STA 522, Sample Exam 1
Date and time: ____________________-

Print Name: ________________________________ ***UB Person ID:*** ____________

Read the following instructions carefully before answering questions:

Instructions:

i. This is a 120 minute exam. There are 5 problems, worth a total of 105 points. You may answer as many questions as you want; however, the maximum you can score is 100.

ii. On the back of the exam you will find formula sheets listing pmfs/pdfs, means, variances and mgfs of some common distributions discussed in class (taken from Casella Berger 2E).

iii. Write your name and UB Person ID number on this cover sheet and on each answer sheet.

iv. Unless specified otherwise, you may quote and use (without proving) any theorem/assertion proved in class or given as homework.

v. You can use a non-programmable calculator. However, you may not use any books, notes, other references, or any other electronic device during exam.

vi. Remember to show your work. Answers lacking adequate justification may not receive full credit.

vii. Once complete, arrange your answer sheets in order. Attach this cover sheet and the question pages to your answers before submitting.

Problem	**Score**	**Maximum**
1		20
2		20
3		20
4		20
5		25
Total		100

1. Let $X_1, ..., X_n$ be iid continuous Uniform$(-1, 1)$ random variables.

   (a) Find $P\left(X_{(1)} > 0.25 \text{ and } X_{(n)} \leq 0.8\right)$. **(10 pts)**

   (b) Does $X_{(n)}$ converge in probability? If so, prove it. **(10 pts)**

2. Suppose $X_1, X_2, \ldots$ is an arbitrary sequence of random variables. Show that if $E\left[\dfrac{X_n^2}{1+X_n^2}\right] \to 0$ as $n \to \infty$ then $X_n \xrightarrow{P} 0$. [**Hint:** Consider Chebyshev's inequality] **(20 pts)**

3. Let X be a single observation from the pmf

$$f(x \mid \theta) = \left(\frac{\theta}{2}\right)^{|x|} (1-\theta)^{1-|x|}, \quad x = -1, 0, 1; \; 0 < \theta < 1$$

   (a) Is $|X|$ sufficient? **(10 pts)**

   (b) Is $|X|$ Complete? [**Hint:** You may save a lot of work by identifying the distribution of $|X|$.] **(10 pts)**

4. Let $X_1, \ldots, X_n$ be a random sample from a pdf

$$f(x \mid \mu, \lambda) = \frac{1}{\lambda} \exp\left[-\frac{1}{\lambda}(x-\mu)\right], \; x > \mu;$$

   where $\mu \in (-\infty, \infty)$ and $\lambda > 0$ are unknown parameters.

   (a) Find sufficient statistics for (μ, λ). **(10 pts)**

   (b) Are the sufficient statistics in part (a) minimal sufficient? **(10 pts)**

5. Answer the following questions.

   (a) Let $X_1, ..., X_n$ be a random sample from a location family. Show that $M - \bar{X}$ is an ancillary statistic, where M is the sample median and $\bar{X}$ is the sample mean. **(10 pts)**

   (b) Suppose $X \sim$ Bernoulli(0.5). Consider the sequence of random variables $Y_n = X$ for $n = 1, 2, \ldots$. Using this sequence or otherwise show that a convergence in distribution does not necessarily imply a convergence in probability. [**Hint:** What is the distribution of $1 - X$?] **(15 pts)**

Table of Common Distributions

Discrete Distributions

Bernoulli(p)

pmf $P(X = x|p) = p^x(1-p)^{1-x}$; $x = 0, 1$; $0 \le p \le 1$

mean and variance $\mathrm{E}X = p$, $\mathrm{Var}\,X = p(1-p)$

mgf $M_X(t) = (1-p) + pe^t$

Binomial(n, p)

pmf $P(X = x|n,p) = \binom{n}{x}p^x(1-p)^{n-x}$; $x = 0, 1, 2, \ldots, n$; $0 \le p \le 1$

mean and variance $\mathrm{E}X = np$, $\mathrm{Var}\,X = np(1-p)$

mgf $M_X(t) = [pe^t + (1-p)]^n$

notes Related to Binomial Theorem (Theorem 3.2.2). The *multinomial* distribution (Definition 4.6.2) is a multivariate version of the binomial distribution.

