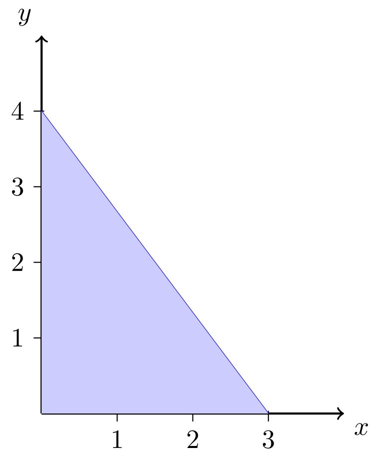
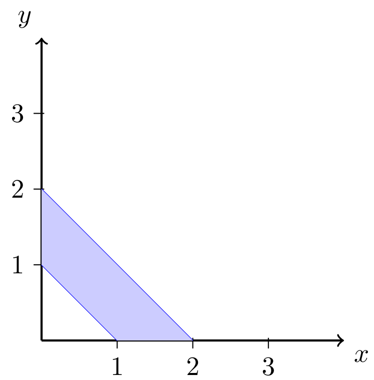
Double integrals and changing the Order of integration

Calculate the area of the following regions using double integrals

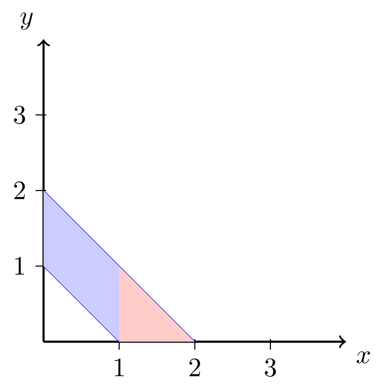
1. 

2. 

Solution

1. We see that the area is enclosed between 0 and 3 along the x-axis. Along the y-axis, it is enclosed between $y = 0$ and $y = -(4/3)x + 4$. This second function is obtained knowing that a line that passes through $(0, 4)$ and $(3, 0)$ is $y = -(4/3)x + 4$. Knowing this, we set up the double integral:
$$\int_0^3 \int_0^{-4/3x+4} dydx = \int_0^3 -4/3x + 4dx$$
The result of this integral is: $\frac{-4}{3}\frac{x^2}{2} + 4x$ And evaluating at the endpoints:
$$-18/3 + 12 = 6$$
Note that the area could also have been calculated as a triangle: $3 * 4/2 = 6$. Or by changing the order of integration, we have the inverted function: $(-3/4)y + 3 = x$
$$\int_0^4 \int_0^{(-3/4)y+3} dxdy = \int_0^4 [(-3/4)y + 3]dy$$
The result of this integral is: $(-3/8)y^2 + 3y$ and evaluating at the endpoints:
$$(-3/8) * 16 + 12 - 0 = 6$$

2. For the second area, we can use the fact that it can be represented as two double integrals. This is because the "floor" changes. From 0 to 1, the floor is a linear function and from 1 to 2, the floor is the x-axis. In one case, we have the area enclosed between 0 and 1 along the x-axis and along the y-axis between the functions $y = -x + 1$ and $y = -x + 2$. While the second region is enclosed between 1 and 2 along the x-axis and along the y-axis between $y = -x + 2$ and $y = 0$. Let's calculate the integrals:
$$\int_0^1 \int_{-x+1}^{-x+2} dydx.$$
The result of the first integral is: $-x + 2 - (-x + 1) = 1$. Integrating this:
$$\int_0^1 1dx = 1 - 0 = 1$$
Now let's calculate the second double integral:
$$\int_1^2 \int_0^{-x+2} dydx.$$
The result of the first integral is: $-x + 2$. Integrating this:
$$\int_1^2 -x + 2dx = -2^2/2 + 4 - (-1/2 + 2) = 1/2$$
Adding the areas: $1 + 1/2 = 3/2$
Graphically:



It is also possible to calculate the area by changing the order of integration. For this, we first express the functions in terms of y:

$$x = 1 - y$$

and

$$x = 2 - y$$

$$\int_0^1 \int_{2-y}^{1-y} dxdy = \int_0^1 [1-y] - [2-y]dy = \int_0^1 1dy = 1$$

And the other integral:

$$\int_1^2 \int_0^{2-y} dxdy = \int_1^2 (2-y)dy$$

The result is: $2y - \frac{y^2}{2}$ Evaluating at the endpoints of the integral:

$$[4-2] - [2-1/2] = 1/2$$

Adding the results: $1/2 + 1 = 3/2$

In graphical terms, it is convenient to rotate the graph: