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Bootstrap-based testing of equality of mean functions or equality of covariance operators for functional data

BY E. PAPARODITIS AND T. SAPATINAS

Department of Mathematics and Statistics, University of Cyprus, P.O. Box 20537, CY 1678 Nicosia, Cyprus

stathisp@ucy.ac.cy t.sapatinas@ucy.ac.cy

SUMMARY

We investigate the properties of a simple bootstrap method for testing the equality of mean functions or of covariance operators in functional data. Theoretical size and power results are derived for certain test statistics, whose limiting distributions depend on unknown infinite-dimensional parameters. Simulations demonstrate good size and power of the bootstrap-based functional tests.

Some key words: Bootstrap; Covariance operator; Functional data; Hypothesis test; Mean function.

1. INTRODUCTION

Functional data are collected in many fields of research; see, e.g., Ramsay & Silverman (2005), Ferraty & Vieu (2006) and Horváth & Kokoszka (2012). When working with more than one population of functional data, testing the equality of certain characteristics of the distributions between the populations, like their mean functions or their covariance operators, is widely discussed in the literature. Benko et al. (2009) and Horváth & Kokoszka (2012, Ch. 5) have developed functional testing procedures for the equality of two or more mean functions. Panaretos et al. (2010), Fremdt et al. (2012) and an unpublished 2014 paper by G. Boente, D. Rodriguez and M. Sued (ArXiv:1404.7080) have developed tests of the equality of two covariance operators. Critical points for these tests are typically obtained by means of asymptotic approximations.

To improve such asymptotic approximations, bootstrap-based functional testing approaches have also been considered. Benko et al. (2009) considered testing the equality of two mean functions and used the bootstrap to obtain critical values, but their procedure is tailored to the test statistic used. Similarly, in the two-sample problem, Zhang et al. (2010) have considered a bootstrap procedure that generates functional pseudo-observations that do not satisfy the null hypothesis and whose validity depends on the test statistic. A different idea for improving asymptotic approximations has been used by the paper by G. Boente et al. (ArXiv:1404.7080) in the context of testing the equality of covariance operators where a bootstrap procedure has been used to calibrate the critical values of the test. Again, this bootstrap is tailored to the test statistic considered. Finally, permutation tests for equality of covariance operators applied to different distance measures between two covariance functions have been considered by Pigoli et al. (2014).

In order to test equality of mean functions or of covariance operators, we investigate the properties of a simple bootstrap-based procedure which is potentially applicable to different test statistics and to several populations. The basic idea has been previously used in the finite-dimensional set-up; see, e.g., Efron & Tibshirani (1993) and Davison & Hinkley (1997). In our functional set-up, we bootstrap the observed functional dataset in such a way that the pseudo-observations satisfy the null hypothesis. This generates pseudo-functional observations within the different populations which have identical mean functions or identical covariance operators, depending on the null hypothesis. A test statistic is then calculated using the pseudo-observations and its distribution is evaluated by Monte Carlo simulation. We show the consistency of the bootstrap procedure in estimating the null distribution of certain two-sample test statistics. In

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particular, we demonstrate the advantages of the bootstrap by focusing on test statistics based on natural global distance measures, the asymptotic distributions of which are awkward due to their dependence on unknown infinite-dimensional parameters; see Benko et al. (2009), Horváth & Kokoszka (2012, Ch. 5) and the paper by G. Boente et al. (ArXiv:1404.7080). This has motivated many researchers to consider test statistics that are based on finite-dimensional projections, leading to more tractable asymptotic distributions; see Horváth & Kokoszka (2012, Ch. 5), Panaretos et al. (2010) and Fremdt et al. (2012). However, our bootstrap procedure does not require the choice of a truncation parameter and the test statistics have good power. Our proposal yields accurate approximations in finite-sample situations. We evaluate the finite-sample behaviour of our functional tests by simulation and compare our results with other proposals.

