STABILITY OF VARIETIES WITH A TORUS ACTION

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Contents

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In this thesis we study several problems related to the existence problem of invariant canonical metrics on Fano orbifolds in the presence of an effective algebraic torus action. The first chapter gives an introduction. The second chapter reviews the existing theory of T-varieties and reviews various stability thresholds and K-stability constructions which we make use of to obtain new results. In the third chapter we find new Kähler-Einstein metrics on some general arrangement varieties. In the fourth chapter we present a new formula for the greatest lower bound on Ricci curvature, an invariant which is now known to coincide with Tian's delta invariant. In the fifth chapter we discuss joint work with my supervisor to find new Kähler-Ricci solitons on smooth Fano threefolds admitting a complexity one torus action.

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Chapter 1

Introduction

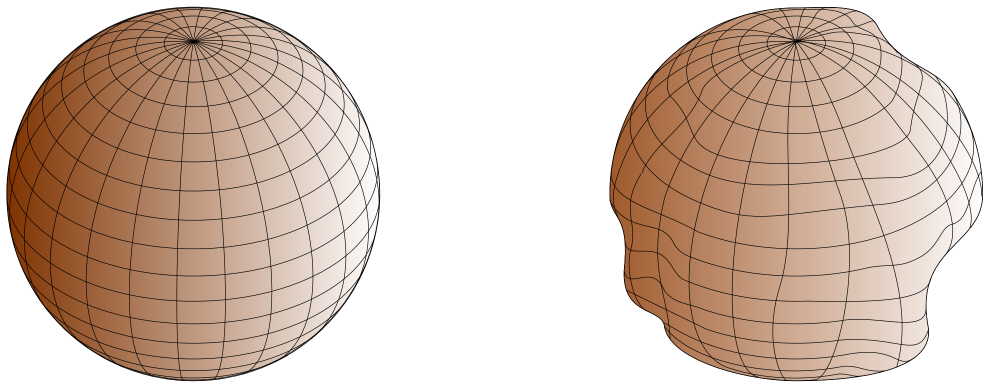
In this thesis we explore several new results relating to the existence of special metrics on certain compact Kähler manifolds admitting an effective algebraic torus action. Our goal is to further the understanding of canonical metrics on these types of manifolds. We will do this by finding new examples, and providing novel effective methods of calculating relevant invariants. Our methods are also a concrete demonstration of the power and utility of the language of T-varieties.

We begin with a brief primer on the theory for the uninitiated reader. A *Kähler manifold* is a smooth manifold X adorned with mutually compatible Riemannian, complex, and symplectic structures. In this situation we call the Riemannian metric g the *Kähler metric* on X, and the symplectic form ω the *Kähler form* of X.

There are many reasons to study Kähler geometry. From the standpoint of algebraic geometry, every smooth complex projective variety inherits a Kähler structure. From a differential geometric perspective, Kähler manifolds are a particularly well-behaved class of Riemannian manifolds, and are a rich enough class to contain many interesting examples. There are also motivations from theoretical physics: The various models of our universe in string theory require extra planck-scale dimensions, and certain Kähler manifolds are the best candidates for the shape of these dimensions.

Historically, it has been an important problem to investigate which Kähler manifolds admit nice Kähler metrics. Generally, what we mean here by "nice" depends on context. For naive motivation consider the real 2-sphere S^2. Most would have in their mind the standard embedding $S^2 = \{x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$, see Figure 1a. There are, however, many choices of smooth embedding, each one corresponding to different choices of Riemannian metrics on S^2, see for example Figure 1b. What sets our favourite embedding apart is that the induced metric has *constant curvature.*

This is a special case of a wider phenonemon if we identify the sphere as the Riemann surface $S^2 \cong \mathbb{P}^1_{\mathbb{C}}$. The uniformization theorem, originally proven by Poincaré [1] and Koebe [2,3], tells us that any Riemann surface admits a metric of constant scalar curvature. The obvious question is then: What happens in higher dimensions?



(a) A metric of constant scalar curvature

(b) A metric of non-constant scalar curvature

Figure 1: Two choices of metric on S^2.

In [4,5], Calabi proved certain results for compact Kähler manifolds which lead to a now famous conjecture. Fix a compact Kähler manifold (X, ω). Recall that the Ricci curvature form $\mathrm{Ric}(\omega)$ is a real $(1,1)$-form and defines a characteristic class $c_1(X) = \frac{1}{2\pi}[\mathrm{Ric}(\omega)]$ of the manifold, known as the first Chern class. Calabi asked whether, given a real $(1,1)$-form η representing the first Chern class of X, can we find a unique Kähler metric ω' in the same cohomology class as ω such that $\mathrm{Ric}(\omega') = 2\pi\eta$?

A related conjecture asks whether all compact Kähler manifolds (X, ω) admit a Kähler-Einstein metric, or more formally whether they admit a Kähler form ω' in the same cohomology class as ω with $\operatorname{Ric}\omega' = \lambda\omega'$, for some real constant λ. This equation is known as the Einstein condition[1], and the Kähler metric corresponding to ω' is called a Kähler-Einstein metric.

It follows by definition that, for X to admit such a metric, $\operatorname{Ric}\omega'$ must be a definite $(1,1)$-form. This separates the problem into three cases: Where $\operatorname{Ric}\omega'$ is positive definite, zero, or negative definite. In the first two cases Kähler-Einstein metrics on X are precisely the metrics of constant scalar curvature, and so one may see this as a direct generalization of the uniformization theorem for Riemann surfaces.

Aubin [6] and Yau [7] settled the negative definite case first. Calabi's conjecture was also proven by Yau in [7], later contributing to him being awarded the Fields Medal. This left the positive definite case, which corresponds to smooth Fano varieties under the Kodaira embedding theorem. It was already, known however, due to Matsushima [8], that not all Fano manifolds were Kähler-Einstein. Thus, it became an objective to find suitable criteria for the existence of a Kähler-Einstein metric on a Fano manifold.

In [9] Futaki introduced a new invariant whose vanishing was also a necessary condition. In [10] Tian introduced a sufficient condition in terms of another invariant, known now as *Tian's alpha invariant.* Tian's alpha invariant is a generalization of the complex singularity exponent of a polynomial $f \in \mathbb{C}[z_1, \ldots, z_n]$, which is defined as follows:

$$c_O(f) := \sup\{\epsilon |\ |f|^{-2\epsilon} \text{ integrable in a neighborhood of } O \in \mathbb{C}^n\}.$$

The Yau-Tian-Donaldson conjecture suggested the notion of *K-stability* as a necessary and sufficient Kähler-Einstein criterion[2]. This was proven in the trilogy of papers [11–13].

A generalization of the notion of a Kähler-Einstein metric is a Kähler-Ricci soliton. To understand how, recall that Kähler Einstein metrics may be seen as generalized fixed point solutions under the Kähler-Ricci flow:

$$\frac{d}{dt}\omega_t = -2\operatorname{Ric}(\omega_t),$$

[1]By analogy to Einstein's field equation for a vacuum.

[2]In full, the YTD talks of csck metrics, which are equivalent to KE in the Fano case

in that under this flow they will remain unchanged up to some scaling factor. A Kähler-Ricci soliton is a generalized fixed point of the flow in the sense that it will remain unchanged up to some biholomorphism.

A further generalization are twisted Kähler-Einstein metrics and twisted Kähler-Ricci solitons. These arise in continuity method arguments, see [14] for example, and depend on a parameter $t \in [0, 1]$. Here we start with a Calabi-Yau type solution ω_0 at $t = 0$, and consider the existence of solutions ω_t along a line segment to the target Kähler-Einstein or Kähler-Ricci soliton equation at $t = 1$ respectively. The supremum of the set of t for which a solution exists turns out to be independent of ω_0, and is of interest as an invariant of X. It is known as *Tian's beta invariant*, or the *greatest lower bound on Ricci curvature*. We will denote this invariant by $R(X)$.

Although K-stability is a criterion for the existence of Kähler-Einstein metrics and their various generalizations, it is not an effective one. In general, the K-stability of a Fano manifold is difficult to calculate. The alpha invariant approach also has limitations in practice. Fortunately, equivariant versions of K-stability and Tian's alpha invariant exist, which, as we will see, provide an effective approach in classes of manifolds and orbifolds with a high degree of symmetry.

In this thesis we will focus our attention on Fano manifolds which are also *T-varieties*. A T-variety is a normal variety which admits an effective action of an algebraic torus $T = (\mathbb{C}^*)^r$. These are a generalization of toric varieties, where $\dim T = \dim X$. In general we call the difference $\dim X - \dim T$ the *complexity* of the torus action.

In the toric case it is well-established that studying X is equivalent to studying some associated combinatorial data: a fan of cones in a vector space built from the cocharacter lattice of T. Thanks to the work of many authors (Altmann, Hausen, Ilten, Petersen, Süß, Vollmert, Liendo to name a few) this combinatorial description extends to higher complexity. We recall some of the theory in Chapter 2.

Equivariant methods have been used to provide some effective criteria for canonical metrics on low complexity T-varieties. If X is a Fano toric variety then the Kähler-Einstein problem is completely solved. In [15] it was shown that X is Kähler-Einstein if and only if the Futaki character vanishes. They also showed that the Futaki character coincides with the barycenter of the lattice Polytope corresponding to X. Wang et al did not use K-stability for this result, but the result was later reproven as an application of the main theorem of [14].

In [16], Süß and Ilten considered the K-stability of T-varieties of complexity one. We recall this in detail in Section 2.4.1. They obtained a combinatorial criterion for K-stability, generalizing the results of [15]. Süß had also used the equivariant version of Tian's alpha invariant to find new Kähler-Einstein metrics on complexity one T-varieties admitting additional symmetries in [17].

In complexity two and above, even equivariant K-stability remains an ineffective criterion. There is, as we shall see, scope for methods like that of [17]. In the next section we will give an overview of the content of this thesis.

1.1 Content of the Thesis

Here we summarize the remaining structure of the thesis. Some of the content of this thesis has been published, and/or submitted to journals. I reference relevant sources clearly when this is the case. I also make clear, in the case of my coauthored work in Chapter 3, the scope of my contribution to the original paper.

Chapter 2 - Preliminaries

We recall definitions and results from Kähler, algebraic, and symplectic geometry to give context to our novel results. We then give a brief introduction to the theory of T-varieties and their equivariant K-stability, which are key in our methods of proof in Chapters 3 and 4.

Chapter 3 - New Kähler-Ricci solitons on Fano threefolds

In Chapter 3 we consider Fano threefolds admitting an effective 2-torus action within the classification of [18]. In [**?**] a not necessarily complete list of such threefolds together with their combinatorial description was given. We extend the results of [16], providing new examples of threefolds admitting a non-trivial Kähler-Ricci soliton. Recall that a Kähler-Ricci soliton on a Fano manifold (X, ω) is a pair (ω', v) satisfying:

$$\mathrm{Ric}(\omega') - L_v \omega' = \omega'$$

We apply some real interval arithmentic approximations to the complexity one formula for the Futaki invariant of Ilten and Süß (see Section 2.4.2) to check the existence criterion [14] (see Section 2.4.1). We include the relevant Sagemath code in Appendix B.

Chapter 4 - $R(X)$ in complexity one

In Chapter 4 we present an explicit effective formula obtained for the greatest lower bound on Ricci curvature $R(X)$ for a complexity one T-variety X. We follow the author's work [19]. These results generalize a result of Li [20], but the proof uses results of G-equivariant K-stability from [14]. The invariant $R(X)$ is often denoted $\beta(X)$ and is referred to as *Tian's beta invariant*. By [] it is now known to coincide with another important invariant, $\delta(X)$.

Chapter 5 - New Kähler-Einstein metrics on symmetric general arrangement varieties

In Chapter 5, we discuss recent results obtaining new Kähler-Einstein metrics on some symmetric complexity two general arrangement varieties. General arrangement varieties are T-varieties where the torus quotient is a projective space, and the critical values of the quotient map form a general arrangement of hyperplanes in that projective space. Smooth general arrangement varieties of complexity and Picard rank 2 were classified according to their Cox ring in [21]. Following the methods of [22], we find three new examples of Kähler-Einstein metrics. As far as we are aware, these are the first examples of Kähler-Einstein metrics found on T-varieties of complexity greater than one by way of equivariant methods.

Chapter 2

Preliminaries

2.1 Kähler geometry

2.1.1 Basic definitions

Kähler manifolds

In this section we recall the basics of Kähler geometry. We then review some important results on the existence of canonical metrics on compact Kähler manifolds. A good reference for the material here is [23]. Let X denote a compact real manifold of dimension $2n$. Suppose we have an almost complex structure J on X, that is an automorphism J of $T_{\mathbb{R}}X$ such that $J^2 = -\mathrm{Id}$. Recall that the complexified tangent bundle, $T_{\mathbb{C}}X := T_{\mathbb{R}}X \otimes \mathbb{C}$, decomposes via eigenspaces of J:

$$T_{\mathbb{C}}X = T^{1,0}X \oplus T^{0,1}X,$$

where $T^{(1,0)}X$ has local generators $\frac{\partial}{\partial z_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} - \sqrt{-1}\frac{\partial}{\partial y_i}\right)$, and $T^{(0,1)}X = \overline{T^{(1,0)}X}$ has local generators $\frac{\partial}{\partial \bar{z}_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} + \sqrt{-1}\frac{\partial}{\partial y_i}\right)$.

Recall also that we have a natural isomorphism of real vector bundles $T_{\mathbb{R}}X \cong T^{(1,0)}$, given by the composition $T_{\mathbb{R}}X \to T_{\mathbb{C}} \to T^{1,0}X$. Note, by definition, the action of J is described by multiplying by $\sqrt{-1}$ on $T^{1,0}X$. The decomposition above induces a decomposition of the complexified cotangent bundle:

$$T^*_{\mathbb{C}}X = T^*_{1,0}X \oplus T^*_{0,1}X,$$

and, moreover, a decomposition:

$$\bigwedge^n T^*_{\mathbb{C}}X = \bigoplus_{p+q=n} \left(\bigwedge^p T^*_{1,0}X \otimes \bigwedge^q T^*_{0,1}X \right) \tag{2.1}$$

We use the following notation for relevant spaces of global sections:

$$A^n(X) := H^0(X, \bigwedge^n T^*_{\mathbb{C}}X) \tag{2.2}$$

$$A^{p,q}(X) := H^0(X,\ (\bigwedge^p T^*_{1,0}X \otimes \bigwedge^q T^*_{0,1}X)\ ). \tag{2.3}$$

A form $\alpha \in A^{p,q}(X)$ is said to be *of type* (p,q). The decomposition (2.1) induces a corresponding decomposition:

$$A^n(X) = \bigoplus_{p+q=n} A^{p,q}(X).$$

Hermitian and Kähler metrics

Roughly speaking, a metric measures distances on a manifold. On real manifolds one usually *Riemannian metrics*. Complex manifolds come with more structure, and the analogous object compatible with this structure, is a *Hermitian metric*. A Hermitian metric is given by a smooth choice of positive definite Hermitian inner product on the fibers of $T^{(1,0)}X$, i.e an element of $H^0(X, T^*_{1,0}X \otimes T^*_{0,1}X)$. Locally we write:

$$h(z) = \sum h_{ij}(z) dz_i \otimes d\bar{z}_j.$$

Given a Hermitian metric h on X, we may consider the real and imaginary parts of h as real tensors on the underlying real manifold via the isomorphisms $T_{\mathbb{R}}X \cong T^{1,0}X \cong \overline{T^{1,0}X}$. The real part $g = \Re h$ is a Riemannian metric on X, called the *induced Riemannian metric* of h. Locally we have:

$$g_z = \sum h_{ij}(z)(dx_i \otimes dx_j + dy_i \otimes dy_j)$$

We may also realize the imaginary part $\omega = -\Im h$, up to sign, as an alternating form on the real tangent bundle $T_{\mathbb{R}}X$, via $T^{1,0}X \cong \overline{T^{1,0}X}$. Set $\omega(v \wedge w) := -\Im h(v, \bar{w}) = -\frac{i}{2}(h - \bar{h})$. We call ω the *associated* $(1,1)$*-form* of h. Locally we have:

$$\omega_z = \sqrt{-1}\sum h_{ij}(z)dz_i \wedge d\bar{z}_j = \sqrt{-1}\sum h_{ij}(z)(dx_i \otimes dy_j - dy_i \otimes dx_j).$$

By definition we have $g(v,w) = g(Jv,Jw)$ and $\omega(v,w) = g(Jv,w)$ for any $v,w \in T_{\mathbb{R}}X$. In fact we may reconstruct h from any Riemannian metric g satisfying $g(v,w) = g(Jv,Jw)$, or alternatively any real $(1,1)$-form ω satisfying the positive definite condition:

$$\omega(v \wedge v) > 0 \text{ for all } v \in T_{\mathbb{R}}X.$$

We may now recall the definition of a Kähler metric.

Definition 1. *A Hermitian metric is Kähler if the associated* $(1,1)$*-form* ω *is closed, i.e* $d\omega = 0$*, where* $d : A^2(X,\mathbb{R}) \to A^3(X,\mathbb{R})$ *is the usual exterior differential.*

We will bow to convention and often refer to ω, instead of g, as a Kähler metric on X in this context. The standard first example of a compact Kähler manifold is the Fubini-Study metric on complex projective space:

Example 1. *Let* s *be a section of the projection map* $\pi : \mathbb{C}^{n+1}\backslash\{0\} \to \mathbb{P}^n$ *over some open set* $U \subset \mathbb{P}^n$*. The Fubini-Study metric* ω_{FS} *is then defined to be*

$$\omega_{FS} := i\partial\bar{\partial}\log||s||^2$$

This is well-defined as any two sections differ on their shared domain by a non-vanishing holomorphic function, $s' = fs$*. It is clearly closed (since* $d = \partial + \bar{\partial}$*). For the standard section on* U_0 *with holomorphic coordinates* $z_1,\dots,z_n$ *we have:*

$$\omega_{FS} := i\partial\bar{\partial}\log(1 + |z_1|^2 + \cdots + |z_n|^2)$$

and at $[1,0,\dots,0] \in U_0$ *we have:*

$$\omega_{FS} = i\sum dz_j \wedge d\bar{z}_j$$

This is positive definite, and so defines a Kähler metric on $\mathbb{P}^n$*.*

This leads to a large class of examples including any smooth projective algebraic variety:

Example 2. *The restriction of* ω_{FS} *to any closed submanifold* $Y \subseteq \mathbb{P}^n$ *induces* *a Kähler structure on Y, as the exterior differential commutes with pulling back differential forms.*

Recall the following important result, telling us that any two Kähler forms of the same class differ by some real-valued function.

$\partial\bar{\partial}$-lemma. *If ω, η are two real $(1,1)$-forms of the same cohomology class then there is a real function $f : X \to \mathbb{R}$ such that $\omega - \eta = \sqrt{-1}\partial\bar{\partial} f$.*

2.1.2 Line bundles and Kodaira Embedding

Recall that we can extend the notion of Hermitian metric to an arbitrary complex vector bundle E: a Hermitian metric on E is defined to be an element $h \in H^0(X, E \otimes \bar{E})^*$. We now recall the notion of a connection and its curvature. A *connection* may be thought of as a way to differentiate tensor fields, and transport data smoothly about a manifold. In our context, a connection is given by a map:

$$\nabla : H^0(X, E) \to H^0(X, E \otimes T^*X)$$

satisfying the Liebniz rule $\nabla(sf) = \nabla sf + s \otimes df$. There is a unique way to extend a connection to an exterior derivative on E-valued differential forms:

$$d^\nabla : \Omega^r(E) \to \Omega^{r+1}(E).$$

The *curvature* of a connection is the 2-form:

$$F^\nabla \in H^0(X, \mathrm{End}(E) \otimes \wedge^2 T^*X),$$

given by:

$$F^\nabla(u, v)(s) := \nabla_u \nabla_v s - \nabla_v \nabla_u s - \nabla_{[u,v]} s.$$

There is a canonical connection on the tangent bundle of any Riemannian manifold known as the *Levi-Civita connection*, satisfying $\nabla g = 0$ and $\nabla_u v - \nabla_v u = [u, v]$. We have a similar situation for any Hermitian vector bundle on a complex manifold:

Example 3. *Let E be a Hermitian vector bundle on a complex manifold X equipped with a holomorphic structure. There is a unique connection ∇ on E such that:*

- *For all sections s we have $\pi_{1,0} \nabla s = \bar{\partial} s$*

- *For any smooth vector field v and sections s, t we have:*

$$v\langle s,t\rangle = \langle \nabla_v s, t\rangle + \langle s, \nabla_v t\rangle$$

This connection is called the Chern connection on E.

Kähler manifolds may be characterized as those manifolds for which the Levi-Civita connection and Chern connection on the tangent bundle coincide. We now recall the definition of the first Chern class of a Hermitian line bundle, which may be used to define notions of positivity.

Definition 2. *The first Chern class of a Hermitian line bundle L is the real cohomology class:*

$$c_1(L) = \frac{1}{2\pi}[-\sqrt{-1}\partial\bar{\partial}\log(h)] \in H^2(X,\mathbb{Z})$$

Example 4. *Suppose (X,g) is a Kähler manifold. Then g induces a Hermitian metric on the holomorphic cotangent bundle $\Omega^{1,0}X$, which in turn induces a Hermitian metric on the canonical line bundle $K_X = \wedge^n\Omega^{1,0}X$, denoted* $\det(g)$*. The curvature of the associated Chern connection to this Hermitian line bundle is called the Ricci curvature form of the manifold, given by:*

$$\mathrm{Ric}(\omega) = -\sqrt{-1}\partial\bar{\partial}\log(\det(g)).$$

The real cohomology class $c_1(K_X) = \frac{1}{2\pi}[-\sqrt{-1}\partial\bar{\partial}\log(\det(g)]$ is called the first Chern class of the Kähler manifold (X,g), and is often denoted just by $c_1(X)$.

We now recall the definition of a positively curved line bundle.

Definition 3. *A real $(1,1)$-form is called positive if the associated symmetric bilinear form defined for real tangent vectors is positive definite. A real cohomology class is called positive if it can be represented by a positive $(1,1)$-form. A line bundle L is called positive if its first Chern class is positive.*

The following theorem, by Kodaira, characterizes smooth projective varieties amongst compact Kähler manifolds. Recall that a line bundle L is very ample if for some global sections $s_0,\ldots,s_n \in H^0(X,L)$ we obtain a well-defined closed embedding into a projective space, given by:

$$\varphi_L : p \mapsto [s_0(p),\ldots,s_n(p)] \in \mathbb{P}^n$$

We say L is ample if some multiple of L is very ample.

Kodaira Embedding Theorem [**?**, Theorem 4]**.** *A holomorphic line bundle over a compact complex manifold is ample if and only if it is positive.*

Paired with the following theorem we can talk either about projective Kähler manifolds or polarized projective algebraic varieties.

Chow's Theorem [24, Theorem 5]**.** *A closed complex submanifold of projective space is a projective algebraic subvariety.*

2.1.3 Canonical metrics on Kähler manifolds

In this section we recall some facts about canonical metrics. A canonical metric is a choice of metric dependent only on the complex structure of the manifold, unique up to biholomorphic automorphisms. The material in this section may be found in [25], for example.

As we touched on in the introduction, Kähler-Einstein metrics are an important class of canonical metric, and the question of which compact Kähler manifolds admit a Kähler-Einstein metric has historically received a lot of attention. Recall the definition of a Kähler-Einstein metric:

Definition 4. *Let X be a Kähler manifold. A Kähler-Einstein metric on X is a Kähler metric $\omega \in 2\pi c_1(X)$ such that $\operatorname{Ric}\omega = \lambda\omega$ for some real constant λ.*

Note if X is Kähler-Einstein then we must have either $K_X = 0$, K_X ample, or $-K_X$ ample. Via Kodaira embedding these correspond to Ricci flat, Ricci positive, and Ricci negative manifolds respectively. We briefly recall the answers to the existence question in the cases of negative and zero Ricci curvature:

Calabi-Yau Theorem [26, Theorem 1]**.** *Let (X,ω) be a compact Kähler manifold. Let α be a real $(1,1)$-form representing $c_1(X)$. Then there exists a real $(1,1)$-form ω' with $[\omega'] = [\omega]$ such that $\operatorname{Ric}(\omega) = 2\pi\alpha$.*

Aubin-Yau Theorem [6,27]**.** *Let X be a compact Kähler manifold with $c_1(X) < 0$. Then there exists a unique Kähler metric $\omega \in -2\pi c_1(X)$ such that $\operatorname{Ric}(\omega) = -\omega$.*

However the following necessary criterion illustrates that the same is not true in the Fano case $c_1(X) > 0$. Recall that a complex algebraic group is reductive if it is the complexification of a compact connected real Lie group.

Matsushima's criterion [8]. *If a Fano manifold X admits a Kähler-Einstein metric, then the holomorphic automorphism group of X is reductive.*

In particular this tells us that the blow up of $\mathbb{P}^1$ in one or two points is not Kähler-Einstein. We end this section by recalling the most general form of canonical metric we will consider. This matches the definition given in [14, Definiton 3].

Definition 5. *A twisted Kähler-Ricci soliton on a Fano manifold (X, ω_0) is a triple (ω, v, t) where $\omega \in 2\pi c_1(X)$ is a Kähler metric, v is a holomorphic vector field, and $t \in [0, 1]$, such that*

$$\mathrm{Ric}(\omega) - \mathcal{L}_v \omega = t\omega + (1-t)\omega_0$$

When $t = 0$ we omit it from the notation and call (ω, v) a Kähler-Ricci soliton. Similarly when v is trivial we call (ω, t) a twisted Kähler-Einstein metric. When both hold then we talk about ω being a Kähler-Einstein metric.

In Section 2.4 we will describe various criteria for the existence of such metrics, but to do so we must first recall some basic tools and language from algebraic and symplectic geometry.

