Cost minimization with a Cobb-Douglas Function and Second Order Conditions

The production function of a company is of the Cobb-Douglas type $f(K, L) = K^{\alpha}L^{\beta}$ where $0 < \alpha < 1$ and $0 < \beta < 1$, K is capital and L labor $(K > 0, L > 0)$. Suppose that the prices of capital and labor are given by $P_k > 0$ and $P_l > 0$ respectively. Find the combination of capital and labor that minimizes the cost when the production must be Q_0 product units $(Q_0 > 0)$

Solution

The problem is to minimize $C = KP_k + LP_l$ subject to the following restriction $Q_0 = K^\alpha L^\beta$. We construct the Lagrangian:

$$L = KP_k + LP_l + \lambda[Q_0 - K^\alpha L^\beta]$$

The first order conditions are:

$$L'_K = P_k - \lambda\alpha K^{\alpha-1}L^\beta = 0$$
$$L'_L = P_l - \lambda\beta K^\alpha L^{\beta-1} = 0$$
$$L'_\lambda = Q_0 - K^\alpha L^\beta = 0$$

From the first two equations we clear λ:

$$\frac{P_k}{\alpha K^{\alpha-1}L^\beta} = \lambda$$
$$\frac{P_l}{\beta K^\alpha L^{\beta-1}} = \lambda$$

We equalize the equations:

$$\frac{P_k}{\alpha K^{\alpha-1}L^\beta} = \frac{P_l}{\beta K^\alpha L^{\beta-1}}$$
$$\frac{P_k}{P_l} = \frac{\alpha K^{\alpha-1}L^\beta}{\beta K^\alpha L^{\beta-1}}$$

This equation shows us that the first order conditions indicate that the slopes of the level curves must be equalized at the same time that the restriction is met. Continuing, we clear L:

$$\frac{P_k}{P_l}\frac{\beta}{\alpha}\frac{1}{K} = L$$

We insert this into the third equation of the first order restrictions:

$$Q_0 - K^\alpha[\frac{P_k}{P_l}\frac{\beta}{\alpha}\frac{1}{K}]^\beta = 0$$

$$Q_0 - K^{\alpha-\beta}[\frac{P_k}{P_l}\frac{\beta}{\alpha}]^\beta = 0$$

We clear K:

$$Q_0[\frac{P_l}{P_k}\frac{\alpha}{\beta}]^\beta = K^{\alpha-\beta}$$

$$\left[Q_0[\frac{P_l}{P_k}\frac{\alpha}{\beta}]^\beta\right]^{\frac{1}{\alpha-\beta}} = K^*$$

Now we insert into the L function

$$\frac{P_k}{P_l}\frac{\beta}{\alpha}K^{-1} = L$$

$$\frac{P_k}{P_l}\frac{\beta}{\alpha}\left[Q_0[\frac{P_l}{P_k}\frac{\alpha}{\beta}]^\beta\right]^{\frac{-1}{\alpha-\beta}} = L$$

Simplifying:

$$Q_0^{\frac{1}{\beta-\alpha}}\frac{P_k}{P_l}\frac{\beta}{\alpha}\left[\frac{P_k}{P_l}\frac{\beta}{\alpha}\right]^{\frac{\beta}{\beta-\alpha}} = L$$

$$Q_0^{\frac{1}{\beta-\alpha}}\left[\frac{P_k}{P_l}\frac{\beta}{\alpha}\right]^{\frac{\alpha}{\beta-\alpha}} = L$$

$$\left[Q_0[\frac{P_k}{P_l}\frac{\beta}{\alpha}]^\alpha\right]^{\frac{1}{\beta-\alpha}} = L^*$$

Now, let's solve the second order conditions: First, we calculate the second derivatives:

$$L''_{KK} = -\lambda\alpha(\alpha-1)K^{\alpha-2}L^{\beta}$$

$$L''_{LL} = -\lambda\beta(\beta-1)K^{\alpha}L^{\beta-2}$$

$$L''_{LK} = -\lambda\beta\alpha K^{\alpha-1}L^{\beta-1}$$

$$L''_{KL} = -\lambda\beta\alpha K^{\alpha-1}L^{\beta-1}$$

Now the derivatives that correspond to the bordered Hessian:

$$g'_K = \alpha K^{\alpha-1}L^{\beta}$$

$$g'_L = \beta K^{\alpha}L^{\beta-1}$$

For the second order conditions, we construct the bordered Hessian. If we are at a minimum, the determinant of the bordered Hessian should be negative.

