

Double Angle Formulas

$$\sin(2x) = 2 \sin(x) \cos(x)$$

$$\cos(2x) = \cos^2(x) - \sin^2(x)$$

$$= 2 \cos^2(x) - 1$$

$$= 1 - 2 \sin^2(x)$$

$$\tan(2x) = \frac{2 \tan(x)}{1 - \tan^2(x)}$$

$$\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$$

$$\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$$

Limits

$$\lim_{x \rightarrow n} f(x) = L$$

Means for any number $\epsilon > 0$ there exists a number $\delta > 0$ such that if $0 < |x - n| < \delta$ then $|f(x) - L| < \epsilon$.

$$\lim_{x \rightarrow a} f(g(x)) = f\left(\lim_{x \rightarrow a} g(x)\right) \text{ if } f(g(x)) \text{ is continuous at } x$$

Standard difference quotient:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Symmetric difference quotient:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x-h)}{2h}$$

Continuity and Differentiability

$f(x)$ is continuous at x if:

$$f(x) = \lim_{k \rightarrow x^+} f(k) = \lim_{k \rightarrow x^-} f(k)$$

$f(x)$ is differentiable at x if:

$$f'_+(x) = f'_-(x)$$

Differentiation Rules

$$\begin{aligned}\frac{d}{dx}cf(x) &= cf'(x) \\ \frac{d}{dx}f(g(x)) &= f'(g(x))g'(x) \\ \frac{d}{dx}(f(x) \pm g(x)) &= f'(x) \pm g'(x) \\ \frac{d}{dx}f(x)^n &= nf(x)^{n-1} \cdot f'(x) \\ \frac{d}{dx}\sqrt{f(x)} &= \frac{f'(x)}{2\sqrt{f(x)}} \\ \frac{d}{dx}(f(x)g(x)) &= f'(x)g(x) + f(x)g'(x) \\ \frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) &= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} \\ \frac{d}{dx}|f(x)| &= \frac{|f(x)|}{f(x)} \cdot f'(x) \\ \frac{d}{dx}b^{f(x)} &= b^{f(x)} \ln b \cdot f'(x) \\ \frac{d}{dx} \log_b(f(x)) &= \frac{f'(x)}{f(x) \ln b}\end{aligned}$$

Trigonometric Differentiation Rules

$$\begin{aligned}\frac{d}{dx} \sin(f(x)) &= \cos(f(x)) \cdot f'(x) \\ \frac{d}{dx} \cos(f(x)) &= -\sin(f(x)) \cdot f'(x) \\ \frac{d}{dx} \tan(f(x)) &= \sec^2(f(x)) \cdot f'(x) \\ \frac{d}{dx} \sec(f(x)) &= \sec(f(x)) \tan(f(x)) \cdot f'(x) \\ \frac{d}{dx} \csc(f(x)) &= -\csc(f(x)) \cot(f(x)) \cdot f'(x) \\ \frac{d}{dx} \cot(f(x)) &= -\csc^2(f(x)) \cdot f'(x)\end{aligned}$$

Inverse Trigonometric Differentiation Rules

$$\begin{aligned}\frac{d}{dx} \sin^{-1}(f(x)) &= \frac{f'(x)}{\sqrt{1-f(x)^2}} \\ \frac{d}{dx} \cos^{-1}(f(x)) &= -\frac{f'(x)}{\sqrt{1-f(x)^2}} \\ \frac{d}{dx} \tan^{-1}(f(x)) &= \frac{f'(x)}{1+f(x)^2}\end{aligned}$$

Mean Value Theorem

If $f(x)$ is continuous on $[a, b]$, then there exists one value c on $[a, b]$ such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Reimann Sums

$$\sum_{k=1}^n f(x_k) \Delta x$$

Assuming regular partition:

$$\begin{aligned} \text{LRAM} &= \sum_{k=1}^n f\left(a + \frac{(k-1)(b-a)}{n}\right) \frac{b-a}{n} \\ \text{MRAM} &= \sum_{k=1}^n f\left(a + \frac{(k-\frac{1}{2})(b-a)}{n}\right) \frac{b-a}{n} \\ \text{RRAM} &= \sum_{k=1}^n f\left(a + \frac{k(b-a)}{n}\right) \frac{b-a}{n} \end{aligned}$$

