# Numerical Simulation of acoustic and electromagnetic scattering problems

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# Contents

# Chapter 1

# The acoustic scattering Problem in full space

## 1.1 Introduction

We study the wave equation in full space $\mathbb{R}^d$, $d \in \{1, 2, 3\}$.

$$\frac{\partial^2}{\partial t^2} p + \gamma \frac{\partial}{\partial t} p = c^2 \Delta p$$

Where

$$\begin{aligned} c &= c(x) \text{ speed of sound} \\ \gamma &= \gamma(x) \text{ damping coefficient} \end{aligned}$$

Assume *time-periodic behaviour*

$$p(x,t) = Re[u(x)e^{-i\omega t}]$$

with frequency $\omega$ and a real-valued function $u$.
Since

$$\begin{aligned} \frac{\partial^2 p(x,t)}{\partial t^2} &= \mathrm{Re}[-\omega^2 u(x) e^{-i\omega t}] \\ \frac{\partial p(x,t)}{\partial t} &= Re[-i\omega u(x) e^{-i\omega t}] \\ \Delta p(x,t) &= \mathrm{Re}[\Delta u(x)] e^{-i\omega t}] \end{aligned}$$

for all times $t > 0$, we infer

$$-\omega^2 u - \gamma i \omega u = c^2 \Delta u, \quad \text{ in } \mathbb{R}^d.$$

This is equivalent to
$$\Delta u + \frac{\omega^2}{c^2}\left(1 + \frac{i\gamma}{\omega}\right) u = 0.$$
We assume, that $c = c_0$ is constant in free space (reference value). We can now define a *wave number*
$$\kappa = \frac{\overbrace{\omega}^{>0}}{\underbrace{c_0}_{>0}} > 0$$
and the *index of refraction*
$$n(x) = \frac{c_0^2}{c(x)^2}\left(1 + i\frac{\gamma}{\omega}\right).$$
This results in the *Helmholtz-Equation*
$$\Delta u + \kappa^2 n u = 0, \quad \text{in } \mathbb{R}^d.$$
We assume that there exists $\alpha > 0$ such that
$$c(x) = c_0 \text{ and } \gamma(x) = 0, \quad \forall |x| > \alpha,$$
that means inhomogeneities of the medium lie inside some bounded region (inside a ball). Furthermore we assume, that outside this ball of radius $\alpha$, $\overline{B_\alpha(0)}$, there are sources that generate plane waves, that is functions of the type
$$u^{in}(x) = e^{ikx\cdot\hat{\theta}},$$
where $|\hat{\theta}| = 1$ for $\hat{\theta} \in \mathbb{R}^d$ and $\cdot$ denotes the inner product in $\mathbb{R}^d$. Then, $u^{in}$ satisfies
$$\Delta u^{in} + k^2 u^{in} = 0.$$
(just calculation). This $u^{in}$ generates a *scattered field* $u^s$. The *total field*
$$u = u^{in} + u^s$$
satisfies
$$\Delta u + \kappa^2 n u = 0.$$

Furthermore we assume *Sommerfeld's radiation condition* $(r = |x|)$:
$$\frac{\partial u^s}{\partial r} - i\kappa u^s(x) = o\left(r^{\frac{1-d}{2}}\right), \quad \text{as } r = |x| \to \infty$$
uniformly in $\frac{x}{|x|}$.

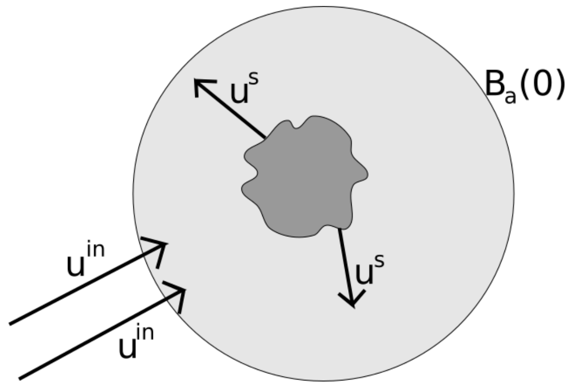


Figure 1.1: Visualization of the problem

## 1.2 Theory for the direct Scattering Problem in $\mathbb{R}^3$

Throughout this section $d = 3$ holds! Furthermore

1. $\hat{\theta} \in \mathbb{R}^3$ with $|\hat{\theta}| = 1$ defines the *indicent field*

$$u^{in}(x) = e^{-i\kappa\hat{\theta}\cdot x}, \quad x \in \mathbb{R}^d$$

2. $\kappa \in \mathbb{R}, \kappa > 0$
3. $n \in L^\infty(\mathbb{R}^3), \mathrm{Re}\, n \geq 0, \mathrm{Im}\, n \geq 0$ and $n(x) = 1$ for all $x \in \mathbb{R}^3 \setminus B_\alpha(0)$ and some $\alpha > 0$

We recall

$$H^1_{loc}(\mathbb{R}^3) := \{u : \mathbb{R}^3 \to \mathbb{C} :\, u|_K \in H^1(K), \quad \text{for every } K = B_R(0) \text{ and any } R > 0\}$$

The *Scattering Problem* reads as follows

**Problem 1.1 (Scattering Problem (S)).** Given $\hat{\theta}, \kappa, n$ as above. Seek $u \in H^1_{loc}(\mathbb{R}^3)$ such that

$$\Delta u + \kappa^2 n u = 0$$

in $\mathbb{R}^3$ in the weak sense, that is

$$\int_{\mathbb{R}^3} \left( \nabla u \cdot \nabla \bar{\Psi} - \kappa n u \bar{\Psi} \right) \, dx = 0$$

for any $\Psi \in H^1(\mathbb{R}^3)$ with compact support, and s.t.

$$u^s = u - u^{in}$$

satisfies Sommerfeld's radiation condition ($d = 3$, sharper version)

$$\frac{\partial u^s}{\partial r} - i\kappa u^s = O\left(\frac{1}{r^2}\right), \quad \text{uniformly in } \hat{x} = \frac{x}{|x|} \text{ as } r = |x| \to \infty$$

*Remark 1.1.* Regularity-theory tells us, that the radiation condition is well-defined, given the assumptions from Problem 1.1.

**Theorem 1.2 (Rellich's Lemma)**
*Let $u$ satisfy $\Delta u + \kappa^2 u$ for every $|x| > \alpha$. The following property*

$$\lim_{R \to \infty} \int_{|x|=R} |u(x)|^2 \, dS = 0$$

*implies, that $u(x) = 0$ for all $|x| > \alpha$.*

*Remark 1.3.* Vice versa, if the property in Theorem 1.2 does not hold, then $u$ cannot vanish for all $|x| > \alpha$.

