Introduction to (Geometric) Clifford Algebras: Applications to classical and quantum relativistic mechanics

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Routines

Algebra of the Physical Space (APS)

Basics

Out[30]=

Scalar	$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
Vector	$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
Bivector	$\sigma_1 \sigma_2 = i \sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ $\sigma_2 \sigma_3 = i \sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ $\sigma_3 \sigma_1 = i \sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
PseudoScalar	$\sigma_{123} = \sigma_1 \sigma_2 \sigma_3 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$

Conjugations

Hermitian Conjugation

Out[31]=

Hermitian	$\sigma_0{}^\dagger = \sigma_0$ $\sigma_k{}^\dagger = \sigma_k$
AntiHermitian	$\sigma_{jk}{}^\dagger = -\sigma_{jk}$ $\sigma_{123}{}^\dagger = -\sigma_{123}$

Clifford Conjugation

Out[32]=

Scalar	$\bar{\sigma}_0 = \sigma_0$ $\bar{\sigma}_{123} = \sigma_{123}$
Vector	$\bar{\sigma}_k = -\sigma_k$ $\bar{\sigma}_{jk} = -\sigma_{jk}$

Extracting Scalar Part

In[33]:= $\texttt{CliffordConjugationSL2C}\left[\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right]$ // MatrixForm

Out[33]//MatrixForm=

$$\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

In[34]:= $\texttt{ScalarPart[x\_]} := \dfrac{\texttt{x + CliffordConjugationSL2C[x]}}{2}$

In[35]:= $\texttt{VectorPart[x\_]} := \dfrac{\texttt{x - CliffordConjugationSL2C[x]}}{2}$

In[36]:= $\texttt{ScalarPart}[a\,\sigma_0 + b\,\sigma_1]$ // MatrixForm

Out[36]//MatrixForm=

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$

In[37]:= $\texttt{VectorPart}[a\,\sigma_0 + b\,\sigma_1]$ // MatrixForm

Out[37]//MatrixForm=

$$\begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}$$

Note 1 : The trace almost reproduces the role of the scalar part

In[38]:= $\frac{1}{2}\,\texttt{Tr}[a\,\sigma_0 + b\,\sigma_1]\,\sigma_0$ // MatrixForm

Out[38]//MatrixForm=

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$

Note 2 : The Clifford conjugation can be used to calculate the determinant

In[39]:=
```
Det@(a11 a12; a21 a22)
```
$$\mathrm{Det}@\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Out[39]= $-a_{12}\, a_{21} + a_{11}\, a_{22}$

In[40]:=
$$1/2\,\mathrm{Tr}\left[\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.\mathrm{CliffordConjugationSL2C}\left[\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right]\right] \mathbin{//} \mathrm{Expand}$$

Out[40]= $-a_{12}\, a_{21} + a_{11}\, a_{22}$

Note 3 : The Clifford conjugation is the product of the inverse with the determinant

In[41]:=
$$\mathrm{CliffordConjugationSL2C}\left[\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right] \mathbin{//} \mathrm{MatrixForm}$$

Out[41]//MatrixForm=
$$\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

In[42]:=
$$\mathrm{Det}\left[\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right] \mathrm{Inverse}\left[\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right] \mathbin{//} \mathrm{MatrixForm}$$

Out[42]//MatrixForm=
$$\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

Extracting Hermitian Part

In[43]:=
$$\mathrm{HermitianPart}[x\_] := \frac{x + x^{\dagger}}{2}$$

In[44]:=
$$\mathrm{AntiHermitianPart}[x\_] := \frac{x - x^{\dagger}}{2}$$

In[45]:=
```
Simplify[HermitianPart[ a σ0 + I b σ1 ],
  Assumptions → {Element[a, Reals], Element[b, Reals]}] // MatrixForm
```

Out[45]//MatrixForm=
$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$

In[46]:=
```
Simplify[AntiHermitianPart[ a σ0 + I b σ1 ],
  Assumptions → {Element[a, Reals], Element[b, Reals]}] // MatrixForm
```

Out[46]//MatrixForm=
$$\begin{pmatrix} 0 & i\, b \\ i\, b & 0 \end{pmatrix}$$

The same calculation can be performed using the bracket

In[47]:=
```
Simplify[
 ⟨a σ0 + I b σ1⟩"H"
 , Assumptions → {Element[a, Reals], Element[b, Reals]}]
```
Out[47]= {{a, 0}, {0, a}}

In[48]:=
```
Simplify[
 ⟨a σ0 + I b σ1⟩"A"
 , Assumptions → {Element[a, Reals], Element[b, Reals]}]
```
Out[48]= $\{\{0, i\, b\}, \{i\, b, 0\}\}$

Out[49]=

Scalar	$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\mathcal{S}$calar $\mathcal{H}$ermitian
Vector	$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\mathcal{V}$ector $\mathcal{H}$ermitian
Bivector	$\sigma_1 \sigma_2 = i\, \sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ $\sigma_2 \sigma_3 = i\, \sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ $\sigma_3 \sigma_1 = i\, \sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\mathcal{V}$ector $\mathcal{A}$nti$\mathcal{H}$ermitian
Pseudo Scalar	$\sigma_{123} = \sigma_1 \sigma_2 \sigma_3 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$	$\mathcal{S}$calar $\mathcal{A}$nti$\mathcal{H}$ermitian

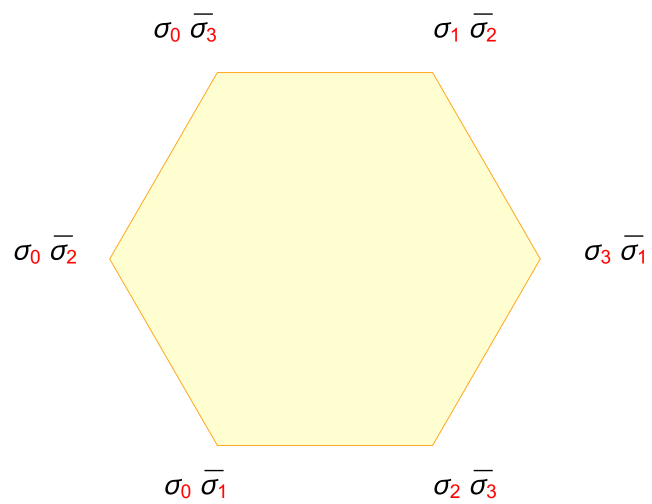
Paravector Space

Out[50]=

Scalar	$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\mathcal{S}$calar $\mathcal{H}$ermitian
Paravector	$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\mathcal{H}$ermitian
Biparavector	$\sigma_0\, \bar{\sigma}_1 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ $\sigma_0\, \bar{\sigma}_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ $\sigma_0\, \bar{\sigma}_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ $\sigma_1\, \bar{\sigma}_2 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$ $\sigma_2\, \bar{\sigma}_3 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$ $\sigma_3\, \bar{\sigma}_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\mathcal{V}$ector
Triparavector	$\sigma_1\, \bar{\sigma}_2\, \sigma_3 = \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix}$ $\sigma_2\, \bar{\sigma}_3\, \sigma_0 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$ $\sigma_3\, \bar{\sigma}_0\, \sigma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\sigma_0\, \bar{\sigma}_1\, \sigma_2 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$	$\mathcal{A}$nti$\mathcal{H}$ermitian
Pseudo Scalar	$\sigma_{123} = \sigma_0\, \bar{\sigma}_1\, \sigma_2\, \bar{\sigma}_3 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$	$\mathcal{S}$calar $\mathcal{A}$nti$\mathcal{H}$ermitian

Biparavector space = Lorentz Lie algebra = SL(2, C)

Out[51]=



Spacetime Paravector

Definition

The spacetime can be represented as a paravector

$$x = x^\mu \sigma_\mu . \tag{1}$$

Sometimes we write

$$x = x^0 \sigma_0 + \mathbf{x}, \tag{2}$$

where the bold symbols stand for standard three-dimensional vectors

In[52]:=
```
(x = x⁰ σ₀ + x¹ σ₁ + x² σ₂ + x³ σ₃) // MatrixForm
```

Symbolize: Warning: The box structure attempting to be symbolized has a similar or identical symbol already defined, possibly overriding previously symbolized box structure.

