Stability of varieties with a torus action

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Stability of varieties with a torus action

Goal of this project: Use equivariant methods to further the understanding of canonical metrics on T-varieties.

Section 1 - Introduction

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- $\mathrm{Ric}(\omega) \in \mathcal{A}^{1,1}(X)$ given by $(u, v) \mapsto r(J(u), v)$
- Kähler-Einstein:

$$\exists \lambda \in \mathbb{R} \quad \mathrm{Ric}(\omega) = \lambda \omega.$$

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Looking for KE metrics:

1. $c_1(X) := \frac{1}{2\pi}[\mathrm{Ric}(\omega)] \in H^2(X, \mathbb{R})$ is independent of ω.

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   - $c_1(X) < 0$ (General type);
   - $c_1(X) = 0$ (Calabi-Yau);
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   - $c_1(X) < 0$ (General type);
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   - $c_1(X) > 0$ (Fano).
3. Obstructions only occur in the Fano case, existence corresponds to *K-stability*

Section 1 - Introduction

Following **Datar and Szkelyhidi '15**:

Let X be a Fano manifold and pick some Kähler metric $\omega_0 \in 2\pi c_1(X)$.

Definition

A twisted Kähler-Ricci soliton is a triple (ω, v, t) where $t \in [0, 1]$, $\omega \in 2\pi c_1(X)$ is a Kähler metric, and v is a holomorphic vector field, such that:

$$\mathrm{Ric}(\omega) - L_v\omega = t\omega + (1-t)\omega_0.$$

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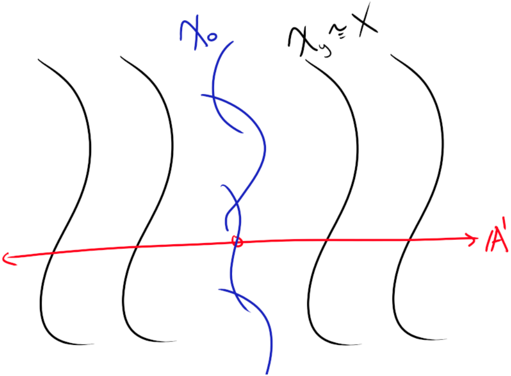
Suppose a reductive group G acts effectively on X.

Special test configuration $(\mathcal{X}, \mathcal{L}, w)$:



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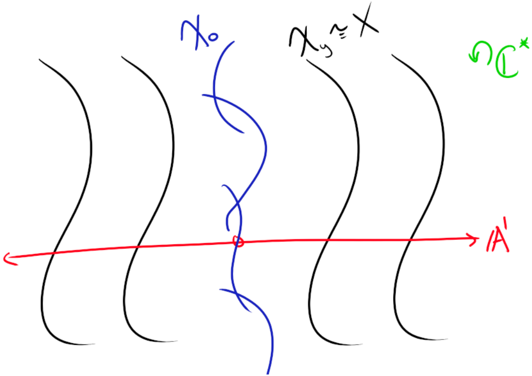

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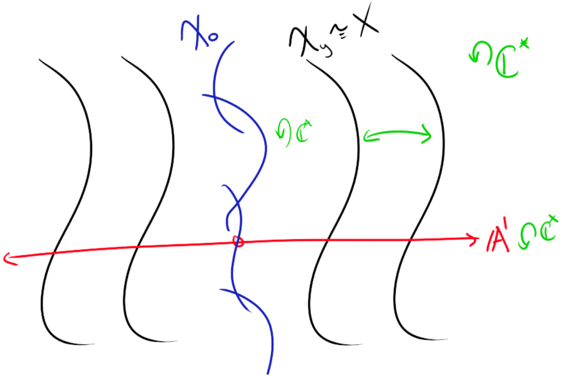

