STABILITY OF VARIETIES WITH A TORUS ACTION

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Contents

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In this thesis we study several problems related to the existence problem of invariant canonical metrics on Fano orbifolds in the presence of an effective algebraic torus action. The first chapter gives an introduction. The second chapter reviews the existing theory of T-varieties and reviews various stability thresholds and K-stability constructions which we make use of to obtain new results. In the third chapter we find new Kähler-Einstein metrics on some general arrangement varieties. In the fourth chapter we present a new formula for the greatest lower bound on Ricci curvature, an invariant which is now known to coincide with Tian's delta invariant. In the fifth chapter we discuss joint work with my supervisor to find new Kähler-Ricci solitons on smooth Fano threefolds admitting a complexity one torus action.

Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

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Chapter 1

Introduction

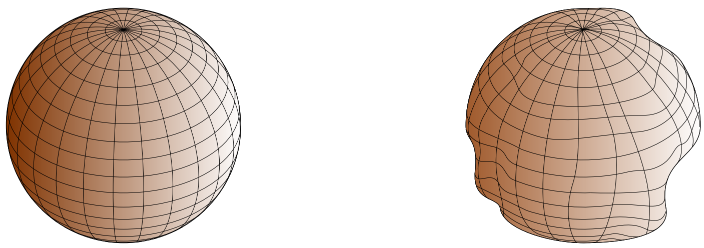
In this thesis we explore several new results relating to the existence of special metrics on certain compact Kähler manifolds admitting an effective algebraic torus action action. Our goal is to provide new examples to work with in the field, and further the understanding of canonical metrics on these types of manifolds. A Kähler manifold is a smooth manifold X adorned with mutually compatible Riemannian, complex, and symplectic structures. In this situation we call the Riemannian metric g the *Kähler metric* on X, and the symplectic form ω the *Kähler form.* An orbifold may be thought of as a generalization of a manifold, where we allow for very mild singularities, and the Kähler condition generalizes easily.

There are many reasons to study Kähler geometry. From the standpoint of algebraic geometry, every smooth complex projective variety inherits a Kähler structure. From a differential geometric perspective, Kähler manifolds are a particularly well-behaved class of Riemannian manifolds, and are a rich enough class to contain many interesting examples. There are also motivations from theoretical physics: the various models of our universe in string theory ask for extra planck-scale dimensions, and certain Kähler manifolds are the best fit for the shape of these dimensions.

Historically it has been an important problem in Kähler geometry to investigate which Kähler manifolds admit nice Kähler metrics. Generally what we mean here by "nice" depends on context. For naive motivation consider the real 2-sphere see Figure 1. Most would have in their mind the standard embedding $S^2 = \{x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$. There are however many choices of smooth embedding, each one corresponding to a different choice of Riemannian metric on

S^2. What sets our favourite embedding apart is that the induced metric is one of constant curvature.

This is a special case of a wider phenonemon if we identify the sphere as a Riemann surface $S^2 \cong \mathbb{P}^1_{\mathbb{C}}$. The uniformization theorem, originally proven by Poincaré [1] and Koebe [2,3], tells us that any Riemann surface admits a metric of constant scalar curvature. The obvious question is then: what happens in higher dimensions?



(a) Constant scalar curvature (b) non-constant scalar curvature

Figure 1.1: two choices of metric on S^2

In [4, 5], Calabi proved certain results for compact Kähler manifolds which lead to a famous conjecture. Fix a compact Kähler manifold (X, ω). Recall that the Ricci curvature form $\mathrm{Ric}(\omega)$ is a real $(1,1)$-form and defines a characteristic class $c_1(X) = \frac{1}{2\pi}[\mathrm{Ric}(\omega)]$ of the manifold, known as the first Chern class. Calibi asked whether, given a real $(1,1)$-form η representing the first Chern class of X, can we find a unique Kähler metric ω' in the same cohomology class as ω such that $\mathrm{Ric}(\omega') = 2\pi\eta$?

A related conjecture asked whether all compact Kähler manifolds (X, ω) admit a Kähler-Einstein metric, or more specifically whether they admit a Kähler form ω' in the same cohomology class as ω with $\mathrm{Ric}\,\omega' = \lambda\omega'$, for some real constant λ. This equation is known as the Einstein condition[1], and the Kähler metric corresponding to ω' is called a Kähler-Einstein metric. (metric of constant scalar curvature) It follows that for X to admit such a metric, $\mathrm{Ric}\,\omega'$ must be a definite $(1,1)$-form. This separates the problem into three cases: positive definite, zero,

[1]as it is analogous to Einstein's field equations in a vacuum.

and negative definite. In the first two cases Kähler-Einstein metrics on X are precisely the metrics of constant scalar curvature, and so one may see this as a direct generalization of the uniformization theorem for Riemann surfaces.

Aubin [6] and Yau [7] settled the negative definite case first. Calibi's conjecture was also proven by Yau in [7], later contributing to him being awarded the fields medal. This left the positive definite case, which correspond to smooth Fano varieties under the Kodeira embedding theorem. It was already known however, due to Matsushima [8], that not all Fano manifolds were Kähler-Einstein. It then became an objective to find suitable criterion for the existence of a Kähler-Einstein metric on a Fano manifold.