Symbolize: Warning: The box structure attempting to be symbolized has a similar or identical symbol already defined, possibly overriding previously symbolized box structure.

Out[52]//MatrixForm=

$$\begin{pmatrix} x^0 + x^3 & x^1 - i\, x^2 \\ x^1 + i\, x^2 & x^0 - x^3 \end{pmatrix}$$

In[53]:= $\frac{1}{2}$ Tr[x.σ_1]

Out[53]= x^1

Similarly, the proper velocity is

$$u = u^\mu \sigma_\mu$$

In[54]:= ($u = u^0\,\sigma_0 + u^1\,\sigma_1 + u^2\,\sigma_2 + u^3\,\sigma_3$) // MatrixForm

Symbolize: Warning: The box structure attempting to be symbolized has a similar or identical symbol already defined, possibly overriding previously symbolized box structure.

Symbolize: Warning: The box structure attempting to be symbolized has a similar or identical symbol already defined, possibly overriding previously symbolized box structure.

Out[54]//MatrixForm=

$$\begin{pmatrix} u^0 + u^3 & u^1 - i\,u^2 \\ u^1 + i\,u^2 & u^0 - u^3 \end{pmatrix}$$

The contraction of paravectors is accomplished as

In[55]:= $\frac{1}{2}$ Tr[x.$\bar{u}$] // Simplify

Out[55]= $u^0\,x^0 - u^1\,x^1 - u^2\,x^2 - u^3\,x^3$

In[56]:= $\frac{1}{2}$ Tr[u.$\bar{u}$] // Simplify

Out[56]= $u^{0\,2} - u^{1\,2} - u^{2\,2} - u^{3\,2}$

The spacetime position has 4 degrees of freedom but the proper velocity has only 3 because it must obey the shell-mass constrain

$$(u^0)^2 - (u^1)^2 - (u^2)^2 - (u^3)^2 = c^2 \tag{3}$$

also written as

$$\frac{1}{2}\,\mathrm{Tr}\,(u.\bar{u}) = c^2 \tag{4}$$

or

$$(u^0)^2 - \boldsymbol{u}^2 = c^2 \tag{5}$$

Lorentz Transformations for paravectors

The Lorentz Boost is defined as the square root of the proper velocity

$$B = \sqrt{\frac{u}{c}}\ , \tag{6}$$

where we note that B is Hermitian.

A useful formula for the square root is

$$B = \frac{u + c\,\sigma_0}{\sqrt{2\,c\,(c + u^0)}}\ . \tag{7}$$

The active Lorentz Boost is carried out by double side multiplication

$$x \to y = \ L\ x\ L^{\dagger}. \tag{8}$$

Considering that the Boost is a Hermitian Lorentz operator we have

$$x \to y = \ \ B\ x\ B. \tag{9}$$

Note : The Lorentz transformation for biparavectors is different and will be treated later.

In[57]:=
```
(BoostSL2C[u⁰ : _, u¹ : _, u² : _, u³ : _] = Module[{}, ((u⁰ σ₀ + u¹ σ₁ + u² σ₂ + u³ σ₃ ) +  c σ₀)/Sqrt[2 c ( c + u⁰)]]) //
  MatrixForm
```

Out[57]//MatrixForm=

$$\begin{pmatrix} \frac{c+u^0+u^3}{\sqrt{2}\sqrt{c\,(c+u^0)}} & \frac{u^1-i\,u^2}{\sqrt{2}\sqrt{c\,(c+u^0)}} \\ \frac{u^1+i\,u^2}{\sqrt{2}\sqrt{c\,(c+u^0)}} & \frac{c+u^0-u^3}{\sqrt{2}\sqrt{c\,(c+u^0)}} \end{pmatrix}$$

Recovering the proper velocity from the Boost to verify the formula (using the shell mass condition)

In[58]:=
```
Simplify[c BoostSL2C[u⁰, u¹, u², u³].BoostSL2C[u⁰, u¹, u², u³]];
U == Simplify[% /. {u¹² + u²² + u³² → u⁰² - c²}]
```

Out[59]= $U == \{\{u^0+u^3,\ u^1-i\,u^2\},\ \{u^1+i\,u^2,\ u^0-u^3\}\}$

The boost operator along x^3 is

In[60]:=
```
BoostSL2C[u⁰, 0, 0, u³] // MatrixForm
```

Out[60]//MatrixForm=

$$\begin{pmatrix} \frac{c+u^0+u^3}{\sqrt{2}\sqrt{c\,(c+u^0)}} & 0 \\ 0 & \frac{c+u^0-u^3}{\sqrt{2}\sqrt{c\,(c+u^0)}} \end{pmatrix}$$

The active boost of the coordinates is

In[61]:=
```
y = Simplify[BoostSL2C[u⁰, 0, 0, u³].x.BoostSL2C[u⁰, 0, 0, u³]]
```

Out[61]=

$$\left\{\left\{\frac{(c+u^0+u^3)^2\,(x^0+x^3)}{2\,c\,(c+u^0)},\ \frac{(c+u^0-u^3)\,(c+u^0+u^3)\,(x^1-i\,x^2)}{2\,c\,(c+u^0)}\right\},\ \left\{\frac{(c+u^0-u^3)\,(c+u^0+u^3)\,(x^1+i\,x^2)}{2\,c\,(c+u^0)},\ \frac{(c+u^0-u^3)^2\,(x^0-x^3)}{2\,c\,(c+u^0)}\right\}\right\}$$

The transformation of coordinates reads

In[184]:=
```
(boostedCoordinates = MapThread[Equal, {{y⁰, y¹, y², y³},
     Simplify[Simplify[1/2 Tr[y.σ#] & /@ {0, 1, 2, 3}] /. {u³² → u⁰² - c²}]}]) // TableForm
```

Out[184]//TableForm=

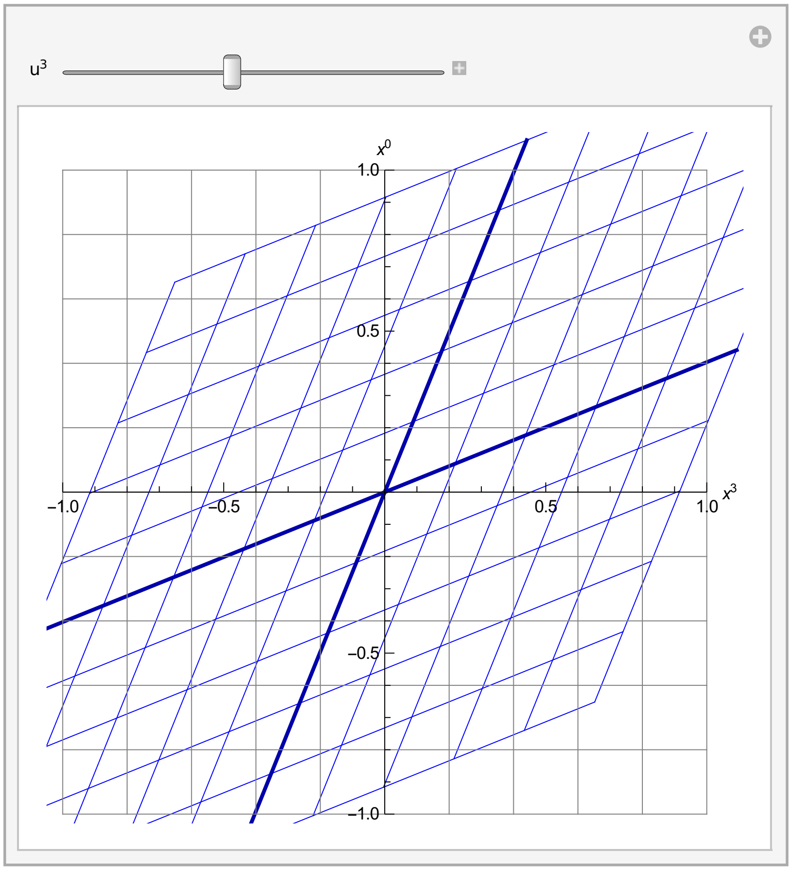
$$y^0 == \frac{u^0\,x^0+u^3\,x^3}{c}$$
$$y^1 == x^1$$
$$y^2 == x^2$$
$$y^3 == \frac{u^3\,x^0+u^0\,x^3}{c}$$

where y^0,y^1,y^2 and y^3 represent the boosted coordinates.

