Agenda

- Impedance
- Examples

Here is a summary of how a signal is represented in the time and the frequency domains.

$$\underbrace{v(t) = A\cos(\omega t + \phi) \rightarrow v(t) = \mathrm{Re}\left\{ A e^{j(\omega t + \phi)} \right\} \rightarrow}_{\text{Time domain}}$$

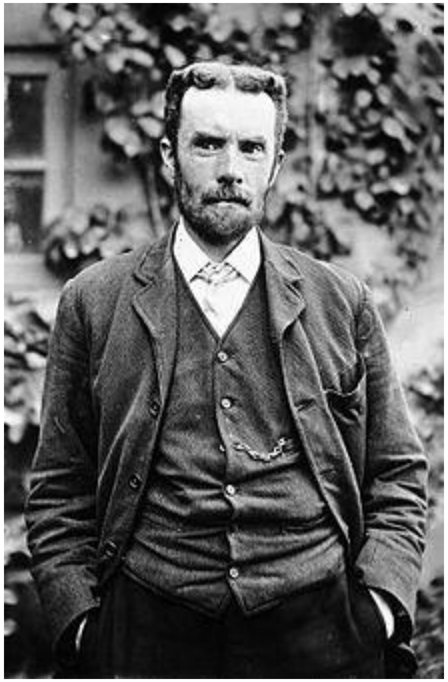
$$\underbrace{\underline{V} = A e^{j\phi} \rightarrow \underline{V} = A\angle\phi \quad \text{(phasor)}}_{\text{Frequency domain}}$$

Recap

OK, let's get started
Z=R+jX, for R, C, and L

The electrical impedance (Z)

- **The measure of the opposition** that a circuit presents to a current when a voltage is applied
- History
  - The term impedance was coined by Oliver Heaviside in July 1886.
  - Arthur Kennelly was the first to represent impedance with complex numbers in 1893.
- Defined as the frequency domain ratio of the voltage to the current.
  - **the voltage–current ratio** for a single complex exponential at **a particular frequency ω.**



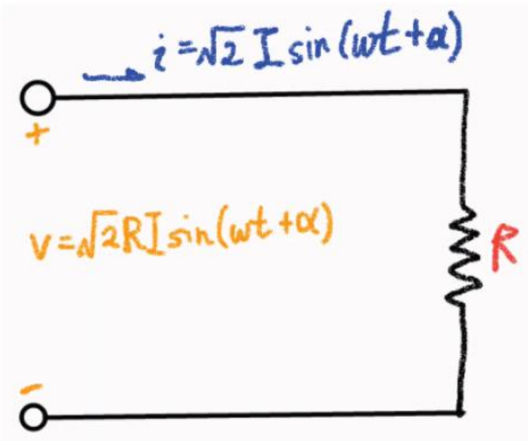
Oliver Heaviside

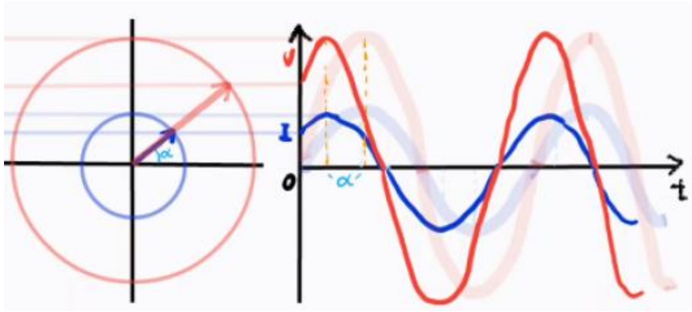
Arthur Kennelly

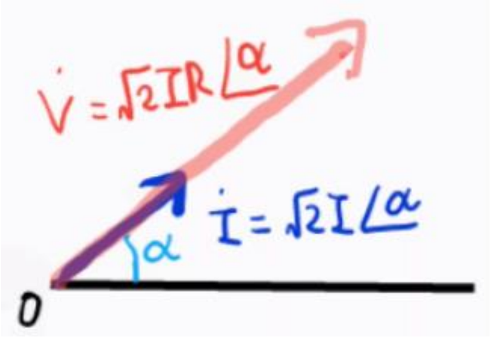
To get the impedance form

- Take a ratio of V and I phasors for each component R, L, and C
  - Z=V/I: voltage across the component when a unit current is applied

Resistors

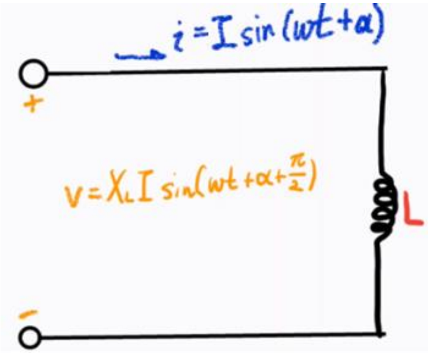


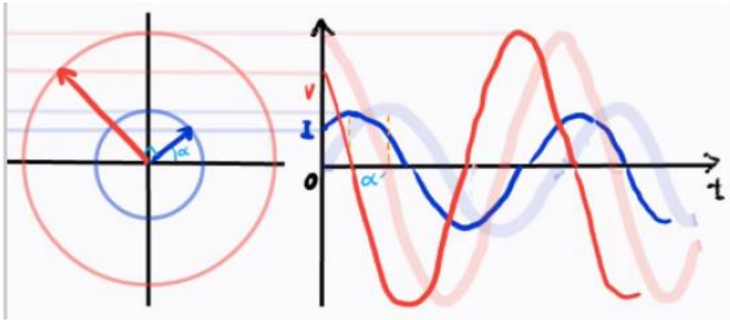


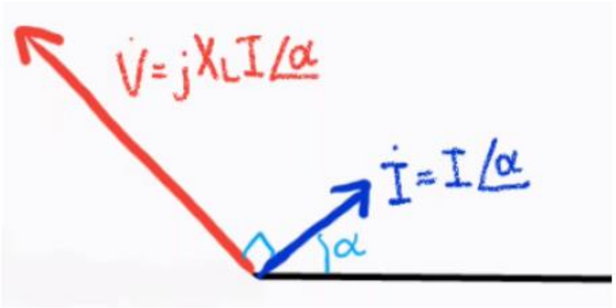


$$Z = \frac{V}{I} = \frac{\sqrt{2} I R \angle \alpha}{\sqrt{2} I \angle \alpha} = R[\Omega]$$

