Maximization and Minimization with Inequalities Without Using Kuhn-Tucker

Calculate the absolute extrema of $f(x, y) = x^2 + 2y^2$ over the region $C = \{(x, y) \in \mathbb{R}/x^2 + y^2 \leq 1\}$

Solution

One way to approach this problem could be using the Kuhn-Tucker method for constrained maximization or minimization problems with inequalities. However, by analyzing the form of the function, we can see that if we want to maximize it, we know that the constraint must hold with equality as there is no reason why we would choose a smaller x or y.

Considering maxima

So, to calculate the maximum, we can set $x^2 + y^2 = 1$. The Lagrangian will then be:

$$L = x^2 + 2y^2 + \lambda(1 - x^2 - y^2)$$

The first-order conditions:

$$L'_x = 2x - \lambda 2x = 0$$
$$L'_y = 4y - \lambda 2y = 0$$
$$L'_\lambda = 1 - x^2 - y^2 = 0$$

Reexpressing the first two:

$$L'_x = 2x(1 - \lambda) = 0$$
$$L'_y = 2y(2 - \lambda) = 0$$

Assuming that $x \neq 0$, from the first equation we can solve for λ:

$$2x = \lambda 2x$$

$$\lambda = 1$$

Then in the second equation:

$$2y = y$$

Which gives us a contradiction. Therefore, let's take a different approach and argue that $x = 0$. In the third equation:

$$1 - 0 - y^2 = 0$$

This gives us $y = 1$ and $y = -1$. Therefore we have two critical points: $(0, 1)$ and $(0, -1)$. Finally, we need to consider the case where $y = 0$. With the third equation:

$$1 - x^2 - 0 = 0$$

$$x^2 = 1$$

Therefore, we have two additional critical points: $(-1, 0)$ and $(1, 0)$. By analyzing the function, we can rule out some points, as if we evaluate the function at these critical points:

- $f(0, 1) = 2$
- $f(0, -1) = 2$
- $f(1, 0) = 1$
- $f(-1, 0) = 1$

Therefore we can rule out the last two points obtained. If we perform the second-order conditions:

$$\bar{H} = \begin{pmatrix} 0 & g'_x & g'_y \\ g'_x & L''_{xx} & L''_{xy} \\ g'_y & L''_{yx} & L''_{yy} \end{pmatrix} = \begin{pmatrix} 0 & 2x & 2y \\ 2x & 2 - 2\lambda & 0 \\ 2y & 0 & 4 - 2\lambda \end{pmatrix}$$

We find the determinant:

$$-2x \begin{vmatrix} 2x & 2y \\ 0 & 4 - 2\lambda \end{vmatrix} + 2y \begin{vmatrix} 2x & 2y \\ 2 - 2\lambda & 0 \end{vmatrix} = -2x(8x - 4x\lambda) - 2(4y - 4y\lambda) = -2x^2(8 - 4\lambda) - 8y(1 - \lambda)$$

Recall that if $x = 0$ and $y = 1$, then $\lambda = 2$ and if $x = 0$ and $y = -1$, then $\lambda = -2$. Evaluating at these points:

$$0 - 8 * (1 - 2) = 8 > 0$$

$$0 + 8 * (1 + 2) = 24 > 0$$

We have two maxima.

Considering minima

We know that both x^2 and y^2 enter the function positively, so if we want to minimize the function, we would want those two numbers to be as small as possible, so the constraint would be $x^2 + y^2 = 0$ since it can never take negative values because x and y are squared. With this, we construct the Lagrangian:

$$L = x^2 + 2y^2 + \lambda(-x^2 - y^2) = 0$$

The first-order conditions:

$$L'_x = 2x - \lambda 2x = 0$$
$$L'_y = 4y - \lambda 2y = 0$$
$$L'_\lambda = -x^2 - y^2 = 0$$

Reexpressing the first two:

$$L'_x = 2x(1 - \lambda) = 0$$
$$L'_y = 2y(2 - \lambda) = 0$$

For the third condition to be satisfied, it is necessary that $y = x = 0$. Therefore, our critical point that satisfies the first-order conditions is unique: $(0, 0)$. Let's proceed with the second-order conditions:

$$\bar{H} = \begin{pmatrix} 0 & g'_x & g'_y \\ g'_x & L''_{xx} & L''_{xy} \\ g'_y & L''_{yx} & L''_{yy} \end{pmatrix} = \begin{pmatrix} 0 & 2x & 2y \\ 2x & 2 - 2\lambda & 0 \\ 2y & 0 & 4 - 2\lambda \end{pmatrix}$$

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Which, evaluated at $(0, 0)$, does not allow us to confirm that the second-order condition is satisfied, but we do not need this because, by analyzing the function, we know that it cannot take any negative number and therefore $(0, 0, 0)$ is an absolute minimum.