STA 522/Solutions to Homework 1

Problem 5.22

From a theorem discussed in class it follows that the pdf of $Z = \min\{X, Y\}$ is given by

$$f_Z(z) = \frac{2!}{(2-1)!}\ \phi(z)\ \left[1 - \Phi(z)\right]^{2-1} = 2\ \phi(z)\ \left[1 - \Phi(z)\right], \quad -\infty < z < \infty$$

where $\phi(z)$ and $\Phi(z)$ denote the pdf and cdf of the standard normal distribution respectively. Note that for any $-\infty < x < \infty$,

$$\Phi(x) = P(X \le x) = P(-X \ge -x) = 1 - P(-X < -x) = 1 - \Phi(-x)$$

where the last equality follows from the fact that $X \sim \mathrm{N}(0,1) \implies -X \sim \mathrm{N}(0,1)$.

Now consider the transformation $U = g(Z) = Z^2$. The support of U is $(0, \infty)$. The inverse transformations are $h_1(u) = \sqrt{u}$ on $(0, \infty)$, and $h_2(u) = -\sqrt{u}$ on $(-\infty, 0)$ with derivatives (Jacobian of transformation) being $\frac{1}{2\sqrt{u}}$ and $-\frac{1}{2\sqrt{u}}$ respectively.

Therefore the pdf of $U = Z^2$ is given by:

$$\begin{aligned}
f_U(u) &= 2\left\{\phi(\sqrt{u})\left[1 - \Phi(\sqrt{u})\right]\ \left|\frac{1}{2\sqrt{u}}\right| + \phi(-\sqrt{u})\left[1 - \Phi(-\sqrt{u})\right]\ \left|-\frac{1}{2\sqrt{u}}\right|\right\} \\
&\overset{(*)}{=} 2\left\{\phi(\sqrt{u})\left[1 - \Phi(\sqrt{u})\right]\ \frac{1}{2\sqrt{u}} + \phi(\sqrt{u})\left[\Phi(\sqrt{u})\right]\ \frac{1}{2\sqrt{u}}\right\} \\
&= 2\ \phi(\sqrt{u})\ \frac{1}{2\sqrt{u}}\ \left[1 - \Phi(\sqrt{u}) + \Phi(\sqrt{u})\right] \\
&= \frac{1}{\sqrt{2\pi}}\ e^{-u/2}\ u^{1/2-1}; \quad 0 < u < \infty
\end{aligned}$$

where $(*)$ follows from the fact that $\phi(\cdot)$ is an even function and that $\Phi(\sqrt{u}) + \Phi(-\sqrt{u}) = 1$ (proved above). The functional form of $f_U(u)$ suggests that $U \sim \chi^2_1$.

Problem 5.26

Part (a): By assumption X_i's are continuous, and $-\infty < u < v < \infty$. Consider the partition $A_1 = (-\infty, u]$, $A_2 = (u, v]$ and $A_3 = (v, \infty)$. Each X_i can lie on exactly one of A_1, A_2 and A_3 with

$$\begin{aligned}
P_X(A_1) &:= P(X_i \in A_1) = P(-\infty < X_i \le u) = F_X(u) \\
P_X(A_1) &:= P(X_i \in A_2) = P(u < X_i \le v) = F_X(v) - F_X(u) \\
P_X(A_3) &:= P(X_i \in A_3) = P(v < X_i < \infty) = 1 - F_X(v)
\end{aligned}$$

Since A_1, A_2 and A_3 forms a partition of the real line, $P_X(A_1) + P_X(A_2) + P_X(A_3) = 1$.

Now associate with each X_i a multinomial trial with 3 possible outcomes: outcome-1, outcome-2 and outcome-3, with outcome-j occurring if $X_i \in A_j$, with probability $P(X_i \in A_i)$; $j = 1, 2, 3$. Then $X_1, X_2, \ldots, X_n$ collectively produce a sequence of n independent multinomial trials, with U, V, and $n - U - V$ measuring the counts/numbers of trials resulting in outcome 1, 2 and 3 respectively. Consequently,

$$\begin{aligned}
(U, V, n - U - V) &\sim \text{Multinomial}(n; P_X(A_1), P_X(A_2), P_X(A_3)) \\
&\equiv \text{Multinomial}(n; F_X(u), F_X(v) - F_X(u), 1 - F_X(v))
\end{aligned}$$

