Bottom-up thermalization in small colliding systems

I. THE BOLTZMANN EQUATION

We plan to study the modification of the bottom-up thermalization [1] in bulk matter with a finite transverse size:

$$\left(\frac{\partial}{\partial\tau}+\vec{v}_\perp\cdot\nabla_\perp-\frac{p_z}{\tau}\frac{\partial}{\partial p_z}\right)f(\tau,\vec{x}_\perp,\vec{p}_\perp,p_z)=C[f] \tag{1}$$

where the collison kernel

$$C[f]=C_{el}[f]+C_{1\to2}[f] \tag{2}$$

with the elastic collision kernel C_{el} given, e.g., in [2]. Here, the typical p_z is expected to change dramatically while the final size of the system is only expected to expand about several times of R, the initial size of the system. So, we shall use a new variable instead of p_z

$$\tilde{p}_z=\frac{\tau}{\tau_0}p_z. \tag{3}$$

Accordingly, we work with

$$\tilde{f}(\tau,\vec{x}_\perp,\vec{p}_\perp,\tilde{p}_z)\equiv f(\tau,\vec{x}_\perp,\vec{p}_\perp,\tau_0\tilde{p}_z/\tau), \tag{4}$$

which satisfies

$$(\partial\tau+\vec{v}_\perp\cdot\nabla_\perp)\,\tilde{f}(\tau,\vec{x}_\perp,\vec{p}_\perp,\tilde{p}_z)=C[f]. \tag{5}$$

The Debye screening mass is given by

$$m_g^2=2g^2N_c\int\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{f}{|\mathbf{p}|}, \tag{6}$$

which gives the thermal mass in thermal equilibrium [3]

$$m_g^2=\frac{g^2N_c}{6}T^2. \tag{7}$$

We first study bulk matter created in the most central collisions, which is rotationally symmetric in the transverse plane. In this case, we can take $\tilde{f}$ as a function of $\tau, r, p_\perp, \phi_r, \tilde{p}_z$ with $\phi_r\equiv\theta-\phi$, θ and ϕ the azimuthal angles in coordinate and momentum spaces, respectively. Using the chain rule one has

$$\begin{aligned}\vec{v}_\perp\cdot\nabla_\perp&=v^x\left(\frac{\partial r}{\partial x}\partial_r-\frac{\partial\theta}{\partial x}\partial_{\phi_r}\right)+v^y\left(\frac{\partial r}{\partial y}\partial_r-\frac{\partial\theta}{\partial y}\partial_{\phi_r}\right)\\&=v_\perp\cos\phi_r\partial_r-v_\perp\sin\phi_r\frac{1}{r}\partial_{\phi_r}.\end{aligned} \tag{8}$$

At $\theta = 0$, one can check that $\partial_r = \partial_x$ and $\frac{1}{r}\partial_{\phi_r} = \frac{1}{r}\partial_\phi = -\partial_y$. Again, we define a new coordinate variable $\tilde{\phi}$ via

$$t_{\phi_r} \equiv \tan\frac{\phi_r(\tau)}{2} = e^{-\frac{1}{rp_\perp}\left(\sqrt{p_\perp^2\tau^2+\tilde{p}_z^2\tau_0^2}-\tau_0\sqrt{p_\perp^2+\tilde{p}_z^2}\right)}\tan\frac{\tilde{\phi}}{2}, \tag{9}$$

which satisfies

$$\dot{\phi}_r = -v_\perp \frac{\sin\phi_r}{r}. \tag{10}$$

Here, the system is also symmetric under $z \to -z$. In this case, $\tilde{\phi}_r$ can be kept within the range $[0, \pi]$, which maps ϕ_r into the same range. And, one has

$$\sin\phi_r = \frac{2t_{\phi_r}}{1+t_{\phi_r}^2}, \qquad \cos\phi_r = \frac{1-t_{\phi_r}^2}{1+t_{\phi_r}^2}. \tag{11}$$

Instead of p_T and $\tilde{p}_z$, we use $\tilde{v}_z$ and $\tilde{p}$, which satisfy

$$v_z = \frac{\tilde{v}_z\frac{\tau_0}{\tau}}{\sqrt{1-\tilde{v}_z^2+\frac{\tau_0^2}{\tau^2}\tilde{v}_z^2}}, \qquad p = \tilde{p}\sqrt{1-\tilde{v}_z^2+\frac{\tau_0^2}{\tau^2}\tilde{v}_z^2} \tag{12}$$

and, accordingly,

$$\begin{aligned}\tan\frac{\phi_r(\tau)}{2} &= e^{-\frac{1}{rp_\perp}(\tau p-\tau_0\tilde{p})}\tan\frac{\tilde{\phi}}{2}\\ &= e^{-\frac{1}{r\sqrt{1-\tilde{v}_z^2}}\left(\tau\sqrt{1-\tilde{v}_z^2+\frac{\tau_0^2}{\tau^2}\tilde{v}_z^2}-\tau_0\right)}\tan\frac{\tilde{\phi}}{2}.\end{aligned} \tag{13}$$

