Linear Algebra - Problem Set 4 - #1 Cuong Trinh

1) Let V be the inner product space $\mathbb{R}[x]_{\leq 2}$ with the inner product defined by $\langle p(x), q(x) \rangle = \int_{-1}^{1} p(x)q(x)\,dx$. Apply the Gram-Schmitch process to the canonical basis $\{1, x, x^2\}$ to produce orthonormal set of vectors spanning V.

Gram-Schmidt Process:

Input $v_1 \ldots v_m \Rightarrow \{1, x, x^2\}$

Set $w_1 = v_1$

1) For $k = 2, \ldots, n$

$$\text{Set } w_k = v_k - \sum_{j=1}^{k-1} \frac{\langle v_j\ w_j \rangle}{\langle v_j\ v_j \rangle} v_j$$

2) For $k = 1, \ldots, n$

$$\text{Set } u_k = \frac{w_k}{\|w_k\|}$$

Output $u_1, \ldots, u_n$

Step 1: Orthogonization

1 : 1 b/c first vector so doesn't count

$$x: \quad x - \frac{\langle x, 1 \rangle}{\langle 1, 1 \rangle} 1 = x - \frac{\int_{-1}^{1} x\,dx}{\int_{-1}^{1} dx} = x - \frac{\left.\frac{x^2}{2}\right|_{-1}^{1}}{\left.x\right|_{-1}^{1}} = \frac{0}{2} = x$$

$$x^2: \quad x^2 - \frac{\langle x^2, 1 \rangle}{\langle 1, 1 \rangle} 1 - \frac{\langle x^2, x \rangle}{\langle x, x \rangle} x = x^2 - \frac{\int_{-1}^{1} x^2\,dx}{\int_{-1}^{1} dx} - \frac{\int_{-1}^{1} x^3\,dx}{\int_{-1}^{1} x^2\,dx} x$$

$$= x^2 - \frac{\left.\frac{x^3}{3}\right|_{-1}^{1}}{\left.x\right|_{-1}^{1}} - \frac{\left.\frac{x^4}{4}\right|_{-1}^{1}}{\left.\frac{x^3}{3}\right|_{-1}^{1}} = x^2 - \frac{\frac{2}{3}}{2} - \frac{0}{\frac{2}{3}} x = x^2 - \frac{1}{3}$$

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Step 2: Orthonormalization $\left\{ 1, \ x, \ x^2 - \frac{1}{3} \right\}$

$$1: \quad \frac{1}{\|1\|} = 1$$

$$x: \quad \frac{x}{\|x\|} = \frac{x}{\sqrt{\langle x, x\rangle}} = \frac{x}{\left(\int_{-1}^{1} x^2\,dx\right)^{\frac{1}{2}}} = \frac{x}{\left(\frac{x^3}{3}\Big|_{-1}^{1}\right)^{\frac{1}{2}}}$$

$$= \frac{x}{\sqrt{\frac{2}{3}}} = \sqrt{\frac{3}{2}}\,x$$

$$x^2 - \frac{1}{3}: \quad \frac{x^2 - \frac{1}{3}}{\|x^2 - \frac{1}{3}\|} = \frac{x^2 - \frac{1}{3}}{\left(\int_{-1}^{1} \left(x^2 - \frac{1}{3}\right)^2 dx\right)^{\frac{1}{2}}}$$

$$\Rightarrow \text{denominator}: \sqrt{\int_{-1}^{1} \left(x^2 - \frac{1}{3}\right)^2 dx} \Rightarrow \sqrt{\int_{-1}^{1} x^4 - \frac{2}{3}x^2 + \frac{1}{9}\,dx}$$

$$\Rightarrow \sqrt{\frac{x^5}{5}\Big|_{-1}^{1} - \frac{2}{9}x^3\Big|_{-1}^{1} + \frac{1}{9}x\Big|_{-1}^{1}}$$

$$\Rightarrow \sqrt{\frac{2}{5} - \frac{4}{9} + \frac{2}{9}}$$

$$\Rightarrow \sqrt{\frac{2}{5} - \frac{2}{9}} = \sqrt{\frac{8}{45}} = \frac{2\sqrt{2}}{3\sqrt{5}}$$

$$x^2 - \frac{1}{3}: \quad \frac{x^2 - \frac{1}{2}}{\frac{2\sqrt{2}}{3\sqrt{5}}} = \frac{3\sqrt{5}}{2\sqrt{2}}\left(x^2 - \frac{1}{3}\right) = \frac{3\sqrt{5}}{2\sqrt{2}}x^2 - \frac{\sqrt{5}}{2\sqrt{2}}$$

$$\Rightarrow \left\{ 1, \ \sqrt{\frac{3}{2}}\,x, \ \frac{3\sqrt{5}}{2\sqrt{2}}x^2 - \frac{\sqrt{5}}{2\sqrt{2}} \right\}$$

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2) Let $T : \mathbb{R}[x]_{\leq 3} \to \mathbb{R}[x]_{\leq 2}$ be the differential map

has basis $\{1, x, x^2, x^3\}$

a) What's $M(T^*)$ in the basis $\{1, x, x^2\}$?

