STA 522 Exam 1 Solutions

Problem 1

Part (a): Since $X_1, X_2, \ldots, X_n$ are iid Uniform(0, 1), the cdf of each X_i is given by:

$$F(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x & \text{if } 0 < x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$

Therefore,

$$\begin{aligned} P\left(X_{(1)} < 0.25\right) &= 1 - \left(X_{(1)} \geq 0.25\right) \\ &= 1 - P\left(X_i \geq 0.25 \text{ for all } i\right) \\ &= 1 - \{1 - F(0.25)\}^n && \text{(iid)} \\ &= 1 - (1 - 0.25)^n = \boxed{1 - (0.75)^n} \end{aligned}$$

and

$$\begin{aligned} P\left(X_{(n)} < 0.25\right) &= P\left(X_i < 0.25 \text{ for all } i\right) \\ &= \{F(0.25)\}^n && \text{(iid)} \\ &= \boxed{(0.25)^n} \end{aligned}$$

Because $X_{(1)} \geq X_{(n)}$, therefore $X_{(n)} < 0.25$ *implies* $X_{(1)} < 0.25$, so that $P(X_{(n)} < 0.25) \leq P(X_{(1)} < 0.25)$.

Part (b): Yes, it does. We'll first show that $X_{(n)} \xrightarrow{P} 1$. This is similar to the solution for Problem 1(b) in the sample exam, with the difference being that here we have a Uniform$(0, 1)$ population instead of a Uniform$(-1, 1)$ population.

Fix $\varepsilon > 0$ small. We have

$$\begin{aligned} P(|X_{(n)} - 1| \geq \varepsilon) &= P(X_{(n)} - 1 \geq \varepsilon) + P(X_{(n)} - 1 < -\varepsilon) \\ &= P(X_{(n)} \geq 1 + \varepsilon) + P(X_{(n)} < 1 - \varepsilon) \\ &= 0 + P(X_i < 1 - \varepsilon, \text{ all } i) \\ &= \{P(X_1 < 1 - \varepsilon)\}^n && \text{(iid)} \\ &= \begin{cases} (1-\varepsilon)^n & \text{if } \varepsilon < 1 \\ 0 & \text{if } \varepsilon \geq 1 \end{cases} \\ &\to 0 \qquad \text{as } n \to \infty \end{aligned}$$

which means $X_{(n)} \xrightarrow{P} 1$.

Now apply the continuous mapping result: if $X_n \xrightarrow{P} X$ then $h(X_n) \xrightarrow{P} X$ for any continuous funciton h. Because here $X_{(n)} \xrightarrow{P} 1$ and $h(x) = x/2$ is a continuous function, therefore, $X_{(n)}/2 \xrightarrow{P} 1/2$.

Problem 2

Fix $\varepsilon > 0$. Then

$$P(|X_n - 0| \geq \varepsilon) = P\left(X_n^2 \geq \varepsilon^2\right) \leq P\left(\frac{X_n^2}{1+X_n^2} \geq \frac{\varepsilon^2}{1+\varepsilon^2}\right) \overset{(*)}{\leq} \frac{\mathrm{E}\left[\frac{X_n^2}{1+X_n^2}\right]}{\frac{\varepsilon^2}{1+\varepsilon^2}} \to 0 \quad \text{as } n \to \infty$$

where $(\star)$ is due to Chebyshev's inequality $\left(\frac{X_n^2}{1+X_n^2} \text{ is non-negative}\right)$. This implies that $X_n \xrightarrow{P} 0$.

Problem 3

Part (a): Sufficiency: The pmf of X is

$$f(x \mid \theta) = \left(\frac{\theta}{2}\right)^{|x|} (1-\theta)^{1-|x|} = \underbrace{\theta^{|x|}(1-\theta)^{1-|x|}}_{=g(T(x)|\theta)} \underbrace{2^{-|x|}}_{=h(x)}$$

where $T(x) = |x|$. Therefore, by the Factorization theorem, $|X|$ is sufficient for θ.

Part (b): Completeness: As suggested in the hint, we first find the pmf of $Y = |X|$. We note that the support of Y is $\{0, 1\}$. Clearly, $P(Y=0) = P(X=0) = \left(\frac{\theta}{2}\right)^0 (1-\theta)^{1-0} = 1-\theta$, and

$$\begin{aligned} P(Y=1) &= P(X=1) + P(X=-1) \\ &= \left(\frac{\theta}{2}\right)^1 (1-\theta)^{1-1} + \left(\frac{\theta}{2}\right)^1 (1-\theta)^{1-1} = \theta \end{aligned}$$

Thus,

$$P(Y=y) = \begin{cases} \theta & y=1 \\ 1-\theta & y=0 \end{cases}$$

for $0 < \theta < 1$, which means $Y \sim \text{Bernoulli}(\theta)$ for $0 < \theta < 1$. Therefore, by the completeness of Binomial family (proved in class) it follows that $Y = |X|$ is complete.

