STABILITY OF VARIETIES WITH A TORUS ACTION

A thesis submitted to the University of Manchester
for the degree of Doctor of Philosophy
in the Faculty of Science and Engineering

2020

By
Jacob Cable
Mathematics

Contents

Word Count: 999,999

List of Tables

List of Figures

In this thesis we study several problems related to the existence problem of invariant canonical metrics on Fano orbifolds in the presence of an effective algebraic torus action. The first chapter gives an introduction. The second chapter reviews the existing theory of T-varieties and reviews various stability thresholds and K-stability constructions which we make use of to obtain new results. In the third chapter we find new Kähler-Einstein metrics on some general arrangement varieties. In the fourth chapter we present a new formula for the greatest lower bound on Ricci curvature, an invariant which is now known to coincide with Tian's delta invariant. In the fifth chapter we discuss joint work with my supervisor to find new Kähler-Ricci solitons on smooth Fano threefolds admitting a complexity one torus action.

Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

Acknowledgements

Chapter 1

Introduction

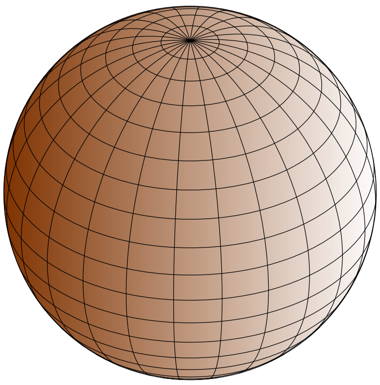
In this thesis we explore several new results relating to the existence of special metrics on certain compact Kähler manifolds admitting an effective algebraic torus action. Our goal is to further the understanding of canonical metrics on these types of manifolds. We will do this by finding new examples, and providing novel effective methods of calculating relevant invariants. Our methods are also a concrete demonstration of the power and utility of the language of T-varieties.

We begin with a brief primer on the theory for the uninitiated reader. A *Kähler manifold* is a smooth manifold X adorned with mutually compatible Riemannian, complex, and symplectic structures. In this situation we call the Riemannian metric g the *Kähler metric* on X, and the symplectic form ω the *Kähler form* of X.

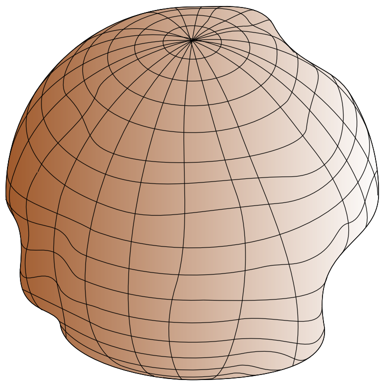
There are many reasons to study Kähler geometry. From the standpoint of algebraic geometry, every smooth complex projective variety inherits a Kähler structure. From a differential geometric perspective, Kähler manifolds are a particularly well-behaved class of Riemannian manifolds, and are a rich enough class to contain many interesting examples. There are also motivations from theoretical physics: The various models of our universe in string theory require extra planck-scale dimensions, and certain Kähler manifolds are the best candidates for the shape of these dimensions.

Historically, it has been an important problem to investigate which Kähler manifolds admit nice Kähler metrics. Generally, what we mean here by "nice" depends on context. For naive motivation consider the real 2-sphere S^2. Most would have in their mind the standard embedding $S^2 = \{x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$, see Figure 1a. There are, however, many choices of smooth embedding, each one corresponding to different choices of Riemannian metrics on S^2, see for example Figure 1b. What sets our favourite embedding apart is that the induced metric has *constant curvature.*

This is a special case of a wider phenonemon if we identify the sphere as the Riemann surface $S^2 \cong \mathbb{P}^1_{\mathbb{C}}$. The uniformization theorem, originally proven by Poincaré [1] and Koebe [2,3], tells us that any Riemann surface admits a metric of constant scalar curvature. The obvious question is then: What happens in higher dimensions?



(a) A metric of constant scalar curvature



(b) A metric of non-constant scalar curvature

Figure 1: Two choices of metric on S^2.

In [4,5], Calabi proved certain results for compact Kähler manifolds which lead to a now famous conjecture. Fix a compact Kähler manifold (X, ω). Recall that the Ricci curvature form $\mathrm{Ric}(\omega)$ is a real $(1,1)$-form and defines a characteristic class $c_1(X) = \frac{1}{2\pi}[\mathrm{Ric}(\omega)]$ of the manifold, known as the first Chern class. Calabi asked whether, given a real $(1,1)$-form η representing the first Chern class of X, can we find a unique Kähler metric ω' in the same cohomology class as ω such that $\mathrm{Ric}(\omega') = 2\pi\eta$?

A related conjecture asks whether all compact Kähler manifolds (X, ω) admit a Kähler-Einstein metric, or more formally whether they admit a Kähler form ω' in the same cohomology class as ω with $\operatorname{Ric}\omega' = \lambda\omega'$, for some real constant λ. This equation is known as the Einstein condition[1], and the Kähler metric corresponding to ω' is called a Kähler-Einstein metric.

It follows by definition that, for X to admit such a metric, $\operatorname{Ric}\omega'$ must be a definite $(1,1)$-form. This separates the problem into three cases: Where $\operatorname{Ric}\omega'$ is positive definite, zero, or negative definite. In the first two cases Kähler-Einstein metrics on X are precisely the metrics of constant scalar curvature, and so one may see this as a direct generalization of the uniformization theorem for Riemann surfaces.

Aubin [6] and Yau [7] settled the negative definite case first. Calabi's conjecture was also proven by Yau in [7], later contributing to him being awarded the Fields Medal. This left the positive definite case, which corresponds to smooth Fano varieties under the Kodaira embedding theorem. It was already, known however, due to Matsushima [8], that not all Fano manifolds were Kähler-Einstein. Thus, it became an objective to find suitable criteria for the existence of a Kähler-Einstein metric on a Fano manifold.

In [9] Futaki introduced a new invariant whose vanishing was also a necessary condition. In [10] Tian introduced a sufficient condition in terms of another invariant, known now as *Tian's alpha invariant.* Tian's alpha invariant is a generalization of the complex singularity exponent of a polynomial $f \in \mathbb{C}[z_1, \dots, z_n]$, which is defined as follows:

$$c_O(f) := \sup\{\epsilon| \ |f|^{-2\epsilon} \text{ integrable in a neighborhood of } O \in \mathbb{C}^n\}.$$

The Yau-Tian-Donaldson conjecture suggested the notion of *K-stability* as a necessary and sufficient Kähler-Einstein criterion[2]. This was proven in the trilogy of papers [11–13].

A generalization of the notion of a Kähler-Einstein metric is a Kähler-Ricci soliton. To understand how, recall that Kähler Einstein metrics may be seen as generalized fixed point solutions under the Kähler-Ricci flow:

$$\frac{d}{dt}\omega_t = -2\operatorname{Ric}(\omega_t),$$

[1] By analogy to Einstein's field equation for a vacuum.

[2] In full, the YTD talks of csck metrics, which are equivalent to KE in the Fano case

in that under this flow they will remain unchanged up to some scaling factor. A Kähler-Ricci soliton is a generalized fixed point of the flow in the sense that it will remain unchanged up to some biholomorphism.

A further generalization are twisted Kähler-Einstein metrics and twisted Kähler-Ricci solitons. These arise in continuity method arguments, see [14] for example, and depend on a parameter $t \in [0, 1]$. Here we start with a Calabi-Yau type solution ω_0 at $t = 0$, and consider the existence of solutions ω_t along a line segment to the target Kähler-Einstein or Kähler-Ricci soliton equation at $t = 1$ respectively. The supremum of the set of t for which a solution exists turns out to be independent of ω_0, and is of interest as an invariant of X. It is known as *Tian's beta invariant*, or the *greatest lower bound on Ricci curvature*. We will denote this invariant by $R(X)$.

Although K-stability is a criterion for the existence of Kähler-Einstein metrics and their various generalizations, it is not an effective one. In general, the K-stability of a Fano manifold is difficult to calculate. The alpha invariant approach also has limitations in practice. Fortunately, equivariant versions of K-stability and Tian's alpha invariant exist, which, as we will see, provide an effective approach in classes of manifolds and orbifolds with a high degree of symmetry.

In this thesis we will focus our attention on Fano manifolds which are also *T-varieties*. A T-variety is a normal variety which admits an effective action of an algebraic torus $T = (\mathbb{C}^*)^r$. These are a generalization of toric varieties, where $\dim T = \dim X$. In general we call the difference $\dim X - \dim T$ the *complexity* of the torus action.

In the toric case it is well-established that studying X is equivalent to studying some associated combinatorial data: a fan of cones in a vector space built from the cocharacter lattice of T. Thanks to the work of many authors (Altmann, Hausen, Ilten, Petersen, Süß, Vollmert, Liendo to name a few) this combinatorial description extends to higher complexity. We recall some of the theory in Chapter 2.

Equivariant methods have been used to provide some effective criteria for canonical metrics on low complexity T-varieties. If X is a Fano toric variety then the Kähler-Einstein problem is completely solved. In [15] it was shown that X is Kähler-Einstein if and only if the Futaki character vanishes. They also showed that the Futaki character coincides with the barycenter of the lattice Polytope corresponding to X. Wang et al did not use K-stability for this result, but the result was later reproven as an application of the main theorem of [14].

In [16], Süß and Ilten considered the K-stability of T-varieties of complexity one. We recall this in detail in Section 2.4.1. They obtained a combinatorial criterion for K-stability, generalizing the results of [15]. Süß had also used the equivariant version of Tian's alpha invariant to find new Kähler-Einstein metrics on complexity one T-varieties admitting additional symmetries in [17].

In complexity two and above, even equivariant K-stability remains an ineffective criterion. There is, as we shall see, scope for methods like that of [17]. In the next section we will give an overview of the content of this thesis.

1.1 Content of the Thesis

Here we summarize the remaining structure of the thesis. Some of the content of this thesis has been published, and/or submitted to journals. I reference relevant sources clearly when this is the case. I also make clear, in the case of my coauthored work in Chapter 3, the scope of my contribution to the original paper.

Chapter 2 - Preliminaries

We recall definitions and results from Kähler, algebraic, and symplectic geometry to give context to our novel results. We then give a brief introduction to the theory of T-varieties and their equivariant K-stability, which are key in our methods of proof in Chapters 3 and 4.

Chapter 3 - New Kähler-Ricci solitons on Fano threefolds

In Chapter **??** we consider Fano threefolds admitting an effective 2-torus action within the classification of [18]. In [19] a not necessarily complete list of such threefolds together with their combinatorial description was given. We extend the results of [16], providing new examples of threefolds admitting a non-trivial Kähler-Ricci soliton. Recall that a Kähler-Ricci soliton on a Fano manifold (X, ω) is a pair (ω', v) satisfying:

$$\mathrm{Ric}(\omega') - L_v\omega' = \omega'$$

We apply some real interval arithmentic approximations to the complexity one formula for the Futaki invariant of Ilten and Süß (see Section 2.4.2) to check the existence criterion [14] (see Section 2.4.1). We include the relevant Sagemath code in Appendix **??**.

Chapter 4 - $R(X)$ in complexity one

In Chapter 3 we present an explicit effective formula obtained for the greatest lower bound on Ricci curvature $R(X)$ for a complexity one T-variety X. We follow the author's work [20]. These results generalize a result of Li [21], but the proof uses results of G-equivariant K-stability from [14]. The invariant $R(X)$ is often denoted $\beta(X)$ and is referred to as *Tian's beta invariant.* By [] it is now known to coincide with another important invariant, $\delta(X)$.

Chapter 5 - New Kähler-Einstein metrics on symmetric general arrangement varieties

In Chapter **??**, we discuss recent results obtaining new Kähler-Einstein metrics on some symmetric complexity two general arrangement varieties. General arrangement varieties are T-varieties where the torus quotient is a projective space, and the critical values of the quotient map form a general arrangement of hyperplanes in that projective space. Smooth general arrangement varieties of complexity and Picard rank 2 were classified according to their Cox ring in [22]. Following the methods of [23], we find three new examples of Kähler-Einstein metrics. As far as we are aware, these are the first examples of Kähler-Einstein metrics found on T-varieties of complexity greater than one by way of equivariant methods.

Chapter 2

Preliminaries

2.1 Kähler geometry

2.1.1 Basic definitions

Kähler manifolds

In this section we recall the basics of Kähler geometry. We then review some important results on the existence of canonical metrics on compact Kähler manifolds. A good reference for the material here is [24]. Let X denote a compact real manifold of dimension $2n$. Suppose we have an almost complex structure J on X, that is an automorphism J of $T_{\mathbb{R}}X$ such that $J^2 = -\mathrm{Id}$. Recall that the complexified tangent bundle, $T_{\mathbb{C}}X := T_{\mathbb{R}}X \otimes \mathbb{C}$, decomposes via eigenspaces of J:

$$T_{\mathbb{C}}X = T^{1,0}X \oplus T^{0,1}X,$$

where $T^{(1,0)}X$ has local generators $\frac{\partial}{\partial z_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} - \sqrt{-1}\frac{\partial}{\partial y_i}\right)$, and $T^{(0,1)}X = \overline{T^{(1,0)}X}$ has local generators $\frac{\partial}{\partial \bar{z}_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} + \sqrt{-1}\frac{\partial}{\partial y_i}\right)$.

