Marshallian demand in a three-period consumption model

Consider a consumer who lives for three periods (periods 0, 1, and 2) The consumer's preferences are represented by the utility function

$$\max_{x_0, x_1, x_2} \quad \ln(x_0) + \beta\delta \ln(x_1) + \beta\delta^2 \ln(x_2)$$

where x_0, x_1, and x_2 denote consumption in periods 0, 1, and 2 respectively, $\beta \in (0, 1)$ is the discount factor, and $\delta > 0$ adjusts the weight on later periods' consumption The consumer faces a budget constraint linking consumption in periods 0 and 1 with period 2 consumption

$$x_2 = w - (1 + r)(x_0 + x_1)$$

where $w > 0$ is initial wealth and $r > -1$ is the interest rate

Find the Marshallian demand functions x_0^*, x_1^*, and x_2^*

Solution

Marshallian demand functions

Since x_2 is determined by x_0 and x_1 via the budget constraint, the problem reduces to maximizing

$$\ln(x_0) + \beta\delta \ln(x_1) + \beta\delta^2 \ln\Big(w - (1+r)(x_0 + x_1)\Big)$$

with respect to x_0 and x_1 (with the usual restrictions ensuring positive consumption) Differentiating the objective function with respect to x_0 yields

$$\frac{1}{x_0} - \beta\delta^2 \frac{(1+r)}{w - (1+r)(x_0 + x_1)} = 0$$

so that

$$\frac{1}{x_0} = \beta\delta^2 \, \frac{1+r}{w - (1+r)(x_0 + x_1)}$$

Similarly, differentiating with respect to x_1 gives

$$\frac{\beta\delta}{x_1} - \beta\delta^2 \frac{(1+r)}{w - (1+r)(x_0 + x_1)} = 0$$

which simplifies to

$$\frac{1}{x_1} = \delta \, \frac{1+r}{w - (1+r)(x_0 + x_1)}$$

By equating the right-hand sides (or dividing the two first-order conditions), we obtain

$$\frac{1/x_0}{1/x_1} = \frac{\beta\delta^2}{\delta} \quad \Longrightarrow \quad \frac{x_1}{x_0} = \beta\delta$$

or equivalently,

$$x_1 = \beta\delta \, x_0$$

Substituting this relation into the budget residual expression Using the first-order condition for x_1

$$w - (1+r)(x_0 + x_1) = \delta(1+r)x_1$$

Replacing x_1 by $\beta\delta \, x_0$, we have

$$w - (1+r)\Big(x_0 + \beta\delta \, x_0\Big) = \delta(1+r)(\beta\delta \, x_0)$$

This simplifies to

$$w - (1+r)(1+\beta\delta)x_0 = \beta\delta^2(1+r)x_0$$

Solving for x_0, we add $(1+r)(1+\beta\delta)x_0$ to both sides to obtain

$$w = (1+r)x_0\Big[(1+\beta\delta) + \beta\delta^2\Big]$$

so that

$$x_0^* = \frac{w}{(1+r)\Big(1 + \beta\delta + \beta\delta^2\Big)}$$

Then

$$x_1^* = \beta\delta \, x_0^* = \frac{\beta\delta \, w}{(1+r)\Big(1 + \beta\delta + \beta\delta^2\Big)}$$

Finally, from the budget constraint

$$x_2^* = w - (1+r)(x_0^* + x_1^*) = w - (1+r)\, x_0^*(1+\beta\delta)$$

Substituting the expression for x_0^*

$$x_2^* = w - (1+r)\frac{w(1+\beta\delta)}{(1+r)\left(1+\beta\delta+\beta\delta^2\right)} = w\left[1 - \frac{1+\beta\delta}{1+\beta\delta+\beta\delta^2}\right]$$

After simplifying

$$x_2^* = w\frac{\beta\delta^2}{1+\beta\delta+\beta\delta^2}$$