Introduction to (Geometric) Clifford Algebras: Applications to classical and quantum relativistic mechanics

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Routines

Algebra of the Physical Space (APS)

The algebra of physical space introduced by W. Baylis (See Refs. [1, 2, 3]) is an elegant algebraic formalism that can be regarded as an extension of the standard vector calculus to deal with the Minkowski space. These lectures slightly depart from the axiomatic approach in the literature by extensively using the Pauli matrix representation. In this way, we lose some mathematical elegance but I hope I provide a more familiar environment constructed on top of standard linear algebra.

There is a second set of lectures treating the Space-Time algebra (STA) developed by Hestenes.

These lectures are written in *Mathematica* including a description of the formalism and the implementation of routines to perform symbolic and numerical calculations. This gives the student the ability to further explore beyond the provided examples and pursue a deeper understanding of the topic. It is like having a live document that is capable to provide answers on the fly.

These lectures are written in *Mathematica* 11.1 and can be alternatively opened by the free Mathematica CDF reader available in https://www.wolfram.com/cdf-player/.

Basics

Scalar	$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
Vector	$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
Bivector	$\sigma_1 \sigma_2 = i \sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ $\sigma_2 \sigma_3 = i \sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ $\sigma_3 \sigma_1 = i \sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
PseudoScalar	$\sigma_{123} = \sigma_1 \sigma_2 \sigma_3 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$

Conjugations

$\mathcal{H}$ermitian $\mathcal{C}$onjugation

Hermitian	$\sigma_0{}^\dagger = \sigma_0$ $\sigma_k{}^\dagger = \sigma_k$
AntiHermitian	$\sigma_{jk}{}^\dagger = -\sigma_{jk}$ $\sigma_{123}{}^\dagger = -\sigma_{123}$

$\mathcal{C}$lifford $\mathcal{C}$onjugation

Scalar	$\bar{\sigma}_0 = \sigma_0$ $\bar{\sigma}_{123} = \sigma_{123}$
Vector	$\bar{\sigma}_k = -\sigma_k$ $\bar{\sigma}_{jk} = -\sigma_{jk}$

Extracting Scalar Part

$$\text{CliffordConjugationSL2C}\left[\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right] \text{ // MatrixForm}$$

$$\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

$$\text{ScalarPart[x\_]} := \frac{\text{x + CliffordConjugationSL2C[x]}}{2}$$

$$\text{VectorPart[x\_]} := \frac{\text{x - CliffordConjugationSL2C[x]}}{2}$$

```
ScalarPart[a σ0 + b σ1] // MatrixForm
```

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$

```
VectorPart[a σ0 + b σ1] // MatrixForm
```

$$\begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}$$

Note 1 : The trace almost reproduces the role of the scalar part

$$\frac{1}{2}\mathrm{Tr}[a\,\sigma_0 + b\,\sigma_1]\,\sigma_0 \text{ // MatrixForm}$$

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$

Note 2 : The Clifford conjugation can be used to calculate the determinant

$$\text{Det@}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$-a_{12}\,a_{21} + a_{11}\,a_{22}$$

$$1/2\,\mathrm{Tr}\left[\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.\text{CliffordConjugationSL2C}\left[\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right]\right] \text{ // Expand}$$

$$-a_{12}\,a_{21} + a_{11}\,a_{22}$$

Note 3 : The Clifford conjugation is the product of the inverse with the determinant

$$\text{CliffordConjugationSL2C}\left[\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right] \text{ // MatrixForm}$$

$$\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

$$\text{Det}\left[\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right]\text{Inverse}\left[\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right] \text{ // MatrixForm}$$

$$\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

Extracting Hermitian Part

$$\text{HermitianPart[x\_] := }\frac{x + x^{\dagger}}{2}$$

$$\text{AntiHermitianPart[x\_] := }\frac{x - x^{\dagger}}{2}$$

```
Simplify[HermitianPart[ a σ0 + I b σ1 ],
  Assumptions → {Element[a, Reals], Element[b, Reals]}] // MatrixForm
```

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$

```
Simplify[AntiHermitianPart[ a σ0 + I b σ1 ],
  Assumptions → {Element[a, Reals], Element[b, Reals]}] // MatrixForm
```

$$\begin{pmatrix} 0 & i\,b \\ i\,b & 0 \end{pmatrix}$$

The same calculation can be performed using the bracket

```
Simplify[
 ⟨a σ0 + I b σ1⟩"H"
 , Assumptions → {Element[a, Reals], Element[b, Reals]}]
```

```
{{a, 0}, {0, a}}
```

```
Simplify[
 ⟨a σ0 + I b σ1⟩"A"
 , Assumptions → {Element[a, Reals], Element[b, Reals]}]
```

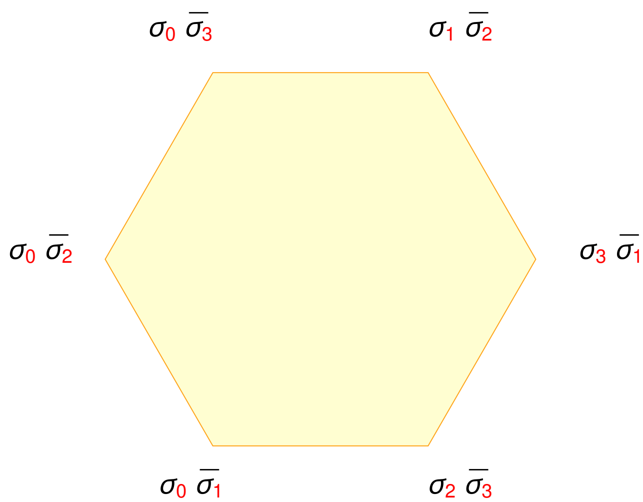
```
{{0, i b}, {i b, 0}}
```

Scalar	$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	Scalar Hermitian
Vector	$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	Vector Hermitian
Bivector	$\sigma_1 \sigma_2 = i\,\sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ $\sigma_2 \sigma_3 = i\,\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ $\sigma_3 \sigma_1 = i\,\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	Vector AntiHermitian
Pseudo Scalar	$\sigma_{123} = \sigma_1 \sigma_2 \sigma_3 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$	Scalar AntiHermitian

Paravector Space

Scalar	$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\mathcal{S}$calar $\mathcal{H}$ermitian
Paravector	$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\mathcal{H}$ermitian
Biparavector	$\sigma_0 \bar{\sigma}_1 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ $\sigma_0 \bar{\sigma}_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ $\sigma_0 \bar{\sigma}_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ $\sigma_1 \bar{\sigma}_2 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$ $\sigma_2 \bar{\sigma}_3 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$ $\sigma_3 \bar{\sigma}_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\mathcal{V}$ector
Triparavector	$\sigma_1 \bar{\sigma}_2 \sigma_3 = \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix}$ $\sigma_2 \bar{\sigma}_3 \sigma_0 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$ $\sigma_3 \bar{\sigma}_0 \sigma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\sigma_0 \bar{\sigma}_1 \sigma_2 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$	$\mathcal{A}$nti$\mathcal{H}$ermitian
Pseudo Scalar	$\sigma_{123} = \sigma_0 \bar{\sigma}_1 \sigma_2 \bar{\sigma}_3 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$	$\mathcal{S}$calar $\mathcal{A}$nti$\mathcal{H}$ermitian

