CET 141: Day 6

Dr. Noori KIM

Objectives (learning points)

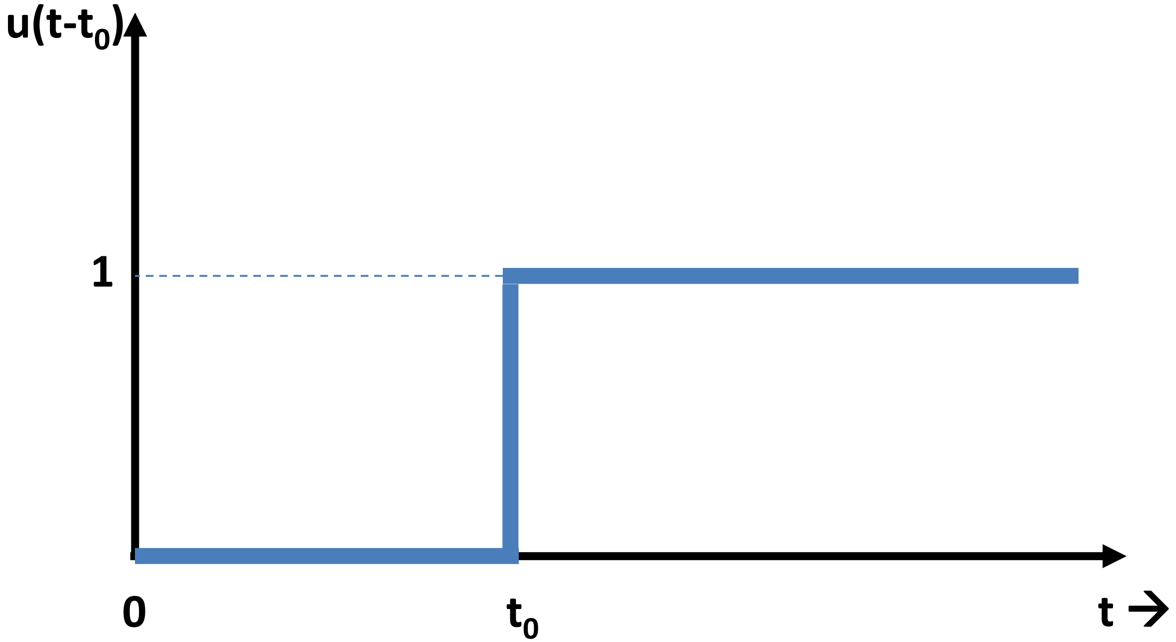
1. Characteristics of RL and RC circuits when there is an on-off switch
   - Analyze when the switch becomes on ←→ off
   - Define a step function to simulate the on-off switch behavior mathematically
2. Definition of "Response" with several concepts
   - i.e., complete, natural, forced….

Agenda

1. The unit step function
2. Initial conditions of switched circuits
3. First-order (1^{st} order) circuits
4. Stability of the 1^{st} order circuit
5. Time constant τ
6. transient R. and steady-state R. for 1^{st}ord. circuits
7. Simple RC, RL circuit analysis (1^{st} ord.)
8. Examples of RL, RC circuits

1. The unit step function

➔ To simulate changes of signal status easily (switching)



$$u(t-t_0)=\begin{cases}0 & t<t_0\\ 1 & t>t_0\end{cases}$$

Normally we define $t_0=0$

2. Initial conditions of switched circuits

- Initial condition was given at t_0
- A switch was on at t_0
- A status before switching ($t<t_0$): steady-state

	Cap response	Ind response
$i(t_0)$	Discontinuous $i_c(t_0^{+}) \neq i_c(t_0^{-})$	Continuous $i_L(t_0^{+}) = i_L(t_0^{-})$ $v_L(t) = L\frac{d}{dt}i_L(t)$
$v(t_0)$	Continuous $v_c(t_0^{+}) = v_c(t_0^{-})$ $i_c(t) = C\frac{d}{dt}v_c(t)$	Discontinuous $v_L(t_0^{+}) \neq v_L(t_0^{-})$

Must be differentiable

- At DC (f=0, t=∞): also steady-state
  - An inductor acts as like a short-circuit (connected wire)
  - A cap acts as like a open-circuit (disconnected wire)

$$\boxed{v_L(t) = L\frac{d}{dt}i_L(t)}$$

If $i_L(t)$ = a Constant (DC)
➔ $V_L(t)=0$ ➔ S.C.

$$\boxed{i_c(t) = C\frac{d}{dt}v_c(t)}$$

If $v_c(t)$ = a Constant (DC)
➔ $i_C(t)=0$ ➔ O.C.

