Expenditure Function and Hicksian Demands for an n-Good Cobb-Douglas Utility Using Duality Conditions

Given the utility function of a Cobb-Douglas form with n goods, where all exponents are equal to 1:

$$u(x_1, x_2, \ldots, x_n) = x_1 x_2 \cdots x_n$$

it is known (from the primal problem) that the Marshallian demands and the indirect utility function are:

$$x_i(p, m) = \frac{m}{n\, p_i}, \quad v(p, m) = \frac{m^n}{n^n \prod_{i=1}^{n} p_i}$$

Using **duality**, find:

1. the *expenditure function* $E(p, u)$
2. the *Hicksian (compensated) demands* $h_i(p, u)$

Solution

1. Expenditure Function $E(p,u)$

The expenditure function $E(p,u)$ is the minimum expenditure required to achieve a given utility level u at prices p. It is the inverse of the indirect utility function $v(p,m)$. From the indirect utility function:

$$v(p,m) = \frac{m^n}{n^n \prod_{i=1}^n p_i},$$

we solve for m in terms of $v(p,m)$:

$$u = \frac{m^n}{n^n \prod_{i=1}^n p_i}.$$

Rearranging for m:

$$m^n = u \cdot n^n \prod_{i=1}^n p_i,$$

$$m = \left(u \cdot n^n \prod_{i=1}^n p_i \right)^{1/n}.$$

Thus, the expenditure function is:

$$E(p,u) = \left(u \cdot n^n \prod_{i=1}^n p_i \right)^{1/n}.$$

2. Hicksian (Compensated) Demands $h_i(p,u)$

The Hicksian demands $h_i(p,u)$ are derived from the expenditure function using Shephard's lemma, which states:

$$h_i(p,u) = \frac{\partial E(p,u)}{\partial p_i}.$$

First, rewrite the expenditure function as:

$$E(p,u) = u^{1/n} \cdot n \cdot \left(\prod_{i=1}^n p_i \right)^{1/n}.$$

Now, take the partial derivative of $E(p,u)$ with respect to p_i:

$$\frac{\partial E(p,u)}{\partial p_i} = u^{1/n} \cdot n \cdot \frac{1}{n} \cdot \left(\prod_{i=1}^n p_i \right)^{1/n-1} \cdot \prod_{j \neq i} p_j.$$

Simplifying:

$$h_i(p,u) = u^{1/n} \cdot \left(\prod_{i=1}^n p_i \right)^{1/n-1} \cdot \prod_{j \neq i} p_j.$$