PROOF. Employs spherical Bessel functions, see e.g. ■

For the proof of uniqueness, we require results on periodic differential equations. We recall *Fourier representations of periodic functions*:
Define

$$Q := (-\pi, \pi)^3 \subseteq \mathbb{R}^3.$$

The functions

$$\{\varphi_j : j \in \mathbb{Z}^3\}$$

with

$$\varphi_j(x) = \frac{1}{(2\pi)^3} e^{ij \cdot x}, \quad \forall x \in Q, j \in \mathbb{Z}^3$$

define a complete orthonormal system (ONS) of $L^2(Q)$ and every $g \in L^2(Q)$ has the expansion

$$g = \sum_{j \in \mathcal{Z}^3} g_j \varphi_j, \quad \text{with } g_j = \int_Q g \varphi_j \, dx, \quad \text{for } j \in \mathcal{Z}^3.$$

*Panceval's inequality* shows

$$||g||_{L^2(Q)} = \sum_{j\in\mathbb{Z}^3} |g_j|^2.$$

Furthermore, if $g \in L^2(Q)$ and $\sum_{j\in\mathcal{Z}^3} |j|^2|g_j|^2 < \infty$, where $|j| = |j_1| + |j_2| + |j_3|$, then $g \in H^1(Q)$.

Define

$$H^1_{per}(Q) \coloneqq \left\{ g \in L^2(Q) : \sum_{j\in\mathbb{Z}^3} (1+|j|)^2|g_j|^2 < \infty \right\}$$

and identify $L^2(Q)$ and $H^1_{per}(Q)$ with the corresponding periodic functions in $\mathbb{R}^3$ by

$$g(2\pi j + x) = g(x), \quad \text{for } x \in Q, j \in \mathbb{Z}^3.$$

**Lemma 1.4**
*Let $p \in \mathbb{R}^3, \alpha \in \mathbb{R}$ and $\hat{e} = \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}$. Then, for every $t > 0$ and every $g \in L^2(Q)$ there exists a unique solution $w = w_t(g) \in H^1_{per}(Q)$ to the differential equation*

$$\Delta w + \lambda_t \cdot \nabla w + \mu_t w = g \tag{*}$$

*understood weakly, that is*

$$\forall \Psi \in C_c^\infty(Q) : \quad \int \left( -\nabla w \cdot \nabla \bar{\Psi} + (\lambda_t \cdot \nabla w + \mu_t w)\, \bar{\Psi} \right) dx = \int_Q g \bar{\Psi}\, dx$$

*for $\lambda_t \coloneqq 2t\hat{e} - ip$, $\mu_t = -(it+\alpha)$. It holds that*

$$\|w\|_{L^2(Q)} \leq \frac{1}{t}\|g\|_{L^2(Q)},$$

*this means that*

$$L_t : L^2(Q) \to L^2(Q) \textit{defined by } g \mapsto w_t(g)$$

*defines a bounded operator*

$$\|L_t\| \leq \frac{1}{t}, \quad \forall t > 0.$$

PROOF. With the Fourier expansions

$$g = \sum_{j\in\mathbb{Z}^3} g_j \varphi_j \quad \text{and} \quad w = \sum_{j\in\mathbb{Z}^3} w_j \varphi_j,$$

Eq. (*) transforms to

$$\forall j \in \mathbb{Z}^3 : \quad c_j w_j = g_j, \quad \text{with } c_j = -|j|^2 + ij \cdot \lambda_t + \mu_t.$$

We have

$$|c_j| \geq |\operatorname{Im} c_j| \overset{insert}{=} |2tj \cdot \hat{e} - t| = t \underbrace{|2\hat{e} \cdot j - 1|}_{\geq 1} \geq t.$$

Thus, the operator

$$(L_t g) := \sum_{j \in \mathbb{Z}^3} \frac{g_j}{c_j} \varphi_j, \quad \text{with } g \in L^2(Q),$$

is well-defined and satisfies

$$\|L_t\| \leq \frac{1}{t}, \quad \forall t > 0.$$

It remains to show, that $w$ is really a solution to the differential equation. In order to do so, let $\Psi \in C_c^\infty(Q)$. As $\Psi$ has compact support, function-value and all derivatives are zero on the boundary of $Q$, and therefore we have also that $\Psi$ is periodic. We can represent

$$\Psi = \sum_{j \in \mathbb{Z}^3} \Psi_j \varphi_j.$$

■

Then

$$\begin{aligned} \int_Q \left( -\nabla w \cdot \nabla \bar{\Psi} + (\lambda_t \nabla w + \mu_t w) \bar{\Psi} \right) dx &= \sum_{j \in \mathbb{Z}^3} c_j w_j \cdot \bar{\Psi}_j \\ &= \sum_{j \in \mathbb{Z}^3} g_j \cdot \bar{\Psi}_j \\ &= \int_Q g \bar{\Psi} \, dx. \end{aligned}$$

**Theorem 1.5 (Unique continuation principle)**
*Let $u \in H^1_{loc}(\mathbb{R}^3)$ solve $\Delta u + \kappa^2 n u = 0$, where $n \in L^\infty(\mathbb{R}^3)$ with $n(x) = 1$ for $|x| > \alpha$, and let $b \geq \alpha$, such that $u(x) = 0$ for all $|x| \geq b$. Then we have $u = 0$ in $\mathbb{R}^3$.*

*Remark 1.6.* Theorem 1.5 holds in a much more general version than we stated here.
PROOF. We introduce the scaling parameter $\varrho = \frac{2b}{\pi}$ and the following function

$$w(x) := e^{\frac{i}{2} x_1 - t\hat{e} \cdot x} u(\varrho x), \quad x \in Q = (-\pi, \pi)^3$$

for some parameter $t > 0$ and now $\hat{e} = \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} \in \mathbb{C}^3$. Since $w(x) = 0$ for all $|x| \geq \frac{pi}{2}$ it can be extended to a $2\pi$-periodic function by

$$w(2\pi j + x) = w(x), \quad \forall j \in \mathbb{Z}^3, x \in Q,$$

with $w \in H^1_{per}(Q)$. It is readily seen that $w$ satisfies

$$\Delta w + \lambda_t \cdot \nabla w + \mu_t w = -\varrho^2 \kappa^2 \tilde{n} w,$$

for

$$\lambda_t := 2t\hat{e} - \begin{bmatrix} i \\ 0 \\ 0 \end{bmatrix}$$
$$\mu_t := -\left(it + \frac{1}{4}\right)$$

and $\tilde{n}$ is the periodic function

$$\tilde{n}(x + 2\pi j) = n(\varrho x), \quad x \in \bar{Q},\, j \in \mathbb{Z}^3$$

(The proof of this is by straightforward differentiation). Lemma 1.4 applies to this situation with $g := -\varrho^2\kappa^2\tilde{n}w$ and yields

$$w = L_t g = -\varrho^2\kappa^2 L_t(\tilde{n}w)$$

with $\|L_t\| \leq \frac{1}{t}$. This means that

$$\begin{aligned} \|w\|_{L^2(Q)} &\leq \frac{1}{t}\varrho^2\kappa^2\|\tilde{n}w\|_{L^2(Q)} \\ &\leq \frac{\varrho^2\kappa^2\|n\|_\infty}{t}\|w\|_{L^2(Q)}. \end{aligned}$$

