STA 522, Spring 2021
Introduction to Theoretical Statistics II

Lecture 3

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AGENDA

- almost sure convergence & SLLN
- convergence in distribution
- central limit theorem
- sufficiency

Review: Sufficiency

- A statistic $T(\underline{x})$ is a **sufficient statistic** for a parameter θ if the conditional distribution of the sample $\underline{X}$ given that $T(\underline{x}) = t$ does not depend on θ.

- **Checking sufficiency:** $T(\underline{x})$ is a sufficient statistic for θ if, for every $\underline{x}$ in the sample space, the ratio $\dfrac{p(\underline{x} \mid \theta)}{q(T(\underline{x}) \mid \theta)}$ is constant as a function of θ. Here $p(\underline{x} \mid \theta) =$ joint pdf/pmf of $\underline{X}$ and $q(t \mid \theta) =$ pdf/pmf of $T(\underline{x})$.

- **Example:** Let $X_1, X_2, \ldots, X_n$ be iid Bernoulli random variables with parameter θ, $0 < \theta < 1$. Then $T(\underline{x}) = \sum_{i=1}^{n} X_i$ is sufficient for θ.

Example (sufficient order statistics). Let $X_1, X_2, \ldots, X_n$ be iid from a distribution with pdf $f(x)$, where we are unable to specify any more information about the pdf (as is the case in nonparametric estimation). Then the order statistics are a sufficient statistic.

To verify this, first let θ be the vector of all parameters in the density f and define $T(\underline{x}) = \left(X_{(1)}, \ldots, X_{(n)}\right)$ where $X_{(i)}$ are the order statistics.

Then $p(\underline{x} \mid \theta) = \prod_{i=1}^{n} f(x_i \mid \theta) = \prod_{i=1}^{n} f(x_{(i)} \mid \theta)$ and

$q(T(\underline{x}) \mid \theta) = n! \prod_{i=1}^{n} f(x_{(i)} \mid \theta)$, which means $\dfrac{p(\underline{x} \mid \theta)}{q(T(\underline{x}) \mid \theta)} = \dfrac{1}{n!}$.

Notes

- this is not much of a reduction, but we shouldn't expect more with so little information about the density
- for specific specific parametric densities substantial reduction is possible.

How to find sufficient statistics?

- The previous theorem allows one to check *if* a statistic is sufficient, but doesn't say *how* to find a sufficient statistic (requires guesswork).
- The following theorem provides *one way* to find a sufficient statistic. The following form is due to Halmos and Savage (1949), but the original idea can be traced back to Neyman and Fisher (1930-39).

Theorem 6.2.6 (Factorization Theorem)

Let $f(\underline{x} \mid \theta)$ denote the joint pdf or pmf of a sample $\underline{X}$. A statistic $T(\underline{x})$ is a sufficient statistic for θ *if and only if* there exist functions $g(t \mid \theta)$ and $h(\underline{x})$ such that, for all sample points $\underline{x}$ and all parameter points θ,

$$f(\underline{x} \mid \theta) = g(T(\underline{x}) \mid \theta) \cdot h(\underline{x}).$$

Notes

1. Thus the Factorization Theorem says that to find a sufficient statistic we first factor f into $g.h$, where (i) h is free of θ, and (ii) g depends on θ and on $\underline{x}$ only through some function $T(\underline{x})$. This function is a sufficient statistic for θ.
2. $h(\underline{x})$ can be 1 in some situations.

Example: Let $X_1, X_2, \ldots, X_n$ be iid Bernoulli random variables with parameter θ, $0 < \theta < 1$. We know that $T(\underline{x}) = \sum_{i=1}^{n} X_i$ is sufficient for θ.
To obtain this through the Factorization Theorem, note that for $x_i \in \{0, 1\}$, $i = 1, \ldots, n$,

$$\begin{aligned} f(\underline{x} \mid \theta) &= \prod_{i=1}^{n} \left\{ \theta^{x_i} \, (1-\theta)^{1-x_i} \right\} \\ &= \theta^{\sum_{i=1}^{n} x_i} \, (1-\theta)^{n - \sum_{i=1}^{n} x_i} = g(T(\underline{x}) \mid \theta) \; h(\underline{x}) \end{aligned}$$

where $h(\underline{x}) = 1$, $T(\underline{x}) = \sum_{i=1}^{n} x_i$, and $g(t \mid \theta) = \theta^t (1-\theta)^{n-t}$.

