# Expenditure Minimization with Stone–Geary Utility

Consider a consumer with the Stone–Geary utility function

$$U(x_1, x_2) = (x_1 - \gamma_1)^{\alpha}(x_2 - \gamma_2)^{1-\alpha},$$

where $x_1 \geq \gamma_1$, $x_2 \geq \gamma_2$, $0 < \alpha < 1$, and $\gamma_1, \gamma_2 > 0$. The prices of goods $x_1$ and $x_2$ are $p_1$ and $p_2$, respectively. Suppose the consumer wishes to achieve a target utility level $U_0 > 0$.

(a) Write down the expenditure minimization problem.

(b) Derive the Hicksian (compensated) demand functions for $x_1$ and $x_2$.

# Solution

## (a) Formulation of the Expenditure Minimization Problem

The consumer minimizes total expenditure while attaining utility $U_0$. That is,

$$\begin{aligned} \min_{x_1,x_2} \quad & p_1x_1 + p_2x_2, \\ \text{s.t.} \quad & (x_1-\gamma_1)^{\alpha}(x_2-\gamma_2)^{1-\alpha} = U_0, \\ & x_1 \geq \gamma_1, \quad x_2 \geq \gamma_2. \end{aligned}$$

For convenience, define the transformed variables

$$y_1 = x_1 - \gamma_1, \quad y_2 = x_2 - \gamma_2,$$

with $y_1, y_2 \geq 0$. Then the utility constraint becomes

$$y_1^{\alpha} y_2^{1-\alpha} = U_0,$$

and the minimization problem is equivalently

$$\begin{aligned} \min_{y_1,y_2} \quad & p_1(y_1+\gamma_1) + p_2(y_2+\gamma_2), \\ \text{s.t.} \quad & y_1^{\alpha} y_2^{1-\alpha} = U_0. \end{aligned}$$

Since $p_1\gamma_1 + p_2\gamma_2$ is constant, we can focus on minimizing

$$p_1y_1 + p_2y_2 \quad \text{subject to} \quad y_1^{\alpha} y_2^{1-\alpha} = U_0.$$

## (b) Derivation of the Hicksian Demands

**Step 1. Set Up the Lagrangian.**

$$\mathcal{L} = p_1y_1 + p_2y_2 + \lambda\Big(U_0 - y_1^{\alpha} y_2^{1-\alpha}\Big).$$

**Step 2. First-Order Conditions.** Differentiate with respect to $y_1$ and $y_2$:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial y_1} &= p_1 - \lambda\alpha\, y_1^{\alpha-1} y_2^{1-\alpha} = 0, \\ \frac{\partial \mathcal{L}}{\partial y_2} &= p_2 - \lambda(1-\alpha)\, y_1^{\alpha} y_2^{-\alpha} = 0. \end{aligned}$$

Dividing the first equation by the second gives:

$$\frac{p_1}{p_2} = \frac{\alpha}{1-\alpha} \cdot \frac{y_2}{y_1}.$$

Thus,

$$y_2 = \frac{p_1(1-\alpha)}{p_2\alpha}\, y_1.$$

**Step 3. Substitute into the Constraint.** Plug the expression for $y_2$ into the utility constraint:

$$y_1^{\alpha}\left(\frac{p_1(1-\alpha)}{p_2\alpha}\,y_1\right)^{1-\alpha} = U_0.$$

Simplify as follows:

$$y_1^{\alpha+1-\alpha}\left(\frac{p_1(1-\alpha)}{p_2\alpha}\right)^{1-\alpha} = y_1\left(\frac{p_1(1-\alpha)}{p_2\alpha}\right)^{1-\alpha} = U_0.$$

Solving for $y_1$ yields:

$$y_1^* = U_0\left(\frac{p_2\alpha}{p_1(1-\alpha)}\right)^{1-\alpha}.$$

**Step 4. Determine $y_2^*$.** Substitute $y_1^*$ back into the relation for $y_2$:

$$y_2^* = \frac{p_1(1-\alpha)}{p_2\alpha}\,y_1^* = U_0\left(\frac{p_1(1-\alpha)}{p_2\alpha}\right)^{\alpha}.$$

**Step 5. Recover the Hicksian Demands for $x_1$ and $x_2$.** Recall that

$$x_1 = y_1 + \gamma_1, \quad x_2 = y_2 + \gamma_2.$$

Thus, the Hicksian (compensated) demands are:

$$x_1^H = U_0\left(\frac{p_2\alpha}{p_1(1-\alpha)}\right)^{1-\alpha} + \gamma_1,$$
$$x_2^H = U_0\left(\frac{p_1(1-\alpha)}{p_2\alpha}\right)^{\alpha} + \gamma_2.$$

Moreover, the corresponding minimum expenditure function is given by

$$e(p_1, p_2, U_0) = p_1\gamma_1 + p_2\gamma_2 + U_0\,\alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)}p_1^{\alpha}p_2^{1-\alpha}.$$