This holds for all $t > 0$. Taking $t >> 1$ results in $\|w\|_{L^2(Q)} = 0$ where $w = 0$. Thus $u \equiv 0$. ■

**Theorem 1.7 (Uniqueness)**
*Problem 1.1 admits at most one solution. That is, $u^{in} \equiv 0$ implies $u \equiv 0$.*

Proof. Let $u^{in} = 0$. Recall the formula

$$|a - i\beta|^2 = |a|^2 + |\beta|^2 + 2\,\mathrm{Im}(\beta\bar{a})$$

(elementary proof). We have

$$\begin{aligned} O\left(\frac{1}{R^2}\right) &= \int_{|x|=R}\left|\frac{\partial u}{\partial r} - i\kappa u\right|^2 dS \\ &= \int_{|x|=R}\left(\left|\frac{\partial u}{\partial r}\right|^2 + \kappa^2|u|^2\right) dS + 2\kappa\,\mathrm{Im}\int_{|x|=R} u\frac{\partial \bar{u}}{\partial r}\, dS \end{aligned}$$

The divergence-theorem implies

$$\begin{aligned} 2\kappa\,\mathrm{Im}\int_{|x|=R} u \underbrace{\frac{\partial \bar{u}}{\partial r}}_{=\nabla\bar{u}\cdot\underbrace{\nu}_{\text{normal}}} dS &= 2\kappa\,\mathrm{Im}\left[\int_{|x|<R}\left(u\Delta\bar{u} + \underbrace{|\nabla u|^2}_{\in\mathbb{R}_0^+}\right) dx\right] \\ &\overset{\text{Helmholtz-eq}}{=} 2\kappa\,\mathrm{Im}\left[\int_{|x|<R}\left(-\kappa^2\bar{n}|u|^2\right) dx\right] \\ &\geq 0, \end{aligned}$$

Since $\operatorname{Im} n \geq 0$. Thus we have for large $R \to \infty$:

$$0 \leq \limsup_{R\to\infty} \int_{|x|=R} \left( \left| \frac{\partial u}{\partial r} \right|^2 + \kappa |u|^2 \right) dS \leq 0,$$

whence

$$\lim_{R\to\infty} \int_{|x|=R} |u|^2 \, dS = 0.$$

Theorem 1.2 (Rellich) implies that $u(x) = 0$ for all $|x| > a$. By the unique continuation principle we have $u \equiv 0$ in $\mathbb{R}^3$. ∎

We have shown uniqueness. For the existence proof we will construct solutions.

**Definition 1.8.** The function

$$\Phi(x,y) := \frac{1}{4\pi|x-y|} e^{i\kappa|x-y|}, \quad x, y \in \mathbb{R}^3, x \neq y$$

is called *fundamental solution* or *free space Green's function.*

$\Phi$ has the following properties

**Proposition 1.9 (Properties of the fundamental solution)**

1. *$\Phi(\cdot, y)$ satisfies the Helmholtz-equation $\Delta u + \kappa^2 u = 0$ in $\mathbb{R}^3 \setminus \{y\}$*
2. *$\Phi$ satisfies the emphradiation condition*

$$\frac{x}{|x|} \cdot \nabla_x \Phi(x,y) - i\kappa \Phi(x,y) = O\left(\frac{1}{|x|^2}\right), \quad \text{as } |x| \to \infty$$

*uniformly in $\hat{x} = \frac{x}{|x|} \in \mathcal{S}^2$ (Sphere)*

3. *$\Phi$ has the asymptotic behaviour*

$$\Phi(x,y) = \frac{e^{i\kappa|x|}}{4\pi|x|} e^{-i\kappa\hat{x}\cdot y} + O\left(\frac{1}{|x|^2}\right), \quad \text{as } |x| \to \infty$$

*uniformly in $\hat{x} = \frac{x}{|x|} \in \mathcal{S}^2$ (Sphere)*

Proof. The first 2 properties are easily verified. We prove the third:

We calculate the the binomial formula

$$
\begin{aligned}
|x-y| - (|x| - \hat{x}\cdot y) &= \frac{|x-y|^2 - (|x| - \hat{x}\cdot y)^2}{|x-y| + (|x| - \hat{x}\cdot y)} \\
&= \frac{|y|^2 - \cancel{2x\cdot y} + \cancel{|x|^2} - \cancel{|x|^2} + \cancel{2x\cdot y} - \overbrace{(\hat{x}\cdot y)^2}^{\leq |y|^2}}{|x|\left(1 + \underbrace{\left|\hat{x} - \frac{y}{|x|}\right|}_{1 - \left|\frac{y}{|x|}\right|} - \frac{\hat{x}\cdot y}{|x|}\right)} \\
&\leq \frac{|y|^2}{2|x|\left(1 - \frac{y}{|x|}\right)}.
\end{aligned}
$$

Hence $|x-y| = |x| - \hat{x}\cdot y + O\left(\frac{1}{|x|}\right)$. Take the exponential of this to get

$$
e^{i\kappa|x-y|} = e^{i\kappa|x|}e^{-i\kappa\hat{x}\cdot y}\left(1 + O\left(\frac{1}{|x|}\right)\right).
$$

By the above formula we also have

$$
\frac{1}{|x-y|} = \frac{1}{|x|} + \left[\frac{|x| - |x-y|}{|x-y||x|}\right] = \frac{1}{|x|} + O\left(\frac{1}{|x|^2}\right).
$$

So

$$
\begin{aligned}
\frac{e^{i\kappa|x-y|}}{|x-y|} &= \frac{e^{i\kappa|x-y|}}{|x|} + O\left(\frac{1}{|x|^2}\right) \\
&= \frac{1}{|x|}e^{i\kappa|x|}e^{-i\kappa\hat{x}\cdot y} + O\left(\frac{1}{|x|^2}\right),
\end{aligned}
$$

as $|x| \to \infty$. ∎

*Remark 1.10.* The previous result states in particular that $\Phi$ is determined by the function

$$
e^{-i\kappa\hat{x}\cdot y}
$$

on $\mathcal{S}^2$ (up to pertubation). We will see that this property holds in a more general context.

With the help of the fundamental solution $\Phi$ we can create

**Theorem 1.11**
*Let $\Omega \subseteq \mathbb{R}^3$ be bounded. For every $\phi \in L^2(\Omega)$, the function*

$$v(x) := \int_\Omega \phi(y)\Phi(x,y)\,dy, \quad x \in \mathbb{R}^3$$

*belongs to $H^1_{loc}(\mathbb{R}^3)$ and satisfies the* Sommerfeld radiation condition*. Moreover, $v$ is the* only *radiating solution (it is the only function in the set of solution that satisfies the Sommerfeld radiation condition) to*

$$\Delta v + \kappa^2 v = -\phi$$

*in the variational sense*

$$\int_{\mathbb{R}^3} \left(\nabla v \cdot \nabla \bar{\Psi} - \kappa^2 v \bar{\Psi}\right) dx = \int_{\mathbb{R}^3} \phi \bar{\Psi}\,dx \tag{*}$$

*for all $\Psi \in H^1(\mathbb{R}^3)$ with compact support. For any $R > 0$ with $\Omega \subseteq B_R(0) =: K$ we have with $c = c(R, \kappa, \Omega)$ that*

$$\|v\|_{H^1(K)} \le c\|\phi\|_{L^2(\Omega)}.$$

*In other words, the mapping $\phi \mapsto v$ is a bounded (continuous linear) operator from $L^2(\Omega)$ to $H^1(K)$.*

Proof. **(1)** Since $\frac{1}{r^2}$ is locally integrable in $\mathbb{R}^3$, the expression on the right-hand side in Eq. (*) is well-defined. Provided $\phi \in C^1(\bar{\Omega})$, then we can interchange integration and differentiation (posible as we have a majorant, cf. PDE theory) and obtain

$$\Delta v + \kappa^2 v = \begin{cases} -\phi & \text{in } \Omega \\ 0 & \text{in } \mathbb{R}^3 \setminus \bar{\Omega} \end{cases}$$

This means that $v$ solves Eq. (*) for any $\kappa \in \mathbb{C}$.