Inductors



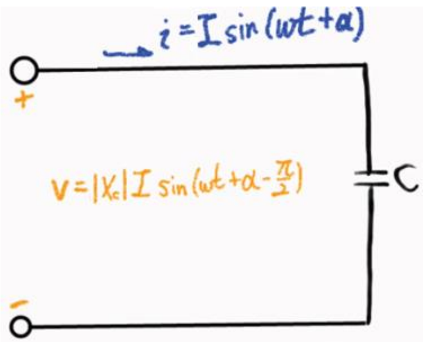


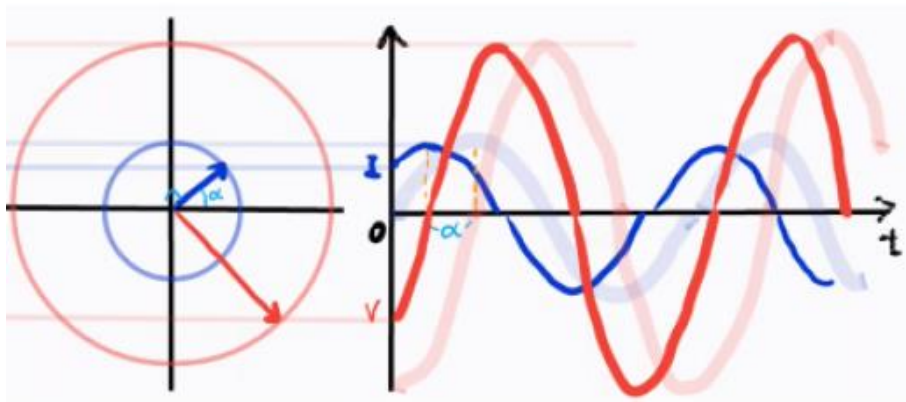


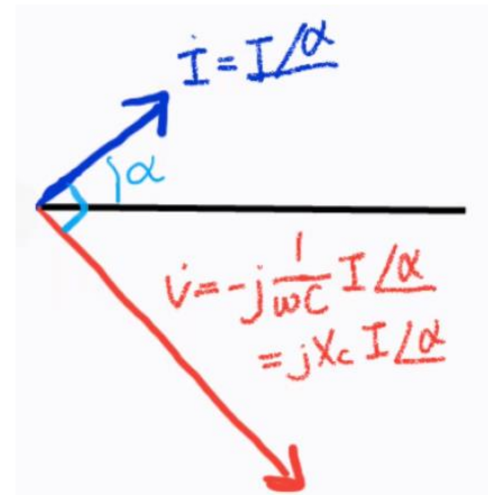
$$Z = \frac{V}{I} = \frac{jX_L I\angle\alpha}{I\angle\alpha} = j{\color{red}X_L} = j{\color{red}\omega L}[\Omega]$$

Where ω=2πf

Capacitors







$$Z = \frac{V}{I} = \frac{jX_c I\angle\alpha}{I\angle\alpha} = jX_c = j\frac{-1}{\omega C} = \frac{1}{j\omega C}\,[\Omega]$$

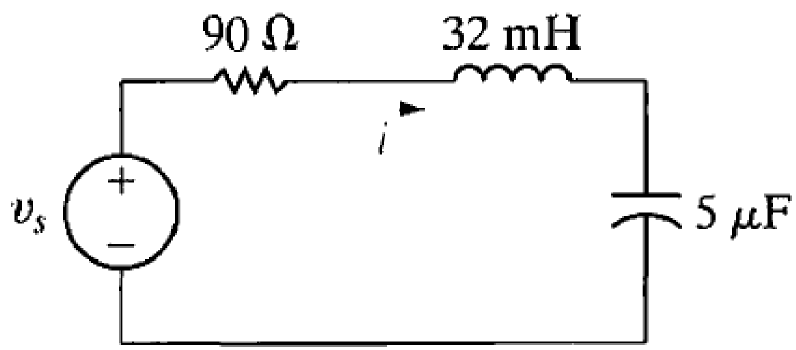
Where ω=2πf

Impedance $Z=R+jX$ [Ω], where

- Series $Z_{eq}=Z_1+Z_2+Z_3$....
  - $Re\{Z_{eq}\}$: from resistors, $Im\{Z_{eq}\}$: inductors and caps
- Parallel $1/Z_{eq}=1/Z_1+1/Z_2+1/Z_3$....
  - $Re\{Z_{eq}\}$ and $Im\{Z_{eq}\}$: combination of R, C, L

Let's do some practice

A 90 Ω resistor, a 32 mH inductor, and a 5 μF capacitor are connected in series across the terminals of a sinusoidal voltage source, as shown in Fig. 9.15. The steady-state expression for the source voltage v_s is $750 \cos(5000t + 30^\circ)$ V.



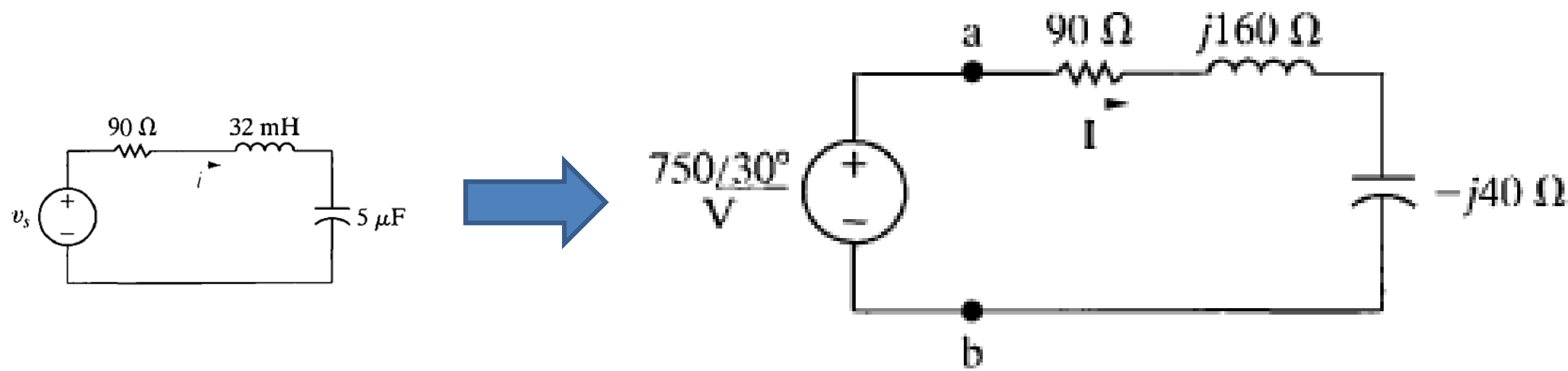
a) Construct the frequency-domain equivalent circuit.