Biparavector space = Lorentz Lie algebra = SL(2, C)



Spacetime Paravector

Definition

The spacetime can be represented as a paravector

$$x = x^\mu \sigma_\mu . \tag{1}$$

Sometimes we write

$$x = x^0 \sigma_0 + \mathbf{x}, \tag{2}$$

where the bold symbols stand for standard three-dimensional vectors

```
(x = x^0 σ0 + x^1 σ1 + x^2 σ2 + x^3 σ3) // MatrixForm
```

$$\begin{pmatrix} x^0 + x^3 & x^1 - i\, x^2 \\ x^1 + i\, x^2 & x^0 - x^3 \end{pmatrix}$$

```
1/2 Tr[x.σ1 ]
```

x^1

Similarly, the proper velocity is

$$u = u^{\mu} \sigma_{\mu}$$

```
(u = u^0 σ_0 + u^1 σ_1 + u^2 σ_2 + u^3 σ_3) // MatrixForm
```

$$\begin{pmatrix} u^0 + u^3 & u^1 - i\, u^2 \\ u^1 + i\, u^2 & u^0 - u^3 \end{pmatrix}$$

The contraction of paravectors is accomplished as

```
1/2 Tr[x.ū] // Simplify
```

$u^0\, x^0 - u^1\, x^1 - u^2\, x^2 - u^3\, x^3$

```
1/2 Tr[u.ū] // Simplify
```

${u^0}^2 - {u^1}^2 - {u^2}^2 - {u^3}^2$

The spacetime position has 4 degrees of freedom but the proper velocity has only 3 because it must obey the shell-mass constrain

$$\left(u^0\right)^2 - \left(u^1\right)^2 - \left(u^2\right)^2 - \left(u^3\right)^2 = c^2 \tag{3}$$

also written as

$$\frac{1}{2} \mathrm{Tr}\,\left(u.\bar{u}\right) = c^2 \tag{4}$$

or

$$\left(u^0\right)^2 - \boldsymbol{u}^2 = c^2 \tag{5}$$

Lorentz Transformations for paravectors

The Lorentz Boost is defined as the square root of the proper velocity

$$B = \sqrt{\frac{u}{c}}\ , \tag{6}$$

where we note that B is Hermitian.

A useful formula for the square root is

$$B = \frac{u + c\,\sigma_0}{\sqrt{2\,c\,(c + u^0)}}\ . \tag{7}$$

The active Lorentz Boost is carried out by double side multiplication

$$x \to y = L\, x\, L^{\dagger}\ . \tag{8}$$

Considering that the Boost is a Hermitian Lorentz operator we have

$$x \to y = B\, x\, B\ . \tag{9}$$

Note : The Lorentz transformation for biparavectors is different and will be treated later.

```
(BoostSL2C[u⁰ : _, u¹ : _, u² : _, u³ : _] = Module[{}, ((u⁰ σ₀ + u¹ σ₁ + u² σ₂ + u³ σ₃) + c σ₀) / Sqrt[2 c (c + u⁰)]]) //
 MatrixForm
```

$$\begin{pmatrix} \frac{c+u^0+u^3}{\sqrt{2}\sqrt{c\,(c+u^0)}} & \frac{u^1-i\,u^2}{\sqrt{2}\sqrt{c\,(c+u^0)}} \\ \frac{u^1+i\,u^2}{\sqrt{2}\sqrt{c\,(c+u^0)}} & \frac{c+u^0-u^3}{\sqrt{2}\sqrt{c\,(c+u^0)}} \end{pmatrix}$$

Recovering the proper velocity from the Boost to verify the formula (using the shell mass condition)

```
Simplify[c BoostSL2C[u⁰, u¹, u², u³].BoostSL2C[u⁰, u¹, u², u³]];
U == Simplify[% /. {u¹² + u²² + u³² → u⁰² - c²}]
```

$U == \{\{u^0 + u^3,\ u^1 - i\,u^2\},\ \{u^1 + i\,u^2,\ u^0 - u^3\}\}$

The boost operator along x^3 is

```
BoostSL2C[u⁰, 0, 0, u³] // MatrixForm
```

$$\begin{pmatrix} \frac{c+u^0+u^3}{\sqrt{2}\sqrt{c\,(c+u^0)}} & 0 \\ 0 & \frac{c+u^0-u^3}{\sqrt{2}\sqrt{c\,(c+u^0)}} \end{pmatrix}$$

The active boost of the coordinates is

```
y = Simplify[BoostSL2C[u⁰, 0, 0, u³].x.BoostSL2C[u⁰, 0, 0, u³]]
```

$$\left\{\left\{\frac{(c+u^0+u^3)^2\,(x^0+x^3)}{2\,c\,(c+u^0)},\ \frac{(c+u^0-u^3)\,(c+u^0+u^3)\,(x^1-i\,x^2)}{2\,c\,(c+u^0)}\right\},\ \left\{\frac{(c+u^0-u^3)\,(c+u^0+u^3)\,(x^1+i\,x^2)}{2\,c\,(c+u^0)},\ \frac{(c+u^0-u^3)^2\,(x^0-x^3)}{2\,c\,(c+u^0)}\right\}\right\}$$

The transformation of coordinates reads

```
MapThread[Equal, {{y⁰, y¹, y², y³},
   Simplify[Simplify[1/2 Tr[y.σ#] & /@ {0, 1, 2, 3}] /. {u³² → u⁰² - c²}]}] // TableForm
```

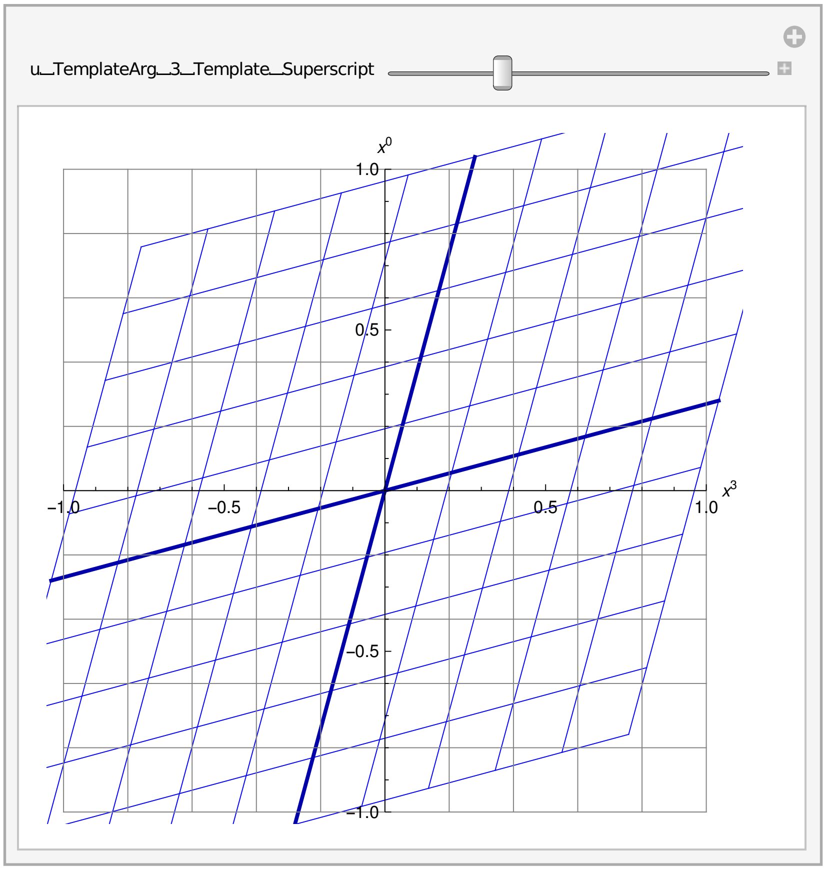
$y^0 == \frac{u^0\,x^0+u^3\,x^3}{c}$
$y^1 == x^1$
$y^2 == x^2$
$y^3 == \frac{u^3\,x^0+u^0\,x^3}{c}$

```
{(u⁰ x⁰ + u³ x³)/c, (u³ x⁰ + u⁰ x³)/c} //. {c → 1, u⁰ → Sqrt[c² + u³²]}
```