By choosing the above free-streaming coordinates, $\tilde{f}$, as a function of τ, $r = x$, $\tilde{\phi}$, $\tilde{p}$ and $\tilde{v}_z$, satisfies

$$(\partial_\tau + v^x\partial_x)\,\tilde{f} = C[f] \tag{14}$$

with

$$v^x = \sqrt{1-v_z^2}\cos\phi_r. \tag{15}$$

Here, we could also choose a free-streaming coordinate for r, although it may slow down the use of GPU due to extensive access to global memory.

II. THE INITIAL CONDITION

At the inital time τ_0, one has

$$n = \int \frac{d^3p}{(2\pi)}f = \frac{1}{2\pi^2}\int_0^\infty dpp^2\int_{-1}^1 \frac{v_z}{2}f(\tau_0, r, p, v_z). \tag{16}$$

III. THE USAGE OF GPUS

Let us solve the distribution function from the Boltzmann equation in the phase space: $(x, \tilde{\phi}, \tilde{p}, \tilde{v}_z)$. Now, since at x, one needs to integrate over momenta. We hence put each spatial points on a block and the momentum distribution is read into shared memory of each block.

Let us take RTX 3070 for example:

Shared Mem/block	# of registers/block	max # of threads/block
48 kb	65536 = 64k	1024

A. Free-streaming solutions

B. Elastic collision kernel

Instead of solving collision kernel, we first try its diffusion approximation [2]

$$D_t f = -\nabla_{\mathbf{p}} \cdot J_g + S_g, \qquad D_t F = -\nabla_{\mathbf{p}} \cdot J_q + S_q \tag{17}$$

where f and F are the distribution functions of gluons and quarks respectively, the currents are given by

$$J_g = -4\pi\alpha_s^2 N_c L \left[I_a \nabla_{\mathbf{p}} f + I_b \frac{\mathbf{p}}{p} f(1+f) \right], \tag{18a}$$

$$J_q = -4\pi\alpha_s^2 C_F L \left[I_a \nabla_{\mathbf{p}} F + I_b \frac{\mathbf{p}}{p} F(1-F) \right], \tag{18b}$$

and the sources by

$$S_g = -\frac{N_f}{C_F} S_q = \frac{4\pi\alpha_s^2 C_F N_f L I_c}{p} \left[F(1+f) - f(1-F) \right], \tag{19}$$

with

$$L \simeq \int_{q_{min}^2}^{q_{max}} \frac{dq}{q} = \frac{1}{2} \ln \frac{\langle p_t^2 \rangle}{m_D^2} \qquad \text{with } q_{min} = m_D, \tag{20}$$

and

$$I_a = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \left[N_c f(1+f) + N_f F(1-F) \right], \tag{21a}$$

$$I_b = 2\int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{p} \left(N_c f + N_f F \right), \tag{21b}$$

$$I_c = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{p} (f + F). \tag{21c}$$

Here, $C_F = (N_c^2-1)/(2N_c)$.

As a first step, we neglect quarks and take $N_f = 0$ and $F = 0$. Now, let us discuss the relation between I_a and $\hat{q}$. Let us define the transverse direction to be orthogonal to $\mathbf{p}$. If one expands around the initial $\mathbf{p}$, one has

$$D_t f = 4\pi\alpha_s^2 N_c L I_a \nabla_{p_t}^2 f. \tag{22}$$

This tells us that

$$\hat{q} = 8\pi\alpha_s^2 N_c I_a \ln\frac{\langle p_t^2\rangle}{m_D^2} = 4\pi\alpha_s^2 \ln\frac{\langle p_t^2\rangle}{m_D^2}\frac{N_c^2}{N_c^2-1} n_g \tag{23}$$

with the gluon number density is

$$n_g \equiv 2(N_c^2-1)\int\frac{d^3\mathbf{p}}{(2\pi)^3} f_{\mathbf{p}}\left(1+f_{\mathbf{p}}\right) . \tag{24}$$

