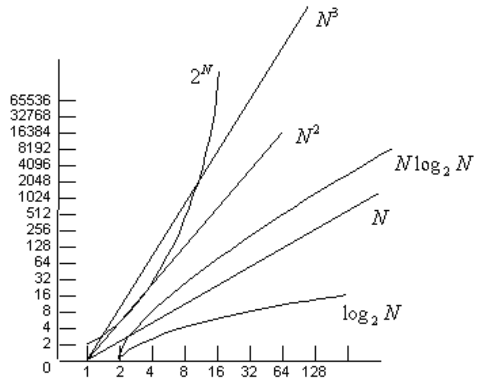
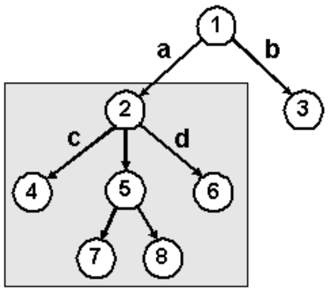
Trees (Part 1)

Introduction to Trees

- Trees are one of the fundamental data structures in computer science.
- Trees are a specific type of *graph*, but much simpler.
- They are constructed so as to retrieve information rapidly.
- Typical search times for trees are *O(lg N)* (Think of the ability to do binary search on a linked list.)



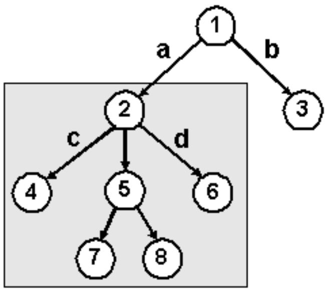
A generic tree:



Terminology

- Trees consist of *vertices* and *edges*.
- Vertex - An object that carries associated information [1, 2, 3]
  - node
- Edge - A connection between two vertices. A *link* from one node to another. [a, b, c]
- Child/Parent - If either the right or left link of **A** is a link to **B**, then **B** is a *child* of **A** and **A** is a *parent* of **B**.
  - 4, 5, and 6 are children of 2; 2 is the parent of 4, 5, and 6
- Sibling - Nodes that have the same parent. [2, 3] have the same parent.
- A node that has no parent is called the *root* of a tree [1]. There is only one root in any given tree.
- Path - A list of vertices [1-2-4]
- Leaf - A node with no children [4, 7]
  - external node
  - terminal node
  - terminal
- Non-leaf - A node with at least one child [1, 2, 5]
  - internal node
  - non-terminal node
  - non-terminal
- Depth (or height) - the length of the longest path from the root to a leaf. [1, 2, 5, 7 = 3]
  - The number of edges in the path is the length. (Or, number of nodes - 1)
  - A tree consisting of 1 node (the root), has a height of 0.
- Subtree - Any given node, with all of its descendants (children). [5, 7, 8] is a subtree, 5 is the root.

- Trees can be ordered or unordered
  - Ordered trees specify the order of the children (examples: parse tree, binary search tree)
  - Unordered trees place no criteria on the ordering of the children (example: file system directories)



- M-ary tree - A tree which must have a specific number of children in a specific order.
  - Binary tree - An M-ary tree where
    - all internal nodes have one or two children
    - all external nodes (leaves) have no children
    - the two children may be sorted and are called the *left child* and *right child*
  - B-tree - An M-ary tree where
    - all internal nodes have between N and N/2 children (where N is generally several hundred)
    - the children are sorted according to some sort order or key

Basic Properties

- A node has at most one edge leading to it.
  - Each node has *exactly* one parent, except the root which has no parent.

- There is at most one path from one node to any other node.
    - If there are multiple paths, it's a graph not a tree.

- There is *exactly* one path from the root to any leaf.

Other Properties

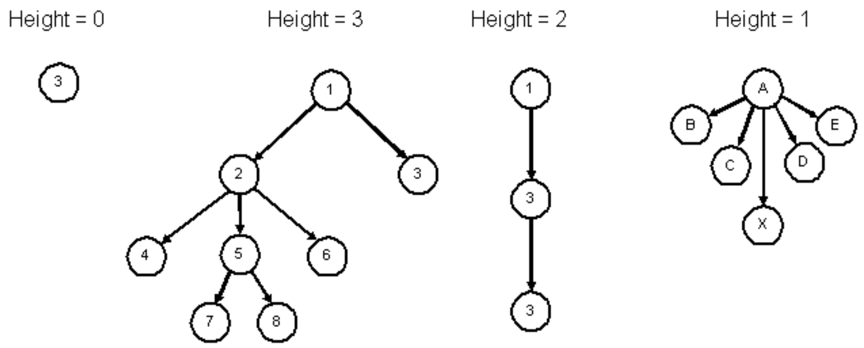
- The **level** of a given node in a tree is defined recursively as:

```
0                   if node is a root
Level(parent) + 1   if node is a child of parent
```

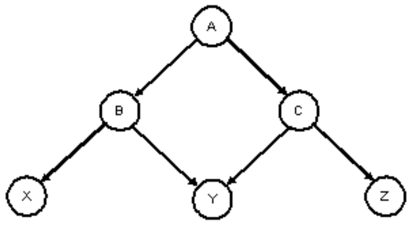
Two interpretations of **height** (or **depth**):

1. The **height** of a tree is the length of the longest path from the root to a leaf.
2. The **height** is the maximum of the levels of the tree's nodes.

These are all trees.



This is not a tree. (Y has 2 parents)



Binary Trees

A **binary tree** is a collection of nodes such that:

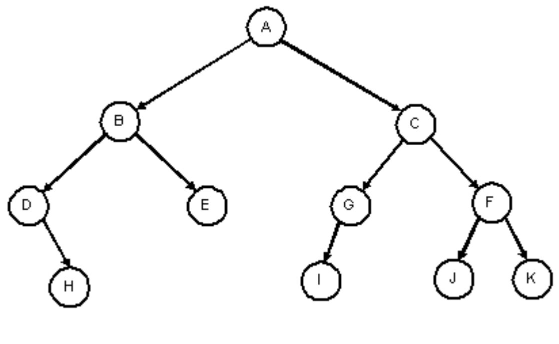
- There are two distinct types of nodes: *internal* and *external*
- An internal node contains *exactly* two links, *left* and *right*
- The two links are *disjoint* binary trees (they have no nodes in common)
- One or both links can be NULL (an empty tree)

- A binary tree with *N* internal nodes has *N + 1* external nodes (some may be empty/NULL)

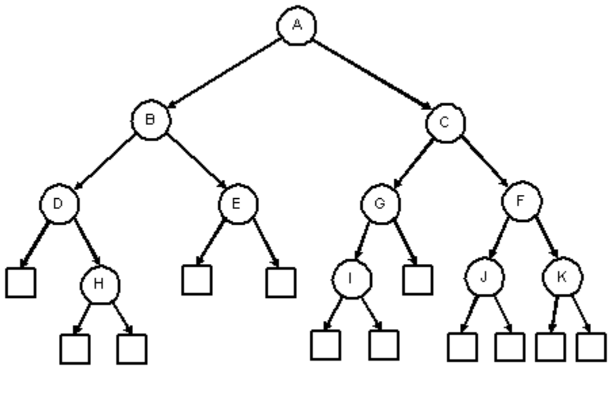
- A binary tree with *N* internal nodes has *2N* links

  Representing binary trees with diagrams: sometimes an empty box is used to indicate an empty child (empty subtree). The squares are external nodes (leaves) and the circles are internal nodes. These diagrams are referred to as extended binary trees:

A binary tree



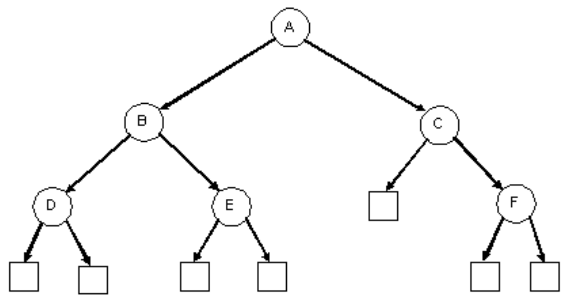
An extended binary tree

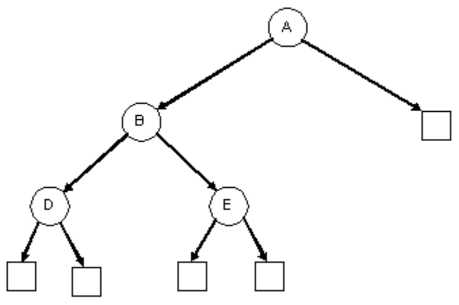


- A *balanced* binary tree (*height-balanced*) is a tree where for ***each node*** the depth of the left and right subtrees differ by no more than 1.

