Nested Utility Function - Perfect Complements and Perfect Substitutes

Consider a consumer with the utility function

$$U(x_1, x_2, x_3) = \min\{x_1,\, 2x_2\} + x_3.$$

The consumer faces prices p_1, p_2, and p_3 for goods x_1, x_2, and x_3, respectively, and has income m.

(a) Write down the consumer's utility maximization problem subject to the budget constraint

$$p_1x_1 + p_2x_2 + p_3x_3 \leq m.$$

(b) Derive the Marshallian demand functions for x_1, x_2, and x_3.

(c) Show that the consumer's optimal allocation depends on the relationship between

$$p^* := p_1 + \frac{p_2}{2}$$

and p_3. In particular, determine the optimal bundle in the cases $p^* < p_3$, $p^* > p_3$, and $p^* = p_3$.

Solution

(a) Formulation of the Problem

The consumer maximizes

$$\max_{x_1,x_2,x_3} U(x_1,x_2,x_3) = \min\{x_1,\, 2x_2\} + x_3$$

subject to the budget constraint

$$p_1x_1 + p_2x_2 + p_3x_3 \leq m, \quad x_1,x_2,x_3 \geq 0.$$

(b) Derivation of the Marshallian Demands

Notice that the utility function is *nested*: the inner aggregator is $\min\{x_1,\, 2x_2\}$, while x_3 enters linearly (that is, as a perfect substitute). To maximize the minimum operator, the consumer will set the two arguments equal (if positive consumption of the composite good is chosen). Thus, if some of the income is allocated to goods x_1 and x_2 so as to obtain a positive composite level, the optimum will satisfy

$$x_1 = 2x_2.$$

Define

$$z := \min\{x_1,\, 2x_2\}.$$

Under the condition $x_1 = 2x_2$, we have $z = x_1$ (and equivalently $x_2 = \dfrac{x_1}{2}$); hence, the utility becomes

$$U = z + x_3.$$

The expenditure on the composite good is

$$\text{Expenditure on } x_1 \text{ and } x_2 = p_1x_1 + p_2x_2 = p_1z + \frac{p_2}{2}z = \left(p_1 + \frac{p_2}{2}\right)z.$$

Define the effective price for one unit of the composite good as

$$p^* = p_1 + \frac{p_2}{2}.$$

Then, if the consumer devotes part of her income to obtaining z and the remainder to x_3, the budget constraint becomes

$$p^*z + p_3x_3 \leq m.$$

Since the utility function in these two "branches" is linear (i.e. perfect substitutes with marginal utility 1 for both z and x_3), the consumer compares the cost per unit of utility from each source:

$$\text{Cost per unit utility from } z = p^*, \quad \text{and from } x_3 = p_3.$$

Thus, the optimal allocation will depend on which of p^* and p_3 is lower.

Case 1: $p^* < p_3$. In this case the composite good (derived from x_1 and x_2) is cheaper. The consumer will spend all her income on obtaining utility via the composite good, setting $x_3 = 0$. The budget constraint becomes

$$p^*z = m, \quad \text{so} \quad z = \frac{m}{p^*}.$$

Recalling that under the optimal condition $x_1 = z$ and $x_2 = \dfrac{z}{2}$, we have:

$$x_1^* = \frac{m}{p^*}, \quad x_2^* = \frac{m}{2p^*}, \quad x_3^* = 0.$$

Case 2: $p^* > p_3$. Here the marginal cost of obtaining utility via x_3 is lower, so the consumer will spend all income on x_3 (and consume zero of the composite good). Thus,

$$x_3^* = \frac{m}{p_3}, \quad x_1^* = 0, \quad x_2^* = 0.$$

Case 3: $p^* = p_3$. In this situation the consumer is indifferent between obtaining utility through the composite good or through x_3. Any allocation (z, x_3) that satisfies

$$p^* z + p_3 x_3 = m \quad \text{with} \quad U = z + x_3 = \frac{m}{p^*}$$

is optimal, provided that if $z > 0$ then the condition $x_1 = 2x_2$ is maintained. In this case the Marshallian demands are not unique.

(c) Discussion

The optimal bundle is determined by comparing the effective cost per unit of utility from the composite good, $p^* = p_1 + \dfrac{p_2}{2}$, with that of x_3, which is p_3:

- If $p^* < p_3$, the composite good is relatively cheaper; the consumer spends all income on x_1 and x_2 (with $x_1 = 2x_2$).
- If $p^* > p_3$, the consumer spends all income on x_3.
- If $p^* = p_3$, any combination that exhausts the budget and yields $U = z + x_3 = \frac{m}{p^*}$ is optimal.