Numerical Simulation of acoustic and electromagnetic scattering problems

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Chapter 1

The acoustic scattering Problem in full space

1.1 Introduction

We study the wave equation in full space $\mathbb{R}^d$, $d \in \{1, 2, 3\}$.

$$\frac{\partial^2}{\partial t^2} p + \gamma \frac{\partial}{\partial t} p = c^2 \Delta p$$

Where

$$\begin{aligned} c &= c(x) \text{ speed of sound} \\ \gamma &= \gamma(x) \text{ damping coefficient} \end{aligned}$$

Assume *time-periodic behaviour*

$$p(x,t) = Re[u(x)e^{-i\omega t}]$$

with frequency ω and a real-valued function u.
Since

$$\begin{aligned} \frac{\partial^2 p(x,t)}{\partial t^2} &= \mathrm{Re}[-\omega^2 u(x) e^{-i\omega t}] \\ \frac{\partial p(x,t)}{\partial t} &= Re[-i\omega u(x) e^{-i\omega t}] \\ \Delta p(x,t) &= \mathrm{Re}[\Delta u(x)] e^{-i\omega t}] \end{aligned}$$

for all times $t > 0$, we infer

$$-\omega^2 u - \gamma i \omega u = c^2 \Delta u, \quad \text{in } \mathbb{R}^d.$$

This is equivalent to

$$\Delta u + \frac{\omega^2}{c^2}\left(1 + \frac{i\gamma}{\omega}\right) u = 0.$$

We assume, that $c = c_0$ is constant in free space (reference value). We can now define a *wave number*

$$\kappa = \frac{\overbrace{\omega}^{>0}}{\underbrace{c_0}_{>0}} > 0$$

and the *index of refraction*

$$n(x) = \frac{c_0^2}{c(x)^2}\left(1 + i\frac{\gamma}{\omega}\right).$$

This results in the *Helmholtz-Equation*

$$\Delta u + \kappa^2 n u = 0, \qquad \text{in } \mathbb{R}^d.$$

We assume that there exists $a > 0$ such that

$$c(x) = c_0 \text{ and } \gamma(x) = 0, \quad \forall |x| > a,$$

that means inhomogeneities of the medium lie inside some bounded region (inside a ball). Furthermore we assume, that outside this ball of radius a, $\overline{B_a(0)}$, there are sources that generate plane waves, that is functions of the type

$$u^{in}(x) = e^{ikx\cdot\hat{\theta}},$$

where $|\hat{\theta}| = 1$ for $\hat{\theta} \in \mathbb{R}^d$ and $\cdot$ denotes the inner product in $\mathbb{R}^d$. Then, u^{in} satisfies

$$\Delta u^{in} + k^2 u^{in} = 0.$$

(just calculation). This u^{in} generates a *scattered field* u^s. The *total field*

$$u = u^{in} + u^s$$

satisfies

$$\Delta u + \kappa^2 n u = 0.$$

Furthermore we assume *Sommerfeld's radiation condition* $(r = |x|)$:

$$\frac{\partial u^s}{\partial r} - i\kappa u^s(x) = o\left(r^{\frac{1-d}{2}}\right), \quad \text{as } r = |x| \to \infty$$

uniformly in $\frac{x}{|x|}$.

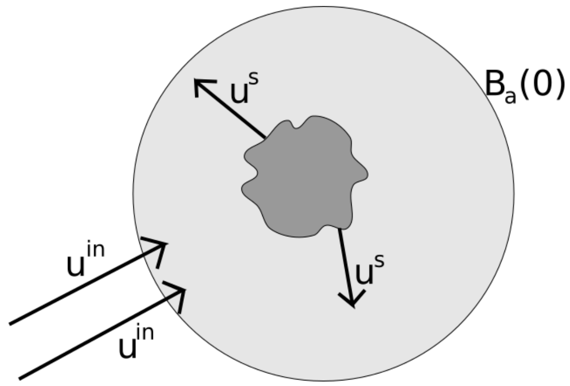

Figure 1.1: Visualization of the problem

1.2 Theory for the direct Scattering Problem in $\mathbb{R}^3$

Throughout this section $d = 3$ holds! Furthermore

1. $\hat{\theta} \in \mathbb{R}^3$ with $|\hat{\theta}| = 1$ defines the *indicent field*

$$u^{in}(x) = e^{-i\kappa\hat{\theta}\cdot x}, \quad x \in \mathbb{R}^d$$

2. $\kappa \in \mathbb{R}, \kappa > 0$
3. $n \in L^\infty(\mathbb{R}^3), \operatorname{Re} n \geq 0, \operatorname{Im} n \geq 0$ and $n(x) = 1$ for all $x \in \mathbb{R}^3 \setminus B_a(0)$ and some $a > 0$

