Problem 2.2

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Abstract

This paper offers a solution to Problem 2.2 posed by Nikolai.

1 Introduction

This paper provides solution for the following problem:

Problem 1.1. *(Problem 2.2 from [1]) Let $A \in \mathbb{R}^{n\times n}$. Show that all eigenvalues of A are inside the circle $\{\lambda \quad | \quad |\lambda| < 1\}$ if and only if for any $x_0 \in \mathbb{R}^n$ and for any bounded sequence $(e_k)_k \subset \mathbb{R}^n$ the solution of the equation*

$$x_{k+1} = Ax_k + e_k, \quad k \geq 0 \tag{1}$$

is bounded in $k > 0$.

By careful observation, one can see that in order to solve Problem 1, we need to consider the spectral radius of the matrix A and apply results about spectral radius.

Problem 1.2. *(Reformulation of Problem 1.1)*
Let $A \in \mathbb{R}^{n\times n}$ with spectral radius $\rho(A)$. Then $\rho(A) < 1$ if and only if solution of (Equation 1) is bounded.

2 Preliminaries

Most of the materials and tools here are from the unit Computational Mathematics 202 by Qun Lin. So the results I can use are mostly from Numerical Linear Algebra.

2.1 Definitions

Definition 2.1. *(Vector Norm).*
A vector norm on $\mathbb{R}^n$ *is a function* $||.|| : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *such that for all* $x, y \in \mathbb{R}^n$

(i) $||x|| \geq 0$.

(ii) $||x|| = 0$ *if and only if* $x = 0$.

(iii) For all $\alpha \in \mathbb{R}$, $||\alpha x|| = |\alpha|||x||$.

(iv) $||x + y|| \leq ||x|| + ||y||$.

Definition 2.2. *(Matrix Norm).*
A Matrix norm on $\mathbb{R}^{n\times n}$ *is a function* $||.|| : \mathbb{R}^{n\times n} \to \mathbb{R}_{\geq 0}$ *such that for all* $A, B \in \mathbb{R}^{n\times n}$

(i) $||A|| \geq 0$.

(ii) $||A|| = 0$ *if and only if* $A = 0$.

(iii) For all $\alpha \in \mathbb{R}$, $||\alpha A|| = |\alpha|||A||$.

(iv) $||A + B|| \leq ||A|| + ||B||$.

(v) $||AB|| \leq ||A||||B||$.

2.2 Boundedness and Convergence of Sequences

Theorem 2.3. *A sequence* $(x_k)_k$ *is bounded in* $\mathbb{R}^n$ *if and only if there exist a non-negative constant real number* M *such that for all* $k \geq 1$, $|x_k| \leq M$.

Theorem 2.4. *Every convergent sequences are bounded.*

Remark 2.5. *The bound* M *is unique and independent of* k*. As a reminder, we are talking about infinite sequences.*

Theorem 2.6. *Every Cauchy Sequence of real or complex numbers are bounded.*

Theorem 2.7. *Every convergent sequence is a Cauchy sequence.*

2.3 Matrix Decomposition

Theorem 2.8. *(Eigen Decomposition).*
If $A_{n\times n}$ has linearly independent eigenvectors, then $A_{n\times n}$ can be written as

$$A = PDP^{-1} \tag{2}$$

where P is a diagonalizable matrix and columns are eigenvectors of A and $D = diag(\lambda_1, \lambda_2, .., \lambda_n)$ is a diagonal matrix where the diagonal entries are the eigenvalues of A.

Remark 2.9. *In (Theorem 2.8), we are asserting that the eigenvectors are linearly independent. What can it tell us about their corresponding eigenvalues?*

Corollary 2.10. *(Matrix Power).* $A^k = PD^kP^{-1}$.

Remark 2.11. *Computation of D^k is simple since this is a diagonal matrix. i.e.* $D = diag(\lambda_1, \lambda_2, .., \lambda_n) \implies D = diag(\lambda_1^k, \lambda_2^k, .., \lambda_n^k)$

Theorem 2.12. *If $\lambda_1, \lambda_2, .., \lambda_n$ are eigenvalues of matrix $A_{n\times n}$, then $\lambda_1^k, \lambda_2^k, ..., \lambda_n^k$ are the eigenvalues of matrix A^k*

Theorem 2.13. *Let $A_{n\times n}$ be an invertible real matrix. If $\lambda_1, \lambda_2, .., \lambda_n$ are the eigenvalues of A, then $\lambda_1^{-1}, \lambda_2^{-1}, ..., \lambda_n^{-1}$ are the eigenvalues of A^{-1}*

2.4 Spectral Radius and Convergent Matrices

Theorem 2.14. *If $A \in \mathbb{R}^{n\times n}$, then*

(i) $||A|| \geq \rho(A)$ *for all natural norm.*

(ii) $\rho(A) < 1 \Leftrightarrow$ *there exists a natural norm such that* $||A|| < 1$.

