The RC/RL circuit (1)

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A sidebar

Linear 1st order with a constant coefficient partial differential equations (PDEs)

→ Linear 1st order P.D.E

1. The form : $\boxed{Ax' + Bx = f(t) \quad \cdots \quad \text{I}}$

   - A and B are constants & $x \Leftrightarrow x(t)$

   - $f(t)$: a known eq. ; i.e., t or $2t$ or $1, 2, 3 \ldots$

2. Solve for I : means find $x(t)$

③ Two types of sols for $x(t)$

- Homogeneous sol = natural response : $\boxed{Ⓐ\ X_h}$

f(t) = "0"

- Particular sol = forced response : $\boxed{Ⓑ\ X_p}$

f(t) = "Something"

Therefore,

f(t) = Ⓐ + Ⓑ = 0 + Something = Something.

(Make sense !!!)

Let's do elaboration, Let's prove it →

Let's prove ③ via $\boxed{\text{SUPERPOSITION}}$ principle

Suppose $\boxed{X_p}$ is a sol to Ⓘ

Again, where $\boxed{Ax' + Bx = f(t) \cdots \text{Ⓘ}}$

⊛ $\rightarrow AX_p' + BX_p = f(t)$

and $\boxed{X_h}$ is a homogeneous sol to Ⓘ

⊛ $\rightarrow AX_h' + BX_h = f(t) = 0$

Then $\underbrace{\boxed{X = X_p + X_h} \text{ is a sol for Ⓘ}}_{\text{What we want to prove}}$

<proof>

Find!!

$$Ax' + Bx \boxed{\overset{?}{=}} f$$

$$A(x_p' + x_h') + B(x_p + x_h) \boxed{\overset{?}{=}} f$$

$$\underbrace{Ax_p' + Bx_p}_{f(t)} + \underbrace{Ax_h' + Bx_h}_{0} \boxed{\overset{?}{=}} f$$

$$\therefore \quad f(t) + 0 = f(t) \quad //$$

Done!

<Example>

$$x' + 4x = \boxed{3}$$

① Homogeneous : $x_h' + 4x_h = 0$

(Everywhere is 0 for this function)

$$x_h' = -4x_h$$

- the Derivative of a function still has $\underbrace{\text{its original form}}^{\text{A scale version of}}$

⇒ An exponential function !!!

$$\therefore \quad x_h = C \cdot e^{-4t}$$

(Check! $x_h' = -4Ce^{-4t}$

$= -4 \cdot Ce^{-4t} = -4x_h$)

② Particular sol : $X_p = A$ (based on $X' + 4x = \boxed{3}$ this) guess a constant.

You plug what you guessed into

① Guess
② Try (substitute)
③ Check

$$X_p' + 4X_p = 3$$
$$(A)' + 4 \cdot A = 3 \Rightarrow 0 + 4A = 3 \quad \therefore A = 3/4.$$

③ As $X = X_p + X_h \Rightarrow 3/4 + c \cdot e^{-4t} = x(t)$

(solve for C via any extra condition given)

Another way to solve

(can we solve the PDE without homo... particular sols?)..

A formal way to solve for $AX' + BX = f(t)$, when f(t)=D (a constant)

$$Ax' + Bx = f(t)$$

$$Ax' + Bx = D$$

$$x' + \frac{x}{\frac{A}{B}} = \frac{D}{A} \quad // \text{ Let } \frac{A}{B} = \tau \text{ and } \boxed{\frac{D}{A}} = K$$

another constant

$$\frac{dx}{dt} + \frac{x}{\tau} = K \Leftrightarrow \frac{dx}{dt} = K - \frac{x}{\tau}$$

$$\frac{dx}{dt} = \frac{K\tau - x}{\tau} \Leftrightarrow \frac{dx}{K\tau - x} = \frac{dt}{\tau}$$

$$\int \frac{dx}{x - k\tau} = -\int \frac{dt}{\tau}$$

$x - k\tau = u$ then $du = dx$

$$\int \frac{du}{u} = \ln|u| + A$$

$$\Downarrow$$

$$\int \frac{dx}{x-k\tau} = -\int \frac{dt}{\tau} \quad \underset{\text{property}}{\overset{\text{Integration}}{\Longleftrightarrow}} \quad \ln(x - k\tau) = -\frac{t}{\tau} + \text{Constant}$$

