Marshallian demand and indirect utility 3

Consider a consumer with the utility function

$$u(q_1, q_2) = q_1 - \frac{1}{q_2},$$

subject to the budget constraint

$$p_1 q_1 + p_2 q_2 = m, \quad q_2 > 0, \quad q_1 \geq 0,$$

where $p_1, p_2 > 0$ and $m > 0$.

Question 1. Derive the Marshallian (uncompensated) demand functions for q_1 and q_2.

Question 2. Derive the corresponding indirect utility function.

Solution

Since the utility function is increasing in both goods (recall that the derivative with respect to q_1 is constant 1 and with respect to q_2 is $\frac{1}{q_2^2} > 0$), the consumer spends all income. However, because the utility is quasilinear in q_1 and nonlinear in q_2, we must check for the possibility of a corner solution.

A. Candidate Interior Solution.
We first assume an interior solution with $q_1 > 0$ and $q_2 > 0$. Form the Lagrangian:

$$\mathcal{L} = q_1 - \frac{1}{q_2} + \lambda\Big(m - p_1 q_1 - p_2 q_2\Big).$$

The first-order conditions (FOCs) are:
<u>For q_1:</u>

$$\frac{\partial \mathcal{L}}{\partial q_1} = 1 - \lambda p_1 = 0 \quad \Longrightarrow \quad \lambda = \frac{1}{p_1}.$$

<u>For q_2:</u>

$$\frac{\partial \mathcal{L}}{\partial q_2} = \frac{1}{q_2^2} - \lambda p_2 = 0 \quad \Longrightarrow \quad \frac{1}{q_2^2} = \lambda p_2.$$

Substitute $\lambda = \frac{1}{p_1}$:

$$\frac{1}{q_2^2} = \frac{p_2}{p_1} \quad \Longrightarrow \quad q_2^2 = \frac{p_1}{p_2} \quad \Longrightarrow \quad q_2^* = \sqrt{\frac{p_1}{p_2}},$$

where we take the positive square root since $q_2 > 0$.
Now, substitute q_2^* into the budget constraint:

$$p_1 q_1 + p_2\sqrt{\frac{p_1}{p_2}} = m \quad \Longrightarrow \quad p_1 q_1 + p_1^{1/2} p_2^{1/2} = m.$$

Solving for q_1 gives

$$q_1^* = \frac{m - p_1^{1/2} p_2^{1/2}}{p_1}.$$

For the interior solution to be feasible we require $q_1^* \geq 0$; that is,

$$\frac{m - \sqrt{p_1 p_2}}{p_1} \geq 0 \quad \Longrightarrow \quad m \geq \sqrt{p_1 p_2}.$$

B. Corner Solution.
If $m < \sqrt{p_1 p_2}$, then the candidate interior solution yields $q_1^* < 0$, which is infeasible. In that case the consumer cannot afford any positive quantity of q_1 while buying q_2 to satisfy the optimality condition. Since utility is linear in q_1 and the term q_1 adds directly to utility, the consumer will choose the corner solution:

$$q_1^* = 0, \quad \text{and allocate the entire budget to } q_2.$$

Then the budget constraint gives:

$$p_2 q_2 = m \quad \Longrightarrow \quad q_2^* = \frac{m}{p_2}.$$

The resulting utility is

$$u(0, q_2^*) = 0 - \frac{1}{m/p_2} = -\frac{p_2}{m}.$$

Final Answers for Question 1 (Marshallian Demands):

$$\text{If } m \geq \sqrt{p_1 p_2}: \quad q_1^* = \frac{m - \sqrt{p_1 p_2}}{p_1}, \quad q_2^* = \sqrt{\frac{p_1}{p_2}},$$

$$\text{If } m < \sqrt{p_1 p_2}: \quad q_1^* = 0, \quad q_2^* = \frac{m}{p_2}.$$

Question 2: Indirect Utility Function.

Case 1: Interior Solution ($m \geq \sqrt{p_1 p_2}$)

Substitute the optimal demands into the utility function:

$$u(q_1^*, q_2^*) = q_1^* - \frac{1}{q_2^*} = \frac{m - \sqrt{p_1 p_2}}{p_1} - \frac{1}{\sqrt{\frac{p_1}{p_2}}}.$$

Note that

$$\frac{1}{\sqrt{\frac{p_1}{p_2}}} = \sqrt{\frac{p_2}{p_1}}.$$

Thus,

$$V(m, p_1, p_2) = \frac{m - \sqrt{p_1 p_2}}{p_1} - \sqrt{\frac{p_2}{p_1}} = \frac{m - 2\sqrt{p_1 p_2}}{p_1}.$$

Case 2: Corner Solution ($m < \sqrt{p_1 p_2}$)

Here, the consumer chooses $q_1^* = 0$ and $q_2^* = \frac{m}{p_2}$, so

$$V(m, p_1, p_2) = u(0, q_2^*) = 0 - \frac{1}{m/p_2} = -\frac{p_2}{m}.$$

Final Answer for Question 2 (Indirect Utility):

$$V(m, p_1, p_2) = \begin{cases} \dfrac{m - 2\sqrt{p_1 p_2}}{p_1}, & \text{if } m \geq \sqrt{p_1 p_2}, \\ -\dfrac{p_2}{m}, & \text{if } m < \sqrt{p_1 p_2}. \end{cases}$$