STA 522, Spring 2022
Introduction to Theoretical Statistics II

Lecture 12

Department of Biostatistics
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AGENDA

- Interval Estimation - inverting hypothesis tests
- pivotal method

Review: Interval Estimation

- An **interval estimate** of a real-valued parameter θ is any pair of functions, $L(\underline{x})$ and $U(\underline{x})$, of a sample that satisfy $L(\underline{x}) \leq U(\underline{x})$ for all $\underline{x} \in \mathcal{X}$.
- For an interval estimator $[L(\underline{X}), U(\underline{X})]$ of a parameter θ, the **coverage probability** of $[L(\underline{X}), U(\underline{X})]$ is the probability that the random interval $[L(\underline{X}), U(\underline{X})]$ covers the true parameter θ.
- For an interval estimator $[L(\underline{X}), U(\underline{X})]$ of a parameter θ, the **confidence coefficient** of $[L(\underline{X}), U(\underline{X})]$ is the infimum of the coverage probabilities, $\inf_\theta P_\theta\left(\theta \in [L(\underline{X}), U(\underline{X})]\right)$. We use **confidence interval** to mean the interval estimator along with its corresponding confidence coefficient.

Review: Interval Estimation

- Methods of finding interval estimators: (a) Invert a test statistic, (b) Use pivotal quantities.
- In general, every confidence interval corresponds to a test, and vice versa. Begin with the acceptance region of a hypothesis test and invert to obtain a confidence interval.
- We considered inversion of a two-sided normal hypothesis test

Correspondence between hypothesis tests and confidence interval

Theorem 9.2.2

For each $\theta_0 \in \Theta$, let $A(\theta_0)$ be the acceptance region of a level α test of $H_0 : \theta = \theta_0$, For each $\underline{x} \in \mathcal{X}$, define a set $C(\underline{x})$ in the parameter space by

$$C(\underline{x}) = \{\theta_0 : \underline{x} \in A(\theta_0)\}$$

Then the random set $C(\underline{X})$ is a $1 - \alpha$ confidence set. Conversely, let $C(\underline{X})$ be a $1 - \alpha$ confidence set. For any $\theta_0 \in \Theta$, define

$$A(\theta_0) = \{\underline{x} : \theta_0 \in C(\underline{X})\}$$

Then $A(\theta_0)$ is the acceptance region of alevel α test of $H_0 : \theta = \theta_0$.

Proof: For the first part, since $A(\theta_0)$ is the acceptance region of a level α test,

$$P_{\theta_0}(\underline{X} \notin A(\theta_0)) \leq \alpha \implies P_{\theta_0}(\underline{X} \in A(\theta_0)) \geq 1 - \alpha$$

Since θ_0 is arbitrary write θ instead of θ_0, so that

$$P_\theta(\theta \in C(\underline{X})) = P_\theta(\underline{X} \in A(\theta)) \geq 1 - \alpha$$

which implies that $C(\underline{X})$ is a $1 - \alpha$ confidence set.

For the second part, observe that

$$P_{\theta_0}(\underline{X} \notin A(\theta_0)) = P_{\theta_0}(\theta \notin C(\underline{X})) \leq \alpha$$

since $C(\underline{X})$ is a $1 - \alpha$ confidence set. This shows that $A(\theta_0)$ is the acceptance region of a level α test.

Remark: Confidence sets vs. intervals

- Note that by inverting a test we get confidence sets, and not necessarily confidence intervals.
- In most cases, one-sided tests give one-sided intervals, two-sided tests give two-sided intervals, and strange-shaped acceptance regions give strange-shaped confidence sets.

Example: (Inverting an LRT) Suppose $X_1, X_2, \ldots, X_n \sim$ iid Exponential(λ). Construct a $1-\alpha$ confidence set for λ.

Consider the test $H_0 : \lambda = \lambda_0$ vs. $\lambda \neq \lambda_0$.

The unrestricted MLE of λ is $\hat{\lambda} = \frac{1}{n}\sum_{i=1}^{n} X_i = \overline{X}$.

