18-660: Numerical Methods for Engineering Design and Optimization

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Electrical & Computer ENGINEERING

Overview

- **Constrained Optimization**
  - Inequality constraint
  - Interior point method
  - Feasibility problem

Inequality Constrained Optimization

$$\begin{aligned} \min_{X} \quad & f(X) \\ \text{S.T.} \quad & g_m(X) \le 0 \quad (m = 1,2,\cdots,M) \\ & AX = B \end{aligned}$$

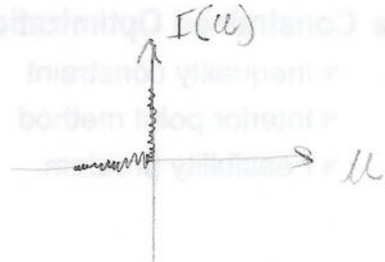
- **Equality constraint can be written as two inequality constraints**

$$g(X) = 0 \quad \Rightarrow \quad \begin{aligned} g(X) &\le 0 \\ -g(X) &\le 0 \end{aligned}$$

Indicator Function

- **Define** **indicator function**

$$\begin{aligned} \min_{X} \quad & f(X) \\ \text{S.T.} \quad & g_m(X) \le 0 \quad (m = 1,2,\cdots,M) \\ & AX = B \end{aligned}$$



$$I(U) = \begin{cases} 0 & (U \le 0) \\ +\infty & (U > 0) \end{cases}$$

$$\begin{aligned} \min_{x} \quad & f(x) + \sum_{M=1}^{M} I[g_m(x)] \\ \text{S.T.} \quad & Ax = B \end{aligned}$$

Indicator Function

$$\min_X \quad f(X)+\sum_{m=1}^{M} I[g_m(X)]$$
$$\text{S.T.} \quad AX = B$$

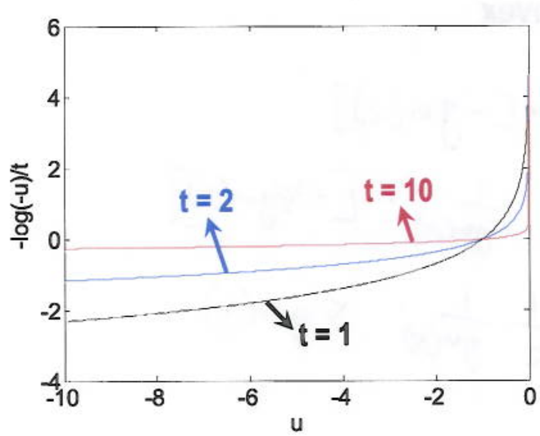
- **Result in a new optimization problem with linear constraints only**
  - However, the indicator function I(•) is not smooth
  - We cannot directly apply Lagrange multiplier and calculate 1st/2nd-order derivatives
- **New idea: approximate I(•) by a smooth function**

Logarithmic Barrier

- **Approximate I(•) by logarithmic barrier**

$$I(u) \approx -1/t \cdot \log(-u) \quad (u \le 0)$$

  - where t > 0 is a user-defined parameter



Logarithmic barrier converges to I(•) iff t → ∞

Logarithmic Barrier

$$\min_X \quad f(X)+\sum_{m=1}^{M} I[g_m(X)] \qquad I(u)\approx -1/t\cdot\log(-u) \quad (u\le 0)$$
$$\text{S.T.} \quad AX=B$$

$$\min \quad f(x) \ \frac{-1}{t} \sum_{M=1}^{M} [\log \tau \cdot g_m(x)]$$

$$\text{S.T.} \quad Ax=B$$

- **Open question: does the new optimization preserve convexity?**

Logarithmic Barrier

$$\min_X \quad f(X)$$
$$\text{S.T.} \quad g_m(X)\le 0 \quad (m=1,2,\cdots,M)$$
$$AX=B$$

$$\Rightarrow \quad \min_X \quad f(X)-\frac{1}{t}\cdot\sum_{m=1}^{M}\log[-g_m(X)]$$
$$\text{S.T.} \quad AX=B$$