$$\boxed{\log_B A = C \Leftrightarrow B^C = A}$$

$$e^{\left(-\frac{t}{\tau} + \text{Constant}\right)} = x - k \cdot \tau$$

// x is a function of t

$$e^{-\frac{t}{\tau}} \cdot e^{\text{Constant}} = x(t) - k\tau$$

$$x(t) = k \cdot \tau + \boxed{e^{\text{Constant}}} \cdot e^{-t/\tau}$$

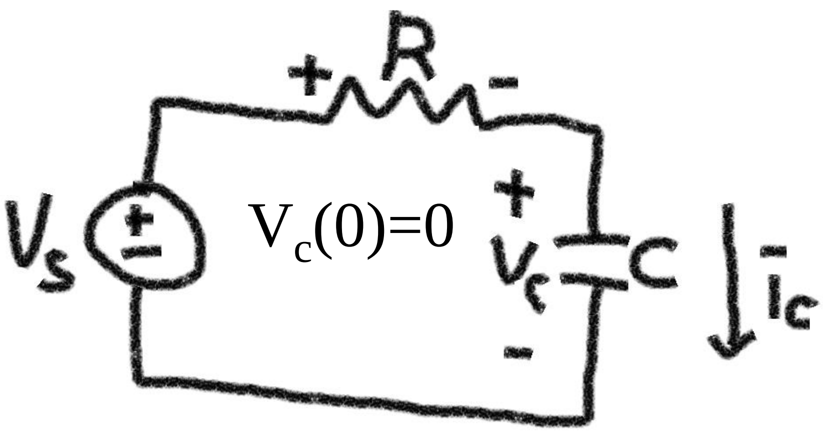
↳ Another constant such as $k_1, k_2 \ldots$

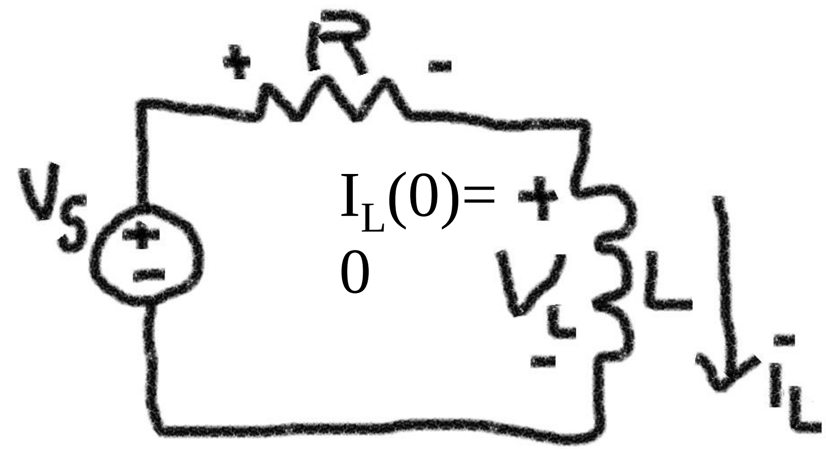
Particular sol./Forced R.

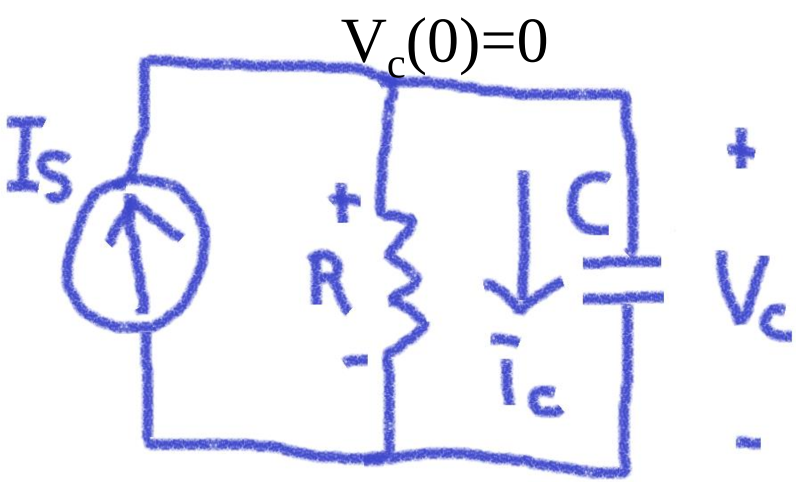
Homogen sol./Natural R.

