Marshallian Demand for a Nested Utility with Min and Substitutes

Consider a consumer with the utility function

$$u(q_1, q_2) = \min\{q_1 + 2q_2,\ 2q_1 + q_2\},$$

and facing the budget constraint

$$p_1 q_1 + p_2 q_2 = m, \quad q_1, q_2 \geq 0,$$

with $p_1, p_2 > 0$ and $m > 0$.

Derive the Marshallian (uncompensated) demand functions for q_1 and q_2.

Derive the corresponding indirect utility function.

Solution

Analysis of the Utility Function

The utility function is given by

$$u(q_1, q_2) = \min\{q_1 + 2q_2,\ 2q_1 + q_2\}.$$

Since utility is determined by the smaller of the two expressions, a natural strategy for the consumer is to "balance" them. Setting

$$q_1 + 2q_2 = 2q_1 + q_2,$$

we obtain

$$q_1 + 2q_2 = 2q_1 + q_2 \quad \Longrightarrow \quad q_2 = q_1.$$

Thus, if an interior (balanced) solution exists, the consumer will choose

$$q_1 = q_2.$$

Substituting $q_2 = q_1$ into either expression yields

$$q_1 + 2q_1 = 3q_1 \quad \text{or} \quad 2q_1 + q_1 = 3q_1,$$

so that the utility is $3q_1$.

Interior (Balanced) Equilibrium

Substituting $q_2 = q_1$ into the budget constraint:

$$p_1q_1 + p_2q_1 = (p_1 + p_2)q_1 = m,$$

which gives

$$q_1^* = q_2^* = \frac{m}{p_1 + p_2}.$$

At this bundle, the utility is

$$u(q_1^*, q_2^*) = 3\frac{m}{p_1 + p_2}.$$

Corner Solutions

If the balanced solution is not optimal, the consumer will specialize in the consumption of one good. We examine two possible corners:

Corner A: Specialization in Good 1 Setting $q_2 = 0$, the utility simplifies to:

$$u(q_1, 0) = \min\{q_1,\, 2q_1\} = q_1.$$

The budget constraint becomes:

$$p_1q_1 = m \quad \Longrightarrow \quad q_1 = \frac{m}{p_1}.$$

Thus, the utility achieved is:

$$u\left(\frac{m}{p_1}, 0\right) = \frac{m}{p_1}.$$

Corner B: Specialization in Good 2 Setting $q_1 = 0$, the utility simplifies to:

$$u(0, q_2) = \min\{2q_2,\, q_2\} = q_2.$$

The budget constraint becomes:

$$p_2q_2 = m \quad \Longrightarrow \quad q_2 = \frac{m}{p_2},$$

and the corresponding utility is:

$$u\left(0, \frac{m}{p_2}\right) = \frac{m}{p_2}.$$

Determining the Optimal Equilibrium

The consumer compares the utility from the interior solution,

$$u^I = 3\frac{m}{p_1 + p_2},$$

with the utilities from the corner solutions:

$$u^A = \frac{m}{p_1} \quad \text{and} \quad u^B = \frac{m}{p_2}.$$

A standard result (which can be verified by analyzing the slopes) shows that the interior solution is optimal if and only if the price ratio satisfies

$$\frac{1}{2} \leq \frac{p_1}{p_2} \leq 2.$$

Thus, the Marshallian demand functions are given by:

$$(q_1^*, q_2^*) = \begin{cases} \left(\frac{m}{p_1}, 0\right), & \text{if } \frac{p_1}{p_2} < \frac{1}{2}, \\ \left(\frac{m}{p_1 + p_2}, \frac{m}{p_1 + p_2}\right), & \text{if } \frac{1}{2} \leq \frac{p_1}{p_2} \leq 2, \\ \left(0, \frac{m}{p_2}\right), & \text{if } \frac{p_1}{p_2} > 2. \end{cases}$$

Indirect Utility Function

The indirect utility function is obtained by substituting the equilibrium demands into the utility function.

For the interior solution (when $\frac{1}{2} \leq \frac{p_1}{p_2} \leq 2$):

$$V(m, p_1, p_2) = u\left(\frac{m}{p_1 + p_2}, \frac{m}{p_1 + p_2}\right) = 3\frac{m}{p_1 + p_2}.$$

For the corner solutions:

$$V(m, p_1, p_2) = \begin{cases} \frac{m}{p_1}, & \text{if } \frac{p_1}{p_2} < \frac{1}{2}, \\ \frac{m}{p_2}, & \text{if } \frac{p_1}{p_2} > 2. \end{cases}$$

Final Answers:

$$(q_1^*, q_2^*) = \begin{cases} \left(\frac{m}{p_1}, 0\right), & \text{if } \frac{p_1}{p_2} < \frac{1}{2}, \\ \left(\frac{m}{p_1 + p_2}, \frac{m}{p_1 + p_2}\right), & \text{if } \frac{1}{2} \leq \frac{p_1}{p_2} \leq 2, \\ \left(0, \frac{m}{p_2}\right), & \text{if } \frac{p_1}{p_2} > 2. \end{cases}$$

$$V(m, p_1, p_2) = \begin{cases} \frac{m}{p_1}, & \text{if } \frac{p_1}{p_2} < \frac{1}{2}, \\ 3\,\frac{m}{p_1 + p_2}, & \text{if } \frac{1}{2} \leq \frac{p_1}{p_2} \leq 2, \\ \frac{m}{p_2}, & \text{if } \frac{p_1}{p_2} > 2. \end{cases}$$