JTR mini-Project

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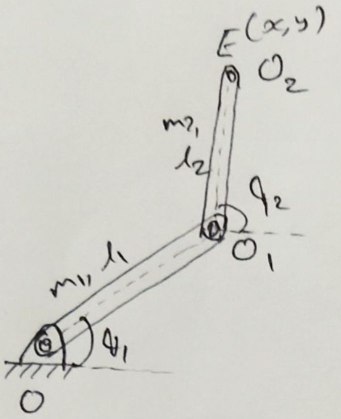
☆ 2R elbow manipulator

two links of masses $m1$ & $m2$
lengths $l1$ & $l2$
& moment of Inertia $I1$ & $I2$



q_1, q_2 - joint angles

#Task-0

→ derivation of 6 key equations with regard to the Kinematics, dynamics & statics of elbow manipulator

→ neat free body diagrams, assumptions & any other explanations

⇒ Forward Kinematics

$$x = l_1 \cos q_1 + l_2 \cos q_2$$
$$y = l_1 \sin q_1 + l_2 \sin q_2$$

Pos O1 (X_1, Y_1)

$X_1 = l_1 \cos q_1$

$Y_1 = l_1 \sin q_1$

→ using simplified notations,

$$\boxed{\begin{aligned} x &= l_1 cq_1 + l_2 cq_2 \\ y &= l_1 sq_1 + l_2 sq_2 \end{aligned}} \quad (1)$$

End-effector (E) position from joint angles

→ End-effector Velocity.
(Differentiating eq (1))

$$\boxed{\begin{aligned} \dot{x} &= -l_1 sq_1 \cdot \dot{q}_1 - l_2 sq_2 \cdot \dot{q}_2 \\ \dot{y} &= +l_1 cq_1 \cdot \dot{q}_1 + l_2 cq_2 \cdot \dot{q}_2 \end{aligned}} \quad (2)$$

End-effector Velocity.

- matrix form

$$\boxed{\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -l_1 sq_1 & -l_2 sq_2 \\ l_1 cq_1 & l_2 cq_2 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}} \quad (2)$$

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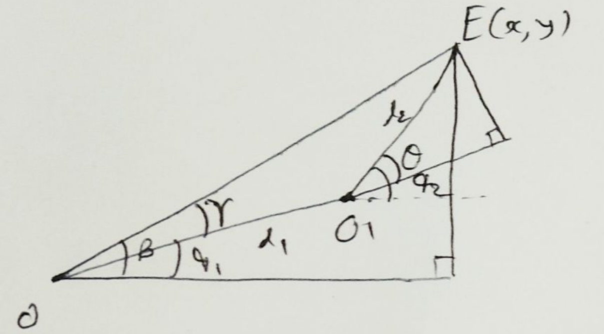
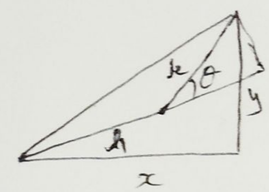
⇒ Inverse Kinematics

(getting q_1, q_2 from x, y position)

Cosine Rule.





~~$\cos\theta = ?$~~

$$x^2+y^2 = l_1^2 + l_2^2 - 2 l_1 l_2 (\pi - \theta)$$

- from cosine rule

$$\boxed{\theta = \cos^{-1}\left(\frac{x^2+y^2-l_1^2-l_2^2}{2 l_1 l_2}\right)}$$

$$q_1 = \beta - \gamma$$

$$\boxed{q_1 = \tan^{-1}\left(\frac{y}{x}\right) - \tan^{-1}\left(\frac{l_2 \sin\theta}{l_1 + l_2\cos\theta}\right)}$$

$$\boxed{q_2 = q_1 + \theta}$$

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Note :- In above derivation, q_2 is in absolute frame. (with reference of origin 'O')

- So as q_1 changes, q_2 also change by default.
  & to keep it ~~const~~ remain same link-2 need some work.

→ To avoid above complexity,
in simulation q_2 taken in relative frame to O1 (link-1)
So if $q_2 = 0$ ⇒

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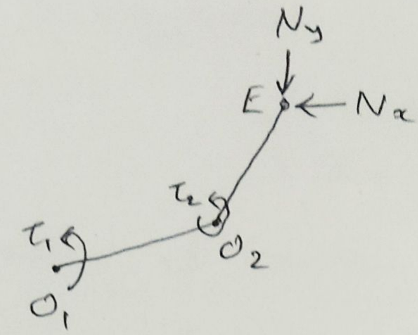
→ Dynamics