**(2)** We cannot directly evaluate $L^2$-integrals of the gradient $\nabla_x \Phi$. To prove stability, we first consider the special case $\kappa = i$ and $\phi \in C^1(\bar{\Omega})$. Then, the fundamental solution

$$\Phi_i(x,y) = \frac{1}{4\pi|x-y|} e^{-|x-y|}$$

decays exponentially for $|x| \to \infty$, ad by approximation arguments Eq. (*) holds for any $\Psi \in H^1(\mathbb{R}^3)$.

Taking $\Psi = v$ we obtain

$$\|v\|^2_{H^1(\mathbb{R}^3)} = \int_\Omega \phi \bar{v} \le \|\phi\|_{L^2(\Omega)} \|v\|_{H^1(\mathbb{R}^3)}$$

(note that Eq. (*) with $\kappa = i$ becomes the $H^1$-Norm). By density, this defines a continuous operator

$$\phi \mapsto v \quad \text{from } L^2(\Omega) \text{ to } H^1(K).$$

**(3)** Let $k > 0$. We define

$$\Psi(x,y) := \Phi_k(x,y) - \Phi_i(x,y) = \frac{1}{4\pi|x-y|}\left[e^{i\kappa|x-y|} - e^{-|x-y|}\right]$$

It is easy to prove that $\Psi$ and $\nabla_x\Psi$ belong to $L^2(K \times \Omega)$. We sketch the crucial part. We calculate

$$4\pi|\nabla_x\Psi(x,y)| = \left|\underbrace{\frac{i\kappa e^{i\kappa|x-y|} - e^{-|x-y|}}{|x-y|}}_{O\left(\frac{1}{|x-y|}\right)} + \frac{e^{i\kappa|x-y|} - e^{-|x-y|}}{|x-y|^3}\right|$$
$$\overset{*}{leq} O\left(\frac{1}{|x-y|}\right) \text{ for small } |x-y|$$

*: The denominator of the right fraction is in $O(|x-y|)$

Thus,

$$\|\nabla_x\Psi\|_{L^2(K\times\Omega)} \leq C\int_K\int_\Omega \frac{1}{|x-y|^2}\,dy\,dx < \infty.$$

**(4)** With **(3)** we see that the mapping

$$\varphi \mapsto \int_\Omega \varphi(y)\Psi(\cdot,y)\,dy$$

is bounded from $L^2(\Omega)$ to $H^1(K)$, because

$$\begin{aligned}\int_K\left|\nabla_x\int_\Omega\varphi(y)\Psi(x,y)\,dy\right|^2 dx &= \int_k\left[\int_\Omega|\varphi(y)\nabla_x\Psi(x,y)|\,dy\right]^2 dx\\ &\leq \int_K \|\varphi\|^2_{L^2(\Omega)}\|\nabla_x\Psi(x,\cdot)\|^2_{L^2(\Omega)}\,dx\\ &\leq \|\varphi\|^2_{L^2(\Omega)}\underbrace{\|\nabla_x\Psi\|^2_{L^2(K\times\Omega)}}_{\overset{(3)}{<}\infty}.\end{aligned}$$

This and **(2)** show that $\varphi \mapsto v$ is also bounded from $L^2(\Omega)$ to $H^1(K)$ for $\kappa > 0$.

**(5)** The radiation condition follows from the radiation condition of $\Phi$:

$$\begin{aligned}\frac{\partial v}{\partial\nu} - i\kappa v &= \int_\Omega\varphi(y)\left(\frac{\partial}{\partial\nu} - i\kappa\right)\Phi(x,y)\,dy\\ &\leq \|\varphi\|_{L^2(\Omega)} O\left(\frac{1}{r}\right)^2.\end{aligned}$$

Uniqueness follows from **(2)**. ∎

*Remark 1.12.* **Proof of Theorem 1.11** (1).
To prove that $\Delta v(x) + \kappa^2 v(x) = -\varphi(x)$, for $x \in \Omega$, it satisfies to verify

$$(\Delta + \kappa^2) \int_{B_\varepsilon(x)} \varphi(y)\Phi(x,y)\, dy = -\varphi(x),$$

for small $\varepsilon > 0$.

**1)** We readily see that

$$\kappa^2 \left| \int_{B_\varepsilon(x)} \varphi(y)\Phi(x,y)\, dy \right| \le \frac{\kappa^2}{4\pi} \left| \int_{B_\varepsilon(x)} \frac{e^{i\kappa|y-x|}}{|x-y|}\, dy \right|$$

and the expression on the right tends to 0 for $\varepsilon \to 0$.

**2)** Since $\Delta_x \Phi(x,y) = \Delta_y \phi(x,y)$, we see

$$\begin{aligned}\Delta_y \int_{B_\varepsilon(x)} \varphi(y)\phi(x,y)\, dy &= \int_{B_\varepsilon(x)} \phi(y)\Delta_y \Phi(x,y)\, dy \\ &= \underbrace{-\int_{B_\varepsilon(x)} \nabla\varphi(y)\cdot\nabla\Phi(x,y)}_{=:A} + \underbrace{\int_{|y-x|=\varepsilon} \varphi(y) \frac{\partial\Phi(x,y)}{\partial\nu}\, dS(y)}_{=:B}.\end{aligned}$$

We have that

$$\nabla_y \Phi(x,y) = \frac{1}{4\pi} e^{i\kappa|x-y|} \left( \frac{1}{|x-y|^2} - \frac{1}{|x-y|^3} \right)(x-y),$$

and thus

$$A \le \|\nabla\varphi\|_{L^\infty} \int_{B_\varepsilon(x)} |\nabla_y \Phi(x,y)|\, dy \to 0, \text{ as } \varepsilon \to 0.$$

