

# **Functional Analysis and PDE**

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# Chapter 0

## Introduction

Functional Analysis is the analysis on infinite dimensional spaces. Applications of FA are in geometry and topology, probability theory, numerical analysis mathematical physics, and PDEs.

**Example.**

$C^2(U)$ ,  $U \subset \mathbb{R}^n$  is the set of functions which are 2 times differentiable and have a bounded derivative.  $U$  should be a bounded, open set. Consider  $f \rightarrow 0$  on  $\partial U$  and

$$\Delta = \sum_{j=1}^n \left( \frac{\partial}{\partial u_j} \right)^2.$$

Fix  $f \in C(U)$  and look for a solution  $u \in C^2(U)$ , s.t.

$$\Delta u = f, \quad u = \delta^{-1} f.$$

What is the inverse operator  $\delta^{-1}$  in this case? That's what we need to study here.

### Program of the lecture

- Structures: We need to define *Topologies*, *Metrics*, *Norms* and *Scalarproducts* on infinite-dimensional spaces
- We introduce *Functional Spaces* as infinite-dimensional vector spaces of functions
- We need to talk about *Linear Operators* to be able to talk about their inverse (see ??)

# Chapter 1

## Structures

We consider *convergence*. We have already seen

$$x_n \rightarrow x \quad \Leftrightarrow \quad \forall \varepsilon > 0 \exists n_0 \forall n > n_0 : x_n \in (x - \varepsilon, x + \varepsilon)$$

In order to do so, we need the definition of a neighbourhood (topology), distances (metric), the length of a vector (norm, Banach spaces), the length and angles (scalar product, Hilbert spaces).

**All vector spaces we consider should base on  $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ !**

### 1.1 Topological spaces

Let  $X$  be a set, let  $2^X$  the set of all possible subsets of  $X$  (including the empty set).

**Definition 1.1 (Topology).** A Topology  $\mathcal{T}$  on the set  $X$  is a family of subsets of  $X$ , that means

$$\mathcal{T} \subseteq 2^X,$$

satisfying

T1  $\emptyset, X \in \mathcal{T}$

T2 Stability under finite intersection:

$$\forall A, B \in \mathcal{T} : A \cap B \in \mathcal{T}$$

T3 Stability under any possible union: Let  $\mathcal{I}$  be some set of indices (may not be countable). Then

$$\forall \{A_i\}_{i \in \mathcal{I}} \subseteq \mathcal{T} : \bigcup_{i \in \mathcal{I}} A_i \in \mathcal{T}$$

Any subset  $A \subset X$  is called an *open set*, if  $A \in \mathcal{T}$ . Else it is called a *closed set*.

$(X, \mathcal{T})$  is called a *topological space*!

*Remark 1.2.* Note that

$$\left( \bigcup_{i \in \mathcal{I}} A_i \right)^c = \bigcap_{i \in \mathcal{I}} A_i^c$$

for all families of open sets  $\{A_i\}_{i \in \mathcal{I}}$  and each index-set  $\mathcal{I}$ .

**Definition 1.3 (Coarser / finer topologies).** Let  $\mathcal{T}_1, \mathcal{T}_2$  be two topologies on  $X$  with  $\mathcal{T}_1 \subset \mathcal{T}_2$ , then we say that

- $\mathcal{T}_1$  is *coarser* / *weaker* than  $\mathcal{T}_2$
- $\mathcal{T}_2$  is *finer* / *stronger* than  $\mathcal{T}_1$

**Example.**

a)  $\mathcal{T} = 2^X$  is a topology on  $X \Rightarrow 2^X$  is the strongest (finest) topology on  $X$ .

Also  $\mathcal{T} = \{\emptyset, X\}$  is a topology on  $X \Rightarrow$  any topology  $\mathcal{T}'$  needs to contain  $\emptyset$  and  $X$ , so  $\mathcal{T} \subset \mathcal{T}'$ . This means that  $\mathcal{T}$  is the weakest / coarsest topology on  $X$

b) On  $\mathbb{R}$  there is a standard topology  $\mathcal{T}_{st}$ :

$$V \in \mathcal{T}_{st} \text{ iff } \forall x \in V \exists \varepsilon > 0 : (x - \varepsilon, x + \varepsilon) \subset V$$

c) relative topology: Let  $A \subset X$ , let  $\mathcal{T}$  be a topology on  $X$ . Then

$$\mathcal{T}_A \{A \cup V : V \in \mathcal{T}\}$$

- d) Intersection of topologies: Let  $\mathcal{I}$  an index set (may be uncountable), let  $(\mathcal{T}_i)_{i \in \mathcal{I}}$  be a family of topologies on  $X$ . Then we can define

$$\bigcap_{i \in \mathcal{I}} \mathcal{T}_i$$

and this is again a topology on  $X$ !