- **If f(X) and g_m(X) are convex**

$$\varphi_m(x) = \frac{-1}{t} \log[-g_m(x)]$$

$$\nabla \varphi_m(x) = \frac{-1}{t} \quad \frac{1}{-g_m(x)} \cdot [-\nabla g_m(x)]$$

$$= \frac{-1}{t} \frac{1}{g_m(x)} \quad \nabla g_m(x)$$

Logarithmic Barrier

$$\min_X \; f(X) \quad \text{S.T.} \quad g_m(X) \le 0 \quad (m = 1,2,\cdots,M), \quad AX = B$$

$$\Rightarrow \quad \min_X \; f(X) - \frac{1}{t}\cdot\sum_{m=1}^{M}\log[-g_m(X)] \quad \text{S.T.} \quad AX = B$$

$Q = [P][P^T]$

- **If f(X) and g_m(X) are convex**

$x^T Q x = (x^T P)(P^T x) = (P^T x)^2$

φ is convex

$$\nabla\varphi_m(X) = -\frac{1}{t}\cdot\frac{1}{g_m(X)}\cdot\nabla g_m(X)$$

$$\underbrace{\nabla^2\varphi_m(X)}_{\text{Positive semi-definite}} = -\frac{1}{t}\cdot\frac{1}{-[g_m(X)]^2}\cdot\nabla g_m(X)\nabla g_m(X)^T - \frac{1}{t}\cdot\frac{1}{g_m(X)}\cdot\nabla^2 g_m(X)$$

$$= \frac{1}{t}\cdot\frac{1}{[g_m(X)]^2}\cdot\underbrace{\nabla g_m(X)\nabla g_m(X)^T}_{\text{Positive semi-definite}} - \frac{1}{t}\cdot\underbrace{\frac{1}{g_m(X)}}_{\text{Negative}}\cdot\underbrace{\nabla^2 g_m(X)}_{\text{Positive semi-definite}}$$

Logarithmic Barrier

$$\min_X \; f(X) \quad \text{S.T.} \quad g_m(X) \le 0 \quad (m = 1,2,\cdots,M), \quad AX = B$$

$$\Rightarrow \quad \min_X \; f(X) - \frac{1}{t}\cdot\sum_{m=1}^{M}\log[-g_m(X)] \quad \text{S.T.} \quad AX = B$$

- **If f(X) and g_m(X) are convex**

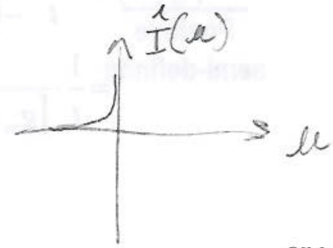
$$\varphi_m(X) = -\frac{1}{t}\cdot\log[-g_m(X)]$$

$$\text{Convex} = \begin{cases} \min\limits_{x} \; f(x) + \sum_{M=1}^{M} \varphi_m(x) \rightarrow \text{Convex} \\ \text{S.T.} \quad Ax = B \end{cases}$$

Interior Point Method

$$\begin{aligned}\min_X \quad & f(X) \\ \text{S.T.} \quad & g_m(X) \le 0 \quad (m = 1,2,\cdots,M) \\ & AX = B\end{aligned} \quad \Rightarrow \quad \begin{aligned}\min_X \quad & f(X) - \frac{1}{t}\cdot\sum_{m=1}^{M}\log[-g_m(X)] \\ \text{S.T.} \quad & AX = B\end{aligned}$$

- **Interior point method is also referred to as barrier method**
  - Step 1: select an initial value of t and an initial guess $X^{(0)}$
  - Step 2: solve linear equality constrained nonlinear optimization to find the optimal solution X^*
  - Step 3: $X^{(0)} = X^*$ and $t = \beta t$ (β is typically 10~20)
  - Repeat Step 2~3 until t is sufficiently large



Feasibility Problem

$$\begin{aligned}\min_X \quad & f(X) \\ \text{S.T.} \quad & g_m(X) \le 0 \quad (m = 1,2,\cdots,M) \\ & AX = B\end{aligned} \quad \Rightarrow \quad \begin{aligned}\min_X \quad & f(X) - \frac{1}{t}\cdot\sum_{m=1}^{M}\log[-g_m(X)] \\ \text{S.T.} \quad & AX = B\end{aligned}$$

