Marshallian demands for a three-good Cobb–Douglas utility function

Consider a consumer with the utility function

$$u(x_1, x_2, x_3) = x_1^{\alpha_1}\, x_2^{\alpha_2}\, x_3^{\alpha_3}$$

where the parameters $\alpha_1, \alpha_2, \alpha_3 > 0$ are generic and need not sum to one The consumer faces prices p_1, p_2, $p_3 > 0$ and has income $m > 0$ The consumer's problem is

$$\max_{x_1,x_2,x_3} \quad x_1^{\alpha_1}\, x_2^{\alpha_2}\, x_3^{\alpha_3} \quad \text{subject to} \quad p_1x_1 + p_2x_2 + p_3x_3 = m$$

(1) Find the consumer's Marshallian demand functions for x_1, x_2, and x_3

(2) Find the indirect utility function

Solution

(1) Marshallian demand functions

The Lagrangian for the constrained maximization problem is

$$\mathcal{L} = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} + \lambda\Big(m - p_1 x_1 - p_2 x_2 - p_3 x_3\Big)$$

First-order conditions
Taking the derivative with respect to x_1

$$\frac{\partial \mathcal{L}}{\partial x_1} = \alpha_1 x_1^{\alpha_1 - 1} x_2^{\alpha_2} x_3^{\alpha_3} - \lambda p_1 = 0$$

which implies

$$\lambda = \frac{\alpha_1 x_1^{\alpha_1 - 1} x_2^{\alpha_2} x_3^{\alpha_3}}{p_1}$$

Similarly, differentiating with respect to x_2 and x_3 yields

$$\lambda = \frac{\alpha_2 x_1^{\alpha_1} x_2^{\alpha_2 - 1} x_3^{\alpha_3}}{p_2}$$

$$\lambda = \frac{\alpha_3 x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3 - 1}}{p_3}$$

Equating the first two expressions for λ and simplifying

$$\frac{\alpha_1}{p_1}\frac{1}{x_1} = \frac{\alpha_2}{p_2}\frac{1}{x_2} \quad \Longrightarrow \quad \frac{x_2}{x_1} = \frac{\alpha_2 p_1}{\alpha_1 p_2}$$

which gives

$$x_2 = \frac{\alpha_2 p_1}{\alpha_1 p_2} x_1$$

Similarly, equating the first and third expressions for λ

$$\frac{x_3}{x_1} = \frac{\alpha_3 p_1}{\alpha_1 p_3}$$

which gives

$$x_3 = \frac{\alpha_3 p_1}{\alpha_1 p_3} x_1$$

Solving for x_1 using the budget constraint
Substituting these into the budget constraint

$$p_1 x_1 + p_2 x_2 + p_3 x_3 = m$$

$$p_1 x_1 + p_2 \frac{\alpha_2 p_1}{\alpha_1 p_2} x_1 + p_3 \frac{\alpha_3 p_1}{\alpha_1 p_3} x_1 = m$$

which simplifies to

$$p_1 x_1 \left(1 + \frac{\alpha_2}{\alpha_1} + \frac{\alpha_3}{\alpha_1}\right) = m$$

Solving for x_1

$$x_1^* = \frac{\alpha_1 m}{p_1(\alpha_1 + \alpha_2 + \alpha_3)}$$

Using the previous expressions for x_2 and x_3

$$x_2^* = \frac{\alpha_2 m}{p_2(\alpha_1 + \alpha_2 + \alpha_3)}$$

$$x_3^* = \frac{\alpha_3 m}{p_3(\alpha_1 + \alpha_2 + \alpha_3)}$$

(2) Indirect utility function

Substituting the optimal demands into the utility function

$$u(x^*) = \left(\frac{\alpha_1 m}{p_1(\alpha_1+\alpha_2+\alpha_3)}\right)^{\alpha_1}\left(\frac{\alpha_2 m}{p_2(\alpha_1+\alpha_2+\alpha_3)}\right)^{\alpha_2}\left(\frac{\alpha_3 m}{p_3(\alpha_1+\alpha_2+\alpha_3)}\right)^{\alpha_3}$$

Rewriting

$$u(x^*) = \frac{\alpha_1^{\alpha_1}\alpha_2^{\alpha_2}\alpha_3^{\alpha_3}}{(\alpha_1+\alpha_2+\alpha_3)^{\alpha_1+\alpha_2+\alpha_3}}\frac{m^{\alpha_1+\alpha_2+\alpha_3}}{p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}}$$

Thus, the indirect utility function is

$$V(m,p_1,p_2,p_3) = \frac{\alpha_1^{\alpha_1}\alpha_2^{\alpha_2}\alpha_3^{\alpha_3}}{(\alpha_1+\alpha_2+\alpha_3)^{\alpha_1+\alpha_2+\alpha_3}}\frac{m^{\alpha_1+\alpha_2+\alpha_3}}{p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}}$$