MIDS 1a

Fundamentals of Linear Algebra



• Explain the geometric motivation behind the inner product notation

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- Define an inner product space

Unit 4: High-Level Objectives

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 - Singular value decomposition
- Explain how to reduce the least squares regression problem to a related linear system

Vector Geometry		

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• Geometric properties of two-dimensional vectors:



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- Note that length, distance, and directional similarity can be represented by a single real number.

• Inner product notation:

Vector Geometry (contd)

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 - Positive definiteness: u u≥0 and u u=0 iff u=0

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• Pythagorean theorem: If x y=0,

||x-y|| 2 = ||x|| 2 + ||y|| 2 = ||x+y|| 2

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 - o Normalization is second.

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 - Matrix of a self-adjoint linear map is symmetric.

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 - Every invertibleT∈L(V) has a factorization S∘ T * T∘U where S and U are isometries.
 - o Given any basis of V, isometries S and U can be found so that $M(T)=M(S)\cdot \Sigma\cdot M(U)$ where Σ is a diagonal matrix with the singular values of T along the diagonal.

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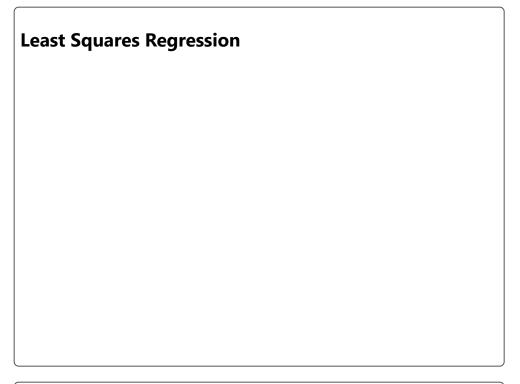
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- Geometric property: proj vu is the unique vector in the direction of v with minimal distance to u



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- What vector x minimizes||Ax-b|| for overdetermined linear system [A|b]?
 - Using projections, any solution x is also a solution to the linear system [A TA| A Tb].
 - The solution to [ATA|ATb] is unique whenever the columns of A are linearly independent.