Spacetime diagram

The spacetime grid for a given velocity boost $\mathbf{u}^3$ (The gray grid represents the rest frame and the blue grid represents the boosted frame)

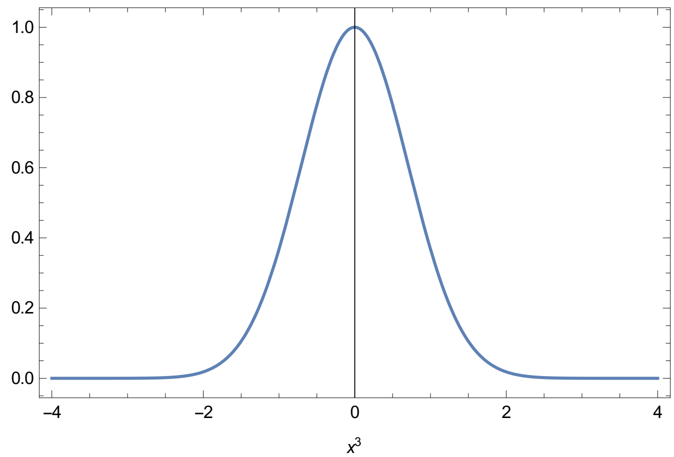
Out[64]=



Lorentz contraction

Let us consider the following Gaussian in the rest frame

In[210]:=
```
Plot[
 Exp[-(x³)²]
 , {x³, -4, 4}, Frame → True, FrameLabel → {"x³",}]
```

Out[210]=


expressing the rest frame coordinates in terms of the boosted coordinates

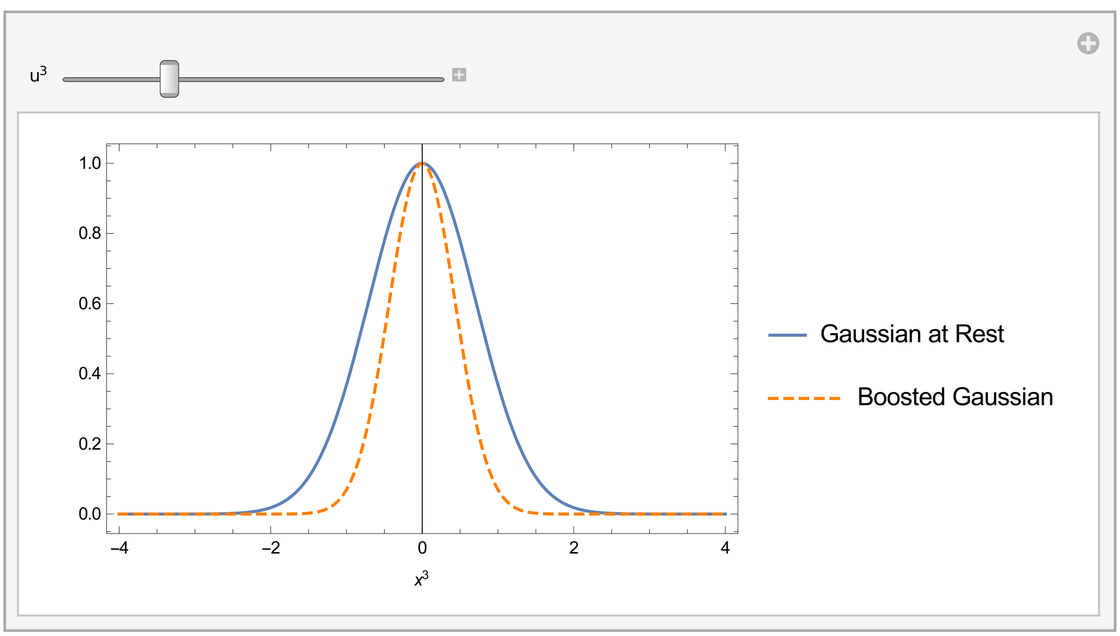
In[225]:=
```
Last[Solve[boostedCoordinates, {x⁰, x¹, x², x³}] /. {u³² → u⁰² - c²}]
```

Out[225]= $\left\{x^0 \to \frac{u^0\, y^0 - u^3\, y^3}{c},\ x^1 \to y^1,\ x^2 \to y^2,\ x^3 \to -\frac{u^3\, y^0 - u^0\, y^3}{c}\right\}$

Then, the boosted Gaussian is

In[226]:=
```
Exp[-(x³)²] //. Last[Solve[boostedCoordinates, {x⁰, x¹, x², x³}]] /. {u³² → u⁰² - c²}
```

Out[226]= $e^{-\frac{(u^3\, y^0 - u^0\, y^3)^2}{c^2}}$

Out[231]=


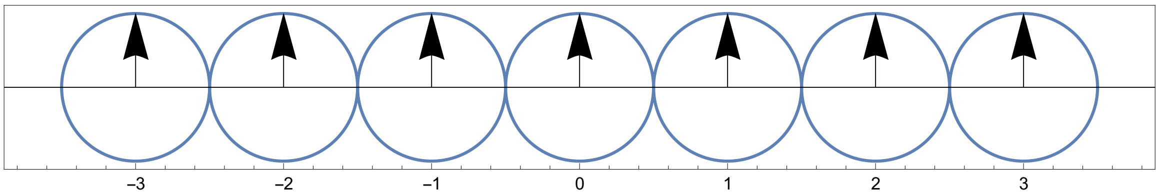
Array of clocks

A drawing of an array of clocks where the clocks in the rest frame are synchronized

The clocks are synchronized in the rest frame

In[379]:=
```
Show @@ Table[
  BoostedSpacetimeClock[u³ /. {u³ → 0}][0, z], {z, -3, 3, 1}]
```

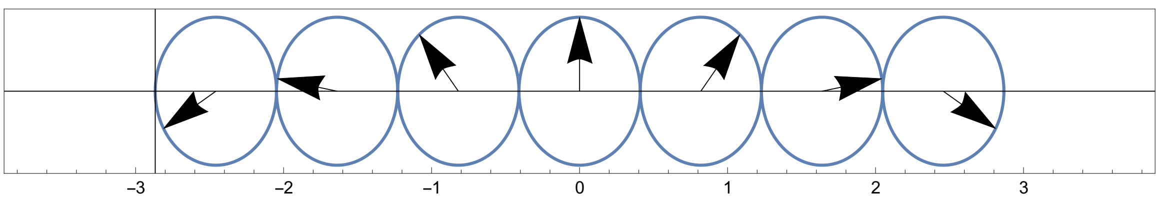
Out[379]=



This system is actively boosted and as a result the clocks are not synchronized anymore as seen by the observer in the original rest frame

In[385]:=
```
Show @@ Table[
  BoostedSpacetimeClock[u³ /. {u³ → 0.7}][0, z], {z, -3, 3, 1}]
```

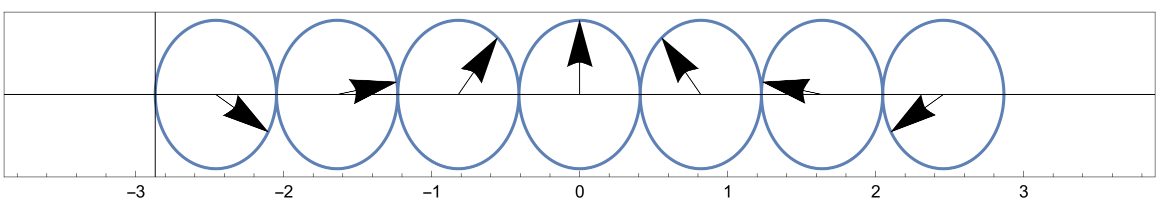
Out[385]=



Given that the clocks are synchronized in the rest frame, what is the perception of an observer in a frame moving with velocity u^3?
The answer follows:

In[384]:=
```
Show @@ Table[
  BoostedSpacetimeClock[u³ /. {u³ → -0.7}][0, z], {z, -3, 3, 1}]
```

Out[384]=



Paragradient

The paragradient is defined as

$\partial = \sigma_\mu \, \partial^\mu = \sigma_0 \, \partial_0 - \sigma_k \, \partial_k.$

The Clifford conjugated paragradient is

$\bar{\partial} = \bar{\sigma}_\mu \, \partial^\mu = \sigma_0 \, \partial_0 + \sigma_k \, \partial_k.$

In[76]:=
```
paraGradient = (σ0.∂x⁰# - σ1.∂x¹# - σ2.∂x²# - σ3.∂x³#) &
```
Out[76]= $\sigma_0.\partial_{x^0}\#1 - \sigma_1.\partial_{x^1}\#1 - \sigma_2.\partial_{x^2}\#1 - \sigma_3.\partial_{x^3}\#1$ &

In[77]:=
```
paraGradientBar = (σ0.∂x⁰# + σ1.∂x¹# + σ2.∂x²# + σ3.∂x³#) &
```
Out[77]= $\sigma_0.\partial_{x^0}\#1 + \sigma_1.\partial_{x^1}\#1 + \sigma_2.\partial_{x^2}\#1 + \sigma_3.\partial_{x^3}\#1$ &

Classical Electrodynamics

Electromagnetic Field

The electromagnetic field as a biparavector is

$$F = F^{\mu\nu} \langle \sigma_\mu \, \overline{\sigma_\mu} \rangle_V \tag{10}$$

which is expanded as

$$F = \mathbb{E} + i\, c\, \mathbb{B} = \sigma_1 \mathbb{E}_1 + \sigma_2 \mathbb{E}_2 + \sigma_3 \mathbb{E}_3 + i\, c\, (\sigma_1 \mathbb{B}_1 + \sigma_2 \mathbb{B}_2 + \sigma_3 \mathbb{B}_3) \tag{11}$$

The function that writes the electromagnetic paravector is

In[78]:=
```
EMF["SL2C"][{𝔼1 : _, 𝔼2 : _, 𝔼3 : _}, {𝔹1 : _, 𝔹2 : _, 𝔹3 : _}] =
  σ1 𝔼1 + σ2 𝔼2 + σ3 𝔼3 + ⅈ c (σ1 𝔹1 + σ2 𝔹2 + σ3 𝔹3);
```

For example we have

In[79]:=
```
(F = EMF["SL2C"][{𝔼1, 𝔼2, 𝔼3}, {𝔹1, 𝔹2, 𝔹3}] // Simplify) // MatrixForm
```
Out[79]//MatrixForm=

$$\begin{pmatrix} i\, c\, \mathbb{B}_3 + \mathbb{E}_3 & c\,(i\, \mathbb{B}_1 + \mathbb{B}_2) + \mathbb{E}_1 - i\, \mathbb{E}_2 \\ i\, c\, \mathbb{B}_1 - c\, \mathbb{B}_2 + \mathbb{E}_1 + i\, \mathbb{E}_2 & -i\, c\, \mathbb{B}_3 - \mathbb{E}_3 \end{pmatrix}$$

In[80]:=
```
EMFUpIndexComponents::usage =
  "EMFUpIndexComponents[ X ], writes the biparavector electromagnetic field
    X as the components of the contravariant tensor components Fμν ";

EMFUpIndexComponents[EMF_] :=
 If[Dimensions[EMF] == {2, 2},
  Simplify[
   Table[
     1 / 2 Tr[ HermitianPart[ EMF.Inverse[(σμ).σ̄ν]]]
     , {μ, 0, 3}, {ν, 0, 3}] /. Conjugate → ForceConjugate
  ]
 ]
```

The (upper index) contravariant tensor components are

In[82]:= `MatrixForm /@ (F → EMFUpIndexComponents[ F]) // CompactFormat`

Out[82]=
$$\begin{pmatrix} \mathrm{i}\, c\, \mathbb{B}_3 + \mathbb{E}_3 & c\,(\mathrm{i}\,\mathbb{B}_1 + \mathbb{B}_2) + \mathbb{E}_1 - \mathrm{i}\,\mathbb{E}_2 \\ \mathrm{i}\, c\,\mathbb{B}_1 - c\,\mathbb{B}_2 + \mathbb{E}_1 + \mathrm{i}\,\mathbb{E}_2 & -\mathrm{i}\, c\,\mathbb{B}_3 - \mathbb{E}_3 \end{pmatrix} \to \begin{pmatrix} 0 & -\mathbb{E}_1 & -\mathbb{E}_2 & -\mathbb{E}_3 \\ \mathbb{E}_1 & 0 & -c\,\mathbb{B}_3 & c\,\mathbb{B}_2 \\ \mathbb{E}_2 & c\,\mathbb{B}_3 & 0 & -c\,\mathbb{B}_1 \\ \mathbb{E}_3 & -c\,\mathbb{B}_2 & c\,\mathbb{B}_1 & 0 \end{pmatrix}$$

Vector potential

The electromagnetic field from the vector potential is

$$F = c \left\langle \partial \bar{A} \right\rangle_V = c\, \mathrm{VectorPart}\left[\partial \bar{A}\right] \tag{12}$$

Defining the paravector potential

In[83]:= $A = A^0[x^0, x^1, x^2, x^3]\,\sigma_0 + A^1[x^0, x^1, x^2, x^3]\,\sigma_1 + A^2[x^0, x^1, x^2, x^3]\,\sigma_2 + A^3[x^0, x^1, x^2, x^3]\,\sigma_3;$

The electromagnetic field is calculated as

In[84]:= $\mathcal{F} = c\,\mathrm{VectorPart}\left[\partial \bar{A}\right]$ `// Simplify;`

Calculating the upper tensor components

In[85]:= `EMFUpIndexComponents[` $\mathcal{F}$ `] // CompactFormat // MatrixForm`

Out[85]//MatrixForm=
$$\begin{pmatrix} 0 & c\,(\partial_1 A^0 + \partial_0 A^1) & c\,(\partial_2 A^0 + \partial_0 A^2) & c\,(\partial_3 A^0 + \partial_0 A^3) \\ -c\,(\partial_1 A^0 + \partial_0 A^1) & 0 & c\,(\partial_2 A^1 - \partial_1 A^2) & c\,(\partial_3 A^1 - \partial_1 A^3) \\ -c\,(\partial_2 A^0 + \partial_0 A^2) & c\,(-\partial_2 A^1 + \partial_1 A^2) & 0 & c\,(\partial_3 A^2 - \partial_2 A^3) \\ -c\,(\partial_3 A^0 + \partial_0 A^3) & c\,(-\partial_3 A^1 + \partial_1 A^3) & c\,(-\partial_3 A^2 + \partial_2 A^3) & 0 \end{pmatrix}$$

Comparing with the given contravariant electromagnetic tensor components

In[86]:= `EMFUpIndexComponents[ F ] // MatrixForm`

Out[86]//MatrixForm=
$$\begin{pmatrix} 0 & -\mathbb{E}_1 & -\mathbb{E}_2 & -\mathbb{E}_3 \\ \mathbb{E}_1 & 0 & -c\,\mathbb{B}_3 & c\,\mathbb{B}_2 \\ \mathbb{E}_2 & c\,\mathbb{B}_3 & 0 & -c\,\mathbb{B}_1 \\ \mathbb{E}_3 & -c\,\mathbb{B}_2 & c\,\mathbb{B}_1 & 0 \end{pmatrix}$$

Energy Density and Poynting vector (Non covariant)