2.2 Algebraic and symplectic tools

In this section we give some definitions from algebraic and symplectic geometry as we will be understanding them throughout the rest of the thesis. In particular we recall some basic geometry invariant theory, which is needed for the arguments in Chapter 5.

2.2.1 The algebraic torus

Fix an algebraic torus $T = (\mathbb{C}^*)^k$. We have mutually dual character and cocharacter lattices:

$$M := \mathrm{Hom}(T, \mathbb{C}^*), \ N = \mathrm{Hom}(\mathbb{C}^*, N),$$

respectively. We denote the associated vector spaces by:

$$M_{\mathbb{K}} := M \otimes_{\mathbb{Z}} \mathbb{K}, \ N_{\mathbb{K}} := N \otimes_{\mathbb{Z}} \mathbb{K},$$

for $\mathbb{K} = \mathbb{Q}, \mathbb{R}$. There is a perfect pairing $M \times N \to \mathbb{Z}$ which extends to a bilinear pairing $M_{\mathbb{K}} \times N_{\mathbb{K}} \to \mathbb{K}$. We may make the identification:

$$T \cong \operatorname{Spec} \mathbb{C}[M] \cong N \otimes \mathbb{C}^*.$$

Finally recall that we may identify the real Lie algebra $\mathfrak{k}$ of the maximal compact subtorus $K \subset T$ as $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$.

2.2.2 Linearizations

Suppose we have an algebraic group G acting algebraically on a scheme or variety X. A lift of the this action to a line bundle L on X is known as a linearization. Linearizations are used in geometric invariant theory to give a good definition of a quotient of X by the G-action.

Definition 6. *Let X be a projective scheme together with an action $\lambda : G \times X \to X$ of a reductive algebraic group G. A linearization of the action λ on L is an action $\tilde{\lambda}$ on L such that:*

- *The projection π is G-equivariant, $\pi \circ \tilde{\lambda} = \lambda \circ \pi$*
- *For $g \in G$ and $x \in X$, the induced map $L_x \mapsto L_{g \cdot x}$ is linear.*

Note a linearization to L naturally induces linearizations to $L^\vee$ and $L^{\otimes r}$ for any $r \in \mathbb{N}$.

Example 5. *A linearization of the trivial bundle on a projective variety X must be of the form*

$$g \cdot (x, z) = (g \cdot x, \chi(x, z) z)$$

for some $\chi \in H^0(G \times X, \mathcal{O}^*_{G \times X}) \cong H^0(G, \mathcal{O}^*_G) = \mathfrak{X}(G)$.

The above example tells us that any two linearizations λ_1, λ_2 of an action to the same line bundle differ by multiplication by some character χ of G: fiberwise we have $\tilde{\lambda}_1 = \chi(x, z)\tilde{\lambda}_2$. Thus, when $G \cong T$ is an algebraic torus we may identify the set of linearizations with the character lattice M.

Example 6. *Recall that an action of G on X induces a canonical linearization on the tangent and cotangent bundles of X, and so induces a canonical linearization on the anti-canonical bundle $-K_X$ as the top exterior power of the cotangent bundle.*

2.2.3 Hamiltonian actions and moment maps

Here we recall some basic notions of Hamiltonian actions and moment maps. We will follow conventions of [28] and [29]. We illustrate the theory with the case of an algebraic torus action. Suppose that (X, ω) is a symplectic manifold.

Definition 7. *Let $\theta : X \to \mathbb{R}$ be a smooth function. A vector field v such that $\iota_v \omega = d\theta$ is called a Hamiltonian vector field, with Hamiltonian function θ.*

Definition 8. *Let K be a real Lie group, with Lie algebra $\mathfrak{k}$, acting smoothly on X. This action is said to be Hamiltonian if there exists a map $\mu : X \to \mathfrak{k}^*$, known as the moment map of the action, such that:*

- *For any $\xi \in \mathfrak{k}$ the map $\mu^\xi : X \to \mathbb{R}$ given by $\mu^\xi(p) := \langle \mu(p), \xi \rangle$ is a hamiltonian function for the vector field v generated by the one-parameter subgroup $\exp(t\xi) \subset K$.*
- *The map μ is equivariant with respect to the action of K on X and with respect to the coadjoint action.*

Example 7. *Let $X \subseteq \mathbb{P}^N$ be a nonsingular complex projective variety and let G be reductive algebraic group acting effectively on $\mathbb{P}^N$, restricting to an action on X. Let K be the maximal compact subgroup in G, with Lie algebra $\mathfrak{k}$. The action of G is given by a representation $\rho : G \to GL(N+1)$, and by choosing appropriate coordinates we may assume K maps to $U(N+1)$ and so preserves the Fubini-Study form. It can be checked that a moment map $\mu : X \to \mathfrak{k}^*$ for the K-action is given by:*

$$\mu([x]) \cdot a := \frac{x^t \rho_*(a) \hat{x}}{|x|^2}, \tag{2.4}$$

where x is any representative of $[x] \in X \subseteq \mathbb{P}^N$. A different choice of linearization in this setting corresponds to multiplying ρ by some character $\chi \in \mathfrak{X}(G)$. Since $\chi(K)$ is compact, it sits inside $S^1 \subset \mathbb{C}^*$, and hence we do not need to change coordinates when considering the effect on the moment map. When we plug this into (2.5) we see that we have translated the moment map by $\chi \in \mathfrak{X}(G) \otimes \mathbb{R} \cong \mathfrak{k}^*$. Moreover, taking the rth power of L corresponds to scaling the moment map by a factor of r. This gives a correspondence between rational elements

$$\chi \in \mathfrak{X}(G) \otimes \mathbb{Q} \subset \mathfrak{k}^*$$

and linearizations of powers of L.

Example 8. *Suppose $G = T$ is an algebraic torus with character and cocharacter lattices M, N respectively. Then ρ is a diagonal matrix of characters $u_0, \ldots, u_N$ and we obtain:*

$$\mu([x]) = \frac{\sum_{j=0}^{N} |x_i|^2 u_i}{|x|^2} \in M$$

Then, by Atiyah, [30], and Guillemin-Sternberg, [31], the image of μ is a convex polytope $P \subset M$. Here we see that for each one-parameter subgroup $w \in N$ we have Hamiltonian function:

$$\theta_w([x]) = \langle \mu([x]), w \rangle$$

2.2.4 Chow and GIT quotients

Here we recall the definition of GIT, Chow, and limit quotients of a projective variety by a reductive algebraic group G. We also explain how, when G is a torus, they may be explicitely calculated via the Kempf-Ness theorem, and recall how GIT quotients behave under smooth blowup. The material here is used in Chapter 5, where we give an exposition of the results of [32]. Thus, the material here may also be found in the preliminary sections of [32].

GIT quotients

Recall the basic setup of Mumford's geometric invariant theory, which provides a method for finding geometric quotients on open subsets of a scheme X when the acting algebraic group G is reductive. In [33] Mumford introduced the notion of a good categorical quotient, which can be shown to be unique if it exists.

Definition 9. *A surjective G-equivariant morphism $\pi : X \to Y$ is a good categorical quotient if the following hold:*

1. *We have $\mathcal{O}_Y = (\pi_* \mathcal{O}_X)^G$;*
2. *if V is a closed G-invariant subset of X then $\pi(V)$ is closed;*
3. *if V, W are closed G-invariant subsets of X and $V \cap W = \emptyset$ then we have $\pi(V) \cap \pi(W) = \emptyset$.*

Good quotients do not always exist for a given scheme X, but we might hope that there exists some dense open subset of X which does admit a good quotient.

Consider the affine case, where $X = \operatorname{Spec} A$. For G reductive then it can be shown that $X /\!\!/ G := \operatorname{Spec} A^G$ is a good categorical quotient.

The same ansatz works in the projective case once we make a choice of a lift of the action to the ring of sections of a given ample line bundle. This choice is known as a linearization of the group action. A linearization u of a group action G on X to L induces an action of G on the ring of sections $R(X, L) := \bigoplus_{j \geq 0} H^0(X, L^{\otimes j})$. Consider the scheme $X /\!\!/_u G := \operatorname{Proj} R(X, L)^G$. Note we have a birational map from X to $X /\!\!/_u G$, defined precisely at $x \in X$ such that there exists some $m > 0$ and $s \in R(X, L)^G_m$ such that $s(x) \neq 0$. Such a point is said to be semi-stable. If in addition $G \cdot x$ is closed and the stabilizer G_x is dimension zero, the point x is said to be stable. The set of semi-stable and stable points will be denoted by $X^{ss}(u)$ and $X^s(u)$ respectively.

Lemma 1 [33, Chapter 1, Section 4]**.** *The canonical morphism $X^{ss}(u) \to X /\!\!/_u G := \operatorname{Proj} R(X, L)^G$ is a good categorical quotient.*

Kempf-Ness approach to GIT quotients

One approach to calculating GIT quotients is via the Kempf-Ness theorem. Let $X \subseteq \mathbb{P}^N$ be a nonsingular complex projective variety and let G be reductive algebraic group acting effectively on $\mathbb{P}^N$, restricting to an action on X. Let K be the maximal compact subgroup in G, with Lie algebra $\mathfrak{k}$. The action of G is given by a representation $\rho : G \to \mathrm{GL}(N+1)$, and by choosing appropriate coordinates we may assume K maps to $U(N+1)$ and so preserves the Fubini-Study form. It can be checked that a moment map $\mu : X \to \mathfrak{k}^*$ is given by:

$$\mu([x]) \cdot a := \frac{x^t \rho_*(a) \hat{x}}{|x|^2} \tag{2.5}$$

Where x is any representative of $[x] \in X \subseteq \mathbb{P}^N$. Note we are now in the situation of the previous subsection, with $L = \mathcal{O}_X(1)$ under the embedding $X \subseteq \mathbb{P}^N$. This moment map is unique up to translations in $\mathfrak{k}^*$. A different choice of linearization in this setting corresponds to multiplying ρ by some character $\chi \in \mathfrak{X}(G)$.

Since $\chi(K)$ is compact it sits inside $S^1 \subset \mathbb{C}^*$, and hence we do not need to change coordinates when considering the effect on the moment map. When we plug this into (2.5) we see that we have translated the moment map by $\chi \in \mathfrak{X}(G) \otimes \mathbb{R} \cong \mathfrak{k}^*$. Moreover, taking the rth power of L corresponds to scaling the moment map by a factor of r. This gives a correspondence between rational elements $\chi \in \mathfrak{X}(G) \otimes \mathbb{Q} \subset \mathfrak{k}^*$ and linearizations of powers of L.

Example 9. *Suppose $G = T$ is an algebraic torus with character and cocharacter lattices M, N respectively. Then ρ is a diagonal matrix of characters $u_0, \ldots, u_N$ and we obtain:*

$$\mu([x]) = \frac{\sum_{j=0}^{N} |x_i|^2 u_i}{|x|^2} \in M$$

Then, by Atiyah, [30], and Guillemin-Sternberg, [31], the image of μ is a convex polytope $P \subset M$.

We will make use of the following theorem of Kempf and Ness. A proof is given in [33, Chapter 8]. See also the original work [34].

Kempf-Ness Theorem [34, Theorem 8.3]**.** *Let $X \subseteq \mathbb{P}^N$ be a nonsingular complex projective variety and let G be reductive algebraic group acting effectively on $\mathbb{P}^N$, restricting to an action on X. Consider a linearization of some power of L corresponding to a rational element $u \in \mathfrak{k}^*$.*

1. *$X^{ss}(u) = \{x \in X | \overline{Gx} \cap \mu^{-1}(u) \neq \emptyset\}$.*

2. *The inclusion of $\mu^{-1}(u)$ into $X^{ss}(u)$ induces a homeomorphism*

$$\mu^{-1}(u)/K \to X /\!\!/_u G$$

where $\mu^{-1}(u)/K$ is endowed with the quotient topology induced from the classical (closed submanifold topology) on $\mu^{-1}(u)$, and $X /\!\!/_u G$ is endowed with its classical (complex manifold) topology

We can use Theorem 2.2.4 to calculate GIT quotients by inspection. To be explicit, suppose $\mu^{-1}(u)/K$ has the structure of a complex projective variety and $q : X^{ss}(u) \to \mu^{-1}(u)/K$ is a G-invariant morphism which restricts to the topological quotient map on the moment fibre, such that $q_*\mathcal{O}_X^G = \mathcal{O}_Y$. The following fact is probably well known, but we prove it here for the reader's convenience.

Lemma 2. *The morphism q is a good categorical quotient, and hence is isomorphic to the GIT quotient map $X \to X /\!\!/_u G$.*

Proof. It is enough to show that q sends closed G-invariant subsets to closed subsets, and disjoint pairs of closed invariant subsets to disjoint pairs of closed subsets.

Firstly suppose that V is a G-invariant Zariski-closed subset of X. Then $q(V) = q(V \cap \mu^{-1}(u))$, and $V \cap \mu^{-1}(u)$ is K-invariant and closed in the classical topology of $\mu^{-1}(u)$. This implies that $q(V)$ is closed in the classical topology on $\mu^{-1}(u)/K \simeq X /\!\!/_u G$. But $q(V)$ is constructable, as the image of a Zariski-closed subset of X, and so we may conclude that $q(V)$ is Zariski-closed in $\mu^{-1}(u)/K \simeq X /\!\!/_u G$.

Now suppose V, W are G-invariant and Zariski-closed in X, with $x \in V$ and $y \in W$ such that $q(x) = q(y)$. By 2.2.4 we may take $x' \in \overline{Gx} \cap \mu^{-1}(u)$, $y' \in \overline{Gy} \cap \mu^{-1}(u)$ such that $q(x') = q(y')$. These two points lie in the same K-orbit. By the G-invariance of V, W we have $V \cap W \neq \emptyset$. □

GIT quotients under smooth blowup

Here we recall some results from [35], which we will use in the proof of Theorem 9. Let G be a reductive group acting on X. Let L be an ample G-invariant line bundle on X. Fix some linearization of the G-action to L. Suppose V be a smooth closed G-stable subvariety of X, defined by some ideal sheaf $\mathcal{J}_V$. Let $f : W \to X$ be the blow-up of X along V.

The goal is to construct a linear action on W lifting the action on X, and describe the GIT quotient $W^{ss} \to W /\!\!/ G$ in terms of $X^{ss} \to X /\!\!/ G$ and f. First let us construct an ample line bundle on W. Let E be the exceptional divisor of the map f. Set $L_d := f^*L^{\otimes d} \otimes \mathcal{O}(-E)$. For sufficiently large d, L_d is ample.

Since $E \cong \mathbb{P}(N_{V,X})$ and $\mathcal{O}(-E)_{|E} \cong \mathcal{O}_{\mathbb{P}(N_{V,X})}(1)$ then the natural action of G on $N_{V,X}$ induces an action on $\mathcal{O}(-E)_{|E}$. We have $W \backslash E \cong X \backslash V$ and $\mathcal{O}(-E)_{|W\backslash E}$ is the trivial line bundle, so admits the product action. The action of G on L lifts to $f^*L^{\otimes d}$, and so we obtain a linear action on L_d. By [35,], for sufficiently large d we have $W^{ss} \subset X^{ss}$. We have the following result:

Lemma 3 [35, Lemma 3.11]. *If d is a sufficiently large multiple of e then the GIT quotient $W /\!\!/ G$ associated to the linearization described above is the blowup of $X /\!\!/ G$ along the image $V /\!\!/ G$ of V in $X /\!\!/ G$. In particular if $V /\!\!/ G$ is a divisor on $X /\!\!/ G$ then $W /\!\!/ G \cong X /\!\!/ G$.*

Chow and limit quotients

Recall the definition of the Chow quotient, as introduced in [36]. If G is any connected linear algebraic group and X is a projective G-variety, then orbit closures of points are generically of the same dimension and degree, and so define points in the corresponding Chow variety. The Chow quotient of the G-action on X is the closure of this set of points.

We now recall the definition of the limit quotient, from [33]. The limit quotient is discussed in detail in [37]. Let G be a reductive algebraic group, and X a projective G-variety. Suppose there are finitely many sets of semi-stable points $X_1, \dots, X_r$ arising from G-linearized ample line bundles on X. Whenever $X_i \subseteq X_j$ holds, there is a dominant projective morphism $X_i /\!\!/ G \to X_j /\!\!/ G$ which turns the set of GIT quotients into an inverse system. The associated inverse limit Y admits a canonical morphism $\bigcap_{i=1}^r X_i \to Y$. The closure of the image of morphism is the limit quotient.

When G is an algebraic torus there are indeed finitely many semi-stable loci. Moreover, by [37, Corollary 2.7], we may calculate the limit quotient by taking the inverse limit of the subsystem obtained by only considering linearizations of powers of one fixed ample line bundle L. In [37, Proposition 2.5] it is shown that the Chow quotient and limit quotient coincide when G is an algebraic torus.

Definition 10. *Let X be a T-variety. Let $\pi : X \dashrightarrow Y$ be the Chow quotient map of X by its torus action. For any prime divisor Z on Y, the generic stabilizer on a component of $\pi^{-1}(Z)$ is a finite abelian group. The maximal order across these components is denoted m_Z. We may then define a boundary divisor for π, given by:*

$$B := \sum_Z \frac{m_Z - 1}{m_Z} \cdot Z \tag{2.6}$$

We call the pair (Y, B) the Chow quotient pair of the T-variety X.

2.3 T-varieties

In this section we briefly recall the theory of complex T-varieties. The best reference for more details is [38]. By a T-variety we will always mean a normal variety with an effective action of an algebraic torus T. Let T, M, N be as described in 2.2.1. Let us fix some additional definitions. By a *polyhedron* we wil mean the intersection of finitely many closed affine halfspaces of $N_{\mathbb{Q}}$, or its dual $M_{\mathbb{Q}}$. By a *cone* we mean the intersection of finitely many closed linear halfspaces of $N_{\mathbb{Q}}$ or its dual $M_{\mathbb{Q}}$. We will assume all cones are generated by primitive elements of their respective lattices.

2.3.1 Toric varieties

First, for context, let us recall the toric situation. A cone $\sigma \subset N_{\mathbb{R}}$ has a dual cone $\sigma^\vee := \{m \in M_{\mathbb{R}} | \langle m, n\rangle \geq 0 \ \forall n \in N_{\mathbb{R}}\}$, and we may construct the normal toric variety $\operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M]$. The torus action is given by the M-grading of the algebra $\mathbb{C}[\sigma^\vee \cap M]$.

Conversely, given a normal affine toric variety X with algebraic torus T, $\mathbb{C}[X]$ is a semigroup subalgebra of $\mathbb{C}[M]$ of the form $\mathbb{C}[\sigma^\vee \cap M]$ for some strongly convex rational polyhedral cone $\sigma \subset N_{\mathbb{R}}$. We write $\mathrm{TV}(\sigma, N) := \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M]$. Face inclusions of cones correspond to equivariant open embeddings of varieties, and so from a complete fan of cones Σ we may construct a normal toric variety X_Σ.

Example 10. *Consider the variety $X = \mathrm{Bl}_z \mathbb{P}^1 \times \mathbb{P}^1$, where $z = ([1, 0], [1, 0])$. We may lift the 2-torus action on $\mathbb{P}^1 \times \mathbb{P}^1$ to X. This action becomes effective once we replace the torus T by $T/\pm Id$. As a toric variety it is given by the fan Σ in Figure 1.*

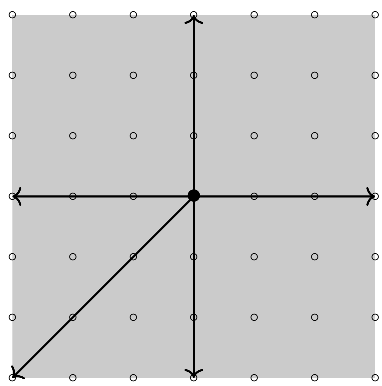


Figure 1: $\Sigma \subset N_{\mathbb{Q}}$

We also recall the description of equivariant polarizations of toric varieties via convex polytopes. Suppose we have a complete fan Σ, with rays $\Sigma(1)$. Any Cartier divisor on $X = \mathrm{TV}(\Sigma)$ is linearly equivalent to a T-equivariant one. Moreover we have the following exact sequence:

$$0 \to M \to \mathrm{CaDiv}_T(X) \cong \mathbb{Z}^{\Sigma(1)} \to \mathrm{Cl}(X) \to 0. \tag{2.7}$$

and relations $\mathrm{div}(\chi^u) = \sum_{\rho \in \Sigma(1)} \langle u, v_\rho \rangle D_\rho$.

To any lattice polytope $P \subset M_{\mathbb{Q}}$ we can associate a normal projective toric variety X_P given by its dual fan, and an ample divisor D_P given by coefficients on the ray generators of $\Sigma(1)$ specified by the equations of halfspaces defining P.

Example 11. *Consider the the following lattice polytope: The normal fan $\mathcal{N}(P)$*

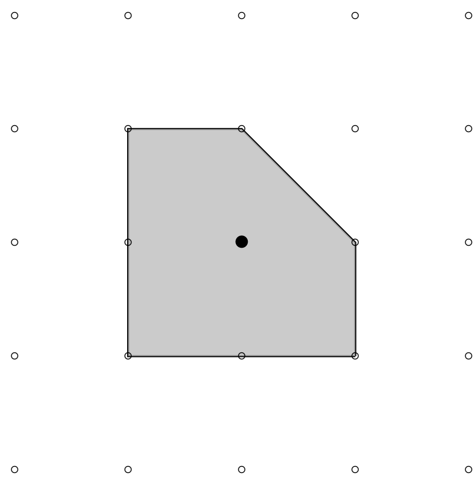


Figure 2: $P \subset M_{\mathbb{Q}}$

is that of $\mathrm{Bl}_z \mathbb{P}^1 \times \mathbb{P}^1$ *as in Example 10. We can calculate the corresponding divisor as*

$$D_P = - \sum_{\rho \in \Sigma(1)} D_\rho \sim -K_X.$$

Conversely the exact sequence 2.7 may be used to construct a polytope from any equivariant polarization of a projective toric variety. Finally, recall that a polytope is called Fano if the origin is contained in its interior, and each vertex is a primitive lattice point of M. Under the correspondence just described, Fano polytopes correspond exactly to anticanonical polarizations of toric varieties.

2.3.2 Higher complexity T-varieties

There is a successful program to extend the combinatorial dictionary of toric varieties to T-varieties of higher complexity. Roughly speaking, the combinatorial data lives over the Chow quotient of X by the T-action, so we have combinatorial data of dimension $\dim T$, and algebreo-geometric data of dimension of the complexity of the torus action.

Recall that one may define an abelian semigroup structure on the set of all polyhedra via Minkowski addition:

$$\Delta + \Delta' := \{v + v' \ |v \in \Delta,\ v' \in \Delta'\}.$$

It is well known that this gives a representation of any polyhedron $\Delta = P + \sigma$ where P is a convex polytope and σ. The cone σ is uniquely specified and is known as the tail cone of Δ. We will write $\operatorname{tail}\Delta = \sigma$, and call Δ a σ-tailed polyhedra in this situation.

The set of σ-tailed polyhedra form a semigroup $\mathrm{Pol}^+_{\mathbb{Q}}(N, \sigma)$ under Minkowski addition. We also include $\emptyset$ here, with $\emptyset + \Delta := \emptyset$ for any Δ. Recall the definition of a polyhedral divisor:

Definition 11. *Let $\sigma \subset N_{\mathbb{R}}$ be a cone, and Y a normal projective variety over $\mathbb{C}$. A polyhedral divisor on (Y, N) with tail cone σ is an element*

$$\mathcal{D} \in \mathrm{Pol}^+_{\mathbb{Q}}(N, \sigma) \otimes \mathrm{CaDiv}^+_{\mathbb{Q}}(Y),$$

where $\mathrm{CaDiv}^+_{\mathbb{Q}}(Y)$ is the semigroup of effective $\mathbb{Q}$-Cartier divisors on Y. We define $\operatorname{tail}\mathfrak{D} = \sigma$.

Let $\operatorname{Loc}\mathfrak{D} := Y \setminus \bigcup_{\mathfrak{D}_Z = \emptyset} Z$. The evaluation of $\mathfrak{D}$ at $u \in \sigma^\vee$ is defined to be the $\mathbb{Q}$-Cartier divisor on Y given by:

$$\mathfrak{D}(u) := \sum_{\mathfrak{D} \neq \emptyset} \min_{v \in \mathfrak{D}_P} \langle v, u \rangle Z_{|\operatorname{Loc}\mathfrak{D}}.$$

Definition 12. *A polyhedral divisor $\mathfrak{D}$, as defined above, is called a p-divisor if $\mathfrak{D}(u)$ is semiample for $u \in \sigma^\vee$ and, in addition, big for $u \in int(\sigma^\vee)$. Note if $\operatorname{Loc}\mathfrak{D}$ affine this is automatically satisfied.*

By [21, Proposition 3.1], p-divisor defines an affine T-variety in the following manner. Note for $u \in \sigma^\vee$ we have $\mathfrak{D}(u) + \mathfrak{D}(u') \leq \mathfrak{D}(u + u')$. Consider the sheaf of N-graded algebras

$$\mathcal{A} := \bigoplus_{w \in \sigma^\vee} \mathcal{O}_{\operatorname{Loc}\mathfrak{D}}(\mathfrak{D}(u))\chi^u.$$

Note the semiample and big conditions in the definition of a p-divisor ensure that the algebra $H^0(\operatorname{Loc}\mathfrak{D}, \mathbb{A})$ is finitely generated. We obtain T-varieties:

$$\tilde{\mathrm{TV}}(\mathfrak{D}) := \operatorname{Spec}_{\operatorname{Loc}\mathfrak{D}} \mathcal{A},\ \mathrm{TV}(\mathfrak{D}) := \operatorname{Spec} H^0(\operatorname{Loc}\mathfrak{D}, \mathcal{A})$$

together with a good quotient $\tilde{\mathrm{TV}}(\mathfrak{D}) \to Y$ (the Chow quotient) of the torus action, and an equivariant contraction $r : \tilde{\mathrm{TV}}(\mathfrak{D}) \to \mathrm{TV}(\mathfrak{D})$. The T-variety $\mathrm{TV}(\mathfrak{D})$ remains unchanged if we pull back $\mathfrak{D}$ by some birational $\varphi : Y' \to Y$. Moreover, modifying $\mathfrak{D}$ by an element in the image of the natural map:

$$N \otimes_{\mathbb{Z}} \mathbb{C}(Y)^* \to \operatorname{Pol}^+_{\mathbb{Q}}(N, \sigma) \otimes \operatorname{CaDiv}^+_{\mathbb{Q}}(Y)$$

does not change $\mathrm{TV}(\mathfrak{D})$. In the converse direction, by [21, Proposition 3.4], any affine T-variety X is of the form $\mathrm{TV}(\mathfrak{D})$ for some p-divisor $\mathfrak{D}$.

