THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1510 Calculus for Engineers (Fall 2021)
Suggested solutions of midterm examination

Short Questions

Each of question 1-18 is worth 3 points.

1. Find the domain of the function

$$f(x) = \ln((2x-7)(3x+5))$$

Answer: $(-\infty, -\frac{5}{3}) \cup (\frac{7}{2}, \infty)$

2. Find the range of the function

$$f(x) = x^2 - 1$$

with domain $[-2, 6]$.

Answer: $[-1, 35]$

3. Which of the following functions have minimum value on the specified intervals? If none of them has minimum value, write NONE.

(a) $f(x) = \dfrac{1}{x}$ on $[1, \infty)$.
(b) $g(x) = (x-3)e^x \sin x$ on $[-3, 7]$.
(c) $h(x) = x$ on $(0, 4]$.

Answer: (b)

4. Let

$$f(x) = \begin{cases} |x| & \text{if } |x| \geq 1; \\ 2x - 3 & \text{if } 0 \leq x < 1; \\ x^2 & \text{if } -1 < x < 0. \end{cases}$$

Write down all the point(s) on $\mathbb{R}$ where $f(x)$ is not continuous. If there is no such point, write NONE.

Answer: $x = 0,\ x = 1$

5. Let

$$f(x) = \begin{cases} ax + 2 & \text{if } x \geq 1; \\ x - a & \text{if } x < 1. \end{cases}$$

Find all the value(s) of a so that $f(x)$ is continuous on $\mathbb{R}$.

Answer: $a = -\dfrac{1}{2}$

6. Find $\dfrac{dy}{dx}$ if

$$y = \frac{x \sin x}{2 - \cos x}$$

Answer: $\dfrac{dy}{dx} = \dfrac{(\sin x + x \cos x)(2 - \cos x) - (x \sin x)(\sin x)}{(2 - \cos x)^2}$

7. Find $\dfrac{dy}{dx}$ if

$$y = \sin(\cos(\sin x))$$

Answer: $\dfrac{dy}{dx} = \cos(\cos(\sin x)) \cdot (-\sin(\sin x)) \cdot \cos x$

8. Find $\dfrac{dy}{dx}$ if

$$y = \arctan x + e^x \arcsin x, \qquad \text{where } x \in (-1, 1)$$

Answer: $\dfrac{dy}{dx} = \dfrac{1}{1 + x^2} + e^x \arcsin x + \dfrac{e^x}{\sqrt{1 - x^2}}$

9. Find $\dfrac{dy}{dx}$ if

$$y = e^{\sin x} \ln x$$

Answer: $\dfrac{dy}{dx} = e^{\sin x}(\cos x) \ln x + e^{\sin x} \dfrac{1}{x}$

10. Let $f(x) = \cos x$. Find $f^{(1510)}\left(\frac{\pi}{6}\right)$.

Answer: $-\frac{\sqrt{3}}{2}$

11. Let $f(x) = (x^2 + x + 1)e^{ax}$ for some $a > 0$. Find $f^{(8)}(x)$.

Answer: $f^{(8)}(x) = (x^2 + x + 1)a^8 e^{ax} + 8(2x + 1)a^7 e^{ax} + 56a^6 e^{ax}$

12. Evaluate

$$\lim_{t\to\infty} \frac{\log_3(t+1)}{\ln(3t)}$$

Answer: $\frac{1}{\ln 3}$

13. Evaluate

$$\lim_{x\to 1} \left(\tan\left(\frac{\pi x}{2}\right) \ln x\right)$$

Answer: $-\frac{2}{\pi}$

14. Evaluate

$$\lim_{x\to\infty} \frac{ae^x + e^{-ax}}{be^x + e^{-bx}},$$

where $a, b > 0$.

Answer: $\frac{a}{b}$

15. Evaluate

$$\lim_{x\to\infty} \frac{(ax)^2 + x\sin(\pi x)}{bx^2 + c},$$

where $a, b, c \neq 0$.

Answer: $\frac{a^2}{b}$

16. Given that

$$\lim_{x\to\infty} f(x) = a, \qquad \lim_{x\to-\infty} f(x) = b, \qquad \lim_{x\to 0^+} f(x) = c, \qquad \lim_{x\to 0^-} f(x) = d$$

for some $a, b, c, d > 0$, evaluate

$$\lim_{x\to 0} f\left(\left|\frac{1}{x}\right|\right)$$

Answer: a

17. Apply linearization of the function

$$f(x) = \sqrt{x}$$

at $x = 16$ to approximate $\sqrt{17}$.

