STA 522 Exam 1 Solutions

Problem 1

Part (a): Since $X_1, X_2, \ldots, X_n$ are iid Uniform(0, 1), the cdf of each X_i is given by:

$$F(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x & \text{if } 0 < x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$

Therefore,

$$\begin{aligned} P\left(X_{(1)} < 0.25\right) &= 1 - \left(X_{(1)} \geq 0.25\right) \\ &= 1 - P\left(X_i \geq 0.25 \text{ for all } i\right) \\ &= 1 - \{1 - F(0.25)\}^n && \text{(iid)} \\ &= 1 - (1 - 0.25)^n = \boxed{1 - (0.75)^n} \end{aligned}$$

and

$$\begin{aligned} P\left(X_{(n)} < 0.25\right) &= P\left(X_i < 0.25 \text{ for all } i\right) \\ &= \{F(0.25)\}^n && \text{(iid)} \\ &= \boxed{(0.25)^n} \end{aligned}$$

Because $X_{(1)} \leq X_{(n)}$, therefore $X_{(n)} < 0.25$ *implies* $X_{(1)} < 0.25$, so that $P(X_{(n)} < 0.25) \leq P(X_{(1)} < 0.25)$.

Part (b): Yes, it does. We'll first show that $X_{(n)} \xrightarrow{P} 1$. This is similar to the solution for Problem 1(b) in the sample exam, with the difference being that here we have a Uniform$(0, 1)$ population instead of a Uniform$(-1, 1)$ population.

Fix $\varepsilon > 0$ small. We have

$$\begin{aligned} P(|X_{(n)} - 1| \geq \varepsilon) &= P(X_{(n)} - 1 \geq \varepsilon) + P(X_{(n)} - 1 < -\varepsilon) \\ &= P(X_{(n)} \geq 1 + \varepsilon) + P(X_{(n)} < 1 - \varepsilon) \\ &= 0 + P(X_i < 1 - \varepsilon, \text{ all } i) \\ &= \{P(X_1 < 1 - \varepsilon)\}^n && \text{(iid)} \\ &= \begin{cases} (1-\varepsilon)^n & \text{if } \varepsilon < 1 \\ 0 & \text{if } \varepsilon \geq 1 \end{cases} \\ &\to 0 \qquad \text{as } n \to \infty \end{aligned}$$

which means $X_{(n)} \xrightarrow{P} 1$.

Now apply the continuous mapping result: if $X_n \xrightarrow{P} X$ then $h(X_n) \xrightarrow{P} h(X)$ for any continuous funciton h. Because here $X_{(n)} \xrightarrow{P} 1$ and $h(x) = x/2$ is a continuous function, therefore, $X_{(n)}/2 \xrightarrow{P} 1/2$.

Problem 2

This is from Lecture 2: see lecture notes. To verify that $F(v) = e^{-1/v} I(v > 0)$ is a cdf, observe:

(i) $F(-\infty) = \lim_{v\to-\infty} F(v) = 0$ and $F(+\infty) = \lim_{v\to\infty} F(v) = \lim_{v\to\infty} e^{-1/v} = 1$.

(ii) F is continuous everywhere on $(-\infty, \infty)$ (actually differentiable everywhere except $v = 0$), and hence is right continuous.

(iii) Note that $F(v)$ is exactly zero for all $v \le 0$, meaning F is non-decreasing in $(-\infty, 0]$. On $(0, \infty)$, F is differentiable, and $\frac{d}{dv} e^{-1/v} = e^{-1/v}\left(-\frac{1}{v^2}\right)(-1) = e^{-1/v}\frac{1}{v^2} > 0$ for all $v > 0$. Hence, $F(v)$ is increasing on $(0, \infty)$. Combining, we see that F is non-decreasing on the entirety of $(-\infty, \infty)$.

These three collectively imply that F is a cdf.

Problem 3

Here $X_i \sim$ iid Bernoulli(θ) for $i = 1, \ldots, n = 10$.

Part (a): The likelihood of θ is

$$L(\theta \mid \underline{x}) = P(\underline{X} = \underline{x} \mid \theta) = \theta^{\sum_{i=1}^{10} x_i}(1-\theta)^{10-\sum_{i=1}^{10} X_i}$$

Part (b): Given that $\sum_{i=1}^{10} X_i = 6$. The likelihood is:

$$L(\theta \mid \sum_{i=1}^{10} X_i = 6) = \theta^6(1-\theta)^4$$

Therefore

$$L(\theta = 0.2 \mid \sum_{i=1}^{10} X_i = 6) = (0.2)^6(0.8)^4 = \boxed{2.61 \times 10^{-5}}$$

and

$$L(\theta = 0.8 \mid \sum_{i=1}^{10} X_i = 6) = (0.8)^6(0.2)^4 = \boxed{4.19 \times 10^{-4}}.$$

This shows that $\theta = 0.8$ has a higher likelihood. Intuitively, the observed data of 6 successes are more compatible with the configuration where population probability of success $\theta = 0.8$ than with $\theta = 0.2$.

