Numerical Simulation of acoustic and electromagnetic scattering problems

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Chapter 1

The acoustic scattering Problem in full space

1.1 Introduction

We study the wave equation in full space $\mathbb{R}^d$, $d \in \{1,2,3\}$.

$$\frac{\partial^2}{\partial t^2}p + \gamma\frac{\partial}{\partial t}p = c^2\Delta p$$

Where

$$\begin{aligned} c &= c(x) \text{ speed of sound} \\ \gamma &= \gamma(x) \text{ damping coefficient} \end{aligned}$$

Assume *time-periodic behaviour*

$$p(x,t) = Re[u(x)e^{-i\omega t}]$$

with frequency ω and a real-valued function u.
Since

$$\begin{aligned} \frac{\partial^2 p(x,t)}{\partial t^2} &= \mathrm{Re}[-\omega^2 u(x)e^{-i\omega t}] \\ \frac{\partial p(x,t)}{\partial t} &= Re[-i\omega u(x)e^{-i\omega t}] \\ \Delta p(x,t) &= \mathrm{Re}[\Delta u(x)]e^{-i\omega t}] \end{aligned}$$

for all times $t > 0$, we infer

$$-\omega^2 u - \gamma i\omega u = c^2\Delta u, \quad \text{ in } \mathbb{R}^d.$$

This is equivalent to
$$\Delta u + \frac{\omega^2}{c^2}\left(1 + \frac{i\gamma}{\omega}\right) u = 0.$$
We assume, that $c = c_0$ is constant in free space (reference value). We can now define a *wave number*
$$\kappa = \frac{\overbrace{\omega}^{>0}}{\underbrace{c_0}_{>0}} > 0$$
and the *index of refraction*
$$n(x) = \frac{c_0^2}{c(x)^2}\left(1 + i\frac{\gamma}{\omega}\right).$$
This results in the *Helmholtz-Equation*
$$\Delta u + \kappa^2 n u = 0, \qquad \text{in } \mathbb{R}^d.$$
We assume that there exists $a > 0$ such that
$$c(x) = c_0 \text{ and } \gamma(x) = 0, \quad \forall |x| > a,$$
that means inhomogeneities of the medium lie inside some bounded region (inside a ball). Furthermore we assume, that outside this ball of radius a, $\overline{B_a(0)}$, there are sources that generate plane waves, that is functions of the type
$$u^{in}(x) = e^{ikx\cdot\hat{\theta}},$$
where $|\hat{\theta}| = 1$ for $\hat{\theta} \in \mathbb{R}^d$ and $\cdot$ denotes the inner product in $\mathbb{R}^d$. Then, u^{in} satisfies
$$\Delta u^{in} + k^2 u^{in} = 0.$$
(just calculation). This u^{in} generates a *scattered field* u^s. The *total field*
$$u = u^{in} + u^s$$
satisfies
$$\Delta u + \kappa^2 n u = 0.$$

Furthermore we assume *Sommerfeld's radiation condition* $(r = |x|)$:
$$\frac{\partial u^s}{\partial r} - i\kappa u^s(x) = o\left(r^{\frac{1-d}{2}}\right), \quad \text{as } r = |x| \to \infty$$
uniformly in $\frac{x}{|x|}$.

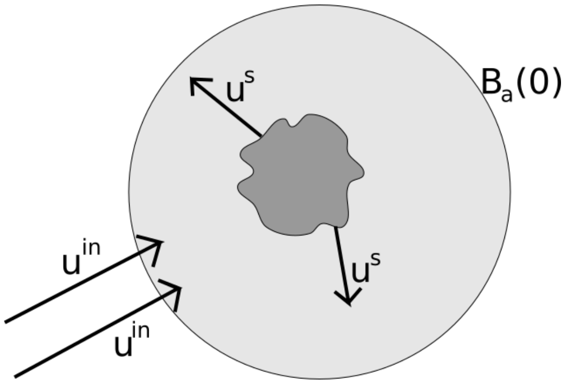

Figure 1.1: Visualization of the problem

1.2 Theory for the direct Scattering Problem in $\mathbb{R}^3$

Throughout this section $d = 3$ holds! Furthermore

1. $\hat{\theta} \in \mathbb{R}^3$ with $|\hat{\theta}| = 1$ defines the *indicent field*