- e) Product topology: Let  $(X, \mathcal{T}_X)$  and  $(Y, \mathcal{T}_Y)$  two topological spaces. Let

$$\mathcal{S} = \{(V, X) \mid V \in \mathcal{T}_X\} \cup \{(X, W) \mid W \in \mathcal{T}_Y\}.$$

The product topology on  $X \times Y$  is the coarsest (weakest) topology on  $X \times Y$ , that contains  $\mathcal{S}$ . In particular it must contain all sets of the form  $U \times V$  for  $U \in \mathcal{T}_X, V \in \mathcal{T}_Y$ .

*Remark 1.4.* If nothing else is said, we consider the standard topology  $\mathcal{T}_{st}$  on  $\mathbb{R}$ !

**Definition 1.5 (Closure / boundary of a set).** Let  $(X, \mathcal{T})$  topological space,  $A \in X$ .

The *interior* of  $A$ ,  $A^\circ$  is the largest open set inside

$$A = \bigcup_{V \in \mathcal{T}, V \subseteq A} V \text{ (open).}$$

The *closure* of  $A$ ,  $\bar{A}$  is the smallest closed set containing

$$A = \bigcup_{V \text{ s.t. } V^c \in \mathcal{T}, V \supseteq A} V \text{ (closed).}$$

The *boundary* of  $A$  is given by

$$\partial A = \bar{A} \setminus A^\circ = \bar{A} \cap \underbrace{(X \setminus A^\circ)}_{(A^c)^c} \text{ (closed).}$$

**Definition 1.6 (dense set / separable set).**  $(X, \mathcal{T})$  topological space in  $X$ . Then

- $A \subset X$  is called *dense in  $X$* , if  $\bar{A} = X$
- $X$  is called *separable*, if there is a countable dense subset of  $X$

**Definition 1.7 (open neighbourhood).**  $(X, \mathcal{T})$  topological space,  $x \in X$ . A subset  $V \subseteq X$  is called an open neighbourhood of  $x$ , if  $V \in \mathcal{T}$  and  $x \in V$ .

**Definition 1.8 (Convergence in topology).** Let  $(\mathcal{T}, X)$  topological space. A sequence  $(x_n)_{n \in \mathbb{N}}$ , i.e. a map

$$\begin{aligned} x : \mathbb{N} &\rightarrow X \\ n &\rightarrow x_n, \end{aligned}$$

converges to  $x^* \in X$ , if

$$\forall V \text{ open neighbourhood of } x^* : \{n \mid x_n \in V^c\} \text{ is finite}$$

(i.e. there is just a finite number of elements that's not contained in  $V$ ). Then we say that  $x^*$  is a limit point for the sequence  $x_n$ .

**Example.**

- a) Let  $\mathcal{T}$  the standard topology on  $\mathbb{R}$ . Then the definition of converges equals the  $\varepsilon$ - $\delta$ -Definition of convergence in Analysis 3.
- b) If  $\mathcal{T} = 2^X$ , then  $x_n \rightarrow x^*$  iff  $x_n$  is constant up to a finite number of terms: As we have  $\mathcal{T} = 2^X$  especially the set  $V = \{x^*\}$  is open. This set gives us the result.
- c) If  $\mathcal{T} = \{\emptyset, X\}$ , then every sequence is convergent! Every point  $x^* \in X$  is a limit point.

**Definition 1.9 (Hausdorff space).** Let  $(X, \mathcal{T})$  topological space. It is called a *Hausdorff space*, if

$$\forall x, y \in X, x \neq y \exists U_x, U_y \in \mathcal{T} \text{ s.t. } x \in U_x, y \in U_y \text{ and } U_x \cap U_y = \emptyset.$$

**Proposition 1.10 (Limits in Hausdorff spaces are unique)**

If  $(X, \mathcal{T})$  is a Hausdorff space, then every sequence has at most one limit, so either the sequence has no limit or it is unique!

PROOF. Proof can be done by contradiction. ■

**Definition 1.11 (Connectedness).** A topological space  $(X, \mathcal{T})$  is *connected*, if we cannot write it as a union of 2 disjoint(!) (nonempty) open sets. So, if  $X$  is connected and  $V \in \mathcal{T}$  is an open set, there is no  $\emptyset \neq W \in \mathcal{T}$  with  $V \cap W = \emptyset$  and  $V \cup W = X$ .

$A \subseteq X$  is *connected*, if  $A$  is connected in  $\mathcal{T}_A$ .

**Definition 1.12 (Continuity).** Let  $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$  topological spaces,  $f : X \rightarrow Y$ .  $f$  is called *continuous*, if

$$\forall V \in \mathcal{T}_Y : f^{-1}(V) \in \mathcal{T}_X.$$

$f$  is *continuous at a point*  $x \in X$ , if

$\forall V$  open neighbourhoods of  $f(x) \exists U$  open neighbourhood of  $x$  in  $\mathcal{T}_X$ , s.t.  $f(U) \subset V$  ( $U \subset f^{-1}(V)$ ).  $f$  is a *homeomorphism* if  $f$  is bijective, and  $f, f^{-1}$  are continuous.

*Remark 1.13.* It holds that

$$\begin{aligned} f^{-1}(A \cup B) &= f^{-1}(A) \cup f^{-1}(B) \\ f^{-1}(A \cap B) &= f^{-1}(A) \cap f^{-1}(B) \\ f^{-1}(Y \setminus A) &= X \setminus f^{-1}(A) \end{aligned}$$

It is also valid that if

$$f_1 : (X, \mathcal{T}_X) \rightarrow (Y, \mathcal{T}_Y), \quad f_2 : (Y, \mathcal{T}_Y) \rightarrow (X, \mathcal{T}_X)$$

and both are continuous, then  $f_2 \circ f_1$  is also continuous!

Moreover

$$f \text{ continuous} \Leftrightarrow f \text{ continuous at every } x \in X$$

**Example.**

a) Let  $\mathcal{T} = 2^X$ ,  $f : X \rightarrow Y$ , let  $\mathcal{T}_Y$  continuous. Any function is then continuous:

$$\forall V \in \mathcal{T}_Y : f^{-1}(V) \in 2^X = \mathcal{T}_X \Rightarrow \text{continuous!}$$

b) Let now  $\mathcal{T}_X = \{\emptyset, X\}$ , then the constant function  $f(x) = y^*, \forall x \in X$  is continuous:

$$\forall V \in \mathcal{T}_Y : f^{-1}(V) = \begin{cases} X & \text{if } y^* \in V \\ \emptyset & \text{if } y^* \notin V \end{cases}$$

c) If  $\mathcal{T}_X = \{\emptyset, X\}$ , and  $Y$  is Hausdorff, then the *emphonly* continuous function is the constant function! (Exercise!)

We may consider the following: Let  $f : (X, \mathcal{T}_X) \rightarrow (Y, \mathcal{T}_Y)$  and  $A \subset X$ . Then we can define the restriction  $f|_A : A \rightarrow Y$ . That's why we also need a topology on  $A$  (that is the induced topology). If  $f$  was continuous, then  $f|_A$  is also continuous as a function mapping between  $(A, \mathcal{T}_A)$  and  $(Y, \mathcal{T}_Y)$ .

**Theorem 1.14 (Intermediate value theorem)**

Let  $(X, \mathcal{T})$  a connected topological space,  $f : (X, \mathcal{T}) \rightarrow \mathbb{R}$  (on  $\mathbb{R}$  we consider the standard topology), and let  $f$  be continuous. Assume there is  $x, y \in X$  s.t.  $f(x) < 0 < f(y)$ . Then there exists a  $z \in X$  s.t.  $f(z) = 0$ .



PROOF. Assume that  $f(z) \neq 0, \forall z \in X$ . This would mean that  $0 \notin f(X)$ . Consider  $V = (0, \infty)$ , which is open in  $\mathcal{T}_{st}$ . Then  $f^{-1}(V)$  is open (as  $f$  is continuous) and is nonempty. We can take the complement of this set:  $X = f^{-1}(V) \cup [f^{-1}(V)]^c$ , and

$$[f^{-1}(V)]^c = f^{-1}(V^c) = f^{-1}((-\infty, 0)) = f^{-1}\left(\underbrace{(-\infty, 0)}_{\text{open}}\right)$$

is an open and nonempty set. As the space  $X$  is connected, this is not possible!

$\Rightarrow$  There must be  $z \in X : f(z) = 0$ . ■

## 1.2 Metric spaces

**Definition 1.15 (Metric).** A function  $d : X \times X \rightarrow [0, \infty)$ ,  $(x, y) \mapsto d(x, y)$  is called a *metric*, if

- M1)  $d(x, y) = 0 \Leftrightarrow x = y$  (Non-negativity)
- M2)  $d(x, y) = d(y, x), \forall x, y \in X$  (Symmetry)
- M3)  $d(x, y) \leq d(x, z) + d(z, y), \forall x, y, z \in X$  (Triangle inequality)

If  $d$  is a metric on  $X$ , then the pair  $(X, d)$  is called a *metric space*.

**Definition 1.16 (Semimetric).** The map  $d : X \times X \rightarrow [0, \infty)$  is called a *semi-metric*, if

- M2)  $d(x, y) = d(y, x), \forall x, y \in X$  (Symmetry)
- M3)  $d(x, y) \leq d(x, z) + d(z, y), \forall x, y, z \in X$  (Triangle inequality)

The non-negativity (which would make  $d$  a metric) is not satisfied!

A semimetric  $d$  can be extended to a metric as follows:

Take equivalence relation  $x \tilde{y}$ , if  $d(x, y) = 0$ , and then take  $\tilde{X} = X_{\sim}$ , so

$$[x] \in \tilde{X} \Rightarrow [x] := \{z \in X \mid z \tilde{x}\} = \{z \in X \mid d(x, z) = 0\}.$$

Set then

$$\tilde{d} : \tilde{X} \times \tilde{X} \rightarrow [0, \infty), \quad \tilde{d}([x], [y]) = d(x, y).$$

Check, that  $\tilde{d}$  is a metric on  $\tilde{X}$ !

