Ordinary differential equation

Solve the following ordinary differential equation:

$$y' = ry(M - y)$$

Solution

$$y' - rMy = -ry^2$$

Solve as Bernoulli (divide the whole expression by y raised to the highest exponent). Then re-express: $z = y^{-n}$. Where n is the highest exponent minus 1.

$$\frac{y'}{y^2} - rMy^{-1} = -r$$

Call $z = y^{-1}$ and $z' = (-1)y^{-2}y'$

$$-z' - rMz = -r$$

Then we have the linear ODE:

$$z' + rMz = r$$

Apply replacement:

$$z = u(t)v(t)$$

and

$$z' = u'v + uv'$$

Where:

$$u' = \frac{du}{dt}$$

$$v' = \frac{dv}{dt}$$

$$u(v' + rmv) + u'v = r$$

$$u(v' + rMv) + u'v = r$$

This leads us to two equations:

$$v' + rMv = 0$$

and

$$u'v = r$$

The first is an ODE that we can solve by separable variables:

$$\frac{dv}{dt} + rMv = 0$$

$$\frac{dv}{v} = -rMdt$$

Integrate:

$$ln(v) = -rMt$$

$$v(t) = e^{-rMt}$$

Solve the other equation using the solution of the first:

$$\frac{du}{dt}e^{-rMt} = r$$

$$du = re^{rMt}dt$$

Integrate (set up substitution): $j = rMt$ and $dj = rMdt$. Also: $dt = dj/rM$

$$u(t) = \int \frac{r}{rM}e^{j}dj$$

$$u(t) = \frac{r}{rM}e^{rMt} + C = \frac{e^{rMt}}{M} + C$$

Now remember that we are looking for $z = uv$:

$$z = uv = (\frac{e^{rMt}}{M} + C)e^{-rMt}$$

$$z = y^{-1} = 1/M + Ce^{-rMt}$$

Solve for y:

$$y = \frac{M}{1 + MCe^{-rMt}}$$