Practical Introduction to (Geometric) Clifford Algebras with applications to Relativity

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Routines

Space-Time Algebra (STA)

The Space-Time Algebra (STA) was introduced by Hestenes in order to describe the Dirac equation without relying on matrices. Compared with Baylis's APS algebra, STA is naturally represented in terms of the gamma matrices. This may be the reason that STA looks more familiar than APS. Again, these lecture notes depart from the axiomatic approach in the literature by the extensive use of the matrix representation. The hope is that the student will learn in a more familiar environment based on standard linear algebra.

Basics

The gamma matrices form the STA algebra. In the Dirac matrix representation we have for example

Scalar	$\mathbb{I} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
Vector	$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ $\gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$ $\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}$ $\gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$
Bivector	$\gamma^0\,\gamma^1$ $\gamma^0\,\gamma^2$ $\gamma^0\,\gamma^3$ $\gamma^1\,\gamma^2$ $\gamma^2\,\gamma^3$ $\gamma^3\,\gamma^1$
Trivector	$\gamma^0\,\gamma^1\,\gamma^2$ $\gamma^0\,\gamma^1\,\gamma^3$ $\gamma^0\,\gamma^2\,\gamma^3$ $\gamma^1\,\gamma^2\,\gamma^3$
PseudoScalar	$\gamma^{0123} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}$

The STA algebra contains the Minkowski metric $g^{\mu\nu}$ according to:

$$\gamma^\mu\,\gamma^\nu + \gamma^\nu\,\gamma^\mu = 2\,g^{\mu\nu}\,\mathbb{I}, \tag{1}$$

valid for any matrix representation. We can verify this as follows

$$\mathbf{Table}\left[\frac{1}{4}\,\mathbf{Tr}\left[\gamma^\mu.\gamma^\nu + \gamma^\nu.\gamma^\mu\right], \{\mu, 0, 3\}, \{\nu, 0, 3\}\right] \,//\, \mathbf{MatrixForm}$$

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

The volume element or pseudoscalar γ^{0123} anticommutes with all vectors

$$\left[\gamma^\mu\,\gamma^{0123},\ \gamma^{0123}\,\gamma^\mu\right]_+ = 0 \tag{2}$$

```
MatrixForm /@ Table[
  γ^μ.(γ^0.γ^1.γ^2.γ^3) == -(γ^0.γ^1.γ^2.γ^3).γ^μ, {μ, 0, 3}]
```

```
{True, True, True, True}
```

The volume element times the complex number is the conventional γ^5

$$\gamma^5 = i\,\gamma^{0123} \tag{3}$$

```
i (γ^0.γ^1.γ^2.γ^3) == γ^5
```

```
True
```

γ^5 anti-commutes with all vectors

$$[\gamma^\mu\,\gamma^5,\ \ \gamma^5\,\gamma^\mu]_+ = 0 \tag{4}$$

```
MatrixForm /@ Table[
  γ^μ.(γ^5) == -(γ^5).γ^μ, {μ, 0, 3}]
```

```
{True, True, True, True}
```

The gamma matrices with down indices are defined as

$$\gamma^\mu = \ (\gamma_\mu)^{-1} \tag{5}$$

and obey

$$\gamma^\nu = \gamma_\mu\, g^{\mu\nu} \tag{6}$$

Reversion Conjugation ~

The reversion conjugation inverts the order of the product of gamma matrices.

