STABILITY OF VARIETIES WITH A TORUS ACTION

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Contents

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In this thesis we study several problems related to the existence problem of invariant canonical metrics on Fano orbifolds in the presence of an effective algebraic torus action. The first chapter gives an introduction. The second chapter reviews the existing theory of T-varieties and reviews various stability thresholds and K-stability constructions which we make use of to obtain new results. In the third chapter we find new Kähler-Einstein metrics on some general arrangement varieties. In the fourth chapter we present a new formula for the greatest lower bound on Ricci curvature, an invariant which is now known to coincide with Tian's delta invariant. In the fifth chapter we discuss joint work with my supervisor to find new Kähler-Ricci solitons on smooth Fano threefolds admitting a complexity one torus action.

Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

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Chapter 1

Preliminaries

1.1 Kähler geometry

1.1.1 Basic definitions

Kähler manifolds

In this section we recall the basics of Kähler geometry. We then review some important results on the existence of canonical metrics on compact Kähler manifolds. A good reference for the material here is [1]. Let X denote a compact real manifold of dimension $2n$. Suppose we have an almost complex structure J on X, that is an automorphism J of $T_{\mathbb{R}}X$ such that $J^2 = -\mathrm{Id}$. Recall that the complexified tangent bundle, $T_{\mathbb{C}}X := T_{\mathbb{R}}X \otimes \mathbb{C}$, decomposes via eigenspaces of J:

$$T_{\mathbb{C}}X = T^{1,0}X \oplus T^{0,1}X,$$

where $T^{(1,0)}X$ has local generators $\frac{\partial}{\partial z_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} - \sqrt{-1}\frac{\partial}{\partial y_i}\right)$, and $T^{(0,1)}X = \overline{T^{(1,0)}X}$ has local generators $\frac{\partial}{\partial \bar{z}_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} + \sqrt{-1}\frac{\partial}{\partial y_i}\right)$.

Recall also that we have a natural isomorphism of real vector bundles $T_{\mathbb{R}}X \cong T^{(1,0)}$, given by the composition $T_{\mathbb{R}}X \to T_{\mathbb{C}} \to T^{1,0}X$. Note, by definition, the action of J is described by multiplying by $\sqrt{-1}$ on $T^{1,0}X$. The decomposition above induces a decomposition of the complexified cotangent bundle:

$$T^*_{\mathbb{C}}X = T^*_{1,0}X \oplus T^*_{0,1}X,$$

and, moreover, a decomposition:

$$\bigwedge^n T^*_{\mathbb{C}}X = \bigoplus_{p+q=n} \left(\bigwedge^p T^*_{1,0}X \otimes \bigwedge^q T^*_{0,1}X \right) \tag{1.1}$$

We use the following notation for relevant spaces of global sections:

$$A^n(X) := H^0(X, \bigwedge^n T^*_{\mathbb{C}}X) \tag{1.2}$$

$$A^{p,q}(X) := H^0(X,\ (\bigwedge^p T^*_{1,0}X \otimes \bigwedge^q T^*_{0,1}X)\ ). \tag{1.3}$$

A form $\alpha \in A^{p,q}(X)$ is said to be *of type* (p,q). The decomposition (1.1) induces a corresponding decomposition:

$$A^n(X) = \bigoplus_{p+q=n} A^{p,q}(X).$$

Hermitian and Kähler metrics

Roughly speaking, a metric measures distances on a manifold. On real manifolds one usually *Riemannian metrics*. Complex manifolds come with more structure, and the analogous object, compatible with this structure, is a *Hermitian metric*. A Hermitian metric is given by a smooth choice of positive definite Hermitian inner product on the fibers of $T^{(1,0)}X$, i.e an element of $H^0(X, T^*_{1,0}X \otimes T^*_{0,1}X)$. Locally we write:

$$h(z) = \sum h_{ij}(z) dz_i \otimes d\bar{z}_j.$$

Given a Hermitian metric h on X, we may consider the real and imaginary parts of h as real tensors on the underlying real manifold via the isomorphisms $T_{\mathbb{R}}X \cong T^{1,0}X \cong \overline{T^{1,0}X}$. The real part $g = \Re h$ is a Riemannian metric on X, called the *induced Riemannian metric* of h. Locally we have:

$$g_z = \sum h_{ij}(z)(dx_i \otimes dx_j + dy_i \otimes dy_j)$$

We may also realize the imaginary part $\omega = -\Im h$, up to sign, as an alternating form on the real tangent bundle $T_{\mathbb{R}}X$, via $T^{1,0}X \cong \overline{T^{1,0}X}$. Set $\omega(v \wedge w) := -\Im h(v, \bar{w}) = -\frac{i}{2}(h - \bar{h})$. We call ω the *associated* $(1,1)$*-form* of h. Locally we have:

$$\omega_z = \sqrt{-1}\sum h_{ij}(z)dz_i \wedge d\bar{z}_j = \sqrt{-1}\sum h_{ij}(z)(dx_i \otimes dy_j - dy_i \otimes dx_j).$$

By definition we have $g(v,w) = g(Jv,Jw)$ and $\omega(v,w) = g(Jv,w)$ for any $v,w \in T_{\mathbb{R}}X$. In fact we may reconstruct h from any Riemannian metric g satisfying $g(v,w) = g(Jv,Jw)$, or alternatively any real $(1,1)$-form ω satisfying the positive definite condition:

$$\omega(v \wedge v) > 0 \text{ for all } v \in T_{\mathbb{R}}X.$$

We may now recall the definition of a Kähler metric.

Definition 1. *A Hermitian metric is Kähler if the associated $(1,1)$-form ω is closed, i.e $d\omega = 0$, where $d : A^2(X,\mathbb{R}) \to A^3(X,\mathbb{R})$ is the usual exterior differential.*

We will bow to convention and often refer to ω, instead of g, as a Kähler metric on X in this context. The standard first example of a compact Kähler manifold is the Fubini-Study metric on complex projective space:

Example 1. *Let s be a section of the projection map $\pi : \mathbb{C}^{n+1}\backslash\{0\} \to \mathbb{P}^n$ over some open set $U \subset \mathbb{P}^n$. The Fubini-Study metric ω_{FS} is then defined to be*

$$\omega_{FS} := i\partial\bar{\partial}\log||s||^2$$

This is well-defined as any two sections differ on their shared domain by a non-vanishing holomorphic function, $s' = fs$. It is clearly closed (since $d = \partial + \bar{\partial}$). For the standard section on U_0 with holomorphic coordinates $z_1, \ldots, z_n$ we have:

$$\omega_{FS} := i\partial\bar{\partial}\log(1 + |z_1|^2 + \cdots + |z_n|^2)$$

and at $[1,0,\ldots,0] \in U_0$ we have:

$$\omega_{FS} = i\sum dz_j \wedge d\bar{z}_j$$

This is positive definite, and so defines a Kähler metric on $\mathbb{P}^n$.

This leads to a large class of examples including any smooth projective algebraic variety:

Example 2. *The restriction of ω_{FS} to any closed submanifold $Y \subseteq \mathbb{P}^n$ produces* *a Kähler structure on Y, as the exterior differential commutes with pulling back differential forms.*

Recall the following important result, telling us that any two Kähler forms of the same class differ by some real-valued function.

$\partial\bar{\partial}$-lemma. *If ω, η are two real $(1,1)$-forms of the same cohomology class then there is a real function $f : X \to \mathbb{R}$ such that $\omega - \eta = \sqrt{-1}\partial\bar{\partial}f$.*

1.1.2 Line bundles and Kodaira Embedding

Recall that we can extend the notion of Hermitian metric to an arbitrary complex vector bundle E: a Hermitian metric on E is defined to be an element $h \in H^0(X, E \otimes \bar{E})^*$. We now recall the notion of a connection and its curvature. A *connection* may be thought of as a way to differentiate tensor fields, and transport data smoothly about a manifold. In our context, a connection is given by a map:

$$\nabla : H^0(X, E) \to H^0(X, E \otimes T^*X)$$

satisfying the Liebniz rule $\nabla(sf) = \nabla sf + s \otimes df$. There is a unique way to extend a connection to an exterior derivative on E-valued differential forms.

$$d^\nabla : \Omega^r(E) \to \Omega^{r+1}(E).$$

The *curvature* of a connection is the 2-form:

$$F^\nabla \in H^0(X, \mathrm{End}(E) \otimes \wedge^2 T^*X),$$

given by:

$$F^\nabla(u, v)(s) := \nabla_u \nabla_v s - \nabla_v \nabla_u s - \nabla_{[u,v]} s.$$

There is a canonical connection on the tangent bundle of any Riemannian manifold known as the *Levi-Civita connection*, satisfying $\nabla g = 0$ and $\nabla_u v - \nabla_v u = [u, v]$. We have a similar situation for any Hermitian vector bundle on a complex manifold:

Example 3. *Let E be a Hermitian vector bundle on a complex manifold X equipped with a holomorphic structure. There is a unique connection ∇ on E such that:*

- *For all sections s we have $\pi_{1,0}\nabla s = \bar{\partial}s$*

- *For any smooth vector field v and sections s, t we have:*

$$v\langle s, t\rangle = \langle \nabla_v s, t\rangle + \langle s, \nabla_v t\rangle$$

This connection is called the Chern connection on E.