$$\bar{H} = \begin{pmatrix} 0 & g'x & g'y \\ g'x & L''_{xx} & L''_{xy} \\ g'y & L''_{yx} & L''_{yy} \end{pmatrix} = \begin{pmatrix} 0 & \alpha K^{\alpha-1}L^{\beta} & \beta K^{\alpha}L^{\beta-1} \\ \alpha K^{\alpha-1}L^{\beta} & -\lambda\alpha(\alpha-1)K^{\alpha-2}L^{\beta} & -\lambda\beta\alpha K^{\alpha-1}L^{\beta-1} \\ \beta K^{\alpha}L^{\beta-1} & -\lambda\beta\alpha K^{\alpha-1}L^{\beta-1} & -\lambda\beta(\beta-1)K^{\alpha}L^{\beta-2} \end{pmatrix}$$

Before replacing at the optimum, we calculate the determinant of the bordered Hessian:

$$\begin{aligned} &-\alpha K^{\alpha-1}L^{\beta}\begin{vmatrix} \alpha K^{\alpha-1}L^{\beta} & \beta K^{\alpha}L^{\beta-1} \\ -\lambda\beta\alpha K^{\alpha-1}L^{\beta-1} & -\lambda\beta(\beta-1)K^{\alpha}L^{\beta-2} \end{vmatrix} \\ &+\beta K^{\alpha}L^{\beta-1}\begin{vmatrix} \alpha K^{\alpha-1}L^{\beta} & \beta K^{\alpha}L^{\beta-1} \\ -\lambda\alpha(\alpha-1)K^{\alpha-2}L^{\beta} & -\lambda\beta\alpha K^{\alpha-1}L^{\beta-1} \end{vmatrix} \end{aligned}$$

Calculating the first term:

$$-\alpha K^{\alpha-1}L^{\beta}\left[(\alpha K^{\alpha-1}L^{\beta})(-\lambda\beta(\beta-1)K^{\alpha}L^{\beta-2}) - (\beta^{\alpha}L^{\beta-1})(-\lambda\beta\alpha K^{\alpha-1}L^{\beta-1})\right]$$

Simplifying the negatives and remembering that $\beta < 1$

$$\underbrace{-\alpha K^{\alpha-1}L^{\beta}\underbrace{\left[\underbrace{(\alpha K^{\alpha-1}L^{\beta})(-\lambda\beta\underbrace{(\beta-1)}_{-}K^{\alpha}L^{\beta-2})}_{+} + (\beta^{\alpha}L^{\beta-1})(\lambda\beta\alpha K^{\alpha-1}L^{\beta-1})\right]}_{+}}_{-}$$

This term is even before inserting the optimal values of K, L or λ since these 3 terms at the optimum are positive and do not affect the previous conclusion. Now let's calculate the second term of the determinant:

$$+\beta K^{\alpha}L^{\beta-1}\left[\alpha K^{\alpha-1}L^{\beta}(-\lambda\beta\alpha K^{\alpha-1}L^{\beta-1}) - \beta K^{\alpha}L^{\beta-1}(-\lambda\alpha(\alpha-1)K^{\alpha-2}L^{\beta})\right]$$

Simplifying the negatives:

$$\underbrace{+\beta K^{\alpha}L^{\beta-1}\left[\underbrace{-\alpha K^{\alpha-1}L^{\beta}(\lambda\beta\alpha K^{\alpha-1}L^{\beta-1})}_{-} + \underbrace{\beta K^{\alpha}L^{\beta-1}(\lambda\alpha\underbrace{(\alpha-1)}_{-}K^{\alpha-2}L^{\beta})}_{-}\right]}_{-}$$

If $\alpha - 1 < 0$ then we have that the determinant of the bordered Hessian is negative and we are at a minimum.