

Double Angle Formulas

$$\sin(2x) = 2 \sin(x) \cos(x)$$

$$\cos(2x) = \cos^2(x) - \sin^2(x)$$

$$= 2 \cos^2(x) - 1$$

$$= 1 - 2 \sin^2(x)$$

$$\tan(2x) = \frac{2 \tan(x)}{1 - \tan^2(x)}$$

$$\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$$

$$\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$$

Limits

$$\lim_{x \rightarrow n} f(x) = L$$

Means for any number $\epsilon > 0$ there exists a number $\delta > 0$ such that if $0 < |x - n| < \delta$ then $|f(x) - L| < \epsilon$.

$$\lim_{x \rightarrow a} f(g(x)) = f\left(\lim_{x \rightarrow a} g(x)\right) \text{ if } f(g(x)) \text{ is continuous at } x$$

Standard difference quotient:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Symmetric difference quotient:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x-h)}{2h}$$

Continuity and Differentiability

$f(x)$ is continuous at x if:

$$f(x) = \lim_{k \rightarrow x^+} f(k) = \lim_{k \rightarrow x^-} f(k)$$

$f(x)$ is differentiable at x if:

$$f'_+(x) = f'_-(x)$$

Differentiation Rules

$$\begin{aligned}\frac{d}{dx}cf(x) &= cf'(x) \\ \frac{d}{dx}f(g(x)) &= f'(g(x))g'(x) \\ \frac{d}{dx}(f(x) \pm g(x)) &= f'(x) \pm g'(x) \\ \frac{d}{dx}f(x)^n &= nf(x)^{n-1} \cdot f'(x) \\ \frac{d}{dx}\sqrt{f(x)} &= \frac{f'(x)}{2\sqrt{f(x)}} \\ \frac{d}{dx}(f(x)g(x)) &= f'(x)g(x) + f(x)g'(x) \\ \frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) &= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} \\ \frac{d}{dx}|f(x)| &= \frac{|f(x)|}{f(x)} \cdot f'(x) \\ \frac{d}{dx}b^{f(x)} &= b^{f(x)} \ln b \cdot f'(x) \\ \frac{d}{dx}\log_b(f(x)) &= \frac{f'(x)}{f(x) \ln b}\end{aligned}$$

Trigonometric Differentiation Rules

$$\begin{aligned}\frac{d}{dx}\sin(f(x)) &= \cos(f(x)) \cdot f'(x) \\ \frac{d}{dx}\cos(f(x)) &= -\sin(f(x)) \cdot f'(x) \\ \frac{d}{dx}\tan(f(x)) &= \sec^2(f(x)) \cdot f'(x) \\ \frac{d}{dx}\sec(f(x)) &= \sec(f(x))\tan(f(x)) \cdot f'(x) \\ \frac{d}{dx}\csc(f(x)) &= -\csc(f(x))\cot(f(x)) \cdot f'(x) \\ \frac{d}{dx}\cot(f(x)) &= -\csc^2(f(x)) \cdot f'(x)\end{aligned}$$

Inverse Trigonometric Differentiation Rules

$$\begin{aligned}\frac{d}{dx}\sin^{-1}(f(x)) &= \frac{f'(x)}{\sqrt{1-f(x)^2}} \\ \frac{d}{dx}\cos^{-1}(f(x)) &= -\frac{f'(x)}{\sqrt{1-f(x)^2}} \\ \frac{d}{dx}\tan^{-1}(f(x)) &= \frac{f'(x)}{1+f(x)^2}\end{aligned}$$

Mean Value Theorem

If $f(x)$ is continuous on $[a, b]$, then there exists one value c on $[a, b]$ such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Reimann Sums

$$\sum_{k=1}^n f(x_k) \Delta x$$

Assuming regular partition:

$$\begin{aligned} \text{LRAM} &= \sum_{k=1}^n f\left(a + \frac{(k-1)(b-a)}{n}\right) \frac{b-a}{n} \\ \text{MRAM} &= \sum_{k=1}^n f\left(a + \frac{(k-0.5)(b-a)}{n}\right) \frac{b-a}{n} \\ \text{RRAM} &= \sum_{k=1}^n f\left(a + \frac{k(b-a)}{n}\right) \frac{b-a}{n} \end{aligned}$$