Answer: $\dfrac{33}{8}$

18. Let

$$\mathcal{C} : x^3 + 5xy - 3y^2 = 3$$

be a curve. Find the equation of the tangent of $\mathcal{C}$ at the point $(1, 1)$.

Answer: $y - 1 = 8(x - 1)$

Long Questions

19. (12 points) Suppose that

$$f(x) = \begin{cases} \tan x - 1 & \text{if } x \in (-\frac{\pi}{2}, 0] \\ x^2 + ax + b & \text{if } x > 0, \end{cases}$$

where a and b are real numbers.

Given that f is differentiable at $x = 0$, without using L'Hôpital's rule, find the values of a and b.

Solution: Since f is differentiable at $x = 0$, it is continuous at $x = 0$. Hence,

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0).$$

In particular,

$$\begin{aligned} \lim_{x \to 0^+} (x^2 + ax + b) &= f(0) = \tan 0 - 1 \\ b &= -1. \end{aligned}$$

Note that

$$\begin{aligned} L' f(0) &= \lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h \to 0^-} \frac{(\tan h - 1) - (-1)}{h} \\ &= \lim_{h \to 0^-} \frac{\sin h}{h} \cdot \frac{1}{\cos h} \\ &= 1, \end{aligned}$$

and

$$\begin{aligned} R' f(0) &= \lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h \to 0^+} \frac{(h^2 + ah - 1) - (-1)}{h} \\ &= \lim_{h \to 0^+} (h + a) \\ &= a. \end{aligned}$$

Since f is differentiable at $x = 0$, w have

$$\begin{aligned} L' f(0) &= R' f(0) \\ 1 &= a. \end{aligned}$$

Therefore, $a = 1$ and $b = -1$.

20. (14 points) Let $\mathcal{C}$ be the curve defined by the equation

$$y^4 - x = xy + \cos x$$

Given that $A = (0, 1)$ is a point on $\mathcal{C}$,

(a) Find $\left.\dfrac{dy}{dx}\right|_A$

(b) Find $\left.\dfrac{d^2y}{dx^2}\right|_A$

Solution:

(a) By implicit differentiation,

$$\begin{aligned}\frac{d}{dx}\left(y^4 - x\right) &= \frac{d}{dx}\left(xy + \cos x\right)\\ 4y^3\frac{dy}{dx} - 1 &= y + x\frac{dy}{dx} - \sin x. \end{aligned} \tag{$*$}$$

At $A = (0, 1)$, we have

$$\begin{aligned}4(1)^3 \left.\frac{dy}{dx}\right|_A - 1 &= 1 + (0)\left.\frac{dy}{dx}\right|_A - \sin 0\\ \left.\frac{dy}{dx}\right|_A &= \frac{1}{2}.\end{aligned}$$

(b) Differentiating both sides of $(*)$ one more time, we have

$$\begin{aligned}\frac{d}{dx}\left(4y^3\frac{dy}{dx} - 1\right) &= \frac{d}{dx}\left(y + x\frac{dy}{dx} - \sin x\right)\\ 12y^2\left(\frac{dy}{dx}\right)^2 + 4y^3\frac{d^2y}{dx^2} &= \frac{dy}{dx} + \frac{dy}{dx} + x\frac{d^2y}{dx^2} - \cos x.\end{aligned}$$

At $A = (0, 1)$, we have

$$\begin{aligned}12(1)^2\left(\frac{1}{2}\right)^2 + 4(1)^3 \left.\frac{d^2y}{dx^2}\right|_A &= 2\left(\frac{1}{2}\right) + (0)\left.\frac{d^2y}{dx^2}\right|_A - \cos 0\\ \left.\frac{d^2y}{dx^2}\right|_A &= -\frac{3}{4}.\end{aligned}$$

21. (10 points) Show that the function

$$f(x) = x^4 - 5x + 1$$

has at least two real roots.

Solution: Clearly $f(x)$ is continuous over its domain $D_f = \mathbb{R}$. Note that

$$\begin{aligned} f(0) &= 0 - 0 + 1 = 1 > 0, \\ f(1) &= 1 - 5 + 1 = -3 < 0, \\ f(2) &= 16 - 10 + 1 = 7 > 0. \end{aligned}$$

Since f is continuous over $[0, 1]$ and $f(0) > 0 > f(1)$, by the Intermediate Value Theorem,