3. 1^{st} order circuits

- Def: Only one capacitor or one inductor circuits
  - Not an absolute number of C or L
  - Simplification by Thevenin or Norton
  - Therefore, kinds of components in the circuit defines RC or RL circuits
- Responses: Circuit outputs

Complete response = Natural R. + Forced R.

Homogeneous sol. (when f=0)

Particular sol.

Recap: 1st order diff eq. solution w/ a constant f(t)

$$x' + \frac{x}{\tau} = K$$

$x' = \frac{d}{dt}x$ x is a function of time → x(t)

τ is a time constant

$$\frac{dx}{dt} = \frac{K\tau - x}{\tau} \Leftrightarrow \frac{dx}{x - K\tau} = -\frac{dt}{\tau}$$

$$\int \frac{dx}{x - K\tau} = \int -\frac{dt}{\tau} \Rightarrow \ln(x - K\tau) = -\frac{t}{\tau} + Const.$$

$$e^{-t/\tau + Const.} = x - K\tau$$

x is a function of time → x(t)

$$\boldsymbol{x(t) = K\tau + A \cdot e^{-t/\tau}}$$

A=$e^{Const.}$ → a constant

Forced R.
Particular sol

Natural R.
Homogeneous sol

4. Stability of the 1st order circuit

(by investigating the natural response)

Homogeneous portion of the answer

$$\boxed{A \cdot e^{-t/\tau}}$$

As t → ∞, if τ>0, this term converges to 0

→ before t< ∞ , this term makes circuits unstable

$A \cdot e^{-t/\tau}$	When τ > 0 , R ≠ 0	When τ < 0
R_{Th} >0 is required to make 1st order circuit stable	t → 0 Stable (the natural R disappears → only forced R remains)	Unstable (without t → 0) (the natural R grows →relatively small forced R.)

| 100%

<Summary of stability>

		t=0	t=τ	t=∞
x(t) 1st PDE	F.R.: Kτ	0%	63.2%	100%
	N.R.: $Ae^{-t/\tau}$	100%	36.7%	0%

Unstable $\longrightarrow$ Stable

Then what is the τ in the circuit responses?

5. The time constant τ

- Def: a special time which represents the speed of a particular system
- Properties:
  - Can respond to **CHANGE**: typically a factor of e^{-1} (i.e., $1-\frac{1}{e}=0.6321\ldots$)
  - 5τ is regarded as its maximum **CHANGE**

- Kinds of τ
  - RC time constant: for RC circuit
    - $\tau_{RC}=RC$ (where $R=R_{th}$)
    - The RC unit: $\Omega\mathrm{F} = \frac{V}{A}\frac{Col}{V} = \frac{sec}{Col}col = sec$
    - Time required to charge a cap (through R) by 63.2% or discharge the cap by 36.8%

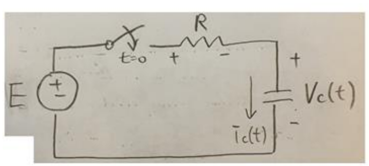
When t=τ=RC

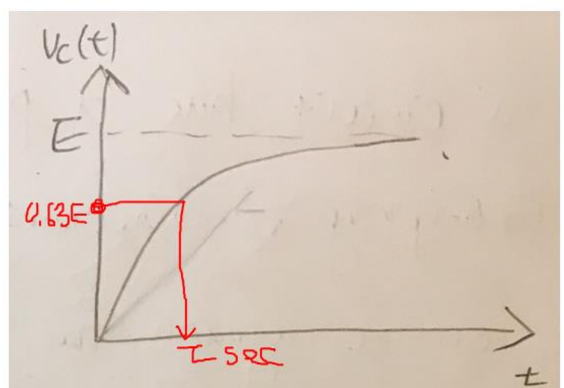
t=τ a special case to define the time constant

$$v_c(t) = E(1\text{-}e^{-t/RC})$$

0.3678....

