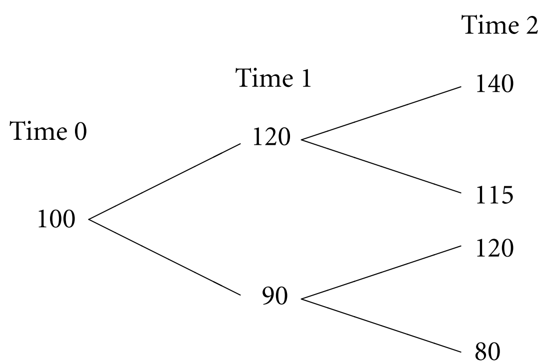
7

Option Pricing

7.1 Discrete time

In the next section we discuss the Black–Scholes formula. To prepare for that, we consider the much simpler problem of pricing options when there are a finite number of time periods and two possible outcomes at each stage. The restriction to two outcomes is not as bad as one might think. One justification for this is that we are looking at the process on a very slow timescale, so at most one interesting event happens (or not) per time period. We begin by considering a very simple special case.



Example 7.1 **Two-period binary tree.** Suppose that a stock price starts at 100 at time 0. At time 1 (one day or one month or 1 year later) it is either worth 120 or 90. If the stock is worth 120 at time 1, then it might be worth 140 or 115 at time 2. If the price is 90 at time 1, then the possibilities at time 2 are 120 and 80. Suppose now that you are offered a **European call option** with **strike price** 100 and **expiry** 2. This means you have an option to buy the stock (but not an obligation to do so) for 100 at time 2, that is, after seeing the outcome of the first and second stages. If the stock price is 80, you do not exercise the option to purchase the stock and your profit is 0. In the other cases you choose to buy the stock at 100 and then immediately sell it at X_2 to get a payoff of $X_2 - 100$, where X_2 is the stock price at time 2. Combining the two cases we can write the payoff in

general as $(X_2 - 100)^+$, where $z^+ = \max\{z, 0\}$ denotes the positive part of z. Our problem is to figure out what is the right price for this option.

At first glance this may seem impossible since we have not assigned probabilities to the various events. However, it is a miracle of "pricing by the absence of arbitrage" that in this case we do not have to assign probabilities to the events to compute the price. To explain this we start by considering a small piece of the tree. When $X_1 = 90$, X_2 will be 120 ("up") or 80 ("down") for a profit of 30 or a loss of 10, respectively. If we pay c for the option then when X_2 is up we make a profit of $20 - c$, but when it is down we make $-c$. The last two sentences are summarized in the following table:

	Stock	Option
Up	30	$20 - c$
Down	-10	$-c$

Suppose we buy x units of the stock and y units of the option, where negative numbers indicate that we sold instead of bought. One possible strategy is to choose x and y so that the outcome is the same if the stock goes up or down:

$$30x + (20 - c)y = -10x + (-c)y$$

Solving, we have $40x + 20y = 0$ or $y = -2x$. Plugging this choice of y into the last equation shows that our profit will be $(-10 + 2c)x$. If $c > 5$, then we can make a large profit with no risk by buying large amounts of the stock and selling twice as many options. Of course, if $c < 5$, we can make a large profit by doing the reverse. Thus, in this case the only sensible price for the option is 5.

A scheme that makes money without any possibility of a loss is called an **arbitrage opportunity**. It is reasonable to think that these will not exist in financial markets (or at least be short-lived), since if and when they exist people take advantage of them and the opportunity goes away. Using our new terminology we can say that the only price for the option that is consistent with absence of arbitrage is $c = 5$, so that must be the price of the option (at time 1 when $X_1 = 90$).

Before we try to tackle the whole tree to figure out the price of the option at time 0, it is useful to look at things in a different way. Generalizing our example, let $a_{i,j}$ be the profit for the ith security when the jth outcome occurs.

Theorem 7.1. *Exactly one of the following holds:*

(i) There is a betting scheme $x = (x_1, x_2, \ldots, x_n)$ so that $\sum_{i=1}^m x_i a_{i,j} \geq 0$ for each j and $\sum_{i=1}^m x_i a_{i,k} > 0$ for some k.

(ii) There is a probability vector $p = (p_1, p_2, \ldots, p_n)$ with $p_j > 0$ so that $\sum_{j=1}^n a_{i,j} p_j = 0$ for all i.

