Show that if $B > A > 0$, where A & B are symmetric real $n \times n$ matrices.

Then $A^{-1} - B^{-1} > 0$.

Before we prove this conjecture, lets list the following theorem for cross-reference.

<u>Th1:</u>

If A and B are positive definite, then $A+B$ is also positive definite.

Th2:

Let A, B, and C be positive-definite Hermitian matrices of same size. If $D := ABC$ is Hermitian, then D is also positive definite.

<u>Th 3:</u>

Let B be a positive-definite matrix and A be non-singular and symmetric. Then ABA is positive-definite.

<u>Remark:</u> Th3 is a special case of Th2 but the matrix A in Th3 need not to be positive definite.

In the conclusion it is necessary to find identities.

$$
\begin{aligned}
A^{-1} - B^{-1} &= (A^{-1} - B^{-1}) I \\
&= (A^{-1} - B^{-1}) \left[AB^{-1} + BB^{-1} - AB^{-1} \right] \\
&= (A^{-1} - B^{-1}) \left[AB^{-1} + (B-A) B^{-1} \right] \\
&= (A^{-1} - B^{-1}) AB^{-1} + (A^{-1} - B^{-1})(B-A) B^{-1} \\
&= \underbrace{B^{-1}(B-A)B^{-1}} + \underbrace{B^{-1}(B-A)A^{-1}(B-A)B^{-1}}
\end{aligned}
$$

Now we have made connection w/ the assumption that $B - A > 0$

By Th1, it is suffice to prove that

$$B^{-1}(B-A)B^{-1} > 0 \qquad (1)$$

$$B^{-1}(B-A)A^{-1}(B-A)B^{-1} > 0 \qquad (2)$$

Now we have to prove that (1) is positive definite. Using Th3, for $B^{-1}(B-A)B^{-1}$, since $B-A>0$ and B^{-1} is nonsingular and symmetric

Proof of B^{-1} is non-singular and symmetric

Since $B>0$ then B^{-1} is also positive definite. Meaning det$(B^{-1})>0$ Thus B^{-1} is non singular. And the inverse of B^{-1} is B. Now, $(B^{-1})^T = (B^T)^{-1} = B^{-1}$. Thus B^{-1} is symmetric. Since we satisfied all the condition of Th3, then $B^{-1}(B-A)B^{-1} > 0$.

Alternative way to prove that (1) is positive-definite is to use Th2. What we need to do is to prove that (1) is symmetric.

$$(B^{-1}(B-A)B^{-1})^T = (B^{-1} - B^{-1}AB^{-1})^T = (B^{-1})^T - (B^{-1}AB^{-1})^T$$
$$= B^{-1} - (AB^{-1})^T(B^{-1})^T = B^{-1}(B^{-1})^T A^T B^{-1},$$
$$= B^{-1} - (B^{-1}AB^{-1}) = B^{-1}(B-A)B^{-1}$$

Hence, $B^{-1}(B-A)B^{-1}$ is positive definite.

Now we will prove that (2) is positive-definite matrix. Let $C_1 = B^{-1}(B-A)$ and $C_2 = (B-A)B^{-1}$. Then we can rewrite (2) as $C_1A^{-1}C_2$. Observe that $C_1^T = C_2$ and $C_2^T = C_1$. Let X to be a non-zero vector in $\mathbb{R}^n$. Also, assume that vector C_2X is non-zero. Since A is positive definite then A^{-1} is also positive definite. By the Definition of Positive definite matrix

$$(C_2X)^T A^{-1}(C_2X) = X^T C_2^T A^{-1} C_2 X = X^T C_1 A^{-1} C_2 X = X^T\left[B^{-1}(B-A)A^{-1}(B-A)B^{-1}\right]X > 0$$

Thus (2) is positive-definite. Now we have prove that (1) and (2) are positive-definite matrices, by Th1, $A^{-1} - B^{-1}$ is also positive definite. $\square$