Marshallian demand and indirect utility for a nested utility

Consider a consumer with the utility function

$$U(q_2, q_3, q_1) = \min\{q_2, q_3\} + q_1,$$

and the budget constraint

$$p_1 q_1 + p_2 q_2 + p_3 q_3 = m, \quad q_1, q_2, q_3 \geq 0,$$

where $p_1, p_2, p_3 > 0$ and $m > 0$.

Derive the Marshallian (uncompensated) demand functions for q_1, q_2, and q_3.

Derive the corresponding indirect utility function.

Solution

Step 1: Consumer's Preferences and Interior Solution

The consumer's preferences are such that good 1 enters additively (linearly), whereas goods 2 and 3 are perfect complements (via the min operator). In an interior solution, the consumer will choose to "balance" the complementary goods by setting

$$q_2 = q_3.$$

At that balanced bundle, the utility simplifies to:

$$U(q_1, q_2, q_3) = q_1 + q_2 \quad (\text{with } q_2 = q_3).$$

The budget constraint then becomes:

$$p_1 q_1 + (p_2 + p_3) q_2 = m.$$

Step 2: Corner Solutions and Utility Maximization

Since the utility function is additive in q_1 and q_2, the consumer may also choose a corner solution by allocating all income to a single category of goods.

Corner A (Spending all income on good 1):

$$q_1 = \frac{m}{p_1}, \quad q_2 = q_3 = 0.$$

The achieved utility is:

$$U_A = \frac{m}{p_1}.$$

Corner B (Spending all income on goods 2 and 3):

$$(p_2 + p_3) q_2 = m \quad \Longrightarrow \quad q_2 = q_3 = \frac{m}{p_2 + p_3}.$$

The achieved utility is:

$$U_B = \frac{m}{p_2 + p_3}.$$

The consumer chooses the bundle yielding the highest utility: - If $p_1 < p_2 + p_3$, the consumer allocates all income to good 1. - If $p_1 > p_2 + p_3$, the consumer allocates all income to the complementary goods q_2 and q_3. - If $p_1 = p_2 + p_3$, the consumer is indifferent among any bundle satisfying $q_2 = q_3$ and $p_1 q_1 + (p_2 + p_3) q_2 = m$.

Step 3: Marshallian Demand Functions

Thus, the Marshallian demands are:

$$(q_1^*, q_2^*, q_3^*) = \begin{cases} \left(\frac{m}{p_1}, 0, 0\right), & \text{if } p_1 < p_2 + p_3, \\ \left(0, \frac{m}{p_2 + p_3}, \frac{m}{p_2 + p_3}\right), & \text{if } p_1 > p_2 + p_3, \\ \text{any } (q_1, q_2, q_3) \text{ satisfying } q_2 = q_3 \text{ and } p_1 q_1 + (p_2 + p_3) q_2 = m, & \text{if } p_1 = p_2 + p_3. \end{cases}$$

Step 4: Indirect Utility Function

The indirect utility function, $V(m, p_1, p_2, p_3)$, is given by the highest attainable utility at the optimal choices:

$$V(m, p_1, p_2, p_3) = \begin{cases} \frac{m}{p_1}, & \text{if } p_1 < p_2 + p_3, \\ \frac{m}{p_2 + p_3}, & \text{if } p_1 > p_2 + p_3, \\ \frac{m}{p_1} = \frac{m}{p_2 + p_3}, & \text{if } p_1 = p_2 + p_3. \end{cases}$$

Final Answers:

<u>Marshallian Demands:</u>

$$(q_1^*, q_2^*, q_3^*) = \begin{cases} \left(\dfrac{m}{p_1}, 0, 0\right), & \text{if } p_1 < p_2 + p_3, \\ \left(0, \dfrac{m}{p_2+p_3}, \dfrac{m}{p_2+p_3}\right), & \text{if } p_1 > p_2 + p_3, \\ \text{any bundle with } q_2 = q_3 \text{ and } p_1 q_1 + (p_2+p_3) q_2 = m, & \text{if } p_1 = p_2 + p_3. \end{cases}$$

<u>Indirect Utility Function:</u>

$$V(m, p_1, p_2, p_3) = \begin{cases} \dfrac{m}{p_1}, & \text{if } p_1 < p_2 + p_3, \\ \dfrac{m}{p_2+p_3}, & \text{if } p_1 > p_2 + p_3, \\ \dfrac{m}{p_1} \quad (\text{or } \dfrac{m}{p_2+p_3}), & \text{if } p_1 = p_2 + p_3. \end{cases}$$