Marshallian Demands for a Nested Stone-Geary Utility Function

Consider a consumer with the following utility function:

$$U(x_1, x_2, x_3) = \left[(x_1 - \bar{x}_1)^{\alpha}(x_2 - \bar{x}_2)^{1-\alpha}\right] + x_3,$$

where $\bar{x}_1 > 0$ and $\bar{x}_2 > 0$ represent minimum consumption levels for goods x_1 and x_2, respectively, and $\alpha \in (0, 1)$ is a parameter indicating the relative preference for x_1 and x_2. The prices of goods x_1, x_2, and x_3 are p_1, p_2, and p_3, respectively, and the consumer has an income of m.

(a) Formulate the utility maximization problem for the consumer subject to the budget constraint:

$$p_1 x_1 + p_2 x_2 + p_3 x_3 \leq m.$$

(b) Derive the Marshallian demands for x_1, x_2, and x_3 as functions of p_1, p_2, p_3, m, and the parameters $\bar{x}_1$, $\bar{x}_2$, and α.

(c) Verify that the derived Marshallian demands satisfy the property of homogeneity of degree zero in prices and income.

(d) Explain how changes in the consumer's income m affect the consumption of x_1, x_2, and x_3.

Solution

(a) The consumer maximizes utility subject to the budget constraint. Since the utility function is defined only for $x_1 \geq \bar{x}_1$ and $x_2 \geq \bar{x}_2$ (to ensure that the expression inside the brackets is defined), we write the problem as

$$\max_{x_1,x_2,x_3} \quad (x_1 - \bar{x}_1)^{\alpha}(x_2 - \bar{x}_2)^{1-\alpha} + x_3$$

subject to

$$p_1 x_1 + p_2 x_2 + p_3 x_3 = m$$

and

$$x_1 \geq \bar{x}_1, \quad x_2 \geq \bar{x}_2, \quad x_3 \geq 0$$

Since utility is strictly increasing in all arguments, the budget constraint will hold with equality at the optimum

(b) It is convenient to define the net consumptions

$$y_1 = x_1 - \bar{x}_1, \qquad y_2 = x_2 - \bar{x}_2$$

so that the utility becomes

$$U = y_1^{\alpha} y_2^{1-\alpha} + x_3$$

and the budget constraint is rewritten as

$$p_1 y_1 + p_2 y_2 + p_3 x_3 = m - p_1\bar{x}_1 - p_2\bar{x}_2 \equiv m'$$

Because the utility is quasilinear in x_3 (with constant marginal utility equal to 1), the allocation to the *composite good* (represented by y_1 and y_2) will be determined by comparing its marginal utility per dollar to that of x_3

Step 1. Cost Minimization for the Composite Good

For a given target level of the composite index

$$z = y_1^{\alpha} y_2^{1-\alpha}$$

the consumer minimizes expenditure on y_1 and y_2 by solving

$$\min_{y_1,y_2} \quad p_1 y_1 + p_2 y_2 \quad \text{subject to} \quad y_1^{\alpha} y_2^{1-\alpha} = z, \quad y_1, y_2 \geq 0$$

where we have defined the net consumptions

$$y_1 = x_1 - \bar{x}_1, \quad y_2 = x_2 - \bar{x}_2.$$

We solve this problem using the method of Lagrange multipliers. Define the Lagrangian as

$$\mathcal{L}(y_1, y_2, \lambda) = p_1 y_1 + p_2 y_2 + \lambda\Big(z - y_1^{\alpha} y_2^{1-\alpha}\Big).$$

Taking the first-order conditions (FOCs) with respect to y_1 and y_2 gives:

$$\frac{\partial \mathcal{L}}{\partial y_1} = p_1 - \lambda\alpha\, y_1^{\alpha-1} y_2^{1-\alpha} = 0, \quad (1)$$

$$\frac{\partial \mathcal{L}}{\partial y_2} = p_2 - \lambda(1-\alpha)\, y_1^{\alpha} y_2^{-\alpha} = 0. \quad (2)$$

Dividing equation (1) by equation (2) to eliminate λ yields:

$$\frac{p_1}{p_2} = \frac{\alpha}{1-\alpha} \cdot \frac{y_2}{y_1}.$$

Solving for y_2, we have:

$$y_2 = \frac{(1-\alpha)p_1}{\alpha\, p_2}\, y_1.$$

Next, substitute this expression for y_2 into the constraint:

$$y_1^{\alpha}\left(\frac{(1-\alpha)p_1}{\alpha\, p_2}\, y_1\right)^{1-\alpha} = z.$$

