Numerical Simulation of acoustic and electromagnetic scattering problems

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Chapter 1

The acoustic scattering Problem in full space

1.1 Introduction

We study the wave equation in full space $\mathbb{R}^d$, $d \in \{1, 2, 3\}$.

$$\frac{\partial^2}{\partial t^2} p + \gamma \frac{\partial}{\partial t} p = c^2 \Delta p$$

Where

$$\begin{aligned} c &= c(x) \text{ speed of sound} \\ \gamma &= \gamma(x) \text{ damping coefficient} \end{aligned}$$

Assume *time-periodic behaviour*

$$p(x,t) = Re[u(x)e^{-i\omega t}]$$

with frequency ω and a real-valued function u.
Since

$$\begin{aligned} \frac{\partial^2 p(x,t)}{\partial t^2} &= \mathrm{Re}[-\omega^2 u(x) e^{-i\omega t}] \\ \frac{\partial p(x,t)}{\partial t} &= Re[-i\omega u(x) e^{-i\omega t}] \\ \Delta p(x,t) &= \mathrm{Re}[\Delta u(x)] e^{-i\omega t}] \end{aligned}$$

for all times $t > 0$, we infer

$$-\omega^2 u - \gamma i \omega u = c^2 \Delta u, \quad \text{ in } \mathbb{R}^d.$$

This is equivalent to

$$\Delta u + \frac{\omega^2}{c^2}\left(1 + \frac{i\gamma}{\omega}\right) u = 0.$$

We assume, that $c = c_0$ is constant in free space (reference value). We can now define a *wave number*

$$\kappa = \frac{\overbrace{\omega}^{>0}}{\underbrace{c_0}_{>0}} > 0$$

and the *index of refraction*

$$n(x) = \frac{c_0^2}{c(x)^2}\left(1 + i\frac{\gamma}{\omega}\right).$$

This results in the *Helmholtz-Equation*

$$\Delta u + \kappa^2 n u = 0, \quad \text{in } \mathbb{R}^d.$$

We assume that there exists $\alpha > 0$ such that

$$c(x) = c_0 \text{ and } \gamma(x) = 0, \quad \forall |x| > \alpha,$$

that means inhomogeneities of the medium lie inside some bounded region (inside a ball). Furthermore we assume, that outside this ball of radius α, $\overline{B_\alpha(0)}$, there are sources that generate plane waves, that is functions of the type

$$u^{in}(x) = e^{ikx\cdot\hat{\theta}},$$

where $|\hat{\theta}| = 1$ for $\hat{\theta} \in \mathbb{R}^d$ and $\cdot$ denotes the inner product in $\mathbb{R}^d$. Then, u^{in} satisfies

$$\Delta u^{in} + k^2 u^{in} = 0.$$

(just calculation). This u^{in} generates a *scattered field* u^s. The *total field*

$$u = u^{in} + u^s$$

satisfies

$$\Delta u + \kappa^2 n u = 0.$$

Furthermore we assume *Sommerfeld's radiation condition* ($r = |x|$):

$$\frac{\partial u^s}{\partial r} - i\kappa u^s(x) = o\left(r^{\frac{1-d}{2}}\right), \quad \text{as } r = |x| \to \infty$$

uniformly in $\frac{x}{|x|}$.

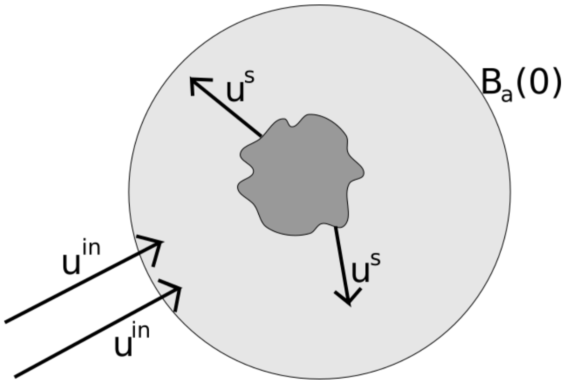

Figure 1.1: Visualization of the problem

1.2 Theory for the direct Scattering Problem in $\mathbb{R}^3$

Throughout this section $d = 3$ holds! Furthermore

1. $\hat{\theta} \in \mathbb{R}^3$ with $|\hat{\theta}| = 1$ defines the *indicent field*

$$u^{in}(x) = e^{-i\kappa\hat{\theta}\cdot x}, \quad x \in \mathbb{R}^d$$

2. $\kappa \in \mathbb{R}, \kappa > 0$
3. $n \in L^\infty(\mathbb{R}^3), \operatorname{Re} n \geq 0, \operatorname{Im} n \geq 0$ and $n(x) = 1$ for all $x \in \mathbb{R}^3 \setminus B_\alpha(0)$ and some $\alpha > 0$

