A Sharp Threshold for minimum bounded-depth and bounded-diameter spanning tree and Steiner trees in random networks

Literature survey on random networks

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Introduction

Background and Motivation

- Given a weighted graph G=(V,E) where edge weights are random and follow some sort of distribution we study minimum spanning tree or Steiner Tree as problem imposes a bound on the diameter or depth from a specific vertex.

MST on Random Graphs

MST on Random Graphs

Lot's of research has been going on to study the properties of MST's on Random Graph. Below two of them are well famous :

- The vertices of Graph G are the points of a **Poisson** point process in euclidean space with the edge weights being their euclidean distance between the points.
- The G is a complete graph with edge weights being i.i.d copies of a random variable. This distribution can be **anything**, maybe an exponential with mean 1 or a uniform number between 0 and 1.

Results

Background

- It can be showed that the expected cost of the minimum tree on the complete graph with edge weights distributed independently and uniformly between 0 and 1 tends to a constant as $n \to \infty$ and the constant is $\zeta(3) = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + ... = 1.202.$ [1].
- It is also shown that distribution of $w[MST(k_n)] \to \mathcal{N}(\zeta(3), \frac{6\zeta(4) - 4\zeta(3)}{n})$ with high probability. [2]

Background Continued...

- Now in the optimal tree most edge weights are close to 0 [as the weight density function that is 1 at weight 0], $w(MST(k_n)) \to \zeta(3)$ with high probability.[1, 3].
- The diameter of $MST(K_n)$ is $\theta(n^{\frac{1}{3}})$ with high probability and $E(diameter MST(k_n)) = \theta(n^{\frac{1}{3}})$ [4]

Theorem 1

For the complete graph K_n with EXP(1) edge weights, if $k = \log_2 log(n) + \omega(1)$ where $\omega(1) \to \infty$ then,
$w[MST_{depth \leq k}(K_n)] \to \zeta(3)$ and $w[MST_{diam \leq 2k}(K_n)] \to \zeta(3)$
in both probability and expectation.
This is tight as in when, $k = log_2 log(n) - \Delta$,
$w[MST_{depth \leq k}(K_n)] = exp(2^{\Delta + \theta(1)})$ and $w[MST_{diam \leq 2k}(K_n)] = exp(2^{\Delta + \theta(1)})$

- There is a sharp cutoff at depth $log_2 log(n) \pm \theta(1)$ and diameter $2 log_2 log(n) \pm \theta(1)$

Remarks

Remarks

- for a given set T of terminal vertices, $w[MST_{diam \leq 2k}(G,T)] \leq w[MST_{depth(r) \leq k}(G,T)]$
- The theorem also holds for other distributions.

Theorem 2

For any number m of terminal vertices $(2 \leq m \leq n)$, if $k \geq log_2 log(n) + \omega(n/(mlog(en/m)))$, and the edge weight probability distribution has density 1 at 0, then

$$\frac{w[MST_{depth \leq k}(K_n, m)]}{w[MST(K_n, m)]} \to 1 \tag{1}$$

and

$$\frac{w[MST_{diam \leq 2k}(K_n, m)]}{w[MST(K_n, m)]} \to 1 \tag{2}$$

in probability and if the expected edge weight cycle is finite, convergence holds in expectation too.

Theorem 2 Continued

- if $m = n$, theorem 2 will get the results as the first part of theorem 1
- But for Steiner tree, with general m we don't know if there exists a sharp bound, but if $m = theta(n)$, the sharp cutoff is $log_2 log(n) \pm \theta(1)$

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- if $m = n$, theorem 2 will get the results as the first part of theorem 1
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Theorem 3

If there are $m = n^{1-o(1)}$ terminal vertices and $2 \leq k \leq log_2 log(n) - log_2 log(en/m) - \omega(1)$ and the edge weight probability distribution has density 1 at 0 then,

$$w[MST_{depth \leq k}(K_n, m)] = (1 - 2^{-k} \pm o(1))\sqrt{\frac{8m}{n}}(\frac{\sqrt{2mn}}{2^k})^{\frac{1}{2^k - 1}} \tag{3}$$

,

$$w[MST_{diam \leq 2k}(K_n, m)] = (1 - 2^{-k} \pm o(1))\sqrt{\frac{8m}{n}}(\frac{\sqrt{2mn}}{2^k})^{\frac{1}{2^k - 1}} \tag{4}$$

and

$$w[MST_{diam \leq 2k+1}(K_n, m)] = (1 - 2^{-k} \pm o(1))\sqrt{\frac{8m}{n}}(\frac{\sqrt{mn/2}}{2^k})^{\frac{1}{2^k - 1}} \tag{5}$$

Theorem 3 Continued..

and if the expected edge weight is finite, convergence holds in expectation too.

- The second part of Theorem 1 follows from this Theorem 3 upon specializing to the case $m = n$ and $k = log_2 log(n) - \Delta$

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