We stay with $B$:

$$\begin{aligned} B &= = \frac{1}{4\pi} \int_{\partial B_\varepsilon(x)} e^{i\kappa\varepsilon} \left( \frac{1}{\varepsilon} - \frac{1}{\varepsilon^2} \right) \varphi(y)\, dS(y) \\ &= \underbrace{\frac{1}{4\pi} e^{i\kappa\varepsilon} \left( \frac{1}{\varepsilon} - \frac{1}{\varepsilon^2} \right) 4\pi\varepsilon^2}_{=e^{i\kappa\varepsilon}(\varepsilon+\varepsilon^{-1})\to -1, \text{ as } \varepsilon\to 0} \underbrace{\fint B_\varepsilon(x)\varphi(y)\, dS(y)}_{\to\varphi(x), \text{ as } \varepsilon\to 0} \end{aligned}$$

**Theorem 1.13 (Lippmann Schwinger integral equation)**
*Let $\alpha > 0$ and $K = \bar{B_\alpha}(0)$. If $u \in H^1$ solves the Problem 1.1, then $u_{|K} \in L^2(K)$ satisfies the* Lippmann Schwinger equation

$$u(x) = u^i n(x) - \kappa^2 \int_{|y|<\alpha} (1 - n(y))\Phi(x,y)u(y)\, dy \tag{LS}$$

*for almost all $x \in K$.*
*Conversely, if $u \in L^2(K)$ satisfies Eq. (LS), then it can be extended by the right-hand side of Eq. (LS) to the solution $u \in H^1_{loc}(\mathbb{R}^3)$ of Problem 1.1.*

Proof. Let $u$ satisfy Problem 1.1 and define

$$v = \int_K \varphi(y)\Phi(\cdot, y)\, dy$$

for $\varphi = \kappa^2(1-n)u \in L^2(K)$. Theorem 1.11 states that $v \in H^1_{loc}(\mathbb{R}^3)$ satisfies $\Delta v + \kappa^2 v = -\varphi$. Since

$$\Delta u + \kappa^2 u = \kappa^2(1-n)u$$

and

$$\Delta u^{in} + \kappa u^{in} = 0,$$

we have (addition of previous equations)

$$\Delta(v + u^s) + \kappa^2(v + u^s) = 0.$$

The uniqueness from Theorem 1.7 shows that $v + u^s = 0$. Therefore

$$u = u^{in} + u^s = u^{in} - v.$$

For the converse direction let $u \in L^2(K)$ satisfy Eq. (LS). Define $v$ as above, such that $u = u^{in} - v$ in $K$ (in the $L^2$-sense). By Theorem 1.11 we know that $v \in H^1_{loc}(\mathbb{R}^3)$ by extending it to $\mathbb{R}^3$, and that $\Delta v + \kappa v = -\varphi$. This implies $u \in H^1_{loc}(\mathbb{R}^3)$. Furthermore

$$\Delta u + \kappa^2 u = \varphi = \kappa^2(1-n)u,$$

that is

$$\Delta u + \kappa^2 n u = 0.$$

This implies that $u^s = -v$. The radiation condition follows again from Theorem 1.11. ∎

We now can derive existence for Problem 1.1.

**Theorem 1.14 (Existence of solutions for Problem 1.1)**
*Let $\kappa, n, \theta$ be as assumed previously. Then, the Lippmann Schwinger equation (LS), and thus Problem 1.1, is uniquely solvable.*

Proof. Define the following operator $T : L^2(B_\alpha(0)) \to L^2(B_\alpha(0))$ by

$$(Tu)(x) = \kappa^2 \int_{|y|<\alpha} (1 - n(y))\Phi(x, y)u(y)\, dy, \text{ for } |x| < \alpha.$$

Theorem 1.11 shows that $T$ is bounded from $L^2(B_\alpha(0))$ to $H^1(B_\alpha(0))$, and by the compact embedding $H^1(B_\alpha(0)) \hookrightarrow L^2(B_\alpha(0))$, the operator $T$ is compact from $L^2(B_\alpha(0))$ into itself. The Lippmann Schwinger equation reads

$$u + Tu = u^{in}$$

and by the Riesz theory (Fredholm alternative) for compact operators, the uniqueness of $u$ (Theorems 1.7 and 1.13) implies its existence. ∎

**Theorem 1.15 (Far-field pattern)**
*Let $u$ solve Problem 1.1. Then we have*

$$u(x) = u^{in}(x) + \frac{e^{i\kappa|x|}}{|x|} u_\infty(\hat{x}) + O\left(\frac{1}{|x|^2}\right), \text{ as } |x| \to \infty,$$

*uniformly in $\hat{x} = \frac{x}{|x|}$ (direction of $x$). The function $u_\infty(\hat{x})$ is given by*

$$u_\infty(\hat{x}) = \frac{\kappa^2}{4\pi} \int_{|y|<\alpha} (n(y) - 1) e^{-i\kappa\hat{x}\cdot y} u(y)\, dy$$

*is called* far-field pattern *or* scattering amplitude

$$u_\infty : \underbrace{\mathcal{S}^2}_{Sphere} \to \mathbb{C}$$

*is analytic on $\mathcal{S}^2$ and determines $u^s$ otuside $B_\alpha(0)$ uniquely ($u_\infty = 0 \Leftrightarrow u^s(x) = 0$ for $|x| > \alpha$).*

The formula for $u(x)$ follows from Proposition 1.9 (iii):

$$\Phi(x, y) = \frac{e^{i\kappa|x|}}{|x|} e^{-i\kappa\hat{x}\cdot y} + O\left(\frac{1}{|x|^2}\right).$$

By Theorem 1.2 (Rellich) we have

$$\lim_{R\to 0} \int_{|x|=R} |u(x)|^2\, dS(x) = 0, \text{ for } |x| > \alpha$$

and this implies uniqueness. The analyticity follows from the formula (calculate derivate w.r.t $\hat{x}$).

*Remark 1.16.*
1. The concept of far-field pattern is <u>fundamental</u> to inverse scattering theory.
2. The analyticity of $u_\infty$ shows that the inverse scattering problem is ill-posed (little pertubations of $u_\infty$ can destroy analyticity of the function!).

**Theorem 1.17 (Green's representation theorem)**
***(i)*** *Any $y \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfies for any $x \in \Omega$*

$$u(x) = \int_{\partial\Omega} \left[ \Phi(x,y) \frac{\partial u(y)}{\partial \nu} - u(y) \frac{\partial \Phi}{\partial \nu(y)} \right] dS(y) - \int_{\Omega} \Phi(x,y)(\kappa^2 u(y) + \Delta u(y))\, dy$$

*(where $\nu$ denotes the outer normal of $\Omega$).*

***(ii)*** *Let $\Omega^c := \mathbb{R}^3 \setminus \Omega$ and let $u \in H^1_{loc}(\mathbb{R}^3)$ satisfy the Sommerfeld radiation condition, and solve*

$$\Delta u + \kappa^2 u = 0, \quad \textit{in } \Omega^c.$$

*Then, the* Green's formula *holds:*

$$u(x) = \int_{\partial\Omega} \left[ u(y) \frac{\partial \Phi(x,y)}{\partial \nu(y)} - \Phi(x,y) \frac{\partial}{\partial \nu} u(y) \right] dy, \quad x \in \mathbb{R}^3 \setminus \overline{\Omega}.$$

***(iii)*** *In the situation of (ii), the far-field pattern of $u$ reads for $\hat{x} = \frac{x}{|x|} \in \mathcal{S}^2$*

$$u_\infty(\hat{x}) = \frac{1}{4\pi} \int_{\partial\Omega} \left[ u(y) \frac{\partial}{\partial \nu(y)} e^{-i\kappa \hat{x}\cdot y} - e^{-i\kappa\hat{x}\cdot y} \frac{\partial u}{\partial \nu}(y) \right] dS(y).$$

PROOF. **(i)** we know from Theorem 1.11 (see addendum) that

$$-u(x) = \int_\Omega u(y)(\kappa^2 + \Delta_y)\Phi(x,y)\, dy.$$

The assertion follows from integrating by parts.

