Homogeneous functions and Euler's theorem

Indicate whether the following functions are homogeneous and, if they are, state the degree. For homogeneous functions, verify Euler's theorem.

1. $f(x, y) = 3x^2 + 5xy - y^2$
2. $f(x, y) = \cos\left(\frac{x}{y}\right)$

Solutions

A function $f : \mathbb{R}^m \to \mathbb{R}$ is said to be **homogeneous of degree n** if for all $\lambda \in \mathbb{R}$ it is verified that:

$$f(\lambda x_1, \lambda x_2, \lambda x_3, \ldots, \lambda x_m) = \lambda^n f(x_1, x_2, x_3, \ldots, x_m)$$

That is, a function is homogeneous of degree n, if when multiplied by a quantity λ to all the variables, the value of the function is multiplied by λ^n.

A **homogeneous function** exhibits a multiplicative scale behavior: if all variables are multiplied by a constant factor, then the function results in the multiplicative factor raised to a power, which turns out to be the degree of homogeneity.

1. $$f(\lambda x, \lambda y) = 3(\lambda x)^2 + 5(\lambda x)(\lambda y) - (\lambda y)^2$$

   It is not homogeneous:

2. $$f(\lambda x, \lambda y) = \cos\left(\frac{\lambda x}{\lambda y}\right) = \cos\left(\frac{x}{y}\right) = \lambda^0 \cos\left(\frac{x}{y}\right)$$

   It is a homogeneous function of degree 0. We verify Euler's theorem:

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 0f(x, y)$$

$$-x\sin\left(\frac{x}{y}\right)\frac{1}{y} + y\sin\left(\frac{x}{y}\right)\frac{x}{y^2} = 0f(x, y)$$

$$0 = 0f(x, y)$$