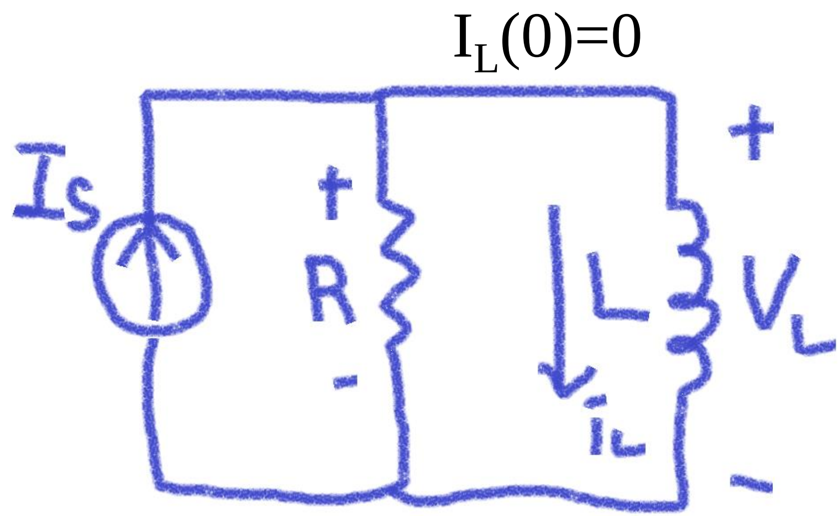
Relation	Resistor (R)	Capacitor (C)	Inductor (L)
v-i:	$v = iR$	$v = \frac{1}{C}\int_{t_0}^{t} i\, dt + v(t_0)$	$v = L\frac{di}{dt}$
i-v:	$i = v/R$	$i = C\frac{dv}{dt}$	$i = \frac{1}{L}\int_{t_0}^{t} v\, dt + i(t_0)$
p or w:	$p = i^2 R = \frac{v^2}{R}$	$w = \frac{1}{2}Cv^2$	$w = \frac{1}{2}Li^2$
Series:	$R_{eq} = R_1 + R_2$	$C_{eq} = \frac{C_1 C_2}{C_1 + C_2}$	$L_{eq} = L_1 + L_2$
Parallel:	$R_{eq} = \frac{R_1 R_2}{R_1 + R_2}$	$C_{eq} = C_1 + C_2$	$L_{eq} = \frac{L_1 L_2}{L_1 + L_2}$
At dc:	Same	Open circuit	Short circuit
Circuit variable that cannot change abruptly:	Not applicable	v	i

Practice examples and solutions 1st order PDEs with a constant source

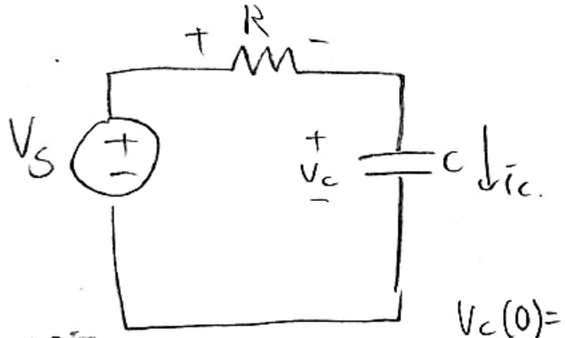








1. Construct a 1st order partial differential equation
2. Solve for 1, based on the given initial condition at t=0



$V_c(0)=0$

PDE

$$\boxed{V_S = RC\cdot\frac{d}{dt}V_c + V_c} \leftarrow \therefore \boxed{\begin{aligned}V_S &= R i_c + V_c \\ V_S &= R\cdot C\cdot\frac{d}{dt}V_c + V_c\end{aligned}}$$

$$\boxed{i_c = C\cdot\frac{d}{dt}V_c}$$

Solve for $V_c(t)$ $\rightarrow V_C(t) = V_h + V_p$.