Kähler manifolds may be characterized as those manifolds for which the Levi-Civita connection and Chern connection on the tangent bundle coincide. We now recall the definition of the first Chern class of a Hermitian line bundle, which may be used to define notions of positivity.

Definition 2. *The first Chern class of a Hermitian line bundle L is the real cohomology class:*

$$c_1(L) = \frac{1}{2\pi}[-\sqrt{-1}\partial\bar{\partial}\log(h)] \in H^2(X, \mathbb{Z})$$

Example 4. *Suppose (X, g) is a Kähler manifold. Then g induces a Hermitian metric on the holomorphic cotangent bundle $\Omega^{1,0}X$, which in turn induces a Hermitian metric on the canonical line bundle $K_X = \wedge^n \Omega^{1,0}X$, denoted $\det(g)$. The curvature of the associated Chern connection to this Hermitian line bundle is called the Ricci curvature form of the manifold, given by:*

$$\mathrm{Ric}(\omega) = -\sqrt{-1}\partial\bar{\partial}\log(\det(g)).$$

The real cohomology class $c_1(K_X) = \frac{1}{2\pi}[-\sqrt{-1}\partial\bar{\partial}\log(\det(g)]$ is called the first Chern class of the Kähler manifold (X, g), and is often denoted just by $c_1(X)$.

We now recall the definition of a positively curved line bundle.

Definition 3. *A real $(1,1)$-form is called positive if the associated symmetric bilinear form defined for real tangent vectors is positive definite. A real cohomology class is called positive if it can be represented by a positive $(1,1)$-form. A line bundle L is called positive if its first Chern class is positive.*

The following theorem characterizes smooth projective varieties amongst compact Kähler manifolds. Recall that a line bundle L is very ample if for some global sections $s_0, \ldots, s_n \in H^0(X, L)$ we obtain a well-defined closed embedding into a projective space, given by:

$$\varphi_L : p \mapsto [s_0(p), \ldots, s_n(p)] \in \mathbb{P}^n$$

We say L is ample if some multiple of L is very ample.

Kodaira Embedding Theorem. *[[2]] A holomorphic line bundle over a compact complex manifold is ample if and only if it is positive.*

Remark 1. *Note that in its original stated form, Kodaira's theorem says This can be seen as equivalent to the above, if we note...*

Paired with the following theorem of Chow, we can talk either about projective Kähler manifolds or polarized projective algebraic varieties.

Chow's Theorem. *[[3]] A closed complex submanifold of projective space is a projective algebraic subvariety.*

1.1.3 Canonical metrics on Kähler manifolds

In this section we recall some facts about canonical metrics. A canonical metric is a choice of metric dependent only on the complex structure of the manifold, unique up to biholomorphic automorphisms. The material in this section may be found in [4], for example.

As we touched on in the introduction, Kähler-Einstein metrics are an important class of canonical metric and the question of which compact Kähler manifolds admit a Kähler-Einstein metric has historically received a lot of attention. Recall the definition of a Kähler-Einstein metric:

Definition 4. *Let X be a Kähler manifold. A Kähler-Einstein metric on X is a Kähler metric $\omega \in 2\pi c_1(X)$ such that $\operatorname{Ric}\omega = \lambda\omega$ for some real constant λ.*

Note if X is Kähler-Einstein then we must have either $K_X = 0$, K_X ample, or $-K_X$ ample. Via Kodaira embedding these correspond to Ricci flat, Ricci positive, and Ricci negative manifolds respectively. We briefly recall the answers to the existence question in the cases of negative and zero Ricci curvature:

Calabi-Yau Theorem. *[[5]] Let (X, ω) be a compact Kähler manifold. Let α be a real $(1,1)$-form representing $c_1(X)$. Then there exists a real $(1,1)$-form ω' with $[\omega'] = [\omega]$ such that $\operatorname{Ric}(\omega) = 2\pi\alpha$.*

Aubin-Yau Theorem. *[[6]] Let X be a compact Kähler manifold with $c_1(X) < 0$. Then there exists a unique Kähler metric $\omega \in -2\pi c_1(X)$ such that $\operatorname{Ric}(\omega) = -\omega$.*

However the following necessary criterion illustrates that the same is not true in the Fano case $c_1(X) > 0$. Recall that a complex algebraic group is reductive if it is the complexification of a compact connected real Lie group.

Matsushima's criterion. *[[7]] If a Fano manifold X admits a Kähler-Einstein metric, then the holomorphic automorphism group of X is reductive.*

In particular this tells us that the blow up of $\mathbb{P}^1$ in one or two points is not Kähler-Einstein. We end this section by recalling the most general form of canonical metric we will consider. This matches the definition given in [8, Definiton 3].

Definition 5. *A twisted Kähler-Ricci soliton on a Fano manifold (X, ω_0) is a triple (ω, v, t) where $\omega \in 2\pi c_1(X)$ is a Kähler metric, v is a holomorphic vector field, and $t \in [0, 1]$, such that*

$$\mathrm{Ric}(\omega) - \mathcal{L}_v \omega = t\omega + (1-t)\omega_0$$

When $t = 0$ we omit it from the notation and call (ω, v) a Kähler-Ricci soliton. Similarly when v is trivial we call (ω, t) a twisted Kähler-Einstein metric. When both hold then we talk about ω being a Kähler-Einstein metric.

In Section 1.4 we will describe various criteria for the existence of such metrics, but to do so we must first recall some basic tools and language from algebraic and symplectic geometry.

1.2 Tools from algebraic and symplectic geometry

In this section we give some definitions from algebraic and symplectic geometry as we will be understanding them throughout the rest of the thesis. In particular we recall some basic geometry invariant theory, which is needed for the arguments Chapter **??**.

1.2.1 The algebraic torus

Fix an algebraic torus $T = (\mathbb{C}^*)^k$. We have mutually dual character and cocharacter lattices:

$$M := \mathrm{Hom}(T, \mathbb{C}^*), \;\; N = \mathrm{Hom}(\mathbb{C}^*, N),$$

respectively. We denote the associated vector spaces by:

$$M_{\mathbb{K}} := M \otimes_{\mathbb{Z}} \mathbb{K},\ N_{\mathbb{K}} := N \otimes_{\mathbb{Z}} \mathbb{K},$$

for $\mathbb{K} = \mathbb{Q}, \mathbb{R}$. There is a perfect pairing $M \times N \to \mathbb{Z}$ which extends to a bilinear pairing $M_{\mathbb{K}} \times N_{\mathbb{K}} \to \mathbb{K}$. We may make the identification:

$$T \cong \operatorname{Spec} \mathbb{C}[M] \cong N \otimes \mathbb{C}^*.$$

Finally recall that we may identify the real Lie algebra $\mathfrak{k}$ of the maximal compact subtorus $K \subset T$ as $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$.

1.2.2 Linearizations

Suppose we have an algebraic group G acting algebraically on a scheme or variety X. A lift of the this action to a line bundle L on X is known as a linearization. Linearizations are used in geometric invariant theory to give a good definition of a quotient of X by the G-action.

Definition 6. *Let X be a projective scheme together with an action $\lambda : G \times X \to X$ of a reductive algebraic group G. A linearization of the action λ on L is an action $\tilde{\lambda}$ on L such that:*

- *The projection π is G-equivariant, $\pi \circ \tilde{\lambda} = \lambda \circ \pi$*
- *For $g \in G$ and $x \in X$, the induced map $L_x \mapsto L_{g \cdot x}$ is linear.*

Note a linearization to L naturally induces linearizations to $L^\vee$ and $L^{\otimes r}$ for any $r \in \mathbb{N}$.

