

Double Angle Formulas

$$\sin(2x) = 2 \sin(x) \cos(x)$$

$$\cos(2x) = \cos^2(x) - \sin^2(x)$$

$$= 2 \cos^2(x) - 1$$

$$= 1 - 2 \sin^2(x)$$

$$\tan(2x) = \frac{2 \tan(x)}{1 - \tan^2(x)}$$

$$\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$$

$$\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$$

Limits

$$\lim_{x \rightarrow c} f(x) = L$$

Means for any number $\epsilon > 0$ there exists a number $\delta > 0$ such that if $0 < |x - c| < \delta$ then $|f(x) - L| < \epsilon$.

$$\lim_{x \rightarrow a} f(g(x)) = f\left(\lim_{x \rightarrow a} g(x)\right) \text{ if } f(g(x)) \text{ is continuous at } x$$

Standard difference quotient:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Symmetric difference quotient:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x-h)}{2h}$$

Continuity and Differentiability

$f(x)$ is continuous at x if:

$$f(x) = \lim_{k \rightarrow x^+} f(k) = \lim_{k \rightarrow x^-} f(k)$$

$f(x)$ is differentiable at x if:

$$f'_+(x) = f'_-(x)$$

Differentiation Rules

$$\begin{aligned}\frac{d}{dx}cf(x) &= cf'(x) \\ \frac{d}{dx}f(g(x)) &= f'(g(x))g'(x) \\ \frac{d}{dx}(f(x) \pm g(x)) &= f'(x) \pm g'(x) \\ \frac{d}{dx}f(x)^n &= nf(x)^{n-1} \cdot f'(x) \\ \frac{d}{dx}\sqrt{f(x)} &= \frac{f'(x)}{2\sqrt{f(x)}} \\ \frac{d}{dx}(f(x)g(x)) &= f'(x)g(x) + f(x)g'(x) \\ \frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) &= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} \\ \frac{d}{dx}|f(x)| &= \frac{|f(x)|}{f(x)} \cdot f'(x) \\ \frac{d}{dx}b^{f(x)} &= b^{f(x)} \ln b \cdot f'(x) \\ \frac{d}{dx}\log_b(f(x)) &= \frac{f'(x)}{f(x) \ln b}\end{aligned}$$

Trigonometric Differentiation Rules

$$\begin{aligned}\frac{d}{dx}\sin(f(x)) &= \cos(f(x)) \cdot f'(x) \\ \frac{d}{dx}\cos(f(x)) &= -\sin(f(x)) \cdot f'(x) \\ \frac{d}{dx}\tan(f(x)) &= \sec^2(f(x)) \cdot f'(x) \\ \frac{d}{dx}\sec(f(x)) &= \sec(f(x))\tan(f(x)) \cdot f'(x) \\ \frac{d}{dx}\csc(f(x)) &= -\csc(f(x))\cot(f(x)) \cdot f'(x) \\ \frac{d}{dx}\cot(f(x)) &= -\csc^2(f(x)) \cdot f'(x)\end{aligned}$$

Inverse Trigonometric Differentiation Rules

$$\begin{aligned}\frac{d}{dx}\sin^{-1}(f(x)) &= \frac{f'(x)}{\sqrt{1-f(x)^2}} \\ \frac{d}{dx}\cos^{-1}(f(x)) &= -\frac{f'(x)}{\sqrt{1-f(x)^2}} \\ \frac{d}{dx}\tan^{-1}(f(x)) &= \frac{f'(x)}{1+f(x)^2}\end{aligned}$$

Mean Value Theorem

If $f(x)$ is continuous on $[a, b]$, then there exists one value c on $[a, b]$ such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Riemann Sums

$$\sum_{k=1}^n f(x_k) \Delta x$$

Assuming regular partition:

$$\begin{aligned} \text{LRAM} &= \sum_{k=1}^n f\left(a + \frac{(k-1)(b-a)}{n}\right) \frac{b-a}{n} \\ \text{MRAM} &= \sum_{k=1}^n f\left(a + \frac{(k-0.5)(b-a)}{n}\right) \frac{b-a}{n} \\ \text{RRAM} &= \sum_{k=1}^n f\left(a + \frac{k(b-a)}{n}\right) \frac{b-a}{n} \end{aligned}$$

Definite Integration

$$\begin{aligned} \int_a^b f(x) \, dx &= \lim_{n \rightarrow \infty} \sum_{k=1}^n f(x_k) \Delta x \\ \int_a^a f(x) \, dx &= 0 \\ \int_a^b f(x) \, dx &= - \int_b^a f(x) \, dx \\ \int_a^b c f(x) \, dx &= c \int_a^b f(x) \, dx \\ \int_a^b (f(x) \pm g(x)) \, dx &= \int_a^b f(x) \, dx \pm \int_a^b g(x) \, dx \\ \int_a^p f(x) \, dx + \int_p^b f(x) \, dx &= \int_a^b f(x) \, dx \\ \int_a^b f(x) \, dx &> \int_a^b g(x) \, dx \text{ if } f(x) > g(x) \text{ on the interval } [a, b] \end{aligned}$$