We recall

$$H^1_{loc}(\mathbb{R}^3) \coloneqq \{u : \mathbb{R}^3 \to \mathbb{C} : \, u|_K \in H^1(K), \quad \text{for every } K = B_R(0) \text{ and any } R > 0\}$$

The *Scattering Problem* reads as follows

Definition 1.1 (Scattering Problem (S)). Given $\hat{\theta}, \kappa, n$ as above. Seek $u \in H^1_{loc}(\mathbb{R}^3)$ such that

$$\Delta u + \kappa^2 n u = 0$$

in $\mathbb{R}^3$ in the weak sense, that is

$$\int_{\mathbb{R}^3} \left(\nabla u \cdot \nabla \bar{\Psi} - \kappa n u \bar{\Psi} \right) dx = 0$$

for any $\Psi \in H^1(\mathbb{R}^3)$ with compact support, and s.t.

$$u^s = u - u^{in}$$

satisfies Sommerfeld's radiation condition ($d = 3$, sharper version)

$$\frac{\partial u^s}{\partial r} - i\kappa u^s = O\left(\frac{1}{r^2}\right), \quad \text{uniformly in } \hat{x} = \frac{x}{|x|} \text{ as } r = |x| \to \infty$$

Remark 1.2. Regularity-theory tells us, that the radiation condition is well-defined, given the assumptions from Definition 1.1.

Theorem 1.3 (Rellich's Lemma)
Let u satisfy $\Delta u + \kappa^2 u$ for every $|x| > a$. The following property

$$\lim_{R\to\infty} \int_{|x|=R} |u(x)|^2 \, dS = 0$$

implies, that $u(x) = 0$ for all $|x| > a$.

Remark 1.4. Vice versa, if the property in Theorem 1.3 does not hold, then u cannot vanish for all $|x| > a$.

PROOF. Employs spherical Bessel functions, see e.g. ∎

For the proof of uniqueness, we require results on periodic differential equations. We recall *Fourier representations of periodic functions*:
Define

$$Q := (-\pi, \pi)^3 \subseteq \mathbb{R}^3.$$

The functions

$$\{\varphi_j : j \in \mathbb{Z}^3\}$$

with

$$\varphi_j(x) = \frac{1}{(2\pi)^3} e^{ij\cdot x}, \quad \forall x \in Q, j \in \mathbb{Z}^3$$

define a complete orthonormal system (ONS) of $L^2(Q)$ and every $g \in L^2(Q)$ has the expansion

$$g = \sum_{j\in\mathcal{Z}^3} g_j \varphi_j, \quad \text{with } g_j = \int_Q g\varphi_j \, dx, \quad \text{for } j \in \mathcal{Z}^3.$$

Panceval's inequality shows

$$||g||_{L^2(Q)} = \sum_{j\in\mathcal{Z}^3} |g_j|^2.$$

Furthermore, if $g \in L^2(Q)$ and $\sum_{j\in\mathcal{Z}^3} |j|^2|g_j|^2 < \infty$, where $|j| = |j_1| + |j_2| + |j_3|$, then $g \in H^1(Q)$.

Define

$$H^1_{per}(Q) := \left\{ g \in L^2(Q) : \sum_{j\in\mathbb{Z}^3} (1+|j|)^2|g_j|^2 < \infty \right\}$$

and identify $L^2(Q)$ and $H^1_{per}(Q)$ with the corresponding periodic functions in $\mathbb{R}^3$ by

$$g(2\pi j + x) = g(x), \quad \text{for } x \in Q, j \in \mathbb{Z}^3.$$

Lemma 1.5
Let $p \in \mathbb{R}^3, a \in \mathbb{R}$ *and* $\hat{e} = \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}$. *Then, for every* $t > 0$ *and every* $g \in L^2(Q)$ *there exists a unique solution* $w = w_t(g) \in H^1_{per}(Q)$ *to the differential equation*

$$\Delta w + \lambda_t \cdot \nabla w + \mu_t w = g \tag{*}$$

understood weakly, that is

$$\forall \Psi \in C_c^\infty(Q) : \quad \int \left(-\nabla w \cdot \nabla\bar{\Psi} + (\lambda_t \cdot \nabla w + \mu_t w)\,\bar{\Psi} \right) dx = \int_Q g\bar{\Psi}\,dx$$

for $\lambda_t := 2t\hat{e} - ip$, $\mu_t = -(it + a)$. *It holds that*

$$\|w\|_{L^2(Q)} \le \frac{1}{t}\|g\|_{L^2(Q)},$$

this means that

$$L_t : L^2(Q) \to L^2(Q) \textit{defined by } g \mapsto w_t(g)$$

defines a bounded operator

$$\|L_t\| \le \frac{1}{t}, \quad \forall t > 0.$$

Proof. With the Fourier expansions

$$g = \sum_{j\in\mathbb{Z}^3} g_j\varphi_j \quad \text{and} \quad w = \sum_{j\in\mathbb{Z}^3} w_j\varphi_j,$$

