Nested Quasilinear Utility: Marshallian and Hicksian Demand

Consider an economy with three goods, x_1, x_2, and x_3, whose prices are p_1, p_2, and p_3, respectively. Assume that $p_1 \neq 2p_2$. In this economy, there is a consumer with the following utility function:

$$U(x_1, x_2, x_3) = \ln(x_1 + 2x_2) + x_3$$

(a) Formulate the utility maximization problem for the agent. Does the budget constraint hold with equality? Justify your answer.

(b) Find the Marshallian demands for x_1, x_2, and x_3, and construct the indirect utility function. Verify that the homogeneity properties of the Marshallian demands are satisfied, as well as the property of budget balance.

(c) Formulate the expenditure minimization problem. Find the Hicksian demands for x_1, x_2, and x_3, and construct the expenditure function. Verify that it satisfies the homogeneity property of the Hicksian demands.

(d) Assume that initially $p_3 = 1$, $p_2 = 5$, $p_1 = 1$, and $m = 10$. Based on this initial situation, compute the substitution effect and the income effect of a change in the price of good 1 to $p_1' = 2$.

Solution

(a) **(a)** The consumer solves

$$\max_{x_1,x_2,x_3} \quad \ln(x_1+2x_2)+x_3$$

subject to

$$p_1x_1+p_2x_2+p_3x_3 \le m$$

and $x_1, x_2, x_3 \ge 0$. Since the partial derivatives

$$\frac{\partial U}{\partial x_1}=\frac{1}{x_1+2x_2}, \quad \frac{\partial U}{\partial x_2}=\frac{2}{x_1+2x_2}, \quad \frac{\partial U}{\partial x_3}=1$$

are all positive, the utility function is strictly increasing in each argument. Therefore, at the optimum the consumer spends all income and the budget constraint holds with equality

(b) Notice that the utility function depends on the goods x_1 and x_2 only through the combination

$$z = x_1 + 2x_2$$

so that

$$U(z, x_3) = \ln(z) + x_3$$

For any given z, the minimum expenditure needed to obtain z is found by solving

$$\min_{x_1,x_2} \; p_1x_1+p_2x_2 \quad \text{subject to} \quad x_1+2x_2=z, \quad x_1,x_2 \ge 0$$

This linear minimization problem yields a corner solution:

- If $p_1 < \frac{p_2}{2}$ (equivalently, $p_2 > 2p_1$), it is optimal to choose $x_1 = z$ and $x_2 = 0$, with cost $p_1 z$
- If $p_1 > \frac{p_2}{2}$ (equivalently, $p_2 < 2p_1$), it is optimal to choose $x_1 = 0$ and $x_2 = \frac{z}{2}$, with cost $\frac{p_2}{2}z$

Define

$$\tilde{p} = \min\left\{p_1, \frac{p_2}{2}\right\}$$

so that the minimum cost to obtain z is $\tilde{p}\,z$. The consumer's problem can now be written as

$$\max_{z,x_3} \quad \ln(z)+x_3 \quad \text{subject to} \quad \tilde{p}\,z + p_3x_3 = m$$

Form the Lagrangian

$$\mathcal{L} = \ln(z) + x_3 + \lambda\Big(m - \tilde{p}\,z - p_3x_3\Big)$$

The first order conditions are

$$\frac{\partial \mathcal{L}}{\partial z} = \frac{1}{z} - \lambda\tilde{p} = 0 \quad \Longrightarrow \quad \lambda = \frac{1}{\tilde{p}\,z}$$

$$\frac{\partial \mathcal{L}}{\partial x_3} = 1 - \lambda p_3 = 0 \quad \Longrightarrow \quad \lambda = \frac{1}{p_3}$$

Equating these expressions gives

$$\frac{1}{\tilde{p}\,z} = \frac{1}{p_3} \quad \Longrightarrow \quad z = \frac{p_3}{\tilde{p}}$$

Substitute $z = \frac{p_3}{\tilde{p}}$ into the budget constraint:

$$\tilde{p}\left(\frac{p_3}{\tilde{p}}\right) + p_3x_3 = p_3 + p_3x_3 = m \quad \Longrightarrow \quad x_3 = \frac{m}{p_3} - 1$$