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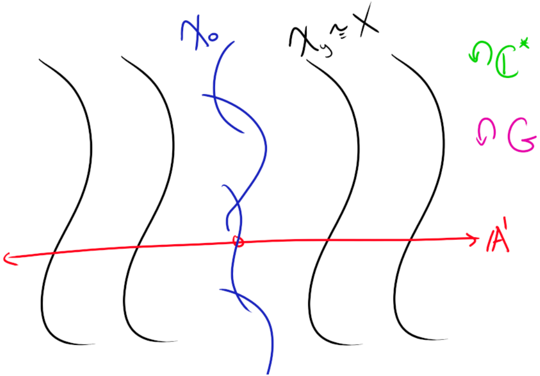

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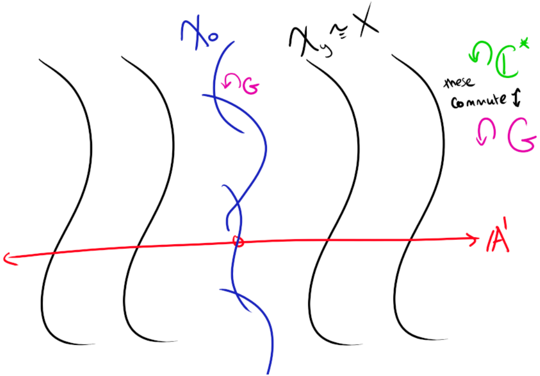

Section 1 - Introduction

Following **Datar and Szkelyhidi '15**:

- Donaldson-Futaki character $\mathrm{DF}_{t,v}(\mathcal{X}) : \mathcal{H} \to \mathbb{R}$.
- (X, v, t) K-semistable if $\mathrm{DF}_{t,v}(\mathcal{X}) \geq 0$ for all special test configurations.
- (X, v, t) is K-stable if, in addition, $\mathrm{DF}_{t,v}(\mathcal{X}) = 0$ holds precisely when $\mathcal{X} \cong X \times \mathbb{A}^1$.

Section 1 - Introduction

Theorem (Berman and Witt-Nyström '14)

$\exists$ a Kähler-Ricci soliton (ω, v) on $X \implies (X, v)$ is K-stable.

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$\exists$ *a Kähler-Ricci soliton* (ω, v) *on* $X \implies (X, v)$ *is* K*-stable.*

Theorem (Datar and Székelyhidi '15)

- (X, v) K*-stable* $\implies \exists$ *a Kähler-Ricci soliton* (ω, v) *on* X*.*
- (X, v, t) K*-semistable* $\implies \exists$ *twisted Kähler-Ricci soliton* (ω_s, v, s) *on* X *for all* $0 \leq s < t$*.*

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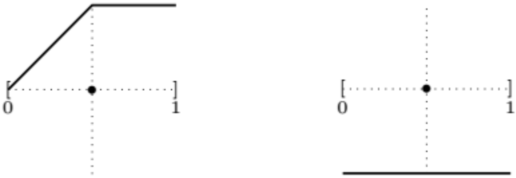
- Complexity is $\dim X - \dim T$
- Character and cocharacter lattices M, N respectively.
- Momentum map $\mu : X \to \Box \subset M_{\mathbb{R}}$.
- Chow quotient $X \to Y$, boundary $B := \sum_Z \frac{m_Z - 1}{m_Z} \cdot Z$

Section 1 - Introduction

Following **Ilten and Süß '17**:

- Fano complexity one T-varieties $\leftrightarrow$ divisorial polytopes on $Y = \mathbb{P}^1$
- Divisorial polytope data: $\Phi = (\square, \Phi_{y_1}, \dots, \Phi_{y_r})$, where $\Phi_i : \square \to \mathbb{Q}$.



$\{0\}$ $\{\infty\}$

Section 1 - Introduction

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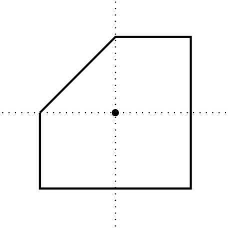
- $\mathcal{X}_0$ for non-product given by $\Delta_y \subset M_{\mathbb{R}} \times \mathbb{R}$.