$$u^{in}(x) = e^{-i\kappa\hat{\theta}\cdot x}, \quad x \in \mathbb{R}^d$$

2. $\kappa \in \mathbb{R}, \kappa > 0$
3. $n \in L^\infty(\mathbb{R}^3), \operatorname{Re} n \geq 0, \operatorname{Im} n \geq 0$ and $n(x) = 1$ for all $x \in \mathbb{R}^3 \setminus B_a(0)$ and some $a > 0$

We recall

$$H^1_{loc}(\mathbb{R}^3) \coloneqq \{u : \mathbb{R}^3 \to \mathbb{C} : \, u|_K \in H^1(K), \quad \text{for every } K = B_R(0) \text{ and any } R > 0\}$$

The *Scattering Problem* reads as follows

Definition 1.1. Scattering Problem (S) Govem $\hat{\theta}, \kappa, n$ as above. Seek $u \in H^1_{loc}(\mathbb{R}^3)$ such that

$$\Delta u + \kappa^2 n u = 0$$

in $\mathbb{R}^3$ in the weak sense, that is

$$\int_{\mathbb{R}^3} \left(\nabla u \cdot \nabla \bar{\Psi} - \kappa n u \bar{\Psi} \right) \, dx = 0$$

for any $\Psi \in H^1(\mathbb{R}^3)$ with compact support, and s.t.

$$u^s = u - u^{in}$$

satisfies Sommerfeld's radiation condition ($d = 3$, sharper version)

$$\frac{\partial u^s}{\partial r} - i\kappa u^s = O\left(\frac{1}{r^2}\right), \quad \text{uniformly in } \hat{x} = \frac{x}{|x|} \text{ as } r = |x| \to \infty$$

Remark 1.2. Regularity-theory tells us, that the radiation condition is well-defined, given the assumptions from Definition 1.1.

Theorem 1.3
Rellich's Lemma Let u satisfy $\Delta u + \kappa^2 u$ for every $|x| > a$. The following property

$$\lim_{R \to \infty} \int_{|x|=R} |u(x)|^2 \, dS = 0$$

implies, that $u(x) = 0$ for all $|x| > a$.

Remark 1.4. Vice versa, if the property in Theorem 1.3 does not hold, then u cannot vanish for all $|x| > a$.

Proof. Employs spherical Bessel functions, see e.g. ■

For the proof of uniqueness, we require results on periodic differential equations. We recall *Fourier representations of periodic functions*:
Define

$$Q := (-\pi, \pi)^3 \subseteq \mathbb{R}^3.$$

The functions

$$\{\varphi_j : j \in \mathbb{Z}^3\}$$

with

$$\varphi_j(x) = \frac{1}{(2\pi)^3} e^{ij \cdot x}, \quad \forall x \in Q, j \in \mathbb{Z}^3$$

define a complete orthonormal system (ONS) of $L^2(Q)$ and every $g \in L^2(Q)$ has the expansion

$$g = \sum_{j \in \mathcal{Z}^3} g_j \varphi_j, \quad \text{with } g_j = \int_Q g \varphi_j \, dx, \quad \text{for } j \in \mathcal{Z}^3.$$

Panceval's inequality shows

$$||g||_{L^2(Q)} = \sum_{j \in \mathcal{Z}^3} |g_j|^2.$$

Furthermore, if $g \in L^2(Q)$ and $\sum_{j \in \mathcal{Z}^3} |j|^2 |g_j|^2 < \infty$, where $|j| = |j_1| + |j_2| + |j_3|$, then $g \in H^1(Q)$.