Example: Let $X_1, X_2, \ldots, X_n$ be iid Weibull$\,(\gamma, \beta)$ with common pdf $f(x \mid \beta) = \dfrac{\gamma}{\beta} x^{\gamma-1} e^{-x^\gamma/\beta}$ for $x > 0$ where $\gamma > 0$ is known and $\beta > 0$ is unknown.

To find a sufficient statistic for β, write for $\underline{x}$ with $x_i > 0$, all i,

$$
\begin{aligned}
f(\underline{x} \mid \beta) &= \prod_{i=1}^{n} \left\{ \frac{\gamma}{\beta}\ x_i^{\gamma-1}\ e^{-x_i^\gamma/\beta} \right\} \\
&= \left(\frac{1}{\beta^n}\ e^{-\frac{1}{\beta}\sum_{i=1}^n x_i^\gamma} \right) \left(\prod_{i=1}^{n} \gamma\ x_i^{\gamma-1} \right) = g(T(\underline{x}) \mid \beta)\ h(\underline{x})
\end{aligned}
$$

where $h(\underline{x}) = \prod_{i=1}^{n} \gamma\ x_i^{\gamma-1}$, $T(\underline{x}) = \sum_{i=1}^{n} x_i^\gamma$, and $g(t \mid \beta) = \dfrac{1}{\beta^n}\ e^{-t/\beta}$.

Therefore, from the Factorization Theorem it follows that $T(\underline{x}) = \sum_{i=1}^{n} X_i^\gamma$ is sufficient for β.

Note

If the support of f involves θ, then we must appropriately define h and g to ensure that the product is 0 where f is 0.

Example: Let $X_1, X_2, \ldots, X_n$ be iid Discrete-Uniform $(1, \ldots, \theta)$, where θ is a positive integer. To find a sufficient statistic for θ, write

$$\begin{aligned} f(\underline{x} \mid \theta) &= \prod_{i=1}^{n} f(x_i \mid \theta) \\ &= \prod_{i=1}^{n} \left\{ \frac{1}{\theta} \; I(x_i \in \{1, \ldots, \theta\}) \right\} \\ &= \frac{1}{\theta^n} \; I\left(\max_{1 \le i \le n} x_i \le \theta \right) \\ &= g(T(\underline{x}) \mid \theta) \; h(\underline{x}) \end{aligned}$$

where $h(\underline{x}) = 1$, $T(\underline{x}) = \max_{1 \le i \le n} x_i$, and $g(t \mid \theta) = \frac{1}{\theta^n} I(t \le \theta)$.

Example: Let $X_1, X_2, \ldots, X_n$ be iid $\mathrm{N}(\mu, \sigma^2)$ random variables, where σ^2 is known. We want a sufficient statistic for μ.

The joint pdf of $\underline{X}$ is given by:

$$\begin{aligned}
f(\underline{x} \mid \mu) &= \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i - \mu)^2\right) \\
&= \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n \exp\left(-\frac{1}{2\sigma^2}\left\{\sum_{i=1}^n (x_i - \overline{x})^2 + n(\overline{x} - \mu)^2\right\}\right) \\
&= \underbrace{\exp\left(-\frac{n}{2\sigma^2}(\overline{x} - \mu)^2\right)}_{=g(T(\underline{x})|\mu)} \underbrace{\left[\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i - \overline{x})^2\right)\right]}_{=h(\underline{x})}
\end{aligned}$$

This means that $T(\underline{X}) = \overline{X}$ is sufficient for μ.

Proof of Theorem 6.2.6

We'll prove this only for discrete cases.

