# Continuous Aggregation for Cobb–Douglas Consumers: Deriving the Market Demand

Consider a market for a good $x$ with price $p$. There is a continuum of consumers with total measure $M$. Each consumer $i$ has a Cobb–Douglas utility function given by

$$U_i(x_i, y_i) = x_i^{\alpha}\, y_i^{1-\alpha},$$

where $0 < \alpha < 1$ and $y_i$ denotes the consumption of the numeraire good (with price normalized to 1). Consumer $i$ earns income $m$, and incomes are heterogeneously distributed in the market according to the probability density function $f(m)$ on the interval $[m_L, m_H]$.

(a) Derive the individual Marshallian demand function for good $x$ for a consumer with income $m$.

(b) Derive the aggregate market demand for good $x$ by integrating the individual demands over the income distribution.

# Solution

## (a) Individual Demand

For a consumer with income $m$, the Cobb–Douglas utility implies that a fixed fraction $\alpha$ of income is allocated to good $x$. Hence, the expenditure on $x$ is:

$$\text{Expenditure on } x = \alpha m.$$

Given that the price of $x$ is $p$, the individual demand for $x$ is:

$$x(m) = \frac{\alpha m}{p}.$$

## (b) Aggregate Market Demand

To derive the aggregate market demand, we integrate the individual demands over the distribution of incomes and then multiply by the total number of consumers $M$. The aggregate demand $Q_x(p)$ is:

$$Q_x(p) = M \int_{m_L}^{m_H} x(m)\, f(m)\, dm = M \int_{m_L}^{m_H} \frac{\alpha m}{p}\, f(m)\, dm.$$

This expression can be simplified by taking out the constant terms:

$$Q_x(p) = \frac{\alpha M}{p} \int_{m_L}^{m_H} m\, f(m)\, dm.$$

The integral

$$\int_{m_L}^{m_H} m\, f(m)\, dm$$

represents the average income (denoted by $\bar{m}$) when $f(m)$ is a probability density function. Thus, the aggregate market demand can be written as:

$$Q_x(p) = \frac{\alpha M\, \bar{m}}{p}.$$

This is the aggregate demand for good $x$ in the market.