- **When we iteratively solve linear equality constrained nonlinear optimization, $X^{(0)}$ must be feasible**

$$AX^{(0)} = B \quad g_m\left[X^{(0)}\right] \le 0 \quad (m = 1,2,\cdots,M)$$

  - Otherwise, $\log\{-g_m[X^{(0)}]\}$ does not have a numerical value
  - We cannot move to the next iteration step

Feasibility Problem

$$\min_X \ f(X) \quad \text{S.T.} \ g_m(X) \le 0 \quad (m = 1,2,\cdots,M), \quad AX = B \quad \Rightarrow \quad \min_X \ f(X) - \frac{1}{t}\cdot\sum_{m=1}^{M}\log[-g_m(X)] \quad \text{S.T.} \ AX = B$$

- **How do we come up with an initial feasible solution?**

$$\underbrace{AX^{(0)} = B}_{\text{Easy}} \quad \underbrace{g_m\left[X^{(0)}\right] \le 0 \quad (m = 1,2,\cdots,M)}_{\text{Difficult}}$$

- **How do we even know that the optimization is feasible?**
  - Not all optimization problems have a feasible solution

Phase I Method

$$\min_X \ f(X) \quad \text{S.T.} \ g_m(X) \le 0 \quad (m = 1,2,\cdots,M), \quad AX = B \quad \Rightarrow \quad \min_X \ f(X) - \frac{1}{t}\cdot\sum_{m=1}^{M}\log[-g_m(X)] \quad \text{S.T.} \ AX = B$$

- **We must do another optimization to decide**
  - Is the optimization feasible?
  - If yes, find one of the feasible solutions

- **This preprocessing step is called phase I, and the interior point method should be applied for phase II**

Phase I Method

$$\min_X \; f(X)$$
$$\text{S.T.} \quad g_m(X) \le 0 \quad (m = 1,2,\cdots,M)$$
$$AX = B$$

Phase II problem

$\min_{x,s}$ s

S.T. $g_m(x) \le s$ $(m=1,2,\ldots M)$

$Ax = B$

Phase I problem

- **Once optimal point [X* s*] is found for phase I problem, we know:**
  - If s* > 0
    - Phase II problem is not feasible
  - If s* ≤ 0
    - Phase II problem is feasible
    - X* is one of the feasible solutions
    - Starting from X*, apply interior point method to solve phase II problem

Phase I Method

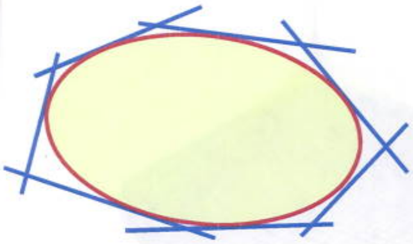
$$\min_X \; f(X) \qquad\qquad \min_{X,s} \; s$$
$$\text{S.T.} \quad g_m(X) \le 0 \quad (m = 1,2,\cdots,M) \quad \Rightarrow \quad \text{S.T.} \quad g_m(X) \le s \quad (m = 1,2,\cdots,M)$$
$$AX = B \qquad\qquad AX = B$$

Phase II problem **Phase I problem**

- **Phase I problem can be easily solved**
  - Select an initial $X^{(0)}$ that satisfies $AX^{(0)} = B$
  - Calculate $g_m[X^{(0)}]$ where m = 1,2,...,M
  - Determine the maximum value of $g_m[X^{(0)}]$, denoted as g_{MAX}
  - Set $s^{(0)} = g_{MAX}$
  - Starting from $[X^{(0)}; s^{(0)}]$, apply interior point method to solve phase I problem and find its optimal solution [X*; s*]

Semidefinite Programming

- **Inequality constraints are not always represented as g(X) ≤ 0**
- **Example: maximum inscribed ellipsoid**
  - Generalized inequality can be solved by semidefinite programming



$$
\begin{aligned}
\max_{W,d} \quad & \det(W) \\
S.T. \quad & \|WB_k\|_2 + B_k^T \cdot d + C_k \le 0 \quad (k = 1, 2, \cdots, K) \\
& W = W^T \\
& \boxed{W \succ= 0}
\end{aligned}
$$

