STA 522, Spring 2021
Introduction to Theoretical Statistics II

Lecture 10

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AGENDA

- Evaluating Tests
- UMP tests
- Neyman Pearson Lemma
- Review for Exam 2

Review: Evaluating Tests

- Let $\mathcal{C}$ be a class of tests for testing $H_0 : \theta \in \Theta_0$ vs. $H_1 : \theta \in \Theta_0^c$. A test in class $\mathcal{C}$, with power function $\beta(\theta)$, is a **uniformly most powerful (UMP) class $\mathcal{C}$ test** if $\beta(\theta) \geq \beta'(\theta)$ for every $\theta \in \Theta_0^c$ and every $\beta'(\theta)$ that is a power function of a test in class $\mathcal{C}$.
- if we take $\mathcal{C}$ to be the class of all level α tests, the test described in the above definition is called a **UMP level α test**.

Neyman-Pearson Lemma

Theorem 8.3.12

Consider testing $H_0 : \theta = \theta_0$ vs. $H_1 : \theta = \theta_1$, where

(1) the pdf or pmf corresponding to θ_i is $f(\underline{x} \mid \theta_i)$ for $i = 0, 1$;

(2) the test has a rejection region R that satisfies

$\underline{x} \in R \quad \text{if} \quad f(\underline{x} \mid \theta_1) > kf(\underline{x} \mid \theta_0)$ and $\underline{x} \in R^c \quad \text{if} \quad f(\underline{x} \mid \theta_1) < kf(\underline{x} \mid \theta_0)$
for some $k \geq 0$; and

(3) $\alpha = P_{\theta_0}(\underline{X} \in R)$.

Then

(a) **(Sufficiency)** any test that satisfies (2) and (3) above is a UMP level α test; and

(b) **(Necessity)** if there exists a test satisfying (2) and (3) above with $k > 0$, then every UMP level α test is a size α test (satisfies (3) above), and every UMP level α test satisfies (2) above, except perhaps on a set A satisfying $P_{\theta_0}(\underline{X} \in A) = P_{\theta_1}(\underline{X} \in A) = 0$.

Proof: Assume that $f(\underline{x} \mid \theta_0)$ and $f(\underline{x} \mid \theta_1)$ are pdfs of continuous random variables.

Note that any test satisfying (3) is a size α and, hence, a level α test:

$$\sup_{\theta \in \Theta_0} P_\theta(\underline{X} \in R) = P_{\theta_0}(\underline{X} \in R) = 0$$

Consider the *test function* $\phi(\underline{x}) = I(\underline{x} \in R)$ of a test satisfying (1) and (2).

Part(a): Let $\phi'(\underline{x})$ be the test function of any other level α test, and let $\beta(\theta)$ and $\beta'(\theta)$ be the power functions for the tests ϕ and ϕ', respectively.

Now consider quantity
$\psi(\underline{x} \mid \theta_0, \theta_1) = (\phi(\underline{x}) - \phi'(\underline{x}))\ (f(\underline{x} \mid \theta_1) - kf(\underline{x} \mid \theta_0))$.

Then $\psi(\underline{x} \mid \theta_0, \theta_1) \geq 0$ for all $\underline{x}$ (since $0 \leq \phi'(\underline{x}) \leq 1$ for all x and $\phi(\underline{x}) = I(\underline{x} \in R)$)

$$\begin{aligned} \Longrightarrow\ 0\ &\leq \int \left[(\phi(\underline{x}) - \phi'(\underline{x}))\ (f(\underline{x} \mid \theta_1) - kf(\underline{x} \mid \theta_0))\right]\ d\underline{x} \\ &= \beta(\theta_1) - \beta'(\theta_1) - k(\beta(\theta_0) - \beta'(\theta_0)) \qquad (\star) \end{aligned}$$

Since $\phi'(\underline{x})$ is a level α test and $\phi(\underline{x})$ is a size α test, therefore $\beta(\theta_0) - \beta'(\theta_0) \geq \alpha - \alpha = 0$. Therefore from $(\star)$,

$$0 \leq \beta(\theta_1) - \beta'(\theta_1) - k(\beta(\theta_0) - \beta'(\theta_0)) \leq \beta(\theta_1) - \beta'(\theta_1)$$

implying $\beta(\theta_1) \geq \beta'(\theta_1)$. This proves part (a).

Part(b): let ϕ' now be the test function for any UMP level α test. By part (a), ϕ, a test satisfying (2) and (3) above, is also a UMP level α test, thus $\beta(\theta_1) = \beta'(\theta_1)$. Since $k \geq 0$, from $(\star)$

$$0 \leq 0 - k(\beta(\theta_0) - \beta'(\theta_0)) \implies \underbrace{\beta(\theta_0)}_{=\alpha} - \beta'(\theta_0) \leq 0 \implies \beta'(\theta_0) \geq \alpha$$

but by assumption $\phi'(\underline{x})$ is a level α test, i.e., $\beta'(\theta_0) \leq \alpha$, which together imply $\beta'(\theta_0) = \alpha$ meaning $\phi'(\underline{x})$ is a size α test, and $(\star)$ is an equality.

However, the non-negative integrand $\psi(\underline{x} \mid \theta_0, \theta_1)$ will have a zero integral only if it satisfies (2), except perhaps on a set A satisfying $P_{\theta_0}(\underline{X} \in A) = P_{\theta_1}(\underline{X} \in A) = 0$. This proves (b).

