Calculus for Engineers

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Contents

Limits Involving Infinity

5.1 Introduction

In this chapter, we will study various limit forms of a function at infinity. Ideas on the horizontal asymptote, slant asymptote, and vertical asymptote are studied via worked examples. Two techniques for examining the existence of various limits of the functions are the dominant test substitution and the factoring principle of dominance.

5.2 What is the limit of a function at infinity?

The topic of the limit of a function at infinity is a very rich and profound one, and it has different interpretations[1]. We will not consider infinity to be a real number, although the real number system can be extended to include infinity.

The symbol for infinity is ∞, while the symbol for negative infinity is $-\infty$. When something approaches ∞ along the real number line, it (generally) increases without bound. Infinity is greater than any defined real number. Similarly, something approaching $-\infty$ (generally) decreases without bound.

5.3 Horizontal asymptotes and $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$

Example 1 Consider a function defined by

$$f(x) = \frac{1}{x}.$$

In Table 5.1, we notice that $\lim\limits_{x\to\infty} f(x) = 0$ and $\lim\limits_{x\to-\infty} f(x) = 0$. The limit of $f(x)$ as x approaches infinity is 0. That is, as x increases without bound, $f(x)$ approaches 0. Similarly, the limit of $f(x)$ as x approaches negative infinity is 0. That is, as x decreases without bound, $f(x)$ approaches 0

[1]

- Infinity is not a number in the usual real number system that we will study in calculus.
- The affinely extended real number system, denoted by $\overline{\mathbb{R}}$ or$[-\infty, \infty]$, includes two points of infinity, one referred to as ∞ (or ∞), and the other referred to as $-\infty$ (We are "adjoining" them to the real number system.) We obtain the two-point compactification of the real numbers. We never refer to ∞ and $-\infty$ as real numbers, though.

As a consequence, in Figure 5.1, the graph of $f(x) = 0$ (that is, the x-axis) is a *horizontal asymptote* for the graph of $f(x)$. An asymptote is a line that a graph approaches; it is usually drawn as a (green) dashed line.

x	$-\infty \leftarrow$	-100	-10	-1	1	10	100	$\rightarrow \infty$
$f(x) = \frac{1}{x}$	$0 \leftarrow$	$-\frac{1}{100}$	$-\frac{1}{10}$	-1	1	$\frac{1}{10}$	$\frac{1}{100}$	$\rightarrow 0$

Table 5.1: Values of $f(x) = \frac{1}{x}$.

□

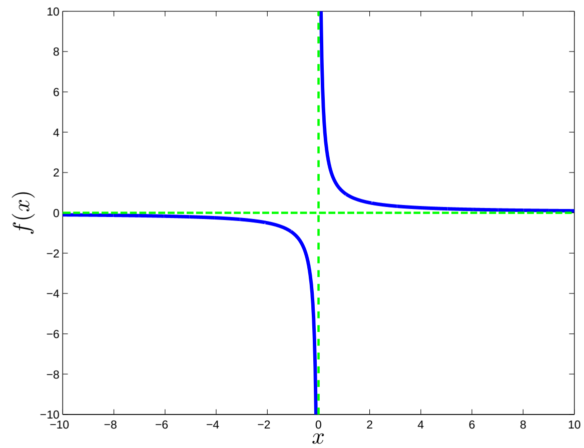

Figure 5.1: Graph of $f(x) = \dfrac{1}{x}$, where $x \in [-10, 10]$.

Definition 1 The graph of $f(x)$ has a horizontal asymptote at $f(x) = A$ (for a real constant A) if and only if $\lim_{x\to\infty} f(x) = A$ and $\lim_{x\to-\infty} f(x) = A$.

□

Example 2 Consider a function defined by

$$f(x) = \frac{|x|}{x}.$$

In Figure 5.2, there are two horizontal asymptotes, at $f(x) = 1$ and $f(x) = -1$. (Usually, when a graph shows this kind of flatness, we do not even bother drawing in the dashed lines.)

□

Example 3 Consider a function defined by

$$f(x) = x + 3.$$

Clearly, both these two limits, $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$, do not exist because

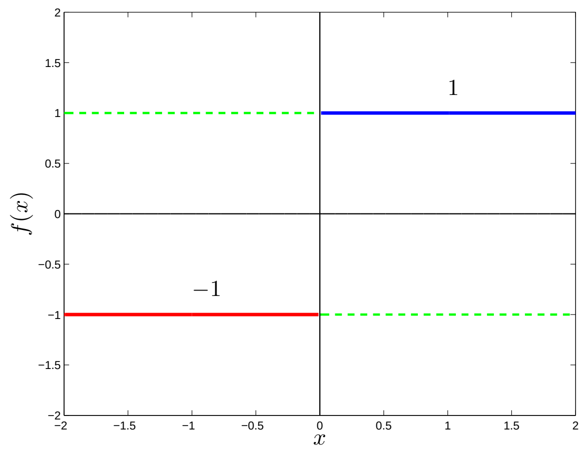

Figure 5.2: Graph of $f(x) = \dfrac{|x|}{x}$, where $x \in [-2, 2]$.

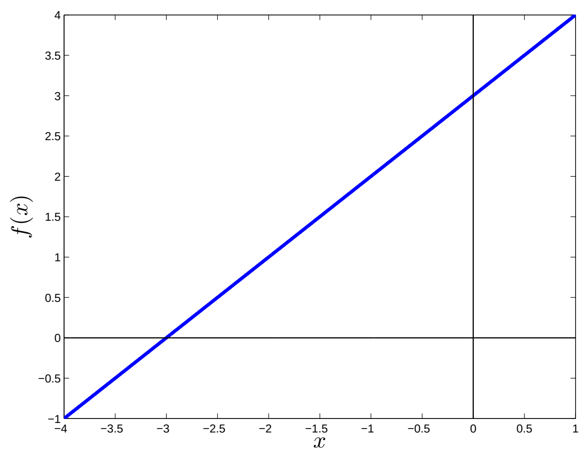

Figure 5.3: Graph of $f(x) = x + 3$, where $x \in [-4, 1]$.

- as x increases without bound, the $f(x)$ function values also (generally) increase without bound, and
- as x decreases without bound, the $f(x)$ function values also (generally) decrease without bound.

□

In Figure 5.3, there are no horizontal asymptotes.

