Absolute Extrema of a Quadratic Function in a Circular Region

Calculate the absolute extrema of $f(x, y) = x^2 + 2y^2$ over the region $C = \{(x, y) \in \mathbb{R}^2/x^2 + y^2 \leq 1\}$

Solution

Setting up the Lagrangian

$$\mathcal{L} = x^2 + 2y^2 + \lambda[1 - x^2 - y^2] \tag{1}$$

The first-order conditions are:

$$\frac{\partial L}{\partial x} = 2x - \lambda 2x = 0 \tag{2}$$

$$\frac{\partial L}{\partial y} = 4y - \lambda 2y = 0 \tag{3}$$

And since the problem involves a constraint that we don't know if it holds with equality, we set up the following constraints:

$$\lambda[x^2 + y^2 - 1] = 0 \tag{4}$$

$$\lambda \geq 0 \tag{5}$$

$$x^2 + y^2 \leq 1 \tag{6}$$

The explanation for 4 is that, it must be true that the constraint holds with equality and then the multiplier can take other values, or the constraint does not hold with equality then, the Lagrange multiplier must be 0 because when maximizing/minimizing, the constraint does not matter (since it is not active, i.e., it does not hold with equality).
For an explanation of 5, see Simon and Blume (1996) Chapter 18, Section 3. Constraint 6 does not require much explanation. From the first two constraints, we set up the following:

$$\frac{\partial L}{\partial x} = 2x(1 - \lambda) = 0 \tag{7}$$

$$\frac{\partial L}{\partial y} = 2y(2 - \lambda) = 0 \tag{8}$$

Since 7 and 8 must be met, but λ cannot take two values at the same time, there are 3 possibilities:

$$x = 0 \wedge y = 0 \tag{9}$$

$$x = 0 \wedge \lambda = 2 \tag{10}$$

$$y = 0 \wedge \lambda = 1 \tag{11}$$

First, we consider 9. We see that for 4 to hold, it is necessary for $\lambda = 0$. On the other hand, 5 and 6 are met. Also, we can see that in this case, we have an absolute minimum because $\forall (x, y) \Rightarrow f(0, 0) \leq f(x, y)$
Now let's consider 10. We see that if $x = 0$, for 8 to hold, it is necessary for $\lambda = 2$. For 4 to hold, it is necessary that

$$2[0^2 + y^2 - 1] = 0 \tag{12}$$

$$y^2 = 1 \tag{13}$$

This leads us to two alternatives:

$$y = 1 \veebar y = -1 \tag{14}$$

We see that both $(0, 1, 2)$ and $(0, -1, 2)$ satisfy all the constraints. Before analyzing whether it is an absolute maximum, we check the last case. Considering 11, we see that we meet 7 and 8. For 4 to hold, it is necessary that:

$$1[x^2 - 0 - 1] = 0 \tag{15}$$

$$x^2 = 1 \tag{16}$$

This leaves us with two alternatives:

$$x = 1 \veebar x = -1 \tag{17}$$

So we have 2 points that meet all the constraints again: $(1, 0, 1)$ and $(-1, 0, 1)$. We can see that $(-1, 0)$ and $(1, 0)$ are not maxima due to the points found previously. Finally, we can see that $(0, 1)$ and $(0, -1)$ are absolute maxima because $\forall (x, y) \Rightarrow f(0, i) \geq f(x, y), i = 0, -1$