Thus, the Marshallian demands are determined by the cost-minimizing choice for z:

$$\begin{cases} \text{If } p_1 < \frac{p_2}{2}: & x_1^* = \frac{p_3}{p_1}, \quad x_2^* = 0, \\ \text{If } p_1 > \frac{p_2}{2}: & x_1^* = 0, \quad x_2^* = \frac{p_3}{p_2}, \end{cases}$$

and in both cases

$$x_3^* = \frac{m}{p_3} - 1$$

Assuming $\frac{m}{p_3} - 1 \geq 0$. The indirect utility function is obtained by plugging the optimal values into the utility function:

$$V(m, p_1, p_2, p_3) = \ln\left(\frac{p_3}{\tilde{p}}\right) + \frac{m}{p_3} - 1$$

Finally, to verify the homogeneity of degree zero of the Marshallian demands, scale all prices and income by a positive scalar t. Then

$$x_1^* = \frac{tp_3}{tp_1} = \frac{p_3}{p_1} \quad \text{or} \quad x_2^* = \frac{tp_3}{tp_2} = \frac{p_3}{p_2},$$

and

$$x_3^* = \frac{tm}{tp_3} - 1 = \frac{m}{p_3} - 1$$

so that each demand remains unchanged. In addition, the budget balance property holds since

$$p_1 x_1^* + p_2 x_2^* + p_3 x_3^* = p_3 + m - p_3 = m$$

Corner Solution for $x_3 = 0$ when $m/p_3 < 1$

Since $m/p_3 < 1$, the interior solution that yields

$$x_3 = \frac{m}{p_3} - 1$$

would imply $x_3 < 0$, which violates the nonnegativity constraint. Therefore, the consumer optimally sets

$$x_3^* = 0$$

and the problem reduces to maximizing the utility obtained from goods x_1 and x_2 only:

$$\max_{x_1, x_2} \quad \ln(x_1 + 2x_2)$$

subject to the budget constraint

$$p_1 x_1 + p_2 x_2 = m$$

Defining the composite good

$$z = x_1 + 2x_2$$

we note that the minimal expenditure to achieve a given z is found by solving

$$\min_{x_1, x_2} \ p_1 x_1 + p_2 x_2 \quad \text{subject to} \quad x_1 + 2x_2 = z$$

which gives a minimal cost of

$$\tilde{p}\, z \quad \text{with} \quad \tilde{p} = \min\left\{p_1, \frac{p_2}{2}\right\}$$

Thus, the budget constraint becomes

$$\tilde{p}\, z = m \quad \Longrightarrow \quad z^* = \frac{m}{\tilde{p}}$$

and the cost-minimizing allocations for x_1 and x_2 are:

$$\begin{cases} \text{if } p_1 < \frac{p_2}{2}: & x_1^* = \frac{m}{p_1}, \quad x_2^* = 0 \\ \text{if } p_1 > \frac{p_2}{2}: & x_1^* = 0, \quad x_2^* = \frac{m}{p_2} \end{cases}$$

The indirect utility function in this corner case is given by

$$V(m, p_1, p_2, p_3) = \ln\left(\frac{m}{\tilde{p}}\right)$$

(c) We start with the given utility function

$$U(x_1, x_2, x_3) = \ln(x_1 + 2x_2) + x_3$$

and a target utility level u. The expenditure minimization problem is

$$\min_{x_1, x_2, x_3} \; p_1x_1 + p_2x_2 + p_3x_3$$

subject to

$$\ln(x_1 + 2x_2) + x_3 \geq u, \quad x_1, x_2, x_3 \geq 0.$$

Since the objective is increasing in all goods, the utility constraint will bind at the optimum:

$$\ln(x_1 + 2x_2) + x_3 = u.$$

Because the utility function is quasilinear in x_3, it is convenient to define

$$z = x_1 + 2x_2.$$

Then the constraint becomes

$$\ln z + x_3 = u \quad \Longrightarrow \quad x_3 = u - \ln z,$$

with the feasibility restriction $x_3 \geq 0$ implying

$$\ln z \leq u \quad \Longrightarrow \quad z \leq e^u.$$

Next, note that for any given z the minimum expenditure on (x_1, x_2) required to achieve $x_1 + 2x_2 = z$ is obtained from

$$\min_{x_1, x_2} \; p_1x_1 + p_2x_2 \quad \text{subject to} \quad x_1 + 2x_2 = z.$$

A standard analysis shows that the optimal allocation is a corner solution:

$$\begin{cases} \text{if } p_1 < \frac{p_2}{2}: & x_1 = z, \; x_2 = 0, \quad \text{with cost } p_1z, \\ \text{if } p_1 > \frac{p_2}{2}: & x_1 = 0, \; x_2 = \frac{z}{2}, \quad \text{with cost } \frac{p_2}{2}z. \end{cases}$$

