STA 522/Solutions to Homework 2

Problem 5.33

Let c be any finite number. Fix $\varepsilon > 0$. We'll show that there exists a positive integer N such that $n \geq N \implies P(X_n + Y_n > c) \geq 1 - \varepsilon$.

To this end, first find a continuity point x_0 of F_X (i.e., find an x_0 where F_X is continuous) such that $F_X(x_0) \leq \varepsilon/3$ (we can always find such a point since $\lim_{t \to -\infty} F_X(t) = 0$, and the total number of points where F_X is discontinuous can at most be countable).

Since $X_n \xrightarrow{d} X$ and x_0 is a continuity point of X, therefore we can find N_1 such that

$$\begin{aligned} n \geq N_1 &\implies |F_{X_n}(x_0) - F_X(x_0)| < \varepsilon/3 \\ &\implies F_{X_n}(x_0) < F_X(x_0) + \varepsilon/3 \leq \varepsilon/3 + \varepsilon/3 = 2\varepsilon/3 \\ &\implies P(X_n > x_0) = 1 - F_{X_n}(x_0) \geq 1 - 2\varepsilon/3 \end{aligned}$$

Since $\lim_{n \to \infty} P(Y_n > c - x_0) \to 1$ as $n \to \infty$, therefore, we can find a positive integer N_2 such that $n \geq N_2 \implies P(Y_n > c - x_0) \geq 1 - \varepsilon/3$. Hence,

$$\begin{aligned} P(X_n + Y_n > c) &\geq P(X_n > x_0,\ Y_n > c - x_0) \\ &\geq P(X_n > x_0) + P(Y_n > c - x_0) - 1 \quad \text{(Bonferroni)} \\ &\geq 1 - 2\varepsilon/3 + 1 - \varepsilon/3 - 1 = 1 - \varepsilon \end{aligned}$$

for all $n \geq N = \max\{N_1, N_2\}$. This completes the proof.

Problem 5.34

Since $\mathrm{E}(\overline{X}_n) = \mu$, we have

$$\mathrm{E}\left(\frac{\sqrt{n}\left(\overline{X}_n - \mu\right)}{\sigma}\right) = \frac{\sqrt{n}}{\sigma}\,\mathrm{E}\left(\overline{X}_n - \mu\right) = 0,$$

and since $\mathrm{Var}(\overline{X}_n) = \dfrac{\sigma^2}{n}$ we have

$$\mathrm{Var}\left(\frac{\sqrt{n}\left(\overline{X}_n - \mu\right)}{\sigma}\right) = \frac{n}{\sigma^2}\,\mathrm{Var}\left(\overline{X}_n - \mu\right) = \frac{n}{\sigma^2}\,\mathrm{Var}\left(\overline{X}_n\right) = \frac{n}{\sigma^2} \cdot \frac{\sigma^2}{n} = 1.$$

Problem 5.39 (Part b)

As discussed in class, consider the sub-sequence corresponding to the "left-most" intervals of the form $[0, 1/k]$ for $k = 1, 2, 3, \ldots$, i.e., consider the subsequence $(X_{n_j}) = X_1, X_2, X_4, X_7, X_{11}, X_{16}, \ldots$. For this subsequence

$$X_{n_j}(s) \to \begin{cases} s + 1 & \text{if } s = 0 \\ s & \text{if } s > 0 \end{cases}$$

i.e., $X_{n_j}(s) \to X(s)$ for all $s \in (0, 1]$. This means $X_{n_j} \xrightarrow{a.s.} X$ since $P((0, 1]) = 1$.

Problem 5.41

As suggested, set $\varepsilon = |x - \mu|$.

Part (a) if $x > \mu$ then $\varepsilon = x - \mu$, so that

$$\begin{aligned} P(|X_n - \mu| \leq \varepsilon) &= P(|X_n - \mu| \leq x - \mu) \\ &= P(\mu - x \leq X_n - \mu \leq x - \mu) \\ &= P(2\mu - x \leq X_n \leq x) \\ &\leq P(X_n \leq x) \end{aligned}$$

On the other hand, if $x < \mu$ then $\varepsilon = \mu - x$ so that

$$\begin{aligned} P(|X_n - \mu| \geq \varepsilon) &= P(|X_n - \mu| \geq \mu - x) \\ &= P(X_n - \mu \geq \mu - x) + P(X_n - \mu \leq x - \mu) \\ &= P(X_n \geq 2\mu - x) + P(X_n \leq x) \\ &\geq P(X_n \leq x) \end{aligned}$$

To prove the $\Longrightarrow$ implication assume $X_n \xrightarrow{P} \mu$ i.e., $P(|X_n - \mu| > \varepsilon) \to 0 \iff P(|X_n - \mu| \leq \varepsilon) \to 1$.

If $x > \mu$ then as $n \to \infty$,

$$1 \geq P(X_n \leq x) \geq P(|X_n - \mu| \leq \varepsilon) \to 1 \implies P(X_n \leq x) \to 1.$$

On the other hand if $x < \mu$ then as $n \to \infty$

$$0 \leq P(X_n \leq x) \leq P(|X_n - \mu| \geq \varepsilon) \to 0 \implies P(X_n \leq x) \to 0.$$

Therefore combining the two cases, we get

$$P(X_n \leq x) \to \begin{cases} 0 & \text{if } x < \mu \\ 1 & \text{if } x > \mu \end{cases}$$

(Note that $x = \mu$ is a discontinuity point of the cdf of a degenerate distribution at μ. Hence, we don't need to show convergence of $F_{X_n}(x) = P(X_n \leq x)$ at $x = \mu$. The limiting cdf is 1 at $x = \mu$.)

Part (b) To prove the $\Longleftarrow$ implication, assume that

$$P(X_n \leq x) \to \begin{cases} 0 & \text{if } x < \mu \\ 1 & \text{if } x \geq \mu \end{cases}$$

holds. Fix $\varepsilon > 0$. As suggested,

$$\begin{aligned} P(|X_n - \mu| > \varepsilon) &= P(X_n - \mu < -\varepsilon) + P(X_n - \mu > \varepsilon) \\ &= P(X_n < \mu - \varepsilon) + P(X_n > \mu + \varepsilon) \\ &= P(X_n < \mu - \varepsilon) + 1 - P(X_n \leq \mu + \varepsilon) \\ &\to 0 + 1 - 1 = 0 \end{aligned}$$

as $n \to \infty$. which completes the proof.