

STA 522, Spring 2021

Introduction to Theoretical Statistics II

Lecture 4

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AGENDA

- ▶ Comments on Exam 1
- ▶ Ancillary Statistics
- ▶ Complete Statistics
- ▶ Likelihood Principle and Equivariance

Review: Factorization Theorem, Minimal Sufficiency

- **Factorization Theorem:** Let $f(\underline{x} \mid \theta)$ denote the joint pdf or pmf of a sample \underline{X} . A statistic $T(\underline{x})$ is a sufficient statistic for θ *if and only if* there exist functions $g(t \mid \theta)$ and $h(\underline{x})$ such that, for all sample points \underline{x} and all parameter points θ ,

$$f(\underline{x} \mid \theta) = g(T(\underline{x}) \mid \theta) \cdot h(\underline{x}).$$

- **Finding minimal sufficient statistics:** Let $f(\underline{x} \mid \theta)$ be the pdf/pmf of a sample \underline{X} . Suppose there exists a function $T(\underline{X})$ such that, for every \underline{x} and \underline{y} , the ratio

$$\frac{f(\underline{x} \mid \theta)}{f(\underline{y} \mid \theta)}$$

is constant as a function of θ if and only if $T(\underline{x}) = T(\underline{y})$. Then $T(\underline{X})$ is a minimal sufficient statistic for θ .

Ancillary Statistics

Definition: A statistic $S(\underline{X})$ whose distribution does not depend on the parameter θ is called an **ancillary statistic**.

- ▶ Alone, an ancillary statistic contains no information on the parameter θ .
- ▶ However, when used in conjunction with other statistics, ancillary statistics sometimes do provide valuable information for inferences about θ .

Example (Uniform Ancillary Statistic): Let X_1, X_2, \dots, X_n be iid Uniform $(\theta - \frac{1}{2}, \theta + \frac{1}{2})$. We saw that $(X_{(1)}, X_{(n)})$ is minimal sufficient for θ . We will show that $R = X_{(n)} - X_{(1)}$ is ancillary for θ by showing that the pdf of R doesn't depend on θ .

To this end we first find the joint pdf of $(X_{(1)}, X_{(n)})$. Note that the common pdf and cdf for each X_i is:

$$f(x | \theta) = I\left(\theta - \frac{1}{2} < x < \theta + \frac{1}{2}\right), \text{ and}$$

$$F(x | \theta) = \begin{cases} 0 & x \leq \theta - \frac{1}{2} \\ x - \theta + \frac{1}{2} & \theta - \frac{1}{2} < x < \theta + \frac{1}{2} \\ 1 & x \geq \theta + \frac{1}{2} \end{cases}$$

From a result discussed in class, we have:

$$\begin{aligned} f_{X_{(1)}, X_{(n)}}(x_1, x_n) &= \frac{n!}{(1-1)!(n-1-1)!(n-n)!} f(x_1 | \theta) f(x_n | \theta) \\ &\quad \times [F(x_1 | \theta)]^{1-1} [F(x_n | \theta) - F(x_1 | \theta)]^{n-1-1} \\ &\quad \times [1 - F(x_n | \theta)]^{n-n} \end{aligned}$$

So, here

$$f_{X_{(1)}, X_{(n)}}(x_1, x_n \mid \theta) = n(n-1) (x_n - x_1)^{n-2} I\left(\theta - \frac{1}{2} < x_1 < x_n < \theta + \frac{1}{2}\right)$$

To find the distribution of $R = X_{(n)} - X_{(1)}$, consider $M = (X_{(1)} + X_{(n)})/2$. The inverse transformation is $X_{(1)} = (2M - R)/2$ and $X_{(n)} = (2M + R)/2$ with Jacobian of transformation being 1.

Joint support for (R, M) :

$\{(r, s) : 0 < r < 1, \theta - \frac{1}{2} + \frac{r}{2} < m < \theta + \frac{1}{2} - \frac{r}{2}\}$. Therefore

$$\begin{aligned} f_R(r \mid \theta) &= \int_{\theta - \frac{1}{2} + \frac{r}{2}}^{\theta + \frac{1}{2} - \frac{r}{2}} n(n-1) r^{n-2} dm \\ &= n(n-1) r^{n-2} (1-r); \quad 0 < r < 1 \end{aligned}$$

This is the pdf of Beta($\alpha = n-1$, $\beta = 2$) distribution, and $f_R(r \mid \theta)$ is the same for all θ .