C. Inelastic collision kernel

The inelastic collision kernel for the $g\to gg$ process is [1]

$$\begin{aligned} C_{\mathrm{g}\to\mathrm{gg}} = \int_0^1 dx\, \frac{d^2 I}{dx\, dt}\Bigg\{ & \frac{1}{x^{5/2}}\left[f_{\frac{\mathbf{p}}{x}}(1+f_{\mathbf{p}})\left(1+f_{\frac{(1-x)}{x}\mathbf{p}}\right) - f_{\mathbf{p}} f_{\frac{(1-x)}{x}\mathbf{p}}\left(1+f_{\frac{\mathbf{p}}{x}}\right)\right] \\ & -\frac{1}{2}\left[f_{\mathbf{p}}(1+f_{x\mathbf{p}})\left(1+f_{(1-x)\mathbf{p}}\right) - f_{x\mathbf{p}} f_{(1-x)\mathbf{p}}(1+f_{\mathbf{p}})\right]\Bigg\} \end{aligned} \tag{25}$$

where the medium-induced gluon spectrum takes the form [4]

$$\frac{d^2 I}{dxdt} = \frac{\alpha_s N_c}{\pi}\frac{(1-x+x^2)^{5/2}}{(x-x^2)^{3/2}}\sqrt{\frac{\hat{q}}{p}}, \tag{26}$$

which is valid when $xp \ll \hat{q}t^2$.

IV. SPATIALLY HOMOGENEOUS CASES

In a homogeneous system, one has

$$\dot{f} = \frac{1}{4}\frac{\hat{q}(t)}{p^2}\left[p^2\left(f' + \frac{1}{T^*(t)} f(1+f)\right)\right]' + C_{g\leftrightarrow gg}[f], \tag{27}$$

with

$$\hat{q} = \frac{4\alpha_s^2 N_c^2}{\pi} \ln \frac{\langle p_t^2 \rangle}{m_D^2} I_a = \frac{4\alpha_s^2 N_c^2}{\pi} \ln \frac{\langle p_t^2 \rangle}{m_D^2} \int_0^\infty dp p^2 f(1+f),$$
$$m_D^2 = \frac{2\alpha_s N_c}{\pi} I_b = \frac{4\alpha_s N_c}{\pi} \int_0^\infty dp p f, \qquad T^* = \frac{\int_0^\infty dp p^2 f(1+f)}{2\int_0^\infty dp p f}. \tag{28}$$

By defining

$$\tau \equiv \frac{\alpha_s^2 N_c^2 t}{\pi} \ln \frac{\langle p_t^2 \rangle}{m_D^2}, \qquad J_g \equiv I_a f' + I_b f(1+f) \tag{29}$$

with

$$I_a \equiv \int_0^\infty dp p^2 f(1+f), \qquad I_b \equiv 2\int_0^\infty dp p f, \tag{30}$$

one can write

$$\begin{aligned} \partial_\tau f = & \frac{1}{p^2}\left(p^2 J_g\right)' + \int_0^1 dx \, \frac{d^2 I}{dx\, dt} \\ & \times \left\{ \frac{1}{x^{5/2}} \left[f_{\frac{p}{x}} (1+f_p) \left(1+f_{\frac{(1-x)}{x}p}\right) - f_p f_{\frac{(1-x)}{x}p} \left(1+f_{\frac{p}{x}}\right) \right] \theta(p-\omega_0)\theta(\frac{1-x}{x}p - \omega_0) \right. \\ & \left. - \frac{1}{2}\left[f_p (1+f_{xp}) \left(1+f_{(1-x)p}\right) - f_{xp} f_{(1-x)p} (1+f_p) \right] \theta(xp-\omega_0)\theta((1-x)p-\omega_0) \right\} \end{aligned} \tag{31}$$

with $\omega_0 = \frac{1}{\pi \ln \frac{\langle p_t^2 \rangle}{m_D^2}} \frac{I_b^2}{I_a}$.

A. Grids

We setup grids at $p_s[i]$ by specifying n, n_{min} and p_{max}. We keep, instead, pf at each $p[i] = (p_s[i] + p_s[i-1])/2$ with $p_s[-1] = 0$. At each time step, we need $p^2 J_q$ at each $p_s[i]$. Here,

$$n[i] \equiv \int_{p_s[i-1]}^{p_s[i]} dp p^2 f \approx \frac{1}{3}(p_s[i]^3 - p_s[i-1]^3) f_i. \tag{32}$$

---

[1] R. Baier, A.H. Mueller, D. Schiff and D.T. Son, *'Bottom up' thermalization in heavy ion collisions*, *Phys. Lett. B* **502** (2001) 51 [hep-ph/0009237].

[2] J.-P. Blaizot, B. Wu and L. Yan, *Quark production, Bose–Einstein condensates and thermalization of the quark–gluon plasma*, *Nucl. Phys. A* **930** (2014) 139 [1402.5049].

[3] M.L. Bellac, *Thermal Field Theory*, Cambridge Monographs on Mathematical Physics, Cambridge University Press (3, 2011), 10.1017/CBO9780511721700.

[4] R. Baier, Y.L. Dokshitzer, A.H. Mueller and D. Schiff, *Medium induced radiative energy loss: Equivalence between the BDMPS and Zakharov formalisms*, *Nucl. Phys. B* **531** (1998) 403 [hep-ph/9804212].