Marshallian demands for a 3-goods Stone–Geary utility function

Consider a three-good setting in which a consumer has a Stone–Geary utility function

$$u(x) = (x_1 - b_1)^{\alpha}(x_2 - b_2)^{\beta}(x_3 - b_3)^{\gamma}$$

where b_1, b_2, $b_3 > 0$ represent the minimum consumption levels of goods 1, 2, and 3 necessary for survival (e.g., calories, water, shelter), and where $\alpha, \beta, \gamma > 0$ are parameters (not necessarily summing to 1) The consumer faces prices p_1, p_2, $p_3 > 0$ and has income $m > 0$

(1) Find the consumer's Marshallian (Walrasian) demand functions for x_1, x_2, and x_3

(2) Find the indirect utility function

Solution

(1) Marshallian demand functions

Define the net (or "surplus") consumption of each good as

$$\tilde{x}_i = x_i - b_i \quad i = 1, 2, 3$$

so that the utility function transforms into

$$u(\tilde{x}) = \tilde{x}_1^{\alpha} \tilde{x}_2^{\beta} \tilde{x}_3^{\gamma}$$

which is a standard Cobb–Douglas function The budget constraint is

$$p_1 x_1 + p_2 x_2 + p_3 x_3 = m$$

Substituting $x_i = \tilde{x}_i + b_i$ for each i, we obtain

$$p_1(\tilde{x}_1 + b_1) + p_2(\tilde{x}_2 + b_2) + p_3(\tilde{x}_3 + b_3) = m$$

which simplifies to

$$p_1\tilde{x}_1 + p_2\tilde{x}_2 + p_3\tilde{x}_3 = m - (p_1 b_1 + p_2 b_2 + p_3 b_3)$$

Define the adjusted income

$$\tilde{m} = m - (p_1 b_1 + p_2 b_2 + p_3 b_3)$$

Thus, the consumer's problem reduces to

$$\max_{\tilde{x}_1, \tilde{x}_2, \tilde{x}_3} \quad \tilde{x}_1^{\alpha} \tilde{x}_2^{\beta} \tilde{x}_3^{\gamma} \quad \text{subject to} \quad p_1\tilde{x}_1 + p_2\tilde{x}_2 + p_3\tilde{x}_3 = \tilde{m}$$

For a Cobb–Douglas utility function of the form $\tilde{x}_1^{\alpha} \tilde{x}_2^{\beta} \tilde{x}_3^{\gamma}$ with arbitrary exponents, the standard solution yields the optimal net demands

$$\tilde{x}_i^* = \frac{a_i}{\alpha + \beta + \gamma} \frac{\tilde{m}}{p_i}, \quad i = 1, 2, 3$$

where $a_1 = \alpha$, $a_2 = \beta$, and $a_3 = \gamma$

Returning to the original variables, the Marshallian demand functions are

$$x_1^* = \frac{\alpha\,(m - p_1 b_1 - p_2 b_2 - p_3 b_3)}{(\alpha + \beta + \gamma) p_1} + b_1$$

$$x_2^* = \frac{\beta\,(m - p_1 b_1 - p_2 b_2 - p_3 b_3)}{(\alpha + \beta + \gamma) p_2} + b_2$$

$$x_3^* = \frac{\gamma\,(m - p_1 b_1 - p_2 b_2 - p_3 b_3)}{(\alpha + \beta + \gamma) p_3} + b_3$$

(2) Indirect utility function

To obtain the indirect utility function, substitute the optimal net demands into the utility function

$$u^* = (\tilde{x}_1^*)^{\alpha} (\tilde{x}_2^*)^{\beta} (\tilde{x}_3^*)^{\gamma} = \left(\frac{\alpha\,\tilde{m}}{(\alpha + \beta + \gamma) p_1} \right)^{\alpha} \left(\frac{\beta\,\tilde{m}}{(\alpha + \beta + \gamma) p_2} \right)^{\beta} \left(\frac{\gamma\,\tilde{m}}{(\alpha + \beta + \gamma) p_3} \right)^{\gamma}$$

which simplifies to

$$V(m, p_1, p_2, p_3) = \frac{\alpha^{\alpha} \beta^{\beta} \gamma^{\gamma}}{(\alpha + \beta + \gamma)^{\alpha+\beta+\gamma}\, p_1^{\alpha} p_2^{\beta} p_3^{\gamma}} \left(m - (p_1 b_1 + p_2 b_2 + p_3 b_3)\right)^{\alpha+\beta+\gamma}$$