Thus the distribution of R does not depend on $\theta \implies R$ is ancillary for θ .

Example (Location Family Ancillary Statistic): Let X_1, X_2, \dots, X_n be iid random variables from a location parameter family with cdf $F(x - \theta)$ for some $\theta \in \mathbb{R}$. Then $R = X_{(n)} - X_{(1)}$ is an ancillary statistic.

To see this, define $Z_i = X_i - \theta$. Then Z_i are iid with common cdf $F(x)$. The cdf of the range statistic R is:

$$\begin{aligned} F_R(r \mid \theta) &= P_\theta(R \leq r) \\ &= P_\theta \left(\max_i X_i - \min_i X_i \leq r \right) \\ &= P_\theta \left(\max_i (Z_i + \theta) - \min_i (Z_i + \theta) \leq r \right) \\ &= P_\theta \left(\max_i Z_i - \min_i Z_i \leq r \right) \end{aligned}$$

The last quantity does not depend on θ as the common cdf of Z_1, \dots, Z_n does not depend on θ .

Hence, R is ancillary for θ .

Example (Scale family ancillary statistic) Let X_1, X_2, \dots, X_n be iid random variables from a scale parameter family with cdf $F(x/\sigma)$ for some $\sigma > 0$. Then any statistic that depends on the sample only through the $n - 1$ values $X_1/X_n, \dots, X_{n-1}/X_n$ is an ancillary statistic.

For example $\frac{X_1 + \dots + X_n}{X_n} = \frac{X_1}{X_n} + \dots + \frac{X_{n-1}}{X_n} + 1$ is an ancillary statistic.

To see this, define $Z_i = X_i/\sigma$ so that Z_i are iid from $F(x)$ (free of σ).

$$\begin{aligned} F_{X_1/X_n, \dots, X_{n-1}/X_n}(y_1, \dots, y_{n-1} \mid \theta) &= P_\sigma(X_1/X_n \leq y_1, \dots, X_{n-1}/X_n \leq y_{n-1}) \\ &= P_\sigma(Z_1/Z_n \leq y_1, \dots, Z_{n-1}/Z_n \leq y_{n-1}) \end{aligned}$$

which does not depend on σ .

Ancillary and Minimal Sufficient Statistics

- ▶ Ancillary statistic by itself does not contain any information on θ while a minimal sufficient statistic contains all information. So it may seem that ancillary and minimal sufficient statistics should be unrelated, or statistically independent. This is not necessarily the case.
- ▶ Consider the Uniform($\theta - \frac{1}{2}, \theta + \frac{1}{2}$) example. Here $\theta \implies (R = X_{(n)} - X_{(1)}, M = (X_{(1)} + X_{(n)})/2)$ is minimal sufficient for θ , but R is ancillary.

Completeness

Definition: Let $f(t \mid \theta)$ be a family of pdfs or pmfs for a statistic $T(\underline{X})$. The family of probability distributions is called **complete** if

$$E_{\theta}(g(T)) = 0 \text{ for all } \theta \implies P_{\theta}(g(T) = 0) = 1 \text{ for all } \theta$$

In this case, $T(\underline{X})$ is called a **complete statistic**.

Notes

- ▶ completeness is a property of a family of distributions, not of a particular distribution. For example, if $X \sim N(0, 1)$ then defining $g(x) = x$, we have that $E[g(X)] = E(X) = 0$, but $g(x)$ satisfies $P(g(X) = 0) = P(X = 0) = 0$ and not 1. However, this is a particular distribution and not a family of distributions.
- ▶ Instead, if we consider $X \sim N(\theta, 1)$, $-\infty < \theta < \infty$, then we will see that no function of X , except one that is 0 with probability 1 for all θ , satisfies $E_{\theta}(g(X)) = 0$ for all θ .

