Calculus for Engineers

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August 2015

Contents

Differentiability

7.1 Introduction

In this chapter we shall study the derivative of a single valued function of a real variable (independent). A one-sided limit and a two-sided limit of a function will be studied. The relationship between the continuity and differentiability of a function will be examined. Some basic theorems on derivatives are given. The derivative of a composite function with the aid of the chain rule is discussed. Finding the derivative of an inverse function is presented. Each concept is explained and illustrated by worked examples.

7.2 Derivative

7.2.1 Definition of derivative

Definition 1 A single valued real function $f(x)$ defined in interval $[a, b]$ is said to be differentiable at any point c of the interval, if

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

or[1]

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h},$$

exists and is finite. This limit is called the derivative of the function f at any point c and is denoted by $f'(c)$ or $\dfrac{d}{dx} f(x)$ at $x = c$.

The process of getting $f'(c)$ is called differentiation.

7.2.2 Definition of the right-hand derivative

Definition 2 The right-hand derivative of the function f at $x = c$ is denoted by $Rf'(c)$, where

$$Rf'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$$

[1] Putting $x = c + h$, where $h \to 0$, we have

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{h \to 0} \frac{f(c+h) - f(c)}{c + h - c} = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}.$$

or[2]

$$\lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h},\ h>0$$

exists and is finite.

7.2.3 Definition of the left-hand derivative

Definition 3 The left-hand derivative of the function f at $x=c$ is denoted by $Lf'(c)$, where

$$Lf'(c) = \lim_{x\to c^-} \frac{f(x)-f(c)}{x-c}$$

or[3]

$$\lim_{h\to 0^-} \frac{f(c+h)-f(c)}{h},\ h>0$$

exists and is finite.

7.2.4 Definition of right- and left-hand derivatives

Corollary 1 A function f is differentiable at a point $x=c$ if both the left-hand derivative and the right-hand derivative exist and are equal, that is,

$$Rf'(c) = Lf'(c) = f'(c).$$

Note 1 If the derivative $Rf'(c)$ or $Lf'(c)$ does not exist, or they exist but are not equal, then the function is not differentiable at (any) point c.

7.2.5 Differentiability of a function in a given interval

Definition 4

(a) In an open interval: A function f defined in an open interval (a,b) is differentiable if f is differentiable at every point of (a,b).

(b) In a closed interval: A function f defined in closed interval $[a,b]$ is differentiable, if

1. $f'(c)$ exists $\forall\ c \in (a,b)$,
2. $Rf'(a)$ exists, and

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[2] Putting $x=c+h$, where $h\to 0^+$, we have

$$\lim_{x\to c^+} \frac{f(x)-f(c)}{x-c} = \lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h},\ h>0.$$

[3] Putting $x=c-h$, where $h\to 0^-$, we have

$$\lim_{x\to c^-} \frac{f(x)-f(c)}{x-c} = \lim_{h\to 0^-} \frac{f(c-h)-f(c)}{h},\ h>0.$$

3. $Lf'(b)$ exists.

Examples

Example 1 Let $\mathbb{R}^+ := \{x \in \mathbb{R} |\ x > 0\}$ be the set of all positive real numbers. Show that a function $f(x) = \sqrt{x}\ \forall x \in \mathbb{R}^+$ is differentiable everywhere.
Solution:

Let c be a real positive constant, then

$$\begin{aligned}
f'(c) &= \lim_{h\to 0} \frac{f(c+h) - f(c)}{h} && \text{Definition of } f' \\
&= \lim_{h\to 0} \frac{\sqrt{c+h} - \sqrt{c}}{h} \cdot \frac{\sqrt{c+h} + \sqrt{c}}{\sqrt{c+h} + \sqrt{c}} && \frac{\sqrt{c+h} + \sqrt{c}}{\sqrt{c+h} + \sqrt{c}} = 1 \\
&= \lim_{h\to 0} \frac{1}{\sqrt{c+h} + \sqrt{c}} && (a-b)(a+b) = a^2 - b^2 \\
&= \frac{1}{2\sqrt{c}}. && \text{Root law with } n = 2
\end{aligned}$$

Since c is an arbitrary constant number, we have $f'(x) = \dfrac{1}{2\sqrt{x}}$. □

Example 2 Show that a function $f(x) = \sin x,\ \forall x \in \mathbb{R}$ is differentiable everywhere.
Solution:

Let c be an arbitrary real number.

Then for $Rf'(c)$

$$\begin{aligned}
f'(c+0) &= \lim_{h\to 0} \frac{f(c+h) - f(c)}{h},\ (h > 0) && \text{Definition of } f' \\
&= \lim_{h\to 0} \frac{\sin(c+h) - \sin c}{h} && \text{Substitute } f(c) = \sin c \\
&= \lim_{h\to 0} \frac{2\cos\left(c + \frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}{h} && \text{Use sum} - \text{to} - \text{product identity} \\
&= \lim_{h\to 0} \cos\left(c + \frac{h}{2}\right) \cdot \lim_{h\to 0} \frac{\sin\frac{h}{2}}{\frac{h}{2}} && \text{Product law} \\
&= \cos c. && \lim_{h\to 0} \frac{\sin\frac{h}{2}}{\frac{h}{2}} = 1 \qquad (7.1)
\end{aligned}$$

Similarly for $Lf'(c)$

$$\begin{aligned}
f'(c-0) &= \lim_{h\to 0}\frac{f(c-h)-f(c)}{-h},\ (h>0) && \text{Definition of } f' \\
&= \lim_{h\to 0}\frac{\sin(c-h)-\sin c}{-h} && \text{Substitute } f(c)=\sin c \\
&= \lim_{h\to 0}\frac{2\cos\left(c-\frac{h}{2}\right)\sin\left(-\frac{h}{2}\right)}{-h} && \text{Use sum}-\text{to}-\text{product identity} \\
&= \lim_{h\to 0}\cos\left(c-\frac{h}{2}\right)\cdot\lim_{h\to 0}\frac{\sin\left(-\frac{h}{2}\right)}{-\frac{h}{2}} && \text{Product law} \\
&= \cos c. && \lim_{h\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}=1 \qquad (7.2)
\end{aligned}$$

From (7.1) and (7.2), we have $f'(c+0)=f'(c-0)=\cos c$. Therefore, $f'(c)=\cos c$. As c is an arbitrary constant number, we have $f'(x)=\cos x,\ \forall x\in\mathbb{R}$. □

Example 3 Show that a function $f(x)=|x|$ is not differentiable at $x=0$.
Solution:

$$\begin{aligned}
Rf'(0) = f'(\underbrace{0}_{=c}+0) &= \lim_{h\to 0}\frac{f(0+h)-f(0)}{h},\ h>0 \\
&= \lim_{h\to 0}\frac{|0+h|-|0|}{h} && \because\ |h|=h \\
&= \lim_{h\to 0}\frac{0+h-0}{h}=1 && (7.3) \\
Lf'(0) = f'(\underbrace{0}_{=c}-0) &= \lim_{h\to 0}\frac{f(0-h)-f(0)}{-h},\ h>0 \\
&= \lim_{h\to 0}\frac{|0-h|-|0|}{-h} && \because\ |-h|=h \\
&= \lim_{h\to 0}\frac{h}{-h}=-1 && (7.4)
\end{aligned}$$

From (7.3) and (7.4), we get

$$f'(0+0)\neq f'(0-0).$$

□

Example 4 Show that the function $f(x)=e^x,\ \forall x\in\mathbb{R}$ is differentiable everywhere.
Solution:

Let c be an arbitrary real number.