The energy density $\mathcal{E}$ and Poynting S vector are obtained as

$$\frac{1}{2}\,\varepsilon_0\, c\, F\, F^{\dagger} = \mathcal{E}\, c + S \tag{13}$$

with

$$\mathcal{E} = \frac{1}{2}\,\varepsilon_0 \left(\mathbb{E}^2 + c^2\, \mathbb{B}^2\right) \tag{14}$$

$$S = \frac{1}{\mu_0}\, \mathbb{E} \times \mathbb{B} \tag{15}$$

Calculating the following expression

In[87]:= $\mathrm{FF} = \mathrm{Simplify}\left[\frac{1}{2} F.(F)^{\dagger}\ /.\ \mathrm{Conjugate} \to \mathrm{ForceConjugate}\right];$

the energy density is

In[88]:=
```
FullSimplify[1/2 Tr[ ε0 FF]] // MatrixForm // CompactFormat
```

Out[88]//MatrixForm=

$$\frac{1}{2}\left(c^2\left(\mathbb{B}_1^2+\mathbb{B}_2^2+\mathbb{B}_3^2\right)+\mathbb{E}_1^2+\mathbb{E}_2^2+\mathbb{E}_3^2\right)\varepsilon_0$$

and the Poynting vector is

In[89]:=
```
Expand@Simplify[Simplify[VectorPart[ε0 c FF]] /. {c^2 → 1/(ε0 μ0)}] // MatrixForm //
  CompactFormat
```

Out[89]//MatrixForm=

$$\begin{pmatrix} \frac{\mathbb{B}_2\mathbb{E}_1}{\mu_0}-\frac{\mathbb{B}_1\mathbb{E}_2}{\mu_0} & \frac{i\,\mathbb{B}_3\mathbb{E}_1}{\mu_0}+\frac{\mathbb{B}_3\mathbb{E}_2}{\mu_0}-\frac{i\,\mathbb{B}_1\mathbb{E}_3}{\mu_0}-\frac{\mathbb{B}_2\mathbb{E}_3}{\mu_0} \\ -\frac{i\,\mathbb{B}_3\mathbb{E}_1}{\mu_0}+\frac{\mathbb{B}_3\mathbb{E}_2}{\mu_0}+\frac{i\,\mathbb{B}_1\mathbb{E}_3}{\mu_0}-\frac{\mathbb{B}_2\mathbb{E}_3}{\mu_0} & -\frac{\mathbb{B}_2\mathbb{E}_1}{\mu_0}+\frac{\mathbb{B}_1\mathbb{E}_2}{\mu_0} \end{pmatrix}$$

Determinant of the electromagnetic field as Lorentz invariant

The fact that $\frac{1}{2}\mathrm{Tr}[F^2]$ is Lorentz invariant implies the invariance of two quantities

In[90]:=
```
FullSimplify[1/2 Tr@HermitianPart[F.F] /. Conjugate → ForceConjugate] // CompactFormat //
  MatrixForm
```

Out[90]//MatrixForm=

$$-c^2\left(\mathbb{B}_1^2+\mathbb{B}_2^2+\mathbb{B}_3^2\right)+\mathbb{E}_1^2+\mathbb{E}_2^2+\mathbb{E}_3^2$$

In[91]:=
```
FullSimplify[1/2 Tr@AntiHermitianPart[F.F] /. Conjugate → ForceConjugate] //
   CompactFormat // MatrixForm
```

Out[91]//MatrixForm=

$$2\,i\,c\,(\mathbb{B}_1\mathbb{E}_1+\mathbb{B}_2\mathbb{E}_2+\mathbb{B}_3\mathbb{E}_3)$$

Moreover

$$\frac{1}{2}\mathrm{Tr}\left[F^2\right] = -\mathrm{Det}[F] \tag{16}$$

In[92]:=
```
-RealPart@Det[F] /. Conjugate → ForceConjugate // Simplify // CompactFormat
```

Out[92]= $-c^2\left(\mathbb{B}_1^2+\mathbb{B}_2^2+\mathbb{B}_3^2\right)+\mathbb{E}_1^2+\mathbb{E}_2^2+\mathbb{E}_3^2$

In[93]:=
```
-ImaginaryPart@Det[F] /. Conjugate → ForceConjugate // Simplify // CompactFormat
```

Out[93]= $2\,i\,c\,(\mathbb{B}_1\mathbb{E}_1+\mathbb{B}_2\mathbb{E}_2+\mathbb{B}_3\mathbb{E}_3)$

Maxwell Equations

The Maxwell equations are

$$\bar{\partial} F = c\mu_0\,\bar{j}. \tag{17}$$

where the para-current is

$$j = c\,\rho + \boldsymbol{j}. \tag{18}$$

The four standard Maxwell equations are

Gauss Law :

$$\left\langle \bar{\partial} F \right\rangle_{SH} = \frac{\rho}{\epsilon_0} \tag{19}$$

Ampere Maxwell Law :

$$\left\langle \bar{\partial} F \right\rangle_{VH} = -c\mu_0 \boldsymbol{j}. \tag{20}$$

No magnetic monopoles:

$$\left\langle \bar{\partial} F \right\rangle_{SA} = 0 \tag{21}$$

Faraday Maxwell:

$$\left\langle \bar{\partial} F \right\rangle_{VA} = 0 \tag{22}$$

Note : The coefficient $c\mu_0$ has units of impedance with value. For the vacuum we have

In[94]:=
```
UnitSimplify[Quantity["SpeedOfLight"] * Quantity["MagneticConstant"]] // N
```

Out[94]= 376.73 Ω

Quantum resistivity

In[95]:=
```
2 Quantity["ElementaryCharge"]^2 / Quantity["ReducedPlanckConstant"]
1 / UnitSimplify[%]
```

Out[95]= $2\, e^2/\hbar$

Out[96]= 2054.118 Ω

Note : The presence of magnetic monopoles would require to add a magnetic current as a **triparavector**, where the magnetic charge would be a pseudoscalar that changes sign by a spatial reflections. Otherwise, electric charges are scalars that do not change under spatial reflections.

Homogeneous case:

$$\bar{\partial} F = 0 \tag{23}$$

Defining the electromagnetic field dependent on the spacetime.

In[97]:=
```
F = EMF["SL2C"][{𝔼1[x⁰, x¹, x², x³], 𝔼2[x⁰, x¹, x², x³], 𝔼3[x⁰, x¹, x², x³]},
    {𝔹1[x⁰, x¹, x², x³], 𝔹2[x⁰, x¹, x², x³], 𝔹3[x⁰, x¹, x², x³]}] // Simplify;
```

The homogeneous Maxwell equations are

In[98]:=
```
MaxwellEqs = ∂̄(F) // Simplify;
```

Gauss Law

$$\left\langle \bar{\partial} F \right\rangle_{SH} = 0 \tag{24}$$

$$\nabla \circ \mathbb{E} = 0 \tag{25}$$

In[99]:=
```
Simplify[ -ScalarPart@HermitianPart[∂̄(F)] /. Conjugate → ForceConjugate] //
  MatrixForm // CompactFormat
```

Out[99]//MatrixForm=

$$\begin{pmatrix} -\partial_1 \mathbb{E}_1 - \partial_2 \mathbb{E}_2 - \partial_3 \mathbb{E}_3 & 0 \\ 0 & -\partial_1 \mathbb{E}_1 - \partial_2 \mathbb{E}_2 - \partial_3 \mathbb{E}_3 \end{pmatrix}$$

Ampere Maxwell

$$\left\langle \bar{\partial} F \right\rangle_{HV} = 0 \tag{26}$$

$$-c\, \nabla \times \mathbb{B} + \frac{\partial \mathbb{E}}{\partial x^0} = 0 \tag{27}$$

In[100]:=
```
Simplify[ VectorPart@HermitianPart[∂̄(F)] /. Conjugate → ForceConjugate] // MatrixForm //
 CompactFormat
```