The reversion conjugation can be computed as

$$\tilde{A} = \gamma^0\, A^\dagger\, \gamma^0 . \tag{7}$$

The reversion conjugation obeys

$$\widetilde{A\,B} = \tilde{B}\ \tilde{A}\ . \tag{8}$$

The reversion conjugation applied to a vector remains the same

$$\tilde{\gamma}^\mu = \gamma^\mu \tag{9}$$

```
Table[
 ReversionConjugationSpin13[γ^μ] == γ^μ
 , {μ, 0, 3}]
```

```
{True, True, True, True}
```

```
Table[ γ̃^μ == γ^μ, {μ, 0, 3}]
```

```
{True, True, True, True}
```

The reversion conjugation of a bivector reverses the order of the product

$$\widetilde{\gamma^\mu\,\gamma^\nu} = \gamma^\nu\,\gamma^\mu \tag{10}$$

```
Table[
  ReversionConjugationSpin13[γ^μ.γ^ν] == γ^ν.γ^μ
  , {μ, 0, 3}, {ν, 0, 3}] // MatrixForm
```

$$\begin{pmatrix} \text{True} & \text{True} & \text{True} & \text{True} \\ \text{True} & \text{True} & \text{True} & \text{True} \\ \text{True} & \text{True} & \text{True} & \text{True} \\ \text{True} & \text{True} & \text{True} & \text{True} \end{pmatrix}$$

Similar result is obtained for trivectors

```
Table[
  ReversionConjugationSpin13[γ^μ.γ^ν.γ^κ] == γ^κ.γ^ν.γ^μ
  , {μ, 0, 3}, {ν, 0, 3}, {κ, 0, 3}] // MatrixForm
```

$$\begin{pmatrix} \begin{pmatrix} \text{True} \\ \text{True} \\ \text{True} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \text{True} \\ \text{True} \\ \text{True} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \text{True} \\ \text{True} \\ \text{True} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \text{True} \\ \text{True} \\ \text{True} \\ \text{True} \end{pmatrix} \\ \begin{pmatrix} \text{True} \\ \text{True} \\ \text{True} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \text{True} \\ \text{True} \\ \text{True} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \text{True} \\ \text{True} \\ \text{True} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \text{True} \\ \text{True} \\ \text{True} \\ \text{True} \end{pmatrix} \\ \begin{pmatrix} \text{True} \\ \text{True} \\ \text{True} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \text{True} \\ \text{True} \\ \text{True} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \text{True} \\ \text{True} \\ \text{True} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \text{True} \\ \text{True} \\ \text{True} \\ \text{True} \end{pmatrix} \\ \begin{pmatrix} \text{True} \\ \text{True} \\ \text{True} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \text{True} \\ \text{True} \\ \text{True} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \text{True} \\ \text{True} \\ \text{True} \\ \text{True} \end{pmatrix} & \begin{pmatrix} \text{True} \\ \text{True} \\ \text{True} \\ \text{True} \end{pmatrix} \end{pmatrix}$$

including the pseudoscalar

```
ReversionConjugationSpin13[γ^0.γ^1.γ^2.γ^3] == γ^3.γ^2.γ^1.γ^0
```

```
True
```

Hermiticity/AntiHermiticity of the bivector space:

$$\left(\gamma^0 \gamma^k\right)^\dagger = \gamma^0 \gamma^k \tag{11}$$

```
Table[
 ConjugateTranspose[γ^0.γ^k] == γ^0.γ^k
 , {k, 1, 3}]
```

```
{True, True, True}
```

AntiHermiticity of the bivector subspace:

$$\begin{aligned} \left(\gamma^1 \gamma^2\right)^\dagger &= -\gamma^1 \gamma^2 \\ \left(\gamma^2 \gamma^3\right)^\dagger &= -\gamma^3 \gamma^4 \\ \left(\gamma^3 \gamma^1\right)^\dagger &= -\gamma^3 \gamma^1 \end{aligned} \tag{12}$$

```
Table[
 ConjugateTranspose[γ¹.γ²] == -γ¹.γ²
 , {μ, 1, 3}]
```

```
{True, True, True}
```

```
Table[
 ConjugateTranspose[γ².γ³] == -γ².γ³
 , {μ, 1, 3}]
```

```
{True, True, True}
```

```
Table[
 ConjugateTranspose[γ³.γ¹] == -γ³.γ¹
 , {μ, 1, 3}]
```

```
{True, True, True}
```