Example 5. *A linearization of the trivial bundle on a projective variety X must be of the form*

$$g \cdot (x, z) = (g \cdot x, \chi(x, z)z)$$

for some $\chi \in H^0(G \times X, \mathcal{O}^*_{G\times X}) \cong H^0(G, \mathcal{O}^*_G) = \mathfrak{X}(G)$.

The above example tells us that any two linearizations λ_1, λ_2 of an action to the same line bundle differ by multiplication by some character χ of G: fiberwise we have $\tilde{\lambda}_1 = \chi(x, z)\tilde{\lambda}_2$. Thus, when $G \cong T$ is an algebraic torus we may identify the set of linearizations with the character lattice M.

Example 6. *Recall that an action of G on X induces a canonical linearization on the tangent and cotangent bundles of X, and so induces a canonical linearization on the anti-canonical bundle $-K_X$ as the top exterior power of the cotangent bundle.*

1.2.3 Hamiltonian actions and moment maps

Here we recall some basic notions of Hamiltonian actions and moment maps. We will follow conventions of [9] and [10]. We illustrate the theory with the case of an algebraic torus action. Suppose (X, ω) is a symplectic manifold.

Definition 7. *Let $\theta : X \to \mathbb{R}$ be a smooth function. A vector field v such that $\iota_v \omega = d\theta$ is called a Hamiltonian vector field, with Hamiltonian function θ.*

Definition 8. *Let K be a real Lie group, with Lie algebra $\mathfrak{k}$, acting smoothly on X. This action is said to be Hamiltonian if there exists a map $\mu : X \to \mathfrak{k}^*$, known as the moment map of the action, such that:*

- *For any $\xi \in \mathfrak{k}$ the map $\mu^\xi : X \to \mathbb{R}$ given by $\mu^\xi(p) := \langle \mu(p), \xi \rangle$ is a hamiltonian function for the vector field v generated by the one-parameter subgroup $\exp(t\xi) \subset K$.*
- *The map μ is equivariant with respect to the action of K on X and with respect to the coadjoint action.*

Example 7. *Let $X \subseteq \mathbb{P}^N$ be a nonsingular complex projective variety and let G be reductive algebraic group acting effectively on $\mathbb{P}^N$, restricting to an action on X. Let K be the maximal compact subgroup in G, with Lie algebra $\mathfrak{k}$. The action of G is given by a representation $\rho : G \to GL(N+1)$, and by choosing appropriate coordinates we may assume K maps to $U(N+1)$ and so preserves the Fubini-Study form. It can be checked that a moment map $\mu : X \to \mathfrak{k}^*$ for the K-action is given by:*

$$\mu([x]) \cdot a := \frac{x^t \rho_*(a) \hat{x}}{|x|^2}, \tag{1.4}$$

where x is any representative of $[x] \in X \subseteq \mathbb{P}^N$. A different choice of linearization in this setting corresponds to multiplying ρ by some character $\chi \in \mathfrak{X}(G)$. Since $\chi(K)$ is compact, it sits inside $S^1 \subset \mathbb{C}^*$, and hence we do not need to change coordinates when considering the effect on the moment map. When we plug this into (1.5) we see that we have translated the moment map by $\chi \in \mathfrak{X}(G) \otimes \mathbb{R} \cong \mathfrak{k}^*$.

Moreover, taking the r*th power of* L *corresponds to scaling the moment map by a factor of* r*. This gives a correspondence between rational elements* $\chi \in \mathfrak{X}(G)\otimes\mathbb{Q} \subset \mathfrak{k}^*$ *and linearizations of powers of* L*.*

Example 8. *Suppose* $G = T$ *is an algebraic torus with character and cocharacter lattices* M, N *respectively. Then* ρ *is a diagonal matrix of characters* $u_0, \ldots, u_N$ *and we obtain:*

$$\mu([x]) = \frac{\sum_{j=0}^{N} |x_i|^2 u_i}{|x|^2} \in M$$

Then, by Atiyah, [11], and Guillemin-Sternberg, [12], the image of μ *is a convex polytope* $P \subset M$*. Here we see that for each one-parameter subgroup* $w \in N$ *we have Hamiltonian function:*

$$\theta_w([x]) = \langle \mu([x]), w \rangle$$

1.2.4 Chow and GIT quotients

Here we recall the definition of GIT, Chow, and limit quotients of a projective variety by a reductive algebraic group G. We also explain how, when G is a torus, they may be explicitely calculated via the Kempf-Ness theorem, and recall how GIT quotients behave under smooth blowup. The material here is used in Chapter **??**, where we give an exposition of the results of [13]. Thus, the material here may also be found in the preliminary sections of [13].

GIT quotients

Recall the basic setup of Mumford's geometric invariant theory, which provides a method for finding geometric quotients on open subsets of a scheme X when the acting algebraic group G is reductive. In [14] Mumford introduced the notion of a good categorical quotient, which can be shown to be unique if it exists.

Definition 9. *A surjective* G*-equivariant morphism* $\pi : X \to Y$ *is a good categorical quotient if the following hold:*

1. *We have* $\mathcal{O}_Y = (\pi_*\mathcal{O}_X)^G$*;*
2. *if* V *is a closed* G*-invariant subset of* X *then* $\pi(V)$ *is closed;*
3. *if* V, W *are closed* G*-invariant subsets of* X *and* $V \cap W = \emptyset$ *then we have* $\pi(V) \cap \pi(W) = \emptyset$*.*

Good quotients do not always exist for a given scheme X, but we might hope that there exists some dense open subset of X which does admit a good quotient. Consider the affine case, where $X = \operatorname{Spec} A$. For G reductive then it can be shown that $X /\!\!/ G := \operatorname{Spec} A^G$ is a good categorical quotient.

The same ansatz works in the projective case once we make a choice of a lift of the action to the ring of sections of a given ample line bundle. This choice is known as a linearization of the group action. A linearization u of a group action G on X to L induces an action of G on the ring of sections $R(X, L) := \bigoplus_{j \geq 0} H^0(X, L^{\otimes j})$. Consider the scheme $X /\!\!/_u G := \operatorname{Proj} R(X, L)^G$. Note we have a birational map from X to $X /\!\!/_u G$, defined precisely at $x \in X$ such that there exists some $m > 0$ and $s \in R(X, L)^G_m$ such that $s(x) \neq 0$. Such a point is said to be semi-stable. If in addition $G \cdot x$ is closed and the stabilizer G_x is dimension zero, the point x is said to be stable. The set of semi-stable and stable points will be denoted by $X^{ss}(u)$ and $X^s(u)$ respectively.

Lemma 1 ([14, Chapter 1, Section 4])**.** *The canonical morphism $X^{ss}(u) \to X /\!\!/_u G := \operatorname{Proj} R(X, L)^G$ is a good categorical quotient.*

Kempf-Ness approach to GIT quotients

One approach to calculating GIT quotients is via the Kempf-Ness theorem. Let $X \subseteq \mathbb{P}^N$ be a nonsingular complex projective variety and let G be reductive algebraic group acting effectively on $\mathbb{P}^N$, restricting to an action on X. Let K be the maximal compact subgroup in G, with Lie algebra $\mathfrak{k}$. The action of G is given by a representation $\rho : G \to \mathrm{GL}(N + 1)$, and by choosing appropriate coordinates we may assume K maps to $U(N + 1)$ and so preserves the Fubini-Study form. It can be checked that a moment map $\mu : X \to \mathfrak{k}^*$ is given by:

$$\mu([x]) \cdot a := \frac{x^t \rho_*(a) \hat{x}}{|x|^2} \tag{1.5}$$

Where x is any representative of $[x] \in X \subseteq \mathbb{P}^N$. Note we are now in the situation of the previous subsection, with $L = \mathcal{O}_X(1)$ under the embedding $X \subseteq \mathbb{P}^N$. This moment map is unique up to translations in $\mathfrak{k}^*$. A different choice of linearization in this setting corresponds to multiplying ρ by some character $\chi \in \mathfrak{X}(G)$.

