Functional Analysis and PDE

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Chapter 1

Introduction

Functional Analysis is the analysis on infinite dimensional spaces. Applications of FA are in geometry and topology, probability theory, numerical analysis mathematical physics, and PDEs.

Example.
$C^2(U)$, $U \subset \mathbb{R}^n$ is the set of functions which are 2 times differentiable and have a bounded derivative. U should be a bounded, open set. Consider $f \to 0$ on ∂U and

$$\Delta = \sum_{j=1}^{n} \left(\frac{\partial}{\partial u_j} \right)^2 .$$

Fix $f \in C(U)$ and look for a solution $u \in C^2(U)$, s.t.

$$\Delta u = f, \quad u = \delta^{-1} f.$$

What is the inverse operator δ^{-1} in this case? That's what we need to study here.

Program of the lecture

- Structures: We need to define *Topologies, Metrics, Norms* and *Scalarproducts* on infinite-dimensional spaces
- We introduce *Functional Spaces* as infinite-dimensional vector spaces of functions
- We need to talk about *Linear Operators* to be able to talk about their inverse (see **??**)

Chapter 2

Structures

We consider *convergence.* We have already seen

$$x_n \to x \quad \Leftrightarrow \quad \forall \varepsilon > 0 \exists n_0 \forall n > n_0 : x_n \in (x - \varepsilon, x + \varepsilon)$$

In order to do so, we need the definition of a neighbourhood (topology), distances (metric), the length of a vector (norm, Banach spaces), the length and angles (scalar product, Hilbert spaces).

All vector spaces we consider should base on $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$!

2.1 Topological spaces

Let X be a set, let 2^X the set of all possible subsets of X (including the empty set).

Definition 2.1 (Topology). A Topology $\mathcal{T}$ on the set X is a family of subsets of X, that means

$$\mathcal{T} \subseteq 2^X,$$

satisfying

T1 $\emptyset, X \in \mathcal{T}$

T2 Stability under finite intersection:

$$\forall A, B \in \mathcal{T}: \quad A \cap B \in \mathcal{T}$$

T3 Stability under any possible union: Let $\mathcal{I}$ be some set of indices (may not be countable). Then

$$\forall \{A_i\}_{i \in \mathcal{I}} \subseteq \mathcal{T}: \quad \bigcup_{i \in \mathcal{I}} A_i \in \mathcal{T}$$

Any subset $A \subset X$ is called an *open set*, if $A \in \mathcal{T}$. Else it is called a *closed set*.

$(X, \mathcal{T})$ is called a *topological space*!

Remark 2.2. Note that

$$\left(\bigcup_{i \in \mathcal{I}} A_i\right)^c = \cap_{i \in \mathcal{I}} A_i^c$$

for all families of open sets $\{A_i\}_{i \in \mathcal{I}}$ and each index-set $\mathcal{I}$.

Definition 2.3 (Coarser / finer topologies). Let $\mathcal{T}_1, \mathcal{T}_2$ be two topologies on X with $\mathcal{T}_1 \subset \mathcal{T}_2$, then we say that

- $\mathcal{T}_1$ is *coarser / weaker* than $\mathcal{T}_2$
- $\mathcal{T}_2$ is *finer /stronger* tthan $\mathcal{T}_1$

Example.

a) $\mathcal{T} = 2^X$ *is a topology on* $X \Rightarrow 2^X$ *is the storngest (finest) topology on* X.

Also $\mathcal{T} = \{\emptyset, X\}$ *is a topology on* $X \Rightarrow$ *any topology* $\mathcal{T}'$ *needs to caontain* $\emptyset$ *and* X, *so* $\mathcal{T} \subset \mathcal{T}'$. *This means that* $\mathcal{T}$ *is the weakest / coarsest topology on* X

b) *On* $\mathbb{R}$ *there is a standard topology* $\mathcal{T}_{st}$*:*

$$V \in \mathcal{T}_{st} \text{ iff } \forall x \in V \, \exists \varepsilon > 0 : (x - \varepsilon, x + \varepsilon) \subset V$$

c) relative topology*: Let* $A \subset X$, *let* $\mathcal{T}$ *be a topology on* X. *Then*

$$\mathcal{T}_A \{A \cup V : V \in \mathcal{T}\}$$

d) Intersection of topologies*: Let $\mathcal{I}$ an index set (may be uncountable), let $(\mathcal{T}_i)_{i\in\mathcal{I}}$ be a family of topologies on X. Then we can define*

$$\bigcap_{i\in\mathcal{I}} \mathcal{T}_i$$

and this is again a topology on X!