**(ii)** Let $R > 0$ and $K = B_R(0)$, with $\overline{\Omega} \subseteq K$. Define $D = K \setminus \overline{\Omega}$. Apply **(i)** to $D$ and obtain, for $x \in D$,

$$u(x) = \int_{\partial\Omega} + \left[ \int_{\partial K} \right] \left[ \Phi(x,y) \frac{\partial u}{\partial \nu}(y) - u(y) \frac{\partial}{\partial \nu(y)} \Phi(x,y) \right] dS(y).$$

We want to show, that the $\int_{\partial K}$-integral vanishes for large $R \to \infty$. From the proof of Theorem 1.7 we know, that $\int_{\partial K} |u|^2\, dS \le O(1)$ as $R \to \infty$. From the Sommerfeld radiation condition of $u$ and $\Phi$, we obtain with the Cauchy-Schwarz inequality

$$\begin{aligned} \int_{\partial K} \dots dS &= \int_{\partial K} \Phi(x,y) \left[ \frac{\partial u}{\partial \nu}(y) - i\kappa u(y) \right] + u(y) \left[ i\kappa \Phi(x,y) - \frac{\partial}{\partial \nu(y)} \Phi(x,y) \right] dS(y) \\ &\overset{C.S.}{\le} \underbrace{\|\Phi\|_{L^2(\partial K, dy)} O\left(\frac{1}{r^2}\right) + \|u\|_{L^2(\partial K)} O\left(\frac{1}{r^2}\right)}_{\to 0,\text{ as } R\to\infty}. \end{aligned}$$

From Proposition 1.9 we know ($\hat{x} = \frac{x}{|x|}$):

$$\Phi(x,y) = \frac{1}{4\pi|x|} e^{i\kappa|x|} e^{-i\kappa\hat{x}\cdot y} + O\left(\frac{1}{|x|^2}\right).$$

We plug this into the formula of **(ii)** and obtain for $|x| >> 1$:

$$u(x) = \frac{1}{4\pi} e^{i\kappa w} \int_{\partial\Omega} \left[ u(y) \frac{\partial}{\partial\nu(y)} e^{-i\kappa\hat{x}\cdot y} - e^{-i\kappa\hat{x}\cdot y} \frac{\partial}{\partial\nu} u(y) \right] dy + O\left(\frac{1}{|x|^2}\right).$$

Uniqueness of the far-field pattern proves **(iii)**. ■

## 1.3 Density results

In this section we collect some density results, useful in the labour study of inverse scattering. Recall: The refraction index $n$ satisfies $|n(x)| = 1$ for $|x| > \alpha$.

**Lemma 1.18**
*Let $\beta > \alpha$, $K := B_\beta(0)$. Then there exist constants $M > 0, C > 0$, such that for any $z \in \mathbb{C}^3$ with $z \cdot z = 0$ and $|z| \geq M$ there exists a solution $u_z \in H^1(K)$ to the equation*

$$\Delta u_z + \kappa^2 n u_z = 0, \quad \textit{in } K,$$

*of the form*

$$u_z(x) = e^{z\cdot x}(1 + v_z(x)), \quad x \in K,$$

*and $v_z$ satisfies*

$$\|v_z\|_{L^2(K)} \leq \frac{C}{|z|}.$$

Proof. The technical lemma will be proven later. ■

**Theorem 1.19 (1st density theorem)**
*Let $\Omega \subseteq \mathbb{R}^3$ be bounded and let $n_1, n_2 \in L^\infty(\Omega)$ such that $(n_1 - 1)$ and $(n_2 - 1)$ have a compact support in $\Omega$. Then, the following linear hall of products is dense in $L^2$:*

$$X := span\{u1u2 : \forall j = 1,2\, u_j \in H^1(\Omega) \textit{ solves } \Delta u_j + \kappa^2 n_j u_j = 0 \textit{ in } \Omega\}$$

Proof. Let $\beta > 0$ such that $\overline{\Omega} \subseteq B_\beta(0)$ and let $g \in L^2(\Omega)$ be $L^2(\Omega)$-Orthogonal to the set $X$. We have to show, that $g$ is zero. Then $X$ is dense.
Fix $y \in \mathbb{R}^3 \setminus \{0\}$ and choose a unit vector $\hat{a} \in \mathbb{R}^3$ ($\hat{}$ means unit length) and some $b \in \mathbb{R}^3$with $|b|^2 = |y|^2 + s^2$ such that $\{y, \hat{x}, b\}$ is an orthogonal system in $\mathbb{R}^3$ and some $s > 0$ (real).
Define the following elements of $\mathbb{C}^3$:

$$z^1 := \frac{1}{2}b - \frac{i}{2}(y + s\hat{a})$$
$$z^2 := -\frac{1}{2}b - \frac{i}{2}(y - s\hat{a}).$$

Then $z^1 \cdot z^1 = 0 = z^2 \cdot z^2$. We have

$$|z^1|^2 = |z^2|^2 = \frac{1}{4}\left[|b|^2 + |y|^2 + s^2\right] \geq \frac{1}{4}s^2.$$

Furthermore $z^1 + z^2 = -iy$. From Lemma 1.18 we obtain solutions to $\Delta u_j + \kappa^2 n_j u_j = 0$ in $B_\beta(0)$ with

$$u_j(x) = e^{z^j \cdot x}\left[1 + v_{z_j}(x)\right].$$

These are in particular solutions when restricted to the smaller domain $\Omega$. The orthogonal property of $g$ yields

$$0 = \int_\Omega e^{(z^1+z^2)x}[1 + v_{z_1}^2 + v_{z_2}^2 + v_{z_1}v_{z_2}]g\,dx.$$

From Lemma 1.18 we obtain

$$\|v_j\|_{L^2(\Omega)} \leq \frac{C}{|z_j|} \leq \frac{2C}{s}.$$

Last inequality follows from a previous equation. From Cauchy's inequality and the limit $s \to 0$ that

$$0 = \int_\Omega e^{-iy\cdot x} g(x)\,dx, \quad \forall y \in \mathbb{R}^3 \setminus \{0\}.$$

So, the Fourier-Transform of $g$ vanishes (it is defined like in our equation). Thus $g = 0$.■

The second density theorem is the following.