Only if Part: Let $T(\underline{x})$ be a sufficient statistic for θ. Choose

$$g(t \mid \theta) = P_\theta\left(T(\underline{X}) = t\right) \text{ and } h(\underline{x}) = P\left(\underline{X} = \underline{x} \mid T(\underline{X}) = T(\underline{x})\right).$$

Since $T(\underline{X})$ is sufficient, the conditional probability $P\left(\underline{X} = \underline{x} \mid T(\underline{X}) = T(\underline{x})\right)$ doesn't depend on θ (definition), and so this choice of h is legitimate.

Therefore

$$\begin{aligned}
f(\underline{x} \mid \theta) &= P_\theta(\underline{X} = \underline{x}) \\
&= P_\theta\left(\underline{X} = \underline{x} \text{ and } T(\underline{X}) = T(\underline{x})\right) \\
&= P_\theta\left(T(\underline{X}) = T(\underline{x})\right) P\left(\underline{X} = \underline{x} \mid T(\underline{X}) = T(\underline{x})\right) \\
&= g(T(\underline{x}) \mid \theta)\ h(\underline{x})
\end{aligned}$$

i.e., factorization holds.

If part: Suppose factorization holds.

Let $q(t \mid \theta)$ be the pmf of $T(\underline{X})$. Define $A_{T(\underline{x})} = \{\underline{y} : T(\underline{y}) = T(\underline{x})\}$.

Then

$$\begin{aligned}
\frac{f(\underline{x} \mid \theta)}{q(T(\underline{X}) \mid \theta)} &= \frac{g(T(\underline{x}) \mid \theta)\ h(\underline{x})}{q(T(\underline{X}) \mid \theta)} \\
&= \frac{g(T(\underline{x}) \mid \theta)\ h(\underline{x})}{\sum_{\underline{y} \in A_T} g(T(\underline{y}) \mid \theta)\ h(\underline{y})} \\
&= \frac{g(T(\underline{x}) \mid \theta)\ h(\underline{x})}{g(T(\underline{x}) \mid \theta)\ \sum_{\underline{y} \in A_T} h(\underline{y})} \\
&= \frac{h(\underline{x})}{\sum_{\underline{y} \in A_T} h(\underline{y})}
\end{aligned}$$

is free of θ.

Therefore, $T(\underline{X})$ is sufficient for θ.

Joint Sufficiency

Definition: Let $X_1, X_2, \ldots, X_n$ be a random sample from the density $f(x \mid \theta)$. The statistics $T_1(\underline{X}), \ldots, T_r(\underline{X})$ are **jointly sufficient for** θ if and only if the conditional distribution of $X_1, X_2, \ldots, X_n$ given $T_1(\underline{X}) = t_1, \ldots, T_r(\underline{X}) = t_r$ does not depend on θ.

Notes

- A set of jointly sufficient statistics may also be referred to as a **vector-valued sufficient statistic**.
- The sample itself, $X_1, X_2, \ldots, X_n$, is always jointly sufficient since the conditional distribution of the sample given the sample does not depend on θ. Also, as seen in a previous example, the order statistics are jointly sufficient as well.
- The Factorization Theorem can still be used to find jointly sufficient statistics.

Example: Let $X_1, X_2, \ldots, X_n$ be iid Uniform $\left(\theta - \frac{1}{2}, \theta + \frac{1}{2}\right)$. We want a sufficient statistic for θ.

$$\begin{aligned} f(\underline{x} \mid \theta) &= \prod_{i=1}^{n} \left\{ I\left(\theta - \frac{1}{2} < x_i < \theta + \frac{1}{2}\right) \right\} \\ &= I\left(\theta - \frac{1}{2} < x_{(1)} < x_{(n)} < \theta + \frac{1}{2}\right) \\ &= g(T_1(\underline{x}), T_2(\underline{x}) \mid \theta)\ h(\underline{x}) \end{aligned}$$

where $T_1(\underline{x}) = x_{(1)}$, $T_2(\underline{x}) = x_{(n)}$, and $h(\underline{x}) = 1$. This shows that $(T_1, T_2) = (X_{(1)}, X_{(n)})$ are jointly sufficient for θ.