Matsushima had shown that necessary condition for a Kähler-Einstein metric was reductivity of the automorphism group of the manifold. In [9] Futaki introduced a new invariant whose vanishing was also a necessary condition. In [10] Tian introduced a sufficient condition in terms of another invariant, known now as *Tian's alpha invariant.* Tian's alpha invariant is a generalization of the complex singularity exponent of a polynomial $f \in \mathbb{C}[z_1, \ldots, z_n]$, which is defined as follows:

$$c_O(f) := \sup\{\epsilon |\ |f|^{-2\epsilon} \text{ integrable in a neighborhood of } O \in \mathbb{C}^n\}.$$

The Yau-Tian-Donaldson conjecture suggested the notion of K-*stability* as a necessary and sufficient Kähler-Einstein criterion[2]. This was proven in the trilogy of papers [11–13].

A generalization of the notion of a Kähler-Einstein metric is a Kähler-Ricci soliton. To understand how, recall that Kähler Einstein metrics may be seen as generalized fixed point solutions under the Kähler-Ricci flow:

$$\frac{d}{dt}\omega_t = -2\,\mathrm{Ric}(\omega_t)$$

In that under this flow they will remain unchanged up to some scaling factor. A Kähler-Ricci soliton is a generalized fixed point of the flow in the sense that it will remain unchanged up to some diffeomorphism (biholomorphism? check). (references for different things here)

A further generalization are twisted Kähler-Einstein metrics and twisted Kähler-Ricci solitons. These arise in continuity method arguments, see [14] for example, and depend on a parameter $t \in [0, 1]$. Here we start with a Calibi-Yau

[2]In full, the YTD talks of csck metrics, which are equivalent to KE in the Fano case

type solution ω_0 at $t = 0$, and consider the existence of solutions ω_t along a line segment to the target Kähler-Einstein or Kähler-Ricci soliton equation at $t = 1$ respectively. The supremum of the set of t for which a solution exists turns out to be independent of ω_0, and of a lot of interest as an invariant of X. It is often called the beta invariant, or the greatest lower bound on Ricci curvature. We will denote this invariant by $R(X)$.

Although K-stability is a criterion for the existence of Kähler-Einstein metrics and various generalizations, it is not an effective one. In general the K-stability of a Fano manifold is difficult to calculate. The alpha invariant approach also has limitations in practice. Fortunately equivariant versions of K-stability and Tian's alpha invariant exist, which, as we will see, provide an effective approach in classes of manifolds and orbifolds with lots of symmetry.

One class in particular we will explore is the class of Fano manifolds and orbifolds which are also T-varieties. A T-variety is a normal variety which admits the effective action of an algebraic torus $T = (\mathbb{C}^*)^r$. These are a generalization of toric varieties, where $\dim T = \dim X$. In general we call the difference $\dim X - \dim T$ the complexity of the torus action.

In the toric case it is well-established that studying X is equivalent to studying some associated combinatorial data: a fan of cones in a vector space built from the cocharacter lattice of T. Thanks to the work of many authors (Altmann, Hausen, Ilten, Petersen, Süß, Vollmert to name a few) this combinatorial description extends to higher complexity.

Equivariant methods have been used to provide some effective criteria for canonical metrics on low complexity T-varieties. If X is a Fano toric variety then the problem is completely solved. In [15] it was shown that X is Kähler-Einstein if and only if the Futaki character vanishes. They showed also that the Futaki character coincides with the barycenter of the lattice Polytope corresponding to X. Wang et al did not use K-stability for this result, but the result was later reproven as an application of the main theorem of [14].

In [16], Suess and Ilten considered the K-stability of T-varieties of complexity one. We recall this in detail in Section **??**. Complexity one Fano T-varieties have a combinatorial description. They obtained a combinatorial criterion for K-stability, generalizing the results of [15]. Süßhad also used the equivariant version of Tian's alpha invariant to find new Kähler-Einstein metrics on complexity one T-varieties admitting additional symmetries in [17].

In complexity two and above, even equivariant K-stability remains an ineffective criterion. In the next section we will give a summary of the new results presented in this thesis, one of which are some new examples of Kähler-Einstein T-manifolds of complexity two. As far as the author is aware, these are first complexity examples to be obtained through equivariant methods.

1.1 Content of the Thesis

Here I list the new results presented in this thesis. Some of the content of this thesis has been published and/or submitted to journals, which I will reference. I also make clear, in the case of my coauthored work, the scope of my contribution to the original paper. Chapters 4 and 5 summarize results obtained in my first and second years of my PhD respectively, and are included for completeness and context.

Chapter 3 - New Kähler-Ricci solitons on Fano threefolds

In Chapter **??** we consider Fano threefolds admitting an effective 2-torus action within the classification of [18]. In [19] a not necessarily complete list of such threefolds together with their combinatorial description was given. We extend the results of [16], providing new examples of threefolds admitting a non-trivial Kähler-Ricci soliton. Recall that a Kähler-Ricci soliton on a Fano manifold (X, ω) is a pair (ω', v) satisfying:

$$\mathrm{Ric}(\omega') - L_v\omega' = \omega'$$

We apply some real interval arithmentic approximations to the complexity one formula for the Futaki invariant of Ilten and Suess (see Section **??**) to check the existence criterion [14] (see Section **??**). We include the relevant Sagemath code in Appendix **??**.