Then

$$\begin{aligned} Rf'(c) = f'(c+0) &= \lim_{h\to 0}\frac{f(c+h)-f(c)}{h},\ h>0 && \text{Definition of derivative} \\ &= \lim_{h\to 0}\frac{e^{c+h}-e^c}{h} && e^{a+b}=e^a\cdot e^b \\ &= \lim_{h\to 0}\frac{e^c(e^h-1)}{h} && ab-a=a(b-1) \\ &= \lim_{h\to 0} e^c \ \lim_{h\to 0}\frac{e^h-1}{h} && \text{Product law} \\ &= e^c. && e^0=1 \end{aligned} \tag{7.5}$$

Similarly

$$\begin{aligned} Lf'(c) = f'(c-0) &= \lim_{h\to 0}\frac{f(c-h)-f(c)}{-h},\ h>0 \\ &= \lim_{h\to 0}\frac{e^{c-h}-e^c}{-h} && e^{a+(-b)}=e^a\cdot e^{-b} \\ &= \lim_{h\to 0} e^c \ \lim_{h\to 0}\frac{e^{-h}-1}{-h} \\ &= e^c. \end{aligned} \tag{7.6}$$

From (7.5) and (7.6), we have $f'(c)=e^c$. But c is an arbitrary number, $f'(x)=e^x$. □

7.2.6 The necessary for the existence of a finite derivative

Theorem 2 If a function f is differentiable at a point c, then it is continuous at c but the converse is not necessarily true.

Proof.

Let a function f defined on $[a,b]$ be differentiable at any point c in $[a,b]$, then

$$f'(c) = \lim_{x\to c}\frac{f(x)-f(c)}{x-c} \tag{7.7}$$

is a finite quantity.

Now we have the identity

$$f(x)-f(c) = \frac{f(x)-f(c)}{(x-c)}\ (x-c),\ x\neq c. \tag{7.8}$$

Taking the limit of (7.8) as x tends to c, we get

$$\lim_{x\to c}[f(x)-f(c)] = \lim_{x\to c}\frac{f(x)-f(c)}{(x-c)}\ \lim_{x\to c}(x-c).$$

From (7.7) and using the product law of limit, we have

$$\lim_{x\to c}[f(x)-f(c)] = f'(c)\ 0.$$

This implies that

$$\lim_{x\to c} f(x) = f(c),$$

i.e. when $x \to c$ then $f(x) \to f(c)$. Hence function f is continuous at $x = c$.

Alternate Proof.

We know that function f is differentiable at any point c. Therefore,

$$Rf'(c) = Lf'(c) = f'(c)$$

or

$$\lim_{h\to 0} \frac{f(c+h) - f(c)}{h} = \lim_{h\to 0} \frac{f(c-h) - f(c)}{-h} = f' \tag{7.9}$$

Now

$$f(c+h) - f(c) = \frac{f(c+h) - f(c)}{h} \cdot h. \tag{7.10}$$

Using the limit of (7.10) as h tends to 0, using (7.9) and using the product law of limit, we have

$$\begin{aligned}
\lim_{h\to 0}[f(c+h) - f(c)] &= \lim_{h\to 0}\left[\frac{f(c+h) - f(c)}{h} \cdot h\right] \\
&= \lim_{h\to 0} \frac{f(c+h) - f(c)}{h} \cdot \lim_{h\to 0} h \\
&= f'(c) \cdot 0 = 0.
\end{aligned}$$

Hence,

$$\lim_{h\to 0} f(c+h) = f(c)$$

or

$$\lim_{x\to c^+} f(x) = f(c). \tag{7.11}$$

Similarly

$$\begin{aligned}
\lim_{h\to 0}[f(c-h) - f(c)] &= \lim_{h\to 0}\left[\frac{f(c-h) - f(c)}{-h} \cdot (-h)\right] \\
&= \lim_{h\to 0} \frac{f(c-h) - f(c)}{-h} \cdot \lim_{h\to 0}(-h) \\
&= f'(c) \cdot 0 = 0
\end{aligned}$$

Hence,

$$\lim_{h\to 0} f(c-h) = f(c)$$

or

$$\lim_{x\to c^-} f(x) = f(c). \tag{7.12}$$

From (7.11) and (7.12), we have $\lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x) = f(c)$. Thus, a function $f(x)$ is continuous at c. $\square$

The converse is not necessarily true (just because the function is continuous at a point, does not necessarily mean that the function is differentiable at that point).

It is worth noting that whenever the graph of a function has a corner at a point, but no break or gap that point, we have a point where the function is continuous, but not differentiable.

Example 5 Show that a function

$$f(x) = |x+1| + |x|, \ \forall x \in \mathbb{R} \tag{7.13}$$

may be continuous at a point but not differentiable there.

Solution:

To test for the continuity of f at $x = -1$: Let us examine the following cases:

- $$f(-1) = 0 + 1 = 1. \tag{7.14}$$

- $$\begin{aligned} f(-1-0) &= \lim_{x\to -1^-} f(x) \\ &= \lim_{x\to -1^-} [-(x+1) + (-x)] \\ &= 1. \end{aligned} \tag{7.15}$$

- $$\begin{aligned} f(-1+0) &= \lim_{x\to -1^+} f(x) \\ &= \lim_{x\to -1^+} [(x+1) + (-x)] \\ &= 1. \end{aligned} \tag{7.16}$$

From (7.14), (7.15) and (7.16), we conclude that

$$f(-1-0) = f(-1+0) = f(-1).$$

Hence f is continuous at $x = -1$.

To test for the differentiability of f at $x = -1$: Let us examine the following cases:

- $$\begin{aligned} Lf'(-1) &= \lim_{x\to -1^-} \frac{f(x) - f(-1)}{x - (-1)} \\ &= \lim_{x\to -1^-} \frac{[-(x+1) + (-x)] - 1}{x+1} \\ &= \lim_{x\to -1^-} \frac{-2(x+1)}{x+1} = -2. \end{aligned} \tag{7.17}$$

•

$$\begin{aligned} Rf'(-1) &= \lim_{x\to -1^+} \frac{f(x)-f(-x)}{x+1} \\ &= \lim_{x\to -1^-} \frac{[(x+1)+(-1)]-1}{x=1} = 0. \end{aligned} \tag{7.18}$$

However, from (7.17) and (7.18), we conclude that

$$Lf'(-1) = -2 \neq 0 = Rf'(-1).$$

Thus, the function f is not differentiable at $x = -1$. □

7.2.7 Some basic theorems on derivatives

Finding the derivatives of some standard functions using Definition 1 will be discussed.