$\left\{\sqrt{1+u^{3\,2}}\ x^0 + u^3\,x^3,\ u^3\,x^0 + \sqrt{1+u^{3\,2}}\ x^3\right\}$

Spacetime diagram

The spacetime grid for a given velocity boost **u³** (The gray grid represents the rest frame and the blue grid represents the boosted frame)

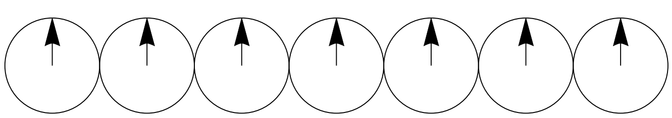


Array of clocks

A drawing of an array of clocks where the clocks in the rest frame are synchronized

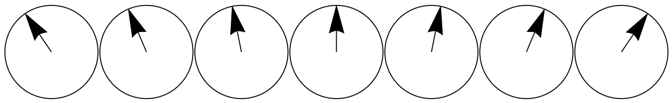
The clocks are synchronized in the rest frame

```
Show @@ Table[
  BoostedSpacetimeClock[u³ /. {u³ → 0}][0, z], {z, -3, 3, 1}]
```



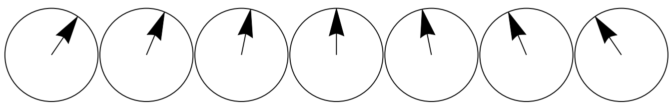
This system is actively boosted and now the clocks are not synchronized anymore as seen by the observer in the original rest frame

```
Show @@ Table[
  BoostedSpacetimeClock[u^3 /. {u^3 → 0.2}][0, z], {z, -3, 3, 1}]
```



Given that the clocks are synchronized in the rest frame, what is the perception of an observer in a frame moving with velocity u^3?
The answer follows:

```
Show @@ Table[
  BoostedSpacetimeClock[u^3 /. {u^3 → -0.2}][0, z], {z, -3, 3, 1}]
```



Time dilation:

```
MapThread[Equal,
  {{y^0, x^1, x^2, 0}, Simplify[Simplify[1/2 Tr[y.σ#] & /@ {0, 1, 2, 3}] /. {u^3^2 → u^0^2 - c^2}]}]
Refine[ Reduce[%, {y^0, x^3}], {x^3 > 0, x^0 > 0, y^0 > 0, c > 0, u^0 > 0}];
First[% /. {u^3^2 → u^0^2 - c^2}]
```

$$\left\{y^0 == \frac{u^0\, x^0 + u^3\, x^3}{c},\ \text{True},\ \text{True},\ 0 == \frac{u^3\, x^0 + u^0\, x^3}{c}\right\}$$

$$y^0 == \frac{c\, x^0}{u^0}$$

Length contraction:

```
MapThread[Equal,
  {{0, x^1, x^2, y^3}, Simplify[Simplify[1/2 Tr[y.σ#] & /@ {0, 1, 2, 3}] /. {u^3^2 → u^0^2 - c^2}]}]
Refine[ Reduce[%, {y^3, x^0}], {x^3 > 0, x^0 > 0, y^0 > 0, c > 0, u^0 > 0}];
First[% /. {u^3^2 → u^0^2 - c^2}]
```

$$\left\{0 == \frac{u^0\, x^0 + u^3\, x^3}{c},\ \text{True},\ \text{True},\ y^3 == \frac{u^3\, x^0 + u^0\, x^3}{c}\right\}$$

$$y^3 == \frac{c\, x^3}{u^0}$$

Paragradient

The paragradient is defined as

$\partial = \sigma_\mu \, \partial^\mu = \sigma_0 \, \partial_0 - \sigma_k \, \partial_k.$

The Clifford conjugated paragradient is

$\bar{\partial} = \bar{\sigma}_\mu \, \partial^\mu = \sigma_0 \, \partial_0 + \sigma_k \, \partial_k.$

```
paraGradient = (σ0.∂x⁰# - σ1.∂x¹# - σ2.∂x²# - σ3.∂x³#) &
```

$\sigma_0 . \partial_{x^0} \#1 - \sigma_1 . \partial_{x^1} \#1 - \sigma_2 . \partial_{x^2} \#1 - \sigma_3 . \partial_{x^3} \#1$ &

```
paraGradientBar = (σ0.∂x⁰# + σ1.∂x¹# + σ2.∂x²# + σ3.∂x³#) &
```

$\sigma_0 . \partial_{x^0} \#1 + \sigma_1 . \partial_{x^1} \#1 + \sigma_2 . \partial_{x^2} \#1 + \sigma_3 . \partial_{x^3} \#1$ &

Classical Electrodynamics

Electromagnetic Field

The electromagnetic field as a biparavector is

$$F = F^{\mu\nu} \langle \sigma_\mu \, \bar{\sigma}_\mu \rangle_V \tag{10}$$

which is expanded as

$$F = \mathbb{E} + i\, c\, \mathbb{B} = \sigma_1 \mathbb{E}_1 + \sigma_2 \mathbb{E}_2 + \sigma_3 \mathbb{E}_3 + i\, c\, (\sigma_1 \mathbb{B}_1 + \sigma_2 \mathbb{B}_2 + \sigma_3 \mathbb{B}_3) \tag{11}$$

The function that writes the electromagnetic paravector is

```
EMF["SL2C"][{𝔼1 : _, 𝔼2 : _, 𝔼3 : _}, {𝔹1 : _, 𝔹2 : _, 𝔹3 : _}] =
  σ1 𝔼1 + σ2 𝔼2 + σ3 𝔼3 + ⅈ c (σ1 𝔹1 + σ2 𝔹2 + σ3 𝔹3);
```