Spacetime vector

Definition

The spacetime can be represented as a vector

$$x = x^{\mu} \gamma_{\mu}. \tag{13}$$

where the bold symbols stand for the standard three-dimensional vectors

```
(x = x⁰ γ₀ + x¹ γ₁ + x² γ₂ + x³ γ₃) // MatrixForm
```

$$\begin{pmatrix} x^0 & 0 & -x^3 & -x^1 + i x^2 \\ 0 & x^0 & -x^1 - i x^2 & x^3 \\ x^3 & x^1 - i x^2 & -x^0 & 0 \\ x^1 + i x^2 & -x^3 & 0 & -x^0 \end{pmatrix}$$

The components can be extracted with the help of the trace

```
1/4 Tr[x.γ¹ ]
```

x^1

Similarly, the proper velocity is

$$u = u^{\mu} \gamma$$

μ

```
(u = u⁰ γ₀ + u¹ γ₁ + u² γ₂ + u³ γ₃) // MatrixForm
```

$$\begin{pmatrix} u^0 & 0 & -u^3 & -u^1 + i u^2 \\ 0 & u^0 & -u^1 - i u^2 & u^3 \\ u^3 & u^1 - i u^2 & -u^0 & 0 \\ u^1 + i u^2 & -u^3 & 0 & -u^0 \end{pmatrix}$$

The contraction of vectors is accomplished as

```
1/4 Tr[x.u] // Simplify
```

$u^0 x^0 - u^1 x^1 - u^2 x^2 - u^3 x^3$

```
1/4 Tr[u.u] // Simplify
```

${u^0}^2 - {u^1}^2 - {u^2}^2 - {u^3}^2$

The spacetime position has 4 degrees of freedom but the proper velocity has only 3 because it must obey the shell-mass constrain

$$\left(u^0\right)^2 - \left(u^1\right)^2 - \left(u^2\right)^2 - \left(u^3\right)^2 = c^2 \tag{14}$$

also written as

$$\frac{1}{4}\mathrm{Tr}[u.u] = c^2 \tag{15}$$

Lorentz Transformations for paravectors

Theory

The active Lorentz Boost is carried out by double side multiplication with the use of the reversion conjugation

$$x \to y = \mathrm{L}\, x\, \tilde{\mathrm{L}}. \tag{16}$$

Utilizing the formula for the reversion we have

$$x \to y = \mathrm{L}\, x\, \gamma^0\, \mathrm{L}^\dagger\, \gamma^0. \tag{17}$$

Using $L = B$ we finally have

$$y = \mathrm{B}\, x\, \gamma^0\, \mathrm{B}\, \gamma^0 = \mathrm{B}\, x\, \mathrm{B}^{-1}. \tag{18}$$

The proper velocity at rest is

$$u_{rest} = c\, \gamma^0 . \tag{19}$$

The boost of the proper velocity at rest leads to an arbitrary proper velocity

$$u = \mathrm{B}\left(c\, \gamma^0\right)\gamma^0\, \mathrm{B}\, \gamma^0 = c\, \mathrm{B}^2\, \gamma^0. \tag{20}$$

Therefore

$$\mathrm{B} = \sqrt{\frac{u\, \gamma^0}{c}}, \tag{21}$$

which can be written as

$$\mathrm{B} = \frac{u\, \gamma^0 + c\, \mathbb{I}}{\sqrt{2\, c\, (c + u^0)}}. \tag{22}$$

The restricted Lorentz group as a double cover is defined as

$$\mathrm{Spin}_+(1,3) = \{ L \mid L\tilde{L} = 1 \}. \tag{23}$$

This group contains all the physically realizable transformations that preserve the direction of time and is isomorphic to the complex special linear group employed by the APS algebra

$$\mathrm{Spin}_+(1,3) \approx SL(2,\mathbb{C}). \tag{24}$$

Evaluations

```
BoostSpin13["Dirac"][Pattern[u⁰, _], Pattern[u¹, _], Pattern[u², _], Pattern[u³, _]] :=
 ((u⁰ γ₀ + u¹ γ₁ + u² γ₂ + u³ γ₃).γ⁰ + c γ⁰.γ⁰) / (Sqrt[2 c (c + u⁰)])
```

