STA 522 Exam 1 Solutions

Problem 1

Part (a): Since $X_1, X_2, \dots, X_n$ are iid, the cdf of each X_i is given by:

$$F(x) = \begin{cases} 0 & \text{if } x \le -1 \\ \frac{x+1}{2} & \text{if } -1 < x < 1 \\ 1 & \text{if } x \ge 1 \end{cases}$$

Therefore,

$$\begin{aligned} P\left(X_{(1)} > 0.25 \text{ and } X_{(n)} \le 0.8\right) &= P\left(X_i > 0.25 \text{ for all } i \text{ and } X_i \le 0.8 \text{ for all } i\right) \\ &= P(0.25 < X_i \le 0.8 \text{ for all } i) \\ &= \{P(0.25 < X_1 \le 0.8)\}^n && \text{(iid)} \\ &= \{F(0.8) - F(0.25)\}^n \\ &= \left\{\frac{0.8+1}{2} - \frac{0.25+1}{2}\right\}^n = (0.55/2)^n = \boxed{(0.275)^n}. \end{aligned}$$

Part (b): Yes, it does. Fix $\varepsilon > 0$. We have

$$\begin{aligned} P(|X_{(n)} - 1| \ge \varepsilon) &= P(X_{(n)} - 1 \ge \varepsilon) + P(X_{(n)} - 1 < -\varepsilon) \\ &= P(X_{(n)} \ge 1 + \varepsilon) + P(X_{(n)} < 1 - \varepsilon) \\ &= 0 + P(X_i < 1 - \varepsilon, \text{ all } i) \\ &= \{P(X_1 < 1 - \varepsilon)\}^n && \text{(iid)} \\ &= \begin{cases} \left(\frac{1-\varepsilon+1}{2}\right)^n = \left(1 - \frac{\varepsilon}{2}\right)^n & \text{if } \varepsilon < 2 \\ 0 & \text{if } \varepsilon \ge 2 \end{cases} \\ &\to 0 \qquad \text{as } n \to \infty \end{aligned}$$

which means $X_{(n)} \xrightarrow{P} 1$.

Problem 2

Part (a): Fix $\varepsilon > 0$. Then

$$P(|X_n - 0| \ge \varepsilon) = P\left(X_n^2 \ge \varepsilon^2\right) \le P\left(\frac{X_n^2}{1+X_n^2} \ge \frac{\varepsilon^2}{1+\varepsilon^2}\right) \overset{(*)}{\le} \frac{\mathrm{E}\left[\frac{X_n^2}{1+X_n^2}\right]}{\frac{\varepsilon^2}{1+\varepsilon^2}} \to 0 \quad \text{as } n \to \infty$$

where $(\star)$ is due to Chebyshev's inequality $\left(\frac{X_n^2}{1+X_n^2} \text{ is non-negative}\right)$. This implies that $X_n \xrightarrow{P} 0$.

Part (b): Fix $\varepsilon > 0$. Consider an arbitrary $\delta > 0$. As suggested in the hint, there exists $k = k_\delta > 0$ such that $P(|Y| > k) < \delta$. Therefore,

$$\begin{aligned}
P(|X_n Y - XY| \geq \varepsilon) &= P(|X_n - X|\ |Y| \geq \varepsilon) \\
&= P(|X_n - X|\ |Y| \geq \varepsilon, |Y| > k) + P(|X_n - X|\ |Y| \geq \varepsilon, |Y| \leq k) \\
&\leq P(|Y| > k) + P(|X_n - X|\ k \geq \varepsilon) \\
&< \delta + P(|X_n - X| \geq \varepsilon/k) && \text{(since } P(|Y| > k) < \delta) \\
&\xrightarrow{n \to \infty} \delta + 0 = \delta && (X_n \xrightarrow{P} X) \\
&\xrightarrow{\delta \to 0} 0. && (\delta \text{ is arbitrary})
\end{aligned}$$

This proves that $X_n Y \xrightarrow{P} XY$.