b) Calculate the steady-state current i by the phasor method.

$$Z_L = j\omega L = j(5000)(32 \times 10^{-3}) = j160\ \Omega,$$

$$Z_C = j\frac{-1}{\omega C} = -j\frac{10^6}{(5000)(5)} = -j40\ \Omega.$$

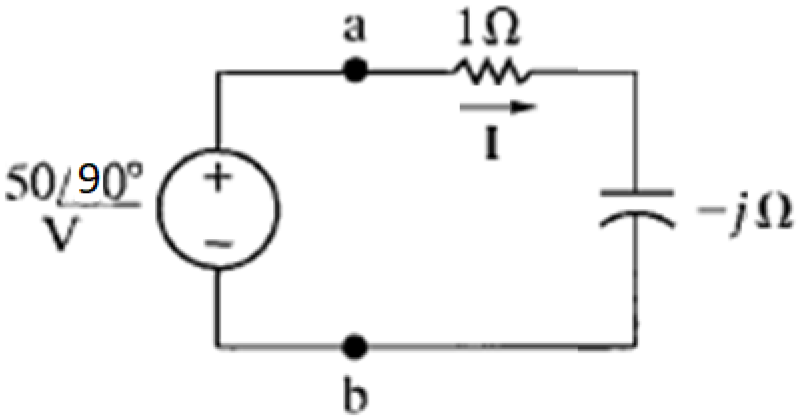
$$\mathbf{V}_s = 750\,\angle 30^\circ\ \mathrm{V}.$$



$$Z_{ab} = 90 + j160 - j40$$
$$= 90 + j120 = 150\,\angle 53.13^\circ\ \Omega.$$

$$\mathbf{I} = \frac{750\,\angle 30^\circ}{150\,\angle 53.13^\circ} = 5\,\angle -23.13^\circ\ \mathrm{A}.$$

$$i = 5\cos(5000t - 23.13^\circ)\ \ \mathrm{A}.$$



Compute i(t) when
ω=1 [rad/sec]

$$\frac{V}{Z} = I = \frac{50\angle 90^\circ}{1-j}$$

$$= \frac{50\angle 90^\circ}{\sqrt{1^2+(-1)^2}\tan^{-1}\frac{-1}{1}}$$

$$= \frac{50\angle 90^\circ}{\sqrt{2}\angle -45^\circ} = 25\sqrt{2}(\angle 90^\circ - \angle -45^\circ)$$

$$= 25\sqrt{2}\ \angle 135^\circ$$

$$= -25\sqrt{2}\angle(135^\circ - 180^\circ)$$

$$= -25\sqrt{2}\angle - 45^\circ$$

$$\angle 135^\circ = e^{j135} = \cos 135^\circ + \mathrm{j}\sin 135^\circ$$

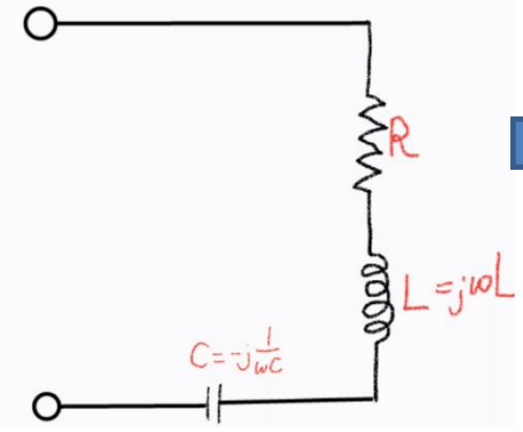
$$= -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$$

$$\angle -45^\circ = e^{j-45} = \cos - 45^\circ + j - \sin 45^\circ$$

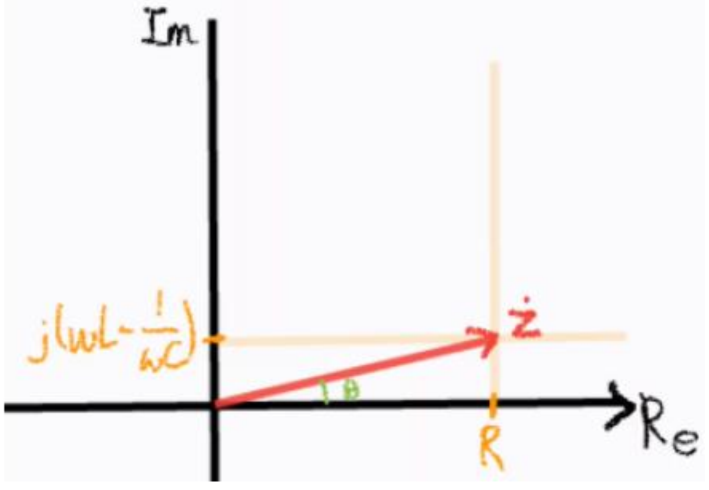
$$= \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$

Complex circuit impedance Ex.