e) Product topology*: Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ two topological spaces. Let*

$$\mathcal{S} = \{(V, X) |\, V \in \mathcal{T}_X\} \cup \{(X, W) |\, W \in \mathcal{T}_Y\}.$$

The product topology on $X \times Y$ is the coarsest (weakest) topology on $X \times Y$, that contains $\mathcal{S}$. In particular it must contain all sets of the form $U \times V$ for $U \in \mathcal{T}_X, V \in \mathcal{T}_Y$.

Remark 2.4. If nothing else is said, we consider the standard topology $\mathcal{T}_{st}$ on $\mathbb{R}$!

Definition 2.5 (Closure / boundary of a set). Let $(X, \mathcal{T})$ topological space, $A \in X$.
The *interior* of A, A° is the largest open set inside

$$A = \bigcup_{V\in\mathcal{T}, V\subseteq A} V \text{ (open)}.$$

The *closure* of A, $\bar{A}$ is the smallest closed set containing

$$A = \bigcup_{V \text{ s.t. } V^c\in\mathcal{T}, V\supseteq A} V \text{ (closed)}.$$

The *boundary* of A is given by

$$\partial A = \bar{A} \setminus A^\circ = \bar{A} \cap \underbrace{(X \setminus A^\circ)}_{(A^c)^c} \text{ (closed)}.$$

Definition 2.6 (dense set / separable set). $(X, \mathcal{T})$ topological space in X. Then

- $A \subset X$ is called *dense in* X, if $\bar{A} = X$
- X is called *separable*, if there is a countable dense subset of X

Definition 2.7 (open neighbourhood). $(X, \mathcal{T})$ topological space, $x \in X$. A subset $V \subseteq X$ is called an open neighbourhood of x, if $V \in \mathcal{T}$ and $x \in V$.

Definition 2.8 (Convergence in topology). Let $(\mathcal{T}, X)$ topological space. A sequence $(x_n)_{n\in\mathbb{N}}$, i.e. a map

$$\begin{aligned} x: \quad & \mathbb{N} \to X \\ n & \to x_n, \end{aligned}$$

emphconverges to $x^* \in X$, if

$$\forall V \text{ open neighbourhood of } x^* : \{n |\, x_n \in V^c\} \text{ is finite}$$

(i.e. there is just a finite number of elements that's not contained in V). Then we say that x^* is a limit point for the sequence x_n.

Example.

a) *Let $\mathcal{T}$ the standard topology on $\mathbb{R}$. Then the definition of converges equals the ε- δ-Definition of convergence in Analysis 3.*

b) *If $\mathcal{T} = 2^X$, then $x_n \to x^*$ iff x_n is constant up to a finite number of terms: As we have $\mathcal{T} = 2^X$ especially the set $V = \{x^*\}$ is open. This set gives us the result.*

c) *If $\mathcal{T} = \{\emptyset, X\}$, then every sequence is convergent! Every point $x^* \in X$ is a limit point.*

Definition 2.9 (Hausdorff space). Let $(X, \mathcal{T})$ topological space. It is called a *Hausdorff space*, if

$$\forall x, y \in X, x \neq y\, \exists U_x, U_y \in \mathcal{T} \text{ s.t. } x \in U_x, y \in U_y \text{ and } U_x \cap U_y = \emptyset.$$

Proposition 2.10 (Limits in Hausdorff spaces are unique)
If $(X, \mathcal{T})$ is a Hausdorff space, then every sequence has at most one limit, so either the sequence has no limit or it is unique!

Proof. Proof can be done by contradiction. ■

Definition 2.11 (Connectedness). A topological space $(X, \mathcal{T})$ is *connected*, if we cannot write it as a union of 2 disjoint(!) (nonempty) open sets. So, if X is connected and $V \in \mathcal{T}$ is an open set, there is no $\emptyset \neq W \in \mathcal{T}$ with $V \cap W = \emptyset$ and $V \cup W = X$.

$A \subseteq X$ is *connected*, if A is connected in $\mathcal{T}_A$.