$$v_c(RC) = E(1\text{-}e^{-RC/RC}) = E(1-e^{-1}) = 0.63E$$

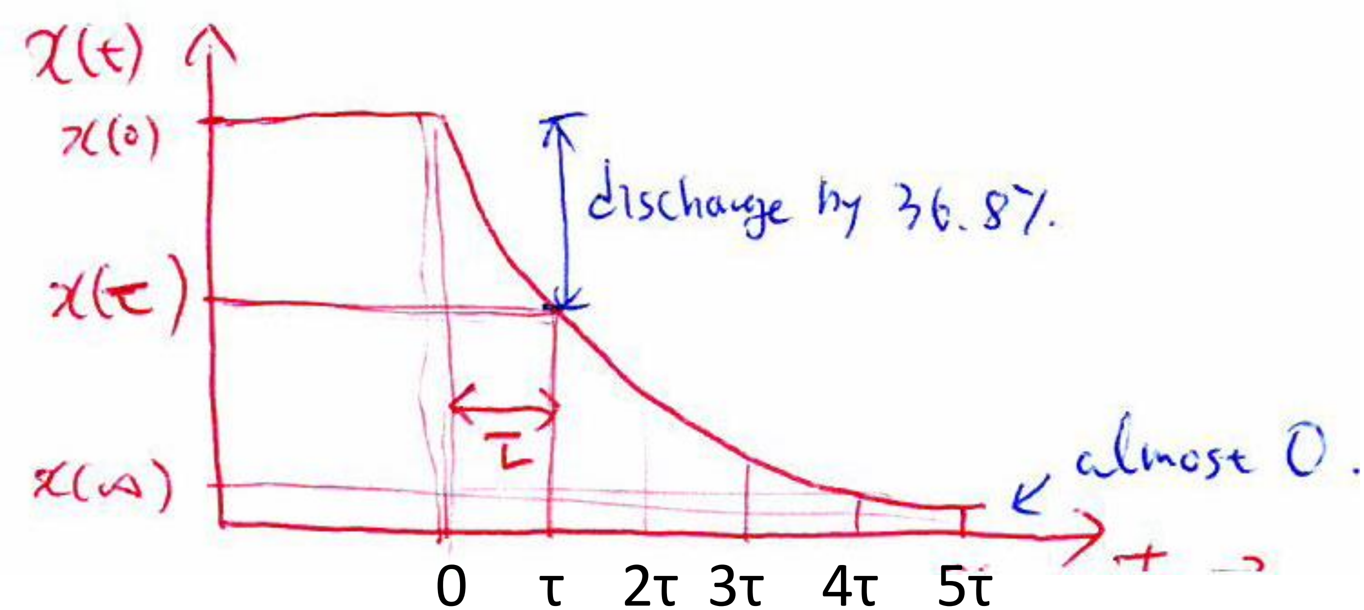




– RL time constant: for RL circuit

- τ_{RL}=L/R (where R=R_{th})
- The L/R unit: $\frac{H}{\Omega} = \frac{V \cdot sec}{A} * \frac{A}{V} = sec$
- Time required to energize an Ind (through R) by 63.2% or de-energize the Ind by 36.8%

- After 5τ ➔considered as 0%



So far, we've talked about..

1. The unit step function
2. Initial conditions of switched circuits
3. First-order circuits
4. Stability of the 1st order circuit
5. Time constant τ

Going back to our original business,
let's talk about the circuit again.

6. 1st ord. circuit: transient R. and steady-state R.
7. Simple RC, RL circuit analysis
8. Examples of RL, RC circuits

In terms of PDE solutions

Complete response = Natural R. + Forced R.

= Homog. S. + Partic. S.

$f(t)=0$ $\quad$ $f(t)=\text{something}$

= <u>**Transient R.**</u> + <u>**Steady-state R.**</u>

Dies out eventually when $t \to \infty$

Dominant response when $t \to \infty$

When we input signals for example

– A sinusoidal function or
– A constant (DC) signal

To circuits, we have

Complete response = Transient R. + Steady-state R

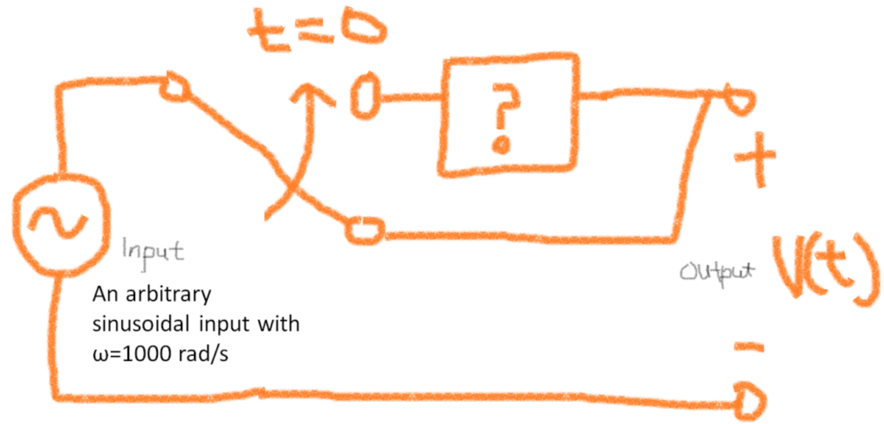
- Existing momentarily due to changing in the circuit (i.e., 0→1 or 1→0)
- Becomes 0 when t→∞