- Determine the total Z



$$Z = R + j\left(\omega L - \frac{1}{\omega C}\right)[\Omega]$$



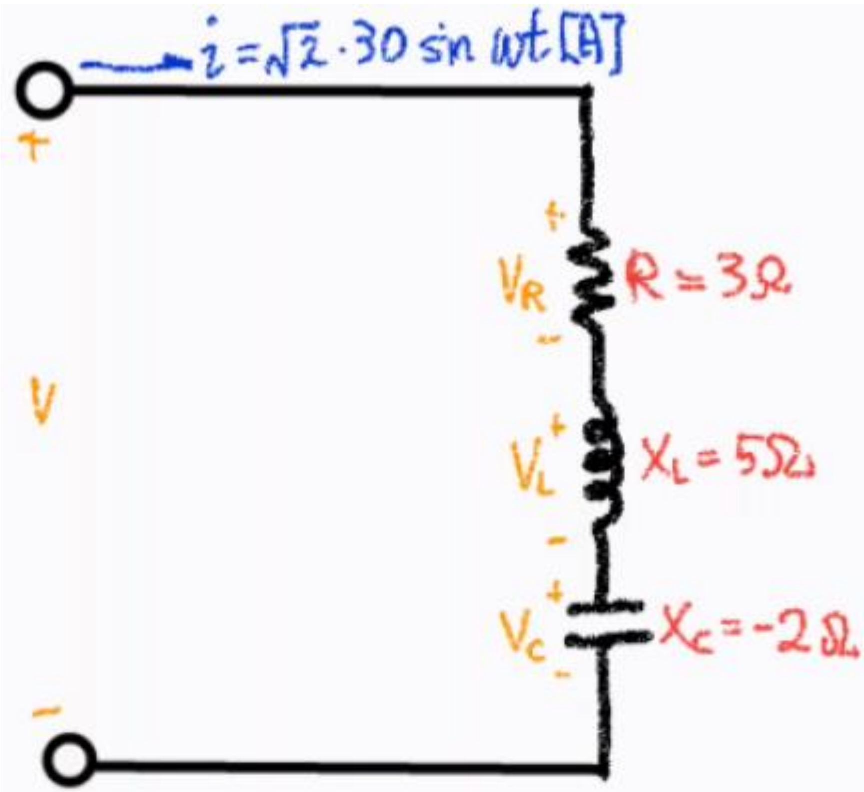
In phasor form

$$\bar{Z} = |Z|e^{j\theta} \text{ where}$$

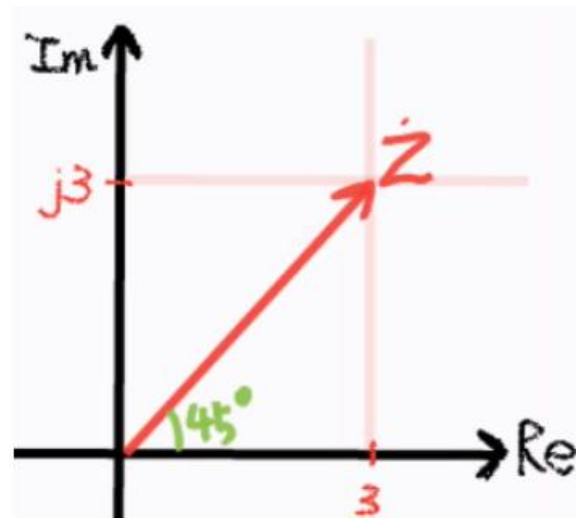
$$\theta = \tan^{-1}\frac{\omega L - \frac{1}{\omega C}}{R}$$

$$|Z| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

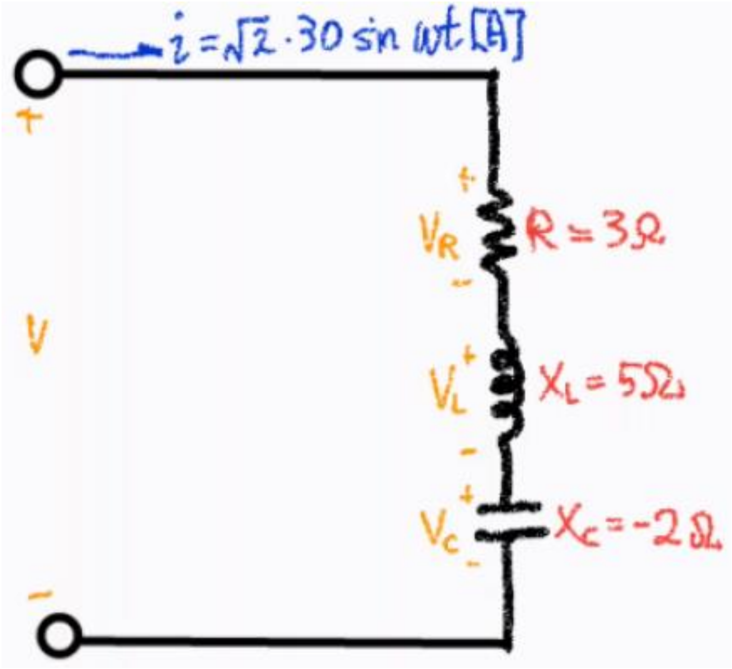
Determine v(t) (using phasor)



$$Z = 3 + j5 - j2$$
$$= 3 + j3\ [\Omega]$$



$$|Z| = \sqrt{3^3 + 3^3} = 3\sqrt{2}$$
$$\theta = \tan^{-1}\frac{3}{3} = 45° = \frac{\pi}{4}\ [rad/s]$$



$$\mathrm{i(t)} = \sqrt{2}\cdot 30\,\boldsymbol{\sin\omega t}\,[A]$$

$$I = 30\sqrt{2}e^{-j\frac{\pi}{2}}[A]$$

$$Z = 3\sqrt{2}e^{j\frac{\pi}{4}}[\Omega]$$

Use V=IZ

- v(t)'s magnitude: I's mag*Z's mag
- v(t)'s ω: same as I
- v(t)'s phase: I's phase + Z's phase

$$\mathrm{V}= 180e^{j(-\frac{\pi}{4})}[\mathrm{V}]$$

$$\sin\omega t = \cos(\omega t - 90°)$$
$$\boldsymbol{\cos\omega t = \sin(\omega t + 90°)} = -\sin(\omega t - 90°)$$

Therefore

$$v(t) = 180\,\boldsymbol{\cos}\,(\boldsymbol{\omega t}-\pi/4)$$
$$= 180\,\boldsymbol{\sin}(\boldsymbol{\omega t} + \tfrac{\pi}{4})\;[V]$$

