Marshallian Demands for an n-Good Perfect-Substitutes Utility

Given a consumer with utility function

$$u(x_1, x_2, \ldots, x_n) \;=\; a_1\, x_1 \;+\; a_2\, x_2 \;+\; \cdots + \; a_n\, x_n$$

where $a_i > 0$ for all i, facing prices $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ and income m, find the Marshallian (ordinary) demand functions $x_i(p, m)$ and the indirect utility function $v(p, m)$.

Solution

1. **Utility maximization problem**
   We want to solve

$$\max_{x_1,\ldots,x_n} \; a_1x_1 + a_2x_2 + \cdots + a_nx_n \quad \text{subject to} \quad \sum_{i=1}^{n} p_i\,x_i \;=\; m \quad \text{and} \quad x_i \geq 0$$

   Because $u(\cdot)$ is linear in the $\{x_i\}$, the maximum is typically attained at a *corner solution*: the consumer spends all income on the good that yields the highest "bang for the buck" ratio, $\frac{a_i}{p_i}$.

2. **Bang-for-the-buck and corner solutions**
   Define

$$\frac{a_i}{p_i} \quad \text{for each good } i$$

   The consumer will choose the good i^* for which $\frac{a_{i^*}}{p_{i^*}}$ is the largest (assuming a unique maximum). In that case, she devotes her entire income m to buying good i^*, so

$$x_{i^*} \;=\; \frac{m}{p_{i^*}} \quad \text{and} \quad x_j \;=\; 0 \quad \text{for all } j \neq i^*$$

3. **Marshallian demand functions**
   Thus, the demand for each good i depends on which good has the highest $\frac{a_i}{p_i}$. If a single ratio is strictly larger than all others, all income is spent on that unique good. Formally, let

$$i^* \;=\; \arg\max_{1 \leq k \leq n} \frac{a_k}{p_k}$$

   Then

$$x_i(p,m) \;=\; \begin{cases} \dfrac{m}{p_{i^*}} & \text{if } i = i^* \text{ and } \frac{a_{i^*}}{p_{i^*}} > \frac{a_j}{p_j} \text{ for all } j \neq i^*, \\ 0 & \text{otherwise} \end{cases}$$

   In case of a tie among multiple goods for the highest ratio, there are infinitely many solutions: the consumer can *split* her income among any goods sharing the same maximal ratio $\frac{a_i}{p_i}$, and achieve the same utility level.

4. **Indirect utility function**
   Substitute the corner solution back into the utility function. If the consumer spends all income on good i^*, then

$$u(x_1,\ldots,x_n) \;=\; a_{i^*}\left(\frac{m}{p_{i^*}}\right) \;=\; m\,\frac{a_{i^*}}{p_{i^*}}$$

   Hence the indirect utility is the maximum of $m\,\frac{a_i}{p_i}$ over all i. Therefore,

$$v(p,m) \;=\; m \max_{1 \leq i \leq n} \frac{a_i}{p_i}$$