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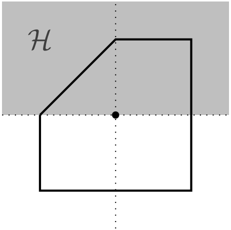

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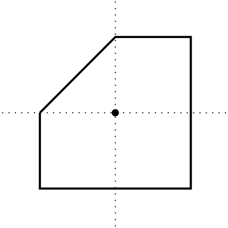
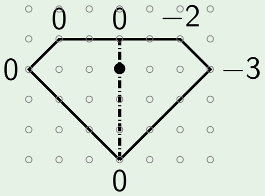


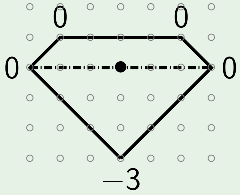
Central fiber Δ_0

- $\mathcal{H} = N_{\mathbb{R}} \times \mathbb{Q}^+$.
- $\mathrm{DF}_{t,v}(\mathcal{X}) = \mathrm{DF}_{t,v}(\Delta_y)$.

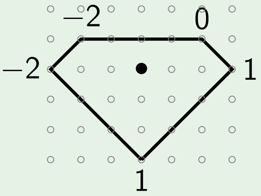
Example (2.30: Blow up of a Quadric in a point)



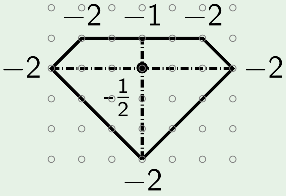
Φ_0



Φ_1



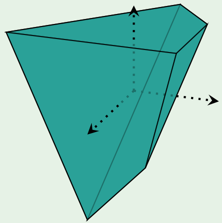
Φ_∞



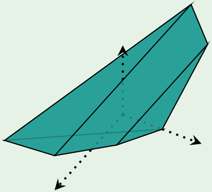
$\deg \Phi$

Section 1 - Introduction

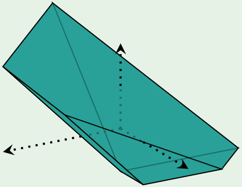
Example (Central fibres for threefold 2.30)



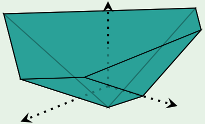
(a) Δ_0



(b) Δ_1



(c) Δ_∞



(d) Δ_{gen}

Section 1 - Introduction

Product configurations:

- formula for $\mathrm{DF}_{t,v}(X \times \mathbb{A}^1)$ in terms of Φ.
- $\mathcal{H} = N_{\mathbb{Q}}$.

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  - Non-product case $\mathrm{DF}_{t,v}(\Delta_y)$;
  - Product case $\mathrm{DF}_{t,v}(X \times \mathbb{A}^1)$.

Section 2 - Results in complexity one

Theorem (C and Süß '18)

The Fano threefolds 2.30, 2.31, 3.18, 3.22 ,3.23, 3.24, 4.8, *from Mori and Mukai's list admit a non-trivial Kähler-Ricci soliton.*

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Combined with **Ilten and Süß '17** and **Przyjalkowski, Cheltsov, and K. A. Shramov '19**, *all* smooth complexity one T-threefolds admit a Kähler-Ricci soliton.

Section 2 - Kähler-Ricci solitons in complexity one

Method:

First we find a candidate for v.

1. Choose $\sigma \in \mathrm{GL}(2)$ such that:

$$\mathrm{DF}_{\sigma^*(v)}(X \times \mathbb{A}^1) = \mathrm{DF}_v(X \times \mathbb{A}^1) \circ \sigma^*,$$

   and a basis e_1, e_2 of N such that $N_{\mathbb{R}}^{\sigma^*} = \mathbb{R}e_2$.

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2. Obtain a closed form for $g(\xi) := \mathrm{vol}(X) \cdot \mathrm{DF}_{\xi e_2}(X \times \mathbb{A}^1)(e_2)$.