Discrete uniform

pmf $P(X = x|N) = \frac{1}{N}$; $x = 1, 2, \ldots, N$; $N = 1, 2, \ldots$

mean and variance $\mathrm{E}X = \frac{N+1}{2}$, $\mathrm{Var}\,X = \frac{(N+1)(N-1)}{12}$

mgf $M_X(t) = \frac{1}{N}\sum_{i=1}^{N} e^{it}$

Geometric(p)

pmf $P(X = x|p) = p(1-p)^{x-1}$; $x = 1, 2, \ldots$; $0 \le p \le 1$

mean and variance $\mathrm{E}X = \frac{1}{p}$, $\mathrm{Var}\,X = \frac{1-p}{p^2}$

mgf $M_X(t) = \frac{pe^t}{1-(1-p)e^t}$, $t < -\log(1-p)$

notes $Y = X - 1$ is negative binomial$(1, p)$. The distribution is *memoryless*: $P(X > s|X > t) = P(X > s - t)$.

Hypergeometric

pmf $P(X = x|N, M, K) = \frac{\binom{M}{x}\binom{N-M}{K-x}}{\binom{N}{K}}$; $x = 0, 1, 2, \ldots, K$;
$M - (N - K) \le x \le M$; $N, M, K \ge 0$

mean and variance $\mathrm{E}X = \frac{KM}{N}$, $\mathrm{Var}\,X = \frac{KM}{N}\frac{(N-M)(N-K)}{N(N-1)}$

notes If $K \ll M$ and N, the range $x = 0, 1, 2, \ldots, K$ will be appropriate.

Negative binomial(r, p)

pmf $P(X = x|r,p) = \binom{r+x-1}{x}p^r(1-p)^x$; $x = 0, 1, \ldots$; $0 \le p \le 1$

mean and variance $\mathrm{E}X = \frac{r(1-p)}{p}$, $\mathrm{Var}\,X = \frac{r(1-p)}{p^2}$

mgf $M_X(t) = \left(\frac{p}{1-(1-p)e^t}\right)^r$, $t < -\log(1-p)$

notes An alternate form of the pmf is given by $P(Y = y|r,p) = \binom{y-1}{r-1}p^r(1-p)^{y-r}$, $y = r, r+1, \ldots$. The random variable $Y = X + r$. The negative binomial can be derived as a gamma mixture of Poissons. (See Exercise 4.34.)

Poisson(λ)

pmf $P(X = x|\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}$; $x = 0, 1, \ldots$; $0 \le \lambda < \infty$

mean and variance $\mathrm{E}X = \lambda$, $\mathrm{Var}\,X = \lambda$

mgf $M_X(t) = e^{\lambda(e^t - 1)}$

Continuous Distributions

Beta(α, β)

pdf	$f(x\|\alpha,\beta) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \quad 0 \le x \le 1, \quad \alpha > 0, \quad \beta > 0$
mean and variance	$\mathrm{E}X = \frac{\alpha}{\alpha+\beta}, \quad \mathrm{Var}\, X = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$
mgf	$M_X(t) = 1 + \sum_{k=1}^{\infty} \left(\prod_{r=0}^{k-1} \frac{\alpha+r}{\alpha+\beta+r} \right) \frac{t^k}{k!}$
notes	The constant in the beta pdf can be defined in terms of gamma functions, $B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$. Equation (3.2.18) gives a general expression for the moments.

Cauchy(θ, σ)

pdf	$f(x\|\theta,\sigma) = \frac{1}{\pi\sigma} \frac{1}{1+\left(\frac{x-\theta}{\sigma}\right)^2}, \quad -\infty < x < \infty; \quad -\infty < \theta < \infty, \quad \sigma > 0$
mean and variance	do not exist
mgf	does not exist
notes	Special case of Student's t, when degrees of freedom = 1. Also, if X and Y are independent $\mathrm{n}(0,1)$, X/Y is Cauchy.

Chi squared(p)

pdf	$f(x\|p) = \frac{1}{\Gamma(p/2)2^{p/2}} x^{(p/2)-1} e^{-x/2}; \quad 0 \le x < \infty; \quad p = 1, 2, \ldots$
mean and variance	$\mathrm{E}X = p, \quad \mathrm{Var}\, X = 2p$
mgf	$M_X(t) = \left(\frac{1}{1-2t}\right)^{p/2}, \quad t < \frac{1}{2}$
notes	Special case of the gamma distribution.