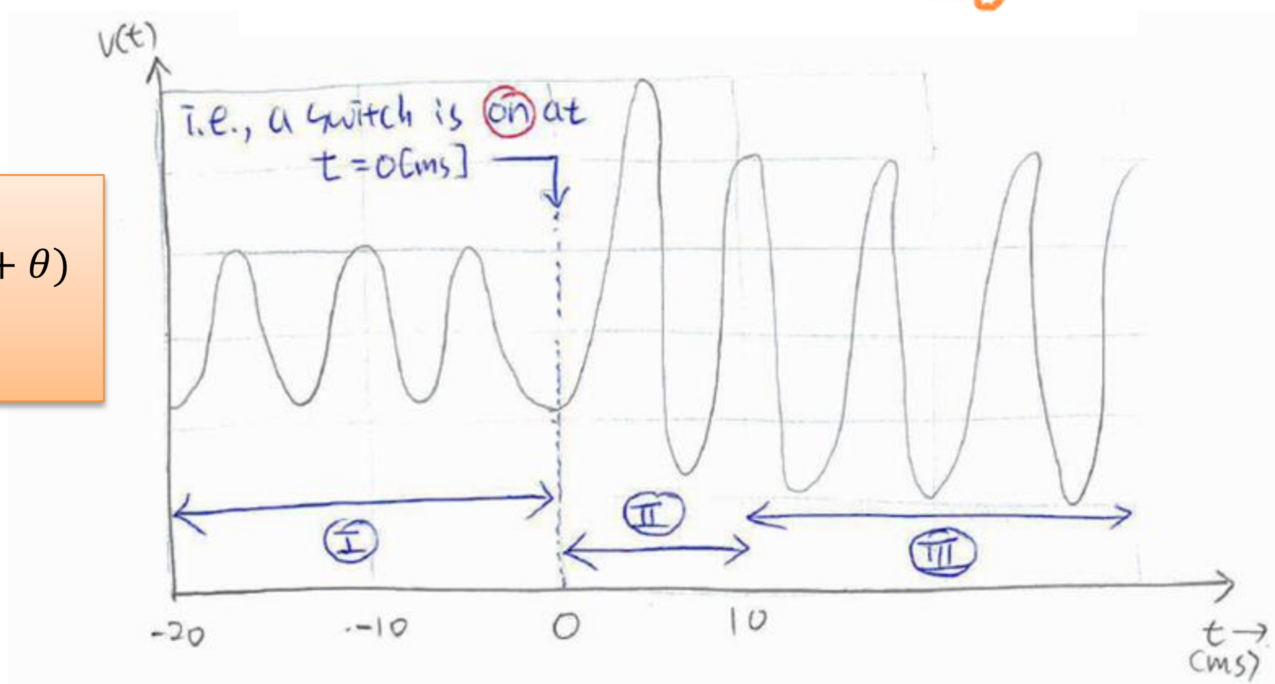
- Stable response
- i.e., before switching /after switching

Note that when the circuit is 1st order, usually,
Transient.R= Natural.R, and Steady-state.R=Forced. R, but not always….