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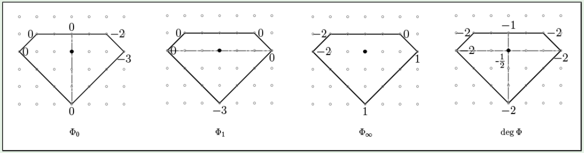
2. Obtain a closed form for $g(\xi) := \mathrm{vol}(X) \cdot \mathrm{DF}_{\xi e_2}(X \times \mathbb{A}^1)(e_2)$.

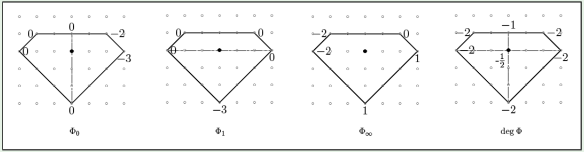
3. Prove that the root of $g(\xi)$ lies in some interval D.

Example



$$g(\xi) = \frac{1}{\xi^4} \cdot \left(\left(2\,\xi^3 - 3\,\xi - 3\right) e^{(4\,\xi)} + 12\,\xi e^{(3\,\xi)} + 3\,\xi + 3 \right) e^{(-3\,\xi)}.$$

Example



We find:

$$0.514 < \xi < 0.515.$$

Section 2 - Kähler-Ricci solitons in complexity one

Method:

Then we check positivity of the $\mathrm{DF}_{t,v}(\Delta_y)$.

1. Extend our basis to a basis for $N_{\mathbb{Q}} \times \mathbb{Q}$ by adjoining $e_3 = (0, 0, 1)$.

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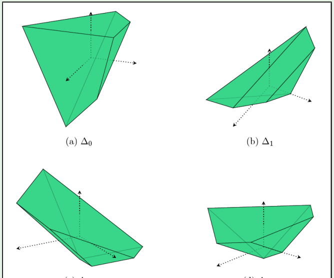
Section 2 - Kähler-Ricci solitons in complexity one

Example



(a) Δ_0 (b) Δ_1 (c) Δ_∞ (d) Δ_{gen}

$$1.087 < h_0(\xi) < 1.458, \qquad 2.178 < h_1(\xi) < 2.470,$$

$$0.446 < h_\infty(\xi) < 0.827, \qquad 4.151 < h_{gen}(\xi) < 4.309.$$

Section 3 - R(X) in complexity one

Recall the invariant

$$R(X) := \sup\{t \in [0,1] \mid \exists \text{ a twisted KE metric } (\omega, t) \text{ on X}\}.$$

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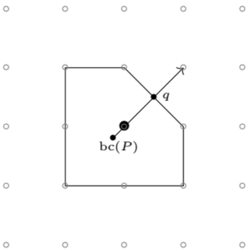
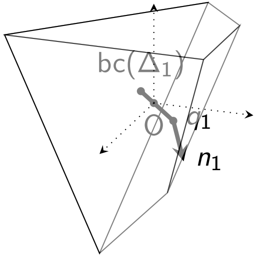
Toric situation:

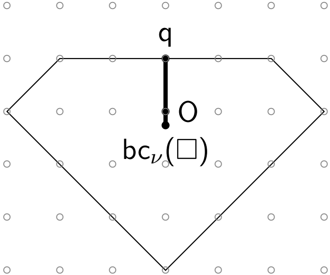


If $\mathrm{bc}(P) = 0$ then $R(X) = 1$, else $R(X) = |q|/|\mathrm{bc}(P) - q|$. (**C.Li, '11**)

Section 3 - R(X) in complexity one

- ν = Duistermaat-Heckman measure on $\square$.
- n_y, n outer normals at q_y, q respectively





Section 3 - R(X) in complexity one

Theorem (C '19)

If $bc(\Delta_y) \in \{0\} \times \mathbb{R}^+$ *for each* y *then* $R(X) = 1$*. Else:*

$$R(X) = \min \left\{ \frac{|q|}{|q - bc_\nu(\square)|} \right\} \cup \left\{ \frac{|q_y|}{|q_y - bc(\Delta_y)|} \right\}_{n_y \in N_{\mathbb{Q}} \times \mathbb{Q}^+}$$

Section 3 - R(X) in complexity one

Method of Proof

1. Rewrite $R(X) = \min_{(\mathcal{X},\mathcal{L})} \inf_w R_{\mathcal{X}}(w)$, where:

$$R_{\mathcal{X}}(w) := \sup(t \mid \mathrm{DF}_t(\mathcal{X},\mathcal{L})w \geq 0).$$

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2. Split into cases: product and non-product configurations.
3. In each case, use formulae for $\mathrm{DF}_t(\mathcal{X},\mathcal{L})w$ to calculate $R_{\mathcal{X}}(w)$
4. Find the w which minimizes $R_{\mathcal{X}}$.

