Marshallian demands for a nonstandard utility function

Consider a consumer with the utility function

$$u(x_1, x_2) = x_1 + \ln(x_1 + x_2)$$

subject to the budget constraint

$$p_1x_1 + p_2x_2 = m, \quad x_1, x_2 \geq 0$$

where $p_1, p_2 > 0$ and $m > 0$ Because the utility function is nonstandard, the optimum may occur either in an interior solution or at a corner We derive the interior solution via the Lagrangian method and then examine conditions under which a corner solution is optimal

(1) Derivation of the interior solution

(2) Corner solutions

(3) Indirect utility function

Solution

(1) Derivation of the interior solution

The consumer's problem is

$$\max_{x_1,x_2}\; x_1 + \ln(x_1 + x_2)$$

subject to

$$p_1x_1 + p_2x_2 = m$$

Setting up the Lagrangian

$$\mathcal{L} = x_1 + \ln(x_1 + x_2) + \lambda\Big(m - p_1x_1 - p_2x_2\Big)$$

First-order conditions

$$\frac{\partial \mathcal{L}}{\partial x_1} = 1 + \frac{1}{x_1 + x_2} - \lambda p_1 = 0$$
$$\frac{\partial \mathcal{L}}{\partial x_2} = \frac{1}{x_1 + x_2} - \lambda p_2 = 0$$

Solving for λ from the second equation

$$\lambda = \frac{1}{p_2(x_1 + x_2)}$$

Substituting this into the first equation

$$1 + \frac{1}{x_1 + x_2} = \frac{p_1}{p_2(x_1 + x_2)}$$

Multiplying both sides by $x_1 + x_2$

$$(x_1 + x_2) + 1 = \frac{p_1}{p_2}$$

Thus

$$x_1 + x_2 = \frac{p_1}{p_2} - 1$$

Define

$$S \equiv x_1 + x_2 = \frac{p_1}{p_2} - 1$$

Using the budget constraint

$$p_1x_1 + p_2x_2 = m$$

Since $x_2 = S - x_1$, substituting gives

$$p_1x_1 + p_2(S - x_1) = m$$

Rearrange

$$(p_1 - p_2)x_1 + p_2S = m$$

Since

$$p_2S = p_2\left(\frac{p_1}{p_2} - 1\right) = p_1 - p_2$$

$$(p_1 - p_2)x_1 + (p_1 - p_2) = m$$

$$(p_1 - p_2)(x_1 + 1) = m$$

Solving for x_1

$$x_1^* = \frac{m}{p_1 - p_2} - 1$$

Since $x_1 + x_2 = S$, the demand for x_2 is

$$x_2^* = \frac{p_1}{p_2} - \frac{m}{p_1 - p_2}$$

(2) Corner solutions

If the conditions for the interior solution are not met, the optimum will occur at a corner

Case A: $m < p_1 - p_2$

Here, x_1^* would be negative Since $x_1 \geq 0$, the consumer will set

$$x_1 = 0$$

Then, the budget constraint implies

$$p_2 x_2 = m \quad \Longrightarrow \quad x_2 = \frac{m}{p_2}$$

Case B: $m > \frac{p_1(p_1-p_2)}{p_2}$

Here, x_2^* would be negative Since $x_2 \geq 0$, the consumer will set

$$x_2 = 0$$

The budget constraint then gives

$$p_1 x_1 = m \quad \Longrightarrow \quad x_1 = \frac{m}{p_1}$$

(3) Indirect utility function

Interior solution: For

$$p_1 - p_2 \leq m \leq \frac{p_1(p_1-p_2)}{p_2}$$

$$V(m, p_1, p_2) = \frac{m}{p_1 - p_2} - 1 + \ln\left(\frac{p_1}{p_2} - 1\right)$$

Corner solution (Case A): For $m < p_1 - p_2$,

$$V(m, p_1, p_2) = \ln\left(\frac{m}{p_2}\right)$$

Corner solution (Case B): For $m > \frac{p_1(p_1-p_2)}{p_2}$,

$$V(m, p_1, p_2) = \frac{m}{p_1} + \ln\left(\frac{m}{p_1}\right)$$

Final Answers:

Marshallian Demands:

$$\text{If } p_1 - p_2 \leq m \leq \frac{p_1(p_1-p_2)}{p_2}: \quad x_1^* = \frac{m}{p_1-p_2} - 1, \quad x_2^* = \frac{p_1}{p_2} - \frac{m}{p_1 - p_2}$$

$$\text{If } m < p_1 - p_2: \quad x_1^* = 0, \quad x_2^* = \frac{m}{p_2}$$

$$\text{If } m > \frac{p_1(p_1-p_2)}{p_2}: \quad x_1^* = \frac{m}{p_1}, \quad x_2^* = 0$$

Indirect Utility Function:

$$V(m, p_1, p_2) = \begin{cases} \ln\left(\frac{m}{p_2}\right), & m < p_1 - p_2 \\ \frac{m}{p_1 - p_2} - 1 + \ln\left(\frac{p_1}{p_2} - 1\right), & p_1 - p_2 \leq m \leq \frac{p_1(p_1-p_2)}{p_2} \\ \frac{m}{p_1} + \ln\left(\frac{m}{p_1}\right), & m > \frac{p_1(p_1-p_2)}{p_2} \end{cases}$$