2) Consistency: Show $\bar{Z}_n$ is consistent estimator for μ

Prove $\bar{Z}_n \xrightarrow{P} \mu$

Using Theorem of Converge in probability

$$\lim_{n \to \infty} P\left(\bar{Z}_n \in \{\mu - \varepsilon, \mu + \varepsilon\}\right) = 1$$

Let μ be the value Z_n converges to

Let ε be some arbitrary small window

=> expanding the Probability

$$P(\bar{Z}_n \in \{\mu - \varepsilon, \mu + \varepsilon\}) = P\left(\mu - \varepsilon \leq \bar{Z}_n \leq \mu + \varepsilon\right)$$

$$= P\left(\bar{Z}_n \geq \mu - \varepsilon \cup \bar{Z}_n \leq \mu + \varepsilon\right)$$

$$= P\left(\bar{Z}_n \geq \mu - \varepsilon\right) + P\left(\bar{Z}_n \leq \mu + \varepsilon\right)$$

=> Here we Note that $P(Z_n \leq p + \varepsilon)$ is just the cdf!

=> following that each Z_i is independent

$$P(\bar{Z}_n \leq p + \varepsilon) = \prod_{i=1}^{n} P(\bar{Z}_i \leq p + \varepsilon) = P(\bar{Z}_i \leq p + \varepsilon)^n$$

Using Probability axioms:

$$P(\bar{Z}_n \geq p + \varepsilon) = 1 - P(\bar{Z}_n \leq p + \varepsilon)$$

$\Rightarrow$ Thus plugging into

$$P(\bar{Z}_n \in \{\mu-\varepsilon, \mu+\varepsilon\}) = P(\bar{Z}_n \leq \mu+\varepsilon) + P(\bar{Z}_n \geq \mu-\varepsilon)$$

$$= P(\bar{Z}_i \leq \mu+\varepsilon)^n +$$

$$(1 - P(\bar{Z}_i \leq \mu+\varepsilon)^n)$$

$$= 1$$

takes the limit

$$\lim_{n\to\infty} P(\bar{Z}_n \in \{\mu-\varepsilon, \mu+\varepsilon\}) = 1$$

$$\lim_{n\to\infty} 1 = 1$$

$$1 = 1$$

Hence $\bar{Z}_n$ converge in probability to μ

2)

* Easier prove: Since I prove that $\bar{Z}_n$ is unbias!
and the problem give $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \sigma_i^2 = 0$

I want to show that $\lim_{n\to\infty} V[\hat{\theta}] = 0$

From property of variance of an estimator

$$V[\bar{Z}_n] = V\left[\frac{1}{n}\left(Z_1 + Z_2 + \dots Z_n\right)\right]$$

$$= \frac{1}{n^2} V\left[Z_1 + Z_2 + \dots + Z_n\right]$$

$$V[\bar{Z}_n] = \frac{1}{n} V[Z_i] = \frac{\sigma_i^2}{n}$$

Plus it into

$$\lim_{n\to\infty} V[\bar{Z}_n] = \underbrace{\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \sigma_i^2 = 0}_{\text{given}}$$

so $\lim_{n\to\infty} V[\bar{Z}_n] = 0 \quad \therefore \quad \bar{Z}_n \xrightarrow{P} \mu$