Simplify by combining the powers of y_1:

$$y_1^{\alpha}\left(\frac{(1-\alpha)p_1}{\alpha\, p_2}\right)^{1-\alpha} y_1^{1-\alpha} = \left(\frac{(1-\alpha)p_1}{\alpha\, p_2}\right)^{1-\alpha} y_1 = z.$$

Thus, the optimal y_1 is:

$$y_1 = z\left(\frac{\alpha\, p_2}{(1-\alpha)p_1}\right)^{1-\alpha}.$$

Substitute back into the expression for y_2 to obtain:

$$y_2 = \frac{(1-\alpha)p_1}{\alpha\, p_2}\, y_1 = z\left(\frac{(1-\alpha)p_1}{\alpha\, p_2}\right)^{\alpha}.$$

Now, compute the minimum cost $C(z)$:

$$C(z) = p_1 y_1 + p_2 y_2.$$

Substitute the expressions for y_1 and y_2:

$$p_1 y_1 = p_1\, z\left(\frac{\alpha\, p_2}{(1-\alpha)p_1}\right)^{1-\alpha} = z\, p_1^{\alpha}\,(\alpha\, p_2)^{1-\alpha}\,(1-\alpha)^{-(1-\alpha)},$$

$$p_2 y_2 = p_2\, z\left(\frac{(1-\alpha)p_1}{\alpha\, p_2}\right)^{\alpha} = z\, p_2^{1-\alpha}\,(1-\alpha\, p_1)^{\alpha}\,\alpha^{-\alpha}.$$

More precisely, rewriting the second term correctly:

$$p_2 y_2 = p_2\, z\left(\frac{(1-\alpha)p_1}{\alpha\, p_2}\right)^{\alpha} = z\, p_2^{1-\alpha}\, p_1^{\alpha}\,\frac{(1-\alpha)^{\alpha}}{\alpha^{\alpha}}.$$

Thus, both terms simplify to the same expression:

$$p_1 y_1 = z\, p_1^{\alpha} p_2^{1-\alpha}\,\frac{\alpha^{1-\alpha}}{(1-\alpha)^{1-\alpha}},$$

$$p_2 y_2 = z\, p_1^{\alpha} p_2^{1-\alpha}\,\frac{(1-\alpha)^{\alpha}}{\alpha^{\alpha}}.$$

Adding these two terms yields:

$$C(z) = z\, p_1^{\alpha} p_2^{1-\alpha}\left[\frac{\alpha^{1-\alpha}}{(1-\alpha)^{1-\alpha}} + \frac{(1-\alpha)^{\alpha}}{\alpha^{\alpha}}\right].$$

It turns out that the bracketed term simplifies to

$$\frac{1}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}},$$

so that

$$C(z) = \frac{p_1^{\alpha}\, p_2^{1-\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}\, z.$$

For brevity, we define:

$$c \equiv \frac{p_1^{\alpha}\, p_2^{1-\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}},$$

so that the minimal expenditure required to achieve the composite level z is

$$C(z) = c\, z.$$

Step 2. Reducing the Problem

Substituting into the original problem, the consumer's maximization becomes

$$\max_{z, x_3} \quad z + x_3 \quad \text{subject to} \quad c\, z + p_3 x_3 = m'$$

This is a linear optimization problem in z and x_3 where the marginal utility per dollar for z is

$$\frac{1}{c}$$

and for x_3 is

$$\frac{1}{p_3}$$

Step 3. Two Cases

Case 1: If

$$\frac{1}{c} > \frac{1}{p_3} \quad \Longrightarrow \quad p_3 > c$$

the composite good yields higher utility per dollar. In this case the consumer spends all available expenditure m' on the composite good so that $x_3^* = 0$. Since

$$c\, z = m'$$

it follows that

$$z^* = \frac{m'}{c} = \frac{m - p_1\bar{x}_1 - p_2\bar{x}_2}{c}$$

Moreover, the cost minimization for the Cobb–Douglas subproblem implies that the optimal spending shares on y_1 and y_2 are α and $1-\alpha$ respectively. Therefore, the demands for y_1 and y_2 are

$$y_1^* = \frac{\alpha\, m'}{p_1}, \qquad y_2^* = \frac{(1-\alpha)\, m'}{p_2}$$

Returning to the original goods we obtain

$$x_1^* = \bar{x}_1 + y_1^* = \bar{x}_1 + \frac{\alpha\,(m - p_1\bar{x}_1 - p_2\bar{x}_2)}{p_1}$$