Example 12. *Let Y be a normal projective variety and D an ample integral Cartier divisor on Y. Let $\mathfrak{D} = [1, \infty) \otimes D$. Then we see*

$$\mathcal{A} = \bigoplus_{n \in \mathbb{Z}^{\geq 0}} \mathcal{O}(nD)\chi^n,$$

and so $\tilde{\mathrm{TV}}(\mathfrak{D})$ is the total space of the line bundle $\mathcal{O}(D)$.

Example 13. *Any toric variety X with torus T may be considered a higher complexity T-variety with respect to any proper subtorus $T' \subset T$. Such a subtorus is given by some surjection of character lattices $p : M \to M'$. Writing M_Y for the kernel of p and denoting the dual surjection $q : N \to N_Y$, we have mutually dual short exact sequences:*

$$\begin{tikzcd} 0 \arrow[r] & N' \arrow[r] & N \arrow[r, "q"] & N_Y \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} 0 & M' \arrow[l] & M \arrow[l, "p"'] & M_Y \arrow[l] & 0 \arrow[l] \end{tikzcd}$$

Suppose $X = \mathrm{TV}(\delta)$ for some cone $\delta \subset \mathbb{N}_{\mathbb{Q}}$. Let $\sigma = \delta \cap N'_{\mathbb{Q}}$. Then the surjection $M \to M'$ induces a surjection $\delta^\vee \to \sigma^\vee$. Let Σ be the coarsest fan which refines all images of faces of δ under q. It may be shown that $Y = \mathrm{TV}(\Sigma)$ is the Chow quotient of X by T' up to normalization. Then X is the T'-variety associated to *the p-divisor:*

$$\mathfrak{D}^\delta := \sum_{a \in \Sigma(1)} \mathfrak{D}_a \otimes \overline{\mathrm{orb}}(a)$$

Where $\mathfrak{D}_a := q^{-1}(a) \cap \delta$, *and* $\overline{\mathrm{orb}}(a)$ *is the torus-invariant divisor on* Y *associated to the ray* a *under the usual orbit-cone correspondence.*

Example 14. *Consider the toric variety given by the following cone:*

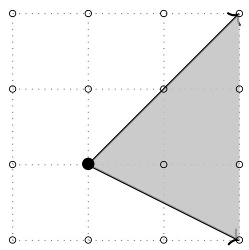
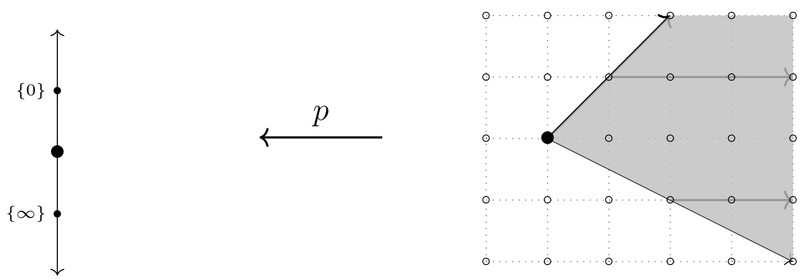


Figure 3: $\sigma \subset N_{\mathbb{Q}}$

and subtorus action given by sublattice $N' := \mathbb{Z}e_2 \subset N$. *We may read off the downgrade p-divisor from the following diagram:*



$\sigma \subset N_{\mathbb{Q}}$ $\sigma \subset N_{\mathbb{Q}}$

and we see this downgraded toric variety is given by the p-divisor:

$$\mathfrak{D} = \{0\} \otimes [1, \infty) + \{\infty\} \otimes [2, \infty).$$

One can define morphisms of p-divisors, and this correspondence turns out to be an equivlance of categories between affine T-varieties and p-divisors up to equivalence via the modifications mentioned above. We will not make use of the general data of a morphism of p-divisors but we will discuss the special case needed for globalization.

By [39] we have a method of gluing p-divisors in a natural way to construct general T-varieties, generalizing the notion of a fan of cones in the toric case.

Suppose $\mathfrak{D}', \mathfrak{D}$ are polyhedral divisors. We write $\mathfrak{D}' \leq \mathfrak{D}$ if $\mathfrak{D}'_Z$ is a face of $\mathfrak{D}_Z$ for each Cartier divisor Z on Y.

Now suppose additionally that $\mathfrak{D}', \mathfrak{D}$ are p-divisors. If $\mathfrak{D}' \leq \mathfrak{D}$ then we obtain graded morphisms of the respective sheaves of algebras $\mathcal{A} \to \mathcal{A}'$ giving a T-equivariant morphism $\mathrm{TV}(\mathfrak{D}') \to \mathrm{TV}(\mathfrak{D})$. Unlike the toric case, this not necessarily an open embedding, as we see in the following example.

Example 15. *Consider the setup in Example () with* $Y = \mathbb{P}^1$ *and* $D = \{\infty\}$. *Then* $r : \tilde{\mathrm{TV}}(\mathfrak{D}) \to \mathrm{TV}(\mathfrak{D})$ *is isomorphic to the blow up of* $\mathbb{A}^2$ *at the origin. If* $\mathfrak{D}' = \emptyset \otimes \{\infty\}$ *then* $\mathfrak{D}' \leq \mathfrak{D}$ *but* $\tilde{TV}(\mathfrak{D}')$ *intersects the exceptional divisor in* $\tilde{\mathrm{TV}}(\mathfrak{D})$, *and so the induced map* $\mathrm{TV}(\mathfrak{D}') \to \mathrm{TV}(\mathfrak{D})$ *is not an open embedding.*

2.3.3 f-divisors and divisorial polytopes

In this section we focus on the complexity one case, where T-equivariant open embeddings may be characterized using degree polyhedra of p-divisors:

$$\deg \mathfrak{D} := \sum_{y \in Y} \mathfrak{D}_y.$$

Note that $\deg \mathfrak{D} \neq \emptyset \iff \mathrm{Loc}\, \mathfrak{D} = Y$. It is also not hard to see that a complexity one polyhedral divisor $\mathfrak{D}$ is a p-divisor if and only if the following two conditions hold:

1. $\deg \mathfrak{D} \subsetneq \mathrm{tail}\, \mathfrak{D}$;

2. $\mathfrak{D}(w)$ has a principal multiple for $w \in (\mathrm{tail}\, \mathfrak{D})^\vee$ such that $w^\perp \cap \deg \mathfrak{D} \neq 0$.

We have the following characterization of open embeddings:

Theorem 1 [40]**.** *Let* Y *be a curve and* $\mathfrak{D}, \mathfrak{D}'$ *polyhedral and* p*-divisors respectively, such that* $\mathfrak{D}' \leq \mathfrak{D}$. *Then* $\mathfrak{D}'$ *is a* p*-divisor and* $\mathrm{TV}(\mathfrak{D}') \to \mathrm{TV}(\mathfrak{D})$ *is an open embedding if and only if* $\deg \mathfrak{D}' = \deg \mathfrak{D} \cap \mathrm{tail}\, \mathfrak{D}'$.

Note that in complexity one $\mathrm{Loc}\, \mathfrak{D}$ is always birational to $\mathbb{P}^1$. Therefore, via pullback, any complexity one normal affine $\mathcal{T}$-variety may be realized as $\mathrm{TV}(\mathfrak{D})$ where $\mathfrak{D}$ is a p-divisor over $Y = \mathbb{P}^1$. Note that in this case that condition 2. above is automatically satisfied.

In complexity one, the object taking the place of a fan of cones is an f-divisor, first introduced in [40]. We recall this construction now. By a *polyhedral* *decomposition* we mean a decomposition of $N_{\mathbb{Q}}$ into a collection of polyhedra, closed under intersection. A polyhedral decomposition has a *tail fan*: a fan comprised of exactly the tail cones of the polyhedra in the decomposition. If $\mathcal{G}$ is a polyhedral decomposition then we write $\operatorname{tail}\mathcal{G}$ for its tail fan.

Definition 13. *An f-divisor is a pair $\mathcal{S} = \left(\sum_{y\in\mathbb{P}^1} S_y \otimes \{y\},\ \mathfrak{deg}\right)$, where:*

1. *S_y are polyhedral subdivisions sharing a tail fan Σ.*
2. *$\mathfrak{deg}$ is some subset of $|\Sigma|$.*
3. *For any full-dimensional marked $\sigma \in \Sigma$ then*

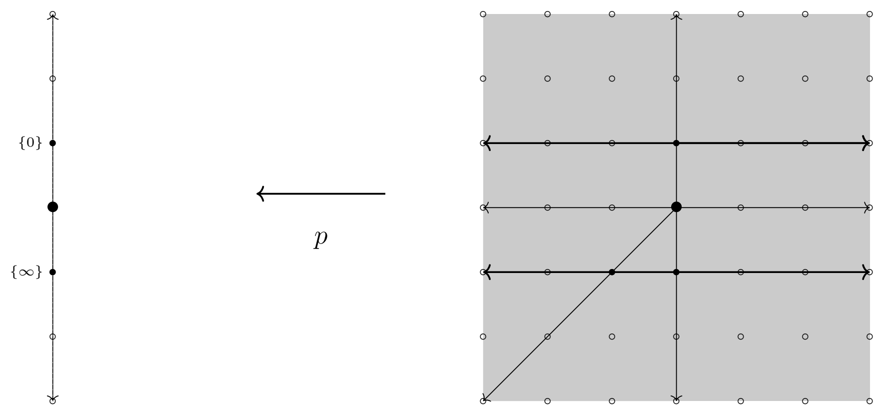
$$\mathfrak{D}^\sigma := \sum \mathfrak{D}^\sigma_y \otimes \{y\}$$

   is a p-divisor, where $\mathfrak{D}^\sigma_y$ is the unique polyhedron in S_y with $\operatorname{tail}(\mathfrak{D}^\sigma_y) = \sigma$, and we have $\deg \mathfrak{D}^\sigma = \mathfrak{deg} \cap \sigma$.

4. *Only finitely many S_y may differ from the tailfan Σ. We call the finite collection of $S_y \neq \Sigma$ the non-trivial slices of $\mathcal{S}$.*

In fact to construct $\mathfrak{deg}$ from the rest of the data it is only necessary to know whether $\deg \mathfrak{D}^\sigma$ is empty or not for each σ. We call those cones σ with $\deg \mathfrak{D}^\sigma \neq \emptyset$ the *marked* tailcones of an f-divisor.

Example 16. *Let us describe the downgrade procedure for the toric variety from Example 10, with respect to the subtorus given by sublattice $N' := \mathbb{Z}e_1 \subset N$. We may read off the downgrade f-divisor from the following diagrams:*



$\Sigma_Y \subset N_Y$ $\Sigma \subset N$

and we see this downgraded toric variety is given by the f-divisor:

Example 17. *Here we give an example of an f-divisor describing the complexity one threefold (2.30) from the list of Mori and Mukai [18].*

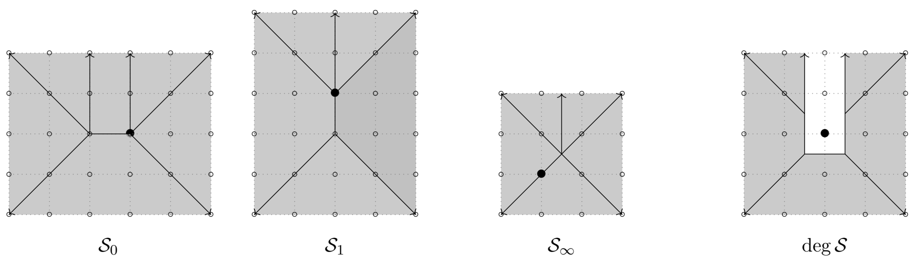


Figure 7: f-divisor of a complexity one threefold

In complexity one there is a generalization of the correspondence between lattice polytopes and polarized projective toric varieties. First we recall the description of Cartier divisors on the T-variety $\mathrm{TV}(\mathcal{S})$ of an f-divisor $\mathcal{S}$ from [41]:

Definition 14. *A Cartier support function on an f-divisor $\mathcal{S}$ is a sum:*

$$h = \sum_{y \in \mathbb{P}^1} h_y \otimes y,$$

where:

1. *For all $y \in \mathbb{P}^1$, $h_y : N_{\mathbb{Q}} \to \mathbb{Q}$ is a piecewise affine function with respect to the decomposition S_y.*
2. *For all $y \in \mathbb{P}^1$ and $v \in N$ if $k \cdot v$ is a lattice point then $k \cdot h_y(v) \in ZZ$*
3. *For each y, the linear part $lin(h_y)$ is independent of y.*
4. *We have $h_y \neq lin(h_y)$ for at most finitely many $y \in \mathbb{P}^1$.*
5. *Writing $h_{|\sigma}(0) = \sum a_y \cdot y$, where a_y is the evaluation of the restriction of h_y to $\mathcal{S}_y^\sigma$ at 0, then for every marked $\sigma \in \operatorname{tail} \mathcal{S}$, the divisor $h_{|\sigma}(0)$ is principal.*

We write $\mathrm{CaSF}(\mathcal{S})$ for the group of Cartier support functions on $\mathcal{S}$.

It is shown in [41] that $\mathrm{CaSF}(\mathcal{S})$ is isomorphic to the group of T-invariant Cartier Divisors on $\mathrm{TV}(\mathcal{S})$. If h is a Cartier support function we denote the corresponding divisor by D_h. We call h ample if D_h is ample.

From what we have recalled so far, an equivariantly polarized T-variety (X, L) may be given by an f-divisor $\mathcal{S}$ and a choice of ample Cartier support function $h \in \mathrm{CaSF}(\mathcal{S})$. We finally recall the construction the novel results of this thesis make most use of, namely the dual picture. A divisorial polytope may be thought of to polarized complexity one T-varieties what a polytope is to a toric variety. Note the following definition differs from that of say [**?**] by a shift of a divisor of degree 2.

Definition 15. *A divisorial polytope is a function Ψ on a lattice polytope $\square \subset M_{\mathbb{R}}$:*

$$\Psi : \square \to \mathrm{Div}_{\mathbb{Q}} \mathbb{P}^1,\ u \mapsto \Sigma_{y \in \mathbb{P}^1} \Psi_y(u) \cdot \{y\},$$

such that:

- *For $y \in \mathbb{P}^1$ the function $\Psi_y : \square \to \mathbb{R}$ is the minimum of finitely many affine functions, and $\Psi_y \equiv 0$ for all but finitely many $y \in \mathbb{P}^1$.*
- *Each Ψ_y takes integral values at the vertices of the polyhedral decomposition its regions of affine linearity induce on $\square$.*
- $\deg \Psi(u) > -2$ *for $u \in int(\square)$;*

Let $\Psi : \square \to \mathrm{CaDiv}_{\mathbb{Q}} Y$ be a divisorial polytope. For $y \in \mathbb{P}^1$ consider the piecewise affine concave function on $N_{\mathbb{Q}}$ given by:

$$\Psi^*_y(u) := \min_{v \in \square}(\langle v, u\rangle - \Psi_P(u)),$$

Let S_y be the polyhedral subdivision induced by Ψ^*_y, with tailfan Σ. Let $\mathfrak{deg}$ be the set of cones $\sigma \in \Sigma$ such that $\deg \circ \Psi_{|F_\sigma} \equiv 0$, where F_σ is the face of $\square$ where $\langle \cdot, v\rangle$ takes its minimum value for $v \in \sigma$. This defines an f-divisor $\mathcal{S} = \sum_{y \in \mathbb{P}^1} S_y \otimes \{y\}$, and Ψ^*_y is seen to be a Cartier support function on $\mathcal{S}$. Moreover may reverse the above construction and obtain a divisoral polytope from any pair $\mathcal{S}, h$ where $\mathcal{S}$ is an f-divisor and h is an ample Cartier support function.

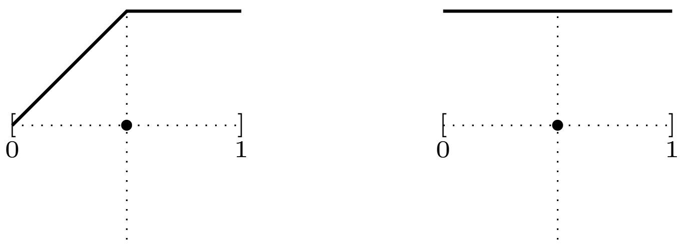
For our results we are particularly interested in Fano T-varieties. We recall the notion of a Fano divisorial polytope. By [], Fano divisorial polytopes correspond to Fano T-varieties $(X, -K_X)$.

Definition 16. *A divisorial polytope $\Psi : \Box \to \mathrm{Div}_{\mathbb{Q}} \mathbb{P}^1$ is said to be Fano if additionally we have that:*

- *The origin is an interior lattice point of $\Box$.*
- *The affine linear pieces of each Ψ_y are of the form $u \mapsto \frac{\langle v,u \rangle - \beta + 1}{\beta}$ for some primitive lattice element $v \in N$;*
- *Every facet F of $\Box$ with $(\deg \circ \Psi_{|F}) \neq -2$ has lattice distance 1 from the origin.*

Example 18. *We can perform a downgrade operation on a polytope describing a toric variety. Given a polytope $P \subset M_{\mathbb{Q}}$ and a complexity one subtorus action given by some surjection $p : M \to M'$, choose a section $s : M' \to M$. Set $\Box = p(P)$ and $\Psi_0(u) := \max(p^{-1}(u) - s(u))$, $\Psi_\infty := \min(p^{-1}(u) - s(u))$. We obtain a divisorial polytope $\Psi := \Psi_0 \otimes \{0\} + \Psi_\infty \otimes \{\infty\}$. The corresponding T-variety is then isomorphic to the toric variety associated to P.*

Example 19. *Consider the toric variety $\mathrm{Bl}_z \mathbb{P}^1 \times \mathbb{P}^1$ with its anticanonical polarization. If we start with the corresponding polytope given in Example 11 and perform the downgrade operation, we end up with divisorial polytope:*



$\{0\}$ $\{\infty\}$

It is easy to see, for example, that the resulting f-divisor from the operation Example 18 coincides with the downgrade f-divisor from Example 16.

We conclide this section with a few pieces of terminology for divisorial polytopes. The push-forward of the measure induced by ω is known as the Duistermaat-Heckman measure, independent of the choice of ω and which we denote by ν. Denote the standard measure on $M_{\mathbb{R}}$ by η.

Definition 17. *Let Ψ be a divisorial polytope.*

- *The degree of* Ψ *is the map* $\deg\Psi : \square \to \mathbb{R}$ *given by* $u \mapsto \deg(\Psi(u))$.
- *The barycenter of* Ψ *is the point* $\mathrm{bc}(\Psi) \in \square$ *such that for all* $v \in N_{\mathbb{R}}$*:*

$$\langle \mathrm{bc}(\Psi), v\rangle = \int_{\square} v \cdot \deg\Psi \; d\eta = \int_{\square} v d\nu.$$

Note by the second equality we see $\mathrm{bc}(\Psi) = \mathrm{bc}_{\nu}(\square)$.

- *The volume of* Ψ *is defined to be:*

$$\mathrm{vol}\,\Psi = \int_{\square} \deg\Psi \; d\eta = \int_{\square} d\nu.$$

2.4 Equivariant K-stability

In this section we recall definitions of K-stability. In summary the K-stability criteria are concerned with the positivity of certain numerical invariants associated to *test configurations* of our original space. We do not go into technical detail about how K-stability relates to the existence of canonical metrics here, but give the definitions and theorems we will rely on later in the thesis.

2.4.1 Twisted equivariant K-stability

Here we recall notions of Twisted equivariant K-stability, following [14]. Let X be a Fano manifold with the action of a complex reductive group G of automorphisms containing a maximal torus T. Fix a T-invariant Kähler form $\omega \in 2\pi c_1(X)$ induced by the Fano condition. First we define test-configurations of X.

Definition 18. *A* G*-equivariant test configuration for* (X, L) *is a* $\mathbb{C}^*$*-equivariant flat family* $\mathcal{X}$ *over the affine line equipped with a relatively ample equivariant* $\mathbb{Q}$*-line bundle* $\mathcal{L}$*, such that:*

1. *The* $\mathbb{C}^*$*-action* λ *on* $(\mathcal{X}, \mathcal{L})$ *lifts the standard action on* $\mathbb{A}^1$*;*
2. *The general fiber is isomorphic to* X *and* $\mathcal{L}$ *is the relative anti-canonical bundle of* $\mathcal{X} \to \mathbb{A}^1$*.*
3. *The action of* G *extends to* $(\mathcal{X}, \mathcal{L})$ *and commutes with the* $\mathbb{C}^*$*-action* λ*.*

A test configuration with $\mathcal{X} \cong X \times \mathbb{A}^1$ is called a product configuration. If such an isomorphism exists and is $\mathbb{C}^*$-equivariant then we call the test configuration *trivial*. Finally a test configuration with normal special fiber is called *special*.

Suppose from now on $G = T$ is a maximal torus in $\mathrm{Aut}(X)$. We then have an induced $T' = T \times \mathbb{C}^*$-action on the special fiber. The canonical lift of T'-action to $-K_{\mathcal{X}_0}$ induces a canonical choice of moment map $\mu : \mathcal{X}_0 \to M'_{\mathbb{R}}$. The restriction of λ to $\mathcal{X}_0$ is generated by the imaginary part of a T'-invariant vector field w, and by an abuse of notation we also write $w \in N'_{\mathbb{R}}$ for the corresponding one-parameter subgroup. The moment map μ then specifies Hamiltonian functions

$$\theta_w := \langle \mu, w \rangle : \mathcal{X}_0 \to \mathbb{R},$$

as we have seen in Section 2.2.3. We now recall a definition of the Donaldson-Futaki invariant in the twisted modified setting.

Definition 19. *The twisted modified Donaldson-Futaki character of a special test configuration $(\mathcal{X}, \mathcal{L})$ is given by:*

$$\mathrm{DF}_{t,\xi}(\mathcal{X}, \mathcal{L}, w) = \mathrm{DF}_{\xi}(\mathcal{X}, \mathcal{L}, w) + \frac{(1-t)}{V} \int_{\mathcal{X}_0} (\max_{\mathcal{X}_0} \theta_w - \theta_w) e^{\theta_\xi}\ \omega^n.$$

where $V = \frac{1}{n!}\int_{\mathcal{X}_0} \omega^n$ is the volume of $\mathcal{X}_0$, and $\mathrm{DF}_{\xi}(\mathcal{X}, \mathcal{L}, w) = \frac{1}{V}\int_{\mathcal{X}_0} \theta_w \omega^n$ is the modified Donaldson-Futaki invariant of the configuration, in the form given in [29, Lemma 3.4].

Note that if $(\mathcal{X}, \mathcal{L})$ is a product configuration then we have $\mathcal{X}_0 \cong X$. Assuming X is non-toric, in this case the maximality of T in $\mathrm{Aut}(X)$ ensures that the restriction of λ to $\mathcal{X}_0$ is a one parameter subgroup of T, given by a choice of $w \in N$. We will use the following definition of K-stability, as given in [14].

Definition 20. *We say the triple (X, t, ξ) is G-equivariantly K-semistable if $\mathrm{DF}_{t,\xi}(\mathcal{X}, \mathcal{L}, w) \geq 0$ for all G-equivariant special configurations $(\mathcal{X}, \mathcal{L}, w)$. We say (X, t, ξ) is K-stable if, in addition, equality holds precisely for product configurations.*

In the non-twisted case K-stability implies the existence of a Kähler-Ricci soliton:

Theorem 2 Berman-Witt-Nystrom**.** *If (X, ξ) admits a Kähler-Ricci soliton then (X, ξ) is K-stable.*

In [14] a result in the converse direction is obtained, which we will make heavy use of in Chapter **??**:

Theorem 3 [14, Proposition 10] **.** *Let X be a polarized Fano manifold, with Kähler form ω. Let $t \in [0,1]$ and ξ be a soliton candidate for X. If (X,t) is G-equivariantly K-semistable then for all $s < t$ there exists $\omega_s \in 2\pi c_1(X)$ such that* $\mathrm{Ric}(\omega_s) - \mathcal{L}_\xi \omega_s = s\omega_s + (1-s)\omega$.

2.4.2 K-stability of T-varieties

Here we review K-stability in complexity one. In [16], Ilten and Suess described non-product special test configurations for a T-variety of complexity one in terms of its divisorial polytope. Let X be a Fano complexity one T-variety, corresponding to the Fano divisorial polytope $\Psi : \square \to \mathbb{P}^1$. Let M, N be the usual character and cocharacter lattices respectively. Let $M' := M \times \mathbb{Z}$ and $N' = \mathrm{Hom}(M', \mathbb{Z})$ with associated vector spaces $M'_{\mathbb{Q}}, N'_{\mathbb{Q}}$.

