Numerical Simulation of acoustic and electromagnetic scattering problems

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Chapter 1

The acoustic scattering Problem in full space

1.1 Introduction

We study the wave equation in full space $\mathbb{R}^d$, $d \in \{1, 2, 3\}$.

$$\frac{\partial^2}{\partial t^2} p + \gamma \frac{\partial}{\partial t} p = c^2 \Delta p$$

Where

$$c = c(x) \text{ speed of sound}$$
$$\gamma = \gamma(x) \text{ damping coefficient}$$

Assume *time-periodic behaviour*

$$p(x, t) = Re[u(x)e^{-i\omega t}]$$

with frequency ω and a real-valued function u.
Since

$$\frac{\partial^2 p(x, t)}{\partial t^2} = \text{Re}[-\omega^2 u(x)e^{-i\omega t}]$$
$$\frac{\partial p(x, t)}{\partial t} = Re[-i\omega u(x)e^{-i\omega t}]$$
$$\Delta p(x, t) = \text{Re}[\Delta u(x)]e^{-i\omega t}]$$

for all times $t > 0$, we infer

$$-\omega^2 u - \gamma i\omega u = c^2 \Delta u, \quad \text{in } \mathbb{R}^d.$$

This is equivalent to

$$\Delta u + \frac{\omega^2}{c^2}\left(1 + \frac{i\gamma}{\omega}\right)u = 0.$$

We assume, that $c = c_0$ is constant in free space (reference value). We can now define a *wave number*

$$\kappa = \overbrace{\frac{\overbrace{\omega}^{>0}}{\underbrace{c_0}_{>0}}}\ > 0$$

and the *index of refraction*

$$n(x) = \frac{c_0^2}{c(x)^2}\left(1 + i\frac{\gamma}{\omega}\right).$$

This results in the *Helmholtz-Equation*

$$\Delta u + \kappa^2 n u = 0, \quad \text{in } \mathbb{R}^d.$$

We assume that there exists $\alpha > 0$ such that

$$c(x) = c_0 \text{ and } \gamma(x) = 0, \quad \forall |x| > \alpha,$$

that means inhomogeneities of the medium lie inside some bounded region (inside a ball). Furthermore we assume, that outside this ball of radius α, $\overline{B_\alpha(0)}$, there are sources that generate plane waves, that is functions of the type

$$u^{in}(x) = e^{ikx\cdot\hat\theta},$$

where $|\hat\theta| = 1$ for $\hat\theta \in \mathbb{R}^d$ and $\cdot$ denotes the inner product in $\mathbb{R}^d$. Then, u^{in} satisfies

$$\Delta u^{in} + k^2 u^{in} = 0.$$

(just calculation). This u^{in} generates a *scattered field* u^s. The *total field*

$$u = u^{in} + u^s$$

satisfies

$$\Delta u + \kappa^2 n u = 0.$$

Furthermore we assume *Sommerfeld's radiation condition* $(r = |x|)$:

$$\frac{\partial u^s}{\partial r} - i\kappa u^s(x) = o\left(r^{\frac{1-d}{2}}\right), \quad \text{as } r = |x| \to \infty$$
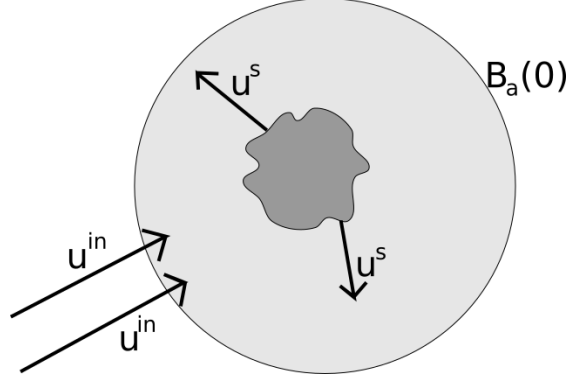
uniformly in $\frac{x}{|x|}$.

Figure 1.1: Visualization of the problem

1.2   Theory for the direct Scattering Problem in $\mathbb{R}^3$

Throughout this section $d = 3$ holds! Furthermore

1. $\hat{\theta} \in \mathbb{R}^3$ with $|\hat{\theta}| = 1$ defines the *indicent field*

$$u^{in}(x) = e^{-i\kappa\hat{\theta}\cdot x}, \quad x \in \mathbb{R}^d$$

2. $\kappa \in \mathbb{R}, \kappa > 0$

3. $n \in L^\infty(\mathbb{R}^3), \operatorname{Re} n \geq 0, \operatorname{Im} n \geq 0$ and $n(x) = 1$ for all $x \in \mathbb{R}^3 \setminus B_\alpha(0)$ and some $\alpha > 0$

We recall

$$H^1_{loc}(\mathbb{R}^3) := \{u : \mathbb{R}^3 \to \mathbb{C} : u|_K \in H^1(K), \quad \text{for every } K = B_R(0) \text{ and any } R > 0\}$$

The *Scattering Problem* reads as follows

Problem 1.1 (Scattering Problem (S)). Given $\hat{\theta}, \kappa, n$ as above. Seek $u \in H^1_{loc}(\mathbb{R}^3)$ such that

$$\Delta u + \kappa^2 n u = 0$$

in $\mathbb{R}^3$ in the weak sense, that is

$$\int_{\mathbb{R}^3} \left(\nabla u \cdot \nabla \bar{\Psi} - \kappa n u \bar{\Psi} \right) dx = 0$$

for any $\Psi \in H^1(\mathbb{R}^3)$ with compact support, and s.t.

$$u^s = u - u^{in}$$

satisfies Sommerfeld's radiation condition ($d = 3$, sharper version)

$$\frac{\partial u^s}{\partial r} - i\kappa u^s = O\left(\frac{1}{r^2}\right), \quad \text{uniformly in } \hat{x} = \frac{x}{|x|} \text{ as } r = |x| \to \infty$$

Remark 1.1. Regularity-theory tells us, that the radiation condition is well-defined, given the assumptions from Problem 1.1.

Theorem 1.2 (Rellich's Lemma)

Let u satisfy $\Delta u + \kappa^2 u$ for every $|x| > \alpha$. The following property

$$\lim_{R \to \infty} \int_{|x|=R} |u(x)|^2 \, dS = 0$$

implies, that $u(x) = 0$ for all $|x| > \alpha$.

Remark 1.3. Vice versa, if the property in Theorem 1.2 does not hold, then u cannot vanish for all $|x| > \alpha$.

PROOF. Employs spherical Bessel functions, see e.g. ■

For the proof of uniqueness, we require results on periodic differential equations. We recall *Fourier representations of periodic functions*:
Define

$$Q := (-\pi, \pi)^3 \subseteq \mathbb{R}^3.$$

The functions

$$\{\varphi_j : j \in \mathbb{Z}^3\}$$

with

$$\varphi_j(x) = \frac{1}{(2\pi)^3} e^{ij \cdot x}, \quad \forall x \in Q, j \in \mathbb{Z}^3$$

define a complete orthonormal system (ONS) of $L^2(Q)$ and every $g \in L^2(Q)$ has the expansion

$$g = \sum_{j \in \mathcal{Z}^3} g_j \varphi_j, \quad \text{with } g_j = \int_Q g \varphi_j \, dx, \quad \text{for } j \in \mathcal{Z}^3.$$

Panceval's inequality shows

$$\|g\|_{L^2(Q)} = \sum_{j \in \mathbb{Z}^3} |g_j|^2.$$

Furthermore, if $g \in L^2(Q)$ and $\sum_{j \in \mathbb{Z}^3} |j|^2 |g_j|^2 < \infty$, where $|j| = |j_1| + |j_2| + |j_3|$, then $g \in H^1(Q)$.

Define

$$H^1_{per}(Q) := \left\{ g \in L^2(Q) : \sum_{j \in \mathbb{Z}^3} (1 + |j|)^2 |g_j|^2 < \infty \right\}$$

and identify $L^2(Q)$ and $H^1_{per}(Q)$ with the corresponding periodic functions in $\mathbb{R}^3$ by

$$g(2\pi j + x) = g(x), \quad \text{for } x \in Q, j \in \mathbb{Z}^3.$$

Lemma 1.4

Let $p \in \mathbb{R}^3, \alpha \in \mathbb{R}$ and $\hat{e} = \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}$. Then, for every $t > 0$ and every $g \in L^2(Q)$ there

exists a unique solution $w = w_t(g) \in H^1_{per}(Q)$ to the differential equation

$$\Delta w + \lambda_t \cdot \nabla w + \mu_t w = g \tag{*}$$

understood weakly, that is

$$\forall \Psi \in C_c^\infty(Q) : \quad \int \left(-\nabla w \cdot \nabla \bar{\Psi} + (\lambda_t \cdot \nabla w + \mu_t w) \bar{\Psi} \right) dx = \int_Q g \bar{\Psi} \, dx$$

for $\lambda_t := 2t\hat{e} - ip$, $\mu_t = -(it + \alpha)$. It holds that

$$\|w\|_{L^2(Q)} \leq \frac{1}{t} \|g\|_{L^2(Q)},$$

this means that

$$L_t : L^2(Q) \to L^2(Q) \, \text{defined by } g \mapsto w_t(g)$$

defines a bounded operator

$$\|L_t\| \leq \frac{1}{t}, \quad \forall t > 0.$$

PROOF. With the Fourier expansions

$$g = \sum_{j \in \mathbb{Z}^3} g_j \varphi_j \quad \text{and} \quad w = \sum_{j \in \mathbb{Z}^3} w_j \varphi_j,$$

Eq. (*) transforms to

$$\forall j \in \mathbb{Z}^3 : \quad c_j w_j = g_j, \quad \text{with } c_j = -|j|^2 + ij \cdot \lambda_t + \mu_t.$$

We have

$$|c_j| \geq |\operatorname{Im} c_j| \overset{insert}{=} |2tj \cdot \hat{e} - t| = t \underbrace{|2\hat{e} \cdot j - 1|}_{\geq 1} \geq t.$$

Thus, the operator

$$(L_t g) := \sum_{j \in \mathbb{Z}^3} \frac{g_j}{c_j} \varphi_j, \quad \text{with } g \in L^2(Q),$$

is well-defined and satisfies

$$\|L_t\| \leq \frac{1}{t}, \quad \forall t > 0.$$

It remains to show, that w is really a solution to the differential equation. In order to do so, let $\Psi \in C_c^\infty(Q)$. As Ψ has compact support, function-value and all derivatives are zero on the boundary of Q, and therefore we have also that Ψ is periodic. We can represent

$$\Psi = \sum_{j \in \mathbb{Z}^3} \Psi_j \varphi_j.$$

$\blacksquare$

Then

$$\int_Q \left(-\nabla w \cdot \nabla \bar{\Psi} + (\lambda_t \nabla w + \mu_t w) \bar{\Psi} \right) dx = \sum_{j \in \mathbb{Z}^3} c_j w_j \cdot \bar{\Psi}_j$$

$$= \sum_{j \in \mathbb{Z}^3} g_j \cdot \bar{\Psi}_j$$

$$= \int_Q g \bar{\Psi} \, dx.$$

Theorem 1.5 (Unique continuation principle)
Let $u \in H^1_{loc}(\mathbb{R}^3)$ solve $\Delta u + \kappa^2 n u = 0$, where $n \in L^\infty(\mathbb{R}^3)$ with $n(x) = 1$ for $|x| > \alpha$, and let $b \geq \alpha$, such that $u(x) = 0$ for all $|x| \geq b$. Then we have $u = 0$ in $\mathbb{R}^3$.

Remark 1.6. Theorem 1.5 holds in a much more general version than we stated here.
PROOF. We introduce the scaling parameter $\varrho = \frac{2b}{\pi}$ and the following function

$$w(x) := e^{\frac{i}{2}x_1 - t\hat{e} \cdot x} u(\varrho x), \quad x \in Q = (-\pi, \pi)^3$$

for some parameter $t > 0$ and now $\hat{e} = \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} \in \mathbb{C}^3$. Since $w(x) = 0$ for all $|x| \geq \frac{pi}{2}$ it can be extended to a 2π-periodic function by

$$w(2\pi j + x) = w(x), \quad \forall j \in \mathbb{Z}^3, \, x \in Q,$$

with $w \in H^1_{per}(Q)$. It is readily seen that w satisfies

$$\Delta w + \lambda_t \cdot \nabla w + \mu_t w = -\varrho^2 \kappa^2 \tilde{n} w,$$

for

$$\lambda_t := 2t\hat{e} - \begin{bmatrix} i \\ 0 \\ 0 \end{bmatrix}$$

$$\mu_t := -\left(it + \frac{1}{4}\right)$$

and $\tilde{n}$ is the periodic function

$$\tilde{n}(x + 2\pi j) = n(\varrho x), \quad x \in \bar{Q}, \ j \in \mathbb{Z}^3$$

(The proof of this is by straightforward differentiation). Lemma 1.4 applies to this situation with $g := -\varrho^2 \kappa^2 \tilde{n} w$ and yields

$$w = L_t g = -\varrho^2 \kappa^2 L_t(\tilde{n} w)$$

with $\|L_t\| \le \frac{1}{t}$. This means that

$$\|w\|_{L^2(Q)} \le \frac{1}{t} \varrho^2 \kappa^2 \|\tilde{n} w\|_{L^2(Q)}$$

$$\le \frac{\varrho^2 \kappa^2 \|n\|_\infty}{t} \|w\|_{L^2(Q)}.$$

This holds for all $t > 0$. Taking $t \gg 1$ results in $\|w\|_{L^2(Q)} = 0$ where $w = 0$. Thus $u \equiv 0$. ∎

Theorem 1.7 (Uniqueness)

Problem 1.1 admits at most one solution. That is, $u^{in} \equiv 0$ implies $u \equiv 0$.

