Lecture Note 13

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MATH1020
General Mathematics

The Cross Product

Find the Cross Product of Two Vectors

For vectors in space, and only for vectors in space, a second product of two vectors defined, called the *cross product*. The cross product of two vectors in space is, in fact also a vector that has applications in both geometry and physics.

Definition 1 If $\mathbf{v} = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$ and $\mathbf{w} = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$ are two vectors in space, the **cross product** $\mathbf{v} \times \mathbf{w}$ is defined as the vector

$$\mathbf{v} \times \mathbf{w} = (b_1c_2 - a_2c_1)\mathbf{i} - (a_1c_2 - a_2c_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}. \tag{1}$$

Notice that the cross product $\mathbf{v} \times \mathbf{w}$ of two vectors is a vector. Because of this, it is sometimes referred to as the **vector product.**

Example 1 Finding Cross Products Using Equation (1)

If $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$ and $\mathbf{w} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, an application (1) gives

$$\begin{aligned}\mathbf{v} \times \mathbf{w} &= (3 \cdot 3 - 2 \cdot 5)\mathbf{i} - (2 \cdot 3 - 1 \cdot 5)\mathbf{j} + (2 \cdot 2 - 1 \cdot 3)\mathbf{k} \\ &= (9 - 10)\mathbf{i} - (6 - 5)\mathbf{j} + (4 - 3)\mathbf{k} \\ &= -\mathbf{i} - \mathbf{j} + \mathbf{k}.\end{aligned}$$

Determinants may be used as an aid in computing cross products. **A 2 by 2 determinant,** symbolized by

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

has the value $a_1 b_2 - a_2 b_1$; that is,

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1 .$$

A 3 by 3 determinant has the value

$$\begin{vmatrix} A & B & C \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} A - \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} B + \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} C.$$

Example 2 Evaluating Determinants

(a) $$\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = 2 \cdot 2 - 1 \cdot 3 = 4 - 3 = 1.$$

(b) $$\begin{vmatrix} A & B & C \\ 2 & 3 & 5 \\ 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 3 & 5 \\ 2 & 3 \end{vmatrix} A - \begin{vmatrix} 2 & 5 \\ 1 & 3 \end{vmatrix} B + \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} C$$
$$= (9 - 10)A - (6 - 5)B + (4 - 3)C$$
$$= -A - B + C.$$

The cross product of the vectors

$$\mathbf{v} = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$$

and

$$\mathbf{w} = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k},$$

that is,

$$\mathbf{v} \times \mathbf{w} = (b_1c_2 - b_2c_1)\mathbf{i} - (a_1c_2 - a_2c_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

may be written symbolically using determinants as

$$\mathbf{u} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\mathbf{k}.$$

Example 3 Using Determinants to Find Cross Products

If $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$ and $\mathbf{w} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, find:

(a) $\mathbf{v} \times \mathbf{w}$

(b) $\mathbf{w} \times \mathbf{v}$

(c) $\mathbf{v} \times \mathbf{v}$

(d) $\mathbf{w} \times \mathbf{w}$

Solution

$$\text{(a) } \mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 5 \\ 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 3 & 5 \\ 2 & 3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 5 \\ 1 & 3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} \mathbf{k} = -\mathbf{i} - \mathbf{j} + \mathbf{k}.$$

$$\text{((b) } \mathbf{w} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 2 & 3 & 5 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} \mathbf{k} = \mathbf{i} + \mathbf{j} - \mathbf{k}.$$

$$
\text{(c) } \mathbf{v} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 5 \\ 2 & 3 & 5 \end{vmatrix} = \begin{vmatrix} 3 & 5 \\ 3 & 5 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 5 \\ 2 & 5 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 3 \\ 2 & 3 \end{vmatrix} \mathbf{k} = 0\mathbf{i} - 0\mathbf{j} + 0\mathbf{k}
$$

$$
= \mathbf{0}.
$$

$$
\text{(d) } \mathbf{w} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 2 & 3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} \mathbf{k} = 0\mathbf{i} - 0\mathbf{j} + 0\mathbf{k}
$$

$$
= \mathbf{0}.
$$

Interpret Algebraic Properties of the Cross Product

Notice in Example 3(a) and (b) that $\mathbf{v} \times \mathbf{w}$ and $\mathbf{w} \times \mathbf{v}$ are negatives of one another.

