Double Angle Formulas

$$\begin{aligned}
\sin(2x) &= 2\sin(x)\cos(x) \\
\cos(2x) &= \cos^2(x) - \sin^2(x) \\
&= 2\cos^2(x) - 1 \\
&= 1 - 2\sin^2(x) \\
\tan(2x) &= \frac{2\tan(x)}{1-\tan^2(x)} \\
\sin^2(x) &= \frac{1}{2}(1-\cos(2x)) \\
\cos^2(x) &= \frac{1}{2}(1+\cos(2x))
\end{aligned}$$

Limits

$$\lim_{x\to n} f(x) = L$$

Means for any number $\epsilon > 0$ there exists a number $\delta > 0$ such that if $0 < |x-n| < \delta$ then $|f(x) - L| < \epsilon$.

$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right)$ if $f(g(x))$ is continuous at x

Standard difference quotient:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Symmetric difference quotient:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x-h)}{2h}$$

Continuity and Differentiability

$f(x)$ is continuous at x if:

$$f(x) = \lim_{k\to x^+} f(k) = \lim_{k\to x^-} f(k)$$

$f(x)$ is differentiable at x if:

$$f'_+(x) = f'_-(x)$$

Differentiation Rules

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}x} c f(x) &= c f'(x) \\
\tfrac{\mathrm{d}}{\mathrm{d}x} f(g(x)) &= f'(g(x)) g'(x) \\
\tfrac{\mathrm{d}}{\mathrm{d}x} (f(x) \pm g(x)) &= f'(x) \pm g'(x) \\
\tfrac{\mathrm{d}}{\mathrm{d}x} f(x)^n &= n f(x)^{n-1} \cdot f'(x) \\
\tfrac{\mathrm{d}}{\mathrm{d}x} \sqrt{f(x)} &= \frac{f'(x)}{2\sqrt{f(x)}} \\
\tfrac{\mathrm{d}}{\mathrm{d}x} (f(x) g(x)) &= f'(x) g(x) + f(x) g'(x) \\
\tfrac{\mathrm{d}}{\mathrm{d}x} \left(\frac{f(x)}{g(x)}\right) &= \frac{f'(x) g(x) - f(x) g'(x)}{g(x)^2} \\
\tfrac{\mathrm{d}}{\mathrm{d}x} |f(x)| &= \frac{|f(x)|}{f(x)} \cdot f'(x) \\
\tfrac{\mathrm{d}}{\mathrm{d}x} b^{f(x)} &= b^{f(x)} \ln b \cdot f'(x) \\
\tfrac{\mathrm{d}}{\mathrm{d}x} \log_b(f(x)) &= \frac{f'(x)}{f(x) \ln b}
\end{aligned}$$

Trigonometric Differentiation Rules

$\tfrac{\mathrm{d}}{\mathrm{d}x} \sin(f(x)) = \cos(f(x)) \cdot f'(x)$

$\tfrac{\mathrm{d}}{\mathrm{d}x} \cos(f(x)) = -\sin(f(x)) \cdot f'(x)$

$\tfrac{\mathrm{d}}{\mathrm{d}x} \tan(f(x)) = \sec^2(f(x)) \cdot f'(x)$

$\tfrac{\mathrm{d}}{\mathrm{d}x} \sec(f(x)) = \sec(f(x)) \tan(f(x)) \cdot f'(x)$

$\tfrac{\mathrm{d}}{\mathrm{d}x} \csc(f(x)) = -\csc(f(x)) \cot(f(x)) \cdot f'(x)$

