Functional Analysis and PDE

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Chapter 0

Introduction

Functional Analysis is the analysis on infinite dimensional spaces. Applications of FA are in geometry and topology, probability theory, numerical analysis mathematical physics, and PDEs.

Example.
$C^2(U)$, $U \subset \mathbb{R}^n$ is the set of functions which are 2 times differentiable and have a bounded derivative. U should be a bounded, open set. Consider $f \to 0$ on ∂U and

$$\Delta = \sum_{j=1}^{n} \left(\frac{\partial}{\partial u_j} \right)^2 .$$

Fix $f \in C(U)$ and look for a solution $u \in C^2(U)$, s.t.

$$\Delta u = f, \quad u = \delta^{-1} f.$$

What is the inverse operator δ^{-1} in this case? That's what we need to study here.

Program of the lecture

- Structures: We need to define *Topologies, Metrics, Norms* and *Scalarproducts* on infinite-dimensional spaces
- We introduce *Functional Spaces* as infinite-dimensional vector spaces of functions
- We need to talk about *Linear Operators* to be able to talk about their inverse (see **??**)

Chapter 1

Structures

We consider *convergence*. We have already seen

$$x_n \to x \quad \Leftrightarrow \quad \forall \varepsilon > 0 \exists n_0 \, \forall n > n_0 : x_n \in (x - \varepsilon, x + \varepsilon)$$

In order to do so, we need the definition of a neighbourhood (topology), distances (metric), the length of a vector (norm, Banach spaces), the length and angles (scalar product, Hilbert spaces).

All vector spaces we consider should base on $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$!

1.1 Topological spaces

Let X be a set, let 2^X the set of all possible subsets of X (including the empty set).

Definition 1.1 (Topology). A Topology $\mathcal{T}$ on the set X is a family of subsets of X, that means

$$\mathcal{T} \subseteq 2^X,$$

satisfying

T1 $\emptyset, X \in \mathcal{T}$

T2 Stability under finite intersection:

$$\forall A, B \in \mathcal{T}: \quad A \cap B \in \mathcal{T}$$

T3 Stability under any possible union: Let $\mathcal{I}$ be some set of indices (may not be countable). Then

$$\forall \{A_i\}_{i \in \mathcal{I}} \subseteq \mathcal{T}: \quad \bigcup_{i \in \mathcal{I}} A_i \in \mathcal{T}$$

Any subset $A \subset X$ is called an *open set*, if $A \in \mathcal{T}$. Else it is called a *closed set*.

$(X, \mathcal{T})$ is called a *topological space*!

Remark 1.2. Note that

$$\left(\bigcup_{i \in \mathcal{I}} A_i \right)^c = \cap_{i \in \mathcal{I}} A_i^c$$

for all families of open sets $\{A_i\}_{i \in \mathcal{I}}$ and each index-set $\mathcal{I}$.

Definition 1.3 (Coarser / finer topologies). Let $\mathcal{T}_1, \mathcal{T}_2$ be two topologies on X with $\mathcal{T}_1 \subset \mathcal{T}_2$, then we say that

- $\mathcal{T}_1$ is *coarser / weaker* than $\mathcal{T}_2$
- $\mathcal{T}_2$ is *finer /stronger* tthan $\mathcal{T}_1$

Example.

a) $\mathcal{T} = 2^X$ *is a topology on* $X \Rightarrow 2^X$ *is the storngest (finest) topology on* X.

Also $\mathcal{T} = \{\emptyset, X\}$ *is a topology on* $X \Rightarrow$ *any topology* $\mathcal{T}'$ *needs to caontain* $\emptyset$ *and* X*, so* $\mathcal{T} \subset \mathcal{T}'$*. This means that* $\mathcal{T}$ *is the weakest / coarsest topology on* X

b) *On* $\mathbb{R}$ *there is a standard topology* $\mathcal{T}_{st}$*:*

$$V \in \mathcal{T}_{st} \text{ iff } \forall x \in V \, \exists \varepsilon > 0 : (x - \varepsilon, x + \varepsilon) \subset V$$

c) relative topology*: Let* $A \subset X$*, let* $\mathcal{T}$ *be a topology on* X*. Then*

$$\mathcal{T}_A \{A \cup V : V \in \mathcal{T}\}$$

d) Intersection of topologies*: Let $\mathcal{I}$ an index set (may be uncountable), let $(\mathcal{T}_i)_{i\in\mathcal{I}}$ be a family of topologies on X. Then we can define*

$$\bigcap_{i\in\mathcal{I}} \mathcal{T}_i$$

and this is again a topology on X!

e) Product topology*: Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ two topological spaces. Let*

$$\mathcal{S} = \{(V, X) | \, V \in \mathcal{T}_X\} \cup \{(X, W) | \, W \in \mathcal{T}_Y\}.$$

The product topology on $X \times Y$ is the coarsest (weakest) topology on $X \times Y$, that contains $\mathcal{S}$. In particular it must contain all sets of the form $U \times V$ for $U \in \mathcal{T}_X, V \in \mathcal{T}_Y$.

Remark 1.4. If nothing else is said, we consider the standard topology $\mathcal{T}_{st}$ on $\mathbb{R}$!

Definition 1.5 (Closure / boundary of a set). Let $(X, \mathcal{T})$ topological space, $A \in X$.
The *interior* of A, A° is the largest open set inside

$$A = \bigcup_{V\in\mathcal{T}, V\subseteq A} V \text{ (open)}.$$

The *closure* of A, $\bar{A}$ is the smallest closed set containing

$$A = \bigcup_{V \text{ s.t. } V^c\in\mathcal{T}, V\supseteq A} V \text{ (closed)}.$$

The *boundary* of A is given by

$$\partial A = \bar{A} \setminus A^\circ = \bar{A} \cap \underbrace{(X \setminus A^\circ)}_{(A^c)^c} \text{ (closed)}.$$

Definition 1.6 (dense set / separable set). $(X, \mathcal{T})$ topological space in X. Then

- $A \subset X$ is called *dense in* X, if $\bar{A} = X$
- X is called *separable*, if there is a countable dense subset of X

Definition 1.7 (open neighbourhood). $(X, \mathcal{T})$ topological space, $x \in X$. A subset $V \subseteq X$ is called an open neighbourhood of x, if $V \in \mathcal{T}$ and $x \in V$.

Definition 1.8 (Convergence in topology). Let $(\mathcal{T}, X)$ topological space. A sequence $(x_n)_{n\in\mathbb{N}}$, i.e. a map

$$\begin{aligned} x: \quad & \mathbb{N} \to X \\ n & \to x_n, \end{aligned}$$

emphconverges to $x^* \in X$, if

$$\forall V \text{ open neighbourhood of } x^* : \{n|\, x_n \in V^c\} \text{ is finite}$$

(i.e. there is just a finite number of elements that's not contained in V). Then we say that x^* is a limit point for the sequence x_n.

Example.

a) *Let $\mathcal{T}$ the standard topology on $\mathbb{R}$. Then the definition of converges equals the ε- δ-Definition of convergence in Analysis 3.*

b) *If $\mathcal{T} = 2^X$, then $x_n \to x^*$ iff x_n is constant up to a finite number of terms: As we have $\mathcal{T} = 2^X$ especially the set $V = \{x^*\}$ is open. This set gives us the result.*

c) *If $\mathcal{T} = \{\emptyset, X\}$, then every sequence is convergent! Every point $x^* \in X$ is a limit point.*

Definition 1.9 (Hausdorff space). Let $(X, \mathcal{T})$ topological space. It is called a *Hausdorff space*, if

$$\forall x, y \in X, x \neq y\, \exists U_x, U_y \in \mathcal{T} \text{ s.t. } x \in U_x, y \in U_y \text{ and } U_x \cap U_y = \emptyset.$$

Proposition 1.10 (Limits in Hausdorff spaces are unique)
If $(X, \mathcal{T})$ is a Hausdorff space, then every sequence has at most one limit, so either the sequence has no limit or it is unique!

PROOF. Proof can be done by contradiction. ■

Definition 1.11 (Connectedness). A topological space $(X, \mathcal{T})$ is *connected*, if we cannot write it as a union of 2 disjoint(!) (nonempty) open sets. So, if X is connected and $V \in \mathcal{T}$ is an open set, there is no $\emptyset \neq W \in \mathcal{T}$ with $V \cap W = \emptyset$ and $V \cup W = X$.

