Recovery of the direct utility

Given the indirect utility function

$$v(\mathbf{p}, m) = \frac{m^2}{4p_1 p_2},$$

for $\mathbf{p} = (p_1, p_2) \gg 0$ and $m > 0$.

Exercise.

1. Find the expenditure function associated with $v(\mathbf{p}, m)$ and, from it (using Roy's Identity), derive the Marshallian demands.

2. "Recover" the direct utility function $u(q_1, q_2)$ that is ordinally equivalent to the utility function generating the indirect utility function $v(\mathbf{p}, m)$.

Solution

1. Expenditure function and Marshallian demands

The indirect utility function is

$$v(\mathbf{p}, m) = \frac{m^2}{4p_1p_2}.$$

The expenditure function $e(\mathbf{p}, u)$ is obtained by solving for m in terms of u. That is, if

$$v(\mathbf{p}, m) = u \quad \Longrightarrow \quad \frac{m^2}{4p_1p_2} = u,$$

then

$$m^2 = 4up_1p_2 \quad \Longrightarrow \quad m = 2\sqrt{up_1p_2}.$$

Thus, the expenditure function is

$$e(\mathbf{p}, u) = 2\sqrt{up_1p_2}.$$

Now, we use Roy's Identity to derive the Marshallian demands:

$$q_i^*(\mathbf{p}, m) = -\frac{\partial v(\mathbf{p}, m)/\partial p_i}{\partial v(\mathbf{p}, m)/\partial m}, \quad i = 1, 2.$$

First, compute the partial derivatives of $v(\mathbf{p}, m)$:

$$\frac{\partial v}{\partial m} = \frac{2m}{4p_1p_2} = \frac{m}{2p_1p_2}.$$

For p_1:

$$\frac{\partial v}{\partial p_1} = -\frac{m^2}{4p_1^2p_2}.$$

Applying Roy's Identity for q_1:

$$q_1^*(\mathbf{p}, m) = -\frac{\partial v/\partial p_1}{\partial v/\partial m} = -\frac{-\frac{m^2}{4p_1^2p_2}}{\frac{m}{2p_1p_2}} = \frac{m}{2p_1}.$$

Similarly, for q_2:

$$\frac{\partial v}{\partial p_2} = -\frac{m^2}{4p_1p_2^2}, \quad \text{and} \quad q_2^*(\mathbf{p}, m) = \frac{m}{2p_2}.$$

Thus, the Marshallian demands are:

$$q_1^*(\mathbf{p}, m) = \frac{m}{2p_1}, \quad q_2^*(\mathbf{p}, m) = \frac{m}{2p_2}.$$

2. Recovering the direct utility function

The indirect utility function relates to the expenditure function through duality:

$$v(\mathbf{p}, m) = \max\{u : e(\mathbf{p}, u) \leq m\}.$$

Since we obtained:

$$e(\mathbf{p}, u) = 2\sqrt{up_1p_2},$$

solving $e(\mathbf{p}, u) = m$ for u gives:

$$2\sqrt{up_1p_2} = m \quad \Longrightarrow \quad \sqrt{u} = \frac{m}{2\sqrt{p_1p_2}},$$
$$\Longrightarrow \quad u = \frac{m^2}{4p_1p_2},$$

which coincides with $v(\mathbf{p}, m)$.

To recover the direct utility function $u(q_1, q_2)$, we need a function that generates the same Marshallian demands when maximized subject to the budget constraint:

$$p_1q_1 + p_2q_2 = m.$$

Since we obtained:

$$q_1^* = \frac{m}{2p_1}, \quad q_2^* = \frac{m}{2p_2},$$

we verify that in equilibrium:

$$p_1\left(\frac{m}{2p_1}\right) + p_2\left(\frac{m}{2p_2}\right) = \frac{m}{2} + \frac{m}{2} = m.$$

Now, consider the following direct utility function:

$$u(q_1, q_2) = 4q_1q_2.$$

Maximizing it under the budget constraint results in a Cobb–Douglas problem (with exponents 1/2 and 1/2 in logarithmic terms), whose solution is:

$$q_1^* = \frac{m}{2p_1}, \quad q_2^* = \frac{m}{2p_2}.$$

The achieved utility level is:

$$u^* = 4\left(\frac{m}{2p_1}\right)\left(\frac{m}{2p_2}\right) = \frac{m^2}{p_1p_2}.$$

Since this is a monotonic transformation of the indirect function:

$$v(\mathbf{p}, m) = \frac{m^2}{4p_1p_2} \quad \Longrightarrow \quad 4\,v(\mathbf{p}, m) = \frac{m^2}{p_1p_2},$$

the direct utility function that we recover is:

$$u(q_1, q_2) = 4q_1q_2.$$

Summary

- The expenditure function is:

$$e(\mathbf{p}, u) = 2\sqrt{up_1p_2}.$$

- The Marshallian demand functions are:

$$q_1^*(\mathbf{p}, m) = \frac{m}{2p_1}, \quad q_2^*(\mathbf{p}, m) = \frac{m}{2p_2}.$$

- The recovered direct utility function is:

$$u(q_1, q_2) = 4q_1q_2.$$