Chapter 4 - The greatest lower bound on Ricci curvature in complexity one

In Chapter 2 we present an explicit effective formula obtained for the greatest lower bound on Ricci curvature $R(X)$ for a complexity one T-variety X. We follow the authors work [20]. These results generalize a result of Li [21], but the proof uses results of G-equivariant K-stability from [14]. The invariant $R(X)$ is often denoted $\beta(X)$ and is referred to as Tian's beta invariant. By [] is now known to coincide with another important invariant, $\delta(X)$.

Chapter 5 - New Kähler-Einstein metrics on symmetric general arrangement varieties

In Chapter **??**, we discuss recent results obtaining new Kähler-Einstein metrics on some symmetric complexity two general arrangement varieties. General arrangement varieties are T-varieties where the torus quotient is a projective space, and the critical values of the quotient map form a general arrangement of hyperplanes in that projective space. Smooth general arrangement varieties of complexity and Picard rank 2 were classified according to their Cox ring in [22]. Following the methods of [23], we find three new examples of Kähler-Einstein metrics. As far as we are aware, these are the first examples of Kähler-Einstein metrics found on T-varieties of complexity greater than one by way of equivariant methods.

Chapter 2

$R(X)$ in complexity one

Recall, as discussed in the introduction, that one approach to the existence of Kähler-Einstein metrics is the study of the continuity path, that is solutions $\omega_t \in 2\pi c_1(X)$ to the equation

$$\mathrm{Ric}(\omega_t) = t\omega_t + (1-t)\omega.$$

for $t \in [0,1]$. By [7] there is always a solution for $t = 0$. However, Tian [24] showed that for some t sufficiently close to 1 there may not be a solution for certain Fano manifolds. It is natural to ask for the supremum of permissible t, which turns out to be independent of the choice of ω.

Definition 1. *Let (X, ω) be a Kähler manifold with $\omega \in 2\pi c_1(X)$. Define:*

$$R(X) := \sup(t \in [0,1] : \exists\ \omega_t \in 2\pi c_1(X)\ \ \mathrm{Ric}(\omega_t) = t\omega_t + (1-t)\omega).$$

This invariant was first discussed, although not explicitly defined, by Tian in [25]. It was first explicitely defined by Rubenstein in [26] and was further studied by Szekelyhidi in [27]. It is sometimes referred to as the greatest lower bound on Ricci curvature.

In [26] Rubenstein showed relation between $R(X)$ and Tian's alpha invariant $\alpha(X)$, and in [28] conjectured that $R(X)$ characterizes the K-semistability of X. This conjecture was later verified by Li in [29].

In [30] Li determined a simple formula for $R(X_\Delta)$, where X_Δ is the polarized toric Fano manifold determined by a reflexive lattice polytope Δ. This result was later recovered in [14], by Datar and Székelyhidi, using notions of G-equivariant

K-stability. Previously $R(X)$ has been calculated for group compactifications by Delcroix [31] and for homogeneous toric bundles by Yao [32]. Let us briefly recall the toric formula.

Theorem 1 (Li)**.** *Suppose X is a smooth Fano toric variety. Let P be the corresponding Fano polytope. If* $\mathrm{bc}(P) = 0$ *then X is Kähler-Einstein and* $R(X) = 1$. *Otherwise let q be the intersection of the ray generated by* $-\mathrm{bc}(P)$ *with the boundary ∂P. We then have:*

$$R(X) = \frac{|q|}{|q - \mathrm{bc}(P)|}.$$

Example 1. *Consider the toric variety* $X = \mathrm{Bl}_z \mathbb{P}^1 \times \mathbb{P}^1$ *from Example* **??**. *It is then easy to calculate $R(X)$ from the polytope P given in Example* **??**. *We have* $\mathrm{bc}(P) = (-2/21, -2/21)$ *and* $q = (1/2, 1/2)$ *so* $R(X) = 21/25$.

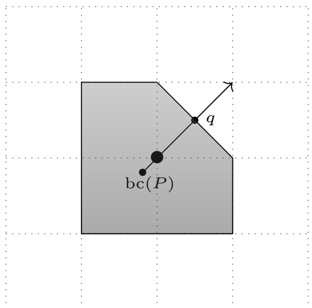


Figure 2.1: $R(X)$ calculation for a toric X

Using similar methods to [14] we obtain an effective formula for manifolds with a torus action of complexity one, in terms its divisorial polytope. Let $\Phi : \Box \to \mathrm{div}\mathbb{P}^1$ be the Fano divisorial polytope corresponding to a smooth Fano complexity one T-variety X. Let $\{\Delta_i\}_{i=1,\dots,r}$ be finite set of degeneration polytopes corresponding to central fibres of the non-product test configurations of X, as described in **??**.

To state our result we must introduce a little more notation. Suppose we have $\mathrm{bc}(\Delta_i) \neq 0$ for some i. Let F_i be the face of Δ_i in which q_i lies, and let S be the set of indices i for which $\mathrm{bc}(\Delta_i) \neq 0$ and all outer normals to F_i lie in H.

Recall the definition of the Duistermatt Heckman measure ν, and associated weighted barycenter of $\Box$ given in **??**. Suppose $\mathrm{bc}_\nu(\Box) \neq 0$. Let q be the intersection of the ray generated by $-\mathrm{bc}_\nu(\Box)$ with $\partial\Box$. Consider the halfspace

$H := N_{\mathbb{R}} \times \mathbb{R}^{+} \subset N'_{\mathbb{R}}$. Let q_i be the point of intersection of $\partial\Delta_i$ with the ray generated by $-\operatorname{bc}(\Delta_i)$.