We recall

$$H^1_{loc}(\mathbb{R}^3) \coloneqq \{u : \mathbb{R}^3 \to \mathbb{C} : u|_K \in H^1(K), \quad \text{for every } K = B_R(0) \text{ and any } R > 0\}$$

The *Scattering Problem* reads as follows

Definition 1.1 (Scattering Problem (S)). Given $\hat{\theta}, \kappa, n$ as above. Seek $u \in H^1_{loc}(\mathbb{R}^3)$ such that

$$\Delta u + \kappa^2 n u = 0$$

in $\mathbb{R}^3$ in the weak sense, that is

$$\int_{\mathbb{R}^3} \left(\nabla u \cdot \nabla \bar{\Psi} - \kappa n u \bar{\Psi} \right) \, dx = 0$$

for any $\Psi \in H^1(\mathbb{R}^3)$ with compact support, and s.t.

$$u^s = u - u^{in}$$

satisfies Sommerfeld's radiation condition ($d = 3$, sharper version)

$$\frac{\partial u^s}{\partial r} - i\kappa u^s = O\left(\frac{1}{r^2}\right), \quad \text{uniformly in } \hat{x} = \frac{x}{|x|} \text{ as } r = |x| \to \infty$$

Remark 1.2. Regularity-theory tells us, that the radiation condition is well-defined, given the assumptions from Definition 1.1.

Theorem 1.3 (Rellich's Lemma)
Let u satisfy $\Delta u + \kappa^2 u$ for every $|x| > \alpha$. The following property

$$\lim_{R \to \infty} \int_{|x|=R} |u(x)|^2 \, dS = 0$$

implies, that $u(x) = 0$ for all $|x| > \alpha$.

Remark 1.4. Vice versa, if the property in Theorem 1.3 does not hold, then u cannot vanish for all $|x| > \alpha$.

PROOF. Employs spherical Bessel functions, see e.g. ■

For the proof of uniqueness, we require results on periodic differential equations. We recall *Fourier representations of periodic functions*:
Define

$$Q := (-\pi, \pi)^3 \subseteq \mathbb{R}^3.$$

The functions

$$\{\varphi_j : j \in \mathbb{Z}^3\}$$

with

$$\varphi_j(x) = \frac{1}{(2\pi)^3} e^{ij \cdot x}, \quad \forall x \in Q, j \in \mathbb{Z}^3$$

define a complete orthonormal system (ONS) of $L^2(Q)$ and every $g \in L^2(Q)$ has the expansion

$$g = \sum_{j \in \mathcal{Z}^3} g_j \varphi_j, \quad \text{with } g_j = \int_Q g \varphi_j \, dx, \quad \text{for } j \in \mathcal{Z}^3.$$

Panceval's inequality shows

$$||g||_{L^2(Q)} = \sum_{j\in\mathbb{Z}^3} |g_j|^2.$$

Furthermore, if $g \in L^2(Q)$ and $\sum_{j\in\mathcal{Z}^3} |j|^2|g_j|^2 < \infty$, where $|j| = |j_1| + |j_2| + |j_3|$, then $g \in H^1(Q)$.