Since $\chi(K)$ is compact it sits inside $S^1 \subset \mathbb{C}^*$, and hence we do not need to change coordinates when considering the effect on the moment map. When we plug this into (1.5) we see that we have translated the moment map by

$\chi \in \mathfrak{X}(G) \otimes \mathbb{R} \cong \mathfrak{k}^*$. Moreover, taking the rth power of L corresponds to scaling the moment map by a factor of r. This gives a correspondence between rational elements $\chi \in \mathfrak{X}(G) \otimes \mathbb{Q} \subset \mathfrak{k}^*$ and linearizations of powers of L.

Example 9. *Suppose $G = T$ is an algebraic torus with character and cocharacter lattices M, N respectively. Then ρ is a diagonal matrix of characters $u_0, \ldots, u_N$ and we obtain:*

$$\mu([x]) = \frac{\sum_{j=0}^{N} |x_i|^2 u_i}{|x|^2} \in M$$

Then, by Atiyah, [11], and Guillemin-Sternberg, [12], the image of μ is a convex polytope $P \subset M$.

We will make use of the following theorem of Kempf and Ness. A proof is given in [14, Chapter 8]. See also the original work [15].

Kempf-Ness Theorem. *[[15, Theorem 8.3]] Let $X \subseteq \mathbb{P}^N$ be a nonsingular complex projective variety and let G be reductive algebraic group acting effectively on $\mathbb{P}^N$, restricting to an action on X. Consider a linearization of some power of L corresponding to a rational element $u \in \mathfrak{k}^*$.*

1. $X^{ss}(u) = \{x \in X | \overline{Gx} \cap \mu^{-1}(u) \neq \emptyset\}$.

2. *The inclusion of $\mu^{-1}(u)$ into $X^{ss}(u)$ induces a homeomorphism*

$$\mu^{-1}(u)/K \to X /\!\!/_u G$$

where $\mu^{-1}(u)/K$ is endowed with the quotient topology induced from the classical (closed submanifold topology) on $\mu^{-1}(u)$, and $X /\!\!/_u G$ is endowed with its classical (complex manifold) topology

We can use Theorem 1.2.4 to calculate GIT quotients by inspection. To be explicit, suppose $\mu^{-1}(u)/K$ has the structure of a complex projective variety and $q : X^{ss}(u) \to \mu^{-1}(u)/K$ is a G-invariant morphism which restricts to the topological quotient map on the moment fibre, such that $q_*\mathcal{O}_X^G = \mathcal{O}_Y$. The following fact is probably well known, but we prove it here for the reader's convenience.

Lemma 2. *The morphism q is a good categorical quotient, and hence is isomorphic to the GIT quotient map $X \to X /\!\!/_u G$.*

Proof. It is enough to show that q sends closed G-invariant subsets to closed subsets, and disjoint pairs of closed invariant subsets to disjoint pairs of closed subsets.

Firstly suppose that V is a G-invariant Zariski-closed subset of X. Then $q(V) = q(V \cap \mu^{-1}(u))$, and $V \cap \mu^{-1}(u)$ is K-invariant and closed in the classical topology of $\mu^{-1}(u)$. This implies that $q(V)$ is closed in the classical topology on $\mu^{-1}(u)/K \simeq X /\!\!/_u G$. But $q(V)$ is constructable, as the image of a Zariski-closed subset of X, and so we may conclude that $q(V)$ is Zariski-closed in $\mu^{-1}(u)/K \simeq X /\!\!/_u G$.

Now suppose V, W are G-invariant and Zariski-closed in X, with $x \in V$ and $y \in W$ such that $q(x) = q(y)$. By 1.2.4 we may take $x' \in \overline{Gx} \cap \mu^{-1}(u)$, $y' \in \overline{Gy} \cap \mu^{-1}(u)$ such that $q(x') = q(y')$. These two points lie in the same K-orbit. By the G-invariance of V, W we have $V \cap W \neq \emptyset$. □

GIT quotients under smooth blowup

Here we recall some results from [16], which we will use in the proof of Theorem **??**. Let G be a reductive group acting on X. Let L be an ample G-invariant line bundle on X. Fix some linearization of the G-action to L. Suppose V be a smooth closed G-stable subvariety of X, defined by some ideal sheaf $\mathcal{J}_V$. Let $f : W \to X$ be the blow-up of X along V.

The goal is to construct a linear action on W lifting the action on X, and describe the GIT quotient $W^{ss} \to W /\!\!/ G$ in terms of $X^{ss} \to X /\!\!/ G$ and f. First let us construct an ample line bundle on W. Let E be the exceptional divisor of the map f. Set $L_d := f^* L^{\otimes d} \otimes \mathcal{O}(-E)$. For sufficiently large d, L_d is ample.

Since $E \cong \mathbb{P}(N_{V,X})$ and $\mathcal{O}(-E)_{|E} \cong \mathcal{O}_{\mathbb{P}(N_{V,X})}(1)$ then the natural action of G on $N_{V,X}$ induces an action on $\mathcal{O}(-E)_{|E}$. We have $W \backslash E \cong X \backslash V$ and $\mathcal{O}(-E)_{|W\backslash E}$ is the trivial line bundle, so admits the product action. The action of G on L lifts to $f^* L^{\otimes d}$, and so we obtain a linear action on L_d. By [16,], for sufficiently large d we have $W^{ss} \subset X^{ss}$. We have the following result:

Lemma 3 ([16, Lemma 3.11]). *If d is a sufficiently large multiple of e then the GIT quotient $W /\!\!/ G$ associated to the linearization described above is the blowup of $X /\!\!/ G$ along the image $V /\!\!/ G$ of V in $X /\!\!/ G$. In particular if $V /\!\!/ G$ is a divisor on $X /\!\!/ G$ then $W /\!\!/ G \cong X /\!\!/ G$.*

Chow and limit quotients

Recall the definition of the Chow quotient, as introduced in [17]. If G is any connected linear algebraic group and X is a projective G-variety, then orbit closures of points are generically of the same dimension and degree, and so define points in the corresponding Chow variety. The Chow quotient of the G-action on X is the closure of this set of points.

We now recall the definition of the limit quotient, from [14]. The limit quotient is discussed in detail in [18]. Let G be a reductive algebraic group, and X a projective G-variety. Suppose there are finitely many sets of semi-stable points $X_1, \dots, X_r$ arising from G-linearized ample line bundles on X. Whenever $X_i \subseteq X_j$ holds, there is a dominant projective morphism $X_i /\!\!/ G \to X_j /\!\!/ G$ which turns the set of GIT quotients into an inverse system. The associated inverse limit Y admits a canonical morphism $\bigcap_{i=1}^r X_i \to Y$. The closure of the image of morphism is the limit quotient.

When G is an algebraic torus there are indeed finitely many semi-stable loci. Moreover, by [18, Corollary 2.7], we may calculate the limit quotient by taking the inverse limit of the subsystem obtained by only considering linearizations of powers of one fixed ample line bundle L. In [18, Proposition 2.5] it is shown that the Chow quotient and limit quotient coincide when G is an algebraic torus.

Definition 10. *Let X be a T-variety. Let $\pi : X \dashrightarrow Y$ be the Chow quotient map of X by its torus action. For any prime divisor Z on Y, the generic stabilizer on a component of $\pi^{-1}(Z)$ is a finite abelian group. The maximal order across these components is denoted m_Z. We may then define a boundary divisor for π, given by:*

$$B := \sum_Z \frac{m_Z - 1}{m_Z} \cdot Z \tag{1.6}$$

We call the pair (Y, B) the Chow quotient pair of the T-variety X.

1.3 T-varieties

In this section we briefly recall the theory of complex T-varieties. By a T-variety we will always mean a normal variety with an effective action of an algebraic torus T. Let T, M, N be as described in 1.2.1. Let us fix some additional definitions. By a *polyhedron* we wil mean the intersection of finitely many closed affine halfspaces of $N_\mathbb{Q}$, or its dual $M_\mathbb{Q}$. By a *cone* we mean the intersection of finitely many closed linear halfspaces of $N_\mathbb{Q}$ or its dual $M_\mathbb{Q}$. We will assume all cones are generated by primitive elements of their respective lattices.