X	R(X)
2.30	23/29
2.31	23/27
3.18	48/55
3.21	76/97
3.22	40/49
3.23	168/221
3.24	21/25
4.5*	64/69
4.8	76/89

Values of $R(X) < 1$ for Fano $(\mathbb{C}^*)^2$-threefolds.

Section 4 - KE in complexity two

Three new examples of Kähler-Einstein manifolds in complexity two:

- The result of a sequence of blowups of a quadric Q^6.

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Three new examples of Kähler-Einstein manifolds in complexity two:

- The result of a sequence of blowups of a quadric Q^6.
- Two hypersurfaces in $\mathbb{P}^3 \times \mathbb{P}^3$ of bidegree $(1, 2)$, $(1, 3)$ respectively.

Section 4 - KE in complexity two

Following **Süß '13**:

Definition

$$\alpha_H(Y,B) := \sup\{\lambda | (Y, B+\lambda D) \text{ log canonical } \forall D \in |-K_X - B|_{\mathbb{Q}}^H\}$$

Section 4 - KE in complexity two

Following **Süß '13**:

Theorem

Suppose $G = HT$ for some discrete H, and $H \curvearrowright X$ is symmetric with respect to $T \curvearrowright X$. If

$$\alpha_H(Y, B) > \frac{\dim X}{\dim X + 1}$$

Then X is Kähler-Einstein.

Section 4 - KE in complexity two

Example (Hypersurfaces in $\mathbb{P}^n \times \mathbb{P}^n$)

$$X = X^{2n-1}_{a,b} := V\left(\sum_{i=0}^{n} x_i^a y_i^b\right) \subseteq \mathbb{P}^n \times \mathbb{P}^n.$$

Set $p = a/\gcd(a,b)$, $q = b/\gcd(a,b)$.

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Set $p = a/\gcd(a,b),\ q = b/\gcd(a,b)$.

- $T = (\mathbb{C}^*)^n \curvearrowright X$ given by weights:

$$\begin{pmatrix} 0 & qI_n & 0 & -pI_n \end{pmatrix}.$$

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Example (Hypersurfaces in $\mathbb{P}^n \times \mathbb{P}^n$)

$$X = X_{a,b}^{2n-1} := V\left(\sum_{i=0}^{n} x_i^a y_i^b\right) \subseteq \mathbb{P}^n \times \mathbb{P}^n.$$

Set $p = a/\gcd(a,b)$, $q = b/\gcd(a,b)$.

- $S_{n+1} \curvearrowright X$ permuting indices, symmetric w.r.t $T \curvearrowright X$.

Section 4 - KE in complexity two

Example (Hypersurfaces in $\mathbb{P}^n \times \mathbb{P}^n$)

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Set $p = a/\gcd(a,b),\ q = b/\gcd(a,b)$.

- $Y = \mathbb{P}^{n-1}$, with boundary γB where B sum of hyperplanes in general position, and:

$$\gamma = \max\left(\frac{p-1}{p}, \frac{q-1}{q}\right)$$

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Example (Blowing up a quadric)

$$Q = Q^{2n} := V\left(\sum_{i=0}^{n} x_{2i}x_{2i+1}\right) \subset \mathbb{P}^{2n+1}.$$

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Example (Blowing up a quadric)

$$Q = Q^{2n} := V\left(\sum_{i=0}^{n} x_{2i}x_{2i+1}\right) \subset \mathbb{P}^{2n+1}.$$

- $T = (\mathbb{C}^*)^{n+1} \curvearrowright Q$, given by $\deg x_{2i} = e_i, \ \deg x_{2i+1} = -e_i$

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Example (Blowing up a quadric)

$$Q = Q^{2n} := V\left(\sum_{i=0}^{n} x_{2i}x_{2i+1}\right) \subset \mathbb{P}^{2n+1}.$$

- $S_{n+1} \curvearrowright Q$ permuting $Z_i := V(x_{2i}, x_{2i+1})$.