Note, by the equation for the Donaldson Futaki invariants **??** and Theorem **??**, we know that $R(X) = 1$ iff $\operatorname{bc}_\nu(\Box) = 0$ and $S = \emptyset$. We may now state our result:

Theorem 2 ([20, Theorem 1.1])**.** *Let X be a complexity one Fano T-variety as above. If $\operatorname{bc}_\nu(\Box) = 0$ and $S = \emptyset$ then $R(X) = 1$. Otherwise:*

$$R(X) = \min \left\{ \frac{|q|}{|q - \operatorname{bc}_\nu(\Box)|} \right\} \cup \left\{ \frac{|q_i|}{|q_i - \operatorname{bc}(\Delta_i)|} \right\}_{i \in S}.$$

Example 2. *Consider the $(\mathbb{C}^*)^2$-threefold 2.30 from example* **??**. *There are 3 normal toric degenerations, given by the polytopes $\Delta_1, \ldots, \Delta_3$. It can be checked in this case that $S = \emptyset$, as for each i there is an outer normal $n_i \notin H$ to the face F_i. See Figure 1 (a) for an example.*

(a) The toric degeneration Δ_1

(b) The moment polytope $\Box$.

Figure 2.2: Determining $R(X)$ for threefold 2.30.

Therefore $R(X)$ is given by the first term in the minimum. We calculate $\operatorname{bc}_\nu(\Box) = (0, -6/23)$ *and $q = (0, 1)$. Then:*

$$R(X) = \frac{1}{1 + 6/23} = \frac{23}{29}.$$

Corollary 1 ([20, Corollary 1.2])**.** *In the table below we calculate $R(X)$ for X a Fano threefold admitting a 2-torus action appearing in the list of Mori and Mukai [18]. We include only those where $R(X) < 1$. Note all admit a Kähler-Ricci soliton by Theorem* **??**.

Table 2.1: Calculations for complexity 1 threefolds appearing in the list of Mori and Mukai for which $R(X) < 1$

X	R(X)
2.30	23/29
2.31	23/27
3.18	48/55
3.21	76/97
3.22	40/49
3.23	168/221
3.24	21/25
4.5*	64/69
4.8	76/89

2.1 A short digression into convex geometry

Let X be a T-variety of complexity one associated to a divisorial polytope $\Psi : \square \to \operatorname{div}(\mathbb{P}^1)$, see Section ??. It follows from Theorem ?? that:

$$R(X) = \inf_{(\mathcal{X},\mathcal{L})} (\sup(t| \operatorname{DF}_t(\mathcal{X},\mathcal{L}) \geq 0)),$$

where $(\mathcal{X},\mathcal{L})$ varies over all special test configurations for (X, L). We have an explicit description of special test configurations and their Donaldson Futaki invariants, see section ??. We will calculate $R(X)$ by considering first the product configurations and then the non-product ones. To calculate the values $\sup(t| \operatorname{DF}_t(\mathcal{X},\mathcal{L}) \geq 0)$ for a given configuration we need to first consider some elementary convex geometry.

Let V be a real vector space and $P \subset V$ be a convex polytope containing the origin, with $\dim P = \dim V$. Fix some point $b \in \operatorname{int}(P)$. Let $q \in \partial P$ be the intersection of ∂P with the ray $\tau = \mathbb{R}^+(-b)$. Suppose $n \in V^\vee$ is an outer normal to a face containing q. For $a \in \partial P$ write $\mathcal{N}(a) = \{w \in V^\vee \mid \langle a, w\rangle = \max_{x\in P}\langle x, c\rangle\}$. For $w \in \mathcal{N}(a)$ let $\Pi(a, w)$ be the affine hyperplane tangent to P at a with normal w. For $w \in \operatorname{int}(\tau^\vee)$ there is a well-defined point of intersection of $\Pi(a, w)$ and τ which we denote p_w. See Figure 2.3 for a schematic.

Lemma 1. *Fix $w \in int(\tau^\vee)\backslash(\mathbb{R}^+n)$. For $s \in [0, 1]$ set $w(s) := sn + (1 - s)w$. As*

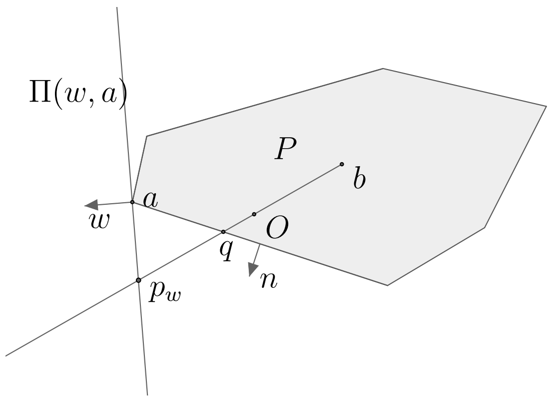


Figure 2.3: An Example in $V \cong \mathbb{R}^2$

$n \in \tau^\vee$ *we may consider* $p(s) := p_{w(s)}$. *For* $0 \leq s' < s \leq 1$ *we then have:*

$$\frac{|p(s)|}{|p(s) - b|} < \frac{|p(s')|}{|p(s') - b|}.$$

Proof. Without loss of generality we may assume $s' = 0$. For $s \in [0, 1]$ the points $p(s), q, b$ are collinear, so $|p(s)| = |p(s) - q| + |q|$ and $|p(s) - b| = |p(s) - q| + |q| + |b|$. Therefore:

$$\frac{|p(s)|}{|p(s) - b|} = \frac{|p(s) - q| + |q|}{|p(s) - q| + |q| + |b|}.$$

Hence it is enough for $|p(s) - q| < |p(0) - q|$ whenever $s > 0$. Since $q \neq 0$ is fixed this is equivalent to:

$$\frac{|p(s) - q|}{|q|} < \frac{|p(0) - q|}{|q|}.$$

For each $s \in [0, 1]$ choose $a(s) \in \partial P$ such that $w(s) \in \mathcal{N}(a(s))$. Write $a = a(0)$ for convenience. We then have:

$$\frac{|p(s) - q|}{|q|} = \frac{\langle a(s) - q, w \rangle}{\langle q, w \rangle}.$$

Note $n \in \mathcal{N}(q)$. Now $\langle a(s) - q, n \rangle \leq 0$ and $\langle a(s) - q, w \rangle \leq \langle a - q, w \rangle$. Clearly we have $\langle q, n\rangle > 0$. Then:

$$\begin{aligned}\frac{\langle a(s)-q, w(s)\rangle}{\langle q, w(s)\rangle} &= \frac{s\langle a(s)-q, n\rangle + (1-s)\langle a(s)-q, w\rangle}{s\langle q, n\rangle + (1-s)\langle q, w\rangle} \\ &\leq \frac{(1-s)\langle a-q, w\rangle}{s\langle q, n\rangle + (1-s)\langle q, w\rangle} \\ &< \frac{\langle a-q, w\rangle}{\langle q, w\rangle}.\end{aligned}$$

□

Corollary 2. *Let V, P, b, q, τ, n be as in the introduction to this section. Fix some open halfspace $H \subset V^\vee$ given by $u \geq 0$ for some $u \in V\backslash\{0\}$. This defines a projection map $\pi : V \to V/\langle u\rangle$. Consider the function $F_b : V^\vee \times [0,1] \to \mathbb{R}$ given by:*

$$F_b(w,t) := t\langle b, w\rangle + (1-t)\max_{x\in P}\langle x, w\rangle$$

For any $W \subseteq V^\vee$ containing n we have:

$$\sup(t \in [0,1] \mid \forall_{w\in W}\ F_b(t,w) \geq 0) = \frac{|q|}{|q-b|}. \tag{2.1}$$

If for some choice of n we have $n \notin H$ then:

$$\sup(t \in [0,1] \mid \forall_{w\in H}\ F_b(t,w) \geq 0) = \frac{|\tilde{q}|}{|\tilde{q}-\pi(b)|}, \tag{2.2}$$

where $\tilde{q}$ is the intersection of the ray $\pi(\tau)$ with the boundary of $\pi(P)$.

Proof. Note that:

$$\sup(t \in [0,1] \mid \forall_{w\in W}\ F_b(t,w) \geq 0) = \inf_{w\in W}\sup(t \in [0,1] \mid F_b(t,w) \geq 0).$$

Moreover $\sup(t \in [0,1] \mid F_b(t,w) \geq 0) = 1 > F_b(t,n)$ for $\langle b, w\rangle \geq 0$, so without loss of generality we may assume $W \subseteq \mathrm{int}(\tau^\vee)$. For $w \in W$ then:

$$\begin{aligned}\sup(t \in [0,1] \mid F_b(t,w) \geq 0) &= \frac{\max_{x\in P}\langle x, w\rangle}{\max_{x\in P}\langle x, w\rangle - \langle b, w\rangle} \\ &= \frac{\langle a, w\rangle}{\langle a, w\rangle - \langle b, w\rangle} \\ &= \frac{\langle p_w, w\rangle}{\langle p_w, w\rangle - \langle b, w\rangle} = \frac{|p_w|}{|p_w - b|}.\end{aligned}$$

Hence:

$$\sup(t \in [0,1] \mid \forall_{w \in W}\ F_b(t,w) \geq 0) = \inf_{w \in W} \frac{|p_w|}{|p_w - b|}.$$

Now for $w \in W$ consider the continuity path $w(s) = sn + (1-s)w$. By Lemma 1 if $n \in W$ then the above infimum is attained when $s = 1$ and we obtain (2.1). Otherwise the infimum is attained at some $w \in \partial W$. For (2.2) restricting F_b to $\partial H \times [0,1]$ gives:

$$F_b(w,t) = t\langle \pi(b), w\rangle + (1-t) \max_{x \in \pi(P)} \langle x, w\rangle.$$

Applying (2.1) to the polytope $\pi(P)$ in the vector space ∂H we obtain (2.2). □

2.2 Proof of Theorem 4

2.2.1 Product Configurations

Recall the formula for the Donaldson-Futaki invariant of a product configuration $\mathcal{X} \cong X \times \mathbb{A}^1$ given in section **??**. In particular the restriction of λ to $\mathcal{X}_0$ is given by a choice of $w \in N$, and we have:

$$\begin{aligned} \mathrm{DF}_t(\mathcal{X},\mathcal{L})(w) &= \mathrm{DF}(\mathcal{X},\mathcal{L})(w) + \frac{(1-t)}{V} \int_X (\max \theta_w - \theta_w)\omega^n \\ &= \langle \mathrm{bc}(\Psi), w\rangle + \frac{(1-t)}{\mathrm{vol}\,\Psi} \int_\Box \max_{x \in \Box} \langle x, w\rangle - \langle \cdot, w\rangle d\eta \\ &= t\langle \mathrm{bc}(\Psi), w\rangle + (1-t) \max_{x \in \Box} \langle x, w\rangle. \end{aligned}$$