$A \subseteq X$ is *connected*, if A is connected in $\mathcal{T}_A$.

Definition 1.12 (Continuity). Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ topological spaces, $f : X \to Y$. f is called *continuous*, if

$$\forall V \in \mathcal{T}_Y : f^{-1}(V) \subset \mathcal{T}_X.$$

f is *continuous at a point* $x \in X$, if

$\forall V$ open neighbourhoods of $f(x)\,\exists U$ open neighbourhood of x in $\mathcal{T}_X$, s.t. $f(U) \subset V$

$(U \subset f^{-1}(V))$. f is a *homeomorphism* if f is bijective, and f, f^{-1} are continuous.

Remark 1.13. It holds that

$$\begin{aligned} f^{-1}(A \cup B) &= f^{-1}(A) \cup f^{-1}(B) \\ f^{-1}(A \cap B) &= f^{-1}(A) \cap f^{-1}(B) \\ f^{-1}(Y \setminus A) &= X \setminus f^{-1}(A) \end{aligned}$$

It is also valid that if

$$f_1 : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y), \quad f_2 : (Y, \mathcal{T}_Y) \to (X, \mathcal{T}_X)$$

and both are continuous, then $f_2 \circ f_1$ is also continuous!
Moreover

$$f \text{ continuous} \Leftrightarrow f \text{ continuous at every } x \in X$$

Example.

a) *Let $\mathcal{T} = 2^X$, $f : X \to Y$, let $\mathcal{T}_Y$ continuous. Any function is then continuous:*

$$\forall V \in \mathcal{T}_Y : f^{-1}(V) \in 2^X = \mathcal{T}_X \Rightarrow \textit{ continuous!}$$

b) *Let now $\mathcal{T}_X = \{\emptyset, X\}$, then the constant function $f(x) = y^*, \forall x \in X$ is contunuous:*

$$\forall V \in \mathcal{T}_Y : f^{-1}(V) = \begin{cases} X & \textit{if } y^* \in V \\ \emptyset & \textit{if } y^* \notin V \end{cases}$$

c) *If $\mathcal{T}_X = \{\emptyset, X\}$, and Y is Hausdorff, then the emphonly continuous function is the constant function! (Exercise!)*

We may consider the following: Let $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ and $A \subset X$. Then we can define the restriction $f_{|A} : A \to Y$. That's why we also need a topology on A (that is the induced topology). If f was continuous, then $f_{|A}$ is also continuous as a function mapping between $(A, \mathcal{T}_A)$ and $(Y, \mathcal{T}_Y)$.

Theorem 1.14 (Intermediate value theorem)
Let $(X, \mathcal{T})$ a connected topological space, $f : (X, \mathcal{T}) \to \mathbb{R}$ (on $\mathbb{R}$ we consider the standard topology), and let f be continuous. Assume there is $x, y \in X$ s.t. $f(x) < 0 < f(y)$. Then there exists a $z \in X$ s.t. $f(z) = 0$.

PROOF. Assume that $f(z) \neq 0, \forall z \in X$. This would mean that $0 \notin f(X)$. Consider $V = (0, \infty)$, which is open in $\mathcal{T}_{st}$. Then $f^{-1}(V)$ is open (as f is continuous) and is nonempty. We can take the complement of this set: $X = f^{-1}(V) \cup [f^{-1}(V)]^c$, and

$$[f^{-1}(V)]^c = f^{-1}(V^c) = f^{-1}((-\infty, 0)) = f^{-1}\left(\underbrace{(-\infty, 0)}_{\text{open}}\right)$$

is an open and nonempty set. As the space X is connected, this is not possible!

$\Rightarrow$ There must be $z \in X : f(z) = 0$. ■

1.2 Metric spaces

Definition 1.15 (Metric). A function $d : X \times X \to [0, \infty)$, $(x, y) \mapsto d(x, y)$ is called a *metric*, if

M1) $d(x, y) = 0 \Leftrightarrow x = y$ (Non-negativity)

M2) $d(x, y) = d(y, x), \forall x, y \in X$ (Symmetry)

M3) $d(x, y) \leq d(x, z) + d(z, y), \forall x, y, z \in X$ (Triangle inequality)

If d is a metric on X, then the pair (X, d) is called a *metric space*.

Definition 1.16 (Semimetric). The map $d : X \times X \to [0, \infty)$ is called a *semimetric*, if

M2) $d(x, y) = d(y, x), \forall x, y \in X$ (Symmetry)

M3) $d(x, y) \leq d(x, z) + d(z, y), \forall x, y, z \in X$ (Triangle inequality)

The non-negativity (which would make d a metric) is <u>not</u> satisfied!

A semimetric d can be extended to a metric as follows:

Take equivalence relation $x\tilde{y}$, if $d(x, y) = 0$, and then take $\tilde{X} = X_{\backslash \tilde{}}$, so

$$[x] \in \tilde{X} \Rightarrow [x] \coloneqq \{z \in X \,|\, z\tilde{x}\} = \{z \in X \,|\, d(x, z) = 0\}.$$

Set then

$$\tilde{d} : \tilde{X} \times \tilde{X} \to [0, \infty), \quad \tilde{d}([x], [y]) = d(x, y).$$

Check, that $\tilde{d}$ is a metric on $\tilde{X}$!

Example.

a) *On* $X = \mathbb{R}^n$*, the map* $d_\infty(x, y) \coloneqq \max_{j=1,\dots,n} |x_j - y_j|$ *is a metric.*

b) *On* $X = \mathbb{R}^n$ *define for* $1 \leq p < \infty$*:*

$$d_p(x, y) \coloneqq \left[\sum_{j=1}^{n} |x_j - y_j|^p\right]^{\frac{1}{p}}$$

is a metric on X*, for* $p = 2$ *it is the euclidean metric.*

c) *Let* $X = \ell_\infty \coloneqq \{a : \mathbb{N} \to \mathbb{R}|$ *boudned sequence*$\}$*. On this space we can define a metric by*

$$d_\infty(a, b) \coloneqq \sup_{j \in \mathbb{N}} |a_j - b_j|.$$

This metric is well-defined as all sequences in ℓ_∞ *are bounded!*

d) *On*

$$\ell_p \coloneqq \{a : \mathbb{N} \to \mathbb{R}| \sum_{j=0}^{\infty} |a_j|^p < \infty\}$$

we can define a metric by

$$d_p(a, b) \coloneqq \left[\sum_{j=0}^{\infty} |a_j - b_j|^p\right]^{\frac{1}{p}}.$$

This expression is finite since we take sequences in ℓ_p*.*

e) Pull-back metric*: Let* $X, (Y, \mathcal{T}_Y)$ *given,* $f : X \to Y$ *injective. Then*

$$d_X(x, y) \coloneqq d_Y(f(x), f(y))$$

is a metric on X*.*
Exercise: *Show that* d_x *is a metric iff* f *is injective and* d_Y *is a metric!*

f) *Let* $x = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$*, and* $Y = [-1, 1]$ *with the standard metric* $d_Y(y_1, y_2) = |y_1 - y_2|$*. Let now* $f : X \to Y$ *given by*

$$f(x) \coloneqq \begin{cases} -1 & x = -\infty \\ \frac{x}{1+|x|} & x \in \mathbb{R} \\ 1 & x = +\infty \end{cases}$$

Then we have that d_x *[...??]*

Some definitions

Definition 1.17. Let (X, d) metric space, $A, B \subset X$. The *diameter* of A is defined as

$$diam(A) \coloneqq \sup_{x,y \in A} d(x, y).$$

The *distance between two sets* is defined as

$$dist(A, B) \coloneqq \inf_{x \in A, y \in B} d(x, y),$$

ald the *distance between a set and a point* is defined as

$$dist(x, A) \coloneqq \inf_{y \in A} d(x, y).$$

A *Neighbourhood of a set* A is given by

$$B_r(A) \coloneqq \{y \in X : d(y, A) < r\}.$$

A *Ball of radius* r *centered at* x is given by

$$B(x, r) \coloneqq \{y \in X | \, d(y, x) < r\}.$$

Proposition 1.18 (Topology induced by a metric)
Let (X, d) *metric space. Define* $\mathcal{T}_d \subset 2^X$ *as*

$$\mathcal{T}_d \coloneqq \{V \in X | \, \forall x \in V \, \exists \varepsilon > 0 : B(x, \varepsilon) \subset V\}.$$

