STA 522 Sample Exam 1 Solutions

Problem 1

$X_1, X_2, \ldots, X_n$ are iid with common cdf

$$F(x \mid \alpha, \beta) = \begin{cases} 0, & x < 0 \\ (x/\beta)^\alpha, & 0 \leq x \leq \beta \\ 1, & x > \beta \end{cases}.$$

Therefore, the common pdf of $X_1, X_2, \ldots, X_n$ is given by:

$$f(x \mid \alpha, \beta) = \frac{d}{dx} F(x \mid \alpha, \beta) = \frac{\alpha}{\beta^\alpha} x^{\alpha-1} I(0 \leq x \leq \beta)$$

Part (a): The joint pdf of $X_1, X_2, \ldots, X_n$ is:

$$\begin{aligned} f(x_1, \ldots, x_n \mid \alpha, \beta) &\overset{\text{iid}}{=} \prod_{i=1}^n f(x_i \mid \alpha, \beta) \\ &= \frac{\alpha^n}{\beta^{n\alpha}} \left(\prod_{i=1}^n x_i \right)^{\alpha-1} \prod_{i=1}^n I(0 \leq x_i \leq \beta) \\ &= \underbrace{\frac{\alpha^n}{\beta^{n\alpha}} \left(\prod_{i=1}^n x_i \right)^{\alpha-1} I(x_{(n)} \leq \beta)}_{=g(T(\underline{x}|\alpha,\beta))} \ \underbrace{I(x_{(1)} > 0)}_{=h(\underline{x})} \end{aligned}$$

where $T(\underline{x}) = (\prod_{i=1}^n x_i, x_{(n)})$. Therefore, by the Factorization theorem, $T(\underline{X}) = (\prod_{i=1}^n X_i, X_{(n)})$ is sufficient for α, β.

Part (b): The joint likelihood for α, β is:

$$L(\alpha, \beta \mid \underline{x}) = \frac{\alpha^n}{\beta^{n\alpha}} \left(\prod_{i=1}^n x_i \right)^{\alpha-1} I(x_{(n)} \leq \beta) \ I(x_{(1)} > 0)$$

For any α, the likelihood function is decreasing in β and is non-zero when $\beta \geq x_{(n)}$. Hence, $\hat{\beta} = X_{(n)}$ is the MLE of β for all α.

The profile likelihood for α is:

$$\log \tilde{L}(\alpha \mid \underline{x}) = \frac{\alpha^n}{x_{(n)}^{n\alpha}} \left(\prod_{i=1}^n x_i \right)^{\alpha-1} \implies \log \tilde{L}(\alpha \mid \underline{x}) = n \log \alpha - n\alpha \log x_{(n)} + (\alpha - 1) \sum_{i=1}^n \log x_i$$

Therefore

$$\frac{\partial}{\partial \alpha} \log \tilde{L}(\alpha \mid \underline{x}) = \frac{n}{\alpha} - n \log x_{(n)} + \sum_{i=1}^{n} \log x_i \gtreqless 0$$

$$\iff \frac{n}{\alpha} \gtreqless n \log x_{(n)} - \sum_{i=1}^{n} \log x_i = \log \left(\frac{x_{(n)}^n}{\prod_{i=1}^n x_i} \right)$$

$$\iff \alpha \lesseqgtr \frac{n}{\log \left(\frac{x_{(n)}^n}{\prod_{i=1}^n x_i} \right)}$$

Hence, the MLE of α is $\hat{\alpha} = \dfrac{n}{\log \left(\frac{X_{(n)}^n}{\prod_{i=1}^n X_i} \right)}$ and the MLE of β is $\hat{\beta} = X_{(n)}$.

Problem 2

Part (a): First find the MLE of θ. This is from Lecture 6:

The likelihood of θ is

$$L(\theta \mid \underline{x}) = \prod_{i=1}^{n} \exp(-\theta) \frac{\theta^{x_i}}{x_i!} \; = \exp(-n\theta) \; \frac{\theta^{\sum_{i=1}^n x_i}}{\prod_{i=1}^n x_i!}$$

The log likelihood is:

$$l(\theta \mid \underline{x}) = \log L(\theta \mid \underline{x}) = -n\theta + \left(\sum_{i=1}^{n} x_i \right) \log \theta - \log \left(\prod_{i=1}^{n} x_i! \right)$$

Therefore,

$$\frac{d \; \log L(\theta \mid \underline{x})}{d\theta} = -n + \left(\sum_{i=1}^{n} x_i \right) \frac{1}{\theta} \gtreqless 0 \;\; \text{according as} \;\; \theta \lesseqgtr \frac{1}{n} \sum_{i=1}^{n} x_i = \overline{x}$$

Therefore, $\hat{\theta} = \overline{x}$ is the MLE of θ. Hence, by the invariance property of the MLE $e^{-\overline{X}}$ is the MLE of $P(X_1 = 0) = e^{-\theta}$.

Part (b): No, the MLE is <u>not</u> unbiased. To see this, first note that $g(x) = -e^{-x}$ is a convex function. Therefore as suggested in the hint,

$$\begin{aligned}
&\mathrm{E}[g(\overline{X})] \overset{\text{Jensen}}{>} g(\mathrm{E}(\overline{X})) \;\; \text{(strict inequality as } g \text{ is strictly convex)} \\
&\text{i.e., } \mathrm{E}[-e^{-\overline{X}}] > -e^{-\mathrm{E}(\overline{X})} = -e^{-\theta} \\
&\iff \mathrm{E}[e^{-\overline{X}}] < e^{-\theta}
\end{aligned}$$