Recall also that we have a natural isomorphism of real vector bundles $T_{\mathbb{R}}X \cong T^{(1,0)}$, given by the composition $T_{\mathbb{R}}X \to T_{\mathbb{C}} \to T^{1,0}X$. Note, by definition, the action of J is described by multiplying by $\sqrt{-1}$ on $T^{1,0}X$. The decomposition above induces a decomposition of the complexified cotangent bundle:

$$T^*_{\mathbb{C}}X = T^*_{1,0}X \oplus T^*_{0,1}X,$$

and, moreover, a decomposition:

$$\bigwedge^n T^*_{\mathbb{C}}X = \bigoplus_{p+q=n} \left(\bigwedge^p T^*_{1,0}X \otimes \bigwedge^q T^*_{0,1}X \right) \tag{2.1}$$

We use the following notation for relevant spaces of global sections:

$$A^n(X) := H^0(X, \bigwedge^n T^*_{\mathbb{C}}X) \tag{2.2}$$

$$A^{p,q}(X) := H^0(X, \ (\bigwedge^p T^*_{1,0}X \otimes \bigwedge^q T^*_{0,1}X) \ ). \tag{2.3}$$

A form $\alpha \in A^{p,q}(X)$ is said to be *of type* (p,q). The decomposition (2.1) induces a corresponding decomposition:

$$A^n(X) = \bigoplus_{p+q=n} A^{p,q}(X).$$

Hermitian and Kähler metrics

Roughly speaking, a metric measures distances on a manifold. On real manifolds one usually *Riemannian metrics*. Complex manifolds come with more structure, and the analogous object compatible with this structure, is a *Hermitian metric*. A Hermitian metric is given by a smooth choice of positive definite Hermitian inner product on the fibers of $T^{(1,0)}X$, i.e an element of $H^0(X, T^*_{1,0}X \otimes T^*_{0,1}X)$. Locally we write:

$$h(z) = \sum h_{ij}(z) dz_i \otimes d\bar{z}_j.$$

Given a Hermitian metric h on X, we may consider the real and imaginary parts of h as real tensors on the underlying real manifold via the isomorphisms $T_{\mathbb{R}}X \cong T^{1,0}X \cong \overline{T^{1,0}X}$. The real part $g = \Re h$ is a Riemannian metric on X, called the *induced Riemannian metric* of h. Locally we have:

$$g_z = \sum h_{ij}(z)(dx_i \otimes dx_j + dy_i \otimes dy_j)$$

We may also realize the imaginary part $\omega = -\Im h$, up to sign, as an alternating form on the real tangent bundle $T_{\mathbb{R}}X$, via $T^{1,0}X \cong \overline{T^{1,0}X}$. Set $\omega(v \wedge w) := -\Im h(v, \bar{w}) = -\frac{i}{2}(h - \bar{h})$. We call ω the *associated* $(1,1)$*-form* of h. Locally we have:

$$\omega_z = \sqrt{-1} \sum h_{ij}(z) dz_i \wedge d\bar{z}_j = \sqrt{-1} \sum h_{ij}(z)(dx_i \otimes dy_j - dy_i \otimes dx_j).$$

By definition we have $g(v, w) = g(Jv, Jw)$ and $\omega(v, w) = g(Jv, w)$ for any $v, w \in T_{\mathbb{R}}X$. In fact we may reconstruct h from any Riemannian metric g satisfying $g(v, w) = g(Jv, Jw)$, or alternatively any real $(1, 1)$-form ω satisfying the positive definite condition:

$$\omega(v \wedge v) > 0 \text{ for all } v \in T_{\mathbb{R}}X.$$

We may now recall the definition of a Kähler metric.

Definition 1. *A Hermitian metric is Kähler if the associated* $(1, 1)$*-form* ω *is closed, i.e* $d\omega = 0$*, where* $d : A^2(X, \mathbb{R}) \to A^3(X, \mathbb{R})$ *is the usual exterior differential.*

We will bow to convention and often refer to ω, instead of g, as a Kähler metric on X in this context. The standard first example of a compact Kähler manifold is the Fubini-Study metric on complex projective space:

Example 1. *Let* s *be a section of the projection map* $\pi : \mathbb{C}^{n+1}\backslash\{0\} \to \mathbb{P}^n$ *over some open set* $U \subset \mathbb{P}^n$*. The Fubini-Study metric* ω_{FS} *is then defined to be*

$$\omega_{FS} := i\partial\bar{\partial} \log ||s||^2$$

This is well-defined as any two sections differ on their shared domain by a non-vanishing holomorphic function, $s' = fs$*. It is clearly closed (since* $d = \partial + \bar{\partial}$*). For the standard section on* U_0 *with holomorphic coordinates* $z_1, \ldots, z_n$ *we have:*

$$\omega_{FS} := i\partial\bar{\partial} \log(1 + |z_1|^2 + \cdots + |z_n|^2)$$

and at $[1, 0, \ldots, 0] \in U_0$ *we have:*

$$\omega_{FS} = i \sum dz_j \wedge d\bar{z}_j$$

This is positive definite, and so defines a Kähler metric on $\mathbb{P}^n$*.*

This leads to a large class of examples including any smooth projective algebraic variety:

Example 2. *The restriction of* ω_{FS} *to any closed submanifold* $Y \subseteq \mathbb{P}^n$ *induces* *a Kähler structure on Y, as the exterior differential commutes with pulling back differential forms.*

Recall the following important result, telling us that any two Kähler forms of the same class differ by some real-valued function.

$\partial\bar{\partial}$-lemma. *If ω, η are two real $(1,1)$-forms of the same cohomology class then there is a real function $f : X \to \mathbb{R}$ such that $\omega - \eta = \sqrt{-1}\partial\bar{\partial}f$.*

2.1.2 Line bundles and Kodaira Embedding

Recall that we can extend the notion of Hermitian metric to an arbitrary complex vector bundle E: a Hermitian metric on E is defined to be an element $h \in H^0(X, E \otimes \bar{E})^*$. We now recall the notion of a connection and its curvature. A *connection* may be thought of as a way to differentiate tensor fields, and transport data smoothly about a manifold. In our context, a connection is given by a map:

$$\nabla : H^0(X, E) \to H^0(X, E \otimes T^*X)$$

satisfying the Liebniz rule $\nabla(sf) = \nabla sf + s \otimes df$. There is a unique way to extend a connection to an exterior derivative on E-valued differential forms:

$$d^\nabla : \Omega^r(E) \to \Omega^{r+1}(E).$$

The *curvature* of a connection is the 2-form:

$$F^\nabla \in H^0(X, \mathrm{End}(E) \otimes \wedge^2 T^*X),$$

given by:

$$F^\nabla(u, v)(s) := \nabla_u \nabla_v s - \nabla_v \nabla_u s - \nabla_{[u,v]} s.$$

There is a canonical connection on the tangent bundle of any Riemannian manifold known as the *Levi-Civita connection*, satisfying $\nabla g = 0$ and $\nabla_u v - \nabla_v u = [u, v]$. We have a similar situation for any Hermitian vector bundle on a complex manifold:

Example 3. *Let E be a Hermitian vector bundle on a complex manifold X equipped with a holomorphic structure. There is a unique connection ∇ on E such that:*

- *For all sections s we have $\pi_{1,0}\nabla s = \bar{\partial}s$*

- *For any smooth vector field v and sections s, t we have:*

$$v\langle s,t\rangle = \langle \nabla_v s, t\rangle + \langle s, \nabla_v t\rangle$$

This connection is called the Chern connection on E.

Kähler manifolds may be characterized as those manifolds for which the Levi-Civita connection and Chern connection on the tangent bundle coincide. We now recall the definition of the first Chern class of a Hermitian line bundle, which may be used to define notions of positivity.

Definition 2. *The first Chern class of a Hermitian line bundle L is the real cohomology class:*

$$c_1(L) = \frac{1}{2\pi}[-\sqrt{-1}\partial\bar{\partial}\log(h)] \in H^2(X,\mathbb{Z})$$

Example 4. *Suppose (X, g) is a Kähler manifold. Then g induces a Hermitian metric on the holomorphic cotangent bundle $\Omega^{1,0}X$, which in turn induces a Hermitian metric on the canonical line bundle $K_X = \wedge^n\Omega^{1,0}X$, denoted* $\det(g)$*. The curvature of the associated Chern connection to this Hermitian line bundle is called the Ricci curvature form of the manifold, given by:*

$$\mathrm{Ric}(\omega) = -\sqrt{-1}\partial\bar{\partial}\log(\det(g)).$$

The real cohomology class $c_1(K_X) = \frac{1}{2\pi}[-\sqrt{-1}\partial\bar{\partial}\log(\det(g)]$ is called the first Chern class of the Kähler manifold (X, g), and is often denoted just by $c_1(X)$.

We now recall the definition of a positively curved line bundle.

Definition 3. *A real $(1,1)$-form is called positive if the associated symmetric bilinear form defined for real tangent vectors is positive definite. A real cohomology class is called positive if it can be represented by a positive $(1,1)$-form. A line bundle L is called positive if its first Chern class is positive.*

The following theorem, by Kodaira, characterizes smooth projective varieties amongst compact Kähler manifolds. Recall that a line bundle L is very ample if for some global sections $s_0, \ldots, s_n \in H^0(X, L)$ we obtain a well-defined closed embedding into a projective space, given by:

$$\varphi_L : p \mapsto [s_0(p), \ldots, s_n(p)] \in \mathbb{P}^n$$

We say L is ample if some multiple of L is very ample.

Kodaira Embedding Theorem [25, Theorem 4]**.** *A holomorphic line bundle over a compact complex manifold is ample if and only if it is positive.*

Paired with the following theorem we can talk either about projective Kähler manifolds or polarized projective algebraic varieties.

Chow's Theorem [26, Theorem 5]**.** *A closed complex submanifold of projective space is a projective algebraic subvariety.*

2.1.3 Canonical metrics on Kähler manifolds

In this section we recall some facts about canonical metrics. A canonical metric is a choice of metric dependent only on the complex structure of the manifold, unique up to biholomorphic automorphisms. The material in this section may be found in [27], for example.

As we touched on in the introduction, Kähler-Einstein metrics are an important class of canonical metric, and the question of which compact Kähler manifolds admit a Kähler-Einstein metric has historically received a lot of attention. Recall the definition of a Kähler-Einstein metric:

Definition 4. *Let X be a Kähler manifold. A Kähler-Einstein metric on X is a Kähler metric $\omega \in 2\pi c_1(X)$ such that $\operatorname{Ric}\omega = \lambda\omega$ for some real constant λ.*

Note if X is Kähler-Einstein then we must have either $K_X = 0$, K_X ample, or $-K_X$ ample. Via Kodaira embedding these correspond to Ricci flat, Ricci positive, and Ricci negative manifolds respectively. We briefly recall the answers to the existence question in the cases of negative and zero Ricci curvature:

Calabi-Yau Theorem [28, Theorem 1]**.** *Let (X, ω) be a compact Kähler manifold. Let α be a real $(1,1)$-form representing $c_1(X)$. Then there exists a real $(1,1)$-form ω' with $[\omega'] = [\omega]$ such that $\operatorname{Ric}(\omega) = 2\pi\alpha$.*

Aubin-Yau Theorem [6,29]**.** *Let X be a compact Kähler manifold with $c_1(X) < 0$. Then there exists a unique Kähler metric $\omega \in -2\pi c_1(X)$ such that $\operatorname{Ric}(\omega) = -\omega$.*

However the following necessary criterion illustrates that the same is not true in the Fano case $c_1(X) > 0$. Recall that a complex algebraic group is reductive if it is the complexification of a compact connected real Lie group.

Matsushima's criterion [8]. *If a Fano manifold X admits a Kähler-Einstein metric, then the holomorphic automorphism group of X is reductive.*

In particular this tells us that the blow up of $\mathbb{P}^1$ in one or two points is not Kähler-Einstein. We end this section by recalling the most general form of canonical metric we will consider. This matches the definition given in [14, Definiton 3].

Definition 5. *A twisted Kähler-Ricci soliton on a Fano manifold (X, ω_0) is a triple (ω, v, t) where $\omega \in 2\pi c_1(X)$ is a Kähler metric, v is a holomorphic vector field, and $t \in [0, 1]$, such that*

$$\mathrm{Ric}(\omega) - \mathcal{L}_v \omega = t\omega + (1 - t)\omega_0$$

When $t = 0$ we omit it from the notation and call (ω, v) a Kähler-Ricci soliton. Similarly when v is trivial we call (ω, t) a twisted Kähler-Einstein metric. When both hold then we talk about ω being a Kähler-Einstein metric.

In Section 2.4 we will describe various criteria for the existence of such metrics, but to do so we must first recall some basic tools and language from algebraic and symplectic geometry.

2.2 Algebraic and symplectic tools

In this section we give some definitions from algebraic and symplectic geometry as we will be understanding them throughout the rest of the thesis. In particular we recall some basic geometry invariant theory, which is needed for the arguments in Chapter **??**.

2.2.1 The algebraic torus

Fix an algebraic torus $T = (\mathbb{C}^*)^k$. We have mutually dual character and cocharacter lattices:

$$M := \mathrm{Hom}(T, \mathbb{C}^*), \ N = \mathrm{Hom}(\mathbb{C}^*, N),$$

respectively. We denote the associated vector spaces by:

$$M_{\mathbb{K}} := M \otimes_{\mathbb{Z}} \mathbb{K}, \ N_{\mathbb{K}} := N \otimes_{\mathbb{Z}} \mathbb{K},$$

for $\mathbb{K} = \mathbb{Q}, \mathbb{R}$. There is a perfect pairing $M \times N \to \mathbb{Z}$ which extends to a bilinear pairing $M_{\mathbb{K}} \times N_{\mathbb{K}} \to \mathbb{K}$. We may make the identification:

$$T \cong \operatorname{Spec} \mathbb{C}[M] \cong N \otimes \mathbb{C}^*.$$

Finally recall that we may identify the real Lie algebra $\mathfrak{k}$ of the maximal compact subtorus $K \subset T$ as $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$.

2.2.2 Linearizations

Suppose we have an algebraic group G acting algebraically on a scheme or variety X. A lift of the this action to a line bundle L on X is known as a linearization. Linearizations are used in geometric invariant theory to give a good definition of a quotient of X by the G-action.

Definition 6. *Let X be a projective scheme together with an action $\lambda : G \times X \to X$ of a reductive algebraic group G. A linearization of the action λ on L is an action $\tilde{\lambda}$ on L such that:*

- *The projection π is G-equivariant, $\pi \circ \tilde{\lambda} = \lambda \circ \pi$*
- *For $g \in G$ and $x \in X$, the induced map $L_x \mapsto L_{g \cdot x}$ is linear.*

Note a linearization to L naturally induces linearizations to $L^\vee$ and $L^{\otimes r}$ for any $r \in \mathbb{N}$.

