Differentiability and level curve slopes

Necessary Condition

A necessary condition for a function $f(x, y)$ to be differentiable at the point (x_0, y_0) is that it be continuous and have partial derivatives at that point.

Sufficient Condition

A sufficient condition for a function $f(x, y)$ to be differentiable at the point (x_0, y_0) is that at least one of its partial derivatives is continuous in a neighborhood of that point.

Properties

- If $f(x, y)$ is differentiable at a point (x_0, y_0), then it is continuous at that point and admits derivatives in any direction and sense.
- If $f(x, y)$ is not continuous or not differentiable at the point (x_0, y_0), then it is not differentiable at that point.
- If $f(x, y)$ does not have continuous partial derivatives, it may or may not be differentiable.

Differential and Rates of Substitution

Suppose we have a differentiable utility function: $u = f(x_1, x_2)$

$$du = f'_{x_1} dx_1 + f'_{x_2} dx_2$$

If we want to see how much an independent variable needs to change in response to changes in the other independent variable such that u remains unchanged, we assume $du = 0$.

$$0 = f'_{x_1} dx_1 + f'_{x_2} dx_2$$

$$-f'_{x_1} dx_1 = f'_{x_2} dx_2$$

$$-\frac{f_{x_1}}{f'_{x_2}} = \frac{dx_2}{dx_1}$$

This ratio of derivatives indicates what is the slope of the level curve associated with the utility function. In other words, it is the slope of the indifference curve. It is usually called the marginal rate of substitution or marginal substitution ratio. The interpretation is as follows:

$$\text{MRS}(x_2/x_1) = |-\frac{f_{x_1}}{f'_{x_2}}|$$

If I increase by an infinitesimal unit the quantity of x_2, the amount that I must increase x_1 so that utility remains constant is: $|-\frac{f_{x_1}}{f'_{x_2}}|$. On the other hand, if we look at the reciprocal:

$$\text{MRS}(x_1/x_2) = |-\frac{f_{x_2}}{f'_{x_1}}|$$

If I increase by an infinitesimal unit the quantity of x_1, the amount that I must increase x_2 so that utility remains constant is: $|-\frac{f_{x_2}}{f'_{x_1}}|$. In the same way, we can interpret the technical marginal rates of substitution, or technical substitution rate. These refer to the slopes of the isoquants of a particular production function:

$$\mathrm{TRS}(x_2/x_1) = |-\frac{f_{x_1}}{f'_{x_2}}|$$

If I increase by an infinitesimal unit the quantity of input x_1, the amount that I must increase x_2 so that the output remains constant is: $|-\frac{f_{x_1}}{f'_{x_2}}|$

$$\mathrm{TRS}(x_1/x_2) = |-\frac{f_{x_2}}{f'_{x_1}}|$$

If I increase by an infinitesimal unit the quantity of input x_2, the amount that I must increase x_1 so that the output remains constant is: $|-\frac{f_{x_2}}{f'_{x_1}}|$ Additionally, interpretations can also be made by speaking of decreases; that is, $\mathrm{MRS}(x_1/x_2)$ indicates how much I must increase x_2 when I decrease x_1 by an infinitesimal amount, so as to keep utility constant.

The reason the interpretation is made with the absolute value of the derivative ratio is because, assuming that there is substitution between X_1 and x_2 (meaning that the level curves have a negative slope), what interests us is the amount in which one good is substituted for another. In cases where we have level curves with a positive slope, the interpretation will vary; in those cases, the formula will indicate how much I must increase one variable when the other variable also increases, in such a way that the value of the dependent variable remains constant.