Solving IS-LM

The acronym refers to 'Investment-Savings' and 'Liquidity preference-Money supply'. It is characterized by the following system:

$$\begin{aligned} Y &= C(Y_d) + I(r) + G \\ M_p &= L(Y; r) \\ Y_d &= Y - T(Y; \theta) \end{aligned}$$

where the endogenous and exogenous variables are:

Endogenous variables:

- Y: Income
- r: Interest rate
- Y_d: Disposable income

Exogenous variables (and parameters):

- G: Public spending
- M_p: Money supply
- θ: Determinant of the tax mass

The variables maintain the following relationship with respect to the sign of the derivatives.

$$0 < \frac{\partial C}{\partial Y_d} < 1, \quad \frac{\partial I}{\partial r} < 0, \quad \frac{\partial L}{\partial Y} > 0, \quad \frac{\partial L}{\partial r} < 0, \quad 0 < \frac{\partial T}{\partial Y} < 1, \quad \frac{\partial T}{\partial \theta} > 0$$

Calculate:

1. How does the equilibrium output change with a variation in public spending?
2. How does the equilibrium interest rate change with a variation in public spending?

Solution

While it is possible to work directly with the system of three equations, it is convenient to internalize the third equation into the first in order to operate with smaller dimension matrices (although doing so makes some derivatives a bit more complex to calculate since the chain rule must be used).

In this way, the system is as follows:

$$\begin{aligned} Y &= c\left(Y - T(Y;\theta)\right) + I(r) + G \\ M_p &= L(Y;r) \end{aligned}$$

where now the endogenous and exogenous variables are:

Endogenous variables:

- Y: Income
- r: Interest rate

Exogenous variables (and parameters):

- G: Public spending
- M_p: Money supply
- θ: Determinant of the tax mass

Rewriting the System

The system is rewritten as:

$$\begin{aligned} Y - C\left(Y - T(Y;\theta)\right) - I(r) - g = 0 \\ M_p - L(Y;r) = 0 \end{aligned}$$

Differentiating the system, we obtain:

$$\begin{aligned} \left(1 - C'_{Y_d} \cdot Y_d{}'_Y\right) dY - I'_r dr - dg - C'_{Y_d} \cdot Y_d{}'_\theta d\theta = 0 \\ dM_p - L'_Y dY - L'_r dr = 0 \end{aligned}$$

Replacing $Y_d{}'_Y$ and $Y_d{}'_\theta$, we have:

$$\begin{aligned} \left(1 - C'_{Y_d}(1 - T'_Y)\right) dY - I'_r dr - dg + C'_{Y_d} \cdot T'_\theta d\theta = 0 \\ dM_p - L'_Y dY - L'_r dr = 0 \end{aligned}$$

Grouping the endogenous variables on the left side of each equation and the exogenous variables (and parameters) on the right side, we get:

$$\begin{aligned} \left(1 - C'_{Y_d}(1 - T'_Y)\right) dY - I'_r dr &= dg - C'_{Y_d} \cdot T'_\theta d\theta \\ -L'_Y dY - L'_r dr &= -dM_p \end{aligned}$$

Rewriting the System in Matrix Form

The system is rewritten in matrix form:

$$\begin{bmatrix} 1 - C'_{Y_d}(1 - T'_Y) & -I'_r \\ -L'_Y & -L'_r \end{bmatrix} \begin{bmatrix} dY \\ dr \end{bmatrix} = \begin{bmatrix} 1 & 0 & -C'_{Y_d} T'_\theta \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} dg \\ dM_p \\ d\theta \end{bmatrix}$$

To proceed with the calculation of the requested derivatives, the Jacobian is first found:

$$|J| = \left| \begin{bmatrix} 1 - C'_{Y_d}(1 - T'_Y) & -I'_r \\ -L'_Y & -L'_r \end{bmatrix} \right| = \underbrace{-\underbrace{(1 - \underbrace{\overbrace{C'_{Y_d}}^{0<\,<1} \underbrace{(1 - \overbrace{T'_Y}^{0<\,<1})}_{>0}}_{0<\,<1})}_{<0} \cdot \overbrace{L'_r}^{<0}}_{>0} - \underbrace{\overbrace{I'_r}^{<0} \overbrace{L'_Y}^{>0}}_{>0} > 0$$

In this way,

$$\frac{\partial Y}{\partial G} = \frac{\begin{bmatrix} 1 & -I'_r \\ 0 & -L'_r \end{bmatrix}}{|J|} = \frac{\overbrace{-L'_r}^{>0}}{-(1 - C'_{Y_d}(1 - T'_Y))L'_r - I'_r L'_Y} > 0$$

$$\frac{\partial r}{\partial G} = \frac{\begin{bmatrix} 1 - C'_{Y_d}(1 - T'_Y) & 1 \\ -L'_Y & 0 \end{bmatrix}}{|J|} = \frac{\overbrace{L'_Y}^{>0}}{-(1 - C'_{Y_d}(1 - T'_Y))L'_r - I'_r L'_Y} > 0$$

With an increase in public spending, both the equilibrium output and the interest rate increase.