$\tfrac{\mathrm{d}}{\mathrm{d}x} \cot(f(x)) = -\csc^2(f(x)) \cdot f'(x)$

Inverse Trigonometric Differentiation Rules

$$\tfrac{\mathrm{d}}{\mathrm{d}x} \sin^{-1}(f(x)) = \frac{f'(x)}{\sqrt{1 - f(x)^2}}$$

$$\tfrac{\mathrm{d}}{\mathrm{d}x} \cos^{-1}(f(x)) = -\frac{f'(x)}{\sqrt{1 - f(x)^2}}$$

$$\tfrac{\mathrm{d}}{\mathrm{d}x} \tan^{-1}(f(x)) = \frac{f'(x)}{1 + f(x)^2}$$

Mean Value Theorem

If $f(x)$ is continuous on $[a, b]$, then there exists one value c on $[a, b]$ such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Reimann Sums

$$\sum_{k=1}^{n} f(x_k)\Delta x$$

Assuming regular partition:

$$\begin{aligned}
\text{LRAM} &= \sum_{k=1}^{n} f\left(a + \frac{(k-1)(b-a)}{n}\right)\frac{b-a}{n} \\
\text{MRAM} &= \sum_{k=1}^{n} f\left(a + \frac{(k-0.5)(b-a)}{n}\right)\frac{b-a}{n} \\
\text{RRAM} &= \sum_{k=1}^{n} f\left(a + \frac{k(b-a)}{n}\right)\frac{b-a}{n}
\end{aligned}$$

Definite Integration

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x$$

$$\int_a^a f(x)\,\mathrm{d}x = 0$$

$$\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x$$

$$\int_a^b cf(x)\,\mathrm{d}x = c\int_a^b f(x)\,\mathrm{d}x$$

$$\int_a^b (f(x) \pm g(x))\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x \pm \int_a^b g(x)\,\mathrm{d}x$$

$$\int_a^p f(x)\,\mathrm{d}x + \int_p^b f(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x$$

$$\int_a^b f(x)\,\mathrm{d}x > \int_a^b g(x)\,\mathrm{d}x \text{ if } f(x) > g(x) \text{ on } [a, b]$$

The mean value of $f(x)$ over $[a, b]$ is:

$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x$$

Mean Value of a Function Theorem

If $f(x)$ is continuous on $[a, b]$, then there exists one value c on $[a, b]$ such that:

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\, \mathrm{d}x$$

Linear Approximation

The linearization of $f(x)$ at a:

$$L(x) = f(a) + f'(a)(x-a)$$

Fundamental Theorem of Calculus

Part 1

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_{a(x)}^{b(x)} f(t)\, \mathrm{d}t = f(b(x))b'(x) - f(a(x))a'(x)$$

Part 2

$$\begin{aligned} F'(x) &= f(x) \\ \int_a^b f(x) &= F(x)\Big|_a^b = F(b) - F(a) \end{aligned}$$

Solving Differential Equations Using A Definite Integral

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= f(x) \\ y &= \int_a^x f(t)\, \mathrm{d}t + C \end{aligned}$$

The easiest solution occurs when $a =$ the given x value and $C =$ the given y value.

Solving Differential Equations Using Antidifferentiation

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= f(x) \\ y &= \int f(x) + C \text{ (general form)} \end{aligned}$$

Specify largest domain of x that includes specified x and does not include any discontinuities.

Euler's Method

Start at (x_0, y_0)

$$\Delta y = \Delta x \cdot \frac{\mathrm{d}y}{\mathrm{d}x}\bigg|_{(x_{n-1}, y_{n-1})}$$
$$(x_n . y_n) = (x_{n-1} + \Delta x, y_{n-1} + \Delta y)$$

Antidifferentiation Rules

$$\int cf(x)\,\mathrm{d}x = c\int f(x)\,\mathrm{d}x$$
$$\int f(x) \pm g(x)\,\mathrm{d}x = \int f(x)\,\mathrm{d}x \pm \int g(x)\,\mathrm{d}x$$
$$\int x^n\,\mathrm{d}x = \frac{x^{n+1}}{n+1} + C$$
$$\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + C$$
$$\int b^x\,\mathrm{d}x = \frac{b^x}{\ln b} + C$$
$$\int \ln x\,\mathrm{d}x = x\ln x - x + C$$