PROOF. Let $u^{in} = 0$. Recall the formula

$$|a - i\beta|^2 = |a|^2 + |\beta|^2 + 2\operatorname{Im}(\beta\bar{a})$$

(elementary proof). We have

$$O\left(\frac{1}{R^2}\right) = \int_{|x|=R} \left|\frac{\partial u}{\partial r} - i\kappa u\right|^2 dS$$

$$= \int_{|x|=R} \left(\left|\frac{\partial u}{\partial r}\right|^2 + \kappa^2 |u|^2\right) dS + 2\kappa \operatorname{Im} \int_{|x|=R} u \frac{\partial \bar{u}}{\partial r} dS$$

The divergence-theorem implies

$$2\kappa \operatorname{Im} \int_{|x|=R} u \underbrace{\frac{\partial \bar{u}}{\partial r}}_{=\nabla\bar{u}\cdot \underbrace{\nu}_{\text{normal}}} dS \quad = \quad 2\kappa \operatorname{Im}\left[\int_{|x|<R}\left(u\Delta\bar{u} + \underbrace{|\nabla u|^2}_{\in\mathbb{R}_0^+}\right) dx\right]$$

$$\overset{\text{Helmholtz-eq}}{=} 2\kappa \operatorname{Im}\left[\int_{|x|<R}\left(-\kappa^2 \bar{n}|u|^2\right) dx\right]$$

$$\ge \quad 0,$$

Since $\operatorname{Im} n \geq 0$. Thus we have for large $R \to \infty$:

$$0 \leq \limsup_{R \to \infty} \int_{|x|=R} \left(\left| \frac{\partial u}{\partial r} \right|^2 + \kappa |u|^2 \right) dS \leq 0,$$

whence

$$\lim_{R \to \infty} \int_{|x|=R} |u|^2 \, dS = 0.$$

Theorem 1.2 (Rellich) implies that $u(x) = 0$ for all $|x| > a$. By the unique continuation principle we have $u \equiv 0$ in $\mathbb{R}^3$. ∎

We have shown uniqueness. For the existence proof we will construct solutions.

Definition 1.8. The function

$$\Phi(x,y) := \frac{1}{4\pi|x-y|} e^{i\kappa|x-y|}, \quad x,y \in \mathbb{R}^3, x \neq y$$

is called *fundamental solution* or *free space Green's function*.

Φ has the following properties

Proposition 1.9 (Properties of the fundamental solution)

1. $\Phi(\cdot, y)$ *satisfies the Helmholtz-equation* $\Delta u + \kappa^2 u = 0$ *in* $\mathbb{R}^3 \setminus \{y\}$

2. Φ *satisfies the emphradiation condition*

$$\frac{x}{|x|} \cdot \nabla_x \Phi(x,y) - i\kappa \Phi(x,y) = O\left(\frac{1}{|x|^2} \right), \quad \text{as } |x| \to \infty$$

   uniformly in $\hat{x} = \frac{x}{|x|} \in \mathcal{S}^2$ *(Sphere)*

3. Φ *has the asymptotic behaviour*

$$\Phi(x,y) = \frac{e^{i\kappa|x|}}{4\pi|x|} e^{-i\kappa\hat{x}\cdot y} + O\left(\frac{1}{|x|^2} \right), \quad \text{as } |x| \to \infty$$

   uniformly in $\hat{x} = \frac{x}{|x|} \in \mathcal{S}^2$ *(Sphere)*

PROOF. The first 2 properties are easily verified. We prove the third:

We calculate the the binomial formula

$$|x-y| - (|x| - \hat{x} \cdot y) = \frac{|x-y|^2 - (|x| - \hat{x} \cdot y)^2}{|x-y| + (|x| - \hat{x} \cdot y)}$$

$$= \frac{|y|^2 - \overcancel{2x \cdot y} + \cancel{|x|^2} - \cancel{|x|^2} + \overcancel{2x \cdot y} - \overbrace{(\hat{x} \cdot y)^2}^{\leq |y|^2}}{|x| \left(1 + \underbrace{\left| \hat{x} - \frac{y}{|x|} \right|}_{1 - \left| \frac{y}{|x|} \right|} - \frac{\hat{x} \cdot y}{|x|} \right)}$$

$$\leq \frac{|y|^2}{2|x| \left(1 - \frac{y}{|x|} \right)}.$$

Hence $|x-y| = |x| - \hat{x} \cdot y + O\left(\frac{1}{|x|} \right)$. Take the exponential of this to get

$$e^{i\kappa|x-y|} = e^{i\kappa|x|} e^{-i\kappa\hat{x}\cdot y} \left(1 + O\left(\frac{1}{|x|} \right) \right).$$

By the above formula we also have

$$\frac{1}{|x-y|} = \frac{1}{|x|} + \left[\frac{|x| - |x-y|}{|x-y||x|} \right] = \frac{1}{|x|} + O\left(\frac{1}{|x|^2} \right).$$

So

$$\frac{e^{i\kappa|x-y|}}{|x-y|} = \frac{e^{i\kappa|x-y|}}{|x|} + O\left(\frac{1}{|x|^2} \right)$$

$$= \frac{1}{|x|} e^{i\kappa|x|} e^{-i\kappa\hat{x}\cdot y} + O\left(\frac{1}{|x|^2} \right),$$

as $|x| \to \infty$.   ∎

Remark 1.10. The previous result states in particular that Φ is determined by the function

$$e^{-i\kappa\hat{x}\cdot y}$$

on $\mathcal{S}^2$ (up to pertubation). We will see that this property holds in a more general context.

With the help of the fundamental solution Φ we can create

Theorem 1.11

Let $\Omega \subseteq \mathbb{R}^3$ be bounded. For every $\phi \in L^2(\Omega)$, the function

$$v(x) := \int_\Omega \phi(y)\Phi(x,y)\,dy, \quad x \in \mathbb{R}^3$$

belongs to $H^1_{loc}(\mathbb{R}^3)$ and satisfies the Sommerfeld radiation condition. *Moreover, v is the* only *radiating solution (it is the only function in the set of solution that satisfies the Sommerfeld radiation condition) to*

$$\Delta v + \kappa^2 v = -\phi$$

in the variational sense

$$\int_{\mathbb{R}^3} \left(\nabla v \cdot \nabla \bar{\Psi} - \kappa^2 v \bar{\Psi} \right) dx = \int_{\mathbb{R}^3} \phi \bar{\Psi}\,dx \tag{*}$$

for all $\Psi \in H^1(\mathbb{R}^3)$ with compact support. For any $R > 0$ with $\Omega \subseteq B_R(0) =: K$ we have with $c = c(R, \kappa, \Omega)$ that

$$\|v\|_{H^1(K)} \leq c\|\phi\|_{L^2(\Omega)}.$$

In other words, the mapping $\phi \mapsto v$ is a bounded (continuous linear) operator from $L^2(\Omega)$ to $H^1(K)$.

PROOF. **(1)** Since $\frac{1}{r^2}$ is locally integrable in $\mathbb{R}^3$, the expression on the right-hand side in Eq. (*) is well-defined. Provided $\phi \in C^1(\bar{\Omega})$, then we can interchange integration and differentiation (posible as we have a majorant, cf. PDE theory) and obtain

$$\Delta v + \kappa^2 v = \begin{cases} -\phi & \text{in } \Omega \\ 0 & \text{in } \mathbb{R}^3 \setminus \bar{\Omega} \end{cases}$$

This means that v solves Eq. (*) for any $\kappa \in \mathbb{C}$.

(2) We cannot directly evaluate L^2-integrals of the gradient $\nabla_x \Phi$. To prove stability, we first consider the special case $\kappa = i$ and $\phi \in C^1(\bar{\Omega})$. Then, the fundamental solution

$$\Phi_i(x,y) = \frac{1}{4\pi|x-y|}e^{-|x-y|}$$

decays exponentially for $|x| \to \infty$, ad by approximation arguments Eq. (*) holds for any $\Psi \in H^1(\mathbb{R}^3)$.

Taking $\Psi = v$ we obtain

$$\|v\|^2_{H^1(\mathbb{R}^3)} = \int_\Omega \phi\bar{v} \leq \|\phi\|_{L^2(\Omega)}\|v\|_{H^1(\mathbb{R}^3)}$$

(note that Eq. (*) with $\kappa = i$ becomes the H^1-Norm). By density, this defines a continuous operator

$$\phi \mapsto v \quad \text{from } L^2(\Omega) \text{ to } H^1(K).$$

(3) Let $k > 0$. We define

$$\Psi(x, y) := \Phi_k(x, y) - \Phi_i(x, y) = \frac{1}{4\pi|x - y|} \left[e^{i\kappa|x-y|} - e^{-|x-y|} \right]$$

It is easy to prove that Ψ and $\nabla_x \Psi$ belong to $L^2(K \times \Omega)$. We sketch the crucial part. We calculate

$$4\pi|\nabla_x \Psi(x, y)| = \left| \underbrace{\frac{i\kappa e^{i\kappa|x-y|} - e^{-|x-y|}}{|x - y|}}_{O\left(\frac{1}{|x-y|}\right)} + \frac{e^{i\kappa|x-y|} - e^{-|x-y|}}{|x - y|^3} \right|$$

$$\overset{*}{\text{leq}} O\left(\frac{1}{|x - y|} \right) \text{ for small } |x - y|$$

*: The denominator of the right fraction is in $O(|x - y|)$

Thus,

$$\|\nabla_x \Psi\|_{L^2(K \times \Omega)} \leq C \int_K \int_\Omega \frac{1}{|x - y|^2} \, dy \, dx < \infty.$$

(4) With **(3)** we see that the mapping

$$\varphi \mapsto \int_\Omega \varphi(y) \Psi(\cdot, y) \, dy$$

is bounded from $L^2(\Omega)$ to $H^1(K)$, because

$$\int_K \left| \nabla_x \int_\Omega \varphi(y) \Psi(x, y) \, dy \right|^2 dx = \int_k \left[\int_\Omega |\varphi(y) \nabla_x \Psi(x, y)| \, dy \right]^2 dx$$

$$\leq \int_K \|\varphi\|_{L^2(\Omega)}^2 \|\nabla_x \Psi(x, \cdot)\|_{L^2(\Omega)}^2 \, dx$$

$$\leq \|\varphi\|_{L^2(\Omega)}^2 \underbrace{\|\nabla_x \Psi\|_{L^2(K \times \Omega)}^2}_{\overset{(3)}{<\infty}}.$$

This and **(2)** show that $\varphi \mapsto v$ is also bounded from $L^2(\Omega)$ to $H^1(K)$ for $\kappa > 0$.

(5) The radiation condition follows from the radiation condition of Φ:

$$\frac{\partial v}{\partial \nu} - i\kappa v = \int_\Omega \varphi(y) \left(\frac{\partial}{\partial \nu} - i\kappa \right) \Phi(x, y) \, dy$$

$$\leq \|\varphi\|_{L^2(\Omega)} O\left(\frac{1}{r} \right)^2.$$

Uniqueness follows from **(2)**. ∎

Remark 1.12. **Proof of Theorem 1.11** (1).
To prove that $\Delta v(x) + \kappa^2 v(x) = -\varphi(x)$, for $x \in \Omega$, it satisfies to verify

$$(\Delta + \kappa^2) \int_{B_\varepsilon(x)} \varphi(y)\Phi(x,y)\,dy = -\varphi(x),$$

for small $\varepsilon > 0$.

1) We readily see that

$$\kappa^2 \left| \int_{B_\varepsilon(x)} \varphi(y)\Phi(x,y)\,dy \right| \leq \frac{\kappa^2}{4\pi} \left| \int_{B_\varepsilon(x)} \frac{e^{i\kappa|y-x|}}{|x-y|}\,dy \right|$$

and the expression on the right tends to 0 for $\varepsilon \to 0$.

2) Since $\Delta_x \Phi(x,y) = \Delta_y \phi(x,y)$, we see

$$\Delta_y \int_{B_\varepsilon(x)} \varphi(y)\phi(x,y)\,dy = \int_{B_\varepsilon(x)} \phi(y)\Delta_y \Phi(x,y)\,dy$$

$$= \underbrace{-\int_{B_\varepsilon(x)} \nabla\varphi(y) \cdot \nabla\Phi(x,y)}_{=:A} + \underbrace{\int_{|y-x|=\varepsilon} \varphi(y)\frac{\partial\Phi(x,y)}{\partial\nu}\,dS(y)}_{=:B}.$$

We have that

$$\nabla_y \Phi(x,y) = \frac{1}{4\pi}e^{i\kappa|x-y|}\left(\frac{1}{|x-y|^2} - \frac{1}{|x-y|^3} \right)(x-y),$$

and thus

$$A \leq \|\nabla\varphi\|_{L^\infty} \int_{B_\varepsilon(x)} |\nabla_y\Phi(x,y)|\,dy \to 0, \text{ as } \varepsilon \to 0.$$

We stay with B:

$$B = = \frac{1}{4\pi} \int_{\partial B_\varepsilon(x)} e^{i\kappa\varepsilon}\left(\frac{1}{\varepsilon} - \frac{1}{\varepsilon^2} \right)\varphi(y)\,dS(y)$$

$$= \underbrace{\frac{1}{4\pi}e^{i\kappa\varepsilon}\left(\frac{1}{\varepsilon} - \frac{1}{\varepsilon^2} \right)4\pi\varepsilon^2}_{=e^{i\kappa\varepsilon}(\varepsilon+\varepsilon^{-1})\to-1,\text{ as }\varepsilon\to 0} \underbrace{\fint B_\varepsilon(x)\varphi(y)\,dS(y)}_{\to\varphi(x),\text{ as }\varepsilon\to 0}$$

Theorem 1.13 (Lippmann Schwinger integral equation)
*Let $\alpha > 0$ and $K = \overline{B_\alpha}(0)$. If $u \in H^1$ solves the Problem 1.1, then $u_{|K} \in L^2(K)$
satisfies the Lippmann Schwinger equation*

$$u(x) = u^i n(x) - \kappa^2 \int_{|y|<\alpha} (1-n(y))\Phi(x,y)u(y)\,dy \qquad \text{(LS)}$$

for almost all $x \in K$.
*Conversely, if $u \in L^2(K)$ satisfies Eq. (LS), then it can be extended by the right-hand
side of Eq. (LS) to the solution $u \in H^1_{loc}(\mathbb{R}^3)$ of Problem 1.1.*

PROOF. Let u satisfy Problem 1.1 and define

$$v = \int_K \varphi(y)\Phi(\cdot, y)\, dy$$

for $\varphi = \kappa^2(1-n)u \in L^2(K)$. Theorem 1.11 states that $v \in H^1_{loc}(\mathbb{R}^3)$ satisfies $\Delta v + \kappa^2 v = -\varphi$. Since

$$\Delta u + \kappa^2 u = \kappa^2(1-n)u$$

and

$$\Delta u^{in} + \kappa u^{in} = 0,$$

we have (addition of previous equations)

$$\Delta(v + u^s) + \kappa^2(v + u^s) = 0.$$

The uniqueness from Theorem 1.7 shows that $v + u^s = 0$. Therefore

$$u = u^{in} + u^s = u^{in} - v.$$

For the converse direction let $u \in L^2(K)$ satisfy Eq. (LS). Define v as above, such that $u = u^{in} - v$ in K (in the L^2-sense). By Theorem 1.11 we know that $v \in H^1_{loc}(\mathbb{R}^3)$ by extending it to $\mathbb{R}^3$, and that $\Delta v + \kappa v = -\varphi$. This implies $u \in H^1_{loc}(\mathbb{R}^3)$. Furthermore

$$\Delta u + \kappa^2 u = \varphi = \kappa^2(1-n)u,$$

that is

$$\Delta u + \kappa^2 n u = 0.$$

This implies that $u^s = -v$. The radiation condition follows again from Theorem 1.11. ∎

We now can derive existence for Problem 1.1.