Example 5. *A linearization of the trivial bundle on a projective variety X must be of the form*

$$g \cdot (x, z) = (g \cdot x, \chi(x, z)z)$$

for some $\chi \in H^0(G \times X, \mathcal{O}^*_{G \times X}) \cong H^0(G, \mathcal{O}^*_G) = \mathfrak{X}(G)$.

The above example tells us that any two linearizations λ_1, λ_2 of an action to the same line bundle differ by multiplication by some character χ of G: fiberwise we have $\tilde{\lambda}_1 = \chi(x, z)\tilde{\lambda}_2$. Thus, when $G \cong T$ is an algebraic torus we may identify the set of linearizations with the character lattice M.

Example 6. *Recall that an action of G on X induces a canonical linearization on the tangent and cotangent bundles of X, and so induces a canonical linearization on the anti-canonical bundle $-K_X$ as the top exterior power of the cotangent bundle.*

2.2.3 Hamiltonian actions and moment maps

Here we recall some basic notions of Hamiltonian actions and moment maps. We will follow conventions of [30] and [31]. We illustrate the theory with the case of an algebraic torus action. Suppose that (X, ω) is a symplectic manifold.

Definition 7. *Let $\theta : X \to \mathbb{R}$ be a smooth function. A vector field v such that $\iota_v \omega = d\theta$ is called a Hamiltonian vector field, with Hamiltonian function θ.*

Definition 8. *Let K be a real Lie group, with Lie algebra $\mathfrak{k}$, acting smoothly on X. This action is said to be Hamiltonian if there exists a map $\mu : X \to \mathfrak{k}^*$, known as the moment map of the action, such that:*

- *For any $\xi \in \mathfrak{k}$ the map $\mu^\xi : X \to \mathbb{R}$ given by $\mu^\xi(p) := \langle \mu(p), \xi \rangle$ is a hamiltonian function for the vector field v generated by the one-parameter subgroup $\exp(t\xi) \subset K$.*
- *The map μ is equivariant with respect to the action of K on X and with respect to the coadjoint action.*

Example 7. *Let $X \subseteq \mathbb{P}^N$ be a nonsingular complex projective variety and let G be reductive algebraic group acting effectively on $\mathbb{P}^N$, restricting to an action on X. Let K be the maximal compact subgroup in G, with Lie algebra $\mathfrak{k}$. The action of G is given by a representation $\rho : G \to GL(N+1)$, and by choosing appropriate coordinates we may assume K maps to $U(N+1)$ and so preserves the Fubini-Study form. It can be checked that a moment map $\mu : X \to \mathfrak{k}^*$ for the K-action is given by:*

$$\mu([x]) \cdot a := \frac{x^t \rho_*(a) \hat{x}}{|x|^2}, \tag{2.4}$$

where x is any representative of $[x] \in X \subseteq \mathbb{P}^N$. A different choice of linearization in this setting corresponds to multiplying ρ by some character $\chi \in \mathfrak{X}(G)$. Since $\chi(K)$ is compact, it sits inside $S^1 \subset \mathbb{C}^*$, and hence we do not need to change coordinates when considering the effect on the moment map. When we plug this into (2.5) we see that we have translated the moment map by $\chi \in \mathfrak{X}(G) \otimes \mathbb{R} \cong \mathfrak{k}^*$. Moreover, taking the rth power of L corresponds to scaling the moment map by a factor of r. This gives a correspondence between rational elements

$$\chi \in \mathfrak{X}(G) \otimes \mathbb{Q} \subset \mathfrak{k}^*$$

and linearizations of powers of L.

Example 8. *Suppose $G = T$ is an algebraic torus with character and cocharacter lattices M, N respectively. Then ρ is a diagonal matrix of characters $u_0, \ldots, u_N$ and we obtain:*

$$\mu([x]) = \frac{\sum_{j=0}^{N} |x_i|^2 u_i}{|x|^2} \in M$$

Then, by Atiyah, [32], and Guillemin-Sternberg, [33], the image of μ is a convex polytope $P \subset M$. Here we see that for each one-parameter subgroup $w \in N$ we have Hamiltonian function:

$$\theta_w([x]) = \langle \mu([x]), w \rangle$$

2.2.4 Chow and GIT quotients

Here we recall the definition of GIT, Chow, and limit quotients of a projective variety by a reductive algebraic group G. We also explain how, when G is a torus, they may be explicitely calculated via the Kempf-Ness theorem, and recall how GIT quotients behave under smooth blowup. The material here is used in Chapter **??**, where we give an exposition of the results of [34]. Thus, the material here may also be found in the preliminary sections of [34].

GIT quotients

Recall the basic setup of Mumford's geometric invariant theory, which provides a method for finding geometric quotients on open subsets of a scheme X when the acting algebraic group G is reductive. In [35] Mumford introduced the notion of a good categorical quotient, which can be shown to be unique if it exists.

Definition 9. *A surjective G-equivariant morphism $\pi : X \to Y$ is a good categorical quotient if the following hold:*

1. *We have $\mathcal{O}_Y = (\pi_* \mathcal{O}_X)^G$;*
2. *if V is a closed G-invariant subset of X then $\pi(V)$ is closed;*
3. *if V, W are closed G-invariant subsets of X and $V \cap W = \emptyset$ then we have $\pi(V) \cap \pi(W) = \emptyset$.*

Good quotients do not always exist for a given scheme X, but we might hope that there exists some dense open subset of X which does admit a good quotient.

Consider the affine case, where $X = \operatorname{Spec} A$. For G reductive then it can be shown that $X /\!\!/ G := \operatorname{Spec} A^G$ is a good categorical quotient.

The same ansatz works in the projective case once we make a choice of a lift of the action to the ring of sections of a given ample line bundle. This choice is known as a linearization of the group action. A linearization u of a group action G on X to L induces an action of G on the ring of sections $R(X, L) := \bigoplus_{j\geq 0} H^0(X, L^{\otimes j})$. Consider the scheme $X /\!\!/_u G := \operatorname{Proj} R(X, L)^G$. Note we have a birational map from X to $X /\!\!/_u G$, defined precisely at $x \in X$ such that there exists some $m > 0$ and $s \in R(X, L)^G_m$ such that $s(x) \neq 0$. Such a point is said to be semi-stable. If in addition $G \cdot x$ is closed and the stabilizer G_x is dimension zero, the point x is said to be stable. The set of semi-stable and stable points will be denoted by $X^{ss}(u)$ and $X^s(u)$ respectively.

Lemma 1 [35, Chapter 1, Section 4]**.** *The canonical morphism* $X^{ss}(u) \to X /\!\!/_u G := \operatorname{Proj} R(X, L)^G$ *is a good categorical quotient.*

Kempf-Ness approach to GIT quotients

One approach to calculating GIT quotients is via the Kempf-Ness theorem. Let $X \subseteq \mathbb{P}^N$ be a nonsingular complex projective variety and let G be reductive algebraic group acting effectively on $\mathbb{P}^N$, restricting to an action on X. Let K be the maximal compact subgroup in G, with Lie algebra $\mathfrak{k}$. The action of G is given by a representation $\rho : G \to \mathrm{GL}(N+1)$, and by choosing appropriate coordinates we may assume K maps to $U(N+1)$ and so preserves the Fubini-Study form. It can be checked that a moment map $\mu : X \to \mathfrak{k}^*$ is given by:

$$\mu([x]) \cdot a := \frac{x^t \rho_*(a)\hat{x}}{|x|^2} \tag{2.5}$$

Where x is any representative of $[x] \in X \subseteq \mathbb{P}^N$. Note we are now in the situation of the previous subsection, with $L = \mathcal{O}_X(1)$ under the embedding $X \subseteq \mathbb{P}^N$. This moment map is unique up to translations in $\mathfrak{k}^*$. A different choice of linearization in this setting corresponds to multiplying ρ by some character $\chi \in \mathfrak{X}(G)$.

Since $\chi(K)$ is compact it sits inside $S^1 \subset \mathbb{C}^*$, and hence we do not need to change coordinates when considering the effect on the moment map. When we plug this into (2.5) we see that we have translated the moment map by $\chi \in \mathfrak{X}(G) \otimes \mathbb{R} \cong \mathfrak{k}^*$. Moreover, taking the rth power of L corresponds to scaling the moment map by a factor of r. This gives a correspondence between rational elements $\chi \in \mathfrak{X}(G) \otimes \mathbb{Q} \subset \mathfrak{k}^*$ and linearizations of powers of L.

Example 9. *Suppose $G = T$ is an algebraic torus with character and cocharacter lattices M, N respectively. Then ρ is a diagonal matrix of characters $u_0, \ldots, u_N$ and we obtain:*

$$\mu([x]) = \frac{\sum_{j=0}^{N} |x_i|^2 u_i}{|x|^2} \in M$$

Then, by Atiyah, [32], and Guillemin-Sternberg, [33], the image of μ is a convex polytope $P \subset M$.

We will make use of the following theorem of Kempf and Ness. A proof is given in [35, Chapter 8]. See also the original work [36].

Kempf-Ness Theorem [36, Theorem 8.3]**.** *Let $X \subseteq \mathbb{P}^N$ be a nonsingular complex projective variety and let G be reductive algebraic group acting effectively on $\mathbb{P}^N$, restricting to an action on X. Consider a linearization of some power of L corresponding to a rational element $u \in \mathfrak{k}^*$.*

1. $X^{ss}(u) = \{x \in X | \overline{Gx} \cap \mu^{-1}(u) \neq \emptyset\}$.

2. *The inclusion of $\mu^{-1}(u)$ into $X^{ss}(u)$ induces a homeomorphism*

$$\mu^{-1}(u)/K \to X /\!\!/_u G$$

where $\mu^{-1}(u)/K$ is endowed with the quotient topology induced from the classical (closed submanifold topology) on $\mu^{-1}(u)$, and $X /\!\!/_u G$ is endowed with its classical (complex manifold) topology

We can use Theorem 2.2.4 to calculate GIT quotients by inspection. To be explicit, suppose $\mu^{-1}(u)/K$ has the structure of a complex projective variety and $q : X^{ss}(u) \to \mu^{-1}(u)/K$ is a G-invariant morphism which restricts to the topological quotient map on the moment fibre, such that $q_* \mathcal{O}_X^G = \mathcal{O}_Y$. The following fact is probably well known, but we prove it here for the reader's convenience.

Lemma 2. *The morphism q is a good categorical quotient, and hence is isomorphic to the GIT quotient map $X \to X /\!\!/_u G$.*

Proof. It is enough to show that q sends closed G-invariant subsets to closed subsets, and disjoint pairs of closed invariant subsets to disjoint pairs of closed subsets.

Firstly suppose that V is a G-invariant Zariski-closed subset of X. Then $q(V) = q(V \cap \mu^{-1}(u))$, and $V \cap \mu^{-1}(u)$ is K-invariant and closed in the classical topology of $\mu^{-1}(u)$. This implies that $q(V)$ is closed in the classical topology on $\mu^{-1}(u)/K \simeq X /\!\!/_u G$. But $q(V)$ is constructable, as the image of a Zariski-closed subset of X, and so we may conclude that $q(V)$ is Zariski-closed in $\mu^{-1}(u)/K \simeq X /\!\!/_u G$.

Now suppose V, W are G-invariant and Zariski-closed in X, with $x \in V$ and $y \in W$ such that $q(x) = q(y)$. By 2.2.4 we may take $x' \in \overline{Gx} \cap \mu^{-1}(u)$, $y' \in \overline{Gy} \cap \mu^{-1}(u)$ such that $q(x') = q(y')$. These two points lie in the same K-orbit. By the G-invariance of V, W we have $V \cap W \neq \emptyset$. □

GIT quotients under smooth blowup

Here we recall some results from [37], which we will use in the proof of Theorem **??**. Let G be a reductive group acting on X. Let L be an ample G-invariant line bundle on X. Fix some linearization of the G-action to L. Suppose V be a smooth closed G-stable subvariety of X, defined by some ideal sheaf $\mathcal{J}_V$. Let $f : W \to X$ be the blow-up of X along V.

The goal is to construct a linear action on W lifting the action on X, and describe the GIT quotient $W^{ss} \to W /\!\!/ G$ in terms of $X^{ss} \to X /\!\!/ G$ and f. First let us construct an ample line bundle on W. Let E be the exceptional divisor of the map f. Set $L_d := f^*L^{\otimes d} \otimes \mathcal{O}(-E)$. For sufficiently large d, L_d is ample.

Since $E \cong \mathbb{P}(N_{V,X})$ and $\mathcal{O}(-E)_{|E} \cong \mathcal{O}_{\mathbb{P}(N_{V,X})}(1)$ then the natural action of G on $N_{V,X}$ induces an action on $\mathcal{O}(-E)_{|E}$. We have $W \backslash E \cong X \backslash V$ and $\mathcal{O}(-E)_{|W\backslash E}$ is the trivial line bundle, so admits the product action. The action of G on L lifts to $f^*L^{\otimes d}$, and so we obtain a linear action on L_d. By [37,], for sufficiently large d we have $W^{ss} \subset X^{ss}$. We have the following result:

Lemma 3 [37, Lemma 3.11]**.** *If d is a sufficiently large multiple of e then the GIT quotient $W /\!\!/ G$ associated to the linearization described above is the blowup of $X /\!\!/ G$ along the image $V /\!\!/ G$ of V in $X /\!\!/ G$. In particular if $V /\!\!/ G$ is a divisor on $X /\!\!/ G$ then $W /\!\!/ G \cong X /\!\!/ G$.*

Chow and limit quotients

Recall the definition of the Chow quotient, as introduced in [38]. If G is any connected linear algebraic group and X is a projective G-variety, then orbit closures of points are generically of the same dimension and degree, and so define points in the corresponding Chow variety. The Chow quotient of the G-action on X is the closure of this set of points.