Example: The boost operator along x^3 is

```
BoostSpin13["Dirac"][u⁰, 0, 0, u³] // MatrixForm
```

$$\begin{pmatrix} \frac{c+u^0}{\sqrt{2}\sqrt{c(c+u^0)}} & 0 & \frac{u^3}{\sqrt{2}\sqrt{c(c+u^0)}} & 0 \\ 0 & \frac{c+u^0}{\sqrt{2}\sqrt{c(c+u^0)}} & 0 & -\frac{u^3}{\sqrt{2}\sqrt{c(c+u^0)}} \\ \frac{u^3}{\sqrt{2}\sqrt{c(c+u^0)}} & 0 & \frac{c+u^0}{\sqrt{2}\sqrt{c(c+u^0)}} & 0 \\ 0 & -\frac{u^3}{\sqrt{2}\sqrt{c(c+u^0)}} & 0 & \frac{c+u^0}{\sqrt{2}\sqrt{c(c+u^0)}} \end{pmatrix}$$

The active boost of the coordinates is

```
(y =
   Simplify[BoostSpin13["Dirac"][u⁰, 0, 0, u³].x.BoostSpin13["Dirac"][u⁰, 0, 0, -u³]] /.
    {u³² → u⁰² - c²} // Simplify) // MatrixForm
```

$$\begin{pmatrix} \frac{u^0 x^0+u^3 x^3}{c} & 0 & -\frac{u^3 x^0+u^0 x^3}{c} & -x^1+i x^2 \\ 0 & \frac{u^0 x^0+u^3 x^3}{c} & -x^1-i x^2 & \frac{u^3 x^0+u^0 x^3}{c} \\ \frac{u^3 x^0+u^0 x^3}{c} & x^1-i x^2 & -\frac{u^0 x^0+u^3 x^3}{c} & 0 \\ x^1+i x^2 & -\frac{u^3 x^0+u^0 x^3}{c} & 0 & -\frac{u^0 x^0+u^3 x^3}{c} \end{pmatrix}$$

The transformation of coordinates reads

```
MapThread[Equal, {{y⁰, y¹, y², y³},
   Simplify[Simplify[1/4 Tr[y.γ#] & /@ {0, 1, 2, 3}] /. {u³² → u⁰² - c²}]}] // TableForm
```

$$y^0 == \frac{u^0 x^0+u^3 x^3}{c}$$
$$y^1 == x^1$$
$$y^2 == x^2$$
$$y^3 == \frac{u^3 x^0+u^0 x^3}{c}$$

The Dirac equation

Theory

The Dirac equation for the column spinor is

$$i\hbar c \partial \psi - ceA \psi - m c^2 \psi = 0. \tag{25}$$

The Dirac-Hestenes equation for the matrix spinor is

$$\hbar c \partial \Psi \gamma^2 \gamma^1 - c A \Psi - m c^2 \Psi \gamma^0 = 0. \tag{26}$$

Defining the projector

$$Q = \frac{1}{4} \left(\mathbb{I} + \gamma^0\right) \left(\mathbb{I} - \gamma^5 \gamma^3\right), \tag{27}$$

and applying to the right side of the Dirac-Hestenes equation

$$\hbar c \partial \Psi \gamma^2 \gamma^1 Q - c A \Psi Q - m c^2 \Psi \gamma^0 Q = 0, \tag{28}$$

we recover the standard Dirac equation identifying

$$\psi = \Psi Q, \tag{29}$$

which basically means that the standard spinor is the first column of the matrix spinor of the Dirac-Hestenes equation.