Note 1 The use of "=" here may be technically inappropriate, but it is commonly accepted. □

Example 4 Consider a function defined by

$$f(x) = \sin x.$$

Clearly, both these two limits, $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$, do not exist because the $f(x)$ function values do not approach a single real constant as x approaches

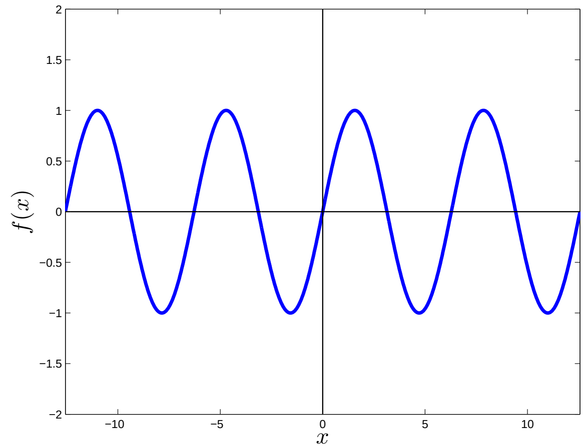

Figure 5.4: Graph of $f(x) = \sin x$, where $x \in [-4, 4]$.

infinity, nor as x approaches negative infinity. The function values oscillate between -1 and 1. We cannot even say that the limit is $+\infty$ or $-\infty$, because the function values are safely bounded between -1 and 1 for all real values of x. In Figure 5.4, no horizontal asymptotes are found.

□

5.4 The limit forms $\frac{1}{\infty}$ and $\frac{1}{-\infty}$.

Let us explore the limit form $\frac{1}{\infty}$ and $\frac{1}{-\infty}$ using the following function

$$f(x) = \frac{1}{x}.$$

In Example 1, we concluded that

$$\lim_{x\to\infty} \frac{1}{x} = 0 \quad \text{and} \quad \lim_{x\to-\infty} \frac{1}{x} = 0.$$

We can also write $\dfrac{1}{x} \to 0$ as $x \to \infty$ and also as $x \to -\infty$.

More generally, it is true that, for functions $n(x)$ and $d(x)$,

$$\frac{n(x)}{d(x)} \to 0 \quad \text{if} \quad n(x) \to 1 \quad \text{and} \quad d(x) \to \infty.$$

This is true whether we are considering all of the indicated limits as $x \to \infty$, or as $x \to -\infty$, or as $x \to c$ (or as $x \to c^+$, or as $x \to c^-$) for some real constant c. Also,

$$\frac{n(x)}{d(x)} \to 0 \quad \text{if} \quad n(x) \to 1 \quad \text{and} \quad d(x) \to -\infty.$$

We will say that the limit forms $\dfrac{1}{\infty}$ and $\dfrac{1}{-\infty}$ yield 0 as a limit.

Example 5 Evaluate

$$\lim_{x\to\infty}\frac{1}{\sqrt[3]{x}}.$$

Solution. Let $n(x) = 1$ and $d(x) = \sqrt[3]{x}$. We observe that

- $n(x) = 1 \to 1$ as $x \to \infty$, and
- $d(x) = \sqrt[3]{x} \to \infty$ as $x \to \infty$.

As x increases without bound, so does its cube root. In Figure 5.1, it helps us to get the limit results:

$$\frac{1}{x} \to 0 \text{ as } x \to \infty \quad \text{and also as } x \to -\infty.$$

Therefore, the limit form of $\frac{1}{\infty}$ applies here, and we have

$$\lim_{x\to\infty}\frac{1}{\sqrt[3]{x}} = 0.$$

□

Example 6 Evaluate

$$\lim_{x\to-\infty}\frac{1}{\sqrt[3]{x}} = 0.$$

Exercise.

Example 7 Evaluate

$$\lim_{x\to-\infty}\frac{1}{\sqrt{x}}.$$

Solution: The limit

$$\lim_{x\to-\infty}\frac{1}{\sqrt{x}}$$

does not exist because $\sqrt{x}$ is not defined as a real quantity whenever $x < 0$. Figure 5.5 show the graphs of $f(x) = \sqrt{x}$ and $f(x) = \dfrac{1}{\sqrt{x}}$. □

5.5 Variations on limit forms

5.5.1 Approaching 0 in different ways

We can write

$$\lim_{x\to\infty}\frac{n(x)}{d(x)} = 0^+.$$

This is because, for functions $n(x)$ and $d(x)$, if $n(x) \to 1$ and $d(x) \to \infty$, then $\dfrac{n(x)}{d(x)} \to 0$

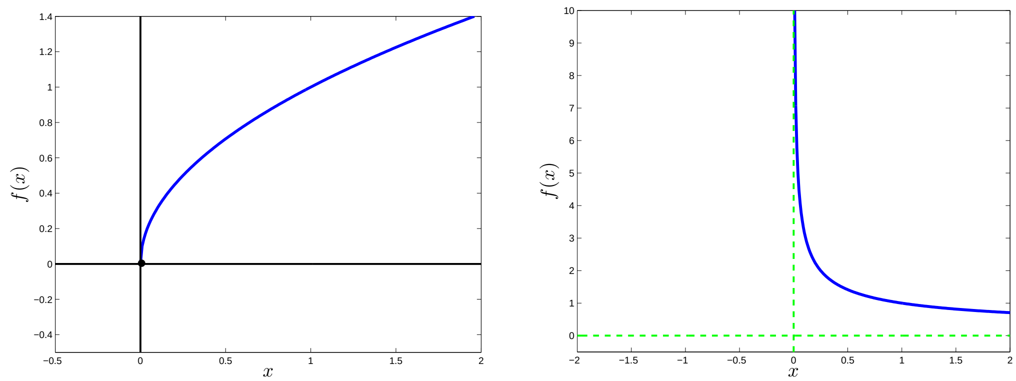

Figure 5.5: (a). Graph of $f(x) = \sqrt{x}$, where $x \in [0, 2]$. (b). Graph of $f(x) = \dfrac{1}{\sqrt{x}}$, where $x \in (0, 2]$, $x \neq 0$.

from the right (or from above, since we are dealing with "$f(x) = \dfrac{n(x)}{d(x)}$" function values).

The 0^+ notation can help us if we want to be more descriptive about how the fraction is approaching 0, or if this limit analysis is part of a larger limit problem.

