Expenditure Function and Compensated Demands for an n-Good Perfect-Complements Utility via Duality

Consider a consumer whose utility function is given by

$$u(x_1, x_2, \ldots, x_n) \;=\; \min\{\, x_1, x_2, \ldots, x_n\}$$

with prices $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ and income m.

Marshallian demands:

$$x_i(p, m) \;=\; \frac{m}{\sum_{j=1}^{n} p_j}\,.$$

Indirect utility function:

$$v(p, m) \;=\; \frac{m}{\sum_{i=1}^{n} p_i}\,.$$

Using duality, derive the following:

1. The **expenditure function** $E(p, u)$, which represents the minimum expenditure required to achieve utility level u at prices p.
2. The **Hicksian (compensated) demands** $h_i(p, u)$, which represent the optimal quantities of goods demanded at prices p to achieve utility level u at minimum cost.

Solution

1. Expenditure Function $E(p, u)$

The expenditure function $E(p, u)$ is the minimum expenditure required to achieve a given utility level u at prices p. It is the inverse of the indirect utility function $v(p, m)$. From the indirect utility function

$$v(p, m) = \frac{m}{\sum_{i=1}^{n} p_i}$$

we solve for m in terms of the utility u (noting $u = v(p, m)$):

$$u = \frac{m}{\sum_{i=1}^{n} p_i}$$

Rearranging for m:

$$m = u \sum_{i=1}^{n} p_i$$

Thus, the expenditure function is:

$$E(p, u) = u \sum_{i=1}^{n} p_i$$

2. Hicksian (Compensated) Demands $h_i(p, u)$

The Hicksian demands $h_i(p, u)$ follow from Shephard's lemma:

$$h_i(p, u) = \frac{\partial E(p, u)}{\partial p_i}$$

Given

$$E(p, u) = u \sum_{i=1}^{n} p_i$$

we take the partial derivative with respect to p_i:

$$\frac{\partial E(p, u)}{\partial p_i} = u$$

Hence, each Hicksian demand is:

$$h_i(p, u) = u$$