Definite Integration

$$\begin{aligned} \int_a^b f(x) \, dx &= \lim_{n \rightarrow \infty} \sum_{k=1}^n f(x_k) \Delta x \\ \int_a^a f(x) \, dx &= 0 \\ \int_a^b f(x) \, dx &= - \int_b^a f(x) \, dx \\ \int_a^b c f(x) \, dx &= c \int_a^b f(x) \, dx \\ \int_a^b (f(x) \pm g(x)) \, dx &= \int_a^b f(x) \, dx \pm \int_a^b g(x) \, dx \\ \int_a^p f(x) \, dx + \int_p^b f(x) \, dx &= \int_a^b f(x) \, dx \\ \int_a^b f(x) \, dx &> \int_a^b g(x) \, dx \text{ if } f(x) > g(x) \text{ on } [a, b] \end{aligned}$$

The mean value of $f(x)$ over $[a, b]$ is:

$$\frac{1}{b-a} \int_a^b f(x) \, dx$$

Mean Value of a Function Theorem

If $f(x)$ is continuous on $[a, b]$, then there exists one value c on $[a, b]$ such that:

$$f(c) = \frac{1}{b-a} \int_a^b f(x) \, dx$$

Linear Approximation

The linearization of $f(x)$ at a :

$$L(x) = f(a) + f'(a)(x - a)$$

Fundamental Theorem of Calculus

Part 1

$$\frac{d}{dx} \int_{a(x)}^{b(x)} f(t) \, dt = f(b(x))b'(x) - f(a(x))a'(x)$$

Part 2

$$F'(x) = f(x)$$
$$\int_a^b f(x) \, dx = F(x) \Big|_a^b = F(b) - F(a)$$

Solving Differential Equations Using A Definite Integral

$$\frac{dy}{dx} = f(x)$$
$$y = \int_a^x f(t) \, dt + C$$

The easiest solution occurs when a = the given x value and C = the given y value.

Solving Differential Equations Using Antidifferentiation

$$\frac{dy}{dx} = f(x)$$
$$y = \int f(x) \, dx + C \text{ (general form)}$$

Specify largest domain of x that includes specified x and does not include any discontinuities.

Euler's Method

Start at (x_0, y_0)

$$\Delta y = \Delta x \cdot \left. \frac{dy}{dx} \right|_{(x_{n-1}, y_{n-1})}$$

$$(x_n, y_n) = (x_{n-1} + \Delta x, y_{n-1} + \Delta y)$$

Antidifferentiation Rules

$$\int cf(x) dx = c \int f(x) dx$$

$$\int f(x) \pm g(x) dx = \int f(x) dx \pm \int g(x) dx$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

$$\int \frac{1}{x} dx = \ln|x| + C$$

$$\int b^x dx = \frac{b^x}{\ln b} + C$$

$$\int \ln x dx = x \ln x - x + C$$

Trigonometric Antidifferentiation Rules

$$\int \sin(x) dx = -\cos(x) + C$$

$$\int \cos(x) dx = \sin(x) + C$$

$$\int \tan(x) dx = -\ln|\cos(x)| + C$$

$$\int \sec(x) dx = \ln|\sec(x) + \tan(x)| + C$$

$$\int \sin^2(x) dx = \frac{1}{2}(x - \sin(x) \cos(x)) + C$$

$$\int \cos^2(x) dx = \frac{1}{2}(x + \sin(x) \cos(x)) + C$$

$$\int \tan^2(x) dx = \tan(x) - x + C$$

$$\int \sec^2(x) dx = \tan(x) + C$$

$$\int \sec(x) \tan(x) dx = \sec(x) + C$$

$$\int \frac{1}{1+x^2} dx = \tan^{-1}(x) + C$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1}(x) + C$$

Inverse Trigonometric Antidifferentiation Rules

$$\int \sin^{-1}(x) = x \sin^{-1}(x) + \sqrt{1-x^2}$$

$$\int \tan^{-1}(x) = x \tan^{-1}(x) - \frac{1}{2} \ln(1+x^2) + C$$

U-Substitution

To differentiate $\int f(g(x)) dx$ where $f(x)$ and $g(x)$ are both antidifferentiable, set $u = g(x)$. Then solve for du and replace dx so that the equation $A \int f(u) du$ results (where A is the necessary “fudge factor”). Antidifferentiate and replace u with $g(x)$.