Out[100]//MatrixForm=

$$\begin{pmatrix} c\,\partial_2 \mathbb{B}_1 - c\,\partial_1 \mathbb{B}_2 + \partial_0 \mathbb{E}_3 & i\,c\,\partial_3 \mathbb{B}_1 + c\,\partial_3 \mathbb{B}_2 - i\,c\,\partial_1 \mathbb{B}_3 - c\,\partial_2 \mathbb{B}_3 + \partial_0 \mathbb{E}_1 - i\,\partial_(\\ -i\,c\,\partial_3 \mathbb{B}_1 + c\,\partial_3 \mathbb{B}_2 + i\,c\,\partial_1 \mathbb{B}_3 - c\,\partial_2 \mathbb{B}_3 + \partial_0 \mathbb{E}_1 + i\,\partial_0 \mathbb{E}_2 & -c\,\partial_2 \mathbb{B}_1 + c\,\partial_1 \mathbb{B}_2 - \partial_0 \mathbb{E}_3 \end{pmatrix}$$

Comparing with the standard formulation

In[101]:=
```
c Curl[ {𝔹1[x0, x1, x2, x3], 𝔹2[x0, x1, x2, x3], 𝔹3[x0, x1, x2, x3]}, {x1, x2, x3}] -
   D[{𝔼1[x0, x1, x2, x3], 𝔼2[x0, x1, x2, x3], 𝔼3[x0, x1, x2, x3]}, x0] //
  TableForm // CompactFormat
```

Out[101]//TableForm=

$$c\,(-\partial_3 \mathbb{B}_2 + \partial_2 \mathbb{B}_3) - \partial_0 \mathbb{E}_1$$
$$c\,(\partial_3 \mathbb{B}_1 - \partial_1 \mathbb{B}_3) - \partial_0 \mathbb{E}_2$$
$$c\,(-\partial_2 \mathbb{B}_1 + \partial_1 \mathbb{B}_2) - \partial_0 \mathbb{E}_3$$

No magnetic monopoles

$$\left\langle \bar{\partial} F \right\rangle_{AS} = 0 \tag{28}$$

$$c\, \nabla \circ \mathbb{B} = 0 \tag{29}$$

In[102]:=
```
Simplify[ ScalarPart@AntiHermitianPart[∂̄(F)] /. Conjugate → ForceConjugate] //
  MatrixForm // CompactFormat
```

Out[102]//MatrixForm=

$$\begin{pmatrix} i\,c\,(\partial_1 \mathbb{B}_1 + \partial_2 \mathbb{B}_2 + \partial_3 \mathbb{B}_3) & 0 \\ 0 & i\,c\,(\partial_1 \mathbb{B}_1 + \partial_2 \mathbb{B}_2 + \partial_3 \mathbb{B}_3) \end{pmatrix}$$

Maxwell Faraday

$$\left\langle \bar{\partial} F \right\rangle_{AV} = 0 \tag{30}$$

$$\nabla \times \mathbb{E} + c\, \frac{\partial \mathbb{B}}{\partial x^0} = 0 \tag{31}$$

In[103]:=
```
ExpandAll[-I VectorPart@AntiHermitianPart[∂̄(F)] /. Conjugate → ForceConjugate] //
  MatrixForm // CompactFormat
```

Out[103]//MatrixForm=

$$\begin{pmatrix} c\,\partial_0\mathbb{B}_3 - \partial_2\mathbb{E}_1 + \partial_1\mathbb{E}_2 & c\,\partial_0\mathbb{B}_1 - i\,c\,\partial_0\mathbb{B}_2 - i\,\partial_3\mathbb{E}_1 - \partial_3\mathbb{E}_2 + i\,\partial_1\mathbb{E}_3 + \partial_2\mathbb{E}_3 \\ c\,\partial_0\mathbb{B}_1 + i\,c\,\partial_0\mathbb{B}_2 + i\,\partial_3\mathbb{E}_1 - \partial_3\mathbb{E}_2 - i\,\partial_1\mathbb{E}_3 + \partial_2\mathbb{E}_3 & -c\,\partial_0\mathbb{B}_3 + \partial_2\mathbb{E}_1 - \partial_1\mathbb{E}_2 \end{pmatrix}$$

Comparing with the standard formulation

In[104]:=
```
Curl[{E1[x0, x1, x2, x3], E2[x0, x1, x2, x3], E3[x0, x1, x2, x3]}, {x1, x2, x3}] +
   c D[{B1[x0, x1, x2, x3], B2[x0, x1, x2, x3], B3[x0, x1, x2, x3]}, x0] //
  CompactFormat // TableForm
```

Out[104]//TableForm=

$$c\,\partial_0\mathbb{B}_1 - \partial_3\mathbb{E}_2 + \partial_2\mathbb{E}_3$$
$$c\,\partial_0\mathbb{B}_2 + \partial_3\mathbb{E}_1 - \partial_1\mathbb{E}_3$$
$$c\,\partial_0\mathbb{B}_3 - \partial_2\mathbb{E}_1 + \partial_1\mathbb{E}_2$$

Lorentz transformations for Biparavectors

The transformation law for the electromagnetic field is different compared with the transformation for paravectors

$$F \to F' = L\,F\,\bar{L}. \tag{32}$$

Let us remind that paravectors transform as

$$x \to x' = L\,x\,L^{\dagger}. \tag{33}$$

In[105]:=
```
Simplify[BoostSL2C[u0, 0, 0, u3].F.CliffordConjugationSL2C@BoostSL2C[u0, 0, 0, u3]];
FullSimplify[EMFUpIndexComponents[%] /. {u3^2 → u0^2 - c^2}];
Simplify[% /. {u3^2 → u0^2 - c^2}] // CompactFormat // MatrixForm
```

Out[107]//MatrixForm=

$$\begin{pmatrix} 0 & -\frac{c\,u^3\mathbb{B}_2 + u^0\mathbb{E}_1}{c} & u^3\,\mathbb{B}_1 - \frac{u^0\mathbb{E}_2}{c} & -\mathbb{E}_3 \\ u^3\,\mathbb{B}_2 + \frac{u^0\mathbb{E}_1}{c} & 0 & -c\,\mathbb{B}_3 & u^0\,\mathbb{B}_2 + \frac{u^3\mathbb{E}_1}{c} \\ -u^3\,\mathbb{B}_1 + \frac{u^0\mathbb{E}_2}{c} & c\,\mathbb{B}_3 & 0 & -u^0\,\mathbb{B}_1 + \frac{u^3\mathbb{E}_2}{c} \\ \mathbb{E}_3 & -\frac{c\,u^0\mathbb{B}_2 + u^3\mathbb{E}_1}{c} & u^0\,\mathbb{B}_1 - \frac{u^3\mathbb{E}_2}{c} & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -\frac{c\,u^3\mathbb{B}_2 + u^0\mathbb{E}_1}{c} & u^3\,\mathbb{B}_1 - \frac{u^0\mathbb{E}_2}{c} & -\mathbb{E}_3 \\ u^3\,\mathbb{B}_2 + \frac{u^0\mathbb{E}_1}{c} & 0 & -c\,\mathbb{B}_3 & u^0\,\mathbb{B}_2 + \frac{u^3\mathbb{E}_1}{c} \\ -u^3\,\mathbb{B}_1 + \frac{u^0\mathbb{E}_2}{c} & c\,\mathbb{B}_3 & 0 & -u^0\,\mathbb{B}_1 + \frac{u^3\mathbb{E}_2}{c} \\ \mathbb{E}_3 & -\frac{c\,u^0\mathbb{B}_2 + u^3\mathbb{E}_1}{c} & u^0\,\mathbb{B}_1 - \frac{u^3\mathbb{E}_2}{c} & 0 \end{pmatrix}$$

The Lorentz transformation does not affect the parallel components of the electromagnetic field