Define

$$H^1_{per}(Q) := \left\{ g \in L^2(Q) : \sum_{j\in\mathbb{Z}^3} (1+|j|)^2 |g_j|^2 < \infty \right\}$$

and identify $L^2(Q)$ and $H^1_{per}(Q)$ with the corresponding periodic functions in $\mathbb{R}^3$ by

$$g(2\pi j + x) = g(x), \quad \text{for } x \in Q, j \in \mathbb{Z}^3.$$

Lemma 1.5
Let $p \in \mathbb{R}^3, \alpha \in \mathbb{R}$ *and* $\hat{e} = \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}$. *Then, for every* $t > 0$ *and every* $g \in L^2(Q)$ *there exists a unique solution* $w = w_t(g) \in H^1_{per}(Q)$ *to the differential equation*

$$\Delta w + \lambda_t \cdot \nabla w + \mu_t w = g \tag{*}$$

understood weakly, that is

$$\forall \Psi \in C_c^\infty(Q) : \quad \int \left(-\nabla w \cdot \nabla\bar{\Psi} + (\lambda_t \cdot \nabla w + \mu_t w)\,\bar{\Psi} \right) dx = \int_Q g\bar{\Psi}\, dx$$

for $\lambda_t := 2t\hat{e} - ip$, $\mu_t = -(it+\alpha)$. *It holds that*

$$\|w\|_{L^2(Q)} \leq \frac{1}{t}\|g\|_{L^2(Q)},$$

this means that

$$L_t : L^2(Q) \to L^2(Q) \textit{defined by } g \mapsto w_t(g)$$

defines a bounded operator

$$\|L_t\| \leq \frac{1}{t}, \quad \forall t > 0.$$

Proof. With the Fourier expansions

$$g = \sum_{j\in\mathbb{Z}^3} g_j\varphi_j \quad \text{and} \quad w = \sum_{j\in\mathbb{Z}^3} w_j\varphi_j,$$

Eq. (*) transforms to

$$\forall j \in \mathbb{Z}^3 : \quad c_j w_j = g_j, \quad \text{with } c_j = -|j|^2 + ij\cdot\lambda_t + \mu_t.$$

We have

$$|c_j| \geq |\operatorname{Im} c_j| \overset{insert}{=} |2tj \cdot \hat{e} - t| = t \underbrace{|2\hat{e} \cdot j - 1|}_{\geq 1} \geq t.$$

Thus, the operator

$$(L_t g) := \sum_{j \in \mathbb{Z}^3} \frac{g_j}{c_j} \varphi_j, \quad \text{with } g \in L^2(Q),$$

is well-defined and satisfies

$$\|L_t\| \leq \frac{1}{t}, \quad \forall t > 0.$$

It remains to show, that w is really a solution to the differential equation. In order to do so, let $\Psi \in C_c^\infty(Q)$. As Ψ has compact support, function-value and all derivatives are zero on the boundary of Q, and therefore we have also that Ψ is periodic. We can represent

$$\Psi = \sum_{j \in \mathbb{Z}^3} \Psi_j \varphi_j.$$

■

Then

$$\begin{aligned} \int_Q \left(-\nabla w \cdot \nabla \bar{\Psi} + (\lambda_t \nabla w + \mu_t w)\, \bar{\Psi} \right) dx &= \sum_{j \in \mathbb{Z}^3} c_j w_j \cdot \bar{\Psi}_j \\ &= \sum_{j \in \mathbb{Z}^3} g_j \cdot \bar{\Psi}_j \\ &= \int_Q g \bar{\Psi}\, dx. \end{aligned}$$

Theorem 1.6 (Unique continuation principle)
Let $u \in H^1_{loc}(\mathbb{R}^3)$ solve $\Delta u + \kappa^2 n u = 0$, where $n \in L^\infty(\mathbb{R}^3)$ with $n(x) = 1$ for $|x| > \alpha$, and let $b \geq \alpha$, such that $u(x) = 0$ for all $|x| \geq b$. Then we have $u = 0$ in $\mathbb{R}^3$.

Remark 1.7. Theorem 1.6 holds in a much more general version than we stated here.
PROOF. We introduce the scaling parameter $\varrho = \frac{2b}{\pi}$ and the following function

$$w(x) := e^{\frac{i}{2}x_1 - t\hat{e}\cdot x} u(\varrho x), \quad x \in Q = (-\pi, \pi)^3$$

for some parameter $t > 0$ and now $\hat{e} = \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} \in \mathbb{C}^3$. Since $w(x) = 0$ for all $|x| \geq \frac{pi}{2}$ it can be extended to a 2π-periodic function by

$$w(2\pi j + x) = w(x), \quad \forall j \in \mathbb{Z}^3, x \in Q,$$

with $w \in H^1_{per}(Q)$. It is readily seen that w satisfies

$$\Delta w + \lambda_t \cdot \nabla w + \mu_t w = -\varrho^2 \kappa^2 \tilde{n} w,$$

for

$$\lambda_t := 2t\hat{e} - \begin{bmatrix} i \\ 0 \\ 0 \end{bmatrix}$$
$$\mu_t := -\left(it + \frac{1}{4}\right)$$

and $\tilde{n}$ is the periodic function

$$\tilde{n}(x + 2\pi j) = n(\varrho x), \quad x \in \bar{Q},\, j \in \mathbb{Z}^3$$