→ W is positive semi-definite

Generalized inequality

Semidefinite Programming

$$
\begin{aligned}
\min_{X} \quad & f(X) \\
& g(X) = \underbrace{x_1F_1 + x_2F_2 + \cdots + x_MF_M}_{\text{Positive semidefinite matrix}} \succ= 0
\end{aligned}
$$

Symmetric

- **Define logarithmic barrier function**

$$
\varphi(X) = -\frac{1}{t} \cdot \log\left[\det\left(x_1F_1 + x_2F_2 + \cdots + x_MF_M\right)\right]
$$

- **φ(X) is convex**
  - log[det(•)] is concave
  - −log[det(•)] is convex

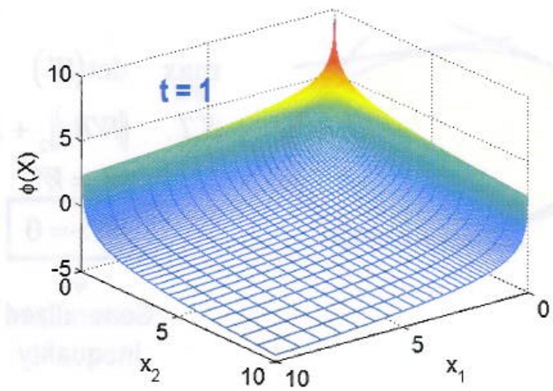
Semidefinite Programming

$$\varphi(X) = -\frac{1}{t}\log\left[\det\left(\underbrace{x_1F_1 + x_2F_2 + \cdots + x_MF_M}_{g(X)}\right)\right]$$

- **φ(X) approaches infinite, if g(X) becomes indefinite**

$$g(X) = \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix}$$

$x_1 \geq 0$ and $x_2 \geq 0$ so that g(X) is positive semidefinite



Semidefinite Programming

$$\min_{X} \quad f(X) \qquad x_1F_1 + \cdots + x_MF_M \succ= 0$$

Phase II problem

$$\min_{X,s} \quad s \qquad x_1F_1 + \cdots + x_MF_M + sI \succ= 0$$

Phase I problem

- **Phase I method**
  - Arbitrarily select an initial $X^{(0)}$
  - Select a sufficiently large $s^{(0)}$ so that phase I constraint is feasible
  - Starting from $[X^{(0)}; s^{(0)}]$, apply interior point method to solve phase I problem and find its optimal solution $[X^*; s^*]$

What value of $s^{(0)}$ is sufficiently large?

If $s^* \leq 0$: Phase II problem is feasible

If $s^* > 0$: phase II problem is infeasible.

Semidefinite Programming

- **A matrix F is diagonally dominant if**

$$F = \begin{bmatrix} F_{11} & F_{12} & F_{13} & \cdots \\ F_{21} & F_{22} & F_{23} & \cdots \\ F_{31} & F_{32} & F_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$\begin{gathered} |F_{11}| \geq |F_{12}| + |F_{13}| + \cdots \\ |F_{22}| \geq |F_{21}| + |F_{23}| + \cdots \\ |F_{33}| \geq |F_{31}| + |F_{32}| + \cdots \\ \vdots \end{gathered}$$

- **A matrix F is positive semidefinite, if**
  - F is symmetric, and
  - F is diagonally dominant, and
  - All diagonal elements are non-negative

Semidefinite Programming

$$\underbrace{x_1^{(0)}F_1 + \cdots + x_M^{(0)}F_M}_{g(X)} + s^{(0)}I \succ= 0$$

- **Select a sufficiently large value of $s^{(0)}$ so that the matrix $g[X^{(0)}] + s^{(0)}I$ is diagonally dominant (hence, positive semidefinite)**

$$\underbrace{\begin{bmatrix} g_{11} & g_{12} & g_{13} & \cdots \\ g_{21} & g_{22} & g_{23} & \cdots \\ g_{31} & g_{32} & g_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}}_{g(x^{(0)})} + \underbrace{\begin{bmatrix} s^{(0)} & & & \\ & s^{(0)} & & \\ & & s^{(0)} & \\ & & & \ddots \end{bmatrix}}_{s^{(0)} \cdot I}$$

Summary

- **Constrained optimization**
  - Inequality constraint
  - Interior point method
  - Feasibility problem

