4.1) Show $|\tilde{\beta}| \leq |\beta|$

a) Compute $\tilde{\beta}$ & β

$$\beta = \frac{Cov[X, Y]}{V[X]}$$

$$Cov[X, Y] = E[XY] - E[X]E[Y]$$

$$\tilde{\beta} = \frac{Cov[S, Y]}{V[S]}$$

$$Cov[S, Y] = E[SY] - E[S]E[Y]$$

$$E[SY] = E[(X+U)Y]$$
$$= E[XY + UY]$$
$$= E[XY] + E[UY]$$

$$E[S] = E[X+U]$$
$$= E[X] + E[U]$$

$$Cov[S, Y] = E[XY] + E[UY] - E[X]E[Y] - E[U]E[Y]$$
$$= Cov[X, Y] + Cov[U, Y]$$

$$\tilde{\beta} = \frac{Cov[X, Y] + Cov[U, Y]}{V[S]}$$

$$\tilde{\beta} = \frac{\mathrm{CoV}[X, Y] + \mathrm{CoV}[U, Y]}{V[S]}$$

The denominator term $\Rightarrow$

$$
\begin{aligned}
V[S] &= E[S^2] - (E[S])^2 \\
&= E[(X+U)^2] - (E[X+U])^2 \\
&= E[X^2 + 2XU + U^2] - (E[X] + E[U])^2 \\
&= E[X^2] + 2E[XU] + E[U^2] \\
&\quad - \underbrace{E[X]^2} - \underbrace{2E[X]E[U]} - \underbrace{E[U]^2} \\
&= V[X] + 2\mathrm{CoV}[X, U] + V[U]
\end{aligned}
$$

$$\tilde{\beta} = \frac{\mathrm{CoV}[X, Y] + \mathrm{CoV}[U, Y]}{V[X] + V[U] + 2\mathrm{CoV}[X, U]}$$

plug this back for comparison

$$|\tilde{\beta}| \leq |\beta|$$

$$\frac{CoV[X,Y] + CoV[U,Y]}{V[X] + V[U] + 2CoV[X,U]} \overset{?}{\leq} \frac{CoV[X,Y]}{V[X]}$$

Here we note that $V[U] > 0$ so the denominator term for $|\tilde{\beta}|$ will be larger than the denominator term for $|\beta|$

$\Rightarrow$ Thus $\dfrac{1}{\text{large value}} < \dfrac{1}{\text{Small value}}$

$\Rightarrow$ Additionally if I multiply the denominator from both side, the RHS will become larger than the LHS

$\Rightarrow$ This implies that $|\tilde{\beta}| \leq |\beta|$ is true!