Definite Integration

$$\begin{aligned} \int_a^b f(x) \, dx &= \lim_{n \rightarrow \infty} \sum_{k=1}^n f(x_k) \Delta x \\ \int_a^a f(x) \, dx &= 0 \\ \int_a^b f(x) \, dx &= - \int_b^a f(x) \, dx \\ \int_a^b c f(x) \, dx &= c \int_a^b f(x) \, dx \\ \int_a^b (f(x) \pm g(x)) \, dx &= \int_a^b f(x) \, dx \pm \int_a^b g(x) \, dx \\ \int_a^p f(x) \, dx + \int_p^b f(x) \, dx &= \int_a^b f(x) \, dx \\ \int_a^b f(x) \, dx &> \int_a^b g(x) \, dx \text{ if } f(x) > g(x) \text{ on } [a, b] \end{aligned}$$

The mean value of $f(x)$ over $[a, b]$ is:

$$\frac{1}{b-a} \int_a^b f(x) \, dx$$

Mean Value of a Function Theorem

If $f(x)$ is continuous on $[a, b]$, then there exists one value c on $[a, b]$ such that:

$$f(c) = \frac{1}{b-a} \int_a^b f(x) \, dx$$

Linear Approximation

The linearization of $f(x)$ at a :

$$L(x) = f(a) + f'(a)(x - a)$$

Fundamental Theorem of Calculus

Part 1

$$\frac{d}{dx} \int_{a(x)}^{b(x)} f(t) \, dt = f(b(x))b'(x) - f(a(x))a'(x)$$

Part 2

$$F'(x) = f(x)$$
$$\int_a^b f(x) \, dx = F(x) \Big|_a^b = F(b) - F(a)$$

Solving Differential Equations Using A Definite Integral

$$\frac{dy}{dx} = f(x)$$
$$y = \int_a^x f(t) \, dt + C$$

The easiest solution occurs when a = the given x value and C = the given y value.

Solving Differential Equations Using Antidifferentiation

$$\frac{dy}{dx} = f(x)$$
$$y = \int f(x) \, dx + C \text{ (general form)}$$

Specify largest domain of x that includes specified x and does not include any discontinuities.

Euler's Method

Start at (x_0, y_0)

$$\Delta y = \Delta x \cdot \left. \frac{dy}{dx} \right|_{(x_{n-1}, y_{n-1})}$$
$$(x_n, y_n) = (x_{n-1} + \Delta x, y_{n-1} + \Delta y)$$

Antidifferentiation Rules

$$\int c f(x) \, dx = c \int f(x) \, dx$$
$$\int f(x) \pm g(x) \, dx = \int f(x) \, dx \pm \int g(x) \, dx$$
$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$$
$$\int \frac{1}{x} \, dx = \ln |x| + C$$
$$\int b^x \, dx = \frac{b^x}{\ln b} + C$$
$$\int \ln x \, dx = x \ln x - x + C$$

Trigonometric Antidifferentiation Rules

$$\begin{aligned}\int \sin(x) \, dx &= -\cos(x) + C \\ \int \cos(x) \, dx &= \sin(x) + C \\ \int \tan(x) \, dx &= -\ln |\cos(x)| + C \\ \int \sec(x) \, dx &= \ln |\sec(x) + \tan(x)| + C \\ \int \sin^2(x) \, dx &= \frac{1}{2}(x - \sin(x) \cos(x)) + C \\ \int \cos^2(x) \, dx &= \frac{1}{2}(x + \sin(x) \cos(x)) + C \\ \int \tan^2(x) \, dx &= \tan(x) - x + C \\ \int \sec^2(x) \, dx &= \tan(x) + C \\ \int \sec(x) \tan(x) \, dx &= \sec(x) + C \\ \int \frac{1}{1+x^2} \, dx &= \tan^{-1}(x) + C \\ \int \frac{1}{\sqrt{1-x^2}} \, dx &= \sin^{-1}(x) + C\end{aligned}$$

Inverse Trigonometric Antidifferentiation Rules

$$\begin{aligned}\int \sin^{-1}(x) \, dx &= x \sin^{-1}(x) + \sqrt{1-x^2} \\ \int \tan^{-1}(x) \, dx &= x \tan^{-1}(x) - \frac{1}{2} \ln(1+x^2) + C\end{aligned}$$

U-Substitution

To differentiate $\int f(g(x)) \, dx$ where $f(x)$ and $g(x)$ are both antiderivable, set $u = g(x)$. Then solve for du and replace dx so that the equation $A \int f(u) \, du$ results (where A is the necessary “fudge factor”). Antidifferentiate and replace u with $g(x)$.