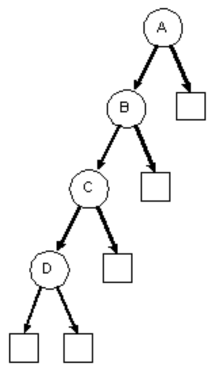
A balanced binary tree

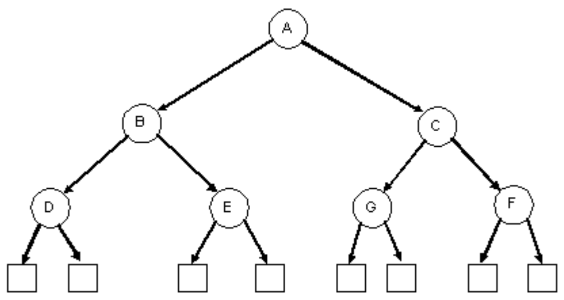
An unbalanced binary tree





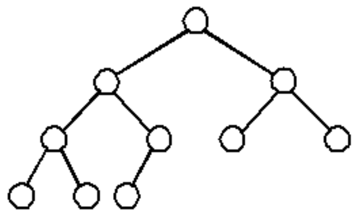
A degenerate binary tree **A balanced binary tree (it's also a *complete* binary tree)**

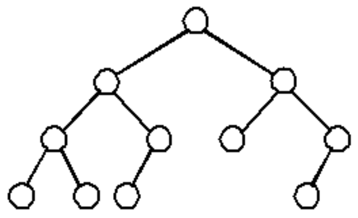


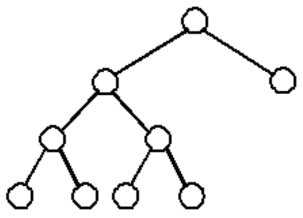


A *complete* binary tree is similar to a balanced binary tree except that all of the leaves must be placed as far to the left as possible. (The leaves must be "filled-in" from left to right, one level at a time.)

A complete binary tree **An incomplete binary tree** **An incomplete binary tree**





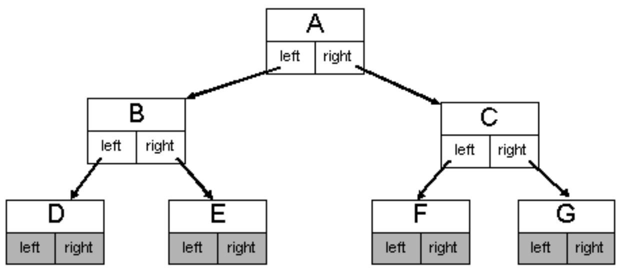


Realize that the two links in a binary tree are not quite the same as the two links in a doubly linked list:

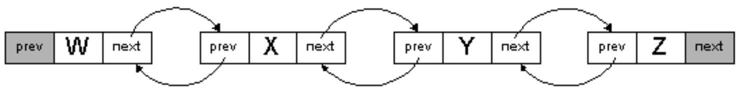
- Trees have a *left* and *right* link.
- Lists have a *previous* and *next* link.
- Both imply ordering, but a different kind of ordering.
- Implementation wise, the nodes can be *structurally equivalent*.

```
struct ListNode           struct TreeNode
{                         {
  ListNode *next;           TreeNode *left;
  ListNode *prev;           TreeNode *right;
  Data *data;               Data *data;
};                        };
```

Tree:



Linked list:



Self-check What is a binary tree? What is a balanced tree? What is a complete tree?

Traversing Binary Trees

Because trees are a recursive data structure, recursive algorithms are quite appropriate. In some cases, iterative (non-recursive) algorithms can be significantly more complicated.

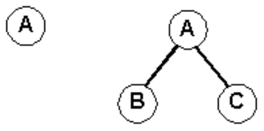
Recursive algorithm

- *Preorder* traversal
  1. **Visit the node**
  2. Traverse (pre-order) the left subtree
  3. Traverse (pre-order) the right subtree
- *Inorder* traversal
  1. Traverse (in-order) the left subtree
  2. **Visit the node**
  3. Traverse (in-order) the right subtree
- *Postorder* traversal
  1. Traverse (post-order) the left subtree
  2. Traverse (post-order) the right subtree
  3. **Visit the node**

Recursive algorithm with base case

- *Preorder* traversal
  If node is not empty
  1. **Visit the node**
  2. Traverse (pre-order) the left subtree
  3. Traverse (pre-order) the right subtree
- *Inorder* traversal
  If node is not empty
  1. Traverse (in-order) the left subtree
  2. **Visit the node**
  3. Traverse (in-order) the right subtree
- *Postorder* traversal
  If node is not empty
  1. Traverse (post-order) the left subtree
  2. Traverse (post-order) the right subtree
  3. **Visit the node**

Given these binary trees:

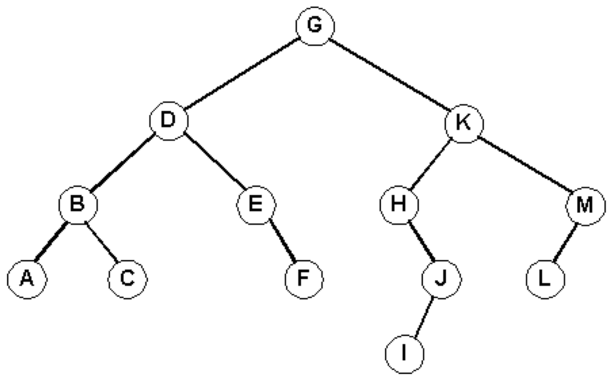


assume that *visiting* a node means printing the letter of the node. The result of the traversing the first tree is **A** in all 3 cases.

For the second tree, we have:

- Preorder traversal: **ABC** (**visit**, traverse left, traverse right)
- Inorder traversal: **BAC** (traverse left, **visit**, traverse right)
- Postorder traversal: **BCA** (traverse left, traverse right, **visit**)

Self-check Perform the three different traversals on the tree below.



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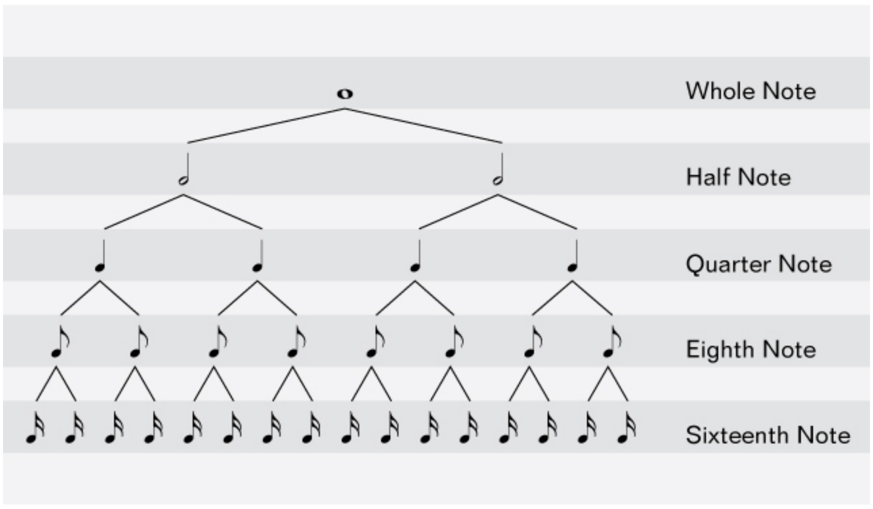
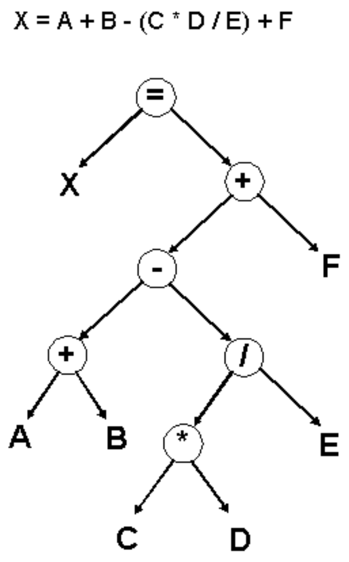
Assuming that *visiting* a node means printing the letter of the node, what is the output for

- a preorder traversal?
- an inorder traversal?
- a postorder traversal?

A (binary) tree showing the relationships between musical notes:



An example of an expression tree: (order is important)



Self-check What kind of traversal would we use to evaluate the expression tree?

Implementing Tree Algorithms

Assume we have these definitions:

```
struct Node
{
  Node *left;
  Node *right;
  int data;
};

Node *MakeNode(int Data)
{
  Node *node = new Node;
  node->data = Data;
  node->left = 0;
  node->right = 0;
  return node;
}

void FreeNode(Node *node)
{
  delete node;
}

typedef Node* Tree;
```

We can construct *random* binary trees by providing a height for the final tree. Assume that the data items are the letters **A, B, C, D**, etc. added in that order.