Carnegie Mellon

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Overview

- Duality
  - Lagrange dual
  - KKT condition

Constrained Nonlinear Optimization

- **Standard form for constrained nonlinear optimization**

$$\begin{aligned} \min_{X} \quad & f(X) \\ S.T. \quad & g_m(X) \le 0 \quad (m = 1,2,\cdots,M) \\ & h_n(X) = 0 \quad (n = 1,2,\cdots,N) \end{aligned}$$

- **We do not write equality constraint h(X) = 0 as two inequality constraints h(X) ≥ 0 and h(X) ≤ 0 in this lecture**
  - Equality and inequality constraints are handled differently in duality theory

Lagrangian

$$\begin{aligned} \min_{X} \quad & f(X) \\ S.T. \quad & g_m(X) \le 0 \quad (m = 1,2,\cdots,M) \\ & h_n(X) = 0 \quad (n = 1,2,\cdots,N) \end{aligned}$$

- **Define the Lagrangian**

$$L(x,U,V) = f(x) + \sum_{M=1}^{M} u_M g_m(x) + \sum_{N=1}^{N} V_N h_N(x)$$

lagrange multipliers

  - L(X,U,V) is a nonlinear function of X, but it is linearly dependent of U and V

Lagrange Dual Function

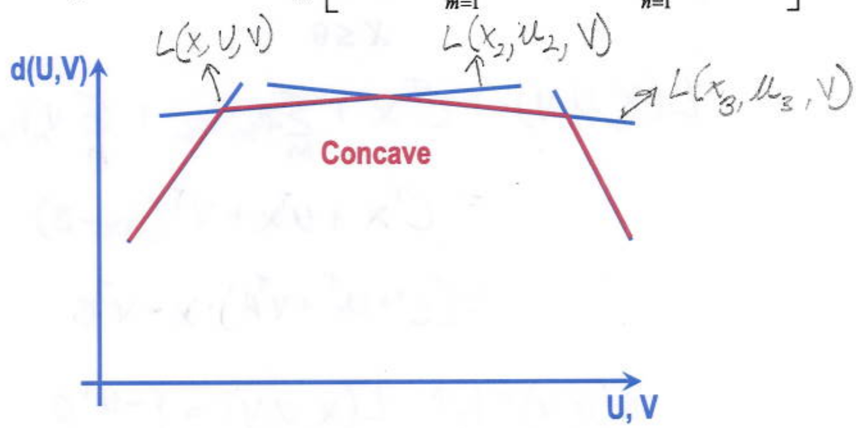
- **Define** **Lagrange dual function**

$$L(U,V) = \inf_X L(X,U,V)$$
$$= \inf \left[f(x) + \sum_{M=1}^{M} u_m g_m(x) + \sum_{N=1}^{N} v_N h_N(x) \right.$$

- **At any given X, L(X,U,V) is a linear function of U and V**
  - d(U,V) is the minimum of an infinite number of linear functions

Lagrange Dual Function

$$d(U,V) = \inf_X L(X,U,V) = \inf_X \left[f(X) + \sum_{m=1}^{M} u_m g_m(X) + \sum_{n=1}^{N} v_n h_n(X) \right]$$



- **For any constrained nonlinear optimization, the Lagrange dual function d(U,V) is concave**

Lower Bound Property

$$\min_X \quad f(X)$$
$$\text{S.T.} \quad g_m(X) \le 0 \quad (m = 1,2,\cdots,M)$$
$$h_n(X) = 0 \quad (n = 1,2,\cdots,N)$$

$$d(U,V) = \inf_X \left[f(X) + \sum_{m=1}^{M} u_m g_m(X) + \sum_{n=1}^{N} v_n h_n(X) \right]$$

- **If X* is the optimal solution and U ≥ 0, then**

$g_m(x^*) \le 0 \qquad h_n(x^*) = 0$

$d(U,V) \le L(x^*, U, V)$

$= f(x^*) + \sum_{m=1}^{M} u_m g_m(x^*) + \sum_{n=1}^{N} v_n \underbrace{h_n(x^*)}_{=0}$

$= f(x^*) + \sum_{m=1}^{M} \underbrace{u_m}_{\ge 0} \underbrace{g_m(x^*)}_{\le 0} \qquad u_m g_m(x^*) \le 0$

