac{1}{4}\sin 2x+\frac{1}{9}\sin 3x$.

(f) $\displaystyle\sum_{r=0}^{n}\frac{(-1)^r}{r!}\cos(2r+1)x$.

Ans: $\displaystyle\sum_{r=0}^{n}\frac{(-1)^r}{r!}\cos(2r+1)x = \cos x-\cos 3x+\frac{1}{2}\cos 5x-\frac{1}{6}\cos 7x+\cdots+\frac{(-1)^n}{n!}\cos(2n+1)x$.

Parametrized Curves

15. Let $(x(t), y(t)) = (\cos t, \sin t)$, for $t\in\mathbb{R}$, be a curve defined on $\mathbb{R}^2$.

(a) Write down the equation of the curve in x and y only.

(b) What is the curve?

Ans:

(a) We have $x = \cos t$ and $y = \sin t$, then $x^2 = \cos^2 t$ and $y^2 = \sin^2 t$. By adding them up, we have $x^2+y^2 = 1$.

(b) The curve is the unit circle (i.e. radius is 1) centered at the origin.

(Remark: If $(x(t), y(t))$ describes a moving point, then as t increases, the point is moving along the circle in counter-clockwise direction.)

16. Let $A = (x_1, y_1)$ and $B = (x_2, y_2)$ be two distinct points on $\mathbb{R}^2$.

Let $(x(t), y(t)) = t(x_2, y_2) + (1-t)(x_1, y_1) = (x_1 + t(x_2 - x_1), y_1 + t(y_2 - y_1))$, for $t \in [0, 1]$, be a curve defined on $\mathbb{R}^2$.

(a) Find the endpoints $(x(0), y(0))$ and $(x(1), y(1))$ of the curve.

(b) Write down the equation of the curve in x and y only.

(c) What is the curve?

Ans:

(a) $(x(0), y(0)) = (x_1, y_1)$ which is the point A and $(x(1), y(1)) = (x_2, y_2)$ which is the point B.

(b) We have $x = x_1 + t(x_2 - x_1)$ and $y_1 + t(y_2 - y_1)$. Since A and B are two distinct points, we have either $x_1 \neq x_2$ or $y_1 \neq y_2$:

- If $x_1 = x_2$, (then $y_1 \neq y_2$), then we have $x = x_1$ which is a vertical line.
- If $y_1 = y_2$, (then $x_1 \neq x_2$), then we have $y = y_1$ which is a horizontal line.
- If $x_1 \neq x_2$ and $y_1 \neq y_2$, then we have $\dfrac{x - x_1}{x_2 - x_1} = t$ and $\dfrac{y - y_1}{y_2 - y_1} = t$. By eliminating t, we have $\dfrac{x - x_1}{x_2 - x_1} = \dfrac{y - y_1}{y_2 - y_1}$, i.e. $\dfrac{y - y_1}{x - x_1} = \dfrac{y_2 - y_1}{x_2 - x_1}$.

(c) The curve is the line segment joining points A and B.

(Remark: If $(x(t), y(t))$ describes a moving point, then as t increases, the point is moving along the line segment starting at A and ending at B.)

17. Let $(x(t), y(t)) = (3\cos t - 2, 3\sin t + 1)$, for $t \in \mathbb{R}$, be a curve defined on $\mathbb{R}^2$.

(a) Write down the equation of the curve in x and y only.

(b) What is the curve?

Ans:

(a) We have $x + 2 = 3\cos t$ and $y - 1 = 3\sin t$, then $(x+2)^2 = 9\cos^2 t$ and $(y-1)^2 = 9\sin^2 t$. By adding them up, we have $(x+2)^2 + (y-1)^2 = 9$.

(b) The curve is the circle centered at $(-2, 1)$ with radius 3.

(Remark: If $(x(t), y(t))$ describes a moving point, then as t increases, the point is moving along the circle in counter-clockwise direction.)

18. Let $(x(t), y(t)) = (t^2, t^3)$, for $t \in \mathbb{R}$, be a curve defined on $\mathbb{R}^2$. Write down the equation of the curve in x and y only.

Ans: We have $x = t^2$ and $y = t^3$, then $x^3 = t^6$ and $y^2 = t^6$. By eliminating t, we have $x^3 = y^2$.

