

STA 522/Solutions to Homework 6

Problem 7.1

Given an x , the MLE is simply the value of θ that maximizes the likelihood. The MLEs corresponding to $x = 0, 1, 2, 3, 4$ are $\hat{\theta} = 1, 1, (2 \text{ or } 3), 3, 3$ respectively.

Problem 7.2

Part (a): Since α is known, the likelihood function for β is given by:

$$L(\beta | \underline{x}) = \prod_{i=1}^n \frac{1}{\Gamma(\alpha)\beta^\alpha} x_i^{\alpha-1} e^{-x_i/\beta} = \frac{1}{\Gamma(\alpha)^n \beta^{n\alpha}} \left(\prod_{i=1}^n x_i^{\alpha-1} \right) \exp \left[-\frac{1}{\beta} \sum_{i=1}^n x_i \right]$$

The log-likelihood function is given by:

$$\log L(\beta | \underline{x}) = -n \log \Gamma(\alpha) - n\alpha \log \beta + (\alpha - 1) \sum_{i=1}^n \log x_i - \frac{1}{\beta} \sum_{i=1}^n x_i$$

To maximize $\log L(\beta | \underline{x})$ we consider the first derivative test:

$$\frac{\partial \log L(\beta | \underline{x})}{\partial \beta} = -\frac{n\alpha}{\beta} + \frac{1}{\beta^2} \sum_{i=1}^n x_i \geq 0 \iff \frac{1}{\beta^2} \sum_{i=1}^n x_i \geq \frac{n\alpha}{\beta} \iff \beta \leq \frac{1}{n\alpha} \sum_{i=1}^n x_i = \frac{1}{\alpha} \bar{x}$$

This shows that $\hat{\beta} = \frac{1}{\alpha} \bar{X}$ is maximum likelihood estimator for α .

Part (b): We shall consider successive optimization. The log-likelihood function for (α, β) is given by (same as in part (a); only α is also unknown here)

$$\log L(\alpha, \beta | \underline{x}) = -n \log \Gamma(\alpha) - n\alpha \log \beta + (\alpha - 1) \sum_{i=1}^n \log x_i - \frac{1}{\beta} \sum_{i=1}^n x_i$$

From part (a), for any α , the log-likelihood is maximized when $\beta = \hat{\beta} = \bar{x}/\alpha$. Plugging $\hat{\beta}$ into $\log L(\alpha, \beta | \underline{x})$ we get the following profile log-likelihood for α :

$$\log \tilde{L}(\alpha | \underline{x}) = \log L(\alpha, \hat{\beta} | \underline{x}) = -n \log \Gamma(\alpha) - n\alpha \log(\bar{x}/\alpha) + (\alpha - 1) \sum_{i=1}^n \log x_i - n\alpha$$

The MLE $\hat{\alpha}$ of α is obtained by numerically maximizing the above log-likelihood. The corresponding MLE of β is $\hat{\beta} = \bar{x}/\hat{\alpha}$.

Problem 7.3

Given the data \underline{x} , let $\hat{\theta}$ be the MLE of $\theta \in \Theta$, where Θ denotes the parameter space. Then

$$L(\hat{\theta} | \underline{x}) \geq L(\theta^* | \underline{x}) \text{ for all } \theta^* \in \Theta \quad (\hat{\theta} \text{ is MLE})$$

$$\iff \log L(\hat{\theta} | \underline{x}) \geq \log L(\theta^* | \underline{x}) \text{ for all } \theta^* \in \Theta \quad (\log \text{ is an increasing function})$$

This completes the proof.