Linked list examples

```
int Count = 0;

Tree BuildBinTreePre(int height)
{
  if (height == -1)
    return 0;

  Node *node = MakeNode('A' + Count++);      // build the node
  node->left = BuildBinTreePre(height - 1);  // build the left tree
  node->right = BuildBinTreePre(height - 1); // build the right tree
  return node;
}

void main()
{
  Tree t = BuildBinTreePre(1);
}
```

This results in a tree that looks like this:

```
  A
 / \
B   C
```

which *order* has this tree been built? Why? How would we construct these trees:

```
  B       C
 / \     / \
A   C   A   B

Tree BuildBinTreeIn(int height)
{
  if (height == -1)
    return 0;

  Node *node = new Node;
  node->left = BuildBinTreeIn(height - 1);  // build left subtree
  node->data = 'A' + Count++;               // build node
  node->right = BuildBinTreeIn(height - 1); // build right subtree
  return node;
}

Tree BuildBinTreePost(int height)
{
  if (height == -1)
    return 0;

  Node *node = new Node;
  node->left = BuildBinTreePost(height - 1);  // build left subtree
  node->right = BuildBinTreePost(height - 1); // build right subtree
  node->data = 'A' + Count++;                 // build node
  return node;
}
```

Self-check: Suppose you used these functions to create a tree with height of 2. This would require 7 nodes and use the letters: **ABCDEFG**. What would the trees look like using: BuildBinTreePre? BuildBinTreeIn? BuildBinTreePost?

More Tree Algorithms

State the recursive algorithms for finding:

- the number of nodes in a tree
- the height of a tree

Definitions and sample implementations:

- Finding the number of nodes in a tree.

  ```
  0                         if the tree is empty
  1 +                       if tree is not empty
  nodes in left subtree +
  nodes in right subtree

  int NodeCount(Tree tree)
  {
    if (tree == 0)
      return 0;
    else
      return 1 + NodeCount(tree->left) + NodeCount(tree->right);
  }
  ```

- Finding the height of a tree.

  ```
  -1                            if the tree is empty (Definition)
   1 + height of left subtree   if height of left subtree > height of right subtree
   1 + height of right subtree  otherwise
  ```

  Implementation #1: (naive, from the definition)

  ```
  int Height(Tree tree)
  {
    if (tree == 0)
      return -1;

    if (Height(tree->left) > Height(tree->right))
      return Height(tree->left) + 1;
    else
      return Height(tree->right) + 1;
  }
  ```

  Implementation #2:

```
int Height(Tree tree)
{
  if (tree == 0)
    return -1;

  int lh = Height(tree->left);
  int rh = Height(tree->right);

  if (lh > rh)
    return lh + 1;
  else
    return rh + 1;
}
```

Implementation #3:

```
int Height(Tree tree)
{
  if (tree == 0)
    return -1;
  else
    return (1 + Max(Height(tree->left), Height(tree->right)));
}
```

Assume that "visting" a node simply means printing out the value of the data element:

```
void VisitNode(Tree tree)
{
  cout << tree->data << endl;
}
```

We can implement three traversal algorithms like this:

Implementation #1

```
void TraversePreOrder(Tree tree)
{
  if (tree == 0)
    return;

  VisitNode(tree);
  TraversePreOrder(tree->left);
  TraversePreOrder(tree->right);
}

void TraverseInOrder(Tree tree)
{
  if (tree == 0)
    return;

  TraverseInOrder(tree->left);
  VisitNode(tree);
  TraverseInOrder(tree->right);
}

void TraversePostOrder(Tree tree)
{
  if (tree == 0)
    return;

  TraversePostOrder(tree->left);
  TraversePostOrder(tree->right);
  VisitNode(tree);
}
```

Implementation #2

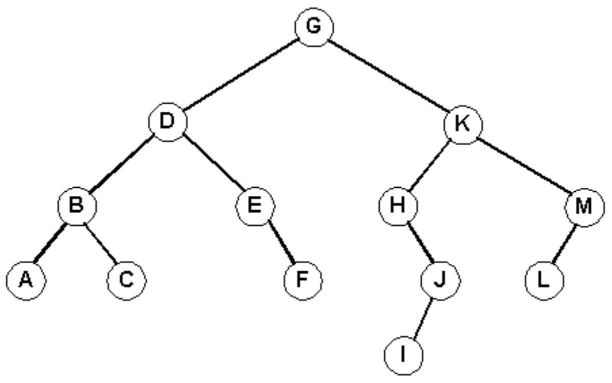
```
void TraversePreOrder(Tree tree)
{
  if (tree)
  {
    VisitNode(tree);
    TraversePreOrder(tree->left);
    TraversePreOrder(tree->right);
  }
}

void TraverseInOrder(Tree tree)
{
  if (tree)
  {
    TraverseInOrder(tree->left);
    VisitNode(tree);
    TraverseInOrder(tree->right);
  }
}

void TraversePostOrder(Tree tree)
{
  if (tree)
  {
    TraversePostOrder(tree->left);
    TraversePostOrder(tree->right);
    VisitNode(tree);
  }
}
```

Self-check Using the implementations above, what is the complexity for each of these traversal orders? In other words, how many nodes are accessed? (How many *times* is each node accessed?)

Level-Order Traversal

Traversing all nodes on level 0, from left to right, then all nodes on level 1 (left to right), then nodes on level 2 (left to right), etc. is *level-order* traversal.

So, a level-order traversal of this tree:



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will result in the nodes being visited in this order:

```
G D K B E H M A C F J L I
```

Traversing in level-order really isn't any more complicated by definition:

The recursive definition:

```
If the level being visited is:   0    Visit the node

If the level being visited is: > 0    Traverse the left subtree
                                      Traverse the right subtree
```

Sample code: Note the use of a *helper* recursive function.

```
void TraverseLevelOrder(Tree tree)
{
 int height = Height(tree);
 for (int i = 0; i <= height; i++)
   TraverseLevelOrder2(tree, i);
}

void TraverseLevelOrder2(Tree tree, int level)
{
  if (level == 0)
    VisitNode(tree);
  else
  {
    TraverseLevelOrder2(tree->left, level - 1);
    TraverseLevelOrder2(tree->right, level - 1);
  }
}
```

Using the implementations above, what is the complexity for level-order traversal? In other words, how many nodes are accessed? (How many *times* is each node accessed?)

Details of the *TraverseLevelOrder2* function above:

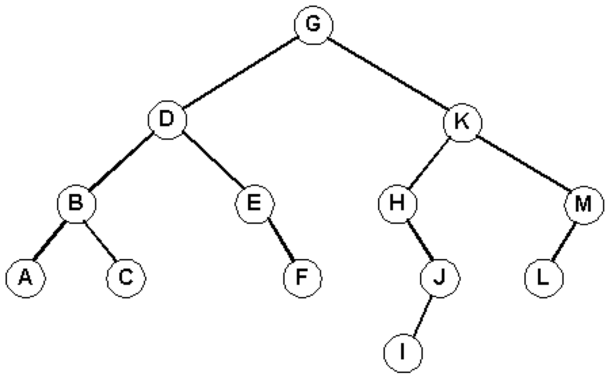
Level	Nodes at level N	Nodes in tree	Node Accesses
0	1	1	1
1	2	3	4
2	4	7	11
3	8	15	26
4	16	31	57
5	32	63	120
6	64	127	247
7	128	255	502
8	256	511	1013
9	512	1023	2036
10	1024	2047	4083
11	2048	4095	8178
12	4096	8191	16369
13	8192	16383	32752
14	16384	32767	65519
15	32768	65535	131054
16	65536	131071	262125
17	131072	262143	524268
18	262144	524287	1048555
19	524288	1048575	2097130

Self Check: Modify the algorithm above so it prints the nodes in reverse level-order: **I L J F C A M H E B K D**

Level-order traversal using a queue

Pseudocode

```
If the tree isn't empty
  Push the node onto the Queue
  While the Queue isn't empty
    Pop a node from the Queue
    Visit the node
    If the node's left child is not NULL
      Push the left child onto the Queue
    If the node's right child is not NULL
      Push the right child onto the Queue
  End While
End If
```



What is the complexity for this level-order traversal?

Self Check: Implement a function similar to *TraverseLevelOrder* that uses a *queue* as an auxiliary data structure. The function won't be recursive.

Self Check: Implement a function similar to *TraverseLevelOrder* that uses a *stack* as an auxiliary data structure. The function won't be recursive. What order is this traversal?

Binary Search Trees

Definition

> *A binary search tree (BST) is a binary tree in which the values in the left subtree of a node are all less than the value in the node, and the values in the right subtree of a node are all greater than the value of the node. The subtrees of a binary search tree must themselves be binary search trees.*

Note that under this definition, a BST never contains duplicate nodes.

Some operations for BSTs:

- InsertItem
- DeleteItem
- ItemExists
- Traverse (Pre, In, Post)
- Count
- Height

Notes:

- Count and Height (Depth) are straight-forward as above.
- The Traverse routines are also as above.
- ItemExists and InsertItem are also trivial:

Using the same definitions from above:

```
struct Node                 Node *MakeNode(int Data)           void FreeNode(Node *node)
{                           {                                  {
  Node *left;                 Node *node = new Node;             delete node;
  Node *right;                node->data = Data;               }
  int data;                   node->left = 0;
};                            node->right = 0;                 typedef Node* Tree;
                              return node;
                            }
```

As always

- State the recursive algorithm (in English) for finding an item in a BST.
- State the recursive algorithm (in English) for inserting an item into a BST.
- Call these two functions ItemExists and InsertItem.
- What are the parameters to these functions?

