Differentiability and continuity

Given the function

$$f(x,y) = \begin{cases} \dfrac{2x^2y}{x^4+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

1. Analyze continuity at the origin
2. Analyze differentiability at the origin
3. Is it differentiable at the origin? Justify

Solutions

1. The function exists at the origin and its value is 0, it only remains to check now whether the limit exists at the origin. For that, we take the parabolic limit: $y = mx^2$

$$lim_{x\to 0}\frac{2x^2mx^2}{x^4+m^2x^4} = lim_{x\to 0}x^4\frac{2m}{x^4(1+m^2)} = \frac{2m}{1+m^2}$$

   Since it depends on m, we can say that the limit does not exist. Therefore, the function is not continuous.

2. We differentiate by definition:

$$f'_x = lim_{h\to 0}\frac{\frac{2(0+h)^2*0}{(0+h)^4+0^2}-0}{h} = lim_{h\to 0}\frac{0/h^4}{h} = 0$$

$$f'_y = lim_{h\to 0}\frac{\frac{2(0)^2*(0+h))}{(0)^4+(0+h)^2}-0}{h} = lim_{h\to 0}\frac{0/h^2}{h} = 0$$

3. The necessary condition for the function to be differentiable is that it is continuous at the point and admits partial derivatives at the point. Since the function is not continuous, it is not differentiable.