Problem 3

Part (a): Sufficiency: There is only one random observation X which constitutes the entire sample. Therefore, X is trivially sufficient.

Completeness: consider

$$\begin{aligned}
\mathrm{E}_\theta(X) &= (-1)\ P_\theta(X = -1) + (0)\ P_\theta(X = 0) + (1)\ P_\theta(X = 1) \\
&= (-1)\left(\frac{\theta}{2}\right)^1 (1-\theta)^{1-1} + (1)\left(\frac{\theta}{2}\right)^1 (1-\theta)^{1-1} = 0 \quad \text{for all } \theta
\end{aligned}$$

Thus the function $g(X) = X$ is such that $E_\theta[g(X)] = 0$ for all θ, but $P_\theta(g(X) = 0) = P_\theta(X = 0) = \left(\frac{\theta}{2}\right)^0 (1-\theta)^{1-0} = 1 - \theta \neq 1$ for any $0 < \theta < 1$. Hence X is NOT complete.

Part (b): Sufficiency: The pmf of X is

$$f(x \mid \theta) = \left(\frac{\theta}{2}\right)^{|x|} (1-\theta)^{1-|x|} = \underbrace{\theta^{|x|}(1-\theta)^{1-|x|}}_{=g(T(x)|\theta)}\ \underbrace{2^{-|x|}}_{=h(x)}$$

where $T(x) = |x|$. Therefore, by the Factorization theorem, $|X|$ is sufficient for θ.

Completeness: As suggested in the hint, we first find the pmf of $Y = |X|$. We note that the support of Y is $\{0, 1\}$. Clearly, $P(Y = 0) = P(X = 0) = \left(\frac{\theta}{2}\right)^0 (1-\theta)^{1-0} = 1 - \theta$, and

$$\begin{aligned}
P(Y = 1) &= P(X = 1) + P(X = -1) \\
&= \left(\frac{\theta}{2}\right)^1 (1-\theta)^{1-1} + \left(\frac{\theta}{2}\right)^1 (1-\theta)^{1-1} = \theta
\end{aligned}$$

Thus,

$$P(Y = y) = \begin{cases} \theta & y = 1 \\ 1 - \theta & y = 0 \end{cases}$$

for $0 < \theta < 1$, which means $Y \sim \text{Bernoulli}(\theta)$ for $0 < \theta < 1$. Therefore, by the completeness of Binomial family (proved in class) it follows that $Y = |X|$ is complete.

Problem 4

Let $\underline{x} = (x_1, \ldots, x_n)$ and $\underline{y} = (y_1, \ldots, y_n)$ be two sample points from the density $f(x \mid \theta)$.

Part(a): We have

$$\frac{f(\underline{x} \mid \theta)}{f(\underline{y} \mid \theta)} = \frac{\prod_{i=1}^n \frac{\gamma}{\theta}\, x_i^{\gamma-1} e^{-x_i^\gamma/\theta}}{\prod_{i=1}^n \frac{\gamma}{\theta}\, y_i^{\gamma-1} e^{-y_i^\gamma/\theta}} = \frac{\prod_{i=1}^n x_i^{\gamma-1}}{\prod_{i=1}^n y_i^{\gamma-1}} \ \exp\left[-\frac{1}{\theta}\left(\sum_{i=1}^n x_i^\gamma - \sum_{i=1}^n y_i^\gamma\right)\right]$$

This is constant as a function of θ <u>if and only if</u> $\sum_{i=1}^n x_i^\gamma = \sum_{i=1}^n y_i^\gamma$. Therefore, $\sum_{i=1}^n X_i^\gamma$ is minimal sufficient for θ.