Then, $\mathcal{T}_d$ *is a topology on* X *and* $(X, \mathcal{T}_d)$ *is a Hausdorff-space.*

Proof. $\mathcal{T}_d$ is a topology (easy exercise).
Let $x \neq y, x, y \in X$. We need to show that there are $U_x, U_y \in \mathcal{T}_d$ s.t. $x \in U_x$, $y \in U_y$ and $U_x \cap U_y = emptyset$.
Try with $U_x = B(x, \varepsilon_1)$, $U_y = B(y, \varepsilon_2)$. What is unknown up till now are the values of $\varepsilon_1, \varepsilon_2$. Define $z \in U_x \cap U_y$. This point exists, iff $d(z, x) < \varepsilon_1$ and $d(z, y) < \varepsilon_2$. This means

$$d(x, y) \leq d(x, z) + d(z, y) < \varepsilon_1 + \varepsilon_2.$$

If $\varepsilon_1 + \varepsilon_2 < d(x, y)$ then $U_x \cap U_y = \emptyset$. This is always possible as we can chose $\varepsilon_1, \varepsilon_2$ so small, that the sum of them is smaller than $d(x, y)$.

Are those balls open in sense of the topology $\mathcal{T}_d$?
It can be checked, that every ball $B(x, r) \in \mathcal{T}_d$ (is open in $\mathcal{T}_d$). ∎

Definition 1.19. $B(x,r)$ is called an *open ball of radius r centered in x*. A *closed ball of radius r centered in x* is given by

$$\bar{B}(x,r) := \{y \in X|\, d(x,y) \leq r\}.$$

The *closure of an open ball $B(x,r)$* is defined as

$$\overline{B(x,r)} = \text{ smallest open set (in the topol.) containing the ball.}$$

In general $\overline{B(x,r)} \subseteq \bar{B}(x,r)$.

Example.
Let $X = \{0,1\}$ and

$$d(x,y) := \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

Then $B(0,1) = \{z \in X|\, d(0,z) < 1\} = \{0\}$ and

$$[B(0,1)]^c = \{1\} = B(1,1).$$

So $B(0,1)$ is open and $B(0,1)^c$ is also open. But $B(0,1)^c$ is the complement of an open set, so it has to be closed as well. Therefore $B(0,1)$ is open and closed at the same time. One sees easily

$$\overline{B(0,1)} = \{0\}, \quad \bar{B}(0,1) = \{0,1\}.$$

Whenever we have a metric space, we can make it a topological space. When does the opposite hold? This property is called *Metrizability* of a topological space

Definition 1.20. A topological space $(X,\mathcal{T})$ s.t. $d : X \times X \to [0,\infty)$ s.t. (X,d) is a metric space, <u>and</u> $\mathcal{T} = \mathcal{T}_d$, then the topological space $(X,\mathcal{T})$ is called *metrizable*.

Remark 1.21. Not all Hausdorff spaces are metrizable!
Remark 1.22. In a Hausdorff-space $(X,\mathcal{T}_d)$ every convergent sequence has a unique limit.

Proposition 1.23
(X,d) metric space (hence $(X,\mathcal{T}_d)$ is Hausdorff-space). Let $x : \mathbb{N} \to X$ a sequence in X. The following are equal

1. $$(x_n)_{n\in\mathbb{N}} \text{ converges to } x^* \in X \text{ in sense of } (X,\mathcal{T}_d)$$

2. $$\forall \varepsilon > 0\, \exists k_0 \geq 0 : \forall k \geq k_0\, x_k \in \underbrace{B(x^*,\varepsilon)}_{d(x^*,x_k)<\varepsilon}$$

Proof. For all V open neighbourhood of x^* all but a finite number of x_k are in V.

$$\exists k_0 > 0\, \forall k \geq k_0\, x_k \in V \text{ iff } V = \text{ball}.$$

For the other direction: V is open neighbourhood of x^*. So there exists $\varepsilon > 0$ s.t.

$$B(x^*, \varepsilon) \subset V.$$

■

Open and closed sets can be characterized using convergent sequences:

Proposition 1.24
Let (X, d) metric space, $X \subset X$. The following are equivalent

1. *A is closed in topology, i.e. $A^c \in \mathcal{T}_d$*
2. *A is* sequentially closed, *i.e. all convergent sequences $x : \mathbb{N} \to A$ converge to a point $x^* \in A$*

Moreover $\forall A \subset X$, the closure $\bar{A}$ (topology) coincides with the sequential closure*:*

$$\bar{A} := \{x^* \in X \,|\, \exists x : \mathbb{N} \to A : \lim_{k \to \infty} x_k = x^*\}.$$

Proof. Exercise! ■

Now we want to speak about continuity in metric spaces.

Proposition 1.25
Let $(X, d_X), (Y, d_Y)$ be two metric spaces, and let $f : X \to Y$. Since we have metric spaces, we can define the corresponding topologies $\mathcal{T}_{d_X}, \mathcal{T}_{d_Y}$, so we can talk about continuity of f in sense of the topological spaces.
The following statements are equivalent

1. *f is continuous (in topology) as a map from $(X, \mathcal{T}_{d_X}) \to (Y, \mathcal{T}_{d_Y})$.*
2. *$\varepsilon - \delta$ continuity (in metric, pointwise), i.e.*

$$\forall x \in X,\, \forall \varepsilon > 0\, \exists \delta > 0 \text{ s.t. } d_x(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon.$$

3. *(Sequential continuity)*

$$\forall x^* \in X\, \forall \text{ sequence } x : \mathbb{N} \to X, k \mapsto x_k : x_k \to x^* \Rightarrow f(x_k) \to f(x^*).$$

(So convergence in topology = convergence in metric)

Proof. Exercise! ■

We now want to be able to compare metrics.

Definition 1.26. Let d_1, d_2 be two metrics on X. We say that d_1 is *stronger* than d_2, if for the corresponding topologies it holds that $\mathcal{T}_{d_1}$ is stronger (finer) than $\mathcal{T}_{d_2}$. Analogeously d_1 is *weaker* than d_2, if $\mathcal{T}_{d_1}$ is weaker (coarser) than $\mathcal{T}_{d_2}$.
d_1 and d_2 are called equivalent, if $\mathcal{T}_{d_1} = \mathcal{T}_{d_2}$.

We can give charakterizations of stronger / weaker metrics in terms of continuity.

Proposition 1.27
Let d_1, d_2 two metrics on X. The following are equivalent:

(1) *d_1 is stronger than d_2 (i.e. $\mathcal{T}_{d_2} \subset \mathcal{T}_{d_1}$).*

(2) *$Id : (X, \mathcal{T}_{d_1}) \to (X, \mathcal{T}_{d_2})$, with $x \mapsto Id(x) = x$ is continuous.*

(3) *Any sequence that is convergent in d_1 must also be convergent in d_2.*

(4)

$$\forall x \in X\, \forall \varepsilon > 0\, \exists \delta > 0 \text{ s.t. } d_1(x, y) < \delta \Rightarrow d_2(x, y) < \varepsilon.$$

PROOF. **(1) $\Leftrightarrow$ (2)** $f : X \to Y$constant $\Leftrightarrow$ $\forall V \subset \mathcal{T}_Y = \mathcal{T}_2\, f^{-1} \in \mathcal{T}_X = \mathcal{T}_1$. Then $\forall V \subset \mathcal{T}_2 \Rightarrow V \in \mathcal{T}_1$. This means for $f = Id$ that

$$f^{-1}(V) = V \in \mathcal{T}_1, \quad \forall V \in \mathcal{T}_2.$$

(2) $\Leftrightarrow$ (3) f is constant at x^*, if $\forall x_k \to x^* \Rightarrow f(x_k) \to f(x^*)$. For $f = Id : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ we have therefore that

$$d_1(x_k, x^*) \to 0 \quad \Rightarrow \quad \underbrace{d_2(\underbrace{f(x_k)}_{=:x_k}, \underbrace{f(x^*)}_{=:x^*})}_{=d_2(x_k, x^*)} \to 0$$

f is constant $\Leftrightarrow$ x_k converges in $d_1 \Rightarrow x_k$ converges in d_2.