For example we have

```
(F = EMF["SL2C"][{𝔼1, 𝔼2, 𝔼3}, {𝔹1, 𝔹2, 𝔹3}] // Simplify) // MatrixForm
```

$$\begin{pmatrix} i\, c\, \mathbb{B}_3 + \mathbb{E}_3 & c\, (i\, \mathbb{B}_1 + \mathbb{B}_2) + \mathbb{E}_1 - i\, \mathbb{E}_2 \\ i\, c\, \mathbb{B}_1 - c\, \mathbb{B}_2 + \mathbb{E}_1 + i\, \mathbb{E}_2 & -i\, c\, \mathbb{B}_3 - \mathbb{E}_3 \end{pmatrix}$$

```
EMFUpIndexComponents::usage =
  "EMFUpIndexComponents[ X ], writes the biparavector electromagnetic field
    X as the components of the contravariant tensor components F^μ ν ";

EMFUpIndexComponents[EMF_] :=
 If[Dimensions[EMF] == {2, 2},
  Simplify[
   Table[
     1/2 Tr[ HermitianPart[ EMF.Inverse[(σ_μ).σ̄_ν]]]
     , {μ, 0, 3}, {ν, 0, 3}] /. Conjugate → ForceConjugate
  ]
 ]
```

The (upper index) contravariant tensor components are

```
MatrixForm /@ (F → EMFUpIndexComponents[ F]) // CompactFormat
```

$$\begin{pmatrix} i\, c\, \mathbb{B}_3 + \mathbb{E}_3 & c\,(i\,\mathbb{B}_1 + \mathbb{B}_2) + \mathbb{E}_1 - i\,\mathbb{E}_2 \\ i\, c\,\mathbb{B}_1 - c\,\mathbb{B}_2 + \mathbb{E}_1 + i\,\mathbb{E}_2 & -i\, c\,\mathbb{B}_3 - \mathbb{E}_3 \end{pmatrix} \to \begin{pmatrix} 0 & -\mathbb{E}_1 & -\mathbb{E}_2 & -\mathbb{E}_3 \\ \mathbb{E}_1 & 0 & -c\,\mathbb{B}_3 & c\,\mathbb{B}_2 \\ \mathbb{E}_2 & c\,\mathbb{B}_3 & 0 & -c\,\mathbb{B}_1 \\ \mathbb{E}_3 & -c\,\mathbb{B}_2 & c\,\mathbb{B}_1 & 0 \end{pmatrix}$$

Vector potential

The electromagnetic field from the vector potential is

$$F = c\,\langle \partial \bar{A} \rangle_V = c\, \mathrm{VectorPart}[\partial \bar{A}] \tag{12}$$

Defining the paravector potential

```
A = A^0[x^0, x^1, x^2, x^3] σ_0 + A^1[x^0, x^1, x^2, x^3] σ_1 + A^2[x^0, x^1, x^2, x^3] σ_2 + A^3[x^0, x^1, x^2, x^3] σ_3;
```

The electromagnetic field is calculated as

```
𝓕 = c VectorPart[∂Ā] // Simplify;
```

Calculating the upper tensor components

```
EMFUpIndexComponents[ 𝓕 ] // CompactFormat // MatrixForm
```

$$\begin{pmatrix} 0 & c\,(\partial_1 A^0 + \partial_0 A^1) & c\,(\partial_2 A^0 + \partial_0 A^2) & c\,(\partial_3 A^0 + \partial_0 A^3) \\ -c\,(\partial_1 A^0 + \partial_0 A^1) & 0 & c\,(\partial_2 A^1 - \partial_1 A^2) & c\,(\partial_3 A^1 - \partial_1 A^3) \\ -c\,(\partial_2 A^0 + \partial_0 A^2) & c\,(-\partial_2 A^1 + \partial_1 A^2) & 0 & c\,(\partial_3 A^2 - \partial_2 A^3) \\ -c\,(\partial_3 A^0 + \partial_0 A^3) & c\,(-\partial_3 A^1 + \partial_1 A^3) & c\,(-\partial_3 A^2 + \partial_2 A^3) & 0 \end{pmatrix}$$

Comparing with the given contravariant electromagnetic tensor components

```
EMFUpIndexComponents[ F ] // MatrixForm
```

$$\begin{pmatrix} 0 & -\mathbb{E}_1 & -\mathbb{E}_2 & -\mathbb{E}_3 \\ \mathbb{E}_1 & 0 & -c\,\mathbb{B}_3 & c\,\mathbb{B}_2 \\ \mathbb{E}_2 & c\,\mathbb{B}_3 & 0 & -c\,\mathbb{B}_1 \\ \mathbb{E}_3 & -c\,\mathbb{B}_2 & c\,\mathbb{B}_1 & 0 \end{pmatrix}$$

Energy Density and Poynting vector (Non covariant)

The energy density $\mathcal{E}$ and Poynting S vector are obtained as

$$\frac{1}{2}\,\varepsilon_0\, c\, F\, F^{\dagger} = \mathcal{E}\, c + S \tag{13}$$

with

$$\mathcal{E} = \frac{1}{2}\,\varepsilon_0\left(\mathbb{E}^2 + c^2\,\mathbb{B}^2\right) \tag{14}$$

$$S = \frac{1}{\mu_0}\,\mathbb{E}\times\mathbb{B} \tag{15}$$

Calculating the following expression

```
FF = Simplify[1/2 F.(F)† /. Conjugate → ForceConjugate];
```

the energy density is

```
FullSimplify[1/2 Tr[ε0 FF]] // MatrixForm // CompactFormat
```

$$\frac{1}{2}\left(c^2\left(\mathbb{B}_1^2+\mathbb{B}_2^2+\mathbb{B}_3^2\right)+\mathbb{E}_1^2+\mathbb{E}_2^2+\mathbb{E}_3^2\right)\varepsilon_0$$

and the Poynting vector is

```
Expand@Simplify[Simplify[VectorPart[ε0 c FF]] /. {c² → 1/(ε0 μ0)}] // MatrixForm //
  CompactFormat
```

$$\begin{pmatrix} \frac{\mathbb{B}_2\mathbb{E}_1}{\mu_0}-\frac{\mathbb{B}_1\mathbb{E}_2}{\mu_0} & \frac{i\,\mathbb{B}_3\mathbb{E}_1}{\mu_0}+\frac{\mathbb{B}_3\mathbb{E}_2}{\mu_0}-\frac{i\,\mathbb{B}_1\mathbb{E}_3}{\mu_0}-\frac{\mathbb{B}_2\mathbb{E}_3}{\mu_0} \\ -\frac{i\,\mathbb{B}_3\mathbb{E}_1}{\mu_0}+\frac{\mathbb{B}_3\mathbb{E}_2}{\mu_0}+\frac{i\,\mathbb{B}_1\mathbb{E}_3}{\mu_0}-\frac{\mathbb{B}_2\mathbb{E}_3}{\mu_0} & -\frac{\mathbb{B}_2\mathbb{E}_1}{\mu_0}+\frac{\mathbb{B}_1\mathbb{E}_2}{\mu_0} \end{pmatrix}$$