Similarly, we can also write

$$\lim_{x\to-\infty} \frac{n(x)}{d(x)} = 0^-.$$

This is because, for functions $n(x)$ and $d(x)$, if $n(x) \to 1$ and $d(x) \to -\infty$, then $\dfrac{n(x)}{d(x)} \to 0$ from the left.

5.5.2 Signs are handled as expected in products and quotients

Example 8 Consider a function defined by

$$f(x) = \frac{-1}{x}.$$

In Example 1, we concluded that $\lim_{x\to\infty} \dfrac{-1}{x} = 0^-$ and $\lim_{x\to-\infty} \dfrac{-1}{x} = 0^+$.

We can also write

$$\frac{-1}{x} \to 0^- \quad \text{as} \quad x \to \infty$$

and

$$\frac{-1}{x} \to 0^+ \quad \text{as} \quad x \to -\infty.$$

□

Let us define the so-called the rescaling property of some limit forms

Definition 2 This property applies to limit forms that do not yield a limit that is a real nonzero constant; that is, the property applies to limit forms that yield: 0, ∞, or $-\infty$ (or, for that matter, the limit does not exist).

If such a limit form is rescaled by multiplying it by a positive real constant, then the resulting limit form will yield the same limit as the original. □

Example 9 Two cases are considered:

- The limit form of $\frac{1}{\infty}$ yields 0 (or, more precisely, 0^+) as a limit, so the $\frac{2}{\infty}$, $\frac{3}{\infty}$ and $\frac{1/2}{\infty}$ limit forms all yield 0 (or 0^+) as a limit, also.
- Likewise, the limit form of $\frac{-1}{\infty}$ yields 0 (or, more precisely, 0^-) as a limit, so the $\frac{-2}{\infty}$, $\frac{-3}{\infty}$ and $\frac{-1/2}{\infty}$ limit forms all yield 0 (or 0^-) as a limit, also.

□

Example 10 The *rescaling property* does not apply to limit forms that yield a limit that is a real nonzero constant. For example, the limit form $\frac{1}{1}$ yields 1, but the limit form $\frac{2}{1}$ yields 2. We will discuss similar limit form properties in a later section. □

5.5.3 The limit form of $(1 \cdot \infty) \Rightarrow \infty$

As a consequence, by our rules for signs and rescaling, we can also write (the $\Rightarrow$ notation means 'implying'):

- Limit form $(b \cdot \infty) \Rightarrow \infty$ for any positive real constant b.
- Limit form $(b \cdot -\infty) \Rightarrow -\infty$ for any positive real constant b.
- Limit form $(b \cdot \infty) \Rightarrow -\infty$ for any negative real constant b.
- Limit form $(b \cdot -\infty) \Rightarrow \infty$ for any negative real constant b.

5.6 Some limit forms involving infinity

Table 5.2 lists some limit forms involving infinity.

5.7 "Classic" indeterminate limit forms

We will discuss these in Chapter 15:

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad 0 \cdot \infty, \quad \infty - \infty, \quad \infty^0, \quad 0^0, \quad 1^\infty.$$

Note 2 The following examples are not indeterminate limit forms:

- Limit form $2^\infty \Rightarrow \infty$ and
- Limit form $\left(\frac{1}{2}\right)^\infty \Rightarrow 0$.

Limit Form $\frac{1}{\infty} \Rightarrow 0^+$
Limit Form $\frac{1}{0^+} \Rightarrow \infty$
Limit Form $\frac{\infty}{1} \Rightarrow \infty$
Limit Form $\frac{\infty}{0^+} \Rightarrow \infty$
Limit Form $\frac{0^+}{\infty} \Rightarrow 0^+$
Limit Form $(1 \cdot \infty) \Rightarrow \infty$
Limit Form $(\infty \cdot \infty) \Rightarrow \infty$
Limit Form $(\infty + b) \Rightarrow \infty$ for any real constant b
Limit Form $(-\infty + b) \Rightarrow -\infty$ for any real constant b
Limit Form $(\infty + \infty) \Rightarrow \infty$
Limit Form $(-\infty - \infty) \Rightarrow -\infty$
Limit Form $\infty^{\infty} \Rightarrow \infty$
Limit Form $\infty^{-\infty} \Rightarrow 0^+$; think of $\infty^{-\infty}$ as $\frac{1}{\infty^{\infty}}$
Limit Form $0^{\infty} \Rightarrow 0$

Table 5.2:

5.8 Limit rules for $\dfrac{b}{x^k}$

Here were some examples:

- In Example 1, we note that $\lim\limits_{x\to\infty} \frac{1}{x} = 0$ and $\lim\limits_{x\to-\infty} \frac{1}{x} = 0$.
- In Example 6 and 7, we note that $\lim\limits_{x\to\infty} \frac{1}{\sqrt[3]{x}} = 0$ and $\lim\limits_{x\to-\infty} \frac{1}{\sqrt[3]{x}} = 0$.
- In Example 8, we note that $\lim\limits_{x\to\infty} \frac{1}{\sqrt{x}} = 0$ but $\lim\limits_{x\to-\infty} \frac{1}{\sqrt{x}}$ does not exist.

More generally, we have

Theorem 1 If b is a real constant, and k is a positive rational constant, then $\lim\limits_{x\to\infty} \frac{b}{x^k} = 0$. Also, $\lim\limits_{x\to-\infty} \frac{b}{x^k} = 0$ if x^k is defined as a real quantity whenever $x < 0$; otherwise, $\lim\limits_{x\to-\infty} \frac{b}{x^k}$ does not exist.

Proof. Assume that b is a real constant (that is, any arbitrary real constant), and k is a positive rational constant.

- Then, $x^k \to \infty$ as $x \to \infty$ and $\lim\limits_{x\to\infty} \frac{b}{x^k}$ has the limit form $\frac{b}{\infty}$, which implies that the limit is 0 (regardless of whether b is positive, negative, or 0).
- Also, $x^k \to -\infty$ as $x \to -\infty$, if x^k is defined as a real quantity whenever $x < 0$. Then, $\lim\limits_{x\to-\infty} \frac{b}{x^k}$ has the limit form $\frac{b}{-\infty}$, which implies that the limit is 0. If x^k is undefined as a real quantity whenever $x < 0$, then $\lim\limits_{x\to-\infty} \frac{b}{x^k}$ x^k does not exist, and $\lim\limits_{x\to-\infty} \frac{b}{x^k}$ does not exist, even if $b = 0$.