Define

$$\tilde{p} = \min\left\{p_1, \frac{p_2}{2}\right\}.$$

Then the minimum cost to produce a level z is $\tilde{p}\,z$. Incorporating the cost for x_3, the overall expenditure (as a function of z) is

$$E(z) = \tilde{p}\,z + p_3\Big(u - \ln z\Big), \qquad 0 < z \leq e^u.$$

To find the optimum, we differentiate $E(z)$ with respect to z:

$$\frac{dE}{dz} = \tilde{p} - \frac{p_3}{z}.$$

Setting $\frac{dE}{dz} = 0$ yields the candidate

$$z^* = \frac{p_3}{\tilde{p}}.$$

However, this candidate is feasible (i.e. $z^* \leq e^u$) only if

$$\frac{p_3}{\tilde{p}} \leq e^u \quad \Longrightarrow \quad u \geq \ln\Big(\frac{p_3}{\tilde{p}}\Big).$$

Thus, we have two cases:

Case 1 (Interior solution): If

$$u \geq \ln\Big(\frac{p_3}{\tilde{p}}\Big),$$

then the optimal choice is

$$z^* = \frac{p_3}{\tilde{p}}, \qquad x_3^* = u - \ln\Big(\frac{p_3}{\tilde{p}}\Big).$$

The cost-minimizing allocation for x_1 and x_2 is determined by the relative sizes of p_1 and $\frac{p_2}{2}$:

$$\begin{cases} \text{if } p_1 < \dfrac{p_2}{2}: \quad x_1^* = z^* = \dfrac{p_3}{p_1}, \quad x_2^* = 0, \\ \text{if } p_1 > \dfrac{p_2}{2}: \quad x_1^* = 0, \quad x_2^* = \dfrac{z^*}{2} = \dfrac{p_3}{p_2}. \end{cases}$$

The expenditure function is then

$$E(u, p_1, p_2, p_3) = \tilde{p}\, z^* + p_3\Big(u - \ln z^*\Big) = p_3 + p_3\Big(u - \ln\Big(\frac{p_3}{\tilde{p}}\Big)\Big) = p_3\Big(u + 1 - \ln\Big(\frac{p_3}{\tilde{p}}\Big)\Big).$$

Case 2 (Corner solution): If

$$u < \ln\Big(\frac{p_3}{\tilde{p}}\Big),$$

then choosing $z^* = \frac{p_3}{\tilde{p}}$ would yield $x_3^* < 0$, which is infeasible. In this case the optimal solution sets

$$x_3^* = 0,$$

so that the utility constraint reduces to

$$\ln(x_1 + 2x_2) = u \quad \Longrightarrow \quad x_1 + 2x_2 = e^u.$$

The cost-minimizing allocation for x_1 and x_2 is

$$\begin{cases} \text{if } p_1 < \dfrac{p_2}{2}: \quad x_1^* = e^u, \quad x_2^* = 0, \\ \text{if } p_1 > \dfrac{p_2}{2}: \quad x_1^* = 0, \quad x_2^* = \dfrac{e^u}{2}. \end{cases}$$

The corresponding expenditure function is

$$E(u, p_1, p_2, p_3) = \tilde{p}\, e^u.$$

Verification of Homogeneity: Hicksian demands must be homogeneous of degree zero in prices. To check this, suppose we scale all prices by a positive factor $t > 0$. Then the effective price for the composite good becomes

$$\tilde{p}' = \min\{tp_1, (tp_2)/2\} = t\tilde{p},$$

and the candidate interior solution becomes

$$z^* = \frac{tp_3}{t\tilde{p}} = \frac{p_3}{\tilde{p}},$$

which is identical to the original z^*. Consequently, the compensated demands

$$x_1^h,\ x_2^h,\ x_3^h$$

remain unchanged when all prices are scaled. A similar argument applies in the corner solution case. Thus, the Hicksian demands are homogeneous of degree zero in prices.

