Marshallian demand and indirect utility function

Consider a consumer with the utility function

$$u(q_1, q_2) = \ln q_1 + 2\ln(q_2 + 2),$$

subject to the budget constraint

$$p_1 q_1 + p_2 q_2 = m, \quad q_1 \geq 0, \quad q_2 \geq 0,$$

where $p_1, p_2 > 0$ and $m > 0$. Assume that the income is sufficiently high so that the solution yields $q_2 \geq 0$.

Question 1. Derive the Marshallian (uncompensated) demand functions for q_1 and q_2.

Question 2. Derive the corresponding indirect utility function.

Solution

(1) Marshallian demand functions.
The consumer maximizes

$$\max_{q_1, q_2} \ln q_1 + 2\ln(q_2 + 2)$$

subject to

$$p_1 q_1 + p_2 q_2 = m.$$

Form the Lagrangian:

$$\mathcal{L} = \ln q_1 + 2\ln(q_2 + 2) + \lambda\Big(m - p_1 q_1 - p_2 q_2\Big).$$

First-order conditions:
FOC with respect to q_1:

$$\frac{\partial \mathcal{L}}{\partial q_1} = \frac{1}{q_1} - \lambda p_1 = 0 \quad \Longrightarrow \quad \lambda = \frac{1}{p_1 q_1}.$$

FOC with respect to q_2:

$$\frac{\partial \mathcal{L}}{\partial q_2} = \frac{2}{q_2 + 2} - \lambda p_2 = 0 \quad \Longrightarrow \quad \lambda = \frac{2}{p_2(q_2 + 2)}.$$

Equating the two expressions for λ:

$$\frac{1}{p_1 q_1} = \frac{2}{p_2(q_2 + 2)}.$$

Solving for q_2:

$$q_2 + 2 = \frac{2p_1}{p_2} q_1 \quad \Longrightarrow \quad q_2 = \frac{2p_1}{p_2} q_1 - 2.$$

Substituting into the budget constraint:

$$p_1 q_1 + p_2 \left(\frac{2p_1}{p_2} q_1 - 2 \right) = m.$$

Simplify:

$$p_1 q_1 + 2p_1 q_1 - 2p_2 = m \quad \Longrightarrow \quad 3p_1 q_1 = m + 2p_2.$$

Thus, the optimal demand for q_1 is

$$q_1^* = \frac{m + 2p_2}{3p_1}.$$

Now, solving for q_2^*:

$$q_2^* = \frac{2(m - p_2)}{3p_2}.$$

Thus, the Marshallian demand functions are:

$$q_1^* = \frac{m + 2p_2}{3p_1}, \qquad q_2^* = \frac{2(m - p_2)}{3p_2}.$$

(2) Indirect utility function.
Substituting the optimal demands into the utility function:

$$V(m, p_1, p_2) = \ln(q_1^*) + 2\ln(q_2^* + 2).$$

We already have

$$q_1^* = \frac{m + 2p_2}{3p_1}.$$

To compute $q_2^* + 2$, note:

$$q_2^* + 2 = \frac{2(m + 2p_2)}{3p_2}.$$

Therefore, the indirect utility function is:

$$V(m, p_1, p_2) = \ln\left(\frac{m + 2p_2}{3p_1}\right) + 2\ln\left(\frac{2(m + 2p_2)}{3p_2}\right).$$

Final answers:

$$q_1^*(m, p_1, p_2) = \frac{m + 2p_2}{3p_1},$$
$$q_2^*(m, p_1, p_2) = \frac{2(m - p_2)}{3p_2}.$$

$$V(m, p_1, p_2) = \ln\left(\frac{m + 2p_2}{3p_1}\right) + 2\ln\left(\frac{2(m + 2p_2)}{3p_2}\right).$$