2. Bootstrap-based functional testing

2·1. *Preliminaries*

We work with random functions X defined on a probability space $(\Omega, \mathcal{A}, P)$ with values in the separable Hilbert space $L^2 = L^2(\mathcal{I}, R)$, the space of square-integrable R-valued functions on the compact interval $\mathcal{I} = [0, 1]$. We assume that $E(\|X\|^2) < \infty$ and denote by $\mu = E(X)$ the mean function of X, i.e., the unique function $\mu \in L^2$ satisfying $E(\langle X, x\rangle) = \langle \mu, x\rangle$, $x \in L^2$. We also denote by $\mathcal{C} = E\{(X - \mu) \otimes (X - \mu)\}$ the covariance operator of X, where the operator $u \otimes v : L^2 \mapsto L^2$ is defined as $(u \otimes v)w = \langle v, w\rangle u$, $u, v \in L^2$. Notice that $\mathcal{C}(f)(t) = \int_{\mathcal{I}} C(t, s) f(s) \mathrm{d}s$, where $C(t, s) = E[\{X(t) - \mu(t)\}\{X(s) - \mu(s)\}]$, $t, s \in \mathcal{I}$, is the covariance function of X, i.e., $\mathcal{C}$ is an integral operator with kernel C; $\mathcal{C}$ is also a Hilbert–Schmidt operator. Denote by $\|\cdot\|_S$ the Hilbert–Schmidt norm.

It is assumed that we have available a collection of random functions satisfying

$$X_{i,j}(t) = \mu_i(t) + \epsilon_{i,j}(t), \quad i = 1, \ldots, K, \; j = 1, \ldots, n_i, \; t \in \mathcal{I},$$

where K $(2 \leqslant K < \infty)$ denotes the number of populations, n_i denotes the number of observations from the ith population and $N = \sum_{i=1}^{K} n_i$ denotes the total number of observations. We also assume that the observations are all independent and, for each $i = 1, \ldots, K$, the errors $\epsilon_{i,1}, \ldots, \epsilon_{i,n_i}$ are L^2-valued random samples with $E(\epsilon_{i,j}) = 0$ and $E(\|\epsilon_{i,j}\|^2) < \infty$. Let $X_N = \{X_{i,j} : i = 1, \ldots, K, \; j = 1, \ldots, n_i\}$ and let $\mathcal{C}_i = E\{(X_{i,j} - \mu_i) \otimes (X_{i,j} - \mu_i)\}$ $(j = 1, \ldots, n_i; i = 1, \ldots, K)$ be the covariance operator of the ith population.

2·2. *Testing the equality of covariance operators*

We are interested in testing the following hypothesis

$$H_0 : \mathcal{C}_1 = \cdots = \mathcal{C}_K \text{ versus } H_1 : \text{there exist } k, l \in \{1, \ldots, K\}, \; k \neq l, \text{ with } \mathcal{C}_k \neq \mathcal{C}_l. \tag{1}$$

The equality under H_0 means that $\|\mathcal{C}_k - \mathcal{C}_l\|_S = 0$ for any pair of indices (k, l), $k \neq l$, while under H_1 we have $\|\mathcal{C}_k - \mathcal{C}_l\|_S > 0$ for at least one pair of indices (k, l), $k \neq l$.

Let T_N be a test statistic for testing H_0. Suppose that T_N rejects H_0 when $T_N > d_{N,\alpha}$, where, for $\alpha \in (0, 1)$, $d_{N,\alpha}$ denotes the critical value of this test. The bootstrap-based functional testing procedure for testing H_0 can then be described as follows.

Step 1. Calculate the sample mean functions $\overline{X}_{i,n_i}(t) = n_i^{-1} \sum_{j=1}^{n_i} X_{i,j}(t)$ $(t \in \mathcal{I})$ and the residual functions $\hat{\epsilon}_{i,j}(t) = X_{i,j}(t) - \overline{X}_{i,n_i}(t)$, $t \in \mathcal{I}$ $(i = 1, \ldots, K, \; j = 1, \ldots, n_i)$.