Notes

- It is likely that a set of jointly sufficient statistics $(T_1(\underline{X}), \ldots, T_r(\underline{X}))$ is needed when the parameter is also a vector, say $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_s)$.
- Usually the sufficient statistic and the parameter vectors are of equal lengths ($r = s$), but different combinations of lengths are possible.

Example (Contd.): Suppose that $X_1, X_2, \ldots, X_n$ are iid $\mathrm{N}(\mu, \sigma^2)$ with both μ and σ^2 unknown. Then

$$\begin{aligned}
f(\underline{x} \mid \mu, \sigma^2) &= \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i - \mu)^2\right) \\
&= \left(\frac{1}{\sqrt{2\pi}}\right)^n \left(\frac{1}{\sigma^2}\right)^{n/2} \exp\left(-\frac{1}{2\sigma^2}\left\{\sum_{i=1}^n (x_i - \overline{x})^2 + n(\overline{x} - \mu)^2\right\}\right) \\
&= \underbrace{\left(\frac{1}{\sigma^2}\right)^{n/2} \exp\left(-\frac{1}{2\sigma^2}\left\{(n-1)s^2 + n(\overline{x} - \mu)^2\right\}\right)}_{=g(T_1(\underline{x}), T_2(\underline{x}) \mid \mu, \sigma^2)} \underbrace{\left(\frac{1}{\sqrt{2\pi}}\right)^n}_{=h(\underline{x})}
\end{aligned}$$

where $T_1(\underline{x}) = \overline{x} = \frac{1}{n}\sum_{i=1}^n x_i$ and $T_2(\underline{x}) = s^2 = \frac{1}{n-1}\sum_{i=1}^n (x_i - \overline{x})^2$. This shows that $(\overline{X}, S^2)$ are jointly sufficient for (μ, σ^2).

Example (Sufficient Statistic for Exponential Family): Let $X_1, X_2, \ldots, X_n$ be iid observations from a pdf/pmf $f(x \mid \boldsymbol{\theta})$ that belongs to an exponential family given by

$$f(x \mid \boldsymbol{\theta}) = h(x)c(\boldsymbol{\theta}) \exp\left(\sum_{i=1}^{k} w_i(\boldsymbol{\theta}) t_i(x)\right),$$

where $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_d)$ and $d \leq k$.

Then

$$T(\underline{X}) = \left(\sum_{j=1}^{n} t_1(X_j), \ldots, \sum_{j=1}^{n} t_k(X_j)\right)$$

is a sufficient statistic for $\boldsymbol{\theta}$.

Invariance Principle for Sufficient Statistics

Theorem

Suppose $T(\underline{X})$ is sufficient for a parameter θ, and let u be a one-to-one function. Then $T^*(\underline{X}) = u(T(\underline{X}))$ is also sufficient for θ.

Proof: Since u is one-to-one, u^{-1} exists as a function and $T(\underline{X}) = u^{-1}(T^*(\underline{X}))$ Then by the Factorization Theorem there exists g and h such that

$$f(\underline{x} \mid \theta) = g(T(\underline{x}) \mid \theta)\, h(\underline{x}) = g(u^{-1}(T^*(\underline{x})) \mid \theta)\, h(\underline{x}) = g^*(T^*(\underline{x}) \mid \theta)\, h(\underline{x})$$

where $g^*(t \mid \theta) = g(u^{-1}(t) \mid \theta)$. Again from the Factorization theorem this shows that $T^*(\underline{X})$ is also sufficient for θ.

Sufficient Statistics are NOT unique

- Based on the Invariance Principle, we see that sufficient statistics are not unique.
- Note that the sample itself, i.e., $T(\underline{X}) = \underline{X}$, is a sufficient statistic. So are the order statistics $T(\underline{X}) = \big(X_{(1)}, \ldots, X_{(n)}\big)$.
- Question: Is one sufficient statistic better than another?
- Recall that the purpose of sufficient statistics is to achieve data reduction without loss of information about the parameter θ.
- A statistic that achieves the most data reduction while still retaining all the information about θ might be preferable.

