Marshallian demands for quadratic utility

Consider a consumer with the quadratic utility function

$$U(x_1, x_2) = x_1 + x_2 - \frac{1}{2}\Big(x_1^2 + x_2^2\Big)$$

where x_1 and x_2 denote the quantities of goods 1 and 2, respectively The consumer faces prices $p_1 > 0$ and $p_2 > 0$ and has income $m > 0$ The budget constraint is given by

$$p_1 x_1 + p_2 x_2 = m$$

(a) Formulate the consumer's problem

(b) Derive the first-order conditions (FOCs)

(c) Solve for the optimal consumption bundle (x_1^*, x_2^*)

Solution

(a) Utility maximization problem

The consumer's problem is

$$\max_{x_1,x_2} \quad U(x_1,x_2) = x_1 + x_2 - \frac{1}{2}\Big(x_1^2 + x_2^2\Big)$$
$$\text{s.t.} \quad p_1x_1 + p_2x_2 = m$$
$$x_1, x_2 \geq 0$$

(b) First-order conditions

Define the Lagrangian

$$\mathcal{L} = x_1 + x_2 - \frac{1}{2}(x_1^2 + x_2^2) + \lambda\Big(m - p_1x_1 - p_2x_2\Big)$$

Taking the partial derivatives with respect to x_1 and x_2

$$\frac{\partial \mathcal{L}}{\partial x_1} = 1 - x_1 - \lambda p_1 = 0 \quad \Longrightarrow \quad x_1 = 1 - \lambda p_1$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = 1 - x_2 - \lambda p_2 = 0 \quad \Longrightarrow \quad x_2 = 1 - \lambda p_2$$

(c) Solving for the optimal consumption bundle

Substituting the expressions for x_1 and x_2 into the budget constraint

$$p_1(1 - \lambda p_1) + p_2(1 - \lambda p_2) = m$$

Simplifying

$$p_1 + p_2 - \lambda(p_1^2 + p_2^2) = m$$

Solving for λ

$$\lambda = \frac{p_1 + p_2 - m}{p_1^2 + p_2^2}$$

Now substituting λ back to obtain the optimal demands

$$x_1^* = 1 - p_1\,\frac{p_1 + p_2 - m}{p_1^2 + p_2^2}$$

$$x_2^* = 1 - p_2\,\frac{p_1 + p_2 - m}{p_1^2 + p_2^2}$$

Interpretation of the solution

Since the quadratic utility function

$$U(x_1,x_2) = x_1 + x_2 - \frac{1}{2}(x_1^2 + x_2^2)$$

attains its maximum (bliss point) at $x_1 = 1$ and $x_2 = 1$ in the absence of a binding budget constraint, the above solution applies when the consumer's income m is insufficient to reach the bliss point, meaning the budget constraint is binding If m is sufficiently large so that the bliss point is affordable, the consumer will choose $(1,1)$ even though it may not exhaust income