

## Double Angle Formulas

$$\sin(2x) = 2 \sin(x) \cos(x)$$

$$\cos(2x) = \cos^2(x) - \sin^2(x)$$

$$= 2 \cos^2(x) - 1$$

$$= 1 - 2 \sin^2(x)$$

$$\tan(2x) = \frac{2 \tan(x)}{1 - \tan^2(x)}$$

$$\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$$

$$\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$$

## Limits

$$\lim_{x \rightarrow c} f(x) = L$$

Means for any number  $\epsilon > 0$  there exists a number  $\delta > 0$  such that if  $0 < |x - c| < \delta$  then  $|f(x) - L| < \epsilon$ .

$$\lim_{x \rightarrow a} f(g(x)) = f\left(\lim_{x \rightarrow a} g(x)\right) \text{ if } f(g(x)) \text{ is continuous at } x$$

Standard difference quotient:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Symmetric difference quotient:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x-h)}{2h}$$

## Continuity and Differentiability

$f(x)$  is continuous at  $x$  if:

$$f(x) = \lim_{k \rightarrow x^+} f(k) = \lim_{k \rightarrow x^-} f(k)$$

$f(x)$  is differentiable at  $x$  if:

$$f'_+(x) = f'_-(x)$$

## Differentiation Rules

$$\begin{aligned}\frac{d}{dx}cf(x) &= cf'(x) \\ \frac{d}{dx}f(g(x)) &= f'(g(x))g'(x) \\ \frac{d}{dx}(f(x) \pm g(x)) &= f'(x) \pm g'(x) \\ \frac{d}{dx}f(x)^n &= nf(x)^{n-1} \cdot f'(x) \\ \frac{d}{dx}\sqrt{f(x)} &= \frac{f'(x)}{2\sqrt{f(x)}} \\ \frac{d}{dx}(f(x)g(x)) &= f'(x)g(x) + f(x)g'(x) \\ \frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) &= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} \\ \frac{d}{dx}|f(x)| &= \frac{|f(x)|}{f(x)} \cdot f'(x) \\ \frac{d}{dx}b^{f(x)} &= b^{f(x)} \ln b \cdot f'(x) \\ \frac{d}{dx}\log_b(f(x)) &= \frac{f'(x)}{f(x) \ln b}\end{aligned}$$

## Trigonometric Differentiation Rules

$$\begin{aligned}\frac{d}{dx}\sin(f(x)) &= \cos(f(x)) \cdot f'(x) \\ \frac{d}{dx}\cos(f(x)) &= -\sin(f(x)) \cdot f'(x) \\ \frac{d}{dx}\tan(f(x)) &= \sec^2(f(x)) \cdot f'(x) \\ \frac{d}{dx}\sec(f(x)) &= \sec(f(x))\tan(f(x)) \cdot f'(x) \\ \frac{d}{dx}\csc(f(x)) &= -\csc(f(x))\cot(f(x)) \cdot f'(x) \\ \frac{d}{dx}\cot(f(x)) &= -\csc^2(f(x)) \cdot f'(x)\end{aligned}$$

## Inverse Trigonometric Differentiation Rules

$$\begin{aligned}\frac{d}{dx}\sin^{-1}(f(x)) &= \frac{f'(x)}{\sqrt{1-f(x)^2}} \\ \frac{d}{dx}\cos^{-1}(f(x)) &= -\frac{f'(x)}{\sqrt{1-f(x)^2}} \\ \frac{d}{dx}\tan^{-1}(f(x)) &= \frac{f'(x)}{1+f(x)^2}\end{aligned}$$

## Mean Value Theorem

If  $f(x)$  is continuous on  $[a, b]$ , then there exists one value  $c$  on  $[a, b]$  such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

## Riemann Sums

$$\sum_{k=1}^n f(x_k) \Delta x$$

Assuming regular partition:

$$\begin{aligned} \text{LRAM} &= \sum_{k=1}^n f\left(a + \frac{(k-1)(b-a)}{n}\right) \frac{b-a}{n} \\ \text{MRAM} &= \sum_{k=1}^n f\left(a + \frac{(k-0.5)(b-a)}{n}\right) \frac{b-a}{n} \\ \text{RRAM} &= \sum_{k=1}^n f\left(a + \frac{k(b-a)}{n}\right) \frac{b-a}{n} \end{aligned}$$