1.3.1 Toric varieties

First, for context, let us recall the toric situation. A cone $\sigma \subset N_\mathbb{R}$ has a dual cone $\sigma^\vee := \{m \in M_\mathbb{R} | \langle m, n\rangle \geq 0 \ \forall n \in N_\mathbb{R}\}$, and we may construct the normal toric variety $\operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M]$. The torus action is given by the M-grading of the algebra $\mathbb{C}[\sigma^\vee \cap M]$.

Conversely, given a normal affine toric variety X with algebraic torus T, $\mathbb{C}[X]$ is a semigroup subalgebra of $\mathbb{C}[M]$ of the form $\mathbb{C}[\sigma^\vee \cap M]$ for some strongly convex rational polyhedral cone $\sigma \subset N_\mathbb{R}$. We write $\operatorname{TV}(\sigma, N) := \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M]$. Face inclusions of cones correspond to equivariant open embeddings of varieties, and so from a complete fan of cones Σ we may construct a normal toric variety X_Σ.

Example 10. *Consider the variety $X = \operatorname{Bl}_z \mathbb{P}^1 \times \mathbb{P}^1$, where $z = ([1, 0], [1, 0])$. We may lift the 2-torus action on $\mathbb{P}^1 \times \mathbb{P}^1$ to X. This action becomes effective once we replace the torus T by $T/\pm Id$. As a toric variety it is given by the fan Σ in Figure 1.*

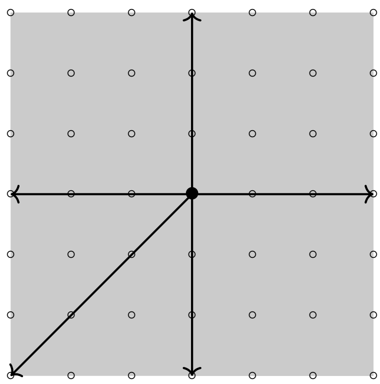


Figure 1: $\Sigma \subset N_\mathbb{Q}$

We also recall the description of equivariant polarizations of toric varieties via convex polytopes. Suppose we have a complete fan Σ, with rays $\Sigma(1)$. Any Cartier divisor on $X = \operatorname{TV}(\Sigma)$ is linearly equivalent to a T-equivariant one. Moreover we have the following exact sequence:

$$0 \to M \to \operatorname{CaDiv}_T(X) \cong \mathbb{Z}^{\Sigma(1)} \to \operatorname{Cl}(X) \to 0. \tag{1.7}$$

and relations $\operatorname{div}(\chi^u) = \sum_{\rho \in \Sigma(1)} \langle u, v_\rho \rangle D_\rho$.

To any lattice polytope $P \subset M_{\mathbb{Q}}$ we can associate a normal projective toric variety X_P given by its dual fan, and an ample divisor D_P given by coefficients on the ray generators of $\Sigma(1)$ specified by the equations of halfspaces defining P.

Example 11. *Consider the the following lattice polytope: The normal fan $\mathcal{N}(P)$*

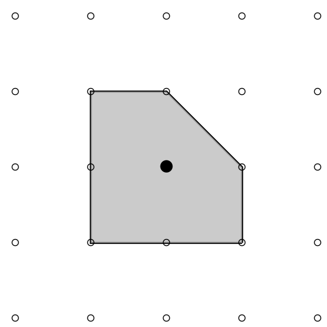


Figure 2: $P \subset M_{\mathbb{Q}}$

is that of $\operatorname{Bl}_z \mathbb{P}^1 \times \mathbb{P}^1$ *as in Example 10. We can calculate the corresponding divisor as*

$$D_P = -\sum_{\rho \in \Sigma(1)} D_\rho \sim -K_X.$$

Conversely the exact sequence 1.7 may be used to construct a polytope from any equivariant polarization of a projective toric variety. Finally, recall that a polytope is called Fano if the origin is contained in its interior, and each vertex is a primitive lattice point of M. Under the correspondence just described, Fano polytopes correspond exactly to anticanonical polarizations of toric varieties.

1.3.2 Higher complexity T-varieties

There is a successful program to extend the combinatorial dictionary of toric varieties to T-varieties of higher complexity. Roughly speaking, the combinatorial data lives over the Chow quotient of X by the T-action, so we have combinatorial data of dimension $\dim T$, and algebreo-geometric data of dimension of the complexity of the torus action.

Recall that one may define an abelian semigroup structure on the set of all polyhedra via Minkowski addition:

$$\Delta + \Delta' := \{v + v' \mid v \in \Delta,\ v' \in \Delta'\}.$$

It is well known that this gives a representation of any polyhedron $\Delta = P + \sigma$ where P is a convex polytope and σ. The cone σ is uniquely specified and is known as the tail cone of Δ. We will write $\operatorname{tail}\Delta = \sigma$, and call Δ a σ-tailed polyhedra in this situation.

The set of σ-tailed polyhedra form a semigroup $\mathrm{Pol}^+_{\mathbb{Q}}(N,\sigma)$ under Minkowski addition. We also include $\emptyset$ here, with $\emptyset + \Delta := \emptyset$ for any Δ. Recall the definition of a polyhedral divisor:

Definition 11. *Let $\sigma \subset N_{\mathbb{R}}$ be a cone, and Y a normal projective variety over $\mathbb{C}$. A polyhedral divisor on (Y, N) with tail cone σ is an element*

$$\mathcal{D} \in \mathrm{Pol}^+_{\mathbb{Q}}(N,\sigma) \otimes \mathrm{CaDiv}^+_{\mathbb{Q}}(Y),$$

where $\mathrm{CaDiv}^+_{\mathbb{Q}}(Y)$ is the semigroup of effective $\mathbb{Q}$-Cartier divisors on Y. We define $\operatorname{tail}\mathfrak{D} = \sigma$.

Let $\operatorname{Loc}\mathfrak{D} := Y \setminus \bigcup_{\mathfrak{D}_Z=\emptyset} Z$. The evaluation of $\mathfrak{D}$ at $u \in \sigma^\vee$ is defined to be the $\mathbb{Q}$-Cartier divisor on Y given by:

$$\mathfrak{D}(u) := \sum_{\mathfrak{D}\neq\emptyset} \min_{v\in\mathfrak{D}_P} \langle v, u\rangle Z_{|\operatorname{Loc}\mathfrak{D}}.$$

Definition 12. *A polyhedral divisor $\mathfrak{D}$, as defined above, is called a p-divisor if $\mathfrak{D}(u)$ is semiample for $u \in \sigma^\vee$ and, in addition, big for $u \in int(\sigma^\vee)$. Note if $\operatorname{Loc}\mathfrak{D}$ affine this is automatically satisfied.*

By [19, Proposition 3.1], p-divisor defines an affine T-variety in the following manner. Note for $u \in \sigma^\vee$ we have $\mathfrak{D}(u) + \mathfrak{D}(u') \le \mathfrak{D}(u+u')$. Consider the sheaf of N-graded algebras

$$\mathcal{A} := \bigoplus_{w\in\sigma^\vee} \mathcal{O}_{\operatorname{Loc}\mathfrak{D}}(\mathfrak{D}(u))\chi^u.$$

Note the semiample and big conditions in the definition of a p-divisor ensure that the algebra $H^0(\operatorname{Loc}\mathfrak{D}, \mathbb{A})$ is finitely generated. We obtain T-varieties

$$\tilde{\mathrm{TV}}(\mathfrak{D}) := \operatorname{Spec}_{\operatorname{Loc}\mathfrak{D}} \mathcal{A},\ \mathrm{TV}(\mathfrak{D}) := \operatorname{Spec} H^0(\operatorname{Loc}\mathfrak{D}, \mathcal{A})$$

together with a good quotient $\tilde{\mathrm{TV}}(\mathfrak{D}) \to Y$ (the Chow quotient) of the torus action, and an equivariant contraction $r : \tilde{\mathrm{TV}}(\mathfrak{D}) \to \mathrm{TV}(\mathfrak{D})$.

The T-variety $\mathrm{TV}(\mathfrak{D})$ remains unchanged if we pull back $\mathfrak{D}$ by some birational $\varphi : Y' \to Y$. Moreover, modifying $\mathfrak{D}$ by an element in the image of the natural map:

$$N \otimes_{\mathbb{Z}} \mathbb{C}(Y)^* \to \mathrm{Pol}^+_{\mathbb{Q}}(N, \sigma) \otimes \mathrm{CaDiv}^+_{\mathbb{Q}}(Y)$$

does not change $\mathrm{TV}(\mathfrak{D})$. In the converse direction, by [19, Proposition 3.4], any affine T-variety X is of the form $\mathrm{TV}(\mathfrak{D})$ for some p-divisor $\mathfrak{D}$.

