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香 港 中 文 大 學
The Chinese University of Hong Kong

二零二零至二一年度上學期科目考試

Course Examination 1st Term, 2020-21

科目編號及名稱 Course Code & Title : **MATH1510A/B/C/D/E/F/G/H/I Calculus for Engineers**

時間 Time allowed : **2** 小時 hours **00** 分鐘 minutes

學號 Student I.D. No : ____________ 座號 Seat No. : ____________

- There are a total of 200 points and 36 questions. Question 1 to 25 are short questions and question 26 to 36 are long questions.
- Write your answers of the **Short Questions** into the given boxes below the corresponding questions **in the question paper**. No step is required. Partial credit is available for some of the questions.
- Write your answers of the **Long Questions in the examination answer book.** If you need extra space to answer questions, raise your hands.
- Please write your answers using a black or blue pen, NOT any other color or a pencil.
- You must return your question paper and examination answer book(s) at the end of the examination.

For internal use only:

1-5		26		31		35					
6-11		27		32a		36a					
12-16		28		32b		36b					
17-19		29		32c							
20-22		30a		33							
23-25		30b		34							
		30c						Total			

Short Questions

Each of question 1-25 is worth 3 points.

1. Find the domain of the function

$$f(x) = \sqrt{x^2 - 2}$$

Answer:

2. Evaluate $\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x$. If the limit does not exist, write "DNE". If it diverges to $\pm\infty$, please indicate so and determine the correct sign.

Answer:

3. Given that $f(0) = 1$, $g(0) = 2$, $\lim_{x\to 0}\frac{f(x)}{x} = 3$, and $\lim_{x\to 0}\frac{g(x)}{x} = 4$, evaluate $\lim_{x\to 0}\frac{f(x)}{g(x)}$. If the limit does not exist, write "DNE". If it diverges to $\pm\infty$, please indicate so and determine the correct sign.

Answer:

4. Find $\frac{dy}{dx}$ if $\quad y = \ln x - \frac{1}{x^2} + \sec x$

Answer:

5. Find $\frac{dy}{dx}$ if $\quad y = \sin\left(\frac{\pi x}{\sqrt{2}+1}\right)$

Answer:

6. Find $\dfrac{dy}{dx}$ if $y = e^x\sqrt{3x+5}$

Answer:

7. Find $\dfrac{dy}{dx}$ if $y = \pi^{\sqrt{x}}$

Answer:

8. Find the approximated value of $\sqrt{15.9}$ by using the linearization of $\sqrt{x}$ at $x = 16$.

Answer:

9. Suppose the area of a square is increasing at a constant rate of 2 unit^2/min. Find the rate of change of its side length when its area is 10 unit.

Answer:

10. Find all the critical point(s) of the function $f(x) = |x^2 + 6x|$.

Answer:

11. Write down the equations of all the asymptotes of the graph $y = \dfrac{\ln(x+1)}{x}$.

Answer:

12. Evaluate $\int (x^3 + 3^x + e^3)\,dx$

Answer:

13. Evaluate $\int \sqrt{3x+1}\,dx$

Answer:

14. Evaluate $\int \sin^3 x\,dx$

Answer:

15. Evaluate $\int \tan x\,dx$

Answer:

16. Evaluate $\int \frac{1}{x^2+2x+2}\,dx$

Answer:

17. Given that $a > 1$, evaluate

$$\lim_{x\to 0^+} \frac{\int_0^x t^a \sin t \, dt}{x^{a+2}}$$

If the limit does not exist, write "DNE". If it diverges to $\pm\infty$, please indicate so and determine the correct sign.

Answer:

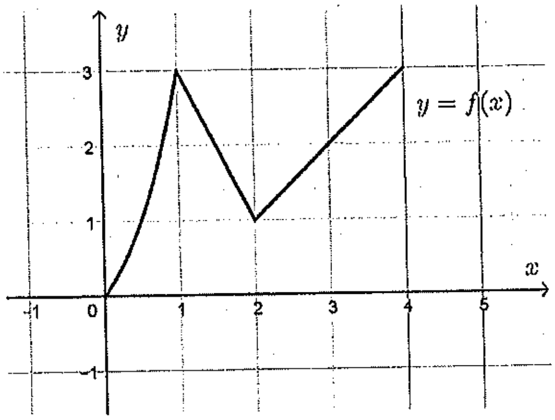
18. Let $\mathcal{R}$ be the region bounded by graphs of

$$y = 4x + 1 \quad \text{and} \quad y = x^2 + 4x - 3$$

Express the area of $\mathcal{R}$ as a definite integral (Do not evaluate).

Answer:

19.



The above diagram shows the graph of $y = f(x)$.

Define $F(x) = \int_2^x f(t)\, dt$. Find $F'(3), F(2)$ and $F(3)$.

Answer:

20. Let $\mathcal{R}$ be the region bounded by the curves

$$y = 2^x - 1, \quad x = 0, \quad y = 1$$

Express the volume of the solid generated by rotating the region $\mathcal{R}$ about the y-axis as a definite integral (Do not evaluate).