Theorem 4 [16]**.** *There exists some $y \in \mathbb{P}^1$ such that for at most one $z \neq y$ the function Ψ_z has non-integral slope at any $u \in \square$, such that $\mathcal{X}_0$ is the toric variety corresponding to the following polytope:*

$$\Delta_y := \left\{ (u,r) \in M'_{\mathbb{Q}} \;\middle|\; u \in \square,\ -1 - \sum_{z \neq y} \Psi_z(u) \leq r \leq 1 + \Psi_y(u) \right\}.$$

Furthermore, the induced $\mathbb{C}^*$-action on $\mathcal{X}_0$ is given by the one-parameter subgroup of $T' = T \times \mathbb{C}^*$ corresponding to $v' = (-mv, m) \in N'$, for some $v \in N$. From the point of view of K-stability it is enough to consider only those configurations with $m = 1$.

By the definition of a divisorial polytope, there are only finitely many distinct polytopes Δ_y: Suppose Ψ corresponds to f-divisor $\mathcal{S}$. For y such that $\mathcal{S}_y \neq \mathrm{tail}\,\mathcal{S}$ we obtain a polytope Δ_y, but for all other y we observe that Δ_y is independent of y, equal to the polytope:

$$\Delta_{\mathrm{gen}} := \left\{ (u,r) \in M'_{\mathbb{Q}} \;\middle|\; u \in \square,\ -1 - \deg \Psi(u) \leq r \leq 1 \right\}$$

Definition 21. *The special configuration polytopes of a Fano T-variety are the polytopes Δ_y (for $\mathcal{S}_y \neq \mathrm{tail}\,\mathcal{S}$) and Δ_{gen} corresponding to normal toric varieties.*

See Figure 2 and Figure 4 for two examples of special configuration polytopes, and Appendix A for the data to construct the special configurations we need for results in this thesis. As observed in [16], we also obtain a description of the (non-twisted) Donaldson-Futaki character of $(\mathcal{X}_0, \xi')$:

$$\mathrm{DF}_{t,\xi'}(\mathcal{X}, \mathcal{L}, v') = \frac{1}{\operatorname{vol} \Delta_y} \left(\int_{\Delta_y} \langle u', v' \rangle \cdot e^{\langle u', \xi' \rangle} du' \right), \tag{2.8}$$

with $\xi', v' \in N_{\mathbb{R}} \times \mathbb{R}$. On the other hand, for $v, \xi \in N_{\mathbb{R}}$ one obtains:

$$\mathrm{DF}_{t,\xi}(\mathcal{X}, \mathcal{L}, v) = \frac{1}{\operatorname{vol} \Phi} \left(\int_{\square} \langle u, v \rangle \cdot \deg \Phi(u) \cdot e^{\langle u, \xi \rangle} \, du \right), \tag{2.9}$$

For non-product configurations the central fiber of a special test configuration is a normal toric variety. The following formula is the toric one obtained in [14].

Lemma 4 [14, theorem...?]**.** *Let $(\mathcal{X}, \mathcal{L}, v')$ be the special test configuration of X corresponding to the polytope Δ_y and the element $v' = (-v, 1) \in N'$, as in Theorem 4. Then:*

$$\mathrm{DF}_t(\mathcal{X}, \mathcal{L}, v') = t \langle \mathrm{bc}(\Delta_y), v' \rangle + (1-t) \max_{x \in \Delta_y} \langle x, v' \rangle.$$

We give the formula for the product configuration case in complexity one:

Lemma 5. *Let $(\mathcal{X}, \mathcal{L}, w)$ be a product configuration of the Fano complexity one T-variety X given by divisorial polytope $\Psi : \square \to \operatorname{Div}_{\mathbb{Q}} \mathbb{P}^1$. Then for $t \in [0,1]$ we have:*

$$\mathrm{DF}_t(\mathcal{X}, \mathcal{L})(w) = t \langle \mathrm{bc}(\Psi), w \rangle + (1-t) \max_{x \in \square} \langle x, w \rangle. \tag{2.10}$$

Proof. In [16] it was shown that $\mathrm{DF}(X \times \mathbb{A}^1, \mathcal{L})(w) = \langle \mathrm{bc}(\Psi), w \rangle$. Note by definition $\theta_w(x) = \langle \mu(x), w \rangle$. We may now push forward the integrals to the image of the moment map $\square$:

$$\begin{aligned}
\mathrm{DF}_t(\mathcal{X}, \mathcal{L}, w) &= \mathrm{DF}(\mathcal{X}, \mathcal{L}, w) + \frac{(1-t)}{V} \int_X (\max \theta_w - \theta_w) \omega^n \\
&= \langle \mathrm{bc}(\Psi), w \rangle + \frac{(1-t)}{\operatorname{vol} \Psi} \int_{\square} \max_{x \in \square} \langle x, w \rangle - \langle \cdot, w \rangle d\eta \\
&= t \langle \mathrm{bc}(\Psi), w \rangle + (1-t) \max_{x \in \square} \langle x, w \rangle.
\end{aligned}$$

$\square$

Chapter 3

Kähler-Ricci solitons on Fano threefolds

In this chapter we prove the following theorem:

Theorem 5 [42, Theorem 1.8]**.** *The Fano threefolds* 2.30*,* 2.31*,* 3.18*,* 3.22 *,*3.23*,* 3.24*,* 4.8*, from Mori and Mukai's classification [18], admit a non-trivial Kähler-Ricci soliton.*

Together with [16, Theorems. 6.1, 6.2] it follows that all known smooth Fano threefolds with an effective complexity-one torus action admit a Kähler-Ricci soliton. We follow the joint work of [42]. Here we describe the contribution of the author of this thesis to this article, namely to perform calculations to test the K-stability of threefolds and thus determine which admit Kähler-Ricci solitons.

3.1 The method of proof

This proof of Theorem 5 is somewhat calculational in nature, and uses some computer assistance. In this section we give a summary of our approach. At the end of this subsection we provide a more formal proof.

Let X be a smooth complexity one Fano T-variety. Recall the definition of a Kähler-Ricci soliton from Section 2.1.3. We test for the existence of a soliton on X using Theorem 3. Recall from Section 2.3 that a Fano complexity one T-variety $(X, -K_X)$ corresponds to a Fano divisorial polytope $\Phi : \square \to \mathrm{Div}_{\mathbb{Q}} \mathbb{P}^1$.

We first use Theorem 2 to find a candidate vector field $\xi \in N_{\mathbb{R}}$ for a soliton.

Any such ξ should satisfy the following equation:

$$\mathrm{DF}_\xi(X \times \mathbb{A}^1, w) = 0 \tag{3.1}$$

for all $w \in N'_\mathbb{R}$. By (2.8) this becomes:

$$\int_\square \langle u, v\rangle \cdot \deg \bar{\Phi}(u) \cdot e^{\langle u,\xi\rangle}\, du = 0 \tag{3.2}$$

By the arguments in [43, Section 3.1] there always exist a unique choice $\xi \in N_\mathbb{R}$ for which this holds. We refer to such a ξ as a *soliton canditate*.

The integral (3.2) may be solved symbolically, outputting an exponential polynomial $g(\xi, e^\xi)$ in ξ. In practice however, the domain P can complicate the calculation for $\dim X > 2$. To deal with this we developed a recursive algorithm, based on results of Barvinok [44], which reduces the integral to evaluations at the vertices of P. We explain this algorithm at the end of this chapter.

For our examples the equation $g(\xi, e^\xi) = 0$ is impossible to solve analytically. To get around this we use *real interval arithmetic* (RIA) estimates to find some hypercube D in which the solution ξ lies. For each special test configuration $(\mathcal{X}, \mathcal{L}, w)$ we then use further RIA to show that $\mathrm{DF}_{\xi'}(\mathcal{X}, , \mathcal{L}, w) > 0$ for $\xi \in D$. In Table 3.1 below we give estimates found for the vector field ξ for each threefold in the list of [**?**]. We can show that our approximations are correct to the nearest 10^{-5}. Note that the threefolds 3.8*, 3.21, 4.5 were shown to admit a non-trivial Kähler-Ricci soliton in [16].

When $\dim X = 2$ the process of finding suitable D via RIA is a simple application of the intermediate value theorem. For $\dim X > 2$ we cannot immediately use the intermediate value theorem to obtain D. In all but one of our examples we are able to make use of additional symmetries to reduce to a one-dimensional problem.

Given an automorphism $\sigma \in \mathrm{GL}(M)$ permuting the vertices of $\square$ such that $\deg(\Phi \circ \sigma) = \deg \Phi$, by (2.9) we have:

$$\mathrm{DF}_{\sigma^*(\xi)}(\mathcal{X}, \mathcal{L}, \cdot) = \mathrm{DF}_{X,\xi}(\mathcal{X}, \mathcal{L}, \cdot) \ \circ\ \sigma^*$$

Since $\xi \in N_\mathbb{R}$ is the unique solution to $F_{X,\xi} = 0$, this gives $\xi \in N_\mathbb{R}^{\sigma^*}$. For $\dim X = 3$ we have $\dim N_\mathbb{R}^{\sigma^*} = 1$ and we are in a situation where intermediate value theorem may be used to find D. Note that in threefold 3.23 there is no such σ. Here we must take another approach, which we explain in the proof below, and in Example 21.

Table 3.1: Fano threefolds and their soliton vector fields in the canonical coordinates coming with the representation of the combinatorial data in [?].

Threefold	ξ
Q	$(0,0)$
2.24*	$(0,0)$
2.29	$(0,0)$
2.30	$(0,0.51489)$
2.31	$(0.28550,0.28550)$
2.32	$(0,0)$
3.8*	$(0,-0.76905)$
3.10*	$(0,0)$
3.18	$(0,0.37970)$
3.19	$(0,0)$
3.20	$(0,0)$
3.21	$(-0.69622,-0.69622)$
3.22	$(0,0.91479)$
3.23	$(0.26618,0.67164)$
3.24	$(0,0.43475)$
4.4	$(0,0)$
4.5*	$(-0.31043,-0.31043)$
4.7	$(0,0)$
4.8	$(0,0.62431)$

We now give a more formal proof of Theorem 5. The complete calculations for the proof are performed using SageMath, and can be found as an online worksheet[1].

Proof of Theorem 5. The data required for this proof is collated in Appendix A. The divisorial polytopes were originally given in [?], although the piecewise affine Ψ discussed there differs from our divisorial polytope Φ by the divisor $D = 2\cdot\{\infty\}$.

[1] `https://cocalc.com/projects/ae8e1663-e2ad-40b8-aec2-30faf4e6a54f/files/threefolds.sagews`

For each of the threefolds 2.30, 2.31, 3.18, 3.22 , 3.24, and 4.8 there exists a non-trivial involution $\sigma \in \mathrm{GL}(M)$ permuting the vertices of $\square$ such that $\deg(\Phi \circ \sigma) = \deg \Phi$. Choose a basis e_1, e_2 of N with $\sigma^*(e_1) = -e_1$ and $\sigma^*(e_2) = e_2$. The soliton candidate $\xi = (\xi_1, \xi_2)$ must be contained in the line $N_{\mathbb{R}}^{\sigma^*} = \mathbb{R}e_2$. For each of these examples we obtain an interval D in which the soliton candidate must lie, via the intermediate value theorem and RIA, see Appendix A.

Recall the description of non-product special test configurations and Donaldson-Futaki invariants from 2.4.2 as integrals over $\Delta_y \subset M'$. Let $e_3 = (0,1) \in N \times \mathbb{Z} =: N'$. In Appendix A we provide closed forms for the functions

$$h_y(\xi_2) := (\operatorname{vol} \Delta_y) \cdot \mathrm{DF}_{\xi_2 e_2}(\mathcal{X}, \mathcal{L}, e_3)$$

for every admissible choice of $y \in \mathbb{P}^1$.

In Appendix A we also give lower bounds on $h_y(D)$ using further RIA, ensuring the positivity of $\mathrm{DF}_{\xi_2 e_2}(\mathcal{X}, \mathcal{L}, e_3)$ and hence of $\mathrm{DF}_{\xi_2 e_2}(\mathcal{X}, \mathcal{L}, v')$ for any special test configuration $(\mathcal{X}, \mathcal{L}, v')$. These threefolds then admit a Kähler-Ricci soliton by Theorem 3. See Example 20 for details of the computation and Appendix B for the implementation in SageMath.

For the case of threefold no. 3.23 there is no involution fixing $\deg \Phi$. In this case we take a more general approach to bound the value of the candidate ξ. Here we make use of some elementary calculus. Note, that ξ is the unique solution to the equation $\nabla_n G = 0$, where

$$G(v) := \int_{\square} \deg \Phi(u) \cdot e^{\langle u, v \rangle}\, du = \int_{\Delta_0} e^{\langle u', (v,0) \rangle}\, du'.$$

We identify a rectangular region $D \subset \mathbb{R}^2$ such that $\nabla_n G > 0$ holds along ∂D, where n is any unit outer normal of the rectangle D. Since D is compact it must contain a local minimum of G, which then cannot lie on ∂D. We then have ξ in the interior of D such that $\nabla_n G = 0$.

To show $\nabla_n G > 0$ along ∂D we have to again use interval arithmetic. We determine a closed form which coincides with $\nabla_n G$ up to a positive constant. We subdivide the faces of the boundary into sufficiently small segments. Using RIA on the closed form for $\nabla_n G(\xi)$ we obtain the positivity result. See Example 21 for details of the computation, and Appendix B for the implementation in SageMath.

$\square$

3.2 Two examples in detail

Here we present the proofs for threefolds 2.30 and 3.23 in detail.

Example 20. *Consider the threefold 2.30 from the classification of Mori and Mukai. The corresponding divisorial polytope* Φ *is given in Figure 1. We first find the unique soliton candidate* $\xi \in N_{\mathbb{R}}$. *We see that* $\deg \Phi$ *is symmetric with respect to reflection* σ *along the vertical axis, and so* $\xi = \xi_2 e_2$ *for some* $\xi_2 \in \mathbb{R}$. *We must find a solution* ξ_2 *to* $\mathrm{DF}_{\xi_2 e_2}(X \times \mathbb{A}^1, \cdot)$, *which is equivalent to* $\mathrm{DF}_{\xi_2 e_2}(X \times \mathbb{A}^1, e_2) = 0$.

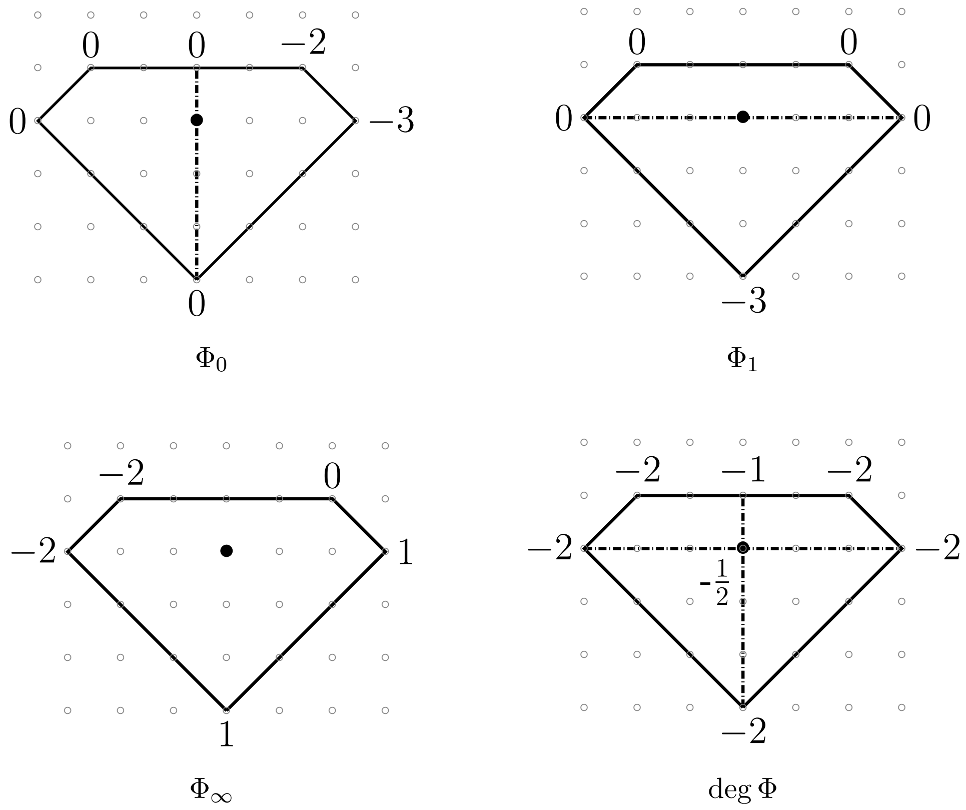


Figure 1: Divisorial polytope for threefold 2.30

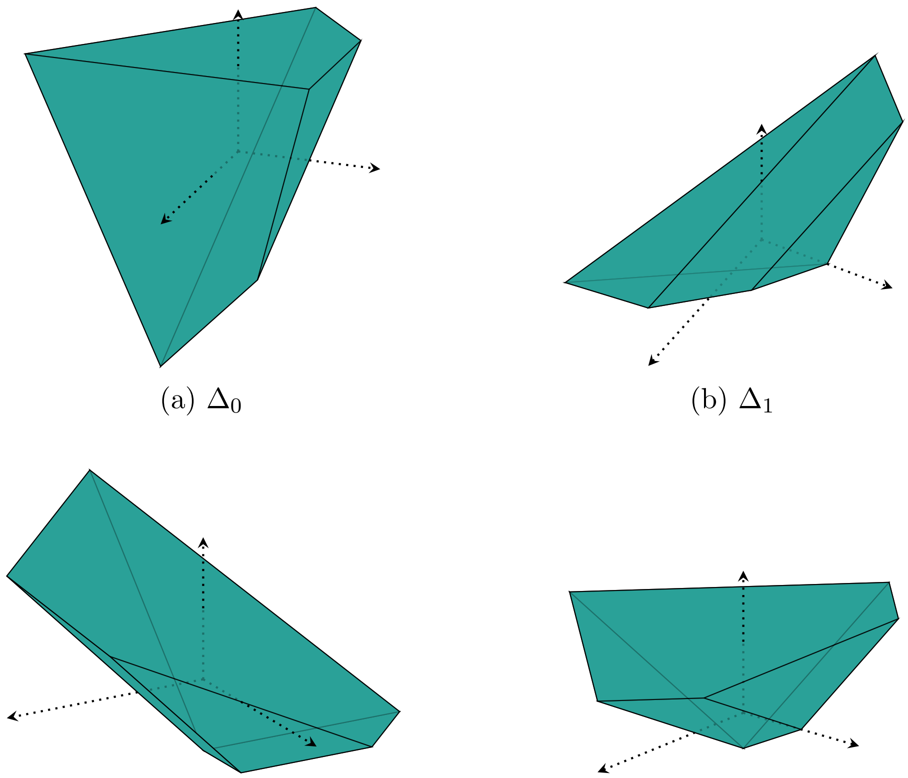
By (2.8) the vanishing of $\mathrm{DF}_{\xi_2 e_2}(X \times \mathbb{A}^1, e_2)$ *is equivalent to that of:*

$$0 = g(\xi_2) := \int_{\square} u_2 \cdot \deg \bar{\Phi}(u) \cdot e^{u_2 \xi_2}\, du = \int_{\Delta_0} u_2 \cdot e^{u_2 \xi_2}\, du,$$

where the integral on the right hand side can be solved analytically. We obtain:

$$g(\xi_2) = \frac{1}{\xi_2^4} \cdot \left(\left(2\,\xi_2^3 - 3\,\xi_2 - 3\right) e^{(4\,\xi_2)} + 12\,\xi_2 e^{(3\,\xi_2)} + 3\,\xi_2 + 3 \right) e^{(-3\,\xi_2)}.$$

Evaluating the exponential functions with a precision of 16 binary digits and using elementary estimations it can be shown that $g(0.514) < 0$ *and* $g(0.515) > 0$*. By the intermediate value theorem then* $0.514 < \xi_2 < 0.515$*. It remains to check the positivity of the Donaldson-Futaki invariant for each toric degeneration appearing as the space of a special test configuration. The corresponding polytopes are given as the convex hull of a finite set of points in Appendix A. See also Figure 2.*



(a) Δ_0 (b) Δ_1 (c) Δ_∞ (d) Δ_{gen}

Figure 2: Special configuration polyhedra for 2.30.

In each case we must check positivity of the Donaldson-Futaki invariant when the induced $\mathbb{C}^*$*-action is given by* $e_3 := (0, 0, 1) \in N \times \mathbb{Z}$*. Denote*

$$h_y(\xi_2) := (\operatorname{vol} \Delta_y) \cdot \mathrm{DF}_{(0,\xi_2)}(\mathcal{X}_y, e_3).$$

Clearly the positivity of h_y *implies that of* $\mathrm{DF}_\xi(\mathcal{X}_{y,0,1})$*. Once more solving the* *integrals appearing in (2.8), we obtain:*

$$
\begin{aligned}
h_0(\xi_2) &= \frac{1}{3\xi_2^4} \cdot \big(\big(2\xi_2^3 - 3\xi_2 - 3\big)e^{4\xi_2} + 3\big(3\xi_2^2 + 2\big)e^{3\xi_2} - 3\xi_2 - 3\big)e^{-3\xi_2} \\
h_1(\xi_2) &= \frac{1}{6\xi_2^4} \cdot \big(\big(8\xi_2^3 + 6\xi_2^2 - 3\big)e^{4\xi_2} - 12\big(3\xi_2^2 - 3\xi_2 + 1\big)e^{3\xi_2} + 12\xi_2 + 15\big)e^{-3\xi_2} \\
h_\infty(\xi_2) &= -\frac{1}{6\xi_2^4} \cdot \big(2\big(2\xi_2^3 - 3\xi_2 - 3\big)e^{4\xi_2} - 3\big(3\xi_2^2 - 12\xi_2 + 2\big)e^{3\xi_2} + 12\xi_2 + 12\big)e^{-3\xi_2} \\
h_{\mathrm{gen}}(\xi_2) &= \frac{1}{6\xi_2^4} \cdot \big(\big(8\xi_2^3 + 6\xi_2^2 - 3\big)e^{4\xi_2} - 3\big(3\xi_2^2 - 2\big)e^{3\xi_2} - 6y - 3\big)e^{-3\xi_2}
\end{aligned}
$$

Using the same precision as above, the evaluations of the exponential functions at the lower and upper bounds for ξ_2 give estimates:

$$
\begin{aligned}
1.087 &< h_0(\xi_2) < 1.458 \\
2.178 &< h_1(\xi_2) < 2.470 \\
0.446 &< h_\infty(\xi_2) < 0.827 \\
4.151 &< h_{\mathrm{gen}}(\xi_2) < 4.309
\end{aligned}
$$

We can therefore conclude that the threefold 2.30 is K-stable, and must admit a non-trivial Kähler-Ricci soliton.

