Calculus for Engineers

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August 2015

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Antiderivatives

19.1 Introduction

In this chapter, we explore the relationship between differentiation and integration. Integration and differentiation are inverse process. Differentiation is the method of finding the derivative of a function. The reverse method is called anti-differentiation.

19.2 Integration: Integral of a function

Definition 1 If the derivative of a function $f(x)$ is $F(x)$, that is, if

$$\frac{d}{dx}f(x) = F(x),$$

we say that $f(x)$ is an antiderivative[1] of the function $F(x)$ and, in symbols, write

$$\int F(x)dx = f(x), \quad \forall x \in [a, b].$$

□

Let us look at some examples to see how it operates in a more specific way.

Example 1 We see that

$$\frac{d}{dx}\sin x = \cos x \quad \text{implies that} \quad \int \cos x dx = \sin x$$

and

$$\frac{d}{dx}\log x = \frac{1}{x} \quad \text{implies that} \quad \int \frac{1}{x}dx = \log x.$$

Note 1

- The process of determining an integral of a function is called *integration*. We are said to integrate $f(x)$ when we find the integral of $f(x)$. The function to be integrated is called *integrand*. Hence $f(x)$ is the integrand.

[1]or *an integral* or a *primitive* of $F(x)$

- The symbol $\int$ denotes the integration. The letter x in dx, denotes that the integration is to be performed with respect to x, where x is called the variable of integration.

- Since integration and differentiation are inverse (or opposite) processes, we have
$$\frac{d}{dx}\left(\int f(x)dx\right) = f(x).$$

The reader may wonder where the constant of integration is. We shall provide details in the following section.

19.3 Indefinite valuedness of integration. General integral. Arbitrary constant.

Definition 2 If $f(x)$ is an antiderivative of $F(x)$, then $f(x)+C$ is also an an antiderivative of $F(x); C$ being a constant whatsoever. Thus,
$$\frac{df(x)}{dx} = F(x) \quad \text{implies that} \quad \frac{d(f(x)+C)}{dx} = \frac{df(x)}{dx} + \frac{dC}{dx} = F(x).$$
□

Again, let $f(x)$ and $\varphi(x)$ be two antiderivatives of $F(x)$ so that we have
$$f'(x) = \varphi'(x) = F(x).$$

As the derivatives of the functions $f(x)$ and $\varphi(x)$ are equal, the functions differ by some constant, that is,
$$f(x) - \varphi(x) = C \quad \Leftrightarrow \quad f(x) = \varphi(x) + C,$$
where C is a constant.

From these considerations, we deduce that the *antiderivative of a function if it exists is not unique* and that if $f(x)$ be any one antiderivative of $F(x)$, then

1. $f(x) + C$ is also an antiderivative; C being any constant;

2. every antiderivative of $F(x)$ can be obtained from $f(x)+C$, by giving some suitable value to C.

Thus if $f(x)$ is *any* one integral of $F(x)$, then $f(x) + C$ is its *general integral.*

From this it follows that *any two antiderivatives of the same function differs by a constant.*

The constant, C, is called the *constant of integration.* The constant of integration will generally be *omitted* and the symbol $\int F(x)dx$ will denote any one antiderivative of $F(x)$. However, it must be remembered that the symbol $\int F(x)dx$ is *really infinite valued.*

It may occur that, by different methods of integration, we obtain different integrals of the same function, but it will always be seen that they differ from each other only by a constant.

Note 2 For illustrative purposes, we may keep the constant of integration, C, in our calculation if necessary.

Example 2 Find the antiderivative of the function $F(x) = \sin x$.
Solution. By Definition 2, let us take

$$\int \cos x dx = \sin x.$$

Also, if C is any constant, then we get

$$\frac{d}{dx}(\sin x + C) = \cos x.$$

Therefore, in general, we have

$$\int \cos x dx = \sin x + C.$$

Example 3 Find the antiderivative of the function $F(x) = \dfrac{1}{x}$.
Solution. By Definition 2, let us take

$$\int \frac{1}{x} dx = \log x.$$

Also, if C is any constant, then we get

$$\frac{d}{dx}\log x = \frac{1}{x}.$$

Therefore, in general, we have

$$\int \frac{1}{x} dx = \log x + C.$$

Theorem 1 If $f(x)$ is an integrable function on x, then

$$\frac{d}{dx}\left(\int f(x)dx\right) = f(x).$$

Proof. Let us consider that

$$\int f(x)dx = g(x) + C. \tag{19.1}$$

By Definition 2, we have

$$\frac{d}{dx}(g(x) + C) = f(x). \tag{19.2}$$

From (19.1) and (19.2), we have

$$\begin{aligned}\frac{d}{dx}\left(\int f(x)dx\right) &= \frac{d}{dx}(g(x)+C)\\ &= \frac{d}{dx}(g(x)) + \frac{d}{dx}(C)\\ &= f(x)+0\\ &= f(x).\end{aligned}$$

Therefore, we have

$$\frac{d}{dx}\left(\int f(x)dx\right) = f(x).$$

19.4 Table of elementary integrals

Some important formulae for integrals that are directly obtained from the derivatives of certain functions are shown in Tables 19.1, 19.2 and 19.3.