$\square$

Example 11 By letting $b=-2$ and $k=3$, we have

$$\lim_{x\to\infty}\frac{-2}{x^3}=0 \quad \text{and} \quad \lim_{x\to-\infty}\frac{-2}{x^3}=0.$$

□

Example 12 By letting $b=\pi$ and $k=\frac{3}{4}$, we have

$$\lim_{x\to\infty}\frac{\pi}{x^{3/4}}=0$$

but

$$\lim_{x\to-\infty}\frac{\pi}{x^{3/4}}=\lim_{x\to-\infty}\frac{\pi}{\left(\sqrt[4]{x}\right)^3}$$

does not exist. □

5.9 How can we tell if a limit is ∞ or $-\infty$?

A useful tool is the so-called "Dominant Term Substitution" test.

Two main ideas must be kept in mind:

1. When analyzing a limit as $x\to\infty$ or $-\infty$, if a and b are integers, then x^a dominates[2] $x^b \Leftrightarrow a>b$.

2. When analyzing a limit as $x\to\infty$ or $-\infty$, the dominant term of a polynomial is the leading term.

The definition of the Dominant Term Substitution test reads as follows:

Definition 3 The limit of an expression is the same as the limit of its dominant term. (By "limit" here, we include ∞ and $-\infty$.) □

Note 3 Do not use dominance if an expression is not defined as a real quantity when considering the limit. This is never an issue with polynomials.

[2] We will say that x^d dominates x^n as $x\to\infty$ for real constants d and $n \Leftrightarrow d>n$. (the $\Leftrightarrow$ notation means 'if and only if'.) This is because the growth of the (the absolute value of) x^d makes the growth of the (the absolute value) of x^n seem negligible by comparison in the long run. More precisely,

$$\lim_{x\to\infty}\frac{x^n}{x^d}=x^{n-d}=0 \;\Leftrightarrow\; d>n.$$

(If $d>n$, the denominator of $\frac{x^n}{x^d}$ behaves more dramatically than the numerator does., and the limit is 0 as $x\to\infty$) Also,

$$\lim_{x\to-\infty}\frac{x^n}{x^d}=0 \;\Leftrightarrow\; d>n.$$

($d>n$, and x^n and x^d are defined as real quantities whenever $x<0$).

5.9.1 The factoring principle of dominance

The factoring principle of dominance applies here:

If the dominant term is factored out of an expression, we obtain the dominant term times something approaching 1. The limit of the expression as $x \to \infty$ or $-\infty$ is therefore the limit of the dominant term, and the factor approaching 1 can be removed when figuring out the "final" limit. This procedure can be applied to the numerator and the denominator of a fraction separately.

Example 13 Evaluate

$$\lim_{x\to\infty}\left(x^{10}-x^{8}\right).$$

Solution:

Method 1: There is a tension between the two terms, x^{10} and $-x^8$, because x^{10} approaches ∞ as $x \to \infty$, while $-x^8$ approaches $-\infty$ as $x \to \infty$. Using the Dominant Term Substitution test, the dominant term here is x^{10}. As $x \to \infty$, x^{10} behaves more dramatically $-x^8$ than does. Therefore, we have

$$\lim_{x\to\infty}\left(x^{10}-x^{8}\right) = \lim_{x\to\infty} x^{10} = \infty.$$

Method 2: Using the limit rules for $\dfrac{b}{x^k}$, we have

$$\begin{aligned}\lim_{x\to\infty}\left(x^{10}-x^{8}\right) &= \lim_{x\to\infty} x^{10}\left(1-\frac{x^8}{x^{10}}\right)\\ &= \lim_{x\to\infty} x^{10}\left(1-\frac{1}{x^2}\right).\end{aligned}$$

As $x \to \infty$, $x^{10} \to \infty$, and $\frac{1}{x^2} \to 0$, thus, $1-\dfrac{1}{x^2} \to 1$, so we have the limit form $(\infty \cdot 1) \Rightarrow \infty$. Therefore, we have

$$\begin{aligned}\lim_{x\to\infty}\left(x^{10}-x^{8}\right) &= \lim_{x\to\infty} \underbrace{\underbrace{x^{10}}_{\to\infty}\underbrace{\left(1-\underbrace{\frac{1}{x^2}}_{\to 0}\right)}_{\to 1}}_{\to\infty}\\ &= \infty.\end{aligned}$$

□

5.10 Limits for rational functions as $x \to \infty$ or $-\infty$

Theorem 2 If $f(x)$ is a rational function with its implied domain, then $\lim_{x\to\infty} f(x) = A$ (for some real constant A) implies $\lim_{x\to-\infty} f(x) = A$, and vice-versa.

If $f(x)$ is not rational, then its graph could have zero, one, or two horizontal asymptotes. □

Example 14 Evaluate

$$\lim_{x\to\infty} \frac{5x^3 + x - 1}{4x^3 - 2x}.$$

Solution:

Method 1: Consider

$$f(x) = \frac{n(x)}{d(x)},$$

where $n(x) = 5x^3+x-1$ is the numerator, which is a polynomial, and $d(x) = 4x^3-2x$ is the denominator, which is a polynomial that is not the 0 polynomial.

Our strategy is to divide (each term of) $n(x)$ and $d(x)$ by the dominant term in $d(x)$. Here, it is x^3. This will ensure that the new denominator will approach a nonzero real constant, namely the leading coefficient of $d(x)$. Here, it is 4.

Using the limit rules for $\dfrac{b}{x^n}$, we have

$$\begin{aligned}
\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{5x^3 + x - 1}{4x^3 - 2x} \\
&= \lim_{x\to\infty} \frac{\frac{5x^3}{x^3} + \frac{x}{x^3} - \frac{1}{x^3}}{\frac{4x^3}{x^3} - \frac{2x}{x^3}} \\
&= \lim_{x\to\infty} \frac{5 + \frac{1}{x^2} - \frac{1}{x^3}}{4 - \frac{2}{x^2}} \\
&= \frac{5 + \lim_{x\to\infty} \frac{1}{x^2} - \lim_{x\to\infty} \frac{1}{x^3}}{4 - 2\lim_{x\to\infty} \frac{1}{x^2}} \\
&= \frac{5 + 0 - 0}{4 - 0} \\
&= \frac{5}{4}.
\end{aligned}$$

Method 2: Consider

$$f(x) = \frac{n(x)}{d(x)},$$

where $n(x) = 5x^3+x-1$ is the numerator, which is a polynomial, and $d(x) = 4x^3-2x$ is the denominator, which is a polynomial that is not the 0 polynomial.