Example 12. *Let Y be a normal projective variety and D an ample integral Cartier divisor on Y. Let $\mathfrak{D} = [1, \infty) \otimes D$. Then we see*

$$\mathcal{A} = \bigoplus_{n \in \mathbb{Z}^{\geq 0}} \mathcal{O}(nD)\chi^n,$$

and so $\tilde{\mathrm{TV}}(\mathfrak{D})$ is the total space of the line bundle $\mathcal{O}(D)$.

Example 13. *Any toric variety X with torus T may be considered a higher complexity T-variety with respect to any proper subtorus $T' \subset T$. Such a subtorus is given by some surjection of character lattices $p : M \to M'$. Writing M_Y for the kernel of p and denoting the dual surjection $q : N \to N_Y$, we have mutually dual short exact sequences:*

$$0 \longrightarrow N' \longrightarrow N \xrightarrow{q} N_Y \longrightarrow 0$$

$$0 \longleftarrow M' \xleftarrow{p} M \longleftarrow M_Y \longleftarrow 0$$

Suppose $X = \mathrm{TV}(\delta)$ for some cone $\delta \subset \mathbb{N}_{\mathbb{Q}}$. Let $\sigma = \delta \cap N'_{\mathbb{Q}}$. Then the surjection $M \to M'$ induces a surjection $\delta^\vee \to \sigma^\vee$. Let Σ be the coarsest fan which refines all images of faces of δ under q. It may be shown that $Y = \mathrm{TV}(\Sigma)$ is the Chow quotient of X by T' up to normalization. Then X is the T'-variety associated to the p-divisor:

$$\mathfrak{D}^\delta := \sum_{a \in \Sigma(1)} \mathfrak{D}_a \otimes \overline{\mathrm{orb}}(a)$$

Where $\mathfrak{D}_a := q^{-1}(a) \cap \delta$, and $\overline{\mathrm{orb}}(a)$ is the torus-invariant divisor on Y associated to the ray a under the usual orbit-cone correspondence.

Example 14. *Consider the toric variety given by the following cone:*

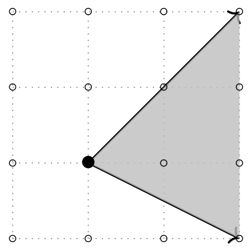
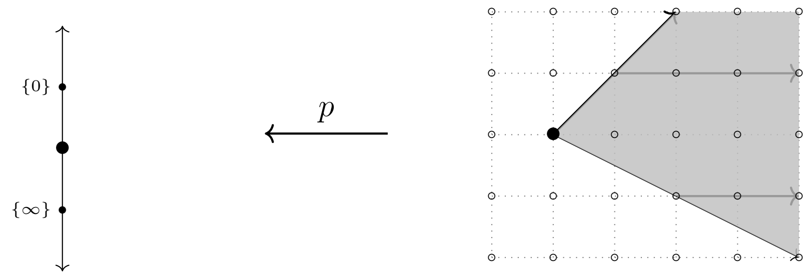


Figure 3: $\sigma \subset N_{\mathbb{Q}}$

and subtorus action given by sublattice $N' := \mathbb{Z}e_2 \subset N$. We may read off the downgrade p-divisor from the following diagram:



$\sigma \subset N_{\mathbb{Q}}$ $\sigma \subset N_{\mathbb{Q}}$

and we see this downgraded toric variety is given by the p-divisor:

$$\mathfrak{D} = \{0\} \otimes [1, \infty) + \{\infty\} \otimes [2, \infty).$$

One can define morphisms of p-divisors, and this correspondence turns out to be an equivlance of categories between affine T-varieties and p-divisors up to equivalence via the modifications mentioned above. We will not make use of the general data of a morphism of p-divisors but we will discuss the special case needed for globalization.

By [20] we have a method of gluing p-divisors in a natural way to construct general T-varieties, generalizing the notion of a fan of cones in the toric case. Suppose $\mathfrak{D}', \mathfrak{D}$ are polyhedral divisors. We write $\mathfrak{D}' \leq \mathfrak{D}$ if $\mathfrak{D}'_Z$ is a face of $\mathfrak{D}_Z$ for each Cartier divisor Z on Y.

Now suppose additionally that $\mathfrak{D}', \mathfrak{D}$ are p-divisors. If $\mathfrak{D}' \leq \mathfrak{D}$ then we obtain graded morphisms of the respective sheaves of algebras $\mathcal{A} \to \mathcal{A}'$ giving a T-equivariant morphism $\mathrm{TV}(\mathfrak{D}') \to \mathrm{TV}(\mathfrak{D})$. Unlike the toric case, this not necessarily an open embedding, as we see in the following example.

Example 15. *Consider the setup in Example () with $Y = \mathbb{P}^1$ and $D = \{\infty\}$. Then $r : \tilde{\mathrm{TV}}(\mathfrak{D}) \to \mathrm{TV}(\mathfrak{D})$ is isomorphic to the blow up of $\mathbb{A}^2$ at the origin. If $\mathfrak{D}' = \emptyset \otimes \{\infty\}$ then $\mathfrak{D}' \le \mathfrak{D}$ but $\tilde{TV}(\mathfrak{D}')$ intersects the exceptional divisor in $\tilde{\mathrm{TV}}(\mathfrak{D})$, and so the induced map $\mathrm{TV}(\mathfrak{D}') \to \mathrm{TV}(\mathfrak{D})$ is not an open embedding.*

1.3.3 f-divisors and divisorial polytopes

In this section we focus on the complexity one case, where T-equivariant open embeddings may be characterized using degree polyhedra of p-divisors:

$$\deg \mathfrak{D} := \sum_{y \in Y} \mathfrak{D}_y.$$

Note that $\deg \mathfrak{D} \neq \emptyset \iff \operatorname{Loc} \mathfrak{D} = Y$. It is also not hard to see that a complexity one polyhedral divisor $\mathfrak{D}$ is a p-divisor if and only if the following two conditions hold:

1. $\deg \mathfrak{D} \subsetneq \operatorname{tail} \mathfrak{D}$;
2. $\mathfrak{D}(w)$ has a principal multiple for $w \in (\operatorname{tail} \mathfrak{D})^\vee$ such that $w^\perp \cap \deg \mathfrak{D} \neq 0$.

We have the following characterization of open embeddings:

Theorem 1 ([21]). *Let Y be a curve and $\mathfrak{D}, \mathfrak{D}'$ polyhedral and p-divisors respectively, such that $\mathfrak{D}' \le \mathfrak{D}$. Then $\mathfrak{D}'$ is a p-divisor and $\mathrm{TV}(\mathfrak{D}') \to \mathrm{TV}(\mathfrak{D})$ is an open embedding if and only if $\deg \mathfrak{D}' = \deg \mathfrak{D} \cap \operatorname{tail} \mathfrak{D}'$.*

Note that in complexity one $\operatorname{Loc} \mathfrak{D}$ is always birational to $\mathbb{P}^1$. Therefore, via pullback, any complexity one normal affine $\mathcal{T}$-variety may be realized as $\mathrm{TV}(\mathfrak{D})$ where $\mathfrak{D}$ is a p-divisor over $Y = \mathbb{P}^1$. Note that in this case that condition 2. above is automatically satisfied. In complexity one, the object taking the place of a fan of cones is an f-divisor, first introduced in [21]. We recall this construction now.

By a *polyhedral decomposition* we mean a decomposition of $N_{\mathbb{Q}}$ into a collection of polyhedra, closed under intersection. A polyhedral decomposition has a *tail fan*: a fan comprised of exactly the tail cones of the polyhedra in the decomposition. If $\mathcal{G}$ is a polyhedral decomposition then we write $\operatorname{tail} \mathcal{G}$ for its tail fan.