Example (Binomial Complete Sufficient Statistic): Suppose that $T \sim \text{Binomial}(n, p)$, where $0 < p < 1$ is an unknown parameter and n is a fixed integer. Then T is complete.

To see this, let g be a function such that $E_p(g(T)) = 0$. Then

$$\begin{aligned} 0 &= E_p(g(T)) = \sum_{t=0}^n g(t) \binom{n}{t} p^t (1-p)^{n-t} \\ &= (1-p)^n \sum_{t=0}^n g(t) \binom{n}{t} \left(\frac{p}{1-p}\right)^t \text{ for all } p \in (0, 1) \\ \implies 0 &= \sum_{t=0}^n g(t) \binom{n}{t} \left(\frac{p}{1-p}\right)^t \text{ for all } p \in (0, 1) \\ \implies 0 &= \sum_{t=0}^n g(t) \binom{n}{t} r^t \text{ for all } r \in (0, \infty) \end{aligned}$$

The last expression is a polynomial of degree n in $r \in (0, \infty)$. For the polynomial to be 0 for all r , each coefficient has to be 0, implying $g(t) = 0$ for $t = 0, \dots, n \implies P_p(g(T) = 0) = 1$ for all p .

Example (Uniform Complete Sufficient Statistics): Suppose X_1, X_2, \dots, X_n iid Uniform($0, \theta$). It follows that $T(\underline{X}) = \max_i X_i$ is a sufficient statistic. We shall show that T is also complete.

Using results on order statistics, the pdf of T is obtained as

$$f(t \mid \theta) = \frac{n}{\theta^n} t^{n-1} I(0 < t < \theta)$$

Suppose $g(t)$ is a function satisfying $E_\theta(g(T)) = 0$, for all $0 < \theta < \infty$. Since $E_\theta(g(T))$ is constant in θ ,

$$\begin{aligned} 0 &= \frac{d}{d\theta} E_\theta(g(T)) = \frac{d}{d\theta} \int_0^\theta g(t) \frac{n}{\theta^n} t^{n-1} dt \\ &= \frac{n}{\theta^n} \frac{d}{d\theta} \left(\int_0^\theta g(t) t^{n-1} dt \right) + \left(\frac{d}{d\theta} \frac{n}{\theta^n} \right) \int_0^\theta g(t) t^{n-1} dt \\ &= \frac{n}{\theta^n} g(\theta) \theta^{n-1} + 0 = \frac{n}{\theta} g(\theta), \quad \text{for all } \theta \end{aligned}$$

Since $\frac{n}{\theta} \neq 0$, we must have $g(\theta) = 0$ for all $\theta > 0$. Hence T is complete.

Complete Statistics in the Exponential Family

Theorem 6.2.25

Let X_1, X_2, \dots, X_n be iid observations from an exponential family with pdf or pmf of the form $f(x | \theta) = h(x)c(\theta) \exp \left(\sum_{j=1}^k w_j(\theta) t_j(x) \right)$, where $\theta = (\theta_1, \dots, \theta_k)$. Then $T(\underline{X}) = \left(\sum_{i=1}^n t_1(X_i), \dots, \sum_{i=1}^n t_k(X_i) \right)$ is a complete statistic for θ as long as the parameter space Θ contains an open set in \mathbb{R}^k .

NOTES

- (a) The dimensions of $\theta = (\theta_1, \dots, \theta_k)$ and $\underline{w} = (w_1(\theta), w_2(\theta), \dots, w_k(\theta))$ must be the same.
- (b) The parameter space Θ doesn't need to be an open set, it just needs to contain an open set in \mathbb{R}^k . Note that this is not possible if the entries in θ are functionally related (i.e., lies on a lower hyper-plane). Example includes the $N(\theta, \theta^2)$ family.

Example (Exercise 6.15) Suppose $X_1, X_2, \dots, X_n \sim \text{iid } N(\theta, \theta^2)$. We argued that the previous result on exponential family cannot be used here. Is this family complete?