(2) $\Leftrightarrow$ (4) f const $\Leftrightarrow$ $\varepsilon - \delta$ continuity:

$$\forall x \in X\, \forall \varepsilon > 0\, \exists \delta > 0 : \underbrace{d_X(x, y)}_{=d_1(x,y)} < \delta \Rightarrow \underbrace{d_Y(f(x), f(y))}_{=d_2(x,y)} < \varepsilon \Rightarrow \Big(d_1(x, y) < \delta \Rightarrow d_2(x, y) < \varepsilon\Big)$$

■

Example.
Let $X = \ell_1 := \{x : \mathbb{N} \to \mathbb{R} | \sum_{j=0}^{\infty} |x_j| < \infty\}$. Define

$$\begin{aligned} d_1(x, y) &:= \sum_{j=0}^{\infty} |x_j - y_j| \\ d_\infty(x, y) &:= \sup_{j \in \mathbb{N}} |x_j - y_j|. \end{aligned}$$

Both metrics are well defined, as no sequence in ℓ_1 can be divergent.

d_∞ is a weaker metric than d_1, if $d_1(x,y) < \delta \Rightarrow \sum_j |x_j - y_j| < \delta$ then $\sup_j |x_j - y_j| < \delta$, i.e. $d_2(x,y) < \delta$.
d_∞ is not equivalent to d_1! Take $x^{(k)} = (\underbrace{\frac{1}{k}, \frac{1}{k}, ..., \frac{1}{k}}_{k}, 0, 0, ..., 0)$. Then

$$d_\infty(x^{(k)}, 0) = \sup_j |x_j^k - 0| = \frac{1}{k} \stackrel{k\to\infty}{\to} 0,$$

that means $x^{(k)} \to 0$ in d_∞! Nevertheless we have in the other metric

$$d_1(x^{(k)}, 0) = \sum_{j=0}^{\infty} |x_j^{(k)} - 0| = k\frac{1}{k} = 1, \quad \forall k,$$

so $x^{(k)} \not\to 0$ in d_1.
$\Rightarrow d_\infty$ cannot be equivalent to d_1!

Definition 1.28 (Cauchy sequence). A sequence $x : \mathbb{N} \to X$ in a metric space (X, d) is a *Cauchy sequence*, if

$$\forall \delta > 0 \, \exists k_0 \geq 0 : \quad d(x_k, x_q) < \delta, \quad \forall k, q \geq k_0.$$

Remark 1.29. Any convergent sequence is a Cauchy sequence! The converse does not need to be true.

Definition 1.30 (Complete space). A metric space (X, d) is a *complete space*, if every Cauchy sequence has a limit in X.

Example.

- *Let $X = \mathbb{Q}$, with the standard metric $d(x,y) = |x - y|$. $(\mathbb{Q}, d)$ is not complete!*
- *(ℓ_p, d_p) for $1 \leq p \leq \infty$ with $d_p(x,y) = (\sum_j |x_j - y_j|^p)^{1/p}$ is complete!*
- *(ℓ_1, d_∞) is not complete! Take for $k \in \mathbb{N}$ the sequence $x^{(k)} = (\underbrace{1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{k+1}}_{k+1}, 0, ..., 0)$,*

  then $d_\infty(x^k, x^{k'}) \sup_j |x_j^k - x_j^{k'}| = \frac{1}{k+2}$, for $k' > k$, and this expression tends to 0 for $k, k' \to \infty$. Therefore it is a Cauchy sequence. However we have that $d_\infty(x^k, x^*) = \frac{1}{k+2} \to 0$.
- *Suppose that (X, d) is complete and let $A \subset X$, s.t. $A \neq X$ and A is dense in X, i.e. $\bar{A} = X$. Then (A, d_A) cannot be complete, because*

  $$\forall x^* \in X \, \exists x : \mathbb{N} \to A : \, x_k \to x^*,$$

  which also has to hold for all $x^* \in X \setminus A$ (by assumption $A \neq X$).

When can we say that (A, d_A) is complete (for $A \subset X$)?

Proposition 1.31
Let (X, d) complete metric space and $A \subset X$ a closed subset of X. Then (A, d_A) is a complete metric space.

Proof. Let $x : \mathbb{N} \to A$ Cauchy sequence. Then x is a Cauchy-sequence in a complete space X. Therefore exists $x^* \in X$ s.t. $x_k \to x^*$ and as A is closed ($\Leftrightarrow A$ sequentially closed) we have that $x^* \in A$. ■

Every noncomplete space can be extended to a complete space, up to an isometry.

Definition 1.32 (Isometry). Let $(X, d_X), (Y, d_Y)$ two metric spaces. A function $f : (X, d_X) \to (Y, d_Y)$ is an *isometry*, if

$$d_X(x, y) = d_Y(f(x), f(y)), \quad \forall x, y \in X.$$

Remark 1.33. If for the isometry f we have that $f(x) = f(y) \Rightarrow d_X(x, y) = 0 \Rightarrow x = y$, so an isometry must always be injective! We can even make it bijective by restricting it's image to $f(X) \subseteq Y$, i.e. $(X, d_X) \to (f(X), d_{Y|_{f(X)}})$.

A sequence x is convergent in X iff $f(x)$ is convergent in $f(X)$.

A sequence x is Cauchy sequence in X iff $f(x)$ is Cauchy sequence in $f(X)$.

Theorem 1.34 (Completion)
Let (X, d) metric space. Then exists a complete space $(\tilde{X}, \tilde{d})$ and an isometry $f : (X, d) \to (\tilde{X}, \tilde{d})$ s.t. $\mathcal{J}(X)$, is dense in $\tilde{X}$.

Remark 1.35. It is not always possible to just complete the space in X. However it works (see Theorem 1.34), if we first map the space using an isometry.
Proof. Do the proof in 3 steps: First construct $(\tilde{X}, \tilde{d})$, then prove that this space is complete. Afterwards construct an embedding of X (deine $\mathcal{J}$).

Step 1: We first define $X^{\mathbb{N}} := \{x : \mathbb{N} \to X\}$ (the set of sequences in X). Now restrict this space to the Cauchy sequences: $\hat{X} := \{x \in X^{\mathbb{N}} | \, x \text{ Cauchy sequence}\}$.
We can now define an equivalence relation $\sim$ by

$$x \sim y \text{ if } \lim_{j \to \infty} d(x_j, y_j) = 0$$

Then define the set of equivalence classes as

$$\tilde{X} = \hat{X}/_{\sim} = \{[x]\}$$

with

$$[x] := \{y \in \hat{X} | \, x \sim y\}.$$

Property of $\hat{X}$: $\forall x, y \in \hat{X}$, $a : \mathbb{N} \to \mathbb{R}_+$, $a_j = d(x_j, y_j)$. Then a is a Cauchy sequence in $\mathbb{R}$, so it is convergent!

$$|a_j - a_{j'}| \to 0, \quad j, j' \to \infty.$$

x, y Cauchy, so $\forall \delta > 0\, \exists j_0, j_0'$ s.t.

$$d(x_j, x_j') < \delta \quad \forall j, j' \geq j_0$$

and

$$d(y_j, y_j') < \delta \quad \forall j, j' \geq j_0'.$$

Then

$$a_j = d(x_j, y_j) \leq \underbrace{d(x_j, x_k)}_{<\delta} + \underbrace{d(x_k, y_k)}_{a_k} + \underbrace{d(y_k, y_j)}_{<\delta}, \quad \forall k, j \geq \max\{j_0, j_0\}$$

So $a_j \leq 2\delta + a_k$, $\forall j, k \geq \max\{j_0, j_0'\}$ and $a_k \leq 2\delta + a_j\, \forall k, k \geq \max\{j_0, j_0'\}$. Therefore $|a_j - a_k| \to 0$, for $j, k \to \infty$.
We can now define

$$\tilde{d}([x], [y]) := \lim_{j \to \infty} d(x_j, y_j).$$

This $\tilde{d}$ is well defined: Let $x' \in [x]$. Then $\lim_{j \to \infty} d(x_j', y_j) \leq \lim d(x_j', x_j) + d(x_j, y_j) = \lim_j d(x_j, y_j)$. It is easy to check that $\tilde{d}$ is a metric on $\tilde{X}$.