Determinant of the electromagnetic field as Lorentz invariant

The fact that $\frac{1}{2}\mathrm{Tr}\left[F^2\right]$ is Lorentz invariant implies the invariance of two quantities

```
FullSimplify[1/2 Tr@HermitianPart[F.F] /. Conjugate → ForceConjugate] // CompactFormat //
  MatrixForm
```

$$-c^2\left(\mathbb{B}_1^2+\mathbb{B}_2^2+\mathbb{B}_3^2\right)+\mathbb{E}_1^2+\mathbb{E}_2^2+\mathbb{E}_3^2$$

```
FullSimplify[1/2 Tr@AntiHermitianPart[F.F] /. Conjugate → ForceConjugate] //
   CompactFormat // MatrixForm
```

$$2\,i\,c\,\left(\mathbb{B}_1\,\mathbb{E}_1+\mathbb{B}_2\,\mathbb{E}_2+\mathbb{B}_3\,\mathbb{E}_3\right)$$

Moreover

$$\frac{1}{2}\mathrm{Tr}\left[F^2\right] = -\mathrm{Det}[F] \tag{16}$$

```
-RealPart@Det[F] /. Conjugate → ForceConjugate // Simplify // CompactFormat
```

$-c^2\left(\mathbb{B}_1^2+\mathbb{B}_2^2+\mathbb{B}_3^2\right)+\mathbb{E}_1^2+\mathbb{E}_2^2+\mathbb{E}_3^2$

```
-ImaginaryPart@Det[F] /. Conjugate → ForceConjugate // Simplify // CompactFormat
```

$2\,i\,c\,(\mathbb{B}_1\,\mathbb{E}_1+\mathbb{B}_2\,\mathbb{E}_2+\mathbb{B}_3\,\mathbb{E}_3)$

Maxwell Equations

The Maxwell equations are

$$\bar{\partial} F = c\mu_0\,\bar{j}. \tag{17}$$

where the para-current is

$$j = c\,\rho + \boldsymbol{j}. \tag{18}$$

The four standard Maxwell equations are

Gauss Law :

$$\left\langle \bar{\partial} F\right\rangle_{SH} = \frac{\rho}{\epsilon_0} \tag{19}$$

Ampere Maxwell Law :

$$\left\langle \bar{\partial} F\right\rangle_{VH} = -c\mu_0\,\boldsymbol{j}. \tag{20}$$

No magnetic monopoles:

$$\left\langle \bar{\partial} F\right\rangle_{SA} = 0 \tag{21}$$

Faraday Maxwell:

$$\left\langle \bar{\partial} F\right\rangle_{VA} = 0 \tag{22}$$

Note : The coefficient $c\mu_0$ has units of impedance with value. For the vacuum we have

```
UnitSimplify[Quantity["SpeedOfLight"] * Quantity["MagneticConstant"]] // N
```

376.73 Ω

Quantum resistivity

```
2 Quantity["ElementaryCharge"]^2 / Quantity["ReducedPlanckConstant"]
1 / UnitSimplify[%]
```

$2\,e^2/\hbar$

2054.118 Ω

Note : The presence of magnetic monopoles would require to add a magnetic current as a **triparavector**, where the magnetic charge would be a pseudoscalar that changes sign by a spatial reflections.

Otherwise, electric charges are scalars that do not change under spatial reflections.

Homogeneous case:

$$\bar{\partial} F = 0 \tag{23}$$

Defining the electromagnetic field dependent on the spacetime.

```
F = EMF["SL2C"][{𝔼1[x⁰, x¹, x², x³], 𝔼2[x⁰, x¹, x², x³], 𝔼3[x⁰, x¹, x², x³]},
    {𝔹1[x⁰, x¹, x², x³], 𝔹2[x⁰, x¹, x², x³], 𝔹3[x⁰, x¹, x², x³]}] // Simplify;
```

The homogeneous Maxwell equations are

```
MaxwellEqs = ∂̄(F) // Simplify;
```

Gauss Law

$$\left\langle \bar{\partial} F \right\rangle_{SH} = 0 \tag{24}$$

$$\nabla \circ \mathbb{E} = 0 \tag{25}$$

```
Simplify[-ScalarPart@HermitianPart[∂̄(F)] /. Conjugate → ForceConjugate] //
  MatrixForm // CompactFormat
```

$$\begin{pmatrix} -\partial_1 \mathbb{E}_1 - \partial_2 \mathbb{E}_2 - \partial_3 \mathbb{E}_3 & 0 \\ 0 & -\partial_1 \mathbb{E}_1 - \partial_2 \mathbb{E}_2 - \partial_3 \mathbb{E}_3 \end{pmatrix}$$

Ampere Maxwell

$$\left\langle \bar{\partial} F \right\rangle_{HV} = 0 \tag{26}$$

$$-c \nabla \times \mathbb{B} + \frac{\partial \mathbb{E}}{\partial x^0} = 0 \tag{27}$$

```
Simplify[VectorPart@HermitianPart[∂̄(F)] /. Conjugate → ForceConjugate] // MatrixForm //
 CompactFormat
```

$$\begin{pmatrix} c\,\partial_2 \mathbb{B}_1 - c\,\partial_1 \mathbb{B}_2 + \partial_0 \mathbb{E}_3 & i\,c\,\partial_3 \mathbb{B}_1 + c\,\partial_3 \mathbb{B}_2 - i\,c\,\partial_1 \mathbb{B}_3 - c\,\partial_2 \mathbb{B}_3 + \partial_0 \mathbb{E}_1 - i\,\partial_(\\ -i\,c\,\partial_3 \mathbb{B}_1 + c\,\partial_3 \mathbb{B}_2 + i\,c\,\partial_1 \mathbb{B}_3 - c\,\partial_2 \mathbb{B}_3 + \partial_0 \mathbb{E}_1 + i\,\partial_0 \mathbb{E}_2 & -c\,\partial_2 \mathbb{B}_1 + c\,\partial_1 \mathbb{B}_2 - \partial_0 \mathbb{E}_3 \end{pmatrix}$$

Comparing with the standard formulation

```
c Curl[ {𝔹1[x⁰, x¹, x², x³], 𝔹2[x⁰, x¹, x², x³], 𝔹3[x⁰, x¹, x², x³]}, {x¹, x², x³}] -
   D[{𝔼1[x⁰, x¹, x², x³], 𝔼2[x⁰, x¹, x², x³], 𝔼3[x⁰, x¹, x², x³]}, x⁰] //
  TableForm // CompactFormat
```

$c\,(-\partial_3 \mathbb{B}_2 + \partial_2 \mathbb{B}_3) - \partial_0 \mathbb{E}_1$

$c\,(\partial_3 \mathbb{B}_1 - \partial_1 \mathbb{B}_3) - \partial_0 \mathbb{E}_2$

$c\,(-\partial_2 \mathbb{B}_1 + \partial_1 \mathbb{B}_2) - \partial_0 \mathbb{E}_3$