Minimal Sufficient Statistic

Definition: A sufficient statistic $T(\underline{X})$ is called a **minimal sufficient statistic** if, for any other sufficient statistic $T'(\underline{X})$, $T(\underline{x})$ is a function of $T'(\underline{x})$, i.e., whenever $T'(\underline{x}) = T'(\underline{y})$, then $T(\underline{x}) = T(\underline{y})$.

Among all sufficient statistics, a minimal sufficient statistic achieves the greatest possible data reduction.

Example: Suppose that $X_1, X_2, \ldots, X_n$ are iid $\mathrm{N}\left(\mu, \sigma^2\right)$, with σ known.

We have seen that $T(\underline{X}) = \overline{X}$ is sufficient for μ.

We can also show that the statistic $T'(\underline{X}) = (\overline{X}, S^2)$ is sufficient for μ.

Since both $T(\underline{X})$ and $T'(\underline{X})$ are sufficient for μ, they each contain the same information about μ. But clearly $T(\underline{X})$ achieves greater data reduction than $T'(\underline{X})$.

If σ were unknown, however, things would be different.

Checking for Minimal Sufficiency

Theorem 6.2.13

Let $f(\underline{x} \mid \theta)$ be the pdf/pmf of a sample $\underline{X}$. Suppose there exists a function $T(\underline{X})$ such that, for every $\underline{x}$ and $\underline{y}$, the ratio

$$\frac{f(\underline{x} \mid \theta)}{f(\underline{y} \mid \theta)}$$

is constant as a function of θ if and only if $T(\underline{x}) = T(\underline{y})$. Then $T(\underline{X})$ is a minimal sufficient statistic for θ.

Proof: To simplify proof assume $f(x \mid \theta) > 0$ for all $x \in \mathcal{X}$ and θ.

$T(\underline{X})$ **is sufficient:** Let $\mathcal{T} = \{t : t = T(\underline{x}) \text{ for some } \underline{x} \in \mathcal{X}\}$ be the image of $\mathcal{X}$ under $T(\underline{x})$.
Define the partition set $A_t = \{\underline{x} : T(\underline{x}) = t\}$, and choose and fix $\underline{x}_t \in A_t$ for each t.
For any $\underline{x} \in \mathcal{X}$, $\underline{x}_{T(\underline{x})}$ is the fixed element that is in the same set, A_t, as $\underline{x}$. Since both $\underline{x} \in A_t$ and $\underline{x}_{T(\underline{x})} \in A_t$, so $T(\underline{x}) = T(\underline{x}_{T(\underline{x})})$, and hence $h(\underline{x}) = f(\underline{x} \mid \theta)/f(\underline{x}_{T(\underline{x})} \mid \theta)$ is a constant function of θ.

Define $g(t \mid \theta) = f(\underline{x}_t \mid \theta)$. Then

$$f(\underline{x} \mid \theta) = \frac{f(\underline{x}_{T(\underline{x})} \mid \theta)\ f(\underline{x} \mid \theta)}{f(\underline{x}_{T(\underline{x})} \mid \theta)} = g(T(\underline{x}) \mid \theta)\ h(\underline{x})$$

which implies sufficiency of $T(\underline{X})$ by the Factorization Theorem.

$T(\underline{X})$ **is minimal:** Let $T'(X)$ be any other sufficient statistic. By the Factorization Theorem, there exist functions g' and h' such that $f(\underline{x} \mid \theta) = g'(T'(\underline{x}) \mid \theta)\ h'(\underline{x})$. Let x and y be two sample points with $T'(\underline{x}) = T'(\underline{y})$. Then

$$\frac{f(\underline{x} \mid \theta)}{f(\underline{y} \mid \theta)} = \frac{g'(T'(\underline{x}) \mid \theta)\ h'(\underline{x})}{g'(T'(\underline{y}) \mid \theta)\ h'(\underline{y})} = \frac{h'(\underline{x})}{h'(\underline{y})}$$

Since this ratio does not depend on θ from the assertion of the theorem, we have $T(\underline{x}) = T(\underline{y})$. So, $T(\underline{x})$ is a function of $T'(\underline{x})$ and $T(\underline{x})$ is a minimal sufficient statistic.

