Second-order differential equation with linear dependence in the solutions

Find the general solution to the following second-order differential equation:

$$y'' - 4y = -e^{2x} + \sin(2x)$$

Solution

First, we solve the homogeneous differential equation:

$$y'' - 4y = 0$$

I propose:

$$y_h = e^{rx}$$

Then $y_h' = re^{rx}$ and $y_h'' = r^2e^{rx}$. Replacing:

$$r^2e^{rx} - 4e^{rx} = 0$$

$$r^2 - 4 = 0$$

The roots are then $r_1 = 2$ and $r_2 = -2$. The homogeneous solution would in this case be:

$$y_h = C_1e^{2x} + C_2e^{-2x}$$

Now solving for the particular solution, as we have e^{2x} we propose Axe^{2x}, but we multiply this term by x since otherwise it is linearly dependent with a part of the homogeneous solution: C_1e^{2x}. We have then Axe^{2x}. And on the other hand, we propose $B\sin(2x)$, our particular solution would be:

$$y_c = Axe^{2x} + B\sin(2x)$$

Deriving to be able to replace in the original equation:

$$y_c' = Ae^{2x} + 2xAe^{2x} + 2B\cos(2x)$$

$$y_c'' = 2Ae^{2x} + 4Axe^{2x} - 4B\sin(2x)$$

Replacing in the original equation:

$$2Ae^{2x} + 4Axe^{2x} - 4B\sin(2x) - 4(Axe^{2x} + B\sin(2x)) = -e^{2x} + \sin(2x)$$

Simplifying we get:

$$2Ae^{2x} - 8B\sin(2x) = -e^{2x} + \sin(2x)$$

Therefore:

$$2Ae^{2x} = -e^{2x}$$

and

$$-8B\sin(2x) = \sin(2x)$$

From which we get $A = -\frac{1}{2}$ and $B = -\frac{1}{8}$. Our particular solution would be:

$$y_c = -\frac{1}{4}xe^{2x} - \frac{1}{8}\sin(2x)$$

And the general solution:

$$y_g = y_h + y_c = C_1e^{2x} + C_2e^{-2x} - \frac{1}{4}xe^{2x} - \frac{1}{8}\sin(2x)$$