# Marshallian demands and indirect utility

Consider a consumer with the utility function

$$u(q_1, q_2) = q_1 q_2 + q_1^2 q_2^2$$

and facing the budget constraint

$$p_1 q_1 + p_2 q_2 = m, \quad q_1, q_2 \geq 0$$

where $p_1, p_2 > 0$ and $m > 0$

**(1) Find the consumer's Marshallian demand functions for $q_1$ and $q_2$**

**(2) Find the indirect utility function**

# Solution

## (1) Marshallian demand functions

The consumer's problem is

$$\max_{q_1,q_2} \ u(q_1,q_2) = q_1q_2 + q_1^2q_2^2$$

subject to

$$p_1q_1 + p_2q_2 = m$$

**Setting up the Lagrangian**

$$\mathcal{L} = q_1q_2 + q_1^2q_2^2 + \lambda\Big(m - p_1q_1 - p_2q_2\Big)$$

**First-order conditions**
Taking the derivative with respect to $q_1$

$$\frac{\partial \mathcal{L}}{\partial q_1} = q_2 + 2q_1q_2^2 - \lambda p_1 = 0$$

Taking the derivative with respect to $q_2$

$$\frac{\partial \mathcal{L}}{\partial q_2} = q_1 + 2q_1^2q_2 - \lambda p_2 = 0$$

To eliminate $\lambda$, divide the first equation by the second

$$\frac{q_2 + 2q_1q_2^2}{q_1 + 2q_1^2q_2} = \frac{\lambda p_1}{\lambda p_2} = \frac{p_1}{p_2}$$

Factor $q_2$ in the numerator and $q_1$ in the denominator

$$\frac{q_2(1+2q_1q_2)}{q_1(1+2q_1q_2)} = \frac{q_2}{q_1} = \frac{p_1}{p_2}$$

Thus, the ratio of optimal quantities is

$$q_2 = \frac{p_1}{p_2}q_1$$

**Substituting into the budget constraint**

$$p_1q_1 + p_2q_2 = m$$

$$p_1q_1 + p_2\left(\frac{p_1}{p_2}q_1\right) = m$$

$$p_1q_1 + p_1q_1 = 2p_1q_1 = m$$

Solving for $q_1$

$$q_1^* = \frac{m}{2p_1}$$

From the ratio relation

$$q_2^* = \frac{m}{2p_2}$$

## (2) Indirect utility function

Substituting $q_1^*$ and $q_2^*$ into the utility function

$$u(q_1^*, q_2^*) = \left(\frac{m}{2p_1}\right)\left(\frac{m}{2p_2}\right) + \left(\frac{m}{2p_1}\right)^2\left(\frac{m}{2p_2}\right)^2$$

Computing each term

$$q_1^* q_2^* = \frac{m^2}{4p_1p_2}$$

$$(q_1^* q_2^*)^2 = \frac{m^4}{16p_1^2p_2^2}$$

Thus, the maximum utility attainable is

$$V(m, p_1, p_2) = \frac{m^2}{4p_1p_2} + \frac{m^4}{16p_1^2p_2^2}$$