Example 6 If $f(x) = b$, where b is a constant, find (a) $f'(1)$, and (b) $f'(x)$.
Solution:

(a) From the definition of a derivative, we obtain

$$f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{b-b}{h} = \lim_{h\to 0} 0 = 0.$$

(b) From the definition of a derivative with $x = b$, we have

$$f'(b) = \lim_{h\to 0} \frac{f(b+h)-f(b)}{h} = \lim_{h\to 0} \frac{b-b}{h} = \lim_{h\to 0} 0 = 0.$$

□

This example illustrates that the derivative of a constant function is 0.

Theorem 3 If b is a real number, then $f'(x) = \dfrac{d}{dx}(b) = 0$.

Theorem 4 If f is differentiable at x and b is a constant, then $\dfrac{d}{dx}(b \cdot f(x)) = b \cdot f'(x)$.

Proof. From the definition of the derivative to the function $b \cdot f(x)$, we obtain

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{bf(x+h)-bf(x)}{h} \\ &= \lim_{h\to 0} \frac{b(f(x+h)-f(x))}{h} \\ &= b \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= bf'(x). \end{aligned} \tag{7.19}$$

Theorem 5 If n is an integer and $f(x) = x^n$, then

$$f'(x) = nx^{n-1} \tag{7.20}$$

provided $x \neq 0$ when $n \leq 0$.

Proof.

Using the definition of a derivative, we have

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{t\to x}\frac{f(t)-f(x)}{t-x} = \lim_{t\to x}\frac{t^n - x^n}{t-x}. \tag{7.21}$$

Case I If n is a positive integer, using the formula

$$t^n - x^n = (t-x)(t^{n-1} + t^{n-2}x + \cdots + tx^{n-2} + x^{n-1}), \tag{7.22}$$

multiplying both sides of (7.22) by $\dfrac{1}{(t-x)}$, and substituting the resultant results into (7.21), we immediately obtain

$$\begin{aligned} f'(x) &= \lim_{t\to x}\underbrace{(t^{n-1} + t^{n-2}x + \cdots + tx^{n-2} + x^{n-1})}_{n \text{ times}} \\ &= nx^{n-1}. \end{aligned} \tag{7.23}$$

Case II If n is a negative integer, we can write $n = -k$ with k positive. Therefore

$$\begin{aligned} f'(x) &= \lim_{t\to x}\frac{f(t)-f(x)}{t-x} \\ &= \lim_{t\to x}\frac{t^{-k}-x^{-k}}{t-x} \\ &= \lim_{t\to x}\frac{\frac{x^k - t^k}{x^k t^k}}{t-x} \\ &= \lim_{t\to x}\frac{(x-t)(x^{k-1} + x^{k-2}t + \cdots + xt^{k-2} + t^{k-1})}{(t-x)x^k t^k} \\ &= -kx^{k-1}/x^{2k} \\ &= -kx^{-k-1} \\ &= nx^{n-1} \end{aligned} \tag{7.24}$$

Case III If $n = 0$, then $f(x) = x^0 = 1$. Therefore $f'(x) = 0 = 0 \cdot x^{0-1}$.

□

Note 2 In essence, the result of Theorem 5 also holds for any real number n; that is,

$$\frac{d}{dx}(x^n) = nx^{n-1}, \tag{7.25}$$

where n is any real number. Formula (7.25) is sometimes called the Power Rule. In words, to differentiate x to a positive integer power, take the power and multiply it by x to the next lower integer power.

Theorem 6 If the functions f and ϕ are defined on $[a,b]$ and are differentiable at any point c of $[a,b]$, then $f\pm\phi$, $f\phi$ and f/ϕ are differentiable at c, and

(i) The Sum and Difference Rules:

$$(f\pm\phi)'(c)=f'(c)\pm\phi'(c); \tag{7.26}$$

(ii) The Product Rule:

$$(f\phi)'(c)=f(c)\phi'(c)+f'(c)\phi(c); \tag{7.27}$$

(iii) The Quotient Rule:

$$\left(\frac{f}{\phi}\right)'(c)=\frac{\phi(c)f'(c)-f(c)\phi'(c)}{[\phi(c)]^2},\ \phi(c)\neq 0. \tag{7.28}$$

Proof.

We know that functions f and ϕ are differentiable at any point c of $[a,b]$; then

$$\lim_{x\to c}\frac{f(x)-f(c)}{x-c}=f'(c),\ \lim_{x\to c}\frac{\phi(x)-\phi(c)}{x-c}=\phi'(c) \tag{7.29}$$

Now

(i)

$$\begin{aligned}(f\pm\phi)'(c)&=\lim_{x\to c}\frac{(f\pm\phi)(x)-(f\pm\phi)(c)}{x-c}\\&=\lim_{x\to c}\frac{[f(x)\pm\phi(x)]-[f(c)\pm\phi(c)]}{x-c}\\&=\lim_{x\to c}\left[\frac{f(x)-f(c)}{x-c}\pm\frac{\phi(x)-\phi(c)}{x-c}\right]\\&=\left[\lim_{x\to c}\frac{f(x)-f(c)}{x-c}\right]\pm\left[\lim_{x\to c}\frac{\phi(x)-\phi(c)}{x-c}\right]\\&=f'(c)\pm\phi'(c). &&\text{from (7.29)}\end{aligned}$$

(ii) Using the identity $ab-ad+ad-cd=a(b-d)+(a-c)d$, the limit of product rule, from (7.29) and $\lim_{x\to c}f(x)=f(c)$, one gets

$$\begin{aligned}(f\phi)'(c)&=\lim_{x\to c}\frac{(f\phi)(x)-(f\phi)(c)}{x-c}\\&=\lim_{x\to c}\frac{f(x)\phi(x)-f(c)\phi(c)}{x-c}\\&=\lim_{x\to c}\frac{f(x)\phi(x)-f(x)\phi(c)+f(x)\phi(c)-f(c)\phi(c)}{x-c}\\&=\lim_{x\to c}\frac{f(x)[\phi(x)-\phi(c)]+\phi(c)[f(x)-f(c)]}{x-c}\\&=\lim_{x\to c}f(x)\lim_{x\to c}\frac{\phi(x)-\phi(c)}{x-c}+\phi(c)\lim_{x\to c}\frac{f(x)-f(c)}{x-c}\\&=f(c)\phi'(c)+f'(c)\phi(c).\end{aligned}$$

Since f is differentiable, f is continuous.