In our limit analysis, we may replace $n(x)$ with its dominant term, $5x^3$, and we may replace $n(x)$ with its dominant term, $4x^3$. This is justified by the Factoring Principle of Dominance (which also gives rise to a rigorous method, but Solution 1 is probably easier):

$$\begin{aligned}\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{5x^3+x-1}{4x^3-2x} \\ &= \lim_{x\to\infty} \frac{5x^3\left(1+\frac{1}{4x^2}-\frac{1}{4x^3}\right)}{4x^3\left(1-\frac{2}{4x^2}\right)} \\ &= \lim_{x\to\infty} \frac{5x^3}{4x^3} \\ &= \frac{5}{4}.\end{aligned}$$

□

Example 15 Evaluate

$$\lim_{x\to\infty} f(x) \quad \text{and} \quad \lim_{x\to-\infty} f(x),$$

where

$$f(x) = \frac{-5+3x^2+6x^3}{1+3x^2}.$$

Solution:

Method 1: Consider

$$f(x) = \frac{n(x)}{d(x)},$$

where $n(x) = -5+3x^2+6x^3$ is the numerator, which is a polynomial, and $d(x) = 1+3x^2$ is the denominator, which is a polynomial that is not the 0 polynomial.

Our strategy is to divide (each term of) $n(x)$ and $d(x)$ by the dominant term in $d(x)$. Here, it is x^2.

Using the limit rules for $\dfrac{b}{x^k}$, we have

$$\begin{aligned}\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{-5+3x^2+6x^3}{1+3x^2} \\ &= \lim_{x\to\infty} \frac{\frac{-5}{x^2}+\frac{3x^2}{x^2}+\frac{6x^3}{x^2}}{\frac{1}{x^2}+\frac{3x^2}{x^2}} \\ &= \lim_{x\to\infty} \frac{\frac{-5}{x^2}+3+6x}{\frac{1}{x^2}+3} \\ &= \frac{\lim\limits_{x\to\infty} \frac{-5}{x^2}+3+6\lim\limits_{x\to\infty} x}{\lim\limits_{x\to\infty} \frac{1}{x^2}+3} \\ &= \infty.\end{aligned}$$

Similarly, we have

$$\lim_{x\to-\infty} f(x) = -\infty.$$

Method 2: Consider

$$f(x) = \frac{n(x)}{d(x)},$$

where $n(x) = -5 + 3x^2 + 6x^3$ is the numerator, which is a polynomial, and $d(x) = 1 + 3x^2$ is the denominator, which is a polynomial that is not the 0 polynomial.

The numerator, $n(x) = -5 + 3x^2 + 6x^3$, has a degree (that is, 3) that is greater than the degree (that is, 2) of the denominator, $d(x) = 1 + 3x^2$. We know from this alone that the graph of $f(x)$ will have no horizontal asymptote. However, because 3 is exactly one more than 2, the graph will have a slant asymptote, also known as an oblique asymptote.

$$\begin{aligned}
\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{-5 + 3x^2 + 6x^3}{1 + 3x^2} \\
&= \lim_{x\to\infty} \frac{6x^3\left(-\frac{5}{6x^3} + \frac{3x^2}{6x^3} + 1\right)}{3x^2\left(\frac{1}{3x^2} + 3\right)} \\
&= \lim_{x\to\infty} \frac{6x^3}{3x^2} \\
&= \lim_{x\to\infty} 2x \\
&= \infty.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\lim_{x\to-\infty} f(x) &= \lim_{x\to-\infty} \frac{-5 + 3x^2 + 6x^3}{1 + 3x^2} \\
&= \lim_{x\to-\infty} \frac{6x^3\left(-\frac{5}{6x^3} + \frac{3x^2}{6x^3} + 1\right)}{3x^2\left(\frac{1}{3x^2} + 3\right)} \\
&= \lim_{x\to-\infty} \frac{6x^3}{3x^2} \\
&= \lim_{x\to-\infty} 2x \\
&= -\infty.
\end{aligned}$$

Figure 5.6 shows the graph of $f(x)$.

$\square$

How can we find an equation for the slant asymptote (SA)?

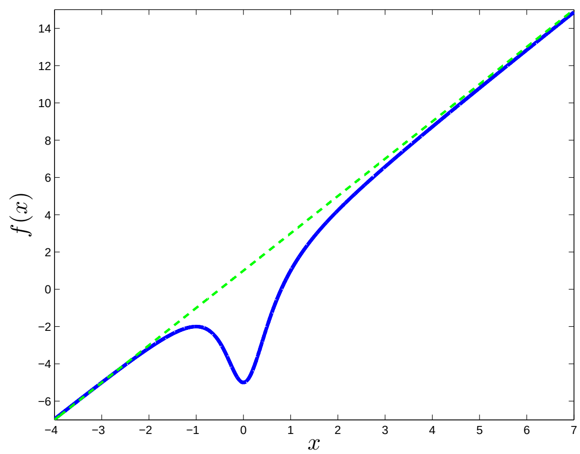

Figure 5.6: Graph of $f(x)$, where $x \in [-4, 7]$.

Using the long division technique, we have

$$\begin{aligned} f(x) &= \frac{-5 + 3x^2 + 6x^3}{1 + 3x^2} \\ &= (2x + 1) + \frac{-2x - 6}{3x^2 + 1}. \end{aligned}$$

The long division technique works as follows:

$$\begin{array}{rr} & 2x + 1 \\ 3x^2 + 0x + 1 & \overline{)\,6x^3 + 3x^2 + 0x - 5} \\ & \underline{6x^3 + 0x^2 + 2x\phantom{{}-5}} \\ & 3x^2 - 2x - 5 \\ & \underline{3x^2 + 0x + 1} \\ & -2x - 6 \end{array}$$

We can stop the division process here, because the degree of the new dividend is less than the degree of the divisor. The degree of $-2x - 6$ is 1, which is less than the degree of $3x^2 + 0x + 1$, which is 2; this means that $-2x - 6$ is our remainder, and $r(x) = \frac{-2x-6}{3x^2+1}$ corresponds to a proper rational function.