$\le f(x^*) \qquad d(U,V)$ is lower bound of $f(x^*)$

Linear Programming Example

$$\min_X \quad C^T X$$
$$\text{S.T.} \quad AX = B \quad \to Ax - B = 0$$
$$X \le 0$$

$L(x, U, V) = C^T x + \sum_m u_m x_m + \sum_n v_n h_n(x)$

$= C^T x + U^T x + V^T (Ax - B)$

$= (C^T + U^T + V^T A) \cdot x - V^T B$

$d(U,V) = \inf_x L(x,U,V) = \begin{cases} -U^T B & (C^T + U^T + V^T A) = 0 \\ -\infty & \text{(otherwise)} \end{cases}$

Lagrange Dual Problem

- **Lagrange dual problem is defined as**

Dual problem

$$\min_X \; f(X)$$
$$\text{S.T.} \;\; g_m(X) \le 0 \quad (m = 1,2,\cdots,M)$$
$$h_n(X) = 0 \quad (n = 1,2,\cdots,N)$$

Primal problem

$$\max_{U,V} \; d(U,V)$$
$$\text{S.T.} \;\; U \ge 0$$
$$d(U^*,V^*) \le f(X^*)$$

- **Linear programming example**

$$\min_X \; C^T X$$
$$\text{S.T.} \;\; AX = B$$
$$X \le 0$$

primal problem

$$\max_{U,V} \; -V^T B$$
$$\text{S.T.} \;\; C^T + U^T + V^T A = 0$$
$$U \ge 0$$

dual problem

Weak Duality

$$\min_X \; f(X)$$
$$\text{S.T.} \;\; g_m(X) \le 0 \quad (m = 1,2,\cdots,M)$$
$$h_n(X) = 0 \quad (n = 1,2,\cdots,N)$$

Primal problem

$$\max_{U,V} \; d(U,V)$$
$$\text{S.T.} \;\; U \ge 0$$

Dual problem

- **Weak duality**
  - X* is primal optimum
  - U* and V* are dual optimum
  - f(X*) ≥ d(U*,V*) (Lagrange dual function is the lower bound)

- **Weak duality holds for any optimization problem (either convex or non-convex)**

Strong Duality

$$\begin{aligned} &\min_X \quad f(X) \\ &\text{S.T.} \quad g_m(X) \le 0 \quad (m = 1,2,\cdots,M) \\ &\qquad\;\; h_n(X) = 0 \quad (n = 1,2,\cdots,N) \end{aligned}$$

Primal problem

$$\begin{aligned} &\max_{U,V} \quad d(U,V) \\ &\text{S.T.} \quad U \ge 0 \end{aligned}$$

Dual problem

- **Strong duality**
  - X* is primal optimum
  - U* and V* are dual optimum
  - f(X*) = d(U*,V*) (duality gap is zero)
- **Strong duality does not hold in general, but it usually holds for convex problems**
  - Conditions that guarantee strong duality in convex problems are referred to as constraint qualifications

Slater's Constraint Qualification

- **Strong duality holds for convex optimization**

$$\begin{aligned} &\min_X \quad f(X) \\ &\text{S.T.} \quad g_m(X) \le 0 \quad (m = 1,2,\cdots,M) \\ &\qquad\;\; AX = B \end{aligned}$$

Equality constraints must be linear

  - if it is strictly feasible, i.e.,

$$\begin{aligned} &g_m(X) < 0 \quad (m = 1,2,\cdots,M) \\ &AX = B \end{aligned}$$

- **Sufficient but not necessary condition**
  - Many other constraint qualifications exist

Quadratic Programming Example

$$\min_X \quad X^T AX + 2B^T X$$
$$\text{S.T.} \quad X^T X \le 1$$

Primal problem

$$\max_{t,u} \quad -t-u$$
$$\text{S.T.} \quad \begin{bmatrix} A+uI & B \\ B^T & t \end{bmatrix} \succ= 0$$
$$u \ge 0$$

Dual problem

- **Primal problem is not convex, if A is not positive semidefinite**
- **Dual problem is convex semidefinite programming**
- **Strong duality holds even if primal problem is not convex**
  - Dual problem can be solved both efficiently and robustly due to convexity

