4.2) Show that $V[Y] = V[\varepsilon] + V[f(x)]$

$$= V[\partial] + V[g(S)]$$

1) Let's first show that

$$V[\varepsilon] + V[f(x)] = V[\partial] + V[g(S)]$$

Note that :

0 by property of BLP

$$V[\varepsilon] = E[\varepsilon^2] - E[\cancelto{0}{\varepsilon}]^2$$

$$V[\partial] = E[\partial^2] - E[\cancelto{0}{\partial}]^2$$

$$E[\varepsilon^2] + V[f(x)] = E[\partial^2] + V[g(S)]$$

$\Rightarrow$ Here all I need to do is show that the second term holds !

Note that

$$V[f(x)] = E[f(x)^2] - (E[f(x)])^2$$

$$= E[(\alpha + \beta X)^2] - (E[\alpha + \beta X])^2$$

$$= E[\alpha^2 + 2\alpha\beta X + \beta^2 X^2]$$

$$- (\alpha + \beta E[X])(\alpha + \beta E[X])$$

$$= \alpha^2 + 2\alpha\beta E[X] + \beta^2 E[X^2] \quad = 0$$

$$-\alpha^2 - 2\alpha\beta E[X] - \beta^2 E[X^2]$$

By the same logic:

$$V[g(S)] = 0$$

So... $E[\varepsilon^2] = E[\partial^2] = V[Y]$

is all that is left!

$$E[\varepsilon^2] = V[Y]$$

$$\Downarrow$$

$$E[(Y - f(X))^2] = V[Y]$$

$$E[Y^2 - 2Yf(X) + f(X)^2] = V[Y]$$

$\Rightarrow$ So I want to show that LHS goto

$$E[Y^2] - E[Y]^2 = V[Y]$$

$$E[Y^2] + E[f(X)^2 - 2Yf(X)]$$

$$+ E[f(X)^2] - 2\alpha E[Y] - 2\beta E[XY]$$

$$- (E[f(X)])^2 + (E[f(X)])^2$$

$$E[Y^2] + V[f(\cancel{X})]^{\,0} + E[f(X)]^2 - 2E[Yf(X)]$$

$$+ E[Y]^2 - E[Y]^2$$

$$E[Y^2] - E[Y]^2 + (E[\varepsilon])^2$$

$$E[Y^2] - (E[Y])^2 = V[Y]$$

By the same Logic: I can do the same here to show that:

$$E[\partial^2] = V[Y]$$

$$\therefore \quad V[Y] = V[\varepsilon^2] + V[f(x)]$$

$$= V[\partial^2] + V[g(s)]$$