Expenditure Function and Hicksian Demands for an n-Good Perfect-Substitutes Utility via Duality

Given a consumer with utility function

$$u(x_1, x_2, \dots, x_n) = a_1\, x_1 + a_2\, x_2 + \dots + a_n\, x_n$$

where $a_i > 0$ for all i, facing prices $\mathbf{p} = (p_1, p_2, \dots, p_n)$ and income m:

Marshallian demands:

$$x_i(p, m) = \begin{cases} \dfrac{m}{p_{i^*}} & \text{if } i = i^* \text{ and } \frac{a_{i^*}}{p_{i^*}} > \frac{a_j}{p_j}\ \forall j \neq i^* \\ 0 & \text{otherwise} \end{cases}$$

where $i^* = \arg\max_k \frac{a_k}{p_k}$.

Indirect utility function:

$$v(p, m) = m \max_{1 \leq i \leq n} \frac{a_i}{p_i}$$

Using duality, derive the following:

1. The **expenditure function** $E(p, u)$, which represents the minimum expenditure required to achieve utility level u at prices p
2. The **Hicksian (compensated) demands** $h_i(p, u)$, which represent the optimal quantities of goods demanded at prices p to achieve utility level u at minimum cost

Solution

1. Expenditure Function $E(p, u)$

The expenditure function is derived by inverting the indirect utility function $v(p, m)$. Starting from:

$$v(p, m) = m \max_{1 \leq i \leq n} \frac{a_i}{p_i}$$

Set $u = v(p, m)$ and solve for m:

$$u = m \max_{1 \leq i \leq n} \frac{a_i}{p_i}$$
$$m = \frac{u}{\max_{1 \leq i \leq n} \frac{a_i}{p_i}}$$

Thus, the expenditure function is:

$$E(p, u) = \frac{u}{\max_{1 \leq i \leq n} \frac{a_i}{p_i}}$$

2. Hicksian (Compensated) Demands $h_i(p, u)$

The Hicksian demands minimize expenditure while achieving utility u. For perfect substitutes, this occurs by spending entirely on the good with the highest $\frac{a_i}{p_i}$. Let $i^* = \arg\max_k \frac{a_k}{p_k}$. Then:

$$h_i(p, u) = \begin{cases} \dfrac{u}{a_{i^*}} & \text{if } i = i^* \\ 0 & \text{otherwise} \end{cases}$$

Formally, using Shephard's lemma:

$$h_i(p, u) = \frac{\partial E(p, u)}{\partial p_i}$$

For $i = i^*$, compute the derivative of $E(p, u) = \frac{u p_{i^*}}{a_{i^*}}$:

$$h_{i^*}(p, u) = \frac{u}{a_{i^*}}$$

For $j \neq i^*$, $\frac{\partial E(p,u)}{\partial p_j} = 0$, so:

$$h_j(p, u) = 0 \quad \text{for } j \neq i^*$$