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Example (Blowing up a quadric)

$$Q = Q^{2n} := V\left(\sum_{i=0}^{n} x_{2i}x_{2i+1}\right) \subset \mathbb{P}^{2n+1}.$$

- $Y = \mathbb{P}^{n-1}$ and trivial boundary.

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Example (Blowing up a quadric)

$$W = W^{2n} := \mathrm{Bl}_{\tilde{Z}_n} \dots \mathrm{Bl}_{\tilde{Z}_1} \mathrm{Bl}_{Z_0} Q^{2n}.$$

- Fano, and independent under permuting Z_i.

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Example (Blowing up a quadric)

$$W = W^{2n} := \mathrm{Bl}_{\tilde{Z}_n} \dots \mathrm{Bl}_{\tilde{Z}_1} \mathrm{Bl}_{Z_0} Q^{2n}.$$

- $T = (\mathbb{C}^*)^n \curvearrowright W$ induced by $W \to Q$.

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Example (Blowing up a quadric)

$$W = W^{2n} := \mathrm{Bl}_{\tilde{Z}_n} \ldots \mathrm{Bl}_{\tilde{Z}_1} \mathrm{Bl}_{Z_0} Q^{2n}.$$

- $X \to Y$ is composition $W \to Q \to \mathbb{P}^{n-1}$, with boundary divisor $\frac{1}{2}B$.

Section 4 - KE in complexity two

Theorem (C '19 (preprint))

$X^5_{1,2}, X^5_{1,3}, W^6$ are Kähler-Einstein.

Section 4 - KE in complexity two

Theorem (C '19 (preprint))

$X^5_{1,2}, X^5_{1,3}, W^6$ are Kähler-Einstein.

Method:

It is enough to show

$$\alpha_{S_4}\left(\mathbb{P}^2, \gamma(\ V(x) + V(y) + V(z) + V(x+y+z)\ )\ \right) \geq 1$$

For $\gamma = \frac{1}{2}, \frac{2}{3}$.

Thank you for listening!

We show $\alpha_{S_4}(\mathbb{P}^2, B/2) \geq 1$. Set $Y = \mathbb{P}^2$.

Proof.

Suppose $D \in |-K_Y - B/2|_{\mathbb{Q}}^{S_4}$.

Suppose $P \in Y$. We will show $(Y, B/2 + D)$ l.c at P.

WLOG $P \notin V(z), V(x+y+z)$.

Remove components of $B/2 + D$ supported at $V(z), V(x+y+z)$ to obtain D'.

D' still invariant under $\sigma : x \leftrightarrow y$.

$D'_0 := \lim_{t \to 0} t \cdot D'$, under $t \cdot [x, y, z] = [tx, ty, z]$.

$$D'_0 = \frac{1}{2}(V(x) + V(y)) + aV(x+y) + bV(x-y) + \sum_i c_i(L_i + \sigma L_i).$$

□

Example (Example of modifying D)

Suppose $f(x,y,z) = x^2 + y^2 + z^2 + (x+y+z)^2$

Set $D = \frac{1}{2}V(f)$.

$$D' = \frac{1}{2}(V(x) + V(y)) + \frac{1}{2}V(f)$$

Limit:

$$\lim_{t \to 0} t \cdot f(x,y,z) = x^2 + y^2 + (x+y)^2$$

$$D'_0 = \frac{1}{2}(V(x) + V(y)) + \frac{1}{2}V(x + \eta y) + \frac{1}{2}V(y + \eta x) \quad \text{(I.c)}$$