Marshallian demand and indirect utility

Consider a consumer with the utility function

$$U(x, y) = x^2 + y^2$$

subject to the budget constraint

$$p_x x + p_y y = m \quad x, y \geq 0$$

with $p_x > 0$, $p_y > 0$, and $m > 0$

(1) Find the Marshallian demand functions x^* **and** y^*

(2) Find the indirect utility function $V(m, p_x, p_y)$

Solution

(1) Marshallian demand functions

Since $U(x, y) = x^2 + y^2$ is strictly increasing in both x and y, the consumer will always want to spend all income However, because this utility function is convex (and not quasi-concave), the maximum utility on the linear budget set is achieved at one of the extreme (corner) points of the budget set The two corners are

$$A: \quad \left(x = \frac{m}{p_x},\, y = 0\right) \quad \text{and} \quad B: \quad \left(x = 0,\, y = \frac{m}{p_y}\right)$$

Evaluating utility at these corners, we have

$$U(A) = \left(\frac{m}{p_x}\right)^2 \quad \text{and} \quad U(B) = \left(\frac{m}{p_y}\right)^2$$

The consumer will choose the corner that yields the higher utility In particular

$$\text{If } \left(\frac{m}{p_x}\right)^2 > \left(\frac{m}{p_y}\right)^2 \quad \Longrightarrow \quad p_x < p_y, \quad \text{the optimal choice is } (x^*, y^*) = \left(\frac{m}{p_x}, 0\right)$$

$$\text{If } \left(\frac{m}{p_x}\right)^2 < \left(\frac{m}{p_y}\right)^2 \quad \Longrightarrow \quad p_x > p_y, \quad \text{the optimal choice is } (x^*, y^*) = \left(0, \frac{m}{p_y}\right)$$

If $p_x = p_y$, then both corners yield the same utility $\left(\frac{m}{p_x}\right)^2$ and the consumer is indifferent

Thus, the Marshallian demand functions are

$$x^*(m, p_x, p_y) = \begin{cases} \dfrac{m}{p_x}, & \text{if } p_x < p_y \\ 0, & \text{if } p_x > p_y \\ \text{any } x \text{ with } p_x x + p_y y = m \text{ and } U(x, y) = \left(\frac{m}{p_x}\right)^2, & \text{if } p_x = p_y \end{cases}$$

$$y^*(m, p_x, p_y) = \begin{cases} 0, & \text{if } p_x < p_y \\ \dfrac{m}{p_y}, & \text{if } p_x > p_y \\ \text{any } y \text{ with } p_x x + p_y y = m \text{ and } U(x, y) = \left(\frac{m}{p_x}\right)^2, & \text{if } p_x = p_y \end{cases}$$

(2) Indirect utility function

The indirect utility function, which gives the maximum utility attainable for given m, p_x, and p_y, is therefore

$$V(m, p_x, p_y) = \max\left\{\left(\frac{m}{p_x}\right)^2, \left(\frac{m}{p_y}\right)^2\right\} = \left(\frac{m}{\min\{p_x, p_y\}}\right)^2$$