THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1510 Calculus for Engineers (Fall 2021)
Homework 4
Deadline: November 20 at 23:00

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Class: MATH1510G

I acknowledge that I am aware of University policy and regulations on honesty in academic work, and of the disciplinary guidelines and procedures applicable to breaches of such policy and regulations, as contained in the website http://www.cuhk.edu.hk/policy/academichonesty/

David — Signature

9-11-2021 — Date

General Guidelines for Homework Submission.

- Please submit your answer to Gradescope through the centralized course MATH1510A-I in Blackboard.
- In Gradescope, for each question, please indicate exactly which page(s) its answer locates. **Answers of incorrectly matched questions will not be graded**.
- **Late submission will NOT be graded and result in zero score.** Any answers showing evidence of plagiarism will also score zero; stronger disciplinary action may also be taken.
- Points will only be awarded for answers with sufficient justifications.
- All questions in **Part A** along with some selected questions in **Part B** will be graded. Question(s) labeled with * are more challenging.

Part A:

1. Evaluate the following indefinite integrals by substitutions.

   (a) $\int (2021x+1)(x-1)^{1510}dx$;

   (b) $\int \frac{(\ln x)^3}{x}dx$.

(a) Let $u = x-1$, then $du = dx$

$\& \; 2021x = 2021u + 2021$

$$\int (2021u+2022)\,u^{1510}\,du$$

$$= (2021u+2022)\frac{u^{1511}}{1511} - \int \frac{u^{1511}}{1511}(2021)\,du.$$

$$= \frac{2021u+2022}{1511}\left(u^{1511}\right) - \frac{2021}{2284632}\left(u^{1512}\right) + \text{Constant}$$

$$= \frac{2021x+1}{1511}(x-1)^{1511} - \frac{2021}{2284632}(x-1)^{1512} + \text{Constant} \;//$$

(b) Let $u = \ln x$, then $du = \frac{1}{x}dx$

$$\int u^3\,du$$

$$= \frac{1}{4}u^4 + \text{Constant}$$

$$= \frac{(\ln x)^4}{4} + \text{Constant} \;//$$

2. Evaluate the following indefinite integrals by integration by parts.

(a) $\displaystyle\int \overset{u}{x^2}\, \overset{dv}{\sin x}\, dx$;

(b) $\displaystyle\int \ln(x+\overset{u}{x^2})\, \overset{dv}{dx}$.

(a)
$$\int x^2 \sin x\, dx = (-\cos x)\, x^2 - \int \overset{dv}{(-\cos x)}\, \overset{u}{2x}\, dx$$
$$= -x^2\cos x + (2x)(\sin x) - 2\int \sin x\, dx$$
$$= -x^2\cos x + 2x\sin x + 2\cos x + \text{Constant} \quad //$$

(b)
$$\int \ln(x+x^2)\, dx = x\ln(x+x^2) - \int x\cdot\frac{1}{x+x^2}\cdot(2x+1)\, dx$$
$$= x\ln(x+x^2) - \int \frac{2x+1}{x+1}\, dx$$
$$= x\ln(x+x^2) - \int \left(2 - \frac{1}{x+1}\right) dx$$
$$= x\ln(x+x^2) - 2x + \ln|x+1|$$
$$+ \text{Constant} \quad //$$

Part B:

3. Evaluate the following indefinite integrals by trigonometric substitutions.

   (a) $\displaystyle\int \frac{1}{x^2\sqrt{x^2-1}}\,dx$ where $x > 1$;

   (b) * $\displaystyle\int \frac{x^3}{\sqrt{4-x^2}}\,dx$ where $0 < x < 2$.

(a) Let $x = \sec\theta$, then $dx = \sec\theta\tan\theta\, d\theta$

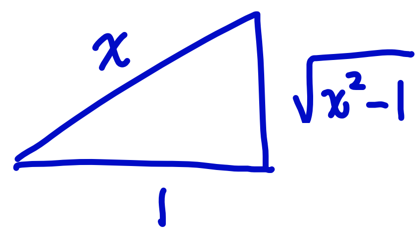
$$\int \frac{1}{\sec^{\cancel{2}}\theta \cdot \cancel{\tan\theta}}\, d\theta\, \cancel{(\sec\theta)}\cancel{(\tan\theta)}$$

$x^2 = 1^2 + ?^2$

$? = \sqrt{x^2-1}$

$$= \int \cos\theta\, d\theta$$

$$= \sin\theta + \text{Constant}$$



$$= \frac{\sqrt{x^2-1}}{x} + \text{Constant} \quad //$$

(b) Let $x = 2\sin\theta$, then $dx = 2\cos\theta\, d\theta$

$$\int \frac{8\sin^3\theta}{\sqrt{4-4\sin^2\theta}}\, d\theta \cdot (2\cos\theta)$$

$$= 8\int \frac{\sin^3\theta\, \cancel{\cos\theta}}{\cancel{\cos\theta}}\, d\theta$$

$$= 8\left(\frac{1}{4}\right)\sin^4\theta + \text{Constant}$$

$$= 2\left(\frac{x}{2}\right)^4 + \text{Constant}$$

$$= \frac{x^4}{8} + \text{Constant} \quad //$$

4. Evaluate the following indefinite integrals by partial fraction decomposition.

(a) $\displaystyle\int \frac{8}{(x-1)(x+1)(x+3)}\,dx$;

(b) $\displaystyle\int \frac{3x^2+7x}{(x-1)(x^2+4x+5)}\,dx$.