Sample code for finding an item in a BST:

```
bool ItemExists(Tree tree, int Data)
{
  if (tree == 0)
    return false;
  else if (Data == tree->data)
    return true;
  else if (Data < tree->data)
    return ItemExists(tree->left, Data);
  else
    return ItemExists(tree->right, Data);
}
```

Sample code for inserting an item into a BST:

```
void InsertItem(Tree &tree, int Data)
{
  if (tree == 0)
    tree = MakeNode(Data);
  else if (Data < tree->data)
    InsertItem(tree->left, Data);
  else if (Data > tree->data)
    InsertItem(tree->right, Data);
  else
    cout << "Error, duplicate item" << endl;
}
```

Self Check: Create a tree using these values (in this order): **12, 22, 8, 19, 10, 9, 20, 4, 2, 6**

What is the height of the resulting tree? What can you say about the tree? (Is it balanced? Is it complete?)

Self Check: Create a tree using the same values but in this order: **2, 4, 6, 10, 8, 22, 12, 9, 19, 20**

What is the height of the resulting tree? What can you say about the tree? (Is it balanced? Is it complete?)

Diagrams of the results.

Self Check: What is the worst case time complexity for searching a BST? Best? What causes the best/worst cases?

Search Times

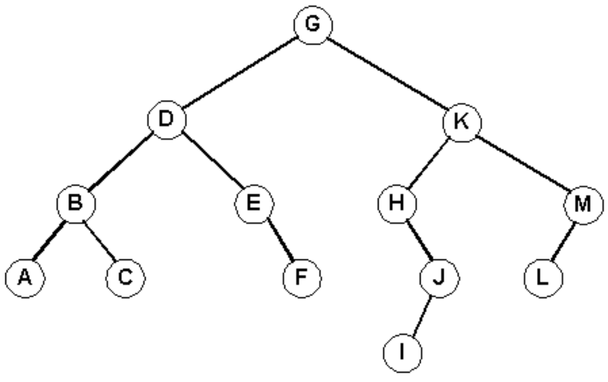
- BST algorithms are highly dependent on the input data.
- One solution is to balance the tree.
- Balancing BSTs can be expensive and the algorithms are more complex than what we've seen so far.
  - Balancing a tree is somewhat analagous to sorting a linked-list.
- One type of balanced tree is AVL (Two Russian Mathematicians, Adel'son-Vel'skii and Landis). More on this later.

Self Check: Given a BST with 10 nodes, what is the maximum and minimum height of the tree?

Self Check: Given a BST of height 2, what is the maximum and minimum number of leaves in the tree?

Deleting A Node

The caveat of deleting a node is that, after deletion, the tree must still be a BST. Using this tree as an example:



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There are four cases to consider:

1. The node to be deleted is a leaf. (Nodes: **A C F I L**)

   This is trivial. Set the parent's pointer to this node to NULL.

2. The node to be deleted has an empty left child but non-empty right child. (Nodes: **E H**)

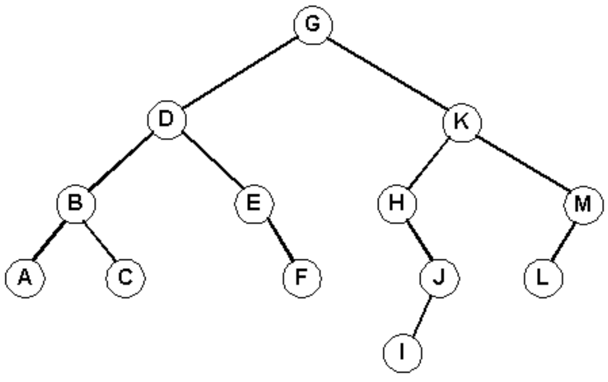
   Replace the deleted node with its right child. Note that this case can be combined with Case #1 by "promoting" the right child. This works even if the right child is NULL.

3. The node to be deleted has an empty right child but non-empty left child. (Nodes: **J M**)

   Similar to #2. Promote the left child.

4. The node to be deleted has both children non-empty. (Nodes: **B D G K**)

   - Replace the data in the deleted node with its predecessor under inorder traversal.
   - Delete the node that held the predecessor.
   - The predecessor will be the *rightmost node in the left subtree of the deleted node.*
   - In the diagram, this is **A** if **B** is deleted, **C** if **D** is deleted, **F** if **G** is deleted, and **J** if **K** is deleted.



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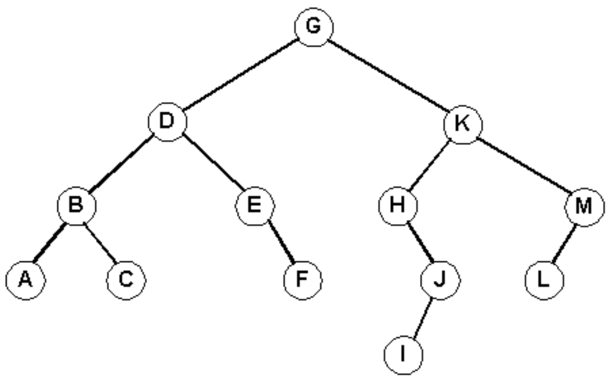
Notes about Case #4

- We are replacing the data in the node, *not* the node itself.
- Because the predecessor comes from the left subtree:
  - it must be less than everything in the right subtree
  - it is an *inorder* predecessor so it must be greater than everything else in the left subtree
- The recursive deletion of the predecessor's node will find the node in one of the simpler cases.

To implement the DeleteItem function, we use a helper function called FindPredecessor, which simply finds the inorder predecessor for a given node.

```
void DeleteItem(Tree &tree, int Data)
{
  if (tree == 0)
    return;
  else if (Data < tree->data)
    DeleteItem(tree->left, Data);
  else if (Data > tree->data)
    DeleteItem(tree->right, Data);
  else // (Data == tree->data)
  {
    if (tree->left == 0)
    {
      Tree temp = tree;
      tree = tree->right;
      FreeNode(temp);
    }
    else if (tree->right == 0)
    {
      Tree temp = tree;
      tree = tree->left;
      FreeNode(temp);
    }
    else
    {
      Tree pred = 0;
      FindPredecessor(tree, pred);
      tree->data = pred->data;
      DeleteItem(tree->left, tree->data);
    }
  }
}

void FindPredecessor(Tree tree, Tree &predecessor)
{
  predecessor = tree->left;
  while (predecessor->right != 0)
    predecessor = predecessor->right;
}
```



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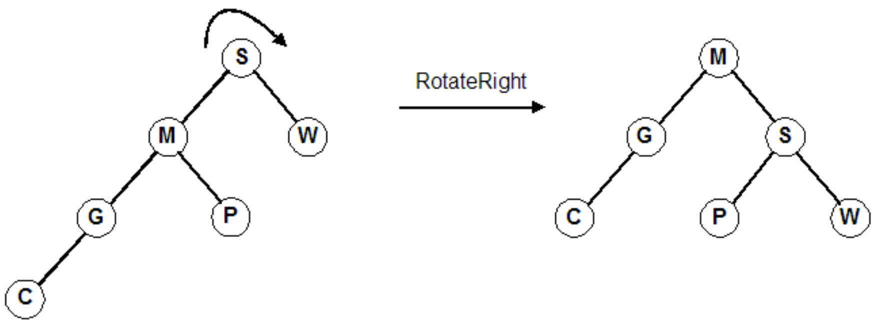
Self Check: What is the resulting tree from deleting the root node (**G**) in the tree above? What does the tree look like if you delete **K** from the tree?

Rotating Nodes

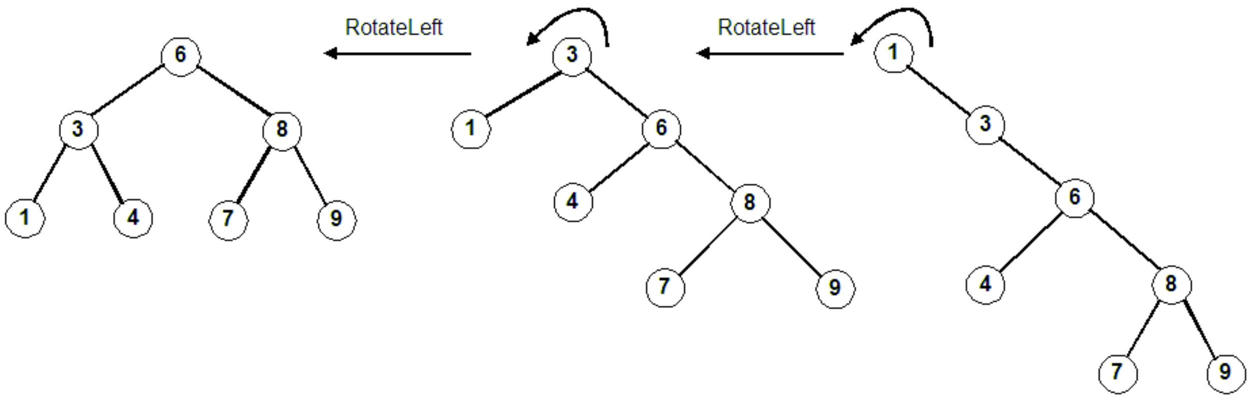
- Rotation is a fundamental technique that is performed on BSTs.
- There are two types of rotations: left and right.
- Rotating a node is the same as promoting one of the node's children.
  - Promoting a child means that it will take the place of its parent.
  - Rotate left means promote the right child.
  - Rotate right means promote the left child.
  - You can't rotate if the necessary child is missing (NULL), which means you can't promote an empty child.
- So, promoting a node is the same as rotating around the node's parent. (There is no direction, promotion is unambiguous.)
- You can rotate about any node that has the necessary child.