Define

$$H^1_{per}(Q) := \left\{ g \in L^2(Q) : \sum_{j \in \mathbb{Z}^3} (1 + |j|)^2 |g_j|^2 < \infty \right\}$$

and identify $L^2(Q)$ and $H^1_{per}(Q)$ with the corresponding periodic functions in $\mathbb{R}^3$ by

$$g(2\pi j + x) = g(x), \quad \text{for } x \in Q, j \in \mathbb{Z}^3.$$

Lemma 1.5
Let $p \in \mathbb{R}^3, a \in \mathbb{R}$ *and* $\hat{e} = \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}$. *Then, for every* $t > 0$ *and every* $g \in L^2(Q)$ *there exists a unique solution* $w = w_t(g) \in H^1_{per}(Q)$ *to the differential equation*

$$\Delta w + \lambda_t \cdot \nabla w + \mu_t w = g \tag{*}$$

understood weakly, that is

$$\forall \Psi \in C_c^\infty(Q) : \quad \int \left(-\nabla w \cdot \nabla \bar{\Psi} + (\lambda_t \cdot \nabla w + \mu_t w) \, \bar{\Psi} \right) dx = \int_Q g \bar{\Psi} \, dx$$

for $\lambda_t := 2t\hat{e} - ip$, $\mu_t = -(it + a)$. *It holds that*

$$\|w\|_{L^2(Q)} \leq \frac{1}{t} \|g\|_{L^2(Q)},$$

this means that

$$L_t : L^2(Q) \to L^2(Q) \textit{defined by } g \mapsto w_t(g)$$

defines a bounded operator

$$\|L_t\| \leq \frac{1}{t}, \quad \forall t > 0.$$

Proof. With the Fourier expansions

$$g = \sum_{j \in \mathbb{Z}^3} g_j \varphi_j \quad \text{and} \quad w = \sum_{j \in \mathbb{Z}^3} w_j \varphi_j,$$

Eq. (*) transforms to

$$\forall j \in \mathbb{Z}^3 : \quad c_j w_j = g_j, \quad \text{with } c_j = -|j|^2 + ij \cdot \lambda_t + \mu_t.$$

We have

$$|c_j| \geq |\operatorname{Im} c_j| \overset{insert}{=} |2tj \cdot \hat{e} - t| = t \underbrace{|2\hat{e} \cdot j - 1|}_{\geq 1} \geq t.$$

Thus, the operator

$$(L_t g) := \sum_{j \in \mathbb{Z}^3} \frac{g_j}{c_j} \varphi_j, \quad \text{with } g \in L^2(Q),$$

is well-defined and satisfies

$$\|L_t\| \leq \frac{1}{t}, \quad \forall t > 0.$$

It remains to show, that w is really a solution to the differential equation. In order to do so, let $\Psi \in C_c^\infty(Q)$. As Ψ has compact support, function-value and all derivatives are zero on the boundary of Q, and therefore we have also that Ψ is periodic. We can represent

$$\Psi = \sum_{j \in \mathbb{Z}^3} \Psi_j \varphi_j.$$

■

Then

$$\begin{aligned}
\int_Q \left(-\nabla w \cdot \nabla \bar{\Psi} + (\lambda_t \nabla w + \mu_t w)\, \bar{\Psi} \right) dx &= \sum_{j \in \mathbb{Z}^3} c_j w_j \cdot \bar{\Psi}_j \\
&= \sum_{j \in \mathbb{Z}^3} g_j \cdot \bar{\Psi}_j \\
&= \int_Q g \bar{\Psi}\, dx.
\end{aligned}$$

Theorem 1.6
Unique continuation principle Let $u \in H^1_{loc}(\mathbb{R}^3)$ solve $\Delta u + \kappa^2 n u = 0$, where $n \in L^\infty(\mathbb{R}^3)$ with $n(x) = 1$ for $|x| > a$, and let $b \geq a$, such that $u(x) = 0$ for all $|x| \geq b$. Then we have $u = 0$ in $\mathbb{R}^3$.

Remark 1.7. Theorem 1.6 holds in a much more general version than we stated here.