Theorem 1.14 (Existence of solutions for Problem 1.1)
Let κ, n, θ be as assumed previously. Then, the Lippmann Schwinger equation (LS), and thus Problem 1.1, is uniquely solvable.

PROOF. Define the following operator $T : L^2(B_\alpha(0)) \to L^2(B_\alpha(0))$ by

$$(Tu)(x) = \kappa^2 \int_{|y|<\alpha} (1 - n(y))\Phi(x, y)u(y)\, dy, \text{ for } |x| < \alpha.$$

Theorem 1.11 shows that T is bounded from $L^2(B_\alpha(0))$ to $H^1(B_\alpha(0))$, and by the compact embedding $H^1(B_\alpha(0)) \hookrightarrow L^2(B_\alpha(0))$, the operator T is compact from $L^2(B_\alpha(0))$ into itself. The Lippmann Schwinger equation reads

$$u + Tu = u^{in}$$

and by the Riesz theory (Fredholm alternative) for compact operators, the uniqueness of u (Theorems 1.7 and 1.13) implies its existence. ∎

Theorem 1.15 (Far-field pattern)

Let u solve Problem 1.1. Then we have

$$u(x) = u^{in}(x) + \frac{e^{i\kappa|x|}}{|x|} u_\infty(\hat{x}) + O\left(\frac{1}{|x|^2}\right), \quad as \ |x| \to \infty,$$

uniformly in $\hat{x} = \frac{x}{|x|}$ (direction of x). The function $u_\infty(\hat{x})$ is given by

$$u_\infty(\hat{x}) = \frac{\kappa^2}{4\pi} \int_{|y|<\alpha} (n(y) - 1)e^{-i\kappa\hat{x}\cdot y} \, u(y) \, dy$$

is called far-field pattern *or* scattering amplitude

$$u_\infty : \underbrace{\mathcal{S}^2}_{Sphere} \to \mathbb{C}$$

is analytic on $\mathcal{S}^2$ and determines u^s otuside $B_\alpha(0)$ uniquely ($u_\infty = 0 \Leftrightarrow u^s(x) = 0$ for $|x| > \alpha$).

The formula for $u(x)$ follows from Proposition 1.9 (iii):

$$\Phi(x, y) = \frac{e^{i\kappa|x|}}{|x|} e^{-i\kappa\hat{x}\cdot y} + O\left(\frac{1}{|x|^2}\right).$$

By Theorem 1.2 (Rellich) we have

$$\lim_{R \to 0} \int_{|x|=R} |u(x)|^2 \, dS(x) = 0, \ \text{for } |x| > \alpha$$

and this implies uniqueness. The analyticity follows from the formula (calculate derivate w.r.t $\hat{x}$).

Remark 1.16.   1. The concept of far-field pattern is <u>fundamental</u> to inverse scattering theory.

   2. The analyticity of u_∞ shows that the inverse scattering problem is ill-posed (little pertubations of u_∞ can destroy analyticity of the function!).

> **Theorem 1.17 (Green's representation theorem)**
> **(i)** Any $y \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfies for any $x \in \Omega$
>
> $$u(x) = \int_{\partial\Omega} \left[\Phi(x,y) \frac{\partial u(y)}{\partial \nu} - u(y) \frac{\partial \Phi}{\partial \nu(y)} \right] dS(y) - \int_{\Omega} \Phi(x,y)(\kappa^2 u(y) + \Delta u(y)) \, dy$$
>
> (where ν denotes the outer normal of Ω).
>
> **(ii)** Let $\Omega^c := \mathbb{R}^3 \setminus \Omega$ and let $u \in H^1_{loc}(\mathbb{R}^3)$ satisfy the Sommerfeld radiation condition, and solve
> $$\Delta u + \kappa^2 u = 0, \quad \text{in } \Omega^c.$$
> Then, the Green's formula holds:
>
> $$u(x) = \int_{\partial\Omega} \left[u(y) \frac{\partial \Phi(x,y)}{\partial \nu(y)} - \Phi(x,y) \frac{\partial}{\partial \nu} u(y) \right] dy, \quad x \in \mathbb{R}^3 \setminus \overline{\Omega}.$$
>
> **(iii)** In the situation of **(ii)**, the far-field pattern of u reads for $\hat{x} = \frac{x}{|x|} \in \mathcal{S}^2$
>
> $$u_\infty(\hat{x}) = \frac{1}{4\pi} \int_{\partial\Omega} \left[u(y) \frac{\partial}{\partial \nu(y)} e^{-i\kappa\hat{x}\cdot y} - e^{-i\kappa\hat{x}\cdot y} \frac{\partial u}{\partial \nu}(y) \right] dS(y).$$

PROOF. **(i)** we know from Theorem 1.11 (see addendum) that

$$-u(x) = \int_{\Omega} u(y)(\kappa^2 + \Delta_y)\Phi(x,y) \, dy.$$

The assertion follows from integrating by parts.

(ii) Let $R > 0$ and $K = B_R(0)$, with $\overline{\Omega} \subseteq K$. Define $D = K \setminus \overline{\Omega}$. Apply **(i)** to D and obtain, for $x \in D$,

$$u(x) = \int_{\partial\Omega} + \left[\int_{\partial K} \right] \left[\Phi(x,y) \frac{\partial u}{\partial \nu}(y) - u(y) \frac{\partial}{\partial \nu(y)} \Phi(x,y) \right] dS(y).$$

We want to show, that the $\int_{\partial K}$-integral vanishes for large $R \to \infty$. From the proof of Theorem 1.7 we know, that $\int_{\partial K} |u|^2 \, dS \leq O(1)$ as $R \to \infty$. From the Sommerfeld radiation condition of u and Φ, we obtain with the Cauchy-Schwarz inequality

$$\int_{\partial K} \dots dS = \int_{\partial K} \Phi(x,y) \left[\frac{\partial u}{\partial \nu}(y) - i\kappa u(y) \right] + u(y) \left[i\kappa \Phi(x,y) - \frac{\partial}{\partial \nu(y)} \Phi(x,y) \right] dS(y)$$

$$\overset{C.S.}{\leq} \underbrace{\|\Phi\|_{L^2(\partial K, dy)} O\left(\frac{1}{r^2}\right) + \|u\|_{L^2(\partial K)} O\left(\frac{1}{r^2}\right)}_{\to 0, \text{ as } R \to \infty}.$$

From Proposition 1.9 we know ($\hat{x} = \frac{x}{|x|}$):

$$\Phi(x,y) = \frac{1}{4\pi|x|} e^{i\kappa|x|} e^{-i\kappa\hat{x}\cdot y} + O\left(\frac{1}{|x|^2}\right).$$

We plug this into the formula of **(ii)** and obtain for $|x| >> 1$:

$$u(x) = \frac{1}{4\pi} e^{i\kappa w} \int_{\partial\Omega} \left[u(y) \frac{\partial}{\partial \nu(y)} e^{-i\kappa\hat{x}\cdot y} - e^{-i\kappa\hat{x}\cdot y} \frac{\partial}{\partial \nu} u(y) \right] dy + O\left(\frac{1}{|x|^2}\right).$$

Uniqueness of the far-field pattern proves **(iii)**. ∎

1.3 Density results

In this section we collect some density results, useful in the labour study of inverse scattering. Recall: The refraction index n satisfies $|n(x)| = 1$ for $|x| > \alpha$.

Lemma 1.18

Let $\beta > \alpha$, $K := B_\beta(0)$. Then there exist constants $M > 0, C > 0$, such that for any $z \in \mathbb{C}^3$ with $z \cdot z = 0$ and $|z| \geq M$ there exists a solution $u_z \in H^1(K)$ to the equation

$$\Delta u_z + \kappa^2 n u_z = 0, \quad in \ K,$$

of the form

$$u_z(x) = e^{z \cdot x}(1 + v_z(x)), \quad x \in K,$$

and v_z satisfies

$$\|v_z\|_{L^2(K)} \leq \frac{C}{|z|}.$$

PROOF. The technical lemma will be proven later. ∎

Theorem 1.19 (1st density theorem)

Let $\Omega \subseteq \mathbb{R}^3$ be bounded and let $n_1, n_2 \in L^\infty(\Omega)$ such that $(n_1 - 1)$ and $(n_2 - 1)$ have a compact support in Ω. Then, the following linear hall of products is dense in L^2:

$$X := span\{u1u2 : \forall j = 1, 2 \ u_j \in H^1(\Omega) \ solves \ \Delta u_j + \kappa^2 n_j u_j = 0 \ in \ \Omega\}$$

PROOF. Let $\beta > 0$ such that $\overline{\Omega} \subseteq B_\beta(0)$ and let $g \in L^2(\Omega)$ be $L^2(\Omega)$-Orthogonal to the set X. We have to show, that g is zero. Then X is dense.

Fix $y \in \mathbb{R}^3 \setminus \{0\}$ and choose a unit vector $\hat{a} \in \mathbb{R}^3$ (^ means unit length) and some $b \in \mathbb{R}^3$ with $|b|^2 = |y|^2 + s^2$ such that $\{y, \hat{x}, b\}$ is an orthogonal system in $\mathbb{R}^3$ and some $s > 0$ (real).

Define the following elements of $\mathbb{C}^3$:

$$z^1 := \frac{1}{2} b - \frac{i}{2}(y + s\hat{a})$$

$$z^2 := -\frac{1}{2} b - \frac{i}{2}(y - s\hat{a}).$$

Then $z^1 \cdot z^1 = 0 = z^2 \cdot z^2$. We have

$$|z^1|^2 = |z^2|^2 = \frac{1}{4}\left[|b|^2 + |y|^2 + s^2\right] \geq \frac{1}{4}s^2.$$

Furthermore $z^1 + z^2 = -iy$. From Lemma 1.18 we obtain solutions to $\Delta u_j + \kappa^2 n_j u_j = 0$ in $B_\beta(0)$ with

$$u_j(x) = e^{z^j \cdot x}\left[1 + v_{z_j}(x)\right].$$

These are in particular solutions when restricted to the smaller domain Ω. The orthogonal property of g yields

$$0 = \int_\Omega e^{(z^1 + z^2)x}[1 + v_{z_1}^2 + v_{z_2}^2 + v_{z_1} v_{z_2}]g\,dx.$$

From Lemma 1.18 we obtain

$$\|v_j\|_{L^2(\Omega)} \le \frac{C}{|z_j|} \le \frac{2C}{s}.$$

Last inequality follows from a previous equation. From Cauchy's inequality and the limit $s \to 0$ that

$$0 = \int_\Omega e^{-iy \cdot x} g(x)\,dx, \quad \forall y \in \mathbb{R}^3 \setminus \{0\}.$$

So, the Fourier-Transform of g vanishes (it is defined like in our equation). Thus $g = 0$. ∎

The second density theorem is the following.