Double exponential(μ, σ)

pdf	$f(x\|\mu,\sigma) = \frac{1}{2\sigma} e^{-\|x-\mu\|/\sigma}, \quad -\infty < x < \infty, \quad -\infty < \mu < \infty, \quad \sigma > 0$
mean and variance	$\mathrm{E}X = \mu, \quad \mathrm{Var}\, X = 2\sigma^2$
mgf	$M_X(t) = \frac{e^{\mu t}}{1-(\sigma t)^2}, \quad \|t\| < \frac{1}{\sigma}$
notes	Also known as the *Laplace* distribution.

Exponential(β)

pdf	$f(x\|\beta) = \frac{1}{\beta} e^{-x/\beta}, \quad 0 \le x < \infty, \quad \beta > 0$
mean and variance	$\mathrm{E}X = \beta, \quad \mathrm{Var}\, X = \beta^2$
mgf	$M_X(t) = \frac{1}{1-\beta t}, \quad t < \frac{1}{\beta}$
notes	Special case of the gamma distribution. Has the *memoryless* property. Has many special cases: $Y = X^{1/\gamma}$ is *Weibull*, $Y = \sqrt{2X/\beta}$ is *Rayleigh*, $Y = \alpha - \gamma \log(X/\beta)$ is *Gumbel*.

F

pdf	$f(x\|\nu_1,\nu_2) = \frac{\Gamma\left(\frac{\nu_1+\nu_2}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)} \left(\frac{\nu_1}{\nu_2}\right)^{\nu_1/2} \frac{x^{(\nu_1-2)/2}}{\left(1+\left(\frac{\nu_1}{\nu_2}\right)x\right)^{(\nu_1+\nu_2)/2}};$ $0 \le x < \infty; \quad \nu_1, \nu_2 = 1, \ldots$
mean and variance	$\mathrm{E}X = \frac{\nu_2}{\nu_2-2}, \quad \nu_2 > 2,$ $\mathrm{Var}\, X = 2\left(\frac{\nu_2}{\nu_2-2}\right)^2 \frac{(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}, \quad \nu_2 > 4$
moments (mgf does not exist)	$\mathrm{E}X^n = \frac{\Gamma\left(\frac{\nu_1+2n}{2}\right)\Gamma\left(\frac{\nu_2-2n}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)} \left(\frac{\nu_2}{\nu_1}\right)^n, \quad n < \frac{\nu_2}{2}$
notes	Related to chi squared ($F_{\nu_1,\nu_2} = \left(\frac{\chi^2_{\nu_1}}{\nu_1}\right) / \left(\frac{\chi^2_{\nu_2}}{\nu_2}\right)$, where the χ^2s are independent) and t ($F_{1,\nu} = t_\nu^2$).

Gamma(α, β)

pdf	$f(x\|\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, \quad 0 \le x < \infty, \quad \alpha, \beta > 0$
mean and variance	$\mathrm{E}X = \alpha\beta, \quad \mathrm{Var}\, X = \alpha\beta^2$
mgf	$M_X(t) = \left(\frac{1}{1-\beta t}\right)^\alpha, \quad t < \frac{1}{\beta}$
notes	Some special cases are exponential ($\alpha = 1$) and chi squared ($\alpha = p/2$, $\beta = 2$). If $\alpha = \frac{3}{2}$, $Y = \sqrt{X/\beta}$ is *Maxwell*. $Y = 1/X$ has the *inverted gamma distribution*. Can also be related to the Poisson (Example 3.2.1).

Logistic(μ, β)

pdf	$f(x\|\mu,\beta) = \frac{1}{\beta} \frac{e^{-(x-\mu)/\beta}}{[1+e^{-(x-\mu)/\beta}]^2}, \quad -\infty < x < \infty, \quad -\infty < \mu < \infty, \quad \beta > 0$
mean and variance	$\mathrm{E}X = \mu, \quad \mathrm{Var}\, X = \frac{\pi^2\beta^2}{3}$

mgf	$M_X(t) = e^{\mu t}\Gamma(1-\beta t)\Gamma(1+\beta t), \quad \lvert t\rvert < \frac{1}{\beta}$
notes	The cdf is given by $F(x\vert\mu,\beta) = \frac{1}{1+e^{-(x-\mu)/\beta}}$.

