Marshallian Demand and Indirect Utility

Consider a consumer with the utility function

$$u(q_1, q_2) = q_1^\alpha + q_2^\alpha, \quad \alpha \in (0, 1],$$

subject to the standard budget constraint

$$p_1 q_1 + p_2 q_2 = m, \quad q_1, q_2 \geq 0,$$

with $p_1, p_2 > 0$ and $m > 0$.

Question 1. Find the consumer's Marshallian (uncompensated) demand functions for q_1 and q_2.

Question 2. Find the indirect utility function corresponding to these demands.

Solution

Marshallian Demands

The consumer's problem is

$$\max_{q_1,q_2} \; q_1^{\alpha} + q_2^{\alpha} \quad \text{subject to} \quad p_1 q_1 + p_2 q_2 = m.$$

Since the utility function is additively separable and concave (for $0 < \alpha < 1$), we use the method of Lagrange. The Lagrangian is

$$\mathcal{L} = q_1^{\alpha} + q_2^{\alpha} + \lambda\Big(m - p_1 q_1 - p_2 q_2\Big).$$

Taking the first–order conditions (FOCs) with respect to q_1 and q_2 (assuming an interior solution with $q_1, q_2 > 0$) yields:

$$\frac{\partial \mathcal{L}}{\partial q_1} = \alpha q_1^{\alpha-1} - \lambda p_1 = 0 \quad \Longrightarrow \quad \lambda = \frac{\alpha\, q_1^{\alpha-1}}{p_1},$$
$$\frac{\partial \mathcal{L}}{\partial q_2} = \alpha q_2^{\alpha-1} - \lambda p_2 = 0 \quad \Longrightarrow \quad \lambda = \frac{\alpha\, q_2^{\alpha-1}}{p_2}.$$

Equating the two expressions for λ:

$$\frac{q_1^{\alpha-1}}{p_1} = \frac{q_2^{\alpha-1}}{p_2}.$$

Rearrange to obtain:

$$\left(\frac{q_1}{q_2}\right)^{\alpha-1} = \frac{p_1}{p_2}.$$

Since $\alpha - 1 < 0$ for $\alpha \in (0,1)$, we can write this as

$$\frac{q_1}{q_2} = \left(\frac{p_1}{p_2}\right)^{\frac{1}{\alpha-1}} = \left(\frac{p_2}{p_1}\right)^{\frac{1}{1-\alpha}}.$$

Define

$$k \equiv \left(\frac{p_2}{p_1}\right)^{\frac{1}{1-\alpha}},$$

so that

$$q_1 = k\, q_2.$$

Now, substitute $q_1 = kq_2$ into the budget constraint:

$$p_1(k\, q_2) + p_2 q_2 = m \quad \Longrightarrow \quad q_2\,(p_1 k + p_2) = m.$$

Thus, the optimal demand for q_2 is

$$q_2^* = \frac{m}{p_1 k + p_2},$$

and the optimal demand for q_1 is

$$q_1^* = k\, q_2^* = \frac{m\, k}{p_1 k + p_2}.$$

Recalling that

$$k = \left(\frac{p_2}{p_1}\right)^{\frac{1}{1-\alpha}},$$

the Marshallian demands are given by:

$$q_1^* = \frac{m\left(\frac{p_2}{p_1}\right)^{\frac{1}{1-\alpha}}}{p_1\left(\frac{p_2}{p_1}\right)^{\frac{1}{1-\alpha}} + p_2},$$

$$q_2^* = \frac{m}{p_1\left(\frac{p_2}{p_1}\right)^{\frac{1}{1-\alpha}} + p_2}.$$

Indirect Utility Function

Substituting the optimal demands into the utility function:

$$u^* = (q_1^*)^\alpha + (q_2^*)^\alpha.$$

Since

$$q_1^* = \frac{m\,k}{p_1 k + p_2} \quad \text{and} \quad q_2^* = \frac{m}{p_1 k + p_2},$$

we have

$$(q_1^*)^\alpha = \left(\frac{m\,k}{p_1 k + p_2}\right)^\alpha, \quad (q_2^*)^\alpha = \left(\frac{m}{p_1 k + p_2}\right)^\alpha.$$

Thus,

$$u^* = \left(\frac{m}{p_1 k + p_2}\right)^\alpha \Big(k^\alpha + 1\Big).$$

Recalling that $k = \left(\frac{p_2}{p_1}\right)^{\frac{1}{1-\alpha}}$, the indirect utility function is:

$$V(m, p_1, p_2) = \left(\frac{m}{p_1 \left(\frac{p_2}{p_1}\right)^{\frac{1}{1-\alpha}} + p_2}\right)^\alpha \left[1 + \left(\frac{p_2}{p_1}\right)^{\frac{\alpha}{1-\alpha}}\right].$$