Integration by Parts

$$\int u \, dv = uv - \int v \, du + C$$

or

$$\begin{array}{rcl}
+ & u & dv \\
- & \frac{du}{dx} & \int dv \, dx \\
+ & \frac{d^2u}{dx^2} & \int (\int dv \, dx) \, dx \\
- & \frac{d^3u}{dx^3} & \int (\int (\int dv \, dx) \, dx) \, dx \\
+ & \dots & \dots \\
& 0 & \dots
\end{array}$$

$$\int u \, dv = \left(u \cdot \int dv \, dx \right) - \left(\frac{du}{dx} \cdot \int \left(\int dv \, dx \right) dx \right) + \left(\frac{d^2u}{dx^2} \cdot \int \left(\int \left(\int dv \, dx \right) dx \right) dx \right) - \dots$$

Partial Fraction Decomposition

Convert fractions of form $\frac{\text{linear or constant}}{\text{factorable quadratic}}$ into its partial fraction.

$$\begin{aligned}
\frac{h(x)}{f(x)g(x)} &= \frac{A}{f(x)} + \frac{B}{g(x)} = \frac{Ag(x)}{f(x)g(x)} + \frac{Bf(x)}{f(x)g(x)} = \frac{Ag(x) + Bf(x)}{f(x)g(x)} \\
h(x) &= Ag(x) + Bf(x)
\end{aligned}$$

Choose a value of x such that $f(x) = 0$ and then $g(x) = 0$ to solve for A and B respectively.

Exponential Growth/Decay

$$\begin{aligned}
\frac{dQ}{dt} &= kQ \\
Q &= Q_0 e^{kt} = Q_0 b^{\frac{t}{p}} \\
k &= -\frac{\ln(b)}{p}
\end{aligned}$$

Half-life

$$k = -\frac{\ln(2)}{p}$$

Logistic Differential Equation

$$\frac{dP}{dt} = kP(m - P) = kPm \left(1 - \frac{P}{m} \right)$$

where P is the population and M is the carrying capacity

$$P(t) = \frac{m}{1 + Ae^{-mkt}}$$

The maximum carrying capacity is reached when $P(t) = m - 0.5$, if $P(0) < m$, or right after $P(t) = m + 0.5$, if $P(0) > m$.

Calculating Area of Regions Bounded by Two Functions

$$A = \int_a^b f(x) - g(x) \, dx$$

where $f(x)$ is the upper function and $g(x)$ is the lower function.

If the functions cross, multiple integrals must be taken with the appropriate upper and lower functions.

Typical Cross-Sectional Areas

Square: $A = b^2 = \frac{1}{2}d^2$

Rectangle: $A = bh$

Equalateral triangle: $A = \frac{\sqrt{3}}{4}b^2$

Right isoceles triangle: $A = \frac{1}{2}l^2 = \frac{1}{4}h^2$

Hexagon: $A = \frac{3\sqrt{3}}{2}s^2$

Circle: $A = \pi r^2 = \frac{1}{4}\pi d^2$

Semicircle: $A = \frac{1}{2}\pi r^2 = \frac{1}{8}\pi d^2$

Parabolic region: $A = \frac{2}{3}bh$

Calculating Volume Using the Disk/Washer Method for Solids of Rotation

$$V = \pi \int_a^b r_{\text{outer}}^2 - r_{\text{inner}}^2 \, dt$$

where dt is the differential thickness of the disk/washer.

Calculating Volume Using the Cylindrical Shell Method for Solids of Rotation

$$V = 2\pi \int_a^b hr \, dt$$

where dt is the differential thickness of the shell.

Calculating Arc Length

$$l = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

L'Hôpital's Rule

In order to evaluate a limit where

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} \Rightarrow \frac{0}{0} \text{ or } \frac{\infty}{\infty}$$

evaluate

$$\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

In order to evaluate a limit where

$$\lim_{x \rightarrow a} f(x)^{g(x)} \Rightarrow "0^0" \text{ or } "1^\infty" \text{ or } "\infty^0"$$

set L equal to the original limit and immediately take the log of both sides

$$L = \lim_{x \rightarrow a} f(x)^{g(x)}$$

$$\ln(L) = \ln\left(\lim_{x \rightarrow a} f(x)^{g(x)}\right)$$

$$\ln(L) = \lim_{x \rightarrow a} g(x) \ln f(x)$$

Evaluate the resulting limit and then raise e to that result

Relative Rates of Growth

To compare rates of $f(x)$ and $g(x)$ evaluate

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \begin{cases} \infty & \text{then } f(x) \text{ grows faster than } g \\ \langle 0, \infty \rangle & \text{then } f(x) \text{ and } g(x) \text{ grow at the same rate} \\ 0 & \text{then } f(x) \text{ grows slower than } g(x) \end{cases}$$

