Forward Projection Model for 3D Multislice Helical CT

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1 Notation

1.1 CT Geometry Parameters

r_{si} (mm) - the distance from the source to the iso-center

r_{sd} (mm) - the distance from the source to detectors

β_0 (rads) - the initial view angle

$\Delta\beta$ (rads) - the angle between adjacent views, rotation should be counter clockwise

u (detector rows/rotation) - pitch, shift of number of detector rows in one scan rotation

i - index of the sinogram, can be translated to (i_r, i_c, i_v)

N_r - number of rows in the detector array, assume even number (TODO)

N_c - number of channels in the detector array, assume even number (TODO)

N_v - number of views

$\Delta\alpha_c$ (rads) - the angle spacing of the detector element in channel direction

$\delta\alpha_c$ (rads) - the angle of offset (naming, TODO), the angle of the ray hitting the center of the detector array relative to the ray passing through iso-center

Δd_r (mm) - the width of the detector element in row direction

1.2 Image Domain Parameters

j - index of the image, can be translated to (j_x, j_y, j_z)

N_x - number of voxels in x direction

N_y - number of voxels in y direction

N_z - number of voxels in z direction

x_c (mm) - center of the center voxel of the image in x direction relative to iso-center

y_c (mm) - center of the center voxel of the image in y direction relative to iso-center
z_c (mm) - center of the center voxel of the image in z direction relative to iso-center
Δ_{xy} (mm) - width of square voxel on XY plane
Δ_z (mm) - voxel width in z direction

1.3 Intermediate Variables

β (rads) - the view angle, i.e. the slope angle of the ray through iso-center
Δz_s (mm) - z shift of the source per view
α_c (rads) - the channel angle relative to the ray through iso-center
α_{c0} (rads) - the channel angle of detector of channel 0 relative to the ray through iso-center
α_j (rads) - the voxel j angle relative to the ray through iso-center
θ (rads) - the ray angle in XY plane, i.e. the slope angle of the ray through the center of voxel in XY plane
$\tilde{\theta}$ (rads) - clipped θ to $[-\pi/4, \pi/4]$
r_{sv} (mm) - the distance from the source to voxel
W_c (rads) - voxel projection width on XY plane
δ_c (rads) - the distance between the center of voxel projection and the center of detector element on XY plane
W_r (mm) - voxel projection width on Z axis
δ_r (mm) - the distance between the center of voxel projection and the center of detector element on Z axis
ϕ (rads) - the ray angle in YZ plane, i.e. the slope angle of the ray through the center of voxel in YZ plane, (TODO)
i_r - row index of the detector element, $i_r \geq 0$
i_c - channel index of the detector element, $i_c \geq 0$
i_v - view index (slowest variable), $i_v \geq 0$
j_x - x-axis index of the voxel, $j_x \geq 0$
j_y - y-axis index of the voxel, $j_y \geq 0$
j_z - z-axis index of the voxel, $j_z \geq 0$
x_0 (mm) - center of the first voxel in x direction relative to iso-center
y_0 (mm) - center of the first voxel in y direction relative to iso-center
z_0 (mm) - center of the first voxel in z direction relative to iso-center
i_c^- - the smallest channel index of detector hit by voxel projection
i_c^+ - the largest channel index of detector hit by voxel projection

i_r^- - the smallest row index of detector hit by voxel projection
i_r^+ - the largest row index of detector hit by voxel projection

$$x_0 = x_c - \Delta_{xy}\left(\frac{N_x - 1}{2}\right) \tag{1}$$

$$y_0 = y_c - \Delta_{xy}\left(\frac{N_y - 1}{2}\right) \tag{2}$$

$$z_0 = z_c - \Delta_{z}\left(\frac{N_z - 1}{2}\right) \tag{3}$$

1.4 Notation of Points

$O = (0, 0)$ - the iso-center on XY-plane.
$S = (x_s, y_s, z_s)$ - the source
$V = (x, y, z)$ - the center of voxel of interest