Trigonometric Antidifferentiation Rules

$$\int \sin(x)\,\mathrm{d}x = -\cos(x) + C$$
$$\int \cos(x)\,\mathrm{d}x = \sin(x) + C$$
$$\int \tan(x)\,\mathrm{d}x = -\ln|\cos(x)| + C$$
$$\int \sec(x)\,\mathrm{d}x = \ln|\sec(x) + \tan(x)| + C$$
$$\int \sin^2(x)\,\mathrm{d}x = \frac{1}{2}(x - \sin(x)\cos(x)) + C$$
$$\int \cos^2(x)\,\mathrm{d}x = \frac{1}{2}(x + \sin(x)\cos(x)) + C$$
$$\int \tan^2(x)\,\mathrm{d}x = \tan(x) - x + C$$
$$\int \sec^2(x)\,\mathrm{d}x = \tan(x) + C$$
$$\int \sec(x)\tan(x)\,\mathrm{d}x = \sec(x) + C$$
$$\int \frac{1}{1+x^2}\,\mathrm{d}x = \tan^{-1}(x) + C$$
$$\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \sin^{-1}(x) + C$$

Inverse Trigonometric Antidifferentiation Rules

$$\int \sin^{-1}(x) = x\sin^{-1}(x) + \sqrt{1-x^2}$$
$$\int \tan^{-1}(x) = x\tan^{-1}(x) - \frac{1}{2}\ln(1+x^2) + C$$

U-Substitution

To differentiate $\int f(g(x))\,\mathrm{d}x$ where $f(x)$ and $g(x)$ are both antidifferentiable, set $u = g(x)$. Then solve for du and replace dx so that the equation $A\int f(u)\,\mathrm{d}u$ results (where A is the necessary "fudge factor"). Antidifferentiate and replace u with $g(x)$.

Integration by Parts

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u + C$$

or

$$\begin{array}{ccl}
+ & u & \mathrm{d}v \\
- & \frac{\mathrm{d}u}{\mathrm{d}x} & \int \mathrm{d}v\,\mathrm{d}x \\
+ & \frac{\mathrm{d}^2u}{\mathrm{d}x^2} & \int \left(\int \mathrm{d}v\,\mathrm{d}x\right)\mathrm{d}x \\
- & \frac{\mathrm{d}^3u}{\mathrm{d}x^3} & \int \left(\int \left(\int \mathrm{d}v\,\mathrm{d}x\right)\mathrm{d}x\right)\mathrm{d}x \\
+ & \cdots & \cdots \\
 & 0 & \cdots
\end{array}$$

$$\int u\,\mathrm{d}v = \left(u\cdot\int \mathrm{d}v\,\mathrm{d}x\right) - \left(\tfrac{\mathrm{d}u}{\mathrm{d}x}\cdot\int\left(\int \mathrm{d}v\,\mathrm{d}x\right)\mathrm{d}x\right) + \left(\tfrac{\mathrm{d}^2u}{\mathrm{d}x^2}\cdot\int\left(\int\left(\int \mathrm{d}v\,\mathrm{d}x\right)\mathrm{d}x\right)\mathrm{d}x\right) - \ldots$$

Partial Fraction Decomposition

Convert fractions of form $\frac{\text{linear or constant}}{\text{factorable quadratic}}$ into its partial fraction.

$$\frac{h(x)}{f(x)g(x)} = \frac{A}{f(x)} + \frac{B}{g(x)} = \frac{Ag(x)}{f(x)g(x)} + \frac{Bf(x)}{f(x)g(x)} = \frac{Ag(x)+Bf(x)}{f(x)g(x)}$$
$$h(x) = Ag(x) + Bf(x)$$

Choose a value of x such that $f(x) = 0$ and then $g(x) = 0$ to solve for A and B respectively.

Exponential Growth/Decay

$\frac{\mathrm{d}Q}{\mathrm{d}t} = kQ$

$$Q = Q_0 e^{kt} = Q_0 b^{\frac{t}{p}}$$

$$k = -\frac{\ln(b)}{p}$$

Half-life

$$k = -\frac{\ln(2)}{p}$$

Logistic Differential Equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP(m - P) = kPm\left(1 - \frac{P}{m}\right)$$

where P is the population and M is the carrying capacity

$$P(t) = \frac{m}{1 + Ae^{-mkt}}$$

The maximum carrying capacity is reached when $P(t) = m - 0.5$, if $P(0) < m$, or right after $P(t) = m + 0.5$, if $P(0) > m$.