Eq. (*) transforms to

$$\forall j \in \mathbb{Z}^3 : \quad c_j w_j = g_j, \quad \text{with } c_j = -|j|^2 + ij\cdot\lambda_t + \mu_t.$$

We have

$$|c_j| \geq |\operatorname{Im} c_j| \overset{insert}{=} |2tj \cdot \hat{e} - t| = t \underbrace{|2\hat{e} \cdot j - 1|}_{\geq 1} \geq t.$$

Thus, the operator

$$(L_t g) := \sum_{j \in \mathbb{Z}^3} \frac{g_j}{c_j} \varphi_j, \quad \text{with } g \in L^2(Q),$$

is well-defined and satisfies

$$\|L_t\| \leq \frac{1}{t}, \quad \forall t > 0.$$

It remains to show, that w is really a solution to the differential equation. In order to do so, let $\Psi \in C_c^\infty(Q)$. As Ψ has compact support, function-value and all derivatives are zero on the boundary of Q, and therefore we have also that Ψ is periodic. We can represent

$$\Psi = \sum_{j \in \mathbb{Z}^3} \Psi_j \varphi_j.$$

■

Then

$$\begin{aligned} \int_Q \left(-\nabla w \cdot \nabla \bar{\Psi} + (\lambda_t \nabla w + \mu_t w)\, \bar{\Psi}\right) dx &= \sum_{j \in \mathbb{Z}^3} c_j w_j \cdot \bar{\Psi}_j \\ &= \sum_{j \in \mathbb{Z}^3} g_j \cdot \bar{\Psi}_j \\ &= \int_Q g \bar{\Psi}\, dx. \end{aligned}$$

Theorem 1.6
Unique continuation principle Let $u \in H^1_{loc}(\mathbb{R}^3)$ solve $\Delta u + \kappa^2 n u = 0$, where $n \in L^\infty(\mathbb{R}^3)$ with $n(x) = 1$ for $|x| > a$, and let $b \geq a$, such that $u(x) = 0$ for all $|x| \geq b$. Then we have $u = 0$ in $\mathbb{R}^3$.

Remark 1.7. Theorem 1.6 holds in a much more general version than we stated here.
PROOF. We introduce the scaling parameter $\varrho = \frac{2b}{\pi}$ and the following function

$$w(x) := e^{\frac{i}{2}x_1 - t\hat{e}\cdot x} u(\varrho x), \quad x \in Q = (-\pi, \pi)^3$$

for some parameter $t > 0$ and now $\hat{e} = \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} \in \mathbb{C}^3$. Since $w(x) = 0$ for all $|x| \geq \frac{pi}{2}$ it can be extended to a 2π-periodic function by

$$w(2\pi j + x) = w(x), \quad \forall j \in \mathbb{Z}^3,\, x \in Q,$$

with $w \in H^1_{per}(Q)$. It is readily seen that w satisfies

$$\Delta w + \lambda_t \cdot \nabla w + \mu_t w = -\varrho^2 \kappa^2 \tilde{n} w,$$

for

$$\lambda_t := 2t\hat{e} - \begin{bmatrix} i \\ 0 \\ 0 \end{bmatrix}$$
$$\mu_t := -\left(it + \frac{1}{4}\right)$$

and $\tilde{n}$ is the periodic function

$$\tilde{n}(x + 2\pi j) = n(\varrho x), \quad x \in \bar{Q},\, j \in \mathbb{Z}^3$$

(The proof of this is by straightforward differentiation). Lemma 1.5 applies to this situation with $g := -\varrho^2\kappa^2\tilde{n}w$ and yields

$$w = L_t g = -\varrho^2\kappa^2 L_t(\tilde{n}w)$$

with $\|L_t\| \leq \frac{1}{t}$. This means that

$$\begin{aligned} \|w\|_{L^2(Q)} &\leq \frac{1}{t}\varrho^2\kappa^2\|\tilde{n}w\|_{L^2(Q)} \\ &\leq \frac{\varrho^2\kappa^2\|n\|_\infty}{t}\|w\|_{L^2(Q)}. \end{aligned}$$

This holds for all $t > 0$. Taking $t >> 1$ results in $\|w\|_{L^2(Q)} = 0$ where $w = 0$. Thus $u \equiv 0$. ■

Theorem 1.8 (Uniqueness)
Definition 1.1 admits at most one solution. That is, $u^{in} \equiv 0$ implies $u \equiv 0$.

Proof. Let $u^{in} = 0$. Recall the formula

$$|a - i\beta|^2 = |a|^2 + |\beta|^2 + 2\,\mathrm{Im}(\beta\bar{a})$$

(elementary proof). We have

$$\begin{aligned} O\left(\frac{1}{R^2}\right) &= \int_{|x|=R}\left|\frac{\partial u}{\partial r} - i\kappa u\right|^2 dS \\ &= \int_{|x|=R}\left(\left|\frac{\partial u}{\partial r}\right|^2 + \kappa^2|u|^2\right) dS + 2\kappa\,\mathrm{Im}\int_{|x|=R} u\frac{\partial \bar{u}}{\partial r}\, dS \end{aligned}$$

The divergence-theorem implies

$$\begin{aligned} 2\kappa\,\mathrm{Im}\int_{|x|=R} u \underbrace{\frac{\partial \bar{u}}{\partial r}}_{=\nabla\bar{u}\cdot\underbrace{\nu}_{\text{normal}}} dS &= 2\kappa\,\mathrm{Im}\left[\int_{|x|<R}\left(u\Delta\bar{u} + \underbrace{|\nabla u|^2}_{\in\mathbb{R}_0^+}\right) dx\right] \\ &\overset{\text{Helmholtz-eq}}{=} 2\kappa\,\mathrm{Im}\left[\int_{|x|<R}\left(-\kappa^2\bar{n}|u|^2\right) dx\right] \\ &\geq 0, \end{aligned}$$

Since $\operatorname{Im} n \geq 0$. Thus we have for large $R \to \infty$:

$$0 \leq \limsup_{R\to\infty} \int_{|x|=R} \left(\left| \frac{\partial u}{\partial r} \right|^2 + \kappa |u|^2 \right) dS \leq 0,$$

whence

$$\lim_{R\to\infty} \int_{|x|=R} |u|^2 \, dS = 0.$$

Theorem 1.3 (Rellich) implies that $u(x) = 0$ for all $|x| > a$. By the unique continuation principle we have $u \equiv 0$ in $\mathbb{R}^3$. ■

We have shown uniqueness. For the existence proof we will construct solutions.

Definition 1.9. The function

$$\Phi(x, y) := \frac{1}{4\pi |x - y|} e^{i\kappa |x-y|}, \quad x, y \in \mathbb{R}^3, x \neq y$$

is called *fundamental solution* or *free space Green's function*.

Φ has the following properties

Proposition 1.10 (Properties of the fundamental solution)

1. *$\Phi(\cdot, y)$ satisfies the Helmholtz-equation $\Delta u + \kappa^2 u = 0$ in $\mathbb{R}^3 \setminus \{y\}$*

2. *Φ satisfies the emphradiation condition*

$$\frac{x}{|x|} \cdot \nabla_x \Phi(x, y) - i\kappa \Phi(x, y) = O\left(\frac{1}{|x|^2}\right), \quad \text{as } |x| \to \infty$$

uniformly in $\hat{x} = \frac{x}{|x|} \in \mathcal{S}^2$ (Sphere)

3. *Φ has the asymptotic behaviour*

$$\Phi(x, y) = \frac{e^{i\kappa|x|}}{4\pi |x|} e^{-i\kappa \hat{x}\cdot y} + O\left(\frac{1}{|x|^2}\right), \quad \text{as } |x| \to \infty$$

uniformly in $\hat{x} = \frac{x}{|x|} \in \mathcal{S}^2$ (Sphere)

Proof. The first 2 properties are easily verified. We prove the third:

We calculate the the binomial formula

$$\begin{aligned}
|x-y|-(|x|-\hat{x}\cdot y) &= \frac{|x-y|^2-(|x|-\hat{x}\cdot y)^2}{|x-y|+(|x|-\hat{x}\cdot y)} \\
&= \frac{|y|^2-\cancel{2x\cdot y}+\cancel{|x|}^2-\cancel{|x|}^2+\cancel{2x\cdot y}-\overbrace{(\hat{x}\cdot y)^2}^{\leq |y|^2}}{|x|\left(1+\underbrace{\left|\hat{x}-\frac{y}{|x|}\right|}_{1-\left|\frac{y}{|x|}\right|}-\frac{\hat{x}\cdot y}{|x|}\right)} \\
&\leq \frac{|y|^2}{2|x|\left(1-\frac{y}{|x|}\right)}.
\end{aligned}$$