2 Calculations

2.1 Determine the current view angle

Input: i_v, β_0, $\Delta\beta$
Output: β

$$\beta = \beta_0 + i_v\Delta\beta \tag{4}$$

2.2 Determine the current source location

Input: β, r_{si}, u, Δd_r
Output: x_s, y_s, z_s

$$x_s = r_{si}\cos\beta \tag{5}$$

$$y_s = r_{si}\sin\beta \tag{6}$$

$$z_s = \frac{u\Delta d_r(\beta - \beta_0)}{2\pi} \tag{7}$$

Here we assume the scan starts at 0 in z-axis and goes postive z direction.

2.3 Determine the location of the voxel

Input: j_x, j_y, j_z, Δ_{xy}, Δ_z, x_0, y_0, z_0
Output: x, y, z

$$x = x_0 + j_x \Delta_{xy} \tag{8}$$
$$y = y_0 + j_y \Delta_{xy} \tag{9}$$
$$z = z_0 + j_z \Delta_z \tag{10}$$

2.4 Calculate the entry of forward projection matrix $A_{i,j}$

In this study, the entry of matrix $A_{i,j}$ is calculated based on the distance driven (DD) method used by Thibault *et al.* [1].

$$A_{i,j} = B_{i,j} \times C_{i,j} \tag{11}$$

where

$$B_{i,j} = \frac{\Delta_{xy}}{\Delta\alpha_c \cos\tilde{\theta}} clip\left[0, \frac{W_c + \Delta\alpha_c}{2} - |\delta_c|, \min(W_c, \Delta\alpha_c)\right] \tag{12}$$
$$C_{i,j} = \frac{1}{\Delta d_r \cos\phi} clip\left[0, \frac{W_r + \Delta d_r}{2} - |\delta_r|, \min(W_r, \Delta d_r)\right] \tag{13}$$

where $clip[a, b, c] = \min(\max(a, b), c)$.

2.5 XY plane

Please see Figure 1 and 2.

(a) Calculation of $\tilde{\theta}$

We can calculate θ using atan2 function, i.e.

$$\theta = atan2(y_s - y, x_s - x) \tag{14}$$

Then we translate θ to $\tilde{\theta}$ using

$$\tilde{\theta} = \left(\theta + \frac{\pi}{4}\right) mod\ \frac{\pi}{2} - \frac{\pi}{4}, \tag{15}$$