In fact, the relation between them (Dirac representation) is one to one according to

$$\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \Longleftrightarrow \begin{pmatrix} \psi_1 & -\psi_2{}^* & \psi_3 & \psi_4{}^* \\ \psi_2 & \psi_1{}^* & \psi_4 & -\psi_3{}^* \\ \psi_3 & \psi_4{}^* & \psi_1 & -\psi_2{}^* \\ \psi_4 & -\psi_3{}^* & \psi_2 & \psi_1{}^* \end{pmatrix}. \tag{30}$$

The current is constructed as

$$J = \Psi \Psi^\dagger \gamma^0. \tag{31}$$

This definition is consistent with the common definition of current

$$j^\mu = \frac{1}{4} \mathrm{Tr}\left[\Psi \Psi^\dagger \gamma^0 \gamma^\mu\right] = \psi^\dagger \gamma^0 \gamma^\mu \psi. \tag{32}$$

The transformation law for spinors is

$$\Psi \to L \Psi \tag{33}$$

Applying to the current J

$$\begin{gathered} J \to \left(L \Psi \gamma^0\right) \left(L \Psi \gamma^0\right)^\dagger \gamma^0 = \\ \left(L \Psi \gamma^0\right) \left(\gamma^0 \Psi^\dagger L^\dagger\right) \gamma^0 = L \Psi \Psi^\dagger L^\dagger \gamma^0 = L \Psi \Psi^\dagger \gamma^0 \gamma^0 L^\dagger \gamma^0 \end{gathered} \tag{34}$$

we can arrive to the conclusion that it transforms properly, i.e. the current is Lorentz covariant

$$J \to L \left(\Psi \Psi^\dagger \gamma^0\right) \tilde{L}. \tag{35}$$

Evaluations

```
DiracColumnToSpin13["Dirac"][{ψ1 : _, ψ2 : _, ψ3 : _, ψ4 : _}] :=
 ( ψ1  -Conjugate[ψ2]  ψ3   Conjugate[ψ4]
   ψ2   Conjugate[ψ1]  ψ4  -Conjugate[ψ3]
   ψ3   Conjugate[ψ4]  ψ1  -Conjugate[ψ2]
   ψ4  -Conjugate[ψ3]  ψ2   Conjugate[ψ1] )
```

```
DiracEquation[ψ_, A_?MatrixQ, m_] := Block[{pre},
  ⅈ ℏ * (γ^0.D[ψ, t] + c γ^1.D[ψ, x^1] + c γ^2.D[ψ, x^2] + c γ^3.D[ψ, x^3]) - c A.ψ - m c^2 ψ
 ]
```

```
DiracEquationSpin13["Dirac"][ψ_?MatrixQ, A_?MatrixQ, m_] :=
 Block[{g12 = γ^2.γ^1, g0 = γ^0, pre},
  ℏ * (γ^0.D[ψ, t] + c γ^1.D[ψ, x^1] + c γ^2.D[ψ, x^2] + c γ^3.D[ψ, x^3]).g12 - c A.ψ - m c^2 ψ.g0
 ]
```

```
(QProjector = 1/4 (𝕀 + γ^0).(𝕀 - (γ^5).(γ^3))) // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The projector has the following properties

$$\gamma^2 \gamma^1 Q = i Q, \tag{36}$$

$$\gamma^0 Q = Q, \tag{37}$$

which can be verified as follows

```
γ^2.γ^1.QProjector // MatrixForm
```

$$\begin{pmatrix} i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

```
γ^0.QProjector // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Constructing the matrix spinor

```
Ψ = DiracColumnToSpin13["Dirac"][{ψ1, ψ2, ψ3, ψ4}]
```

$\{\{\psi_1, -\psi_2^*, \psi_3, \psi_4^*\}, \{\psi_2, \psi_1^*, \psi_4, -\psi_3^*\}, \{\psi_3, \psi_4^*, \psi_1, -\psi_2^*\}, \{\psi_4, -\psi_3^*, \psi_2, \psi_1^*\}\}$