Step 2. Generate bootstrap functional pseudo-observations $X^*_{i,j}$ $(i = 1, \ldots, K, \; j = 1, \ldots, n_i)$ according to

$$X^*_{i,j} = \overline{X}_{i,n_i} + \epsilon^*_{i,j}, \text{ where } \epsilon^*_{i,j} = \hat{\epsilon}_{I,J}. \tag{2}$$

Here, I is a discrete random variable with probability $\mathrm{pr}(I = i) = n_i/N$ $(i = 1, \ldots, K)$ and, given $I = i$, the discrete random variable J satisfies $\mathrm{pr}(J = j \mid I = i) = n_i^{-1}$ $(i = 1, \ldots, K; \; j = 1, \ldots, n_i)$.

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Step 3. Let T_N^* be T_N calculated using the $X_{i,j}^*$ $(i = 1, \ldots, K;\ j = 1, \ldots, n_i)$. Denote by $D_{N,T}^*$ the distribution function of T_N^*. For any given $\alpha \in (0, 1)$, reject H_0 if and only if $T_N > d_{N,\alpha}^*$, where $d_{N,\alpha}^*$ denotes the α-quantile of $D_{N,T}^*$, i.e., $D_{N,T}^*(d_{N,\alpha}^*) = 1 - \alpha$.

Clearly, since the random functions $\epsilon_{i,j}^*$ are generated independently from each other, for any two different pairs of indices, say (i_1, j_1) and (i_2, j_2), the corresponding pseudo-observations X_{i_1,j_1}^* and X_{i_2,j_2}^* are independent. Furthermore, the $X_{i,j}^*$ satisfy $E^*(X_{i,j}^*) = \bar{X}_{i,n_i}$ and have covariance operators $\mathcal{C}_i^* = E^*\{(X_{i,j}^* - \bar{X}_{i,n_i}) \otimes (X_{i,j}^* - \bar{X}_{i,n_i})\} = N^{-1}\sum_{i=1}^{K} n_i \hat{\mathcal{C}}_i = \hat{\mathcal{C}}_N$. Here, E^* refers to expectation with respect to the bootstrap distribution, $\hat{\mathcal{C}}_i = n_i^{-1}\sum_{j=1}^{n_i}(X_{i,j} - \bar{X}_{i,n_i}) \otimes (X_{i,j} - \bar{X}_{i,n_i})$ is the sample estimator of the covariance operator $\mathcal{C}_i$, and $\hat{\mathcal{C}}_N$ is the corresponding pooled estimator of the covariance operator. Thus, conditional on $\mathcal{X}_N$, the $X_{i,j}^*$ have, within each population i, the same mean function $\bar{X}_{i,n_i}$, which may be different for different populations. Furthermore, the covariance operator in each population i equals the pooled sample covariance operator $\hat{\mathcal{C}}_N$. That is, the $X_{i,j}^*$ satisfy H_0 in (1).

Consider the case $K = 2$. It is natural to compare the covariance operators in the two populations by evaluating the Hilbert–Schmidt norm of their differences. Such an approach has been recently proposed by the paper by G. Boente et al. (ArXiv:1404.7080) by using the test statistic

$$T_N = N\|\hat{\mathcal{C}}_1 - \hat{\mathcal{C}}_2\|_S^2.$$

If $n_1/N \to \theta \in (0, 1)$, $E(\|X_{i,1}\|^4) < \infty$ $(i = 1, 2)$, and H_0 given in (1) with $K = 2$ is true, then, as n_1, $n_2 \to \infty$, Theorem 3.1 of the paper by G. Boente et al. (ArXiv:1404.7080) shows that T_N converges weakly to $\sum_{l=1}^{\infty} \lambda_l Z_l^2$, where Z_l are independent standard Gaussian random variables and λ_l are the eigenvalues of the operator $\mathcal{B} = \theta^{-1}\mathcal{B}_1 + (1-\theta)^{-1}\mathcal{B}_2$. Here, $\mathcal{B}_i$ is the covariance operator of the limiting Gaussian random element U_i to which $n_i^{1/2}(\hat{\mathcal{C}}_i - \mathcal{C}_i)$ converges weakly as $n_i \to \infty$. Since the limiting distribution of T_N depends on an infinite number of unknown eigenvalues, implementation of this asymptotic result for calculating critical values is difficult. Let

$$T_N^* = N\|\hat{\mathcal{C}}_1^* - \hat{\mathcal{C}}_2^*\|_S^2,$$

where $\hat{\mathcal{C}}_i^* = n_i^{-1}\sum_{j=1}^{n_i}(X_{i,j}^* - \bar{X}_{i,n_i}^*) \otimes (X_{i,j}^* - \bar{X}_{i,n_i}^*)$ $(i = 1, 2)$. Theorem 1 shows that this bootstrap procedure leads to consistent estimation of the critical values of interest.