Definition 13. *An f-divisor is a pair $\left(\sum_{y \in \mathbb{P}^1} S_y \otimes \{y\},\ \deg \mathcal{S}\right)$, such that:*

1. *For each point $y \in \mathbb{P}^1$, S_y is a polyhedral subdivision with tail fan Σ*

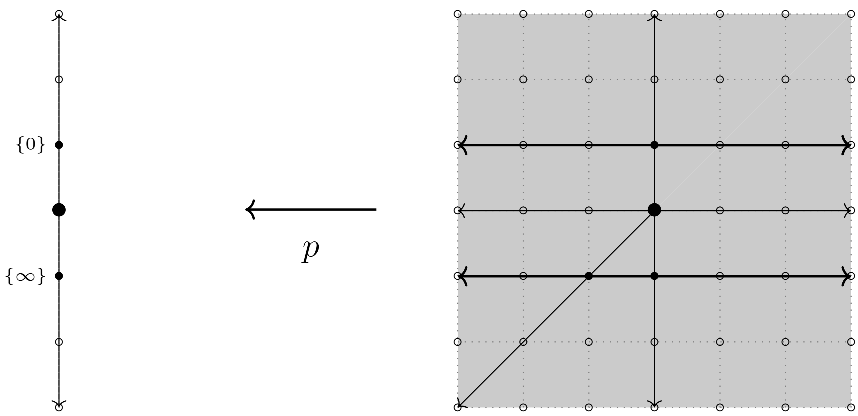
2. *For any full-dimensional* $\sigma \in \Sigma$, *if* $\sigma \cap \deg \mathcal{S} \neq \emptyset$ *then*

$$\mathfrak{D}^\sigma := \sum \mathfrak{D}^\sigma_y \otimes \{y\}$$

is a p-divisor, where $\mathfrak{D}^\sigma_y$ *is the unique polyhedron in* S_y *with* $\operatorname{tail}(\mathfrak{D}^\sigma_y) = \sigma$.

3. *Only finitely many* S_y *may differ from the tailfan* Σ. *We call the finite collection of* $S_y \neq \Sigma$ *the non-trivial slices of* $\mathcal{S}$.

Example 16. *Let us describe the downgrade procedure for the toric variety from Example 10, with respect to the subtorus given by sublattice* $N' := \mathbb{Z}e_1 \subset N$. *We may read off the downgrade* f*-divisor from the following diagrams:*

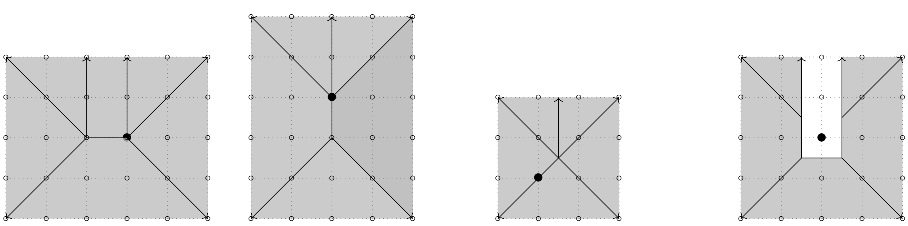


$\Sigma_Y \subset N_Y$ $\qquad$ $\Sigma \subset N$

and we see this downgraded toric variety is given by the f*-divisor:*

S_0 $\qquad$ $S_\infty = \deg S$

Example 17. *Here we give an example of an* f*-divisor describing the complexity one threefold (2.30) from the list of Mori and Mukai [22].*



$\mathcal{S}_0$ $\qquad$ $\mathcal{S}_1$ $\qquad$ $\mathcal{S}_\infty$ $\qquad$ $\deg \mathcal{S}$

In complexity one there is a generalization of the correspondence between lattice polytopes and polarized projective toric varieties. We recall the definition of a divisorial polytope, which is to polarized complexity one T-varieties what the associated polytope is to an equivariantly polarized toric variety.

Definition 14. *A divisorial polytope is a function Ψ on a lattice polytope $\Box \subset M_{\mathbb{R}}$:*

$$\Psi : \Box \to \operatorname{Div}_{\mathbb{R}} \mathbb{P}^1,\ u \mapsto \Sigma_{y \in \mathbb{P}^1} \Psi_y(u) \cdot \{y\},$$

such that:

- *For $y \in \mathbb{P}^1$ the function $\Psi_y : \Box \to \mathbb{R}$ is the minimum of finitely many affine functions, and $\Psi_y \equiv 0$ for all but finitely many $y \in \mathbb{P}^1$.*
- *Each Ψ_y takes integral values at the vertices of the polyhedral decomposition its regions of affine linearity induce on $\Box$.*
- $\deg \Psi(u) > -2$ *for $u \in int(\Box)$;*

A divisorial polytope is said to be Fano if additionally we have that:

- *The origin is an interior lattice point of $\Box$.*
- *The affine linear pieces of each Ψ_y are of the form $u \mapsto \frac{\langle v,u\rangle - \beta + 1}{\beta}$ for some primitive lattice element $v \in N$;*
- *Every facet F of $\Box$ with $(\deg \circ \Psi_{|F}) \neq -2$ has lattice distance 1 from the origin.*

Let Ψ be a divisorial polytope. We may construct a complexity one polarized T-variety from the graded ring S given by:

$$S_k := \bigoplus_{u \in \Box \cap \frac{1}{k} M} H^0(\mathbb{P}^1, \mathcal{O}(\lfloor k \cdot (\Psi(u) + D) \rfloor),$$

where D is some integral divisor of degree 2. We may also recover a divisorial polytope Ψ from any polarized complexity one T-variety (X, L) such that $H^0(X, L^k) = S_k$.

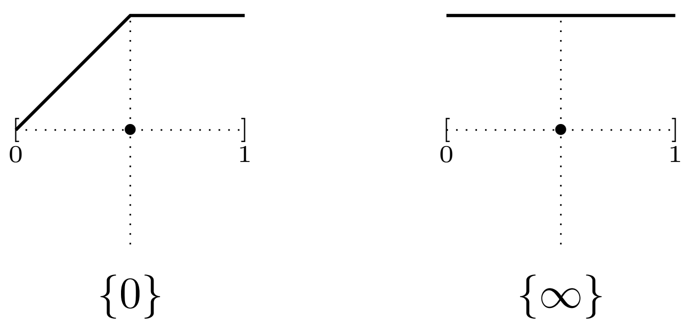
From a divisorial polytope Ψ and $P \in \mathbb{P}^1$ consider the piecewise affine concave function $\Psi^*_P(u) := \min_{v \in \Box}(\langle v, u\rangle - \Psi_P(u))$. Let S_P be the polyhedral subdivision induced by Ψ^*_P, with tailfan Σ. Let $\mathfrak{deg}$ be the set of cones $\sigma \in \Sigma$ such that

$\deg \circ \Psi_{|F_\sigma} \equiv 0$, where F_σ is the face of $\square$ where $\langle \cdot, v\rangle$ takes its minimum value for $v \in \sigma$.

Moreover, Fano divisorial polytopes correspond to Fano T-varieties $(X, -K_X)$. Thus all Fano complexity one T-varieties may be described in this way. For more details of this construction and the correspondence see [23].

Example 18. *We can perform a downgrade operation on a polytope describing a toric variety. Given a polytope $P \subset M_{\mathbb{Q}}$ and a complexity one subtorus action given by some surjection $p : M \to M'$, choose a section $s : M' \to M$. Set $\square = p(P)$ and $\Psi_0(u) := \max(p^{-1}(u) - s(u))$, $\Psi_\infty := \min(p^{-1}(u) - s(u))$. We obtain a divisorial polytope $\Psi := \Psi_0 \otimes \{0\} + \Psi_\infty \otimes \{\infty\}$. The corresponding T-variety is then isomorphic to the toric variety associated to P.*

Example 19. *Consider the toric downgrade we described in (). If we start with the polytope () and perform the downgrade operation, we end up with divisorial polytope:*



It is easy to see that the resulting f-divisor from the operation () coincides with the downgrade f-divisor from ().

We now recall some basic terminology for divisorial polytopes, which we will make use of in later sections. The push-forward of the measure induced by ω is known as the Duistermaat-Heckman measure, independent of the choice of ω and which we denote by ν. Denote the standard measure on $M_{\mathbb{R}}$ by η.

Definition 15. *Let Ψ be a divisorial polytope.*

- *The degree of Ψ is the map $\deg \Psi : \square \to \mathbb{R}$ given by $u \mapsto \deg(\Psi(u))$.*
- *The barycenter of Ψ is $\mathrm{bc}(\Psi) \in \square$, such that for all $v \in N_{\mathbb{R}}$:*

$$\langle \mathrm{bc}(\Psi), v\rangle = \int_{\square} v \cdot \deg \Psi \; d\eta = \int_{\square} v d\nu.$$

Note by the second equality we see $\mathrm{bc}(\Psi) = \mathrm{bc}_\nu(\Box)$.

