

## Expenditure Minimization with Stone–Geary Utility

Consider a consumer with the Stone–Geary utility function

$$U(x_1, x_2) = (x_1 - \gamma_1)^\alpha (x_2 - \gamma_2)^{1-\alpha},$$

where  $x_1 \geq \gamma_1$ ,  $x_2 \geq \gamma_2$ ,  $0 < \alpha < 1$ , and  $\gamma_1, \gamma_2 > 0$ . The prices of goods  $x_1$  and  $x_2$  are  $p_1$  and  $p_2$ , respectively. Suppose the consumer wishes to achieve a target utility level  $U_0 > 0$ .

- (a) Write down the expenditure minimization problem.
- (b) Derive the Hicksian (compensated) demand functions for  $x_1$  and  $x_2$ .

## Solution

### (a) Formulation of the Expenditure Minimization Problem

The consumer minimizes total expenditure while attaining utility  $U_0$ . That is,

$$\begin{aligned} \min_{x_1, x_2} \quad & p_1 x_1 + p_2 x_2, \\ \text{s.t.} \quad & (x_1 - \gamma_1)^\alpha (x_2 - \gamma_2)^{1-\alpha} = U_0, \\ & x_1 \geq \gamma_1, \quad x_2 \geq \gamma_2. \end{aligned}$$

For convenience, define the transformed variables

$$y_1 = x_1 - \gamma_1, \quad y_2 = x_2 - \gamma_2,$$

with  $y_1, y_2 \geq 0$ . Then the utility constraint becomes

$$y_1^\alpha y_2^{1-\alpha} = U_0,$$

and the minimization problem is equivalently

$$\begin{aligned} \min_{y_1, y_2} \quad & p_1(y_1 + \gamma_1) + p_2(y_2 + \gamma_2), \\ \text{s.t.} \quad & y_1^\alpha y_2^{1-\alpha} = U_0. \end{aligned}$$

Since  $p_1\gamma_1 + p_2\gamma_2$  is constant, we can focus on minimizing

$$p_1 y_1 + p_2 y_2 \quad \text{subject to} \quad y_1^\alpha y_2^{1-\alpha} = U_0.$$

### (b) Derivation of the Hicksian Demands

**Step 1. Set Up the Lagrangian.**

$$\mathcal{L} = p_1 y_1 + p_2 y_2 + \lambda (U_0 - y_1^\alpha y_2^{1-\alpha}).$$

**Step 2. First-Order Conditions.** Differentiate with respect to  $y_1$  and  $y_2$ :

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial y_1} &= p_1 - \lambda \alpha y_1^{\alpha-1} y_2^{1-\alpha} = 0, \\ \frac{\partial \mathcal{L}}{\partial y_2} &= p_2 - \lambda (1-\alpha) y_1^\alpha y_2^{-\alpha} = 0. \end{aligned}$$

Dividing the first equation by the second gives:

$$\frac{p_1}{p_2} = \frac{\alpha}{1-\alpha} \cdot \frac{y_2}{y_1}.$$

Thus,

$$y_2 = \frac{p_1(1-\alpha)}{p_2\alpha} y_1.$$

**Step 3. Substitute into the Constraint.** Plug the expression for  $y_2$  into the utility constraint:

$$y_1^\alpha \left( \frac{p_1(1-\alpha)}{p_2\alpha} y_1 \right)^{1-\alpha} = U_0.$$

Simplify as follows:

$$y_1^{\alpha+1-\alpha} \left( \frac{p_1(1-\alpha)}{p_2\alpha} \right)^{1-\alpha} = y_1 \left( \frac{p_1(1-\alpha)}{p_2\alpha} \right)^{1-\alpha} = U_0.$$

Solving for  $y_1$  yields:

$$y_1^* = U_0 \left( \frac{p_2\alpha}{p_1(1-\alpha)} \right)^{1-\alpha}.$$

**Step 4. Determine  $y_2^*$ .** Substitute  $y_1^*$  back into the relation for  $y_2$ :

$$y_2^* = \frac{p_1(1-\alpha)}{p_2\alpha} y_1^* = U_0 \left( \frac{p_1(1-\alpha)}{p_2\alpha} \right)^{\alpha}.$$

**Step 5. Recover the Hicksian Demands for  $x_1$  and  $x_2$ .** Recall that

$$x_1 = y_1 + \gamma_1, \quad x_2 = y_2 + \gamma_2.$$

Thus, the Hicksian (compensated) demands are:

$$\begin{aligned} x_1^H &= U_0 \left( \frac{p_2\alpha}{p_1(1-\alpha)} \right)^{1-\alpha} + \gamma_1, \\ x_2^H &= U_0 \left( \frac{p_1(1-\alpha)}{p_2\alpha} \right)^{\alpha} + \gamma_2. \end{aligned}$$

Moreover, the corresponding minimum expenditure function is given by

$$e(p_1, p_2, U_0) = p_1\gamma_1 + p_2\gamma_2 + U_0 \alpha^{-\alpha} (1-\alpha)^{-(1-\alpha)} p_1^{\alpha} p_2^{1-\alpha}.$$