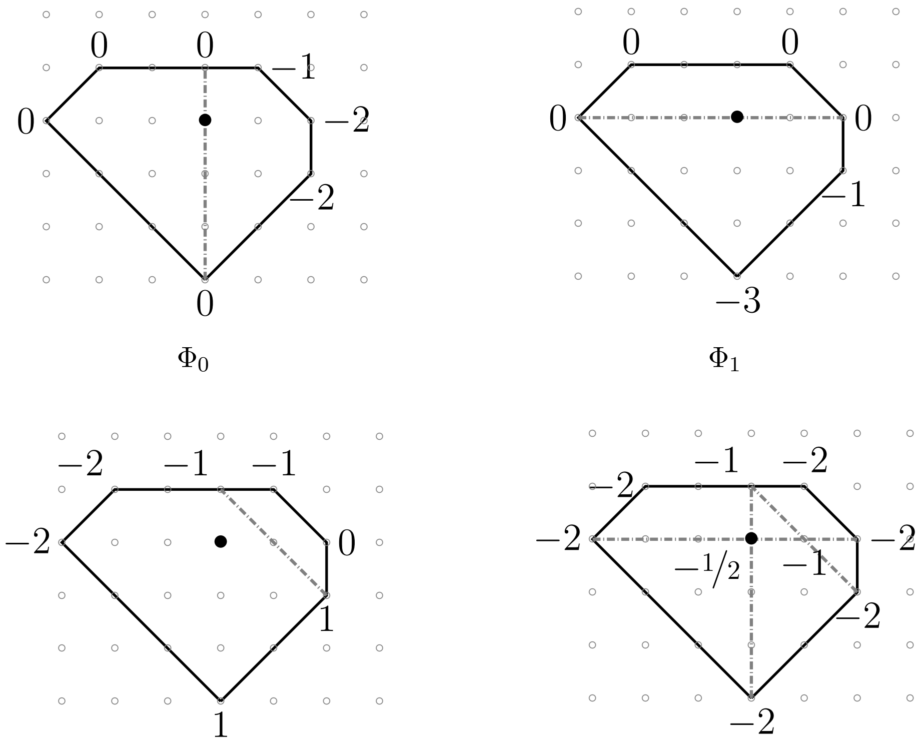
Example 21. *Consider the threefold 3.23. We follow the calculations outlined in the above proof of Theorem 5. As before we first have to find a closed form for* $\mathrm{DF}_\xi(n)$*, or* $\nabla_n G(\xi)$*, respectively. See Figure 3 for the divisorial polytope in this case. Then numerically we can find an approximation to ξ as the point:*

$$(x_0, x_1) = (0.26617786, 0.67164063).$$

Setting $\epsilon = 10^{-5}$, consider the square containing our approximation, given by:

$$D = [x_0 - \epsilon, x_0 + \epsilon] \times [x_1 - \epsilon, x_1 + \epsilon]$$

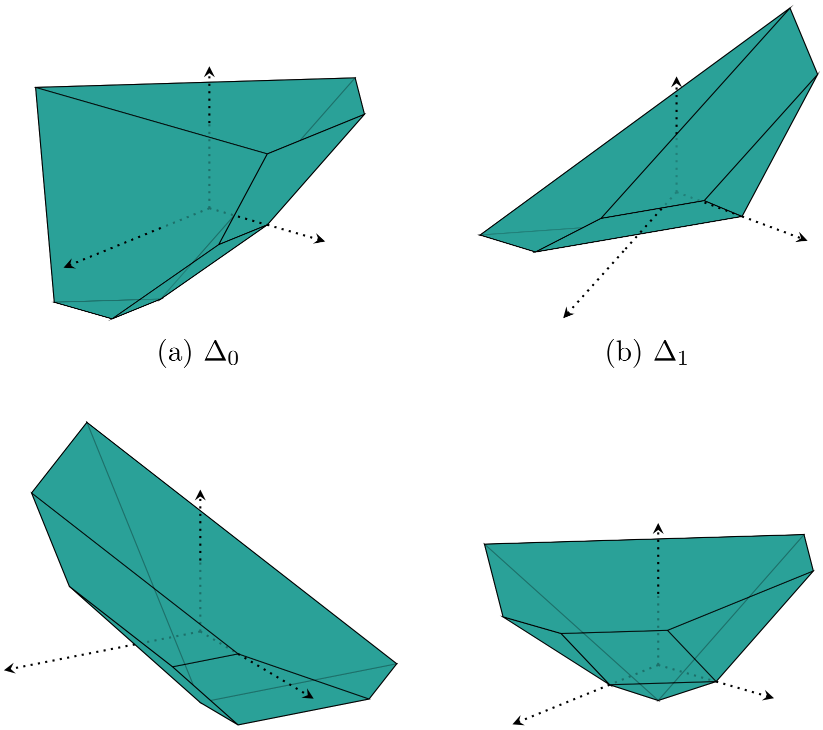
Subdividing each edge of the boundary ∂D into line segments of length $\epsilon/1500$, we use interval arithmetic to verify that the gradient of h is positive in the outer normal direction for each of these segments, in fact $\nabla_n G > 5.536 \cdot 10^{-6}$ along ∂D. Once again it remains to check the positivity of the Donaldson-Futaki invariant for each degeneration. The degenerations of this threefold correspond to the polytopes:



Φ_0 Φ_1

Φ_∞ $\deg \Phi$

Figure 3: Divisorial polytope for threefold 2.30



(a) Δ_0 (b) Δ_1

(c) Δ_∞ (d) Δ_{gen}

Figure 4: Special configuration polyhedra for 3.23

Interval arithmetic gives the following lower bounds on the Donaldson-Futaki invariants:

$$
\begin{aligned}
h_0(\xi_2) &> 1.2766 \\
h_1(\xi_2) &> 1.8401 \\
h_\infty(\xi_2) &> 0.1004 \\
h_{\text{gen}}(\xi_2) &> 3.4443
\end{aligned}
$$

We can therefore conclude that the threefold 3.23 is K-stable, and must admit a non-trivial Kähler-Ricci soliton. See also Appendix B for the SageMath code of the calculations.

3.3 Barvinok Integration

Here we describe the recursive algorithm used to symbolically calculate closed forms of the integrals used in the proofs of the previous section. We are interested in integrals of the following form:

$$\int_P l_1(x)e^{l_2(x)} \tag{3.3}$$

for some linear functions l_1, l_2 and some polytope P. If we were working with surfaces, as is done earlier in [42], then P is just an interval and the integral can be computed easily by hand. In theory, one could subdivide and parameterize the domain for higher dimensions, but this quickly makes the process of integration a very time-consuming and computation-heavy task, for even mildly complicated P.

In [44] Stoke's theorem is applied to a polytope domain to obtain formulae for integrals of a similar form. In particular we make use of the following:

Lemma 6 [44]**.** *Let $\{F_i\}_i$ be the set of all facets of a polytope P, and μ_i be the Lebesgue measure on the affine hull of F_i induced from dx on $\mathbb{R}^n$. Denote by n_i the outer unit normal to F_i. Let $c \in \mathbb{C}^n$ and $\Lambda \in \mathbb{R}^n$ such that $\langle \lambda, c\rangle \neq 0$. Then*

$$\int_P e^{\langle c,x\rangle}dx = \frac{1}{\langle c, \lambda\rangle} \sum_i \langle n_i, \lambda\rangle \int_{F_i} e^{\langle c,x\rangle} d\mu_i. \tag{3.4}$$

We now outline how we implement this in the form of a recursive algorithm. Care must be taken to ensure the correct scaling is used when applying Lemma **??**

with induced Duistermaat Heckmann measures ν_i instead of induced Lebesgue measures μ_i, but this amounts to using lattice-primitive normals rather than Lebesgue unit normals. We will only describe how our algorithm works for $l_1 = 1$, as the general case may be obtained through integration by parts.

First we describe certain constructions needed in the recursion process. Any facet F of a polytope P is given by the intersection of P with some halfspace:

$$\langle a, x\rangle = \langle a, u_0\rangle$$

for some primitive element $a \in N$ and a choice of vertex $u_0 \in \mathcal{V}(F)$. Consider the sublattice $M_F := \ker a$. Consider the inclusion:

$$\iota : M_F \to M.$$

Pick a section:

$$s : M \to M_F$$

given by a matrix with rows forming a basis of primitive lattice elements of M_F. We may identify $F \cong P_F := F - u_0$. Set $c_F = \iota^*(c)$, and $\lambda_F = s(\lambda)$.

We can now use the formula 3.4 to define a recursive function Barv to compute the integral $\int_P e^{\langle c, \rangle x} dx$. It takes input (N, F, c, λ). We have the following relation:

$$\mathrm{Barv}(M, F, c, \lambda) = \sum_F \frac{e^{\langle c, u_0\rangle}}{\langle c, \lambda\rangle} \cdot \langle \lambda, n_F\rangle \mathrm{Barv}(M_F, P_F, c_F, \lambda_F)$$

We give our algorithm in pseudocode below. Caching was later added, by Süß, to reduce computation time. The full SageMath code is included in Appendix B.

Algorithm 1: Barv(M, P, c, λ)

```
input  : A tuple (N, P, c, λ) as described above, with λ sufficiently
         general.
output : The integral ∫_P e^{⟨c,x⟩} dx

I ← 0;
if c = 0 then
    if dim P = 0 then
        return 1;
    else
        return vol_{ν_P}(P);
else
    for F a facet of P do
        u_0 ← a vertex of F;
        M_F ← the ambient lattice of F;
        n_F ← the primitive outer normal to M_F;
        s ← an appropriate section of M_F ⊂ M;
        P_F ← F − u_0;
        c_F ← ι*(c);
        λ_F ← s(λ);
        coeff_F ← (e^{⟨c,u_0⟩} / ⟨c,λ⟩) · ⟨λ, n_F⟩;
        I ← I + coeff · Barvinok(N_F, P_F, c_F, λ_F);
return I
```

Example 22. *Suppose $P = \operatorname{conv}(\{(\pm 1, \pm 1)\})$, $c = (0, 1)$. We have facets $F_1, \ldots, F_4$ as labelled in Figure 5. In this case $\lambda = (1, 1)$ is sufficiently general. The first face we consider is F_1. We pick $u_0 = (1, -1)$, and we have $M_F = \mathbb{Z}$, $n_F = (1, 0)$. Pick $s = (0, 1)$. Now $P_{F_1} = [0, 2] \subset M_F \otimes \mathbb{R}$, $c_F = 1$, $\lambda_F = 1$. We calculate coeff$_F = e^{-1}$.*

For this branch of the recursion we then calculate:

$$\mathit{Barv}(M_F, P_F, c_F, \lambda_F) = \int_0^2 e^x dx.$$

Applying the algorithm again we obtain $e^2 - e$. In sum then F_1 contributes $e^{-1} * (e^2 - e) = e - 1$ to the integral $\int_P e^y dx$.

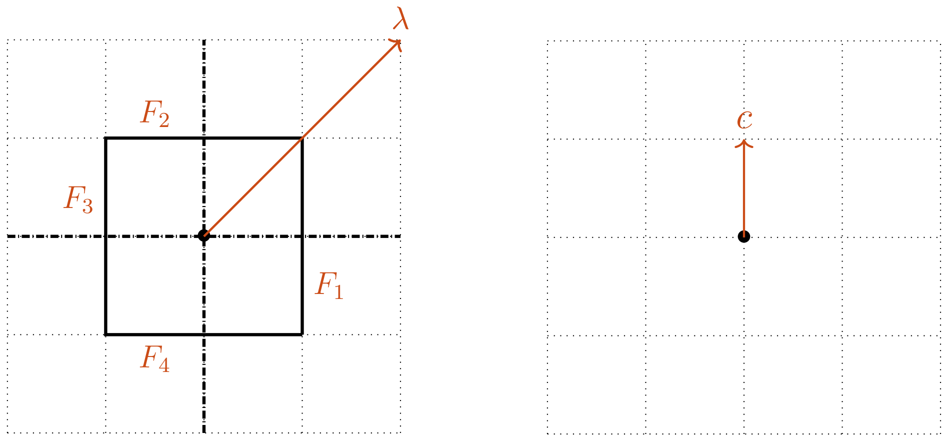


Figure 5: First level of recursion

Figure 6: Second level of recursion for the branch F_1

Chapter 4

$R(X)$ in complexity one

Recall, as discussed in the introduction, that one approach to the existence of Kähler-Einstein metrics is the study of the continuity path, that is solutions $\omega_t \in 2\pi c_1(X)$ to the equation

$$\mathrm{Ric}(\omega_t) = t\omega_t + (1-t)\omega.$$

for $t \in [0,1]$. By [7] there is always a solution for $t = 0$. However, Tian [45] showed that for some t sufficiently close to 1 there may not be a solution for certain Fano manifolds. It is natural to ask for the supremum of permissible t, which turns out to be independent of the choice of ω.

Definition 22. *Let (X, ω) be a Kähler manifold with $\omega \in 2\pi c_1(X)$. Define:*

$$R(X) := \sup(t \in [0,1] : \exists\ \omega_t \in 2\pi c_1(X)\ \ \mathrm{Ric}(\omega_t) = t\omega_t + (1-t)\omega).$$

This invariant was first discussed, although not explicitly defined, by Tian in [46]. It was first explicitely defined by Rubenstein in [47] and was further studied by Szekelyhidi in [48]. It is sometimes referred to as the greatest lower bound on Ricci curvature.

In [47] Rubenstein showed relation between $R(X)$ and Tian's alpha invariant $\alpha(X)$, and in [49] conjectured that $R(X)$ characterizes the K-semistability of X. This conjecture was later verified by Li in [50].

In [51] Li determined a simple formula for $R(X_\Delta)$, where X_Δ is the polarized toric Fano manifold determined by a reflexive lattice polytope Δ. This result was later recovered in [14], by Datar and Székelyhidi, using notions of G-equivariant

K-stability. Previously $R(X)$ has been calculated for group compactifications by Delcroix [52] and for homogeneous toric bundles by Yao [53]. Let us briefly recall the toric formula.

Theorem 6 [20, Theorem 1?]**.** *Suppose X is a smooth Fano toric variety. Let P be the corresponding Fano polytope. If* $\mathrm{bc}(P) = 0$ *then X is Kähler-Einstein and* $R(X) = 1$*. Otherwise let q be the intersection of the ray generated by* $-\mathrm{bc}(P)$ *with the boundary ∂P. We then have:*

$$R(X) = \frac{|q|}{|q - \mathrm{bc}(P)|}.$$

Example 23. *Consider the toric variety* $X = \mathrm{Bl}_z \mathbb{P}^1 \times \mathbb{P}^1$ *from Example 10. It is then easy to calculate $R(X)$ from the polytope P given in Example 11. We have* $\mathrm{bc}(P) = (-2/21, -2/21)$ *and* $q = (1/2, 1/2)$ *so* $R(X) = 21/25$.

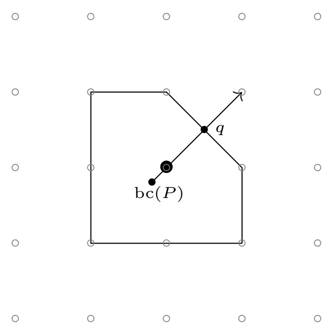


Figure 1: $R(X)$ calculation for a toric X

Using similar methods to [14] we obtain an effective formula for manifolds with a torus action of complexity one, in terms its divisorial polytope. Let $\Phi : \square \to \mathrm{Div}_{\mathbb{Q}} \mathbb{P}^1$ be the Fano divisorial polytope corresponding to a smooth Fano complexity one T-variety X. Let $\{\Delta_y\}_{y \in \mathbb{P}^1}$ be the finite set of degeneration polytopes corresponding to central fibres of the non-product test configurations of X, as in Definition 21.

To state our result we must introduce a little more notation. Suppose we have $\mathrm{bc}(\Delta_y) \neq 0$ for some i. Consider the halfspace $H := N_{\mathbb{R}} \times \mathbb{R}^{\geq 0} \subset N'_{\mathbb{R}}$. Let F_y be the face of Δ_y in which q_y lies, and let S be the set of points y for which $\mathrm{bc}(\Delta_y) \neq 0$ and all outer normals to F_y lie in H. Note, by definition of Δ_y, we have:

$$S = \{y \in \mathbb{P}^1 |\, \mathrm{bc}(\Delta_y) \notin H\}$$

Recall the definition of the Duistermatt Heckman measure ν, and associated weighted barycenter of $\square$ given in Definition 17. Suppose $\mathrm{bc}_\nu(\square) \neq 0$. Let q be the intersection of the ray generated by $-\mathrm{bc}_\nu(\square)$ with $\partial\square$. Let q_y be the point of intersection of $\partial\Delta_y$ with the ray generated by $-\mathrm{bc}(\Delta_y)$.

Note, by the equation for the Donaldson Futaki invariants (2.8) and Theorem 3, we know that $R(X) = 1$ if $\mathrm{bc}(\Delta_y) \in \{0\} \times \mathbb{R}^+$ for each y. We may now state our result:

Theorem 7 [19, Theorem 1.1]**.** *Let X be a complexity one Fano T-variety as above. If* $\mathrm{bc}(\Delta_y) \in \{0\} \times \mathbb{R}^+$. *Otherwise:*

$$R(X) = \min \left\{ \frac{|q|}{|q - \mathrm{bc}_\nu(\square)|} \right\} \cup \left\{ \frac{|q_y|}{|q_y - \mathrm{bc}(\Delta_y)|} \right\}_{\mathrm{bc}(\Delta_y) \notin H}. \tag{4.1}$$

Example 24. *To see that this formula is truly a generalization of Li's result, consider the situation of a toric downgrade. Suppose X is a Fano toric variety given by a polytope P. Let $\Phi = \Phi_0 \otimes \{0\} + \Phi \otimes \{\infty\}$ be the divisorial polytope obtained through the downgrade procedure in Example 18. Under the associated surjection $p : M' \to M$ we have $p(\mathrm{bc}(\Delta_0)) = p(\mathrm{bc}(\Delta_\infty)) = \mathrm{bc}_\nu(\square)$. It is easy to see $\emptyset \neq S \subset \{0, \infty\}$. By convexity of P then for $y \in S$ we have $|q| \geq |p(q_y)|$, and moreover:*

$$\frac{|q|}{|q - \mathrm{bc}_\nu(\square)|} \geq \frac{|p(q_y)|}{|p(q_y) - \mathrm{bc}_\nu(\square)|} = \frac{|p(q_y)|}{|p(q_y - \mathrm{bc}(\Delta_y))|} = \frac{|q_y|}{|q_y - \mathrm{bc}(\Delta_y)|}$$

Clearly then the minimum in (4.1) is obtained at one of $y \in \{0, \infty\}$

Example 25. *Consider the $(\mathbb{C}^*)^2$-threefold 2.30 from Example 20. There are 3 normal toric degenerations, given by the polytopes $\Delta_0, \Delta_1, \Delta_\infty$. It can easily be checked in this case that $S = \emptyset$, see Figure 2a for example, or Appendix A. Therefore $R(X)$ is given by the first term in the minimum. We calculate* $\mathrm{bc}_\nu(\square) = (0, -6/23)$ *and $q = (0, 1)$. Then:*

$$R(X) = \frac{1}{1 + 6/23} = \frac{23}{29}.$$

Corollary 1 [19, Corollary 1.2]**.** *In Table 4.1 below we give $R(X)$ for X a Fano threefold admitting a 2-torus action appearing in the list of Mori and Mukai [18]. We include only those where $R(X) < 1$. Note all admit a Kähler-Ricci soliton by Theorem 5.*

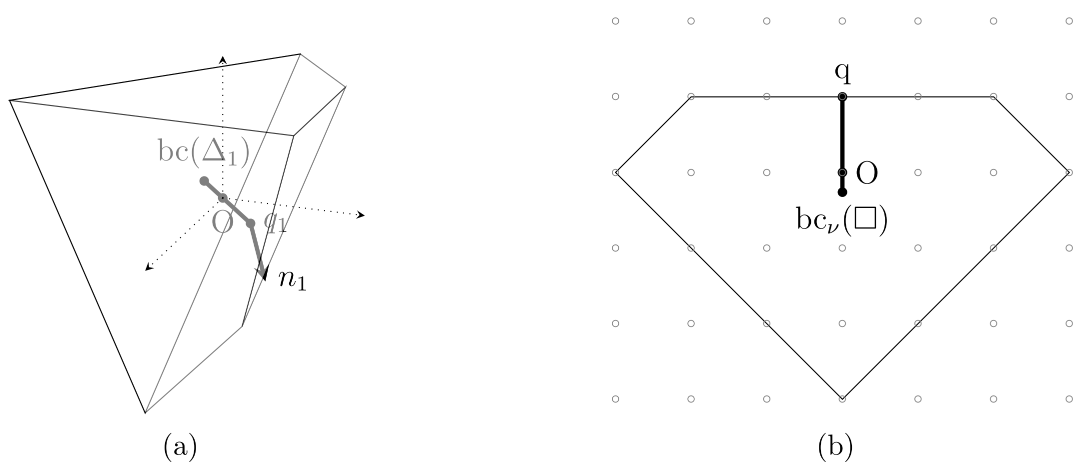


Figure 2: Some of the calculation of $R(X)$ for threefold 2.30. (a) Degeneration polytope Δ_1 with barycenter $\mathrm{bc}(\Delta_1)$ and q_1, n_1 shown, (b) Moment polytope $\square$ with Duistermaat-Heckmann barycenter $\mathrm{bc}_\nu(\square)$ and q shown.

X	R(X)
2.30	23/29
2.31	23/27
3.18	48/55
3.21	76/97
3.22	40/49
3.23	168/221
3.24	21/25
4.5*	64/69
4.8	76/89

Table 4.1: Calculations for complexity 1 threefolds appearing in the list of Mori and Mukai for which $R(X) < 1$

Let X be a T-variety of complexity one associated to a divisorial polytope $\Psi : \square \to \mathrm{Div}_{\mathbb{Q}}(\mathbb{P}^1)$, see Section 2.3. It follows from Theorem 3 that:

$$R(X) = \inf_{(\mathcal{X},\mathcal{L})} (\sup(t | \, \mathrm{DF}_t(\mathcal{X}, \mathcal{L}, \cdot) \geq 0)), \tag{4.2}$$

where $(\mathcal{X}, \mathcal{L})$ varies over all special test configurations for (X, L). We have an explicit description of special test configurations and their Donaldson Futaki invariants, see Section 2.4. We will calculate $R(X)$ by considering first the product configurations and then the non-product configurations. To calculate the values $\sup(t|\,\mathrm{DF}_t(\mathcal{X},\mathcal{L}) \geq 0)$ for a given configuration we need to first consider some elementary convex geometry.

4.1 A short digression into convex geometry

Let V be a real vector space and $P \subset V$ be a full dimensional convex polytope containing the origin. Fix some point $b \in \mathrm{int}(P)$. Let $q \in \partial P$ be the intersection of ∂P with the ray $\tau = \mathbb{R}^+(-b)$. Suppose $n \in V^\vee$ is an outer normal to a face containing q. For $a \in \partial P$ write $\mathcal{N}(a) = \{w \in V^\vee \,|\langle a, w\rangle = \max_{x\in P}\langle x, c\rangle\}$. For $w \in \mathcal{N}(a)$ let $\Pi(a, w)$ be the affine hyperplane tangent to P at a with normal w. For $w \in \mathrm{int}(\tau^\vee)$ there is a well-defined point of intersection of $\Pi(a, w)$ and τ which we denote r_w. See Figure 3 for a schematic.

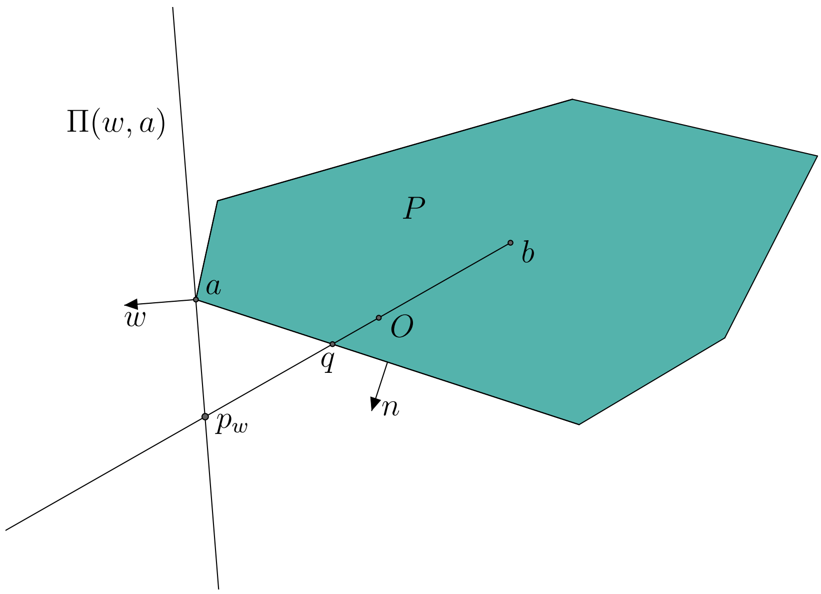


Figure 3: An Example in $V \cong \mathbb{R}^2$

Lemma 7. *Fix $w \in int(\tau^\vee)\backslash(\mathbb{R}^+n)$. For $s \in [0,1]$ set $w(s) := sn + (1-s)w$. As $n \in \tau^\vee$ we may consider $r(s) := r_{w(s)}$. For $0 \leq s' < s \leq 1$ we then have:*

$$\frac{|r(s)|}{|r(s) - b|} < \frac{|r(s')|}{|r(s') - b|}.$$

Proof. Without loss of generality we may assume $s' = 0$. For $s \in [0,1]$ the points

$r(s), q, b$ are collinear, so $|r(s)| = |r(s)-q|+|q|$ and $|r(s)-b| = |r(s)-q|+|q|+|b|$. Therefore:

$$\frac{|r(s)|}{|r(s)-b|} = \frac{|r(s)-q|+|q|}{|r(s)-q|+|q|+|b|}.$$

Hence it is enough for $|r(s)-q| < |r(0)-q|$ for $s > 0$. Since $q \neq 0$ is fixed this is equivalent to:

$$\frac{|r(s)-q|}{|q|} < \frac{|r(0)-q|}{|q|}.$$

For each $s \in [0,1]$ choose $a(s) \in \partial P$ such that $w(s) \in \mathcal{N}(a(s))$. Write $a = a(0)$ for convenience. We then have:

$$\frac{|r(s)-q|}{|q|} = \frac{\langle a(s)-q, w\rangle}{\langle q, w\rangle}.$$

Note $n \in \mathcal{N}(q)$. Now $\langle a(s)-q, n\rangle \leq 0$ and $\langle a(s)-q, w\rangle \leq \langle a-q, w\rangle$. Clearly we have $\langle q, n\rangle > 0$. Then:

$$\begin{aligned}
\frac{|r(s)-q|}{|q|} &= \frac{\langle a(s)-q, w(s)\rangle}{\langle q, w(s)\rangle} \\
&= \frac{s\langle a(s)-q, n\rangle + (1-s)\langle a(s)-q, w\rangle}{s\langle q, n\rangle + (1-s)\langle q, w\rangle} \\
&\leq \frac{(1-s)\langle a-q, w\rangle}{s\langle q, n\rangle + (1-s)\langle q, w\rangle} \\
&< \frac{\langle a-q, w\rangle}{\langle q, w\rangle} \\
&= \frac{|r(0)-q|}{|q|}.
\end{aligned}$$

□

Let V, P, b, q, τ, n be as in the introduction to this section. Fix some open halfspace $H \subset V^\vee$ given by $u \geq 0$ for some $u \in V\backslash\{0\}$. This defines a projection map $p : V \to V/\langle u\rangle$. Consider the function $F_b : V^\vee \times [0,1] \to \mathbb{R}$ given by:

$$F_b(w,t) := t\langle b, w\rangle + (1-t)\max_{x\in P}\langle x, w\rangle$$

Corollary 2. *For any $W \subseteq V^\vee$ containing n we have:*

$$\sup(t \in [0,1] \mid \forall_{w \in W}\ F_b(t,w) \geq 0) = \frac{|q|}{|q-b|}. \tag{4.3}$$

If for some choice of n we have $n \notin H$ then:

$$\sup(t \in [0,1] \mid \forall_{w \in H}\ F_b(t,w) \geq 0) = \frac{|\tilde{q}|}{|\tilde{q}-p(b)|}, \tag{4.4}$$

where $\tilde{q}$ is the intersection of the ray $p(\tau)$ with the boundary of $p(P)$.

