Exchange Economy with Externalities and Complex Utility Functions

Consider a pure exchange economy with two agents and two goods. Let x_{ij} be the consumption of good i by agent j

The agents' utility functions are:

$$u_1(x_{11}, x_{21}) = \min\{\, x_{11} + 3x_{21},\ 4x_{11} + x_{21}\}$$

$$u_2(x_{12}, x_{22}) = 3x_{12} + x_{21}$$

The initial endowments are:

$$\omega_1 = (0,\, \beta) \quad \text{and} \quad \omega_2 = (1,\, 0)$$

(a) Find the set of Pareto-optimal allocations in terms of β

(b) Solve the utility maximization problem for both agents to determine their demands as functions of prices and β

Solution

(a) Pareto Optimal Allocations

First, observe that the total endowments are

$$x_{11} + x_{12} = 1 \quad \text{and} \quad x_{21} + x_{22} = \beta$$

so any feasible allocation must satisfy those equations and non-negativity of consumptions

Agent 1's utility:

$$u_1(x_{11}, x_{21}) = \min\{\, x_{11} + 3x_{21},\ 4x_{11} + x_{21}\}$$

Agent 2's utility:

$u_2(x_{12}, x_{22}) = 3x_{12} + x_{21}$ (note that x_{21} is Agent 1's consumption of good 2 according to the problem statement)

Checking Pareto improvements from the initial endowment

The initial endowment is

$$\omega_1 = (0, \beta), \quad \omega_2 = (1, 0)$$

which implies the allocation

$$(x_{11}, x_{21}) = (0, \beta), \quad (x_{12}, x_{22}) = (1, 0)$$

Under this allocation

$$u_1(0, \beta) = \min\{\, 0 + 3\beta,\ 4 \cdot 0 + \beta\} = \min\{3\beta, \beta\} = \beta \quad (\text{assuming } \beta > 0)$$

$$u_2(1, 0) = 3 \cdot 1 + (\text{Agent 1 consumes } \beta \text{ of good 2}) = 3 + \beta$$

Hence, at the initial endowment,

$$u_1 = \beta \quad \text{and} \quad u_2 = 3 + \beta$$

A direct examination shows that any attempt to reallocate goods in a way that increases one agent's utility strictly ends up decreasing the other's utility. For example:

- Giving good 1 from Agent 2 to Agent 1 reduces x_{12} and thus reduces $3x_{12}$ in u_2
- Giving good 2 from Agent 1 to Agent 2 reduces x_{21}, which directly appears as $+x_{21}$ in u_2, so that also hurts Agent 2
- Interior reallocations end up lowering either u_1 or u_2 below their initial values

Therefore, the initial endowment itself is already Pareto optimal

Generalizing to all possible reallocations

To see the full range of Pareto-optimal allocations, consider a general allocation $(x_{11}, x_{21}; x_{12}, x_{22})$ subject to

$$x_{11} + x_{12} = 1 \quad x_{21} + x_{22} = \beta \quad x_{ij} \geq 0$$

We have

$$u_1(x_{11}, x_{21}) = \min\{\, x_{11} + 3x_{21},\ 4x_{11} + x_{21}\} \quad \text{and} \quad u_2(x_{12}, x_{21}) = 3\, x_{12} + x_{21}$$

(i) All of good 2 goes to Agent 1. In $u_2 = 3\, x_{12} + x_{21}$, the partial derivative with respect to x_{21} is $+1$, so Agent 2 strictly prefers that any amount of good 2 be in the hands of Agent 1 (since it appears positively in u_2). Meanwhile, Agent 1 is not harmed if x_{21} is larger, because x_{21} also enters positively in both expressions inside the $\min\{\}$ of u_1. Consequently, any amount of good 2 given to Agent 2 can be reallocated to Agent 1 to strictly increase u_2 without lowering u_1. Thus Pareto efficiency requires

$$x_{21} = \beta \quad \text{and} \quad x_{22} = 0$$

(ii) Splitting good 1. Since $x_{21} = \beta$ is fixed for Pareto efficiency, the only remaining choice is how to split the total of 1 unit of good 1 between the two agents. Let $t \in [0,1]$ be the fraction of good 1 that goes to Agent 1:

$$x_{11} = t \quad x_{12} = 1 - t$$

so the allocation is

$$(x_{11}, x_{21};\, x_{12}, x_{22}) = (t, \beta;\, 1-t, 0)$$

Their utilities become

$$u_1(t, \beta) = \min\{\, t + 3\beta,\ 4t + \beta\}$$

$$u_2(1-t, \beta) = 3\,(1-t) + \beta = 3 + \beta - 3t$$

No further reallocation of good 2 is beneficial because of the positive effect on both agents of having $x_{21} = \beta$. Thus the entire Pareto frontier is spanned by $t \in [0,1]$.

Hence, for any $\beta > 0$, the full set of Pareto-optimal allocations is $\{\,(t, \beta;\, 1-t, 0) : 0 \le t \le 1\}$, and the initial endowment corresponds to $t = 0$.