- *The volume of* Ψ *is defined to be:*

$$\mathrm{vol}\,\Psi = \int_\Box \deg \Psi \; d\eta = \int_\Box d\nu.$$

1.4 Equivariant K-stability

In this section we recall definitions of K-stability. In summary the K-stability criteria are concerned with the positivity of certain numerical invariants associated to *test configurations* of our original space. We do not go into detail about how or why K-stability should relate to the existence of canonical metrics here, but give the definitions and theorems we will rely on later in the thesis.

1.4.1 Twisted equivariant K-stability

Here we recall notions of Twisted equivariant K-stability, following [8]. Let X be a Fano manifold with the action of a complex reductive group G of automorphisms containing a maximal torus T. Fix a T-invariant Kähler form $\omega \in 2\pi c_1(X)$ induced by the Fano condition. Recall that the Lie algebra $\mathfrak{t}$ of the maximal compact torus in T may be identified with $N_\mathbb{R} = N \otimes \mathbb{R}$.

Definition 16. *A G-equivariant test configuration for (X, L) is a $\mathbb{C}^*$-equivariant flat family $\mathcal{X}$ over the affine line equipped with a relatively ample equivariant $\mathbb{Q}$-line bundle $\mathcal{L}$ such that:*

1. *The $\mathbb{C}^*$-action λ on $(\mathcal{X}, \mathcal{L})$ lifts the standard action on $\mathbb{A}^1$;*
2. *The general fiber is isomorphic to X and $\mathcal{L}$ is the relative anti-canonical bundle of $\mathcal{X} \to \mathbb{A}^1$.*
3. *The action of G extends to $(\mathcal{X}, \mathcal{L})$ and commutes with the $\mathbb{C}^*$-action λ.*

A test configuration with $\mathcal{X} \cong X \times \mathbb{A}^1$ is called a product configuration. If such an isomorphism exists and is $\mathbb{C}^*$-equivariant then we call the test configuration trivial. Finally a test configuration with normal special fiber is called special.

We work with $G = T$ being a maximal torus in $\mathrm{Aut}(X)$. We then have an induced $T' = T \times \mathbb{C}^*$-action on the special fiber. The canonical lift of T'-action to

$-K_{\mathcal{X}_0}$ induces a canonical choice of moment map $\mu : \mathcal{X}_0 \to M'_{\mathbb{R}}$. The restriction of λ to $\mathcal{X}_0$ is generated by the imaginary part of a T'-invariant vector field w, and by an abuse of notation we also write $w \in N'_{\mathbb{R}}$ for the corresponding one-parameter subgroup. The moment map μ then specifies Hamiltonian functions $\theta_w := \langle \mu, w \rangle : \mathcal{X}_0 \to \mathbb{R}$, as we have seen in Section 1.2.3.

Definition 17. *The twisted Donaldson-Futaki character of a special test configuration $(\mathcal{X}, \mathcal{L})$ is given by:*

$$\mathrm{DF}_{t,\xi}(\mathcal{X}, \mathcal{L}, w) = \mathrm{DF}_{\xi}(\mathcal{X}, \mathcal{L}, w) + \frac{(1-t)}{V} \int_{\mathcal{X}_0} (\max_{\mathcal{X}_0} \theta_w - \theta_w) e^{\theta_\xi} \; \omega^n.$$

where $V = \frac{1}{n!} \int_{\mathcal{X}_0} \omega^n$ is the volume of $\mathcal{X}_0$, and $\mathrm{DF}_{\xi}(\mathcal{X}, \mathcal{L}, w) = \frac{1}{V} \int_{\mathcal{X}_0} \theta_w \omega^n$ is the modified Donaldson-Futaki invariant of the configuration, in the form given in [10, Lemma 3.4].

Note if $(\mathcal{X}, \mathcal{L})$ is a product configuration then we have $\mathcal{X}_0 \cong X$. Assuming X is non-toric, the maximality of T in $\mathrm{Aut}(X)$ ensures that the restriction of λ to $\mathcal{X}_0$ is a one parameter subgroup of T, given by a choice of $w \in N$.

Definition 18. *We say the triple (X, t, ξ) is G-equivariantly K-semistable if $\mathrm{DF}_{t,\xi}(\mathcal{X}, \mathcal{L}, w) \geq 0$ for all G-equivariant special configurations $(\mathcal{X}, \mathcal{L}, w)$. We say (X, t, ξ) is K-stable if in addition equality holds precisely for product configurations.*

We will use the following theorem later:

Theorem 2 (Berman-Witt-Nystrom). *If (X, ξ) admits a Kähler-Ricci soliton then (X, ξ) is K-stable.*

From (Datar and Sze) we have a result in the converse direction:

Theorem 3 ([8, Proposition 10]). *Let X be a polarized Fano manifold, with Kähler form ω. Let $t \in [0, 1]$ and ξ be a soliton candidate for X. Then (X, t) is G-equivariantly K-semistable only if for all $s < t$ there exists $\omega_s \in 2\pi c_1(X)$ such that* $\mathrm{Ric}(\omega_s) - \mathcal{L}_\xi \omega_s = s\omega_s + (1-s)\omega$.

1.4.2 K-stability of T-varieties

Here we review K-stability in complexity one. In [24] Ilten and Suess described non-product special test configurations for a T-variety of complexity one in terms of its divisorial polytope. We first recall, from [24], the description of special fibers of non-product special configurations.

Let X be a Fano T-variety of complexity 1, corresponding to the Fano divisorial polytope $\Psi : \Box \to Y$. Without loss of generality we may assume $Y = \mathbb{P}^1$. Then there exists some $y \in \mathbb{P}^1$, with at most one of Ψ_z having non-integral slope at any $u \in \Box$ for $z \neq y$, such that $\mathcal{X}_0$ is the toric variety corresponding to the following polytope:

$$\Delta_y := \left\{ (u, r) \in M_{\mathbb{R}} \times \mathbb{R} \;\middle|\; u \in \Box,\ -1 - \sum_{z \neq y} \Psi_z(u) \leq r \leq 1 + \Psi_y(u) \right\}.$$

Furthermore, the induced $\mathbb{C}^*$-action on $\mathcal{X}_0$ is given by the one-parameter subgroup of $T' = T \times \mathbb{C}^*$ corresponding to $v' = (-mv, m) \in N \times \mathbb{Z}$, for some $v \in N$. In fact it turns out, from [24], it is enough to consider those configurations with $m = 1$. As observed in [24], we obtain a description of the (non-twisted) Donaldson-Futaki character of $(\mathcal{X}_0, \xi')$:

$$\mathrm{DF}_{\mathcal{X}_0, \xi'}(v') = \frac{1}{\operatorname{vol} \Delta_y} \left(\int_{\Delta_y} \langle u', v' \rangle \cdot e^{\langle u', \xi' \rangle} du' \right), \tag{1.8}$$

with $\xi', v' \in N_{\mathbb{R}} \times \mathbb{R}$. On the other hand, for $v, \xi \in N_{\mathbb{R}}$ one obtains:

$$\mathrm{DF}_{X,\xi}(v) = F_{\mathcal{X}_0, (\xi, 0)}((v, 0)) = \frac{1}{\int_{\Box} \deg \bar{\Phi}(u)\, du} \left(\int_{\Box} \langle u, v \rangle \cdot \deg \bar{\Phi}(u) \cdot e^{\langle u, \xi \rangle}\, du \right), \tag{1.9}$$

Lemma 4.

$$\mathrm{DF}_t(\mathcal{X}, \mathcal{L}) = t \langle \mathrm{bc}(\Delta_y), v' \rangle + (1 - t) \max_{x \in \Delta_y} \langle x, v' \rangle.$$

Proof. In [24] the formula $\mathrm{DF}(\mathcal{X}, \mathcal{L}) = \langle \mathrm{bc}(\Delta_y), v' \rangle$ is given. Note that the Hamiltonian function, by definition, satisfies $\theta_w(x) = \langle \mu(x), w \rangle$. We may then calculate the remaining integrals on the image of the moment map, Δ_y. $\Box$

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