No magnetic monopoles

$$\left\langle \bar{\partial} F \right\rangle_{AS} = 0 \tag{28}$$

$$c \nabla \circ \mathbb{B} = 0 \tag{29}$$

```
Simplify[ ScalarPart@AntiHermitianPart[∂̄(F)] /. Conjugate → ForceConjugate] //
  MatrixForm // CompactFormat
```

$$\begin{pmatrix} i\,c\,(\partial_1 \mathbb{B}_1+\partial_2 \mathbb{B}_2+\partial_3 \mathbb{B}_3) & 0 \\ 0 & i\,c\,(\partial_1 \mathbb{B}_1+\partial_2 \mathbb{B}_2+\partial_3 \mathbb{B}_3) \end{pmatrix}$$

Maxwell Faraday

$$\left\langle \bar{\partial} F \right\rangle_{AV} = 0 \tag{30}$$

$$\nabla \times \mathbb{E} + c\,\frac{\partial \mathbb{B}}{\partial x^0} = 0 \tag{31}$$

```
ExpandAll[ -I VectorPart@AntiHermitianPart[∂̄(F)] /. Conjugate → ForceConjugate] //
  MatrixForm // CompactFormat
```

$$\begin{pmatrix} c\,\partial_0 \mathbb{B}_3 - \partial_2 \mathbb{E}_1 + \partial_1 \mathbb{E}_2 & c\,\partial_0 \mathbb{B}_1 - i\,c\,\partial_0 \mathbb{B}_2 - i\,\partial_3 \mathbb{E}_1 - \partial_3 \mathbb{E}_2 + i\,\partial_1 \mathbb{E}_3 + \partial_2 \mathbb{E}_3 \\ c\,\partial_0 \mathbb{B}_1 + i\,c\,\partial_0 \mathbb{B}_2 + i\,\partial_3 \mathbb{E}_1 - \partial_3 \mathbb{E}_2 - i\,\partial_1 \mathbb{E}_3 + \partial_2 \mathbb{E}_3 & -c\,\partial_0 \mathbb{B}_3 + \partial_2 \mathbb{E}_1 - \partial_1 \mathbb{E}_2 \end{pmatrix}$$

Comparing with the standard formulation

```
Curl[ {𝔼1[x⁰, x¹, x², x³], 𝔼2[x⁰, x¹, x², x³], 𝔼3[x⁰, x¹, x², x³]}, {x¹, x², x³}] +
   c D[{𝔹1[x⁰, x¹, x², x³], 𝔹2[x⁰, x¹, x², x³], 𝔹3[x⁰, x¹, x², x³]}, x⁰] //
  CompactFormat // TableForm
```

$c\,\partial_0 \mathbb{B}_1 - \partial_3 \mathbb{E}_2 + \partial_2 \mathbb{E}_3$
$c\,\partial_0 \mathbb{B}_2 + \partial_3 \mathbb{E}_1 - \partial_1 \mathbb{E}_3$
$c\,\partial_0 \mathbb{B}_3 - \partial_2 \mathbb{E}_1 + \partial_1 \mathbb{E}_2$

Lorentz transformations for Biparavectors

The transformation law for the electromagnetic field is different compared with the transformation for paravectors

$$F \to F' = L\,F\,\bar{L}. \tag{32}$$

Let us remind that paravectors transform as

$$x \to x' = L\,x\,L^{\dagger}. \tag{33}$$

```
Simplify[BoostSL2C[ u⁰, 0, 0, u³].F.CliffordConjugationSL2C@BoostSL2C[ u⁰, 0, 0, u³]];
FullSimplify[EMFUpIndexComponents[%] /. {u³² → u⁰² - c²}];
Simplify[% /. {u³² → u⁰² - c²}] // CompactFormat // MatrixForm
```

$$\begin{pmatrix} 0 & -\frac{c\,u^3\,\mathbb{B}_2+u^0\,\mathbb{E}_1}{c} & u^3\,\mathbb{B}_1-\frac{u^0\,\mathbb{E}_2}{c} & -\mathbb{E}_3 \\ u^3\,\mathbb{B}_2+\frac{u^0\,\mathbb{E}_1}{c} & 0 & -c\,\mathbb{B}_3 & u^0\,\mathbb{B}_2+\frac{u^3\,\mathbb{E}_1}{c} \\ -u^3\,\mathbb{B}_1+\frac{u^0\,\mathbb{E}_2}{c} & c\,\mathbb{B}_3 & 0 & -u^0\,\mathbb{B}_1+\frac{u^3\,\mathbb{E}_2}{c} \\ \mathbb{E}_3 & -\frac{c\,u^0\,\mathbb{B}_2+u^3\,\mathbb{E}_1}{c} & u^0\,\mathbb{B}_1-\frac{u^3\,\mathbb{E}_2}{c} & 0 \end{pmatrix}$$

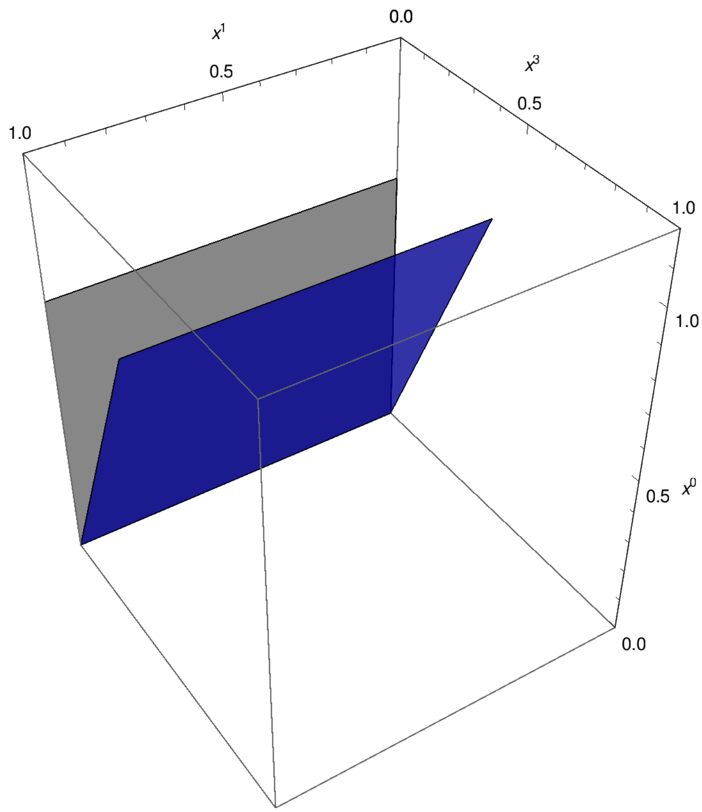
The Lorentz transformation does not affect the parallel components of the electromagnetic field

```
{Tr[EMF["SL2C"][{0, 0, 𝔼₃}, {0, 0, 𝔹₃}].σ₃ / 2] →
    ExpandAll@Tr[BoostSL2C[ u⁰, 0, 0, u³].EMF["SL2C"][{0, 0, 𝔼₃}, {0, 0, 𝔹₃}].
        CliffordConjugationSL2C@BoostSL2C[ u⁰, 0, 0, u³].
        σ₃ / 2] /. {u³² → u⁰² - c²} // Simplify}
```