(iii) $\forall \epsilon > 0$, $\rho(A) \leq ||A|| \leq \rho(A) + \epsilon$

Theorem 2.15. *If $A \in \mathbb{R}^{n\times n}$, then the following are equivalent*

(i) A is a convergent matrix.

(ii) $lim_{k\longrightarrow\infty}||A^k|| = 0$, *for all natural norms.*

(iii) $\rho(A) < 1$.

(iv) $lim_{k \longrightarrow \infty} A^k x = 0$*, for all* $x \in \mathbb{R}^{n \times n}$*.*

Theorem 2.16. *(Neumann Lemma).*
If $\rho(A) < 1$*, then*

$$(I - A)^{-1} = lim_{k \longrightarrow \infty} \sum_{i=0}^{k} A^i$$

and

$$(I - A)$$

is non-singular.

Corollary 2.17. *If there is a natural norm such that* $||A|| < 1$*, then*

$$||(I - A)^{-1}|| \leq \frac{1}{1 - ||A||}$$

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3 Simpler Cases

3.1 Case $n = 1$ and $e_k = 0$

Difference Equation: $x_{k+1} = ax_k$
The constructed difference equation solution is

$$x_k = a^k x_0$$

Now the problem is to prove that x_k is bounded (i.e. bounded above and below).

Claim 3.1. *If* $|a| < 1$*, then there exists* $m, M \in \mathbb{R}$ *such that* $m \leq a^k x_0 \leq M$ *for all* $k > 0$ *and for all* $x_0 \in \mathbb{R}$

Proof. Let $k \geq 1$ be an integer and let x_0 be any real number. Assume that $|a| \leq 1$. Then

$$\begin{aligned} |a^k x_0| &= |a^k||x_0| \\ &= |a|^k |x_0| \\ &< |x_0| \end{aligned}$$

Since $|x_0| \geq 0$ and constant, by (Theorem 2.3), the sequence $a^k x_0$ is bounded. $\square$

3.2 Case $n = 1$ and $e_k = b$

Difference Equation: $x_{k+1} = ax_k + b$
Just observing and using geometric series (finite), we can see that the solution for the difference equation is

$$x_k = a^k x_0 + \frac{b(a^k - 1)}{a - 1}$$

Claim 3.2. *If* $|a| < 1$*, then*

$$x_k = a^k x_0 + \frac{b(a^k - 1)}{a - 1}$$

is bounded.

Proof. Let $k \geq 1$ be an integer and let x_0 and b be any real number. Assume that $|a| \leq 1$. Then

$$\begin{aligned} |x_k| &= |a^k x_0 + \frac{b(a^k - 1)}{a - 1}| \\ &\leq |a^k x_0| + |\frac{b(a^k - 1)}{a - 1}|, \quad \text{by triangle inequality} \\ &< |x_0| + |\frac{b(a^k - 1)}{a - 1}|, \quad \text{by Proof from Problem 2.1} \end{aligned}$$

$\frac{b(a^k-1)}{a-1}$ approaches $\frac{1}{1-a}$, as k goes to ∞, since $|a| \in (0, 1)$.

Since $|x_0| + \frac{1}{1-a} \geq 0$, by (Theorems 2.3 and 2.4), x_k is bounded. □

3.3 Case $n = 1$ and any bounded sequence e_k

Difference Equation: $x_{k+1} = ax_k + e_k$

The solution to the difference equation is

$$x_k = Ca^k + q_k$$

where q_k is the particular solution and C is constant (which can be solved by using the initial value x_0)

There are few possible expression for the bounded sequence e_k. It could be:

- List
- Closed-form (i.e. a function)
- Recurrence Equation[1]

Question 3.3. *How is the particular solution q_k related to the (assumed) bounded sequence e_k?*

Question 3.4. *Is the particular solution q_k a bounded sequence?*

3.4 Case $n \in \mathbb{N}$ and $e_k = 0$ (i.e. zero constant vector)

The solution of the difference equation of the main problem can be written as

$$x_k = A^k x_0 \tag{3}$$

Claim 3.5. *Let x_k be a sequence in $\mathbb{R}^n$. Then x_k is bounded if and only if there exists $M \in \mathbb{R}_{\geq 0}$ such that $||x_k|| \leq M$ for all k.*

Proof. This follows directly from the definition of bounded sequence in a Normed Vector Space. □

Claim 3.6. *(Special Case for (Equation 3).*
If $\rho(A) < 1$, then

$$x_k = A^k x_0 = PD^kP^{-1}x_0$$

is bounded

Proof.

$$\begin{aligned}
||x_k|| &= ||A^k x_0|| \\
&= ||AA \cdots Ax_0|| \\
&\leq ||A||||A|| \cdots ||A||||x_0|| \\
&= ||A||^k ||x_0|| \\
&< ||x_0||, \quad \text{Since } ||.|| \text{ is a natural norm and } ||A|| < 1 \text{ by Theorem 5.10 (ii)}
\end{aligned}$$

Since $||x_0|| \geq 0$, (by Claim 3.5) x_k is bounded. □

[1] Can be solve using Maple's rsolve command to get close-form formula.