## Definite Integration

$$\begin{aligned} \int_a^b f(x) \, dx &= \lim_{n \rightarrow \infty} \sum_{k=1}^n f(x_k) \Delta x \\ \int_a^a f(x) \, dx &= 0 \\ \int_a^b f(x) \, dx &= - \int_b^a f(x) \, dx \\ \int_a^b c f(x) \, dx &= c \int_a^b f(x) \, dx \\ \int_a^b (f(x) \pm g(x)) \, dx &= \int_a^b f(x) \, dx \pm \int_a^b g(x) \, dx \\ \int_a^p f(x) \, dx + \int_p^b f(x) \, dx &= \int_a^b f(x) \, dx \\ \int_a^b f(x) \, dx &> \int_a^b g(x) \, dx \text{ if } f(x) > g(x) \text{ on the interval } [a, b] \end{aligned}$$

The mean value of  $f(x)$  over  $[a, b]$  is:

$$\frac{1}{b-a} \int_a^b f(x) \, dx$$

## Mean Value of a Function Theorem

If  $f(x)$  is continuous on  $[a, b]$ , then there exists one value  $c$  on  $[a, b]$  such that:

$$f(c) = \frac{1}{b-a} \int_a^b f(x) \, dx$$

## Linear Approximation

The linearization of  $f(x)$  at  $a$ :

$$L(x) = f(a) + f'(a)(x - a)$$

## Fundamental Theorem of Calculus

### Part 1

$$\frac{d}{dx} \int_{a(x)}^{b(x)} f(t) \, dt = f(b(x))b'(x) - f(a(x))a'(x)$$

### Part 2

$$F'(x) = f(x)$$
$$\int_a^b f(x) \, dx = F(x) \Big|_a^b = F(b) - F(a)$$

## Solving Differential Equations Using A Definite Integral

$$\frac{dy}{dx} = f(x)$$
$$y = \int_a^x f(t) \, dt + C$$

The easiest solution occurs when  $a$  = the given  $x$  value and  $C$  = the given  $y$  value.

## Solving Differential Equations Using Antidifferentiation

$$\frac{dy}{dx} = f(x)$$
$$y = \int f(x) \, dx + C \text{ (general form)}$$

Specify largest domain of  $x$  that includes specified  $x$  and does not include any discontinuities.

## Euler's Method

Start at  $(x_0, y_0)$

$$\Delta y = \Delta x \cdot \left. \frac{dx}{dy} \right|_{(x_{n-1}, y_{n-1})}$$
$$(x_n, y_n) = (x_{n-1} + \Delta x, y_{n-1} + \Delta y)$$

## Antidifferentiation Rules

$$\int c f(x) dx = c \int f(x) dx$$
$$\int f(x) \pm g(x) dx = \int f(x) dx \pm \int g(x) dx$$
$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$
$$\int \frac{1}{x} dx = \ln |x| + C$$
$$\int b^x dx = \frac{b^x}{\ln b} + C$$
$$\int \ln x dx = x \ln x - x + C$$

## Trigonometric Antidifferentiation Rules

$$\begin{aligned}\int \sin(x) \, dx &= -\cos(x) + C \\ \int \cos(x) \, dx &= \sin(x) + C \\ \int \tan(x) \, dx &= -\ln |\cos(x)| + C \\ \int \sec(x) \, dx &= \ln |\sec(x) + \tan(x)| + C \\ \int \sin^2(x) \, dx &= \frac{1}{2}(x - \sin(x) \cos(x)) + C \\ \int \cos^2(x) \, dx &= \frac{1}{2}(x + \sin(x) \cos(x)) + C \\ \int \tan^2(x) \, dx &= \tan(x) - x + C \\ \int \sec^2(x) \, dx &= \tan(x) + C \\ \int \sec(x) \tan(x) \, dx &= \sec(x) + C \\ \int \frac{1}{1+x^2} \, dx &= \tan^{-1}(x) + C \\ \int \frac{1}{\sqrt{1-x^2}} \, dx &= \sin^{-1}(x) + C\end{aligned}$$

## Inverse Trigonometric Antidifferentiation Rules

$$\begin{aligned}\int \sin^{-1}(x) \, dx &= x \sin^{-1}(x) + \sqrt{1-x^2} \\ \int \tan^{-1}(x) \, dx &= x \tan^{-1}(x) - \frac{1}{2} \ln(1+x^2) + C\end{aligned}$$

## U-Substitution

To differentiate  $\int f(g(x)) \, dx$  where  $f(x)$  and  $g(x)$  are both antiderivable, set  $u = g(x)$ . Then solve for  $du$  and replace  $dx$  so that the equation  $\int f(u) \, du$  results. Antidifferentiate and replace  $u$  with  $g(x)$ .