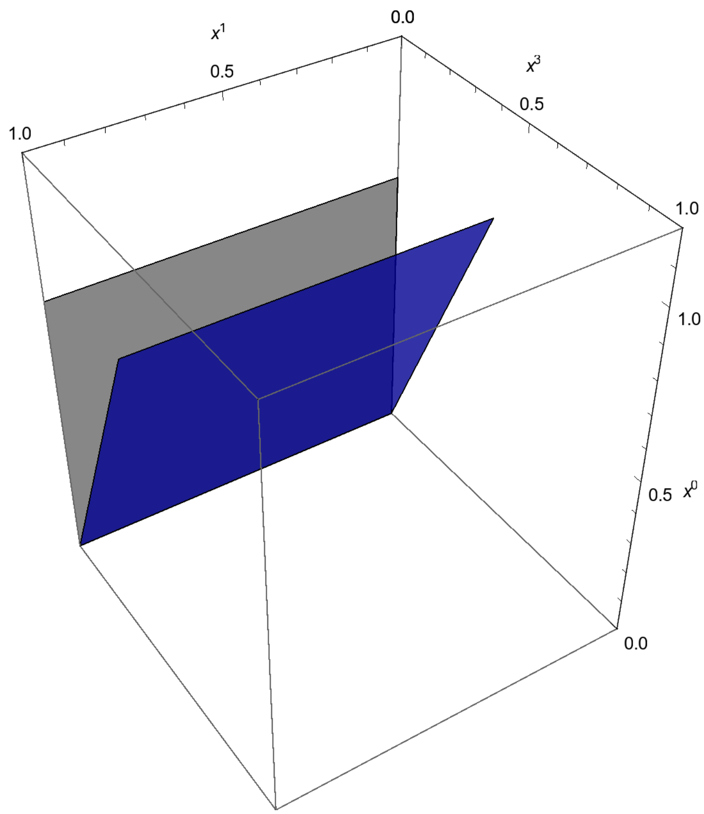
In[108]:=
```
{Tr[EMF["SL2C"][{0, 0, 𝔼₃}, {0, 0, 𝔹₃}].σ₃/2] →
    ExpandAll@Tr[BoostSL2C[u⁰, 0, 0, u³].EMF["SL2C"][{0, 0, 𝔼₃}, {0, 0, 𝔹₃}].
        CliffordConjugationSL2C@BoostSL2C[u⁰, 0, 0, u³].
        σ₃/2] /. {u³^2 → u⁰^2 - c^2} // Simplify}
```

Out[108]= $\{ i\, c\, \mathbb{B}_3 + \mathbb{E}_3 \to i\, c\, \mathbb{B}_3 + \mathbb{E}_3 \}$

The following figure represents an electric field $\mathbb{E}_1$ in a rest frame as a gray plane $x^0 - x^1$ and the electromagnetic field (blue) as a result of a boost along u^3. The boosted electromagnetic field is composed of an electric field $\mathbb{E}'_1$ and an additional a magnetic field $\mathbb{B}'_2$ spanning the plane $x^1 - x^3$

In[110]:=
```
Show[
 BoostedE1[0.8, 0., Gray],
 BoostedE1[0.8, 0.5, Blue]
]
```

Out[110]=



Lorentz Force

The Lorentz force is calculated as

$$m \frac{d}{d\tau} u^{\mu} = \frac{e}{c} F^{\mu\nu} u_{\nu} = \left\langle \frac{e}{c} F\, u \right\rangle_H \tag{34}$$

In[111]:= $\text{Simplify}\left[\frac{1}{c}\langle F.u\rangle_{\text{"H"}}\ /.\ \text{Conjugate}\to\text{ForceConjugate}\right]$ // Simplify // CompactFormat // MatrixForm

Out[111]//MatrixForm=

$$\begin{pmatrix} \frac{-c\,u^2\,\mathbb{B}_1+c\,u^1\,\mathbb{B}_2+u^1\,\mathbb{E}_1+u^2\,\mathbb{E}_2+u^0\,\mathbb{E}_3+u^3\,\mathbb{E}_3}{c} & -i\,u^3\,\mathbb{B}_1-u^3\,\mathbb{B}_2+i\,u^1\,\mathbb{B}_3+u^2\,\mathbb{B}_3+\frac{u^0\,\mathbb{E}_1}{c}-\frac{i\,u^0\,\mathbb{E}_2}{c} \\ i\,u^3\,\mathbb{B}_1-u^3\,\mathbb{B}_2-i\,u^1\,\mathbb{B}_3+u^2\,\mathbb{B}_3+\frac{u^0\,\mathbb{E}_1}{c}+\frac{i\,u^0\,\mathbb{E}_2}{c} & \frac{c\,u^2\,\mathbb{B}_1-c\,u^1\,\mathbb{B}_2+u^1\,\mathbb{E}_1+u^2\,\mathbb{E}_2-u^0\,\mathbb{E}_3+u^3\,\mathbb{E}_3}{c} \end{pmatrix}$$

Taking components we recover the standard formulas

In[112]:= $\frac{1}{2}\text{Tr}\left[\left(\frac{1}{c}\langle F.u\rangle_{\text{"H"}}\right).\sigma_0\right]$ /. Conjugate → ForceConjugate // Simplify

Out[112]= $\frac{1}{c}\left(u^1\,\mathbb{E}_1[x^0, x^1, x^2, x^3]+u^2\,\mathbb{E}_2[x^0, x^1, x^2, x^3]+u^3\,\mathbb{E}_3[x^0, x^1, x^2, x^3]\right)$

In[113]:= $\frac{1}{2}\text{Tr}\left[\left(\frac{1}{c}\langle F.u\rangle_{\text{"H"}}\right).\sigma_1\right]$ /. Conjugate → ForceConjugate // Simplify

Out[113]= $-u^3\,\mathbb{B}_2[x^0, x^1, x^2, x^3]+u^2\,\mathbb{B}_3[x^0, x^1, x^2, x^3]+\frac{u^0\,\mathbb{E}_1[x^0, x^1, x^2, x^3]}{c}$

In[114]:= $\frac{1}{2}\text{Tr}\left[\left(\frac{1}{c}\langle F.u\rangle_{\text{"H"}}\right).\sigma_1\right]$ /. Conjugate → ForceConjugate // Simplify

Out[114]= $-u^3\,\mathbb{B}_2[x^0, x^1, x^2, x^3]+u^2\,\mathbb{B}_3[x^0, x^1, x^2, x^3]+\frac{u^0\,\mathbb{E}_1[x^0, x^1, x^2, x^3]}{c}$

In[115]:= $\frac{1}{2}\text{Tr}\left[\left(\frac{1}{c}\langle F.u\rangle_{\text{"H"}}\right).\sigma_1\right]$ /. Conjugate → ForceConjugate // Simplify

Out[115]= $-u^3\,\mathbb{B}_2[x^0, x^1, x^2, x^3]+u^2\,\mathbb{B}_3[x^0, x^1, x^2, x^3]+\frac{u^0\,\mathbb{E}_1[x^0, x^1, x^2, x^3]}{c}$

Spinorial Classical Dynamics

The proper velocity u can be written in terms of the associated spinor Λ (also known as proper spinor or eigenspinor)

$$u = c\,\Lambda\,\Lambda^{\dagger}. \tag{35}$$

The most general equation of motion for the spinor can be written in terms of a biparavector Ω

$$\frac{d}{d\tau}\Lambda = \frac{1}{2}\,\Omega\,\Lambda\,, \tag{36}$$

where we can identify Ω in terns of the electromagnetic field F as

$$\Omega = \frac{e}{m\,c}\,F\,, \tag{37}$$

yielding the dynamical equation

$$\frac{d}{d\tau}\Lambda = \frac{e}{2\,m\,c}\,F\,\Lambda\,. \tag{38}$$

In general, the complete dynamics is determined by the coupled equations:

$$\begin{aligned} \frac{d}{d\tau}\Lambda &= \frac{e}{2\,m\,c}\,F\,\Lambda\,, \\ \frac{d}{d\tau}x &= c\,\Lambda\,\Lambda^{\dagger}. \end{aligned} \tag{39}$$

However, it is possible to show that for directed electromagnetic waves, these equations decouple and lead to the classical Volkov solutions.