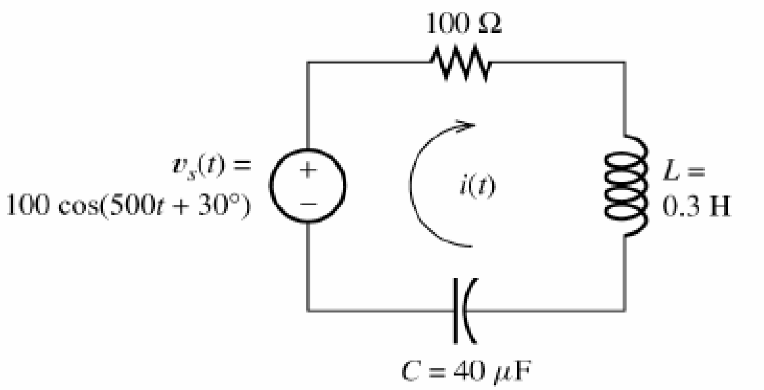
General steps for Circuit Analysis Using Phasors and Impedances

1. Replace the time descriptions of the voltage and current sources with the corresponding phasors.
   - All of the sources must have the same frequency.
   - In general circuit analysis, the standard of the phasor is a cosine function

2. Replace inductances by their complex impedances $Z_L = j\omega L$. Replace capacitances by their complex impedances $Z_C = 1/(j\omega C)$. Resistances have impedances equal to their resistances.

3. Analyze the circuit using any of the circuit analysis techniques, performing the calculations with complex arithmetic.

Example 1



A Phasor form?

Find *i(t)*:

$$V_S = 100\cos(500t + 30^\circ) \rightarrow \mathbf{V}_S = 100\angle 30^\circ$$

$$Z_L = j\omega L = j(500)(0.3) = j150\Omega$$

$$Z_C = -j\frac{1}{\omega C} = -j\frac{1}{(500)(40)(10^{-6})} = -j50\Omega$$

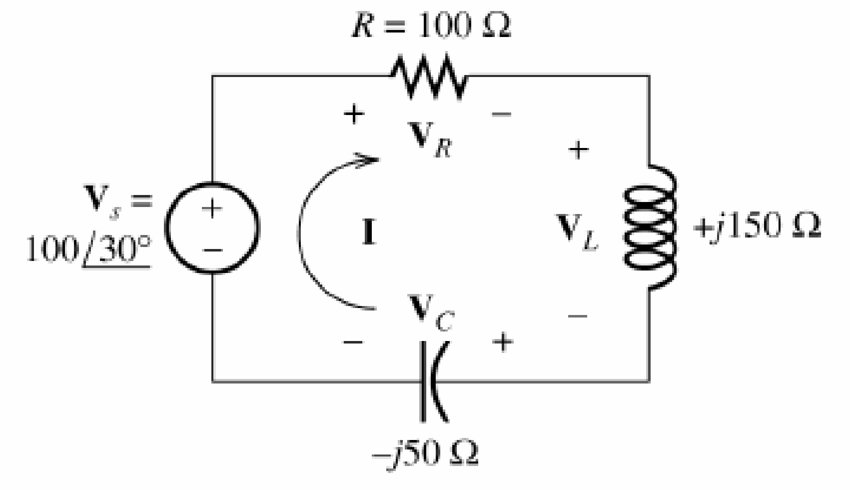
$$Z_{eq} = R + Z_L + Z_C = 100 + j150 - j50 = 100 + j100$$

$$= 141.4\angle 45^\circ$$

$$\mathbf{I} = \frac{\mathbf{V}_S}{Z} = \frac{100\angle 30^\circ}{141.4\angle 45^\circ} = 0.707\angle -15^\circ \rightarrow i(t) = 0.707\cos(500t - 15^\circ)$$

Example 2

Find the phasor voltage across each element:



$$\mathbf{V}_S = 100\angle 30^\circ$$

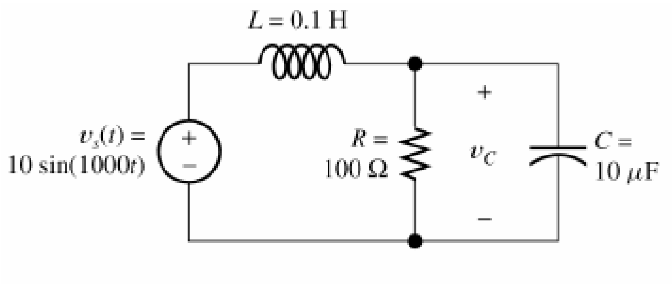
$$\mathbf{I}_S = 0.707\angle -15^\circ$$

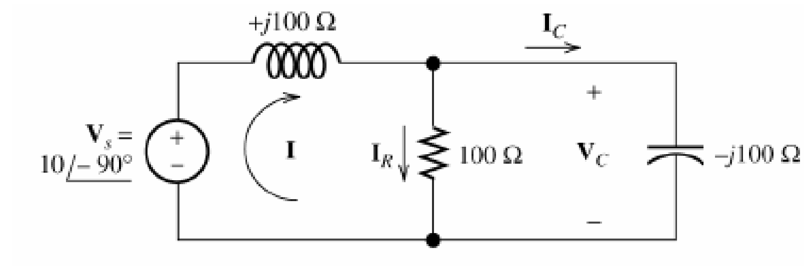
$$\mathbf{V}_R = R\mathbf{I} = (100)(0.707\angle -15^\circ)$$

$$\mathbf{V}_L = (jwL)\mathbf{I} = (\omega L\angle 90^\circ)\mathbf{I} = (\omega L\angle 90^\circ)(0.707\angle -15^\circ) = 106.1\angle 75^\circ$$

$$\mathbf{V}_C = \left(-j\frac{1}{\omega C}\right)\mathbf{I} = \left(\frac{1}{\omega C}\angle -90^\circ\right)\mathbf{I} = (50\angle -90^\circ)(0.707\angle -15^\circ) = 35.4\angle -105^\circ$$

Example 3





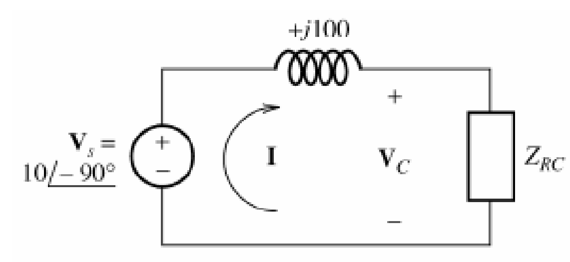
Find the voltage $v_c(t)$ in steady state:

$$v_s(t) = 10\sin(1000t) = 10\cos(1000t - 90) \rightarrow \mathbf{V}_s = 10\angle -90$$