Integration by Parts

$$\int u dv = uv - \int v du + C$$

or

$$\begin{array}{rcl} + & u & dv \\ - & \frac{du}{dx} & \int dv dx \\ + & \frac{d^2u}{dx^2} & \int (\int dv dx) dx \\ - & \frac{d^3u}{dx^3} & \int (\int (\int dv dx) dx) dx \\ + & \dots & \dots \\ & 0 & \dots \end{array}$$

$$\int u dv = \left(u \cdot \int dv dx \right) - \left(\frac{du}{dx} \cdot \int (\int dv dx) dx \right) + \left(\frac{d^2u}{dx^2} \cdot \int (\int (\int dv dx) dx) dx \right) - \dots$$

Partial Fraction Decomposition

Convert fractions of form $\frac{\text{linear or constant}}{\text{factorable quadratic}}$ into its partial fraction.

$$\frac{h(x)}{f(x)g(x)} = \frac{A}{f(x)} + \frac{B}{g(x)} = \frac{Ag(x)}{f(x)g(x)} + \frac{Bf(x)}{f(x)g(x)} = \frac{Ag(x) + Bf(x)}{f(x)g(x)}$$

$$h(x) = Ag(x) + Bf(x)$$

Choose a value of x such that $f(x) = 0$ and then $g(x) = 0$ to solve for A and B respectively.

Exponential Growth/Decay

$$\frac{dQ}{dt} = kQ$$

$$Q = Q_0 e^{kt} = Q_0 b^{\frac{t}{p}}$$

$$k = -\frac{\ln(b)}{p}$$

Half-life

$$k = -\frac{\ln(2)}{p}$$

Logistic Differential Equation

$$\frac{dP}{dt} = kP(M - P) = kPM\left(1 - \frac{P}{M}\right)$$

where P is the population and M is the carrying capacity

$$P(t) = \frac{M}{1 + Ae^{-Mkt}}$$

The maximum carrying capacity is reached when $P(t) = M - 0.5$, if $P(0) < M$, or right after $P(t) = M + 0.5$, if $P(0) > M$.

Calculating Area of Regions Bounded by Two Functions

$$A = \int_a^b f(x) - g(x) \, dx$$

where $f(x)$ is the upper function and $g(x)$ is the lower function.

If the functions cross, multiple integrals must be taken with the appropriate upper and lower functions.

Typical Cross-Sectional Areas

Square: $A = b^2 = \frac{1}{2}d^2$

Rectangle: $A = bh$

Equalateral triangle: $A = \frac{\sqrt{3}}{4}b^2$

Right isoceles triangle: $A = \frac{1}{2}l^2 = \frac{1}{4}h^2$

Hexagon: $A = \frac{3\sqrt{3}}{2}s^2$

Circle: $A = \pi r^2 = \frac{1}{4}\pi d^2$

Semicircle: $A = \frac{1}{2}\pi r^2 = \frac{1}{8}\pi d^2$

Parabolic region: $A = \frac{2}{3}bh$

Calculating Volume Using the Disk/Washer Method for Solids of Rotation

$$V = \pi \int_a^b r_{\text{outer}}^2 - r_{\text{inner}}^2 dx$$

where dx is the differential thickness of the disk/washer.

Calculating Volume Using the Cylindrical Shell Method for Solids of Rotation

$$V = 2\pi \int_a^b hr dx$$

where dx is the differential thickness of the shell.

Calculating Arc Lengh

$$l = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

L'Hôpitals Rule

In order to evaluate a limit where

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} \Rightarrow \frac{0}{0} \text{ or } \frac{\infty}{\infty}$$

evaluate

$$\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

In order to evaluate a limit where

$$\lim_{x \rightarrow a} f(x)^{g(x)} \Rightarrow "0^0" \text{ or } "1^\infty" \text{ or } "\infty^0"$$

set L equal to the original limit and immediately take the log of both sides

$$\begin{aligned} L &= \lim_{x \rightarrow a} f(x)^{g(x)} \\ \ln(L) &= \ln\left(\lim_{x \rightarrow a} f(x)^{g(x)}\right) \\ \ln(L) &= \lim_{x \rightarrow a} g(x) \ln f(x) \end{aligned}$$

Evaluate the resulting limit and then raise e to that result

Relative Rates of Growth

To compare rates of $f(x)$ and $g(x)$ evaluate

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \begin{cases} \infty & \text{then } f(x) \text{ grows faster than } g \\ \langle 0, \infty \rangle & \text{then } f(x) \text{ and } g(x) \text{ grow at the same rate} \\ 0 & \text{then } f(x) \text{ grows slower than } g(x) \end{cases}$$