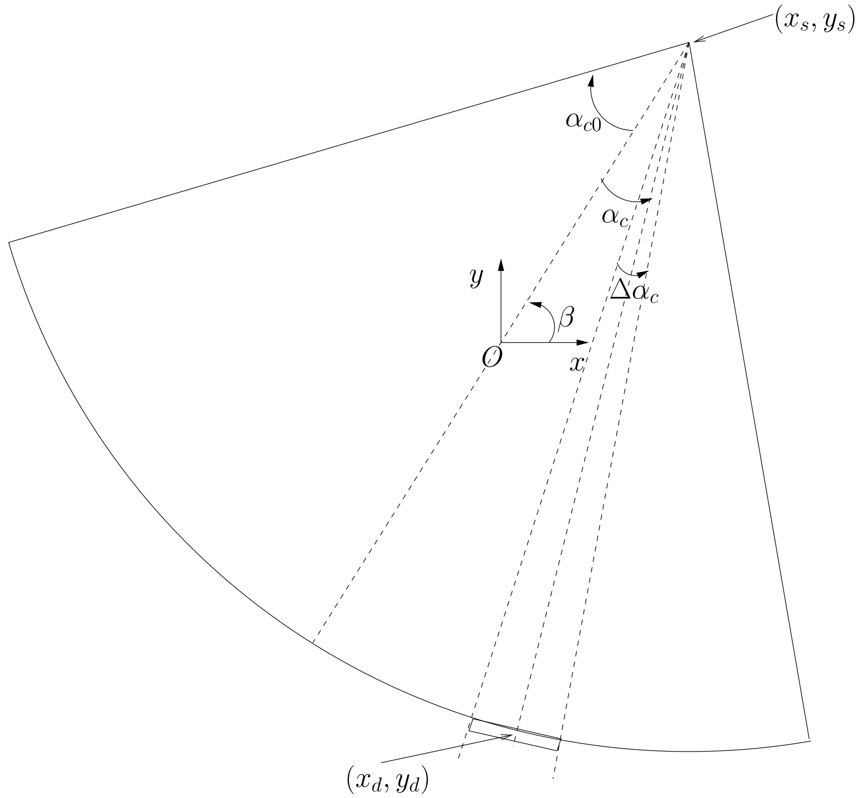

Figure 1: geometry in XY plane (a)

which results in $|\tilde{\theta}| < \frac{\pi}{4}$.

(b) Calculation of W_c and δ_c

We will measure all the quantities in XY plane in radians.

For W_c, we approximately have

$$W_c = \frac{\Delta_{xy} \cos \tilde{\theta}}{\sqrt{(x_s - x)^2 + (y_s - y)^2}} \tag{16}$$