Theorem 1.20 (2nd density theorem)
Let $n \in L^\infty(\mathbb{R}^3)$ with $n(x) = 1$ for $|x| > \alpha$. Let $\beta > \alpha$ and define

$$H := \left\{v \in H^1(B_\beta(0))|\, \Delta v + \kappa^2 n v = 0 \text{ in } B_\beta(0)\right\}$$

the set of Helmholtz-solutions. Let for any $\hat\theta \in \mathcal{S}^2$, $u(\cdot, \hat\theta)$ denote the total field corresponding to the incident plain wave $e^{i\kappa\hat\theta \cdot x}$ (plain wave from direction $\hat\theta$). Then, the span of all plain wave solutions,

$$span\{u(\cdot, \hat\theta) : \hat\theta \in \mathcal{S}^2\},$$

is dense

$$H|_{B_\alpha(0)} \text{ in } L^2(B_\alpha(0)) \text{ - norm.}$$

PROOF. Let $K = B_\alpha(0)$ and $\langle v, w \rangle$ the L^2 product over K, so

$$\langle v, w \rangle := \int_K v\overline{w}\,dx.$$

Let v be in the closure of H with the following property:

$$\langle v, u(\cdot, \hat\theta) \rangle = 0$$

for all $\hat\theta \in \mathcal{S}^2$. Recall the operator

$$T : v \mapsto \kappa^2 \int_{|x|<\alpha} (1 - n(y))\Phi(x, y)w(y)\,dy$$

and the Lippmann-Schwinger equation

$$u(\cdot, \theta) = (1 + T)^{-1} u^{in}(\cdot, \theta).$$

With the adjoint operator T^*, that satisfies

$$\langle T^* v, w \rangle = \langle v, Tw \rangle, \quad \forall v, w \in X,$$

and the fact that

$$[[1 + T]^{-1}]^* = (1 + T^*)^{-1},$$

we obtain that

$$\begin{aligned} 0 &= \langle v, (1 + T)^{-1} u^{in}(\cdot, \hat{\theta}) \rangle \\ &= \langle (1 + T^*)^{-1} v, u^{in}(\cdot, \hat{\theta}) \rangle, \end{aligned}$$

for all $\hat{\theta} \in \mathcal{S}^2$. We compute T^*:
Since, for all $w_1, w_1 \in L^2(K)$ we have

$$\begin{aligned} \frac{1}{\kappa^2} \langle T^* w_1, w_2 \rangle &= \frac{1}{\kappa^2} \langle w_1, Tw_2 \rangle \\ &= \int_K \int_K w_1(x) \overline{(1 - n(y)) \Phi(x,y) w_2(y)} \, dy \, dx \\ &= \langle (1 - \overline{n}) \int_K w_1(x) \overline{\Phi(x, \cdot)} \, dx, w_2 \rangle, \end{aligned}$$

we have

$$T^* w_1 = \kappa^2 \overline{(1 - n)} \int_K w_1(y) \overline{\Phi(\cdot, y)} \, dy.$$

Thus $w := (1 + T^*)^{-1} v$ satisfies

$$v(x) = w(x) + \kappa^2 (1 - \overline{n(x)}) \int_K \overline{\Phi(x,y)} w(y) \, dy.$$

We now set

$$\tilde{w}(x) := \int_K \overline{w(y)} \Phi(x,y) \, dy, \quad x \in \mathbb{R}^3.$$

Then, $\tilde{w}$ is a volume potential for $\overline{w}$ in the sense of [Theorem 1.11](): $w \in H^1_{loc}(\mathbb{R}^3)$ and

$$\int_{\mathbb{R}^3} \left(\nabla \tilde{w} \cdot \nabla \overline{\Psi} - \kappa^2 \tilde{w} \overline{\Psi} \right) dx = \int_K \overline{w} \overline{\Psi} \, dx, \quad \forall \Psi \in H^1(\mathbb{R}^3) \text{ with compact support.}$$

The far-field pattern $\tilde{w}_\infty$ vanishes. Indeed, as [Theorem 1.17]() (iii) and the divergence theorem imply for any $\hat{\Theta} \in \mathcal{S}^2$:

$$\overline{\tilde{w}_\infty(\hat{\theta})} = \int_K w(y) e^{i\kappa\hat{\theta} \cdot y} \, dy = 0,$$

since w is orthogonal to the plain waves. Rellichs theorem implies then, that $\tilde{w} = 0$ outside of K! Consider a sequence $v_j \in H$ with $v_j \to v$ in $L^2(H)$ as $j \to \infty$. From

$$v = w + \overline{\kappa^2(1-n)\tilde{w}}$$

we get

$$\int_K \overline{v}v_j \, dx = \int_K \overline{w}v_j \, dx + \kappa^2 \int_K (1-n)\tilde{w}v_j \, dx. \tag{*}$$

Since $\tilde{w}$ vanishes outside K, we obtain from $v_j \in H$

$$\int_{|x|<\beta} (\nabla v_j \cdot \nabla \tilde{w} - \kappa^2 v_j \cdot \tilde{w}) \, dx = -\kappa^2 \int_K (1-n)v_j\tilde{w} \, dx.$$

We extend v_j to a $H^1(\mathbb{R}^3)$ function with compact support. The property of $\tilde{w}$ implies

$$\int_{|x|<\beta} (\nabla \tilde{w} \cdot \nabla v_j - \kappa^2 \tilde{w} \cdot v_j) \, dx = \int_K (1-n)v_j\tilde{w} \, dx.$$

The previous formulas imply

$$-\kappa^2 \int_K (1-n)v_j\tilde{w} \, dx = \int_K \overline{w}v_j \, dx.$$

This and Eq. (*) give

$$\int_K \overline{v}v_j = 0, \quad \forall j \in \mathbb{N}.$$

In the limit $j \to \infty$ we obtain

$$\|v\|_{L^2(K)} = 0 \iff v = 0.$$

This proves density. ∎

Chapter 2

Discretization of the direct scattering problem

2.1 Dealing with infinite domains

Recall the direct scattering Problem 1.1 from Chapter 1:
Given some incident wave $u^{in}(x)$ of plain-wave type, i.e.

$$u^{in}(x) = ce^{i\kappa\hat{\theta}\cdot x},$$

with direction $\hat{\theta} \in \mathcal{S}^{d-1}$ (unit sphere in $\mathbb{R}^d$) and (real) frequency $\kappa > 0$, find the total field $u \in H^1_{loc}(\mathbb{R}^d)$ such that for all $\Psi \in H^1(\mathbb{R}^d)$ with compact support it holds that

$$\int_{\mathbb{R}^d} (\nabla u \cdot \nabla\overline{\Psi} - \kappa^2 n u\overline{\Psi})\, dx = 0,$$

where $n \in L^\infty(\Omega)$ function with $\Re n, \Im n \geq 0$. Additionaly the scattered field $u^s := u - u^{in}$ should solve the Sommerfeld radiation condition

$$\frac{\partial u^s(x)}{\partial r} - i\kappa u^s(x) = O\left(\frac{1-d}{2}\right),$$

for $|x| = r \to \infty$ uniformly in $\hat{x} = \frac{x}{|x|} \in \mathcal{S}^{d-1}$.

The scattering problem is an infinite-dimensional variational problem on an infinite (unbounded) domain.

Aim of this chapter: Approximate Problem 1.1 in a finite computational complexity and time.

This section deals with the restriction / truncation to bounded domains.

Common approach: Simply replace $\mathbb{R}^d$ by some bounded domain Ω and put some artificial absorbing boundary condition on $\partial\Omega$. The principle is to allow all waves to leave the domain without any artifical scattering (i.e. the initial wave, that may be scattered by some objects in Ω, should be able to leave the domain without any harm). At the boundary of the domain, nothing that's leaving should be scattered inside again (as Ω is just used for being able to compute something).

Reference: Lecture-notes of Olof Runborg, KTH Stockholm.

2.1.1   One-dimensional illustration

In 1d (and only in 1d), it is possible to design exact local absorbing boundary conditions. W.l.o.g. let the artificial domain $\Omega = B_L(0)$ be clutered and let $\overline{B_\alpha(0)} \subseteq B_L(0)$ ($L > \alpha$, where α is the parameter from Chapter 1, $n|_{B_\alpha(0)^c} = 1$). Close to $\partial\Omega = \{-L, L\}$, the medium is homogeneous and hence the scattered wave $u^s := u - u^{in}$ satisfies the Helmholtz-equation

$$u^s_{xx} - \kappa^2 u^s = 0,$$

for $|x \pm L|$ sufficiently small.

Therefore

$$u^s(x) = \underbrace{c_0 e^{-i\kappa x}}_{A} + \underbrace{c_1 e^{i\kappa x}}_{B}, \tag{2.1}$$

with $c_0, c_1 \in \mathbb{R}$ and x close to $\pm L$. The part A of the solutions travels to the left of the interval, the part B travels to the right of the interval. As stated above: If we were sitting on the boundary, we would like to be able to allow waves to pass the boundary from the inside. If a wave comes from the outside, we don't let it pass. Therefore we want to design the absorbing boundary condition at $-L$ such that it accepts any c_0, but prevents $c_1 \neq 0$. This is archieved by applying the operator $\frac{\partial}{\partial x} + i\kappa$. Why this? Because:

$$\left(\frac{\partial}{\partial x} + i\kappa\right) u^s = -c_0 i\kappa e^{-i\kappa x} + c_1 i\kappa e^{i\kappa x} + c_0 i\kappa e^{-i\kappa x} + c_1 i\kappa e^{i\kappa x}$$

$$= 2c_1 i\kappa e^{i\kappa x},$$

and this shows that $\left(\frac{\partial}{\partial x} + i\kappa\right)$ doesn't 'see' waves leaving Ω (i.e. the c_0 part of the solution), and incoming waves can be avoided by the boundary condition

$$\left(\frac{\partial}{\partial x} u^s + i\kappa u^s\right)(-L) = 0.$$

Similarily, at $x = +L$ we impose

$$\left(\frac{\partial}{\partial x} u^s - i\kappa u^s\right)(+L) = 0$$

(calculation shows this again). Introducing the concept of an outer normal ν, i.e. $\nu(\pm L) = \pm 1$, we can rewrite both conditions as

$$\frac{\partial u^s}{\partial \nu}(x) - i\kappa u^s(x) = 0,$$

for $x \in \partial\Omega = \{-L, L\}$. This leads to the following inhomogeneous boundary condition for the total wave $u = u^s + u^{in}$:

$$\left(\frac{\partial u}{\partial \nu} - i\kappa u\right)(x) = \left(\frac{\partial u^{in}}{\partial \nu} - i\kappa u^{in}\right)(x), \quad x \in \partial\Omega = \{-L, L\}. \tag{2.2}$$

Example.
$u^{in} = e^{-i\kappa x}$ *is a wave travelling from right to left. Then the absorbing boundary condition reads*

$$\left(\frac{\partial u}{\partial \nu} - i\kappa u\right)(-L) = 0,$$
$$\left(\frac{\partial u}{\partial \nu} - i\kappa u\right)(L) = 2i\kappa u^{in}(L).$$

We shall emphasize that the scattering problem on Ω with absorbing boundary condition Eq. (2.2),

$$\Delta u + i\kappa n u = 0, \quad \text{in } \Omega$$
$$\frac{\partial u}{\partial \nu} - i\kappa u = \frac{\partial u^{in}}{\partial \nu} - i\kappa u^{in}, \quad \text{on } \partial\Omega,$$

reproduces the solution of Problem 1.1 in Ω.

2.1.2   Multi-dimensional case

In higher dimensions, there is no simple form like Eq. (*) for this scattered wave close to the artificial boundary $\partial\Omega$. Exact absorbing boundary conditions will be necesarrily non-local and computationally demanding.

A cheaper and popular approach is the following:
Assume that the scattered wave in the full problem is close to a spherical wave in a vicinity of the artificial boundary. If we choose $\Omega = B_L(0)$, then the 1d-arguments can be applied. If $u^s(x) \approx e^{i\kappa|x|}$ (i.e. the wave is approximately spherical; spherical waves have no angular dependence) close to $\partial\Omega$, then $\frac{\partial u^s}{\partial \nu} - i\kappa u^s \approx i\kappa e^{ik|x|} - i\kappa e^{i\kappa|x|} = 0$ on $\partial\Omega$.
This motivates the simplest absorbing bc in 2 and 3 dimensions:

$$\frac{\partial u}{\partial \nu} - i\kappa u = \frac{\partial u^{in}}{\partial \nu} - i\kappa u^{in}, \quad \text{on } \partial\Omega. \tag{2.3}$$

We have seen in Theorem 1.15 that u^s is actually spherical, up to some perturbation of order $O\left(\frac{1}{|x|^2}\right)$:

$$u^s(x) = \frac{e^{i\kappa|x|}}{|x|} u_\infty(\hat{x}) + O\left(\frac{1}{|x|^2}\right), \quad \text{as } |x| \to \infty,$$

i.e. 'the scattered wave is spherical if we are far enough away from all obstacles'.

Hence, for sufficiently large L $(\Omega = B_L(0))$, is absorbing boundary condition (2.3) appears reasonable.

For the remaining part of this chapter, our model problem reads

---

Problem 2.1.

$$\Delta u + \kappa^2 n u = 0, \text{ in } \Omega$$
$$\frac{\partial u}{\partial \nu} - i\kappa u = \frac{\partial u^{in}}{\partial \nu} - i\kappa u^{in}, text on \partial\Omega.$$

Remark 2.1. The absorbing boundary condition (2.3) can be seen as applying the Sommerfeld radiation condition at $\partial B_L(0)$ instead of ∞.

Note, that the choice of the boundary condition we made was just the easiest one. There are many alternatives. The most popular approach is the so called *Perfectly Matched Layer* (PML). In this approach no bc on $\partial\Omega$ are imposed. The idea is to put around Ω some thin layer of artificial material (by changing n / making it imaginary). This can also be used to absorb outgoing waves.

2.2 Well-posedness of the scattering problem in bounded domains

Recall the scattering problem

$$\begin{aligned}
\Delta u + \kappa^2 n u &= 0, & \text{in } \Omega \\
\frac{\partial u^s}{\partial \nu} - i\kappa u^s &= 0, & \text{on } \partial\Omega
\end{aligned} \tag{2.4}$$

in a boudned domain $\Omega \supseteq B_\alpha(0) \supseteq \text{supp}(1-n)$ with absorbing boundary condition Eq. (2.3) for the scattered wave $u^s := u - u^{in}$. W.l.o.g. we assume that $\Omega \subseteq \mathbb{R}^3$ as in Chapter 1. To be compatible with linear finite elements, we assume that Ω is polyhedral. Moreover we choose Ω to be convex, which seems to be reasonable with regard to the desired absorption properties (in this way no inner boundaries may reflect incoming waves).

Since it is easier for the numerical analysis of finite element methods, we will now switch to the equivalent problem of finding the scattered wave, i.e.

Problem 2.2. Given the incident wave $u^{in} \in H^1(\Omega)$, we seek $u^s \in H^1(\Omega)$ such that

$$
\begin{aligned}
\Delta u^s + \kappa^2 n u^s &= \overbrace{-\Delta u^{in} - \kappa^2 n u^{in}}^{\text{known data}}, \quad \text{in } \Omega \\
\frac{\partial u^s}{\partial \nu} - i\kappa u^2 &= 0, \quad \text{on } \partial\Omega.
\end{aligned} \tag{2.5}
$$

For $u^{in} = c e^{i\kappa\hat{\Theta}\cdot x}, \hat{\Theta} \in \mathcal{S}^2(\mathbb{R}^3)$ the right hand side of the first equation becodes $\kappa^2(1-n)u^{in}$ (in Ω).

