Unconstrained and Constrained Optimization

Unconstrained Extrema[1]

First Order Conditions

The necessary conditions to find an extremum are:

- $f'x|_{x_0,y_0} = 0$
- $f'y|_{x_0,y_0} = 0$

Second Order Conditions

The Hessian matrix for a function of two variables is defined as:

$$H = \begin{pmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{pmatrix}$$

- if $|H|_{x_0,y_0}| > 0$, relative extremum
  - if $f''_{xx}(x_0, y_0) < 0$, relative maximum (negative definite Hessian)
  - if $f''_{xx}(x_0, y_0) > 0$, relative minimum (positive definite Hessian)
- if $|H|_{x_0,y_0}| < 0$, saddle point
- if $|H|_{x_0,y_0}| = 0$, ambiguous case, there may be a maximum, minimum, or no extrema.

Constrained Extrema

When maximizing or minimizing functions subject to a constraint, we construct the Lagrangian. Let's suppose we want to find the maximum or minimum of $f(x, y)$ subject to the following constraint: $g(x, y) = 0$. Then we construct the following function:

$$L(x, y, \lambda) = f(x, y) + \lambda[g(x, y)]$$

We get the first order conditions just as in the case of unconstrained extrema:

- $L'x = 0$
- $L'y = 0$
- $L'\lambda = 0$

And for the second order conditions, we construct the bordered Hessian:

$$\bar{H} = \begin{pmatrix} 0 & g'x & g'y \\ g'x & L''_{xx} & L''_{xy} \\ g'y & L''_{yx} & L''_{yy} \end{pmatrix}$$

We evaluate the determinant of the bordered Hessian at the point and follow these 3 conditions:

- if $|\bar{H}|_{x_0,y_0}| > 0$, maximum
- if $|\bar{H}|_{x_0,y_0}| < 0$, minimum
- if $|\bar{H}|_{x_0,y_0}| = 0$, ambiguous case, there may be a maximum, minimum, or no extrema.

[1] All these conditions refer to relative maxima or minima, either for unconstrained or constrained extrema.

Examples

Unconstrained Extrema 1

$$f(x, y) = z = xy + 1/x + 1/y$$

First order conditions:

$$f'x = y - x^{-2} = 0$$
$$f'y = x - y^{-2} = 0$$

Solving the first equation gives:

$$y = x^{-2}$$

Inserting into the second equation:

$$x - [x^{-2}]^{-2} = 0$$
$$x - x^4 = 0$$
$$x(1 - x^3) = 0$$

This leads me to the condition $x = 1$ or $x = 0$. However, $x = 0$ does not satisfy $y = x^{-2}$. But $x = 1$ leads to $y = 1^{-2} = 1$.

$$f(1, 1) = 3$$

Let's then evaluate the point $(1, 1, 3)$ to see if it satisfies the sufficient conditions for a maximum or minimum.

$$H|_{1,1} = \begin{pmatrix} 2x^{-3} & 1 \\ 1 & 2y^{-3} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

We calculate the determinant: $2 * 2 - 1 * 1 = 3 > 0$. So we are dealing with an extremum and as $f''_{xx}|_{1,1} = x^{-3} = 1 > 0$, we are dealing with a minimum.

Unconstrained Extrema 2

$$f(x, y) = z = (x - y)^4 + (y - 1)^4$$

I calculate the first order conditions:

$$f'x = 4(x - y)^3 = 0$$
$$f'y = -4(x - y)^3 + 4(y - 1)^3 = 0$$

From the first equation, I obtain that: $x = y$. Using this in the second equation:

$$-4(y - y)^3 + (y - 1)^3 = 0$$
$$-4(0)^3 + 4(y - 1)^3 = 0$$
$$(y - 1)^3 = 0$$

Then $y = 1$ and therefore $x = 1$. Moving to the second order conditions:

$$H|_{1,1} = \begin{pmatrix} 12(x - y)^2 & -12(x - y)^2 \\ -12(x - y)^2 & 12(x - y)^2 + 12(y - 1)^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

The determinant is 0. However, note that the function can never take negative values as it is the sum of two positive numbers raised to a power. The minimum value it can take is 0. The point we found is precisely $(1, 1, 0)$. Therefore, we are dealing with a minimum.

Constrained Extrema

$$f(x, y) = x^2 + y^2$$

Subject to:

$$x = y$$

I construct the Lagrangian:

$$L = x^2 + y^2 + \lambda(y - x)$$

I calculate the necessary conditions:

$$L'x = 2x - \lambda = 0$$

$$L'y = 2y + \lambda = 0$$

$$L'\lambda = y - x = 0$$

From the first two equations, I get the value of λ:

$$\lambda = 2x$$

$$\lambda = -2y$$

I equate and obtain: $-2y = 2x$, that is, $x = -y$. Replacing this in the third condition

$$y + y = 0$$

Therefore $2y = 0$, which is only satisfied if $y = 0$ and this leads me to $x = 0$. Moving to the second order condition:

$$\bar{H}_{(0,0)} = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

Calculating the determinant:

$$0(2 * 2 - 0 * 0) + 1(-1 * 2 - 2 * 1) + 1(-1 * 0 - 2 * 1) = -2 < 0$$

We are dealing with a constrained minimum.