For δ_c, we have

$$\delta_c = (\alpha_j - \alpha_c + \pi) \ mod \ 2\pi - \pi \tag{17}$$

where

$$\alpha_j = \theta - \beta \tag{18}$$

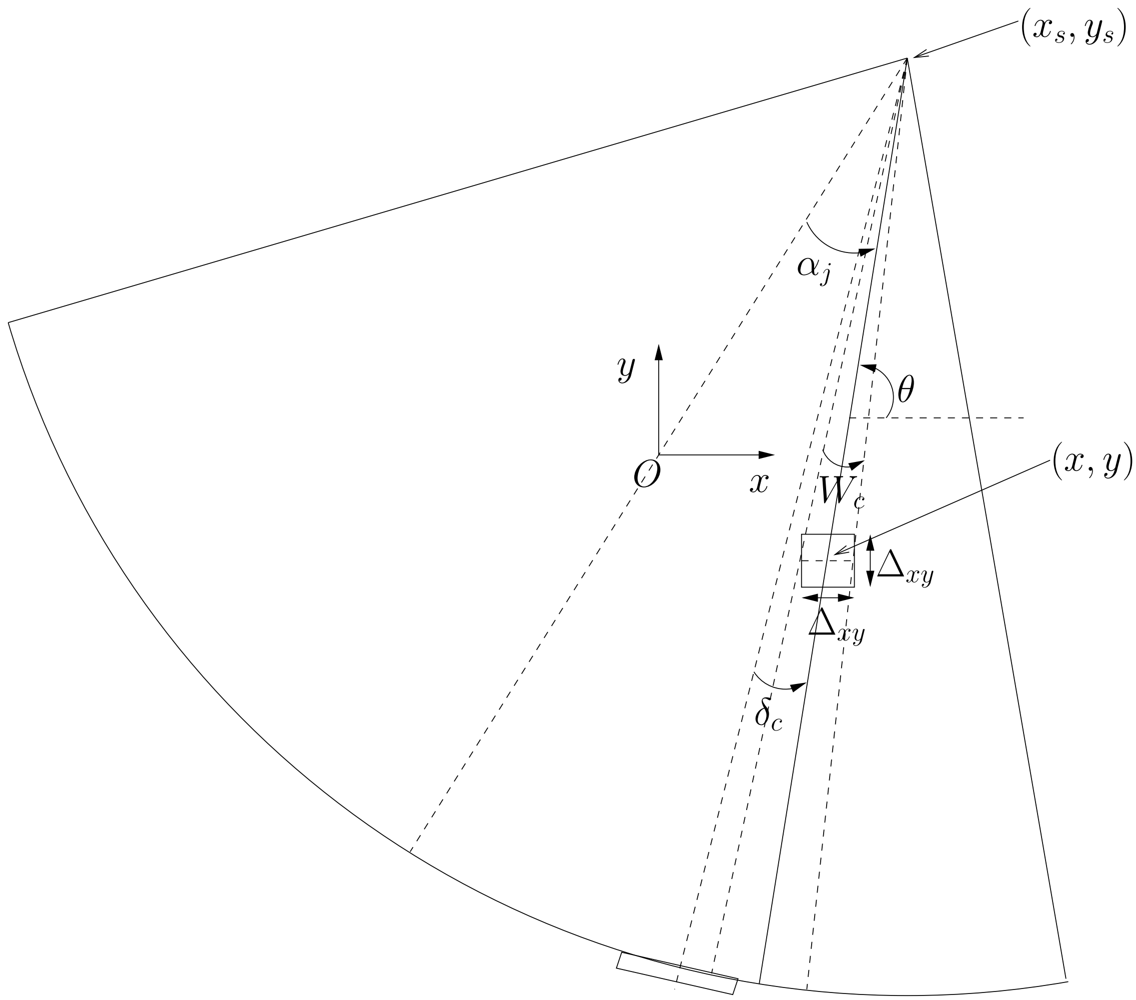

Figure 2: geometry in XY plane (b)

and

$$\alpha_c = i_c \Delta\alpha_c + \alpha_{c0} \tag{19}$$

$$\alpha_{c0} = -\frac{N_c - 1}{2}\Delta\alpha_c + \delta\alpha_c \tag{20}$$

2.6 YZ plane

Please see Figure 3.

(a) Calculate ϕ

TODO, can we assume $|\phi| < \frac{\pi}{4}$ because the detector in row direction is thin?

