Functional Analysis and PDE

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Chapter 1

Introduction

Functional Analysis is the analysis on infinite dimensional spaces. Applications of FA are in geometry and topology, probability theory, numerical analysis mathematical physics, and PDEs.

Example.
$C^2(U)$, $U \subset \mathbb{R}^n$ is the set of functions which are 2 times differentiable and have a bounded derivative. U should be a bounded, open set. Consider $f \to 0$ on ∂U and

$$\Delta = \sum_{j=1}^{n} \left(\frac{\partial}{\partial u_j} \right)^2 .$$

Fix $f \in C(U)$ and look for a solution $u \in C^2(U)$, s.t.

$$\Delta u = f, \quad u = \delta^{-1} f.$$

What is the inverse operator δ^{-1} in this case? That's what we need to study here.

Program of the lecture

- Structures: We need to define *Topologies, Metrics, Norms* and *Scalarproducts* on infinite-dimensional spaces
- We introduce *Functional Spaces* as infinite-dimensional vector spaces of functions
- We need to talk about *Linear Operators* to be able to talk about their inverse (see **??**)

Chapter 2

Structures

We consider *convergence*. We have already seen

$$x_n \to x \quad \Leftrightarrow \quad \forall \varepsilon > 0 \exists n_0 \forall n > n_0 : x_n \in (x - \varepsilon, x + \varepsilon)$$

In order to do so, we need the definition of a neighbourhood (topology), distances (metric), the length of a vector (norm, Banach spaces), the length and angles (scalar product, Hilbert spaces).

All vector spaces we consider should base on $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$!

2.1 Topological spaces

Let X be a set, let 2^X the set of all possible subsets of X (including the empty set).

Definition 2.1 (Topology). A Topology $\mathcal{T}$ on the set X is a family of subsets of X, that means

$$\mathcal{T} \subseteq 2^X,$$

satisfying

T1 $\emptyset, X \in \mathcal{T}$

T2 Stability under finite intersection:

$$\forall A, B \in \mathcal{T}: \quad A \cap B \in \mathcal{T}$$

T3 Stability under any possible union: Let $\mathcal{I}$ be some set of indices (may not be countable). Then

$$\forall \{A_i\}_{i \in \mathcal{I}} \subseteq \mathcal{T}: \quad \bigcup_{i \in \mathcal{I}} A_i \in \mathcal{T}$$

Any subset $A \subset X$ is called an *open set*, if $A \in \mathcal{T}$. Else it is called a *closed set.*

$(X, \mathcal{T})$ is called a *topological space*!

Remark 2.2. Note that

$$\left(\bigcup_{i \in \mathcal{I}} A_i \right)^c = \cap_{i \in \mathcal{I}} A_i^c$$

for all families of open sets $\{A_i\}_{i \in \mathcal{I}}$ and each index-set $\mathcal{I}$.

Definition 2.3 (Coarser / finer topologies). Let $\mathcal{T}_1, \mathcal{T}_2$ be two topologies on X with $\mathcal{T}_1 \subset \mathcal{T}_2$, then we say that

- $\mathcal{T}_1$ is *coarser / weaker* than $\mathcal{T}_2$
- $\mathcal{T}_2$ is *finer /stronger* tthan $\mathcal{T}_1$

Example.

a) $\mathcal{T} = 2^X$ *is a topology on* $X \Rightarrow 2^X$ *is the storngest (finest) topology on* X.

Also $\mathcal{T} = \{\emptyset, X\}$ *is a topology on* $X \Rightarrow$ *any topology* $\mathcal{T}'$ *needs to caontain* $\emptyset$ *and* X*, so* $\mathcal{T} \subset \mathcal{T}'$. *This means that* $\mathcal{T}$ *is the weakest / coarsest topology on* X

b) *On* $\mathbb{R}$ *there is a standard topology* $\mathcal{T}_{st}$*:*

$$V \in \mathcal{T}_{st} \text{ iff } \forall x \in V\, \exists \varepsilon > 0 : (x - \varepsilon, x + \varepsilon) \subset V$$

c) relative topology*: Let* $A \subset X$*, let* $\mathcal{T}$ *be a topology on* X. *Then*

$$\mathcal{T}_A \{A \cup V : V \in \mathcal{T}\}$$

d) Intersection of topologies*: Let $\mathcal{I}$ an index set (may be uncountable), let $(\mathcal{T}_i)_{i\in\mathcal{I}}$ be a family of topologies on X. Then we can define*

$$\bigcap_{i\in\mathcal{I}} \mathcal{T}_i$$

and this is again a topology on X!