Complementary Slackness

$$\min_X \quad f(X)$$
$$\text{S.T.} \quad g_m(X) \le 0 \quad (m = 1,2,\cdots,M)$$
$$h_n(X) = 0 \quad (n = 1,2,\cdots,N)$$

Primal problem

$$\max_{U,V} \quad d(U,V)$$
$$\text{S.T.} \quad U \ge 0$$

Dual problem

- **Assume that strong duality holds, X* is primal optimum, and U* and V* are dual optimum**

$$d(U^*, V^*) = \inf_X \left[f(x) + \sum_m u_m^* g_m(x) + \sum_n V_n^* h_n(x) \right]$$
$$\parallel$$
$$f(x^*) \le L(x^*, U^*, V^*)$$
$$= f(x^*) + \sum_m \underbrace{u_m^*}_{\ge 0} \underbrace{g_m(x^*)}_{\le 0} + \sum_n V_n^* \underbrace{h_n(x^*)}_{=0}$$
$$= f(x^*) + \sum_m u_m^* g_m(x^*)$$
$$\le f(x^*)$$

Complementary Slackness

$$\min_X \quad f(X)$$
$$\text{S.T.} \quad g_m(X) \le 0 \quad (m = 1,2,\cdots,M)$$
$$h_n(X) = 0 \quad (n = 1,2,\cdots,N)$$

Primal problem

$$\max_{U,V} \quad d(U,V)$$
$$\text{S.T.} \quad U \ge 0$$

Dual problem

$$f(X^*) \le f(X^*) + \sum_{m=1}^{M} u_m^* g_m(X^*) \le f(X^*)$$

$\sum_m u_m^* g_m(x^*) = 0$ $\qquad u_m^* \ge 0$

$g_m(x^*) \le 0$

$\rightarrow u_m^* g_m(x^*) = 0$ $\qquad u_m^* g_m(x^*) \le 0$

- $u_m^* > 0 \rightarrow g_m(X^*) = 0$ (active constraint)
- $g_m(X^*) < 0 \rightarrow u_m^* = 0$ (inactive constraint)

Karush-Kuhn-Tucker (KKT) Condition

$$\min_X \quad f(X)$$
$$\text{S.T.} \quad g_m(X) \le 0 \quad (m = 1,2,\cdots,M)$$
$$h_n(X) = 0 \quad (n = 1,2,\cdots,N)$$

Primal problem

$$\max_{U,V} \quad d(U,V)$$
$$\text{S.T.} \quad U \ge 0$$

Dual problem

- **If strong duality holds and X*, U* and V* are optimal, then**

$$g_m(X^*) \le 0 \quad (m = 1,2,\cdots,M)$$
$$h_n(X^*) = 0 \quad (n = 1,2,\cdots,N)$$

Primal constraints

$U^* \ge 0$ Dual constraints

$u_m^* g_m(X^*) = 0 \quad (m = 1,2,\cdots,M)$ Complementary slackness $\qquad \nabla L(x^*, u^*, v^*) = 0$

$$\nabla f(X^*) + \sum_{m=1}^{M} u_m^* \cdot \nabla g_m(X^*) + \sum_{n=1}^{N} v_n^* \cdot \nabla h_n(X^*) = 0$$

X* minimizes L(X,U*,V*)

KKT Condition for Convex Problem

$$\begin{aligned}\min_{X}\quad & f(X)\\ \text{S.T.}\quad & g_m(X)\le 0 \quad (m=1,2,\cdots,M)\\ & h_n(X)=0 \quad (n=1,2,\cdots,N)\end{aligned}$$

Primal problem

$$\Rightarrow$$

$$\begin{aligned}\max_{U,V}\quad & d(U,V)\\ \text{S.T.}\quad & U\ge 0\end{aligned}$$

Dual problem

- **Given a convex problem with strong duality, X*, U* and V* are optimal if and only if they satisfy the KKT condition**
- **Many convex programming algorithms are derived from KKT**

Boyd and Vandenberghe, "Convex Optimization," Cambridge University Press, 2004

Summary

- **Duality**
  - Lagrange dual
  - KKT condition