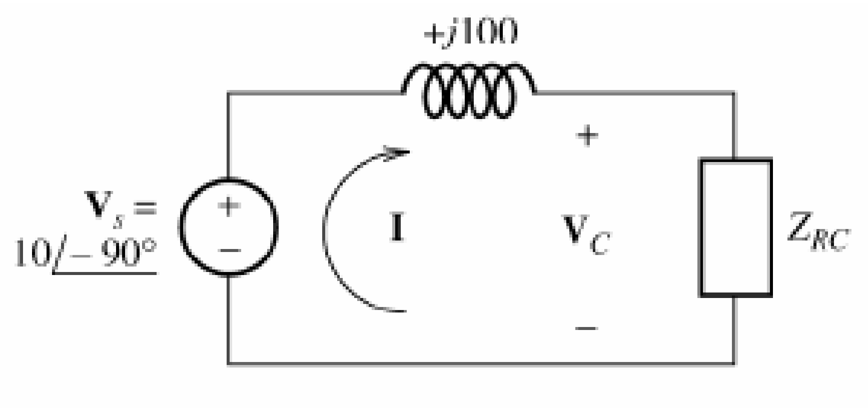
$$Z_L = jwL = j(1000)(0.1) = j100\Omega$$

$$Z_C = -j\frac{1}{\omega C} = -j\frac{1}{(1000)(10x10^{-6})} = -j100\Omega$$

$$Z_{RC} = \frac{1}{\frac{1}{R} + \frac{1}{Z_C}} = \frac{1}{\frac{1}{100} + \frac{1}{-j100}} = \frac{1}{0.01 + j0.01}$$



$$= \frac{1\angle 0}{0.01414\angle 45} = 70.71\angle -45 = 50 - j50$$



$$V_C = \left(\frac{Z_{RC}}{Z_{RC} + Z_L} \right) \mathbf{V}_S = \left(\frac{70.71\angle -45^\circ}{(50 - j50) + j100} \right) (10\angle -90^\circ)$$

$$= \left(\frac{70.71\angle -45^\circ}{50 + j50} \right) (10\angle -90^\circ)$$

$$= \left(\frac{70.71\angle -45^\circ}{70.71\angle 45^\circ} \right) (10\angle -90^\circ) = (1\angle -90^\circ)(10\angle -90^\circ) = 10\angle -180^\circ$$

$$v_C(t) = 10\cos(1000t - 180^\circ) = -10\cos(1000t)$$

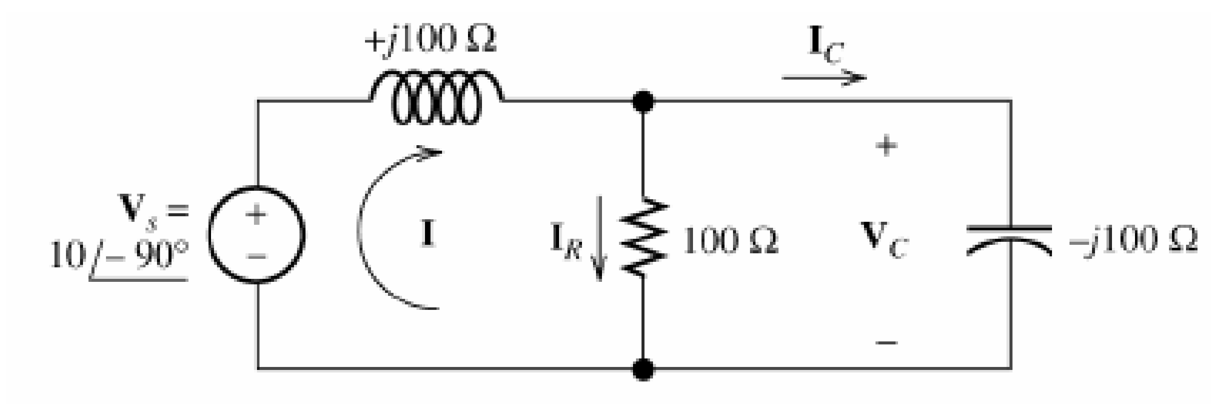
Example 4

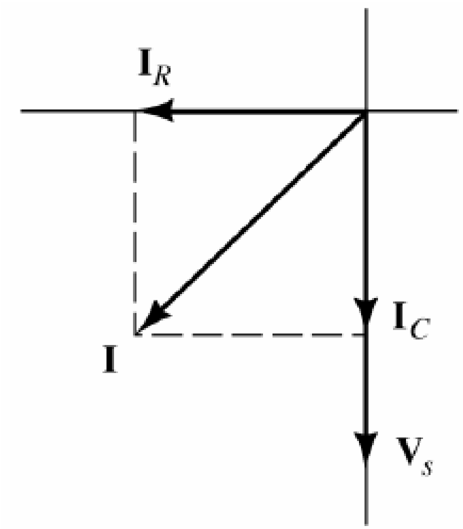
Find the phasor current through each element:

$$\mathbf{I} = \frac{\mathbf{V}_S}{Z_L + Z_{RC}} = \frac{10\angle -90^\circ}{j100 + (50 - j50)} = \frac{10\angle -90^\circ}{50 + j50} = \frac{10\angle -90^\circ}{70.71\angle 45^\circ} = 0.1414\angle -135$$

$$\mathbf{I}_R = \frac{\mathbf{V}_C}{R} = \frac{10\angle -180^\circ}{100} = 0.1\angle -180^\circ$$

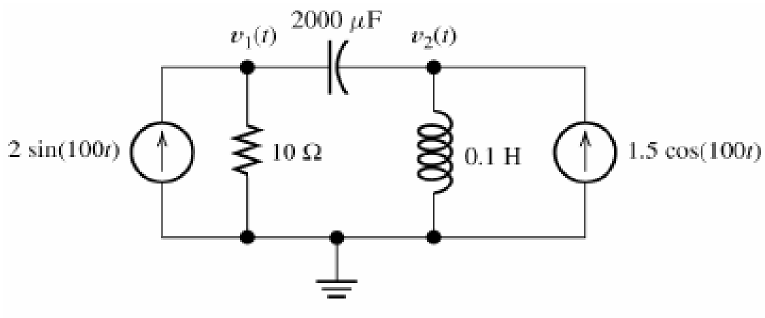
$$\mathbf{I}_C = \frac{\mathbf{V}_C}{Z_C} = \frac{10\angle -180^\circ}{-j100} = \frac{10\angle -180^\circ}{100\angle -90} = 0.1\angle -90^\circ$$

