Homogeneity and homotheticity of preferences

Discuss the homogeneity and homotheticity of the preferences represented by the following utility functions:

1. $$u(q_1, q_2) = q_1 q_2 + q_1^2 q_2^2$$

2. $$u(q_1, q_2) = q_1^{\alpha} q_2^{\beta}, \quad \text{with } \alpha, \beta > 0.$$

3. $$u(q_1, q_2) = \min\{\alpha q_1, \beta q_2\}, \quad \text{where } \alpha, \beta > 0.$$

4. $$u(q_1, q_2) = \alpha q_1 + \beta q_2, \quad \text{with } \alpha, \beta > 0.$$

Solution

The preferences are homothetic if the MRS between any two goods—goods 1 and 2, in the present case—is a function that depends only on the consumption ratio $\frac{q_2}{q_1}$, and conversely, not on the absolute quantities of the goods. Consequently, if it doubles, triples, etc., the amount of both goods, the MRS is not modified.

Preferences are homothetic when: (a) the isoquants are straight lines that start from the origin, and (b) the Marshallian demand functions resulting have an income elasticity equal to 1.

The known preference sets of homothetic preferences are the preferences for substitute goods, the preferences for complementary goods, the Cobb-Douglas type preferences, the CES preferences, etc. On the contrary, the quasilinear preferences are not homothetic.

1. This utility function represents quadratic preferences. Clearly, $u(\cdot)$ is not a homogeneous function, but it is homothetic: $\text{MRS}_{q_1}(q_1, q_2) = \frac{-q_1(1+2q_1q_2)}{q_2(1+2q_1q_2)} = \frac{-q_1}{q_2}$ is immediate to verify that it (for example) doubles the quantities of q_1 and q_2, the MRS does not vary. In other words, the slope of the indifference curves is a homogeneous function of degree 0 in q_1 and q_2, with which curves that have the same slope in all points of a ray-vector that starts from the origin. **In conclusion, the quadratic preferences are not homogeneous, but they are homothetic.**

2. In this case we have Cobb-Douglas preferences. **These preferences are homogeneous of degree $\alpha + \beta$ in q_1 and q_2**, and therefore are also homothetic. Indeed, considering that the marginal rate of substitution $\text{MRS}_{q_1}(q_1, q_2) = -\frac{\alpha q_1^{\alpha-1} q_2^{\beta}}{\beta q_1^{\alpha} q_2^{\beta-1}}$, and given that this is a homogeneous function of degree 0 in q_1 and q_2, $\text{MRS}_{q_1}(tq_1, tq_2) = -\frac{\alpha (tq_1)^{\alpha-1}(tq_2)^{\beta}}{\beta (tq_1)^{\alpha}(tq_2)^{\beta-1}} = t^0 \cdot \text{MRS}_{q_1}(q_1, q_2)$, **it is proven that these preferences are homothetic.**

3. These are preferences for complementary goods. $u(.)$ is a homogeneous of degree 1. To determine their possible homotheticity, we cannot use the criterion of degree 0 homogeneity of the MRS because such MRS is not defined over the entire domain. Instead, we use the result according to which preferences are homothetic if, and only if, they give rise to Marshallian demands that present Engel curves (or income-consumption curves) that are linear functions increasing and passing through the origin, which is the same as Marshallian demands with an income elasticity equal to 1.

   Given that, in equilibrium, it must happen that $q\alpha 1 = \beta q_2$ and $p_1 q_1 + p_2 q_2 = m$, the resulting Marshallian demands are $q_1^m(p, m) = \frac{3\alpha m}{3\beta p_1 + \alpha p_2}$ and $q_2^m(p, m) = \frac{\alpha m}{3\beta p_1 + \alpha p_2}$. **From here it is immediate to check that the preferences for complementary goods are homothetic preferences**, since $\frac{\partial q_1^m}{\partial m} \frac{m}{q_1} = \frac{\partial q_2^m}{\partial m} \frac{m}{q_2} = 1$.

4. This utility function represents the preference ordering for substitute goods. **It deals with homogeneous preferences of degree 1, which are also homothetic.** Indeed, the slope of the indifference curves, $\text{MRS}_{q_1}(q_1, q_2) = -\frac{u_{q_2}}{u_{q_1}} = -\frac{a}{b}$, is a homogeneous function of degree 0 in q_1 and q_2, $\text{MRS}_{q_1}(tq_1, tq_2) = t^0 \cdot \text{MRS}_{q_1}(q_1, q_2)$.