18-660: Numerical Methods for Engineering Design and Optimization

Homework 3

Issued: September 12
Due: September 19 (midnight, Pittsburgh time)

Please submit the PDF file of your solution to the course web site before midnight on the due day.

Problem 1: Linear Equation Solver

Apply Gaussian elimination without pivoting to solve the following linear equation by hand calculation. Next, apply LU decomposition to solve the same linear equation by hand calculation. Show all steps in detail.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 15 \\ 25 \end{bmatrix} \quad (1)$$

$x_1 + 2x_2 + 3x_3 = 6$
$4x_1 + 5x_2 + 6x_3 = 15$
$7x_1 + 8x_2 + 10x_3 = 25$

Gaussian Elimination w/o Pivoting:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 6 \\ 0 & 6 & 11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \\ 17 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 6 & 11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 17 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$x_1 = 1$
$x_2 = 1$
$x_3 = 1$

LU Decomposition:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix} = \begin{bmatrix} l_{11} & & \\ l_{21} & l_{22} & \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ & 1 & u_{23} \\ & & 1 \end{bmatrix}$$

$1 = l_{11} \cdot 1$
$4 = l_{21} \cdot 1$
$7 = l_{31} \cdot 1$

$$\begin{bmatrix} \boxed{1} & 2 & 3 \\ \boxed{4} & 3 & 6 \\ \boxed{7} & 8 & 10 \end{bmatrix}$$

$2 = l_{11} \cdot u_{12} = u_{12} \quad u_{12} = 2$
$3 = l_{11} \cdot u_{13} = u_{13} \quad u_{13} = 3$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix} \Rightarrow$$

$5 = l_{21} u_{12} + l_{22} = 4 \cdot 2 + l_{22} \quad l_{22} = -3$
$8 = l_{31} \cdot u_{12} + l_{32} = 7 \cdot 2 + l_{32} \quad l_{32} = -6$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 4 & -3 & 6 \\ 7 & -6 & 10 \end{bmatrix}$$

$6 = l_{21} \cdot u_{13} + l_{22} \cdot u_{23} = 4 \cdot 3 + 3 \cdot u_{23} \quad u_{23} = 2$
$10 = l_{31} u_{13} + l_{32} \cdot u_{23} + l_{33} = 7 \cdot 3 - 6 \cdot 2 + l_{33} \quad l_{33} = 1$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 4 & -3 & 2 \\ 7 & -6 & 1 \end{bmatrix}$$

LU Decomposition (Continued):

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & -3 & 2 \\ 7 & -6 & 1 \end{bmatrix}$$

$$U \cdot x = V$$

$$LV = B$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} V_1 - 2V_2 \\ V_2 \\ V_3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} V_1 - 2V_2 + V_3 \\ V_2 - 2V_3 \\ V_3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 4 & -3 & 0 \\ 7 & -6 & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 15 \\ 25 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & -6 & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -9 \\ -17 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 1 \end{bmatrix} \quad \begin{matrix} V_1 = 6 \\ V_2 = 3 \\ V_3 = 1 \end{matrix}$$

$$\Rightarrow \begin{matrix} x_1 = V_1 - 2V_2 + V_3 \\ x_2 = V_2 - 2V_3 \\ x_3 = V_3 \end{matrix} \qquad \begin{matrix} x_1 = 6 - 2 \cdot 3 + 1 = 1 \\ x_2 = 3 - 2 = 1 \\ x_3 = 1 \end{matrix} \qquad x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \checkmark$$