Hence $|x-y|=|x|-\hat{x}\cdot y+O\left(\frac{1}{|x|}\right)$. Take the exponential of this to get

$$e^{i\kappa|x-y|}=e^{i\kappa|x|}e^{-i\kappa\hat{x}\cdot y}\left(1+O\left(\frac{1}{|x|}\right)\right).$$

By the above formula we also have

$$\frac{1}{|x-y|}=\frac{1}{|x|}+\left[\frac{|x|-|x-y|}{|x-y||x|}\right]=\frac{1}{|x|}+O\left(\frac{1}{|x|^2}\right).$$

So

$$\begin{aligned}
\frac{e^{i\kappa|x-y|}}{|x-y|} &= \frac{e^{i\kappa|x-y|}}{|x|}+O\left(\frac{1}{|x|^2}\right) \\
&= \frac{1}{|x|}e^{i\kappa|x|}e^{-i\kappa\hat{x}\cdot y}+O\left(\frac{1}{|x|^2}\right),
\end{aligned}$$

as $|x|\to\infty$. ■

Remark 1.11. The previous result states in particular that Φ is determined by the function

$$e^{-i\kappa\hat{x}\cdot y}$$

on $\mathcal{S}^2$ (up to pertubation). We will see that this property holds in a more general context.

With the help of the fundamental solution Φ we can create

Theorem 1.12
Let $\Omega \subseteq \mathbb{R}^3$ be bounded. For every $\phi \in L^2(\Omega)$, the function

$$v(x) := \int_\Omega \phi(y)\Phi(x,y)\,dy, \quad x \in \mathbb{R}^3$$

belongs to $H^1_{loc}(\mathbb{R}^3)$ and satisfies the Sommerfeld radiation condition*. Moreover, v is the* only *radiating solution (it is the only function in the set of solution that satisfies the Sommerfeld radiation condition) to*

$$\Delta v + \kappa^2 v = -\phi$$

in the variational sense

$$\int_{\mathbb{R}^3} \left(\nabla v \cdot \nabla \bar{\Psi} - \kappa^2 v \bar{\Psi}\right)\,dx = \int_{\mathbb{R}^3} \phi \bar{\Psi}\,dx \tag{*}$$

for all $\Psi \in H^1(\mathbb{R}^3)$ with compact support. For any $R > 0$ with $\Omega \subseteq B_R(0) =: K$ we have with $c = c(R, \kappa, \Omega)$ that

$$\|v\|_{H^1(K)} \le c\|\phi\|_{L^2(\Omega)}.$$

In other words, the mapping $\phi \mapsto v$ is a bounded (continuous linear) operator from $L^2(\Omega)$ to $H^1(K)$.

Proof. **(1)** Since $\frac{1}{r^2}$ is locally integrable in $\mathbb{R}^3$, the expression on the right-hand side in Eq. (*) is well-defined. Provided $\phi \in C^1(\bar{\Omega})$, then we can interchange integration and differentiation (posible as we have a majorant, cf. PDE theory) and obtain

$$\Delta v + \kappa^2 v = \begin{cases} -\phi & \text{in } \Omega \\ 0 & \text{in } \mathbb{R}^3 \setminus \bar{\Omega} \end{cases}$$

This means that v solves Eq. (*) for any $\kappa \in \mathbb{C}$.

(2) We cannot directly evaluate L^2-integrals of the gradient $\nabla_x \Phi$. To prove stability, we first consider the special case $\kappa = i$ and $\phi \in C^1(\bar{\Omega})$. Then, the fundamental solution

$$\Phi_i(x,y) = \frac{1}{4\pi|x-y|} e^{-|x-y|}$$

decays exponentially for $|x| \to \infty$, ad by approximation arguments Eq. (*) holds for any $\Psi \in H^1(\mathbb{R}^3)$.

Taking $\Psi = v$ we obtain

$$\|v\|^2_{H^1(\mathbb{R}^3)} = \int_\Omega \phi \bar{v} \le \|\phi\|_{L^2(\Omega)} \|v\|_{H^1(\mathbb{R}^3)}$$

(note that Eq. (*) with $\kappa = i$ becomes the H^1-Norm). By density, this defines a continuous operator

$$\phi \mapsto v \quad \text{from } L^2(\Omega) \text{ to } H^1(K).$$