From Examples 3(c) and (d), we might conjecture that the cross product of a vector with itself is the zero vector.

These and other algebraic properties of the cross product are given next.

Theorem 1 **Algebraic Properties of the Cross Product**

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in space and if α is a scalar, then

$$\mathbf{u} \times \mathbf{u} = \mathbf{0}. \tag{2}$$

$$\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}. \tag{3}$$

$$\alpha(\mathbf{u} \times \mathbf{v}) = (\alpha\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (\alpha\mathbf{v}). \tag{4}$$

$$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}. \tag{5}$$

Proof:

We only prove properties (2) and (4) here.

To prove property (2), we let $\mathbf{u} = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$. then

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \mathbf{k}$$
$$= 0\mathbf{i} - 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}.$$

To prove property (4), we let $\mathbf{u} = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$ and $\mathbf{v} = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$. Then

$$\begin{aligned} \alpha(\mathbf{u} \times \mathbf{v}) &= \alpha[(b_1c_2 - a_2c_1)\mathbf{i} - (a_1c_2 - a_2c_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}] && \text{Apply (1)} \\ &= \alpha(b_1c_2 - b_2c_1)\mathbf{i} - \alpha(a_1c_2 - a_2c_1)\mathbf{j} + \alpha(a_1b_2 - a_2b_1)\mathbf{k}. \end{aligned} \tag{6}$$

Since $\alpha\mathbf{u} = \alpha(a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k})$, we have

$$\begin{aligned} (\alpha\mathbf{u}) \times \mathbf{v} &= \alpha(\alpha b_1c_2 - b_2\alpha c_1)\mathbf{i} - \alpha(\alpha a_1c_2 - a_2\alpha c_1)\mathbf{j} + \alpha(\alpha a_1b_2 - a_2\alpha b_1)\mathbf{k} \\ &= \alpha(b_1c_2 - a_2c_1)\mathbf{i} - \alpha(a_1c_2 - a_2c_1)\mathbf{j} + \alpha(a_1b_2 - a_2b_1)\mathbf{k}. \end{aligned} \tag{7}$$

Based on equations (6) and (7), the first part of property (4) follows. The second part can be proved in a similar fashion.

Interpret Geometric Properties of the Cross Product The cross product has several interesting geometric properties.

Theorem 2 Geometric Properties of the Cross Product

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in space.

$$\mathbf{u} \times \mathbf{v} \text{ is orthogonal to both } \mathbf{u} \text{ and } \mathbf{v}. \tag{8}$$

$$||\mathbf{u} \times \mathbf{v}|| = ||\mathbf{u}||||\mathbf{v}|| \sin\theta, \text{ where } \theta \text{ is the angle between } \mathbf{u} \text{ and } \mathbf{v}. \tag{9}$$

$$||\mathbf{u} \times \mathbf{v}|| \text{ is the area of the parallelogram having } \mathbf{u} \neq \mathbf{0} \text{ and } \mathbf{v} \neq \mathbf{0} \text{ as adjacent sides.} \tag{10}$$

$$\mathbf{u} \times \mathbf{v} = \mathbf{0} \text{ if and only if } \mathbf{u} \text{ and } \mathbf{v} \text{ are parallel.} \tag{11}$$

Proof of Property (8)

Let $\mathbf{u} = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$ and $\mathbf{v} = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$. Then

$$\mathbf{u} \times \mathbf{v} = (b_1c_2 - b_2c_1)\mathbf{i} - (a_1c_2 - a_2c_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}.$$

Now we compute the dot product $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v})$.

$$\begin{aligned}\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) &= (a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k})[(b_1c_2 - b_2c_1)\mathbf{i} - (a_1c_2 - a_2c_1)\mathbf{j} \\ &\quad + (a_1b_2 - a_2b_1)\mathbf{k}] \\ &= a_1(b_1c_2 - b_2c_1) - b_1(a_1c_2 - a_2c_1) + c_1(a_1b_2 - a_2b_1) = 0.\end{aligned}$$

Since two vectors are orthogonal if their dot product is aero, it follows that $\mathbf{u}$ and $\mathbf{u} \times \mathbf{v}$ are orthogonal.