$V_h \rightarrow RC\cdot\frac{d}{dt}V_h + V_h = 0$

$$V_h = -RC\cdot\frac{d}{dt}V_h$$

$$\boxed{\therefore V_h = K_1\cdot e^{-t/RC}}$$

$$V_h' = -\frac{1}{RC}\cdot K_1\cdot e^{-t/RC}$$

$$= -\frac{K_1}{RC}e^{-t/RC}$$

$$V_h = -RC\underbrace{\boxed{\frac{-K_1}{RC}e^{-t/RC}}}_{V_h'} = K_1\cdot e^{-t/RC} = V_h.$$

①

$V_p \rightarrow X$ (a constant)

$\therefore RC\cdot\frac{d}{dt}V_p + V_p = V_S$

$$RC\cdot\cancel{\frac{d}{dt}X} + X = V_S$$

0

$$\therefore \boxed{X = V_S = V_p}$$

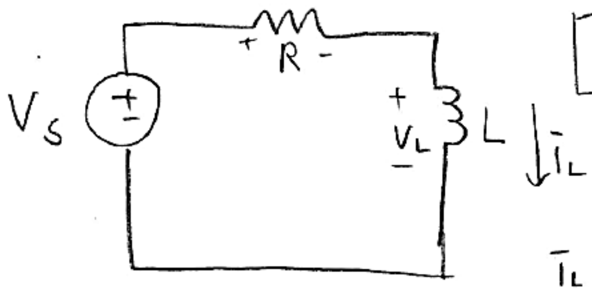
$$V_c(t) = V_h + V_p = K_1\cdot e^{-t/RC} + V_S.$$

as $V_c(0) = 0$

$$V_c(0) = K_1\cdot\underbrace{e^{-0/RC}}_{1} + V_s = 0$$

$$K_1 + V_S = 0,\quad K_1 = -V_S.$$

$$\therefore V_c(t) = V_S(1 - e^{-t/RC})\ [V].$$



$\boxed{V_C = L\cdot\frac{d}{dt} i_L}$

$i_L(0) = 0$ ⊛

$V_S = R\cdot i_L + V_L$

PDE

$$\boxed{V_s = R\cdot i_L + L\cdot\frac{d}{dt} i_L}$$

Solve for $i_L(t) \rightarrow i_L(t) = i_h(t) + i_p(t)$

$i_h \rightarrow R\cdot i_h + L\cdot\frac{d}{dt} i_h = 0$

$i_h = -\frac{L}{R}\frac{d}{dt} i_h$

$\therefore \boxed{i_h(t) = K_1\cdot e^{-t/L/R} = K_1\cdot e^{-\frac{R}{L}t}}$

Check

$i_h' = K_1 \left(-\frac{R}{L}\right) e^{-R/L\, t}$

$i_h = -\frac{L}{R} i_h' = -\frac{L}{R}\cdot K_1\left(-\frac{R}{L}\right)e^{-\frac{R}{L}t} = K_1 e^{-R/L\, t}$

$i_p \rightarrow X$ (a constant)

$V_S = R\, i_p + L\cdot\frac{d}{dt} i_p$

$= R\cdot X + L\cdot\underbrace{\cancel{\frac{d}{dt}X}}_{0}$

$\therefore X = \frac{V_s}{R} = i_p$

$i_L(t) = i_h + i_p = K_1\cdot e^{-\frac{R}{L}t} + \frac{V_s}{R}$

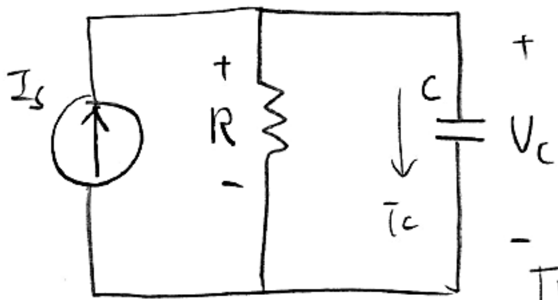
as $i_L(0) = 0$ (given ⊛)

$K_1\cdot \underbrace{\cancel{e^{-\frac{R}{L}\cdot 0}}}_{1} + \frac{V_s}{R} = 0$

$\therefore K_1 = -\frac{V_s}{R}$

$\therefore i_L(t) = \frac{V_s}{R}\left(1 - e^{-\frac{R}{L}t}\right)$ [A]