Step 2: Prove that $(\tilde{X}, \tilde{d})$ is complete.
Let $[x^{(k)}]$ (each $x^{(k)} \in \hat{X}$ is Cauchy sequence) be a Cauchy sequence in $\tilde{X}$. Then

$$\forall \delta > 0\, \exists k_0 \geq 0 \text{ s.t. } \tilde{d}([x^{(k)}], [x^{(k')}]) < \delta, \quad \forall k, k' \geq k_0.$$

Construction of the limit: $x^{(k)}$ (for fixed k) is Cauchy in X, so

$$\forall \delta\, \exists k_0 \text{ s.t. } d(x_j^{(k)}, x_j^{(k')}) < \delta, \quad \forall j, j' \geq k_0.$$

Choose now $\delta = \frac{1}{k}$, and $j_k - k_0$. Then

$$d(x_j^{(k)}, d_{j_k}^{(k)}) < \frac{1}{k}, \quad \forall j \geq j_k.$$

Define

$$y_k := \underbrace{x_{j_k}^{(k)}}_{\text{one element in the seq. } x^{(k)}}$$

and

$$y := (y_k)_{k \in \mathbb{N}} \in X^{\mathbb{N}}.$$

We constructed a sequence in X. We need to prove that $y \in \hat{X}$ (then we can define $[y]$) and that $\tilde{d}([x^{(k)}], [y]) \to 0$ for $k \to \infty$.
We first show that $y \in \hat{X}$ is Cauchy sequence. Look at

$$d(x_k, y_{k'}) < \delta, \quad \forall k, k' \geq j_0.$$

For our sequence this is

$$
\begin{aligned}
d(x^k_{j_k}, x^{(k)}_{j_{k'}}) &\le \underbrace{\overbrace{d(x^{(k)}_{j_k}, x^{(k)}_j)}^{\text{same seq.}}}_{<\frac{1}{k}} + \overbrace{d(x^{(k)}_j, x^{(k')}_j)}^{\text{same ele., diff. seq.}} + \underbrace{d(x^{(k')}_j, x^{(k)}_{j'_k})}_{<\frac{1}{k'}} \\
&\le \underbrace{\frac{1}{k} + d(x^{(k)}_j, x^{(k')}_j) + \frac{1}{k'}}_{\forall j \ge \max\{j_k, j_{k'}\}} \\
&\le \frac{1}{k} + \frac{1}{k} + \underbrace{\lim_{j\to\infty} d(x^{(k)}_j, x^{(k')}_j)}_{\tilde{d}([x^k],[x^{k'}]) \overset{\text{Cauchy}}{\to} 0, k,k'\to\infty} \quad .
\end{aligned}
$$

Therefore

$$d(y_k, y_{k'}) \to 0, \quad k, k' \to \infty$$

so $y \in \hat{X}$.

Step 3: Consider

$$
\begin{aligned}
\tilde{d}([x^k], [y]) &= \lim_{j\to\infty} d(x^k_l, y_l) \\
&= \lim_{j\to\infty} d(x^{(k)}_l, y^{(l)}_{j_l}) \\
&\le \lim_{l\to\infty} \underbrace{d(x^k_l, x^k_{j_k})}_{\le \frac{1}{k}, f.l \ge j_k} + \underbrace{d(x^k_{j_k}, x^l_{j_l})}_{d(y_k, y_l)\to 0\, y \text{ Cauchy}}
\end{aligned}
$$

And therefore $[x^k] \to [y]$ in $\tilde{X}$.
$(\tilde{X}, \tilde{d})$ is complete. Define $\mathcal{J} : X \to \tilde{X}$ as $\mathcal{J}(x) = [\bar{x}]$, $\bar{x}$ sequence with $\bar{x}_j = x$ for all j. Then

$$\tilde{d}(\mathcal{J}(x), \mathcal{J}(x')) = \tilde{d}([\bar{x}], [\bar{x'}]) = \lim_{j\to\infty} d(\bar{x}_j, \bar{x}_{j'}) = d(x, x').$$

Therefore $\mathcal{J}$ is an isometry!
For $[x] \in \tilde{X}$ define $\bar{x}^{(k)}_j = (x_k), \quad \forall j$. Then $\bar{x}^k$ is a sequence in $f(X)$ and $\tilde{d}([x], [x^k]) \to 0$ for $k \to \infty$, so $\mathcal{J}(X)$ is dense in $\tilde{X}$. ∎

1.3 Normed spaces

To define a *normed space*, the set X must at least be a vector space (always over $\mathbb{R}$ or $\mathbb{C}$).

Definition 1.36 (Normed space). Let X be a $\mathbb{K}$-vector space, for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. A map $\|\cdot\| : X \to [0, \infty)$ with $x \mapsto \|x\|$ is called a *norm*, if

N_1) Definiteness $\|X\| = 0 \Rightarrow x = 0$

N_2) Homogenity $\|\alpha x\| = |\alpha| \, \|x\|, \forall x \in X, \forall \alpha \in \mathbb{K}$

N_3) Triangle inequality $\|x + y\| \leq \|x\| + \|y\|, \quad \forall x, y \in X$

In this case, the pair $(X, \|\cdot\|)$ is called a *normed space.*

Remark 1.37. 1. $\|0\| = \|0x\| = 0\|x\| = 0$.

2. A *seminorm* just satisfies properties N_2 and N_3, but not N_1. We can extend a seminorm $\|\cdot\|$ to a norm by taking

$$X/\sim, \quad x \sim y \text{ if } \|x - y\| = 0$$

analogeously to what we did for a semimetric in order to extend it to a metric.

3. Let $(X, \|\cdot\|)$ be a normed space. The function

$$\begin{aligned} d : &X \times X \to [0, \infty) \\ &(x, y) \mapsto \; d(x, y) = \|x - y\| \end{aligned}$$

then defines a metric on X.

Notions of convergence, continuity, completeness are defined on normed spaces, using the metric d, induced by the norm $\|\cdot\|$ (see remark above).

We have an additional property:

$$\|x - y\| \geq | \, \|x\| - \|y\| \, |$$

This holds because of putting the following 2 estimates together:

$$\begin{aligned} \|x\| &\leq \|x - y\| + \|y\|, \\ \|y\| &\leq \|x - y\| + \|x\|; \end{aligned}$$

Definition 1.38 (Banach space). A normed space $(X, \|\cdot\|)$ is called a *Banach space*, if X is complete under the induced metric d (by the norm $\|\cdot\|$).

Example.
The spaces

$$\begin{aligned} \ell_p &:= \{x : \mathbb{N} \to \mathbb{R} | \sum_{j \in \mathbb{N}} |x_j|^p < \infty\}, \quad 1 \leq p < \infty \\ \ell_\infty &:= \{x : \mathbb{N} \to \mathbb{R} | \sup_{j \in \mathbb{N}} |x_j| < \infty\} \end{aligned}$$

can be equipped with the norm

$$\|x\|_p := \left(\sum_{j\in\mathbb{N}|x_j|^p}\right)^{\frac{1}{p}},$$

respectively

$$\|x\|_\infty := \sup_{j\in\mathbb{N}} |x_j|.$$

The spaces $(\ell_p, \|\cdot\|_p)$ *is a Banach space for* $1 \le p \le \infty$*.*

We also want to be able to compare norms.

Definition 1.39. Let $\|\cdot\|_1, \|\cdot\|_2$ be two norms on X. We say that $\|\cdot\|_1$ is *stronger* than $\|\cdot\|_2$, if d_1 is stronger than d_2 (i.e. for the corresponding topologies we have $\mathcal{T}_1 \subset \mathcal{T}_2$).
$\|\cdot\|_1$ is *weaker* than $\|\cdot\|_2$ if d_1 is weaker than d_2.
$\|\cdot\|_1$ and $\|\cdot\|_2$ are *equivalent*, if d_1, d_2 are equivalent.