Let $q \in N_{\mathbb{R}}$ be the point of intersection of the ray generated by $-\mathrm{bc}(\Psi)$ with $\partial\Box$. Applying (2.1), we obtain:

$$\sup(t | \mathrm{DF}_t(\mathcal{X},\mathcal{L}) \geq 0) = \frac{|q|}{|q - \mathrm{bc}_\nu(\Box)|}.$$

2.2.2 Non-Product Configurations

Recall the description of special non-product test configurations of X from **??**. In particular each such configuration $(\mathcal{X},\mathcal{L})$ has toric central fiber X_{Δ_i} Δ_i where Δ_i is one of the degeneration polytopes of **??**. The restriction of λ to $\mathcal{X}_0$ may be assumed to be given by some $v' = (-v, 1) \in N_{\mathbb{Q}}$. Recall the formula for the twisted Donaldson-Futaki from **??**:

$$\mathrm{DF}_t(\mathcal{X}, \mathcal{L}) = t\langle \mathrm{bc}(\Delta_y), v' \rangle + (1-t) \max_{x \in \Delta_y} \langle x, v' \rangle.$$

Set $H := N_{\mathbb{R}} \times \mathbb{R}^+$.

Proposition 1. *For any non-product configuration $(\mathcal{X}, \mathcal{L})$ with special fiber one of the Δ_i above, let σ_i be the cone of outer normals to Δ_i at the unique point of intersection of $\partial\Delta_i$ with the ray generated by $-\mathrm{bc}(\Delta_i)$. Denote this point of intersection by q_i. Then:*

$$\sup(t | \mathrm{DF}_t(\mathcal{X}, \mathcal{L}) \geq 0) = \begin{cases} \frac{|q_i|}{|q_i - \mathrm{bc}(\Delta_i)|} & \sigma_i \cap H \neq \emptyset; \\ \\ \frac{|q|}{|q - \mathrm{bc}_\nu(\square)|} & \sigma_i \cap H = \emptyset. \end{cases}$$

Proof. Extend $\mathrm{DF}_t(\mathcal{X}, \mathcal{L})$ linearly to the whole of $N_{\mathbb{R}} \times \mathbb{R}$. In the case $\sigma_i \cap H \neq \emptyset$ we may apply (1) from Corollary 2 with $P = \Delta_i$ and $b = \mathrm{bc}(\Delta_i)$. Otherwise we may apply (2.2), noting that $\pi(\Delta_i) = \square$ and $\pi(\mathrm{bc}(\Delta_i)) = \mathrm{bc}_\nu(\square)$. □

Proof of Theorem 2. With Remark 2 in mind, observe that a special test configuration must either be product or non-product. Any non-product configurations Δ_i with $\sigma_i \cap H \neq \emptyset$ have their contribution to the infimum already accounted for and we may exclude them. The result follows. □

of Corollary 1. We calculate unit outer normals n_i of F_i for every degeneration polytope of each threefold in this list, see A. In each case we observe $n_i \notin H$. The divisorial polytopes and Duistermaat-Heckman measures were originally given in [19], and may be also be found in Appendix A. We may then calculate $R(X)$ using just the base polytope $\square$ and its Duistermaat-Heckman barycenter. □

Appendix A

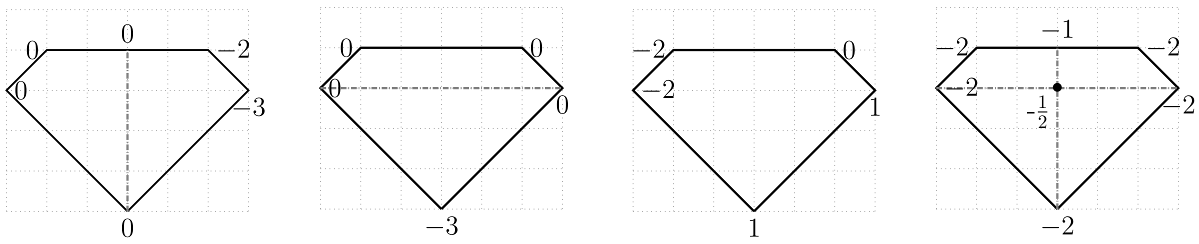
Threefold Data

In this Appendix we...