Proof. Note that:

$$\sup(t \in [0,1] \mid \forall_{w \in W}\ F_b(t,w) \geq 0) = \inf_{w \in W} \sup(t \in [0,1] \mid F_b(t,w) \geq 0).$$

Moreover $\sup(t \in [0,1] \mid F_b(t,w) \geq 0) = 1 > F_b(t,n)$ for $\langle b,w\rangle \geq 0$, so without loss of generality we may assume $W \subseteq \operatorname{int}(\tau^\vee)$. For $w \in W$ then:

$$\begin{aligned}\sup(t \in [0,1] \mid F_b(t,w) \geq 0) &= \frac{\max_{x \in P}\langle x,w\rangle}{\max_{x \in P}\langle x,w\rangle - \langle b,w\rangle}\\ &= \frac{\langle a,w\rangle}{\langle a,w\rangle - \langle b,w\rangle}\\ &= \frac{\langle p_w,w\rangle}{\langle p_w,w\rangle - \langle b,w\rangle} = \frac{|p_w|}{|p_w-b|}.\end{aligned}$$

Hence:

$$\sup(t \in [0,1] \mid \forall_{w \in W}\ F_b(t,w) \geq 0) = \inf_{w \in W} \frac{|p_w|}{|p_w-b|}.$$

Now for $w \in W$ consider the continuity path $w(s) = sn + (1-s)w$. By Lemma 7 if $n \in W$ then the above infimum is attained when $s = 1$ and we obtain (4.3). Otherwise the infimum is attained at some $w \in \partial W$. For (4.4) restricting F_b to $\partial H \times [0,1]$ gives:

$$F_b(w,t) = t\langle p(b),w\rangle + (1-t)\max_{x \in p(P)}\langle x,w\rangle.$$

Applying (4.3) to the polytope $p(P)$ in the vector space ∂H we obtain (4.4). $\square$

4.2 Proof of Theorem 4

4.2.1 Product Configurations

Recall the formula (2.10) for the twisted Donaldson-Futaki invariant of a product configuration $\mathcal{X} \cong X \times \mathbb{A}^1$. Let $q \in N_{\mathbb{R}}$ be the point of intersection of the ray generated by $-\operatorname{bc}(\Psi)$ with $\partial\Box$. Applying (4.3), from Corollary 2, to (4.2) we obtain:

$$\sup(t|\operatorname{DF}_t(\mathcal{X},\mathcal{L},\cdot) \geq 0) = \frac{|q|}{|q - \operatorname{bc}_\nu(\Box)|}.$$

4.2.2 Non-Product Configurations

Recall the description of special non-product test configurations of X from Section 2.4, and in particular the formula for the twisted Donaldson-Futaki in Lemma 4. Set $H := N_{\mathbb{Q}} \times \mathbb{R}^+$.

Proposition 1. *For any non-product configuration $(\mathcal{X},\mathcal{L})$, with special fiber one of the Δ_y, let σ_y be the cone of outer normals to Δ_y at the unique point of intersection of $\partial\Delta_y$ with the ray generated by $-\operatorname{bc}(\Delta_y)$. Denote this point of intersection by q_y. Then:*

$$\sup(t|\operatorname{DF}_t(\mathcal{X},\mathcal{L},\cdot) \geq 0) = \begin{cases} \frac{|q_y|}{|q_y - \operatorname{bc}(\Delta_y)|} & \sigma_y \cap H \neq \emptyset; \\ \\ \frac{|q|}{|q - \operatorname{bc}_\nu(\Box)|} & \sigma_y \cap H = \emptyset. \end{cases}$$

Proof. Extend $\operatorname{DF}_t(\mathcal{X},\mathcal{L},\cdot)$ linearly to the whole of $N_{\mathbb{R}} \times \mathbb{R}$. In the case $\sigma_y \cap H \neq \emptyset$ we may apply (1) from Corollary 2 with $P = \Delta_y$ and $b = \operatorname{bc}(\Delta_y)$. Otherwise we may apply (4.4) from Corollary 2, noting that $p(\Delta_y) = \Box$ and $p(\operatorname{bc}(\Delta_y)) = \operatorname{bc}_\nu(\Box)$. □

Proof of Theorem 7. With Remark 2 in mind, observe that a special test configuration must either be product or non-product. Any non-product configurations Δ_y with $\sigma_y \cap H \neq \emptyset$ have their contribution to the infimum already accounted for and we may exclude them. The result follows. □

of Corollary 1. We observe $\operatorname{bc}(\Delta_y) \in H$ for every special test configuration polytope Δ_y of each threefold in this list, see A. We may then calculate $R(X)$ using just the base polytope $\Box$ and its Duistermaat-Heckman barycenter. The divisorial polytopes and Duistermaat-Heckman measures were originally given in [**?**], and may be also be found in Appendix A. □

Chapter 5

Kähler-Einstein metrics in complexity two

In this chapter we use equivariant methods to find new Kähler-Einstein metrics on some complexity two T-varieties. The first examples we are interested in are some hypersurfaces of bidegree (a, b). Consider the following varieties:

$$X_{a,b}^{2n-1} := V\left(\sum_{i=0}^{n} x_i^a y_i^b\right) \subseteq \mathbb{P}^n \times \mathbb{P}^n$$

Let $p = a/d, q = b/d$, where $d = \gcd(a, b)$. There is an effective $T = (\mathbb{C}^*)^n$-action on $X_{a,b}^{2n-1}$ specified by weights $(0|qI_n|0| - pI_n)$ on the homogeneous coordinates. Our first result is the following:

Theorem 8 [32, Theorem 1.1]**.** *$X_{1,2}^5$ and $X_{1,3}^5$ admit T-invariant Kähler-Einstein metrics.*

Note that $X_{1,2}^5$ and $X_{1,3}^5$ appear in the classification of [21] as varieties $4E, 4F$ respectively. Note also that $X_{1,1}^{2n-1}$ is the flag manifold of type $(1, n-1)$ and is known to be Kähler-Einstein as a homogeneous manifold, see remarks immediately preceeding [54, Theorem 3] for example. Our method of proof also allows us to calculate the topological orbit space of the compact torus action on these varieties:

Corollary 3 [32, corollary 1.5]**.** *Let K be the maximal compact torus of T. There is a homeomorphism:*

$$X_{a,b}^{2n-1}/K \cong S^{n-1} * \mathbb{P}^{n-1}.$$

Where the later is the topological join of the $(n-1)$-sphere and complex projective

$(n-1)$-space. In particular, this shows that the K-orbit space of the flag manifold $F(1, n-1, \mathbb{C}^n) = X^{2n-1}_{1,1}$ is of this form.

Our third example is an iterated blow-up of the even-dimensional quadric hypersurface. Consider the following representation:

$$Q^{2n} := V\left(\sum_{i=0}^{n} x_{2i}x_{2i+1}\right) \subset \mathbb{P}^{2n+1}.$$

Let $Z_i := V(x_{2i}, x_{2i+1}) \subset Q^{2n}$ for $i = 0, \ldots, n$. Let W^{2n} denote the wonderful compactification of the arrangement of subvarieties of Q built by $Z_0, \ldots, Z_n$. We will show that W^{2n} is Fano in Section 5.1.2. Our second result is the following:

Theorem 9 [32, Theorem 1.2]**.** *W^6 admits a T'-invariant Kähler-Einstein metric.*

Note that these examples admit additional symmetries: there is a natural S_{n+1}-action on $X^{2n-1}_{a,b}$ permuting the indices of variables, and by results of [55], the S_{n+1}-action on Q^{2n} permuting the Z_i induces a well-defined action on W^{2n}.

5.1 Chow quotient calculations

We now calculate the Chow quotient pairs of our examples. As discussed in 2.2.4, the Chow quotient coincides with the GIT limit quotient.

5.1.1 Bidegree (a, b) hypersurfaces

For the varieties $X^{2n-1}_{a,b}$ we use the Kempf-Ness theorem to calculate GIT quotients. The inverse system is simple enough in this case to then deduce the Chow quotient pair and in addition prove Corollary 3. Fix natural numbers $n, a, b > 0$ and for brevity let $X = X^{2n-1}_{a,b}$.

Let K denote the maximal compact torus in T. First we calculate our GIT and Chow quotients. Let L be the restriction of $\mathcal{O}(1,1)$ to X. Using (2.5) we can explicitely give a moment map for the torus action:

$$\mu([x],[y]) = \frac{\sum |x_i y_j|^2 (qe_i - pe_j)}{\sum |x_i y_j|^2}.$$

Where we define $e_0 := 0$. The moment image polytope P is the convex hull of the vectors $\{qe_i - pe_j\}_{i,j}$. Consider a boundary point $u \in \partial P$. In this case we show that the moment fibre of u is contained in one T-orbit, and thus the GIT quotient is just contraction to a point. The key observation here is the following:

Lemma 8. *Suppose $\mu([x],[y]) = \mu([x'],[y'])$ and for each j we have*

$$x_j y_j = x'_j y'_j = 0.$$

Then for each j we have $x_i = 0 \iff x'_i = 0$ and $y_i = 0 \iff y'_i = 0$.

Proof. Suppose first $i > 0$. For some positive real constants A, B, C, D we have:

$$A \sum_{j=0}^{n} |x_i y_j|^2 - B \sum_{j=0}^{n} |x_j y_i|^2 = C \sum_{j=0}^{n} |x'_i y'_j|^2 - D \sum_{j=0}^{n} |x'_j y'_i|^2.$$

The conclusion follows by considering signs. Suppose now $i = 0$. By applying the affine linear functional $l(w) := w \cdot (\sum e_j) - (q-p)$ to the equation $\mu([x],[y]) = \mu([x'],[y'])$, we obtain:

$$E \sum_{j=1}^{n} |x_j y_0|^2 - F \sum_{j=0}^{n} |x_0 y_j|^2 = G \sum_{j=1}^{n} |x'_j y'_0|^2 - H \sum_{j=0}^{n} |x'_0 y'_j|^2.$$

For some positive real constants E, F, G, H. Again by signs we obtain the result. □

Lemma 9. *For $u \in \partial P$ the moment fibre $\mu^{-1}(u)$ is contained in one T-orbit.*

Proof. Suppose $\mu([x],[y]) = \mu([x'],[y']) \in \partial P$. Since $(q-p)e_i \in P^\circ$ then $x_i y_i = x'_i y'_i = 0$ for each $i > 0$. By the defining equation of X then also $x_0 y_0 = x_0 y_0 = 0$. Applying Lemma 8 we are done. □

In particular Lemma 9 implies that the GIT quotient associated to $u \in \partial\Box$ is a contraction to a point. Now consider moment fibres of points in the interior of P. We calculate the associated GIT quotient by selecting an appropriate rational map, as in Lemma 2.

Lemma 10. *For $u \in P^\circ$ the topological quotient $\mu^{-1}(u) \to \mu^{-1}(u)/K$ is:*

$$\mu^{-1}(u) \to \mathbb{P}^{n-1}; \quad ([x],[y]) \mapsto (x_1^p y_1^q : \cdots : x_n^p y_n^q).$$

Proof. Clearly the map is K-invariant. If $(x, y), (x', y') \in \mu^{-1}(u)$ then for any representatives x, y, x', y' we have:

$$(x_1^p y_1^q : \cdots : x_n^p y_n^q) = ({x'_1}^p {y'_1}^q : \cdots : {x'_n}^p {y'_n}^q)$$

Fix a representative x of $[x]$. Pick a representative y of $[y]$ such that $|x||y| = 1$. By Lemma 8 we may pick a representative x' such that $x_0 = x'_0$. For any representatives x', y' of $[x'], [y']$ respectively, there is $\zeta \in \mathbb{C}^*$ such that $x_i^a y_i^b = \zeta {x'_i}^a {y'_i}^b$ for $i > 0$. Pick a representative y' so that we have $\zeta = 1$.

Note that rescaling our chosen y' by an element of S^1 does not change anything here. By the defining equation of X then $x_0^p y_0^q = {x'_0}^p {y'_0}^q$. Applying Lemma 8 we have $\nu \in \mathbb{C}^*$ such that $y_0 = \nu {y'}_0$. As $x_0 = {x'}_0$ then $\nu \in S^1$, and we may rescale y' by $1/\nu$ so that ${y'}_0 = y_0$.

If $x_i y_i = 0$ then by Lemma 8 we can pick $t \in \mathbb{C}^*$ such that $x_i = t^q x'_i,\ y_i = t^{-p} y'_i$. Suppose now $x_i y_i \neq 0$. Then $x_i, y_i, x'_i, y'_i \neq 0$. Pick t such that $t^p = y'_i / y_i$. Now $x_i^p = t^{pq} {x'_i}^p$. Hence there exists some pth root of unity, say ξ, such that $x_i = \xi t^q x'_i$. Since p, q are coprime we may pick another pth root of unity γ such that $\gamma^q = \xi$. Picking s such that $s^d = \gamma t$ we obtain $x_i = s^q {x'}_i$ and $y_i = s^{-p} y'_i$. We then have

$$([x], [y]) \in T \cdot ([x'], [y']) \ \cap \ \mu^{-1}(u) = S \cdot ([x'], [y']).$$

Thus we have described a closed map with fibres precisely the K-orbits of $\mu^{-1}(u)$. This must be the topological quotient of the K-action. □

By Lemma 2, we see that for any $u \in P^\circ$ the associated GIT quotient is given by the map:

$$([x], [y]) \mapsto (x_1^p y_1^q : \cdots : x_n^p y_n^q).$$

This implies the Chow quotient is given by the same formula. We may now calculate the boundary divisor of this quotient. Fix some homogeneous coordinates $z_1, \ldots, z_n$ on $\mathbb{P}^{n-1}$. For $\gamma \in \mathbb{Q}$ define the $\mathbb{Q}$-divisor

$$B_\gamma := \gamma \sum_i H_i, \tag{5.1}$$

where $H_1, \ldots, H_n$ are the coordinate hyperplanes of $\mathbb{P}^{n-1}$ and H_0 is the hyperplane $V(\sum_{i=1}^n z_i)$. We will prove the following:

Lemma 11. *The Chow quotient pair of $X_{a,b}^{2n-1}$ by T is $(\mathbb{P}^{n-1}, B_\gamma)$ with*

$$\gamma = \max\left(\frac{p-1}{p}, \frac{q-1}{q}\right).$$

Proof. From the above discussion we know the Chow quotient map is given by:

$$X \to \mathbb{P}^{n-1}; \quad \pi : ([x],[y]) \mapsto (x_1^p y_1^q : \cdots : x_n^p y_n^q)$$

Suppose that Z is a prime divisor on the quotient, and D is a component of $\pi^{-1}(Z)$. If D intersects the open set where $x_i, y_i \neq 0$, then $t_i^p = t_i^q = 1$ for any t in the generic stabilizer of D. As p, q are coprime this would imply $t_i = 1$. Suppose D is a component of $\pi^{-1}(Z)$ for some Z not of the form H_j. Then for each i, D intersects the open set where $x_i, y_i \neq 0$, so D has trivial generic stabilizer.

Now consider the prime divisor H_j on the quotient, for some fixed j. The irreducible components of $\pi^{-1}(H_i)$ are given by the homogeneous ideals (x_j^p), (y_j^q). The generic stabilizer of the first is a cyclic group of order p, generated by the element $t \in T$ with $t_i = 1$ for $i \neq j$ and t_j a primitive pth root of unity. By symmetry the generic stabilizer of the second is a cyclic group of order q. This gives the required boundary divisor. □

Proof of Corollary 3. By Lemma 9 and Lemma 10 we see that the T-action on $X_{a,b}^{2n-1}$ has almost trivial variation of GIT, as defined in [56, Definition 2.7], with $Y = \mathbb{P}^{n-1}$. The result follows by [56, Proposition 2.9]. □

5.1.2 A wonderful compactification on the quadric

Using results of [35], we may obtain the Chow quotient of W^{2n} from that of Q^{2n}. We construct W^{2n} as a *wonderful compactification* of an arrangement on an even dimensional quadric. We show that this compactification is Fano and we calculate the Chow quotient pair with respect to an induced torus action. First recall the notion of wonderful compactifications of arrangements of subvarieties, as introduced in [55].

Definition 23. *Let X be a nonsingular algebraic variety. An arrangement of subvarieties of X is a finite collection $\mathcal{A}$ of subvarieties closed under pairwise scheme-theoretic intersection. A building set of $\mathcal{A}$ is a subset $\mathcal{B} \subset \mathcal{A}$ such that for any $A \in \mathcal{A} \backslash \mathcal{B}$ the minimal elements of $\{B \in \mathcal{B} | B \supset A\}$ intersect transversally* *and the intersection is A. We will say that $\mathcal{A}$ is built by $\mathcal{B}$ if $\mathcal{B}$ is a building set for $\mathcal{A}$.*

Theorem 10 [55, Theorem 1.3]**.** *Let X be a nonsingular projective variety, and $V_1, \ldots, V_k$ a collection of subvarieties such that any non-empty subset of $\{V_1, \ldots, V_k\}$ forms a building set for an arrangement of subvarieties. Consider the iterated blowup*

$$\tilde{X} := \mathrm{Bl}_{\tilde{V}_k} \mathrm{Bl}_{\tilde{V}_{k-1}} \ldots \mathrm{Bl}_{\tilde{V}_2} \mathrm{Bl}_{V_1} X$$

Where $\tilde{V}_i$ represents the strict transform of V_i under the composition

$$\mathrm{Bl}_{\tilde{V}_{i-1}} \mathrm{Bl}_{\tilde{V}_{i-2}} \ldots \mathrm{Bl}_{\tilde{V}_2} \mathrm{Bl}_{V_1} X \to X.$$

Then:

- *Each blowup is along a nonsingular variety;*
- *$\tilde{X}$ is isomorphic to the blowup along the ideal $I_1 I_2 \cdots I_k$, where I_i is the homogeneous ideal corresponding to V_i for each i.*

Following [55], in the situation of Theorem 10 we will call $\tilde{X}$ the wonderful compactification of the arrangement built by $V_1, \ldots, V_k$. Note that the composition of blowups $\tilde{X} \to X$ is independent of the ordering of the V_i.

Let W be the wonderful compactification of the arrangement of subvarieties of Q built by $Z_0, \ldots, Z_n$, where $Z_i := V(x_{2i}, x_{2i+1}) \subseteq Q$.

Lemma 12. *W is Fano.*

Proof. By adjunction it is enough to show that $-K_B - W$ is ample, where B is the wonderful compactification of the arrangement of subvarieties of $\mathbb{P}^{2n+1}$ built by $V_0, \ldots, V_n$, where $V_i := V(x_{2i}, x_{2i+1}) \subseteq \mathbb{P}^{2n+1}$.

For each i pick $\sigma_i \in S_n$ such that $\sigma_i(1) = i$. Each σ_i corresponds to a sequence of blowups whose composition is independent of i, as in Theorem 10. Denote by $\psi_i : \mathrm{Bl}_{V_i} \to \mathbb{P}^{2n+1}$ the first blowup of this sequence, and $f_i : B \to \mathrm{Bl}_{V_i}$ the composition of the remaining blowups, so that the wonderful compactification is given by the composition $f_i \circ \psi_i$. Denote the exceptional divisor of ψ_i by E_i.

Consider the divisor $D_i := \psi_i^* \mathcal{O}(1) - E_i$ on $\mathrm{Bl}_{V_i} \mathbb{P}^{2n+1}$. Note that D_i is nef, since for any curve C in $\mathrm{Bl}_{V_i} \mathbb{P}^{2n+1}$ we may pick a hyperplane $H \subset \mathbb{P}^{2n+1}$ such that $C \not\subset \tilde{H}$ but $Z_i \subset \tilde{H}$, whereupon $(f^*\mathcal{O}(1) - E_i) \cdot C = \tilde{H} \cdot C \geq 0$. Now

$$-K_B - W \sim \sum_{i=0}^{n-1} f_i^* D_i + (f_0 \circ \psi_0)^* \mathcal{O}(n).$$

It is easy to see that the divisors $(f \circ \psi)^*\mathcal{O}(n),\ f_0^* D_0, \ldots, f_{n-1}^* D_{n-1}$ span a full dimensional subcone of the nef cone of B, and that $-K_B - W$ is clearly on the interior of this cone. $\square$

By construction there is a natural morphism $f : W \to Q$ which is a composition of blowups, each centered at a smooth subvariety by Theorem 10. Fix the line bundle $L = \mathcal{O}(1)_{|Q}$ on Q. Recall that there is an n-torus T acting on Q prescribed by $\deg x_{2i} = e_{i+1}, \deg x_{2i+1} = -e_{i+1}$. This torus action may be extended to the compactification W. These torus actions are not effective, but we may quotient by the global stabilizer, a cyclic group of order two generated by $-\mathrm{Id} = (-1, \ldots, -1) \in T$, to obtain the action of an effective torus T' on Q and W. Quotienting does not affect the calculation of GIT quotients. In [56] the GIT quotients $\pi : Q \to Q /\!\!/ T'$ were determined. They are either trivial contractions to a point, or of the following form:

$$Q \to \mathbb{P}^{n-1}; \quad [x] \mapsto (x_1 x_2 : \cdots : x_{2n-1} x_{2n}). \tag{5.2}$$

The Chow quotient is also then given by (5.2). Following [35], there is an ample line bundle $\tilde{L}$ on W such that any linearization of $\tilde{L}$ is a lift of a linearization of L. Moreover, given a linearization, it can be shown that $W^{ss} \subset f^{-1}(X^{ss})$. By [35, Lemma 3.11] the GIT quotients of W given by a linearization of $\tilde{L}$ are precisely the restrictions of compositions $\pi \circ f$, where π is the GIT quotient map for Q given by the corresponding linearization of L. We can conclude that the Chow quotient is the restriction of a composition of the blowup map f followed by the map (5.2).

We now calculate the boundary divisor of this quotient. Recall the definition of the $\mathbb{Q}$-divisor B_γ on $\mathbb{P}^{n-1}$ from (5.1).

Lemma 13. *The Chow quotient pair of W^{2n} by its T'-action is $(\mathbb{P}^{n-1}, B_{1/2})$.*

Proof. From the formula (5.2) it is easy to calculate that the boundary divisor of the Chow quotient pair of Q is trivial. Therefore the only chance for $m_Z > 1$ occurs at the exceptional loci of blowups. If we construct W with the following sequence of blowups

$$W = \mathrm{Bl}_{\tilde{Z}_n} \dots \mathrm{Bl}_{\tilde{Z}_1} \mathrm{Bl}_{Z_0} Q$$

The exceptional divisor of the composition of blowup maps is of the form $E_{n-1} + \dots + E_0$, where E_i is the exceptional divisor of the $(i+1)$th blowup in the sequence. By symmetry it is enough to calculate the generic stabilizer of E_0. Consider $\mathrm{Bl}_{Z_0} Q$, realized as a subvariety of $Q \times \mathbb{P}^1$, given by the additional equation $vx_0 - ux_1$, where u, v are the homogeneous variables in the second factor.

There is an induced T-action on Bl_{Z_0}, under which the equation $vx_0 - ux_1$ must be homogeneous with respect to the induced grading of the character lattice of T. This implies that $\deg u = \deg v + 2e_1$, and we see that the generic stabilizer of the $T' = T/\langle \pm \mathrm{Id} \rangle$-action on the exceptional divisor must be a cyclic group of order 2, generated by the element $(-1, 1, \dots, 1) + \langle \pm \mathrm{Id} \rangle$. $\square$

5.2 Log canonical thresholds and Tian's criterion

To prove Theorem 8 and Theorem 9 we will use a version of *Tian's criterion*. This criterion, mentioned briefly in the introduction to the thesis, is a sufficient condition for the existence of a Kähler-Einstein metric on a Kähler manifold. In order to use this criterion we must first recall the definition of the log canonical threshold, Tian's alpha invariant, and their relation.

5.2.1 Log canonical thresholds

Here we recall the definition of the global log canonical threshold of of a log pair. Recall that a log pair (Y, D) consists of a normal variety Y and a $\mathbb{Q}$-divisor D, where the coefficients of the irreducible components of D lie in $[0, 1]$. The canonical divisor of such a pair is $K_Y + D$. A pair (Y, D) is called smooth if Y is smooth and D is a simple normal crossings divisor. A log resolution of a log pair (Y, D) is a birational map $\varphi : \tilde{Y} \to Y$ such that $(\tilde{Y}, \varphi^* D)$ is smooth.

Suppose $\varphi : \tilde{Y} \to Y$ is a log resolution of a log pair (Y, D). Write $D = \sum a_i D_i$ for prime D_i and rational a_i. Then:

$$\varphi^*(K_Y + D) - K_{\tilde{Y}} \sim_{\mathbb{Q}} \sum_i a_i \tilde{D}_i + \sum_j b_j E_j$$

where $\tilde{D}_i$ is the proper transform of D_i and the E_j are the φ-exceptional divisors.

Definition 24. *We say (Y, D) is log canonical at $P \in Y$ if we have $a_i \leq 1$ for $P \in D_i$, and $b_j \leq 1$ for E_j such that $\varphi(E_j) = P$. This condition is independent of the choice of resolution. If (Y, D) is log canonical at all $P \in Y$ then we say (Y, D) is (globally) log canonical.*

Example 26. *Consider the pair $Y = \mathbb{P}^2$ and $D = \sum a_i L_i$ where L_i are all lines through a point $P \in Y$. Blowing up at P we obtain the following:*

$$\varphi^*(K_Y + D) - K_{\tilde{Y}} \sim_{\mathbb{Q}} (\deg D - 1)E + \sum a_i \tilde{L}_i$$

Where E is the exceptional divisor of the blow-up. Therefore (Y, D) is log-canonical whenever we have $\deg D \leq 2$ and all $a_i \leq 1$.

Recall the following consequence of the main theorem of [57], as stated in the proof of [58, Lemma 5.1]. This allows us to degenerate a pair under a $\mathbb{C}^*$-action if we want to show it is log canonical.