## Integration by Parts

$$\int u \, dv = uv - \int v \, du + C$$

or

$$\begin{array}{ll}
 u & dv \\
 \frac{du}{dx} & \int dv \, dx \\
 \frac{d^2u}{dx^2} & \int \left( \int dv \, dx \right) dx \\
 \frac{d^3u}{dx^3} & \int \left( \int \left( \int dv \, dx \right) dx \right) dx \\
 \dots & \dots \\
 0 & \dots
 \end{array}$$

$$\int u \, dv = \left( u \cdot \int dv \, dx \right) - \left( \frac{du}{dx} \cdot \int \left( \int dv \, dx \right) dx \right) + \left( \frac{d^2u}{dx^2} \cdot \int \left( \int \left( \int dv \, dx \right) dx \right) dx \right) - \dots$$

## Partial Fraction Decomposition

Convert fractions of form  $\frac{\text{linear or constant}}{\text{factorable quadratic}}$  into its partial fraction.

$$\begin{aligned}
 \frac{h(x)}{f(x)g(x)} &= \frac{A}{f(x)} + \frac{B}{g(x)} = \frac{Ag(x)}{f(x)g(x)} + \frac{Bf(x)}{f(x)g(x)} = \frac{Ag(x) + Bf(x)}{f(x)g(x)} \\
 h(x) &= Ag(x) + Bf(x)
 \end{aligned}$$

Choose a value of  $x$  such that  $f(x) = 0$  and then  $g(x) = 0$  to solve for  $A$  and  $B$  respectively.

## Exponential Growth/Decay

$$\begin{aligned}
 \frac{dQ}{dt} &= kQ \\
 Q &= Q_0 e^{kt} = Q_0 b^{\frac{t}{p}} \\
 k &= -\frac{\ln(b)}{p}
 \end{aligned}$$

**Half-life**

$$k = -\frac{\ln(2)}{p}$$

## Logistic Differential Equation

$$\frac{dP}{dt} = kP(m - P) = kPm \left( 1 - \frac{P}{m} \right)$$

where  $P$  is the population and  $M$  is the carrying capacity

$$P(t) = \frac{m}{1 + Ae^{-mkt}}$$

The maximum carrying capacity is reached when  $P(t) = m - 0.5$ , if  $P(0) < m$ , or right after  $P(t) = m + 0.5$ , if  $P(0) > m$ .

## Calculating Area of Regions Bounded by Two Functions

$$A = \left| \int_a^b f(x) - g(x) \, dx \right|$$

## Typical Cross-Sectional Areas

Square:  $A = b^2 = \frac{1}{2}d^2$

Rectangle:  $A = bh$

Equalateral triangle:  $A = \frac{\sqrt{3}}{4}b^2$

Right isoceles triangle:  $A = \frac{1}{2}l^2 = \frac{1}{4}h^2$

Hexagon:  $A = \frac{3\sqrt{3}}{2}s^2$

Circle:  $A = \pi r^2 = \frac{1}{4}\pi d^2$

Semicircle:  $A = \frac{1}{2}\pi r^2 = \frac{1}{8}\pi d^2$

Parabolic region:  $A = \frac{2}{3}bh$

## Calculating Volume Using the Disk/Washer Method for Solids of Rotation

$$V = \pi \int_a^b r_{\text{outer}}^2 - r_{\text{inner}}^2 \, dt$$

where  $dt$  is the differential thickness of the disk/washer.

## Calculating Volume Using the Cylindrical Shell Method for Solids of Rotation

$$V = 2\pi \int_a^b hr \, dt$$

where  $dt$  is the differential thickness of the shell.