We now recall the definition of the limit quotient, from [35]. The limit quotient is discussed in detail in [39]. Let G be a reductive algebraic group, and X a projective G-variety. Suppose there are finitely many sets of semi-stable points $X_1, \ldots, X_r$ arising from G-linearized ample line bundles on X. Whenever $X_i \subseteq X_j$ holds, there is a dominant projective morphism $X_i /\!\!/ G \to X_j /\!\!/ G$ which turns the set of GIT quotients into an inverse system. The associated inverse limit Y admits a canonical morphism $\bigcap_{i=1}^{r} X_i \to Y$. The closure of the image of morphism is the limit quotient.

When G is an algebraic torus there are indeed finitely many semi-stable loci. Moreover, by [39, Corollary 2.7], we may calculate the limit quotient by taking the inverse limit of the subsystem obtained by only considering linearizations of powers of one fixed ample line bundle L. In [39, Proposition 2.5] it is shown that the Chow quotient and limit quotient coincide when G is an algebraic torus.

Definition 10. *Let X be a T-variety. Let $\pi : X \dashrightarrow Y$ be the Chow quotient map of X by its torus action. For any prime divisor Z on Y, the generic stabilizer on a component of $\pi^{-1}(Z)$ is a finite abelian group. The maximal order across these components is denoted m_Z. We may then define a boundary divisor for π, given by:*

$$B := \sum_Z \frac{m_Z - 1}{m_Z} \cdot Z \tag{2.6}$$

We call the pair (Y, B) the Chow quotient pair of the T-variety X.

2.3 T-varieties

In this section we briefly recall the theory of complex T-varieties. The best reference for more details is [40]. By a T-variety we will always mean a normal variety with an effective action of an algebraic torus T. Let T, M, N be as described in 2.2.1. Let us fix some additional definitions. By a *polyhedron* we wil mean the intersection of finitely many closed affine halfspaces of $N_{\mathbb{Q}}$, or its dual $M_{\mathbb{Q}}$. By a *cone* we mean the intersection of finitely many closed linear halfspaces of $N_{\mathbb{Q}}$ or its dual $M_{\mathbb{Q}}$. We will assume all cones are generated by primitive elements of their respective lattices.

2.3.1 Toric varieties

First, for context, let us recall the toric situation. A cone $\sigma \subset N_{\mathbb{R}}$ has a dual cone $\sigma^{\vee} := \{m \in M_{\mathbb{R}} | \langle m, n\rangle \geq 0 \ \forall n \in N_{\mathbb{R}}\}$, and we may construct the normal toric variety $\operatorname{Spec} \mathbb{C}[\sigma^{\vee} \cap M]$. The torus action is given by the M-grading of the algebra $\mathbb{C}[\sigma^{\vee} \cap M]$.

Conversely, given a normal affine toric variety X with algebraic torus T, $\mathbb{C}[X]$ is a semigroup subalgebra of $\mathbb{C}[M]$ of the form $\mathbb{C}[\sigma^{\vee} \cap M]$ for some strongly convex rational polyhedral cone $\sigma \subset N_{\mathbb{R}}$. We write $\mathrm{TV}(\sigma, N) := \operatorname{Spec} \mathbb{C}[\sigma^{\vee} \cap M]$. Face inclusions of cones correspond to equivariant open embeddings of varieties, and so from a complete fan of cones Σ we may construct a normal toric variety X_{Σ}.

Example 10. *Consider the variety $X = \mathrm{Bl}_z \mathbb{P}^1 \times \mathbb{P}^1$, where $z = ([1,0],[1,0])$. We may lift the 2-torus action on $\mathbb{P}^1 \times \mathbb{P}^1$ to X. This action becomes effective once we replace the torus T by $T/\pm Id$. As a toric variety it is given by the fan Σ in Figure 1.*

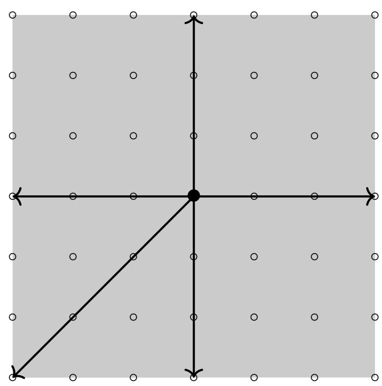


Figure 1: $\Sigma \subset N_{\mathbb{Q}}$

We also recall the description of equivariant polarizations of toric varieties via convex polytopes. Suppose we have a complete fan Σ, with rays $\Sigma(1)$. Any Cartier divisor on $X = \mathrm{TV}(\Sigma)$ is linearly equivalent to a T-equivariant one. Moreover we have the following exact sequence:

$$0 \to M \to \mathrm{CaDiv}_T(X) \cong \mathbb{Z}^{\Sigma(1)} \to \mathrm{Cl}(X) \to 0. \tag{2.7}$$

and relations $\mathrm{div}(\chi^u) = \sum_{\rho \in \Sigma(1)} \langle u, v_\rho \rangle D_\rho$.

To any lattice polytope $P \subset M_{\mathbb{Q}}$ we can associate a normal projective toric variety X_P given by its dual fan, and an ample divisor D_P given by coefficients on the ray generators of $\Sigma(1)$ specified by the equations of halfspaces defining P.

Example 11. *Consider the the following lattice polytope: The normal fan $\mathcal{N}(P)$*

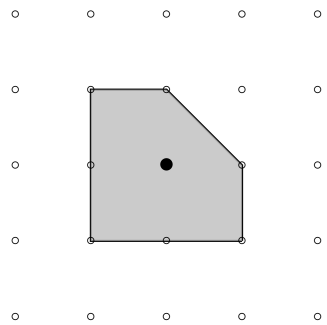


Figure 2: $P \subset M_{\mathbb{Q}}$

is that of $\mathrm{Bl}_z \mathbb{P}^1 \times \mathbb{P}^1$ *as in Example 10. We can calculate the corresponding divisor as*

$$D_P = - \sum_{\rho \in \Sigma(1)} D_\rho \sim -K_X.$$

Conversely the exact sequence 2.7 may be used to construct a polytope from any equivariant polarization of a projective toric variety. Finally, recall that a polytope is called Fano if the origin is contained in its interior, and each vertex is a primitive lattice point of M. Under the correspondence just described, Fano polytopes correspond exactly to anticanonical polarizations of toric varieties.

2.3.2 Higher complexity T-varieties

There is a successful program to extend the combinatorial dictionary of toric varieties to T-varieties of higher complexity. Roughly speaking, the combinatorial data lives over the Chow quotient of X by the T-action, so we have combinatorial data of dimension $\dim T$, and algebreo-geometric data of dimension of the complexity of the torus action.

Recall that one may define an abelian semigroup structure on the set of all polyhedra via Minkowski addition:

$$\Delta + \Delta' := \{v + v' \;| v \in \Delta,\; v' \in \Delta'\}.$$

It is well known that this gives a representation of any polyhedron $\Delta = P + \sigma$ where P is a convex polytope and σ. The cone σ is uniquely specified and is known as the tail cone of Δ. We will write $\operatorname{tail}\Delta = \sigma$, and call Δ a σ-tailed polyhedra in this situation.

The set of σ-tailed polyhedra form a semigroup $\mathrm{Pol}^+_{\mathbb{Q}}(N,\sigma)$ under Minkowski addition. We also include $\emptyset$ here, with $\emptyset + \Delta := \emptyset$ for any Δ. Recall the definition of a polyhedral divisor:

Definition 11. *Let $\sigma \subset N_{\mathbb{R}}$ be a cone, and Y a normal projective variety over $\mathbb{C}$. A polyhedral divisor on (Y, N) with tail cone σ is an element*

$$\mathcal{D} \in \mathrm{Pol}^+_{\mathbb{Q}}(N,\sigma) \otimes \mathrm{CaDiv}^+_{\mathbb{Q}}(Y),$$

where $\mathrm{CaDiv}^+_{\mathbb{Q}}(Y)$ *is the semigroup of effective $\mathbb{Q}$-Cartier divisors on Y. We define* $\operatorname{tail}\mathfrak{D} = \sigma$.

Let $\operatorname{Loc}\mathfrak{D} := Y \setminus \bigcup_{\mathfrak{D}_Z = \emptyset} Z$. The evaluation of $\mathfrak{D}$ at $u \in \sigma^\vee$ is defined to be the $\mathbb{Q}$-Cartier divisor on Y given by:

$$\mathfrak{D}(u) := \sum_{\mathfrak{D} \neq \emptyset} \min_{v \in \mathfrak{D}_P} \langle v, u \rangle Z_{|\operatorname{Loc}\mathfrak{D}}.$$

Definition 12. *A polyhedral divisor $\mathfrak{D}$, as defined above, is called a p-divisor if $\mathfrak{D}(u)$ is semiample for $u \in \sigma^\vee$ and, in addition, big for $u \in \mathrm{int}(\sigma^\vee)$. Note if* $\operatorname{Loc}\mathfrak{D}$ *affine this is automatically satisfied.*

By [22, Proposition 3.1], p-divisor defines an affine T-variety in the following manner. Note for $u \in \sigma^\vee$ we have $\mathfrak{D}(u) + \mathfrak{D}(u') \leq \mathfrak{D}(u + u')$. Consider the sheaf of N-graded algebras

$$\mathcal{A} := \bigoplus_{w \in \sigma^\vee} \mathcal{O}_{\operatorname{Loc}\mathfrak{D}}(\mathfrak{D}(u))\chi^u.$$

Note the semiample and big conditions in the definition of a p-divisor ensure that the algebra $H^0(\operatorname{Loc}\mathfrak{D}, \mathbb{A})$ is finitely generated. We obtain T-varieties:

$$\tilde{\mathrm{TV}}(\mathfrak{D}) := \operatorname{Spec}_{\operatorname{Loc}\mathfrak{D}} \mathcal{A},\ \mathrm{TV}(\mathfrak{D}) := \operatorname{Spec} H^0(\operatorname{Loc}\mathfrak{D}, \mathcal{A})$$

together with a good quotient $\tilde{\mathrm{TV}}(\mathfrak{D}) \to Y$ (the Chow quotient) of the torus action, and an equivariant contraction $r : \tilde{\mathrm{TV}}(\mathfrak{D}) \to \mathrm{TV}(\mathfrak{D})$. The T-variety $\mathrm{TV}(\mathfrak{D})$ remains unchanged if we pull back $\mathfrak{D}$ by some birational $\varphi : Y' \to Y$. Moreover, modifying $\mathfrak{D}$ by an element in the image of the natural map:

$$N \otimes_{\mathbb{Z}} \mathbb{C}(Y)^* \to \operatorname{Pol}^+_{\mathbb{Q}}(N, \sigma) \otimes \operatorname{CaDiv}^+_{\mathbb{Q}}(Y)$$

does not change $\mathrm{TV}(\mathfrak{D})$. In the converse direction, by [22, Proposition 3.4], any affine T-variety X is of the form $\mathrm{TV}(\mathfrak{D})$ for some p-divisor $\mathfrak{D}$.

Example 12. *Let Y be a normal projective variety and D an ample integral Cartier divisor on Y. Let $\mathfrak{D} = [1, \infty) \otimes D$. Then we see*

$$\mathcal{A} = \bigoplus_{n \in \mathbb{Z}^{\geq 0}} \mathcal{O}(nD)\chi^n,$$

and so $\tilde{\mathrm{TV}}(\mathfrak{D})$ is the total space of the line bundle $\mathcal{O}(D)$.

Example 13. *Any toric variety X with torus T may be considered a higher complexity T-variety with respect to any proper subtorus $T' \subset T$. Such a subtorus is given by some surjection of character lattices $p : M \to M'$. Writing M_Y for the kernel of p and denoting the dual surjection $q : N \to N_Y$, we have mutually dual short exact sequences:*

$$0 \longrightarrow N' \longrightarrow N \xrightarrow{q} N_Y \longrightarrow 0$$

$$0 \longleftarrow M' \xleftarrow{p} M \longleftarrow M_Y \longleftarrow 0$$

Suppose $X = \mathrm{TV}(\delta)$ for some cone $\delta \subset \mathbb{N}_{\mathbb{Q}}$. Let $\sigma = \delta \cap N'_{\mathbb{Q}}$. Then the surjection $M \to M'$ induces a surjection $\delta^{\vee} \to \sigma^{\vee}$. Let Σ be the coarsest fan which refines all images of faces of δ under q. It may be shown that $Y = \mathrm{TV}(\Sigma)$ is the Chow quotient of X by T' up to normalization. Then X is the T'-variety associated to the p-divisor:

$$\mathfrak{D}^{\delta} := \sum_{a \in \Sigma(1)} \mathfrak{D}_a \otimes \overline{\operatorname{orb}}(a)$$

Where $\mathfrak{D}_a := q^{-1}(a) \cap \delta$, *and* $\overline{\mathrm{orb}}(a)$ *is the torus-invariant divisor on* Y *associated to the ray* a *under the usual orbit-cone correspondence.*

Example 14. *Consider the toric variety given by the following cone:*

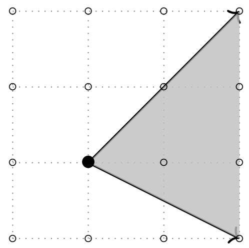
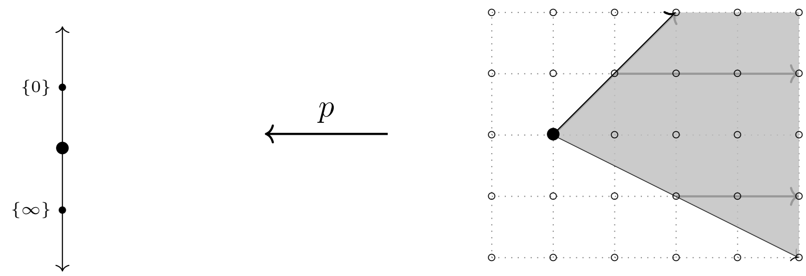


Figure 3: $\sigma \subset N_{\mathbb{Q}}$

and subtorus action given by sublattice $N' := \mathbb{Z}e_2 \subset N$. *We may read off the downgrade* p*-divisor from the following diagram:*



$\sigma \subset N_{\mathbb{Q}}$ $\sigma \subset N_{\mathbb{Q}}$

and we see this downgraded toric variety is given by the p*-divisor:*

$$\mathfrak{D} = \{0\} \otimes [1, \infty) + \{\infty\} \otimes [2, \infty).$$

One can define morphisms of p-divisors, and this correspondence turns out to be an equivlance of categories between affine T-varieties and p-divisors up to equivalence via the modifications mentioned above. We will not make use of the general data of a morphism of p-divisors but we will discuss the special case needed for globalization.