Note: An important property of rotation is that after the rotation, the sort order is preserved. This is important, because the resulting tree must still be a BST.

Rotate **right** about the root, **S**. (Same as promoting the **left** child, **M**)



Rotate left twice about the root. (Far right diagram) First rotate about **1**, then rotate about **3**. (Same as promoting **3** and then promoting **6**)



Using the defintions above. **Note the parameter to each function is a *reference to a pointer*.**

Rotating a tree right

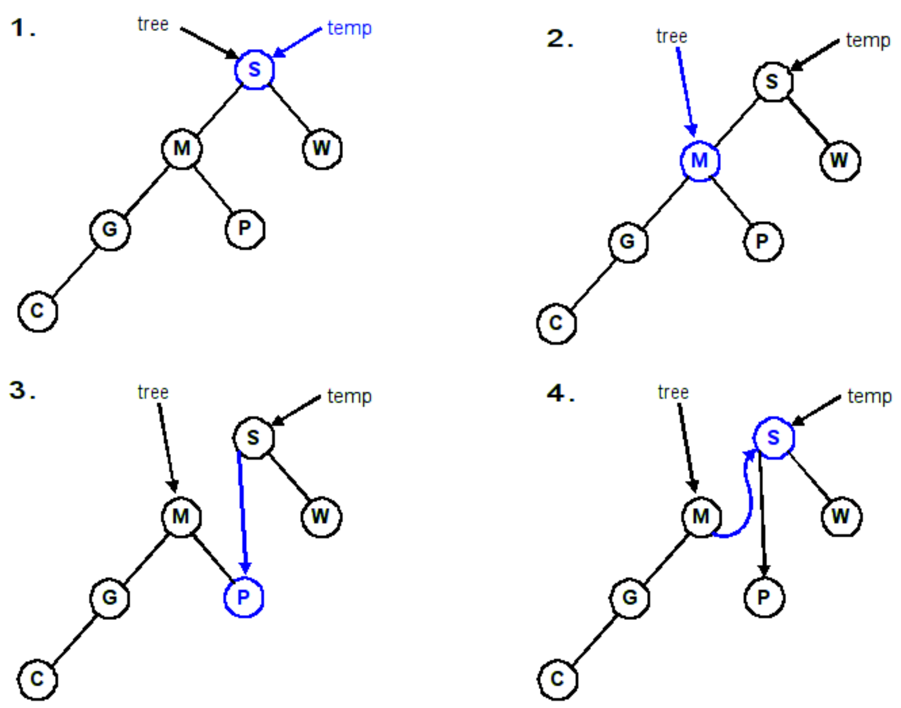
```
void RotateRight(Tree &tree)
{
  Tree temp = tree;
  tree = tree->left;
  temp->left = tree->right;
  tree->right = temp;
}
```

Rotating a tree left

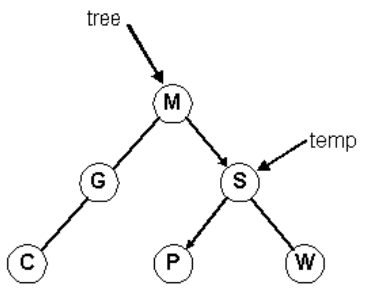
```
void RotateLeft(Tree &tree)
{
  Tree temp = tree;
  tree = tree->right;
  temp->right = tree->left;
  tree->left = temp;
}
```

Follow the four lines of code in this example. We are rotating right about **S** (promoting **M**).

```
1. temp = Tree;              // temp ===> S
2. tree = temp->left;        // tree ===> M
3. temp->left = tree->right; // temp->left ===> P
4. tree->right = temp;       // tree->right ===> S
```



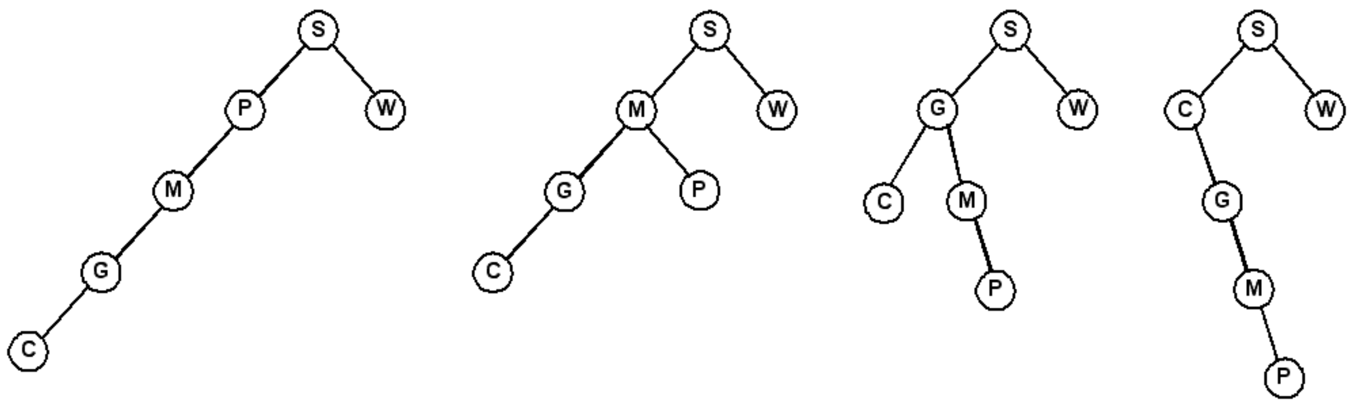
Adjusting the diagram:



> You can easily see why we passed a reference (or pointer) to the root of the tree. If you just pass the pointer itself (by value), after the rotation `tree` still points at node `S`, which is wrong. Keep this in mind when you are implementing the tree functions.

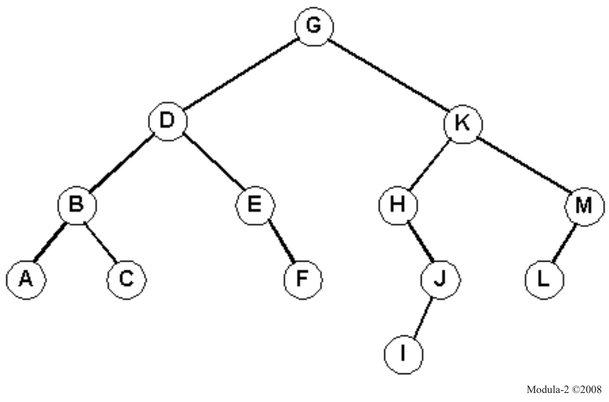
Note that these four trees below all contain the same data.

One **Two** **Three** **Four**



Can you explain why there are four different representations for the same data?

Self Check: What is the resulting tree from rotating left about the root node (**G**) in the tree below? How about rotating right about the root? Rotating left about a node means promoting the node's left child. Rotating right about a node means promoting the node's left child.



Self Check: After rotating right about **G** above, the tree is unbalanced because 1 or more nodes is unbalanced. Specifically, which nodes are unbalanced?

Self Check: Insert the letters: **P I N K F L O Y D E R S** into a BST. What is the height of the tree? The tree is NOT balanced. Which nodes in the tree are unbalanced? Rotate about the root (**P**) node. Now what is the height of the tree? Which nodes are unbalanced now? Finally, delete node **P** from the *original* tree. Check your work with this program.

Self Check: Insert the letters: **K E Y B O A R D I S T** into a BST. What is the height of the tree? The tree is NOT balanced. Which nodes in the tree are unbalanced? Rotate about the root (**K**) node. Now what is the height of the tree? Which nodes are unbalanced now? Finally, delete node **E** from the *rotated* tree. Check your work with this program.

Self Check: Insert the letters: **K E Y B O A R D I S T** into a BST. What is the sequence of letters when doing a Pre-order traversal? An In-order traversal? A Post-order traversal?

BST/AVL program showing balance factors and node counts. You can use this program to observe how a binary tree is built and how rotation changes a tree. You can ignore the AVL tree for now.

Splay Trees

Invented by D.D. Sleator and R.E. Tarjan in 1985.

- Most of the time, we don't make any assumptions about the data.
- Usually assume equal distribution of data and random values.
- Non-random data can lead to worst-case situations (building a BST from already-sorted data).
- Non-uniform distribution of data can also lead to worst-case situations.
- If we know how the data is distributed, we can choose better data structures.

A splay tree uses this knowledge to an advantage.

