Derivatives of implicit systems of equations

1. Compute the derivatives $\dfrac{dy}{dx}$ and $\dfrac{dz}{dx}$

$$\begin{cases} x^3 + 2y^3 - 3z - 13 = 0 \\ x - 6y + z^3 + 5 = 0 \end{cases}$$

2. Compute the derivatives $\dfrac{\partial u}{\partial x}$, $\dfrac{\partial u}{\partial y}$, $\dfrac{\partial v}{\partial x}$ and $\dfrac{\partial v}{\partial y}$

$$\begin{cases} x^2 - y^2 - u^3 + v^2 + 4 = 0 \\ 2xy + y^2 - 2u^2 + 3v^4 + 8 = 0 \end{cases}$$

Solution

1. Fully differentiate the two equations:

$$\begin{cases} 3x^2dx + 6y^2dy - 3dz = 0 \\ dx - 6dy + 3z^2dz = 0 \end{cases}$$

Separate dependent from independent variables:

$$\begin{cases} 6y^2dy - 3dz = -3x^2dx \\ -6dy + 3z^2dz = -dx \end{cases}$$

$$\begin{cases} 6y^2\dfrac{dy}{dx} - 3\dfrac{dz}{dx} = -3x^2 \\ -6\dfrac{dy}{dx} + 3z^2\dfrac{dz}{dx} = -1 \end{cases}$$

Express the system in matrix form:

$$\begin{pmatrix} 6y^2 & -3 \\ -6 & 3z^2 \end{pmatrix} \begin{pmatrix} \dfrac{dy}{dx} \\ \dfrac{dz}{dx} \end{pmatrix} = \begin{pmatrix} -3x^2 \\ -1 \end{pmatrix}$$

Solve using Cramer's method:

$$\frac{dy}{dx} = \frac{\begin{vmatrix} -3x^2 & -3 \\ -1 & 3z^2 \end{vmatrix}}{\begin{vmatrix} 6y^2 & -3 \\ -6 & 3z^2 \end{vmatrix}} = \frac{-9x^2z^2 - 3}{18z^2y^2 - 18} = -\frac{9x^2z^2 + 3}{18z^2y^2 - 18}$$

$$\frac{dz}{dx} = \frac{\begin{vmatrix} 6y^2 & -3x^2 \\ -6 & -1 \end{vmatrix}}{\begin{vmatrix} 6y^2 & -3 \\ -6 & 3z^2 \end{vmatrix}} = \frac{-6y^2 - 18x^2}{18z^2y^2 - 18} = -\frac{6y^2 + 18x^2}{18z^2y^2 - 18}$$

2. Fully differentiate the equations:

$$\begin{cases} 2xdx - 2ydy - 3u^2du + 2vdv = 0 \\ 2ydx + (2x + 2y)dy - 4udu + 12v^3dv = 0 \end{cases}$$

Separate dependent from independent variables:

$$\begin{cases} -3u^2du + 2vdv = -2xdx + 2ydy \\ -4udu + 12v^3dv = -2ydx - (2x + 2y)dy \end{cases}$$

First, compute derivatives with respect to x, thus set $dy = 0$:

$$\begin{cases} -3u^2\dfrac{du}{dx} + 2v\dfrac{dv}{dx} = -2x \\ -4u\dfrac{du}{dx} + 12v^3\dfrac{dv}{dx} = -2y \end{cases}$$

Express the system in matrix form:

$$\begin{pmatrix} -3u^2 & 2v \\ -4u & 12v^3 \end{pmatrix} \begin{pmatrix} \dfrac{du}{dx} \\ \dfrac{dv}{dx} \end{pmatrix} = \begin{pmatrix} -2x \\ -2y \end{pmatrix}$$

Solve using Cramer's method:

$$\frac{\partial u}{\partial x} = \frac{\begin{vmatrix} -2x & 2v \\ -2y & 12v^3 \end{vmatrix}}{\begin{vmatrix} -3u^2 & 2v \\ -4u & 12v^3 \end{vmatrix}} = \frac{-24xv^3 + 4vy}{-36u^2v^3 + 8vu}$$

$$\frac{\partial v}{\partial x} = \frac{\begin{vmatrix} -3u^2 & -2x \\ -4u & -2y \end{vmatrix}}{\begin{vmatrix} -3u^2 & 2v \\ -4u & 12v^3 \end{vmatrix}} = \frac{6u^2y - 8xu}{-36u^2v^3 + 8vu}$$

Next, consider the system with $dx = 0$:

$$\begin{cases} -3u^2\dfrac{du}{dy} + 2v\dfrac{dv}{dy} = 2y \\ -4u\dfrac{du}{dy} + 12v^3\dfrac{dv}{dy} = -2x - 2y \end{cases}$$

Express the system in matrix form:

$$\begin{pmatrix} -3u^2 & 2v \\ -4u & 12v^3 \end{pmatrix} \begin{pmatrix} \dfrac{du}{dy} \\ \dfrac{dv}{dy} \end{pmatrix} = \begin{pmatrix} 2y \\ -2x - 2y \end{pmatrix}$$

Solve using Cramer's method:

$$\frac{\partial u}{\partial y} = \frac{\begin{vmatrix} 2y & 2v \\ -2x - 2y & 12v^3 \end{vmatrix}}{\begin{vmatrix} -3u^2 & 2v \\ -4u & 12v^3 \end{vmatrix}} = \frac{-24yv^3 + 4vy}{-36u^2v^3 + 8vu}$$

$$\frac{\partial v}{\partial y} = \frac{\begin{vmatrix} -3u^2 & 2y \\ -4u & -2x - 2y \end{vmatrix}}{\begin{vmatrix} -3u^2 & 2v \\ -4u & 12v^3 \end{vmatrix}} = \frac{6u^2x + 8yu}{-36u^2v^3 + 8vu}$$