Nested Utility Function with Cobb-Douglas and Perfect Complements

Consider a consumer with the utility function

$$U(x_1, x_2, x_3) = \min\left\{ x_1^{\alpha} x_2^{1-\alpha},\ x_3 \right\},$$

where $0 < \alpha < 1$. The prices of goods x_1, x_2, and x_3 are p_1, p_2, and p_3, respectively, and the consumer has income m.

(a) Write down the consumer's utility maximization problem subject to the budget constraint

$$p_1 x_1 + p_2 x_2 + p_3 x_3 = m.$$

(b) Derive the Marshallian demand functions for x_1, x_2, and x_3.

Solution

(a) Formulation of the Problem

The consumer's problem is

$$\max_{x_1,x_2,x_3} \quad U(x_1,x_2,x_3) = \min\left\{x_1^{\alpha}x_2^{1-\alpha},\ x_3\right\},$$
$$\text{s.t.} \quad p_1x_1 + p_2x_2 + p_3x_3 = m,$$
$$x_1, x_2, x_3 \geq 0.$$

Because utility is determined by the smaller of $x_1^{\alpha}x_2^{1-\alpha}$ and x_3, the highest possible utility is attained when these two expressions are equal. That is, in any optimal bundle we have

$$x_1^{\alpha}x_2^{1-\alpha} = x_3.$$

(b) Derivation of the Marshallian Demands

Step 1. Using the Equality Condition. Since the consumer maximizes

$$U = \min\left\{x_1^{\alpha}x_2^{1-\alpha},\ x_3\right\},$$

at an optimum we set

$$x_1^{\alpha}x_2^{1-\alpha} = x_3.$$

Then the utility achieved is $U = x_3$ and the budget constraint becomes

$$p_1x_1 + p_2x_2 + p_3x_3 = m.$$

Our goal is now to choose x_1 and x_2 (and hence x_3) to maximize x_3 while satisfying both the budget constraint and the condition

$$x_1^{\alpha}x_2^{1-\alpha} = x_3.$$

Step 2. Solving the Cobb–Douglas Subproblem. Let z denote the output of the Cobb–Douglas aggregator:

$$z := x_1^{\alpha}x_2^{1-\alpha}.$$

In the optimal solution we have $z = x_3$. Suppose the consumer allocates an amount m' of her income to purchase x_1 and x_2. Then she spends the remaining income $m - m'$ on x_3; that is,

$$x_3 = m - m'.$$

Given the subbudget $p_1x_1 + p_2x_2 = m'$, the standard Cobb–Douglas maximization problem

$$\max_{x_1,x_2} \quad x_1^{\alpha}x_2^{1-\alpha} \quad \text{subject to} \quad p_1x_1 + p_2x_2 = m'$$

yields the well-known demands

$$x_1 = \frac{\alpha\, m'}{p_1}, \quad x_2 = \frac{(1-\alpha)\, m'}{p_2},$$

and the maximum value achieved is

$$x_1^{\alpha}x_2^{1-\alpha} = \left(\frac{\alpha\, m'}{p_1}\right)^{\alpha}\left(\frac{(1-\alpha)\, m'}{p_2}\right)^{1-\alpha} = m' \cdot \frac{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}{p_1^{\alpha}p_2^{1-\alpha}}.$$

Define the constant

$$K := \frac{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}{p_1^{\alpha}p_2^{1-\alpha}}.$$

Then the Cobb–Douglas aggregator yields

$$z = m'\, K.$$

Step 3. Equalizing the Two Sources of Utility. At optimum we require

$$z = x_3,$$

so that

$$m'K = m - m'.$$

Solving for m':

$$m'(K+1) = m \quad \Longrightarrow \quad m' = \frac{m}{K+1}.$$

Then the optimal value of the composite is

$$x_3 = z = m'K = \frac{mK}{K+1}.$$

Step 4. Marshallian Demands. Recalling the demands for x_1 and x_2 in the Cobb–Douglas problem,

$$x_1^* = \frac{\alpha\, m'}{p_1} = \frac{\alpha}{p_1} \cdot \frac{m}{K+1},$$

$$x_2^* = \frac{(1-\alpha)\, m'}{p_2} = \frac{1-\alpha}{p_2} \cdot \frac{m}{K+1},$$

and

$$x_3^* = \frac{mK}{K+1},$$

with

$$K = \frac{\alpha^\alpha (1-\alpha)^{1-\alpha}}{p_1^\alpha p_2^{1-\alpha}}.$$

Summary. The Marshallian demands for the consumer are:

$$x_1^* = \frac{\alpha\, m}{p_1\left(1 + \dfrac{\alpha^\alpha (1-\alpha)^{1-\alpha}}{p_1^\alpha p_2^{1-\alpha}}\right)},$$

$$x_2^* = \frac{(1-\alpha)\, m}{p_2\left(1 + \dfrac{\alpha^\alpha (1-\alpha)^{1-\alpha}}{p_1^\alpha p_2^{1-\alpha}}\right)},$$

$$x_3^* = \frac{m\, \dfrac{\alpha^\alpha (1-\alpha)^{1-\alpha}}{p_1^\alpha p_2^{1-\alpha}}}{1 + \dfrac{\alpha^\alpha (1-\alpha)^{1-\alpha}}{p_1^\alpha p_2^{1-\alpha}}}.$$

These demands are derived under the assumption that the consumer equalizes the two arguments of the min{} operator (i.e., $x_1^\alpha x_2^{1-\alpha} = x_3$) so as to maximize utility.