Part (b): Note on the outset that $U = \sum_{k=1}^n I(X_k \leq u)$ (= number of k's such that $X_k \leq u$) and $V = \sum_{k=1}^n I(u < X_k \leq v)$. Hence,

$$U + V = \sum_{k=1}^n [I(X_k \leq u) + I(u < X_k \leq v)] = \sum_{k=1}^n I(X_k \leq v)$$

i.e., $U + V$ is the number of k's such that $X_k \leq v$. Consequently,

$$\begin{aligned}\{X_{(i)} \leq u, X_{(j)} \leq v\} &= \{\text{out of the } n\ X_k'\text{s at least } i \text{ are} \leq u,\ \underline{\text{and}} \text{ at least } j \text{ are} \leq v\} \\ &= \{\text{the no. of } k's \text{ such that } \{X_k \leq u\} \text{ is } \geq i, \\ &\qquad \underline{\text{and}} \text{ the no. of } k's \text{ such that } \{X_k \leq v\} \text{ is } \geq j\} \\ &= \{U \geq i,\ U + V \geq j\}\end{aligned}$$

This implies

$$\begin{aligned}P\left(X_{(i)} \leq u, X_{(j)} \leq v\right) &= P\left(U \geq i,\ U+V \geq j\right) \\ &\overset{(*)}{=} P\left(U \geq i,\ U+V \geq j,\ U < j\right) + P\left(U \geq i,\ U+V \geq j,\ U \geq j\right) \\ &\overset{(**)}{=} P(i \leq U < j,\ U+V \geq j) + P(U \geq j) \\ &= \sum_{k=i}^{j-1} \sum_{m=j-k}^{n-k} P(U = k, V = m) + P(U \geq j) \\ &= \sum_{k=i}^{j-1} \sum_{m=j-k}^{n-k} \frac{n!}{k!\ m!\ (n-k-m)!}\ [F_X(u)]^k\ [F_X(v) - F_X(u)]^m [1 - F_X(v)]^{n-k-m} \\ &\qquad + P(U \geq j)\end{aligned}$$

where $(*)$ is due to the theorem of total probability and $(**)$ follows from the fact that $\{U \geq j\} \subseteq \{U \geq i\}$ and $\{U \geq j\} \subseteq \{U + V \geq j\}$ which implies $\{U \geq i\} \cap \{U + V \geq j\} \cap \{U \geq j\} = \{U \geq j\}$.

Problem 5.32

Part (a.1): Fix $\varepsilon > 0$.

$$\begin{aligned}P(\left|Y_n - \sqrt{a}\right| > \varepsilon) &= P(\left|\sqrt{X_n} - \sqrt{a}\right| > \varepsilon) \\ &= P(\left|\sqrt{X_n} - \sqrt{a}\right| \left|\sqrt{X_n} + \sqrt{a}\right| > \varepsilon \left|\sqrt{X_n} + \sqrt{a}\right|) \\ &= P(|X_n - a| > \varepsilon \left|\sqrt{X_n} + \sqrt{a}\right|) \\ &\overset{(*)}{=} P(|X_n - a| > \varepsilon \left|\sqrt{X_n} + \sqrt{a}\right|, X_n > 0) \\ &\leq P(|X_n - a| > \varepsilon\ \sqrt{a}) \to 0\end{aligned}$$

as $n \to \infty$, since $X_n \xrightarrow{P} a$, where $(*)$ follows from the fact that for any set A, $P(A) = P(A \cap B)$ if $P(B) = 1$. This means $Y_n = \sqrt{X_n} \xrightarrow{P} \sqrt{a}$.