The current is obtained as

```
J = Ψ.Ψ^†.γ^0;
```

We can then verify that it coincides with the standard definition of the current

```
Simplify[1/4 Tr[J.γ^0]] == Conjugate[{ψ1, ψ2, ψ3, ψ4}].γ^0.γ^0.{ψ1, ψ2, ψ3, ψ4}
```

```
True
```

```
Simplify[1/4 Tr[J.γ^1]] == Conjugate[{ψ1, ψ2, ψ3, ψ4}].γ^0.γ^1.{ψ1, ψ2, ψ3, ψ4}
```

```
True
```

```
Simplify[1/4 Tr[J.γ^2]] == Simplify[Conjugate[{ψ1, ψ2, ψ3, ψ4}].γ^0.γ^2.{ψ1, ψ2, ψ3, ψ4}]
```

```
True
```

```
Simplify[1/4 Tr[J.γ^3]] == Simplify[Conjugate[{ψ1, ψ2, ψ3, ψ4}].γ^0.γ^3.{ψ1, ψ2, ψ3, ψ4}]
```

```
True
```

Spin Operator

Theory

The boomerang can be defined as [See Lounesto's Book]

$$Z = \psi \psi^{\dagger} \gamma^0. \tag{38}$$

This expression is Relativistic covariant and contains all the information to recover the spinor. The boomerang in terms of the matrix spinor is

$$Z = \Psi Q \Psi^{\dagger} \gamma^0. \tag{39}$$

$$Z = \frac{1}{4} \Psi \left(\mathbb{I} + \gamma^0 - \gamma^5 \gamma^3 - \gamma^0 \gamma^5 \gamma^3\right) \Psi^{\dagger} \gamma^0. \tag{40}$$

This can be manipulated to

$$Z = \frac{1}{4} \left(\Psi \Psi^{\dagger} \gamma^0 + \Psi \gamma^0 \Psi^{\dagger} \gamma^0 - \Psi \gamma^5 \gamma^3 \Psi^{\dagger} \gamma^0 - \Psi \gamma^0 \gamma^5 \gamma^3 \Psi^{\dagger} \gamma^0\right). \tag{41}$$

For the case

$$\Psi = \sqrt{\rho}\, L, \tag{42}$$

where L is an element of the double cover of the restricted Lorentz group

$$Z = \frac{1}{4} \left(J + \sqrt{\rho} - \gamma^5 \Psi \gamma^3 \Psi^{\dagger} \gamma^0 - \gamma^5 \Psi \gamma^3 \gamma^0 \Psi^{\dagger} \gamma^0\right), \tag{43}$$

$$Z = \frac{1}{4} \left(J + \sqrt{\rho} - \gamma^5 \left(\Psi \gamma^3 \gamma^0 \tilde{\Psi}\right) - \gamma^5 \left(\Psi \gamma^3 \tilde{\Psi}\right)\right). \tag{44}$$

This , means that we can identify the spin as the vector

$$S = - \Psi \gamma^3 \tilde{\Psi} \tag{45}$$

while there is one additional associate bivector

$$F = \Psi \gamma^3 \gamma^0 \tilde{\Psi}. \tag{46}$$

Therefore, the boomerang can be written as

$$Z = \frac{1}{4}\left(J + \sqrt{\rho} - \gamma^5 F + \gamma^5 S\right). \tag{47}$$

This means that the spin components can be recovered as

$$S^\mu = \mathrm{Tr}\left[\, Z\ \gamma^\mu \gamma^5\right]. \tag{48}$$

Using the decomposition in terms of Boosts (Hermitian) and Rotations (Unitary)

$$\Psi = \sqrt{\rho}\ B\,R, \tag{49}$$

the boomerang is

$$Z = \rho\, B\ (R\,Q\,R^\dagger)\ B\,\gamma^0. \tag{50}$$

The rotation of the projector is in general

$$Q_R = R\,Q\,R^\dagger = \frac{1}{4}\left(\mathbb{I} + \gamma^0\right).\left(\mathbb{I} + s^3\gamma^5\gamma^3 + s^2\gamma^5\gamma^2 + s^1\gamma^5\gamma^1\right) \tag{51}$$