Lemma 14. *Let (Y, D) be a log pair. Suppose $\{D_t | t \in \mathbb{C}\}$ is a family of $\mathbb{Q}$-divisors such that $D_t \sim_{\mathbb{Q}} D$, $D_1 = D$, and for $t \neq 0$ there exists $\phi_t \in \mathrm{Aut}(X)$ such that $D_t = \phi_t(D)$. Then (Y, D) is log canonical if (Y, D_0) is.*

Now we recall the definition of the global log canonical threshold of a pair, as given in [56].

Definition 25. *The global G-equivariant log canonical threshold of a log pair (Y, B) is defined to be:*

$$\mathrm{glct}_G(Y, B) := \sup\{\lambda | (Y, B + \lambda D) \text{ log canonical } \forall D \in |-K_X - B|_{\mathbb{Q}}^G\}$$

When B is trivial we will suppress it in our notation, writing $\mathrm{glct}_G(X)$ for the G-equivariant log canonical threshold of a normal variety X.

5.2.2 Tian's alpha invariant and criterion

Here we recall the definition of Tian's alpha invariant and its relation to the global log canonical threshold. We extend a result by Demailley finite groups to a finite group semi-direct product an algebraic torus, which we will need for calculations later in this chapter. The paper [58] serves as a good reference for the definitions in this section.

Tian's alpha invariant is a generalization of the complex singularity exponent of a polynomial $f \in \mathbb{C}[z_1, \ldots, z_n]$, defined as follows:

$$c_O(f) := \sup\{\epsilon |\ |f|^{-2\epsilon} \text{ integrable in a neighborhood of } O \in \mathbb{C}^n\}.$$

Let X be a complex manifold. Let $G \subset \mathrm{Aut}(X)$ be a compact group of automorphisms acting on X. Let L be a G-invariant line bundle on X, equipped with a G-invariant singular Hermitian metric h. Locally $L \cong U \times \mathbb{C}$ and on U we can write $||\xi||_h^2 = |\xi|^2 e^{-2\phi(x)}$ for $z \in U, \xi \in L_z$, where $\xi \in L_z \cong \mathbb{C}$. Assume ϕ is a locally integrable function for the Lebesgue measure, and that the curvature form $\Theta L, h := \frac{i}{\pi}\partial\bar{\partial}\phi$ is non-negative as a $(1,1)$-current. We say that locally $h = e^{-2\phi}$.

Definition 26. *For any compact G-stable subset $K \subset X$, the complex singularity exponent of h is defined to be:*

$$c_K(h) = \sup\{\epsilon |\ \forall x \in K\ h^{\epsilon} = e^{-2\epsilon\phi} \textit{ is integrable in a neighbourhood of } x\}$$

Tian's alpha invariant is then the value

$$\alpha_{G,K}(L) := \inf_{\{h \textit{ is } G-\textit{equivariant}:\ \Theta_{L,h} \geq 0\}} c_K(h)$$

Where h runs over all Hermitian metrics on L such that $\Theta_{L,h} \geq 0$.

Recall Tian's criterion:

Theorem 11 Tian's Criterion**.** *Let X be a Fano manifold and $G \subset \mathrm{Aut}(X)$ reductive group of symmetries. If*

$$\alpha_G(X) > \frac{\dim(X)}{\dim(X)+1}$$

Then X admits a G-invariant Kähler-Einstein metric.

In Demailley's appendix of [58] it is shown that $\mathrm{glct}_G(X) = \alpha_G(X)$ for $G \subseteq \mathrm{Aut}(X)$ a finite subgroup. The same proof may be easily extended to our setting, where G is the semidirect product of a torus T and a finite subgroup H of the normalizer of T in $\mathrm{Aut}(X)$. We outline one way of doing this in the following lemma.

Lemma 15. *Suppose that X is a T-variety and H is a finite subgroup of the normalizer $\mathcal{N}_{\mathrm{Aut}(X)}(T)$. Then* $\mathrm{glct}_{HT}(X) = \alpha_{HT}(X)$.

Proof. One may define the log canonical threshold of a linear system $|\Sigma| \subset |mL|$ for any Hermitian line bundle L on X, see remarks succeeding [58, Definition A.2]. Note by definition if $D \in |\Sigma|$ then $\mathrm{g}\mathbf{lct}(\frac{1}{m}D) \leq \mathrm{g}\mathbf{lct}(\frac{1}{m}|\Sigma|)$ with equality when Σ is one-dimensional. As stated in [58, (A.1)], we have:

$$\alpha_{HT}(L) = \inf_{m \in \mathbb{Z}_{>0}} \inf_{|\Sigma| \subset |mL|, \Sigma^{HT} = \Sigma} \mathrm{g}\mathbf{lct}(\frac{1}{m}|\Sigma|)$$

Clearly we have the inequality:

$$\alpha_{HT}(L) \leq \inf_{m \in \mathbb{Z}_{>0}} \inf_{D \in |mL|^{HT}} \mathrm{g}\mathbf{lct}(\frac{1}{m}D)$$

Now suppose $|\Sigma| \subset |mL|$ such that $|\Sigma|^{HT} = |\Sigma|$. Take $D \in |\Sigma|$. We may repeatedly degenerate D along $\mathbb{C}^*$-actions to obtain $D' \in |mL|^T$, with $\mathrm{g}\mathbf{lct}(\frac{1}{m}D') \leq \mathrm{g}\mathbf{lct}(\frac{1}{m}D) \leq \mathrm{g}\mathbf{lct}(\frac{1}{m}|\Sigma|)$ by Lemma 14. Let $r = |H|$. Since H normalizes T, we may take $D'' := \sum_{h \in H} h \cdot D''$, and then:

$$\mathrm{g}\mathbf{lct}(\frac{1}{mr}D'') \leq \mathrm{g}\mathbf{lct}(\frac{1}{mr}|r\Sigma|) = \mathrm{g}\mathbf{lct}(\frac{1}{m}|\Sigma|).$$

Then we have:

$$\alpha_{HT}(L) = \inf_{m \in \mathbb{Z}_{>0}} \inf_{D \in |mL|^{HT}} \mathrm{g}\mathbf{lct}(\frac{1}{m}D).$$

In particular when $L = -K_X$ the left hand side is equal to $\mathrm{glct}_{HT}(X)$. □

As a consequence we may check Tian's criterion on the Chow quotient pair, for a symmetric T-variety. Let X be a symmetric T-variety. Let $\pi : X \dashrightarrow Y$ be the Chow quotient map by the torus action with boundary divisor $B := \sum_Z \frac{m_Z - 1}{m_Z} \cdot Z.$, see 10. Since H is in the normalizer of T, then its action descends to Y. The action of H extends to the whole of Y. Süß, showed that the global log canonical threshold of X with respect to HT coincides with that of the pair (Y, B) with respect to H:

Theorem 12 [22, Theorem 1.2]. *Let X be a symmetric log terminal Fano T-variety. Assume that the Chow quotient $\pi : X \dashrightarrow Y$ is surjective. Then:*

$$\mathrm{glct}_{HT}(X) = \min\{1, \mathrm{glct}_H(Y, B)\}.$$

5.2.3 Log canonical threshold bounds

We now turn to our examples. We begin by calculating a bound on the log canonical threshold of a candidate log pair for the Chow quotient pair of our examples. We do this by degenerating along a $\mathbb{C}^*$-action.

Let $Y = \mathbb{P}^2$, with projective coordinates x_1, x_2, x_3, and consider the boundary divisor $B_\gamma := \gamma \sum_{i=0}^{3} H_i$, where H_1, H_2, H_3 are the coordinate hyperplanes, and $H_0 = V(\sum_i x_i)$. Consider the subgroup $G \cong S_4$ of $\mathrm{Aut}(Y)$ permuting the hyperplanes $H_0, \ldots, H_3$.

We now provide the following lower bound on the global log canonical threshold of the pair $(\mathbb{P}^2, B_\gamma)$, by considering degenerations under a $\mathbb{C}^*$-action.

Lemma 16. *Consider a log pair $(\mathbb{P}^2, B_\gamma)$, where $B_\gamma = \gamma \sum_i H_i$. We then have:*

$$\mathrm{glct}_{S_4}(\mathbb{P}^2, B_\gamma) \geq \begin{cases} \infty, & \textit{for } \gamma \geq \frac{3}{4}; \\ \frac{2(1-\gamma)}{3-4\gamma}, & \textit{for } \frac{3}{4} \geq \gamma \geq \frac{1}{2}; \\ \frac{1}{3-4\gamma} & \textit{for } \frac{1}{2} \geq \gamma. \end{cases}$$

Proof.

To show $\mathrm{glct}_G(Y, B_\gamma) \geq \lambda$ it is sufficient to show that $B_\gamma + \lambda D$ is log canonical for any $D \in |-K_Y - B_\gamma|_{\mathbb{Q}}^G$. Fix such a D and take $P \in Y$. At most two of the H_i pass through P, so without loss of generality suppose H_0, H_3 do not. Modify $B_\gamma + \lambda D$ by removing any components supported at H_0, H_3, to obtain a divisor D'. Note $B_\gamma + \lambda D$ is log canonical at P if D' is globally log canonical. Note also that although D' may not be G-invariant, it is still invariant under the involution σ swapping x_1 and x_2. Finally note $D' \geq \gamma H_1 + \gamma H_2$.

Consider the $\mathbb{C}^*$-action $t \cdot [x_1 : x_2 : x_3] = [tx_1 : tx_2 : x_3]$. By (14), D' is log canonical if $D'_0 := \lim_{t \to 0} (t \cdot D')$ is log canonical. This $\mathbb{C}^*$-action commutes with σ, and so D'_0 is invariant under σ. Moreover is is clear that each component of D'_0 must be a line through the point $[0, 0, 1]$. By σ-invariance D'_0 must be of the form:

$$D'_0 = \gamma(H_1 + H_2) + aV(x_1 + x_2) + bV(x_1 - x_2) + \sum_i c_i(L_i + \sigma L_i)$$

Where $a + b + \sum_i 2c_i \leq 2\gamma + 3\lambda - 4\gamma\lambda$. This is a divisor of the form described in

Example 26. It is therefore log canonical iff the following inequalities hold:

$$
\begin{aligned}
2\gamma + 3\lambda - 4\gamma\lambda &< 2 \\
3\gamma - 4\gamma\lambda &\leq 1
\end{aligned}
$$

Basic manipulation of inequalities gives our bounds on the global threshold. □

Proof of Theorem 8. By Lemma 11 the Chow quotient pairs of $X^5_{1,2}, X^5_{1,3}$ by their torus action are $(\mathbb{P}^2, B_{1/2}), (\mathbb{P}^2, B_{2/3})$ respectively. By Lemma 16 and Theorem 12 then $\alpha_{S_4}(X^5_{1,2}), \alpha_{S_4}(X^5_{1,3}) > 5/6$. Apply Theorem 11. □

Proof of Theorem 9. By Lemma 11 the Chow quotient pair of W^6 by its effective torus action is $(\mathbb{P}^2, B_{1/2})$. By Lemma 16 and Theorem 12 then $\alpha_{S_4}(W^6) > 6/7$. Apply Theorem 11. □

Chapter 6

Conclusions and further work

In this brief final chapter we summarize the work in this thesis and describe how one might pursue further related results. As outlined in the introduction to this thesis, we set out with a goal of pushing further the understanding of canonical metrics on Fano T-varieties, and constructing examples as evidence of equivariant methods in this context.

We achieved this goal. Our results on Kähler-Ricci solitons in Chapter 3 provided new examples, and give new insights into modified K-stability in complexity one. In Chapter 4 we give an effective formula for an invariant defined precisely in terms of the existence of certain canonical metrics. Finally, we provide new examples of Kähler-Einstein metrics in complexity two, showing that equivariant methods are not restricted to actions of low complexity.

Further work

Kähler-Ricci solitons

All Fano toric orbifolds admit a Kähler-Ricci soliton in [59]. However, as shown in [42], not all complexity one Fano T-varieties do. Nevertheless all *smooth* examples so far have admitted a Kähler-Ricci soliton. This obviously leads to the question of whether it holds true in general. On the one hand it seems like the same approach as the toric case cannot work, as the proof for toric orbifolds seems very similar to the proof in the smooth case. The first step would be to calculate more examples and see if there exists a counterexample in higher dimensions.

R(X)

It is interesting that only special test configurations with negative classical Donaldson-Futaki character contribute to the infimum (4.2) for complexity one T-varieties, and it would be of interest to see if this held true in any more general context. If more divisorial polytopes are calculated for T-varieties in higher dimensions, $R(X)$ is easy to calculate from Theorem 7.

General arrangment varieties

The proof of the results in Chapter 5 relies on the bound for $\mathrm{glct}_{S_4}(\mathbb{P}^2, B_\gamma)$. This is the only obstruction in extending the results to higher dimensonal elements of the families: Taking limits along a $\mathbb{C}^*$-action in higher dimensions is not useful, as the resulting divisor may be more complicated. In one dimension higher, for example, the most we are able to say about the components of such a limit is that they are ruled surfaces intersecting at least at one common point.

With more knowledge or methods of calculating such alpha invariants in higher dimensions, we would obtain more Kähler-Einstein metrics and in particular they would be in higher complexity.

Appendix A

Threefold Data

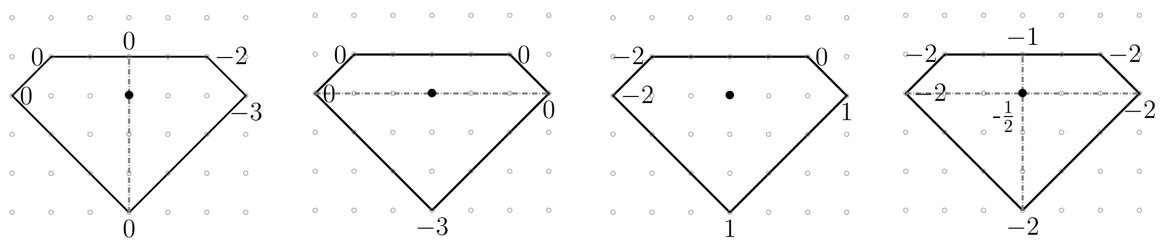
In this Appendix we include data needed for the results in Chapters 3 and 4. We include the key below for ease of use.

Key:

- "Φ" - the Fano divisorial polytope $\Phi : \Box \to \operatorname{Div}_{\mathbb{Q}} \mathbb{P}^1$, specified in diagram form as values on the base polytope $\Box$.
- "σ" - an automorphism which permutes the vertices of $\Box$ such that $\deg \Phi \circ \sigma = \deg \Phi$.
- "$g(\xi)$" - the integral $\int_{\Box} \langle u, v \rangle \cdot \deg \Phi(u) \cdot e^{\langle u, \xi \rangle} du$, a positive multiple of $\mathrm{DF}(X \times \mathbb{A}^1, e_2)$.
- "$R(X)$" - the calculated greatest lower bound on Ricci curvature (Tian's beta invariant) in Chapter 4.
- "$\xi \sim$ " - an approximation of the Kähler-Ricci soliton found in Chapter 3, representated an element of $N_{\mathbb{R}}$.
- "$\mathrm{Vert}(\Delta_y)$" - the degeneration polytope Δ_y is given by convex hull of the columns of this matrix.
- "$h_y(\xi)$" - a positive multiple of the Donaldson-Futaki invariant of the test cofiguration corresponding to Δ_y with $\mathbb{C}^*$-action given by $e_3 \in N'$.
- "$h_y(\xi_2) >$" - a lower bound on $h_y(\xi)$, found using real interval arithmetic.

Name:	2.30	**Description:**	Blow up of Q in a point

$\sigma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $\quad g(\xi) = \frac{1}{\xi_2^4} \cdot \left((2\,\xi_2^3 - 3\,\xi_2 - 3)e^{(4\,\xi_2)} + 12\,\xi_2 e^{(3\,\xi_2)} + 3\,\xi_2 + 3\right) e^{(-3\,\xi_2)}$, $\quad R(X) = 23/29$, $\quad \xi \sim (0, 0.51489)$

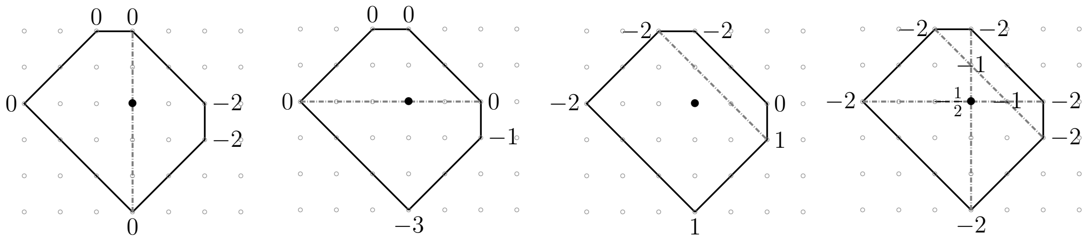


Φ_0 $\qquad$ Φ_1 $\qquad$ Φ_∞ $\qquad$ $\deg\Phi$

y	$\mathrm{Vert}(\Delta_y)$	$h_y(\xi_2)$	$h_y(\xi_2) >$
0	$\begin{pmatrix} 0 & 3 & -3 & 2 & -2 & 0 \\ -3 & 0 & 0 & 1 & 1 & 1 \\ 1 & -2 & 1 & -1 & 1 & 1 \end{pmatrix}$	$\frac{\left(\left(2\,\xi_2^3 - 3\,\xi_2 - 3\right)e^{(4\,\xi_2)} + 3\left(3\,\xi_2^2 + 2\right)e^{(3\,\xi_2)} - 3\,\xi_2 - 3\right)e^{(-3\,\xi_2)}}{3\,\xi_2^4}$	1.087
1	$\begin{pmatrix} -3 & 0 & 3 & 2 & 0 & -2 \\ 0 & -3 & 0 & 1 & 1 & 1 \\ 1 & -2 & 1 & 1 & 0 & 1 \end{pmatrix}$	$\frac{\left(\left(8\,\xi_2^3 + 6\,\xi_2^2 - 3\right)e^{(4\,\xi_2)} - 12\left(3\,\xi_2^2 - 3\,\xi_2 + 1\right)e^{(3\,\xi_2)} + 12\,\xi_2 + 15\right)e^{(-3\,\xi_2)}}{6\,\xi_2^4}$	2.178
∞	$\begin{pmatrix} -3 & 0 & 2 & 0 & 0 & -2 & 3 \\ 0 & -3 & 1 & 0 & 1 & 1 & 0 \\ -1 & 2 & 1 & -1 & -1 & -1 & 2 \end{pmatrix}$	$-\frac{\left(2\left(2\,\xi_2^3 - 3\,\xi_2 - 3\right)e^{(4\,\xi_2)} - 3\left(3\,\xi_2^2 - 12\,\xi_2 + 2\right)e^{(3\,\xi_2)} + 12\,\xi_2 + 12\right)e^{(-3\,\xi_2)}}{6\,\xi_2^4}$	0.4465
$\notin \{0, 1, \infty\}$	$\begin{pmatrix} 0 & 0 & -3 & 2 & 0 & -2 & 3 \\ 0 & -3 & 0 & 1 & 1 & 1 & 0 \\ -\frac{1}{2} & 1 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}$	$\frac{\left(\left(8\,\xi_2^3 + 6\,\xi_2^2 - 3\right)e^{(4\,\xi_2)} - 3\left(3\,\xi_2^2 - 2\right)e^{(3\,\xi_2)} - 6\,\xi_2 - 3\right)e^{(-3\,\xi_2)}}{6\,\xi_2^4}$	4.151

Name:	2.31	**Description:**	Blow up of Q in a line

$$\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad g(\xi_1, \xi_1) = \mapsto -\frac{\left(9\,\xi_1^2 - 8\left(\xi_1^2 - \xi_1\right)e^{(5\,\xi_1)} + \left(3\,\xi_1^2 - 5\,\xi_1 - 3\right)e^{(4\,\xi_1)} + 9\,\xi_1 + 3\right)e^{(-3\,\xi_1)}}{2\,\xi_1^4}, \quad R(X) = 23/27\,, \quad \xi \sim (0.28550, 0.28550)$$

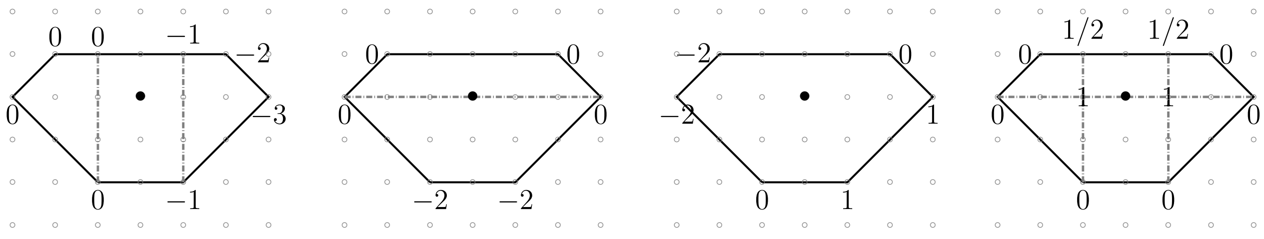


Φ_0 Φ_1 Φ_∞ $\deg \Phi$

y	$\mathrm{Vert}(\Delta_y)$	$h_y(\xi_2)$	$h_y(\xi_2) >$
0	$\begin{pmatrix} 0 & 1 & 2 & -3 & 0 & 2 & -1 \\ -3 & 0 & -1 & 0 & 2 & 0 & 2 \\ 1 & -1 & -1 & 1 & 1 & -1 & 1 \end{pmatrix}$	$\frac{\left(12\,\xi_2^2 + \left(4\,\xi_2^2 + 2\,\xi_2 - 3\right)e^{(4\,\xi_2)} + 10\,\xi_2 + 3\right)e^{(-3\,\xi_2)}}{8\,\xi_2^4}$	1.9509
1	$\begin{pmatrix} -3 & 0 & -1 & 2 & 0 & 2 & 0 \\ 0 & -3 & 2 & -1 & 2 & 0 & 1 \\ 1 & -2 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}$	$-\frac{(6\xi_2^2 - 16(\xi_2^2 - \xi_2)e^{(5\xi_2)} + (2\xi_2^2 - 12\xi_2 - 3)e^{(4\xi_2)} + 8\xi_2 + 3)e^{(-3\xi_2)}}{8\xi_2^4}$	1.9410
∞	$\begin{pmatrix} -3 & 0 & 0 & 2 & 0 & -1 & 2 \\ 0 & -3 & 2 & 0 & 0 & 2 & -1 \\ -1 & 2 & -1 & 1 & -1 & -1 & 2 \end{pmatrix}$	$\frac{(3\xi_2^2 - (3\xi_2^2 + 4\xi_2 + 2)e^{(4\xi_2)} + 4\xi_2 + 4e^{(5\xi_2)} - 4e^{(3\xi_2)} + 2)e^{(-3\xi_2)}}{4\xi_2^4}$	0.4632
$\notin \{0, 1, \infty\}$	$\begin{pmatrix} 0 & 0 & -3 & 2 & 2 & 1 & 0 & -1 & 0 \\ 0 & -3 & 0 & -1 & 0 & 0 & 2 & 2 & 1 \\ -\frac{1}{2} & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}$	$\frac{(6\xi_2^2 + 4(2\xi_2^2 - 2\xi_2 + 1)e^{(5\xi_2)} - (2\xi_2^2 - 3\xi_2 + 2)e^{(4\xi_2)} + 5\xi_2 - 4\,e^{(3\,\xi_2)} + 2)e^{(-3\,\xi_2)}}{4\,\xi_2^4}$	4.370

Name:	3.18	**Description:**	Blow up of Q in a point and a conic

$$\sigma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad g(0,\xi_2) = -\frac{\left(4\,\xi_2^2-\left(3\,\xi_2^3+\xi_2^2-6\,\xi_2-6\right)e^{(3\,\xi_2)}-24\,\xi_2 e^{(2\,\xi_2)}-6\right)e^{(-2\,\xi_2)}}{2\,\xi_2^4}, \quad R(X) = 48/55\,, \qquad \xi \sim (0, 0.37970)$$