The weak formulation of Problem 2.2 (and also the concept of a weak solution) is based on the sesquilinearform

$$
\begin{aligned}
a: \quad H^1(\Omega) \times H^1(\Omega) &\to \mathbb{C} \\
(u,v) &\mapsto \int_\Omega \nabla u \cdot \nabla \overline{v} \, dx - \kappa^2 \int_\Omega n u \overline{v} \, dx + i\kappa \int_{\partial\Omega} u\overline{v} \, dS_x \tag{2.6}
\end{aligned}
$$

and the anti-linear form

$$
\begin{aligned}
F: \quad H^1(\Omega) &\to \mathbb{C} \\
v &\mapsto \kappa^2 \int_\Omega (1-n) u^{in} \overline{v} \, dx \tag{2.7}
\end{aligned}
$$

(satisfies $F(\lambda v) = \overline{\lambda} F(v)$).

The weak formulation reads: Given $F \in (H^1(\Omega))'$ (space of bounded anti-linear functionals), e.g. the particular F of Section 2.2, seek $u^s \in H^1(\Omega)$ such that

$$
a(u^s, v) = F(v), \quad \forall v \in H^1(\Omega). \tag{2.8}
$$

We call any solution u^s of Eq. (2.8) a *weak solution of Problem 2.2* and $u = u^{in} + u^s$ *weak solution of Section 2.2* .

The sesquilinear form a is bounded:

$$
|a(u,v)| \leq C\|u\|_\kappa \|v\|_\kappa,
$$

with respect to the κ-dependent norm

$$
\|\cdot\|_\kappa := \sqrt{\|\nabla \cdot\|^2_{L^2(\Omega)} + \kappa^2 \|n^{\frac{1}{2}} \cdot\|^2_{L^2(\Omega)}},
$$

which is equivalent to the classical H^1-norm. The boundedness follows from Cauchy-Schwarz inequality and the trace inequality

$$\|v\|^2_{L^2(\partial\Omega)} \leq C'(\Omega)\|v\|_{L^2(\Omega)}\|v\|_{H^1(\Omega)}.$$

The constant $C'(\Omega)$ (and also the constant $C(\Omega)$) are independent of $\kappa \geq 1$. The regime $\kappa \to 0$ would requiere special treatment here and in what follows. Since this is not our regime of interest, we shall assumme that κ is sufficiently large, i.e. $\kappa \geq 1$. All results remain valid for $0 < \kappa < 1$. However, constants may blow up with $\kappa \to 0$.

More explanation: When $\kappa \to 0$ we are mainly killing the $\kappa^2 nu$ term of our equation, and also the κ term in the boundary condition. This leads to the Laplace-Problem with Neumann-B.C. However, there is more than one solution (e.g. the constant function is a solution).

The sesquilinear form a is symetric: $a(u,v) = a(v,u)$, $\forall u, v \in H^1(\Omega, \mathbb{R})$. However, it is not hermitian in general:

$$a(u,v) \neq \overline{a(u,v)}.$$

Moreover a is not coercive (that is, a is far away from being a scalar product), but a satisfies

$$\underbrace{\operatorname{Re} a(u,u) + 2\kappa^2\|n^{\frac{1}{2}}u\|^2_{L^2(\Omega)} = \|u\|^2_{\kappa}, \quad \forall u \in H^1(\Omega).}_{\text{'Garding's inequality}}$$

due to the compactness of the embedding $H^1(\Omega) \subseteq L^2(\Omega)$.

The sesquilinear form a can be cast in the form "coercive + compact perturbation". Hence, the well-posedness of Eq. (2.8) follows from uniqueness by *Fredholm's alternative*.

To show uniqueness, assume that $u \in H^1(\Omega)$ satisfies Eq. (2.8) with right-hand side $F = 0$, i.e.

$$a(u,v) = 0, \quad \forall v \in H^1(\Omega).$$

Choosing $v = u$ yields

$$\operatorname{Im} a(u,u) = \kappa\|u\|^2_{L^2(\partial\Omega)} = 0,$$

i.e. $u \in H^1_0(\Omega)$. Hence we can extend u to whole $\mathbb{R}^d$, and the extension

$$\tilde{u}(x) := \begin{cases} u(x) & x \in \Omega \\ 0 & \text{elsewhere} \end{cases}$$

is in $H^1_{loc}(\mathbb{R}^d)$ and satisfies the assumptions of Theorem 1.5 (unique continuation principle). Hence

$$\Delta u + \kappa nu = 0$$
$$u(x) = 0, \quad \forall|x| \geq \beta \geq \alpha$$

implies $u \equiv 0$.

This proves that for any $F \in (H^1(\Omega))'$ there exists a unique solution $u^s \in H^1(\Omega)$ of Eq. (2.8) and

$$\|u^s\|_\kappa \leq C_{st}(\Omega, n, \kappa)\|F\|_{(H^1(\Omega))'},$$

with some generic (st = stability) constant $C_{st}(\Omega, n, \kappa)$, that only depends on Ω, n, κ and especially <u>not</u> on F, u^s! For the particular F of Section 2.2, this implies

$$\|u^s\|_\kappa \leq C_{st}(\Omega, n, \kappa)\|1 - n\|_{L^\infty(\Omega)}\|u^{in}\|_{\kappa,\text{supp}(1-n)}$$

(the scattered wave is bounded by the incident wave).

The dependence of C_{st} on $|kappa$ is not known in general. Actually, very little is known. With regard to [Bet] an exponential growth w.r. to κ is possible for non generic coefficients n.

If this is the case, then numerical approximation will hardly be reliable (there is no theoretical evidence in our methods), because perturbations caused e.g. by the Galerkin method, or quadrature errors, or numerical linear algebra (precision of arithmetics), etc. may be amplified by C_{st}. There is really not much that we can do apart from "hoping" that our computations reflect the true solution. In order to analyse the numerical method, we will assume a moderate growth.

Assumption 2.2 (Polynomial-in-κ stability)
Given $n \in L^\infty(\Omega)$ with $\text{supp}(1-n) \subseteq B_\alpha(0) \subseteq \Omega$, we assume that there are constants $C(\Omega, n)$ and $\beta(\Omega, n)$ such that for all $k \geq 1$ and $F \in (H^1(\Omega))'$, the unique solution $u^s \in H^1(\Omega)$ of Eq. (2.8) satisfies

$$\|u^s\|_\kappa \underbrace{C(\Omega, n)}_{C_{st}(\Omega, n, k)} k^{\beta(\Omega, n)}\|F\|_{(H^1(\Omega))'}.$$

Recall that Ω is convex!
Remark 2.3. For constant n, it is known that Assumption 2.2 is satisfied with $C(\Omega, n)$ and $\beta(\Omega, n) = 1$. However, for the particular F of Section 2.2, this is not relevant, because $n = const$ implies $u^s = 0$. For non-constant coeffientes, almost no results are available in literature.

See the literature for a broader overview on regularity estimates. LITERATUR FEHLT!!!!!!!!

Assumption 2.2 will allow us to quantify rates of convergence of the Galerkin method as the mesh-size tends to zero. As usual, the accuracy of the methods depends on the

regularity of u^s. If the right-han side functional of Eq. (2.8) is in L^2, then the classical regularity theory for $-\Delta$ shows that $u^s \in H^2(\Omega)$ and

$$\|D^2 u^s\|_{L^2(\Omega)} \le C(\Omega) \left(\|\Delta u^s\|_{L^2(\Omega)} + \|\frac{\partial u^s}{\partial \nu}\|_{H^{\frac{1}{2}}(\partial\Omega)} \right)$$

and we can use that $\|\Delta u^s = -\kappa^2 u^s + F$ and $\frac{\partial u^s}{\partial \nu} = i\kappa u^s$. Therefore we can estimate

$$
\begin{aligned}
\|D^2 u^s\|_{L^2(\Omega)} &\le C(\Omega) \left(\|\Delta u^s\|_{L^2(\Omega)} + \|\frac{\partial u^s}{\partial \nu}\|_{H^{\frac{1}{2}}(\partial\Omega)} \right) \\
&\le C(\Omega,n) \left(\|F\|_{L^2(\Omega)} + \underbrace{k^2 \|u^s\|_{L^2(\Omega)}}_{=\kappa \|u^s\|_\kappa} + \kappa \|u^s\|_\kappa \right) \\
&\le C(\Omega,n) (\|F\|_{L^2(\Omega)} + \kappa^{\beta(\Omega,n)+1} \underbrace{\|F\|_{(H^1(\Omega))}}_{\le c(n)\kappa^{-1}\|F\|_{L^2(\Omega)}})
\end{aligned}
$$

so in the end

$$\|D^2 u\|_{L^2(\Omega)} \le C(\Omega,n)\kappa^{\beta(\Omega,n)} \|F\|_{L^2(\Omega)}. \tag{2.9}$$

2.3 Finite element discretization

We study the FE discretization of the model variational problem Eq. (2.8). Recall that Ω is assumed to be polyhedral. Let $\{\mathcal{T}_h\}_{h>0}$ be a family of *regular* subdivisions of Ω into tetrahedra. For the sake of simplicity, we consider only quasi-uniform triagulations and shape-regular families of triangulations, that is there are constants $c, C > 0$ such that

$$\forall h > 0 \, \forall T \in \mathcal{T}_h : \quad ch^d \le |T| \le Ch^d, \tag{2.10}$$

where $h := \max_{T \in \mathcal{T}_h} \operatorname{diam} T$.

Remark 2.4. Regular means, that any 2 distinct elements of $\mathcal{T}_h$ are either disjoint or share exactly one node, or one face.

We consider linear finite elements, i.e. we are seeking an approximation of the unknown scattered wave u^s in the subspace

$$V_h := \left\{ v_h \in C^0(\Omega) \, | \, \forall T \in \mathcal{T}_h : \, v_h|_T \text{ is affine} \right\} \subseteq H^1(\Omega),$$

of continuous, piecewise affine functions.

The approximate scattered field $u_h^s \in V_h$ is selected by Galerkin projection:
Find $u_h^s \in V_h$ such that

$$a(u_h^s, v_h) = F(v_h), \quad \forall v_h \in V_h \tag{2.11}$$

Note that the well-posedness of the discrete problem is not guaranteed for a general mesh-size h.

The clear condition

$$h \lesssim \kappa^{-1} \tag{2.12}$$

is not sufficient! The well-posedness of the standard FEM requires a stronger condition (*resolution condition*) of the form

$$h \lesssim \kappa^{-(\beta(\Omega,n)+1)},$$

where β is the exponent from Assumption 2.2. We will decude the (conditional) stability / well-posedness of the method from the following error estimate:

Theorem 2.5 (Quasi-optimality of Galerkin FEM)

Let Ω and n are chosen s.t. Assumption 2.2 holds true. Furthermore let $\kappa^{\beta(\Omega,n)+1}h \leq 1$ and sufficiently small. Then there is a constant $C(\Omega,n)$ such that the solution $u^s \in H^1(\Omega)$ and its galerkin approximation $u_h^s \in V_h$ satisfy

$$\|u^s - u_h^s\|_\kappa \leq C(\Omega,n) \min_{v_h \in V_h} \|u^s - v_h\|_\kappa \tag{2.13}$$

PROOF. Define $e := u^s - u_h^s \in H^1(\Omega)$ and the following auxiliary problem of finding $z \in H^1(\Omega)$ such that

$$a(z,w) = \kappa^2 \underbrace{\int_\Omega n\bar{e}\bar{w}\,dS}_{\in L^2(\Omega)}, \quad \forall w \in H^1(\Omega).$$

As we know from the end of Section 2.2, Eq. (2.9), $z \in H^2(\Omega)$ and

$$\|D^2 z\|_{L^2(\Omega)} + \|z\|_\kappa \leq C(\Omega,n)\kappa^{\beta(\Omega,n)+1}\|e\|_\kappa,$$

where $\beta(\Omega,n)$ is the exponent in Assumption 2.2, and $C(\Omega,n)$ is some generic constant independent of κ. This shows that

$$\kappa^2\|n^{\frac{1}{2}}e\|_{L^2(\Omega)}^2 = a(z,\bar{e}) = a(e,\bar{z}) \overset{\text{Galerkin property } a(e,\bar{z_h})=0}{=} a(e,\bar{z}-\bar{z_h}).$$

Choosing $z_h = I_h z$, the nodal interpolation, then

$$\|z - z_h\|_\kappa \leq C(\Omega,n)h\kappa^{\beta(\Omega,n)+1}\|e\|_\kappa.$$

In this estimate we are using that $h\kappa^{\beta(\Omega,h)+1} \leq 1$.
Hence,

$$\kappa^2\|n^{\frac{1}{2}}e\|_{L^2(\Omega)}^2 \leq C(\Omega,n)h\kappa^{\beta(\Omega,n)+1}\|e\|_\kappa^2.$$

Using the Galerkin property once more yields, for any $v_h \in V_h$

$$\begin{aligned}
\|e\|_\kappa^2 &= \operatorname{Re} a(e,e) + 2\kappa^2\|n^{\frac{1}{2}}e\|_{L^2(\Omega)}^2 \\
&= \operatorname{Re} a(e, u^s - \cancel{u_h^s} - v_h) + 2\kappa^2\|n^{\frac{1}{2}}e\|_{L^2(\Omega)} \\
&\leq C(\Omega,n)\|e\|_\kappa\|u^s - v_h\|_\kappa + \tilde{C}(\Omega,n)h\kappa^{\beta(\Omega,n)+1}\|e\|_\kappa^2,
\end{aligned}$$

for sufficiently small h, i.e. $h \leq \kappa^{\beta(\Omega,n)+1}$. This proves the assertion. ∎

Remark 2.6 (Resolution condition and stability). Theorem 2.5 states quasi-optimality and stability of the Finite Element Method, provided

$$h\kappa^{\beta(\Omega,n)+1} \leq C(\Omega, n)$$

is sufficiently small. In particular, we have stability in the followiung sense

$$\|u_h\|_\kappa \leq C(\Omega, n)\|u\|_\kappa \leq C(\Omega, n)\kappa^{\beta(\Omega,n)+1}\|F\|_{H^1(\Omega)}.$$

While this resolution condition is rather sharp for quasi-optimality, it can be relaxed if we are merely interested in existence. However, the natural resolution condition

$$\kappa h \leq C$$

is by no means sufficient for stability of quasi-optimality. There is even a proof that

$$\frac{\|u - u_h\|_\kappa}{\min_{v_h \in V_h} \|u - v_h\|_\kappa} \overset{\kappa \to \infty}{\to} \infty.$$

This is referred to as the *pollution effect.*
Reference: Babuska, Sauter, SIAM Rev 2000.
Remark 2.7. Under the resolution condition **??** and H^2 regularity of u, the approximation properties of V_h show that

$$\|u - u_h\|_\kappa \lesssim h\kappa\|F\|_{L^2(\Omega)}.$$

In particular, for F from Section 2.2, we have

$$\|u - u_h\|_\kappa \lesssim h\kappa\|u^{in}\|_{\kappa,\text{supp}(n)}.$$

Chapter 3

Inverse Problems

3.1 Basic concepts and examples

In many mathematical problems, one is interested in determing a solution of some problem $Px = f$, for given f. This is referred to as the <u>forward</u> or <u>direct problem</u>.
The <u>inverse problem</u> is to reconstruct f from given solution x. This "definition" is rather vague. Consider the following examples.