Problem 4

Let $\underline{x} = (x_1, \ldots, x_n)$ and $\underline{y} = (y_1, \ldots, y_n)$ be two sample points from the density $f(x \mid \theta)$.

Part (a): Sufficiency: We'll use the Factorization theorem on the joint density:

$$\begin{aligned} f(\underline{x} \mid \lambda) &= \prod_{i=1}^n \frac{1}{\lambda} \exp\left[-\frac{1}{\lambda}(x_i - \mu)\right] I(x_i > \mu) \\ &= \exp\left[-\frac{1}{\lambda}\sum_{i=1}^n x_i\right] I(x_{(1)} > \mu) \\ &= g(T_1(\underline{x}), T_2(\underline{x}) \mid \lambda, \mu)\ h(\underline{x}) \end{aligned}$$

where $T_1(\underline{x}) = \sum_{i=1}^n x_i$, $T_2(\underline{x}) = x_{(1)}$, $g(t_1, t_2 \mid \lambda, \mu) = \exp(-t_1/\lambda)\ I(t_2 > \mu)$. Therefore, by the Factorization theorem, $(\sum_{i=1}^n X_i, X_{(1)})$ is jointly sufficient for (λ, μ).

Part (b): Minimal Sufficiency: We have

$$\begin{aligned}\frac{f(\underline{x} \mid \mu, \lambda)}{f(\underline{y} \mid \mu, \lambda)} &= \frac{\prod_{i=1}^{n} \frac{1}{\lambda} \exp\left[-\frac{1}{\lambda}(x_i - \mu)\right] I(x_i > \mu)}{\prod_{i=1}^{n} \frac{1}{\lambda} \exp\left[-\frac{1}{\lambda}(y_i - \mu)\right] I(y_i > \mu)} \\ &= \exp\left[-\frac{1}{\lambda}\left(\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} y_i\right)\right] \frac{\prod_{i=1}^{n} I(x_i > \mu)}{\prod_{i=1}^{n} I(y_i > \mu)} \\ &= \exp\left[-\frac{1}{\lambda}\left(\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} y_i\right)\right] \frac{I(x_{(1)} > \mu)}{I(y_{(1)} > \mu)}\end{aligned}$$

This is constant as a function of (μ, λ) if and only if $x_{(1)} = y_{(1)}$ and $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$. Therefore, $(X_{(1)}, \sum_{i=1}^{n} X_i)$ is minimal sufficient for (μ, λ).

Problem 5

Part (a): Let us denote by $f(x - \mu)$ the common location family density of $X_1, X_2, \ldots, X_n$. Then there exist iid observations $Z_1, \ldots, Z_n$ from the density $f(x)$ (the standard density of the family) such that $Z_i = X_i - \mu$, i.e., $X_i = Z_i + \mu$.

Note that the sample median is:

$$\begin{aligned}M(X_1, X_2, \ldots, X_n) &= \begin{cases} X_{\left(\frac{n+1}{2}\right)} & n \text{ is odd} \\ \frac{X_{\left(\frac{n}{2}\right)} + X_{\left(\frac{n}{2}+1\right)}}{2} & n \text{ is even} \end{cases} \\ &= \begin{cases} \mu + Z_{\left(\frac{n+1}{2}\right)} & n \text{ is odd} \\ \mu + \frac{Z_{\left(\frac{n}{2}\right)} + Z_{\left(\frac{n}{2}+1\right)}}{2} & n \text{ is even} \end{cases} \\ &= \mu + M(Z_1, \ldots, Z_n)\end{aligned}$$

Again,

$$\overline{X} = \frac{1}{n} \sum_{i=1}^{n} X_i = \frac{1}{n} \sum_{i=1}^{n} (\mu + Z_i) = \mu + \overline{Z}$$

Hence

$$M(X_1, X_2, \ldots, X_n) - \overline{X} = M(Z_1, \ldots, Z_n) - \overline{Z}$$

where the RHS contains random variables whose distribution does not depend on the parameter μ. Hence $M - \overline{X}$ is an ancillary statistic.

Part (b): As suggested in the hint consider a sequence Y_n where $Y_n = X$, and $W = 1 - X$, where $X \sim \text{Binomial}(1, 0.5)$. Then X and $W = 1 - X$ have the same distribution. Then $Y_n \xrightarrow{d} X$ (trivially; all have the same distribution) which means $Y_n \xrightarrow{d} W$ as X and W have the same distribution.

However, for any $0 < \varepsilon < 1$,

$$\begin{aligned}P(|Y_n - W| \geq \varepsilon) = P(|X - 1 + X| \geq \varepsilon) = P(|2X - 1| \geq \varepsilon) &= P(2X \geq 1 + \varepsilon) + P(2X \leq 1 - \varepsilon) \\ &= P\left(X \geq \frac{1+\varepsilon}{2}\right) + P\left(X \leq \frac{1-\varepsilon}{2}\right) \\ &= P(X = 1) + P(X = 0) = 1 \not\to 0\end{aligned}$$

as $n \to \infty$. Hence $Y_n \stackrel{P}{\not\to} W$.