Consider the sufficient statistic $T(\underline{X}) = (\bar{X}, S^2)$. Note that

$$E_{\theta}(\bar{X}^2) = \text{Var}_{\theta}(\bar{X}) + (E_{\theta}(\bar{X}))^2 = \theta^2/n + \theta^2 = \frac{n+1}{n}\theta^2$$

and

$$E_{\theta}(S^2) = \theta^2$$

.

Therefore if we define

$$g(T(\underline{X})) = \frac{n}{n+1} T_1(\underline{X})^2 - T_2(\underline{X})$$

Then $E_{\theta}(g(T(\underline{X}))) = 0$ for all θ but $P_{\theta}(g(T(\underline{X})) = 0) = 0$.

Hence, the family is not complete.

Basu's Theorem

Theorem 6.2.24

If $T(\underline{X})$ is a complete and minimal sufficient statistic, then $T(\underline{X})$ is independent of every ancillary statistic.

Proof: (Only for discrete distributions.) Let $S(\underline{X})$ be any ancillary statistic for the parameter θ . Then $P(S(\underline{X}) = s)$ does not depend on θ .

Again, since $T(\underline{X})$ is sufficient,

$P(S(\underline{X}) = s \mid T(\underline{X}) = t) = P(\underline{X} \in \{\underline{x} : S(\underline{x}) = s\} \mid T(\underline{X}) = t)$ does not depend on θ .

So enough to show that

$$P(S(\underline{X}) = s \mid T(\underline{X}) = t) = P(S(\underline{X}) = s) \text{ for all } t \in \mathcal{T}.$$

We have

$$P(S(\underline{X}) = s) = \sum_{t \in \mathcal{T}} P(S(\underline{X}) = s \mid T(\underline{X}) = t) P_{\theta}(T(\underline{X}) = t)$$

and from $\sum_{t \in \mathcal{T}} P_{\theta}(T(\underline{X}) = t) = 1$,

$$P(S(\underline{X}) = s) = \sum_{t \in \mathcal{T}} P(S(\underline{X}) = s \mid T(\underline{X}) = t) P_{\theta}(T(\underline{X}) = t)$$

Define $g(t) = P(S(\underline{X}) = s \mid T(\underline{X}) = t) - P(S(\underline{X}) = s)$. Then from the above two equations we have

$$E_{\theta}(g(T)) = \sum_{t \in \mathcal{T}} g(t) P_{\theta}(T(\underline{X}) = t) \text{ for all } \theta$$

$$\implies g(t) = 0 \text{ for all } t \in \mathcal{T} \quad (T \text{ is complete})$$

$$\implies P(S(\underline{X}) = s \mid T(\underline{X}) = t) = P(S(\underline{X}) = s) \text{ for all } t \in \mathcal{T}$$

Using Basu's Theorem

Example: Let X_1, X_2, \dots, X_n be iid exponential observations with parameter θ . Find the expected value of

$$g(\underline{X}) = \frac{X_n}{X_1 + \dots + X_n}.$$

Note on the outset that exponential is a scale family, and so from a previous example it follows that $g(\underline{X})$ is ancillary for θ .

From the results on exponential family, it follows that $T(\underline{X}) = \sum_{i=1}^n X_i$ is complete and sufficient (verify).

Hence, by Basu's Theorem, $T(\underline{X})$ and $g(\underline{X})$ are independent, meaning $E_\theta[g(\underline{X})T(\underline{X})] = E[g(\underline{X})]E_\theta[T(\underline{X})]$. Here $E_\theta[g(\underline{X})T(\underline{X})] = E_\theta[X_n] = \theta$, and $E_\theta[T(\underline{X})] = E_\theta[\sum_{i=1}^n X_i] = n\theta$.

This implies $E[g(X)] = 1/n$.

Example: Let X_1, X_2, \dots, X_n be iid observations from $N(\mu, \sigma^2)$ distribution. We can establish the independence between \bar{X} and S^2 using Basu's theorem.

First fix σ^2 to some arbitrary value say σ_0^2 .

Then \bar{X} is a complete sufficient statistic for μ .

For any fixed σ_0^2 , the family $N(\mu, \sigma_0^2)$ is a location family with location parameter μ . It can be shown that (homework) S^2 , a statistic based on $X_1 - \bar{X}, \dots, X_n - \bar{X}$, is ancillary for μ .