THEOREM 1. *If* $E(\|X_{i,1}\|^4) < \infty$, $i \in \{1, 2\}$, *and* $n_1/N \to \theta \in (0, 1)$, *then, as* $n_1, n_2 \to \infty$,

$$\sup_{x \in R}\left|\mathrm{pr}\left(T_N^* \leqslant x \mid \mathcal{X}_N\right) - \mathrm{pr}_{H_0}\left(T_N \leqslant x\right)\right| \to 0,$$

in probability, where $\mathrm{pr}_{H_0}(T_N \leqslant \cdot)$ *is the distribution function of* T_N *when* H_0 *in* (1) *with* $K = 2$ *is true and* $\mathcal{B}_1 = \mathcal{B}_2$.

Remark 1. If H_1 is true, i.e., if $\|\mathcal{C}_1 - \mathcal{C}_2\|_S > 0$, then, as $n_1, n_2 \to \infty$, $T_N \to \infty$, in probability. Theorem 1 and Slutsky's theorem then imply that the test T_N based on the bootstrap critical values obtained from the distribution of T_N^* is consistent, that is, its power approaches unity.

2·3. *Testing the equality of mean functions*

We are interested in testing the following hypothesis

$$H_0: \mu_1 = \cdots = \mu_K \text{ versus } H_1: \text{there exist } k, l \in \{1, \ldots, K\},\ k \neq l,\ \text{with } \mu_k \neq \mu_l. \tag{3}$$

As in the previous section, equality under H_0 means that $\|\mu_k - \mu_l\| = 0$ for any pair of indices (k, l), with $k \neq l$, while under H_1 we have $\|\mu_k - \mu_l\| > 0$ for at least one pair of indices (k, l), $k \neq l$.

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This testing problem can be addressed by changing Step 2 of the bootstrap resampling algorithm of § 2·2. In particular, we replace equation (2) by

$$X^{+}_{i,j} = \bar{X}_N + \epsilon^{+}_{i,j}, \tag{4}$$

where $\bar{X}_N = N^{-1} \sum_{i=1}^{K} \sum_{j=1}^{n_i} X_{i,j}$ is the pooled mean estimator and $\epsilon^{+}_{i,j} = \hat{\epsilon}_{i,J}$. Here, J is a discrete random variable satisfying $\mathrm{pr}(J = j) = 1/n_i$ $(j = 1, \ldots, n_i;\ i = 1, \ldots, K)$. This ensures that the covariance structure of the functional observations in each population i is retained by the bootstrap algorithm. This covariance structure may be different for different populations, although the bootstrap procedure generates K populations of functional pseudo-observations having identical mean functions. In particular, conditional on X_N, we have $E^{+}(X^{+}_{i,j}) = \bar{X}_N$ and the $X^{+}_{i,j}$ have covariance operators $\hat{\mathcal{C}}_i$, where E^{+} refers to expectation with respect to the bootstrap distribution.

Remark 2. If, instead of equation (4), we use $X^{\circ}_{i,j} = \bar{X}_N + \epsilon^{*}_{i,j}$ $(j = 1, \ldots, n_i;\ i = 1, \ldots, K)$ to generate pseudo-observations and $\varepsilon^{*}_{i,j}$ defined as in Step 2 of the algorithm in § 2·2, then the $X^{\circ}_{i,j}$ will have an identical mean function equal to $\bar{X}_N$ and an identical covariance operator equal to $\hat{\mathcal{C}}_N$. This could be applied for testing simultaneously the equality of mean functions and covariance operators.