(The proof of this is by straightforward differentiation). Lemma 1.5 applies to this situation with $g := -\varrho^2\kappa^2\tilde{n}w$ and yields

$$w = L_t g = -\varrho^2\kappa^2 L_t(\tilde{n}w)$$

with $\|L_t\| \leq \frac{1}{t}$. This means that

$$\begin{aligned}\|w\|_{L^2(Q)} &\leq \frac{1}{t}\varrho^2\kappa^2\|\tilde{n}w\|_{L^2(Q)} \\ &\leq \frac{\varrho^2\kappa^2\|n\|_\infty}{t}\|w\|_{L^2(Q)}.\end{aligned}$$

This holds for all $t > 0$. Taking $t >> 1$ results in $\|w\|_{L^2(Q)} = 0$ where $w = 0$. Thus $u \equiv 0$. ■

Theorem 1.8 (Uniqueness)
Definition 1.1 admits at most one solution. That is, $u^{in} \equiv 0$ implies $u \equiv 0$.

Proof. Let $u^{in} = 0$. Recall the formula

$$|a - i\beta|^2 = |a|^2 + |\beta|^2 + 2\,\mathrm{Im}(\beta\bar{a})$$

(elementary proof). We have

$$\begin{aligned}O\left(\frac{1}{R^2}\right) &= \int_{|x|=R}\left|\frac{\partial u}{\partial r} - i\kappa u\right|^2 dS \\ &= \int_{|x|=R}\left(\left|\frac{\partial u}{\partial r}\right|^2 + \kappa^2|u|^2\right) dS + 2\kappa\,\mathrm{Im}\int_{|x|=R} u\frac{\partial\bar{u}}{\partial r}\, dS\end{aligned}$$

The divergence-theorem implies

$$\begin{aligned}2\kappa\,\mathrm{Im}\int_{|x|=R} u \underbrace{\frac{\partial\bar{u}}{\partial r}}_{=\nabla\bar{u}\cdot\underbrace{\nu}_{\text{normal}}} dS &= 2\kappa\,\mathrm{Im}\left[\int_{|x|<R}\left(u\Delta\bar{u} + \underbrace{|\nabla u|^2}_{\in\mathbb{R}_0^+}\right) dx\right] \\ &\overset{\text{Helmholtz-eq}}{=} 2\kappa\,\mathrm{Im}\left[\int_{|x|<R}\left(-\kappa^2\bar{n}|u|^2\right) dx\right] \\ &\geq 0,\end{aligned}$$

Since $\operatorname{Im} n \geq 0$. Thus we have for large $R \to \infty$:

$$0 \leq \limsup_{R\to\infty} \int_{|x|=R} \left(\left| \frac{\partial u}{\partial r} \right|^2 + \kappa |u|^2 \right) dS \leq 0,$$

whence

$$\lim_{R\to\infty} \int_{|x|=R} |u|^2 \, dS = 0.$$

Theorem 1.3 (Rellich) implies that $u(x) = 0$ for all $|x| > a$. By the unique continuation principle we have $u \equiv 0$ in $\mathbb{R}^3$. ■

We have shown uniqueness. For the existence proof we will construct solutions.

Definition 1.9. The function

$$\Phi(x, y) := \frac{1}{4\pi|x-y|} e^{i\kappa|x-y|}, \quad x, y \in \mathbb{R}^3, x \neq y$$

is called *fundamental solution* or *free space Green's function*.