By [41] we have a method of gluing p-divisors in a natural way to construct general T-varieties, generalizing the notion of a fan of cones in the toric case. Suppose $\mathfrak{D}', \mathfrak{D}$ are polyhedral divisors. We write $\mathfrak{D}' \leq \mathfrak{D}$ if $\mathfrak{D}'_Z$ is a face of $\mathfrak{D}_Z$ for each Cartier divisor Z on Y.

Now suppose additionally that $\mathfrak{D}', \mathfrak{D}$ are p-divisors. If $\mathfrak{D}' \leq \mathfrak{D}$ then we obtain graded morphisms of the respective sheaves of algebras $\mathcal{A} \to \mathcal{A}'$ giving a T-equivariant morphism $\mathrm{TV}(\mathfrak{D}') \to \mathrm{TV}(\mathfrak{D})$. Unlike the toric case, this not necessarily an open embedding, as we see in the following example.

Example 15. *Consider the setup in Example () with $Y = \mathbb{P}^1$ and $D = \{\infty\}$. Then $r : \tilde{\mathrm{TV}}(\mathfrak{D}) \to \mathrm{TV}(\mathfrak{D})$ is isomorphic to the blow up of $\mathbb{A}^2$ at the origin. If $\mathfrak{D}' = \emptyset \otimes \{\infty\}$ then $\mathfrak{D}' \leq \mathfrak{D}$ but $\tilde{TV}(\mathfrak{D}')$ intersects the exceptional divisor in $\tilde{\mathrm{TV}}(\mathfrak{D})$, and so the induced map $\mathrm{TV}(\mathfrak{D}') \to \mathrm{TV}(\mathfrak{D})$ is not an open embedding.*

2.3.3 f-divisors and divisorial polytopes

In this section we focus on the complexity one case, where T-equivariant open embeddings may be characterized using degree polyhedra of p-divisors:

$$\deg \mathfrak{D} := \sum_{y \in Y} \mathfrak{D}_y.$$

Note that $\deg \mathfrak{D} \neq \emptyset \iff \mathrm{Loc}\,\mathfrak{D} = Y$. It is also not hard to see that a complexity one polyhedral divisor $\mathfrak{D}$ is a p-divisor if and only if the following two conditions hold:

1. $\deg \mathfrak{D} \subsetneq \mathrm{tail}\,\mathfrak{D}$;

2. $\mathfrak{D}(w)$ has a principal multiple for $w \in (\mathrm{tail}\,\mathfrak{D})^\vee$ such that $w^\perp \cap \deg \mathfrak{D} \neq 0$.

We have the following characterization of open embeddings:

Theorem 1 [42]**.** *Let Y be a curve and $\mathfrak{D}, \mathfrak{D}'$ polyhedral and p-divisors respectively, such that $\mathfrak{D}' \leq \mathfrak{D}$. Then $\mathfrak{D}'$ is a p-divisor and $\mathrm{TV}(\mathfrak{D}') \to \mathrm{TV}(\mathfrak{D})$ is an open embedding if and only if $\deg \mathfrak{D}' = \deg \mathfrak{D} \cap \mathrm{tail}\,\mathfrak{D}'$.*

Note that in complexity one $\mathrm{Loc}\,\mathfrak{D}$ is always birational to $\mathbb{P}^1$. Therefore, via pullback, any complexity one normal affine $\mathcal{T}$-variety may be realized as $\mathrm{TV}(\mathfrak{D})$ where $\mathfrak{D}$ is a p-divisor over $Y = \mathbb{P}^1$. Note that in this case that condition 2. above is automatically satisfied.

In complexity one, the object taking the place of a fan of cones is an f-divisor, first introduced in [42]. We recall this construction now. By a *polyhedral decomposition* we mean a decomposition of $N_{\mathbb{Q}}$ into a collection of polyhedra, closed under intersection. A polyhedral decomposition has a *tail fan*: a fan comprised of exactly the tail cones of the polyhedra in the decomposition. If $\mathcal{G}$ is a polyhedral decomposition then we write $\operatorname{tail}\mathcal{G}$ for its tail fan.

Definition 13. *An f-divisor is a pair $\mathcal{S} = \left(\sum_{y\in\mathbb{P}^1} S_y \otimes \{y\},\ \mathfrak{deg}\right)$, where:*

1. *S_y are polyhedral subdivisions sharing a tail fan Σ.*
2. *$\mathfrak{deg}$ is some subset of $|\Sigma|$.*
3. *For any full-dimensional marked $\sigma \in \Sigma$ then*

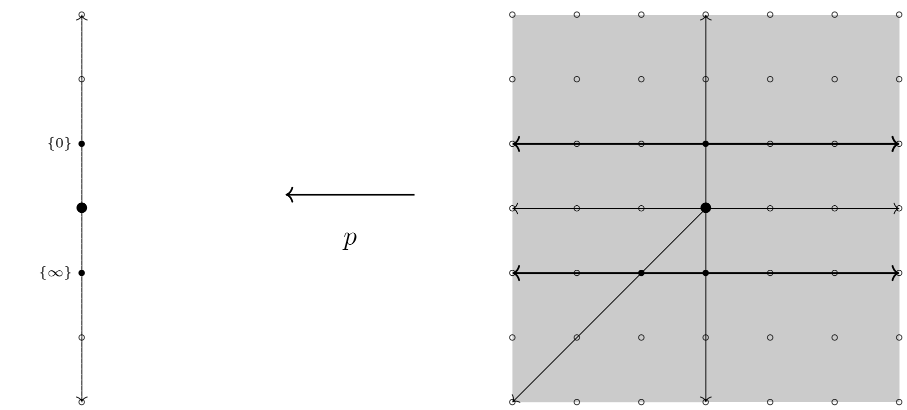
$$\mathfrak{D}^\sigma := \sum \mathfrak{D}^\sigma_y \otimes \{y\}$$

   is a p-divisor, where $\mathfrak{D}^\sigma_y$ is the unique polyhedron in S_y with $\operatorname{tail}(\mathfrak{D}^\sigma_y) = \sigma$, and we have $\deg \mathfrak{D}^\sigma = \mathfrak{deg} \cap \sigma$.

4. *Only finitely many S_y may differ from the tailfan Σ. We call the finite collection of $S_y \neq \Sigma$ the non-trivial slices of $\mathcal{S}$.*

In fact to construct $\mathfrak{deg}$ from the rest of the data it is only necessary to know whether $\deg \mathfrak{D}^\sigma$ is empty or not for each σ. We call those cones σ with $\deg \mathfrak{D}^\sigma \neq \emptyset$ the *marked* tailcones of an f-divisor.

Example 16. *Let us describe the downgrade procedure for the toric variety from Example 10, with respect to the subtorus given by sublattice $N' := \mathbb{Z}e_1 \subset N$. We may read off the downgrade f-divisor from the following diagrams:*



$\Sigma_Y \subset N_Y$ $\Sigma \subset N$

and we see this downgraded toric variety is given by the f-divisor:

S_0 $S_\infty = \deg S$

Example 17. *Here we give an example of an f-divisor describing the complexity one threefold (2.30) from the list of Mori and Mukai [18].*

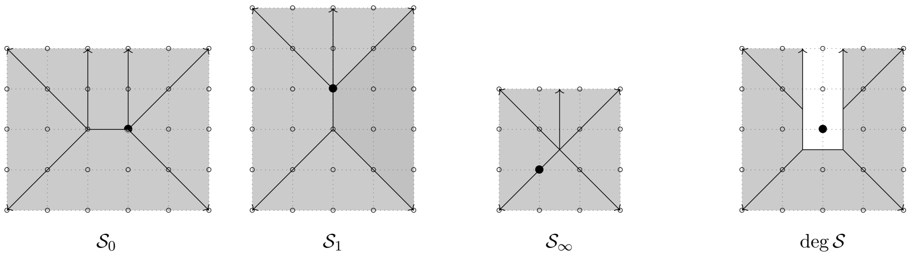


Figure 7: f-divisor of a complexity one threefold

In complexity one there is a generalization of the correspondence between lattice polytopes and polarized projective toric varieties. First we recall the description of Cartier divisors on the T-variety $\mathrm{TV}(\mathcal{S})$ of an f-divisor $\mathcal{S}$ from [43]:

Definition 14. *A Cartier support function on an f-divisor $\mathcal{S}$ is a sum:*

$$h = \sum_{y \in \mathbb{P}^1} h_y \otimes y,$$

where:

1. *For all $y \in \mathbb{P}^1$, $h_y : N_{\mathbb{Q}} \to \mathbb{Q}$ is a piecewise affine function with respect to the decomposition S_y.*
2. *For all $y \in \mathbb{P}^1$ and $v \in N$ if $k \cdot v$ is a lattice point then $k \cdot h_y(v) \in ZZ$*
3. *For each y, the linear part $lin(h_y)$ is independent of y.*
4. *We have $h_y \neq lin(h_y)$ for at most finitely many $y \in \mathbb{P}^1$.*
5. *Writing $h_{|\sigma}(0) = \sum a_y \cdot y$, where a_y is the evaluation of the restriction of h_y to $\mathcal{S}_y^\sigma$ at 0, then for every marked $\sigma \in \operatorname{tail} \mathcal{S}$, the divisor $h_{|\sigma}(0)$ is principal.*

We write $\mathrm{CaSF}(\mathcal{S})$ *for the group of Cartier support functions on $\mathcal{S}$.*

It is shown in [43] that $\mathrm{CaSF}(\mathcal{S})$ is isomorphic to the group of T-invariant Cartier Divisors on $\mathrm{TV}(\mathcal{S})$. If h is a Cartier support function we denote the corresponding divisor by D_h. We call h ample if D_h is ample.

From what we have recalled so far, an equivariantly polarized T-variety (X, L) may be given by an f-divisor $\mathcal{S}$ and a choice of ample Cartier support function

$h \in \mathrm{CaSF}(\mathcal{S})$. We finally recall the construction the novel results of this thesis make most use of, namely the dual picture. A divisorial polytope may be thought of to polarized complexity one T-varieties what a polytope is to a toric variety. Note the following definition differs from that of say [19] by a shift of a divisor of degree 2.

Definition 15. *A divisorial polytope is a function Ψ on a lattice polytope $\Box \subset M_{\mathbb{R}}$:*

$$\Psi : \Box \to \mathrm{Div}_{\mathbb{Q}} \mathbb{P}^1, \ u \mapsto \Sigma_{y \in \mathbb{P}^1} \Psi_y(u) \cdot \{y\},$$

such that:

- *For $y \in \mathbb{P}^1$ the function $\Psi_y : \Box \to \mathbb{R}$ is the minimum of finitely many affine functions, and $\Psi_y \equiv 0$ for all but finitely many $y \in \mathbb{P}^1$.*
- *Each Ψ_y takes integral values at the vertices of the polyhedral decomposition its regions of affine linearity induce on $\Box$.*
- $\deg \Psi(u) > -2$ *for $u \in int(\Box)$;*

Let $\Psi : \Box \to \mathrm{CaDiv}_{\mathbb{Q}} Y$ be a divisorial polytope. For $y \in \mathbb{P}^1$ consider the piecewise affine concave function on $N_{\mathbb{Q}}$ given by:

$$\Psi_y^*(u) := \min_{v \in \Box}(\langle v, u \rangle - \Psi_P(u)),$$

Let S_y be the polyhedral subdivision induced by Ψ_y^*, with tailfan Σ. Let $\mathfrak{deg}$ be the set of cones $\sigma \in \Sigma$ such that $\deg \circ \Psi_{|F_\sigma} \equiv 0$, where F_σ is the face of $\Box$ where $\langle \cdot, v \rangle$ takes its minimum value for $v \in \sigma$. This defines an f-divisor $\mathcal{S} = \sum_{y \in \mathbb{P}^1} S_y \otimes \{y\}$, and Ψ_y^* is seen to be a Cartier support function on $\mathcal{S}$. Moreover may reverse the above construction and obtain a divisoral polytope from any pair $\mathcal{S}, h$ where $\mathcal{S}$ is an f-divisor and h is an ample Cartier support function.

For our results we are particularly interested in Fano T-varieties. We recall the notion of a Fano divisorial polytope. By [], Fano divisorial polytopes correspond to Fano T-varieties $(X, -K_X)$.