Consider again the case $K = 2$. For testing the equality of two mean functions, a natural approach is to compute the L^2-distance between the sample mean functions $\bar{X}_{1,n_1}$ and $\bar{X}_{2,n_2}$. This approach was considered by Benko et al. (2009) and Horváth & Kokoszka (2012, Ch. 5) using the test statistic

$$S_N = \frac{n_1 n_2}{N} \| \bar{X}_{1,n_1} - \bar{X}_{2,n_2} \|^2 .$$

If $n_1/N \to \theta \in (0, 1)$, $E(\|X_{i,1}\|^4) < \infty$, $i \in \{1, 2\}$, and H_0 given in (3) with $K = 2$ is true, then, as n_1, $n_2 \to \infty$, Theorem 5.1 of Horváth & Kokoszka (2012) shows that S_N converges weakly to $\int_{\mathcal{I}} \Gamma^2(t)\mathrm{d}t$, where $\Gamma^2(t) = (1-\theta)\Gamma_1^2(t) + \theta \Gamma_2^2(t)$ and $\{\Gamma_1(t), t \in [0, 1]\}$ and $\{\Gamma_2(t), t \in [0, 1]\}$ are two independent Gaussian processes with mean zero and covariance operators $\mathcal{C}_1$ and $\mathcal{C}_2$. The limiting distribution of S_N depends on the unknown infinite-dimensional parameters $\mathcal{C}_1$ and $\mathcal{C}_2$, so analytical calculation of critical values is difficult. Let

$$S^{+}_N = \frac{n_1 n_2}{N} \| \bar{X}^{+}_{1,n_1} - \bar{X}^{+}_{2,n_2} \|^2 .$$

Theorem 2 shows that the bootstrap consistently estimates the critical values of the test S_N.

Theorem 2. *If $E(\|X_{i,1}\|^4) < \infty$, $i \in \{1, 2\}$, and $n_1/N \to \theta \in (0, 1)$, then, as $n_1, n_2 \to \infty$,*

$$\sup_{x \in R} \left| \mathrm{pr}\left(S^{+}_N \leqslant x \mid X_N\right) - \mathrm{pr}_{H_0}\left(S_N \leqslant x\right) \right| \ \to\ 0,$$

in probability, where $\mathrm{pr}_{H_0}(S_N \leqslant \cdot)$ is the distribution function of S_N when H_0 given in (3) with $K = 2$ is true.

Remark 3. Under the same assumptions as in Theorem 5.2 of Horváth & Kokoszka (2012) and if H_1 is true, i.e., $\|\mu_1 - \mu_2\| > 0$, then, as $n_1, n_2 \to \infty$, $S_N \to \infty$, in probability. Thus, Theorem 2 and Slutsky's theorem imply consistency of the test S_N using the bootstrap critical values obtained from the distribution of S^{+}_N, that is, its power approaches unity.

3. Numerical results

3·1. *Simulations*

We investigate below the size and power of the bootstrap-based tests and compare them with corresponding projection-based tests. Motivated by Kraus & Panaretos (2012), we generate non-Gaussian curves X_1 and X_2, via $X_i(t) = \sum_{k=1}^{10} \{2^{1/2} k^{-1/2} \sin(\pi k t) V_{i,k} + k^{-1/2} \cos(2\pi k t) W_{i,k}\}$, $t \in \mathcal{I}$, $(i = 1, 2)$, where $V_{i,k}$ and $W_{i,k}$ $(i = 1, 2;\ k = 1, 2, \ldots, 10)$ are independent t_5-distributed random variables. All curves were simulated at 500 equidistant points in the interval $\mathcal{I}$, and transformed into functional objects using the Fourier

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Table 1. *Empirical size and power (%) of $T_{2,N}$ and T_N^* for the equality of two covariance operators*