The mean value of $f(x)$ over $[a, b]$ is:

$$\frac{1}{b-a} \int_a^b f(x) \, dx$$

Mean Value of a Function Theorem

If $f(x)$ is continuous on $[a, b]$, then there exists one value c on $[a, b]$ such that:

$$f(c) = \frac{1}{b-a} \int_a^b f(x) dx$$

Fundamental Theorem of Calculus

Part 1

$$\frac{d}{dx} \int_{a(x)}^{b(x)} f(t) dt = f(b(x))b'(x) - f(a(x))a'(x)$$

Part 2

$$F'(x) = f(x)$$
$$\int_a^b f(x) = F(x) \Big|_a^b = F(b) - F(a)$$

Solving Differential Equations Using A Definite Integral

$$\frac{dy}{dx} = f(x)$$
$$y = \int_a^x f(t) dt + C$$

The easiest solution occurs when a = the given x value and C = the given y value.

Solving Differential Equations With Antidifferentiation

$$\frac{dy}{dx} = f(x)$$
$$y = \int f(x) + C \text{ (general form)}$$

Specify largest domain of x that includes specified x and does not include any discontinuities.

Euler's Method

Start at (x_0, y_0)

$$\Delta y = \Delta x \cdot \left. \frac{dx}{dy} \right|_{(x_{n-1}, y_{n-1})}$$
$$(x_n, y_n) = (x_{n-1} + \Delta x, y_{n-1} + \Delta y)$$

Antidifferentiation Rules

$$\begin{aligned}\int cf(x) \, dx &= c \int f(x) \, dx \\ \int f(x) \pm g(x) \, dx &= \int f(x) \, dx \pm \int g(x) \, dx \\ \int x^n \, dx &= \frac{x^{n+1}}{n+1} + C \\ \int \frac{1}{x} \, dx &= \ln|x| + C \\ \int b^x \, dx &= \frac{b^x}{\ln b} + C \\ \int \ln x \, dx &= x \ln x - x + C\end{aligned}$$

Trigonometric Antidifferentiation Rules

$$\begin{aligned}\int \sin(x) \, dx &= -\cos(x) + C \\ \int \cos(x) \, dx &= \sin(x) + C \\ \int \tan(x) \, dx &= -\ln|\cos(x)| + C \\ \int \sin^2(x) \, dx &= \frac{1}{2}(x - \sin(x)\cos(x)) + C \\ \int \cos^2(x) \, dx &= \frac{1}{2}(x + \sin(x)\cos(x)) + C \\ \int \tan^2(x) \, dx &= \tan(x) - x + C \\ \int \sec^2(x) \, dx &= \tan(x) + C \\ \int \sec(x)\tan(x) \, dx &= \sec(x) + C \\ \int \frac{1}{1+x^2} \, dx &= \tan^{-1}(x) + C \\ \int \frac{1}{\sqrt{1-x^2}} \, dx &= \sin^{-1}(x) + C\end{aligned}$$

Inverse Trigonometric Antidifferentiation Rules

$$\begin{aligned}\int \sin^{-1}(x) \, dx &= x \sin^{-1}(x) + \sqrt{1-x^2} \\ \int \tan^{-1}(x) \, dx &= x \tan^{-1}(x) - \frac{1}{2} \ln(1+x^2) + C\end{aligned}$$

U-Substitution

To differentiate $\int f(g(x)) dx$ where $f(x)$ and $g(x)$ are both antiderifferentiable, set $u = g(x)$. Then solve for du and replace dx so that the equation $\int f(u) du$ results. Antidifferentiate and replace u with $g(x)$.

Integration by Parts

$$\int u dv = uv - \int v du + C$$

or

$$u \quad dv$$

$$\frac{du}{dx} \quad \int dv dx$$

$$\frac{d^2u}{dx^2} \quad \int (\int dv dx) dx$$

$$\frac{d^3u}{dx^3} \quad \int (\int (\int dv dx) dx) dx$$

$$\dots \quad \dots$$

$$0 \quad \dots$$

$$\int u dv = \left(u \cdot \int dv dx \right) - \left(\frac{du}{dx} \cdot \int \left(\int dv dx \right) dx \right) + \left(\frac{d^2u}{dx^2} \cdot \int \left(\int \left(\int dv dx \right) dx \right) dx \right) - \dots$$

Partial Fraction Decomposition

Convert fractions of form $\frac{\text{linear or constant}}{\text{factorable quadratic}}$ into its partial fraction.

$$\frac{h(x)}{f(x)g(x)} = \frac{A}{f(x)} + \frac{B}{g(x)} = \frac{Ag(x)}{f(x)g(x)} + \frac{Bf(x)}{f(x)g(x)} = \frac{Ag(x) + Bf(x)}{f(x)g(x)}$$

$$h(x) = Ag(x) + Bf(x)$$

Choose a value of x such that $f(x) = 0$ and then $g(x) = 0$ to solve for A and B respectively.