Therefore, $r(x) \to 0$ as $x \to \infty$ and as $x \to -\infty$; see Case 1 in our following summary.

(The proper rational part "decays" in the long run process.) Therefore, the graph of $f(x)$ approaches the graph of $p(x)$ as $x \to \infty$ as $x \to -\infty$.

Here, the graph of $f(x)$ approaches the graph of $f(x) = 2x + 1$, which is the equation for our slant asymptote (or oblique asymptote) for the graph of $f(x)$.

5.10.1 Summary

Assume $f(x) = \dfrac{n(x)}{d(x)}$ on its implied domain, where $n(x)$ and $d(x)$ are polynomials, and $d(x)$ is not the zero polynomial.

Let $\deg(n)$ and $\deg(d)$ be their respective degrees.

Case 1: If $\deg(n(x)) < \deg(d(x))$, then $f(x)$ is a proper rational function, and the $x-$axis $(y = 0)$ is the only horizontal asymptote of its graph.

$$\lim_{x\to\infty} f(x) = 0, \quad \text{and} \quad \lim_{x\to-\infty} f(x) = 0.$$

Case 2: If $\deg(n(x)) = \deg(d(x))$, then $y = A$ is the only horizontal asymptote of the graph of $f(x)$, where

$$A = \frac{\text{the leading coefficient of } n\ (x)}{\text{the leading coefficient of } d\ (x)}$$

where A can be considered as the ratio of the leading coefficients.

$$\lim_{x\to\infty} f(x) = A, \quad \text{and} \quad \lim_{x\to-\infty} f(x) = A.$$

Case 3: If $\deg(n(x)) > \deg(d(x))$, then the graph of $f(x)$ has no horizontal asymptotes.

In particular,

$$\lim_{x\to\infty} f(x) = \infty \quad \text{or} \quad -\infty, \quad \text{and} \quad \lim_{x\to-\infty} f(x) = \infty \quad \text{or} \quad -\infty.$$

If $\deg(n(x)) = \deg(d(x)) + 1$, then the graph does have a slant asymptote, also known as an oblique asymptote.

5.11 Limits for algebraic function as $x \to \infty$ or $-\infty$

If we are considering a limit as $x \to \infty$, then we will say that, for rational constants a and b, x^a a dominates $x^b \Leftrightarrow a > b$.

The same is true as $x \to -\infty$, provided that x^a and x^b are defined as real quantities whenever $x < 0$.

Example 16 Evaluate

$$\lim_{x\to\infty} \left(5x^{9/2} + x^3 + 2 + x^{-2}\right).$$

Solution: We have

$$\lim_{x\to\infty} \left(5x^{9/2} + x^3 + 2 + x^{-2}\right) = \lim_{x\to\infty} 5x^{9/2} = \infty.$$

This analysis is appropriate, because $5x^{9/2}$ is the dominant term as $x \to \infty$ (It helps us to think of 1 as x^0 , even though 0^0 is typically considered to be undefined.)

□

Example 17 Evaluate

$$\lim_{x\to-\infty}\left(5x^{9/2}+x^3+2+x^{-2}\right).$$

Solution: We have

$$\lim_{x\to-\infty}\left(5x^{9/2}+x^3+2+x^{-2}\right)$$

which does not exist, not even as ∞ or $-\infty$. This is because $5x^{9/2}$, also written as $5\left(\sqrt{x}\right)^7$, is undefined as a real quantity whenever $x<0$.

□

Example 18 Evaluate

$$\lim_{x\to\infty}\frac{4x^3-\sqrt{x^{10}-8}}{(x+5)^2}.$$

Solution:

Analysis. Although the radicand, $x^{10}-8$, can be negative in value, we have a different situation from the previous example.

It "eventually stays nonnegative" as $x\to\infty$, in the sense that $x^{10}-8\geq 0$ on the $x-$interval (k,∞) for some real constant k. Therefore, the radical "eventually" yields real values as $x\to\infty$.

In the radicand, $x^{10}-8$, x^{10} dominates -8.

In the power-base, $x+5$, x dominates 5.

Solution. Using the dominant term substitution test, we have

$$\begin{aligned}\lim_{x\to\infty}\frac{4x^3-\sqrt{x^{10}-8}}{(x+5)^2}&=\lim_{x\to\infty}\frac{4x^3-\sqrt{x^{10}}}{(x)^2}\\&=\lim_{x\to\infty}\frac{4x^3-x^5}{x^2}\\&=\lim_{x\to\infty}\frac{-x^5}{x^2}\\&=\lim_{x\to\infty}-x^3\\&=-\infty.\end{aligned}$$

□

Note 4 The Factoring Principle of Dominance should not be applied locally to the radicand, $x^{10}-8$. Example 19 will show how that approach can fail. □

The Dominant Term Substitution test can be easily abused. We can use the Dominant Term Substitution test if, at every step in our solution, there is a clearly dominant term in the expression you are finding the limit of (or, if analyzing a fraction, if there is a clearly dominant term in the numerator and a clearly dominant term in the denominator).

Example 19 Evaluate
$$\lim_{x\to\infty}\left(x-\sqrt{x^2+x}\right).$$

Solution:

Incorrect Solution: Using the Dominant Term Substitution test, we have

$$\lim_{x\to\infty}\left(x-\sqrt{x^2+x}\right)=\lim_{x\to\infty}\left(x-\underbrace{x}_{\text{wrong!}}\right)=\lim_{x\to\infty}(0)=0.$$

Clearly, the dominant term substitution test can not be used here, because neither term (neither x nor $-\sqrt{x^2+x}$ is dominant; they are both "on the order of x."

Correct solution:

$$\begin{aligned}
\lim_{x\to\infty}\left(x-\sqrt{x^2+x}\right) &= \lim_{x\to\infty}\frac{x-\sqrt{x^2+x}}{1}\cdot\frac{x+\sqrt{x^2+x}}{x+\sqrt{x^2+x}} && \text{rationalize}\\
&= \lim_{x\to\infty}\frac{x^2-(x^2+x)}{x+\sqrt{x^2+x}}\\
&= \lim_{x\to\infty}\frac{-x}{x+\sqrt{x^2+x}}\\
&= \lim_{x\to\infty}\frac{\frac{-x}{x}}{\frac{x}{x}+\frac{\sqrt{x^2+x}}{x}}\\
&= \lim_{x\to\infty}\frac{-1}{1+\sqrt{\frac{x^2+x}{x^2}}} && \text{becase } \sqrt{x^2}=|x|=x \text{ for } x>0\\
&= \lim_{x\to\infty}\frac{-1}{1+\sqrt{1+\frac{1}{x}}}\\
&= -\frac{1}{2}.
\end{aligned}$$

□

Note 5 Given $\lim_{x\to-\infty}\left(x-\sqrt{x^2+x}\right)=-\infty$.

