STA 522 Sample Exam 1 Solutions

Problem 1

Part (a): Since $X_1, X_2, \ldots, X_n$ are iid, the cdf of each X_i is given by:

$$F(x) = \begin{cases} 0 & \text{if } x \leq -1 \\ \frac{x+1}{2} & \text{if } -1 < x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$

Therefore,

$$\begin{aligned} P\left(X_{(1)} > 0.25 \text{ and } X_{(n)} \leq 0.8\right) &= P\left(X_i > 0.25 \text{ for all } i \text{ and } X_i \leq 0.8 \text{ for all } i\right) \\ &= P(0.25 < X_i \leq 0.8 \text{ for all } i) \\ &= \{P(0.25 < X_1 \leq 0.8)\}^n & \text{(iid)} \\ &= \{F(0.8) - F(0.25)\}^n \\ &= \left\{\frac{0.8+1}{2} - \frac{0.25+1}{2}\right\}^n = (0.55/2)^n = \boxed{(0.275)^n}. \end{aligned}$$

Part (b): Yes, it does. Fix $\varepsilon > 0$. We have

$$\begin{aligned} P(|X_{(n)} - 1| \geq \varepsilon) &= P(X_{(n)} - 1 \geq \varepsilon) + P(X_{(n)} - 1 < -\varepsilon) \\ &= P(X_{(n)} \geq 1 + \varepsilon) + P(X_{(n)} < 1 - \varepsilon) \\ &= 0 + P(X_i < 1 - \varepsilon, \text{ all } i) \\ &= \{P(X_1 < 1 - \varepsilon)\}^n & \text{(iid)} \\ &= \begin{cases} \left(\frac{1-\varepsilon+1}{2}\right)^n = \left(1 - \frac{\varepsilon}{2}\right)^n & \text{if } \varepsilon < 2 \\ 0 & \text{if } \varepsilon \geq 2 \end{cases} \\ &\to 0 \qquad \text{as } n \to \infty \end{aligned}$$

which means $X_{(n)} \xrightarrow{P} 1$.

Problem 2

Fix $\varepsilon > 0$. Then

$$P(|X_n - 0| \geq \varepsilon) = P\left(X_n^2 \geq \varepsilon^2\right) \leq P\left(\frac{X_n^2}{1+X_n^2} \geq \frac{\varepsilon^2}{1+\varepsilon^2}\right) \overset{(*)}{\leq} \frac{\mathrm{E}\left[\frac{X_n^2}{1+X_n^2}\right]}{\frac{\varepsilon^2}{1+\varepsilon^2}} \to 0 \quad \text{as } n \to \infty$$

where $(\star)$ is due to Chebyshev's inequality $\left(\frac{X_n^2}{1+X_n^2} \text{ is non-negative}\right)$. This implies that $X_n \xrightarrow{P} 0$.

Problem 3

Part (a): <u>Sufficiency:</u> The pmf of X is

$$f(x \mid \theta) = \left(\frac{\theta}{2}\right)^{|x|} (1-\theta)^{1-|x|} = \underbrace{\theta^{|x|}(1-\theta)^{1-|x|}}_{=g(T(x)|\theta)} \underbrace{2^{-|x|}}_{=h(x)}$$

where $T(x) = |x|$. Therefore, by the Factorization theorem, $|X|$ is sufficient for θ.

Part (b): Completeness: As suggested in the hint, we first find the pmf of $Y = |X|$. We note that the support of Y is $\{0, 1\}$. Clearly, $P(Y = 0) = P(X = 0) = \left(\frac{\theta}{2}\right)^0 (1-\theta)^{1-0} = 1 - \theta$, and

$$
\begin{aligned}
P(Y = 1) &= P(X = 1) + P(X = -1) \\
&= \left(\frac{\theta}{2}\right)^1 (1-\theta)^{1-1} + \left(\frac{\theta}{2}\right)^1 (1-\theta)^{1-1} = \theta
\end{aligned}
$$

Thus,

$$
P(Y = y) = \begin{cases} \theta & y = 1 \\ 1 - \theta & y = 0 \end{cases}
$$

for $0 < \theta < 1$, which means $Y \sim \text{Bernoulli}(\theta)$ for $0 < \theta < 1$. Therefore, by the completeness of Binomial family (proved in class) it follows that $Y = |X|$ is complete.

Problem 4

Let $\underline{x} = (x_1, \ldots, x_n)$ and $\underline{y} = (y_1, \ldots, y_n)$ be two sample points from the density $f(x \mid \theta)$.