(iii)

$$
\begin{aligned}
\left(\frac{f}{\phi}\right)(c) &= \lim_{x\to c}\frac{\left(\frac{f}{\phi}\right)(x)-\left(\frac{f}{\phi}\right)(c)}{x-c}\\
&= \lim_{x\to c}\frac{\frac{f(x)}{\phi(x)}-\frac{f(c)}{\phi(c)}}{x-c}\\
&= \lim_{x\to c}\frac{\frac{f(x)\phi(c)-f(c)\phi(x)}{\phi(c)\phi(x)}}{(x-c)}\\
&= \lim_{x\to c}\frac{f(x)\phi(c)-f(c)\phi(c)+f(c)\phi(c)-f(c)\phi(x)}{(x-c)\phi(c)\phi(x)}\\
&= \lim_{x\to c}\frac{\phi(c)[f(x)-f(c)]-f(c)[\phi(x)-\phi(c)]}{(x-c)\phi(c)\phi(x)}\\
&= \lim_{x\to c}\frac{1}{\phi(c)}\frac{1}{\phi(x)}\left[\phi(c)\lim_{x\to c}\frac{f(x)-f(c)}{x-c}-f(c)\lim_{x\to c}\frac{\phi(x)-\phi(c)}{x-c}\right]\\
&= \frac{\phi(c)f'(c)-f(c)\phi'(c)}{[\phi(c)]^2}.
\end{aligned}
$$

By (7.29) and since the function ϕ is differentiable at point c, we conclude that the function ϕ is continuous at c. Therefore, $\lim_{x\to c}\phi(x)=\phi(c)$.

□

Note 3 We note that

- The derivative of the sum (or the difference) of two differentiable functions is the sum (or the difference) of their derivatives.
- The derivative of the product of two differentiable functions equals the first function times the derivative of the second function plus the second function times the derivative of the first function.
- The derivative of the quotient of two differentiable functions is equal to the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

7.2.8 Derivative of composite functions

Theorem 7 If f and ϕ are two functions such that the range of f is contained in the domain of ϕ, and f is differentiable at $x=c$ and ϕ is differentiable at $f(c)$, then ϕ is differentiable at $x=c$, and

$$(\phi\circ f)'(c)=\phi'[f(c)]f'(c).$$

Proof.

As the range of f is contained in the domain of ϕ, therefore the domain of f and of the composite function ϕ will be same.

Now as a function f is differentiable at any point $x = c$, then

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}. \tag{7.30}$$

Also a function ϕ is differentiable at $f(c)$

$$\phi'[f(c)] = \lim_{f(x) \to f(c)} \frac{\phi[f(x)] - \phi[f(c)]}{f(x) - f(c)}. \tag{7.31}$$

Now

$$\begin{aligned}
(\phi \circ f)'(c) &= \lim_{x \to c} \frac{(\phi \circ f)(x) - (\phi \circ f)(c)}{x - c} \\
&= \lim_{x \to c} \frac{\phi[f(x)] - \phi[f(c)]}{x - c} \\
&= \lim_{x \to c} \frac{\phi[f(x)] - \phi[f(c)]}{f(x) - f(c)} \cdot \frac{f(x) - f(c)}{x - c} \\
&= \lim_{x \to c} \frac{\phi[f(x)] - \phi[f(c)]}{f(x) - f(c)} \cdot \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.
\end{aligned}$$

We know that a function f is differentiable at $x = c$, therefore, it will be continuous at $x = c$; that is, $\lim_{x \to c} f(x) = f(c)$. Now

$$\begin{aligned}
(\phi \circ f)'(c) &= \underbrace{\lim_{f(x) \to f(c)} \frac{\phi[f(x)] - \phi[f(c)]}{f(x) - f(c)}}_{(7.31)} \cdot \underbrace{\lim_{x \to c} \frac{f(x) - f(c)}{x - c}}_{(7.30)} \\
&= \phi'[f(c)] f'(c).
\end{aligned}$$

□

7.2.9 Derivative of the inverse function

Theorem 8 If $f(x)$ is a strictly monotonic (one-one) function with domain $[a, b]$ and it is differentiable at c with $f'(c) \neq 0$, then f^{-1}, the inverse of f, is differentiable at $f(c)$; and

$$(f^{-1})'[f(c)] = \frac{1}{f'(c)}, \text{ whenever } f'(c) \neq 0.$$

Proof. If the function f is differentiable at point c, then the function f is continuous at c. Thus, the function is one-one in the neighbourhood of c. And the inverse of function f exists and

$$[f^{-1} \circ f](x) = x.$$

Now

$$\begin{aligned}(f^{-1})'[f(c)] &= \lim_{f(x)\to f(c)} \frac{f^{-1}[f(x)] - f^{-1}[f(c)]}{f(x)-f(c)} \\ &= \lim_{f(x)\to f(c)} \frac{x-c}{f(x)-f(c)} \\ &= \lim_{f(x)\to f(c)} \frac{1}{\dfrac{f(x)-f(c)}{x-c}}.\end{aligned}$$

Since the function f is continuous at point c, this implies that f^{-1} is also continuous at c, and

we have

$$\lim_{f(x)\to f(c)} f^{-1}[f(x)] = f^{-1}[f(c)]$$

or

$$\lim_{f(x)\to f(c)} x = c.$$

Therefore, we have

$$\begin{aligned}(f^{-1})'[f(c)] &= \lim_{x\to c} \frac{1}{\dfrac{f(x)-f(c)}{x-c}} = \frac{1}{\displaystyle\lim_{x\to c}\dfrac{f(x)-f(c)}{x-c}} \\ &= \frac{1}{f'(c)}.\end{aligned}$$

□

Note 4 Let $y = f(x)$, where x is differentiable in its domain. Also let $x = \phi(y)$, the inverse function of $f(x)$ exist. Now we will establish the relationship between $f'(x)$ and $\phi'(y)$ such that

$$\frac{\Delta y}{\Delta x}\cdot\frac{\Delta x}{\Delta y} = 1 \quad \text{or} \quad \frac{\Delta x}{\Delta y} = \frac{1}{\Delta y/\Delta x}. \tag{7.32}$$

Taking the limits of both sides of (7.32) when $\Delta x \to 0$, we get

$$\lim_{\Delta x\to 0}\left(\frac{\Delta y}{\Delta x}\cdot\frac{\Delta x}{\Delta y}\right) = 1 \quad \text{or} \quad \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}\cdot\lim_{\Delta x\to 0}\frac{\Delta x}{\Delta y} = 1$$

or

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}\cdot\lim_{\Delta y\to 0}\frac{\Delta x}{\Delta y} = 1, \quad \text{since } \Delta y \to 0 \text{ as} \Delta x \to 0$$

or

$$\frac{dx}{dy} = \frac{1}{dy/dx} \quad \text{or} \quad \frac{dy}{dx}\cdot\frac{dx}{dy} = 1$$

or

$$f'(x)\cdot\phi'(y) = 1.$$

Hence $\dfrac{dy}{dx}$ is the inverse of $\dfrac{dx}{dy}$.

7.3 Worked Examples

Example 7 Test the continuity and differentiability of the function f defined by:

$$f(x) = \begin{cases} x\left(\dfrac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}\right); & \text{if } x \neq 0 \\ 0; & \text{if } x = 0. \end{cases}$$

at $x = 0$.