We observe that $\sqrt{x^2}=|x|=-x$ when $x<0$.

When considering the limit as $x\to-\infty$, one may assume that $x<0$ in the limit analysis. (Here, assume $x\le 1$ due to the domain issue.)

□

5.12 Vertical asymptotes and infinite limits at a point

Example 20 Consider a function defined by

$$f(x)=\frac{1}{x}.$$

x	-1	$-\frac{1}{10}$	$-\frac{1}{100}$	$\to 0^-$	$0^+ \leftarrow$	$\frac{1}{100}$	$\frac{1}{10}$	1
$f(x) = \frac{1}{x}$	-1	-10	-100	$\to -\infty$	$\infty \leftarrow$	100	10	1

Table 5.3: Values of $f(x) = \dfrac{1}{x}$.

Let us experiment with a table (Table 5.3):

Here, we can write:

$$\lim_{x \to 0^+} f(x) = \infty. \tag{5.1}$$

(5.1) reads "the limit of $f(x)$ as x approaches 0 from the right is infinity." That is, as x approaches 0 from higher numbers, the function values $f(x)$ (generally) increase without bound.

Similarly, we can write:

$$\lim_{x \to 0^-} f(x) = -\infty. \tag{5.2}$$

(5.2) reads "the limit of $f(x)$ as x approaches 0 from the left is negative infinity." That is, as x approaches 0 from lower numbers, the function values $f(x)$ (generally) decrease without bound.

Therefore,

$$\lim_{x \to 0} f(x)$$

does not exist, not even as ∞ or $-\infty$.

Therefore,

$$\lim_{x \to 0^+} f(x) = \infty \quad \text{or} \quad \lim_{x \to 0^-} f(x) = -\infty.$$

(either alone would have been sufficient), the graph of $x = 0$ (that is, the $y-$axis) is a vertical asymptote for the graph of $y = f(x)$, as shown in Figure 5.7.

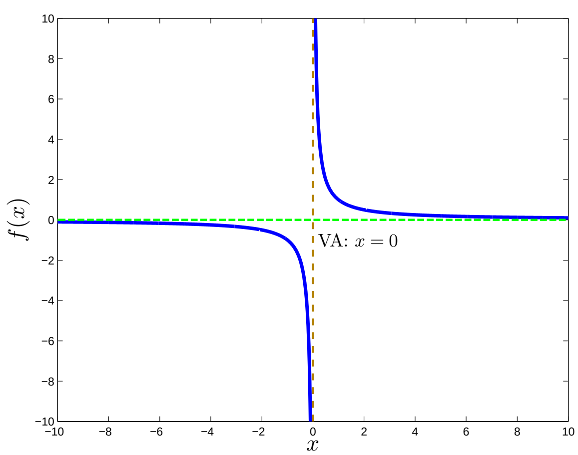

Figure 5.7: Graph of $f(x) = \frac{1}{x}$, where $x \in [-10, 10]$.

□

Definition 4 The graph of $f(x)$ has a **vertical asymptote** at $x = c$ (for a real constant c) $\Leftrightarrow$

$$\lim_{x \to c^+} f(x) = \infty \quad \text{or} \quad -\infty,$$

or

$$\lim_{x \to c^-} f(x) = \infty \quad \text{or} \quad -\infty.$$

As x approaches c from the left or the right, $f(x)$ "explodes" in the sense that it approaches ∞ or $-\infty$.

Note 6

- The graph of $f(x)$ for a function $f(x)$ can have any nonnegative integer number of vertical asymptotes, or infinitely many (remember the graph of $f(x) = \tan x$, for example).
- A polynomial graph has no vertical asymptotes.
- The graph of a rational function has a nonnegative integer number of vertical asymptotes.

□

5.13 The limit forms $\frac{1}{0^+}$ and $\frac{1}{0^-}$

Example 1 showed us that

$$\lim_{x \to 0^+} \frac{1}{x} = \infty,$$

and

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty,$$

More generally, it is true that, for functions $n(x)$ and $d(x)$,

$$\frac{n(x)}{d(x)} \to \infty \quad \text{if} \quad n(x) \to 1 \quad \text{and} \quad d(x) \to 0^+.$$

This is true whether we are considering all of the indicated limits as $x \to \infty$, as $x \to -\infty$, or as $x \to c$ (or as $x \to c^+$, or as $x \to c^-$) for some real constant c.

Also,

$$\frac{n(x)}{d(x)} \to -\infty \quad \text{if} \quad n(x) \to 1 \quad \text{and} \quad d(x) \to 0^-.$$

Here are a few results:

$$\text{Limit form } \frac{1}{0^+} \Rightarrow \infty$$

and

$$\text{Limit form } \frac{1}{0^-} \Rightarrow -\infty.$$

More generally, if b is a real constant:

$$\text{Limit form } \frac{b}{0^+} \Rightarrow \infty, \quad \text{if} \quad b > 0$$

and

$$\text{Limit form } \frac{b}{0^+} \Rightarrow -\infty, \quad \text{if} \quad b > 0.$$

The sign variants are as expected:

$$\text{Limit form } \frac{-1}{0^+} \Rightarrow -\infty$$

and

$$\text{Limit form } \frac{-1}{0^-} \Rightarrow \infty.$$

More generally, if b is a real constant:

$$\text{Limit form } \frac{b}{0^+} \Rightarrow -\infty, \quad \text{if} \quad b < 0$$

and

$$\text{Limit form } \frac{b}{0^+} \Rightarrow \infty, \quad \text{if} \quad b < 0.$$

5.14 Rational functions

Example 21 Evaluate

$$\lim_{x \to 0^+} \frac{1}{x^2}, \quad \lim_{x \to 0^-} \frac{1}{x^2}, \quad \text{and} \quad \lim_{x \to 0} \frac{1}{x^2}.$$

Solution: Because $x^2 > 0$ for all non-zero values of x, all three give the limit form $\frac{1}{0^+} \Rightarrow \infty$. Figure 5.8 shows the vertical asymptote at $x = 0$.