(a) By partial decomposition, we have:

$$\int \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+3}\,dx$$

where A, B, C are integers.

$A(x^2+4x+3)$

$B(x^2+2x+3)$

$C(x^2-1)$

$$A+B+C = 0 \quad \dots \quad ①$$

$$4A+2B = 0 \quad \dots \quad ②$$

$$3A+3B-C = 8 \quad \dots \quad ③$$

$① + ③ : \quad 4A+4B = 8 \quad \dots \quad ④$

$④ - ② : \quad 2B = 8$

$B = 4, \quad A = -2, \quad C = -2$ //

$\therefore$ We have:

$$-2\int \frac{1}{x-1}\,dx + 4\int \frac{1}{x+1}\,dx - 2\int \frac{1}{x+3}\,dx$$

$$= -2\ln|x-1| + 4\ln|x+1| - 2\ln|x+3| + \text{Constant}$$

$$= 2\ln\left|\frac{(x+1)^2}{(x-1)(x+3)}\right| + \text{Constant} \quad //$$

(b) By partial decomposition, we have:

$$\int \frac{A}{x-1} + \frac{Bx+C}{x^2+4x+5}\,dx \quad , \text{ where } A, B, C \text{ are integers.}$$

$$\begin{aligned} A+B \quad\quad &= 3 \;\ldots \text{①} \\ 4A-B+C &= 7 \;\ldots \text{②} \\ 5A \quad\; -C &= 0 \;\ldots \text{③} \end{aligned}$$

$$\text{②}+\text{③} : \; 9A - B = 7 \;\ldots \text{④}$$

$$\text{④}+\text{①} : \; 10A = 10$$

$$A = 1, \quad B = 2, \quad C = 5$$

$\therefore$ We have $\displaystyle\int \frac{1}{x-1} + \frac{2x+5}{x^2+4x+5}\,dx$

$$= \int \frac{1}{x-1}\,dx + \int \frac{2x+4+1}{x^2+4x+5}\,dx$$

$$= \ln|x-1| + \ln(x^2+4x+5) + \int \frac{1}{(x+2)^2+1}\,dx$$

$$= \ln|x-1| + \ln(x^2+4x+5) + \tan^{-1}(x+2) + \text{Constant} \quad //$$

5. Evaluate the following indefinite integrals by t-substitution.

(a) $\displaystyle\int \frac{1}{2\sin x + \cos x + 1}\,dx$;

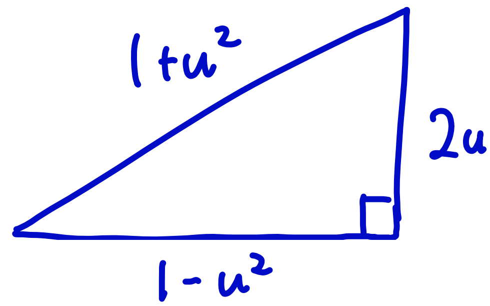
(b) $\displaystyle\int \frac{1}{2+\cos x}\,dx$.

(a) Let $u = \tan\frac{x}{2}$,

$$du = \sec^2\left(\frac{x}{2}\right)\cdot\frac{1}{2}\,dx$$

We have $\displaystyle \tan x = \frac{2\tan\frac{x}{2}}{1-\tan^2\left(\frac{x}{2}\right)}$

$$= \frac{2u}{1-u^2}$$



$$\sin x = \frac{2u}{1+u^2}$$

$$\cos x = \frac{1-u^2}{1+u^2}$$

$$\therefore\ 2\int \frac{1}{4u+1-u^2+1+u^2}\,du$$

$$= \int \frac{1}{2u+1}\,du$$

$$= \frac{1}{2}\ln|2u+1| + \text{Constant}$$

$$= \frac{1}{2}\ln\left|2\tan\frac{x}{2}+1\right| + \text{Constant.} \quad //$$

(b) Let $u = \tan\frac{x}{2}$.

$$du = \sec^2\left(\frac{x}{2}\right)\cdot\frac{1}{2}\,dx$$

By obtaining (a), we have:

$$\int \frac{1}{2+\cos x}\,dx = 2\int \frac{1}{2+2u^2+1-u^2}\,du$$

$$= 2\int \frac{1}{u^2+3}\,du$$

$$= \frac{2\sqrt{3}}{3}\int \frac{1}{\frac{u^2}{3}+1}\,d\left(\frac{u}{\sqrt{3}}\right)$$

$$= \frac{2\sqrt{3}}{3}\tan^{-1}\left(\frac{u}{\sqrt{3}}\right) + \text{Constant}$$

$$= \frac{2\sqrt{3}}{3}\tan^{-1}\left(\frac{\sqrt{3}\tan\frac{x}{2}}{3}\right)$$

$$+ \text{Constant} \quad //$$

6. Derive a reduction formula for

$$I_n = \int x^n \sin x \, dx$$

where n is an integer, $n \geq 2$. Hence, compute I_4.