**Theorem 1.20 (2nd density theorem)**
*Let $n \in L^\infty(\mathbb{R}^3)$ with $n(x) = 1$ for $|x| > \alpha$. Let $\beta > \alpha$ and define*

$$H := \left\{ v \in H^1(B_\beta(0)) |\, \Delta v + \kappa^2 n v = 0 \text{ in } B_\beta(0) \right\}$$

*the set of Helmholtz-solutions. Let for any $\hat\theta \in \mathcal{S}^2$, $u(\cdot, \hat\theta)$ denote the total field corresponding to the incident plain wave $e^{i\kappa\hat\theta\cdot x}$ (plain wave from direction $\hat\theta$). Then, the span of all plain wave solutions,*

$$span\{u(\cdot,\hat\theta) : \hat\theta \in \mathcal{S}^2\},$$

*is dense*

$$H|_{B_\alpha(0)} \text{ in } L^2(B_\alpha(0)) \text{ - norm.}$$

Proof. Let $K = B_\alpha(0)$ and $\langle v, w\rangle$ the $L^2$ product over $K$, so

$$\langle v, w\rangle := \int_K v\overline{w}\,dx.$$

Let $v$ be in the closure of $H$ with the following property:

$$\langle v, u(\cdot, \hat\theta)\rangle = 0$$

for all $\hat\theta \in \mathcal{S}^2$. Recall the operator

$$T : v \mapsto \kappa^2 \int_{|x|<\alpha} (1 - n(y))\Phi(x,y)w(y)\,dy$$

and the Lippmann-Schwinger equation

$$u(\cdot,\theta) = (1+T)^{-1}u^{in}(\cdot,\theta).$$

With the adjoint operator $T^*$, that satisfies

$$\langle T^*v, w\rangle = \langle v, Tw\rangle, \quad \forall v, w \in X,$$

and the fact that

$$[[1+T]^{-1}]^* = (1+T^*)^{-1},$$

we obtain that

$$\begin{aligned} 0 &= \langle v, (1+T)^{-1}u^{in}(\cdot,\hat\theta)\rangle \\ &= \langle (1+T^*)^{-1}v, u^{in}(\cdot,\hat\theta)\rangle, \end{aligned}$$

for all $\hat\theta \in \mathcal{S}^2$. We compute $T^*$:
Since, for all $w_1, w_1 \in L^2(K)$ we have

$$\begin{aligned} \frac{1}{\kappa^2}\langle T^* w_1, w_2\rangle &= \frac{1}{\kappa^2}\langle w_1, T w_2\rangle \\ &= \int_K\int_K w_1(x)\overline{(1-n(y))\Phi(x,y)w_2(y)}\,dy\,dx \\ &= \langle (1-\overline{n})\int_K w_1(x)\overline{\Phi(x,\cdot)}\,dx, w_2\rangle, \end{aligned}$$

we have

$$T^*w_1 = \kappa^2\overline{(1-n)}\int_K w_1(y)\overline{\Phi(\cdot,y)}\,dy.$$

Thus $w := (1+T^*)^{-1}v$ satisfies

$$v(x) = w(x) + \kappa^2(1-\overline{n(x)})\int_K \overline{\Phi(x,y)}w(y)\,dy.$$

We now set

$$\tilde w(x) := \int_K \overline{w(y)}\Phi(x,y)\,dy, \quad x \in \mathbb{R}^3.$$

Then, $\tilde w$ is a volume potential for $\bar w$ in the sense of Theorem 1.11: $w \in H^1_{loc}(\mathbb{R}^3)$ and

$$\int_{\mathbb{R}^3}\left(\nabla\tilde w\cdot\nabla\overline{\Psi} - \kappa^2\tilde w\overline{\Psi}\right)dx = \int_K \overline{w}\overline{\Psi}\,dx, \quad \forall\Psi \in H^1(\mathbb{R}^3) \text{ with compact support.}$$

The far-field pattern $\tilde w_\infty$ vanishes. Indeed, as Theorem 1.17 (iii) and the divergence theorem imply for any $\hat\Theta \in \mathcal{S}^2$:

$$\overline{\tilde w_\infty(\hat\theta)} = \int_K w(y)e^{i\kappa\hat\theta\cdot y}\,dy = 0,$$

since $w$ is orthogonal to the plain waves. Rellichs theorem implies then, that $\tilde{w} = 0$ outside of $K$! Consider a sequence $v_j \in H$ with $v_j \to v$ in $L^2(H)$ as $j \to \infty$. From

$$v = w + \overline{\kappa^2(1-n)\tilde{w}}$$

we get

$$\int_K \overline{v} v_j \, dx = \int_K \overline{w} v_j \, dx + \kappa^2 \int_K (1-n)\tilde{w} v_j \, dx. \tag{*}$$

Since $\tilde{w}$ vanishes outside $K$, we obtain from $v_j \in H$

$$\int_{|x|<\beta} (\nabla v_j \cdot \nabla \tilde{w} - \kappa^2 v_j \cdot \tilde{w}) \, dx = -\kappa^2 \int_K (1-n) v_j \tilde{w} \, dx.$$

We extend $v_j$ to a $H^1(\mathbb{R}^3)$ function with compact support. The property of $\tilde{w}$ implies

$$\int_{|x|<\beta} (\nabla \tilde{w} \cdot \nabla v_j - \kappa^2 \tilde{w} \cdot v_j) \, dx = \int_K (1-n) v_j \tilde{w} \, dx.$$

The previous formulas imply

$$-\kappa^2 \int_K (1-n) v_j \tilde{w} \, dx = \int_K \overline{w} v_j \, dx.$$

This and Eq. (*) give

$$\int_K \overline{v} v_j = 0, \quad \forall j \in \mathbb{N}.$$

In the limit $j \to \infty$ we obtain

$$\|v\|_{L^2(K)} = 0 \Longleftrightarrow v = 0.$$

This proves density. ∎

# Chapter 2

# Discretization of the direct scattering problem

## 2.1 Dealing with infinite domains

Recall the direct scattering Problem 1.1 from Chapter 1:
Given some incident wave $u^{in}(x)$ of plain-wave type, i.e.

$$u^{in}(x) = ce^{i\kappa\hat{\theta}\cdot x},$$

with direction $\hat{\theta} \in \mathcal{S}^{d-1}$ (unit sphere in $\mathbb{R}^d$) and (real) frequency $\kappa > 0$, find the total field $u \in H^1_{loc}(\mathbb{R}^d)$ such that for all $\Psi \in H^1(\mathbb{R}^d)$ with compact support it holds that

$$\int_{\mathbb{R}^d} (\nabla u \cdot \nabla\overline{\Psi} - \kappa^2 nu\overline{\Psi})\, dx = 0,$$

where $n \in L^\infty(\Omega)$ function with $\Re n, \Im n \geq 0$. Additionaly the scattered field $u^s := u - u^{in}$ should solve the Sommerfeld radiation condition

$$\frac{\partial u^s(x)}{\partial r} - i\kappa u^s(x) = O\left(\frac{1-d}{2}\right),$$

for $|x| = r \to \infty$ uniformly in $\hat{x} = \frac{x}{|x|} \in \mathcal{S}^{d-1}$.