Part (a): Sufficiency: We'll use the Factorization theorem on the joint density:

$$
\begin{aligned}
f(\underline{x} \mid \lambda) &= \prod_{i=1}^{n} \frac{1}{\lambda} \exp\left[-\frac{1}{\lambda}(x_i - \mu)\right] I(x_i > \mu) \\
&= \exp\left[-\frac{1}{\lambda} \sum_{i=1}^{n} x_i\right] I(x_{(1)} > \mu) \\
&= g(T_1(\underline{x}), T_2(\underline{x}) \mid \lambda, \mu) \; h(\underline{x})
\end{aligned}
$$

where $T_1(\underline{x}) = \sum_{i=1}^n x_i$, $T_2(\underline{x}) = x_{(1)}$, $g(t_1, t_2 \mid \lambda, \mu) = \exp(-t_1/\lambda) \; I(t_2 > \mu)$. Therefore, by the Factorization theorem, $(\sum_{i=1}^n X_i, X_{(1)})$ is jointly sufficient for (λ, μ).

Part (b): Minimal Sufficiency: We have

$$
\begin{aligned}
\frac{f(\underline{x} \mid \mu, \lambda)}{f(\underline{y} \mid \mu, \lambda)} &= \frac{\prod_{i=1}^n \frac{1}{\lambda} \exp\left[-\frac{1}{\lambda}(x_i - \mu)\right] I(x_i > \mu)}{\prod_{i=1}^n \frac{1}{\lambda} \exp\left[-\frac{1}{\lambda}(y_i - \mu)\right] I(y_i > \mu)} \\
&= \exp\left[-\frac{1}{\lambda}\left(\sum_{i=1}^n x_i - \sum_{i=1}^n y_i\right)\right] \frac{\prod_{i=1}^n I(x_i > \mu)}{\prod_{i=1}^n I(y_i > \mu)} \\
&= \exp\left[-\frac{1}{\lambda}\left(\sum_{i=1}^n x_i - \sum_{i=1}^n y_i\right)\right] \frac{I(x_{(1)} > \mu)}{I(y_{(1)} > \mu)}
\end{aligned}
$$

This is constant as a function of (μ, λ) if and only if $x_{(1)} = y_{(1)}$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$. Therefore, $(X_{(1)}, \sum_{i=1}^n X_i)$ is minimal sufficient for (μ, λ).

Problem 5

Part (a): Let us denote by $f(x - \mu)$ the common location family density of $X_1, X_2, \ldots, X_n$. Then there exist iid observations $Z_1, \ldots, Z_n$ from the density $f(x)$ (the standard density of the family) such that $Z_i = X_i - \mu$, i.e., $X_i = Z_i + \mu$.

Note that the sample median is:

$$
\begin{aligned}
M(X_1, X_2, \ldots, X_n) &= \begin{cases} X_{\left(\frac{n+1}{2}\right)} & n \text{ is odd} \\ \frac{X_{\left(\frac{n}{2}\right)} + X_{\left(\frac{n}{2}+1\right)}}{2} & n \text{ is even} \end{cases} \\
&= \begin{cases} \mu + Z_{\left(\frac{n+1}{2}\right)} & n \text{ is odd} \\ \mu + \frac{Z_{\left(\frac{n}{2}\right)} + Z_{\left(\frac{n}{2}+1\right)}}{2} & n \text{ is even} \end{cases} \\
&= \mu + M(Z_1, \ldots, Z_n)
\end{aligned}
$$

Again,

$$
\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i = \frac{1}{n}\sum_{i=1}^{n}(\mu + Z_i) = \mu + \overline{Z}
$$

Hence

$$
M(X_1, X_2, \ldots, X_n) - \overline{X} = M(Z_1, \ldots, Z_n) - \overline{Z}
$$

where the RHS contains random variables whose distribution does not depend on the parameter μ. Hence $M - \overline{X}$ is an ancillary statistic.

Part (b): As suggested in the hint consider a sequence Y_n where $Y_n = X$, and $W = 1 - X$, where $X \sim \text{Binomial}(1, 0.5)$. Then X and $W = 1 - X$ have the same distribution. Then $Y_n \xrightarrow{d} X$ (trivially; all have the same distribution) which means $Y_n \xrightarrow{d} W$ as X and W have the same distribution.

However, for any $0 < \varepsilon < 1$,

$$
\begin{aligned}
P(|Y_n - W| \geq \varepsilon) = P(|X - 1 + X| \geq \varepsilon) &= P(|2X - 1| \geq \varepsilon) = P(2X \geq 1 + \varepsilon) + P(2X \leq 1 - \varepsilon) \\
&= P\left(X \geq \frac{1+\varepsilon}{2}\right) + P\left(X \leq \frac{1-\varepsilon}{2}\right) \\
&= P(X = 1) + P(X = 0) = 1 \not\to 0
\end{aligned}
$$

as $n \to \infty$. Hence $Y_n \overset{P}{\not\to} W$.