Part (a.2): Fix $0 < \varepsilon < 1$.

$$\begin{aligned}
P\left(\left|\frac{a}{X_n} - 1\right| < \varepsilon\right) &= P\left(1 - \varepsilon < \frac{a}{X_n} < 1 + \varepsilon\right) \\
&= P\left(\frac{a}{1+\varepsilon} < X_n < \frac{a}{1-\varepsilon}\right) \\
&= P\left(\frac{a}{1+\varepsilon} - a < X_n - a < \frac{a}{1-\varepsilon} - a\right) \\
&= P\left(-\frac{a\varepsilon}{1+\varepsilon} < X_n - a < \frac{a\varepsilon}{1-\varepsilon}\right) \\
&\geq P\left(-\frac{a\varepsilon}{1+\varepsilon} < X_n - a < \frac{a\varepsilon}{1+\varepsilon}\right) \qquad \left(\frac{a\varepsilon}{1+\varepsilon} < \frac{a\varepsilon}{1-\varepsilon}\right) \\
&= P\left(|X_n - a| < \frac{a\varepsilon}{1+\varepsilon}\right) \to 1
\end{aligned}$$

as $n \to \infty$ since $X_n \xrightarrow{P} a$. This means $Y_n' = \frac{a}{X_n} \xrightarrow{P} 1$.

Part (b): Given $S_n^2 \xrightarrow{P} \sigma^2$. From part a.1 we have $S_n = \sqrt{S_n^2} \xrightarrow{P} \sqrt{\sigma^2} = \sigma$, which together with part a.2 implies $\sigma/S_n \xrightarrow{P} 1$.

Problem 5.38

Part (a): First note that $X_1, X_2, \ldots, X_n$ being iid implies

$$M_{S_n}(t) = \mathrm{E}(e^{tS_n}) = [\mathrm{E}(e^{tX_1})]^n = [M_X(t)]^n.$$

Now for $0 < t < h$

$$P(S_n > a) = P(S_n t > at) = P\left(e^{S_n t} > e^{at}\right) \overset{(*)}{\leq} \frac{\mathrm{E}\left(e^{S_n t}\right)}{e^{at}} = e^{-at}\ M_{S_n}(t) = e^{-at}\ [M_X(t)]^n$$

and for $-h < t < 0$

$$P(S_n \leq a) = P(S_n t \geq at) = P\left(e^{S_n t} \geq e^{at}\right) \overset{(**)}{\leq} \frac{\mathrm{E}\left(e^{S_n t}\right)}{e^{at}} = e^{-at}\ M_{S_n}(t) = e^{-at}\ [M_X(t)]^n$$

where both $(*)$ and $(**)$ are both due to the Chebyshev inequality ($e^{S_n t}$ is a non-negative random variable regardless of whether $t < 0$ or $t > 0$).

Part (c): As suggested, for $Y_i = X_i - \mu - \varepsilon$ with $\varepsilon > 0$ we have $E(Y_i) = \mu - \mu - \varepsilon = -\varepsilon < 0$. Therefore from part (b) we have for some $0 < c_1 < 1$

$$c_1^n \geq P\left(\sum_{i=1}^n Y_i > 0\right) = P\left(n\overline{X}_n - n\mu - n\varepsilon > 0\right) = P\left(\overline{X}_n - \mu > \varepsilon\right)$$

Part (d): For $Y_i = -X_i + \mu - \varepsilon$, $E(Y_i) = -\varepsilon < 0$. Therefore, from part (b) we have for some $0 < c_2 < 1$

$$c_2^n \geq P\left(\sum_{i=1}^n Y_i > 0\right) = P\left(-n\overline{X}_n + n\mu - n\varepsilon > 0\right) = P\left(\overline{X}_n - \mu < -\varepsilon\right)$$

Therefore from part (c)

$$P\left(\left|\overline{X}_n - \mu\right| > \varepsilon\right) = P\left(\overline{X}_n - \mu > \varepsilon\right) + P\left(\overline{X}_n - \mu < -\varepsilon\right) \leq c_1^n + c_2^n \leq c^n + c^n = 2c^n$$

where $c = \max\{c_1, c_2\}$ so that $0 < c < 1$.

Problem 5.39 (part a)

Let X and $X_1, X_2, \dots$ be random variables with $X_n \xrightarrow{P} X$. Fix $\varepsilon > 0$. As suggested in the hint, due to continuity of h we can find a $\delta = \delta(\varepsilon) > 0$ such that

$$|x_n - x| < \delta \implies |h(x_n) - h(x)| < \varepsilon$$

This means

$$P(|h(X_n) - h(X)| < \varepsilon) \geq P(|X_n - X| < \delta) \to 1$$

as $n \to \infty$ (the limit is due to the in probability convergence of X_n). This means that $h(X_n) \xrightarrow{P} h(X)$.