Φ has the following properties

Proposition 1.10 (Properties of the fundamental solution)

1. *$\Phi(\cdot, y)$ satisfies the Helmholtz-equation $\Delta u + \kappa^2 u = 0$ in $\mathbb{R}^3 \setminus \{y\}$*

2. *Φ satisfies the emphradiation condition*

$$\frac{x}{|x|} \cdot \nabla_x \Phi(x, y) - i\kappa \Phi(x, y) = O\left(\frac{1}{|x|^2}\right), \quad \text{as } |x| \to \infty$$

   uniformly in $\hat{x} = \frac{x}{|x|} \in \mathcal{S}^2$ (Sphere)

3. *Φ has the asymptotic behaviour*

$$\Phi(x, y) = \frac{e^{i\kappa|x|}}{4\pi|x|} e^{-i\kappa \hat{x}\cdot y} + O\left(\frac{1}{|x|^2}\right), \quad \text{as } |x| \to \infty$$

   uniformly in $\hat{x} = \frac{x}{|x|} \in \mathcal{S}^2$ (Sphere)

Proof. The first 2 properties are easily verified. We prove the third:

We calculate the the binomial formula

$$
\begin{aligned}
|x-y| - (|x| - \hat{x}\cdot y) &= \frac{|x-y|^2 - (|x| - \hat{x}\cdot y)^2}{|x-y| + (|x| - \hat{x}\cdot y)} \\
&= \frac{|y|^2 - \cancel{2x\cdot y} + \cancel{|x|}^2 - \cancel{|x|}^2 + \cancel{2x\cdot y} - \overbrace{(\hat{x}\cdot y)^2}^{\le |y|^2}}{|x|\left(1 + \underbrace{\left|\hat{x} - \frac{y}{|x|}\right|}_{1-\left|\frac{y}{|x|}\right|} - \frac{\hat{x}\cdot y}{|x|}\right)} \\
&\le \frac{|y|^2}{2|x|\left(1 - \frac{y}{|x|}\right)}.
\end{aligned}
$$

Hence $|x-y| = |x| - \hat{x}\cdot y + O\left(\frac{1}{|x|}\right)$. Take the exponential of this to get

$$
e^{i\kappa|x-y|} = e^{i\kappa|x|}e^{-i\kappa\hat{x}\cdot y}\left(1 + O\left(\frac{1}{|x|}\right)\right).
$$

By the above formula we also have

$$
\frac{1}{|x-y|} = \frac{1}{|x|} + \left[\frac{|x| - |x-y|}{|x-y||x|}\right] = \frac{1}{|x|} + O\left(\frac{1}{|x|^2}\right).
$$

So

$$
\begin{aligned}
\frac{e^{i\kappa|x-y|}}{|x-y|} &= \frac{e^{i\kappa|x-y|}}{|x|} + O\left(\frac{1}{|x|^2}\right) \\
&= \frac{1}{|x|}e^{i\kappa|x|}e^{-i\kappa\hat{x}\cdot y} + O\left(\frac{1}{|x|^2}\right),
\end{aligned}
$$

as $|x| \to \infty$. ■

Remark 1.11. The previous result states in particular that Φ is determined by the function

$$
e^{-i\kappa\hat{x}\cdot y}
$$

on $\mathcal{S}^2$ (up to pertubation). We will see that this property holds in a more general context.

With the help of the fundamental solution Φ we can create

Theorem 1.12
Let $\Omega \subseteq \mathbb{R}^3$ be bounded. For every $\phi \in L^2(\Omega)$, the function

$$v(x) := \int_\Omega \phi(y)\Phi(x,y)\,dy, \quad x \in \mathbb{R}^3$$

belongs to $H^1_{loc}(\mathbb{R}^3)$ and satisfies the Sommerfeld radiation condition. *Moreover, v is the* only *radiating solution (it is the only function in the set of solution that satisfies the Sommerfeld radiation condition) to*

$$\Delta v + \kappa^2 v = -\phi$$

in the variational sense

$$\int_{\mathbb{R}^3} \left(\nabla v \cdot \nabla \bar{\Psi} - \kappa^2 v \bar{\Psi}\right)\,dx = \int_{\mathbb{R}^3} \phi \bar{\Psi}\,dx \tag{*}$$

for all $\Psi \in H^1(\mathbb{R}^3)$ with compact support. For any $R > 0$ with $\Omega \subseteq B_R(0) =: K$ we have with $c = c(R, \kappa, \Omega)$ that

$$\|v\|_{H^1(K)} \leq c\|\phi\|_{L^2(\Omega)}.$$

In other words, the mapping $\phi \mapsto v$ is a bounded (continuous linear) operator from $L^2(\Omega)$ to $H^1(K)$.