Example.
<u>Forward problem (FP)</u>: Find the zeros $(x_1, ..., x_n)$ of a given polynomial p of degree n
<u>Inverse problem (IP)</u>: Determine p with zeros $(x_1, ..., x_n)$
$\Rightarrow$ The solution of (IP) is given by $p(x) = \prod_{i=1}^{n}(x - x_i)$. Here (IP) is easier than (FP).
Example (Lagrange interpolation).
<u>(FP)</u>: Evaluate a polynomial p at $(x_1, ..., x_n)$
<u>(IP)</u>: Find a polynomial p of degree n with prescribed values $(y_1, ..., y_n)$ at $(x_1, ..., x_n)$.
Example (Inverse scattering).
<u>(FP)</u>: Given κ, the refractive index n and a plane wave $u^{in} = e^{-i\kappa\Theta\cdot x}$, compute the far field patterns $\hat{u}_\infty$
<u>(IP)</u>: Given far-field patterns $\hat{u}_\infty(\hat{\Theta})$ for direction(s) $\hat{\Theta} \in \mathcal{S}^2$, reconstruct the refractive index n

We have already seen, that $\hat{u}_\infty$ is analytic and, hence, the inverse scattering problem is "ill-posed".

A problem is referred to as well-posed, if it has a unique solution, which depends continuously on the data.

> **Definition 3.1 (Well-posedness and ill-posedness).** Let X, Y be normed spaces
> with a mapping $K : X \to Y$. The equation $Kx = y$ is *well-posed*, if
>
> 1. For every $y \in Y$ there exists $x \in X$ with $Kx = y$ ("existence")
>
> 2. For every $y \in Y$ there exists at most one $x \in X$ with $Kx = y$ ("uniqueness")
>
> 3. For every sequence $\{x_n\} \in X^{\mathbb{N}}$ with $Kx_n \to Kx$ as $n \to \infty$, it holds that
> $x_n \to x$ as $n \to \infty$ ("Stability")
>
> If one of the properties does not hold, then the problem is *ill-posed*.

Remark 3.2. The first 2 conditions are algebraic conditions. The last states continuity
of K^{-1} and, therefore, depends on the topologies on X, Y.

Example (Hadamard, 1923).
*We want to solve $\Delta u = 0$ in $\mathbb{R} \times [0, \infty)$, subject to $u(x,0) = f(x), \frac{\partial u}{\partial y}(x,0) = g(x)$
$(x \in \mathbb{R})$. Let $f = 0$ and $g(x) = \frac{1}{n} \sin(nx)$. Then the unique solution is*

$$u(x, y) = \frac{1}{n^2} \sin(nx) \sinh(ny), \quad (x, y) \in \mathbb{R} \times [0, \infty).$$

Therefore

$$\sup_{x \in \mathbb{R}}\{|f(x)|, |g(x)|\} = \frac{1}{n} \overset{n \to \infty}{\to} 0,$$

but $\sup_{x \in \mathbb{R}} |u(x,y)| = \frac{1}{h^2} \sin(ny) \to \infty$, as $n \to \infty$. Therefore, this "initial value problem" is ill-posed.

Example (Differentiation).
Let $X = Y = C([0,1])$ with max-norm $\| \cdot \|_\infty$.
(FP) Given $x \in X$, find the antiderivative $y \in Y$ with $y(0) = 0$. This is, compute

$$y(t) = \int_0^t x(t)\, dt, \quad \forall t \in [0,1].$$

*(IP) Given some $y \in Y$ with $y(0) = 0$, determine the derivative $x = y'$. This means:
Solve the integral equation $Kx = y$ where*

$$(Kx)(t) = \int_0^t x(t)\, dt, \quad t \in [0,1].$$

*To see, that (IP) is ill-posed, consider a continuous and differentiable $y \in Y$, with $x = y'$.
For the perturbation δ define*

$$\tilde{y}(t) := y + \delta \sin\left(\frac{t}{\delta^2}\right).$$

*(We would expect that the difference between the derivatives of y and $\tilde{y}$ is small, for
small δ)*
We have $\|y - \tilde{y}\|_\infty = \delta$. But since

$$\tilde{x}(t) = x(t) = \frac{1}{\delta} \cos\left(\frac{t}{\delta^2}\right)$$

solves $K\tilde{x} = \tilde{y}$, and

$$\|x - \tilde{x}\|_\infty = \mathcal{O}\left(\frac{1}{\delta}\right),$$

so the problem is ill-posed.

If we now let

$$Y = \{y \in C^1[0,1] : y(0) = 0\} \text{ with } \|y\|_{C^1} = \|y'\|_\infty,$$

the problem is well-posed. Indeed

$$\|x - \tilde{x}\|_\infty = \|y'\tilde{y}'\|_{C^1}.$$

The previous example shows, that the "worst-case error" for some given tolerance δ may be ∞. Let now

$$z := \tilde{y} - y$$

and for simplicity assume that $z(0) = 0 = z'(0)$ and
tildez ≥ 0. Then

$$\begin{aligned}
|x(t) - \tilde{x}(t)|^2 &= |z'(t)|^2 \\
&= \int_0^2 \frac{d}{ds}(z'(s))^2 \, ds \\
&= 2\int_0^t z'(s)z''(s)ds \\
&\leq 2\|z''\|_\infty z(t) \\
&\leq 2\|y''\tilde{y}''\| \, \|\tilde{y} - y\|_\infty.
\end{aligned}$$

This means, under additional information

$$\max\{\|y''\|_\infty, \|\tilde{y}''\|_\infty\} \leq E,$$

we have the following bound

$$\|x - \tilde{x}\|_\infty \leq 4E\|y\tilde{y}\|_\infty.$$

This leads to the following concept

Definition 3.3 (Worst-case error). Let $K : X \to Y$ be a linear bounded operator between Banach spaces X, Y, and let $X_1 \subseteq X$ be a subspace of X, with a stronger "stronger" norm $\|\cdot\|_1$, i.e.

$$\|\cdot\| \leq c\|\cdot\|_1,$$

for some $C > 0$. We define the *worst-case error* for the errorδ in the data and the a priori information

$$\|x\|_1 \leq E,$$

by

$$\mathcal{F}(\delta, E, \|\cdot\|_1) := \sup\{\|x\|_X : x \in X_1, \|Kx\| \leq \delta, \|x\|_1 \leq E\}.$$

In practical examples, K is often compact. This leads typically to ill-posed problems (inversion of compact operators leads to ill-posed problems). In particular we have the following two results

---

Theorem 3.4

Let $K : X \to Y$ be a linear, bounded, compact operator between Banach spaces X, Y, with the property

$$\dim \left[X_{|\ker(K)} \right] = \infty$$

(quotient space).

Then there exists a sequence $\{x_n\} \in X^{\mathbb{N}}$ such that $Kx_n \to 0$, but $\|x_n\| \to \infty$ as $n \to \infty$. In particular, if K is injective, the inverse $K^{-1} : \operatorname{Im}(K) \to X$ is unbounded.

PROOF. We write $\mathcal{N} := \ker(K)$. Since $\mathcal{N}$ is closed, we can build $X_{|\mathcal{N}}$, which is a normed space:

$$\|[x]\| := \inf_{z \in \mathcal{N}} \|x + z\|.$$

The induced operator $\tilde{K} : X_{|\mathcal{N}} \to Y$, with $\tilde{K}([x]) := K(x)$ $(x \in X)$, is injective and compact. Since $X_{|\mathcal{N}}$ is infinite-dimensional, the identity $id : X_{|\mathcal{N}} \to X_{|\mathcal{N}}$ cannot be compact. But since $id = \tilde{K}^{-1}\tilde{K}$ and $\tilde{K}$ is compact, $\tilde{K}^{-1}$ cannot be continuous. Otherwise the identity would be compact as composition of continuous and compact operators. Since $\tilde{K}^{-1}$ is unbounded, there exists a sequence $\{[z_n]\} \in (X_{|\mathcal{N}})^{\mathbb{N}}$ with $\|[z_n]\| = 1$ and $Kz_n \to 0$ as $n \to \infty$. By shifting with $v_n \in \mathcal{N}$ such that $\|z_n + v_n\| \geq \frac{1}{2}$, and defining

$$x_n := \frac{z_n + v_n}{\sqrt{\|Kz_n\|}},$$

we obtain

$$\|Kx_n\| = \frac{\|Kz_n\|}{\sqrt{\|Kz_n\|}} \to 0 \quad \text{and} \quad \|x_n\| \to \infty. \qquad \blacksquare$$

Recall the worst-case error

$$\mathcal{F}(\delta, E, \| \cdot \|_1) = \sup\{\|x\| : \|Kx\| \leq \delta, \|x\|_1 \leq E, x \in X_1\}.$$

Lemma 3.5

Let $K : X \to Y$ be linear, compact and continuous, and assume that $X_{|\ker(K)}$ to be infinite-dimensional. Then, for every a priori information $E > 0$ there exists a constant $c > 0$ and a tolerance $\delta_0 > 0$ such that

$$\mathcal{F}(E, \delta, \| \cdot \|) \geq c, \quad \forall 0 < \delta \leq \delta_0.$$

PROOF. For contradiction assume that there is a sequence $\{\delta_n\}$, $\delta_n \to 0$, such that $\mathcal{F}(E, \delta_n, \| \cdot \|) \to 0$ as $n \to \infty$. Consider again

$$\tilde{K} : X_{|\ker K} \to Y$$

and let $\{x_m\} \subset X^{\mathbb{N}}$ with $Kx_m \to 0$ as $m \to \infty$. Then there exists a subsequence $\{x_{m_n}\}_n$ with

$$\|Kx_{m_n}\| \leq \delta_n, \quad \forall n.$$

Define

$$z_n := \begin{cases} x_{m_n}, & \|x_{m_n}\| \leq E \\ E\|x_{m_n}\|^{-1}x_{m_n}, & \|x_{m_n}\| > E \end{cases}$$

Then $\|z_n\| \leq E$ for all n, and $\|Kz_n\| \leq \delta_n$. Since the worst-case error converges to zero, also $\|z_n\| \to 0$ has to hold for $n \to \infty$. Moreover, by definition we have $z_n = x_{m_n}$, if $n >> 1$ (sufficiently large). (If we are in the second case, the norm is always E. Therefore sooner or later we will get into the first case, as $\|z_n\| \to 0$). This procedure cannot only be applied to $\{x_m\}$, but also to any subsequence. Hence $x_m \to 0$ as $m \to \infty$. This implies that $\tilde{K}^{-1}$ is continuous, which contradicts Theorem 3.4. ∎

3.2 Regularizing filters

In this section, we study one (of many possible) regularization strategy. Throughout this section, let $K : X \to Y$ be linear, bounded, compact operator between Hilbert spaces X, Y and assume that K is injective. We assume that for $y \in Y$ there is a (unique) solution $x \in X$ with $Kx = y$, that is $y \in \operatorname{Im} K$. In practice, one has perturbed data y^δ with

$$\|y - y^\delta\| \leq \delta.$$

We have seen that $Kx^\delta = y^\delta$ may not be solvable. The aim is to determine at least some approximation $x^\delta \in X$ such that $\|x - x^\delta\|$ is comparable to the worst-case error. We seek a bounded approximation

$$R : Y \to X, \quad \text{of } K^{-1}, \quad \mathcal{R}(K) \to X.$$

Definition 3.6. A family of linear and bounded operators $(R_\alpha)_{\alpha>0}$ such that $\lim_{\alpha\to 0} R_\alpha K(x) \to x$, for all $x \in X$, is called a *regularization strategy*.

Remark 3.7. Definition 3.6 states <u>pointwise</u> convergence towards the identity. It can be shown that uniform convergence cannot be expected in general.

Let now $y^\delta \in Y$ be "measured data" with $\|y - y^\delta\| < \delta$ and define

$$x^{\alpha,\delta} := R_\alpha y^\delta.$$

Then

$$\|x^{\alpha,\delta} - x\| \leq \|R_\alpha y^\delta - R_\alpha y\| + \|R_\alpha y - x\| \leq \|R_\alpha\| \underbrace{\|y^\delta - y\|}_{\leq \delta} + \|R_\alpha y - x\|$$

so

$$\|x^{\alpha,\delta} - x\| \leq \|R_\alpha\|\delta + \|R_\alpha y - x\|. \tag{3.1}$$

We expect that $\|R_\alpha\| \to \infty$ as $\alpha \to 0$. Therefore we need a stratetgy for minimizing the RHS of Eq. (3.1).

Definition 3.8. A regularization strategy $\alpha = \alpha(\delta)$ is *admissible*, if for $\delta \to 0$ we have $\alpha \to 0$ and

$$\sup\{\|R_{\alpha(\delta)}y^\delta - x\| : y^\delta \in Y \text{ with } \|Kx - y^\delta\| < \delta\} \to 0$$

for every $x \in X$.

We now briefly recall the *singular value decomposition (SVD)* in Hilbert spaces. Here $J \subseteq \mathbb{N}$ be an index set (finite or infinite).