Solution:

To test for the continuity of f at $x = 0$:

Here

- $$f(0) = 0. \tag{7.33}$$

- $$\begin{aligned} f(0-0) &= \lim_{x\to 0^-} f(x) \\ &= \lim_{h\to 0} f(0-h) = \lim_{h\to 0^-} f(-h) \\ &= \lim_{h\to 0}(-h)\frac{e^{-1/h} - e^{1/h}}{e^{-1/h} + e^{1/h}},\ h > 0 \\ &= \lim_{h\to 0}(-h)\frac{e^{-2/h} - 1}{e^{-2/h} + 1} = (-0)\left(\frac{0-1}{0+1}\right) = 0. \end{aligned} \tag{7.34}$$

- $$\begin{aligned} f(0+0) &= \lim_{x\to 0^+} f(x) \\ &= \lim_{h\to 0} f(0+h) = \lim_{h\to 0^+} f(h) \\ &= \lim_{h\to 0} h\frac{e^{1/h} - e^{-1/h}}{e^{1/h} + e^{-1/h}},\ h > 0 \\ &= \lim_{h\to 0} h\frac{1 - e^{-2/h}}{1 + e^{-2/h}} = (0)\left(\frac{1-0}{1+0}\right) = 0. \end{aligned} \tag{7.35}$$

From (7.33), (7.34) and (7.35)

$$f(0) = f(0+0) = f(0-0).$$

Hence the given function f is continuous at $x = 0$.

To test for the differentiability of f at $x = 0$:

$$\begin{aligned}
Rf'(0) = f'(0+0) &= \lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} \\
&= \lim_{h\to 0} \frac{f(0+h)-f(0)}{(0+h)-0} = \lim_{h\to 0} \frac{f(h)-0}{h} \\
&= \lim_{h\to 0} \frac{e^{1/h}-e^{-1/h}}{e^{1/h}+e^{-1/h}}, \ h > 0 \\
&= \lim_{h\to 0} \frac{1-e^{-2/h}}{1+e^{-2/h}} = \frac{1-0}{1+0} = 1. \qquad (7.36) \\
Lf'(0) = f'(0-0) &= \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} \\
&= \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} \frac{f(-h)-0}{-h} \\
&= \lim_{h\to 0} \frac{e^{-1/h}-e^{1/h}}{e^{-1/h}+e^{1/h}} \\
&= \lim_{h\to 0} \frac{e^{-2/h}-1}{e^{-2/h}+1} = \frac{0-1}{0+1} = -1. \qquad (7.37)
\end{aligned}$$

From (7.36) and (7.37), we get

$$Rf'(0) \neq Lf'(0).$$

Hence the given function $f(x)$, is not differentiable at $x = 0$. □

Example 8 Examine the continuity and differentiability of the following function:

$$f(x) = \begin{cases} 1+\sin x; & 0 < x < \frac{\pi}{2}; \\ 2+\left(x-\frac{\pi}{2}\right)^2; & x \geq \frac{\pi}{2}. \end{cases}$$

Solution:

To test for the continuity of f at $x = \frac{\pi}{2}$:

Here

- $$f\left(\frac{\pi}{2}\right) = 2. \qquad (7.38)$$

- $$\begin{aligned}
f\left(\frac{\pi}{2}-0\right) &= \lim_{x\to \frac{\pi}{2}^-} f(x) \\
&= \lim_{h\to 0} f\left(\frac{\pi}{2}-h\right) \\
&= \lim_{h\to 0} \left[1+\sin\left(\frac{\pi}{2}-h\right)\right] = 1+\sin\frac{\pi}{2} \\
&= 2. \qquad (7.39)
\end{aligned}$$

•

$$\begin{aligned} f\left(\frac{\pi}{2}+0\right) &= \lim_{h\to\frac{\pi}{2}^+} f(x) \\ &= \lim_{h\to 0} f\left(\frac{\pi}{2}+h\right) \\ &= \lim_{h\to 0}\left[2+\left(\frac{\pi}{2}+h-\frac{\pi}{2}\right)^2\right] \\ &= \lim_{h\to 0}(2+h^2) = 2. \end{aligned} \tag{7.40}$$

From (7.38), (7.39) and (7.40)

$$f\left(\frac{\pi}{2}\right) = f\left(\frac{\pi}{2}-0\right) = f\left(\frac{\pi}{2}+0\right).$$

Hence the given function is continuous at $x = \pi/2$.

To test for the differentiability of f at $x = \frac{\pi}{2}$:

$$\begin{aligned} Rf'\left(\frac{\pi}{2}\right) &= f'\left(\frac{\pi}{2}+0\right) \\ &= \lim_{x\to\frac{\pi}{2}^+} \frac{f(x)-f\left(\frac{\pi}{2}\right)}{x-\frac{\pi}{2}} \\ &= \lim_{h\to 0} \frac{f\left(\frac{\pi}{2}+h\right)-f\left(\frac{\pi}{2}\right)}{h} \\ &= \lim_{h\to 0} \frac{2+\left(\frac{\pi}{2}+h-\frac{\pi}{2}\right)^2-2}{h} = \lim_{h\to 0}\frac{h^2}{h} = 0 \end{aligned} \tag{7.41}$$

$$\begin{aligned} Lf'\left(\frac{\pi}{2}\right) &= f'\left(\frac{\pi}{2}-0\right) \\ &= \lim_{x\to\frac{\pi}{2}^-} \frac{f(x)-f\left(\frac{\pi}{2}\right)}{x-\frac{\pi}{2}} \\ &= \lim_{h\to 0} \frac{f\left(\frac{\pi}{2}-h\right)-f\left(\frac{\pi}{2}\right)}{-h} \\ &= \lim_{h\to 0} \frac{1+\sin\left(\frac{\pi}{2}-h\right)-2}{-h} \\ &= \lim_{h\to 0} \frac{\sin\frac{\pi}{2}\cos h-\cos\frac{\pi}{2}\sin h-1}{-h} \\ &= \lim_{h\to 0} \frac{\cos h-1}{-h} = \lim_{h\to 0}\frac{1-2\sin^2\frac{h}{2}-1}{-h} \\ &= \lim_{h\to 0} \frac{\sin^2\frac{h}{2}}{\left(\frac{h}{2}\right)^2}\cdot\frac{h}{2} \\ &= \lim_{\frac{h}{2}\to 0}\left(\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)^2\cdot\lim_{h\to 0}\frac{h}{2} = 1\cdot 0 = 0. \end{aligned} \tag{7.42}$$

From (7.41) and (7.42)

$$Rf'\left(\frac{\pi}{2}\right) = Lf'\left(\frac{\pi}{2}\right).$$

Hence, the given function f is differentiable at $x = \frac{\pi}{2}$. $\square$

Example 9 A function $f(x)$ is defined as follows:

$$f(x) = \begin{cases} x^m \sin \frac{1}{x}; & x \neq 0 \\ 0; & x = 0 \end{cases}$$

What condition should be imposed on m so that at $x = 0$:

(a) $f(x)$ is continuous.