Definition 16. *A divisorial polytope $\Psi : \Box \to \mathrm{Div}_{\mathbb{Q}} \mathbb{P}^1$ is said to be Fano if additionally we have that:*

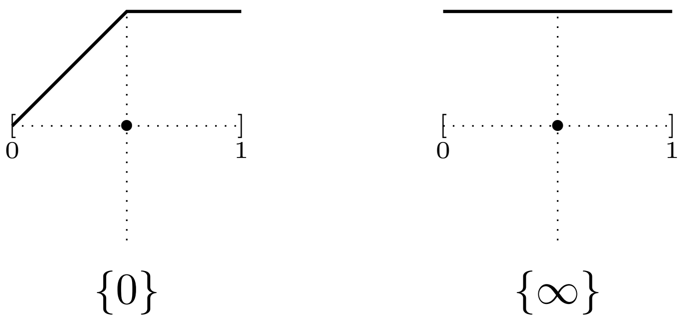
- *The origin is an interior lattice point of $\Box$.*

- *The affine linear pieces of each Ψ_y are of the form $u \mapsto \frac{\langle v,u\rangle-\beta+1}{\beta}$ for some primitive lattice element $v \in N$;*

- *Every facet F of $\Box$ with $(\deg \circ \Psi_{|F}) \neq -2$ has lattice distance 1 from the origin.*

Example 18. *We can perform a downgrade operation on a polytope describing a toric variety. Given a polytope $P \subset M_{\mathbb{Q}}$ and a complexity one subtorus action given by some surjection $p : M \to M'$, choose a section $s : M' \to M$. Set $\Box = p(P)$ and $\Psi_0(u) := \max(p^{-1}(u) - s(u))$, $\Psi_\infty := \min(p^{-1}(u) - s(u))$. We obtain a divisorial polytope $\Psi := \Psi_0 \otimes \{0\} + \Psi_\infty \otimes \{\infty\}$. The corresponding T-variety is then isomorphic to the toric variety associated to P.*

Example 19. *Consider the toric variety $\mathrm{Bl}_z\, \mathbb{P}^1 \times \mathbb{P}^1$ with its anticanonical polarization. If we start with the corresponding polytope given in Example 11 and perform the downgrade operation, we end up with divisorial polytope:*



It is easy to see, for example, that the resulting f-divisor from the operation Example 18 coincides with the downgrade f-divisor from Example 16.

We conclude this section with a few pieces of terminology for divisorial polytopes. The push-forward of the measure induced by ω is known as the Duistermaat-Heckman measure, independent of the choice of ω and which we denote by ν. Denote the standard measure on $M_{\mathbb{R}}$ by η.

Definition 17. *Let Ψ be a divisorial polytope.*

- *The degree of Ψ is the map $\deg \Psi : \Box \to \mathbb{R}$ given by $u \mapsto \deg(\Psi(u))$.*

- *The barycenter of Ψ is the point $\mathrm{bc}(\Psi) \in \Box$ such that for all $v \in N_{\mathbb{R}}$:*

$$\langle \mathrm{bc}(\Psi), v\rangle = \int_{\Box} v \cdot \deg \Psi \; d\eta = \int_{\Box} v d\nu.$$

Note by the second equality we see $\mathrm{bc}(\Psi) = \mathrm{bc}_\nu(\Box)$.

- *The volume of* Ψ *is defined to be:*

$$\operatorname{vol}\Psi = \int_\Box \deg\Psi \; d\eta = \int_\Box d\nu.$$

2.4 Equivariant K-stability

In this section we recall definitions of K-stability. In summary the K-stability criteria are concerned with the positivity of certain numerical invariants associated to *test configurations* of our original space. We do not go into technical detail about how K-stability relates to the existence of canonical metrics here, but give the definitions and theorems we will rely on later in the thesis.

2.4.1 Twisted equivariant K-stability

Here we recall notions of Twisted equivariant K-stability, following [14]. Let X be a Fano manifold with the action of a complex reductive group G of automorphisms containing a maximal torus T. Fix a T-invariant Kähler form $\omega \in 2\pi c_1(X)$ induced by the Fano condition. First we define test-configurations of X.

Definition 18. *A* G*-equivariant test configuration for* (X, L) *is a* $\mathbb{C}^*$*-equivariant flat family* $\mathcal{X}$ *over the affine line equipped with a relatively ample equivariant* $\mathbb{Q}$*-line bundle* $\mathcal{L}$*, such that:*

1. *The* $\mathbb{C}^*$*-action* λ *on* $(\mathcal{X}, \mathcal{L})$ *lifts the standard action on* $\mathbb{A}^1$*;*

2. *The general fiber is isomorphic to* X *and* $\mathcal{L}$ *is the relative anti-canonical bundle of* $\mathcal{X} \to \mathbb{A}^1$*.*

3. *The action of* G *extends to* $(\mathcal{X}, \mathcal{L})$ *and commutes with the* $\mathbb{C}^*$*-action* λ*.*

A test configuration with $\mathcal{X} \cong X \times \mathbb{A}^1$ is called a product configuration. If such an isomorphism exists and is $\mathbb{C}^*$-equivariant then we call the test configuration *trivial*. Finally a test configuration with normal special fiber is called *special*.

Suppose from now on $G = T$ is a maximal torus in $\mathrm{Aut}(X)$. We then have an induced $T' = T \times \mathbb{C}^*$-action on the special fiber. The canonical lift of T'-action to $-K_{\mathcal{X}_0}$ induces a canonical choice of moment map $\mu : \mathcal{X}_0 \to M'_{\mathbb{R}}$ as in **??**. The restriction of λ to $\mathcal{X}_0$ is generated by the imaginary part of a T'-invariant vector field w, and by an abuse of notation we also write $w \in N'_{\mathbb{R}}$ for the corresponding one-parameter subgroup. The moment map μ then specifies Hamiltonian functions

$$\theta_w := \langle \mu, w \rangle : \mathcal{X}_0 \to \mathbb{R},$$

as we have seen in Section 2.2.3. We now recall a definition of the Donaldson-Futaki invariant in the twisted modified setting.

Definition 19. *The twisted modified Donaldson-Futaki character of a special test configuration* $(\mathcal{X}, \mathcal{L})$ *is given by:*

$$\mathrm{DF}_{t,\xi}(\mathcal{X}, \mathcal{L}, w) = \mathrm{DF}_{\xi}(\mathcal{X}, \mathcal{L}, w) + \frac{(1-t)}{V} \int_{\mathcal{X}_0} (\max_{\mathcal{X}_0} \theta_w - \theta_w) e^{\theta_\xi}\ \omega^n.$$

where $V = \frac{1}{n!} \int_{\mathcal{X}_0} \omega^n$ *is the volume of* $\mathcal{X}_0$*, and* $\mathrm{DF}_{\xi}(\mathcal{X}, \mathcal{L}, w) = \frac{1}{V} \int_{\mathcal{X}_0} \theta_w \omega^n$ *is the modified Donaldson-Futaki invariant of the configuration, in the form given in [31, Lemma 3.4].*

Note that if $(\mathcal{X}, \mathcal{L})$ is a product configuration then we have $\mathcal{X}_0 \cong X$. Assuming X is non-toric, in this case the maximality of T in $\mathrm{Aut}(X)$ ensures that the restriction of λ to $\mathcal{X}_0$ is a one parameter subgroup of T, given by a choice of $w \in N$. We will use the following definition of K-stability, as given in [14].

Definition 20. *We say the triple* (X, t, ξ) *is* G*-equivariantly* K*-semistable if* $\mathrm{DF}_{t,\xi}(\mathcal{X}, \mathcal{L}, w) \geq 0$ *for all* G*-equivariant special configurations* $(\mathcal{X}, \mathcal{L}, w)$*. We say* (X, t, ξ) *is* K*-stable if, in addition, equality holds precisely for product configurations.*

In the non-twisted case K-stability implies the existence of a Kähler-Ricci soliton:

Theorem 2 Berman-Witt-Nystrom**.** *If* (X, ξ) *admits a Kähler-Ricci soliton then* (X, ξ) *is* K*-stable.*

In [14] a result in the converse direction is obtained, which we will make heavy use of in Chapter **??**:

Theorem 3 [14, Proposition 10] **.** *Let* X *be a polarized Fano manifold, with Kähler form* ω*. Let* $t \in [0, 1]$ *and* ξ *be a soliton candidate for* X*. If* (X, t) *is* G*-equivariantly* K*-semistable then for all* $s < t$ *there exists* $\omega_s \in 2\pi c_1(X)$ *such that* $\mathrm{Ric}(\omega_s) - \mathcal{L}_\xi \omega_s = s\omega_s + (1-s)\omega$.

2.4.2 K-stability of T-varieties

Here we review K-stability in complexity one. In [16], Ilten and Suess described non-product special test configurations for a T-variety of complexity one in terms of its divisorial polytope. Let X be a Fano complexity one T-variety, corresponding to the Fano divisorial polytope $\Psi : \square \to \mathbb{P}^1$. Let M, N be the usual character and cocharacter lattices respectively. Let $M' := M \times \mathbb{Z}$ and $N' = \mathrm{Hom}(M', \mathbb{Z})$ with associated vector spaces $M'_{\mathbb{Q}}, N'_{\mathbb{Q}}$.

Theorem 4 [16]**.** *There exists some $y \in \mathbb{P}^1$ such that for at most one $z \neq y$ the function Ψ_z has non-integral slope at any $u \in \square$, such that $\mathcal{X}_0$ is the toric variety corresponding to the following polytope:*

$$\Delta_y := \left\{ (u, r) \in M'_{\mathbb{Q}} \;\middle|\; u \in \square,\ -1 - \sum_{z \neq y} \Psi_z(u) \leq r \leq 1 + \Psi_y(u) \right\}.$$

Furthermore, the induced $\mathbb{C}^*$-action on $\mathcal{X}_0$ is given by the one-parameter subgroup of $T' = T \times \mathbb{C}^*$ corresponding to $v' = (-mv, m) \in N'$, for some $v \in N$. From the point of view of K-stability it is enough to consider only those configurations with $m = 1$.

By the definition of a divisorial polytope, there are only finitely many distinct polytopes Δ_y: Suppose Ψ corresponds to f-divisor $\mathcal{S}$. For y such that $\mathcal{S}_y \neq \operatorname{tail} \mathcal{S}$ we obtain a polytope Δ_y, but for all other y we observe that Δ_y is independent of y, equal to the polytope:

$$\Delta_{\text{gen}} := \left\{ (u, r) \in M'_{\mathbb{Q}} \;\middle|\; u \in \square,\ -1 - \deg \Psi(u) \leq r \leq 1 \right\}$$

Definition 21. *The special configuration polytopes of a Fano T-variety are the polytopes Δ_y (for $\mathcal{S}_y \neq \operatorname{tail} \mathcal{S}$) and Δ_{gen} corresponding to normal toric varieties.*

See Figure **??** and Figure **??** for two examples of special configuration polytopes, and Appendix **??** for the data to construct the special configurations we need for results in this thesis. As observed in [16], we also obtain a description of the (non-twisted) Donaldson-Futaki character of $(\mathcal{X}_0, \xi')$:

$$\mathrm{DF}_{\mathcal{X}_0, \xi'}(v') = \frac{1}{\operatorname{vol} \Delta_y} \left(\int_{\Delta_y} \langle u', v' \rangle \cdot e^{\langle u', \xi' \rangle} du' \right), \tag{2.8}$$

with $\xi', v' \in N_\mathbb{R} \times \mathbb{R}$. On the other hand, for $v, \xi \in N_\mathbb{R}$ one obtains:

$$\mathrm{DF}_{X,\xi}(v) = \frac{1}{\operatorname{vol}\Phi}\left(\int_\square \langle u, v\rangle \cdot \deg\Phi(u) \cdot e^{\langle u,\xi\rangle}\, du\right), \tag{2.9}$$

For non-product configurations the central fiber of a special test configuration is a normal toric variety. The following formula is the toric one obtained in [14].

Lemma 4 [14]**.** *Let $(\mathcal{X}, \mathcal{L}, v')$ be the special test configuration of X corresponding to the polytope Δ_y and the element $v' = (-v, 1) \in N'$, as in Theorem 4. Then:*

$$\mathrm{DF}_t(\mathcal{X}, \mathcal{L}, v') = t\langle \mathrm{bc}(\Delta_y), v'\rangle + (1-t)\max_{x\in\Delta_y}\langle x, v'\rangle.$$

We give the formula for the product configuration case in complexity one:

Lemma 5. *Let $(\mathcal{X}, \mathcal{L}, w)$ be a product configuration of the Fano complexity one T-variety X given by divisorial polytope $\Psi : \square \to \operatorname{Div}_\mathbb{Q}\mathbb{P}^1$. Then for $t \in [0,1]$ we have:*

$$\mathrm{DF}_t(\mathcal{X}, \mathcal{L})() = t\langle \mathrm{bc}(\Psi), w\rangle + (1-t)\max_{x\in\square}\langle x, w\rangle.$$

Proof. In [16] it was shown that $\mathrm{DF}(X \times \mathbb{A}^1, \mathcal{L})(w) = \langle \mathrm{bc}(\Psi), w\rangle$. Note by definition $\theta_w(x) = \langle \mu(x), w\rangle$. We may now push forward the integrals to the image of the moment map $\square$:

$$\begin{aligned}\mathrm{DF}_t(\mathcal{X}, \mathcal{L})(w) &= \mathrm{DF}(\mathcal{X}, \mathcal{L})(w) + \frac{(1-t)}{V}\int_X (\max\theta_w - \theta_w)\omega^n \\ &= \langle \mathrm{bc}(\Psi), w\rangle + \frac{(1-t)}{\operatorname{vol}\Psi}\int_\square \max_{x\in\square}\langle x, w\rangle - \langle \cdot, w\rangle d\eta \\ &= t\langle \mathrm{bc}(\Psi), w\rangle + (1-t)\max_{x\in\square}\langle x, w\rangle.\end{aligned}$$

□

Chapter 3

$R(X)$ in complexity one

Recall, as discussed in the introduction, that one approach to the existence of Kähler-Einstein metrics is the study of the continuity path, that is solutions $\omega_t \in 2\pi c_1(X)$ to the equation

$$\mathrm{Ric}(\omega_t) = t\omega_t + (1-t)\omega.$$

for $t \in [0,1]$. By [7] there is always a solution for $t = 0$. However, Tian [44] showed that for some t sufficiently close to 1 there may not be a solution for certain Fano manifolds. It is natural to ask for the supremum of permissible t, which turns out to be independent of the choice of ω.

Definition 22. *Let (X, ω) be a Kähler manifold with $\omega \in 2\pi c_1(X)$. Define:*

$$R(X) := \sup(t \in [0,1] : \exists\ \omega_t \in 2\pi c_1(X)\ \ \mathrm{Ric}(\omega_t) = t\omega_t + (1-t)\omega).$$

This invariant was first discussed, although not explicitly defined, by Tian in [45]. It was first explicitely defined by Rubenstein in [46] and was further studied by Szekelyhidi in [47]. It is sometimes referred to as the greatest lower bound on Ricci curvature.