$\{i\, c\, \mathbb{B}_3 + \mathbb{E}_3 \to i\, c\, \mathbb{B}_3 + \mathbb{E}_3\}$

The following figure represents an electric field $\mathbb{E}_1$ in a rest frame as a gray plane $x^0 - x^1$ and the electromagnetic field (blue) as a result of a boost along u^3. The boosted electromagnetic field is composed of an electric field $\mathbb{E}'_1$ and an additional a magnetic field $\mathbb{B}'_2$ spanning the plane $x^1 - x^3$

```
Show[
 BoostedE1[0.8, 0., Gray],
 BoostedE1[0.8, 0.5, Blue]
]
```



Lorentz Force

The Lorentz force is calculated as

$$m \frac{d}{d\tau} u^{\mu} = \frac{e}{c} F^{\mu\nu} u_{\nu} = \left\langle \frac{e}{c} F\, u \right\rangle_H \tag{34}$$

```
Simplify[1/c ⟨F.u⟩"H" /. Conjugate → ForceConjugate] // Simplify // CompactFormat //
 MatrixForm
```

$$\begin{pmatrix} \frac{-c\, u^2\, \mathbb{B}_1 + c\, u^1\, \mathbb{B}_2 + u^1\, \mathbb{E}_1 + u^2\, \mathbb{E}_2 + u^0\, \mathbb{E}_3 + u^3\, \mathbb{E}_3}{c} & -i\, u^3\, \mathbb{B}_1 - u^3\, \mathbb{B}_2 + i\, u^1\, \mathbb{B}_3 + u^2\, \mathbb{B}_3 + \frac{u^0\, \mathbb{E}_1}{c} - \frac{i\, u^0\, \mathbb{E}_2}{c} \\ i\, u^3\, \mathbb{B}_1 - u^3\, \mathbb{B}_2 - i\, u^1\, \mathbb{B}_3 + u^2\, \mathbb{B}_3 + \frac{u^0\, \mathbb{E}_1}{c} + \frac{i\, u^0\, \mathbb{E}_2}{c} & \frac{c\, u^2\, \mathbb{B}_1 - c\, u^1\, \mathbb{B}_2 + u^1\, \mathbb{E}_1 + u^2\, \mathbb{E}_2 - u^0\, \mathbb{E}_3 + u^3\, \mathbb{E}_3}{c} \end{pmatrix}$$

Taking components we recover the standard formulas

```
1/2 Tr[(1/c ⟨F.u⟩"H").σ0] /. Conjugate → ForceConjugate // Simplify
```

$$\frac{1}{c}\left(u^1\, \mathbb{E}_1[x^0, x^1, x^2, x^3] + u^2\, \mathbb{E}_2[x^0, x^1, x^2, x^3] + u^3\, \mathbb{E}_3[x^0, x^1, x^2, x^3]\right)$$

```
1/2 Tr[(1/c ⟨F.u⟩"H").σ1] /. Conjugate → ForceConjugate // Simplify
```

$$-u^3\, \mathbb{B}_2[x^0, x^1, x^2, x^3] + u^2\, \mathbb{B}_3[x^0, x^1, x^2, x^3] + \frac{u^0\, \mathbb{E}_1[x^0, x^1, x^2, x^3]}{c}$$

```
1/2 Tr[(1/c ⟨F.u⟩"H").σ1] /. Conjugate → ForceConjugate // Simplify
```

$$-u^3\, \mathbb{B}_2[x^0, x^1, x^2, x^3] + u^2\, \mathbb{B}_3[x^0, x^1, x^2, x^3] + \frac{u^0\, \mathbb{E}_1[x^0, x^1, x^2, x^3]}{c}$$

```
1/2 Tr[(1/c ⟨F.u⟩"H").σ1] /. Conjugate → ForceConjugate // Simplify
```

$$-u^3\, \mathbb{B}_2[x^0, x^1, x^2, x^3] + u^2\, \mathbb{B}_3[x^0, x^1, x^2, x^3] + \frac{u^0\, \mathbb{E}_1[x^0, x^1, x^2, x^3]}{c}$$

Spinorial Classical Dynamics

The proper velocity u can be written in terms of the associated spinor Λ (also known as proper spinor or eigenspinor)

$$u = c\,\Lambda\,\Lambda^{\dagger}. \tag{35}$$

The most general equation of motion for the spinor can be written in terms of a biparavector Ω

$$\frac{d}{d\tau}\Lambda = \frac{1}{2}\,\Omega\,\Lambda\,, \tag{36}$$

where we can identify Ω in terns of the electromagnetic field F as

$$\Omega = \frac{e}{m\,c}\,F\,, \tag{37}$$

yielding the dynamical equation

$$\frac{d}{d\tau}\Lambda = \frac{e}{2\,m\,c}\,F\,\Lambda\,. \tag{38}$$

In general, the complete dynamics is determined by the coupled equations:

$$\begin{aligned} \frac{d}{d\tau}\Lambda &= \frac{e}{2\,m\,c}\,F\,\Lambda\,, \\ \frac{d}{d\tau}x &= c\,\Lambda\,\Lambda^{\dagger}\,. \end{aligned} \tag{39}$$

However, it is possible to show that for directed electromagnetic waves, these equations decouple and lead to the classical Volkov solutions.

Charged particle in a constant electric field

Let us take the case of an electric field along the x direction

$$F = \mathbb{E}_0\,\sigma_1$$

The integration of Eq (21) gives

$$\Lambda = e^{\frac{e\,\mathbb{E}_0}{2\,m\,c}\sigma_1\,\tau}\;\Lambda\,(0) \tag{40}$$

The initial the condition

$$\Lambda\,(0) = 1 \tag{41}$$

results in

$$\Lambda\text{Sol} = \left\{\Lambda \to \text{MatrixExp}\left[\frac{e\,\mathbb{E}_0}{2\,m\,c}\,\sigma_1\,\tau\right]\right\} \;//\; \text{ExpToTrig}$$

$$\left\{\Lambda \to \left\{\left\{\text{Cosh}\left[\frac{e\,\mathbb{E}_0\,\tau}{2\,c\,m}\right],\,\text{Sinh}\left[\frac{e\,\mathbb{E}_0\,\tau}{2\,c\,m}\right]\right\},\,\left\{\text{Sinh}\left[\frac{e\,\mathbb{E}_0\,\tau}{2\,c\,m}\right],\,\text{Cosh}\left[\frac{e\,\mathbb{E}_0\,\tau}{2\,c\,m}\right]\right\}\right\}\right\}$$

The proper velocity is

$$u = \Lambda\,\Lambda^{\dagger} = e^{\frac{e\,\mathbb{E}_0}{m\,c}\sigma_1\,\tau} \tag{42}$$

$$\left(\text{uSol} = \left\{u \to \Lambda.\Lambda \;\;/.\; \Lambda\text{Sol} \;//\; \text{ExpToTrig} \;//\; \text{Simplify}\right\}\right);$$