19. Let $(x(t), y(t)) = (a\cos t, b\sin t)$, for $t \in \mathbb{R}$, $a, b > 0$, be a curve defined on $\mathbb{R}^2$. Write down the equation of the curve in x and y only.

    Ans: We have $(x/a)^2 = \cos^2 t$ and $(y/b)^2 = \sin^2 t$, then by eliminating t, we have $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.
    (Remark: The curve is the ellipse which passes through $(\pm a, 0)$ and $(0, \pm b)$.)

Sequences

20. A sequence $\{a_n\}$ is defined recursively by the following equations:

$$\begin{cases} a_1 = 2 \\ a_{n+1} = a_n^2 + 1 \text{ for } n \geq 1 \end{cases}$$

    Find the first 4 terms of the sequence.

    Ans: We have

$$\begin{aligned} a_1 &= 2 \\ a_2 &= a_1^2 + 1 = 2^2 + 1 = 5 \\ a_3 &= a_2^2 + 1 = 5^2 + 1 = 26 \\ a_4 &= a_3^2 + 1 = 26^2 + 1 = 677 \end{aligned}$$

21. A sequence $\{a_n\}$ is defined recursively by the following equations:

$$\begin{cases} a_1 = 1 \text{ and } a_2 = 2 \\ a_n = 2a_{n-1} + a_{n-2} \text{ for } n \geq 3 \end{cases}$$

    Find a_4.

    Ans: We have

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 2 \\ a_3 &= 2a_2 + a_1 = 2 \times 2 + 1 = 5 \\ a_4 &= 2a_3 + a_2 = 2 \times 5 + 2 = 12 \end{aligned}$$

22. Let $\{a_n\}$ be a sequence defined by $a_n = \dfrac{2n+1}{n+3}$ for any positive integer n.

    Complete the following table.

n	10	100	1000	10000
a_n				

    By observation, when n is getting bigger and bigger, what value does a_n get closer and closer to? Hence, guess the value of $\lim_{n\to\infty} a_n$.

Ans:

n	10	100	1000	10000
a_n	1.6154	1.9515	1.9950	1.9995

By observation, when n is getting bigger and bigger, a_n gets closer and closer to 2. Hence, we guess $\lim_{n\to\infty} a_n = 2$.

(Remark: This table only gives an idea why $\lim_{n\to\infty} a_n = 2$, but it is not a formal proof.)

23. For each of the following sequences, find $\lim_{n\to\infty} a_n$, if it exists.

(a) $a_n = \left(\frac{1}{3}\right)^n$;

Ans: $\lim_{n\to\infty} a_n = 0$.

(b) $a_n = (-1)^n$;

Ans: The limit does not exist (it is an oscillating sequence).

(c) $a_n = 3^n$;

Ans: The limit does not exist (it diverges to infinity).

(d) $a_n = \dfrac{n^2 - n + 3}{3n^2 + 2n}$;

Ans: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{n^2 - n + 3}{3n^2 + 2n} = \lim_{n\to\infty} \dfrac{1 - \frac{1}{n} + \frac{3}{n^2}}{3 + \frac{2}{n}} = \dfrac{1 - 0 + 0}{3 + 0} = \dfrac{1}{3}$.

(e) $a_n = \dfrac{6n + 3}{2n^2 + 9n - 5}$;

Ans: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{6n + 3}{2n^2 + 9n - 5} = \lim_{n\to\infty} \dfrac{\frac{6}{n} + \frac{3}{n^2}}{2 + \frac{9}{n} - \frac{5}{n^2}} = \dfrac{0 + 0}{2 + 0 - 0} = 0$.

(f) $a_n = \dfrac{n^2 + n}{n + 7}$;

Ans: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{n^2 + n}{n + 7} = \lim_{n\to\infty} \dfrac{n + 1}{1 + \frac{7}{n}}$

When n goes to infinity, the denominator $1 + \dfrac{7}{n}$ goes to 1 while the numerator $n + 1$ goes to infinity, so the limit does not exist (it diverges to infinity).

(g) $a_n = \dfrac{\sqrt{4n^2 + 3}}{2n + 7}$;

Ans: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{\sqrt{4n^2 + 3}}{2n + 7} = \lim_{n\to\infty} \dfrac{\sqrt{4 + \frac{3}{n^2}}}{2 + \frac{7}{n}} = \dfrac{\sqrt{4}}{2} = 1$.