The LR statistic is given by

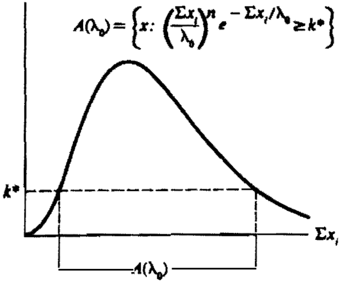
$$\begin{aligned}
\frac{L(\lambda_0 \mid \underline{x})}{\sup_{\Theta} L(\lambda \mid \underline{x})} &= \frac{\frac{1}{\lambda_0^n} \ \exp\left(-\sum_{i=1}^{n} x_i/\lambda_0\right)}{\sup_{\lambda>0} \frac{1}{\lambda^n} \ \exp\left(-\sum_{i=1}^{n} x_i/\lambda\right)} \\
&= \frac{\frac{1}{\lambda_0^n} \ \exp\left(-\sum_{i=1}^{n} x_i/\lambda_0\right)}{\frac{1}{\overline{x}^n} \ \exp\left(-\sum_{i=1}^{n} x_i/\overline{x}\right)} \\
&= \left(\frac{\sum_{i=1}^{n} x_i}{\lambda_0}\right)^n e^n \ e^{-\sum_{i=1}^{n} x_i/\lambda_0}
\end{aligned}$$

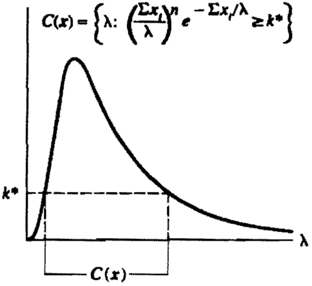
For fixed λ_0, the acceptance region of the LR test is given by:

$$A(\lambda_0) = \left\{ \underline{x} : \left(\frac{\sum_{i=1}^n x_i}{\lambda_0} \right)^n e^{-\sum_{i=1}^n x_i/\lambda_0} \geq k^* \right\}$$

where k^* is a constant chosen to satisfy $P_{\lambda_0}(\underline{X} \in A(\lambda_0)) = 1 - \alpha$.
Inverting this acceptance region gives the following $1 - \alpha$ confidence set:

$$C(\underline{x}) = \left\{ \lambda : \left(\frac{\sum_{i=1}^n x_i}{\lambda} \right)^n e^{-\sum_{i=1}^n x_i/\lambda} \geq k^* \right\}$$





Note that $C(\underline{x})$ depends on $\underline{x}$ only through $\sum_{i=1}^{n} x_i$. So the confidence set can be expressed in the form:

$$C\left(\sum_{i=1}^{n} x_i\right) = \left\{\lambda : L\left(\sum_{i=1}^{n} x_i\right) \leq \lambda \leq U\left(\sum_{i=1}^{n} x_i\right)\right\}$$

where $L = L(\sum_{i=1}^{n} x_i)$ and $U = U(\sum_{i=1}^{n} x_i)$ are functions such that $P_{\lambda_0}(\underline{X} \in A(\lambda_0)) = 1 - \alpha$ and

$$\left(\frac{\sum_{i=1}^{n} x_i}{L}\right)^n e^{-\sum_{i=1}^{n} x_i / L} = \left(\frac{\sum_{i=1}^{n} x_i}{U}\right)^n e^{-\sum_{i=1}^{n} x_i / U}$$

Call $\frac{\sum_{i=1}^{n} x_i}{L} = a$ and $\frac{\sum_{i=1}^{n} x_i}{U} = b$ with $a > b$, then the above equation becomes $a^n e^{-a} = b^n e^{-b}$. Thus a $1 - \alpha$ confidence interval becomes $\left\{\lambda : \frac{1}{a} \sum_{i=1}^{n} X_i \leq \lambda \leq \frac{1}{b} \sum_{i=1}^{n} X_i\right\}$, where a and b satisfy:

(1) $P_\lambda\left(\frac{1}{a}\sum_{i=1}^{n} X_i \leq \lambda \leq \frac{1}{b}\sum_{i=1}^{n} X_i\right) = P_\lambda\left(b \leq \frac{\sum_{i=1}^{n} X_i}{\lambda} \leq a\right) = 1 - \alpha$

(2) $a^n e^{-a} = b^n e^{-b}$

Example: (Normal one-sided confidence bound)
$X_1, X_2, \ldots, X_n \sim$ iid $\mathrm{N}(\mu, \sigma^2)$. Consider constructing a $1-\alpha$ upper confidence bound for μ, i.e., we want a confidence interval of the form $C(x) = (-\infty, U(\underline{x})]$.