Problem 4

The joint density of $\underline{X}$ is:

$$f(\underline{x} \mid \theta, \gamma) = \prod_{i=1}^{n}\left(\frac{\gamma}{\theta}\, x_i^{\gamma-1} e^{-x_i^\gamma/\theta}\right) = \left(\frac{\gamma}{\theta}\right)^n \prod_{i=1}^{n} x_i^{\gamma-1} \; \exp\left(-\frac{1}{\theta}\sum_{i=1}^{n} x_i^\gamma\right)$$

Part (a): If $\gamma > 0$ is known, then

$$f(\underline{x} \mid \theta) = \underbrace{\left(\frac{\gamma}{\theta}\right)^n \exp\left(-\frac{1}{\theta}\sum_{i=1}^{n} x_i^\gamma\right)}_{=g(T(\underline{x})|\theta)} \underbrace{\prod_{i=1}^{n} x_i^{\gamma-1}}_{=h(\underline{x})}$$

Therefore, by the Factorization theorem, $T(\underline{X}) = \sum_{i=1}^{n} X_i^\gamma$ is sufficient for θ.

Part (b): It follows from the joint density above that a factorization based on any univariate or bivariate or lower (than n) dimensional statistic is not feasible when both θ and γ are unknown. Therefore, by the necessity half of the factorization theorem (i.e., we need to have a factorization of joint pdf for a lower dimensional sufficient statistic to exit) it follows that no univariate sufficient statistic exist in this case. A (semi-trivial) sufficient statistic would be the order statistics: $(X_{(1)}, \ldots, X_{(n)})$.

Problem 5

Part (a): Because $X_1, \ldots, X_n$ are iid from the scale family $\frac{1}{\sigma} f(x/\sigma)$, we can construct iid observations $Z_1, \ldots, Z_n$ from the density $f(x)$ (the standard density of the family which is free of σ) such that $Z_i = X_i/\sigma$, i.e., $X_i = \sigma Z_i$.

Note that the sample median is:

$$
\begin{aligned}
M(X_1, X_2, \ldots, X_n) &= \begin{cases} X_{\left(\frac{n+1}{2}\right)} & n \text{ is odd} \\ \frac{X_{\left(\frac{n}{2}\right)} + X_{\left(\frac{n}{2}+1\right)}}{2} & n \text{ is even} \end{cases} \\
&= \begin{cases} \sigma \ Z_{\left(\frac{n+1}{2}\right)} & n \text{ is odd} \\ \sigma \ \frac{Z_{\left(\frac{n}{2}\right)} + Z_{\left(\frac{n}{2}+1\right)}}{2} & n \text{ is even} \end{cases} \\
&= \sigma M(Z_1, \ldots, Z_n)
\end{aligned}
$$

Again,

$$
\overline{X} = \frac{1}{n} \sum_{i=1}^{n} X_i = \frac{1}{n} \sum_{i=1}^{n} (\sigma Z_i) = \sigma \overline{Z}
$$

Hence

$$
\log M - \log \overline{X} = \log \left(\frac{M(X_1, X_2, \ldots, X_n)}{\overline{X}} \right) = \log \left(\frac{\sigma M(Z_1, \ldots, Z_n)}{\sigma \overline{Z}} \right) = \log \left(\frac{M(Z_1, \ldots, Z_n)}{\overline{Z}} \right)
$$

where the RHS contains random variables whose distribution does not depend on the parameter σ. Hence $\log M - \log \overline{X}$ is an ancillary statistic.

Part(b): [NOTE: Recall that for Uniform$(0, \theta)$, $T = X_{(n)}$ is sufficient, and not $\sum_{i=1}^{n} X_i$. So we can't use Basu's theorem directly. However, for this specific types of problem, there is a much quicker way, shown as follows.]

As suggested in the hint, the random variables $\dfrac{X_i}{X_1 + \cdots + X_n}$ all have the same distribution and hence mean as X_i's are iid.

Therefore, for some constant $k(>0)$, $E\left[\dfrac{X_i}{X_1 + \cdots + X_n}\right] = k$ for all $i = 1, \ldots, n$.

So,

$$
\begin{aligned}
& E\left[\frac{X_1}{X_1 + \cdots + X_n}\right] + \cdots + E\left[\frac{X_n}{X_1 + \cdots + X_n}\right] = \underbrace{k + \cdots + k}_{n \text{ many}} = nk \\
\implies & \underbrace{E\left[\frac{X_1}{X_1 + \cdots + X_n} + \cdots + \frac{X_n}{X_1 + \cdots + X_n}\right]}_{= E\left[\frac{X_1 + \cdots + X_n}{X_1 + \cdots + X_n}\right] = E(1) = 1} = nk \\
& \text{i.e., } nk = 1 \implies k = \frac{1}{n}
\end{aligned}
$$

Therefore, $E\left[\dfrac{X_n}{X_1 + \cdots + X_n}\right] = k = \dfrac{1}{n}$.