Key:

Name:	2.30C	**Description:**	Blow up of Q in a line

$$\sigma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad g(\xi) = \frac{1}{\xi_2^4} \cdot \left((2\,\xi_2^3 - 3\,\xi_2 - 3)e^{(4\,\xi_2)} + 12\,\xi_2 e^{(3\,\xi_2)} + 3\,\xi_2 + 3 \right) e^{(-3\,\xi_2)}, \quad R(X) = 23/29\,, \quad \xi \sim (0, 0.51489)$$



Φ_0 Φ_1 Φ_∞ $\deg\Phi$

y	$\mathrm{Vert}(\Delta_y)$	n_y	$h_y(\xi_2)$	$h_y(\xi_2) \in$
0	$\begin{pmatrix} -3 & -2 & 2 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -3 & 1 \\ 1 & 1 & -1 & -2 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} a \\ b \\ c \end{pmatrix}$	$\frac{1}{3\xi_2^4} \cdot (\ (2\xi_2^3 - 3\xi_2 - 3)e^{4\xi_2} + 12\xi_2 e^{3\xi_2} + 3\xi_2 + 3\ )e^{-3\xi_2}$	$(1.087, 1.458)$
1	$\begin{pmatrix} -3 & -2 & 2 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -3 & 1 \\ 1 & 1 & -1 & -2 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} a \\ b \\ c \end{pmatrix}$	$\frac{1}{3\xi_2^4} \cdot (\ (2\xi_2^3 - 3\xi_2 - 3)e^{4\xi_2} + 12\xi_2 e^{3\xi_2} + 3\xi_2 + 3\ )e^{-3\xi_2}$	$(1.087, 1.458)$
∞	$\begin{pmatrix} -3 & -2 & 2 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -3 & 1 \\ 1 & 1 & -1 & -2 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} a \\ b \\ c \end{pmatrix}$	$\frac{1}{3\xi_2^4} \cdot (\ (2\xi_2^3 - 3\xi_2 - 3)e^{4\xi_2} + 12\xi_2 e^{3\xi_2} + 3\xi_2 + 3\ )e^{-3\xi_2}$	$(1.087, 1.458)$

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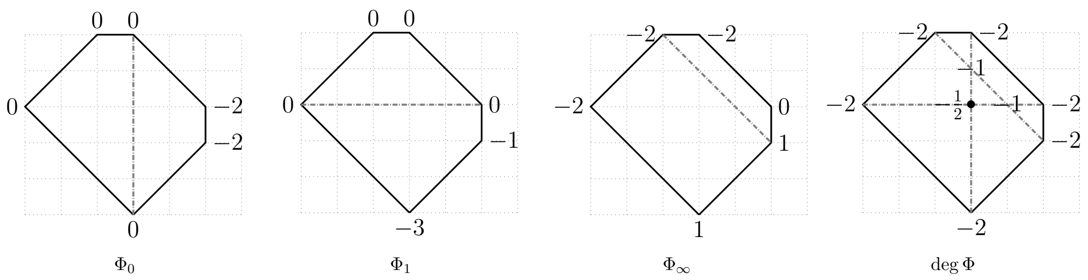

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0	$\begin{pmatrix} -3 & -2 & 2 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -3 & 1 \\ 1 & 1 & -1 & -2 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} a \\ b \\ c \end{pmatrix}$	$\frac{1}{3\xi_2^4} \cdot (\ (2\xi_2^3 - 3\xi_2 - 3)e^{4\xi_2} + 12\xi_2 e^{3\xi_2} + 3\xi_2 + 3\ )e^{-3\xi_2}$	$(1.087, 1.458)$
1	$\begin{pmatrix} -3 & -2 & 2 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -3 & 1 \\ 1 & 1 & -1 & -2 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} a \\ b \\ c \end{pmatrix}$	$\frac{1}{3\xi_2^4} \cdot (\ (2\xi_2^3 - 3\xi_2 - 3)e^{4\xi_2} + 12\xi_2 e^{3\xi_2} + 3\xi_2 + 3\ )e^{-3\xi_2}$	$(1.087, 1.458)$
∞	$\begin{pmatrix} -3 & -2 & 2 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -3 & 1 \\ 1 & 1 & -1 & -2 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} a \\ b \\ c \end{pmatrix}$	$\frac{1}{3\xi_2^4} \cdot (\ (2\xi_2^3 - 3\xi_2 - 3)e^{4\xi_2} + 12\xi_2 e^{3\xi_2} + 3\xi_2 + 3\ )e^{-3\xi_2}$	$(1.087, 1.458)$

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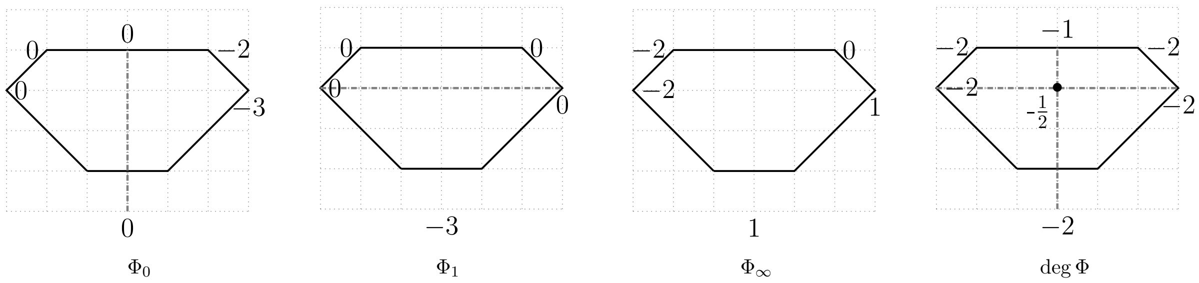