such that, the spin at rest is

$$0 = \mathrm{Tr}\left[\left(Q_R\,\gamma^0\right)\gamma^0\right] \tag{52}$$

$$s^k = \mathrm{Tr}\left[\left(Q_R\,\gamma^0\right)\gamma^k\right] \tag{53}$$

The spin at boosted reference frames can be calculated by applying a Lorentz boost to $Q_R\,\gamma^0$

Evaluations

Testing the equivalence between the definitions of the boomerang

```
Outer[Times, {ψ1, ψ2, ψ3, ψ4}, Conjugate[{ψ1, ψ2, ψ3, ψ4}]].γ⁰ - Ψ.QProjector.Ψ†.γ⁰
{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}
```

The rotated projector is defined as

```
(QRotatedProjector = 1/4 (𝕀 + γ⁰).(𝕀 + s³ (γ⁵).(γ³) + s² (γ⁵).(γ²) + s¹ (γ⁵).(γ¹))) //
 MatrixForm
```

$$\begin{pmatrix} \frac{1}{2}\left(1-s^3\right) & \frac{1}{2}\left(-s^1 + i\,s^2\right) & 0 & 0 \\ \frac{1}{2}\left(-s^1 - i\,s^2\right) & \frac{1}{2}\left(1+s^3\right) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

```
Simplify[Tr[(QRotatedProjector.γ⁰).γ⁵.γ⁰]]
0

Simplify[Tr[(QRotatedProjector.γ⁰).γ⁵.γ¹]]
s¹

Simplify[Tr[(QRotatedProjector.γ⁰).γ⁵.γ²]]
s²
```

```
Simplify[Tr[(QRotatedProjector.γ⁰).γ⁵.γ³]]
```

s^3

The boomerang for a boosted spin

```
Z = Simplify[
   BoostSpin13["Dirac"][ √(c² + (u¹)² + (u²)² + (u³)²), u¹, u², u³].QRotatedProjector.
    BoostSpin13["Dirac"][ √(c² + (u¹)² + (u²)² + (u³)²), (u¹), (u²), (u³)].(γ⁰)];
```

```
Simplify[Tr[Z.γ⁵.γ⁰]]
```

$$\frac{u^1\, s^1 + u^2\, s^2 + u^3\, s^3}{c}$$

```
Simplify[Tr[Z.γ⁵.γ¹]]
```

$$\left(\left(c^2 + u^{1\,2} + c\,\sqrt{c^2 + u^{1\,2} + u^{2\,2} + u^{3\,2}}\right) s^1 + u^1\,\left(u^2\, s^2 + u^3\, s^3\right)\right) \Big/ \left(c\,\left(c + \sqrt{c^2 + u^{1\,2} + u^{2\,2} + u^{3\,2}}\right)\right)$$

```
Simplify[Tr[Z.γ⁵.γ²]]
```

$$\left(u^1\, u^2\, s^1 + \left(c^2 + u^{2\,2} + c\,\sqrt{c^2 + u^{1\,2} + u^{2\,2} + u^{3\,2}}\right) s^2 + u^2\, u^3\, s^3\right) \Big/ \left(c\,\left(c + \sqrt{c^2 + u^{1\,2} + u^{2\,2} + u^{3\,2}}\right)\right)$$

```
Simplify[Tr[Z.γ⁵.γ³]]
```

$$\left(u^1\, u^3\, s^1 + u^2\, u^3\, s^2 + \left(c^2 + u^{3\,2} + c\,\sqrt{c^2 + u^{1\,2} + u^{2\,2} + u^{3\,2}}\right) s^3\right) \Big/ \left(c\,\left(c + \sqrt{c^2 + u^{1\,2} + u^{2\,2} + u^{3\,2}}\right)\right)$$

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