3.5 Case $n \in \mathbb{N}$ and $e_k = b \in \mathbb{R}^n$

Difference Equation: $x_{k+1} = Ax_k + b$.

The solution to the above difference equation is

$$x_k = A^k x_0 + \sum_{i=0}^{k-1} A^i b$$

Remember, if the eigenvectors of A are linearly independent, then A could be written as $A = PDP^{-1}$ by Eigen Decomposition. The solution could be simplified to

$$x_k = PD^k P^{-1} x_0 + P\tilde{D}P^{-1} b$$

where $\tilde{D} = diag(\frac{\lambda_1^k - 1}{\lambda_1 - 1}, \frac{\lambda_2^k - 1}{\lambda_2 - 1}, ..., \frac{\lambda_n^k - 1}{\lambda_n - 1})$ and $\lambda_1, \lambda_2, ..., \lambda_n$ are the eigenvalues of matrix A.

Claim 3.7. *Let $\rho(A)$ be the spectral radius of matrix A. If $\rho(A) < 1$, then*

$$x_k = A^k x_0 + \sum_{i=0}^{k-1} A^i b$$

is bounded.

Proof. Assume that $\rho(A) < 1$, then

$$\begin{aligned}
||x_k|| &= ||A^k x_0 + \sum_{i=0}^{k-1} A^i b|| \\
&\leq ||A^k x_0|| + ||\sum_{i=0}^{k-1} A^i b|| \\
&\leq ||A^k|| ||x_0|| + ||\sum_{i=0}^{k-1} A^i|| + ||b|| \\
&\leq ||A||^k ||x_0|| + ||I + A + A^2 + ... + A^{k-2} + A^{k-1}|| ||b|| \\
&\leq ||A||^k ||x_0|| + (||I|| + ||A|| + ||A||^2 + ... + ||A||^{k-2} + ||A||^{k-1}) ||b||
\end{aligned}$$

Since $\rho(A) < 1$, then, by Theorem 5.10 (ii), $||A|| < 1$. Thus,

$$||x_k|| \leq \|A\| \, \|x_0\| + \frac{\|b\|}{1 - \|A\|}$$

Therefore, x_k is bounded. $\square$

4 Bounded Solution

In this section we will prove that the solution of Equation 1 is bounded.

Strategies we can use to prove ($\Rightarrow$) of the problem:

1. Construct an *explicit* (i.e. Closed-form) formula for the solution x_k (optional).
2. Prove directly $||x_k|| \leq ... \leq (...) \in \mathbb{R}_{>0}$. This "(...)" must be a constant!!!
3. Prove that x_k is convergent.
4. Prove that x_k is a *Cauchy sequence.*

We know that the solution for Equation 1 is

$$x_k = A^k x_0 + \sum_{i=0}^{k-1} A^{k-1-i} e_i \tag{4}$$

Let $S(k) := A^k x_0 + \sum_{i=0}^{k-1} A^{k-1-i} e_i$, then taking the limit of Equation 4

$$\lim_{k\to\infty} x_k = \lim_{k\to\infty} A^k x_0 + \lim_{k\to\infty} S(k)$$

Because A^k is convergent[2], we have that $\lim_{k\to\infty} A^k x_0 = 0 \in \mathbb{R}^n$. Now the problem is to prove that

$$\lim_{k\to\infty} S(k) = \lim_{k\to\infty} \sum_{i=0}^{k-1} A^{k-1-i} e_i$$

[2] In this section, we are proving the "If" part of the problem. So we are assuming that all eigenvalues of the matrix A to be inside the unit circle.

is convergent. Notice that this resembles the formula in Theorem 2.16. But that above involves e_{k-i}.

4.1 Direct Approach

The following proof uses direct approach where I would try to construct an upperbound for $\|x_k\|$, and make sure that this upperbound is a positive real constant.