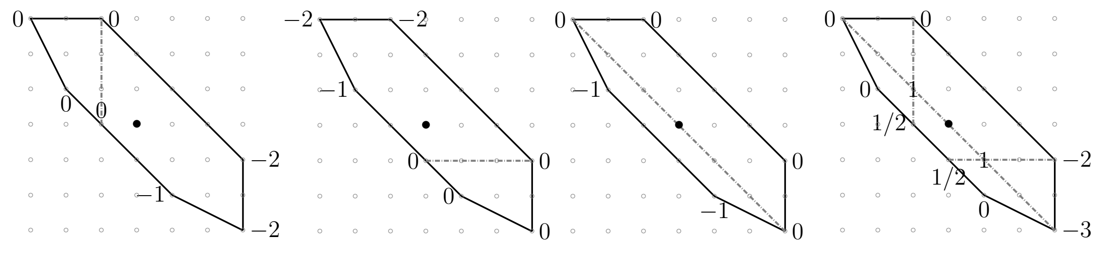


Φ_0 Φ_1 Φ_∞ $\deg\Phi$

y	$\mathrm{Vert}(\Delta_y)$	$h_y(\xi_2)$	$h_y(\xi_2) >$
0	$\begin{pmatrix} -1 & 3 & -3 & 1 & 1 & -1 & 2 & -2 \\ -2 & 0 & 0 & -2 & 1 & 1 & 1 & 1 \\ 1 & -2 & 1 & 0 & 0 & 1 & -1 & 1 \end{pmatrix}$	$\frac{\left(3\,\xi_2^2+2\left(\xi_2^3-3\,\xi_2-3\right)e^{(3\,\xi_2)}+6\left(3\,\xi_2^2+2\right)e^{(2\,\xi_2)}-6\right)e^{(-2\,\xi_2)}}{6\,\xi_2^4}$	0.3681
1	$\begin{pmatrix} -3 & -1 & 3 & 2 & 1 & 1 & -2 & -1 \\ 0 & -2 & 0 & 1 & 1 & -2 & 1 & 1 \\ 1 & -1 & 1 & 1 & \frac{1}{2} & -1 & 1 & \frac{1}{2} \end{pmatrix}$	$\frac{\left(12\,\xi_2^2-\left(14\,\xi_2^3+15\,\xi_2^2-6\right)e^{(3\,\xi_2)}+24\left(3\,\xi_2^2-3\,\xi_2+1\right)e^{(2\,\xi_2)}+6\,\xi_2-30\right)e^{(-2\,\xi_2)}}{12\,\xi_2^4}$	2.516
∞	$\begin{pmatrix} -3 & -1 & 1 & 2 & -1 & 1 & 1 & -1 & -2 & 3 \\ 0 & -2 & -2 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ -1 & 1 & 2 & 1 & -1 & 0 & 0 & -1 & -1 & 2 \end{pmatrix}$	$\frac{\left(9\,\xi_2^2-2\left(\xi_2^3-3\,\xi_2-3\right)e^{(3\,\xi_2)}+6\left(\xi_2^2-6\,\xi_2+1\right)e^{(2\,\xi_2)}-12\right)e^{(-2\,\xi_2)}}{6\,\xi_2^4}$	1.013
$\notin \{0,1,\infty\}$	$\begin{pmatrix} -1 & -1 & -3 & 2 & 1 & 3 & 1 & -2 & -1 & 1 \\ 0 & -2 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & -2 \\ 0 & 1 & 1 & 1 & \frac{1}{2} & 1 & 0 & 1 & \frac{1}{2} & 1 \end{pmatrix}$	$\frac{\left(12\,\xi_2^2+\left(14\,\xi_2^3+15\,\xi_2^2-6\right)e^{(3\,\xi_2)}-12\left(2\,\xi_2^2-1\right)e^{(2\,\xi_2)}-6\,\xi_2-6\right)e^{(-2\,\xi_2)}}{12\,\xi_2^4}$	3.946

Name:	3.21	**Description:**	Blow up of $\mathbb{P}^1 \times \mathbb{P}^2$ in curve of bidegree $(2,1)$.

$$\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad g(\xi_1,\xi_1) = \frac{\left(\xi_1^3 - \xi_1^2 + 4\left(\xi_1^2 - \xi_1\right)e^{(3\,\xi_1)} + 6\,(2\,\xi_1 - 1)e^{\xi_1} - 2\,\xi_1 + 6\right)e^{(-\xi_1)}}{\xi_1^4}, \quad R(X) = 76/97\,, \qquad \xi \sim (-0.69622, -0.69622)$$

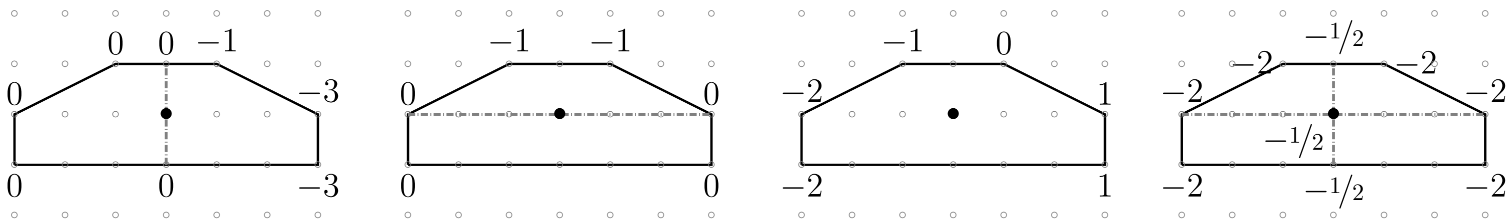


Φ_0 Φ_1 Φ_∞ $\deg \Phi$

y	$\mathrm{Vert}(\Delta_y)$	$h_y(\xi_2)$	$h_y(\xi_2) \in$
0	$\begin{pmatrix} -2 & 1 & -3 & 3 & 1 & -1 & 3 & -1 & 0 \\ 1 & -1 & 3 & -1 & -2 & 3 & -3 & 0 & -1 \\ 1 & -1 & 1 & -1 & 0 & 1 & -1 & 1 & 0 \end{pmatrix}$	$-\frac{\left(\xi_2^3 - \xi_2^2 - 2\left(2\,\xi_2^2 - 4\,\xi_2 + 3\right)e^{\xi_2} - 2\,\xi_2 + 6\right)e^{(-\xi_2)}}{2\,\xi_2^4}$	0.6910
1	$\begin{pmatrix} -3 & -2 & 3 & 1 & 3 & -1 & -1 & 0 & -1 \\ 3 & 1 & -3 & -2 & -1 & 1 & 3 & -1 & 0 \\ -1 & 0 & 1 & 1 & 1 & -1 & -1 & 1 & 0 \end{pmatrix}$	$-\frac{\left(\xi_2^3 - \xi_2^2 - 2\left(2\,\xi_2^2 - 4\,\xi_2 + 3\right)e^{\xi_2} - 2\,\xi_2 + 6\right)e^{(-\xi_2)}}{2\,\xi_2^4}$	0.6910
∞	$\begin{pmatrix} -3 & -2 & -1 & 3 & 1 & 3 & 0 & -1 \\ 3 & 1 & 3 & -3 & -2 & -1 & -1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}$	$\frac{\left(5\,\xi_2^3 + 9\,\xi_2^2 + 6\left(8\,\xi_2^2 - 4\,\xi_2 + 1\right)e^{(3\,\xi_2)} - 24\left(6\,\xi_2^2 - 8\,\xi_2 + 5\right)e^{\xi_2} - 66\,\xi_2 + 114\right)e^{(-\xi_2)}}{24\,\xi_2^4}$	0.6910
∞	$\begin{pmatrix} -1 & -3 & -2 & 0 & 3 & 1 & -1 & 3 & 1 & -1 \\ 1 & 3 & 1 & -1 & -1 & -1 & 3 & -3 & -2 & 0 \\ 0 & 1 & 1 & \frac{1}{2} & 1 & 0 & 1 & 1 & 1 & \frac{1}{2} \end{pmatrix}$	$-\frac{\left(19\,\xi_2^3 - 33\,\xi_2^2 - 6\left(8\,\xi_2^2 - 4\,\xi_2 + 1\right)e^{(3\,\xi_2)} + 24\left(2\,\xi_2^2 - 1\right)e^{\xi_2} + 18\,\xi_2 + 30\right)e^{(-\xi_2)}}{24\,\xi_2^4}$	3.907

Name:	3.22	**Description:**	Blow up of $\mathbb{P}^1 \times \mathbb{P}^2$ in $\{0\} \times \mathbb{P}^2$

$$\sigma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad g(0,\xi_2) = \frac{\left(9\,\xi_2^3+9\,\xi_2^2+\left(\xi_2^3+3\,\xi_2^2-24\right)e^{(2\,\xi_2)}+24\,(\xi_2+1)e^{\xi_2}\right)e^{(-\xi_2)}}{2\,\xi_2^4}, \quad R(X) = 40/49\,, \quad \xi \sim (0, 0.91479)$$

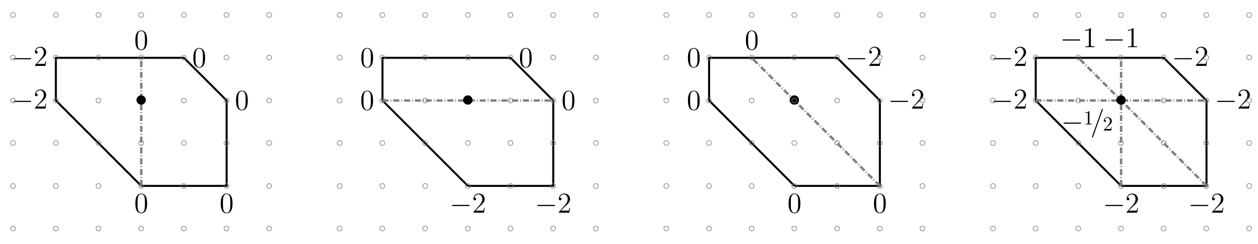


Φ_0 Φ_1 Φ_∞ $\deg \Phi$

y	$\mathrm{Vert}(\Delta_y)$	$h_y(\xi_2)$	$h_y(\xi_2) >$
0	$\begin{pmatrix} 3 & -3 & 3 & -3 & 1 & 0 & 0 & -1 \\ 0 & 0 & -1 & -1 & 1 & -1 & 1 & 1 \\ -2 & 1 & -2 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$	$\frac{9\,\xi_2^2+\left(\xi_2^3+3\,\xi_2^2-24\right)e^{\xi_2}+24\,\xi_2+24}{3\,\xi_2^4}$	0.8323
1	$\begin{pmatrix} -3 & -1 & -3 & 3 & 1 & 3 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & -1 & -1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}$	$-\frac{\left(27\,\xi_2^3+\left(\xi_2^3-24\,\xi_2-96\right)e^{(2\,\xi_2)}+24\left(3\,\xi_2^2+5\,\xi_2+4\right)e^{\xi_2}\right)e^{(-\xi_2)}}{12\,\xi_2^4}$	2.164
∞	$\begin{pmatrix} -3 & -3 & 3 & 1 & -1 & 0 & 0 & 0 & 3 \\ 0 & -1 & 0 & 1 & 1 & 0 & -1 & 1 & -1 \\ -1 & -1 & 2 & 1 & 0 & -1 & -1 & 0 & 2 \end{pmatrix}$	$\frac{9\,\xi_2^2+\left(\xi_2^3+3\,\xi_2^2-24\right)e^{\xi_2}+24\,\xi_2+24}{6\,\xi_2^4}$	0.4161
$\notin \{0,1,\infty\}$	$\begin{pmatrix} 0 & -3 & -3 & 3 & 1 & 0 & -1 & 3 & 0 \\ 0 & -1 & 0 & -1 & 1 & 1 & 1 & 0 & -1 \\ -\frac{1}{2} & 1 & 1 & 1 & 1 & \frac{1}{2} & 1 & 1 & -\frac{1}{2} \end{pmatrix}$	$-\frac{\left(27\,\xi_2^3-\left(5\,\xi_2^3+18\,\xi_2^2+24\,\xi_2-48\right)e^{(2\,\xi_2)}+6\left(3\,\xi_2^2-4\,\xi_2-8\right)e^{\xi_2}\right)e^{(-\xi_2)}}{12\,\xi_2^4}$	3.419

Name:	3.24	**Description:**	Blow up of W in $(0:0:1:*:*:0)$.

$\sigma = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$, $\quad g_1(0,\xi_2) = \xi_2 \mapsto \frac{2\left(\left(\xi_2^2-2\right)e^{(3\,\xi_2)}-2\,\xi_2+4\,e^{(2\,\xi_2)}-2\right)e^{(-2\,\xi_2)}}{\xi_2^3}$, $\quad R(X) = 21/25$, $\quad \xi \sim (0, 0.43475)$

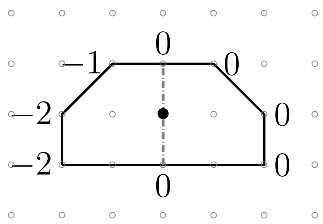


Φ_0 $\quad$ Φ_1 $\quad$ Φ_∞ $\quad$ $\deg \Phi$

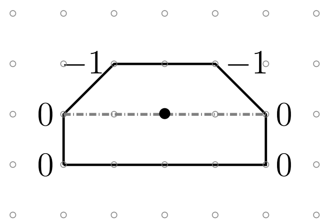
y	$\mathrm{Vert}(\Delta_y)$	$h_y(\xi_2)$	$h_y(\xi_2) >$
0	$\begin{pmatrix} -3 & -2 & 2 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -3 & 1 \\ 1 & 1 & -1 & -2 & 1 & 1 \end{pmatrix}$	$\frac{2\left((\xi_2-1)e^{(2\,\xi_2)}+\xi_2+1\right)e^{(-2\,\xi_2)}}{\xi_2^3}$	0.8793
1	$\begin{pmatrix} -3 & -2 & 2 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -3 & 1 \\ 1 & 1 & -1 & -2 & 1 & 1 \end{pmatrix}$	$-\frac{\left(12\,\xi_2^2-\left(7\,\xi_2^3+3\,\xi_2^2-6\,\xi_2+6\right)e^{(3\,\xi_2)}+12\left(2\,\xi_2^2-2\,\xi_2+1\right)e^{(2\,\xi_2)}+12\,\xi_2-6\right)e^{(-2\,\xi_2)}}{6\,\xi_2^4}$	1.2944
∞	$\begin{pmatrix} -3 & -2 & 2 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -3 & 1 \\ 1 & 1 & -1 & -2 & 1 & 1 \end{pmatrix}$	$\frac{2\left((\xi_2-1)e^{(2\,\xi_2)}+\xi_2+1\right)e^{(-2\,\xi_2)}}{\xi_2^3}$	0.8793
$\notin \{0,1,\infty\}$	$\begin{pmatrix} -3 & -2 & 2 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -3 & 1 \\ 1 & 1 & -1 & -2 & 1 & 1 \end{pmatrix}$	$\frac{\left(12\,\xi_2^2+\left(7\,\xi_2^3+3\,\xi_2^2-6\,\xi_2+6\right)e^{(3\,\xi_2)}+12\,\xi_2-12\,e^{(2\,\xi_2)}+6\right)e^{(-2\,\xi_2)}}{6\,\xi_2^4}$	3.0544

Name: 4.8 **Description:** Blow up of $(\mathbb{P}^1)^3$ in a curve.

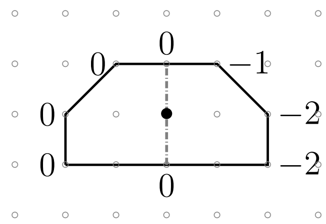
$$\sigma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad g(0,\xi_2) = \frac{\left(4\,\xi_2^3+4\,\xi_2^2+\left(\xi_2^3+\xi_2^2-2\,\xi_2-6\right)e^{(2\,\xi_2)}+2\,(4\,\xi_2+3)e^{\xi_2}\right)e^{(-\xi_2)}}{\xi_2^4}, \quad R(X) = 76/89\,, \quad \xi \sim (0, 0.62431)$$



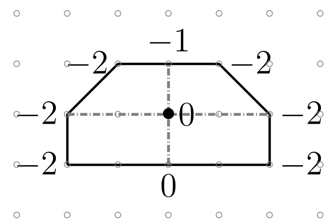
Φ_0



Φ_1



Φ_∞

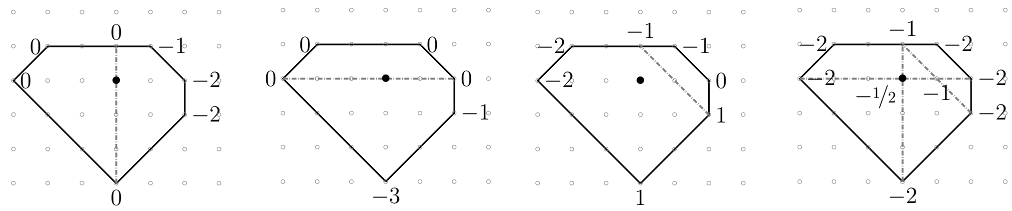


$\deg\Phi$

y	$\mathrm{Vert}(\Delta_y)$	$h_y(\xi_2)$	$h_y(\xi_2) >$
0	$\begin{pmatrix} -3 & -2 & 2 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -3 & 1 \\ 1 & 1 & -1 & -2 & 1 & 1 \end{pmatrix}$	$\frac{4\,\xi_2^2+\left(\xi_2^3+\xi_2^2-2\,\xi_2-6\right)e^{\xi_2}+8\,\xi_2+6}{2\,\xi_2^4}$	0.6636
1	$\begin{pmatrix} -3 & -2 & 2 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -3 & 1 \\ 1 & 1 & -1 & -2 & 1 & 1 \end{pmatrix}$	$\frac{4\,\xi_2^2+\left(\xi_2^3+\xi_2^2-2\,\xi_2-6\right)e^{\xi_2}+8\,\xi_2+6}{2\,\xi_2^4}$	0.6636
∞	$\begin{pmatrix} -3 & -2 & 2 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -3 & 1 \\ 1 & 1 & -1 & -2 & 1 & 1 \end{pmatrix}$	$-\frac{\left(4\,\xi_2^3+\left(\xi_2^3-6\,\xi_2-12\right)e^{(2\,\xi_2)}+6\,\left(2\,\xi_2^2+3\,\xi_2+2\right)e^{\xi_2}\right)e^{(-\xi_2)}}{3\,\xi_2^4}$	1.103
$\notin \{0,1,\infty\}$	$\begin{pmatrix} -3 & -2 & 2 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -3 & 1 \\ 1 & 1 & -1 & -2 & 1 & 1 \end{pmatrix}$	$-\frac{\left(4\,\xi_2^3-\left(2\,\xi_2^3+3\,\xi_2^2-6\right)e^{(2\,\xi_2)}-6\,(\xi_2+1)e^{\xi_2}\right)e^{(-\xi_2)}}{3\,\xi_2^4}$	2.442

Name: 3.23 **Description:** Blow up of Q in a point and strict transform of line passing through

$R(X) = 168/221$, $\xi \sim (0.26618, 0.67164)$



Φ_0 Φ_1 Φ_∞ $\deg \Phi$

y	$\mathrm{Vert}(\Delta_y)$	$h_y(\xi_1, \xi_2) >$
0	$\begin{pmatrix} 0 & 2 & 1 & -3 & -2 & 0 & 1 & 0 & 2 \\ -3 & -1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & -1 & -1 & 1 & 1 & 1 & 0 & 0 & -1 \end{pmatrix}$	1.2766
1	$\begin{pmatrix} -3 & 0 & -2 & 2 & 1 & 2 & 0 \\ 0 & -3 & 1 & 0 & 1 & -1 & 1 \\ 1 & -2 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}$	1.8402
∞	$\begin{pmatrix} -3 & 0 & 0 & 2 & 1 & 0 & -2 & 0 & 2 \\ 0 & -3 & 0 & 0 & 1 & 1 & 1 & 1 & -1 \\ -1 & 2 & -1 & 1 & 0 & -1 & -1 & 0 & 2 \end{pmatrix}$	0.1005
$\notin \{0, 1, \infty\}$	$\begin{pmatrix} 0 & 0 & -3 & 2 & 1 & 1 & 2 & -2 & 0 \\ 0 & -3 & 0 & -1 & 1 & 0 & 0 & 1 & 1 \\ -\frac{1}{2} & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}$	3.4443

Appendix B

SageMath Code

Here we include the library of functions written to obtain the results in Chapter 3. We give a brief description of the purpose of each function, and an explanation on the nature of inputs and outputs.

First we have some auxillary functions. The function projection returns maps used to descend from a polytope P to a facet F in the recursive integration algorithm described in section 3.3.

```
def projection(P,F):
    A = F.as_polyhedron().equations()[0].A()
    b = F.as_polyhedron().equations()[0].b()
    A = list(A)
    B = vector(QQ,A)
    B = denominator(B)*B
    A = list(B)
    B = matrix(ZZ,B)
    K = B.right_kernel()
    if P.dimension() == 1:
        K = matrix(ZZ,[0])
    else:
        K = matrix(ZZ,K.basis())
    return [K,A]
```

The functions acoeff and bcoeff calculate specific coefficients used at points where the Barvinok algorithm terminates. They take as input a face F of a polytope P, together with a vector $c \in N_{\mathbb{R}}$ such that Y.

```
def acoeff(F,c):
    v_0 = F.vertices()[0].vector(); v_0
    return exp(c.dot_product(v_0))

def bcoeff(F,v):
    u_0 = F.vertices()[0].vector()
    return v.dot_product(u_0)
```

The function relvolume is simply a modification of the SageMath polytope volume method so that we may use it to calculate relative volumes of the faces of P.

```
def relvolume(P):
    if P.dimension() == 0:
        return 1
    else:
        return P.volume()
```

The function intexp calculates the integral of $e^{\langle c,x\rangle}$ over the polyhedron P recursively using the Barvinok method. It takes as input a polyhedron P, a sufficiently general vector L, and a vector c for the integrand. To optimize performance values obtained at faces are cached along the way.

```
def intexp(P,L,c,face=None, cache=None):
    if face is None: face = tuple(range(P.n_vertices()))
    if cache is None:
        cache = {}
    if cache.has_key(frozenset(face)):
        return cache[frozenset(face)]
    I = 0
    if c.is_zero():
        cache[face]= relvolume(P)
        return relvolume(P)
    else:
        for F in P.faces(P.dimension() -1):
            ProjMatrix,A = projection(P,F)
            n_F = -F.ambient_Hrepresentation()[0].A()
            n_F = n_F/gcd(n_F)
            coeff = acoeff(F,c)*(1/L.dot_product(c))*(L.dot_product(n_F))
            c_F = ProjMatrix*c
            L_F = ProjMatrix*L
            v_0 = F.vertices()[0].vector(); v_0
            Vert = [ProjMatrix.transpose().solve_right(v.vector() - v_0) for v in F.vertices() ]
            face_F = tuple([face[i] for i in F._ambient_Vrepresentation_indices])
            P_F = Polyhedron(vertices = Vert, backend="cdd")
            I = I + coeff*intexp(P_F,L_F,c_F,face_F,cache)
    cache[frozenset(face)]=I
    return I
```

The function intxexp calculates the integral of $\langle x, v\rangle \cdot e^{\langle c,x\rangle}$ over the polyhedron P recursively using the Barvinok method. The input is a polyhedron P, a sufficiently general vector L, and vectors c and v for the integrand. To optimize performance values obtained at faces are again cached along the way.

```
def intxexp(P,L,c,v,face=None,cache=None,cache2=None):
    if face is None: face = tuple(range(P.n_vertices()))
    if cache2 is None:
        cache2 = {}
    if cache is None:
        cache = {}
```

```
    if cache.has_key(frozenset(face)):
        return cache[frozenset(face)]
    I = 0
    if c.is_zero():
        if v.is_zero() : return 0
        for T in list(P.triangulate()):
            T = [P.Vrepresentation()[ZZ(t)].vector() for t in T]
            bary = sum(T)/len(T)
            T = Polyhedron(vertices = T)
            I = I + T.volume()*bary.dot_product(v)
    else:
        for F in P.faces(P.dimension() -1):
                ProjMatrix = projection(P,F)
                L_F = ProjMatrix[0].insert_row(0,vector(ZZ,ProjMatrix[1])).transpose().solve_right(L)[1:]
                c_F = ProjMatrix[0]*c
                v_F = ProjMatrix[0]*v
                v_0 = F.vertices()[0].vector(); v_0
                Vert = [ProjMatrix[0].transpose().solve_right(w.vector() - v_0) for w in F.vertices() ]
                face_F = tuple([face[i] for i in F._ambient_Vrepresentation_indices])
                P_F = Polyhedron(vertices = Vert,backend="cdd")
                n_F = -F.ambient_Hrepresentation()[0].A()
                n_F = n_F/gcd(n_F)
                I = I + (L.dot_product(n_F)/L.dot_product(c))*acoeff(F,c)*(intxexp(P_F,L_F,c_F,v_F,face_F,
                     ↪ cache,cache2) + bcoeff(F,v)*intexp(P_F,L_F,c_F,face_F,cache2)) - L.dot_product(
                     ↪ n_F)*(v.dot_product(L)/(L.dot_product(c)^2))*acoeff(F,c)*intexp(P_F,L_F,c_F,face_F
                     ↪ ,cache2)
    cache[frozenset(face)]=I
    return I
```

The function degenerations returns the list of degeneration polytopes (normal or otherwise) of a T-variety specified by a divisorial polytope. It takes inputs L and B, where L is a list of matrices and B is the base polytope, referred to as $\square$ in this thesis. Each matrix in L represents a piecewise affine function on B in the following way: Suppose v_j are the columns of M in the list L. We then form the piecewise affine function:

$$\psi_M(x_1, \ldots, x_n) := \min_j \{(1, x_1, x_2, \ldots, x_n) \cdot v_j\}$$

```
def degenerations(L,B):
    dim = L[0].nrows()
    genericfunction = matrix(QQ, dim, 1, lambda i, j: 0)
    L = L +[genericfunction]
    def shift(A,r):
        shift = matrix(QQ, A.transpose().nrows() , 1, lambda i, j: r);
        zeroes = matrix(QQ, A.transpose().nrows(), A.nrows() -1 , lambda i, j: 0);
        return A + shift.augment(zeroes).transpose()
    def minsum(L):
        msCurrent = L[0]
        def minsum_step(A1,A2):
```

```
            C = matrix(QQ,A2.nrows(),1,lambda i, j: 0)
            for c in A1.columns():
                for d in A2.columns():
                    C = C.augment(d+c);
            return C.delete_columns([0])
        for l in L[1:]:
            msCurrent = minsum_step(msCurrent,l)
        return msCurrent
    def degenerations_step(L,k,B):
        up = shift(L[k],1)
        base = matrix(QQ, B.inequalities_list()[:]);
        low = L[:]; low.pop(k);
        low = minsum(low).transpose();
        up = up.transpose()
        base = base.augment(matrix(QQ, base.nrows() , 1, lambda i, j: 0)).transpose()
        up = up.augment(matrix(QQ, up.nrows(), 1, lambda i, j: -1)).transpose()
        low = low.augment(matrix(QQ, low.nrows(), 1, lambda i, j: 1)).transpose()
        low = shift(low,1)
        P = Polyhedron(ieqs = [list(v) for v in up.augment(low.augment(base)).columns()])
        return P;
    Polyhedra = [];
    for k in range(len(L)):
        Polyhedra = Polyhedra + [degenerations_step(L,k,B)];
    return Polyhedra
```

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