STA 522 / Solution to Homework 8

Problem 7.24

Part (a): The likelihood function for λ is:

$$L(\lambda \mid \underline{x}) = \prod_{i=1}^{n} e^{-\lambda} \frac{\lambda^{x_i}}{x_i!} = e^{-n\lambda} \frac{\lambda^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i!}$$

The prior density for λ is:

$$f_\lambda(\lambda) = \frac{1}{\Gamma(\alpha)\beta^\alpha} \lambda^{\alpha-1} e^{-\lambda/\beta}, \quad \lambda \geq 0$$

Hence, the joint pdf of $\underline{X}, \lambda$ is:

$$f_{\underline{X},\lambda}(\underline{x}, \lambda) = e^{-n\lambda} \frac{\lambda^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i!} \frac{1}{\Gamma(\alpha)\beta^\alpha} \lambda^{\alpha-1} e^{-\lambda/\beta}$$

$$= \frac{1}{\prod_{i=1}^{n} x_i!} \frac{1}{\Gamma(\alpha)\beta^\alpha} e^{-\lambda(n+1/\beta)} \lambda^{\alpha + \sum_{i=1}^{n} x_i - 1}$$

Note that for fixed $\underline{x}$,

$$f_{\lambda \mid \underline{x}}(\lambda \mid \underline{x}) = \frac{f_{\underline{x},\lambda}(\underline{x}, \lambda)}{f_{\underline{x}}(\underline{x})} \propto f_{X,\lambda}(\underline{x}, \lambda) \quad [\text{as a function of } \lambda]$$

Therefore, identifying the terms involving λ in $f_{\underline{X},\lambda}(\underline{x},\lambda)$ we get

$$f_{\lambda \mid \underline{x}}(\lambda \mid \underline{x}) \propto e^{-\lambda(n+1/\beta)} \lambda^{\alpha + \sum_{i=1}^{n} x_i - 1}$$

with the right hand side being the kernel of

$$\text{Gamma}\left(\alpha + \sum_{i=1}^{n} x_i, \frac{1}{n + 1/\beta}\right) \equiv \text{Gamma}\left(\alpha + \sum_{i=1}^{n} x_i, \frac{\beta}{n\beta + 1}\right)$$

distribution. Therefore, the posterior dist of λ is

$$\text{Gamma}\left(\alpha + \sum_{i=1}^{n} x_i, \frac{\beta}{n\beta + 1}\right)$$

Part (b): Using results for Gamma distribution, we have

$$E(\lambda \mid \underline{x}) = \left(\alpha + \sum_{i=1}^{n} x_i\right) \cdot \frac{\beta}{n\beta + 1}$$

$$\text{Var}(\lambda \mid \underline{x}) = \left(\alpha + \sum_{i=1}^{n} x_i\right) \cdot \frac{\beta^2}{(n\beta + 1)^2}$$

Problem 7.41

Part (a): We have

$$E_\mu\left(\sum_{i=1}^n a_i X_i\right) = \sum_{i=1}^n a_i E_\mu(X_i) \overset{\substack{X_i \text{ iid with } E(X_1)=\mu}}{=} \sum_{i=1}^n a_i \mu = \mu\left(\sum_{i=1}^n a_i\right)$$

Therefore, $\sum_{i=1}^n a_i X_i$ is unbiased for μ if

$$E_\mu\left(\sum_{i=1}^n a_i X_i\right) = \mu\left(\sum_{i=1}^n a_i\right) = \mu \quad \text{for all } \mu \Rightarrow \sum_{i=1}^n a_i = 1$$

Part (b): We have

$$\mathrm{Var}_{\sigma^2}\left(\sum_{i=1}^n a_i X_i\right) \overset{X_i\text{'s are indep}}{=} \sum_{i=1}^n a_i^2 \mathrm{Var}_{\sigma^2}(X_i) \overset{X_i \text{ iid with } \mathrm{Var}_{\sigma^2}(X_1)=\sigma^2}{=} \sum_{i=1}^n a_i^2 \sigma^2 = \sigma^2\left(\sum_{i=1}^n a_i^2\right)$$

From Cauchy-Schwarz inequality, we have

$$\left(\sum_{i=1}^n a_i^2\right)^2 \underbrace{\left(\sum_{i=1}^n 1^2\right)}_{=n} \geqslant \left(\sum_{i=1}^n a_i \cdot 1\right)^2 = \left(\sum_{i=1}^n a_i\right)^2 \overset{\text{part (a)}}{=} 1 \quad \Rightarrow \sum_{i=1}^n a_i^2 \geqslant \frac{1}{n}$$

which implies $\mathrm{Var}_{\sigma^2}\left(\sum_{i=1}^n a_i X_i\right) = \sigma^2\left(\sum_{i=1}^n a_i^2\right) \geqslant \sigma^2/n$

Equality in the above Cauchy-Schwarz inequality is attained if

$a_i = k \cdot 1$ for all i, for some constant k. This, together with $\sum_{i=1}^n a_i = 1$ from part (a) implies that $\sum_{i=1}^n k = 1 \Rightarrow k = 1/n$.

Therefore, $\mathrm{Var}_{\sigma^2}\left(\sum_{i=1}^n a_i X_i\right)$ is the minimum when equality is attained in the above Cauchy Schwarz inequality. In that case, the estimator is $\sum_{i=1}^n (1/n) X_i = \bar{X}$.

Problem 7.49

Part (a):

From the theory of order statistics, the pdf of $Y = \min\{X_1, \cdots, X_n\}$ is obtained as:

$$f_Y(y|\lambda) = n\, f_X(y|\lambda)\left[1-F_X(y|\lambda)\right]^{n-1} = n\,\frac{1}{\lambda}e^{-y/\lambda}\left(1-(1-e^{-y/\lambda})\right)^{n-1}$$

$$= \frac{n}{\lambda}e^{-ny/\lambda}\ ; \quad y>0$$

This implies that $Y \sim \text{Exponential}\left(\frac{1}{n/\lambda}\right) \equiv \text{Exponential}(\lambda/n)$

Therefore. $E_\lambda(Y) = \lambda/n \Rightarrow E_\lambda(nY) = \lambda$ for all $\lambda > 0$.

Hence, nY is an unbiased estimator of λ.

Part (b):

Since $X_1, \cdots, X_n \sim$ iid Exponential (λ), and Exponential (λ) is a member of the exponential family, therefore, $Z = \sum_{i=1}^{n} X_i$ is Complete sufficient for λ. Hence, using Rao-Blackwell Theorem and Lehmann-Scheffe theorem, it follows that $\phi(\underset{\sim}{x}) = E(Y|Z)$ is the UMVUE of λ.

Again, since $E_\lambda(Z) = E_\lambda\left(\sum_{i=1}^{n} x_i\right) = n\lambda$, therefore $\psi(\underset{\sim}{x}) = Z/n$ is an unbiased estimator for λ which is based on the complete sufficient statistic Z. Hence $\psi(\underset{\sim}{x})$ is UMVUE for λ.

By uniqueness of UMVUE, we must have $\phi(\underset{\sim}{x}) = \psi(\underset{\sim}{x})$ which implies $\phi(\underset{\sim}{x}) = E(Y|Z) = Z/n = \frac{1}{n}\sum_{i=1}^{n} x_i$.

That Z/n is the UMVUE, implies that Z/n is better than nY. (another unbiased estimator.