$$I_n = \int \overset{u}{x^n} \, \overset{dv}{\sin x \, dx}$$

$$= x^n(-\cos x) - \int \overset{u}{n x^{n-1}} \, \overset{dv}{(-\cos x)\, dx}$$

$$= -x^n \cos x + n x^{n-1} \sin x - (n^2 - n) \int x^{n-2} \sin x \, dx$$

$$= -x^n \cos x + n x^{n-1} \sin x - n(n-1) I_{n-2} \quad //$$

To find I_4:

$$= -x^4 \cos x + 4x^3 \sin x - 12 I_2$$

$$= -x^4 \cos x + 4x^3 \sin x$$
$$- 12(-x^2 \cos x + 2x \sin x - 2 I_0)$$

$$= -x^4 \cos x + 4x^3 \sin x + 12x^2 \cos x$$
$$- 24x \sin x + 2 \cos x \quad //$$

7. * Evaluate the following indefinite integrals.

(a) $\displaystyle\int \frac{\sin\sqrt{x}}{\sqrt{x}\cos^3\sqrt{x}}dx$;

(b) $\displaystyle\int \frac{3\sin x}{2-\cos x-\cos^2 x}\,dx$;

(c) $\displaystyle\int \frac{2-\sqrt{x}}{x+1}dx$;

(d) $\displaystyle\int \frac{2}{x(x^{1/3}+2)}\,dx$;

(e) $\displaystyle\int \frac{\sqrt{x}}{e^{\sqrt{x}}}dx$.

(a) Let $u=\sqrt{x}$, $du=\dfrac{1}{2\sqrt{x}}dx$

We have: $\displaystyle 2\int \frac{\sin u}{\cos^3 u}\,du$

$$= -2\int \frac{1}{\cos^3 u}\,d(\cos u)$$

$$= \frac{1}{\cos^2 u} + \text{Constant}$$

$$= \frac{1}{\cos^2\sqrt{x}} + \text{Constant} \quad //$$

(b) $\displaystyle -3\int \frac{\sin x}{\cos^2 x+\cos x-2}\,dx$

Let $u=\cos x$, $du=-\sin x\,dx$

$$3\int \frac{1}{u^2+u-2}\,dx$$

$$= 3\int \frac{1}{(u+2)(u-1)}\,dx$$

$$= \int \frac{-1}{u+2} + \frac{1}{u-1}\,dx$$

$$= \ln\left|\frac{u-1}{u+2}\right| + \text{Constant}$$

$$= \ln\left|\frac{\cos x-1}{\cos x+2}\right| + \text{Constant} \quad //$$

(c) $\displaystyle\int \frac{2-\sqrt{x}}{x+1}\,dx$

Let $u=\sqrt{x}$, $du = \dfrac{1}{2\sqrt{x}}\,dx$

$$\int \frac{2-u}{u^2+1}\cdot(2u)\,du$$

$$= 2\int \frac{2u}{u^2+1} - \frac{u^2}{u^2+1}\,du$$

$$= 2\int \frac{1}{u^2+1}\,d(u^2+1) - 2\int 1-\frac{1}{u^2+1}\,du$$

$$= 2\ln(u^2+1) - 2u + \tan^{-1}(u) + \text{Constant}$$

$$= 2\ln(x+1) - 2\sqrt{x} + \tan^{-1}(\sqrt{x}) + \text{Constant} \quad //$$

(d) $\displaystyle\int \frac{2}{x(x^{\frac{1}{3}}+2)}\,dx$

Let $u = x^{\frac{1}{3}}$, $du = \frac{1}{3}x^{-\frac{2}{3}}\,dx$

$$6\int \frac{1}{u(u+2)}\,du$$

$$= 3\int \frac{1}{u} + \frac{-1}{u+2}\,du$$

$$= 3\ln\left|\frac{u}{u+2}\right| + \text{Constant} \quad //$$

(e) $\displaystyle\int \frac{\sqrt{x}}{e^{\sqrt{x}}}\,dx$

Let $u = \sqrt{x}$, $du = \frac{1}{2\sqrt{x}}\,dx$

$$2\int u^2 e^{-u}\, du$$

$$= 2\left[u^2(-e^{-u}) - \int 2u(-e^{-u})\, du\right]$$

$$= 2\left[-u^2e^{-u} + 2ue^{-u} - \int 2e^{-u}\, du\right]$$

$$= 2\left(-u^2e^{-u} + 2ue^{-u} + 2e^{-u}\right) + \text{Constant}$$

$$= 4e^{-\sqrt{x}} + 4\sqrt{x}e^{-\sqrt{x}} - 2xe^{-\sqrt{x}} + \text{Constant}$$