No.	Differentiation Formulae Already Known to us $\frac{d}{dx}f(x) = f'(x)$	Corresponding Formulae for Integrals $\int f'dx = f(x)+C$ (Antiderivative with Arbitrary Constants)
1.	$\frac{d}{dx}(x^n) = nx^{n-1}, \quad n \in \mathbb{R}$	$\int nx^{n-1}dx = x^n + C, \quad n \in \mathbb{R}$
2.	$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n,\ n \neq -1$	$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad ,n \neq -1, \quad n \in \mathbb{R}$
3.	$\frac{d}{dx}(e^x) = e^x$	$\int e^x dx = e^x + C$
4.	$\frac{d}{dx}(a^x) = a^x \cdot \log_e a\ (a>0)$	$\int a^x \cdot \log_e a dx = a^x + C, \quad a > 0$
	or $\frac{d}{dx}\left(\frac{a^x}{\log_e a}\right) = a^x\ (a>0)$	$\int a^x dx = \frac{a^x}{\log_e a} + C$
5.	$\frac{d}{dx}(\log_e x) = \frac{1}{x}\ (x>0)$	$\int \frac{1}{x}dx = \log_e \lvert x\rvert + C, \quad x \neq 0$

Table 19.1: Table of derivatives and corresponding integrals: basic rules

Example 4 Prove that

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, \text{ when } n \neq -1,$$

where C is the constant of integration.

Solution. We have

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \frac{(n+1)x^{n+1-1}}{(n+1)} = x^n.$$

No.	Differentiation Formulae Already Known to us $\frac{d}{dx}f(x) = f'(x)$	Corresponding Formulae for Integrals $\int f'dx = f(x) + C$ (Antiderivative with Arbitrary Constants)
6.	$\frac{d}{dx}(\sin x) = \cos x, \quad n \in \mathbb{R}$	$\int \cos x dx = \sin x + C$
7.	$\frac{d}{dx}(\cos x) = -\sin x$	$\int (-\sin x)dx = \cos x + C$
		or $\int (\sin x)dx = -\cos x + C$
8.	$\frac{d}{dx}(e^x) = e^x$	$\int e^x dx = e^x + C$
9.	$\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$	$\int \left(-\text{cosec}^2 x\right) dx = \cot x + C$
		or $\int \left(\text{cosec}^2 x\right) dx = -\cot x + C$
10.	$\frac{d}{dx}(\sec x) = \sec x \cdot \tan x$	$\int \sec x \cdot \tan x dx = \sec x + C$
11.	$\frac{d}{dx}(\text{cosec} x) = -\text{cosec} x \cdot \cot x$	$\int (-\text{cosec} x \cdot \cot x)dx = \text{cosec} x + C$
		or $\int (\text{cosec} x \cdot \cot x)dx = -\text{cosec} x + C$

Table 19.2: Table of derivatives and corresponding integrals: trigonometric functions

Therefore, we get

$$\int x^n dx = \frac{x^{n+1}}{n+1}, \text{ when } n \neq -1.$$

□

It is important to notice that when $n \neq -1$, the integral of x^n is obtained by increasing the index n by 1 and dividing by the increased index, $n + 1$. Thus, we may write, for example,

$$\int x^{1/2} dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} = \frac{2}{3}x^{\frac{3}{2}} + C,$$

$$\int \frac{1}{x^2} dx = \int x^{-2} dx = \frac{x^{-2+1}}{-2+1} = -\frac{1}{x} + C.$$

Example 5 Prove that

$$\int \frac{1}{x} dx = \log|x| + C,$$

where $x \neq 0$ and $|x|$ denotes the absolute value of x.