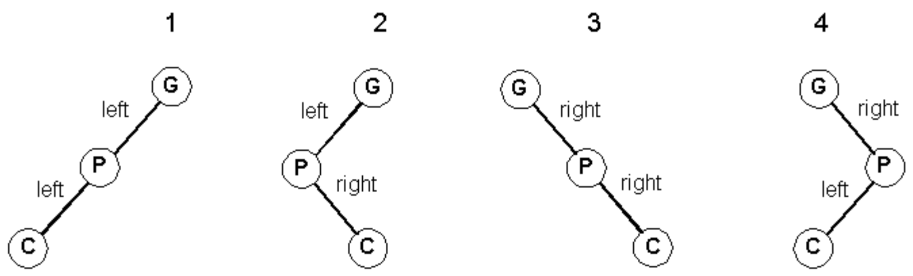
- A splay tree is a binary search tree.
- Newly inserted items are propagated (promoted) to the root.
- This propagation occurs when writing (inserting) and reading (accessing) an item.
- We call this propagation of a node *splaying*.

The idea behind splay trees is that *frequently* accessed data is always near the top.

- Splay trees are not guaranteed to be balanced
- Worst-case is not guaranteed to be "good"
- Average time may be excellent (this may be more important than worst case)
- Algorithms for splaying a node are simple (only require rotation)
- The algorithm is a variation of the more general BST root insertion

Splaying algorithm

- We want to splay a node two levels at a time.
- This means we want to promote the node to the position of its grandparent (parent's parent)
- The algorithm depends on the node's orientation to its grandparent (1 of 4 orientations)

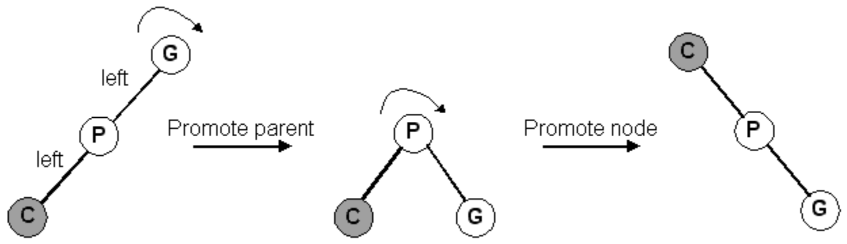


1. left-left, promote the parent, promote the node
2. left-right, promote the node, promote the node (node is promoted twice)
3. right-right, promote the parent, promote the node
4. right-left, promote the node, promote the node (node is promoted twice)

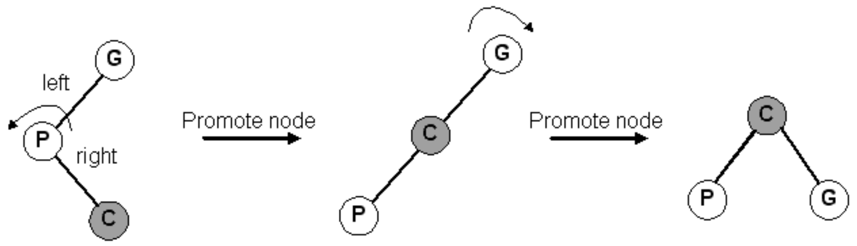
Promoting a node simply means rotating about the node's parent (which we've done). Promoting a node doesn't require you to specify left or right. The direction is implied.

- If node is a right-child, rotate parent LEFT
- If node is a left-child, rotate parent RIGHT

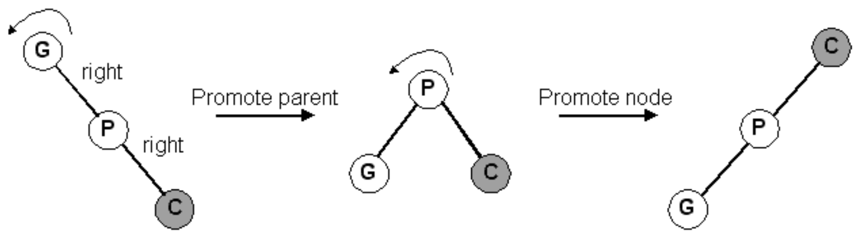
Left-Left orientation (zig-zig)



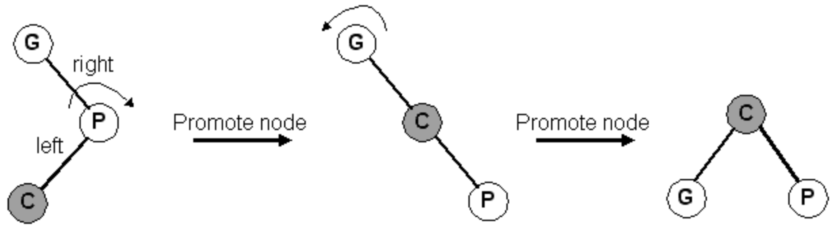
Left-Right orientation (zig-zag)



Right-Right orientation (zig-zig)



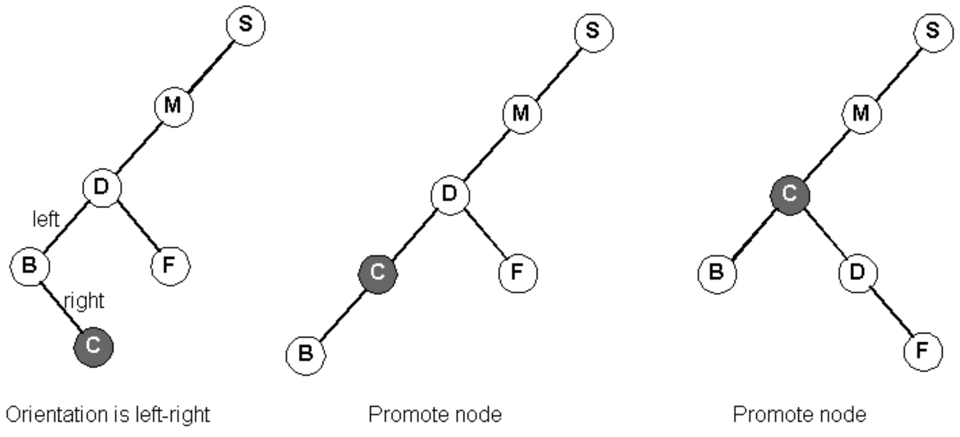
Right-Left orientation (zig-zag)



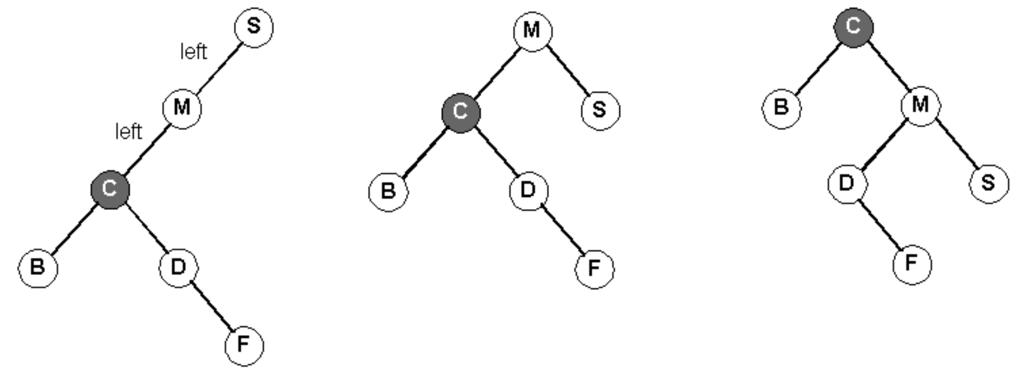
- We continue to promote until we reach the root.
- The "special case" is if our parent is the root.
- If the parent is the root, simply perform a rotation to bring the node to the root.

In the example below, we want to splay node **C** to the root.

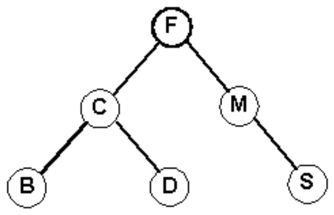
Our orientation with our grandparent is left-right at first:



Now, our orientation with our grandparent is left-left:



The result of splaying **F** to the root:



Additional Notes:

- Good *amortized* performance: The average cost is efficient.
- Acts sort of like a built-in caching mechanism.
- Works well with non-uniform access patterns over a small working set.
- Simple to implement.

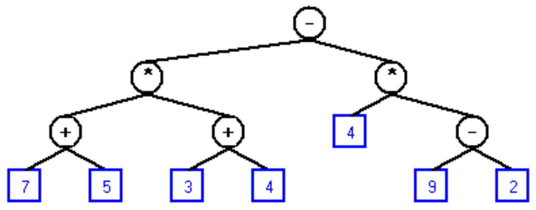
Expression Trees

Expression trees are like binary trees, but they are not "sorted" in the usual way.