Example: Let $X_1, X_2, \ldots, X_n$ be iid $\mathrm{N}\left(\mu, \sigma^2\right)$, with both μ and σ^2 unknown. Given a sample $\underline{X}$ we saw that $(\overline{X}, S^2)$ is sufficient for (μ, σ^2). To show that $(\overline{X}, S^2)$ is minimal sufficient, consider two sample points $\underline{x}$ and $\underline{y}$ with sample mean and variances $(\overline{x}, s_x^2)$ and $(\overline{x}, s_x^2)$ respectively. Then

$$\frac{f(\underline{x} \mid \mu, \sigma^2)}{f(\underline{y} \mid \mu, \sigma^2)} = \frac{(2\pi\sigma^2)^{-n/2} \exp\left(-[n(\overline{x} - \mu)^2 + (n-1)s_x^2]/(2\sigma^2)\right)}{(2\pi\sigma^2)^{-n/2} \exp\left(-[n(\overline{y} - \mu)^2 + (n-1)s_y^2]/(2\sigma^2)\right)} = \frac{h'(\underline{x})}{h'(\underline{y})}$$
$$= \exp\left([-n(\overline{x}^2 - \overline{y}^2) + 2n\mu(\overline{x} - \overline{y}) - (n-1)(s_x^2 - s_y^2)]/(2\sigma^2)\right)$$

is constant as a function of (μ, σ^2) if and only if $\overline{x} = \overline{y}$ and $s_x^2 = s_y^2$.

Example: Let $X_1, X_2, \ldots, X_n$ be iid beta (α, β). Show that $\left(\sum_{i=1}^{n} \log X_i, \sum_{i=1}^{n} \log(1 - X_i)\right)$ is a minimal sufficient statistic for (α, β).

Verify sufficiency using Factorization Theorem:

$$
\begin{aligned}
f(\underline{x} \mid \alpha, \beta) &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)} \prod_{i=1}^{n} \left\{ x_i^{\alpha-1} (1 - x_i)^{\beta-1} \right\} \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)}\ \exp\left[(\alpha - 1) \sum_{i=1}^{n} \log x_i + (\beta - 1) \sum_{i=1}^{n} \log(1 - x_i) \right] \times 1 \\
&= g\left(\sum_{i=1}^{n} \log x_i, \sum_{i=1}^{n} \log(1 - x_i) \mid \alpha, \beta \right)\ h(\underline{x})
\end{aligned}
$$

Show minimal sufficiency:

$$\frac{f(\underline{x} \mid \alpha,\beta)}{f(\underline{y} \mid \alpha,\beta)} = \frac{\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\ \Gamma(\beta)} \exp\left[(\alpha-1)\sum_{i=1}^n \log x_i + (\beta-1)\sum_{i=1}^n \log(1-x_i)\right]}{\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\ \Gamma(\beta)} \exp\left[(\alpha-1)\sum_{i=1}^n \log y_i + (\beta-1)\sum_{i=1}^n \log(1-y_i)\right]}$$

$$= \exp\Bigg[(\alpha-1)\left(\sum_{i=1}^n \log x_i - \sum_{i=1}^n \log y_i\right) + (\beta-1)\left(\sum_{i=1}^n \log(1-x_i) - \sum_{i=1}^n \log(1-y_i)\right)\Bigg].$$

This is constant in (α,β) if and only if $\sum_{i=1}^n \log x_i = \sum_{i=1}^n \log y_i$ and $\sum_{i=1}^n \log(1-x_i) = \sum_{i=1}^n \log(1-y_i)$.

Invariance Principle for Minimal Sufficient Statistics

Theorem

Suppose $T(\mathbf{X})$ is a minimal sufficient statistic for a parameter θ, and let u be a one-to-one function. Then $T^*(\mathbf{X}) = u(T(\mathbf{X}))$ is also a minimal sufficient statistic for θ.

Example: $\left(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} X_i^2\right)$ is a minimal sufficient statistic for (μ, σ^2).

Homework

- Convergence: Read p. 235 – 240 Exercises 5.33, 5.34, 5.39b, 5.41.
- Sufficiency: Read p. 271 – 274.