(d) We use the utility function

$$U(x_1, x_2, x_3) = \ln(x_1 + 2x_2) + x_3,$$

and recall that the consumer's optimal choice is determined by first minimizing expenditure for a given utility level. For convenience we define the composite good

$$z = x_1 + 2x_2.$$

Step 1. Initial Optimal Bundle

With initial prices

$$p_1 = 1, \quad p_2 = 5, \quad p_3 = 1, \quad m = 10,$$

the cost-minimization for the z part involves the effective price

$$\tilde{p} = \min\left\{p_1, \frac{p_2}{2}\right\} = \min\left\{1, \frac{5}{2}\right\} = 1.$$

In the utility maximization problem (with the binding budget),

$$z^* = \frac{p_3}{\tilde{p}} = \frac{1}{1} = 1,$$

and the optimal allocation is

$$x_1^0 = z^* = 1, \quad x_2^0 = 0, \quad x_3^0 = \frac{m}{p_3} - 1 = 10 - 1 = 9.$$

Thus, the initial bundle is

$$(x_1^0, x_2^0, x_3^0) = (1, 0, 9),$$

with utility

$$U(1, 0, 9) = \ln(1) + 9 = 9.$$

Step 2. New (Uncompensated) Bundle after p_1 Increases to 2

With the new price $p_1' = 2$ and the other parameters unchanged,

$$p_1' = 2, \quad p_2 = 5, \quad p_3 = 1, \quad m = 10,$$

the effective price is now

$$\tilde{p}' = \min\left\{p_1', \frac{p_2}{2}\right\} = \min\left\{2, \frac{5}{2}\right\} = 2.$$

Then the optimal z becomes

$$z^* = \frac{p_3}{\tilde{p}'} = \frac{1}{2} = 0.5,$$

with the cost-minimizing allocation (since $p_1' < \frac{p_2}{2}$) given by

$$x_1' = z^* = 0.5, \quad x_2' = 0,$$

and the remainder of income is spent on good 3:

$$p_3 x_3' = m - \tilde{p}' z^* = 10 - 2(0.5) = 10 - 1 = 9 \quad \Longrightarrow \quad x_3' = 9.$$

Thus, the new bundle is

$$(x_1', x_2', x_3') = (0.5, 0, 9),$$

with utility

$$U(0.5, 0, 9) = \ln(0.5) + 9 \approx -0.6931 + 9 \approx 8.3069.$$

Step 3. Compensated (Hicksian) Bundle

To isolate the substitution effect we find the bundle that minimizes expenditure at the new prices while keeping utility at the initial level $u = 9$. The expenditure minimization problem is

$$\min_{x_1, x_2, x_3} \; 2x_1 + 5x_2 + x_3 \quad \text{subject to} \quad \ln(x_1 + 2x_2) + x_3 = 9.$$

Defining $z = x_1 + 2x_2$ and writing the constraint as

$$x_3 = 9 - \ln z,$$

the minimum cost for achieving z in the x_1, x_2 sector is

$$\tilde{p}'z = 2z \quad (\text{since } \tilde{p}' = 2),$$

and the expenditure becomes

$$E(z) = 2z + p_3(9 - \ln z) = 2z + 9 - \ln z.$$

Minimizing with respect to z,

$$\frac{dE}{dz} = 2 - \frac{1}{z} = 0 \quad \Longrightarrow \quad z^* = \frac{1}{2} = 0.5.$$

Then the Hicksian demands are (using the fact that $p_1' < \frac{p_2}{2}$):

$$x_1^h = z^* = 0.5, \quad x_2^h = 0, \quad x_3^h = 9 - \ln(0.5) = 9 + \ln 2 \approx 9.6931.$$

This compensated bundle yields the original utility level $u = 9$ at the new prices.

Step 4. Decomposition into Substitution and Income Effects

For Good 1:

$$\text{Substitution Effect (SE)} = x_1^h - x_1^0 = 0.5 - 1 = -0.5.$$

$$\text{Income Effect (IE)} = x_1' - x_1^h = 0.5 - 0.5 = 0.$$

For Good 3:

$$\text{Substitution Effect (SE)} = x_3^h - x_3^0 = (9 + \ln 2) - 9 = \ln 2 \approx 0.6931.$$

$$\text{Income Effect (IE)} = x_3' - x_3^h = 9 - (9 + \ln 2) = -\ln 2 \approx -0.6931.$$

For Good 2: Since $x_2^0 = x_2^h = x_2' = 0$, both effects are zero.

Summary:

$$\begin{aligned} &\text{Total effect on } x_1: && -0.5 \quad (\text{all due to substitution}) \\ &\text{Total effect on } x_3: && 0.6931 + (-0.6931) = 0 \quad (\text{substitution and income effects cancel}) \\ &\text{Total effect on } x_2: && 0. \end{aligned}$$

Thus, the rise in p_1 from 1 to 2 causes a 0.5 unit decrease in good 1 entirely from the substitution effect, while the adjustment in good 3 is completely due to the income effect (with the substitution effect increasing x_3 by approximately 0.6931 and the