Marshallian Demand and Indirect Utility for a Three-Good Utility

Consider a consumer with the utility function

$$u(q_0, q_1, q_2) = q_0 + q_1^{\alpha} q_2^{\beta},$$

where $\alpha, \beta > 0$. The consumer faces the budget constraint

$$p_0 q_0 + p_1 q_1 + p_2 q_2 = m, \quad q_0, q_1, q_2 \geq 0,$$

with $p_0, p_1, p_2 > 0$ and $m > 0$. The problem may yield an interior solution for the nonlinear part or a corner solution if the consumer cannot afford to purchase any of the composite goods.

Questions

1. Derive the Marshallian (uncompensated) demand functions for q_0, q_1, and q_2, considering both the interior and the relevant corner solutions.
2. Derive the indirect utility function corresponding to these demands.

Solution

1. Marshallian Demand Functions

The consumer's problem is:

$$\max_{q_0,q_1,q_2} \quad u(q_0,q_1,q_2) = q_0 + q_1^\alpha q_2^\beta$$

subject to

$$p_0q_0 + p_1q_1 + p_2q_2 = m.$$

(A) Interior Solution

Assuming an interior solution with $q_1 > 0$ and $q_2 > 0$, the Lagrangian is:

$$\mathcal{L} = q_0 + q_1^\alpha q_2^\beta + \lambda(m - p_0q_0 - p_1q_1 - p_2q_2).$$

The first-order conditions are:

$$\frac{\partial \mathcal{L}}{\partial q_0} = 1 - \lambda p_0 = 0 \quad \Rightarrow \quad \lambda = \frac{1}{p_0}.$$

$$\frac{\partial \mathcal{L}}{\partial q_1} = \alpha q_1^{\alpha-1} q_2^\beta - \lambda p_1 = 0 \quad \Rightarrow \quad \alpha q_1^{\alpha-1} q_2^\beta = \frac{p_1}{p_0}.$$

$$\frac{\partial \mathcal{L}}{\partial q_2} = \beta q_1^{\alpha} q_2^{\beta-1} - \lambda p_2 = 0 \quad \Rightarrow \quad \beta q_1^{\alpha} q_2^{\beta-1} = \frac{p_2}{p_0}.$$

Dividing the FOC for q_1 by that for q_2:

$$\frac{\alpha q_1^{\alpha-1} q_2^\beta}{\beta q_1^{\alpha} q_2^{\beta-1}} = \frac{p_1}{p_2} \quad \Rightarrow \quad \frac{\alpha}{\beta}\frac{q_2}{q_1} = \frac{p_1}{p_2}.$$

Thus, we obtain:

$$q_2 = \frac{p_1\beta}{p_2\alpha} q_1.$$

Substituting this into the FOC for q_1:

$$\alpha q_1^{\alpha-1} \left(\frac{p_1\beta}{p_2\alpha} q_1\right)^\beta = \frac{p_1}{p_0}.$$

Simplifying:

$$q_1^{\alpha+\beta-1} = \frac{p_1^{1-\beta} p_2^\beta \alpha^{\beta-1}}{p_0\beta^\beta}.$$

Taking the $\frac{1}{\alpha+\beta-1}$ power:

$$q_1^* = \left(\frac{p_1^{1-\beta} p_2^\beta \alpha^{\beta-1}}{p_0\beta^\beta}\right)^{\frac{1}{\alpha+\beta-1}}.$$

Substituting for q_2^*:

$$q_2^* = \frac{p_1\beta}{p_2\alpha} q_1^*.$$

Using the budget constraint:

$$q_0^* = \frac{m - \frac{p_1(\alpha+\beta)}{\alpha} q_1^*}{p_0}.$$

An interior solution exists if and only if $q_0^* \geq 0$, i.e.,

$$m \geq \frac{p_1(\alpha+\beta)}{\alpha} q_1^*.$$

(B) Corner Solution

If the consumer's income is insufficient to afford q_1^* and q_2^*, then the optimal choice is to spend all income on the numeraire good:

$$q_0^* = \frac{m}{p_0}, \quad q_1^* = 0, \quad q_2^* = 0.$$

2. Indirect Utility Function

(A) Interior Solution

For $m \geq \frac{p_1(\alpha+\beta)}{\alpha} q_1^*$, the indirect utility function is:

$$V(m, p_0, p_1, p_2) = q_0^* + (q_1^*)^\alpha (q_2^*)^\beta.$$

Since

$$(q_1^*)^\alpha (q_2^*)^\beta = (q_1^*)^{\alpha+\beta} \left(\frac{p_1 \beta}{p_2 \alpha}\right)^\beta,$$

we obtain:

$$V(m, p_0, p_1, p_2) = \frac{m - \frac{p_1(\alpha+\beta)}{\alpha} q_1^*}{p_0} + \left(\frac{p_1 \beta}{p_2 \alpha}\right)^\beta (q_1^*)^{\alpha+\beta}.$$

(B) Corner Solution

If $m < \frac{p_1(\alpha+\beta)}{\alpha} q_1^*$, then the indirect utility is:

$$V(m, p_0, p_1, p_2) = \frac{m}{p_0}.$$

Final Answers

Marshallian demand functions

$$q_1^* = \left(\frac{p_1^{1-\beta} p_2^\beta \alpha^{\beta-1}}{p_0 \beta^\beta}\right)^{\frac{1}{\alpha+\beta-1}}, \quad q_2^* = \frac{p_1 \beta}{p_2 \alpha} q_1^*, \quad q_0^* = \frac{m - \frac{p_1(\alpha+\beta)}{\alpha} q_1^*}{p_0}.$$

Indirect utility function

$$V(m, p_0, p_1, p_2) = \begin{cases} \frac{m}{p_0}, & m < \frac{p_1(\alpha+\beta)}{\alpha} q_1^*, \\ \frac{m - \frac{p_1(\alpha+\beta)}{\alpha} q_1^*}{p_0} + \left(\frac{p_1 \beta}{p_2 \alpha}\right)^\beta (q_1^*)^{\alpha+\beta}, & m \geq \frac{p_1(\alpha+\beta)}{\alpha} q_1^*. \end{cases}$$

Remarks

This solution assumes an interior optimum unless income is too low, in which case the consumer purchases only the numeraire good.