Lognormal(μ, σ^2)

pdf	$f(x\vert\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma}\frac{e^{-(\log x-\mu)^2/(2\sigma^2)}}{x}, \quad 0 \le x < \infty, \quad -\infty < \mu < \infty,$ $\sigma > 0$
mean and variance	$\mathrm{E}X = e^{\mu+(\sigma^2/2)}, \quad \mathrm{Var}\,X = e^{2(\mu+\sigma^2)} - e^{2\mu+\sigma^2}$
moments (mgf does not exist)	$\mathrm{E}X^n = e^{n\mu+n^2\sigma^2/2}$
notes	Example 2.3.5 gives another distribution with the same moments.

Normal(μ, σ^2)

pdf	$f(x\vert\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma}e^{-(x-\mu)^2/(2\sigma^2)}, \quad -\infty < x < \infty, \quad -\infty < \mu < \infty,$ $\sigma > 0$
mean and variance	$\mathrm{E}X = \mu, \quad \mathrm{Var}\,X = \sigma^2$
mgf	$M_X(t) = e^{\mu t+\sigma^2 t^2/2}$
notes	Sometimes called the *Gaussian* distribution.

Pareto(α, β)

pdf	$f(x\vert\alpha,\beta) = \frac{\beta\alpha^\beta}{x^{\beta+1}}, \quad a < x < \infty, \quad \alpha > 0, \quad \beta > 0$
mean and variance	$\mathrm{E}X = \frac{\beta\alpha}{\beta-1}, \quad \beta > 1, \quad \mathrm{Var}\,X = \frac{\beta\alpha^2}{(\beta-1)^2(\beta-2)}, \quad \beta > 2$
mgf	does not exist

t

pdf	$f(x\vert\nu) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\frac{1}{\sqrt{\nu\pi}}\frac{1}{\left(1+\left(\frac{x^2}{\nu}\right)\right)^{(\nu+1)/2}}, \quad -\infty < x < \infty, \quad \nu = 1, \ldots$
mean and variance	$\mathrm{E}X = 0, \quad \nu > 1, \quad \mathrm{Var}\,X = \frac{\nu}{\nu-2}, \quad \nu > 2$
moments (mgf does not exist)	$\mathrm{E}X^n = \frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{\nu-n}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\nu}{2}\right)}\nu^{n/2}$ if $n < \nu$ and even, $\mathrm{E}X^n = 0$ if $n < \nu$ and odd.
notes	Related to F $(F_{1,\nu} = t_\nu^2)$.

Uniform(a, b)

pdf	$f(x\vert a,b) = \frac{1}{b-a}, \quad a \le x \le b$
mean and variance	$\mathrm{E}X = \frac{b+a}{2}, \quad \mathrm{Var}\,X = \frac{(b-a)^2}{12}$
mgf	$M_X(t) = \frac{e^{bt}-e^{at}}{(b-a)t}$
notes	If $a = 0$ and $b = 1$, this is a special case of the beta $(\alpha = \beta = 1)$.

Weibull(γ, β)

pdf	$f(x\vert\gamma,\beta) = \frac{\gamma}{\beta}x^{\gamma-1}e^{-x^\gamma/\beta}, \quad 0 \le x < \infty, \quad \gamma > 0, \quad \beta > 0$
mean and variance	$\mathrm{E}X = \beta^{1/\gamma}\Gamma\left(1+\frac{1}{\gamma}\right), \quad \mathrm{Var}\,X = \beta^{2/\gamma}\left[\Gamma\left(1+\frac{2}{\gamma}\right) - \Gamma^2\left(1+\frac{1}{\gamma}\right)\right]$
moments	$\mathrm{E}X^n = \beta^{n/\gamma}\Gamma\left(1+\frac{n}{\gamma}\right)$
notes	The mgf exists only for $\gamma \ge 1$. Its form is not very useful. A special case is exponential $(\gamma = 1)$.