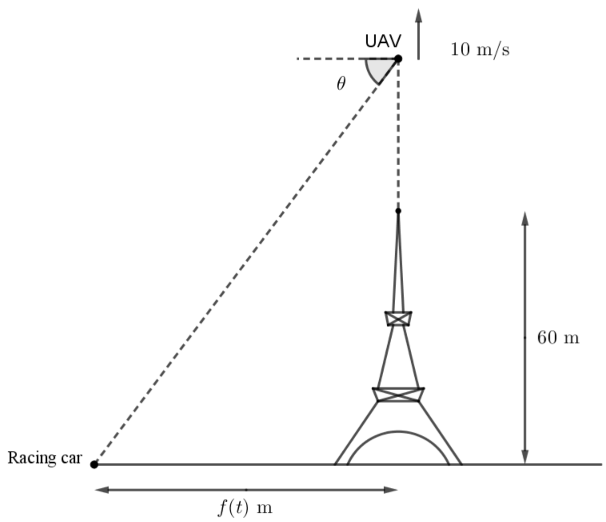
$$f(c_1) = 0 \qquad \text{for some } c_1 \in (0, 1).$$

Since f is continuous over $[1, 2]$ and $f(2) > 0 > f(1)$, by the Intermediate Value Theorem,

$$f(c_2) = 0 \qquad \text{for some } c_2 \in (1, 2).$$

As $c_1 < 1 < c_2$, $f(x)$ has at least two real roots.

22. (16 points) A racing car moves away from a tower, and the horizontal distance between the racing car and the tower after $t > 0$ seconds is given by a differentiable function $f(t)$ meter. To monitor the racing car, when it starts to move, an unmanned aerial vehicle (UAV) flies vertically upwards with constant speed 10 m/s from the top of a tower 60 meters in height.



Given that at $t = 4$, the angle of depression θ from the UAV to the racing car is $\frac{\pi}{4}$ radian.

(a) Find $f(4)$.

(b) At $t = 4$, the angle of depression θ decreases at a rate of 0.15 radian/s. Find the speed of the racing car at that moment.

Solution:

(a) At $t = 4$, height of UAV $= 60 + 4(10) = 100$ m. Hence

$$f(4) = \frac{100}{\tan \frac{\pi}{4}} = 100.$$

(b) Note that

$$f(t) = \frac{60 + 10t}{\tan \theta} = (60 + 10t) \cot \theta.$$

Hence

$$f'(t) = (10) \cot \theta + (60 + 10t)(-\csc^2 \theta)\theta'(t).$$

At $t = 4$, $\theta = \frac{\pi}{4}$, $\theta'(4) = -0.15$ radian/s, and hence

$$\begin{aligned} f'(4) &= 10 \cot \frac{\pi}{4} + (60 + 10(4))(-\csc^2 \frac{\pi}{4})(-0.15) \\ &= 40. \end{aligned}$$

Thus the speed of the racing car at $t = 4$ is 40 m/s.

23. (14 points) Suppose

$$f(x) = \begin{cases} 2x - \sin x & \text{if } x \leq 0 \\ x^2 \cos\left(\dfrac{2}{x}\right) & \text{if } x > 0 \end{cases}$$

Find $D_{f'}$ (the domain of f') and $f'(x)$ for any $x \in D_{f'}$.

Solution: Clearly f is differentiable at any $x \neq 0$, and

$$f'(x) = \begin{cases} 2 - \cos x & \text{if } x < 0 \\ 2x \cos\left(\dfrac{2}{x}\right) - x^2 \sin\left(\dfrac{2}{x}\right)\left(-\dfrac{2}{x^2}\right) & \text{if } x > 0. \end{cases}$$

For $x = 0$, note that $f(0) = 2(0) - \sin 0 = 0$,

$$\begin{aligned} L'f(0) &= \lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h \to 0^-} \frac{(2h - \sin h) - 0}{h} \\ &= \lim_{h \to 0^-} \left(2 - \frac{\sin h}{h}\right) \\ &= 1, \end{aligned}$$

and

$$\begin{aligned} R'f(0) &= \lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h \to 0^+} \frac{h^2 \cos\left(\frac{2}{h}\right) - 0}{h} \\ &= \lim_{h \to 0^+} h \cos\left(\frac{2}{h}\right) \\ &= 0, \end{aligned}$$

by the Squeeze Theorem, since $-h \leq h \cos\left(\frac{2}{h}\right) \leq h$ for $h > 0$ and

$$\lim_{h \to 0^+} (-h) = \lim_{h \to 0^+} h = 0.$$

Since $L'f(0) = 1 \neq 0 = R'f(0)$, f is not differentiable at $x = 0$.