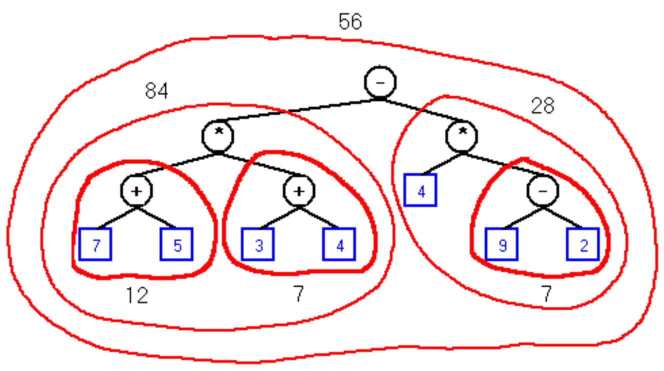
Given this expression:

```
(7 + 5) * (3 + 4) - (4 * (9 - 2))
```

The result after evaluation is 56. The tree that represents it looks like this:



Evaluating the tree gives the same result. Evaluating an expression tree simply means reducing each subtree by post-order traversal. Why post-order?



Self-Check: Perform a post-order traversal on the expression tree above. This will create a postfix expression. Using the stack method from here, evaluate it and verify that you get the value 56.

Grammar

A simple definition for the *grammar*, or language, that defines an expression looks something like this:

```
<expression> ::= <term> { <addop> <term> }
<term>       ::= <factor> { <mulop> <factor> }
```

```
<factor>     ::= ( <expression> ) | <identifier> | <literal>
<addop>      ::= + | -
<mulop>      ::= * | /
<identifier> ::= a | b | c | ... | z | A | B | C | ... | Z
<literal>    ::= 0 | 1 | 2 | ... | 9
```

Note that the grammar is (indirectly) recursive. The vertical bars are read as "OR", and the curly braces means that the item inside can be repeated 0 or more times.

Our *"language"* consists of the following tokens:

```
Valid tokens: ()+-*/abcdefghijklmnopqrstuvwxyzABCDEFGHIJKLMNOPQRSTUVWXYZ0123456789
```

Examples:

```
 Valid expressions: A, B, 1, A + 2, A + B, A * (B), A * (B - 2), (1)
Invalid constructs: AB, 3A, 123, A(3), A + (), A * -3
```

Given any infix valid expression within the language, we can evaluate or reduce the expression in a two-step process:

1. Construct a parse tree from the infix expression.
2. Simplify the parse tree by evaluating sub-trees (sub-expressions).

Step 1: Pseudocode for Parsing

```
<expression> ::= <term> { <addop> <term> }
<term>       ::= <factor> { <mulop> <factor> }
<factor>     ::= ( <expression> ) | <identifier> | <literal>
```

```
MakeExpression(Tree)
  Make a term, setting Tree to point to it

  while the next token is '+' or '-'
    Make an operator node, setting left child to Tree and right to NULL. (Tree points to new node)
    Get the next token.
    Make a term, setting the right child of Tree to point to it.
  end while
End MakeExpression

MakeTerm(Tree)
  Make a factor, setting Tree to point to it

  while the next token is '*' or '/'
    Make an operator node, setting left child to Tree and right to NULL. (Tree points to new node)
    Get the next token.
    Make a factor, setting the right child of Tree to point to it.
  end while
End MakeTerm

MakeFactor(Tree)
  if current token is '(', then
    Get the next token
    Make an expression, setting Tree to point to it
  else if current token is an IDENTIFIER
    Make an identifier node, set Tree to point to it, set left/right children to NULL.
  else if current token is a LITERAL
    Make a literal node, set Tree to point to it, set left/right children to NULL.
  end if

  Get the next token
End MakeFactor


GetNextToken
  while whitespace
    Increment CurrentPosition
  end while

  CurrentToken = Expression[CurrentPosition]

  Increment CurrentPosition
End GetNextToken
```

```
<expression> ::= <term> { <addop> <term> }
<term>       ::= <factor> { <mulop> <factor> }
<factor>     ::= ( <expression> ) | <identifier> | <literal>
```

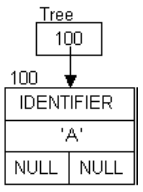
Definitions

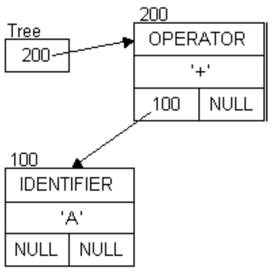
Some implementations

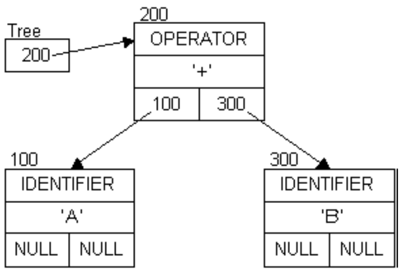
Diagrams for the expression: `A + B` (All addresses are arbitrary, but represent the order the nodes were created.)

1. Make an IDENTIFIER (EXPRESSION → TERM → FACTOR → IDENTIFIER) node and set **`Tree`** to point to this term.
2. Make an OPERATOR node, set left child to **`Tree`** and right child to NULL. (Tree now points to this new operator node)
3. Make an IDENTIFIER (EXPRESSION → TERM → FACTOR → IDENTIFIER) node and set right child of **`Tree`** to point to this term.

`A` **`A +`** **`A + B`**



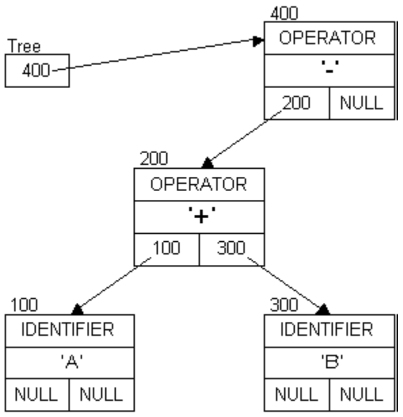




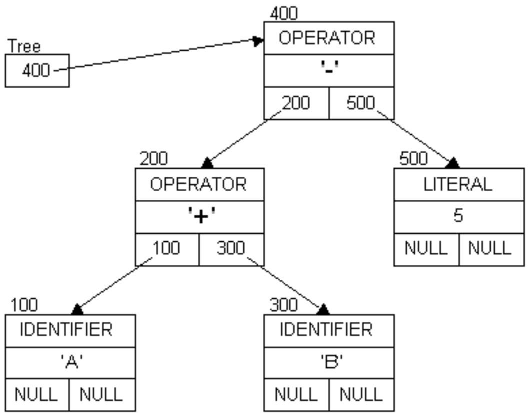
Extending the expression to: `A + B - 5`

1. Make an OPERATOR node, set left child to **`Tree`** (from above) and right child to NULL. (Tree now points to this new operator node)
2. Make a LITERAL (EXPRESSION → TERM → FACTOR → LITERAL) node and set right child of **`Tree`** to point to this term.

`A + B -`

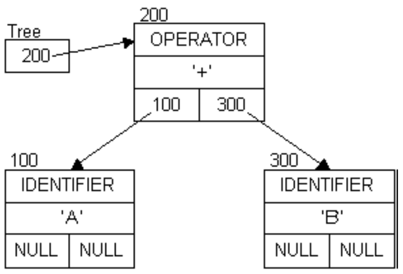


`A + B - 5`



Diagrams for the expression: `A + B * C`

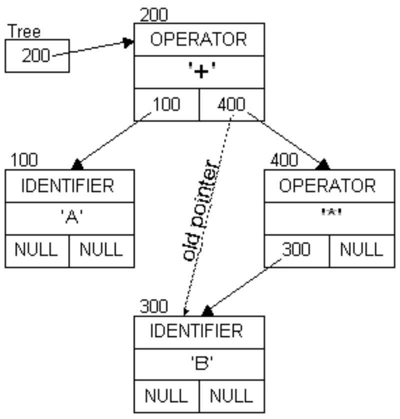
`A + B` as before:



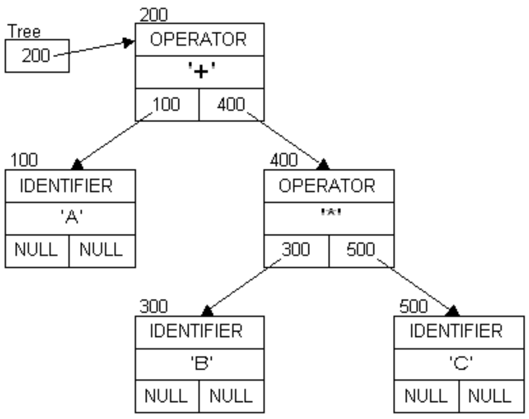
Adding: **`* C`**

1. Make an OPERATOR node, set left child to **`Tree`** (from above, tree is the right pointer of '+') and right child to NULL. (Tree now points to this new operator node)
2. Make a LITERAL (TERM → FACTOR → LITERAL) node and set right child of **`Tree`** to point to this term.

