# First-order ordinary differential equations: exact equations

We say that an ODE is **EXACT** if its general representation is:

$$P(x,y) \cdot dx + Q(x,y) \cdot dy = 0$$

There must exist a function $U(x,y)$, called the *potential function*, such that:

$$dU(x,y) = U'_x\, dx + U'_y\, dy \quad \Rightarrow \quad U'_x = P(x,y) \quad \wedge \quad U'_y = Q(x,y)$$

Since $dU(x,y) = 0$ when the functions $P(x,y)$ and $Q(x,y)$ are equal to the partial derivatives of $U(x,y)$:

$$U(x,y) = C$$

For the equation to be **Exact**, it must satisfy the *symmetry condition*: $P'_y = Q'_x$

$$\begin{cases} \frac{\partial U}{\partial x} = P(x,y) \Rightarrow \int dU = \int P(x,y)dx = F(x,y) + \alpha(y) \\ \frac{\partial U}{\partial y} = Q(x,y) \Rightarrow \int dU = \int Q(x,y)dy = F(x,y) + \beta(x) \end{cases}$$

$$U(x,y) = F(x,y) + \alpha(y) + \beta(x) = C$$

## Example

$$(2x^3 + y) \cdot dx + (x + 2y^2) \cdot dy = 0$$

For the equation to be **Exact**, it must satisfy the *symmetry condition*:

$$P'_y = Q'_x \quad \Rightarrow \quad 1 = 1$$

$$\begin{cases} \frac{\partial U}{\partial x} = (2x^3 + y) \Rightarrow \int dU = \int (2x^3 + y) dx = 2\frac{x^4}{4} + xy = \frac{x^4}{2} + xy \\ \frac{\partial U}{\partial y} = (x + 2y^2) \Rightarrow \int dU = \int (x + 2y^2) dy = xy + 2\frac{y^3}{3} = xy + \frac{2y^3}{3} \end{cases}$$

$$\Rightarrow \quad U(x, y) = F(x, y) + \alpha(y) + \beta(x) = xy + \frac{x^4}{2} + \frac{2y^3}{3} = C$$

$$\text{Solution:} \quad xy + \frac{x^4}{2} + \frac{2y^3}{3} = C$$