Marshallian Demands for an n-Good Cobb-Douglas

Given the utility function of a Cobb-Douglas form with n goods, where all exponents are equal to 1:

$$u(x_1, x_2, \ldots, x_n) = x_1 x_2 \cdots x_n$$

find the Marshallian demands $x_i(p, m)$ and the indirect utility function $v(p, m)$, where $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ is the vector of prices, and m is the consumer's income.

Solution

1. **Utility maximization problem**: The consumer solves the following optimization problem:

$$\max_{x_1, x_2, \ldots, x_n} \quad x_1 x_2 \cdots x_n \quad \text{subject to} \quad \sum_{i=1}^{n} p_i x_i = m$$

Using the method of Lagrange, the Lagrangian is:

$$\mathcal{L} = x_1 x_2 \cdots x_n + \lambda \left(m - \sum_{i=1}^{n} p_i x_i \right)$$

2. **First-order conditions**: Taking partial derivatives with respect to x_i and λ, we have:

$$\frac{\partial \mathcal{L}}{\partial x_i} = \frac{x_1 x_2 \cdots x_n}{x_i} - \lambda p_i = 0, \quad \forall i$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = m - \sum_{i=1}^{n} p_i x_i = 0$$

From the first condition:

$$\lambda = \frac{x_1 x_2 \cdots x_n}{x_i p_i}, \quad \forall i$$

Equating λ for two goods x_i and x_j, we find:

$$\frac{x_1 x_2 \cdots x_n}{x_i p_i} = \frac{x_1 x_2 \cdots x_n}{x_j p_j} \quad \Longrightarrow \quad \frac{x_j}{x_i} = \frac{p_i}{p_j}$$

where we assume that no good is demanded in a quantity of zero ($x_i > 0 \, \forall i$). This ensures all goods are consumed in strictly positive quantities.

Thus, any two goods' demands must satisfy

$$x_i \propto \frac{1}{p_i}$$

Since

$$x_i = k \, \frac{1}{p_i} \quad \text{for some constant } k,$$

the budget constraint

$$\sum_{i=1}^{n} p_i \, x_i = m$$

becomes

$$\sum_{i=1}^{n} p_i \Big(k \, \frac{1}{p_i} \Big) = k \sum_{i=1}^{n} \Big(p_i \, \frac{1}{p_i} \Big) = k \, n = m$$

Solving for k gives

$$k = \frac{m}{n}$$

Therefore,

$$x_i = k \, \frac{1}{p_i} = \frac{m}{n} \, \frac{1}{p_i} = \frac{m}{n \, p_i}$$

3. **Marshallian demand functions**: The demand for each good i is:

$$x_i(p, m) = \frac{m}{n p_i}$$

4. **Indirect utility function**: Substituting $x_i(p, m)$ into the utility function:

$$v(p, m) = \prod_{i=1}^{n} \frac{m}{np_i}$$

Simplifying:

$$v(p, m) = \frac{m^n}{n^n \prod_{i=1}^{n} p_i}$$

The indirect utility function is:

$$v(p, m) = \frac{m^n}{n^n \prod_{i=1}^{n} p_i}$$