Directional Derivatives

1. Calculate by definition the directional derivative of $f(x, y) = 3x^2 + 2y$ at $(x_0, y_0) = (-1, 1)$ for:
   (a) $U = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
   (b) $U = (1, 0)$
   (c) $U = (0, 1)$
2. Calculate the partial derivatives of f at the point $(x_0, y_0) = (-1, 1)$. Compare with the result obtained earlier.
3. Calculate the gradient vector of function f.
4. Verify the results obtained in the first part using the calculation formula.
5. Calculate the maximum directional derivative and the unit vector for which this condition is met.

Solutions

1. (a) First, we need to check if the vector is a unit vector by calculating its norm:

$$\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}=\sqrt{\frac{1}{4}+\frac{3}{4}}=\sqrt{\frac{4}{4}}=1$$

Since the vector is a unit vector, we can perform the calculation with the following formula:

$$D_{\vec{v}}f=\lim_{h\to 0}\frac{f(x+v_xh,y+v_yh)-f(x,y)}{h}$$

Where $v_x=\frac{1}{2}$ and $v_y=\frac{\sqrt{3}}{2}$ With point $(-1,1)$

$$\lim_{h\to 0}\frac{3(-1+\frac{1}{2}h)^2+2(1+\frac{\sqrt{3}}{2}h)-(3+2)}{h}$$

$$\lim_{h\to 0}\frac{3(1-h+\frac{h^2}{4})+2+\sqrt{3}h-5}{h}=\lim_{h\to 0}\frac{3-3h+\frac{3h^2}{4}+2+\sqrt{3}h-5}{h}$$

$$\lim_{h\to 0}\frac{h(-3+\frac{3h}{4}+\sqrt{3})}{h}=-3+\sqrt{3}$$

(b) We evaluate the following vector $(1;0)$

$$\sqrt{1^2+0^2}=1$$

The vector is a unit vector, so we proceed with the calculation:

$$\lim_{h\to 0}\frac{3(-1+1h)^2+2(1+0h)-(3+2)}{h}=\lim_{h\to 0}\frac{3(1-2h+h^2)+2-3-2}{h}$$

$$\lim_{h\to 0}\frac{3-6h+3h^2-3}{h}=\lim_{h\to 0}\frac{-6+3h}{1}=-6$$

(c) Now with the following vector: $(0;1)$

$$\sqrt{0^2+1^2}=1$$

Being a unit vector, we calculate as before:

$$\lim_{h\to 0}\frac{3(-1+0h)^2+2(1+1h)-(3+2)}{h}=\lim_{h\to 0}\frac{3+2+2h-5}{h}=2$$

2. We calculate the partial derivatives by definition:

$$f'_x=\lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}=\lim_{h\to 0}\frac{3(x+h)^2+2y-3x^2-2y}{h}$$

$$f'_x=\lim_{h\to 0}\frac{3(x^2+2hx+h^2)-3x^2}{h}=\lim_{h\to 0}\frac{6hx+3h^2}{h}=-6$$

Now we calculate the other partial derivative by definition:

$$f'_y=\lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}=\lim_{h\to 0}\frac{3x^2+2(y+h)-3x^2-2y}{h}$$

$$\lim_{h\to 0}\frac{3x^2+2y+2h-3x^2-2y}{h}=2$$

The partial derivatives are equal to the directional derivatives $(0,1)$ and $(0,1)$ since these vectors mean moving only in terms of x and y.

3. The gradient vector is given by the first-order partial derivatives:

$$\nabla f(x,y) = (f'_x, f'_y) = (6x, 2)$$

4. Without using the definition, we can calculate the directional derivatives as follows:

$$D_{\vec{v}} f = f'_x v_x + f'_y v_y$$

For the first vector $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ with $f'_x = -6$ and $f'_y = 2$:

$$D_{\vec{v}} f = -6\frac{1}{2} + 2\frac{\sqrt{3}}{2} = -3 + \sqrt{3}$$

Without using the definition, we can calculate the directional derivatives in the following way:

$$D_{\vec{v}} f = f'_x v_x + f'_y v_y = 6x\frac{1}{2} + 2\frac{\sqrt{3}}{2}$$

Evaluating at the point:

$$-3 + \sqrt{3}$$

For the other two vectors:

$$D_{\vec{v}} f = f'_x v_x + f'_y v_y = 6x \cdot 1 + 2 \cdot 0 = 6x = -6$$

$$D_{\vec{v}} f = f'_x v_x + f'_y v_y = 6x \cdot 0 + 2 = 2$$

5. The maximum value of the directional derivative at the point $(-1, 1)$ is the norm of the gradient vector evaluated at that point.

$$\sqrt{(6x)^2 + 2^2} = \sqrt{36x^2 + 4} = \sqrt{40}$$

And the unit vector then is the gradient vector evaluated at the point divided by the norm that we obtained before:

$$U = \left(-\frac{6}{\sqrt{40}}, \frac{2}{\sqrt{40}}\right)$$