Recap

MIDS 1a

Fundamentals of Linear Algebra

Unit 4: High-Level Objectives

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  - Singular value decomposition
- Explain how to reduce the least squares regression problem to a related linear system

Vector Geometry

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- Note that length, distance, and directional similarity can be represented by a single real number.

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- Pythagorean theorem: If x y=0 ,

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  - Every invertibleT∈L(V) has a factorization S∘ T * T∘U where S and U are isometries.
  - Given any basis of V, isometries S and U can be found so that M(T)=M(S)·Σ·M(U) where Σ is a diagonal matrix with the singular values of T along the diagonal.

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- Geometric property: proj vu is the unique vector in the direction of v with minimal distance to u

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  - The solution to [A TA| A Tb] is unique whenever the columns of A are linearly independent.