Marshallian demand and indirect utility for a CES utility function

Consider a consumer with the CES utility function

$$U(x_1, x_2) = \left(\alpha x_1^{\rho} + (1-\alpha) x_2^{\rho}\right)^{\frac{1}{\rho}}$$

with parameters $\alpha \in (0, 1)$ and $\rho \leq 1,\ \rho \neq 0$ The consumer faces prices $p_1 > 0$ and $p_2 > 0$ and has income $m > 0$ The consumer's problem is

$$\begin{aligned} \max_{x_1, x_2} \quad & U(x_1, x_2) = \left(\alpha x_1^{\rho} + (1-\alpha) x_2^{\rho}\right)^{\frac{1}{\rho}} \\ \text{s.t.} \quad & p_1 x_1 + p_2 x_2 = m \\ & x_1, x_2 > 0 \end{aligned}$$

(a) Derive the Marshallian demand functions

(b) Derive the indirect utility function

Solution

(a) Marshallian demand functions

Step 1: First-order conditions (FOCs)

The partial derivatives of the utility function are

$$\frac{\partial U}{\partial x_1} = \alpha\, x_1^{\rho-1}\Big(\alpha x_1^\rho + (1-\alpha)x_2^\rho\Big)^{\frac{1}{\rho}-1}$$

$$\frac{\partial U}{\partial x_2} = (1-\alpha)\, x_2^{\rho-1}\Big(\alpha x_1^\rho + (1-\alpha)x_2^\rho\Big)^{\frac{1}{\rho}-1}$$

At the optimum, the marginal rate of substitution (MRS) equals the price ratio

$$\frac{\frac{\partial U}{\partial x_1}}{\frac{\partial U}{\partial x_2}} = \frac{\alpha\, x_1^{\rho-1}}{(1-\alpha)\, x_2^{\rho-1}} = \frac{p_1}{p_2}$$

Rearranging

$$\frac{x_1^{\rho-1}}{x_2^{\rho-1}} = \frac{p_1(1-\alpha)}{p_2\alpha} \quad\implies\quad \frac{x_1}{x_2} = \left(\frac{p_1(1-\alpha)}{p_2\alpha}\right)^{\frac{1}{\rho-1}}$$

Define

$$k \;=\; \left(\frac{p_1(1-\alpha)}{p_2\alpha}\right)^{\frac{1}{\rho-1}}$$

so that

$$x_1 = k\, x_2$$

Step 2: Substituting into the budget constraint

Substituting $x_1 = k\, x_2$ into the budget constraint

$$p_1(k\, x_2) + p_2 x_2 = m \quad\implies\quad x_2\,(p_1 k + p_2) = m$$

Thus, the Marshallian demand functions are

$$x_2^*(p_1,p_2,m) = \frac{m}{p_1 k + p_2}$$

$$x_1^*(p_1,p_2,m) = k\, x_2^*(p_1,p_2,m) = \frac{k\, m}{p_1 k + p_2}$$

(b) Indirect utility function

Substituting the optimal demands into the utility function

$$V(p_1,p_2,m) = U(x_1^*,x_2^*) = (\alpha(x_1^*)^\rho + (1-\alpha)(x_2^*)^\rho)^{\frac{1}{\rho}}$$

Substituting $x_1^* = k\, x_2^*$ and $x_2^* = \dfrac{m}{p_1 k + p_2}$

$$\alpha(x_1^*)^\rho + (1-\alpha)(x_2^*)^\rho = \alpha\left(\frac{k\, m}{p_1 k + p_2}\right)^\rho + (1-\alpha)\left(\frac{m}{p_1 k + p_2}\right)^\rho$$

Factoring out

$$= \left(\frac{m}{p_1 k + p_2}\right)^\rho \Big[\alpha k^\rho + (1-\alpha)\Big]$$

Taking the $1/\rho$ power

$$V(p_1,p_2,m) = \frac{m}{p_1 k + p_2}\Big[\alpha k^\rho + (1-\alpha)\Big]^{\frac{1}{\rho}}$$

where

$$k \;=\; \left(\frac{p_1(1-\alpha)}{p_2\alpha}\right)^{\frac{1}{\rho-1}}$$