To obtain such an interval we'll consider the one sided tests $H_0 : \mu = \mu_0$ vs. $H_1 : \mu < \mu_0$.

[Note $H_1 : \mu < \mu_0$ specifies "large values" of μ_0, so the confidence interval, which is obtained from inverting the acceptance region (favorable to H_0) will contain "small" values of μ_0].

The size α LRT of H_0 versus H_1 rejects H_0 if

$$\frac{\overline{X} - \mu_0}{S/\sqrt{n}} < -t_{n-1,\alpha}$$

The acceptance region of the test is

$$A(\mu_0) = \left\{ \underline{x} : \overline{x} \geq \mu_0 - t_{n-1,\alpha} \frac{s}{\sqrt{n}} \right\}$$

Inverting we get the following confidence interval:

$$\begin{aligned} C(\underline{x}) &= \{\mu_0 : \underline{x} \in A(\mu_0)\} \\ &= \left\{ \mu_0 : \mu_0 \leq \overline{x} + t_{n-1,\alpha} \frac{s}{\sqrt{n}} \right\} \\ &\equiv \left(-\infty, \overline{x} + t_{n-1,\alpha} \frac{s}{\sqrt{n}} \right] \end{aligned}$$

Pivotal Quantities

Definition: A random variable $Q(\underline{X}, \theta)$ is a **pivotal quantity** (or **pivot**) if and only if the distribution of $Q(\underline{X}, \theta)$ is independent of all parameters. That is, if $\underline{X} \sim F(\underline{x} \,|\, \theta)$, then $Q(\underline{X}, \theta)$ has the same distribution for all values of θ.

- The function $Q(\underline{X}, \theta)$ will usually explicitly contain both parameters and statistics, but for any set $\mathcal{A}$, $P_\theta(Q(\underline{X}, \theta) \in \mathcal{A})$ cannot depend on θ.
- The technique of constructing confidence sets from pivots relies on being able to find a pivot and a set $\mathcal{A}$ the set $\{\theta : Q(\underline{X}, \theta) \in \mathcal{A}\}$ is a set estimate of θ.

Examples of Pivotal Quantities

(a) If $\overline{X}$ is the mean of a random sample of size n from a normal population with mean μ and variance σ^2, then

$$Y = \overline{X} - \mu \quad \text{and} \quad Z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}}$$

are pivotal quantities.

(b) If $\overline{X}$ and S^2 are the mean and variance of a random sample of size n from a normal population with mean μ and variance σ^2, then

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1} \quad \text{and} \quad T = \frac{\overline{X} - \mu}{S/\sqrt{n}} \sim t_{n-1}$$

are pivotal quantities.

(c) If $X_1, X_2, \ldots, X_n \sim$ iid Gamma$(1, \beta) \equiv$ Exponential(β), then

$$Y = \frac{2}{\beta} \sum_{i=1}^{n} X_i \sim \chi^2_{2n} \equiv \text{Gamma}(n, 2)$$

is a pivotal quantity.

Proof: Homework. Use mgf.

(d) If $X_1, X_2, \ldots, X_n \sim$ iid uniform$(0, \theta)$ and $Y_n = X_{(n)}$, then

$$T_n = \frac{Y_n}{\theta}$$

is a pivotal quantity.

Proof: The cdf of T_n is

$$F_{T_n}(t) = P(T_n \le t) = P(Y_n \le t\theta) = \{F_X(t\theta)\}^n = t^n \; I(0 \le t \le 1)$$

(e) **Pivotal quantities for Location-Scale families.** Suppose $X_1, X_2, \dots, X_n$ is a random sample from a family of pdfs f. Then (i) If f is a location family of the form $f(x-\mu)$ then $\overline{X}-\mu$ is a pivotal quantity. (ii) If f is a scale family of the form $\frac{1}{\sigma} f\left(\frac{x}{\sigma}\right)$, then $\overline{X}/\sigma$ is a pivotal quantity. (iii) If f is a location-scale family of the form $\frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right)$ then $\frac{(\overline{X}-\mu)}{S_X}$ is a pivotal quantity.