Exponential Growth/Decay

$$\frac{dQ}{dt} = kQ$$

$$Q = Q_0 e^{kt} = Q_0 b^{\frac{t}{p}}$$

$$k = -\frac{\ln(b)}{p}$$

Half-life

$$k = -\frac{\ln(2)}{p}$$

Logistic Differential Equation

$$\frac{dP}{dt} = kP(m - P) = kPm \left(1 - \frac{P}{m}\right)$$

where P is the population and M is the carrying capacity

$$P(t) = \frac{m}{1 + Ae^{-mkt}}$$

The maximum carrying capacity is reached when $P(t) = m - 0.5$, if $P(0) < m$, or right after $P(t) = m + 0.5$, if $P(0) > m$.

Calculating Area of Regions Bounded by Two Functions

$$A = \left| \int_a^b f(x) - g(x) \, dx \right|$$

Typical Cross-Sectional Areas

Square: $A = b^2 = \frac{1}{2}d^2$

Rectangle: $A = bh$

Equalateral triangle: $A = \frac{\sqrt{3}}{4}b^2$

Right isoceles triangle: $A = \frac{1}{2}l^2 = \frac{1}{4}h^2$

Hexagon: $A = \frac{3\sqrt{3}}{2}s^2$

Circle: $A = \pi r^2 = \frac{1}{4}\pi d^2$

Semicircle: $A = \frac{1}{2}\pi r^2 = \frac{1}{8}\pi d^2$

Parabolic region: $A = \frac{2}{3}bh$

Calculating Volume Using the Disk/Washer Method for Solids of Rotation

$$V = \pi \int_a^b r_{\text{outer}}^2 - r_{\text{inner}}^2 \, dt$$

where dt is the differential thickness of the disk/washer.

Calculating Volume Using the Cylindrical Shell Method for Solids of Rotation

$$V = 2\pi \int_a^b hr \, dt$$

where dt is the differential thickness of the shell.

Calculating Arc Length

$$l = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

Sequences

Explicit Definition

$$x_n = f(n)$$

where $n \in \{1, 2, 3, \dots\}$.

Recursive Definition

$$x_n = f(x_{n-1})$$

where x_1 is defined and $n \in \{2, 3, 4, \dots\}$.

Arithmetic

$$x_n = d(n - 1) + x_1$$

$$x_n = x_{n-1} + d$$

Geometric

$$x_n = x_1 r^{n-1}$$

$$x_n = x_{n-1} r$$

Convergence/Divergence

A sequence converges if the limit $\lim_{n \rightarrow \infty} x_n$ is a finite number else it diverges.

L'Hôpital's Rule

In order to evaluate a limit where

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} \Rightarrow \frac{0}{0} \text{ or } \frac{\infty}{\infty}$$

instead evaluate

$$\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

In order to evaluate a limit where

$$\lim_{x \rightarrow a} f(x)^{g(x)} \Rightarrow "0^0" \text{ or } "1^\infty" \text{ or } "\infty^0"$$

instead evaluate

$$e^{-\lim_{x \rightarrow a} \frac{f'(x)g(x)^2}{f(x)g'(x)}}$$

Relative Rates of Growth

To compare rates of $f(x)$ and $g(x)$ evaluate

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \begin{cases} \infty & \text{then } f(x) \text{ grows faster than } g \\ 0 < i < \infty & \text{then } f(x) \text{ and } g(x) \text{ grow at the same rate} \\ 0 & \text{then } f(x) \text{ grows slower than } g(x) \end{cases}$$

Improper Integrals

Type 1

$$\int_a^\infty f(x) \, dx = \lim_{n \rightarrow \infty} \int_a^n f(x) \, dx$$

$$\int_{-\infty}^b f(x) \, dx = \lim_{n \rightarrow \infty} \int_{-n}^b f(x) \, dx$$

$$\int_{-\infty}^\infty f(x) \, dx = \lim_{n \rightarrow \infty} \int_{-n}^n f(x) \, dx$$

Type 2

$$\int_a^b f(x)$$

such that there is at least one vertical-asymptotic discontinuity on $[a, b]$.

$$\begin{aligned}\int_a^b f(x) \, dx \text{ (where discontinuity at } a) &= \lim_{n \rightarrow a^+} \int_n^b f(x) \, dx \\ \int_a^b f(x) \, dx \text{ (where discontinuity at } b) &= \lim_{n \rightarrow b^-} \int_a^n f(x) \, dx \\ \int_a^b f(x) \, dx \text{ (where discontinuity } d \text{ on } \langle a, b \rangle) &= \lim_{n \rightarrow d^-} \int_a^n f(x) \, dx + \lim_{n \rightarrow d^+} \int_n^b f(x) \, dx\end{aligned}$$

Comparison Tests

For type 1 improper integrals ($a = \infty$ and/or $b = \infty$) where the integrand is not easily antiderivable:

$$\int_a^b f(x) \, dx$$

Establish bound functions which can be antiderivated:

$$l(x) \leq f(x) \leq u(x)$$

Convergence if $\int_a^b f(x) \, dx \leq \int_a^b u(x) \, dx$ where $\int_a^b u(x) \, dx$ converges

Divergence if $\int_a^b f(x) \, dx \geq \int_a^b l(x) \, dx$ where $\int_a^b l(x) \, dx$ diverges