**Example.**

a) On  $X = \mathbb{R}^n$ , the map  $d_\infty(x, y) := \max_{j=1, \dots, n} |x_j - y_j|$  is a metric.

b) On  $X = \mathbb{R}^n$  define for  $1 \leq p < \infty$ :

$$d_p(x, y) := \left[ \sum_{j=1}^n |x_j - y_j|^p \right]^{\frac{1}{p}}$$

is a metric on  $X$ , for  $p = 2$  it is the euclidean metric.

c) Let  $X = \ell_\infty := \{a : \mathbb{N} \rightarrow \mathbb{R} \mid \text{bounded sequence}\}$ . On this space we can define a metric by

$$d_\infty(a, b) := \sup_{j \in \mathbb{N}} |a_j - b_j|.$$

This metric is well-defined as all sequences in  $\ell_\infty$  are bounded!

d) On

$$\ell_p := \{a : \mathbb{N} \rightarrow \mathbb{R} \mid \sum_{j=0}^{\infty} |a_j|^p < \infty\}$$

we can define a metric by

$$d_p(a, b) := \left[ \sum_{j=0}^{\infty} |a_j - b_j|^p \right]^{\frac{1}{p}}.$$

This expression is finite since we take sequences in  $\ell_p$ .

e) Pull-back metric: Let  $X, (Y, \mathcal{T}_Y)$  given,  $f : X \rightarrow Y$  injective. Then

$$d_X(x, y) := d_Y(f(x), f(y))$$

is a metric on  $X$ .

**Exercise:** Show that  $d_x$  is a metric iff  $f$  is injective and  $d_Y$  is a metric!

f) Let  $x = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ , and  $Y = [-1, 1]$  with the standard metric  $d_Y(y_1, y_2) = |y_1 - y_2|$ . Let now  $f : X \rightarrow Y$  given by

$$f(x) := \begin{cases} -1 & x = -\infty \\ \frac{x}{1+|x|} & x \in \mathbb{R} \\ 1 & x = +\infty \end{cases}$$

Then we have that  $d_x$  [...??]

Some definitions

**Definition 1.17.** Let  $(X, d)$  metric space,  $A, B \subset X$ . The *diameter* of  $A$  is defined as

$$\text{diam}(A) := \sup_{x, y \in A} d(x, y).$$

The *distance between two sets* is defined as

$$\text{dist}(A, B) := \inf_{x \in A, y \in B} d(x, y),$$

and the *distance between a set and a point* is defined as

$$\text{dist}(x, A) := \inf_{y \in A} d(x, y).$$

A *Neighbourhood of a set*  $A$  is given by

$$B_r(A) := \{y \in X : d(y, A) < r\}.$$

A *Ball of radius  $r$  centered at  $x$*  is given by

$$B(x, r) := \{y \in X : d(y, x) < r\}.$$

**Proposition 1.18 (Topology induced by a metric)**

Let  $(X, d)$  metric space. Define  $\mathcal{T}_d \subset 2^X$  as

$$\mathcal{T}_d := \{V \in 2^X \mid \forall x \in V \exists \varepsilon > 0 : B(x, \varepsilon) \subset V\}.$$

Then,  $\mathcal{T}_d$  is a topology on  $X$  and  $(X, \mathcal{T}_d)$  is a Hausdorff-space.

PROOF.  $\mathcal{T}_d$  is a topology (easy exercise).

Let  $x \neq y, x, y \in X$ . We need to show that there are  $U_x, U_y \in \mathcal{T}_d$  s.t.  $x \in U_x, y \in U_y$  and  $U_x \cap U_y = \emptyset$ .

Try with  $U_x = B(x, \varepsilon_1), U_y = B(y, \varepsilon_2)$ . What is unknown up till now are the values of  $\varepsilon_1, \varepsilon_2$ . Define  $z \in U_x \cap U_y$ . This point exists, iff  $d(z, x) < \varepsilon_1$  and  $d(z, y) < \varepsilon_2$ . This means

$$d(x, y) \leq d(x, z) + d(z, y) < \varepsilon_1 + \varepsilon_2.$$

If  $\varepsilon_1 + \varepsilon_2 < d(x, y)$  then  $U_x \cap U_y = \emptyset$ . This is always possible as we can choose  $\varepsilon_1, \varepsilon_2$  so small, that the sum of them is smaller than  $d(x, y)$ .

Are those balls open in sense of the topology  $\mathcal{T}_d$ ?