Part (b): We have

$$\begin{aligned}\frac{f(\underline{x} \mid \mu, \lambda)}{f(\underline{y} \mid \mu, \lambda)} &= \frac{\prod_{i=1}^n \frac{1}{\lambda} \exp\left[-\frac{1}{\lambda}(x_i - \mu)\right] I(x_i > \mu)}{\prod_{i=1}^n \frac{1}{\lambda} \exp\left[-\frac{1}{\lambda}(y_i - \mu)\right] I(y_i > \mu)} \\ &= \exp\left[-\frac{1}{\lambda}\left(\sum_{i=1}^n x_i - \sum_{i=1}^n y_i\right)\right] \frac{\prod_{i=1}^n I(x_i > \mu)}{\prod_{i=1}^n I(y_i > \mu)} \\ &= \exp\left[-\frac{1}{\lambda}\left(\sum_{i=1}^n x_i - \sum_{i=1}^n y_i\right)\right] \frac{I(x_{(1)} > \mu)}{I(y_{(1)} > \mu)}\end{aligned}$$

This is constant as a function of (μ, λ) <u>if and only if</u> $x_{(1)} = y_{(1)}$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$. Therefore, $(X_{(1)}, \sum_{i=1}^n X_i)$ is minimal sufficient for (μ, λ).

Part (c): We have

$$\frac{f(\underline{x} \mid \theta)}{f(\underline{y} \mid \theta)} = \frac{\prod_{i=1}^n \frac{1}{\theta} I(\theta < x_i < 2\theta)}{\prod_{i=1}^n \frac{1}{\theta} I(\theta < y_i < 2\theta)} = \frac{I(\theta < x_{(1)} < x_{(n)} < 2\theta)}{I(\theta < x_{(1)} < x_{(n)} < 2\theta)} = \frac{I(x_{(n)}/2 < \theta < x_{(1)})}{I(y_{(n)}/2 < \theta < y_{(1)})}$$

This is constant as a function of θ <u>if and only if</u> $x_{(1)} = y_{(1)}$ and $x_{(n)} = y_{(n)}$. Therefore, $(X_{(1)}, X_{(n)})$ is minimal sufficient for θ.

Problem 5

Part (a): Let us denote by $f(x - \mu)$ the common location family density of $X_1, X_2, \ldots, X_n$. Then there exist iid observations $Z_1, \ldots, Z_n$ from the density $f(x)$ (the standard density of the family) such that $Z_i = X_i - \mu$, i.e., $X_i = Z_i + \mu$.

Note that the sample median is:

$$\begin{aligned}M(X_1, X_2, \ldots, X_n) &= \begin{cases} X_{\left(\frac{n+1}{2}\right)} & n \text{ is odd} \\ \frac{X_{\left(\frac{n}{2}\right)} + X_{\left(\frac{n}{2}+1\right)}}{2} & n \text{ is even}\end{cases} \\ &= \begin{cases} \mu + Z_{\left(\frac{n+1}{2}\right)} & n \text{ is odd} \\ \mu + \frac{Z_{\left(\frac{n}{2}\right)} + Z_{\left(\frac{n}{2}+1\right)}}{2} & n \text{ is even}\end{cases} \\ &= \mu + M(Z_1, \ldots, Z_n)\end{aligned}$$

Again,

$$\overline{X} = \frac{1}{n}\sum_{i=1}^n X_i = \frac{1}{n}\sum_{i=1}^n (\mu + Z_i) = \mu + \overline{Z}$$

Hence

$$M(X_1, X_2, \ldots, X_n) - \overline{X} = M(Z_1, \ldots, Z_n) - \overline{Z}$$

where the RHS contains random variables whose distribution does not depend on the parameter μ. Hence $M - \overline{X}$ is an ancillary statistic.

Part (b): Note at the outset that conditional on $N = n$, $X \mid N = n \sim \text{Binomial}(n, \theta)$.

