Marshallian demands for a quasilinear utility function

Consider a consumer with the quasilinear utility function

$$U(x_1, x_2) = x_1 + \sqrt{x_2}$$

where x_1 is the numeraire good and x_2 is the non-numeraire good The consumer faces prices $p_1 > 0$ and $p_2 > 0$ and has income $m > 0$ The budget constraint is given by

$$p_1 x_1 + p_2 x_2 = m$$

(a) Derive the Marshallian demand functions x_1^* and x_2^*

(b) Derive the indirect utility function $V(p_1, p_2, m)$

Solution

(a) Marshallian demand functions

The consumer's problem is

$$\begin{aligned} \max_{x_1,x_2} \quad & U(x_1,x_2) = x_1 + \sqrt{x_2} \\ \text{s.t.} \quad & p_1x_1 + p_2x_2 = m \\ & x_1, x_2 \geq 0 \end{aligned}$$

Step 1: Form the Lagrangian

$$\mathcal{L} = x_1 + \sqrt{x_2} + \lambda\Big(m - p_1x_1 - p_2x_2\Big)$$

Step 2: First-order conditions
Differentiate with respect to x_1

$$\frac{\partial \mathcal{L}}{\partial x_1} = 1 - \lambda p_1 = 0 \quad \Longrightarrow \quad \lambda = \frac{1}{p_1}$$

Differentiate with respect to x_2

$$\frac{\partial \mathcal{L}}{\partial x_2} = \frac{1}{2\sqrt{x_2}} - \lambda p_2 = 0 \quad \Longrightarrow \quad \frac{1}{2\sqrt{x_2}} = \lambda p_2$$

Substituting $\lambda = \frac{1}{p_1}$

$$\frac{1}{2\sqrt{x_2}} = \frac{p_2}{p_1} \quad \Longrightarrow \quad \sqrt{x_2} = \frac{p_1}{2p_2}$$

Thus, the optimal demand for x_2 is

$$x_2^* = \frac{p_1^2}{4p_2^2}$$

Step 3: Solve for x_1^*
Substituting x_2^* into the budget constraint

$$p_1x_1 + p_2\left(\frac{p_1^2}{4p_2^2}\right) = m \quad \Longrightarrow \quad p_1x_1 + \frac{p_1^2}{4p_2} = m$$

Solving for x_1

$$x_1^* = \frac{m}{p_1} - \frac{p_1}{4p_2}$$

(b) Indirect utility function

Substituting the optimal demands into the utility function

$$V(p_1,p_2,m) = x_1^* + \sqrt{x_2^*} = \left(\frac{m}{p_1} - \frac{p_1}{4p_2}\right) + \sqrt{\frac{p_1^2}{4p_2^2}}$$

Noting that

$$\sqrt{\frac{p_1^2}{4p_2^2}} = \frac{p_1}{2p_2}$$

we obtain

$$V(p_1,p_2,m) = \frac{m}{p_1} - \frac{p_1}{4p_2} + \frac{p_1}{2p_2} = \frac{m}{p_1} + \frac{p_1}{4p_2}$$

Final answers

$$x_2^*(p_1, p_2, m) = \frac{p_1^2}{4p_2^2}$$

$$x_1^*(p_1, p_2, m) = \frac{m}{p_1} - \frac{p_1}{4p_2}$$

$$V(p_1, p_2, m) = \frac{m}{p_1} + \frac{p_1}{4p_2}$$