Proposition 1.40
Let $\|\cdot\|_1, \|\cdot\|_2$ *be two norms on* X*. Then* $\|\cdot\|_1$ *is stronger than* $\|\cdot\|_2$*, if there is a constant* $c > 0$*, s.t.*

$$\|x\|_2 \le C\|x\|_1, \quad \forall x \in X.$$

$\|\cdot\|_1, \|\cdot\|_2$ *are equivalent, if there are* $c, C > 0$ *s.t.*

$$c\|x\|_1 \le \|x\|_2 \le C\|x\|_1, \quad \forall x \in X.$$

PROOF. $\|\cdot\|_1$ is stronger than $\|\cdot\|_2$. This means that

$$\forall x \in X\, \forall \varepsilon > 0\, \exists \delta_{x,\varepsilon} > 0$$

s.t.

$$\underbrace{d_1(x,y)}_{\|x-y\|_1} < \delta_{x,\varepsilon} \Rightarrow \underbrace{d_2(x,y)}_{\|x-y\|_2} < \varepsilon.$$

So, for fix $x = 0, \varepsilon = 1$ there is $\delta > 0$ s.t.

$$\|y\|_1 < \delta \Rightarrow \|y\|_2 < 1.$$

For any $y \in X$ let $z_\varepsilon := y\frac{\delta}{\|y\|_1+\varepsilon}$, for $\varepsilon > 0$. Using this we have

$$\|z_\varepsilon\|_1 \le \delta \Rightarrow \|z_\varepsilon\|_2 < 1$$

so

$$\|y\|_2 \le \frac{1}{\delta}\left(\|y\|_1 + \varepsilon\right), \quad \forall \varepsilon > 0.$$

For $\varepsilon \to 0$ we have then that $\|y\|_2 \le \frac{1}{\delta}\|y\|_1$. ■

1.4 Hilbert spaces

We want to be able to measure angles between vectors, i.e. a scalar product.

Definition 1.41 (Sesquilinear form). Let X be a $\mathbb{K}$-vector space. A map

$$\langle \cdot, \cdot \rangle : X \times X \to \mathbb{K}, \quad (x, y) \mapsto \langle x, y \rangle$$

is called a *sesquilinear form*, if it is linear in the first, and antilinear in the second argument, i.e.

$$\begin{aligned} \langle \alpha x, y \rangle &= \alpha \langle x, y \rangle \\ \langle x, \alpha y \rangle &= \bar{\alpha} \langle x, y \rangle \\ \langle x + x', y \rangle &= \langle x, y \rangle + \langle x', y \rangle \\ \langle x, y + y' \rangle &= \langle x, y \rangle + \langle x, y' \rangle. \end{aligned}$$

A sesquilinear form is called

- *symmetric*, if $\langle x, y \rangle = \langle x \bar{,} y \rangle, \quad \forall x, y \in X$;
- *positive semidefinite*, if $\langle x, x \rangle \geq 0, \quad \forall x \in X$;
- *positive definite*, if $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ iff $x = 0$.

Notations for the sesquilinear form: $\langle \cdot, \cdot \rangle, (\cdot, \cdot), (\cdot, \cdot)_X$ depending on the situation.

In the following we assume the symmetry and positive semidefiniteness!

Lemma 1.42
Let X be a $\mathbb{K}$-vector space and $\langle \cdot, \cdot \rangle$ be a sesquilinearform, symmetrical and positive semidefinite. Set

$$\| \cdot \| : X \to [0, \infty), \quad x \mapsto \|x\| := \sqrt{\langle x, x \rangle}.$$

This is well defined, since $\langle x, x \rangle \geq 0$, $\forall x \in X$.
Then we have

1. *Homogenity* $\|\alpha x\| = |\alpha| \, \|x\|, \quad \forall x \in X, \forall \alpha \in \mathbb{C}$
2. *Cauchy-Schwarz inequality*
$$|\langle x, y \rangle| \leq \|x\| \, \|y\|, \quad \forall x, y \in X$$
3. *Triangle inequality*
$$\|x + y\| \leq \|x\| + \|y\|, \quad \forall x, y \in X$$
4. *Parallelogram identity*
$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2, \quad \forall x, y \in X$$

From above properties 1 and 3 we know that $\| \cdot \|$ is a seminorm. It is even a norm, if $\langle \cdot, \cdot \rangle$ is positive definite.
Proof. Exercise! ∎

Definition 1.43. A positive definite, symmetrical sesquilinear form on X is called a *scalar product.*
The pair $(X, \langle \cdot, \cdot \rangle)$ is called a *Pre-Hilbert space.*

Remark 1.44. Scalar product $\Rightarrow$ norm $\Rightarrow$ distance $\Rightarrow$ Topology.

Lemma 1.45
Let $(X, \| \cdot \|)$ normed space. Then there exists a scalar product $\langle \cdot, \cdot \rangle$ on X, s.t.

$$\|x\| = \sqrt{\langle x, x \rangle}$$

iff $\| \cdot \|$ satisfies the parallelogram identity

$$\|x + y\| + \|x - y\| = 2\|x\|^2 + 2\|y\|^2, \quad \forall x, y \in X.$$

Proof. Exercise! ∎

Definition 1.46 (Hilbert space). A *Hilbert space* is a Pre-Hilbert space $(X, \langle\cdot,\cdot\rangle)$ (with a scalar-product $\langle\cdot,\cdot\rangle$) that is complete under the norm (i.e. the metric) induced by the scalar product.

Chapter 2

Function spaces

2.1 Bounded functions

We consider functions $f : X \to Y$, where X is some set and Y is at least a normed space.

Definition 2.1. Let X be a set and Y a normed $\mathbb{K}$-vector space $(X, \|\cdot\|)$. The *space of bounded functions* is defined as

$$B(X;Y) \coloneqq \{f : X \to Y | \sup_{x \in X} \|f(x)\| < \infty\}.$$

Remark 2.2. 1. Y is a vector space, so $B(X;Y)$ is a $\mathbb{K}$-vector space, with

$$(f+g)(x) = f(x) + g(x), \quad (\alpha f)(x) = \alpha f(x)$$

2. Let $\|\cdot\| : B(X;Y) \to [0,\infty)$ with $f \mapsto \|f\| \coloneqq \sup_{x \in X} \|f(x)\|$. This defines a norm on $B(X;Y)$ (Exercise!), so $(B(X;Y), \|\cdot\|)$ is a normed space!

Is the space $(B(X;Y), \|\cdot\|)$ complete?

Proposition 2.3
Let Y be a Banach space. Then $(B(X;Y), \|\cdot\|)$ is a Banach space.

Proof. We need to show that every Cauchy sequence has a limit in $B(X;Y)$. Let f_n be a Cauchy sequence in $B(X;Y)$. This means that

$$\forall \varepsilon > 0 \, \exists k_0 \geq 0 : \quad \forall n, m \geq k_0 \, \|f_n - f_m\| < \varepsilon.$$

This shows that

$$\sup_x \|f_n(x) - f_m(x)\| < \varepsilon,$$

and therefore $f_n(x)$ is a Cauchy-sequence in Y, for all $x \in X$. As Y is complete (as it is Banach space), for each $x \in X$ there is $f(x) \in Y$ s.t.

$$f_n(x) \to f(x).$$

Remark: We know that f_n is Cauchy-sequence. Thus, for $\varepsilon = 1$ there is $k_0 \geq 0$ s.t.

$$\|f_n - f_m\| < 1, \quad \forall n, m \geq k_0,$$

and

$$\|f_n\| \leq \overbrace{\|f_n - f_{k_0}\|}^{<1} + \overbrace{\|f_{k_0}\|}^{M}, \quad \forall n \geq k_0.$$

$\Rightarrow \|f_n\| \leq 1 + M, \quad \forall n \geq k_0$ which means that $\|f_n(x)\| \leq 1 + M$ for all $x \in X, n \geq k_0$.

Using this remark, we can do the following

$$\|f(x)\| \leq \underbrace{\|f(x) - f_n(x)\|}_{\to 0,\text{ for } n \to \infty} + \underbrace{\|f_n(x)\|}_{\leq 1+M}.$$

There is n s.t. $\|f(x) - f_n(x)\| < 1 \Rightarrow \|f(x)\| \leq 2 + M \Rightarrow \|f\| = \sup_x \|f(x)\| \leq 2 + M$. Therefore $f \in B(X;Y)$.