Let's say: $T(p(x)) = p'(x)$
- where $p(x)$ is some polynomial defined by $\mathbb{R}[x]_{\leq 3}$
- and $p'(x)$ is the first order derivative of $p(x)$

$\Rightarrow$ So for the basis $\{1, x, x^2\}$;

$$T(a + bx + cx^2 + dx^3) = (a + bx + cx^2 + dx^3)' = b + 2cx + 3dx^2$$

$\Rightarrow$ Thus ve can represent $M(T) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$.

$\Rightarrow$ Recall that, for any given basis $v_1 \dots v_n$ in V and for any given $w_1 \dots w_m$ in W,

$$M(T, \vec{v}, \vec{w}) = [M(T^*, \vec{w}, \vec{v})]^T$$

So $M(T^*) = M(T)^T$

$$M(T^*) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

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b) What is $T^*(2+3x+5x^2)$

$$M(T^*) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$T^*(2+3x+5x^2) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 6 \\ 15 \end{bmatrix}$$

$$= 2x + 6x^2 + 15x^3$$

Note $\Rightarrow T^*(2+3x+5x^3) \longrightarrow$ something in $\mathbb{R}[x]_{\leq 3}$ w/ basis $\{1, x, x^2, x^3\}$

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3) Find singular value of T where $M(T) = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$

$$M(T^*) = [M(T)]^T = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$$

Singular value of $T = \sqrt{T^*T}$

$$\left(\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \right)^{\frac{1}{2}} = \left(\begin{bmatrix} (1+1) & (2+1) \\ (2+1) & (4+1) \end{bmatrix} \right)^{\frac{1}{2}}$$

Now ve calculate the eigenvalues of

$$= \left(\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} \right)^{\frac{1}{2}}$$

$$\det \left| \begin{bmatrix} 2-\lambda & 3 \\ 3 & 5-\lambda \end{bmatrix} \right|^{\frac{1}{2}} = (2-\lambda)(5-\lambda) - 9 = 0$$

$$= (10 - 7\lambda + \lambda^2) - 9 = 0$$

$$\frac{49}{4} + -1 = \left(\lambda^2 - 7\lambda + \frac{49}{4}\right)$$

$$\left(\lambda - \frac{7}{2}\right)^2 = \frac{45}{4}$$

$$(\lambda)^{\frac{1}{2}} = \left(\frac{7}{2} \pm \frac{3\sqrt{5}}{2}\right)^{\frac{1}{2}}$$

Singular values of T:

$$\boxed{\lambda_1 = \frac{\sqrt{7}}{\sqrt{2}} + \frac{\sqrt{3\sqrt{5}}}{\sqrt{2}} = \frac{\sqrt{2}\sqrt{7+3\sqrt{5}}}{2} \qquad \lambda_2 = \frac{\sqrt{7}}{\sqrt{2}} - \frac{\sqrt{3\sqrt{5}}}{\sqrt{2}} = \frac{\sqrt{2}\sqrt{7-3\sqrt{5}}}{2}}$$

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4) Least Square Approx. to overdeterminant matrix $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 0 \end{array}\right]$

$$\Rightarrow \left[\begin{array}{c|c} A^T A & A^T b \end{array}\right] \Rightarrow A^T = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} (1+1+4) & (0+1+2) \\ (0+1+2) & (0+1+1) \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 3 & 2 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Least Square approx: $\left[\begin{array}{cc|c} 6 & 3 & 0 \\ 3 & 2 & 0 \end{array}\right]$

RREF $\left[\begin{array}{cc|c} 6 & 3 & 0 \\ 3 & 2 & 0 \end{array}\right] \xrightarrow{R_1 \leftarrow \frac{1}{6} R_1} \left[\begin{array}{cc|c} 1 & \frac{1}{2} & 0 \\ 3 & 2 & 0 \end{array}\right]$

$\xrightarrow{R_2 \leftarrow R_2 - 3R_1} \left[\begin{array}{cc|c} 1 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \end{array}\right] \xrightarrow{R_1 \leftarrow R_1 - R_2} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{array}\right]$

$\xrightarrow{R_2 \leftarrow 2R_2} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$

approx: $x_1 = 0$

$x_2 = 0$