Answer:

21. Find the radius of convergence of the power series

$$\sum_{n=1}^{\infty} (-1)^n \frac{3^n + n}{2^n} (2x+1)^n.$$

Answer:

22. Which of the following Maclaurin series is/are correct? Write **NONE** if none of them is correct.

(a) $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

(b) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

(c) $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$

Answer:

23. Suppose $f(x)$ and $g(x)$ have Maclaurin series

$$\sum_{n=0}^{\infty}(n+1)x^n = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

$$\sum_{n=0}^{\infty}(-1)^n 2^{n+1} x^n = 2 - 4x + 8x^2 - 16x^3 + \cdots$$

respectively. Find the second order Maclaurin polynomial of $f(x)g(x)$.

Answer:

24. Suppose $f(x)$ and $g(x)$ have Maclaurin series

$$\sum_{n=0}^{\infty}\frac{x^n}{n+1} = 1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \cdots$$

$$\sum_{n=1}^{\infty}(-1)^{n+1} 2^n x^n = 2x - 4x^2 + 8x^3 - \cdots$$

respectively. Find the second order Maclaurin polynomial of $(f \circ g)(x)$.

Answer:

25. Suppose $f(x)$ has Maclaurin series

$$\sum_{n=0}^{\infty}\frac{x^n}{n+1}.$$

Find the value of $f^{(10)}(0)$.

Answer:

Long Questions

26. (9 points) Let

$$f(x)=\begin{cases} x(1-x) & \text{if } x\geq 0,\\ x^2+ax+b & \text{if } x<0 \end{cases}$$

Find the value(s) of a, b such that f is differentiable at $x=0$

27. (4 points) Find the equation of the tangent of the curve

$$x\sin y = 2\ln x + y^3$$

at the point $P=(1,0)$.

28. (4 points) Evaluate

$$\lim_{x\to\infty}\frac{x\ln x-1}{e^x+x-2}$$

If the limit does not exist, write "DNE". If it diverges to $\pm\infty$, please indicate so and determine the correct sign.

29. (14 points) Consider the function

$$f(x)=x^3-3x^2-24x-2$$

(a) Find the largest interval(s) over which $f(x)$ is decreasing.

(b) Find all the critical point(s) of $f(x)$. For each critical point(s), determine if it's a local maximum, local minimum or neither.

(c) Find the global maximum and minimum value of $f(x)$ over $[-4,8]$.

30. (20 points) Evaluate the following integrals.

(a) $\displaystyle\int_1^2 \frac{1}{x^2-x}\,dx$;

(b) $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}(x+1)}\,dx$;

(c) $\displaystyle\int \frac{1}{\cos x+2}\,dx$.

31. (9 points) Let n be a non-negative integer and

$$I_n=\int_0^\pi x^n\sin x\,dx$$

Find a reduction formula for I_n. Hence, evaluate I_6.

32. (15 points)

(a) Find the fourth order Maclaurin polynomial of

$$f(x) = \cos^2 x - \sin^2 x.$$

(b) Find the Taylor series with center $a = 2$ of

$$f(x) = e^x.$$

(c) Find the Maclaurin series of

$$f(x) = \frac{1}{3x+2}.$$

33. (20 points) Let C be the curve defined by

$$y = f(x) = \begin{cases} x^2 - 16 & \text{if } x \in [4,6] \\ 0 & \text{if } x \in [0,4] \end{cases}$$

A vase is formed by revolving C about the y-axis. Suppose the vase is empty at the beginning. Then water is poured into the vase at a constant rate of 40π unit^3/s.

(a) When the depth of the water inside is h unit ($0 \le h \le 20$), show that its volume is given by

$$V = \frac{1}{2}\pi(h^2 + 32h) \quad (\text{unit}^3)$$

(b) How long does it take to fill the whole vase?

(c) Find the rate of change of the depth of the water (in unit/s) when $h = 10$ unit.

34. (12 points)

(a) By considering Lagrange remainder, show that

$$\left|\ln(1+x^2) - \left(x^2 - \frac{1}{2}x^4 + \frac{1}{3}x^6\right)\right| \le x^8$$

for any $x \in (-0.5, 0.5)$.

(b) Using the result in part (a), show that

$$\lim_{x\to 0} \frac{\ln(x^4 + 2x^2 + 1) - 2x^2 + x^4}{x^3 \sin^3 x} = \frac{2}{3}$$

35. (8 points) Let $f(x)$ be a continuous function on $[0,2]$ with $f(0)=f(2)$. Show that there exist $x,y\in[0,2]$ such that $y-x=1$ and $f(x)=f(y)$.

36. (10 points)

(a) Given that the series

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(n+1)2^n}=\frac{1}{2(2)}-\frac{1}{3(2^2)}+\frac{1}{4(2^3)}-\frac{1}{5(2^4)}+\cdots$$

converges, find its exact value.

(b) Given that the limit

$$\lim_{n\to\infty}\sum_{k=1}^{n}\ln\left(\sqrt[n]{\frac{n+k}{n}}\right)=\lim_{n\to\infty}\left(\ln\sqrt[n]{\frac{n+1}{n}}+\ln\sqrt[n]{\frac{n+2}{n}}+\cdots+\ln\sqrt[n]{\frac{n+n}{n}}\right)$$

exists, find its exact value.

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