(b) $f(x)$ is differentiable.

(c) $f'(x)$ continuous.

Solution:

(a) To test for the continuity at $x = 0$:

Here

- $$f(0) = 0 \tag{7.43}$$

- $$\begin{aligned} f(0-0) &= \lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h) \\ &= \lim_{h\to 0} (-h)^m \sin\left(\frac{1}{-h}\right) \quad h > 0 \\ &= \lim_{h\to 0} (-1)^{m+1} h^m \sin\left(\frac{1}{h}\right) \end{aligned} \tag{7.44}$$

- $$\begin{aligned} f(0+0) &= \lim_{x\to 0^+} f(x) = \lim_{h\to w0} f(0+h) = \lim_{h\to 0} f(h) \\ &= \lim_{h\to 0} (h)^m \sin\frac{1}{h} \; [\because \; h > 0] \end{aligned} \tag{7.45}$$

As $f(0) = 0$, then function will be continuous at $x = 0$, if using equations (7.44) and (7.45) the limits are also zero. We know that $\sin\left(\frac{1}{h}\right)$ is a finite quantity between -1 and 1, therefore by equations (7.44) and (7.45) the limits will be zero if $m > 0$.

(b) To test for the differentiability at $x = 0$:

$$\begin{aligned}
Rf'(0) = f'(0+0) &= \lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} \\
&= \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} \\
&= \lim_{h\to 0} h^{m-1}\sin\frac{1}{h},\ \ h>0 \qquad (7.46) \\
Lf'(0) = f'(0-0) &= \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0)} \\
&= \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h},\ \ h>0 \\
&= \lim_{h\to 0} \frac{h^m(-1)^{m+1}\sin\frac{1}{h}}{-h} \\
&= \lim_{h\to 0} (-1)^m h^{m-1}\sin\frac{1}{h}. \qquad (7.47)
\end{aligned}$$

A function $f(x)$ is differentiable at $x=0$, if

$$Rf'(0) = Lf'(0)$$

because $\lim\limits_{h\to 0} h^{m-1}\sin\frac{1}{h}$ exist when $m-1>0$ or $m>1$.

From (7.46) and (7.47) a function $f(x)$ is differentiable at $x=0$, if $m>1$.

(c) To test for the continuity of $f'(x)$ at $x=0$:

Differentiating $f(x) = x^m \sin\frac{1}{x}$, $\quad x\neq 0$ with respect to x with the aid of the product rule, we have

$$\begin{aligned}
f'(x) &= mx^{m-1}\sin\frac{1}{x} + x^m(\left(\frac{-1}{x^2}\right)\cos\frac{1}{x}, \quad x\neq 0 \\
&= x^{m-2}\left(mx\sin\frac{1}{x} - \cos\frac{1}{x}\right).
\end{aligned}$$

Clearly $f'(x)$, will be continuous at $x=0$, if $m>2$.

□

Example 10 A function

$$f(x) = \begin{cases} 1+x & \text{if } x<2 \\ 5-x & \text{if } x\geq 2 \end{cases}$$

is continuous but not differentiable at $x=2$.

Solution:

To test for the continuity of f at $x=2$:

Here

- $$f(2) = 3. \qquad (7.48)$$

•
$$\begin{aligned} f(2-0) &= \lim_{x\to 2^-} f(x) = \lim_{h\to 0} f(2-h) \\ &= \lim_{h\to 0} 1+(2-h) \\ &= 3. \end{aligned} \tag{7.49}$$

•
$$\begin{aligned} f(2+0) &= \lim_{x\to 2^+} f(x) = \lim_{h\to 0} f(2+h) \\ &= \lim_{h\to 0} 5-(2+h) \\ &= 3. \end{aligned} \tag{7.50}$$

From (7.48), (7.49) and (7.50)
$$f(2-0) = f(2) = f(2+0)$$
Hence f is continuous at $x = 2$.
To test for the differentiability of f at $x = 2$:
$$\begin{aligned} Rf'(2) &= \lim_{x\to 2^+} \frac{f(x)-f(2)}{x-2} \\ &= \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} \\ &= \lim_{h\to 0} \frac{(5-(2+h))-3}{h} = -1 \end{aligned} \tag{7.51}$$
$$\begin{aligned} Lf'(2) &= \lim_{x\to 2^-} \frac{f(x)-f(2)}{x-2} \\ &= \lim_{h\to 0} \frac{f(2-h)-f(2)}{-h} \\ &= \lim_{h\to 0} \frac{(1+(2-h))-3}{-h} = 1. \end{aligned} \tag{7.52}$$
From (7.51) and (7.52)
$$Rf'(2) \neq Lf'(2).$$
Hence the given function is not differentiable at $x = 2$. □

Example 11 Examine for continuity and differentiability of the function f defined by:
$$f(x) = \begin{cases} -x^2; & \text{if } x \le 0 \\ 5x-4; & \text{if } 0 < x \le 1 \\ 4x^2-3x; & \text{if } 1 < x < 2 \\ 3x+4; & \text{if } x \ge 2 \end{cases}$$
at $x = 0, 1, 2$.
Solution:

Figure 7.1 describes the graph of the function f.

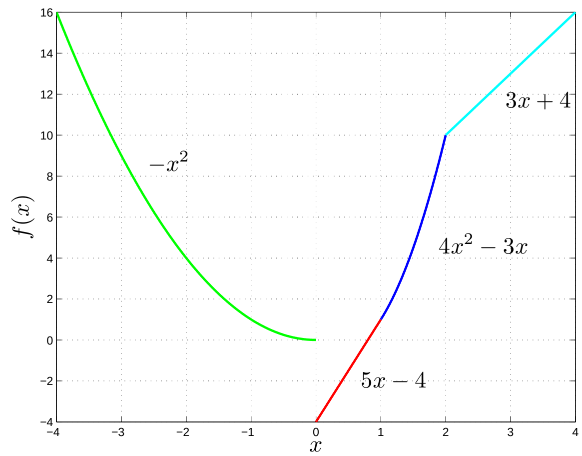

Figure 7.1: $f(x)$

1. **To test for the continuity of f at $x = 0$:**

   Here

   - $$f(0) = 0. \tag{7.53}$$

   - $$\begin{aligned} f(0-0) &= \lim_{x\to 0^-} f(x) \\ &= \lim_{h\to 0^-} f(0-h) \\ &= \lim_{h\to 0}(-h^2) = 0. \end{aligned} \tag{7.54}$$

   - $$\begin{aligned} f(0+0) &= \lim_{x\to 0^+} f(x) \\ &= \lim_{h\to 0} f(0+h) \\ &= \lim_{h\to 0} 5h - 4 = -4. \end{aligned} \tag{7.55}$$

   From equation (7.53), (7.54) and (7.55).

   $$f(0-0) = 0 = f(0) \neq -4 = f(0+0).$$

   Hence f is not continuous at $x = 0$.