Solution. Clearly either $x > 0$ or $x < 0$. Therefore, there may be two cases:

Case I. When $x > 0$, then $|x| = x$. So

$$\frac{d}{dx}(\log|x|) = \frac{d}{dx}(\log x) = \frac{1}{x}.$$

No.	Differentiation Formulae Already Known to us $\frac{d}{dx}f(x) = f'(x)$	Corresponding Formulae for Integrals $\int f'dx = f(x) + C$ (Antiderivative with Arbitrary Constants)
12.	$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$
	$\frac{d}{dx}\left(\cos^{-1} x\right) = \frac{-1}{\sqrt{1-x^2}}$	$\int \frac{dx}{\sqrt{1-x^2}} = -\cos^{-1} x + C$
13.	$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$	$\int \frac{dx}{1+x^2} = \tan^{-1} x + C$
	$\frac{d}{dx}\left(\cos^{-1} x\right) = \frac{-1}{1+x^2}$	$\int \frac{dx}{1+x^2} = -\cot^{-1} x + C$
14.	$\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{x \cdot \sqrt{x^2-1}}$	$\int \frac{1}{x \cdot \sqrt{x^2-1}}dx = \sec^{-1} x + C$
	$\frac{d}{dx}\left(\mathrm{cosec}^{-1} x\right) = \frac{-1}{x \cdot \sqrt{x^2-1}}$	$\int \frac{1}{x \cdot \sqrt{x^2-1}}dx = -\mathrm{cosec}^{-1} x + C$

Table 19.3: Table of derivatives and corresponding integrals: inverse trigonometric functions

Therefore, in this case, we get

$$\int \frac{1}{x}dx = \log|x| + C.$$

Case II. When $x < 0$, then $|x| = -x$. So

$$\frac{d}{dx}\left(\log|x|\right) = \frac{d}{dx}\left(\log(-x)\right) = \frac{1}{-x}(-1) = \frac{1}{x}.$$

Therefore, in this case, we get

$$\int \frac{1}{x}dx = \log|x| + C.$$

Thus, from both the cases, we have

$$\int \frac{1}{x}dx = \log|x| + C.$$

Note 3 The inverse trigonometric functions in the above table are single-valued functions as defined in our later chapter. Thus

$$\sin^{-1} x, \tan^{-1} x, \cot^{-1} x, \text{ and } \mathrm{cosec}^{-1} x$$

are the angles, lying between $-\pi/2$ and $\pi/2$, whose sine, tangent, cotangent and cosecant are x. Also

$$\cos^{-1} x \quad \text{and} \quad \sec^{-1} x,$$

are the angles, lying between 0 and π, whose cosine and secant are x.

Note 4 From the above table, we see that both $\sin^{-1} x$ and $-\cos^{-1} x$ are integrals of $\dfrac{1}{\sqrt{1-x^2}}$. From this we cannot deduce the equality of $\sin^{-1} x$ and $-\cos^{-1} x$. The only legitimate conclusion is that they differ by some constant.

From the result of the elementary trigonometry, we know that

$$\sin^{-1} x - (-\cos^{-1} x) = \sin^{-1} x + \cos^{-1} x = \frac{1}{2}\pi.$$

19.5 Two simple theorems

Theorem 2 The integral of the product of a constant and a function is equal to the product of the constant and the integral of the function, that is,

$$\int kf(x)dx = k\int f(x)dx, \quad k \in \mathbb{R}. \tag{19.3}$$

Proof. The proof will follow from the corresponding theorem of Differential Calculus which states that the derivative of the product of a constant and a function is equal to the product of a constant and the derivative of the function.

Differentiating the RHS of (19.3) with respect to x, we obtain

$$\frac{d}{dx}\left(k\int f(x)dx\right) = k\frac{d}{dx}\int f(x)dx = kf(x),$$

which implies that

$$\int kf(x)dx = k\int f(x)dx.$$

□

Theorem 3 The integral of the sum or difference of two functions is equal to the sum or difference of their integrals, that is,

$$\int (f(x) \pm g(x))\,dx = \int f(x)dx \pm \int g(x)dx. \tag{19.4}$$

Proof. The proof will follow from the corresponding theorem of Differential Calculus which states that the derivative of the sum or difference of two functions is equal to the sum or difference of their derivatives.

Differentiating the RHS of (19.4) with respect to x, we obtain

$$\begin{aligned}\frac{d}{dx}\left(\int f(x)dx \pm \int g(x)dx\right) &= \frac{d}{dx}\int f(x)dx \pm \frac{d}{dx}\int g(x)dx \\ &= f(x) \pm g(x)\end{aligned}$$

which implies that

$$\int (f(x) \pm g(x))\,dx = \int f(x)dx \pm \int g(x)dx.$$

□

Remark 1

1. Theorem 3 can easily be generalised to the case of the algebraic sum of a *finite* number of functions so that we have

$$\begin{aligned}&\int (f_1(x) \pm f_2(x) \pm f_3(x) \pm \cdots \pm f_n(x))\,dx \\ &= \int f_1(x)dx \pm \int f_2(x)dx \pm \cdots \pm \int f_n(x)dx.\end{aligned}$$

2. The derivation to show that the indefinite integral of the difference of two functions is equal to the difference of their integrals can be easily done.