Does $f_n \to f$ hold? I.e. $\forall \varepsilon > 0\, \exists n_0 \geq 0$ s.t.

$$\sup_x \|f_n(x) - f(x)\| = \|f_n - f\| < \varepsilon, \quad \forall n \geq n_0.$$

As f_n is Cauchy sequence there is n_0 independent of x, s.t.

$$\forall n, m \geq n_0 : \quad \sup_x \|f_n(x) - f_m(x)\| < \frac{\varepsilon}{2}.$$

$\forall x \in X : f_n(x) \to f(x)$, hence $\exists n_x \geq 0$ s.t. $\forall n \geq n_x$ it holds that

$$\|f_n(x) - f(x)\| < \frac{\varepsilon}{2}.$$

Take any $n \geq n_0$. Then for $n \geq n_x \Rightarrow \|f_n(x) - f(x)\| < \frac{\varepsilon}{2}$. Moreover we have for $n_0 \leq n < n_x$ that

$$\|f_n(x) - f(x)\| \leq \|f_m(x) - f_n(x)\|_{\|} f_m(x) - f(x)\|, \quad \forall m.$$

As m is freely chosen, we can especially take $m \geq n_x > n_0$. Then

$$\|f_n(x) - f(x)\| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This means that $\forall x \in X : \|f_n(x) - f(x)\| < \varepsilon$, so $\|f_n - f\| < \varepsilon$ and therefore $f_n \to f$. ■

Important special case: $Y = \mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Then we write $B(X;\mathbb{K}) = B(X)$.

We now want to know, when $B(X;Y)$ is separable. In general, this is not the case.

Proposition 2.4
The space $B(X;\mathbb{R})$ is separable iff X is finite. In particular

$$\ell_\infty = B(\mathbb{N};\mathbb{R}) = \text{ set of bounded sequences}$$

is not separable!

Proof. Exercise! ■

2.2 Continuous functions

We again consider functions of the form $f : X \to Y$, where Y is a vector space. For considering continuous functions, we need at least a topology on X.

Definition 2.5. Let $(X, \mathcal{T})$ be a topological, and $(Y, \|\cdot\|)$ a normed space. The *space of continuous functions* $X \to Y$ is defined as

$$C(X;Y) \coloneqq \{f : X \to Y \mid f \text{ continuous}\}.$$

The *set of bounded continuous functions* $X \to Y$ is defined as

$$C_b(X;Y) \coloneqq B(X;Y) \cap C(X;Y).$$

Remark 2.6. $(B(X;Y), \|\cdot\|)$ is complete. Now we have $(C_b(X;Y), \|\cdot\|)$ (with the norm of $B(X;Y)$ restricted to continuous functions), which is again a normed space.

Is the space $(C(X;Y), \|\cdot\|)$ complete?

Theorem 2.7
Let Y be a Banach space. Then, $(C(X,Y,\|\cdot\|)$ is Banach space too.

Proposition 2.8
Let $U \subset \mathbb{R}^n$ open and let $(K_i)_{i\in\mathbb{N}}$ family of compact sets $K_i \subset \mathbb{R}^n$, s.t.

$$K_i \subseteq K_{i+1}, \quad \forall i$$

with

$$U = \bigcup_{i=0}^{\infty} K_i.$$

We assume, that $\forall x \in U \exists i_k \in \mathbb{N}, r_k > 0$ s.t. $B(x, r_k) \subset K_{i_k}$, $(\forall i \geq ik)$. We define for any $f \in C(U;Y)$

$$\|f\|_i := \sup_{x\in K_i} \|f(x)\| < \infty, \qquad \text{since } K_i \text{ compact}, \quad \forall i \in \mathbb{N}.$$

Moreover, $\forall f, g \subset C(U;Y)$ let

$$d(f,g) := \sum_{i=1}^{\infty} 2^{-1} \underbrace{\frac{\|f-g\|_i}{1+\|f-g\|_i}}_{\leq 1} \leq \sum_{i=1}^{\infty} 2^{-1} = 1.$$

Then: $\|\cdot\|_i$ *is a* seminorm *on* $C(U;Y)$, $\forall i$ *and* d *ia a* metric *on* $C(U;Y)$.

Remark 2.9. Above metric d is called the *Fréchet metric*, generated by the family of seminorms $(\|\cdot\|_i)_{i=1}^{\infty}$.
Proof. Exercise! ■

Proposition 2.10
Let $f : \mathbb{N} \to C(U;Y)$ be a sequence of fuctions in $C(U;Y)$, and let U as in Proposition 2.8. The following statements are equivalent

(a) $g \in C(U;Y)$ *s.t.* $\lim_{n\to\infty} d(f_n, g) \to 0$ *(i.e. $f_n \to g$ in d).*

(b) $\forall i \geq 1 : \lim_{n\to\infty} \|f_n - g\|_i = 0$.

(c) $\forall K \subset U$ *compact, it holds that* $\sup_{x\in K} \|f_n(x) - g(x)\| \overset{n\to\infty}{\to} 0$.

Proof. $(a) \Rightarrow (b)$

$$d(f_n, g) = \sum_{i=1}^{\infty} 2^{-1} \frac{\|f_n - g\|_i}{1+\|f_n-g\|_i} \Rightarrow \Big[d \to 0 \Rightarrow \|f_n - g\|_i \to 0, \quad \forall i \Big].$$

$(b) \Rightarrow (a)$

$$\|\cdot\|_i \to 0, \quad \forall i$$

implies that

$$\forall \varepsilon > 0 \exists i_0(\varepsilon) \text{ s.t. } \sum_{i\geq i_0}^{\infty} 2^{-2} \frac{\|\cdot\|_i}{1+\|\cdot\|_i} \leq \sum_{i\geq i_0}^{\infty} 2^{-1} < \varepsilon,$$

and this implies

$$d(f_n, g) \leq \sum_{i=1}^{i_0-1} 2^{-1} \frac{\|f_n - g\|_i}{1 + \|f_n - g\|_i} + \varepsilon$$

so there is n s.t. $\|f_n - g\|_i < \varepsilon, \quad \forall n \geq n_0, \quad \forall i \in [1, .., i_0]$. Therefore $d(f_n, g) \leq 2\varepsilon, \quad \forall n \geq n_0$, so $d \to 0$.

$(b) \Leftarrow (c)$ Easy.

$(b) \Rightarrow (c)$ $K \subset U$ compact, therefore $\forall x \in K \exists i_k \in \mathbb{N}$ and $r_x > 0$ s.t.

$$B(x, r_x) \subset K_{i_x}.$$

$K \subset \bigcup_{x \in K} B(x, r_x)$ and as K compact there are $x_1, ..., x_q$ with

$$K \subset \bigcup_{j=1}^{q} B(x_j, r_{x_j}) \subset \bigcup_{j=1}^{q} K_{i_{x_j}} \subset K_{\bar{i}},$$

where $\bar{i} = \max_{j=1,\ldots,k}\{i_{x_j}\}$. Then

$$\|f_n - g\|_i \to 0 \Rightarrow \sup_{x \in K} \|f_n(x) - g(x)\| \to 0.$$

∎

Theorem 2.11
*Let $X = U \subset \mathbb{R}^n$ open (as in **??**). If $(Y, \|\cdot\|)$ is a Banach space, then $(C(X; Y), d)$ is a complete metric space. If $(Y, \|\cdot\|)$ is a separable normed space, then $(C(X; Y), d)$ is a separable metric space.*

Remark 2.12. The space above is just a complete metric, not a Banach space (there is no norm defined).