Definition 2.12 (Continuity). Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ topological spaces, $f : X \to Y$. f is called *continuous*, if

$$\forall V \in \mathcal{T}_Y : f^{-1}(V) \subset \mathcal{T}_X.$$

f is *continuous at a point* $x \in X$, if

$\forall V$ open neighbourhoods of $f(x)\, \exists U$ open neighbourhood of x in $\mathcal{T}_X$, s.t. $f(U) \subset V$

$(U \subset f^{-1}(V))$. f is a *homeomorphism* if f is bijective, and f, f^{-1} are continuous.

Remark 2.13. It holds that

$$\begin{aligned} f^{-1}(A \cup B) &= f^{-1}(A) \cup f^{-1}(B) \\ f^{-1}(A \cap B) &= f^{-1}(A) \cap f^{-1}(B) \\ f^{-1}(Y \setminus A) &= X \setminus f^{-1}(A) \end{aligned}$$

It is also valid that if

$$f_1 : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y), \quad f_2 : (Y, \mathcal{T}_Y) \to (X, \mathcal{T}_X)$$

and both are continuous, then $f_2 \circ f_1$ is also continuous!
Moreover

$$f \text{ continuous} \Leftrightarrow f \text{ continuous at every } x \in X$$

Example.

a) *Let $\mathcal{T} = 2^X$, $f : X \to Y$, let $\mathcal{T}_Y$ continuous. Any function is then continuous:*

$$\forall V \in \mathcal{T}_Y : f^{-1}(V) \in 2^X = \mathcal{T}_X \Rightarrow \textit{ continuous!}$$

b) *Let now $\mathcal{T}_X = \{\emptyset, X\}$, then the constant function $f(x) = y^*, \forall x \in X$ is contunuous:*

$$\forall V \in \mathcal{T}_Y : f^{-1}(V) = \begin{cases} X & \textit{if } y^* \in V \\ \emptyset & \textit{if } y^* \notin V \end{cases}$$

c) *If $\mathcal{T}_X = \{\emptyset, X\}$, and Y is Hausdorff, then the emphonly continuous function is the constant function! (Exercise!)*

We may consider the following: Let $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ and $A \subset X$. Then we can define the restriction $f_{|A} : A \to Y$. That's why we also need a topology on A (that is the induced topology). If f was continuous, then $f_{|A}$ is also continuous as a function mapping between $(A, \mathcal{T}_A)$ and $(Y, \mathcal{T}_Y)$.

Theorem 2.14 (Intermediate value theorem)
Let $(X, \mathcal{T})$ a connected topological space, $f : (X, \mathcal{T}) \to \mathbb{R}$ (on $\mathbb{R}$ we consider the standard topology), and let f be continuous. Assume there is $x, y \in X$ s.t. $f(x) < 0 < f(y)$. Then there exists a $z \in X$ s.t. $f(z) = 0$.

Proof. Assume that $f(z) \neq 0, \forall z \in X$. This would mean that $0 \notin f(X)$. Consider $V = (0, \infty)$, which is open in $\mathcal{T}_{st}$. Then $f^{-1}(V)$ is open (as f is continuous) and is nonempty. We can take the complement of this set: $X = f^{-1}(V) \cup [f^{-1}(V)]^c$, and

$$[f^{-1}(V)]^c = f^{-1}(V^c) = f^{-1}((-\infty, 0)) = f^{-1}\left(\underbrace{(-\infty, 0)}_{\text{open}}\right)$$

is an open and nonempty set. As the space X is connected, this is not possible!

$\Rightarrow$ There must be $z \in X : f(z) = 0$. ∎

2.2 Metric spaces

Definition 2.15 (Metric). A function $d : X \times X \to [0, \infty)$, $(x, y) \mapsto d(x, y)$ is called a *metric*, if

M1) $d(x, y) = 0 \Leftrightarrow x = y$ (Non-negativity)

M2) $d(x, y) = d(y, x)$, $\forall x, y \in X$ (Symmetry)

M3) $d(x, y) \leq d(x, z) + d(z, y)$, $\forall x, y, z \in X$ (Triangle inequality)

If d is a metric on X, then the pair (X, d) is called a *metric space*.