Proof: (i), (ii) homework. For (iii) consider the standard member $f(z)$ of the family, and let $Z_1, \dots, Z_n \sim \text{iid } f(z)$ such that $X_i = \mu + \sigma Z_i$. We have

$$\frac{(\overline{X}-\mu)}{S_X} = \frac{(\mu + \sigma \overline{Z} - \mu)}{S_{\mu+\sigma Z}} = \frac{\sigma \overline{Z}}{\sigma S_Z} = \frac{\overline{Z}}{S_Z}$$

whose distribution is free of μ, σ as the common pdf $f(z)$ of Z_i is free of μ and σ.

Checking if a pivot exists

Theorem

Suppose that T is a real-valued statistic. Suppose that $Q(t, \theta)$ is a monotone function of t for each value of $\theta \in \Theta$. If the pdf $f(t \mid \theta)$ of T can be written in the form

$$f(t \mid \theta) = g(Q(t, \theta)) \left| \frac{\partial}{\partial t} Q(t, \theta) \right|$$

for some function g, then T is a pivot.

Proof: Homework. (Problem 9.10a).

The Pivotal Method of Finding Confidence Sets

Theorem

To construct a $(1-\alpha)100\%$ confidence interval for θ:

Step 1: Find a pivotal quantity Q that is a monotone function of θ.

Step 2: Find l and u such that

$$P_\theta(l < Q < u) = 1 - \alpha.$$

Note that there are an infinite number of solutions, hence we will use the equal-tails confidence interval by letting $l = 100\left(\frac{\alpha}{2}\right)$ percentile of Q and $u = 100\left(1 - \frac{\alpha}{2}\right)$ percentile of Q.

Step 3: Solve the inequality $l < Q < u$ for θ to obtain statistics $\hat{\Theta}_1$ and $\hat{\Theta}_2$ such that $P\left(\hat{\Theta}_1 < \theta < \hat{\Theta}_2\right) = 1 - \alpha$.

Example: Let $X_1, X_2, \dots, X_n \sim$ iid Exponential(θ). Use the pivotal quantity

$$Y = \frac{2}{\theta}\sum_{i=1}^{n} X_i$$

to obtain a 95% confidence interval for θ.

As discussed before we have $Y = \frac{2}{\theta}\sum_{i=1}^{n} X_i \sim \chi^2_{2n}$, so that

$$P\left(l < Y < u\right) = P\left(l < \chi^2_{2n} < u\right) = 1 - \alpha$$

There are infinitely many l and u that satisfy the above. The equal tail CI will have $l = \chi^2_{2n,1-\alpha/2}$ and $u = \chi^2_{2n,\alpha/2}$. This means

$$P_\theta\left(\chi^2_{2n,1-\alpha/2} < \frac{2}{\theta}\sum_{i=1}^{n} X_i < \chi^2_{2n,\alpha/2}\right) = 1 - \alpha$$

Hence, a $1 - \alpha$ confidence interval for θ is given by:

$$\left(\frac{2\sum_{i=1}^{n} X_i}{\chi^2_{2n,\alpha/2}}, \frac{2\sum_{i=1}^{n} X_i}{\chi^2_{2n,1-\alpha/2}}\right)$$

Example: Let $X_1, X_2, \ldots, X_n \sim$ iid Uniform$(0, \theta)$. Use the pivotal quantity

$$\frac{Y_n}{\theta}$$

to obtain a 95% confidence interval for θ.

As discussed, $T_n = \frac{Y_n}{\theta}$ has cdf $F_{T_n}(t) = t^n I(0 < t < 1)$. Therefore

$$P(l < T_n < u) = F_{T_n}(u) - F_{T_n}(l) = u^n - l^n = 1 - \alpha$$

To find the upper α point of the distribution, note that

$$P(T_n > t) = 1 - F_{T_n}(t) = 1 - t^n \stackrel{\text{set}}{=} \alpha \implies t = (1-\alpha)^{1/n}$$

Therefore, the equal tails confidence interval is obtained from:

$$P\left((1 - 1 + \alpha/2)^{1/n} < T_n < (1 - \alpha/2)^{1/n}\right) = 1 - \alpha$$

implying that a $1 - \alpha$ confidence interval is given by:

$$\left(\frac{X_{(n)}}{(1-\alpha/2)^{1/n}}, \frac{X_{(n)}}{(\alpha/2)^{1/n}}\right)$$