It can be checked, that every ball  $B(x, r) \in \mathcal{T}_d$  (is open in  $\mathcal{T}_d$ ). ■

**Definition 1.19.**  $B(x, r)$  is called an *open ball of radius  $r$  centered in  $x$* . A *closed ball of radius  $r$  centered in  $x$*  is given by

$$\bar{B}(x, r) := \{y \in X \mid d(x, y) \leq r\}.$$

The *closure of an open ball  $B(x, r)$*  is defined as

$$\overline{B(x, r)} = \text{smallest open set (in the topol.) containing the ball.}$$

In general  $\overline{B(x, r)} \subseteq \bar{B}(x, r)$ .

**Example.**

Let  $X = \{0, 1\}$  and

$$d(x, y) := \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

Then  $B(0, 1) = \{z \in X \mid d(0, z) < 1\} = \{0\}$  and

$$[B(0, 1)]^c = \{1\} = B(1, 1).$$

So  $B(0, 1)$  is open and  $B(0, 1)^c$  is also open. But  $B(0, 1)^c$  is the complement of an open set, so it has to be closed as well. Therefore  $B(0, 1)$  is open and closed at the same time. One sees easily

$$\overline{B(0, 1)} = \{0\}, \quad \bar{B}(0, 1) = \{0, 1\}.$$

Whenever we have a metric space, we can make it a topological space. When does the opposite hold? This property is called *Metrizability* of a topological space

**Definition 1.20.** A topological space  $(X, \mathcal{T})$  s.t.  $d : X \times X \rightarrow [0, \infty)$  s.t.  $(X, d)$  is a metric space, and  $\mathcal{T} = \mathcal{T}_d$ , then the topological space  $(X, \mathcal{T})$  is called *metrizable*.

*Remark 1.21.* Not all Hausdorff spaces are metrizable!

*Remark 1.22.* In a Hausdorff-space  $(X, \mathcal{T}_d)$  every convergent sequence has a unique limit.

**Proposition 1.23**

$(X, d)$  metric space (hence  $(X, \mathcal{T}_d)$  is Hausdorff-space). Let  $x : \mathbb{N} \rightarrow X$  a sequence in  $X$ . The following are equal

1.

$$(x_n)_{n \in \mathbb{N}} \text{ converges to } x^* \in X \text{ in sense of } (X, \mathcal{T}_d)$$

2.

$$\forall \varepsilon > 0 \exists k_0 \geq 0 : \forall k \geq k_0 \ x_k \in \underbrace{B(x^*, \varepsilon)}_{d(x^*, x_k) < \varepsilon}$$

PROOF. For all  $V$  open neighbourhood of  $x^*$  all but a finite number of  $x_k$  are in  $V$ .

$$\exists k_0 > 0 \forall k \geq k_0 x_k \in V \text{ iff } V = \text{ball}.$$

For the other direction:  $V$  is open neighbourhood of  $x^*$ . So there exists  $\varepsilon > 0$  s.t.

$$B(x^*, \varepsilon) \subset V. \quad \blacksquare$$

Open and closed sets can be characterized using convergent sequences:

**Proposition 1.24**

Let  $(X, d)$  metric space,  $X \subset X$ . The following are equivalent

1.  $A$  is closed in topology, i.e.  $A^c \in \mathcal{T}_d$
2.  $A$  is sequentially closed, i.e. all convergent sequences  $x : \mathbb{N} \rightarrow A$  converge to a point  $x^* \in A$

Moreover  $\forall A \subset X$ , the closure  $\bar{A}$  (topology) coincides with the sequential closure:

$$\bar{A} := \{x^* \in X \mid \exists x : \mathbb{N} \rightarrow A : \lim_{k \rightarrow \infty} x_k = x^*\}.$$

PROOF. Exercise! ■

Now we want to speak about continuity in metric spaces.

**Proposition 1.25**

Let  $(X, d_X), (Y, d_Y)$  be two metric spaces, and let  $f : X \rightarrow Y$ . Since we have metric spaces, we can define the corresponding topologies  $\mathcal{T}_{d_X}, \mathcal{T}_{d_Y}$ , so we can talk about continuity of  $f$  in sense of the topological spaces.

The following statements are equivalent

1.  $f$  is continuous (in topology) as a map from  $(X, \mathcal{T}_{d_X}) \rightarrow (Y, \mathcal{T}_{d_Y})$ .
2.  $\varepsilon - \delta$  continuity (in metric, pointwise), i.e.

$$\forall x \in X, \forall \varepsilon > 0 \exists \delta > 0 \text{ s.t. } d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon.$$

3. (Sequential continuity)

$$\forall x^* \in X \forall \text{ sequence } x : \mathbb{N} \rightarrow X, k \mapsto x_k : x_k \rightarrow x^* \Rightarrow f(x_k) \rightarrow f(x^*).$$

(So convergence in topology = convergence in metric)

PROOF. Exercise! ■

We now want to be able to compare metrics.