Proof. We know that

$$x_k = A^k x_0 + \sum_{i=0}^{k-1} A^{k-1-i} e_i$$

, then

$$\|x_k\| = \left\| A^k x_0 + \sum_{i=0}^{k-1} A^{k-1-i} e_i \right\| \leq \|A\|^k \|x_0\| + \|S(k)\|$$

where

$$\begin{aligned} \|S(k)\| &= \left\| \sum_{i=0}^{k-1} A^{k-1-i} e_i \right\| \\ &\leq \|A\|^{k-1} e_0 + \|A\|^{k-2} e_1 + ... + \|A\|^1 e_{k-2} + \|A\|^0 e_{k-1} \end{aligned}$$

Let e_k be a bounded sequence in $\mathbb{R}^n$ i.e. there exist $M \in \mathbb{R}_{>0}$ such that $\|e_k\| \leq M$ for all $k \geq 0$. Assume that the spectral radius of matrix A is less than 1. Then $\|A\| \in (0, 1)$ by Theorem 2.14. Thus,

$$\begin{aligned} \|S(k)\| &\leq \|A\|^{k-1} M + \|A\|^{k-2} M + ... + \|A\|^1 M + \|A\|^0 M \\ &= M(\|A\|^{k-1} + \|A\|^{k-2} + ... + \|A\|^1 + \|A\|^0) \end{aligned}$$

Using finite geometric series we have that

$$\|S(k)\| \leq M \frac{\|A\|^k - 1}{\|A\| - 1}$$

Therefore,

$$\|x_k\| \leq \|A\|^k \|x_0\| + M \frac{\|A\|^k - 1}{\|A\| - 1}$$

Note that, for $\|A\|^k \|x_0\|$, we see that $\|A\|^k \|x_0\| \longrightarrow 0$ as $k \longrightarrow \infty$ since $\|A\| \in (0,1)$. So in order to achieve an upperbound, k must be 1. Thus,

$$0 \leq \|A\|^k \|x_0\| \leq \|A\| \|x_0\|$$

[3] Now, for $M\frac{\|A\|^k-1}{\|A\|-1}$, observe that this becomes $\frac{M}{1-\|A\|}$ as $k \longrightarrow \infty$. Since $\|A\| \in (0,1)$, we have

$$M\frac{\|A\|^k - 1}{\|A\| - 1} \leq \frac{M}{1 - \|A\|}$$

Combining the upperbounds, we get

$$\|x_k\| \leq \|A\| \|x_0\| + \frac{M}{1 - \|A\|}$$

Since $\|A\| \|x_0\| + \frac{M}{1-\|A\|}$ is a positive real constant, we can conclude that x_k is bounded. $\square$

4.2 Convergence approach

Lets take the norm of x_k and then take the limit as $k \longrightarrow \infty$.

$$\lim_{k \longrightarrow \infty} ||x_k|| = \lim_{k \longrightarrow \infty} ||A^k x_0 + \sum_{i=0}^{k-1} A^{k-1-i} e_i||$$

Conjecture 4.1. *Let (x_k) be a sequence in a normed vector space. If* $\lim_{k \longrightarrow \infty} ||x_k||$ *exist for all norm $||.||$, then x_k is a convergent sequence. Moreover, x_k is bounded.*

Conjecture 4.2. $\lim_{k \longrightarrow \infty} ||A^k x_0|| = 0$

Proof. Suppose that the spectral radius of matrix A is less than one, then by Theorem 2.15, the limit will be zero. $\square$

Conjecture 4.3. *Let $S(k) := \sum_{i=0}^{k-1} A^{k-1-i} e_i$. Then $S(k)$ is convergent as $k \longrightarrow \infty$.*

[3] In fact, since $\|A\| < 1$, $\|A\|^k \|x_0\| < \|x_0\|$

4.3 Induction on k

4.4 Cauchy Sequence approach

5 Spectral radius and unit circle

Now we will consider the "only if" part of Problem 1.1 (i.e. we will show that the eigenvalues of matrix A are inside the unit circle from complex plane).

As a note, I do not think that we need the assumption that x_k is bounded, but I do require that the sequence e_k to be bounded. The key ingredients here are

- Finite Geometric Series
- Theorem 2.14

Proof. Suppose that e_k is bounded i.e. there exists $M \in \mathbb{R}_{>0}$ such that $\|e_k\| \leq M$.

$$\begin{aligned}\left\|\sum_{i=0}^{k-1} A^{k-1-i} e_i\right\| &\leq \sum_{i=0}^{k-1} \|A\|^{k-1-i} \|e_i\| \\ &\leq M \sum_{i=0}^{k-1} \|A\|^{k-1-i}\end{aligned}$$

Using Finite Geometric Series, we have that

$$\left\|\sum_{i=0}^{k-1} A^{k-1-i} e_i\right\| \leq M \frac{\|A\|^k - 1}{\|A\| - 1}$$

The inequality holds when $\|A\| < 1$. Thus $\rho(A) < 1$ i.e. the eigenvalues are inside the unit circle. $\square$

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References

[1] N. Dokuchaev. Challenging Problems for Students. http://maths.curtin.edu.au/local/docs/nikolai/probs-nd.pdf