		$n_1 = n_2 = 25$			$n_1 = n_2 = 50$		
γ	Test	$\alpha = 1\%$	5%	10%	$\alpha = 1\%$	5%	10%
1·0	$T_{2,N}$	0	0·6	2·2	0	1·6	4·1
	T_N^*	0·3	2·5	8·2	0·6	3·2	7·6
1·2	$T_{2,N}$	0	1·6	3·9	0·3	2·6	7·2
	T_N^*	0·5	5·0	14·7	0·8	9·8	23·1
1·4	$T_{2,N}$	0	1·1	5·2	0·2	6·5	22·1
	T_N^*	1·6	16·8	36·8	12·8	46·1	67·6
1·6	$T_{2,N}$	0	1·0	9·5	1·4	28·5	55·9
	T_N^*	4·7	33·8	61·2	37·0	79·6	90·3
1·8	$T_{2,N}$	0	3·6	23·0	6·6	57·4	82·1
	T_N^*	10·4	55·7	82·3	61·2	91·5	96·6
2·0	$T_{2,N}$	0	7·0	40·5	24·5	83·6	95·7
	T_N^*	17·7	66·6	89·2	74·2	93·7	97·7

basis with 49 basis functions. We considered 2000 replications and took sample sizes of $n_1 = n_2 = 25$ or $n_1 = n_2 = 50$ random curves.

First, consider the problem of testing the equality of two covariance operators. We considered the bootstrap test based on T_N^* of § 2·2 and the asymptotic test based on $T_{p,N}$ using p functional principal components considered in Fremdt et al. (2012); see also Panaretos et al. (2010) and Kraus & Panaretos (2012). Three nominal levels, $\alpha = 1\%$, 5%, 10%, are considered and all bootstrap calculations are based on 1000 repetitions. To evaluate the power of the tests, we modified the curves generated in the second group according to $X_2(t) = \gamma X_1(t)$ $(t \in \mathcal{I})$ for selected values of the scaling parameter γ; $\gamma = 1$ corresponds to the null hypothesis of equality of covariance operators. Table 1 gives results for T_N^* as well as the best results obtained for $T_{p,N}$ that corresponds to the choice $p = 2$. The test based on $T_{p,N}$ heavily underestimates the size whereas that based on T_N^* shows much better behaviour. Furthermore, T_N^* has, overall, higher power than $T_{p,N}$. Results for other choices of p are reported in the Supplementary Material; the quality of the asymptotic $\chi^2_{p(p+1)/2}$ approximation of $T_{p,N}$ under H_0 becomes worse as p increases.

Consider next testing the equality of two mean functions. Here, we investigate the size and power of S_N^+ and $S_{p,N}^{(1)}$ and $S_{p,N}^{(2)}$ based on p functional principal components; see Horváth & Kokoszka (2012, Ch. 5). The curves are generated using the above-mentioned non-Gaussian simulation set-up with $X_2(t) = \delta + X_1(t)$ $(t \in \mathcal{I})$ for selected values of the shift parameter δ; $\delta = 0$ corresponds to the null hypothesis of equality of mean functions. The simulation results are reported in Table 2 where the best results obtained for the projection-based tests are shown and which correspond to the case $p = 2$. These results demonstrate the improved size and power of S_N^+. Furthermore, for testing the equality of means, the results obtained using $S_{p,N}^{(1)}$ and $S_{p,N}^{(2)}$ seem to be less sensitive to the choice of p than for testing equality of covariance operators.

3·2. *Mediterranean fruit flies*

We apply the tests to egg-laying trajectories of Mediterranean fruit flies, *Ceratitis capitata*, or medflies for short; see the Supplementary Material for more details. The tests have been applied to smooth curves obtained using a Fourier basis with 49 basis functions, with bootstrap calculations based on 1000 replications. Table 3 shows the p-values for the absolute and the relative egg-laying curves of the tests for the equality of the two covariance operators, using $T_{p,N}$ with different values of p and T_N^*. Both tests lead

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Table 2. *Empirical size and power (%) of $S^{(1)}_{2,N}$, $S^{(2)}_{2,N}$ and S^{+}_{N} for the equality of two mean functions*

