Utility maximization perfect complements

Problem Statement

Consider a consumer who derives utility from consuming two goods, x_1 and x_2. The consumer's utility function is given by:

$$U(x_1, x_2) = \min\{ax_1, bx_2\}, \tag{1}$$

where a and b are positive constants. The consumer has a budget constraint given by:

$$p_1 x_1 + p_2 x_2 = M, \tag{2}$$

where p_1 and p_2 are the prices of good 1 and good 2, respectively, and M is the consumer's income. Find the optimal consumption bundle and the optimal utility for the consumer.

Solution

Optimal consumption bundle

Since the utility function represents perfect complements, the consumer will consume the goods in fixed proportions determined by the constants a and b. The ratio of consumption for the two goods must be:

$$\frac{x_1}{x_2} = \frac{b}{a}. \tag{3}$$

Now, we substitute the expression for the consumption ratio back into the budget constraint:

$$p_1 \left(\frac{b}{a}\right) x_2 + p_2 x_2 = M. \tag{4}$$

Solving for x_2, we find:

$$x_2^* = \frac{aM}{abp_1 + a^2 p_2}. \tag{5}$$

Now, we can find the optimal x_1^* using the expression for the consumption ratio:

$$x_1^* = \frac{b}{a} x_2^* = \frac{bM}{abp_1 + a^2 p_2}. \tag{6}$$

Optimal utility

Finally, we substitute the optimal consumption bundle into the utility function to find the optimal utility:

$$U^* = \min\{ax_1^*, bx_2^*\} = \min\left\{a\frac{bM}{abp_1 + a^2 p_2}, b\frac{aM}{abp_1 + a^2 p_2}\right\}. \tag{7}$$

Since the goods are perfect complements, the consumer will always consume them in equal proportions, so both terms in the min function will be equal:

$$U^* = \frac{abM}{abp_1 + a^2 p_2}. \tag{8}$$

Thus, the optimal consumption bundle for the consumer is $x_1^* = \frac{bM}{abp_1 + a^2 p_2}$ and $x_2^* = \frac{aM}{abp_1 + a^2 p_2}$, and the optimal utility is $U^* = \frac{abM}{abp_1 + a^2 p_2}$.