Problem 7.7

First find the likelihood function of θ . Here $\theta \in \Theta = \{0, 1\}$, with

$$L(\theta = 0 \mid \underline{x}) = \prod_{i=1}^n I(0 < x_i < 1) = I(0 < x_{(1)} < x_{(n)} < 1)$$

and

$$L(\theta = 1 \mid \underline{x}) = \prod_{i=1}^n \frac{1}{2\sqrt{x_i}} I(0 < x_i < 1) = \frac{1}{2^n \prod_{i=1}^n \sqrt{x_i}} I(0 < x_{(1)} < x_{(n)} < 1)$$

Therefore

$$\frac{L(\theta = 1 \mid \underline{x})}{L(\theta = 0 \mid \underline{x})} = \frac{1}{2^n \prod_{i=1}^n \sqrt{x_i}} \geq 1 \iff 1 \geq 2^n \prod_{i=1}^n \sqrt{x_i}$$

Thus, the MLE of θ is

$$\hat{\theta} = \begin{cases} 1 & \text{if } 2^n \prod_{i=1}^n \sqrt{x_i} < 1 \\ 0 & \text{if } 2^n \prod_{i=1}^n \sqrt{x_i} > 1 \\ 0 \text{ or } 1 & \text{if } 2^n \prod_{i=1}^n \sqrt{x_i} = 1 \end{cases}$$

Problem 7.9

Here $X_1, X_2, \dots, X_n \sim \text{iid Uniform}(\theta)$, $\theta > 0$. We have seen in class (lecture 5 & 6) that the method of moments and the method of maximum likelihood estimators of θ are $\hat{\theta}_{MM} = 2\bar{X}$ and $\hat{\theta} = X_{(n)}$ respectively. We shall compare the two estimators using their mean squared errors.

For $\hat{\theta}_{MM}$ we have

$$E_{\theta}(\hat{\theta}_{MM}) = 2E_{\theta}(\bar{X}) \stackrel{\text{iid}}{=} 2E(X_1) = 2\frac{\theta}{2} = \theta \text{ for all } \theta$$

i.e., $\hat{\theta}_{MM}$ is unbiased for θ . Hence,

$$\text{MSE}_{\theta}(\hat{\theta}_{MM}) = \text{Var}_{\theta}(\hat{\theta}_{MM}) = 4 \text{Var}_{\theta}(\bar{X}) \stackrel{\text{iid}}{=} \frac{4}{n} \text{Var}_{\theta}(X_1) = \frac{4}{n} \cdot \frac{\theta^2}{12} = \frac{\theta^2}{3n}.$$

For $\hat{\theta}$, we have (from lecture 7)

$$E_{\theta}(\hat{\theta}) = \frac{n}{n+1}\theta \text{ and } \text{Var}_{\theta}(\hat{\theta}) = \left(\frac{n}{n+1}\right)^2 \text{Var}_{\theta}\left(\frac{n+1}{n} X_{(n)}\right) = \frac{n}{(n+1)^2(n+2)}\theta^2$$

Therefore,

$$\begin{aligned} \text{MSE}_{\theta}(\hat{\theta}) &= \left(\text{Bias}_{\theta}(\hat{\theta})\right)^2 + \text{Var}_{\theta}(\hat{\theta}) \\ &= \left(\frac{n}{n+1}\theta - \theta\right)^2 + \frac{n}{(n+1)^2(n+2)}\theta^2 \\ &= \frac{1}{(n+1)^2}\theta^2 + \frac{n}{(n+1)^2(n+2)}\theta^2 \\ &= \frac{2n+2}{(n+1)^2(n+2)}\theta^2 = \frac{2\theta^2}{(n+1)(n+2)} \end{aligned}$$

Thus,

$$\begin{aligned} \text{MSE}_{\theta}(\hat{\theta}_{MM}) - \text{MSE}_{\theta}(\hat{\theta}) &= \frac{\theta^2}{3n} - \frac{2\theta^2}{(n+1)(n+2)} \\ &= \frac{n^2 - 3n + 2}{3n(n+1)(n+2)} \theta^2 = \frac{(n-1)(n-2)}{3n(n+1)(n+2)} \theta^2 \end{aligned}$$

Hence $\text{MSE}_{\theta}(\hat{\theta}_{MM}) = \text{MSE}_{\theta}(\hat{\theta})$ for all θ when $n = 1, 2$ and $\text{MSE}_{\theta}(\hat{\theta}_{MM}) > \text{MSE}_{\theta}(\hat{\theta})$ for all θ for $n \geq 3$. Hence, in terms of having a smaller MSE, $\hat{\theta}$ is preferred.