Proof. Exercise: Use Proposition 2.10 ∎

2.3 Differentiable functions

Definition 2.13. Let $U \subset \mathbb{R}^n$ open. Then we define

$$C^k(U; \mathbb{R}^m) := \{f : U \to \mathbb{R}^m | \, f \; k \text{ times differentiable and } D^l f \text{ continuous, } \forall 0 \leq l \leq k\}$$

$$C^\infty(U, \mathbb{R}^m) := \{f : U \to \mathbb{R}^m | \, f \textbf{ smooth}\}.$$

Remark 2.14. For $l = 2$:

$$D^l f(x) = \left\{\frac{\partial}{\partial x_{j_1}} \frac{\partial}{\partial x_{j_2}} f\right\}_{j_1, j_2 = 1}^{n} = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} ... \partial_{x_n}^{\alpha_n} f = \partial^\alpha f,$$

with the multi-index α satisfying $|\alpha| = \sum_{j=1}^{n} \alpha_j = 2$, and $\alpha_j \in \mathbb{N}$, $\forall j$.

$D^l f(x)$ is an l-multilinear form (= matrix is a tensor).
On a finite dimensional space, all norms are equivalent.
Euclidean norm: $\|D^l f(x)\| = \sqrt{\sum_{\alpha,|\alpha|=l} \|\partial^\alpha f(x)\|^2}$. If the space is $\mathbb{R}^n$, the norm is often just written as $|\cdot|$.
$C^k \subset C(U;\mathbb{R}^m) \Rightarrow$ Use the metric d? (Yes, if U is bounded!) However, nothing guarantees the completeness of the space C^k with this metric d. We need to define a more suitable norm. If U is bounded, then $\overline{U}$ is compact. We restrict ourselves to functions that have a "nice" behaviour on $\overline{U}$, i.e. that do not explode on the boundary of U.

Definition 2.15. Let $U \subset \mathbb{R}^n$ open and bounded. Define

$$C^k(\overline{U};\mathbb{R}^m) := \{f \in C^k(U;\mathbb{R}^m) |\, D^l f \text{ has a continuous extension to } \overline{U}\}$$

We do the same with smooth functions:

$$C^\infty(\overline{U};\mathbb{R}^m) := \bigcap_{k=c}^{\infty} C^k(\overline{U};\mathbb{R}^m)$$

Remark 2.16. If $\overline{U}$ is closed and bounded in $\mathbb{R}^n$, then $\overline{U}$ is compact, so

$$\sup_{x\in U} \|D^l f(x)\| \leq \infty, \quad \forall 0 \leq l \leq k$$

Using those definitions above, we can define norms on the spaces.

Definition 2.17. We define

$$[f]_{C^l} := \sup_{x\in\overline{U}} \|D^l f(x)\|, \quad 0 \leq l \leq k, \quad f \in C^k(\overline{U};\mathbb{R}^m)$$

As $f \in C^k(\overline{U};\mathbb{R}^m)$ has a continuous extension on $\overline{U}$, and $\overline{U}$ is compact, the supremum exists. Therefore

$$[f]_{C^k} < \infty.$$

Next, we define

$$\|f\|_{C^k} := \sum_{l=0}^{k} [f]_{C^l}$$

and

$$d(f,g) := \sum_{i=0}^{\infty} 2^{-i} \frac{[f-g]_{C^i}}{1+[f-g]_i}.$$

This metric is welldefined for all $f, g \in C^\infty(\overline{U};\mathbb{R}^m)$.

Theorem 2.18
For all $0 \leq l \leq k$ *we have that* $[\cdot]_{C^l}$ *is a seminorm on* $C^k(\overline{U};\mathbb{R}^m)$*. Moreover,* $\|\cdot\|_{C^k}$ *is a norm on* $C^k(\overline{U};\mathbb{R}^m)$ *and* $d(\cdot,\cdot)$ *is a metric on* $C^\infty(\overline{U};\mathbb{R}^m)$*.*
In this case, we have that $Y = \mathbb{R}^m$*, and this space is complete and separable. Therefore* $(C^k(\overline{U},\mathbb{R}^m), \|\cdot\|_{C^k})$ *is a separable Banach space. Analogeously, the space* $(C^\infty(\overline{U};\mathbb{R}^m), d)$ *is a complete, separable, metric space.*

Theorem 2.19
Exercise!
Hint: *For the last part of the claim consider* $D^l f_n \to g_j$*, and show that* $g_l = D^l g_0$*.*

Remark 2.20. Consider a bounded subset $U \subset \mathbb{R}^m$. Then we can always construct a sequence $(K_i)_{i\in\mathbb{N}}$ of compact subsets, s.t.

$$\bigcup_i K_i = U.$$

Thus every $f \in C^k(U;\mathbb{R}^m)$ when restricted to a compact set, i.e. $f_{|K_i}$, implies that we can define $\|f_{|K_i}\|_{C^k}$, as $f \in C^k(K_i;\mathbb{R}^m)$. Using this, we can define a metric on $C^k(U;\mathbb{R}^m)$ (that is NOT a norm).

Next, we will talk about the *compact support* of a function.

Definition 2.21. Let $(X,\mathcal{T}), (Y,\|\cdot\|)$, and let $f : X \to Y$. The *support of* f is defined as

$$\operatorname{supp} f := \overline{\{x \in X |\, f(x) \neq 0\}}.$$

The emohspace of continuous functions with compact support is then defined as

$$C_c(X;Y) := \{f \in C(X;Y) |\ \operatorname{supp} f \text{ is compact}\}.$$

For all $U \subset \mathbb{R}^n$ that are open, and $Y = \mathbb{R}^m$, we define

$$C_c^k(U;\mathbb{R}^m) := \{f \in C^k(U;\mathbb{R}^m) |\ \operatorname{supp} f \text{ compact}\},$$

and

$$\mathcal{D}(U;\mathbb{R}^m) := C_c^\infty(U;\mathbb{R}^m) = C^\infty(U;\mathbb{R}^m) \cap C_c(U;\mathbb{R}^m).$$

In many cases the notion C_0^k instead of C_c^k is used.
If $Y = \mathbb{R}$, then we write $C^k(U), C_c^k(U)$ instead of $C^k(U;\mathbb{R}), C_c^k(U;\mathbb{R})$.
For $U \subset \mathbb{R}^n$, we have that $\operatorname{supp} f$ is compact $\Leftrightarrow$ $\operatorname{supp} f$ is bounded.

Example.
Let $f : [0,\infty) \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} 1-x & 0 \leq x \leq 1 \\ 0 & x \geq 1 \end{cases}$$

Then

$$f \in C_c([0,\infty),\mathbb{R}); \tag{2.1}$$
$$f \in C_c([0,\infty],\mathbb{R}); \tag{2.2}$$
$$f \notin C_c((0,1),\mathbb{R}). \tag{2.3}$$

Let's talk about weaker continuity.

Definition 2.22 (Hölder continuity). Let $\alpha \in (0,1]$, and let $A \subset \mathbb{R}^n$. We say that $f : A \to Y$ is *Hölder continuous with parameter* α, if

$$[f]_\alpha \coloneqq \sup_{x,y\in A, x\neq y} \frac{\|f(x)-f(y)\|}{\|x-y\|^\alpha} < \infty.$$

Remark 2.23. f is continuous in the classical sense, but not necessary differentiable. An easy counterexample can be constructed using the square-root, as

$$\frac{\sqrt{x}}{x^\alpha} < \infty \Rightarrow \alpha = \frac{1}{2}.$$

Definition 2.24. Let $\alpha \in (0,1]$, and $A \subset \mathbb{R}^n$. We define

$$C^0,\alpha(A;\mathbb{R}^m) \coloneqq \{f \in C_b(A;\mathbb{R}^m) |\ \underbrace{f \text{ is } \alpha\text{-Hölder continuous}}_{[f]_\alpha<\infty}\},$$

and let

$$\|f\|_\alpha \coloneqq \sup_{x\in A} \|f(x)\| + \underbrace{[f]_\alpha}_{\text{Quasiderivative}}.$$

Let now $U \subset \mathbb{R}^n$ open and bounded. Define

$$C^{k,\alpha}(U;\mathbb{R}^m) \coloneqq \{f \in C^k(\overline{U};\mathbb{R}^m) |\, [D^k f]_\alpha < \infty\}$$

(those functions have normal derivatives up to order k, and this k-th derivative is nearly again differentiable, but in fact just the quasiderivative exists). On this space define a norm

$$\|f\|_{k,\alpha} \coloneqq \|f\|_k + \underbrace{[D^k f]_\alpha}_{D^k f \text{ is 'almost' diff.}}$$