Improper Integrals

Type 1

$$\int_a^\infty f(x) dx = \lim_{n \rightarrow \infty} \int_a^n f(x) dx$$

$$\int_{-\infty}^b f(x) dx = \lim_{n \rightarrow \infty} \int_{-n}^b f(x) dx$$

$$\int_{-\infty}^\infty f(x) dx = \lim_{n \rightarrow \infty} \int_{-n}^n f(x) dx$$

Type 2

$$\int_a^b f(x)$$

such that there is at least one vertical-asymptotic discontinuity on $[a, b]$.

$$\begin{aligned}\int_a^b f(x) \, dx \text{ (where discontinuity at } a) &= \lim_{n \rightarrow a^+} \int_n^b f(x) \, dx \\ \int_a^b f(x) \, dx \text{ (where discontinuity at } b) &= \lim_{n \rightarrow b^-} \int_a^n f(x) \, dx \\ \int_a^b f(x) \, dx \text{ (where discontinuity } d \text{ on } \langle a, b \rangle) &= \lim_{n \rightarrow d^-} \int_a^n f(x) \, dx + \lim_{n \rightarrow d^+} \int_n^b f(x) \, dx\end{aligned}$$

P-Integrals

An integral that fits the pattern:

$$\int_1^\infty \frac{1}{x^p} \, dx \text{ where } p \text{ is positive}$$

$$p > 1 \Rightarrow \text{the integral converges to } \frac{1}{p-1}.$$

$$0 < p \leq 1 \Rightarrow \text{the integral diverges.}$$

Comparison Tests

For type 1 improper integrals ($a = \infty$ and/or $b = \infty$) where the integrand is not easily antidifferentiable:

$$\int_a^b f(x) \, dx$$

Establish bound functions which can be antidifferentiated:

$$l(x) \leq f(x) \leq u(x)$$

$$\text{Convergence if } \int_a^b f(x) \, dx \leq \int_a^b u(x) \, dx \text{ and } \int_a^b u(x) \, dx \text{ converges}$$

$$\text{Divergence if } \int_a^b f(x) \, dx \geq \int_a^b l(x) \, dx \text{ and } \int_a^b l(x) \, dx \text{ diverges}$$

Sequences

Explicit Definition

$$x_k = f(k)$$

where $k \in \{1, 2, 3, \dots\}$.

Recursive Definition

$$x_k = m(x_{k-1})$$

where x_1 is defined and $k \in \{2, 3, 4, \dots\}$.

Arithmetic

$$x_k = d(k - 1) + x_1$$

$$x_k = t_{k-1} + d$$

Geometric

$$x_k = x_1 r^{k-1}$$

$$x_k = x_{k-1} r$$

Convergence/Divergence

A sequence converges if the limit $\lim_{n \rightarrow \infty} x_n$ is a finite number else it diverges.

Infinite Series

$$\sum_{k=0}^{\infty} x_k$$

Convergence

Consider the sequence of “partial sums” (s) where:

$$s_n = \sum_{k=0}^n x_k$$

If $\{s_n\}$ converges to S then the infinite series converges to S .

Power Series

$$\sum_{k=0}^{\infty} c_k (x - a)^k$$

Taylor/Maclaurin Series

Given an infinitely differentiable function $f(x)$, the Taylor series centered around a is equal to the power series with a specific c_k :

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x - a)^k$$

A Maclaurin series is a Taylor series centered around 0 ($a = 0$).

The n -order polynomial function $P_n(x)$ that best fits $f(x)$ is equal to the Taylor series evaluated with upper limit n :

$$P_n(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x - a)^k$$

Common Maclaurin Series

$$\frac{a}{1-x} = \sum_{k=0}^{\infty} ax^k \quad x \in \langle -1, 1 \rangle$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \quad x \in \mathbb{R}$$

$$\sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \quad x \in \mathbb{R}$$

$$\cos(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \quad x \in \mathbb{R}$$

$$\tan^{-1}(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} \quad x \in [-1, 1]$$

$$\ln(x+1) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k} \quad x \in \langle -1, 1 \rangle$$

Calculating Error for a Taylor Polynomial

$$P_{\infty}(x) = P_n(x) + R_n(x)$$

where P_{∞} is the entire infinite Taylor series and R_n is the remainder.

$$|P_n(x) - f(x)| = \text{magnitude of error}$$