Intertemporal Choice with Borrowing Constraints

A consumer receives an income of

$$y_1 = 60 \quad \text{in period 1} \quad \text{and} \quad y_2 = 90 \quad \text{in period 2.}$$

The consumer has the utility function

$$U(c_1, c_2) = \ln(c_1) + \ln(c_2),$$

and can save or borrow at an interest rate r. The standard intertemporal budget constraint is

$$c_1 + \frac{c_2}{1+r} = y_1 + \frac{y_2}{1+r}.$$

However, the consumer faces a borrowing constraint that limits the amount that can be borrowed in period 1. Specifically, if we denote by B the maximum amount the consumer is allowed to borrow, then the borrowing constraint implies

$$c_1 \leq y_1 + B.$$

Answer the following:

(a) Write down (i) the standard intertemporal budget constraint and (ii) the modified budget set that incorporates the borrowing constraint.

(b) Derive the optimal consumption bundle under the unconstrained case and under the case when the borrowing constraint is binding.

Solution

(a) Budget Constraints

Standard Budget Constraint: Without borrowing restrictions, the consumer's intertemporal budget constraint is:

$$c_1 + \frac{c_2}{1+r} = 60 + \frac{90}{1+r}.$$

Modified Budget Constraint (with Borrowing Constraint): Let $s = y_1 - c_1$ denote savings in period 1. If $s < 0$, the consumer is borrowing. A borrowing constraint stipulates that the amount borrowed, $-s$, cannot exceed B; equivalently,

$$s \geq -B \implies y_1 - c_1 \geq -B \implies c_1 \leq 60 + B.$$

Thus, the consumer's choice set is given by:

$$\begin{cases} c_1 + \dfrac{c_2}{1+r} = 60 + \dfrac{90}{1+r}, \\ c_1 \leq 60 + B. \end{cases}$$

(b) Optimal Consumption Bundle

Unconstrained Case: The consumer maximizes

$$\max_{c_1, c_2} \quad \ln(c_1) + \ln(c_2)$$

subject to

$$c_1 + \frac{c_2}{1+r} = 60 + \frac{90}{1+r}.$$

Since the utility is Cobb–Douglas (with equal weights), the optimal solution splits the present-value income equally between current and discounted future consumption. Let

$$Y = 60 + \frac{90}{1+r}.$$

Then the optimal expenditure (in present-value terms) on c_1 is $\frac{Y}{2}$ and on c_2 is also $\frac{Y}{2}$. However, because the cost of one unit of c_2 in present-value terms is $\frac{1}{1+r}$, the optimal choices are:

$$c_1^* = \frac{Y}{2}, \qquad c_2^* = (1+r) \cdot \frac{Y}{2}.$$

That is, in the unconstrained case the solution is:

$$c_1^* = \frac{1}{2}\left(60 + \frac{90}{1+r}\right), \quad c_2^* = \frac{1+r}{2}\left(60 + \frac{90}{1+r}\right).$$

Constrained Case: Suppose the unconstrained optimal c_1^* exceeds the borrowing limit, i.e., if

$$c_1^* > 60 + B.$$

Then the borrowing constraint binds and the consumer cannot choose c_1 greater than $60 + B$. In that case, the consumer chooses

$$c_1 = 60 + B.$$

Substituting $c_1 = 60 + B$ into the intertemporal budget constraint to solve for c_2:

$$(60 + B) + \frac{c_2}{1+r} = 60 + \frac{90}{1+r} \implies \frac{c_2}{1+r} = \frac{90}{1+r} - B.$$

Thus,

$$c_2 = (1+r)\left(\frac{90}{1+r} - B\right) = 90 - B(1+r).$$

So, when the borrowing constraint is binding, the optimal bundle is:

$$c_1^c = 60 + B, \quad c_2^c = 90 - B(1+r).$$

Note that feasibility requires $c_2^c \geq 0$, i.e., $90 \geq B(1+r)$.