Optimal consumption in a two-period intertemporal choice problem with income in both periods

Consider a consumer who lives for two periods. In period 1, the consumer chooses consumption c_1 and in period 2 chooses c_2 The consumer's preferences are represented by the Cobb–Douglas utility function

$$U(c_1, c_2) = c_1^{\alpha} c_2^{1-\alpha} \quad 0 < \alpha < 1$$

The consumer earns income $y_1 > 0$ in period 1 and income $y_2 > 0$ in period 2 The consumer can save or borrow at a gross interest rate $1 + r$ (with $r \geq 0$) The intertemporal budget constraint is

$$c_1 + \frac{c_2}{1 + r} = y_1 + \frac{y_2}{1 + r}$$

(a) Formulate the consumer's utility maximization problem

(b) Derive the first-order conditions and solve for the optimal consumption levels c_1^* and c_2^*

Solution

(a) Utility maximization problem

The consumer's problem is

$$\begin{aligned} \max_{c_1, c_2} \quad & U(c_1, c_2) = c_1^{\alpha} c_2^{1-\alpha} \\ \text{s.t.} \quad & c_1 + \frac{c_2}{1+r} = y_1 + \frac{y_2}{1+r} \\ & c_1, c_2 \geq 0 \end{aligned}$$

(b) Derivation of the optimal consumption levels

Step 1: Form the Lagrangian

$$\mathcal{L} = c_1^{\alpha} c_2^{1-\alpha} + \lambda \left(y_1 + \frac{y_2}{1+r} - c_1 - \frac{c_2}{1+r} \right)$$

Step 2: First-order conditions

$$\frac{\partial \mathcal{L}}{\partial c_1} = \alpha\, c_1^{\alpha-1} c_2^{1-\alpha} - \lambda = 0$$

$$\frac{\partial \mathcal{L}}{\partial c_2} = (1-\alpha)\, c_1^{\alpha} c_2^{-\alpha} - \frac{\lambda}{1+r} = 0$$

Step 3: Solve for the optimal consumption ratio Dividing the first FOC by the second yields

$$\frac{\alpha\, c_1^{\alpha-1} c_2^{1-\alpha}}{(1-\alpha)\, c_1^{\alpha} c_2^{-\alpha}} = 1+r$$

Simplifying

$$\frac{\alpha}{1-\alpha} \cdot \frac{c_2}{c_1} = 1+r$$

which implies

$$\frac{c_2}{c_1} = \frac{(1-\alpha)(1+r)}{\alpha} \quad \Longrightarrow \quad c_2 = \frac{(1-\alpha)(1+r)}{\alpha}\, c_1$$

Step 4: Substitute into the budget constraint

$$c_1 + \frac{c_2}{1+r} = y_1 + \frac{y_2}{1+r}$$

Substituting $c_2 = \frac{(1-\alpha)(1+r)}{\alpha}\, c_1$

$$c_1 + \frac{1}{1+r} \left(\frac{(1-\alpha)(1+r)}{\alpha}\, c_1 \right) = c_1 + \frac{(1-\alpha)}{\alpha}\, c_1$$

This simplifies to

$$c_1 \left(1 + \frac{1-\alpha}{\alpha} \right) = c_1 \left(\frac{\alpha + 1 - \alpha}{\alpha} \right) = \frac{c_1}{\alpha}$$

Setting the left-hand side equal to the right-hand side of the budget constraint, we have

$$\frac{c_1}{\alpha} = y_1 + \frac{y_2}{1+r}$$

Thus, the optimal consumption in period 1 is

$$c_1^* = \alpha \left(y_1 + \frac{y_2}{1+r} \right)$$

Step 5: Solve for c_2^* Using the relation between c_2 and c_1

$$c_2^* = \frac{(1-\alpha)(1+r)}{\alpha} c_1^* = \frac{(1-\alpha)(1+r)}{\alpha} \left[\alpha \left(y_1 + \frac{y_2}{1+r}\right)\right]$$

Simplifying

$$c_2^* = (1-\alpha)(1+r)\left(y_1 + \frac{y_2}{1+r}\right)$$

Final answers

$$c_1^* = \alpha \left(y_1 + \frac{y_2}{1+r}\right)$$

$$c_2^* = (1-\alpha)(1+r)\left(y_1 + \frac{y_2}{1+r}\right)$$