3. The results of Theorems 2 and 3 can be generalised to the form

$$\begin{aligned}&\int (k_1 f_1(x) \pm k_2 f_2(x) \pm k_3 f_3(x) \pm \cdots \pm k_n f_n(x))\, dx \\ &\quad = \int k_1 f_1(x) dx \pm \int k_2 f_2(x) dx \pm \cdots \pm \int k_n f_n(x) dx, \quad k_1, \cdots, k_n \in \mathbb{R}.\end{aligned}$$

   That is, the integration of the linear combination of a finite number of functions is equal to the linear combination of their integrals.

Note 5 Theorems 2 and 3 prove useful when the integrand can be decomposed into the sum of a number of functions whose integrals are known. In fact this *decomposition of an integrand into the sum of a number of functions with known integrals* constitutes an important technique of integration as will be seen later on.

19.6 Worked examples

The following are a number of worked examples:

Example 6

$$\int 5x^3 dx = 5 \int x^3 dx = \frac{5x^4}{4} + C.$$

□

Example 7

$$\begin{aligned}&\int (a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n) dx \\ &= \int a_0 dx + \int a_1 x dx + \int a_2 x^2 dx + \cdots + \int a_n x^n dx \\ &= a_0 \int 1 dx + a_1 \int x dx + a_2 \int x^2 dx + \cdots + a_n \int x^n dx \\ &= a_0 x + C_0 + a_1 \frac{x^2}{2} + C_1 + a_2 \frac{x^3}{3} + C_3 + \cdots + a_n \frac{x^{n+1}}{n+1} + C_n \\ &= a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} + \cdots + a_n \frac{x^{n+1}}{n+1} + C,\end{aligned}$$

where $C = C_0 + C_1 + \cdots + C_n$. □

Example 8

$$\begin{aligned}\int \left(\cos x + \frac{2}{x} - e^x\right) dx &= \int \cos x dx + 2 \int \frac{1}{x} dx - \int e^x dx \\ &= \sin x + 2 \log|x| - e^x + C.\end{aligned}$$

□

Example 9

$$
\begin{aligned}
\int \frac{3-5x^2+7x^4-9x^6}{x^6}dx &= \int \left(\frac{3}{x^6}-\frac{5}{x^4}+\frac{7}{x^2}-9\right)dx \\
&= \int \frac{3}{x^6}dx - \int \frac{5}{x^4}dx + \int \frac{7}{x^2}dx - \int 9dx \\
&= 3\int \frac{1}{x^6}dx - 5\int \frac{1}{x^4}dx + 7\int \frac{1}{x^2}dx - 9\int 1dx \\
&= -\frac{3}{5x^5}+\frac{5}{3x^3}-\frac{7}{x}-9x \\
&= \frac{-9+25x^2-105x^4-135x^6}{15x^5}+C.
\end{aligned}
$$

□

Example 10

$$
\begin{aligned}
\int \frac{x^2}{1+x^2}dx &= \int \frac{(x^2+1)-1}{x^2+1}dx \\
&= \int \left(1-\frac{1}{x^2+1}\right)dx \\
&= \int 1dx - \int \frac{1}{x^2+1}dx \\
&= x-\tan^{-1}x+C.
\end{aligned}
$$

□

Example 11

$$
\begin{aligned}
\int \frac{x^4}{x^2+1}dx &= \int \frac{x^4-1+1}{x^2+1}dx \\
&= \int \left(x^2-1+\frac{1}{x^2+1}\right)dx \\
&= \frac{x^3}{3}-x+\tan^{-1}x+C.
\end{aligned}
$$

□

Example 12

$$
\begin{aligned}
\int \frac{\sin^3 x+\cos^3 x}{\sin^2 x\cos^2 x} &= \int \frac{\sin^3 x}{\sin^2 x\cos^2 x}dx + \int \frac{\cos^3 x}{\sin^2 x\cos^2 x}dx \\
&= \int \tan x\sec x dx + \int \cot x \operatorname{cosec} x dx \\
&= \sec x - \operatorname{cosec} x + C.
\end{aligned}
$$

□

Example 13

$$\begin{aligned}\int (1-\cos 2x)dx &= \int \sqrt{2\sin^2 x}dx \\ &= \int \sqrt{2}\sin x dx \\ &= \sqrt{2}\int \sin x dx \\ &= -\sqrt{2}\cos x + C.\end{aligned}$$

$\square$