So, for any fixed σ_0^2 , \bar{X} and S^2 are independent (Basu's theorem).

Since σ_0^2 is arbitrary, therefore, \bar{X} and S^2 are independent for any (μ, σ^2)

The Likelihood Function

Definition: Let $f(\underline{x} \mid \theta)$ denote the joint pdf or pmf of the sample $\underline{X} = (X_1, X_2, \dots, X_n)$. Then, given that $\underline{X} = \underline{x}$ is observed, the function of θ defined by

$$L(\theta \mid \underline{x}) = f(\underline{x} \mid \theta)$$

is called the **likelihood function**.

If \underline{X} is a discrete random vector, then $L(\theta \mid \underline{x}) = P_{\theta}(\underline{X} = \underline{x})$. If for two parameter points θ_1 and θ_2 ,

$P_{\theta_1}(\underline{X} = \underline{x}) = L(\theta_1 \mid \underline{x}) > L(\theta_2 \mid \underline{x}) = P_{\theta_2}(\underline{X} = \underline{x})$, then \underline{x} is more likely to have occurred if $\theta = \theta_1$ than if $\theta = \theta_2$.

Likelihood Principle: If \underline{x} and \underline{y} are two sample points such that $L(\theta \mid \underline{x})$ is proportional to $L(\theta \mid \underline{y})$, that is, there exists a constant $C(\underline{x}, \underline{y})$ such that $L(\theta \mid \underline{x}) = C(\underline{x}, \underline{y}) L(\theta \mid \underline{y})$ for all θ , then the conclusions drawn from \underline{x} and \underline{y} should be identical.

Example (Negative Binomial Likelihood): Let X have a negative binomial distribution with $r = 3$ and success probability p . Find the likelihood function if $x = 2$ is observed, and also for general $X = x$.

If $x = 2$, the likelihood function for $0 \leq p \leq 1$ is

$$L(p \mid 2) = P_p(X = 2) = \binom{4}{2} p^3 (1 - p)^2$$

For general $X = x$ the likelihood function is:

$$L(p \mid x) = P_p(X = x) = \binom{3 + x - 1}{x} p^3 (1 - p)^x$$

Example (Poisson Likelihood): Let $\underline{X} = (X_1, X_2, \dots, X_n)$ denote a random sample from a Poisson distribution with mean λ . The likelihood function for $0 < \lambda < \infty$ is given by:

$$L(\lambda \mid \underline{x}) = P_{\lambda}(X = \underline{x}) = \exp(-n\lambda) \frac{\lambda^{\sum_{i=1}^n x_i}}{\prod_{i=1}^n x_i!}$$

Example (Normal Likelihood): Let $\underline{X} = (X_1, X_2, \dots, X_n)$ denote a random sample from a $N(\mu, \sigma^2)$ distribution. The likelihood function for $-\infty < \mu < \infty$ and $\sigma > 0$ is given by

$$L(\mu, \sigma \mid \underline{x}) = f(\underline{x} \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}} (\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu)^2\right)$$

Equivariance

Equivariance Principle: If $\underline{Y} = g(\underline{X})$ is a change of measurement scale such that the model for \underline{Y} has the same formal structure as the model for \underline{X} , then an inference procedure should be both measurement equivariant and formally equivariant.

Example (Binomial equivariance): Suppose $X \sim \text{Binomial}(n, p)$ and we want to “estimate” p using x , say using the statistic $T(x)$.

Now $X \sim \text{Binomial}(n, p) \implies Y = n - X \sim \text{Binomial}(n, q = 1 - p)$, so $T(y)$ should be an estimator for $q = 1 - p$.

Since $p + q = 1$, it is reasonable to ensure that their estimator also satisfies this relationship, i.e.,

$$T(x) + T(y) = 1 \implies T(x) = 1 - T(y) = 1 - T(n - x)$$

If we only consider estimator satisfying this relationship, then we get a greatly reduced and simplified set of estimators.

Homework

- ▶ Read p. 282 – 291.
- ▶ Exercises: TBA.