(3)



$\boxed{i_c = C \cdot \frac{d}{dt} V_c}$

$\boxed{V_c(0) = 0}$

$$I_s = \frac{V_c}{R} + i_c = \frac{V_c}{R} + C \cdot \frac{d}{dt} V_c$$

PDE

$$\Rightarrow \boxed{I_s = \frac{V_c}{R} + C \cdot \frac{d}{dt} V_c}$$

solve for $V_c(t)$ where

$$V_c(t) = V_p(t) + V_h(t) \,/\!/$$

$$V_h \rightarrow \frac{V_h}{R} + C \cdot \frac{d}{dt} V_h = 0$$

$$V_h = -RC \frac{d}{dt} V_h \quad \therefore V_h = K_1 e^{-\frac{t}{RC}}$$

(check)

$$V_h = -RC V_h' = -RC \left(\frac{-1}{RC}\right) K_1 \cdot e^{-t/RC} = K_1 e^{-t/RC} = V_h$$

$V_p = X$ (a constant)

$$\frac{V_p}{R} + C \cdot \frac{d}{dt} V_p = I_s$$

$$\rightarrow \frac{X}{R} + \cancel{C \cdot \frac{d}{dt} X}^{\,0} = I_s$$

$$\boxed{X = R \cdot I_s = V_p}$$

$$V_c(t) = V_p + V_h = R \cdot I_s + K_1 e^{-t/RC}$$

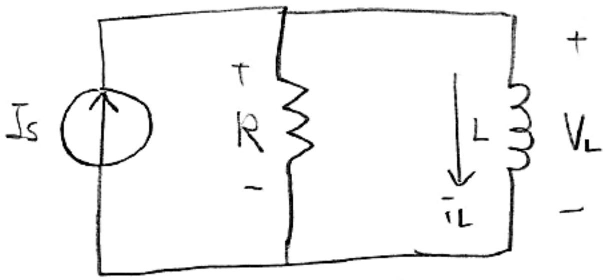
as $V_c(0) = 0$.

$$V_c(0) = R \cdot I_s + K_1 \cdot \underbrace{e^{-0/RC}}_{1} = 0$$

$$\therefore \boxed{K_1 = -R I_s}$$

$$\therefore V_c(t) = R I_s - R I_s e^{-t/RC}$$

$$= R I_s (1 - e^{-t/RC}) \, [V]$$



$\boxed{V_L = L \cdot \frac{d}{dt} i_L}$

$i_L(0) = 0$

$I_S = \frac{V_L}{R} + i_L$

PDE $\rightarrow$ $\boxed{I_S = \frac{L}{R}\frac{d}{dt} i_L + i_L}$

Solve for $i_L(t)$.

$i_L(t) = i_h(t) + i_p(t)$

$i_h \rightarrow \frac{L}{R}\frac{d}{dt} i_h + i_h = 0.$

$i_h = -\frac{L}{R}\frac{d}{dt} i_h.$ $\boxed{\therefore i_h(t) = K_1 \cdot e^{-\frac{R}{L}t}}$

check $\boxed{\begin{aligned} i_h = -\frac{L}{R}\frac{d}{dt} i_h &= -\frac{\cancel{L}}{\cancel{R}} \cdot K_1 \cdot \left(\frac{-\cancel{R}}{\cancel{L}}\right) \cdot e^{-R/L\,t} \\ &= K_1 \cdot e^{-R/L\,t} = i_h \end{aligned}}$

$i_p = X$ (a constant)

$I_S = \frac{L}{R}\frac{d}{dt} i_p + i_p = \underbrace{\cancel{\frac{L}{R}\frac{d}{dt} X}}_{0} + X$

$\boxed{X = \underline{\underline{I_S}} = i_p}$

$i_L(t) = i_h + i_p = K_1 \cdot e^{-\frac{R}{L}t} + I_S$

as $i_L(0) = 0.$

$i_L(0) = K_1 \cdot \underbrace{e^{-\frac{R}{L}\cdot 0}}_{1} + I_S = K_1 + I_S = 0$

$\therefore K_1 = -I_S$

$i_L(t) = I_S(1 - e^{-R/L \cdot t})$ [A]