Let $K : X \to Y$ be linear and compact with $K^* : Y \to X$. Then $K^*K : X \to X$ is self-adjoint, compact, and has real nonnegative eigenvalues $(\lambda_j)_{j\in J}$. The numbers

$$\mu_j := \sqrt{\lambda_j}$$

are called the **singular values** of K. They can be ordered:

$$\mu_1 \geq \mu_2 \geq ... \geq 0.$$

The SVD states:

Theorem 3.9 (Singular-value decomposition)
There exist orthonormal systems $(x_j)_{j\in J} \in X^J$ and $\{y_j\}_{j\in J} \in Y^J$ such that

$$Kx_j = \mu_j y_j, \quad \forall j \in J,$$

and $K^*y_j = \mu_j x_j$. We call $(\mu_j, x_j, y_j)_j$ a singular system for K. Every $x \in X$ has the singular value decomposition

$$x = x_0 + \sum_{j\in J}(x, x_j)x_j, \quad \text{for some } x_0 \in \ker K$$

($(\cdot, \cdot)$ denotes the scalar product), and

$$Kx = \sum_{j\in J}\mu_j(x, x_j)x_j.$$

If $kx = y$ is solvable, then

$$x = \sum_{j\in J}\frac{1}{\mu_j}(y, y_j)x_j.$$

PROOF. See Functional Analysis I. ∎

By damping the factors $\frac{1}{\mu_j}$ we obtain a class of regularization strategies.

Theorem 3.10 (Regularizing filter)

Let $K : X \to Y$ be linear and compact, with singular system $(x_j, y_j, \mu_j)_{j \in J}$ and let

$$q : (0, \infty) \times (0, \|K\|) \to \mathbb{R}$$

be a function satisfying

(i) $q(\alpha, \mu) \leq 1$ for all $(\alpha, \mu) \in (0, \infty) \times (0, \|K\|)$

(ii) For any $\alpha > 0$ there exists a constant $c(\alpha) > 0$ such that

$$|q(\alpha, \mu)| \leq c(\alpha)\mu, \quad \forall 0 < \mu \leq \|K\|$$

(iii) $\lim_{\alpha \to 0} q(\alpha, \mu) \to 1$ pointwise, for every $0 < \mu \leq \|K\|$

Then $R_\alpha : Y \to X$ defined by

$$R_\alpha(y) = \sum_{j \in J} \frac{q(\alpha, \mu_j)}{\mu_j}(y, y_j)x_j$$

for $y \in Y$ is a regularization strategy with

$$\|R_\alpha\| \leq c(\alpha).$$

The strategy $\alpha = \alpha(\delta)$ is admissible if $\alpha(\delta) \to 0$ and $\delta c(\alpha(\delta)) \to 0$ as $\delta \to 0$. If in addition, for some $c_1 > 0$, we have

(iv) $|q(\alpha, \mu)| \leq c_1 \frac{\sqrt{\alpha}}{\mu}$, for all $\alpha > 0$ and $0 < \mu \leq \|K\|$ then

$$\|R_\alpha K x - x\| \leq c_1 \sqrt{\alpha}\|z\|$$

provided $x \in \mathcal{R}(K^*)$ with $x = K^* z$.

PROOF (ADDENDUM TO THE PROOF OF **??**). To see that for $\delta \to 0$, $\alpha(\delta) \to 0$ and $\delta c(\alpha(\delta)) \to 0$ implies admissibility, we see for

$$\|Kx - y^\delta\|_Y < \delta$$

that

$$\|R_{\alpha(\delta)}y^\delta - x\|_X \leq \underbrace{\|R_{\alpha(\delta)}(y^\delta - Kx)\|_X}_{\leq c(\alpha(\delta))\delta} + \underbrace{\|R_{\alpha(\delta)}Kx - x\|_X}_{\to 0 \text{ by } ??}.$$

$\Rightarrow$ This is admissibility! ∎

From formula **??** we see that we seek $x^\alpha = R_\alpha y$ with

$$R_\alpha : (\alpha Id + K^*K)^{-1}K^* : Y \to X.$$

Assume K is compact and linear wit hsingular system (μ_j, x_j, y_j). Then

$$
\begin{aligned}
R_\alpha y &= (\alpha Id + K^*K)^{-1} K^* y \\
&= \sum_{j=1}^{\infty} \frac{1}{\alpha + \mu_j^2} (K^* y, x_j) x_j \\
&= \sum_{j=1}^{\infty} \frac{1}{\alpha + \mu_j^2} (y, y_j) x_j \\
&\stackrel{q(\alpha,\mu_j) = \frac{\mu_j^2}{\alpha + \mu_j^2}}{=} \sum_{j=1}^{\infty} \frac{q(\alpha, \mu_j)}{\mu_j} (y, y_j) x_j
\end{aligned}
$$

The function q satisfies

$$
|q(\alpha, \mu) - 1| = \frac{\alpha}{\alpha + \mu^2} = \frac{\sqrt{\alpha}}{\mu} \frac{\sqrt{\alpha}\mu}{\alpha + \mu^2} \leq \frac{1}{2} \frac{\sqrt{\alpha}}{\mu} \tag{**}
$$

The last estimate follows using the elementary formula

$$
2\sqrt{\alpha}\mu \leq \alpha + \mu^2.
$$

This is (iv) of **??**. Hence we obtain

Theorem 3.11 (Tikhonov regularization method)
Let $K : X \to Y$ be linear and compact, and $\alpha > 0$.

(i) *The operator $\alpha Id + K^*K$ is boundedly invertible. $(R_\alpha)_\alpha$ forms a regularization strategy with*
$$
\|R_\alpha\| \leq \frac{1}{2\sqrt{\alpha}}.
$$
$x^{\alpha,\delta} = R_\alpha y^\delta$ is the unique solution to
$$
\alpha x^{\alpha,\delta} + K^*K x^{\alpha,\delta} = K^* y^\delta.
$$
Every choice $\alpha(\delta)$ with $\alpha(\delta) \to 0$ and $\frac{\delta^2}{\alpha(\delta)} \to 0$ as $\delta \to 0$ is admissible!

- *Let $x \in K^*z \in \operatorname{Im} K^*$ and $\|z\|_Y \leq E$. The choice $\alpha(\delta) = c\frac{\delta}{E}$ for some $c > 0$ leads to*
$$
\|x^{\alpha(\delta),\delta} - x\|_X \leq \frac{1}{2}\left(\frac{1}{\sqrt{c}} + \sqrt{c}\right)\sqrt{\delta E}.
$$

PROOF. (i) Follows from the above discussion. Indeed we have $2\sqrt{\alpha}\mu \leq \alpha + \mu^2$ and hence

$$
q(\alpha, \mu) = \frac{\mu^2}{\alpha + \mu^2} \leq \frac{\mu}{2\sqrt{a}},
$$

which verifies (ii) of **??** for $c(\alpha) = \frac{1}{2\sqrt{\alpha}}$.

(ii) follows from **??**:

$$\|x^{\alpha,\delta} - x\|_X \leq \delta\|R_\alpha\| + \|R_\alpha Kx - x\|$$

and **??** (iv) and $(**)$, namely

$$\|R_\alpha Kx - x\|_X \leq \frac{\sqrt{\alpha}}{2}\|z\|_y, \quad \text{for } c_1 = \frac{1}{sqrt2}.$$

Then

$$\|x^{\alpha,\delta} - x\|_X \leq \frac{\delta}{2\sqrt{\alpha}} + \frac{\sqrt{\alpha}}{2}\|z\|_Y \quad \text{since } \|R_\alpha\| \leq \frac{1}{2\alpha} \text{ from (i).} \qquad \blacksquare$$

With $\alpha(\delta) = c\frac{\delta}{E}$ we obtain

$$\|x^{\alpha,\delta} - x\|_X \leq \frac{\sqrt{\delta}\sqrt{E}}{2\sqrt{c}} + \frac{\sqrt{c}\sqrt{\delta}}{2\sqrt{E}}\underbrace{\|z\|_Y}_{\leq E}$$

$$\leq \left(\frac{1}{2\sqrt{c}} + \frac{\sqrt{c}}{2}\right)\sqrt{\delta E}.$$

Example.
$X = \mathbb{R}$, $Y = \mathbb{R}^2$. Let

$$K : X \to Y, \quad K(x) = \begin{bmatrix} x \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{=K} x.$$

Hence

$$\alpha Id + K^*K = 1 + \alpha,$$

and so for

$$\begin{bmatrix} y_1^\delta \\ y_2^\delta \end{bmatrix} \in Y$$

the regularized problem is

$$(1 + x^{\alpha,\delta}) = y_1^\delta.$$

Thus: $x^{\alpha,\delta} = \frac{y_1^\delta}{1+\alpha}$.

Remark 3.12. The bound from Theorem 3.11 (ii) can for the above model problem even exactly computed.

Chapter 4

The inverse scattering Problem

In this chapter we applly the ideas of regularization theory to the inverse scattering problem.

4.1 Uniqueness for the inverse scattering problem

We revisit the Helmholtz equation and adopt the notation of Chapter 1. We ask the following question:

Given the far-field patterns $u_\infty : \mathcal{S}^2 \to \mathbb{C}$ arising the solution u of the forward scattering Problem 1.1, can we determine the refractive index n?

The answer is, in principle, positive (in principle means we require infinitely many measurements), as the following result shows.

Theorem 4.1 (Uniqueness)

Let $n_1, n_2 \in L^\infty(\mathbb{R})$ be two refractive indices with $n_1(x) = n_2(x) = 1$ for all $|x| \geq \alpha$, and let $u_{1,\infty}, u_{2,\infty}$ be the corresponding far-field patterns. If

$$u_{1,\infty}(\hat{x}, \hat{\Theta}) = u_{2,\infty}(\hat{x}, \hat{\Theta}), \quad \forall \hat{x} \in \mathcal{S}^2 \, \forall \hat{\Theta} \in \mathcal{S}^2$$

then $n_1 = n_2$ (almost everywhere)!

That is, the knowledge of all possible far-field patterns for $\hat{\Theta} \in \mathcal{S}^2$ uniquely determines the refractive index.

PROOF. Let $b > a$ and let $v_1 \in H^1(B_b(0))$ be any solution to $\Delta v_1 + \kappa^2 n_1 v_1 = 0$. Let

$v_2 = u_2(\cdot, \hat{\Theta})$ be the (unique!) solution of Problem 1.1 with

$$u^{in}(x) = e^{i\kappa\hat{\Theta}\cdot x}$$

and the refractive index n_2, with $\hat{\Theta} \in \mathcal{S}^2$ arbitrary. Let $u_1(\cdot, \hat{\Theta})$ solve Problem 1.1 with u^{in} and refractive index n_1. Set

$$u := u_1(\cdot, \hat{\Theta}) - u_2(\cdot, \hat{\Theta}) = u_1^s(\cdot, \hat{\Theta}) - u_2^s(\cdot, \hat{\Theta})$$

(just split the solution into incoming and scattered part).

We assume $u_{1,\infty}(\cdot, \hat{\Theta}) = u_{2,\infty}(\cdot, \hat{\Theta})$. From Rellich's lemma (Theorem 1.2) we know that $u = 0$ outside $B_a(0)$. Furthermore, by the definition of u we have

$$\Delta u + \kappa^2 n_1 u = \kappa^2(n_2 - n_1)v_2,$$

in the weak sense

$$\int_{\mathbb{R}^3} (\nabla u \cdot \nabla \bar{\Psi} - \kappa^2 n_1 u \bar{\Psi}) \, dx = -\kappa^2 \int_{\mathbb{R}^3} (n_2 - n_1) v_2 \bar{\Psi} \, dx.$$

As u vanishes outside of $B_a(0)$ we can restrict the integrals to this ball:

$$\int_{B_a(0)} (\nabla u \cdot \nabla \bar{\Psi} - \kappa^2 n_1 u \bar{\Psi}) \, dx = -\kappa^2 \int_{B_a(0)} (n_2 - n_1) v_2 \bar{\Psi} \, dx.$$

for all Ψ with compact support. We choose $\Psi = \overline{v_1 \phi}$ for some smooth function ϕ with compact support, that is 1 in $B_a(0)$.

This leads to

$$\kappa^2 \int_{B_a(0)} (n_1 - n_2) v_1 v_2 \, dx = \int_{B_a(0)} (\nabla u \cdot \nabla v_1 - \kappa^2 n_1 u v_1) \, dx$$
$$= 0$$

because $\Delta v_1 + \kappa^2 n_1 v_1 = 0$ in $B_b(0)$ and the test function u vanishes outside $B_a(0)$. Thus, we have shown

$$\int_{B_a(0)} (n_1 - n_2) v_1 v_2 \, dx = 0. \tag{*}$$

By the density result of Theorem 1.20, (*) holds for all v_2 that solve

$$\Delta v_2 + \kappa^2 n_2 v_2 = 0, \quad \text{in } B_b(0). \qquad \blacksquare$$

4.2   The regularized Problem

In this section we consider refractive indices $n \in L^2(D)$ for some bounded region $D \subseteq B_a(0)$.

Bibliography: For inverse problems in general and scattering theory see [**?**] or [**?**].

Motivated by Theorem 4.1 we study the "contrast-to-measurement operator", where the contrast q is defined as $q = n - 1$.

$$T : L^2(D) \to L^2(\mathcal{S}^2 \times \mathcal{S}^2), \quad q \mapsto \left[(\hat{x}, \hat{\Theta}) \mapsto u_{\infty, 1+q}(\hat{x}, \hat{\Theta}) \right].$$

Here, for the plane wave direction $\hat{\Theta} \in \mathcal{S}^2$ we denote the far-field pattern of the solution to Problem 1.1 with refractive index $n = 1 + q$ as $u_{\infty, 1+q}(\cdot, \hat{\Theta})$. The inverse problem reads

$$Tq = f \tag{4.1}$$

for given $f \in L^2(\mathcal{S}^2 \times \mathcal{S}^2)$. Here f represents far-field measurements for all $\hat{\Theta} \in \mathcal{S}^2$. We have already discussed, that Eq. (4.1) is ill-posed. We now want to apply the ideas of the Tikhonov regularization to this setting. For theoretical justification see [**?**].
For given $\alpha > 0$, the Tikhonov functional reads, for $q \in L^2(D)$:

$$J_\alpha(q) := \|Tq - f\|^2_{L^2(\mathcal{S}^2 \times \mathcal{S}^2)} + \alpha \|q\|^2_{L^2(D)}.$$

Heuristically, the term $\alpha \|q\|^2_{L^2(D)}$ models the a priori information that q is not very large (example: material deffects). The practical minimization of J_α requires the discretization in steps.