Proof. **(1)** Since $\frac{1}{r^2}$ is locally integrable in $\mathbb{R}^3$, the expression on the right-hand side in Eq. (*) is well-defined. Provided $\phi \in C^1(\bar{\Omega})$, then we can interchange integration and differentiation (posible as we have a majorant, cf. PDE theory) and obtain

$$\Delta v + \kappa^2 v = \begin{cases} -\phi & \text{in } \Omega \\ 0 & \text{in } \mathbb{R}^3 \setminus \bar{\Omega} \end{cases}$$

This means that v solves Eq. (*) for any $\kappa \in \mathbb{C}$.

(2) We cannot directly evaluate L^2-integrals of the gradient $\nabla_x \Phi$. To prove stability, we first consider the special case $\kappa = i$ and $\phi \in C^1(\bar{\Omega})$. Then, the fundamental solution

$$\Phi_i(x,y) = \frac{1}{4\pi|x-y|}e^{-|x-y|}$$

decays exponentially for $|x| \to \infty$, ad by approximation arguments Eq. (*) holds for any $\Psi \in H^1(\mathbb{R}^3)$.

Taking $\Psi = v$ we obtain

$$\|v\|^2_{H^1(\mathbb{R}^3)} = \int_\Omega \phi \bar{v} \leq \|\phi\|_{L^2(\Omega)} \|v\|_{H^1(\mathbb{R}^3)}$$

(note that Eq. (*) with $\kappa = i$ becomes the H^1-Norm). By density, this defines a continuous operator

$$\phi \mapsto v \quad \text{from } L^2(\Omega) \text{ to } H^1(K).$$

(3) Let $k > 0$. We define

$$\Psi(x,y) := \Phi_k(x,y) - \Phi_i(x,y) = \frac{1}{4\pi|x-y|}\left[e^{i\kappa|x-y|} - e^{-|x-y|}\right]$$

It is easy to prove that Ψ and $\nabla_x\Psi$ belong to $L^2(K \times \Omega)$. We sketch the crucial part. We calculate

$$4\pi|\nabla_x\Psi(x,y)| = \left|\underbrace{\frac{i\kappa e^{i\kappa|x-y|} - e^{-|x-y|}}{|x-y|}}_{O\left(\frac{1}{|x-y|}\right)} + \frac{e^{i\kappa|x-y|} - e^{-|x-y|}}{|x-y|^3}\right|$$
$$\overset{*}{leq} O\left(\frac{1}{|x-y|}\right) \text{ for small } |x-y|$$

*: The denominator of the right fraction is in $O(|x-y|)$

Thus,

$$\|\nabla_x\Psi\|_{L^2(K\times\Omega)} \leq C\int_K\int_\Omega \frac{1}{|x-y|^2}\,dy\,dx < \infty.$$

(4) With **(3)** we see that the mapping

$$\varphi \mapsto \int_\Omega \varphi(y)\Psi(\cdot,y)\,dy$$

is bounded from $L^2(\Omega)$ to $H^1(K)$, because

$$\begin{aligned}\int_K\left|\nabla_x\int_\Omega \varphi(y)\Psi(x,y)\,dy\right|^2 dx &= \int_k\left[\int_\Omega|\varphi(y)\nabla_x\Psi(x,y)|\,dy\right]^2 dx\\ &\leq \int_K \|\varphi\|^2_{L^2(\Omega)}\|\nabla_x\Psi(x,\cdot)\|^2_{L^2(\Omega)}\,dx\\ &\leq \|\varphi\|^2_{L^2(\Omega)}\underbrace{\|\nabla_x\Psi\|^2_{L^2(K\times\Omega)}}_{\overset{(3)}{<}\infty}.\end{aligned}$$

This and **(2)** show that $\varphi \mapsto v$ is also bounded from $L^2(\Omega)$ to $H^1(K)$ for $\kappa > 0$.