For a particular initial condition, the spacetime trajectory can be integrated as

$$x = \frac{m\,c}{e\,\mathbb{E}_0}\;e^{\frac{e\,\mathbb{E}_0}{m\,c}\sigma_1\,\tau}\;\sigma_1 \tag{43}$$

$$\text{xSol} = \left\{X \to \frac{m\,c}{e\,\mathbb{E}_0}\,\text{MatrixExp}\left[\frac{e\,\mathbb{E}_0}{2\,m\,c}\,\sigma_1\,\tau\right].\sigma_1\right\}$$

$$\left\{X \to \left\{\left\{\frac{c\left(-\frac{1}{2}e^{-\frac{e\,\mathbb{E}_0\,\tau}{2\,c\,m}}+\frac{1}{2}e^{\frac{e\,\mathbb{E}_0\,\tau}{2\,c\,m}}\right)m}{e\,\mathbb{E}_0},\,\frac{c\left(\frac{1}{2}e^{-\frac{e\,\mathbb{E}_0\,\tau}{2\,c\,m}}+\frac{1}{2}e^{\frac{e\,\mathbb{E}_0\,\tau}{2\,c\,m}}\right)m}{e\,\mathbb{E}_0}\right\},\right.\right.$$
$$\left.\left.\left\{\frac{c\left(\frac{1}{2}e^{-\frac{e\,\mathbb{E}_0\,\tau}{2\,c\,m}}+\frac{1}{2}e^{\frac{e\,\mathbb{E}_0\,\tau}{2\,c\,m}}\right)m}{e\,\mathbb{E}_0},\,\frac{c\left(-\frac{1}{2}e^{-\frac{e\,\mathbb{E}_0\,\tau}{2\,c\,m}}+\frac{1}{2}e^{\frac{e\,\mathbb{E}_0\,\tau}{2\,c\,m}}\right)m}{e\,\mathbb{E}_0}\right\}\right\}\right\}$$

The spacetime trajectory parametrized by the proper time can be extracted as

$$\text{ct} \;\text{->}\; \text{Tr}\left[X.\sigma_0 \;/.\; \text{xSol}\right] \;//\; \text{ExpToTrig}$$

$$\text{ct} \to \frac{2\,c\,m\,\text{Sinh}\left[\frac{e\,\mathbb{E}_0\,\tau}{2\,c\,m}\right]}{e\,\mathbb{E}_0}$$

```
X¹ → Tr[X.σ₁ /. xSol] // ExpToTrig
```

$$X^1 \to \frac{2\, c\, m\, \mathrm{Cosh}\left[\frac{e\, \mathbb{E}_0\, \tau}{2\, c\, m}\right]}{e\, \mathbb{E}_0}$$

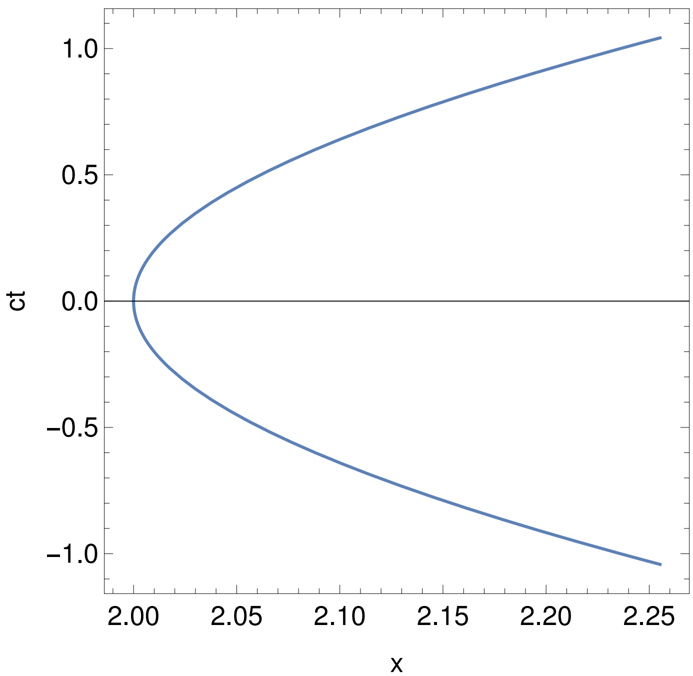
```
X² → Tr[X.σ₂ /. xSol] // ExpToTrig
```

$X^2 \to 0$

```
X³ → Tr[X.σ₃ /. xSol] // ExpToTrig
```

$X^3 \to 0$

```
ParametricPlot[
 Evaluate[{Tr[X.σ₁ /. xSol], Tr[X.σ₀ /. xSol]} /. {e → 1, 𝔼₀ → 1, m → 1, c → 1}]
 , {τ, -1, 1}, Frame → True, FrameLabel → {"x", "ct"}, AspectRatio → 1,
 BaseStyle → {FontSize → 14}]
```



The determinant of the classical spinor is always 1

```
Det[Λ /. ΛSol] // Simplify
```

1

The Dirac Equation

The Dirac equation can be written as

$$i\, c\, \hbar\, \overline{\partial}\, \Psi\, \sigma_3 - c\, e\, \overline{A}\, \Psi = m\, c^2\, \Psi, \qquad (44)$$

where the spinor Ψ is related to the standard Dirac spinor (Dirac basis) as

$$\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \Longleftrightarrow \begin{pmatrix} \psi_1 + \psi_3 & -\psi_2{}^* + \psi_4{}^* \\ \psi_2 + \psi_4 & \psi_1{}^* - \psi_3{}^* \end{pmatrix}. \tag{45}$$

In the Weyl column representation we have

$$\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \Longleftrightarrow \sqrt{2} \begin{pmatrix} \psi_3 & -\psi_2{}^* \\ \psi_4 & \psi_1{}^* \end{pmatrix}. \tag{46}$$

The current is given by

$$J = \Psi\Psi^\dagger, \tag{47}$$

which is similar to the formula for the proper velocity

$$u = \Lambda\Lambda^\dagger, \tag{48}$$

but in the case of the Dirac spinor we have

$$\mathrm{Det}[\Psi] = e^{i\beta}, \tag{49}$$

where β is the Yvon-Takabayashi angle where β=0 for particles and β = -π for antiparticles.

The vector potential can be solved as

$$e\bar{A} = \left(i\,\hbar\bar{\partial}\,\Psi\sigma_3 - m\,c\Psi\right)\Psi^{-1}. \tag{50}$$

```
DiracEquationSL2C[ψ_?MatrixQ, A_?MatrixQ, m_] := Module[{K, ψc},
  ψc = ConjugateTranspose[CliffordConjugationSL2C[ψ]];
  K = ⅈ ℏ (σ0.∂t ψ + c σ1.∂x1 ψ + c σ2.∂x2 ψ + c σ3.∂x3 ψ).σ3 ;
   K - c Ā.ψ - m c^2 ψc
 ]
```

References

1. Baylis, W.E., Electrodynamics : a modern geometric approach, Birkhauser, 1999
2. Baylis, W.E., Clifford (Geometric) Algebras : with applications to physics, mathematics, and engineering, Birkhauser, 1996
3. Baylis, W.E., Classical eigenspinors and the Dirac equation, Phys. Rev. A, **45**, 4293, 1992