Calculating Area of Regions Bounded by Two Functions

$$A = \int_a^b f(x) - g(x)\,\mathrm{d}x$$

where $f(x)$ is the upper function and $g(x)$ is the lower function.

If the functions cross, multiple integrals must be taken with the appropriate upper and lower functions.

Typical Cross-Sectional Areas

Square:	$A = b^2 = \frac{1}{2}d^2$
Rectangle:	$A = bh$
Equalateral triangle:	$A = \frac{\sqrt{3}}{4}b^2$
Right isoceles triangle:	$A = \frac{1}{2}l^2 = \frac{1}{4}h^2$
Hexagon:	$A = \frac{3\sqrt{3}}{2}s^2$
Circle:	$A = \pi r^2 = \frac{1}{4}\pi d^2$
Semicircle:	$A = \frac{1}{2}\pi r^2 = \frac{1}{8}\pi d^2$
Parabolic region:	$A = \frac{2}{3}bh$

Calculating Volume Using the Disk/Washer Method for Solids of Rotation

$$V = \pi \int_a^b r_{\text{outer}}^2 - r_{\text{inner}}^2 \, \mathrm{d}t$$

where $\mathrm{d}t$ is the differential thickness of the disk/washer.

Calculating Volume Using the Cylindrical Shell Method for Solids of Rotation

$$V = 2\pi \int_a^b hr \, \mathrm{d}t$$

where $\mathrm{d}t$ is the differential thickness of the shell.

Calculating Arc Lengh

$$l = \int_a^b \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \, \mathrm{d}x = \int_c^d \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2} \, \mathrm{d}y$$

L'Hôpitals Rule

In order to evaluate a limit where

$$\lim_{x \to a} \frac{f(x)}{g(x)} \Rightarrow \text{“}\frac{0}{0}\text{”} \text{ or } \text{“}\frac{\infty}{\infty}\text{”}$$

evaluate

$$\lim_{x \to a} \frac{f'(x)}{g'(x)}$$

In order to evaluate a limit where

$$\lim_{x\to a} f(x)^{g(x)} \Rightarrow \text{“}0^0\text{” or “}1^\infty\text{” or “}\infty^0\text{”}$$

set L equal to the original limit and immediately take the log of both sides

$$\begin{aligned} L &= \lim_{x\to a} f(x)^{g(x)} \\ \ln(L) &= \ln\left(\lim_{x\to a} f(x)^{g(x)}\right) \\ \ln(L) &= \lim_{x\to a} g(x)\ln f(x) \end{aligned}$$

Evaluate the resulting limit and then raise e to that result

Relative Rates of Growth

To compare rates of $f(x)$ and $g(x)$ evaluate

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \begin{cases} \infty & \text{then } f(x) \text{ grows faster than } g \\ \langle 0,\infty\rangle & \text{then } f(x) \text{ and } g(x) \text{ grow at the same rate} \\ 0 & \text{then } f(x) \text{ grows slower than } g(x) \end{cases}$$

Improper Integrals

Type 1

$$\begin{aligned} \int_a^\infty f(x)\,\mathrm{d}x &= \lim_{n\to\infty}\int_a^n f(x)\,\mathrm{d}x \\ \int_{-\infty}^b f(x)\,\mathrm{d}x &= \lim_{n\to\infty}\int_{-n}^b f(x)\,\mathrm{d}x \\ \int_{-\infty}^\infty f(x)\,\mathrm{d}x &= \lim_{n\to\infty}\int_{-n}^n f(x)\,\mathrm{d}x \end{aligned}$$

Type 2

$$\int_a^b f(x)$$

such that there is at least one vertical-asymptotic discontinuity on $[a,b]$.

