4.3) Min & max values for $\text{CoV}[X,Y]$

$$\text{CoV}[X,Y] = E[XY] - E[X]E[Y]$$

Case of minimum covariance:

$\Rightarrow$ minimize covariance means correlation $= -1$

$$\rho = \frac{\text{CoV}[X,Y]}{\sigma[X]\,\sigma[Y]}$$

; $\rho = -1$ for minimum
$\rho = +1$ for maximum

$$\sigma[X]\,\sigma[Y] = \text{CoV}[X,Y]$$

$$\Rightarrow \sigma[X] = \sqrt{V[X]}$$

$$V[X] = E[X^2] - E[X]^2$$

$$= \sum_{X=0}^{1} x^2 f(x) - \left(\sum_{X=0}^{1} x f(x)\right)^2 \qquad \text{LOTUS}$$

$$\left(0\left(\tfrac{1}{3}\right) + 1\left(\tfrac{2}{3}\right)\right) - \left(0\left(\tfrac{1}{3}\right) + 1\left(\tfrac{2}{3}\right)\right)^2$$

$$V[X] = \left(\tfrac{2}{3}\right) - \left(\tfrac{4}{9}\right) = \frac{2}{9}$$

$$V[Y] = E[Y^2] - E[Y]^2$$

$$= \sum_{y=0}^{1} y^2 f(y) - \left(\sum_{y=0}^{1} y\, f(y) \right)^2$$

$$= \left(0\left(\tfrac{1}{2}\right) + 1\left(\tfrac{1}{2}\right)\right) - \left(0\left(\tfrac{1}{2}\right) + 1\left(\tfrac{1}{2}\right)\right)^2$$

$$= \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$\sigma_x = \sqrt{V[X]} = \sqrt{\frac{2}{9}} \qquad \sigma_y = \sqrt{V[Y]} = \sqrt{\frac{1}{4}}$$

For Minimimum $CoV[X, Y] \Rightarrow \rho = -1$

$$-\sigma_x \sigma_y = CoV[X, Y]$$

$$CoV[X, Y] = -\sqrt{\frac{2}{9}}\sqrt{\frac{1}{4}} = \sqrt{\frac{2}{36}}$$

$$CoV[X, Y] = -0.24 \Rightarrow \text{minimum}$$

By the same logic correlation at maximum is +1
So the max covariance is

$$CoV[X, Y] = 0.24 \Rightarrow \text{maximum}$$