Q4. Buffon Needle

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given: Needle of lenght ℓ

Some parallel line w/ distant t apart
divided into white / gray region

Throw stick at random onto the table

Let $X \Rightarrow$ R.V. for horizontal distance b/t midpt of stick and nearest parallel line (surface change color)

$\theta \Rightarrow$ R.V. for acute angle b/t the line & parallel rule

Find P(Stick intersect one of the rule)

1) Start w/ the pdf

distance : since line are equidistance

$$f(x) = \begin{cases} \frac{2}{t} ; & 0 \le x \le \frac{t}{2} \\ 0 ; & \text{otherwise} \end{cases}$$

angle : since it is acute $0 \le \theta \le \frac{\pi}{2}$

$$f(\theta) = \begin{cases} \frac{2}{\pi} ; & 0 \le \theta \le \frac{\pi}{2} \\ 0 ; & \text{otherwise} \end{cases}$$

& since $X \perp\!\!\!\perp \Theta$

$$f(x, \theta) = \begin{cases} \frac{4}{t\pi} ; & 0 \le x \le \frac{t}{2} ; \ 0 \le x \le \frac{\pi}{2} \\ 0 ; & \text{Otherwise} \end{cases}$$

So to find probability, we integrate the pdf!

$$P = \int_{\theta}\int_{x} f(x,\theta)\, dx\, d\theta$$

⇒ What's the unit of integration?

⇒ The needle cross the line in many way

1) mid point cross the line while stick is flat

.... other cases

2) Stick cross the line while ∥ to the line ☺

↳ dependence on θ & the length of the stick

$$x \le \frac{l}{2}\sin\theta$$

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{l}{2}\sin\theta} \frac{4}{\pi}\, dx\, d\theta = \int_0^{\frac{\pi}{2}} \frac{4}{\pi} \times \Big|_0^{\frac{l}{2}\sin\theta} d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{2\, l \sin\theta}{\pi}\, d\theta$$

$$P\left(X \le \frac{l}{2}\sin\theta\right) = \frac{2l}{\pi}\int_0^{\frac{\pi}{2}} \sin\theta\, d\theta \quad (\sin\theta \to 1)$$

$$= \frac{2l}{\pi}$$