$$\begin{aligned} \int_a^b f(x)\,\mathrm{d}x \text{ (where discontinuity at } a) &= \lim_{n\to a^+}\int_n^b f(x)\,\mathrm{d}x \\ \int_a^b f(x)\,\mathrm{d}x \text{ (where discontinuity at } b) &= \lim_{n\to b^-}\int_a^n f(x)\,\mathrm{d}x \\ \int_a^b f(x)\,\mathrm{d}x \text{ (where discontinuity } d \text{ on } \langle a,b\rangle) &= \lim_{n\to d^-}\int_a^n f(x)\,\mathrm{d}x + \lim_{n\to d^+}\int_n^b f(x)\,\mathrm{d}x \end{aligned}$$

P-Integrals

An integral that fits the pattern:

$\int_1^\infty \frac{1}{x^p}\,\mathrm{d}x$ where p is positive

$p > 1 \Rightarrow$ the integral converges to $\frac{1}{p-1}$.

$0 < p \leq 1 \Rightarrow$ the integral diverges.

Comparison Tests

For type 1 improper integrals ($a = \infty$ and/or $b = \infty$) where the integrand is not easily antidifferentiable:

$\int_a^b f(x)\,\mathrm{d}x$

Establish bound functions which can be antidifferentiated:

$l(x) \leq f(x) \leq u(x)$

Convergence if $\int_a^b f(x)\,\mathrm{d}x \leq \int_a^b u(x)\,\mathrm{d}x$ and $\int_a^b u(x)\,\mathrm{d}x$ converges

Divergence if $\int_a^b f(x)\,\mathrm{d}x \geq \int_a^b l(x)\,\mathrm{d}x$ and $\int_a^b l(x)\,\mathrm{d}x$ diverges

Sequences

Explicit Definition

$x_k = f(k)$

where $k \in \{1, 2, 3, \dots\}$.

Recursive Definition

$x_k = m(x_{k-1})$

where x_1 is defined and $k \in \{2, 3, 4, \dots\}$.

Arithmetic

$x_k = d(k-1) + x_1$

$x_k = t_{k-1} + d$

Geometric

$x_k = x_1 r^{k-1}$

$x_k = x_{k-1} r$

Convergence/Divergence

A sequence converges if the limit $\lim_{n\to\infty} x_n$ is a finite number else it diverges.

Infinite Series

$$\sum_{k=0}^{\infty} x_k$$

Convergence

Consider the sequence of "partial sums" (s) where:

$$s_n = \sum_{k=0}^{n} x_k$$

If $\{s_n\}$ converges to S then the infinite series converges to S.

Power Series

$$\sum_{k=0}^{\infty} c_k (x-a)^k$$

Taylor/Maclaurin Series

Given an infinitely differentiable function $f(x)$, the Taylor series centered around a is equal to the power series with a specific c_k:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

A Maclaurin series is a Taylor series centered around 0 $(a = 0)$.

The n-order polynomial function $P_n(x)$ that best fits $f(x)$ is equal to the Taylor series evaluated with upper limit n:

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Common Maclaurin Series

$$\begin{aligned}
\frac{a}{1-x} &= \sum_{k=0}^{\infty} ax^k && x \in \langle -1, 1 \rangle \\
e^x &= \sum_{k=0}^{\infty} \frac{x^k}{k!} && x \in \mathbb{R} \\
\sin(x) &= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} && x \in \mathbb{R} \\
\cos(x) &= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} && x \in \mathbb{R} \\
\tan^{-1}(x) &= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} && x \in [-1, 1] \\
\ln(x+1) &= \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k} && x \in \langle -1, 1]
\end{aligned}$$

Calculating Error for a Taylor Polynomial

$P_\infty(x) = P_n(x) + R_n(x)$

where P_∞ is the entire infinite Taylor series and R_n is the remainder (note that the remainder series starts with $k = n + 1$).

$$\begin{aligned}
R_n(x) &= \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \\
|R_n(x)| &= \left| \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \right| \\
|R_n(x)| &= \frac{\left| f^{(n+1)}(c) \right|}{(n+1)!}|x-a|^{n+1}
\end{aligned}$$

For an unknown $c \in [0, x]$.

To bound the error:

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

Where M is a positive number greater than or equal to every possible value $f^{(n+1)}(t)$ where t is in the interval of study.

The series then converges if:

$$\lim_{n\to\infty} |R_n(x)| = 0$$