**Definition 1.26.** Let  $d_1, d_2$  be two metrics on  $X$ . We say that  $d_1$  is *stronger* than  $d_2$ , if for the corresponding topologies it holds that  $\mathcal{T}_{d_1}$  is stronger (finer) than  $\mathcal{T}_{d_2}$ . Analogously  $d_1$  is *weaker* than  $d_2$ , if  $\mathcal{T}_{d_1}$  is weaker (coarser) than  $\mathcal{T}_{d_2}$ .  $d_1$  and  $d_2$  are called equivalent, if  $\mathcal{T}_{d_1} = \mathcal{T}_{d_2}$ .

We can give characterizations of stronger / weaker metrics in terms of continuity.

**Proposition 1.27**

Let  $d_1, d_2$  two metrics on  $X$ . The following are equivalent:

- (1)  $d_1$  is stronger than  $d_2$  (i.e.  $\mathcal{T}_{d_2} \subset \mathcal{T}_{d_1}$ ).
- (2)  $Id : (X, \mathcal{T}_{d_1}) \rightarrow (X, \mathcal{T}_{d_2})$ , with  $x \mapsto Id(x) = x$  is continuous.
- (3) Any sequence that is convergent in  $d_1$  must also be convergent in  $d_2$ .
- (4)

$$\forall x \in X \forall \varepsilon > 0 \exists \delta > 0 \text{ s.t. } d_1(x, y) < \delta \Rightarrow d_2(x, y) < \varepsilon.$$

PROOF. (1)  $\Leftrightarrow$  (2)  $f : X \rightarrow Y$  constant  $\Leftrightarrow \forall V \subset \mathcal{T}_Y = \mathcal{T}_2 f^{-1} \in \mathcal{T}_X = \mathcal{T}_1$ . Then  $\forall V \subset \mathcal{T}_2 \Rightarrow V \in \mathcal{T}_1$ . This means for  $f = Id$  that

$$f^{-1}(V) = V \in \mathcal{T}_1, \quad \forall V \in \mathcal{T}_2.$$

(2)  $\Leftrightarrow$  (3)  $f$  is constant at  $x^*$ , if  $\forall x_k \rightarrow x^* \Rightarrow f(x_k) \rightarrow f(x^*)$ . For  $f = Id : (X, \mathcal{T}_1) \rightarrow (X, \mathcal{T}_2)$  we have therefore that

$$d_1(x_k, x^*) \rightarrow 0 \quad \Rightarrow \quad \underbrace{d_2(\underbrace{f(x_k)}_{=:x_k}, \underbrace{f(x^*)}_{=:x^*})}_{=:d_2(x_k, x^*)} \rightarrow 0$$

$f$  is constant  $\Leftrightarrow x_k$  converges in  $d_1 \Rightarrow x_k$  converges in  $d_2$ .

(2)  $\Leftrightarrow$  (4)  $f$  const  $\Leftrightarrow \varepsilon - \delta$  continuity:

$$\forall x \in X \forall \varepsilon > 0 \exists \delta > 0 : \underbrace{d_X(x, y)}_{=:d_1(x, y)} < \delta \Rightarrow \underbrace{d_Y(f(x), f(y))}_{=:d_2(x, y)} < \varepsilon \Rightarrow \left( d_1(x, y) < \delta \Rightarrow d_2(x, y) < \varepsilon \right)$$

**Example.**

Let  $X = \ell_1 := \{x : \mathbb{N} \rightarrow \mathbb{R} \mid \sum_{j=0}^{\infty} |x_j| < \infty\}$ . Define

$$\begin{aligned} d_1(x, y) &:= \sum_{j=0}^{\infty} |x_j - y_j| \\ d_{\infty}(x, y) &:= \sup_{j \in \mathbb{N}} |x_j - y_j|. \end{aligned}$$

Both metrics are well defined, as no sequence in  $\ell_1$  can be divergent.

$d_\infty$  is a weaker metric than  $d_1$ , if  $d_1(x, y) < \delta \Rightarrow \sum_j |x_j - y_j| < \delta$  then  $\sup_j |x_j - y_j| < \delta$ , i.e.  $d_2(x, y) < \delta$ .

$d_\infty$  is not equivalent to  $d_1$ ! Take  $x^{(k)} = (\underbrace{\frac{1}{k}, \frac{1}{k}, \dots, \frac{1}{k}}_k, 0, 0, \dots, 0)$ . Then

$$d_\infty(x^{(k)}, 0) = \sup_j |x_j^{(k)} - 0| = \frac{1}{k} \xrightarrow{k \rightarrow \infty} 0,$$

that means  $x^{(k)} \rightarrow 0$  in  $d_\infty$ ! Nevertheless we have in the other metric

$$d_1(x^{(k)}, 0) = \sum_{j=0}^{\infty} |x_j^{(k)} - 0| = k \frac{1}{k} = 1, \quad \forall k,$$

so  $x^{(k)} \not\rightarrow 0$  in  $d_1$ .

$\Rightarrow d_\infty$  cannot be equivalent to  $d_1$ !