Here a vector x satisfying (i) is an arbitrage opportunity. We never lose any money but for at least one outcome we gain a positive amount. Turning to (ii), the vector p is called a martingale measure since if the probability of the jth outcome is p_j, then the expected change in the price of the ith stock is equal to 0. Combining the two interpretations we can restate Theorem 7.1 as

Theorem 7.2. *There is no arbitrage if and only if there is a strictly positive probability vector so that all the stock prices are martingale.*

Why is this true? One direction is easy. If (i) is true, then for any strictly positive probability vector $\sum_{i=1}^m \sum_{j=1}^n x_i a_{i,j} p_j > 0$, so (ii) is false.

Suppose now that (i) is false. The linear combinations $\sum_{i=1}^m x_i a_{i,j}$ when viewed as vectors indexed by j form a linear subspace of n-dimensional Euclidean space. Call it $\mathcal{L}$. If (i) is false, this subspace intersects the positive orthant $\mathcal{O} = \{y \colon y_j \geq 0 \text{ for all } j\}$ only at the origin. By linear algebra we know that $\mathcal{L}$ can be extended to an $(n-1)$-dimensional subspace $\mathcal{H}$ that only intersects $\mathcal{O}$ at the origin.

Since $\mathcal{H}$ has dimension $n-1$, it can be written as $\mathcal{H} = \{y : \sum_{j=1}^n y_j p_j = 0\}$. Since for each fixed i the vector $a_{i,j}$ is in $\mathcal{L} \subset \mathcal{H}$, (ii) holds. To see that all the $p_j > 0$, we leave it to the reader to check that if not, there would be a nonzero vector in $\mathcal{O}$ that would be in $\mathcal{H}$. □

To apply Theorem 7.1 to our simplified example we begin by noting that in this case $a_{i,j}$ is given by

		$j=1$	$j=2$
Stock	$i=1$	30	-10
Option	$i=2$	$20-c$	$-c$

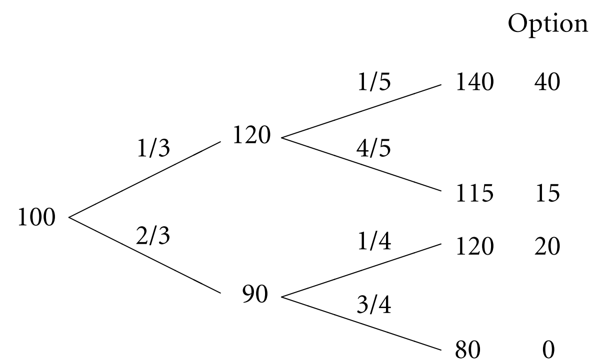
By Theorem 7.2 if there is no arbitrage, then there must be an assignment of probabilities p_j so that

$$30p_1 - 10p_2 = 0 \qquad (20-c)p_1 + (-c)p_2 = 0$$

From the first equation we conclude that $p_1 = 1/4$ and $p_2 = 3/4$. Rewriting the second we have

$$c = 20p_1 = 20 \cdot (1/4) = 5$$

To generalize from the last calculation to finish our example we note that the equation $30p_1 - 10p_2 = 0$ says that under p_j the stock price is a martingale (that is, the average value of the change in price is 0), while $c = 20p_1 + 0p_2$ says that the price of the option is then the expected value under the martingale probabilities. Using these ideas we can quickly complete the computations in our example. When $X_1 = 120$, the two possible scenarios lead to a change of $+20$ or -5, so the relative probabilities of these two events should be 1/5 and 4/5. When $X_0 = 100$, the possible price changes on the first step are $+20$ and -10, so their relative probabilities are 1/3 and 2/3. Drawing a picture of the possibilities, we have



so the value of the option is

$$\frac{1}{15} \cdot 40 + \frac{4}{15} \cdot 15 + \frac{1}{6} \cdot 20 = \frac{80 + 120 + 100}{30} = 10 \qquad \square$$

The last derivation may seem a little devious, so we now give a second derivation of the price of the option. In the scenario described above, our investor has four possible actions:

A_0. Put \$1 in the bank and end up with \$1 in all possible scenarios.
A_1. Buy one share of stock at time 0 and sell it at time 1.
A_2. Buy one share at time 1 if the stock is at 120 and sell it at time 2.
A_3. Buy one share at time 1 if the stock is at 90 and sell it at time 2.