Improper Integrals

Type 1

$$\begin{aligned} \int_a^\infty f(x) \, dx &= \lim_{n \rightarrow \infty} \int_a^n f(x) \, dx \\ \int_{-\infty}^b f(x) \, dx &= \lim_{n \rightarrow \infty} \int_{-n}^b f(x) \, dx \\ \int_{-\infty}^\infty f(x) \, dx &= \lim_{n \rightarrow \infty} \int_{-n}^n f(x) \, dx \end{aligned}$$

Type 2

$$\int_a^b f(x)$$

such that there is at least one vertical-asymptotic discontinuity on $[a, b]$.

$$\begin{aligned} \int_a^b f(x) \, dx \text{ (where discontinuity at } a) &= \lim_{n \rightarrow a^+} \int_n^b f(x) \, dx \\ \int_a^b f(x) \, dx \text{ (where discontinuity at } b) &= \lim_{n \rightarrow b^-} \int_a^n f(x) \, dx \\ \int_a^b f(x) \, dx \text{ (where discontinuity } d \text{ on } \langle a, b \rangle) &= \lim_{n \rightarrow d^-} \int_a^n f(x) \, dx + \lim_{n \rightarrow d^+} \int_n^b f(x) \, dx \end{aligned}$$

P-Integrals

An integral that fits the pattern:

$$\int_1^{\infty} \frac{1}{x^p} dx \text{ where } p \text{ is positive}$$

$$p > 1 \Rightarrow \text{the integral converges to } \frac{1}{p-1}.$$

$$0 < p \leq 1 \Rightarrow \text{the integral diverges.}$$

Comparison Tests

For positive type 1 improper integrals ($a = \infty$ and/or $b = \infty$) where the integrand is not easily antidifferentiable:

$$\int_a^b f(x) dx$$

Establish bound functions which can be antidifferentiated:

$$l(x) \leq f(x) \leq u(x)$$

$$\text{Convergence if } \int_a^b f(x) dx \leq \int_a^b u(x) dx \text{ and } \int_a^b u(x) dx \text{ converges}$$

$$\text{Divergence if } \int_a^b f(x) dx \geq \int_a^b l(x) dx \text{ and } \int_a^b l(x) dx \text{ diverges}$$

Sequences

Explicit Definition

$$x_k = f(k)$$

where $k \in \{1, 2, 3, \dots\}$.

Recursive Definition

$$x_k = m(x_{k-1})$$

where x_1 is defined and $k \in \{2, 3, 4, \dots\}$.

Arithmetic

$$x_k = d(k-1) + x_1$$

$$x_k = t_{k-1} + d$$

Geometric

$$x_k = x_1 r^{k-1}$$

$$x_k = x_{k-1} r$$

Convergence/Divergence

A sequence converges if the limit $\lim_{n \rightarrow \infty} x_n$ is a finite number else it diverges.

Infinite Series

$$\sum_{k=0}^{\infty} x_k$$

Convergence

Consider the sequence of “partial sums” (s) where:

$$s_n = \sum_{k=0}^n x_k$$

If $\{s_n\}$ converges to S then the infinite series converges to S .

Power Series

$$\sum_{k=0}^{\infty} c_k (x - a)^k$$

Taylor/Maclaurin Series

Given an infinitely differentiable function $f(x)$, the Taylor series centered around a is equal to the power series with a specific c_k :

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x - a)^k$$

A Maclaurin series is a Taylor series centered around 0 ($a = 0$).

The n -order polynomial function $P_n(x)$ that best fits $f(x)$ is equal to the Taylor series evaluated with upper limit n :

$$P_n(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x - a)^k$$

Common Maclaurin Series

$$\frac{a}{1-x} = \sum_{k=0}^{\infty} ax^k \quad x \in \langle -1, 1 \rangle$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \quad x \in \mathbb{R}$$

$$\sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \quad x \in \mathbb{R}$$

$$\cos(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \quad x \in \mathbb{R}$$

$$\tan^{-1}(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} \quad x \in [-1, 1]$$

$$\ln(x+1) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k} \quad x \in \langle -1, 1 \rangle$$

Calculating Error for a Taylor Polynomial

$$P_{\infty}(x) = P_n(x) + R_n(x)$$

where P_{∞} is the entire infinite Taylor series and R_n is the remainder (note that the remainder series starts with $k = n + 1$).

Lagrange Error:

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x-a)^{n+1} \quad \text{for an unknown } c \in \langle a, x \rangle.$$

$$|R_n(x)| = \left| \frac{f^{(n+1)}(c)}{(n+1)!} (x-a)^{n+1} \right|$$

$$|R_n(x)| = \frac{|f^{(n+1)}(c)|}{(n+1)!} |x-a|^{n+1}$$

To bound the error:

$$|R_n(x)| \leq \frac{M}{(n+1)!} |x-a|^{n+1}$$

Where M is a positive number greater than or equal to every possible value $f^{(n+1)}(t)$ where t is in the interval of study.