In [46] Rubenstein showed relation between $R(X)$ and Tian's alpha invariant $\alpha(X)$, and in [48] conjectured that $R(X)$ characterizes the K-semistability of X. This conjecture was later verified by Li in [49].

In [50] Li determined a simple formula for $R(X_\Delta)$, where X_Δ is the polarized toric Fano manifold determined by a reflexive lattice polytope Δ. This result was later recovered in [14], by Datar and Székelyhidi, using notions of G-equivariant

K-stability. Previously $R(X)$ has been calculated for group compactifications by Delcroix [51] and for homogeneous toric bundles by Yao [52]. Let us briefly recall the toric formula.

Theorem 5 Li**.** *Suppose X is a smooth Fano toric variety. Let P be the corresponding Fano polytope. If* $\mathrm{bc}(P) = 0$ *then X is Kähler-Einstein and $R(X) = 1$. Otherwise let q be the intersection of the ray generated by $-\mathrm{bc}(P)$ with the boundary ∂P. We then have:*

$$R(X) = \frac{|q|}{|q - \mathrm{bc}(P)|}.$$

Example 20. *Consider the toric variety $X = \mathrm{Bl}_z \mathbb{P}^1 \times \mathbb{P}^1$ from Example* **??**. *It is then easy to calculate $R(X)$ from the polytope P given in Example* **??**. *We have* $\mathrm{bc}(P) = (-2/21, -2/21)$ *and* $q = (1/2, 1/2)$ *so* $R(X) = 21/25$.

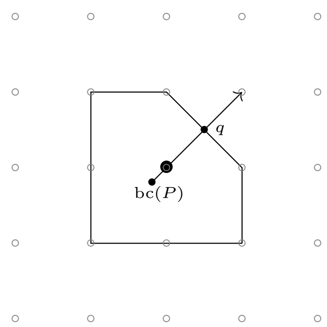


Figure 1: $R(X)$ calculation for a toric X

Using similar methods to [14] we obtain an effective formula for manifolds with a torus action of complexity one, in terms its divisorial polytope. Let $\Phi : \square \to \mathrm{Div}_{\mathbb{Q}} \mathbb{P}^1$ be the Fano divisorial polytope corresponding to a smooth Fano complexity one T-variety X. Let $\{\Delta_i\}_{i=1,\ldots,r}$ be the finite set of degeneration polytopes corresponding to central fibres of the non-product test configurations of X, as in Definition **??**.

To state our result we must introduce a little more notation. Suppose we have $\mathrm{bc}(\Delta_i) \neq 0$ for some i. Let F_i be the face of Δ_i in which q_i lies, and let S be the set of indices i for which $\mathrm{bc}(\Delta_i) \neq 0$ and all outer normals to F_i lie in H. Recall the definition of the Duistermatt Heckman measure ν, and associated weighted barycenter of $\square$ given in Section **??**. Suppose $\mathrm{bc}_\nu(\square) \neq 0$. Let q be the intersection of the ray generated by $-\mathrm{bc}_\nu(\square)$ with $\partial\square$. Consider the halfspace

$H := N_{\mathbb{R}} \times \mathbb{R}^+ \subset N'_{\mathbb{R}}$. Let q_i be the point of intersection of $\partial\Delta_i$ with the ray generated by $-\operatorname{bc}(\Delta_i)$.

Note, by the equation for the Donaldson Futaki invariants (2.8) and Theorem 3, we know that $R(X) = 1$ iff $\operatorname{bc}_\nu(\Box) = 0$ and $S = \emptyset$. We may now state our result:

Theorem 6 [20, Theorem 1.1]**.** *Let X be a complexity one Fano T-variety as above. If $\operatorname{bc}_\nu(\Box) = 0$ and $S = \emptyset$ then $R(X) = 1$. Otherwise:*

$$R(X) = \min \left\{ \frac{|q|}{|q - \operatorname{bc}_\nu(\Box)|} \right\} \cup \left\{ \frac{|q_i|}{|q_i - \operatorname{bc}(\Delta_i)|} \right\}_{i \in S}.$$

Remark 1. *To see that this formula is truly a generalization of Li's result, consider the situation of a toric downgrade. Here we have*

Example 21. *Consider the $(\mathbb{C}^*)^2$-threefold 2.30 from Example* **??**. *There are 3 normal toric degenerations, given by the polytopes $\Delta_0, \Delta_1, \Delta_\infty$. It can be checked in this case that $S = \emptyset$, as for each i there is an outer normal $n_i \notin H$ to the face F_i, see Figure 2a for example, or Appendix* **??** *for calculations of the normals*

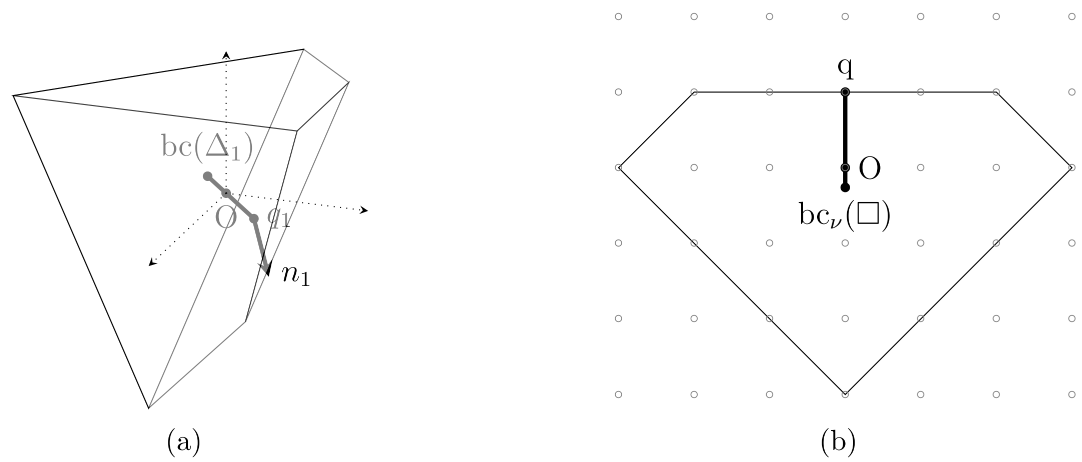


Figure 2: Some of the calculation of $R(X)$ for threefold 2.30. (a) Degeneration polytope Δ_1 with barycenter $\operatorname{bc}(\Delta_1)$ and q_1, n_1 shown, (b) Moment polytope $\Box$ with Duistermaat-Heckmann barycenter $\operatorname{bc}_\nu(\Box)$ and q shown.

Therefore $R(X)$ is given by the first term in the minimum. We calculate $\operatorname{bc}_\nu(\Box) = (0, -6/23)$ and $q = (0,1)$. Then:

$$R(X) = \frac{1}{1 + 6/23} = \frac{23}{29}.$$

Corollary 1 [20, Corollary 1.2]**.** *In Table* **??** *below we give $R(X)$ for X a Fano threefold admitting a 2-torus action appearing in the list of Mori and Mukai [18]. We include only those where $R(X) < 1$. Note all admit a Kähler-Ricci soliton by Theorem* **??***.*

X	R(X)
2.30	23/29
2.31	23/27
3.18	48/55
3.21	76/97
3.22	40/49
3.23	168/221
3.24	21/25
4.5*	64/69
4.8	76/89

Table 3.1: Calculations for complexity 1 threefolds appearing in the list of Mori and Mukai for which $R(X) < 1$

Let X be a T-variety of complexity one associated to a divisorial polytope $\Psi : \square \to \mathrm{Div}_{\mathbb{Q}}(\mathbb{P}^1)$, see Section 2.3. It follows from Theorem **??** that:

$$R(X) = \inf_{(\mathcal{X},\mathcal{L})} (\sup(t|\,\mathrm{DF}_t(\mathcal{X},\mathcal{L}) \geq 0)),$$

where $(\mathcal{X},\mathcal{L})$ varies over all special test configurations for (X, L). We have an explicit description of special test configurations and their Donaldson Futaki invariants, see Section **??**. We will calculate $R(X)$ by considering first the product configurations and then the non-product configurations. To calculate the values $\sup(t|\,\mathrm{DF}_t(\mathcal{X},\mathcal{L}) \geq 0)$ for a given configuration we need to first consider some elementary convex geometry.

3.1 A short digression into convex geometry

Let V be a real vector space and $P \subset V$ be a convex polytope containing the origin, with $\dim P = \dim V$. Fix some point $b \in \mathrm{int}(P)$. Let $q \in \partial P$ be the intersection of ∂P with the ray $\tau = \mathbb{R}^+(-b)$. Suppose $n \in V^\vee$ is an outer normal to a face containing q. For $a \in \partial P$ write $\mathcal{N}(a) = \{w \in V^\vee \mid \langle a, w\rangle = \max_{x\in P}\langle x, c\rangle\}$. For $w \in \mathcal{N}(a)$ let $\Pi(a, w)$ be the affine hyperplane tangent to P at a with normal w. For $w \in \text{int}(\tau^\vee)$ there is a well-defined point of intersection of $\Pi(a, w)$ and τ which we denote p_w. See Figure 3 for a schematic.

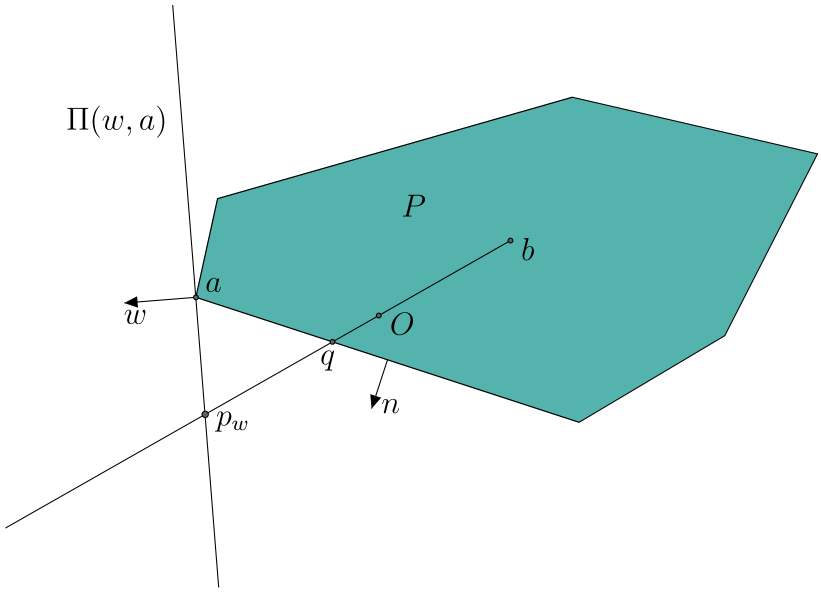


Figure 3: An Example in $V \cong \mathbb{R}^2$

Lemma 6. *Fix $w \in int(\tau^\vee)\backslash(\mathbb{R}^+n)$. For $s \in [0, 1]$ set $w(s) := sn + (1 - s)w$. As $n \in \tau^\vee$ we may consider $p(s) := p_{w(s)}$. For $0 \le s' < s \le 1$ we then have:*

$$\frac{|p(s)|}{|p(s) - b|} < \frac{|p(s')|}{|p(s') - b|}.$$

Proof. Without loss of generality we may assume $s' = 0$. For $s \in [0, 1]$ the points $p(s), q, b$ are collinear, so $|p(s)| = |p(s)-q|+|q|$ and $|p(s)-b| = |p(s)-q|+|q|+|b|$. Therefore:

$$\frac{|p(s)|}{|p(s) - b|} = \frac{|p(s) - q| + |q|}{|p(s) - q| + |q| + |b|}.$$

Hence it is enough for $|p(s) - q| < |p(0) - q|$ whenever $s > 0$. Since $q \neq 0$ is fixed this is equivalent to:

$$\frac{|p(s) - q|}{|q|} < \frac{|p(0) - q|}{|q|}.$$

For each $s \in [0, 1]$ choose $a(s) \in \partial P$ such that $w(s) \in \mathcal{N}(a(s))$. Write $a = a(0)$

for convenience. We then have:

$$\frac{|p(s)-q|}{|q|} = \frac{\langle a(s)-q, w\rangle}{\langle q, w\rangle}.$$

Note $n \in \mathcal{N}(q)$. Now $\langle a(s)-q, n\rangle \leq 0$ and $\langle a(s)-q, w\rangle \leq \langle a-q, w\rangle$. Clearly we have $\langle q, n\rangle > 0$. Then:

$$\begin{aligned} \frac{\langle a(s)-q, w(s)\rangle}{\langle q, w(s)\rangle} &= \frac{s\langle a(s)-q, n\rangle + (1-s)\langle a(s)-q, w\rangle}{s\langle q, n\rangle + (1-s)\langle q, w\rangle} \\ &\leq \frac{(1-s)\langle a-q, w\rangle}{s\langle q, n\rangle + (1-s)\langle q, w\rangle} \\ &< \frac{\langle a-q, w\rangle}{\langle q, w\rangle}. \end{aligned}$$

□

Corollary 2. *Let V, P, b, q, τ, n be as in the introduction to this section. Fix some open halfspace $H \subset V^\vee$ given by $u \geq 0$ for some $u \in V\backslash\{0\}$. This defines a projection map $\pi : V \to V/\langle u\rangle$. Consider the function $F_b : V^\vee \times [0,1] \to \mathbb{R}$ given by:*

$$F_b(w,t) := t\langle b, w\rangle + (1-t)\max_{x\in P}\langle x, w\rangle$$

For any $W \subseteq V^\vee$ containing n we have:

$$\sup(t \in [0,1] \mid \forall_{w\in W}\ F_b(t,w) \geq 0) = \frac{|q|}{|q-b|}. \tag{3.1}$$

If for some choice of n we have $n \notin H$ then:

$$\sup(t \in [0,1] \mid \forall_{w\in H}\ F_b(t,w) \geq 0) = \frac{|\tilde{q}|}{|\tilde{q}-\pi(b)|}, \tag{3.2}$$

where $\tilde{q}$ is the intersection of the ray $\pi(\tau)$ with the boundary of $\pi(P)$.