## Calculating Arc Lengh

$$l = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy$$



## L'Hôpital's Rule

In order to evaluate a limit where

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} \Rightarrow \frac{0}{0} \text{ or } \frac{\infty}{\infty}$$

instead evaluate

$$\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

In order to evaluate a limit where

$$\lim_{x \rightarrow a} f(x)^{g(x)} \Rightarrow 0^0 \text{ or } 1^\infty \text{ or } \infty^0$$

instead evaluate

$$e^{-\lim_{x \rightarrow a} \frac{f'(x)g(x)^2}{f(x)g'(x)}}$$

## Relative Rates of Growth

To compare rates of  $f(x)$  and  $g(x)$  evaluate

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \begin{cases} \infty & \text{then } f(x) \text{ grows faster than } g \\ \langle 0, \infty \rangle & \text{then } f(x) \text{ and } g(x) \text{ grow at the same rate} \\ 0 & \text{then } f(x) \text{ grows slower than } g(x) \end{cases}$$

## Improper Integrals

### Type 1

$$\int_a^\infty f(x) \, dx = \lim_{n \rightarrow \infty} \int_a^n f(x) \, dx$$

$$\int_{-\infty}^b f(x) \, dx = \lim_{n \rightarrow \infty} \int_{-n}^b f(x) \, dx$$

$$\int_{-\infty}^\infty f(x) \, dx = \lim_{n \rightarrow \infty} \int_{-n}^n f(x) \, dx$$

### Type 2

$$\int_a^b f(x)$$

such that there is at least one vertical-asymptotic discontinuity on  $[a, b]$ .

$$\begin{aligned}\int_a^b f(x) \, dx \text{ (where discontinuity at } a) &= \lim_{n \rightarrow a^+} \int_n^b f(x) \, dx \\ \int_a^b f(x) \, dx \text{ (where discontinuity at } b) &= \lim_{n \rightarrow b^-} \int_a^n f(x) \, dx \\ \int_a^b f(x) \, dx \text{ (where discontinuity } d \text{ on } \langle a, b \rangle) &= \lim_{n \rightarrow d^-} \int_a^n f(x) \, dx + \lim_{n \rightarrow d^+} \int_n^b f(x) \, dx\end{aligned}$$

## P-Integrals

An integral that fits the pattern:

$$\int_1^\infty \frac{1}{x^p} \, dx \text{ where } p \text{ is positive}$$

$$p > 1 \Rightarrow \text{the integral converges to } \frac{1}{p-1}.$$

$$0 < p \leq 1 \Rightarrow \text{the integral diverges.}$$

## Comparison Tests

For type 1 improper integrals ( $a = \infty$  and/or  $b = \infty$ ) where the integrand is not easily antiderivable:

$$\int_a^b f(x) \, dx$$

Establish bound functions which can be antiderivated:

$$l(x) \leq f(x) \leq u(x)$$

Convergence if  $\int_a^b f(x) \, dx \leq \int_a^b u(x) \, dx$  and  $\int_a^b u(x) \, dx$  converges

Divergence if  $\int_a^b f(x) \, dx \geq \int_a^b l(x) \, dx$  and  $\int_a^b l(x) \, dx$  diverges

## Sequences

### Explicit Definition

$$x_k = f(k)$$

where  $k \in \{1, 2, 3, \dots\}$ .

### Recursive Definition

$$x_k = m(x_{k-1})$$

where  $x_1$  is defined and  $k \in \{2, 3, 4, \dots\}$ .

## Arithmetic

$$x_k = d(k - 1) + x_1$$

$$x_k = t_{k-1} + d$$

## Geometric

$$x_k = x_1 r^{k-1}$$

$$x_k = x_{k-1} r$$

## Convergence/Divergence

A sequence converges if the limit  $\lim_{n \rightarrow \infty} x_n$  is a finite number else it diverges.

## Infinite Series

$$\sum_{k=1}^{\infty} x_k$$

### Convergence

Consider the sequence of “partial sums” ( $s$ ) where:

$$s_n = \sum_{k=1}^n x_k$$

If  $\{s_n\}$  converges to  $S$  then the infinite series converges to  $S$ .

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r} \text{ for } -1 < r < 1$$

### Power Series

$$\sum_{k=0}^{\infty} c_k (x - a)^k$$

### Taylor/Maclaurin Series

Given an infinitely differentiable function  $f(x)$ , the Taylor series centered around  $a$  is equal to the power series where

$$c_k = \frac{f^{(k)}(a)}{k!}$$

A Maclaurin series is a Taylor series centered around 0 ( $a = 0$ ).