`A + B *`



`A + B * C`



Step 2: Simplifying the Parse Tree

Algorithm to recursively simplify a tree:

1. Recursively simplify left subtree
2. Recursively simplify right subtree
3. Simplfy the node (perform the arithmetic, no recursion)

Some simplification examples:

```
4 * (2 + 3) → 20      A * (2 + 3) → A * 5      A * (3 - 4 + 1) + B → B        A + 2 * 3 → A + 6
```

Simplification Rules:

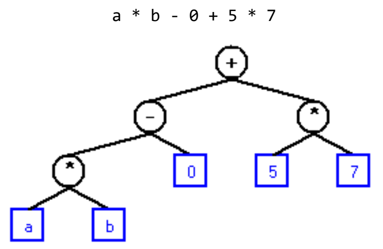
Condition	Action
Both children are LITERAL	Evaluate the expression and promote the result to the node that contained the operator. 0 / 0 ⟶ (exception)

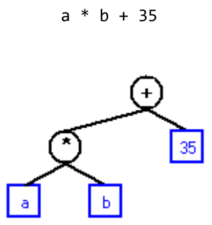
The left child is a LITERAL and the right child is an IDENTIFIER or OPERATOR (expression).	If expression is one of these forms, it can be simplified and the result promoted:
	`0 + E → E      1 * E → E`
	`0 * E → 0      0 / E → 0`

The right child is a LITERAL and the left child is an IDENTIFIER or OPERATOR (expression).	If expression is one of these forms, it can be simplified and the result promoted:
	`E + 0 → E      E - 0 → E`
	`E * 0 → 0      E * 1 → E`
	`E / 1 → E      E / 0 → (exception)`

Both children are IDENTIFIER.	If expression is one of these forms, it can be simplified and the result promoted: `I - I → 0      I / I → 1`

Example of the form: E - 0 ⟶ E





Caveats

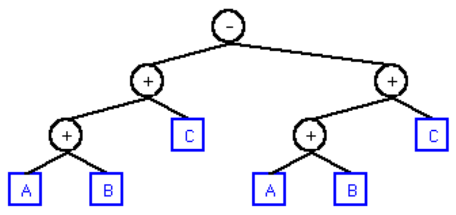
This technique will not be able to simplify all expressions. For example, these can not be simplified:

```
(A + 7) / (A + 7)           # should be 1
(A + 7) - (A + 7)           # should be 0
(A + 7) - (7 + A)           # same
(3 + A + 4) - (A + 7)       # same
(A + B + C) - (C + B + A)   # same
```

To simplify these expressions, more complex algorithms are needed. One step is to *normalize* the expressions such that different operand orderings can be dealt with. For example, sorting the identifiers in alphabetical order:

```
(A + C + B)  ==> (A + B + C)
(C + B + A)  ==> (A + B + C)
(B + A + C)  ==> (A + B + C)
```

The resulting tree:



Then, you'd have to recognize that the entire left subtree of the root is identical to the entire right subtree of the root. You would do this with a recursive algorithm that determines if two trees are identical. The Poor-Man's Way™ is to simply perform a traversal (pre-, post-, in-order, doesn't matter) and compare the outputs.

A possibly more elegant solution would be a simple recursive function. Here's a prototype:

```
// From the example code above
struct Node
{
  Node *left;
  Node *right;
  int data;
}

bool isIdentical(const Node *left_tree, const Node *right_tree);
```

And one possible algorithm (psuedocode):

- If left_tree is NULL AND right_tree is NULL, return true. (Base case)
- If one of the pointers is NULL but the other is not, return false. (Base case. Can't be identical)
- Else both pointers are valid.
    - If data in both pointers is different, return false (Base case. Can't be identical)
    - Else recursively check the left subtree AND recursively check the right subtree. (Recursive case)

Of course, this is complicated by the fact that different operators have different precedence and commutative properties and you can't ignore that:

```
(A * C + B)  ==> (A * B + C)  # incorrect
(C - B - A)  ==> (A - B - C)  # incorrect
```

This requires much more sophisticated logic (in the general case) to make sure that the transformations do not change the underlying meaning. This is non-trivial for dealing with arbitrarily complex expressions.

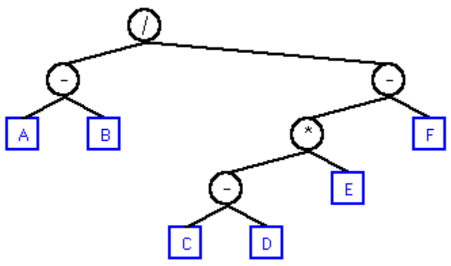
Expression Tree Viewer

Self-check Build the parse tree for these expressions:

```
A + B + C
A + B * C
A * B + C
(A + B) * C
A * (B + C)
```

Substitute integers for A, B, and C and then evaluate the tree using postfix traversal.

Self-check Given the expression tree below, show the postfix expression. Hint: perform a *post-order traversal*.



Real World: Compilers and Constant Folding

This is how compilers do constant folding. They simply make a tree out of the expression and perform a post-order traversal to evaluate it at *compile time*.

Given this code (from the link above):

```
int main()
{
  int a = 30;
  int b = 9 - (a / 5); // b is 3
  int c;

  c = b * 4;           // c is 12
  if (c > 10)          // 12 > 10 (true)
  {
     c = c - 10;       // c is 12 - 10, which is 2
  }

  return c * (60 / a); // return 2 * (60 /30) ==> 2 * 2 ==> 4
}
```

Compiler-generated assembly code (GNU g++ 5.1):

No optimization	Optimization -O
`  .file "main.cpp"`	`  .file "main.cpp"`
`  .text`	`  .text`
`  .globl  main`	`  .globl  main`
`  .type main, @function`	`  .type main, @function`
`main:`	`main:`
`.LFB0:`	`.LFB0:`
`  pushq %rbp`	`  movl  $4, %eax`
`  movq  %rsp, %rbp`	`  ret`
`  movl  $30, -8(%rbp)`	`.LFE0:`
`  movl  -8(%rbp), %ecx`	`  .size main, .-main`
`  movl  $1717986919, %edx`	`  .ident  "GCC: (Mead custom build) 5.1.0"`
`  movl  %ecx, %eax`	`  .section  .note.GNU-stack,"",@progbits`
`  imull %edx`	
`  sarl  %edx`	
`  movl  %ecx, %eax`	
`  sarl  $31, %eax`	
`  subl  %eax, %edx`	
`  movl  %edx, %eax`	
`  movl  $9, %edx`	
`  subl  %eax, %edx`	
`  movl  %edx, %eax`	
`  movl  %eax, -12(%rbp)`	
`  movl  -12(%rbp), %eax`	
`  sall  $2, %eax`	
`  movl  %eax, -4(%rbp)`	
`  cmpl  $10, -4(%rbp)`	

```
  jle .L2
  subl  $10, -4(%rbp)
.L2:
  movl  $60, %eax
  cltd
  idivl -8(%rbp)
  imull -4(%rbp), %eax
  popq  %rbp
  ret
.LFE0:
  .size main, .-main
  .ident  "GCC: (Mead custom build) 5.1.0"
  .section  .note.GNU-stack,"",@progbits
```

If `a`, `b`, and `c` were declared global instead of local, the compiler won't optimize it. For example:

```
int a = 30;
int b = 9 - (a / 5); // b is 3
int c;

int main()
{
  c = b * 4;           // c is 12
  if (c > 10)          // 12 > 10 (true)
  {
     c = c - 10;       // c is 12 - 10, which is 2
  }

  return c * (60 / a); // return 2 * (60 /30) ==> 2 * 2 ==> 4
}
```

Why is that?

If we declare them as static:

```
static int a = 30;
static int b = 9 - (a / 5);
static int c = b * 4;

int main()
{
  if (c > 10)
  {
    c = c - 10;
  }

  return c * (60 / a);
}
```

and compile with `-O`, we see this (all of the "noise" in the assembly code has been removed and comments added)

```
main:
  movl  _ZL1c(%rip), %eax   ; put c in eax
  cmpl  $10, %eax           ; compare c with 10
  jle .L2                   ; if less-than or equal, jump to L2
  subl  $10, %eax           ; c is greater than 10, so subtract 10
  movl  %eax, _ZL1c(%rip)   ; put eax back into c
.L2:
  movl  _ZL1c(%rip), %eax   ; put c in eax
  addl  %eax, %eax          ; c + c, same as c * 2
  ret
```

If we compile with `-O2`, we see this slightly more optimized version (without the redundant load of `c`):

```
main:
.LFB0:
  movl  _ZL1c(%rip), %eax
  cmpl  $10, %eax
  jle .L2
  subl  $10, %eax
  movl  %eax, _ZL1c(%rip)
.L2:
  addl  %eax, %eax
  ret
```

However, if I add this before `main`

```
void foo()
{
  extern int i;
  a = i;
}
```

All optimizations are removed. But, if I marked `foo` as static:

```
static void foo()
{
  extern int i;
  a = i;
}
```

all optimizations are back on. But, wait, there's more! If I actually *call* `foo`:

```
int main()
{
  foo();
```

```
    if (c > 10)
    {
      c = c - 10;
    }

    return c * (60 / a);
  }
```

all of the optimizations are off again! Aren't modern compilers a Wonderful Thing™!