Proof. Note that:

$$\sup(t \in [0,1] \mid \forall_{w\in W}\ F_b(t,w) \geq 0) = \inf_{w\in W} \sup(t \in [0,1] \mid F_b(t,w) \geq 0).$$

Moreover $\sup(t \in [0,1] \mid F_b(t,w) \geq 0) = 1 > F_b(t,n)$ for $\langle b, w\rangle \geq 0$, so without loss of generality we may assume $W \subseteq \operatorname{int}(\tau^\vee)$. For $w \in W$ then:

$$\begin{aligned}\sup(t \in [0,1] \mid F_b(t,w) \geq 0) &= \frac{\max_{x\in P}\langle x, w\rangle}{\max_{x\in P}\langle x, w\rangle - \langle b, w\rangle} \\ &= \frac{\langle a, w\rangle}{\langle a, w\rangle - \langle b, w\rangle} \\ &= \frac{\langle p_w, w\rangle}{\langle p_w, w\rangle - \langle b, w\rangle} = \frac{|p_w|}{|p_w - b|}.\end{aligned}$$

Hence:

$$\sup(t \in [0,1] \mid \forall_{w\in W}\ F_b(t,w) \geq 0) = \inf_{w\in W} \frac{|p_w|}{|p_w - b|}.$$

Now for $w \in W$ consider the continuity path $w(s) = sn + (1-s)w$. By Lemma 6 if $n \in W$ then the above infimum is attained when $s = 1$ and we obtain (3.1). Otherwise the infimum is attained at some $w \in \partial W$. For (3.2) restricting F_b to $\partial H \times [0,1]$ gives:

$$F_b(w,t) = t\langle \pi(b), w\rangle + (1-t) \max_{x\in\pi(P)} \langle x, w\rangle.$$

Applying (3.1) to the polytope $\pi(P)$ in the vector space ∂H we obtain (3.2). $\square$

3.2 Proof of Theorem 4

3.2.1 Product Configurations

Recall the formula **??** for the twisted Donaldson-Futaki invariant of a product configuration $\mathcal{X} \cong X \times \mathbb{A}^1$. Let $q \in N_{\mathbb{R}}$ be the point of intersection of the ray generated by $-\operatorname{bc}(\Psi)$ with $\partial\square$. Applying (3.1) to **??**, we obtain:

$$\sup(t|\operatorname{DF}_t(\mathcal{X},\mathcal{L}) \geq 0) = \frac{|q|}{|q - \operatorname{bc}_\nu(\square)|}.$$

3.2.2 Non-Product Configurations

Recall the description of special non-product test configurations of X from **??**, and in particular the formula for the twisted Donaldson-Futaki **??**. Set $H := N_{\mathbb{Q}} \times \mathbb{R}^+$.

Proposition 1. *For any non-product configuration $(\mathcal{X},\mathcal{L})$ with special fiber one of the Δ_i above, let σ_i be the cone of outer normals to Δ_i at the unique point* *of intersection of $\partial\Delta_i$ with the ray generated by $-\operatorname{bc}(\Delta_i)$. Denote this point of intersection by q_i. Then:*

$$\sup(t|\operatorname{DF}_t(\mathcal{X},\mathcal{L})\geq 0)=\begin{cases}\frac{|q_i|}{|q_i-\operatorname{bc}(\Delta_i)|} & \sigma_i\cap H\neq\emptyset;\\ \\ \frac{|q|}{|q-\operatorname{bc}_\nu(\Box)|} & \sigma_i\cap H=\emptyset.\end{cases}$$

Proof. Extend $\operatorname{DF}_t(\mathcal{X},\mathcal{L})$ linearly to the whole of $N_{\mathbb{R}}\times\mathbb{R}$. In the case $\sigma_i\cap H\neq\emptyset$ we may apply (1) from Corollary 2 with $P=\Delta_i$ and $b=\operatorname{bc}(\Delta_i)$. Otherwise we may apply (3.2), noting that $\pi(\Delta_i)=\Box$ and $\pi(\operatorname{bc}(\Delta_i))=\operatorname{bc}_\nu(\Box)$. $\Box$

Proof of Theorem 6. With Remark 2 in mind, observe that a special test configuration must either be product or non-product. Any non-product configurations Δ_i with $\sigma_i\cap H\neq\emptyset$ have their contribution to the infimum already accounted for and we may exclude them. The result follows. $\Box$

of Corollary 1. We calculate outer normals n_i of F_i for every special test configuration polytope of each threefold in this list, see **??**. In each case we verify $n_i\notin H$. The divisorial polytopes and Duistermaat-Heckman measures were originally given in [19], and may be also be found in Appendix **??**. We may then calculate $R(X)$ using just the base polytope $\Box$ and its Duistermaat-Heckman barycenter. $\Box$

Bibliography

[1] H. Poincaré, “Sur l’uniformisation des fonctions analytiques,” *Acta mathematica*, vol. 31, no. 1, pp. 1–63, 1908.

[2] P. Koebe, “Über die uniformisierung der algebraischen kurven. i,” *Mathematische Annalen*, vol. 67, no. 2, pp. 145–224, 1909.

[3] P. Koebe, “Über die uniformisierung der algebraischen kurven. ii,” *Mathematische Annalen*, vol. 69, no. 1, pp. 1–81, 1910.

[4] E. Calabi, “The space of Kähler metrics,” *Proceedings of the International Congress of Mathematicians*, vol. 2, pp. 206–207, 1956.

[5] E. Calabi, “On Kähler manifolds with vanishing canonical class,” *Princeton Mathematical Series*, no. 12, pp. 78–89, 1957.

[6] T. Aubin, “Equations du type monge-ampere sur la varietes kahleriennes compactes,” *CR Acad. Sci. Paris*, vol. 283, pp. 119–121, 1976.

[7] S.-T. Yau, “Calabi’s conjecture and some new results in algebraic geometry,” *Proceedings of the National Academy of Sciences*, vol. 74, no. 5, pp. 1798–1799, 1977.

[8] Y. Matsushima, “Sur la structure du groupe dhoméomorphismes analytiques dune certaine variété kaehlérinne,” *Nagoya Mathematical Journal*, vol. 11, pp. 145–150, 1957.

[9] A. Futaki, “An obstruction to the existence of einstein kähler metrics,” *Inventiones mathematicae*, vol. 73, no. 3, pp. 437–443, 1983.

[10] G. Tian, “On kähler-einstein metrics on certain kähler manifolds withc 1 (m)¿ 0,” *Inventiones mathematicae*, vol. 89, no. 2, pp. 225–246, 1987.

[11] X. Chen, S. Donaldson, and S. Sun, “Kähler-einstein metrics on fano manifolds. i: Approximation of metrics with cone singularities,” *Journal of the American Mathematical Society*, vol. 28, no. 1, pp. 183–197, 2015.

[12] X. Chen, S. Donaldson, and S. Sun, “Kähler-einstein metrics on fano manifolds. ii: Limits with cone angle less than 2,” *Journal of the American Mathematical Society*, vol. 28, no. 1, pp. 199–234, 2015.

[13] X. Chen, S. Donaldson, and S. Sun, “Kähler-einstein metrics on fano manifolds. iii: Limits as cone angle approaches 2 and completion of the main proof,” *Journal of the American Mathematical Society*, vol. 28, no. 1, pp. 235–278, 2015.

[14] V. Datar and G. Székelyhidi, “Kähler–einstein metrics along the smooth continuity method,” *Geometric and Functional Analysis*, vol. 26, no. 4, pp. 975–1010, 2016.

[15] X.-J. Wang and X. Zhu, “Kähler–Ricci solitons on toric manifolds with positive first Chern class,” *Advances in Mathematics*, vol. 188, no. 1, pp. 87–103, 2004.

[16] N. Ilten and H. Süß, “K-stability for Fano manifolds with torus action of complexity 1,” *Duke Math. J.*, vol. 166, no. 1, pp. 177–204, 2017.

[17] H. Süß, “Kähler–einstein metrics on symmetric fano t-varieties,” *Advances in Mathematics*, vol. 246, pp. 100–113, 2013.

[18] S. Mori and S. Mukai, “Classification of fano 3-folds with b 2 2,” *manuscripta mathematica*, vol. 36, no. 2, pp. 147–162, 1981.

[19] H. Süß, “Fano threefolds with 2-torus action-a picture book,” *arXiv preprint arXiv:1308.2379*, 2013.

[20] J. Cable, “Greatest lower bounds on ricci curvature for fano T-manifolds of complexity one,” *Bulletin of the London Mathematical Society*, vol. 51, no. 1, pp. 34–42, 2019.

[21] C. Li, “Greatest lower bounds on ricci curvature for toric fano manifolds,” *arXiv preprint arXiv:0909.3443*, 2009.

[22] J. Hausen, C. Hische, and M. Wrobel, "On torus actions of higher complexity," *arXiv preprint arXiv:1802.00417*, 2018.

[23] H. Süß, "Kähler-Einstein metrics on symmetric Fano T-varieties," *Advances in Mathematics*, vol. 246, pp. 100 – 113, 2013.

[24] P. Griffiths and J. Harris, "Principles of algebraic geometry, pure and applied mathematics," 1978.

[25] K. Kodaira, "On kahler varieties of restricted type an intrinsic characterization of algebraic varieties," *Annals of Mathematics*, pp. 28–48, 1954.

[26] W.-L. Chow, "On compact complex analytic varieties," *American Journal of Mathematics*, vol. 71, no. 4, pp. 893–914, 1949.

[27] G. Székelyhidi, *An introduction to extremal Kähler metrics*, vol. 152. American Mathematical Soc., 2014.

[28] S.-T. Yau, "Calabi's conjecture and some new results in algebraic geometry," *Proceedings of the National Academy of Sciences*, vol. 74, no. 5, pp. 1798–1799, 1977.

[29] S.-T. Yau, "Harmonic functions on complete riemannian manifolds," *Communications on Pure and Applied Mathematics*, vol. 28, no. 2, pp. 201–228, 1975.

[30] A. C. da Silva, "Symplectic geometry," in *Handbook of differential geometry*, vol. 2, pp. 79–188, Elsevier, 2006.

[31] R. J. Berman and D. W. Nystrom, "Complex optimal transport and the pluripotential theory of k\" ahler-ricci solitons," *arXiv preprint arXiv:1401.8264*, 2014.

[32] M. F. Atiyah, "Convexity and commuting Hamiltonians," *Bulletin of the London Mathematical Society*, vol. 14, no. 1, pp. 1–15, 1982.

[33] V. Guillemin and S. Sternberg, "Convexity properties of the moment mapping," *Inventiones mathematicae*, vol. 67, no. 3, pp. 491–513, 1982.

[34] J. Cable, "Kähler-Einstein metrics on symmetric general arrangement varieties." 2019.

[35] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory*, vol. 34. Springer Science & Business Media, 1994.

[36] G. Kempf and L. Ness, "The length of vectors in representation spaces," in *Algebraic geometry*, pp. 233–243, Springer, 1979.

[37] F. C. Kirwan, "Partial desingularisations of quotients of nonsingular varieties and their Betti numbers," *Annals of mathematics*, vol. 122, no. 1, pp. 41–85, 1985.

[38] M. M. Kapranov, "Chow quotients of Grassmannians. I," *Adv. Soviet Math*, vol. 16, no. 2, pp. 29–110, 1993.

[39] H. Bäker, J. Hausen, and S. Keicher, "On Chow quotients of torus actions," *Michigan Math. J.*, vol. 64, no. 3, pp. 451–473, 2015.

[40] K. Altmann and J. Hausen, "Polyhedral divisors and algebraic torus actions," *Mathematische Annalen*, vol. 334, no. 3, pp. 557–607, 2006.

[41] K. Altmann, J. Hausen, and H. Süss, "Gluing affine torus actions via divisorial fans," *Transformation Groups*, vol. 13, no. 2, pp. 215–242, 2008.

[42] N. O. Ilten and H. Süß, "Polarized complexity-one t-varieties," *arXiv preprint arXiv:0910.5919*, 2009.

[43] L. Petersen and H. Süß, "Torus invariant divisors," *Israel Journal of Mathematics*, vol. 182, no. 1, pp. 481–504, 2011.

[44] G. Tian, "On stability of the tangent bundles of Fano varieties," *Internat. J. Math*, vol. 3, no. 3, pp. 401–413, 1992.

[45] G. Tian, "On Kähler-Einstein metrics on certain Kähler manifolds with $c_1(m) > 0$," *Inventiones mathematicae*, vol. 89, no. 2, pp. 225–246, 1987.

[46] Y. A. Rubinstein, "Some discretizations of geometric evolution equations and the ricci iteration on the space of kähler metrics," *Advances in Mathematics*, vol. 218, no. 5, pp. 1526–1565, 2008.

[47] G. Székelyhidi, "Greatest lower bounds on the ricci curvature of fano manifolds," *Compositio Mathematica*, vol. 147, no. 1, pp. 319–331, 2011.

[48] Y. Rubinstein, "On the construction of nadel multiplier ideal sheaves and the limiting behavior of the ricci flow," *Transactions of the American Mathematical Society*, vol. 361, no. 11, pp. 5839–5850, 2009.

[49] C. Li *et al.*, "K-semistability is equivariant volume minimization," *Duke Mathematical Journal*, vol. 166, no. 16, pp. 3147–3218, 2017.

[50] C. Li, "Greatest lower bounds on ricci curvature for toric fano manifolds," *Advances in Mathematics*, vol. 226, no. 6, pp. 4921–4932, 2011.

[51] T. Delcroix, "Kähler–einstein metrics on group compactifications," *Geometric and Functional Analysis*, vol. 27, no. 1, pp. 78–129, 2017.

[52] Y. Yao, "Greatest lower bounds on ricci curvature of homogeneous toric bundles," *International Journal of Mathematics*, vol. 28, no. 04, p. 1750024, 2017.