Normal distribution analysis and probability calculations

A building is composed of 20 apartments. In 8 of them, 2 people live per apartment, and in the remaining 12, 3 people live per apartment. The monthly water consumption *per person* is modeled with a normal distribution with mean $\mu = 11\,\mathrm{m}^3$ and standard deviation $\sigma = 2\,\mathrm{m}^3$. The building has a single meter that records the total consumption of the building. The rate is \$200 per m^3 of water consumed.

a) Calculate the probability that the total monthly payment for the building exceeds \$110,000.

b) Calculate the probability that the consumption of an apartment with 3 people exceeds the average consumption per apartment.

Solution

a) Probability that the total monthly payment for the building exceeds $110,000

First, we calculate the total consumption corresponding to a payment of $110,000:

$$\text{Total consumption (in m}^3) = \frac{\$110,000}{\$200/\text{m}^3} = 550\,\text{m}^3$$

Now, we determine the expected total consumption of the building and its standard deviation.

Total number of people in the building

- Apartments with 2 people: $n_2 = 8$ apartments
- Apartments with 3 people: $n_3 = 12$ apartments
- Total number of people:

$$N_{\text{people}} = (8 \times 2) + (12 \times 3) = 16 + 36 = 52 \text{ people}$$

Total consumption of the building

Since the consumption per person is a normal random variable $X \sim N(\mu = 11, \sigma = 2)\,\text{m}^3$, the total consumption of the building T is the sum of 52 independent normal variables.

Expected total consumption:

$$E[T] = N_{\text{people}} \times \mu = 52 \times 11 = 572\,\text{m}^3$$

Variance of the total consumption:

$$\text{Var}(T) = N_{\text{people}} \times \sigma^2 = 52 \times 2^2 = 52 \times 4 = 208\,\text{m}^6$$

Standard deviation of the total consumption:

$$\sigma_T = \sqrt{\text{Var}(T)} = \sqrt{208} \approx 14.42\,\text{m}^3$$

Calculation of the probability

We want to calculate:

$$P(\text{Total payment} > \$110,000) = P(T > 550\,\text{m}^3)$$

Calculating the z-score:

$$z = \frac{550 - E[T]}{\sigma_T} = \frac{550 - 572}{14.42} = \frac{-22}{14.42} \approx -1.525$$

Looking up $P(Z > -1.525)$ in the standard normal distribution table, we find:

$$P(Z > -1.525) = 1 - P(Z < -1.525) = 1 - 0.0637 = 0.9363$$

Conclusion: The probability that the total monthly payment for the building exceeds $110,000 is approximately 93.63%.

b) Probability that the consumption of an apartment with 3 people exceeds the average consumption per apartment

Average consumption per apartment

The expected total consumption of the building is $E[T] = 572\,\text{m}^3$, and there are $n_{\text{apts}} = 20$ apartments. Therefore, the average consumption per apartment is:

$$\mu_{\text{apt}} = \frac{E[T]}{n_{\text{apts}}} = \frac{572}{20} = 28.6\,\text{m}^3$$

Consumption of an apartment with 3 people

The consumption of an apartment with 3 people is the sum of 3 independent normal random variables $X_i \sim N(11, 2^2)\,\mathrm{m}^3$.

Mean consumption of the apartment:

$$\mu_{3\mathrm{p}} = 3 \times \mu = 3 \times 11 = 33\,\mathrm{m}^3$$

Variance of the apartment consumption:

$$\sigma_{3\mathrm{p}}^2 = 3 \times \sigma^2 = 3 \times 4 = 12\,\mathrm{m}^6$$

Standard deviation:

$$\sigma_{3\mathrm{p}} = \sqrt{\sigma_{3\mathrm{p}}^2} = \sqrt{12} \approx 3.464\,\mathrm{m}^3$$

Calculation of the probability

We want to calculate:

$$P(\text{Apartment consumption} > \mu_{\mathrm{apt}}) = P(X > 28.6)$$

Calculating the z-score:

$$z = \frac{28.6 - \mu_{3\mathrm{p}}}{\sigma_{3\mathrm{p}}} = \frac{28.6 - 33}{3.464} = \frac{-4.4}{3.464} \approx -1.270$$

Looking up $P(Z > -1.270)$:

$$P(Z > -1.270) = 1 - P(Z < -1.270) = 1 - 0.1020 = 0.8980$$

Conclusion: The probability that the consumption of an apartment with 3 people exceeds the average consumption per apartment is approximately 89.80%.