		$n_1 = n_2 = 25$			$n_1 = n_2 = 50$		
δ	Test	$\alpha = 1\%$	5%	10%	$\alpha = 1\%$	5%	10%
0·0	$S^{(1)}_{2,N}$	0·8	3·5	7·6	1·0	3·7	8·4
	$S^{(2)}_{2,N}$	0·7	3·6	7·5	0·8	4·0	8·4
	S^{+}_{N}	1·2	5·8	11·8	0·7	4·4	8·5
0·2	$S^{(1)}_{2,N}$	0·8	6·4	11·9	3·0	9·6	17·4
	$S^{(2)}_{2,N}$	0·8	6·6	11·6	3·3	9·3	16·7
	S^{+}_{N}	2·4	8·8	15·8	4·0	13·0	20·6
0·4	$S^{(1)}_{2,N}$	5·1	15·3	24·7	13·5	31·9	42·2
	$S^{(2)}_{2,N}$	4·3	14·4	23·5	12·3	29·6	41·1
	S^{+}_{N}	5·7	20·6	32·0	18·2	40·6	54·0
0·6	$S^{(1)}_{2,N}$	14·7	30·2	41·5	40·6	62·0	72·6
	$S^{(2)}_{2,N}$	13·4	28·8	39·9	36·3	60·6	72·2
	S^{+}_{N}	21·6	44·8	59·3	54·2	77·9	86·8
0·8	$S^{(1)}_{2,N}$	29·8	53·4	64·9	71·0	84·0	91·0
	$S^{(2)}_{2,N}$	27·8	52·0	62·1	67·9	83·6	90·2
	S^{+}_{N}	47·3	71·2	81·3	86·1	96·1	98·2
1·0	$S^{(1)}_{2,N}$	50·1	69·7	78·3	87·6	93·7	96·3
	$S^{(2)}_{2,N}$	46·0	67·1	79·2	85·6	93·8	96·3
	S^{+}_{N}	73·7	90·7	95·2	98·0	99·5	99·7

Table 3. *p-values of $T_{p,N}$ and T^{*}_{N} for the equality of the two covariance operators applied to the absolute and relative egg-laying curves*

	Absolute					Relative				
T^{*}_{N}	0·179					0·003				
$T_{p,N}$	$p=4$ 0·253	$p=5$ 0·211	$p=6$ 0·385	$p=7$ 0·545	$p=8$ 0·520	$p=5$ 0·004	$p=6$ 0·021	$p=7$ 0·064	$p=8$ 0·130	$p=9$ 0·121
f_p	0·940	0·958	0·974	0·982	0·989	0·845	0·912	0·949	0·974	0·985

to similar conclusions. At the commonly used α-levels, the null hypothesis cannot be rejected. However, this is not the case for the relative egg-laying curves where for $T_{p,N}$, the conclusion depends on p. In this case, T^{*}_{N} does not reject the null hypothesis. The value $f_p = (\sum_{k=1}^{p} \hat{\lambda}_k)/(\sum_{k=1}^{N} \hat{\lambda}_k)$ reported in Table 3 describes the fraction of the sample variance explained by the p-first functional principal components. Table 4 shows the p-values for the absolute egg-laying curves for testing the equality of the two mean functions. The behaviour of $S^{(1)}_{p,N}$ depends on the value of p while that of $S^{(2)}_{p,N}$ is more stable. On the other hand, S^{+}_{N} rejects the null hypothesis. For the relative egg-laying curves, $S^{(1)}_{p,N}$, $S^{(2)}_{p,N}$ and S^{+}_{N} reject the null hypothesis, hence we do not report these results. This example demonstrates the advantages of the bootstrap which allows the use of test statistics based on natural global distance measures and which avoids selecting truncation parameters, like the number of functional principal components.

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Table 4. *p-values of $S^{(1)}_{p,N}$, $S^{(2)}_{p,N}$ and S^{+}_{N} for the equality of the two mean functions applied to the absolute egg-laying curves*

S^{+}_{N}	0·011							
	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$	$p=7$	$p=8$	$p=9$
$S^{(1)}_{p,N}$	0·021	0·029	0·056	0·099	0·154	0·054	0·025	0·040
$S^{(2)}_{p,N}$	0·007	0·008	0·009	0·009	0·010	0·010	0·010	0·010
f_p	0·837	0·899	0·939	0·958	0·973	0·982	0·989	0·994

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Supplementary material

Supplementary material available at *Biometrika* online includes the proofs of Theorems 1 and 2, additional simulation results, and details for the Mediterranean fruit flies dataset.

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