(h) $a_n = \cos \dfrac{n\pi}{2}$;

Ans: Note that the sequence is $0, -1, 0, 1, \ldots$ and it repeats every four terms, so the limit does not exist (it is an oscillating sequence).

(i) $a_n = \dfrac{\sin n}{n}$. (Hint: Use the sandwich theorem.)

Ans: Note that for any positive integer n, we have $-1 \leq \sin n \leq 1$ and so $-\dfrac{1}{n} \leq \dfrac{\sin n}{n} \leq \dfrac{1}{n}$.

Also $\lim\limits_{n\to\infty} -\dfrac{1}{n} = \lim\limits_{n\to\infty} \dfrac{1}{n} = 0$.

By the sandwich theorem, $\lim\limits_{n\to\infty} \dfrac{\sin n}{n} = 0$.

24. Prove that $\lim\limits_{n\to\infty} \dfrac{\sin n + 100}{2n + (-1)^n} = 0$.

Ans: Note that for any positive integer n, we have $-1 \leq \sin n \leq 1$ and $-1 \leq (-1)^n \leq 1$, so $\dfrac{99}{2n+1} \leq \dfrac{\sin n + 100}{2n + (-1)^n} \leq \dfrac{101}{2n-1}$.

Also $\lim\limits_{n\to\infty} \dfrac{99}{2n+1} = \lim\limits_{n\to\infty} \dfrac{101}{2n-1} = 0$.

By the sandwich theorem, $\lim\limits_{n\to\infty} \dfrac{\sin n + 100}{2n + (-1)^n} = 0$.

25. (Challenge) Let $\alpha > 0$. Prove that $\lim\limits_{n\to\infty} \dfrac{\alpha^n}{n!} = 0$.

Ans:

- If $0 < \alpha < 1$, then we have $\lim\limits_{n\to\infty} \alpha^n = 0$ and $\lim\limits_{n\to\infty} \dfrac{1}{n!} = 0$, therefore

$$\lim_{n\to\infty} \frac{\alpha^n}{n!} = \left(\lim_{n\to\infty} \alpha^n\right)\left(\frac{1}{n!}\right) = 0 \cdot 0 = 0.$$

- If $\alpha > 1$, (We cannot repeat the above argument since $\lim\limits_{n\to\infty} \alpha^n$ does not exist, and that is why this case is more difficult) we let K be a positive integer such that $\alpha < K$. Then, for any $n > K$, we have

$$\begin{aligned}
\frac{\alpha^n}{n!} &= \frac{\alpha}{1}\cdot\frac{\alpha}{2}\cdot\frac{\alpha}{3}\cdots\frac{\alpha}{K-1}\cdot\frac{\alpha}{K}\cdot\frac{\alpha}{K+1}\cdots\frac{\alpha}{n} \\
&\leq \left(\frac{\alpha}{1}\cdot\frac{\alpha}{2}\cdot\frac{\alpha}{3}\cdots\frac{\alpha}{K-1}\right)\cdot\frac{\alpha}{K}\cdot\frac{\alpha}{K}\cdots\frac{\alpha}{K} \qquad \left(\because \frac{\alpha}{r} < \frac{\alpha}{K} \text{ for } K < r\right) \\
&= M\cdot\left(\frac{\alpha}{K}\right)^{n-K+1} \\
&\quad \left(\text{Let } M = \frac{\alpha}{1}\cdot\frac{\alpha}{2}\cdot\frac{\alpha}{3}\cdots\frac{\alpha}{K-1} \text{ which is independent from } n\right)
\end{aligned}$$

Therefore, for any $n > K$, we have $0 \leq \dfrac{\alpha^n}{n!} \leq M\cdot\left(\dfrac{\alpha}{K}\right)^{n-K+1}$.

Also, note that $\dfrac{\alpha}{K} < 1$, so $\lim\limits_{n\to\infty}\left(\dfrac{\alpha}{K}\right)^{n-K+1} = 0$.

Then, we have $\lim\limits_{n\to\infty} 0 = \lim\limits_{n\to\infty} M\cdot\left(\dfrac{\alpha}{K}\right)^{n-K+1}$.

By the sandwich theorem, we have $\lim\limits_{n\to\infty} \dfrac{\alpha^n}{n!} = 0$.