Charged particle in a constant electric field

Let us take the case of an electric field along the x direction

$$F = \mathbb{E}_0\, \sigma_1$$

The integration of Eq (21) gives

$$\Lambda = e^{\frac{e \mathbb{E}_0}{2 m c} \sigma_1 \tau} \Lambda(0) \tag{40}$$

The initial the condition

$$\Lambda(0) = 1 \tag{41}$$

results in

In[116]:= $\Lambda\text{Sol} = \left\{\Lambda \to \text{MatrixExp}\left[\frac{e\, \mathbb{E}_0}{2\, m\, c}\, \sigma_1\, \tau\right]\right\}$ // ExpToTrig

Out[116]= $\left\{\Lambda \to \left\{\left\{\text{Cosh}\left[\frac{e\, \mathbb{E}_0\, \tau}{2\, c\, m}\right], \text{Sinh}\left[\frac{e\, \mathbb{E}_0\, \tau}{2\, c\, m}\right]\right\}, \left\{\text{Sinh}\left[\frac{e\, \mathbb{E}_0\, \tau}{2\, c\, m}\right], \text{Cosh}\left[\frac{e\, \mathbb{E}_0\, \tau}{2\, c\, m}\right]\right\}\right\}\right\}$

The proper velocity is

$$u = \Lambda \Lambda^{\dagger} = e^{\frac{e \mathbb{E}_0}{m c} \sigma_1 \tau} \tag{42}$$

In[117]:= (uSol = {u → Λ.Λ /. ΛSol // ExpToTrig // Simplify});

For a particular initial condition, the spacetime trajectory can be integrated as

$$X = \frac{m\, c}{e\, \mathbb{E}_0}\, e^{\frac{e \mathbb{E}_0}{m c} \sigma_1 \tau}\, \sigma_1 \tag{43}$$

In[118]:= $\text{xSol} = \left\{X \to \frac{m\, c}{e\, \mathbb{E}_0}\, \text{MatrixExp}\left[\frac{e\, \mathbb{E}_0}{2\, m\, c}\, \sigma_1\, \tau\right].\sigma_1\right\}$

Out[118]= $\left\{X \to \left\{\left\{\frac{c\left(-\frac{1}{2} e^{-\frac{e \mathbb{E}_0 \tau}{2 c m}} + \frac{1}{2} e^{\frac{e \mathbb{E}_0 \tau}{2 c m}}\right) m}{e\, \mathbb{E}_0}, \frac{c\left(\frac{1}{2} e^{-\frac{e \mathbb{E}_0 \tau}{2 c m}} + \frac{1}{2} e^{\frac{e \mathbb{E}_0 \tau}{2 c m}}\right) m}{e\, \mathbb{E}_0}\right\}, \left\{\frac{c\left(\frac{1}{2} e^{-\frac{e \mathbb{E}_0 \tau}{2 c m}} + \frac{1}{2} e^{\frac{e \mathbb{E}_0 \tau}{2 c m}}\right) m}{e\, \mathbb{E}_0}, \frac{c\left(-\frac{1}{2} e^{-\frac{e \mathbb{E}_0 \tau}{2 c m}} + \frac{1}{2} e^{\frac{e \mathbb{E}_0 \tau}{2 c m}}\right) m}{e\, \mathbb{E}_0}\right\}\right\}\right\}$

The spacetime trajectory parametrized by the proper time can be extracted as

In[119]:= ct -> Tr[X.σ_0 /. xSol] // ExpToTrig

Out[119]= $ct \to \frac{2\, c\, m\, \text{Sinh}\left[\frac{e\, \mathbb{E}_0\, \tau}{2\, c\, m}\right]}{e\, \mathbb{E}_0}$

In[120]:= X^1 → Tr[X.σ_1 /. xSol] // ExpToTrig

Out[120]= $X^1 \to \frac{2\, c\, m\, \text{Cosh}\left[\frac{e\, \mathbb{E}_0\, \tau}{2\, c\, m}\right]}{e\, \mathbb{E}_0}$

In[121]:= X^2 → Tr[X.σ_2 /. xSol] // ExpToTrig

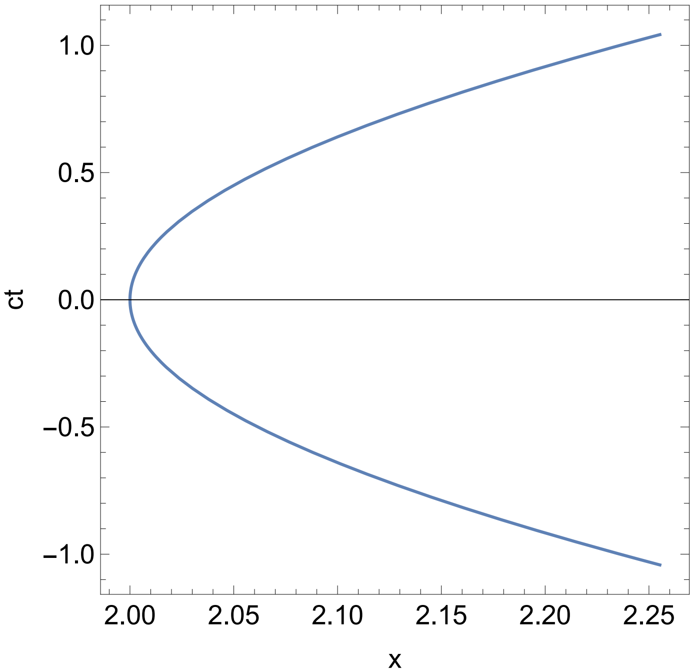
Out[121]= $X^2 \to 0$

In[122]:= X^3 → Tr[X.σ_3 /. xSol] // ExpToTrig

Out[122]= $X^3 \to 0$

In[123]:=
```
ParametricPlot[
 Evaluate[ {Tr[ X.σ₁ /. xSol], Tr[ X.σ₀ /. xSol] }  /. {e → 1, 𝔼₀ → 1, m → 1, c → 1}]
 , {τ, -1, 1}, Frame → True, FrameLabel → {"x", "ct"}, AspectRatio → 1,
 BaseStyle → {FontSize → 14}]
```

Out[123]=



The determinant of the classical spinor is always 1

In[124]:=
```
Det[ Λ /. ΛSol ] // Simplify
```

Out[124]= 1

The Dirac Equation

The Dirac equation can be written as

$$i\, c\, \hbar\, \bar{\partial}\, \Psi\, \sigma_3 - c\, e\, \bar{A}\, \Psi = m\, c^2\, \Psi, \tag{44}$$

where the spinor Ψ is related to the standard Dirac spinor (Dirac basis) as

$$\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \Longleftrightarrow \begin{pmatrix} \psi_1 + \psi_3 & -\psi_2{}^* + \psi_4{}^* \\ \psi_2 + \psi_4 & \psi_1{}^* - \psi_3{}^* \end{pmatrix}. \tag{45}$$

In the Weyl column representation we have

$$\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \Longleftrightarrow \sqrt{2}\, \begin{pmatrix} \psi_3 & -\psi_2{}^* \\ \psi_4 & \psi_1{}^* \end{pmatrix}. \tag{46}$$

The current is given by

$$J = \Psi\Psi^{\dagger}, \tag{47}$$

which is similar to the formula for the proper velocity

$$u = \Lambda\Lambda^{\dagger}, \tag{48}$$

but in the case of the Dirac spinor we have

$$\mathrm{Det}[\Psi] = e^{i\beta}, \tag{49}$$

where β is the Yvon-Takabayashi angle where β=0 for particles and $\beta = -\pi$ for antiparticles.

The vector potential can be solved as

$$e\bar{A} = \left(i\,\hbar\bar{\partial}\,\Psi\sigma_3 - m\,c\,\Psi\right)\Psi^{-1}. \tag{50}$$

In[133]:=
```
DiracEquationSL2C[ψ_?MatrixQ, A_?MatrixQ, m_] := Module[{K, ψc},
  ψc = ConjugateTranspose[CliffordConjugationSL2C[ψ]];
  K = ⅈ ℏ (σ0.∂t ψ + c σ1.∂x1 ψ + c σ2.∂x2 ψ + c σ3.∂x3 ψ).σ3 ;
   K - c Ā.ψ - m c^2 ψc
  ]
```

References

1. Baylis, W.E., Electrodynamics : a modern geometric approach, Birkhauser, 1999
2. Baylis, W.E., Clifford (Geometric) Algebras : with applications to physics, mathematics, and engineering, Birkhauser, 1996
3. Baylis, W.E., Classical eigenspinors and the Dirac equation, Phys. Rev. A, **45**, 4293, 1992