   To test for the differentiability of f at $x = 0$:

   $$\begin{aligned} Lf'(0) = f'(0-0) &= \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} \\ &= \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} \\ &= \lim_{h\to 0} \frac{-h^2-0}{-h} = 0 \end{aligned} \tag{7.56}$$

$$\begin{aligned} Rf'(0) = f'(0+0) &= \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} \\ &= \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h \to 0} \frac{5h - 4 - 0}{h} \\ &= -\infty. \end{aligned} \tag{7.57}$$

From (7.56) and (7.57)

$$Lf'(0) \neq Rf'(0).$$

Note 5 If the function is differentiable at any point, then it is continuous at that point. Because f is not continuous at $x = 0$ that implies that f is not differentiable at $x = 0$.

2. **To test for the continuity of f at $x = 1$:**

   Here

   - $$f(1) = 5 \cdot 1 - 4 = 1. \tag{7.58}$$

   - $$\begin{aligned} f(1-0) &= \lim_{x \to 1^-} f(x) \\ &= \lim_{h \to 0} f(1-h) \\ &= \lim_{h \to 0} 5(1-h) - 4 \\ &= 1. \end{aligned} \tag{7.59}$$

   - $$\begin{aligned} f(1+0) &= \lim_{x \to 1^+} f(x) \\ &= \lim_{h \to 0} f(1+h) \\ &= \lim_{h \to 0} 4(1+h)^2 - 3(1+h) \\ &= 1. \end{aligned} \tag{7.60}$$

   From (7.58), (7.59) and (7.60)

   $$f(1) = f(1-0) = f(1+0).$$

   Hence f is continuous at $x = 1$.

   To test for the differentiability of f at $x = 1$:

$$
\begin{aligned}
Lf'(1) = f'(1-0) &= \lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1} \\
&= \lim_{h\to 0} \frac{f(1-h)-f(1)}{-h} \\
&= \lim_{h\to 0} \frac{5(1-h)-4-1}{-h} = 5 \qquad (7.61) \\
Rf'(1) = f'(1+0) &= \lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1} \\
&= \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} \\
&= \lim_{h\to 0} \frac{4(1+h)^2-3(1+h)-1}{h} \\
&= \lim_{h\to 0} 4h+5 = 5 \qquad (7.62)
\end{aligned}
$$

From (7.61) and (7.62)

$$Lf'(1) = 5 = Rf'(1)$$

Hence the function is differentiable at $x = 1$.

3. **To test for the continuity of f at $x = 2$:**

   Here

   - $$f(2) = 3 \cdot 2 + 4 = 10. \qquad (7.63)$$

   - $$
\begin{aligned}
f(2-0) &= \lim_{x\to 2^-} f(x) \\
&= \lim_{h\to 0} f(2-h) \\
&= \lim_{h\to 0} 4(2-h)^2 - 3(2-h) \\
&= \lim_{h\to 0} 10 - 13h + 4h^2 = 10. \qquad (7.64)
\end{aligned}
$$

   - $$
\begin{aligned}
f(2+0) &= \lim_{x\to 2^+} f(x) = \lim_{h\to 0} f(2+h) \\
&= \lim_{h\to 0} 3(2+h)+4 \\
&= 10. \qquad (7.65)
\end{aligned}
$$

   From (7.63), (7.64) and (7.65), f is continuous at $x = 2$.

   To test for the differentiability of f at $x = 2$:

$$
\begin{aligned}
Lf'(2) = f'(2-0) &= \lim_{x \to 2^-} \frac{f(x) - f(2)}{x-2} \\
&= \lim_{h \to 0} \frac{f(2-h) - f(2)}{-h} \\
&= \lim_{h \to 0} \frac{[4(2-h)^2 - 3(2-h)] - 10}{-h} \\
&= \lim_{h \to 0} 13 - 4h = 13 \qquad (7.66) \\
Rf'(2) = f'(2+0) &= \lim_{x \to 2^+} \frac{f(x) - f(2)}{x-2} \\
&= \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} \\
&= \lim_{h \to 0} \frac{3(2+h) + 4 - 10}{h} = 3. \qquad (7.67)
\end{aligned}
$$

From (7.66) and (7.67)

$$Lf'(2) \neq Rf'(2).$$

Hence f is not differentiable at $x = 2$.

□

Example 12 Test the continuity and differentiability of the following function in $[1, 4]$:

$$f(x) = |x-2| + 2|x-3|.$$

Solution:

The given function can be defined as:

$$
f(x) = \begin{cases} 8 - 3x & \text{if } 1 < x < 2; \\ 4 - x & \text{if } 2 \leq x \leq 3; \\ 3x - 8 & \text{if } 3 < x < 4, \end{cases}
$$

as illustrated in Figure 7.2.

Let a be any real number in the open interval $(1, 2)$.

Now, we will test the continuity of the function at $x = a$.

Here

$$f(a) = 8 - 3a. \qquad (7.68)$$

$$
\begin{aligned}
f(a-0) &= \lim_{x \to a^-} f(x) \\
&= \lim_{h \to 0} f(a-h) \\
&= \lim_{h \to 0} [8 - 3(a-h)] = 8 - 3a. \qquad (7.69)
\end{aligned}
$$

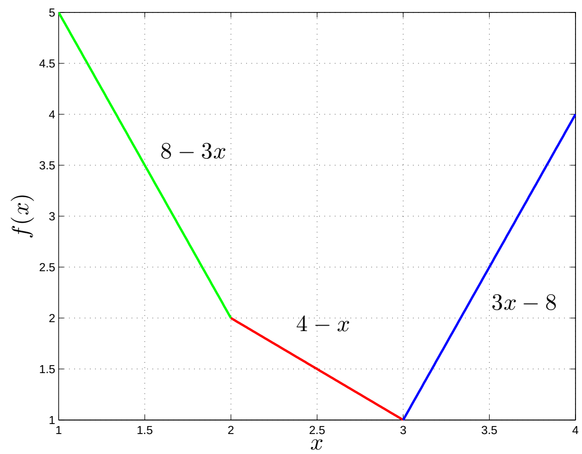

Figure 7.2: $f(x)$

$$
\begin{aligned}
f(a+0) &= \lim_{x \to a^+} f(x) \\
&= \lim_{h \to 0} f(a+h) \\
&= \lim_{h \to 0} [8-3(a+h)] = 8-3a.
\end{aligned} \tag{7.70}
$$

From (7.68), (7.69) and (7.70), we get

$$f(a) = f(a-0) = f(a+0).$$

Hence the function is continuous at $x = a$, that is, the function is continuous at every point in the interval $(1, 2)$.

Now, we will test the continuity and differentiability at $x = 2$ and $x = 3$.