**Definition 1.28 (Cauchy sequence).** A sequence  $x : \mathbb{N} \rightarrow X$  in a metric space  $(X, d)$  is a *Cauchy sequence*, if

$$\forall \delta > 0 \exists k_0 \geq 0 : \quad d(x_k, x_q) < \delta, \quad \forall k, q \geq k_0.$$

*Remark 1.29.* Any convergent sequence is a Cauchy sequence! The converse does not need to be true.

**Definition 1.30 (Complete space).** A metric space  $(X, d)$  is a *complete space*, if every Cauchy sequence has a limit in  $X$ .

**Example.**

- Let  $X = \mathbb{Q}$ , with the standard metric  $d(x, y) = |x - y|$ .  $(\mathbb{Q}, d)$  is not complete!
- $(\ell_p, d_p)$  for  $1 \leq p \leq \infty$  with  $d_p(x, y) = (\sum_j |x_j - y_j|^p)^{1/p}$  is complete!
- $(\ell_1, d_\infty)$  is not complete! Take for  $k \in \mathbb{N}$  the sequence  $x^{(k)} = (1, \frac{1}{2}, \frac{1}{3}, \dots, \underbrace{\frac{1}{k+1}}_{k+1}, 0, \dots, 0)$ ,

then  $d_\infty(x^k, x^{k'}) \sup_j |x_j^k - x_j^{k'}| = \frac{1}{k+2}$ , for  $k' > k$ , and this expression tends to 0 for  $k, k' \rightarrow \infty$ . Therefore it is a Cauchy sequence. However we have that  $d_\infty(x^k, x^*) = \frac{1}{k+2} \rightarrow 0$ .

- Suppose that  $(X, d)$  is complete and let  $A \subset X$ , s.t.  $A \neq X$  and  $A$  is dense in  $X$ , i.e.  $\bar{A} = X$ . Then  $(A, d_A)$  cannot be complete, because

$$\forall x^* \in X \exists x : \mathbb{N} \rightarrow A : x_k \rightarrow x^*,$$

which also has to hold for all  $x^* \in X \setminus A$  (by assumption  $A \neq X$ ).

When can we say that  $(A, d_A)$  is complete (for  $A \subset X$ )?

**Proposition 1.31**

Let  $(X, d)$  complete metric space and  $A \subset X$  a closed subset of  $X$ . Then  $(A, d_A)$  is a complete metric space.

PROOF. Let  $x : \mathbb{N} \rightarrow A$  Cauchy sequence. Then  $x$  is a Cauchy-sequence in a complete space  $X$ . Therefore exists  $x^* \in X$  s.t.  $x_k \rightarrow x^*$  and as  $A$  is closed ( $\Leftrightarrow A$  sequentially closed) we have that  $x^* \in A$ . ■

Every noncomplete space can be extended to a complete space, up to an isometry.

**Definition 1.32 (Isometry).** Let  $(X, d_X), (Y, d_Y)$  two metric spaces. A function  $f : (X, d_X) \rightarrow (Y, d_Y)$  is an *isometry*, if

$$d_X(x, y) = d_Y(f(x), f(y)), \quad \forall x, y \in X.$$

*Remark 1.33.* If for the isometry  $f$  we have that  $f(x) = f(y) \Rightarrow d_X(x, y) = 0 \Rightarrow x = y$ , so an isometry must always be injective! We can even make it bijective by restricting it's image to  $f(X) \subseteq Y$ , i.e.  $(X, d_X) \rightarrow (f(X), d_Y|_{f(X)})$ .

A sequence  $x$  is convergent in  $X$  iff  $f(x)$  is convergent in  $f(X)$ .

A sequence  $x$  is Cauchy sequence in  $X$  iff  $f(x)$  is Cauchy sequence in  $f(X)$ .

**Theorem 1.34 (Completion)**

Let  $(X, d)$  metric space. Then exists a complete space  $(\tilde{X}, \tilde{d})$  and an isometry  $f : (X, d) \rightarrow (\tilde{X}, \tilde{d})$  s.t.  $\mathcal{J}(X)$ , is dense in  $\tilde{X}$ .

*Remark 1.35.* It is not always possible to just complete the space in  $X$ . However it works (see [Theorem 1.34](#)), if we first map the space using an isometry.

PROOF. Do the proof in 3 steps: First construct  $(\tilde{X}, \tilde{d})$ , then prove that this space is complete. Afterwards construct an embedding of  $X$  (define  $\mathcal{J}$ ).

**Step 1:** We first define  $X^{\mathbb{N}} := \{x : \mathbb{N} \rightarrow X\}$  (the set of sequences in  $X$ ). Now restrict this space to the Cauchy sequences:  $\hat{X} := \{x \in X^{\mathbb{N}} | x \text{ Cauchy sequence}\}$ .