These actions produce the following payoffs in the indicated outcomes:

Time 1	Time 2	A_0	A_1	A_2	A_3	Option
120	140	1	20	20	0	40
120	115	1	20	-5	0	15
90	120	1	-10	0	30	20
90	80	1	-10	0	-10	0

Noting that the payoffs from the four actions are themselves vectors in four-dimensional space, it is natural to think that by using a linear combination of these actions we can reproduce the option exactly. To find the coefficients we write four equations in four unknowns:

$$\begin{aligned} z_0 + 20z_1 + 20z_2 &= 40 \\ z_0 + 20z_1 - 5z_2 &= 15 \\ z_0 - 10z_1 + 30z_3 &= 20 \\ z_0 - 10z_1 - 10z_3 &= 0 \end{aligned} \tag{7.1}$$

Subtracting the second equation from the first and the fourth from the third gives $25z_2 = 25$ and $40z_3 = 20$, so $z_2 = 1$ and $z_3 = 1/2$. Plugging in these values, we have two equations in two unknowns:

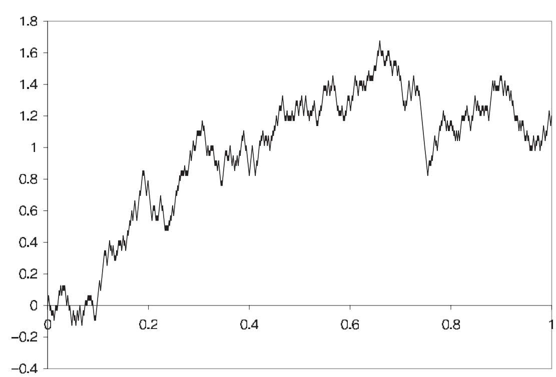
$$z_0 + 20z_1 = 20 \qquad z_0 - 10z_1 = 5$$

Taking differences, we conclude that $30z_1 = 15$, so $z_1 = 1/2$ and $z_0 = 10$.

The reader may have already noticed that $z_0 = 10$ is the option price. This is no accident. What we have shown is that with \$10 cash we can buy and sell shares of stock to produce the outcome of the option in all cases. In the terminology of Wall Street, $z_1 = 1/2$, $z_2 = 1$, $z_3 = 1/2$ is a **hedging strategy** that allows us to **replicate the option**. Once we can do this it follows that the fair price must be \$10. To do this note that if we could sell it for \$12, then we can take \$10 of the cash to replicate the option and have a sure profit of \$2.

7.2 Continuous time

To do option pricing in continuous time we need a model of the stock price, and for this we have to first explain **Brownian motion**. Let $X_1, X_2, \ldots,$ be independent and take the values 1 and -1 with probability 1/2 each. $EX = 0$ and $EX^2 = 1$, so if we let $S_n = X_1 + \cdots + X_n$, then $S_n/\sqrt{n}$ converges to χ a standard normal distribution. Intuitively, Brownian motion is what results when we look not only at time n but also at how the process got there. To be precise, we let $t \geq 0$ and consider $S_{[nt]}/\sqrt{n}$, where $[nt]$ is the largest integer $\leq nt$. In words, we multiply n by t and then round down to the nearest whole number. When $n = 1{,}000$ the picture looks like



To understand the nature of the limit process we note that

$$\frac{S_{[nt]}}{\sqrt{n}} = \frac{S_{[nt]}}{\sqrt{[nt]}} \cdot \frac{\sqrt{[nt]}}{\sqrt{n}}$$

The first term approaches a standard normal distribution and the second $\sqrt{t}$, so $S_n/\sqrt{n}$ converges to $\sqrt{t}\chi$, a normal with mean 0 and variance t. Repeating the reasoning in the last paragraph we can see that if $s < t$, then $(S_{[nt]} - S_{[ns]})/\sqrt{n}$ converges to a normal with mean zero and variance $t - s$. Noting that $(S_{[nt]} - S_{[ns]})$ is independent of $S_{[ns]}$ suggests the following definition of the limiting process that we call **Brownian motion**.