(5) The radiation condition follows from the radiation condition of Φ:

$$\begin{aligned}\frac{\partial v}{\partial\nu} - i\kappa v &= \int_\Omega \varphi(y)\left(\frac{\partial}{\partial\nu} - i\kappa\right)\Phi(x,y)\,dy\\ &\leq \|\varphi\|_{L^2(\Omega)}O\left(\frac{1}{r}\right)^2.\end{aligned}$$

Uniqueness follows from **(2)**. ■

Remark 1.13. **Proof of Theorem 1.12** (1).
To prove that $\Delta v(x) + \kappa^2 v(x) = -\varphi(x)$, for $x \in \Omega$, it satisfies to verify

$$(\Delta + \kappa^2) \int_{B_\varepsilon(x)} \varphi(y)\Phi(x,y)\, dy = -\varphi(x),$$

for small $\varepsilon > 0$.

1) We readily see that

$$\kappa^2 \left| \int_{B_\varepsilon(x)} \varphi(y)\Phi(x,y)\, dy \right| \le \frac{\kappa^2}{4\pi} \left| \int_{B_\varepsilon(x)} \frac{e^{i\kappa|y-x|}}{|x-y|}\, dy \right|$$

and the expression on the right tends to 0 for $\varepsilon \to 0$.

2) Since $\Delta_x \Phi(x,y) = \Delta_y \phi(x,y)$, we see

$$\begin{aligned} \Delta_y \int_{B_\varepsilon(x)} \varphi(y)\phi(x,y)\, dy &= \int_{B_\varepsilon(x)} \phi(y)\Delta_y \Phi(x,y)\, dy \\ &= \underbrace{-\int_{B_\varepsilon(x)} \nabla\varphi(y) \cdot \nabla\Phi(x,y)}_{=:A} + \underbrace{\int_{|y-x|=\varepsilon} \varphi(y) \frac{\partial \Phi(x,y)}{\partial \nu}\, dS(y)}_{=:B}. \end{aligned}$$

We have that

$$\nabla_y \Phi(x,y) = \frac{1}{4\pi} e^{i\kappa|x-y|} \left(\frac{1}{|x-y|^2} - \frac{1}{|x-y|^3} \right)(x-y),$$

and thus

$$A \le \|\nabla\varphi\|_{L^\infty} \int_{B_\varepsilon(x)} |\nabla_y \Phi(x,y)|\, dy \to 0, \text{ as } \varepsilon \to 0.$$

We stay with B:

$$\begin{aligned} B &= = \frac{1}{4\pi} \int_{\partial B_\varepsilon(x)} e^{i\kappa\varepsilon} \left(\frac{1}{\varepsilon} - \frac{1}{\varepsilon^2} \right) \varphi(y)\, dS(y) \\ &= \underbrace{\frac{1}{4\pi} e^{i\kappa\varepsilon} \left(\frac{1}{\varepsilon} - \frac{1}{\varepsilon^2} \right) 4\pi\varepsilon^2}_{=e^{i\kappa\varepsilon}(\varepsilon+\varepsilon^{-1}) \to -1, \text{ as } \varepsilon \to 0} \underbrace{⨍ B_\varepsilon(x)\varphi(y)\, dS(y)}_{\to \varphi(x), \text{ as } \varepsilon \to 0} \end{aligned}$$

Theorem 1.14 (Lippmann Schwinger integral equation)
Let $\alpha > 0$ and $K = \overline{B_\alpha}(0)$. If $u \in H^1$ solves the Definition 1.1, then $u_{|K} \in L^2(K)$ satisfies the Lippmann Schwinger equation

$$u(x) = u^i n(x) - \kappa^2 \int_{|y|<\alpha} (1 - n(y))\Phi(x,y)u(y)\, dy \qquad \text{(LS)}$$

for almost all $x \in K$.
Conversely, if $u \in L^2(K)$ satisfies Eq. (LS), then it can be extended by the right-hand side of Eq. (LS) to the solution $u \in H^1_{loc}(\mathbb{R}^3)$ of Definition 1.1.