The scattering problem is an infinite-dimensional variational problem on an infinite (unbounded) domain.

**Aim of this chapter:** Approximate Problem 1.1 in a finite computational complexity and time.

This section deals with the restriction / truncation to bounded domains.

**Common approach:** Simply replace $\mathbb{R}^d$ by some bounded domain $\Omega$ and put some artificial absorbing boundary condition on $\partial\Omega$. The principle is to allow all waves to leave the domain without any artifical scattering (i.e. the initial wave, that may be scattered by some objects in $\Omega$, should be able to leave the domain without any harm). At the boundary of the domain, nothing that's leaving should be scattered inside again (as $\Omega$ is just used for being able to compute something).

**Reference:** Lecture-notes of Olof Runborg, KTH Stockholm.

### 2.1.1 One-dimensional illustration

In 1d (and only in 1d), it is possible to design exact local absorbing boundary conditions. W.l.o.g. let the artificial domain $\Omega = B_L(0)$ be clutered and let $\overline{B_\alpha(0)} \subseteq B_L(0)$ ($L > \alpha$, where $\alpha$ is the parameter from Chapter 1, $n|_{B_\alpha(0)^c} = 1$). Close to $\partial\Omega = \{-L, L\}$, the medium is homogeneous and hence the scattered wave $u^s := u - u^{in}$ satisfies the Helmholtz-equation

$$u^s_{xx} - \kappa^2 u^s = 0,$$

for $|x \pm L|$ sufficiently small.

Therefore

$$u^s(x) = \underbrace{c_0 e^{-i\kappa x}}_{A} + \underbrace{c_1 e^{i\kappa x}}_{B}, \tag{2.1}$$

with $c_0, c_1 \in \mathbb{R}$ and $x$ close to $\pm L$. The part $A$ of the solutions travels to the left of the interval, the part $B$ travels to the right of the interval. As stated above: If we were sitting on the boundary, we would like to be able to allow waves to pass the boundary from the inside. If a wave comes from the outside, we don't let it pass. Therefore we want to design the absorbing boundary condition at $-L$ such that it accepts any $c_0$, but prevents $c_1 \neq 0$. This is archieved by applying the operator $\frac{\partial}{\partial x} + i\kappa$. Why this? Because:

$$\begin{aligned}\left(\frac{\partial}{\partial x} + i\kappa\right) u^s &= -c_0 i\kappa e^{-i\kappa x} + c_1 i\kappa e^{i\kappa x} + c_0 i\kappa e^{-i\kappa x} + c_1 i\kappa e^{i\kappa x} \\ &= 2c_1 i\kappa e^{i\kappa x},\end{aligned}$$

and this shows that $\left(\frac{\partial}{\partial x} + i\kappa\right)$ doesn't 'see' waves leaving $\Omega$ (i.e. the $c_0$ part of the solution), and incoming waves can be avoided by the boundary condition

$$\left(\frac{\partial}{\partial x} u^s + i\kappa u^s\right)(-L) = 0.$$

Similarily, at $x = +L$ we impose

$$\left(\frac{\partial}{\partial x} u^s - i\kappa u^s\right)(+L) = 0$$

(calculation shows this again). Introducing the concept of an outer normal $\nu$, i.e. $\nu(\pm L) = \pm 1$, we can rewrite both conditions as

$$\frac{\partial u^s}{\partial \nu}(x) - i\kappa u^s(x) = 0,$$

for $x \in \partial\Omega = \{-L, L\}$. This leads to the following inhomogeneous boundary condition for the total wave $u = u^s + u^{in}$:

$$\left(\frac{\partial u}{\partial \nu} - i\kappa u\right)(x) = \left(\frac{\partial u^{in}}{\partial \nu} - i\kappa u^{in}\right)(x), \quad x \in \partial\Omega = \{-L, L\}. \tag{2.2}$$

**Example.**
*$u^{in} = e^{-i\kappa x}$ is a wave travelling from right to left. Then the absorbing boundary condition reads*

$$\left(\frac{\partial u}{\partial \nu} - i\kappa u\right)(-L) = 0,$$
$$\left(\frac{\partial u}{\partial \nu} - i\kappa u\right)(L) = 2i\kappa u^{in}(L).$$

We shall emphasize that the scattering problem on $\Omega$ with absorbing boundary condition Eq. (2.2),

$$\Delta u + i\kappa n u = 0, \quad \text{in } \Omega$$
$$\frac{\partial u}{\partial \nu} - i\kappa u = \frac{\partial u^{in}}{\partial \nu} - i\kappa u^{in}, \quad \text{on } \partial\Omega,$$

reproduces the solution of Problem 1.1 in $\Omega$.

### 2.1.2 Multi-dimensional case

In higher dimensions, there is no simple form like Eq. (*) for this scattered wave close to the artificial boundary $\partial\Omega$. Exact absorbing boundary conditions will be necesarrily non-local and computationally demanding.

A cheaper and popular approach is the following:
Assume that the scattered wave in the full problem is close to a spherical wave in a vicinity of the artificial boundary. If we choose $\Omega = B_L(0)$, then the 1d-arguments can be applied. If $u^s(x) \approx e^{i\kappa|x|}$ (i.e. the wave is approximately spherical; spherical waves have no angular dependence) close to $\partial\Omega$, then $\frac{\partial u^s}{\partial \nu} - i\kappa u^s \approx i\kappa e^{ik|x|} - i\kappa e^{i\kappa|x|} = 0$ on $\partial\Omega$.
This motivates the simplest absorbing bc in 2 and 3 dimensions:

$$\frac{\partial u}{\partial \nu} - i\kappa u = \frac{\partial u^{in}}{\partial \nu} - i\kappa u^{in}, \quad \text{on } \partial\Omega. \tag{2.3}$$

We have seen in Theorem 1.15 that $u^s$ is actually spherical, up to some perturbation of order $O\left(\frac{1}{|x|^2}\right)$:

$$u^s(x) = \frac{e^{i\kappa|x|}}{|x|}u_\infty(\hat{x}) + O\left(\frac{1}{|x|^2}\right), \quad \text{as } |x| \to \infty,$$

i.e. 'the scattered wave is spherical if we are far enough away from all obstacles'.
Hence, for sufficiently large $L$ ($\Omega = B_L(0)$), is absorbing boundary condition (2.3) appears reasonable.
For the remaining part of this chapter, our model problem reads

**Problem 2.1.**

$$\begin{aligned} &\Delta u + \kappa^2 nu = 0, \text{ in } \Omega \\ &\frac{\partial u}{\partial \nu} - i\kappa u = \frac{\partial u^{in}}{\partial \nu} - i\kappa u^{in}, text on \partial\Omega. \end{aligned}$$

*Remark 2.1.* The absorbing boundary condition (2.3) can be seen as applying the Sommerfeld radiation condition at $\partial B_L(0)$ instead of $\infty$.
Note, that the choice of the boundary condition we made was just the easiest one. There are many alternatives. The most popular approach is the so called *Perfectly Matched Layer* (PML).