- B_t has a normal distribution with mean 0 and variance t.
- If $0 < t_1 < \cdots < t_n$ then $B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}$ are independent

In modeling stock prices it is natural to assume that the daily percentage changes in the price are independent. For this reason and the mundane fact that stock prices must be >0, we model the stock as what is called **geometric Brownian motion.**

$$X_t = X_0 \cdot \exp(\mu t + \sigma B_t) \tag{7.2}$$

μ is the exponential growth rate of the stock and σ its volatility. In writing the model we have assumed that the growth rate and volatility of the stock are constant. If we also assume that the interest rate r is constant, then the discounted stock price is

$$e^{-rt} X_t = X_0 \cdot \exp((\mu - r)t + \sigma B_t)$$

Here, we have to multiply by e^{-rt}, since \$1 at time t has the same value as e^{-rt} dollars today.

Our problem is to determine the fair price of a European call option $(X_t - K)^+$ with strike price K and expiry t. Extrapolating wildly from Theorem 7.2, we can say that any consistent set of prices must come from a martingale measure. This implies

$$\mu = r - \sigma^2/2 \tag{7.3}$$

To compute the value of the call option, we need to compute its value in the model in (7.2) for this special value of μ. Using the fact that $\log(X_t/X_0)$ has a normal$(\mu t, \sigma^2 t)$ distribution, one can show

Black–Scholes formula. *The price of the European call option $(X_T - K)^+$ is given by*

$$X_0 \Phi(\sigma\sqrt{t} - \alpha) - e^{-rt} K \Phi(-\alpha)$$

where Φ is the distribution function of a standard normal and

$$\alpha = \{\log(K/X_0 e^{\mu t})\}/\sigma\sqrt{t}$$

To try to come to grips with this ugly formula note that $K/X_0 e^{\mu t}$ is the ratio of the strike price to the expected value of the stock at time t under the martingale probabilities, while $\sigma\sqrt{t}$ is the standard deviation of $\log(X_t/X_0)$.

Example 7.2 **Microsoft call options.** The February 23, 1998, *Wall Street Journal* listed the following prices for July call options on Microsoft stock.

Strike	75	80	85
Price	11	$8\frac{1}{8}$	$5\frac{1}{2}$

On this date Microsoft stock was trading at $81\frac{5}{8}$, while the annual interest rate was about 4% per year. Should you buy the call option with strike 80?

Solution. The answer to this question depends on your opinion of the volatility of the market over the period. Suppose that we follow a traditional rule of thumb and decide that $\sigma = 0.3$; that is, over a 1-year period a stock's price might change by about 30% of its current value. In this case the drift rate for the martingale measure is

$$\mu = r - \sigma^2/2 = 0.04 - (0.09)/2 = 0.04 - 0.045 = -0.005$$

and so the log ratio is

$$\log(K/X_0 e^{\mu t}) = \log(80/(81.625 e^{-0.005(5/12)}) = \log(80/81.455) = -0.018026$$

Five months corresponds to $t = 5/12$, so the standard deviation

$$\sigma\sqrt{t} = 0.3\sqrt{5/12} = 0.19364$$

and $\alpha = -0.018026/0.19364 = -0.09309$. Plugging in now, we have a price of

$$\begin{aligned}
&81.625\Phi(0.19365 + 0.09309) - e^{-0.04(5/12)} 80\Phi(0.09309)\\
&= 81.625\Phi(0.28674) - 78.678\Phi(0.09309)\\
&= 81.625(0.6128) - 78.678(0.5371) = 50.02 - 42.25 = 7.76
\end{aligned}$$

This is somewhat lower than the price quoted in the paper. There are two reasons for this. First, the options listed in the *Wall Street Journal* are

American call options. The holder has the right to exercise at any time during the life of the option. Since one can ignore the additional freedom to exercise early, American options are at least as valuable as their European counterparts. Second, and perhaps more importantly, we have not spent much effort on our estimate of r and σ. Nonetheless, as the next example shows the predictions of the formula are in rough agreement with the observed process.

Example 7.3

Intel call options. Again consulting the *Wall Street Journal* for February 23, 1998, we find the following prices listed for July call options on Intel stock, which was trading at $94\frac{3}{16}$.

Strike	70	75	80	85	90	95	100	105
Price	26	22	18	$14\frac{1}{2}$	$11\frac{3}{8}$	$8\frac{3}{4}$	$6\frac{1}{2}$	$4\frac{3}{8}$
Formula	25.65	21.16	17.01	13.59	10.11	7.11	5.39	4.13