Definition 2.16 (Semimetric). The map $d : X \times X \to [0, \infty)$ is called a *semimetric*, if

M2) $d(x, y) = d(y, x)$, $\forall x, y \in X$ (Symmetry)

M3) $d(x, y) \leq d(x, z) + d(z, y)$, $\forall x, y, z \in X$ (Triangle inequality)

The non-negativity (which would make d a metric) is <u>not</u> satisfied!

A semimetric d can be extended to a metric as follows:

Take equivalence relation $x\tilde{y}$, if $d(x, y) = 0$, and then take $\tilde{X} = X_{\backslash \tilde{}}$, so

$$[x] \in \tilde{X} \Rightarrow [x] := \{z \in X |\, z\tilde{x}\} = \{z \in X |\, d(x, z) = 0\}.$$

Set then

$$\tilde{d} : \tilde{X} \times \tilde{X} \to [0, \infty), \quad \tilde{d}([x], [y]) = d(x, y).$$

Check, that $\tilde{d}$ is a metric on $\tilde{X}$!

Example.

a) *On $X = \mathbb{R}^n$, the map $d_\infty(x, y) \coloneqq \max_{j=1,\dots,n} |x_j - y_j|$ is a metric.*

b) *On $X = \mathbb{R}^n$ define for $1 \le p < \infty$:*

$$d_p(x, y) \coloneqq \left[\sum_{j=1}^{n} |x_j - y_j|^p\right]^{\frac{1}{p}}$$

is a metric on X, for $p = 2$ it is the euclidean metric.

c) *Let $X = \ell_\infty \coloneqq \{a : \mathbb{N} \to \mathbb{R}|$ boudned sequence$\}$. On this space we can define a metric by*

$$d_\infty(a, b) \coloneqq \sup_{j \in \mathbb{N}} |a_j - b_j|.$$

This metric is well-defined as all sequences in $\ell_i nfty$ are bounded!

d) *On*

$$\ell_p \coloneqq \{a : \mathbb{N} \to \mathbb{R}| \sum_{j=0}^{\infty} |a_j|^p < \infty\}$$

we can define a metric by

$$d_p(a, b) \coloneqq \left[\sum_{j=0}^{\infty} |a_j - b_j|^p\right]^{\frac{1}{p}}.$$

This expression is finite since we take sequences in ℓ_p.

e) Pull-back metric*: Let $X, (Y, \mathcal{T}_Y)$ given, $f : X \to Y$ injective. Then*

$$d_X(x, y) \coloneqq d_Y(f(x), f(y))$$

is a metric on X.
Exercise: *Show that d_x is a metric iff f is injective and d_Y is a metric!*

f) *Let $x = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$, and $Y = [-1, 1]$ with the standard metric $d_Y(y_1, y_2) = |y_1 - y_2|$. Let now $f : X \to Y$ given by*

$$f(x) \coloneqq \begin{cases} -1 & x = -\infty \\ \frac{x}{1+|x|} & x \in \mathbb{R} \\ 1 & x = +\infty \end{cases}$$

Then we have that d_x

Some definitions

Definition 2.17. Let (X, d) metric space, $A, B \subset X$. The *diameter* of A is defined as

$$diam(A) \coloneqq \sup_{x,y \in A} d(x, y).$$

The *distance between two sets* is defined as

$$dist(A, B) \coloneqq \inf_{x \in A, y \in B} d(x, y),$$

ald the *distance between a set and a point* is defined as

$$dist(x, A) \coloneqq \inf_{y \in A} d(x, y).$$

A *Neighbourhood of a set* A is given by

$$B_r(A) \coloneqq \{y \in X : d(y, A) < r\}.$$

A *Ball of radius* r *centered at* x is given by

$$B(x, r) \coloneqq \{y \in X |\, d(y, x) < r\}.$$

Proposition 2.18 (Topology induced by a metric)
Let (X, d) *metric space. Define* $\mathcal{T}_d \subset 2^X$ *as*

$$\mathcal{T}_d \coloneqq \{V \in X |\, \forall x \in V\, \exists \varepsilon > 0 : B(x, \varepsilon) \subset V\}.$$

Then, $\mathcal{T}_d$ *is a topology on* X *and* $(X, \mathcal{T}_d)$ *is a Hausdorff-space.*