{FBD of 2R manipulator Appling constant force on wall (static equilibrium)}

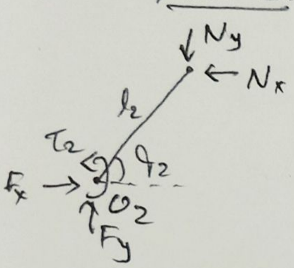
→ Under static equilibrium

$$\Sigma F_x = 0$$
$$\Sigma F_y = 0$$
$$\Sigma M = 0$$

- force applied by manipulator

$$F_x = -N_x$$
$$F_y = -N_y$$

→ For link-2 :-

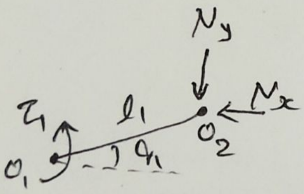


$\Sigma M_{O_2} = 0$ (CCW +ve)

$$\tau_2 - N_y l_2 c q_2 + N_x l_2 s q_2 = 0$$

$$\therefore \boxed{\tau_2 = N_y l_2 c q_2 - N_x l_2 s q_2}$$

→ For link-1 :-



$\Sigma M_{O_1} = 0$ (CCW +ve)

$$\tau_1 - N_y l_1 c q_1 + N_x l_1 s q_1 = 0$$

$$\therefore \boxed{\tau_1 = N_y l_1 c q_1 - N_x l_1 s q_1}$$

$$\therefore \boxed{\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} -l_1 s q_1 & l_2 c q_1 \\ -l_2 s q_2 & l_2 c q_2 \end{bmatrix} \begin{bmatrix} N_x \\ N_y \end{bmatrix}}$$

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⇒ Dynamics

→ Using Lagrangian

$L = K - V$

$\begin{cases} K - \cancel{\text{Pot}}\ \text{Kinetic Energy} \\ V - \text{Potential Energy} \end{cases}$

→ Equation of motion

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \tau_i}$$

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τ_i is a generalised force (torque) at i^{th} joint using principle of Virtual work

→ Kinetic Energy,

$$K = \underbrace{\frac{1}{2}\left(\frac{1}{3} m_1 l_1^2\right)\dot{q}_1^2}_{\text{Pure rotation energy of link-1}} + \underbrace{\frac{1}{2}\left(\frac{1}{3} m_2 l_2^2\right)\dot{q}_2^2}_{\text{Pure rotation energy of link-2}} + \frac{1}{2} m_2 V_{c_2}^2$$

$\left(\frac{1}{2} I \omega^2\right)$

$$V_{c_2}^2 = (l_1 \dot{q}_1)^2 + \left(\frac{l_2}{2}\dot{q}_2\right)^2 + 2 l_1 \dot{q}_1 \frac{l_2}{2}\dot{q}_2 \cos(q_2 - q_1)$$

→ Potential Energy

$$V = m_1 g \frac{l_1}{2} s q_1 + m_2 g \left(l_1 s q_1 + \frac{l_2}{2} s q_2\right)$$

⇒ Deriving (4)(5) using above expressions

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⇒ For joint-1, $i=1$ & Also for joint-2, $i=2$

$$\frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_1}(K-V)\right]-\frac{\partial}{\partial q_1}(K-V)=\tau_1$$

$$\frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_2}(K-V)\right]-\frac{\partial}{\partial q_2}(K-V)=\tau_2$$

$$\boxed{\begin{aligned}&\frac{1}{3}m_1 l_1^2\ddot{q}_1+m_2 l_2^2\ddot{q}_1+m_2\frac{l_1 l_2}{2}\ddot{q}_2\cos(q_2-q_1)\\&-m_2\frac{l_1 l_2}{2}\dot{q}_2(\dot{q}_2-\dot{q}_1)\sin(q_2-q_1)\\&-m_1 g\frac{l_1}{2}cq_1-m_2 g l_1 cq_1=\tau_1\end{aligned}}$$

$$\boxed{\begin{aligned}&\frac{1}{3}m_2 l_2^2\ddot{q}_2+\frac{1}{4}m_2 l_2^2\ddot{q}_2+m_2\frac{l_1 l_2}{2}\ddot{q}_1\cos(q_2-q_1)\\&-m_2\frac{l_1 l_2}{2}\dot{q}_1(\dot{q}_2-\dot{q}_1)\sin(q_2-q_1)\\&-m_2 g\frac{l_2}{2}cq_2=\tau_2\end{aligned}}$$

⑥

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→ Note that equation-(4) is valid for any forces (not just wall force)

So, for the robot to behave like a spring.

forces, $F_x = K_x (x - x_0)$

$F_y = K_y (y - y_0)$

$\Rightarrow F_x = K(l_1 cq_1 + l_2 cq_2)$

$F_y = K(l_1 sq_1 + l_2 sq_2)$

Assumptions

→ hidden damping not included (Ideal spring)

→ End effector at origin $(x_0, y_0) = (0, 0)$

→ $K_x = K_y = K$

- using equation-(4)

$$\begin{bmatrix} \tau_{1i} \\ \tau_{2i} \end{bmatrix} = \begin{bmatrix} -l_1 sq_1 & l_2 cq_1 \\ -l_2 sq_2 & l_2 cq_2 \end{bmatrix} \begin{bmatrix} K(l_1 cq_1 + l_2 cq_2) \\ K(l_1 sq_1 + l_2 sq_2) \end{bmatrix}$$

$$\boxed{\begin{bmatrix} \tau_{1i} \\ \tau_{2i} \end{bmatrix} = \begin{bmatrix} K(l_1 sq_1 + l_2 sq_2)\cdot l_2 cq_1 - K(l_1 cq_1 + l_2 cq_2)\cdot l_1 sq_1 \\ K(l_1 sq_1 + l_2 sq_2)\cdot l_2 cq_2 - K(l_1 cq_1 + l_2 cq_2)\cdot l_2 sq_2 \end{bmatrix}}$$ (7)