Second-order Taylor polynomial

Calculate an approximate value of the Cobb-Douglas function $f(x, y) = x^{\frac{1}{4}} y^{\frac{3}{4}}$ at the point $(1.1, 0.9)$ using a second-order Taylor polynomial.

Solution

The formula for the second-order Taylor polynomial is:

$$P_2(x,y) = f(a,b)+f_x(a,b)(x-a)+f_y(a,b)(y-b)+\frac{1}{2}f_{xx}(a,b)(x-a)^2+f_{xy}(a,b)(x-a)(y-b)+\frac{1}{2}f_{yy}(a,b)(y-b)^2$$

Let's calculate the derivatives:

$$f'_x = \frac{1}{4}x^{\frac{-3}{4}}y^{\frac{3}{4}}$$

$$f'_y = \frac{3}{4}x^{\frac{1}{4}}y^{\frac{-1}{4}}$$

$$f'_{xx} = -\frac{3}{16}x^{\frac{-7}{4}}y^{\frac{3}{4}}$$

$$f'_{yy} = -\frac{3}{16}x^{\frac{1}{4}}y^{\frac{-5}{4}}$$

$$f'_{yx} = f'_{xy} = \frac{3}{16}x^{\frac{-3}{4}}y^{\frac{-1}{4}}$$

Substituting the corresponding values:

$$P_2 = 1^{\frac{1}{4}}1^{\frac{3}{4}}+\frac{1}{4}1^{\frac{-3}{4}}1^{\frac{3}{4}}(0.1)+\frac{3}{4}1^{\frac{1}{4}}1^{\frac{-1}{4}}(-0.1)-\frac{1}{2}\frac{3}{16}1^{\frac{-7}{4}}1^{\frac{3}{4}}(0.1)^2+\frac{3}{16}1^{\frac{-3}{4}}1^{\frac{-1}{4}}(0.1)(-0.1)-\frac{1}{2}\frac{3}{16}1^{\frac{1}{4}}1^{\frac{-5}{4}}(-0.1)^2 \approx 0.946$$

Notice that if we evaluate the function at the point $(1.1, 0.9)$:

$$f(1.1, 0.9) = 1.1^{1/4}0.9^{3/4} \approx 0.946$$

The result is similar