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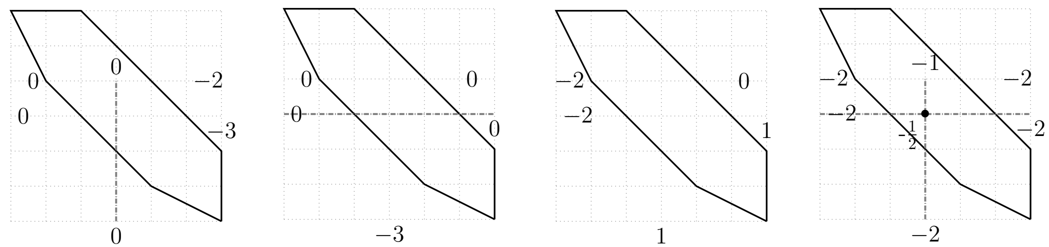

Φ_0 Φ_1 Φ_∞ $\deg \Phi$

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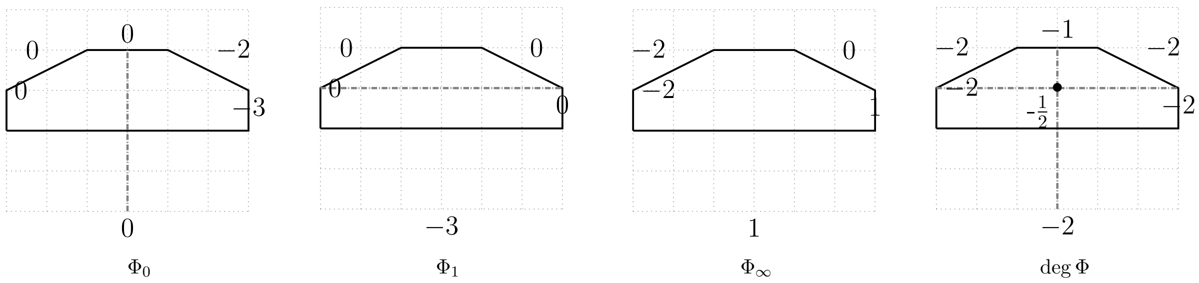

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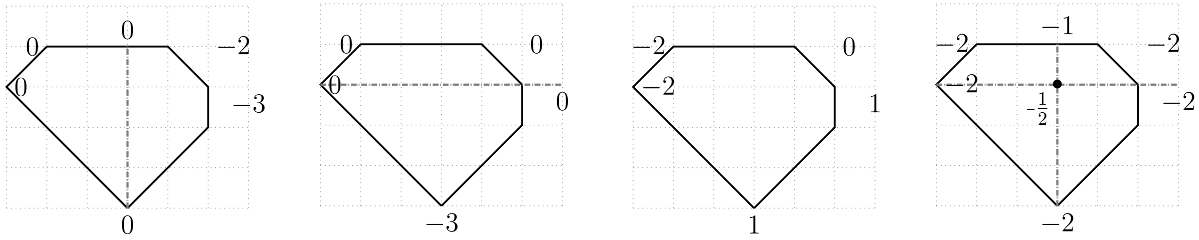

Φ_0 $\qquad$ Φ_1 $\qquad$ Φ_∞ $\qquad$ $\deg\Phi$

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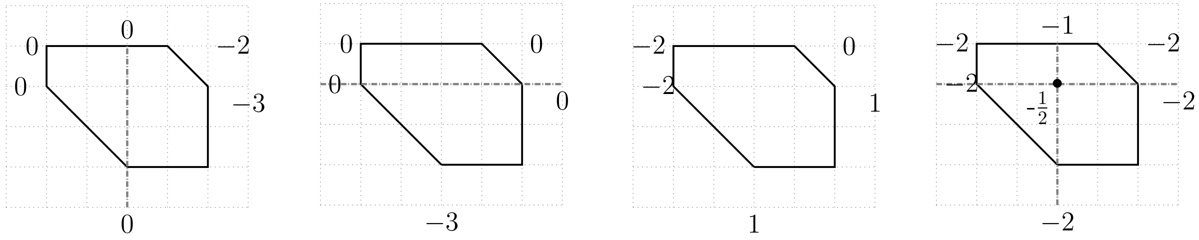

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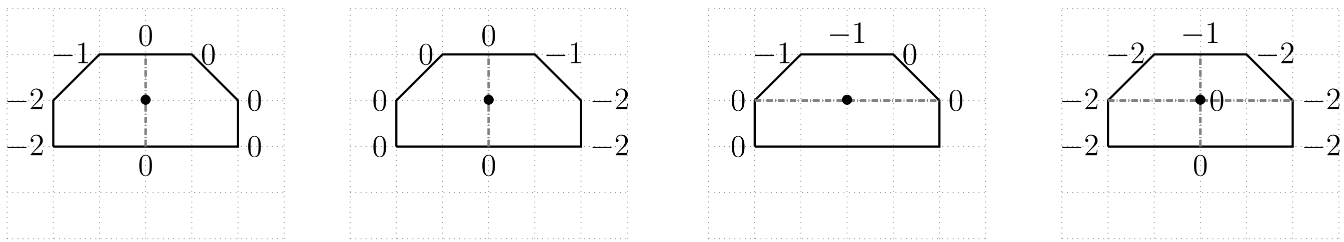

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1	$\begin{pmatrix} -3 & -2 & 2 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -3 & 1 \\ 1 & 1 & -1 & -2 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} a \\ b \\ c \end{pmatrix}$	$\frac{1}{3\xi_2^4} \cdot (\ (2\xi_2^3 - 3\xi_2 - 3)e^{4\xi_2} + 12\xi_2 e^{3\xi_2} + 3\xi_2 + 3\ )e^{-3\xi_2}$	$(1.087, 1.458)$
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