Proof. Let u satisfy Definition 1.1 and define

$$v = \int_K \varphi(y)\Phi(\cdot, y)\, dy$$

for $\varphi = \kappa^2(1-n)u \in L^2(K)$. Theorem 1.12 states that $v \in H^1_{loc}(\mathbb{R}^3)$ satisfies $\Delta v + \kappa^2 v = -\varphi$. Since

$$\Delta u + \kappa^2 u = \kappa^2(1-n)u$$

and

$$\Delta u^{in} + \kappa u^{in} = 0,$$

we have (addition of previous equations)

$$\Delta(v + u^s) + \kappa^2(v + u^s) = 0.$$

The uniqueness from Theorem 1.8 shows that $v + u^s = 0$. Therefore

$$u = u^{in} + u^s = u^{in} - v.$$

For the converse direction let $u \in L^2(K)$ satisfy Eq. (LS). Define v as above, such that $u = u^{in} - v$ in K (in the L^2-sense). By Theorem 1.12 we know that $v \in H^1_{loc}(\mathbb{R}^3)$ by extending it to $\mathbb{R}^3$, and that $\Delta v + \kappa v = -\varphi$. This implies $u \in H^1_{loc}(\mathbb{R}^3)$. Furthermore

$$\Delta u + \kappa^2 u = \varphi = \kappa^2(1-n)u,$$

that is

$$\Delta u + \kappa^2 n u = 0.$$

This implies that $u^s = -v$. The radiation condition follows again from Theorem 1.12. ∎

We now can derive existence for Definition 1.1.

Theorem 1.15 (Existence of solutions for Definition 1.1)
Let κ, n, θ be as assumed previously. Then, the Lippmann Schwinger equation (LS), and thus Definition 1.1, is uniquely solvable.

Proof. Define the following operator $T : L^2(B_\alpha(0)) \to L^2(B_\alpha(0))$ by

$$(Tu)(x) = \kappa^2 \int_{|y|<\alpha} (1 - n(y))\Phi(x, y)u(y)\, dy, \text{ for } |x| < \alpha.$$

Theorem 1.12 shows that T is bounded from $L^2(B_\alpha(0))$ to $H^1(B_\alpha(0))$, and by the compact embedding $H^1(B_\alpha(0)) \hookrightarrow L^2(B_\alpha(0))$, the operator T is compact from $L^2(B_\alpha(0))$ into itself. The Lippmann Schwinger equation reads

$$u + Tu = u^{in}$$

and by the Riesz theory (Fredholm alternative) for compact operators, the uniqueness of u (Theorems 1.8 and 1.14) implies its existence. ∎

Theorem 1.16 (Far-field pattern)
Let u solve Definition 1.1. Then we have

$$u(x) = u^{in}(x) + \frac{e^{i\kappa|x|}}{|x|} u_\infty(\hat{x}) + O\left(\frac{1}{|x|^2}\right), \text{ as } |x| \to \infty,$$

uniformly in $\hat{x} = \frac{x}{|x|}$ (direction of x). The function $u_\infty(\hat{x})$ is given by

$$u_\infty(\hat{x}) = \frac{\kappa^2}{4\pi} \int_{|y|<\alpha} (n(y) - 1) e^{-i\kappa \hat{x}\cdot y} u(y)\, dy$$

is called far-field pattern *or* scattering amplitude

$$u_\infty : \underbrace{\mathcal{S}^2}_{\textit{Sphere}} \to \mathbb{C}$$

is analytic on $\mathcal{S}^2$ and determines u^s otuside $B_\alpha(0)$ uniquely
($u_\infty = 0 \Leftrightarrow u^s(x) = 0$ for $|x| > \alpha$).

The formula for $u(x)$ follows from Proposition 1.10 (iii):

$$\Phi(x, y) = \frac{e^{i\kappa|x|}}{|x|} e^{-i\kappa\hat{x}\cdot y} + O\left(\frac{1}{|x|^2}\right).$$

By Theorem 1.3 (Rellich) we have

$$\lim_{R\to 0} \int_{|x|=R} |u(x)|^2\, dS(x) = 0, \text{ for } |x| > \alpha$$

and this implies uniqueness. The analyticity follows from the formula (calculate derivate w.r.t $\hat{x}$).

Remark 1.17.

1. The concept of far-field pattern is fundamental to inverse scattering theory.
2. The analyticity of u_∞ shows that the inverse scattering problem is ill-posed (little pertubations of u_∞ can destroy analyticity of the function!).