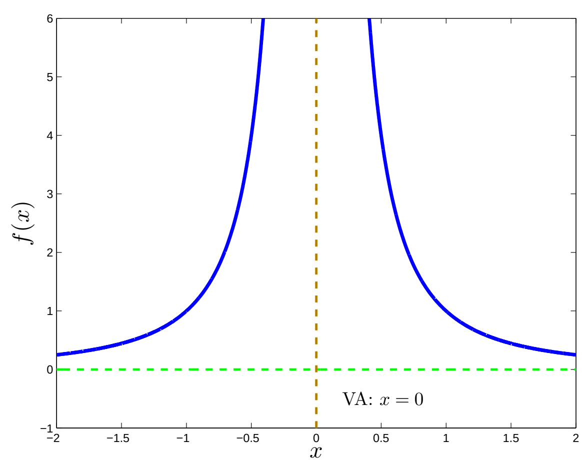

Figure 5.8: Graph of $f(x) = \frac{1}{x^2}$, where $x \in [-2, 2]$.

□

Example 22 Let

$$f(x) = \frac{x+1}{x^2+4x}.$$

Evaluate

$$\lim_{x \to -4^+} f(x), \quad \lim_{x \to -4^-} f(x), \quad \text{and} \quad \lim_{x \to -4} f(x).$$

Solution:

Analysis: We observe that

- $\lim\limits_{x \to -4}(x+1) = -4+1 = -3;$
- $\lim\limits_{x \to -4}(x^2+2x) = (-4)^2 + 4(-4) = 0.$

All three problems give the limit form $\frac{-3}{0}$. However, we need to know how the denominator approaches 0. Since it is easier to analyze signs of products than of sums (for example, do you automatically know the sum of a and b if $a > 0$ and $b < 0$?), we will factor the denominator.

Solution:

$$\begin{aligned}
\lim_{x \to -4^+} f(x) &= \lim_{x \to -4^+} \frac{x+1}{x^2+4x} \\
&= \frac{\underbrace{\lim_{x \to -4^+}(x+1)}_{\to -3}}{\underbrace{\lim_{x \to -4^+} x}_{\to -4}\underbrace{\lim_{x \to -4^+}(x+4)}_{\to 0^+}} \qquad \text{Limit form } \frac{-3}{0^-} \\
&= \infty
\end{aligned}$$

$$\begin{aligned}
\lim_{x \to -4^-} f(x) &= \lim_{x \to -4^-} \frac{x+1}{x^2+4x} \\
&= \frac{\underbrace{\lim_{x \to -4^-}(x+1)}_{\to -3}}{\underbrace{\lim_{x \to -4^-} x}_{\to -4}\underbrace{\lim_{x \to -4^-}(x+4)}_{\to 0^-}} \qquad \text{Limit form } \frac{-3}{0^+} \\
&= -\infty
\end{aligned}$$

Therefore, the two-sided limit

$$\lim_{x \to -4} f(x)$$

does not exist, not even as ∞ or $-\infty$.

$\square$

Finding vertical asymptotes for graphs of rational functions (expressed in simplified form)

Definition 5 If

- $f(x)$ is rational and written in the form $f(x) = \frac{n(x)}{d(x)}$,
- $n(x)$ and $d(x)$ are polynomials,
- $d(x) \neq 0$ (that is, the zero polynomial), and
- $n(x)$ and $d(x)$ have no real zeros in common; that is, they have no variable factors in common.

then
The graph of $f(x)$ has a vertical asymptote at $x = c$ and

$$\lim_{x \to c^+} f(x) = \infty \quad \text{or} \quad -\infty$$

and

$$\lim_{x \to c^-} f(x) = \infty \quad \text{or} \quad -\infty.$$

$\Leftrightarrow c$ is a real zero of $d(x)$.

□

Example 23 Let

$$f(x) = \frac{x+1}{x^2+4x}.$$

Find the equations of the vertical asymptotes of the graph of $f(x)$ in the xy-plane. Justify your answer using limits.

Solution: We observe that the numerator and the denominator have no variable factors (and no real zeros) in common. That is,

$$\frac{x+1}{x^2+4x} = \frac{x+1}{x(x+4).}$$

Therefore, the real zeros of the denominator correspond to the vertical asymptotes of the graph. The vertical asymptotes are at $x = 0$ and $x = -4$, as shown in Figure 5.9
To justify the vertical asymptote at $x = 0$ using limits, show one of the following:

- $$\lim_{x \to 0^-} f(x) = \infty \quad \text{or} \quad -\infty$$

  (it turns out to be ∞), or

- $$\lim_{x \to 0^+} f(x) = \infty \quad \text{or} \quad -\infty$$

  (it turns out to be $-\infty$).

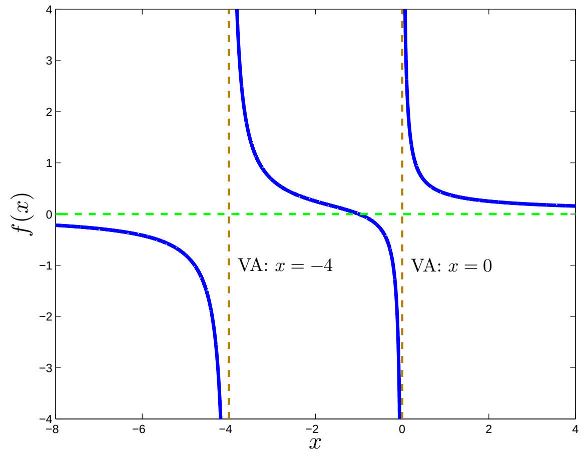

Figure 5.9: Graph of $f(x) = \dfrac{x+1}{x^2+4x}$, where $x \in [-8, 4]$.

To justify the vertical asymptote at $x = -4$ using limits, show one of the following:

- $$\lim_{x \to -4^+} f(x) = \infty \quad \text{or} \quad -\infty$$
  (it turns out to be ∞), or
- $$\lim_{x \to -4^-} f(x) = \infty \quad \text{or} \quad -\infty$$
  (it turns out to be $-\infty$).

$\square$