The series then converges if:

$$\lim_{n \rightarrow \infty} |R_n(x)| = 0$$

Infinite Series Tests

nth-Term Test

If $\lim_{n \rightarrow \infty} a_n \neq 0$, the series $\sum_{n=0}^{\infty} a_n$ diverges.

Comparison Test

To conclude whether a series S converges or diverges establish a known series T .

Convergence if $S \leq T$ and T converges

Divergence if $S \geq T$ and T diverges

Alternating Test

Any alternating series that passes the n^{th} -term test where the magnitude of successive terms decreases will converge.

A series which depends on alternation has conditional convergence. A series which converges whether all terms are positive or negative has absolute convergence.

Additionally the error is less than the next term in the sequence for an alternating sequence.

P-Series Test

If the series is in the form $a_n = \frac{1}{n^p}$, then if $p > 1$ the series converges, else it diverges.

Telescoping Series Test

For a series where subsequent terms cancel out previous terms such as:

$$\sum_{k=1}^{\infty} \frac{1}{(k)(k+1)}$$

Separate into partial fractions:

$$\sum_{k=1}^{\infty} \frac{1}{k} - \frac{1}{k+1} = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}$$

Ratio Test

For a series:

$$\sum_{k=0}^{\infty} t_k \text{ where } t_k \geq 0$$

consider

$$\lim_{k \rightarrow \infty} \frac{t_{k+1}}{t_k} = \begin{cases} [0, 1) & \text{the series converges} \\ 1 & \text{the test is inconclusive} \\ \langle 1, \infty \rangle \text{ or } \infty & \text{the series is divergent} \end{cases}$$

To evaluate for a power series take the absolute value of the ratio:

$$\lim_{k \rightarrow \infty} \left| \frac{c_{k+1}(x-a)^{k+1}}{c_k(x-a)^k} \right| = \lim_{k \rightarrow \infty} \left| \frac{c_{k+1}}{c_k} \right| |x-a|$$

Then to find radius of convergence:

$$l = \lim_{k \rightarrow \infty} \left| \frac{c_{k+1}}{c_k} \right| = \begin{cases} 0 & \text{the series converges for } x \in \mathbb{R} \\ \langle 0, \infty \rangle & \text{the series converges for } |x-a| < R = \frac{1}{l} \\ \infty & \text{the series converges only for } x = a \end{cases}$$

Integral Test

Find positive, continuous, and decreasing function where $f(n) = a_n$. If $\int_0^{\infty} f(x) dx$ converges the series will converge; if it diverges the series will diverge.

Parametric Functions

Both x and y defined in terms of third variable:

$$x = f(t)$$

$$y = g(t)$$

Eliminating the Parameter

Solve the for t in one of the equations and substitute or find another way to combine the two equations to eliminate t ;

Taking Derivatives

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt} \left(\frac{dy}{dx} \right)}{\frac{dx}{dt}}$$

Calculating Arc Length

$$\int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2} dt$$

Polar Functions

Points are identified by radius r and angle θ : (r, θ) . There exist infinitely many polar representations for the same point.

Equations are usually r in terms of θ : $r = f(\theta)$.

Relationships With Cartesian Points

$$r = \sqrt{x^2 + y^2}$$

$$x = r \cos(\theta)$$

$$y = r \sin(\theta)$$

Calculating Instantaneous Slope

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{d}{d\theta}(r \sin(\theta))}{\frac{d}{d\theta}(r \cos(\theta))} = \frac{\frac{dr}{d\theta} \sin(\theta) + r \cos(\theta)}{\frac{dr}{d\theta} \cos(\theta) - r \sin(\theta)}$$

Calculating Area Bounded by Curve

$$\frac{1}{2} \int_{\alpha}^{\beta} (r(\theta))^2 d\theta$$

Vector Functions

$$\vec{s}(t) = \langle f(t), g(t) \rangle$$

$$\vec{s}'(t) = \vec{v}(t) = \langle f'(t), g'(t) \rangle$$

$$\vec{s}''(t) = \vec{v}'(t) = \vec{a}(t) = \langle f''(t), g''(t) \rangle$$

$$\text{speed} = |\vec{v}| = \sqrt{(f'(t))^2 + (g'(t))^2}$$