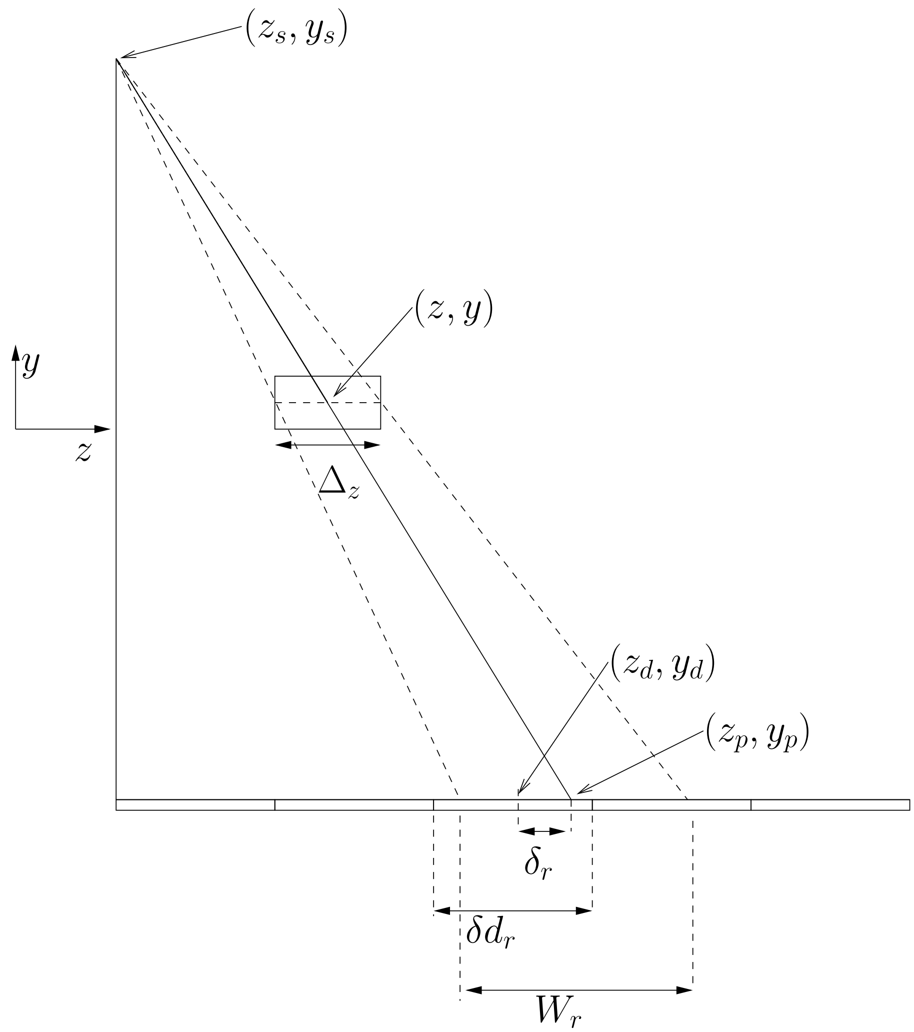

Figure 3: geometry in YZ plane

If this is the case, we don't need to do the adjustment.

Actually, all we need to know is $\cos\phi$, which is

$$\cos\phi = \frac{|y_s - y|}{\sqrt{(y_s - y)^2 + (z_s - z)^2}} \tag{21}$$

(b) Calculate W_r and δ_r

We will measure all of them in millimeter.

Define the magnification factor M as

$$M = \frac{r_{sd}}{\sqrt{(x_s - x)^2 + (y_s - y)^2}} \tag{22}$$

For W_r, we have

$$W_r = M\Delta_z \tag{23}$$

For δ_r, we have

$$\delta_r = z_p - z_d \tag{24}$$

where

$$z_p = z_s + M(z - z_s) \tag{25}$$

and

$$z_d = z_s + \left(i_r - \frac{N_r - 1}{2}\right)\Delta d_r \tag{26}$$

2.7 Identify voxel projection range on XY-plane

Consider a projection of voxel j onto XY plane. Two channel indicies of detector elements which rays hit through each of two voxel edges are denoted as i_c^- and i_c^+. Our approach is to calculate $B_{i,j}$ only for the channel indicies from i_c^- to i_c^+. The terms i_c^- and i_c^+ can be approximated as:

$$\alpha_j = \theta - \beta \tag{27}$$

$$\tilde{\alpha}_j = (\alpha_j + \pi)\ mod\ 2\pi - \pi \tag{28}$$

$$i_c^- = \left\lfloor \frac{\tilde{\alpha}_j - \frac{W_c}{2} - \alpha_{c0} + \frac{\Delta\alpha_c}{2}}{\Delta\alpha_c} \right\rfloor \tag{29}$$

$$i_c^+ = \left\lfloor \frac{\tilde{\alpha}_j + \frac{W_c}{2} - \alpha_{c0} + \frac{\Delta\alpha_c}{2}}{\Delta\alpha_c} \right\rfloor \tag{30}$$

2.8 Identify voxel projection range on YZ-plane

$$z_p = z_s + M(z - z_s) \tag{31}$$

$$i_r^- = \left\lfloor \frac{z_p - \frac{M\Delta_z}{2} - d_{r0} + \frac{\Delta d_r}{2}}{\Delta d_r} \right\rfloor \tag{32}$$

$$i_r^+ = \left\lfloor \frac{z_p + \frac{M\Delta_z}{2} - d_{r0} + \frac{\Delta d_r}{2}}{\Delta d_r} \right\rfloor \tag{33}$$

$$d_{r0} = z_s - \frac{N_r - 1}{2}\Delta d_r \tag{34}$$

References

[1] Jean-Baptiste Thibault, Ken Sauer, Charles Bouman, and Jiang Hsieh. A three-dimensional statistical approach to improved image quality for multi-slice helical ct. *Medical Physics*, 34(11):4526–4544, 2007.