Proof. $\mathcal{T}_d$ is a topology (easy exercise).
Let $x \neq y, x, y \in X$. We need to show that there are $U_x, U_y \in \mathcal{T}_d$ s.t. $x \in U_x$, $y \in U_y$ and $U_x \cap U_y = emptyset$.
Try with $U_x = B(x, \varepsilon_1)$, $U_y = B(y, \varepsilon_2)$. What is unknown up till now are the values of $\varepsilon_1, \varepsilon_2$. Define $z \in U_x \cap U_y$. This point exists, iff $d(z, x) < \varepsilon_1$ and $d(z, y) < \varepsilon_2$. This means

$$d(x, y) \leq d(x, z) + d(z, y) < \varepsilon_1 + \varepsilon_2.$$

If $\varepsilon_1 + \varepsilon_2 < d(x, y)$ then $U_x \cap U_y = \emptyset$. This is always possible as we can chose $\varepsilon_1, \varepsilon_2$ so small, that the sum of them is smaller than $d(x, y)$.

Are those balls open in sense of the topology $\mathcal{T}_d$?
It can be checked, that every ball $B(x, r) \in \mathcal{T}_d$ (is open in $\mathcal{T}_d$). ∎

Definition 2.19. $B(x,r)$ is called an *open ball of radius r centered in x*. A *closed ball of radius r centered in x* is given by

$$\bar{B}(x,r) := \{y \in X \,|\, d(x,y) \leq r\}.$$

The *closure of an open ball $B(x,r)$* is defined as

$$\overline{B(x,r)} = \text{ smallest open set (in the topol.) containing the ball.}$$

In general $\overline{B(x,r)} \subseteq \bar{B}(x,r)$.

Example.
Let $X = \{0,1\}$ and

$$d(x,y) := \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

Then $B(0,1) = \{z \in X \,|\, d(0,z) < 1\} = \{0\}$ and

$$[B(0,1)]^c = \{1\} = B(1,1).$$

So $B(0,1)$ is open and $B(0,1)^c$ is also open. But $B(0,1)^c$ is the complement of an open set, so it has to be closed as well. Therefore $B(0,1)$ is open and closed at the same time. One sees easily

$$\overline{B(0,1)} = \{0\}, \quad \bar{B}(0,1) = \{0,1\}.$$

Whenever we have a metric space, we can make it a topological space. When does the opposite hold? This property is called *Metrizability* of a topological space

Definition 2.20. A topological space $(X,\mathcal{T})$ s.t. $d : X \times X \to [0,\infty)$ s.t. (X,d) is a metric space, <u>and</u> $\mathcal{T} = \mathcal{T}_d$, then the topological space $(X,\mathcal{T})$ is called *metrizable.*

Remark 2.21. Not all Hausdorff spaces are metrizable!
Remark 2.22. In a Hausdorff-space $(X,\mathcal{T}_d)$ every convergent sequence has a unique limit.

Proposition 2.23
(X,d) metric space (hence $(X,\mathcal{T}_d)$ is Hausdorff-space). Let $x : \mathbb{N} \to X$ a sequence in X. The following are equal

1. $$(x_n)_{n\in\mathbb{N}} \text{ converges to } x^* \in X \text{ in sense of } (X,\mathcal{T}_d)$$

2. $$\forall \varepsilon > 0 \,\exists k_0 \geq 0 : \forall k \geq k_0 \; x_k \in \underbrace{B(x^*,\varepsilon)}_{d(x^*,x_k)<\varepsilon}$$

PROOF. For all V open neighbourhood of x^* all but a finite number of x_k are in V.

$$\exists k_0 > 0 \, \forall k \geq k_0 \, x_k \in V \text{ iff } V = \text{ball}.$$

For the other direction: V is open neighbourhood of x^*. So there exists $\varepsilon > 0$ s.t.

$$B(x^*, \varepsilon) \subset V.$$

Open and closed sets can be characterized using convergent sequences:

Proposition 2.24
Let (X, d) metric space, $X \subset X$. The following are equivalent

1. *A is closed in topology, i.e. $A^c \in \mathcal{T}_d$*
2. *A is* sequentially closed*, i.e. all convergent sequences $x : \mathbb{N} \to A$ converge to a point $x^* \in A$*

Moreover $\forall A \subset X$, the closure $\bar{A}$ (topology) coincides with the sequential closure*:*

$$\bar{A} := \{x^* \in X \,|\, \exists x : \mathbb{N} \to A : \lim_{k \to \infty} x_k = x^*\}.$$

PROOF. Exercise! ∎