(b) Utility Maximization and Demands

Agent 2's problem

Agent 2 has the endowment $(1, 0)$, so their income is

$$I_2 = p_1 \cdot 1 + p_2 \cdot 0 = p_1$$

Agent 2 maximizes

$$u_2(x_{12}, x_{22}) = 3\,x_{12} + x_{21}$$

subject to

$$p_1\,x_{12} + p_2\,x_{22} \le p_1, \quad x_{12}, x_{22} \ge 0$$

Notice that x_{21} is *not* chosen by Agent 2; it is Agent 1's consumption of good 2. Therefore, for a standard Walrasian demand, only the term $3\,x_{12}$ is actually controlled by Agent 2. From Agent 2's perspective, x_{21} is a constant (an externality determined by Agent 1)

Focusing on the part they *do* control, Agent 2 solves

$$\max_{x_{12},\, x_{22} \ge 0} 3\,x_{12}$$

subject to

$$p_1\,x_{12} + p_2\,x_{22} \le p_1$$

This is a linear program in (x_{12}, x_{22}). Since only x_{12} yields positive marginal utility (equal to 3), Agent 2 spends the entire budget on good 1. Hence

$$x_{12} = \frac{p_1}{p_1} = 1, \quad x_{22} = 0$$

Thus, the demand for Agent 2 is

$$\text{Agent 2's Demand: } (x_{12}^*,\, x_{22}^*) = (1,\, 0)$$

Agent 1's problem

We want to solve the following utility maximization problem:

$$\max_{x_{11},\, x_{21} \ge 0} \min\{\, x_{11} + 3x_{21},\ 4x_{11} + x_{21}\} \quad \text{subject to} \quad p_1 x_{11} \ + \ p_2 x_{21} \ \le \ p_2\,\beta$$

Step 1: Check for an interior solution

For an interior solution that makes both expressions inside the $\min\{\cdot,\cdot\}$ equal, we set:

$$x_{11} + 3x_{21} \;=\; 4x_{11} + x_{21} \;\implies\; 3x_{21} - x_{21} \;=\; 4x_{11} - x_{11} \;\implies\; 2x_{21} \;=\; 3x_{11} \;\implies\; x_{21} \;=\; \frac{3}{2}\,x_{11}$$

Plug this relation into the budget constraint:

$$p_1\,x_{11} \;+\; p_2\,(\frac{3}{2}x_{11}) \;=\; x_{11}\left(p_1 \;+\; \frac{3}{2}\,p_2\right) \;\leq\; p_2\,\beta$$

At optimum (assuming an interior solution is indeed optimal), the budget is fully spent, so

$$x_{11} \;=\; \frac{p_2\,\beta}{p_1 \;+\; \frac{3}{2}p_2} \quad , \quad x_{21} \;=\; \frac{3}{2}\,\frac{p_2\,\beta}{p_1 \;+\; \frac{3}{2}p_2}$$

Step 2: Corner solutions

Because utility is the minimum of two linear expressions, we must also check corners where the consumer spends all income on a single good:

1. **Corner 1:** $x_{21} = 0$. Then
$$u_1 = \min\{\,x_{11},\;4x_{11}\} = x_{11}$$
and the budget constraint becomes $p_1\,x_{11} \leq p_2\,\beta$. Hence
$$x_{11} = \frac{p_2\,\beta}{p_1} \quad \text{and} \quad u_1 = \frac{p_2\,\beta}{p_1}$$

2. **Corner 2:** $x_{11} = 0$. Then
$$u_1 = \min\{\,3x_{21},\;x_{21}\} = x_{21}$$
(because $x_{21} \leq 3x_{21}$ for $x_{21} \geq 0$), and the budget constraint is $p_2\,x_{21} \leq p_2\,\beta$. Thus
$$x_{21} = \beta \quad \text{and} \quad u_1 = \beta$$

Step 3: Compare utilities

Let

$$U_{\text{int}} \;=\; \min\Big\{x_{11} + 3x_{21},\;4x_{11} + x_{21}\Big\} \;\text{(under the interior solution)} \;=\; \frac{11}{2}\,x_{11} \;=\; \frac{11\,p_2\,\beta}{2\,p_1 \;+\; 3\,p_2}$$

$$U_{\text{corner1}} \;=\; \frac{p_2\,\beta}{p_1} \quad , \quad U_{\text{corner2}} \;=\; \beta$$

By comparing U_{int} with U_{corner1} and U_{corner2}, we find the Marshallian demands are piecewise according to the ratio $\frac{p_1}{p_2}$:

$$\text{If } \frac{p_1}{p_2} < \frac{1}{3}, \quad \text{Corner 1 is better}$$

$$\text{If } \frac{1}{3} \;\leq\; \frac{p_1}{p_2} \;\leq\; 4, \quad \text{Interior solution is better}$$

$$\text{If } \frac{p_1}{p_2} > 4, \quad \text{Corner 2 is better}$$

Step 4: Final Marshallian demand

$$x_{11}^*(p_1, p_2, \beta) = \begin{cases} \dfrac{p_2\,\beta}{p_1} & \text{if } \dfrac{p_1}{p_2} < \dfrac{1}{3} \\ \dfrac{p_2\,\beta}{p_1 \ + \ \dfrac{3}{2}p_2} & \text{if } \dfrac{1}{3} \leq \dfrac{p_1}{p_2} \leq 4 \\ 0 & \text{if } \dfrac{p_1}{p_2} > 4 \end{cases}$$

$$x_{21}^*(p_1, p_2, \beta) = \begin{cases} 0 & \text{if } \dfrac{p_1}{p_2} < \dfrac{1}{3} \\ \dfrac{3}{2}\,\dfrac{p_2\,\beta}{p_1 + \dfrac{3}{2}p_2} & \text{if } \dfrac{1}{3} \leq \dfrac{p_1}{p_2} \leq 4 \\ \beta & \text{if } \dfrac{p_1}{p_2} > 4 \end{cases}$$

Therefore, the Marshallian demand for the agent with utility $\min\{x_{11} + 3x_{21},\, 4x_{11} + x_{21}\}$ is given by the above piecewise functions