e) Product topology*: Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ two topological spaces. Let*

$$\mathcal{S} = \{(V, X) |\, V \in \mathcal{T}_X\} \cup \{(X, W) |\, W \in \mathcal{T}_Y\}.$$

The product topology on $X \times Y$ is the coarsest (weakest) topology on $X \times Y$, that contains $\mathcal{S}$. In particular it must contain all sets of the form $U \times V$ for $U \in \mathcal{T}_X, V \in \mathcal{T}_Y$.

Remark 2.4. If nothing else is said, we consider the standard topology $\mathcal{T}_{st}$ on $\mathbb{R}$!

Definition 2.5 (Closure / boundary of a set). Let $(X, \mathcal{T})$ topological space, $A \in X$.
The *interior* of A, A° is the largest open set inside

$$A = \bigcup_{V\in\mathcal{T}, V\subseteq A} V \text{ (open)}.$$

The *closure* of A, $\bar{A}$ is the smallest closed set containing

$$A = \bigcup_{V \text{ s.t. } V^c\in\mathcal{T}, V\supseteq A} V \text{ (closed)}.$$

The *boundary* of A is given by

$$\partial A = \bar{A} \setminus A^\circ = \bar{A} \cap \underbrace{(X \setminus A^\circ)}_{(A^c)^c} \text{ (closed)}.$$

Definition 2.6 (dense set / separable set). $(X, \mathcal{T})$ topological space in X. Then

- $A \subset X$ is called *dense in* X, if $\bar{A} = X$
- X is called *separable*, if there is a countable dense subset of X

Definition 2.7 (open neighbourhood). $(X, \mathcal{T})$ topological space, $x \in X$. A subset $V \subseteq X$ is called an open neighbourhood of x, if $V \in \mathcal{T}$ and $x \in V$.

Definition 2.8 (Convergence in topology). Let $(\mathcal{T}, X)$ topological space. A sequence $(x_n)_{n\in\mathbb{N}}$, i.e. a map

$$\begin{aligned} x: \quad \mathbb{N} &\to X \\ n &\to x_n, \end{aligned}$$

emphconverges to $x^* \in X$, if

$$\forall V \text{ open neighbourhood of } x^* : \{n | \, x_n \in V^c\} \text{ is finite}$$

(i.e. there is just a finite number of elements that's not contained in V). Then we say that x^* is a limit point for the sequence x_n.

Example.

a) *Let $\mathcal{T}$ the standard topology on $\mathbb{R}$. Then the definition of converges equals the ε- δ-Definition of convergence in Analysis 3.*

b) *If $\mathcal{T} = 2^X$, then $x_n \to x^*$ iff x_n is constant up to a finite number of terms: As we have $\mathcal{T} = 2^X$ especially the set $V = \{x^*\}$ is open. This set gives us the result.*

c) *If $\mathcal{T} = \{\emptyset, X\}$, then every sequence is convergent! Every point $x^* \in X$ is a limit point.*

Definition 2.9 (Hausdorff space). Let $(X, \mathcal{T})$ topological space. It is called a *Hausdorff space*, if

$$\forall x, y \in X, x \neq y \, \exists U_x, U_y \in \mathcal{T} \text{ s.t. } x \in U_x, y \in U_y \text{ and } U_x \cap U_y = \emptyset.$$

Proposition 2.10 (Limits in Hausdorff spaces are unique)
If $(X, \mathcal{T})$ is a Hausdorff space, then every sequence has at most one limit, so either the sequence has no limit or it is unique!

PROOF. Proof can be done by contradiction. ∎

Definition 2.11 (Connectedness). A topological space $(X, \mathcal{T})$ is *connected*, if we cannot write it as a union of 2 disjoint(!) (nonempty) open sets. So, if X is connected and $V \in \mathcal{T}$ is an open set, there is no $\emptyset \neq W \in \mathcal{T}$ with $V \cap W = \emptyset$ and $V \cup W = X$.

$A \subseteq X$ is connected, if A is connected in $\mathcal{T}_A$.