1. In practical situations, only finitely many measurements $\hat{\Theta} \in \Xi_h \subseteq \mathcal{S}^2$ are known.

2. The space $L^2(D)$ over which we optimize must be discretized. We choose a mesh / triangulation D_h of D and choose the ansatz space $P_0(D_h)$ of piecewise constants.

3. For the computation of the far-field patterns we have to discretize the Helmholtz equation with the techniques of Chapter 2. We truncate the domain to some bounded $\Omega \subseteq \mathbb{R}^3$ with a triangulation Ω_h. Then we solve Section 2.2 with

$$F_{\hat{\Theta}} = -\kappa^2 q u^{in}(\cdot, \hat{\Theta})$$

and obtain the far-field approximation $u^h_{\infty, 1+q}$.

Altough the discrete Tikhonov functional reads, for $q \in P_0(D_h)$,

$$J_\alpha^h(q) = \sum_{\hat{\Theta} \in \Xi_h} \|u_{\infty,1+q}^h(\cdot, \hat{\Theta}) - f(\cdot, \hat{\Theta})\|_{L^2(\mathcal{S}^2)}^2 + \alpha \|q\|_{L^2(D)}^2.$$

The discrete inverse scattering problem is

$$\boxed{\text{minimize } J_\alpha^h \text{ over } P_0(D_h)}$$

Suppose $N = dim(P_0(D_h))$, then this is a nonlinear minimization problem over $\mathbb{R}^N$. Several methods are available for the practical approx. solution. A necessary optimality condition is $DJ_\alpha(q_*) = 0$ for a minimizer q_* and the (Fréchet) derivative D. The chain rule shows that the crucial part in the computation of DJ_α^h is the computation of

$$\frac{\partial}{\partial q_j} u_{\infty,1+q}^h,$$

that is the derivative of the Helmholtz solution w.r.t q. We write $q = \sum_{T \in \Omega_h} q_T \mathbf{1}_T$. We now identify q with $(q_j)_{j=1}^N, q = \sum_{j=1}^N q_j \mathbf{1}_j$ for the characteristic function $\mathbf{1}_j$ of $T_j \in D_h$. We want to compute

$$\frac{\partial}{\partial q_j} u_{h,1+q}$$

(We surpress the parameter $\hat{\Theta}$ and far-field transformation). We denote by A the finite element stiffness matrix, by M the boundary mass matrix. Then [Section 2.2](#) reads

$$Au_h^s - \kappa^2 M((1+q) * u_h^s) - i\kappa B u_h^s = -\kappa M(q * u^{in}).$$

Notation: For $p \in P_0(D_h)$ and $v \in P_1(\Omega_h) \cap H^1(\Omega)$ we write $(p * v)$ for the function

$$(p * v)_{|T_j} = p_j v_{|T_j}$$

(assume that D_h, Ω, h are properly aligned (i.e. same meshes are used and D_h is part of the mesh of Ω_h)).
Warning: We deal with real-valued basis functions, so A, M, C are symmetric. In general this is not true.

Now, a formal derivative with the chain rule gives

$$A\frac{\partial}{\partial q_j}u_h^s - \kappa^2 M((1+q) * \frac{\partial}{\partial q_j}u_h^s) - \kappa^2 M(e_j * u_h^s) - i\kappa B\frac{\partial u_h^s}{\partial q_j} = -\kappa M(e_j * u^{in})$$

for the j-th standard cartesian basis vector e_j. Wit

$$w_h := \frac{\partial}{\partial q_j}u_h^s$$

this means

$$Aw_h - \kappa^2 M((1+q) * w_h) - i\kappa B w_h = -\kappa M(e_j * u^{in}) + \kappa^2 (Me_j * u_h^s).$$

Here u_h^s is the discrete Helmholtz solution for the contrast p where we want to compute the derivative. This means w_h is the solution of a (fara...?) Helmholtz problem, and the computation of DJ_α^h requires the solution of $\#\Xi_h \times N$ Helmholtz problems. In Newton's method we require the Hessian matrix $D^2 J_\alpha^h$. For

$$z_h = \frac{\partial}{\partial q_j} \frac{\partial}{\partial q_l} u_h^s$$

a similar calculation yields

$$Az_h - \kappa^2 M((1+q) * z_h) - i\kappa B z_h = \kappa^2 M(e_j * \frac{\partial u_h}{\partial q_l} + e_l * \frac{\partial u_h}{\partial q_j}).$$

The right-hand side stems from the computation of DJ_α^h, so it is known. We have $O(N^2)$ problems to solve.

By reduced integration of the right-hand side (quadrature formula) we can reduce it to N problems: First solve for all unit vectors e_j and then scale by the average of $\frac{\partial u_h}{\partial q_j}$ etc. This corresponds to

$$\int_T q_h v_h \approx \fint_T v_h q_h \operatorname{meas} T$$

This procedure may be less accurate but much cheaper!

This discussion shows that an efficient solution method for the (forward) Helmholtz problem is very important.

4.3   The factorization method

We still consider the nonlinear inverse problem of previous paragraphs, but we restrict ourselves to the more modest problem to determine only the shape of the region, where the index of refraction n differs from 1. In other words, we are looking for the support of the contrast $q = 1 - n$.

The *factorization method* of Kirsch [**?**] provides an explicit characterization of $D :=$ $\operatorname{supp} q$ given far-field measurements (as before) for all incoming wave directions. The approach requires a few more assumptions.

Assumption 4.2

c4se3ass1 There exist finitely many domains D_j, $j = 1, ..., M$ s.t.

$$\overline{D_j} \cap \overline{D_k} = \emptyset, \quad for \ j \neq k$$

and

$$\mathbb{R}^3 \setminus \bigcup_{j=1}^{M} \overline{D_j} \quad is \ connected.$$

Moreover, $n \in L^\infty(\mathbb{R}^3; \mathbb{R})$ such that $q = (1 - n) \geq q_0 > 0$ on $D := \bigcup_{j=1}^{m} D_j$ for some $q_0 > 0$ and $q = 0$ (resp. $n = 1$) on $\mathbb{R}^3 \setminus D$.

The factorization approach is related to the far-field operator, defined by

$$F_q : L^2(\mathcal{S}^2) \rightarrow L^2(\mathcal{S}^2)$$
$$F_q(g)(\hat{x}) := \int_{\mathcal{S}^2} u_{\infty,q}(\hat{x}, \hat{\Theta}) g(\hat{\Theta}) \, dS(\hat{\Theta}), \quad \hat{s} \in \mathcal{S}^2$$

Up to now the far-field u_∞ of u was related to some specific direction of incoming wave $\hat{\Theta}$. Now we give this as a variable argument.

Here $u_{\infty,q}(\cdot, \hat{\Theta}) : \mathcal{S}^2 \rightarrow \mathbb{C}$ denotes the far-field pattern of the scattered field u^s caused by an incident plane wave $u^{in} = e^{i\kappa \hat{\Theta} \cdot x}$ hits the inhomogenity of the material q.

The far-field operator is a compact linear integral operator, that is assumed to be known (at leats in some approximate way, maybe for a given set of directions), because $u_{\infty,q}$ is known data. This section concerns the following variant of the inverse scattering problem:

Problem 4.1. Given the far-field operator F_q, determine the support of the scatterer $D = \text{supp} \, q$.

Theorem 4.3 (Factorization of F_q)
The far-field operator F_q has the factorization

$$F_q = H^* G_q H$$

with bounded linear operator $H : L^2(\mathcal{S}^2) \to L^2(D)$ and its adjoint operator $H^* : L^2(D) \to L^2(\mathcal{S}^2)$ defined by

$$(Hg)(x) = \int_{\mathcal{S}^2} g(\hat{\Theta}) e^{i\kappa \hat{\Theta} \cdot x} \, dS(\hat{\Theta}), \quad x \in D$$

and

$$(H^* \varphi)(\hat{x}) := \int_D \varphi(y) e^{-i\kappa \hat{x} \cdot y} \, dy, \quad \hat{x} \in \mathcal{S}^2$$

and $G_q : L^2(D) \to L^2(D)$ defined by

$$G_q f := \kappa^2 q(f + \kappa^2 v_{|D})$$

where $v \in H^1_{loc}(\mathbb{R}^3)$ that solve

$$\Delta v + \kappa^2 n v = \kappa^2 g \overbrace{f}^{\text{incident wave field}}, \quad in \ \mathbb{R}^3$$

subject to Sommerfelds radiation condition (c.f. Problem 2.2).

PROOF. The factorization follows by construction, the linearity of the scattering problem, Theorem 1.11, Proposition 1.9 (iii). ∎

It turns out that there is a link between the domain D and the range of the adjoing of the so called *Herglotz operator H* defined in the theorem abvoe.

Theorem 4.4
(Let $\mathbb{R}^3 \setminus \overline{D}$) connected. Define for any $z \in \mathbb{R}^3$ a far-field $\phi_z \in L^2(\mathcal{S}^2)$ by

$$\phi_z(\hat{x}) := e^{-i\kappa \hat{x} \cdot z}, \quad \hat{x} \in \mathcal{S}^2.$$

Then it holds that
$$x \in D \quad \Longleftrightarrow \quad \phi_z \in \mathcal{R}(H^*)$$

($\mathcal{R}$ is the range (image)).

PROOF. $\Rightarrow$ Let $z \in D$ and chooose $\tilde{\Phi} \in C^\infty(\mathbb{R}^3)$ with

$$\tilde{\Phi}(y) = \Phi(y, z) \text{outside } D.$$

Moreover set

$$\varphi(y) := -\Delta \tilde{\Phi} - \kappa^2 \tilde{\Phi}.$$

By Theorem 1.11 we have

$$\tilde{\Phi} = \int \varphi(y)\Phi(\cdot, y)\, dy \quad \text{in } \mathbb{R}^3.$$

The far-field of $\tilde{\Phi}$, hence, equals $H^*\varphi$ and also the far-field pattern of $\Phi(\cdot, z)$ (i.e. ϕ_z).

$\Leftarrow$ Observe that $\phi_z \in \mathcal{R}(H^*)$ if and only if there is $\varphi \in L^2(D)$ s.t.

$$\int_D \varphi(y)\Phi(x, y)\, dy = \Phi(x, z), \quad x \in (D \cup \{z\})^C.$$

For $z \notin D$ and $x \to z$ the right-hand side blows up while the left-hand side remains bounded (Cauchy-Schwarz,...). ∎

Theorem 4.5
Let κ^2 is <u>not</u> an interior transmission eigenvalue and $q(x) \geq q_0 > 0$ in D (doesn't change sign). Then
$$\mathcal{R}(H^*) = \mathcal{R}\left((F^*F)^{\frac{1}{4}}\right).$$

Definition 4.6 (Interior transmission eigenvalue). $\kappa > 0$ is called interior transmission eigenvalue, if there is a corresponding eigenfunction $u \in H_0^2(D)$ s.t.

$$\int_D \frac{1}{n-1}(\Delta u + \kappa^2 u)(\Delta \bar{v} + \kappa^2 n \bar{v})\, dx = 0, \quad \forall v \in H_0^2(D)$$

Remark 4.7. κ is almost always <u>not</u> an interior transmission eigenvalue, just for some specific wave numbers.

For further information see also [**?**].

PROOF. The proof follows some abstract range identities and in particular from the fact that, under our assumptions, F_q is one-to-one, normal, $id + \frac{i\kappa}{2\pi}F_q$ is unitary, G_q is a compact perturbation of a coercive operator and

$$\Im(\varphi, G_q\varphi)_{L^2(Q)} > 0, \quad \forall \varphi \in cl(\mathcal{R}(H)), \varphi \neq -.$$

For all details, we refer to [**?**, Sec. 6.5]. ∎

Finally, Picard's criterion ([**?**, Thm A.54]) leads the method.

Theorem 4.8

$$z \in D \iff \phi_z \in \mathcal{R}\left((F^*F)^{\frac{1}{4}}\right).$$

Let $\{\lambda_j, j \in \mathbb{N}\} \subset \mathcal{C}$ be the eigenvalues of F_q with normalized eigenfunctions $\Psi_j \in L^2(\mathcal{S}^2)$, for $j \in \mathbb{N}$. Then

$$z \in D \iff \sum_{j \in \mathbb{N}} \frac{|(\phi_z, \Psi_j)_{L^2(\mathcal{S}^2)}|^2}{|\lambda_j|} < \infty$$

$$\iff \left[\sum_{j \in \mathbb{N}} \frac{|(\phi_z, \Psi_j)_{L^2(\mathcal{S}^2)}|^2}{|\lambda_j|}\right]^{-1} > 0$$

(using the convention $\frac{1}{\infty} = 0$)

Remark 4.9. Theorem 4.8 shows that

$$\omega(z) := \sum_{j \in \mathbb{N}} \left[\frac{|(\phi_z, \Psi_j)_{L^2(\mathcal{S}^2)}|^2}{|\lambda_j|}\right]^{-1} \geq 0,$$

we have that the characteristic function of D equals $sign(\Omega)$.

Theorem 4.8 and the previous remark suggest the following algorithm:

1: *Choose some discrete set of (grid) point (pixels) in a certain test domain*
2: *For any such point z plot some finite approximation*

$\omega_{L,h}(z) := \sum_{j=1}^{L} \left[\frac{|(\phi_z, \Psi_{j,h})_{L^2(\mathcal{S}^2)}|^2}{|\lambda_j|}\right]^{-1}$

   where L is some truncation parameter and h represents the discretization of the far-field operator

---

The hope is, that one is able to identify D from large values of $\omega_{L,h}$.

Note that the ill-posedness of this inverse scattering problem affects this imaging process, because we divide by small numbers $|\lambda_j|$. This requires regularization! See e.g. [**?**] Coltan, Coule, Monk Recent developments in inverse accoustic scat. Siam review 42, 2000, For further information on the factorization method see [**?**]. Lechleiter Factorization method in inverse gc, prepring universität bremen, 2016,

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