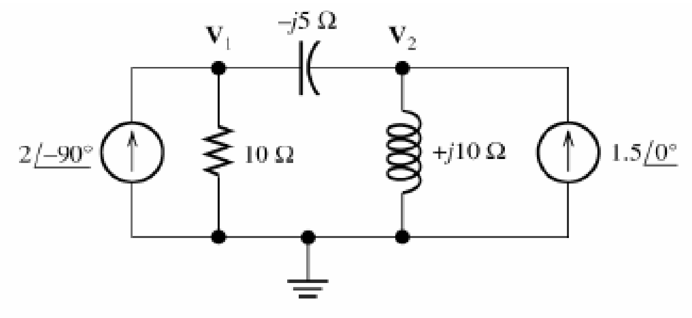




Example 5

Use the node voltage technique to find $v_1(t)$:





$$KCL\ at\ node\ 1:$$

$$\frac{\mathbf{V}_1}{10}+\frac{\mathbf{V}_1-\mathbf{V}_2}{-j5}=2\angle-90^\circ \rightarrow (0.1+j0.2)\mathbf{V}_1-j0.2\mathbf{V}_2=-j2$$

$$KCL\ at\ node\ 2:$$

$$\frac{\mathbf{V}_2}{j10}=\frac{\mathbf{V}_1-\mathbf{V}_2}{-j5}+1.5\angle0^\circ \rightarrow \qquad -j0.2\mathbf{V}_1+j0.1\mathbf{V}_2=1.5$$

$$(0.1+j0.2)\mathbf{V}_1 - j0.2\mathbf{V}_2 = -j2$$
$$-j0.2\mathbf{V}_1 + j0.1\mathbf{V}_2 = 1.5$$

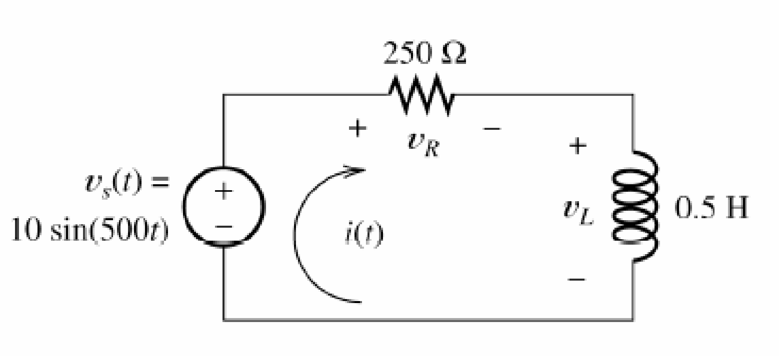
$$(0.1+j0.2)\mathbf{V}_1 - j0.2\mathbf{V}_2 = -j2$$
$$-j0.4\mathbf{V}_1 + j0.2\mathbf{V}_2 = 3$$

$Adding:$

$$(0.1-0.2j)\mathbf{V}_1 = 3-2j \rightarrow \mathbf{V}_1 = \frac{3-2j}{0.1-0.2j} = 14+8j = 16.12\angle 29.74^\circ$$

$$v_1(t) = 16.1\cos(100t + 29.7^\circ)$$

Exercise 6



Find *i(t):*

$$v_S(t) = 10\sin(500t) = 10\cos(500t - 90^\circ) \rightarrow \mathbf{V}_S = 10\angle -90$$

$$Z_R = 250\Omega$$

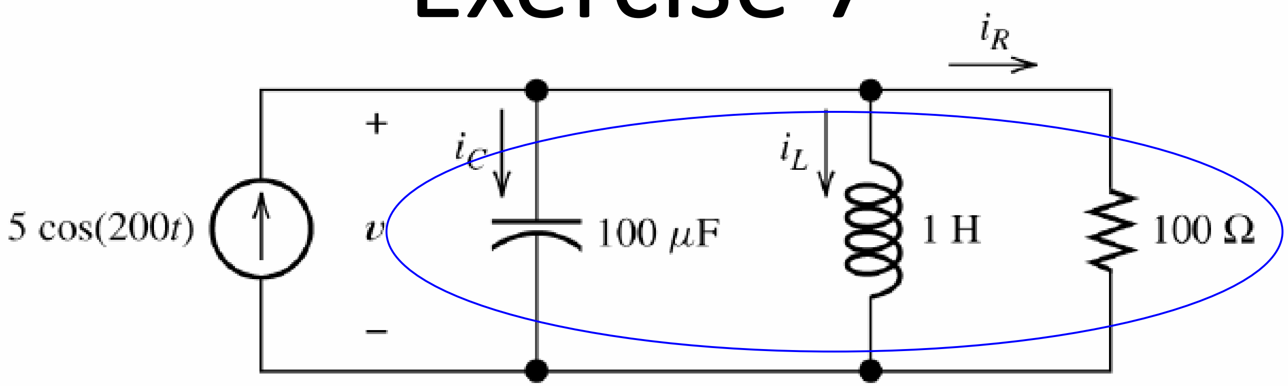
$$Z_L = j\omega L = j(500)(0.5) = 250j$$

$$KVL:$$

$$\mathbf{I}Z_R + \mathbf{I}Z_L = \mathbf{V}_S \rightarrow \mathbf{I} = \frac{\mathbf{V}_S}{Z_R + Z_L} = \frac{\mathbf{V}_S}{250 + j250} = \frac{10\angle -90^\circ}{353.6\angle 45^\circ} = 28.4x10^{-3}\angle -135^\circ$$

$$i(t) = 28.4x10^{-3}\cos(500t - 135^\circ)$$

Exercise 7



Find the phasor voltage and phasor current through each element:

$$\frac{1}{Z_{eff}} = \frac{1}{Z_C} + \frac{1}{Z_L} + \frac{1}{Z_R}$$

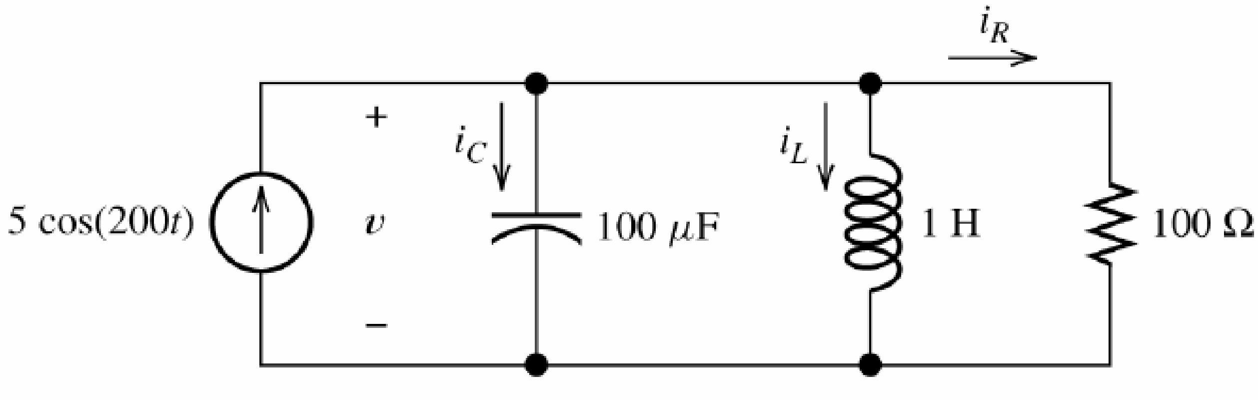
$$Z_C = -j\frac{1}{\omega C} = -j\frac{1}{(200)(100x10^{-6})} = -50j$$

$$Z_L = j\omega L = j(200)(1) = j200$$

$$\frac{1}{Z_{eff}} = \frac{1}{-50j} + \frac{1}{j200} + \frac{1}{100} = \frac{j}{50} - \frac{j}{200} + \frac{1}{100} = \frac{2+j3}{200}$$

$$Z_{eff} = \frac{200}{2+j3} = 30.77 - j46.15 = 55.47\angle -56.31^\circ$$

Exercise 8



Find the phasor voltage and current through each element:

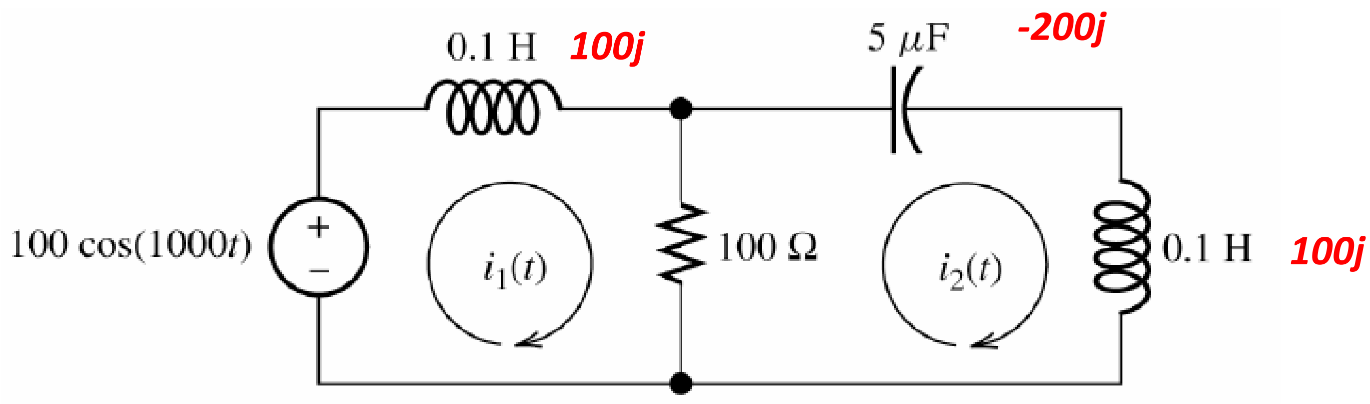
$$\mathbf{V}_C = \mathbf{V}_L = \mathbf{V}_R = \mathbf{I}Z_{eff} = (5\angle 0^\circ)(55.47\angle -56.31^\circ) = 277.3\angle -56.31^\circ$$

$$\mathbf{I}_C = \frac{\mathbf{V}_C}{Z_C} = \frac{277.3\angle -56.31^\circ}{50\angle -90^\circ} = 5.55\angle 33.69^\circ$$

$$\mathbf{I}_L = \frac{\mathbf{V}_L}{Z_L} = \frac{277.3\angle -56.31^\circ}{200\angle 90^\circ} = 1.39\angle -146.3^\circ$$

$$\mathbf{I}_R = \frac{\mathbf{V}_R}{Z_R} = \frac{277.3\angle -56.31^\circ}{100\angle 0^\circ} = 2.77\angle -56.31^\circ$$

Exercise 9



Solve for one of the mesh currents, i1(t):

$$I_1 Z_{L_1} + (I_1 - I_2) Z_R = 100$$

$$I_2 Z_C + I_2 Z_{L_2} = (I_1 - I_2) Z_R$$

$$(100 + 100j) I_1 - 100 I_2 = 100$$

$$-100 I_1 + (100 - j100) I_2 = 0$$

$$(100+100j)I_1 - 100I_2 = 100$$
$$-100I_1 + (100-j100)I_2 = 0$$

$$(141\angle 45)I_1 - (100\angle 0)I_2 = 100\angle 0$$
$$(-100\angle 0)I_1 + (141\angle -45)I_2 = 0$$

$$I_1 = \frac{(100\angle 0)(141\angle -45) - (0)(-100\angle 0)}{(141\angle 45)(141\angle -45) - (-100\angle 0)(-100\angle 0)}$$
$$= \frac{14100\angle -45}{(141)^2\angle 0 - (100)^2\angle 0} = \frac{9.97x10^3 - j9.97x10^3}{9881} = 1.009 - j1.009 = \sqrt{2}\angle -45$$

$$i_1(t) = 1.41\cos(1000t - 45^\circ)$$