Constrained optimization

Given the function $f(x, y) = x^2 + y^2 - x - y$ subject to $x^2 + y^2 = 9$:

a) Find the extrema of the function.

b) If this were an economic function, indicate the extrema, calculate the value of λ, and provide its economic interpretation.

Solution

a) Finding the extrema of the function

To find the extrema of $f(x, y)$ subject to the constraint $x^2 + y^2 = 9$, we use the method of Lagrange multipliers.

Define the Lagrangian:

$$\mathcal{L}(x, y, \lambda) = x^2 + y^2 - x - y - \lambda(x^2 + y^2 - 9)$$

Compute the partial derivatives and set them equal to zero:

- $\dfrac{\partial \mathcal{L}}{\partial x} = 2x - 1 - 2\lambda x = 0$
- $\dfrac{\partial \mathcal{L}}{\partial y} = 2y - 1 - 2\lambda y = 0$
- $\dfrac{\partial \mathcal{L}}{\partial \lambda} = -(x^2 + y^2 - 9) = 0$

From the first two equations:

$$\begin{cases} 2x(1-\lambda) = 1 & (1) \\ 2y(1-\lambda) = 1 & (2) \end{cases}$$

Simplify:

- From (1): $x = \dfrac{1}{2(1-\lambda)}$
- From (2): $y = \dfrac{1}{2(1-\lambda)}$

Thus, $x = y$.

Substitute $x = y$ into the constraint:

$$x^2 + x^2 = 9 \implies 2x^2 = 9 \implies x^2 = \frac{9}{2} \implies x = \pm\frac{3}{\sqrt{2}} = \pm\frac{3\sqrt{2}}{2}$$

Therefore, the critical points are:

- $\left(\dfrac{3\sqrt{2}}{2}, \dfrac{3\sqrt{2}}{2}\right)$
- $\left(-\dfrac{3\sqrt{2}}{2}, -\dfrac{3\sqrt{2}}{2}\right)$

Evaluate $f(x, y)$ at these points:

- At $\left(\dfrac{3\sqrt{2}}{2}, \dfrac{3\sqrt{2}}{2}\right)$:

$$f = \left(\frac{3\sqrt{2}}{2}\right)^2 + \left(\frac{3\sqrt{2}}{2}\right)^2 - \frac{3\sqrt{2}}{2} - \frac{3\sqrt{2}}{2} = 9 - 3\sqrt{2}$$

- At $\left(-\dfrac{3\sqrt{2}}{2}, -\dfrac{3\sqrt{2}}{2}\right)$:

$$f = \left(-\frac{3\sqrt{2}}{2}\right)^2 + \left(-\frac{3\sqrt{2}}{2}\right)^2 - \left(-\frac{3\sqrt{2}}{2}\right) - \left(-\frac{3\sqrt{2}}{2}\right) = 9 + 3\sqrt{2}$$

Conclusion:

- **Minimum at** $\left(\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$ **with value** $f = 9 - 3\sqrt{2}$**.**
- **Maximum at** $\left(-\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}\right)$ **with value** $f = 9 + 3\sqrt{2}$**.**

b) Economic interpretation and calculation of λ

Calculate λ at the critical points using:

$$\lambda = 1 - \frac{1}{2x}$$

- At $x = \frac{3\sqrt{2}}{2}$:

$$\lambda = 1 - \frac{1}{2\left(\frac{3\sqrt{2}}{2}\right)} = 1 - \frac{1}{3\sqrt{2}} \approx 1 - 0.2357 = 0.7643$$

- At $x = -\frac{3\sqrt{2}}{2}$:

$$\lambda = 1 - \frac{1}{2\left(-\frac{3\sqrt{2}}{2}\right)} = 1 + \frac{1}{3\sqrt{2}} \approx 1 + 0.2357 = 1.2357$$

Economic interpretation of λ:

In economics, the Lagrange multiplier λ represents the **rate of change of the optimal value of the objective function with respect to changes in the constraint**. This means it indicates how much the optimal value of $f(x, y)$ would change if the constraint were relaxed by one unit.

- A $\lambda > 0$ means that **relaxing the constraint increases the optimal value of the function**.
- A $\lambda < 0$ means that **relaxing the constraint decreases the optimal value of the function**.

In our case:

- At the **maximum** ($\lambda \approx 1.2357$), an increase in the constraint (e.g., from 9 to $9 + \delta$) would increase the maximum value of $f(x, y)$ by approximately 1.2357δ.
- At the **minimum** ($\lambda \approx 0.7643$), an increase in the constraint would increase the minimum value of $f(x, y)$ by approximately 0.7643δ.