Therefore, $D_{f'} = \mathbb{R} \backslash \{0\}$ and

$$f'(x) = \begin{cases} 2 - \cos x & \text{if } x < 0 \\ 2x \cos\left(\dfrac{2}{x}\right) + 2 \sin\left(\dfrac{2}{x}\right) & \text{if } x > 0. \end{cases}$$

24. (20 points)

(a) Let k be a positive integer. Show that

$$\frac{1}{k+1} < \ln(k+1) - \ln(k) < \frac{1}{k}$$

(b) Let n be a positive integer. Show that

$$\ln(n+1) < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < 1 + \ln n$$

(c) By part (b), evaluate

$$\lim_{n\to\infty} \frac{1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}}{\ln n}.$$

Solution:

(a) Let $f(x) = \ln x$, which is differentiable over $(0, \infty)$.
By Lagrange Mean Value Theorem on f over $[k, k+1]$,

$$\begin{aligned} \frac{f(k+1) - f(k)}{k+1-k} &= f'(c) \qquad \text{for some } c \in (k, k+1) \\ \ln(k+1) - \ln(k) &= \frac{1}{c}. \end{aligned}$$

Since $\dfrac{1}{x}$ is a strictly decreasing function over $(0, \infty)$,

$$\begin{aligned} & \frac{1}{k+1} < \frac{1}{c} < \frac{1}{k} \\ \Longrightarrow \quad & \frac{1}{k+1} < \ln(k+1) - \ln(k) < \frac{1}{k}. \end{aligned}$$

(b) By (a), we have

$$\sum_{k=1}^{n-1} \frac{1}{k+1} < \sum_{k=1}^{n-1} (\ln(k+1) - \ln(k)) = \ln(n) - \ln 1 = \ln n,$$

and

$$\sum_{k=1}^{n} \frac{1}{k} > \sum_{k=1}^{n} (\ln(k+1) - \ln(k)) = \ln(n+1) - \ln(1) = \ln(n+1).$$

Hence,

$$\ln(n+1) < \sum_{k=1}^{n} \frac{1}{k} = \frac{1}{1} + \sum_{k=1}^{n-1} \frac{1}{k+1} < 1 + \ln n.$$

That is

$$\ln(n+1) < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < 1 + \ln n.$$

(c) By (b), for any positive integer n,

$$\frac{\ln(n+1)}{\ln n} < \frac{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}}{\ln n} < \frac{1}{\ln n}+1.$$

Since

$$\lim_{n\to\infty}\frac{\ln(n+1)}{\ln n} = \lim_{n\to\infty}\frac{\ln n+\ln(1+\frac{1}{n})}{\ln n} = \lim_{n\to\infty}\left(1+\frac{\ln(1+\frac{1}{n})}{\ln n}\right) = 1+0=1,$$

and

$$\lim_{n\to\infty}\left(\frac{1}{\ln n}+1\right) = 0+1=1,$$

it follows from the Squeeze Theorem that

$$\lim_{n\to\infty}\frac{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}}{\ln n} = 1.$$

25. (10 points) Given that

$$\lim_{x\to 0}(\sin(x^3)f(ax)) = \lim_{x\to 0}(\sin^3(x)g(bx)) = 1$$

for some $a, b \neq 0$, evaluate

$$\lim_{x\to 0}\frac{f(x)}{g(x)}$$

Solution: Letting $y = ax$, the first limit becomes

$$\lim_{y\to 0}\sin\left(\frac{y^3}{a^3}\right)f(y) = 1.$$

Letting $y = bx$, the second limit becomes

$$\lim_{y\to 0}\sin^3\left(\frac{y}{b}\right)g(y) = 1.$$

Hence,

$$\begin{aligned}
\lim_{x\to 0}\frac{f(x)}{g(x)} &= \lim_{x\to 0}\frac{\sin\left(\frac{x^3}{a^3}\right)f(x)}{\sin\left(\frac{x^3}{a^3}\right)}\cdot\frac{\sin^3\left(\frac{x}{b}\right)}{\sin^3\left(\frac{x}{b}\right)g(x)}\\
&= \lim_{x\to 0}\left[\sin\left(\frac{x^3}{a^3}\right)f(x)\right]\cdot\frac{\frac{x^3}{a^3}}{\sin\left(\frac{x^3}{a^3}\right)}\cdot\frac{1}{\sin^3\left(\frac{x}{b}\right)g(x)}\cdot\left(\frac{\sin\left(\frac{x}{b}\right)}{\frac{x}{b}}\right)^3\cdot\frac{a^3}{b^3}\\
&= 1\cdot 1\cdot 1\cdot 1\cdot\frac{a^3}{b^3}\\
&= \frac{a^3}{b^3}.
\end{aligned}$$