We can now define an equivalence relation  $\sim$  by

$$x \sim y \text{ if } \lim_{j \rightarrow \infty} d(x_j, y_j) = 0$$

Then define the set of equivalence classes as

$$\tilde{X} = \hat{X} / \sim = \{[x]\}$$

with

$$[x] := \{y \in \hat{X} | x \sim y\}.$$



Property of  $\hat{X}$ :  $\forall x, y \in \hat{X}$ ,  $a : \mathbb{N} \rightarrow \mathbb{R}_+$ ,  $a_j = d(x_j, y_j)$ . Then  $a$  is a Cauchy sequence in  $\mathbb{R}$ , so it is convergent!

$$|a_j - a_{j'}| \rightarrow 0, \quad j, j' \rightarrow \infty.$$

$x, y$  Cauchy, so  $\forall \delta > 0 \exists j_0, j'_0$  s.t.

$$d(x_j, x_{j'}) < \delta \quad \forall j, j' \geq j_0$$

and

$$d(y_j, y_{j'}) < \delta \quad \forall j, j' \geq j'_0.$$

Then

$$a_j = d(x_j, y_j) \leq \underbrace{d(x_j, x_k)}_{< \delta} + \underbrace{d(x_k, y_k)}_{a_k} + \underbrace{d(y_k, y_j)}_{< \delta}, \quad \forall k, j \geq \max\{j_0, j'_0\}$$

So  $a_j \leq 2\delta + a_k$ ,  $\forall j, k \geq \max\{j_0, j'_0\}$  and  $a_k \leq 2\delta + a_j \forall k, j \geq \max\{j_0, j'_0\}$ . Therefore  $|a_j - a_k| \rightarrow 0$ , for  $j, k \rightarrow \infty$ .

We can now define

$$\tilde{d}([x], [y]) := \lim_{j \rightarrow \infty} d(x_j, y_j).$$

This  $\tilde{d}$  is well defined: Let  $x' \in [x]$ . Then  $\lim_{j \rightarrow \infty} d(x'_j, y_j) \leq \lim_{j \rightarrow \infty} d(x'_j, x_j) + d(x_j, y_j) = \lim_{j \rightarrow \infty} d(x_j, y_j)$ . It is easy to check that  $\tilde{d}$  is a metric on  $\tilde{X}$ .

**Step 2:** Prove that  $(\tilde{X}, \tilde{d})$  is complete.

Let  $[x^{(k)}]$  (each  $x^{(k)} \in \hat{X}$  is Cauchy sequence) be a Cauchy sequence in  $\tilde{X}$ . Then

$$\forall \delta > 0 \exists k_0 \geq 0 \text{ s.t. } \tilde{d}([x^{(k)}], [x^{(k')}]) < \delta, \quad \forall k, k' \geq k_0.$$

Construction of the limit:  $x^{(k)}$  (for fixed  $k$ ) is Cauchy in  $X$ , so

$$\forall \delta \exists k_0 \text{ s.t. } d(x_j^{(k)}, x_{j'}^{(k)}) < \delta, \quad \forall j, j' \geq k_0.$$

Choose now  $\delta = \frac{1}{k}$ , and  $j_k - k_0$ . Then

$$d(x_j^{(k)}, x_{j_k}^{(k)}) < \frac{1}{k}, \quad \forall j \geq j_k.$$

Define

$$y_k := \underbrace{x_{j_k}^{(k)}}_{\text{one element in the seq. } x^{(k)}}$$

and

$$y := (y_k)_{k \in \mathbb{N}} \in X^{\mathbb{N}}.$$

We constructed a sequence in  $X$ . We need to prove that  $y \in \hat{X}$  (then we can define  $[y]$ ) and that  $\tilde{d}([x^{(k)}], [y]) \rightarrow 0$  for  $k \rightarrow \infty$ .

We first show that  $y \in \hat{X}$  is Cauchy sequence. Look at

$$d(x_k, y_{k'}) < \delta, \quad \forall k, k' \geq j_0. \quad \blacksquare$$

For our sequence this is

$$\begin{aligned}
 d(x_{j_k}^k, x_{j_{k'}}^{(k)}) &\leq \underbrace{d(x_{j_k}^{(k)}, x_j^{(k)})}_{< \frac{1}{k}} + \underbrace{d(x_j^{(k)}, x_j^{(k')})}_{\text{same ele., diff. seq.}} + \underbrace{d(x_j^{(k')}, x_{j_k}^{(k)})}_{< \frac{1}{k'}} \\
 &\leq \underbrace{\frac{1}{k} + d(x_j^{(k)}, x_j^{(k')}) + \frac{1}{k'}}_{\forall j \geq \max\{j_k, j_{k'}\}} \\
 &\leq \frac{1}{k} + \frac{1}{k} + \underbrace{\lim_{j \rightarrow \infty} d(x_j^{(k)}, x_j^{(k')})}_{\substack{\bar{d}([x^k], [x^{k'}]) \xrightarrow{\text{Cauchy}} 0, k, k' \rightarrow \infty}}.
 \end{aligned}$$

Therefore

$$d(y_k, y_{k'}) \rightarrow 0, \quad k, k' \rightarrow \infty$$

so  $y \in \hat{X}$ .

**Step 3:**

■