Minimal Sufficiency: Let (x, n) and (y, m) be two sample points from the joint pmf of (X, N). Then

$$\frac{P_\theta(X = x, N = n)}{P_\theta(X = y, N = m)} = \frac{P_\theta(X = x \mid N = n)\ P(N = n)}{P_\theta(X = y \mid N = m)\ P(N = m)} = \frac{\binom{n}{x}\theta^x(1-\theta)^{n-x}\ p_n}{\binom{m}{y}\theta^y(1-\theta)^{m-y}\ p_m} = \frac{\binom{n}{x}p_n}{\binom{m}{y}p_m}\ \theta^{x-y}(1-\theta)^{(n-x)-(m-y)}$$

This is constant as a function in θ if and only if $x = y$ and $n - x = m - y$, i.e., $x = y$ and $n = m$. Hence (X, N) is minimal sufficient for θ.

N is ancillary: Clearly, the pmf of N is $P(N = n) = p_n$ which is free of θ. Hence N is ancillary.

Problem 6

Part (a): The derivation is essentially similar to that for a binomial distribution, which has been derived in class.

Sufficiency: The joint pmf of $X_1, \ldots, X_n$ is

$$f(\underline{x} \mid \lambda) = \prod_{i=1}^{n} e^{-\lambda}\frac{\lambda^{x_i}}{x_i!} = \underbrace{e^{-n\lambda}\lambda^{\sum_{i=1} x_i}}_{=g(T(\underline{x}|\lambda))}\underbrace{\prod_{i=1}^{n}\frac{1}{x_i!}}_{=h(\underline{x})}$$

where $T(\underline{x}) = \sum_{i=1}^n x_i$. Therefore, by the factorization theorem, $\sum_{i=1}^n X_i$ is sufficient for λ.

Completeness: First obtain the pmf of $Y = \sum_{i=1}^n X_i$. The mgf of each X_i is $M(t) = e^{\lambda(e^t-1)}$. The mgf of $Y = \sum_{i=1}^n X_i$ is

$$M_Y(t) = \mathrm{E}\left(e^{tY}\right) = \mathrm{E}\left(e^{t\sum_{i=1}^n X_i}\right) \overset{\text{iid}}{=} \left[\mathrm{E}(e^{tX_1})\right]^n = M(t)^n = e^{n\lambda(e^t-1)}$$

which is the mgf of Poisson$(n\lambda)$. Therefore, by the uniqueness of mgf, $\sum_{i=1}^n X_i \sim \text{Poisson}(n\lambda)$.

Consider any function $g(y)$ of $y = \sum_{i=1}^n$. Then

$$\mathrm{E}_\lambda(g(Y)) = \sum_{y=0}^{\infty} g(y)\ P_\lambda(Y = y) = \sum_{y=0}^{\infty} g(y)\quad \underbrace{e^{-n\lambda}\frac{(n\lambda)^y}{y!}}_{>0,\text{ for all }\lambda>0}$$

Therefore $E_\lambda(g(Y)) = 0$ for all $\lambda > 0$ means $g(y) = 0$ for $y = 0, 1, \ldots$, i.e., $P_\lambda(g(Y) = 0) = 1$. This means that Y is complete.

Part (b): As suggested in the hint consider a sequence X_n where $X_n = X$, and $Y = 1 - X$, where $X \sim \text{Binomial}(1, 0.5)$. Then X and $Y = 1 - X$. Then $X_n \xrightarrow{d} X$ (trivially; all have the same distribution) which means $X_n \xrightarrow{d} Y$ as X and Y have the same distribution.

However, for any $0 < \varepsilon < 1$,

$$\begin{aligned}
P(|X_n - Y| \geq \varepsilon) = P(|X - 1 + X| \geq \varepsilon) = P(|2X - 1| \geq \varepsilon) &= P(2X \geq 1 + \varepsilon) + P(2X \leq 1 - \varepsilon) \\
&= P\left(X \geq \frac{1+\varepsilon}{2}\right) + P\left(X \leq \frac{1-\varepsilon}{2}\right) \\
&= P(X = 1) + P(X = 0) = 1 \not\to 0
\end{aligned}$$

as $n \to \infty$. Hence $X_n \overset{P}{\not\to} Y$.