Intertemporal choice with logarithmic utility

A consumer has an intertemporal utility function

$$U(c_1, c_2) = \ln(c_1) + 0.9 \ln(c_2),$$

and receives an income of $y_1 = 100$ in period 1 and $y_2 = 150$ in period 2. The consumer can save or borrow at a gross interest rate $1 + r$. The intertemporal budget constraint is given by

$$c_1 + \frac{c_2}{1+r} = y_1 + \frac{y_2}{1+r}.$$

Answer the following:

(a) Write the consumer's intertemporal budget constraint explicitly for $r = 0.10$ and for $r = 0.20$.

(b) Compute the period-1 (horizontal) and period-2 (vertical) intercepts for both cases.

(c) Sketch both budget lines on the (c_1, c_2) plane and explain how an increase in the interest rate affects the slope and intercepts.

(d) Solve the consumer's utility maximization problem by deriving the Marshallian demand functions for c_1 and c_2, and comment on how a change in the interest rate affects the optimal consumption bundle.

Solution

(a) Writing the budget constraint:
The general intertemporal budget constraint is

$$c_1 + \frac{c_2}{1+r} = 100 + \frac{150}{1+r}.$$

For $r = 0.10$:

$$c_1 + \frac{c_2}{1.10} = 100 + \frac{150}{1.10} \quad \Longrightarrow \quad c_1 + \frac{c_2}{1.10} = 236.36.$$

For $r = 0.20$:

$$c_1 + \frac{c_2}{1.20} = 100 + \frac{150}{1.20} \quad \Longrightarrow \quad c_1 + \frac{c_2}{1.20} = 225.$$

(b) Computing the intercepts:
For $r = 0.10$:

- Horizontal intercept: $c_1 = 236.36$.
- Vertical intercept: $c_2 = 260.0$.

For $r = 0.20$:

- Horizontal intercept: $c_1 = 225$.
- Vertical intercept: $c_2 = 270$.

(c) Sketch and analysis of budget lines:
The budget line has the form

$$c_1 + \frac{c_2}{1+r} = \text{Total Present Value (TPV)}.$$

For $r = 0.10$, TPV ≈ 236.36; for $r = 0.20$, TPV $= 225$. The slope of the budget line is $-(1+r)$, given by:

$$c_2 = (1+r)\Big[\text{TPV} - c_1\Big].$$

Thus:

- For $r = 0.10$, the slope is -1.10.
- For $r = 0.20$, the slope is -1.20.

(d) Utility Maximization and Marshallian Demands:

Step 1: Set Up the Lagrangian
The consumer maximizes:

$$\max_{c_1, c_2} \quad \ln(c_1) + 0.9\ln(c_2)$$

subject to

$$c_1 + \frac{c_2}{1+r} = Y.$$

The Lagrangian is:

$$\mathcal{L} = \ln(c_1) + 0.9\ln(c_2) + \lambda\left(Y - c_1 - \frac{c_2}{1+r}\right).$$

Step 2: Derive the First-Order Conditions (FOCs)

Differentiate with respect to c_1:

$$\frac{\partial \mathcal{L}}{\partial c_1} = \frac{1}{c_1} - \lambda = 0 \quad \Longrightarrow \quad \lambda = \frac{1}{c_1}.$$

Differentiate with respect to c_2:

$$\frac{\partial \mathcal{L}}{\partial c_2} = \frac{0.9}{c_2} - \lambda \frac{1}{1+r} = 0 \quad \Longrightarrow \quad \lambda = \frac{0.9(1+r)}{c_2}.$$

Step 3: Equate the Expressions for λ and Solve

Setting the two expressions for λ equal, we have:

$$\frac{1}{c_1} = \frac{0.9(1+r)}{c_2} \quad \Longrightarrow \quad c_2 = 0.9(1+r)c_1.$$

Step 4: Substitute into the Budget Constraint

$$c_1 + \frac{c_2}{1+r} = Y.$$

Substituting $c_2 = 0.9(1+r)c_1$:

$$(1+0.9)c_1 = 1.9c_1 = Y.$$

Solving for c_1:

$$c_1^* = \frac{Y}{1.9}.$$

Then,

$$c_2^* = (1+r)\frac{0.9Y}{1.9}.$$

Step 5: Numerical Examples

Case 1: $r = 0.10$

$$c_1^* \approx 124.38, \quad c_2^* \approx 123.20.$$

Case 2: $r = 0.20$

$$c_1^* \approx 118.42, \quad c_2^* \approx 127.90.$$

An increase in the interest rate r reduces the present-value income Y, which lowers the optimal current consumption c_1^*. However, the optimal future consumption c_2^* is multiplied by the factor $(1+r)$, so its response depends on the interplay of income and substitution effects.