To test for the continuity of f at $x = 2$:

Here

$$f(2) = 4-2 = 2. \tag{7.71}$$

$$
\begin{aligned}
f(2+0) &= \lim_{x \to 2^+} f(x) \\
&= \lim_{h \to 0} f(2+h) \\
&= \lim_{h \to 0} 4-(2+h) = 2.
\end{aligned} \tag{7.72}
$$

$$
\begin{aligned}
f(2-0) &= \lim_{x \to 2^-} f(x) \\
&= \lim_{h \to 0} f(2-h) \\
&= \lim_{h \to 0} 8-3(2-h) = 2.
\end{aligned} \tag{7.73}
$$

From (7.71), (7.72) and (7.73)

$$f(2) = f(2+0) = f(2-0) = 2.$$

Hence f is continuous at $x = 2$.
Similarly, we can prove the continuity of function at $x = 3$.
To test for the differentiability of f at at $x = a$:
Let a be a real number in the open interval $(1, 2)$.
Here

$$\begin{aligned}
Lf'(a) = f'(a-0) &= \lim_{x\to a^-} \frac{f(x)-f(a)}{x-a} \\
&= \lim_{h\to 0} \frac{f(a-h)-f(a)}{-h} \\
&= \lim_{h\to 0} \frac{[8-3(a-h)]-[8-3a]}{-h} = -3 \qquad (7.74) \\
Rf'(a) = f'(a+0) &= \lim_{x\to a^+} \frac{f(x)-f(a)}{x-a} \\
&= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \\
&= \lim_{h\to 0} \frac{[8-3(a+h)]-(8-3a)}{h} = -3. \qquad (7.75)
\end{aligned}$$

From (7.74) and (7.75)

$$Rf'(a) = Lf'(a).$$

Therefore, the given function is differentiable at every point in open interval $(1, 2)$. Similarly, we can prove differentiability of the function in the interval $(2, 3)$ and $(3, 4)$.
To test for the differentiability of f at at $x = 2$:

$$\begin{aligned}
Lf'(2) = f'(2-0) &= \lim_{x\to 2^-} \frac{f(x)-f(2)}{x-2} \\
&= \lim_{h\to 0} \frac{f(2-h)-f(2)}{-h} \\
&= \lim_{h\to 0} \frac{[8-3(2-h)]-2}{-h} = -3 \qquad (7.76) \\
Rf'(2) = f'(2+0) &= \lim_{x\to 2^+} \frac{f(x)-f(2)}{x-2} \\
&= \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} \\
&= \lim_{h\to 0} \frac{[4-(2+h)]-2}{h} = -1. \qquad (7.77)
\end{aligned}$$

From (7.76) and (7.77), we get

$$Rf'(2) \neq Lf'(2).$$

Clearly f is not derivable at $x = 2$.
Similarly we can prove f is not differentiable at $x = 3$.
Hence, f is continuous in $[1, 4]$ but not differentiable. □

Example 13 Test the following function for continuity and differentiability:

$$f(x) = |x| + |x-1| \text{ at } x = 0, 1.$$

Solution:

The given function is equivalent to

$$f(x) = \begin{cases} 1-2x, & x < 0; \\ 1, & 0 \le x \le 1; \\ 2x-1, & x > 1, \end{cases}$$

as illustrated in Figure 7.3.

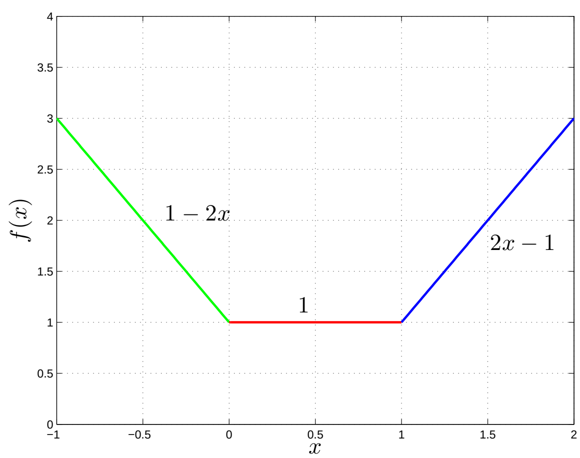

Figure 7.3: $f(x)$

To test for the continuity of f at $x = 0$:
Here

$$f(0) \;=\; 1. \tag{7.78}$$

$$\begin{aligned} f(0+0) &= \lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) \\ &= \lim_{h\to 0} 1 = 1. \end{aligned} \tag{7.79}$$

$$\begin{aligned} f(0-0) &= \lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) \\ &= \lim_{h\to 0} 1-2(0-h) = 1. \end{aligned} \tag{7.80}$$

From (7.78), (7.79) and (7.80), we have

$$f(0) = f(0+0) = f(0-0).$$

Hence f is continuous at $x = 0$.
To test for the differentiability of f at at $x = 0$:

$$\begin{aligned} Rf'(0) = f'(0+0) &= \lim_{x\to 0^+} \frac{f(x) - f(0)}{x} \\ &= \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h\to 0} \frac{1-1}{h} = 0 \qquad (7.81) \\ Lf'(0) = f'(0-0) &= \lim_{x\to 0^-} \frac{f(x) - f(0)}{x} \\ &= \lim_{h\to 0} \frac{f(0-h) - f(0)}{-h} \\ &= \lim_{h\to 0} \frac{[1-2(0-h)] - 1}{-h} = -2 \qquad (7.82) \end{aligned}$$

From (7.81) and (7.82), we have

$$Rf'(0) \neq Lf'(0).$$

Hence f is not differentiable at $x = 0$
To test for the continuity of f at $x = 1$:
Here

$$f(1) = 1. \qquad (7.83)$$

$$\begin{aligned} f(1+0) &= \lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h) \\ &= \lim_{h\to 0} 2(1+h) - 1 = 1 \qquad (7.84) \end{aligned}$$

$$\begin{aligned} f(1-0) &= \lim_{x\to 1^-} f(x) \\ &= \lim_{h\to 0} f(1-h) \\ &= \lim_{h\to 0} 1 = 1. \qquad (7.85) \end{aligned}$$

From (7.83), (7.84) and (7.85), we have

$$f(1) = f(1-0) = f(1+0).$$

Hence f is continuous at $x = 0$.
To test for the differentiability of f at at $x = 1$:

$$
\begin{aligned}
Rf'(1) = f'(1+0) &= \lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1} \\
&= \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} \\
&= \lim_{h\to 0} \frac{[2(1+h)-1]-1}{h} = 2 \qquad (7.86) \\
Lf'(1) = f'(1-0) &= \lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1} \\
&= \lim_{h\to 0} \frac{f(1-h)_f(1)}{-h} \\
&= \lim_{h\to 0} \frac{1-1}{-h} = 0. \qquad (7.87)
\end{aligned}
$$

From (7.86) and (7.87), we have

$$Rf'(1) \neq Lf'(1).$$

Hence the function is continuous at $x = 0$ and $x = 1$ but not differentiable at these points. From the graph, it is also clear that the function is continuous but not differentiable at $x = 0$ and at $x = 1$.

□