Tests Based on Sufficient Statistics

Corollary 8.3.13

Consider testing $H_0 : \theta = \theta_0$ vs. $H_1 : \theta = \theta_1$. Suppose $T(\underline{X})$ is a sufficient statistic for θ, and let $g(t \mid \theta_i)$ be the pdf or pmf of T corresponding to θ_i for $i = 0, 1$. Then any test based on T with rejection region S (a subset of the sample space of T) is a UMP level α test if it satisfies

(1) for some $k \geq 0$,

$$t \in S \quad \text{if} \quad g(t \mid \theta_1) > kg(t \mid \theta_0)$$

and

$$t \in S^c \quad \text{if} \quad g(t \mid \theta_1) < kg(t \mid \theta_0)$$

and

(2) $\alpha = P_{\theta_0}(T \in S)$.

Proof: Use factorization theorem. Reading exercise. See p. 390 in the textbook.

Example: Suppose $X \sim \text{binomial}(2, \theta)$, and we are testing $H_0 : \theta = \frac{1}{2}$ vs. $H_1 : \theta = \frac{3}{4}$. Determine the UMP level α tests for $\alpha = 0, \frac{1}{4}, \frac{3}{4}, 1$.

At the outset note that a "larger" value of X favors H_1, and a smaller value of X favors H_0.

We have $f(x \mid \theta) = \binom{2}{x} \, \theta^x \, (1-\theta)^{2-x}; \; x = 0, 1, 2$. Consider the ratio

$$\frac{f\left(x \mid \theta = \frac{3}{4}\right)}{f\left(x \mid \theta = \frac{1}{2}\right)} = \frac{\binom{2}{x} \; \left(\frac{3}{4}\right)^x \; \left(\frac{1}{4}\right)^{2-x}}{\binom{2}{x} \; \left(\frac{1}{2}\right)^x \; \left(\frac{1}{2}\right)^{2-x}} = \left(\frac{3}{2}\right)^x \; \left(\frac{1}{2}\right)^{2-x}; \quad x = 0, 1, 2$$

Therefore,

$$\frac{f\left(0 \mid \theta = \frac{3}{4}\right)}{f\left(0 \mid \theta = \frac{1}{2}\right)} = \frac{1}{4}; \quad \frac{f\left(1 \mid \theta = \frac{3}{4}\right)}{f\left(1 \mid \theta = \frac{1}{2}\right)} = \frac{3}{4}; \quad \frac{f\left(2 \mid \theta = \frac{3}{4}\right)}{f\left(2 \mid \theta = \frac{1}{2}\right)} = \frac{9}{4}.$$

(a) If we choose $\frac{3}{4} < k < \frac{9}{4}$ then NP Lemma says that the test that rejects H_0 if $X = 2$ is the UMP level $\alpha = P\left(X = 2 \mid \theta = \frac{1}{2}\right) = \frac{1}{4}$ test.

(b) If we choose $\frac{1}{4} < k < \frac{3}{4}$ then NP Lemma says that the test that rejects H_0 if $X = 1$ or $X = 2$ is the UMP level $\alpha = P\left(X = 1 \text{ or } 2 \mid \theta = \frac{1}{2}\right) = \frac{3}{4}$ test

(c) Choosing $k < \frac{1}{4}$ or $k > \frac{3}{4}$ produces UMP level 1 or level 0 tests tests respectively.

- If $k = \frac{3}{4}$, NP lemma says that we must reject H_0 when $X = 2$ and accept H_0 but leaves the action for $X = 1$ undetermined.
  – If we accept H_0 for $X = 1$, we get the UMP level $\alpha = \frac{1}{4}$ test as above (case (a)).
  – If we reject H_0 for $X = 1$, we get the UMP level $\alpha = \frac{3}{4}$ test as above (case (b)).

Example (UMP Normal test): Let $X_1, X_2, \ldots, X_n \sim$ iid $N(\theta, \sigma^2)$ population, σ^2 known. Consider testing $H_0 : \theta = \theta_0$ vs. $H_1 : \theta = \theta_1$, where $\theta_0 > \theta_1$. Find the UMP test.

The sample mean $\overline{X}$ is a sufficient statistic for θ. So we'll use the corollary of NP lemma with sufficient statistic.

Here

$$g(\overline{x} \mid \theta_1) > k \; g(\overline{x} \mid \theta_0)$$

is equivalent to (HW, use $\theta_1 - \theta_0 < 0$)

$$\overline{x} < \frac{\frac{2\sigma^2 \log k}{n} - (\theta_0^2 - \theta_1^2)}{2(\theta_1 - \theta_0)}$$

i.e., of the form $\overline{x} < c$. Therefore, by the (corollary to) the NP lemma, a test that rejects H_0 when $\overline{x} < c$ is a UMP size α test, where c is obtained from

$$\alpha = P_{\theta_0}(\overline{X} < c) = P_{\theta_0}\left(\frac{\overline{X} - \theta_0}{\sigma/\sqrt{n}} < \frac{c - \theta_0}{\sigma/\sqrt{n}}\right) \implies \frac{c - \theta_0}{\sigma/\sqrt{n}} = z_{1-\alpha} = -z_\alpha$$

i.e., $c = \theta_0 - z_\alpha \frac{\sigma}{\sqrt{n}}$

Homework

- Read p. 387 – 392.
- Exercises: TBA.