Expenditure Function and Hicksian Demands for an n-Good Max Utility via Duality

Consider a consumer whose utility function is

$$u(x_1, x_2, \ldots, x_n) = \max\{\, x_1,\ x_2,\ \ldots,\ x_n\},$$

with prices $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ and income m:

Marshallian demands:

$$x_{i^*}(p, m) = \frac{m}{p_{i^*}}, \quad x_j(p, m) = 0 \ \text{ for all } j \neq i^*$$

where $p_{i^*} = \min\{p_1, \ldots, p_n\}$.

Indirect utility function:

$$v(p, m) = \frac{m}{\min_i \, p_i}$$

Using duality, derive the following:

1. The **expenditure function** $E(p, u)$, which represents the minimum expenditure required to achieve utility level u at prices p
2. The **Hicksian (compensated) demands** $h_i(p, u)$, which represent the optimal quantities of goods demanded at prices p to achieve utility level u at minimum cost

Solution

1. Expenditure Function $E(p,u)$

The expenditure function is derived by inverting the indirect utility function $v(p,m)$. Starting from:

$$v(p,m) = \frac{m}{\min_i\, p_i}$$

Set $u = v(p,m)$ and solve for m:

$$u = \frac{m}{\min_i\, p_i}$$

$$m = u \cdot \min_i\, p_i$$

Thus, the expenditure function is:

$$E(p,u) = u \cdot \min_i\, p_i$$

2. Hicksian (Compensated) Demands $h_i(p,u)$

The Hicksian demands minimize expenditure while achieving utility u. For the max utility function, this occurs by spending entirely on the cheapest good. Let $p_{i^*} = \min_i\, p_i$. Then:

$$h_i(p,u) = \begin{cases} u & \text{if } i = i^* \\ 0 & \text{otherwise} \end{cases}$$

Formally, using Shephard's lemma:

$$h_i(p,u) = \frac{\partial E(p,u)}{\partial p_i}$$

For $i = i^*$, compute the derivative of $E(p,u) = u \cdot p_{i^*}$:

$$h_{i^*}(p,u) = u$$

For $j \neq i^*$, $\frac{\partial E(p,u)}{\partial p_j} = 0$, so:

$$h_j(p,u) = 0 \quad \text{for } j \neq i^*$$