$$x_2^* = \bar{x}_2 + y_2^* = \bar{x}_2 + \frac{(1-\alpha)(m - p_1\bar{x}_1 - p_2\bar{x}_2)}{p_2}$$

and

$$x_3^* = 0$$

Case 2: If

$$\frac{1}{c} < \frac{1}{p_3} \quad \Longrightarrow \quad p_3 < c$$

then the linear good x_3 provides higher utility per dollar. In this situation the consumer does not spend any extra income on the composite good beyond the minimum required consumption. Hence, the optimal choice is to take

$$y_1^* = 0, \qquad y_2^* = 0$$

so that

$$x_1^* = \bar{x}_1, \qquad x_2^* = \bar{x}_2$$

and all the remaining income is allocated to x_3

$$p_3 x_3^* = m - p_1\bar{x}_1 - p_2\bar{x}_2 \quad \Longrightarrow \quad x_3^* = \frac{m - p_1\bar{x}_1 - p_2\bar{x}_2}{p_3}$$

Summary of Marshallian Demands

Define

$$m' \equiv m - p_1\bar{x}_1 - p_2\bar{x}_2, \qquad c \equiv \frac{p_1^\alpha p_2^{1-\alpha}}{\alpha^\alpha(1-\alpha)^{1-\alpha}}$$

Then the Marshallian demands are given by

If $p_3 > c$:

$$x_1^* = \bar{x}_1 + \frac{\alpha\,(m - p_1\bar{x}_1 - p_2\bar{x}_2)}{p_1}, \quad x_2^* = \bar{x}_2 + \frac{(1-\alpha)(m - p_1\bar{x}_1 - p_2\bar{x}_2)}{p_2}, \quad x_3^* = 0$$

If $p_3 < c$:

$$x_1^* = \bar{x}_1, \quad x_2^* = \bar{x}_2, \quad x_3^* = \frac{m - p_1\bar{x}_1 - p_2\bar{x}_2}{p_3}$$

If $p_3 = c$, the consumer is indifferent among bundles satisfying

$$p_1(x_1 - \bar{x}_1) + p_2(x_2 - \bar{x}_2) + p_3 x_3 = m - p_1\bar{x}_1 - p_2\bar{x}_2$$

with the additional requirement that the ratio

$$\frac{x_2 - \bar{x}_2}{x_1 - \bar{x}_1} = \frac{p_1(1-\alpha)}{p_2\alpha}$$

holds

(c) *Homogeneity of Degree Zero*

Notice that in both cases the demands depend on income and prices only through the combination $m - p_1\bar{x}_1 - p_2\bar{x}_2$ and the ratios m/p_1, m/p_2, and m/p_3. If all prices and income are scaled by a positive factor t then

$$m' \to t(m - p_1\bar{x}_1 - p_2\bar{x}_2), \quad p_i \to tp_i$$

and for instance in Case 1

$$x_1^* = \bar{x}_1 + \frac{\alpha\,(tm - t(p_1\bar{x}_1 + p_2\bar{x}_2))}{tp_1} = \bar{x}_1 + \frac{\alpha\,(m - p_1\bar{x}_1 - p_2\bar{x}_2)}{p_1}$$

with similar results for x_2^* and x_3^* in Case 2. Thus, the Marshallian demands are homogeneous of degree zero in prices and income

(d) *Effect of Changes in Income*

Because the utility function is quasilinear in x_3, the way additional income is allocated depends on which case applies

In Case 1 ($p_3 > c$): The consumer allocates all available extra income to the composite good (i.e. to x_1 and x_2). Increases in income m lead to increases in x_1 and x_2 as follows:

$$\Delta x_1 = \frac{\alpha \, \Delta m}{p_1}, \qquad \Delta x_2 = \frac{(1-\alpha) \, \Delta m}{p_2}$$

with x_3 remaining zero

In Case 2 ($p_3 < c$): The consumer does not purchase any extra of goods 1 and 2 beyond the subsistence levels $\bar{x}_1$ and $\bar{x}_2$; all additional income is devoted to x_3

$$\Delta x_3 = \frac{\Delta m}{p_3}$$

Thus, changes in income affect the consumption of the composite good only when it is chosen (Case 1), while in Case 2 the consumption of x_1 and x_2 remains fixed and x_3 absorbs all income changes.