Similarly, $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$, so $\mathbf{v}$ and $\mathbf{u} \times \mathbf{v}$ are orthogonal.

Find a Vector Orthogonal to Two Given Vectors

As long as the vectors $\mathbf{u}$ and $\mathbf{v}$ are not parallel, they will form a place in space. See Figure 1.

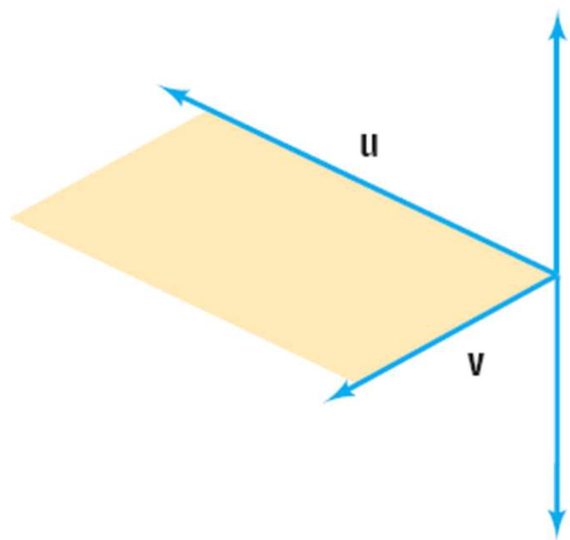

Figure 1:

Based on property (8), the vector $\mathbf{u} \times \mathbf{v}$ is normal to this plane.

As Figure 1 illustrates, there are essentially (without regard to magnitude) two vectors normal to the plane containing $\mathbf{u}$ and $\mathbf{v}$. It can be shown that the vector

$\mathbf{u} \times \mathbf{v}$ is the one determined by the thumb of the right hand when the other fingers of the right hand are cupped so that they point in a direction from $\mathbf{u}$ to $\mathbf{v}$. See Figure 2.

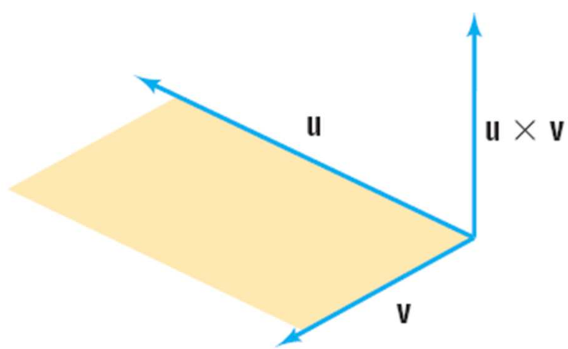

Figure 2:

Example 4 Finding a Vector Orthogonal to Two Given Vectors

Find a vector that is orthogonal to $\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = -\mathbf{i} + 3\mathbf{j} - \mathbf{k}$.

Solution:

Based on property (8), such a vector is $\mathbf{u} \times \mathbf{v}$.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 5 \\ 1 & 2 & 3 \end{vmatrix} = (2-3)\mathbf{i} - [-3-(-1)]\mathbf{j} + (9-2)\mathbf{k} = -\mathbf{i} + 2\mathbf{j} + 7\mathbf{k}.$$

The vector $-\mathbf{i} + 2\mathbf{j} + 7\mathbf{k}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

Check: Two vectors are orthogonal if their dot product is zero.

$$\mathbf{u} \times (-\mathbf{i} + 2\mathbf{j} + 7\mathbf{k}) = (3\mathbf{i} - 2\mathbf{j} + \mathbf{k}) \cdot (-\mathbf{i} + 2\mathbf{j} + 7\mathbf{k}) = -3 - 4 + 7 = 0.$$

$$\mathbf{v} \times (-\mathbf{i} + 2\mathbf{j} + 7\mathbf{k}) = (-\mathbf{i} + 3\mathbf{j} - \mathbf{k}) \cdot (-\mathbf{i} + 2\mathbf{j} + 7\mathbf{k}) = 1 + 6 - 7 = 0.$$

Proof of Property (10)

Suppose that $\mathbf{u}$ and $\mathbf{v}$ are adjacent sides of a parallelogram.
See Figure 3.

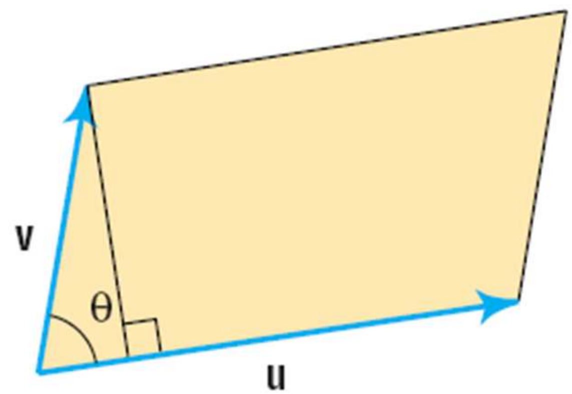

Figure 3:

Then the lengths of these sides are $||\mathbf{u}||$ and $||\mathbf{v}||$.

If θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then the height of the parallelogram is $||\mathbf{v}|| \sin \theta$

and its area is

$$\begin{aligned}
\text{Area of parallelogram} &= \text{Base} \times \text{Height} \\
&= ||\mathbf{u}||[||\mathbf{v}||\sin\theta] \qquad \text{Property (9)} \\
&= ||\mathbf{u} \times \mathbf{v}||.
\end{aligned}$$

Find the Area of a Parallelogram

Example 5 Find the Area of a Parallelogram

Find the area of the parallelogram whose vertices are
$P_1 = (0, 0, 0)$, $P_2 = (3, -2, 1)$, $P_3 = (-1, 3, -1)$, and $P_4 = (2, 1, 0)$.

Solution:

Two adjacent sides of this parallelogram are

$$\mathbf{u} = \overrightarrow{P_1P_2} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$$

and

$$\mathbf{v} = \overrightarrow{P_1P_3} = -\mathbf{i} + 3\mathbf{j} - \mathbf{k}.$$

Since $\mathbf{u} \times \mathbf{v} = -\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ (Example 4), the area of the parallelogram is

$$\begin{aligned}\text{Area of parallelogram} &= ||\mathbf{u}||||\mathbf{v}|| \\ &= \sqrt{1 + 4 + 49} = \sqrt{54} = 3\sqrt{6} \text{ square units.}\end{aligned}$$

Proof of Property (11)

The proof requires two parts.

If $\mathbf{u}$ and $\mathbf{v}$ are parallel, then there is a scalar α such that $\mathbf{u} = \alpha\mathbf{v}$. Then

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= (\alpha\mathbf{v}) \times \mathbf{v} && \text{Property (4)} \\ &= \alpha(\mathbf{v} \times \mathbf{v}) && \text{Property (2)} \\ &= \mathbf{0}. \end{aligned}$$

If $\mathbf{u} \times \mathbf{v} = \mathbf{0}$, then , by property (9), we have

$$||\mathbf{u} \times \mathbf{v}|| = ||\mathbf{u}||||\mathbf{v}|| \sin\theta = 0.$$

Since $\mathbf{u} \neq \mathbf{0}$ and $\mathbf{v} \neq \mathbf{0}$, we must have $\sin\theta = 0$, so $\theta = 0$ or $\theta = \pi$. In either case, since θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then $\mathbf{u}$ and $\mathbf{v}$ are parallel.