6. The 1st order circuit: T. R. and SS R.

Complete R. = **Transient R.** + **Steady-state R**



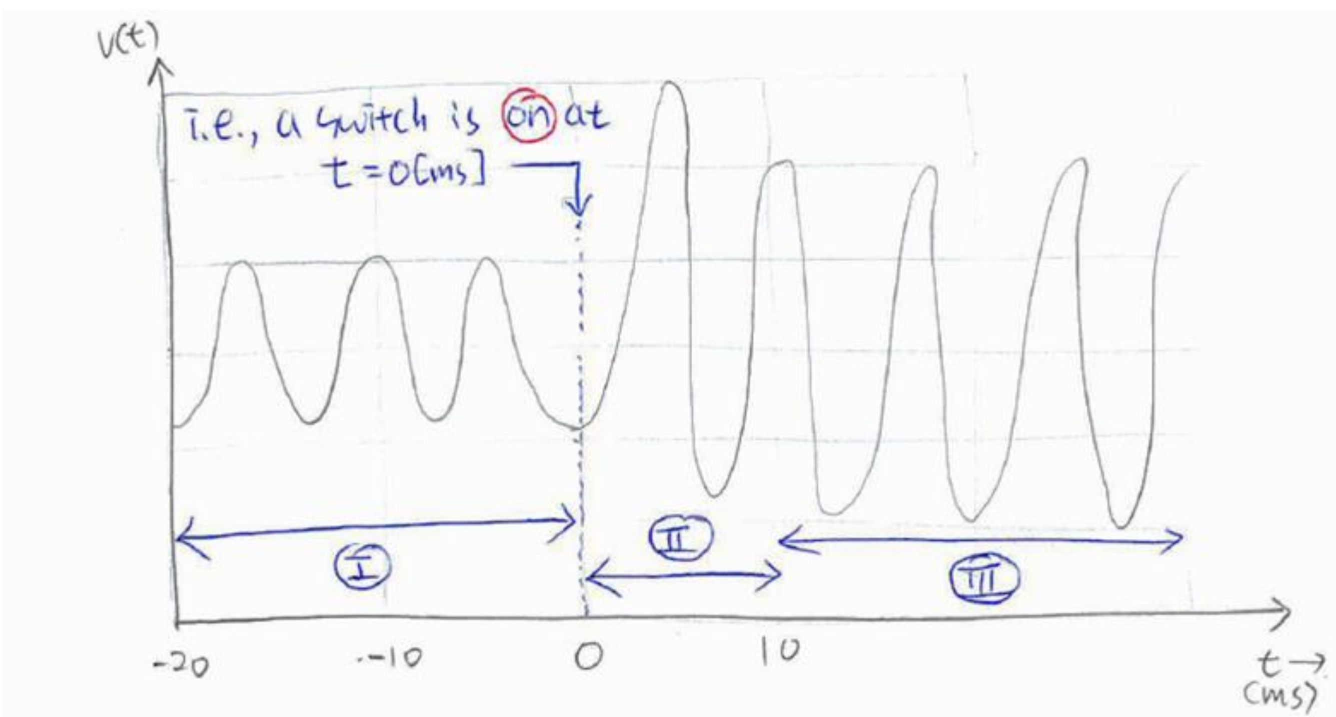


Before switching
$v(t) = A\cos(1000t + \theta)$
at switchin $(\mathrm{t} = 0)$
$\mathrm{v}(0) = \mathrm{A}\cos\theta$

I. A steady state when the switch is opened (not yet closed)

II. A transient state right after the switch is closed
   - A momentarily status due to changing its status

III. A steady state when the switch is closed



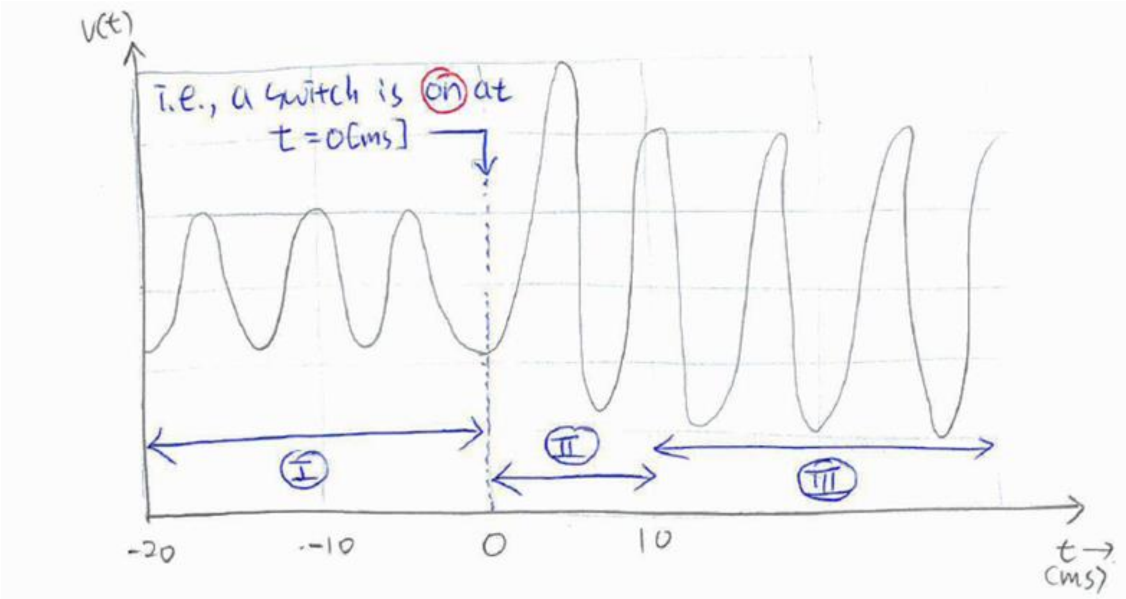
A sidebar

Partic. Sol. forms for the 1st order PDE

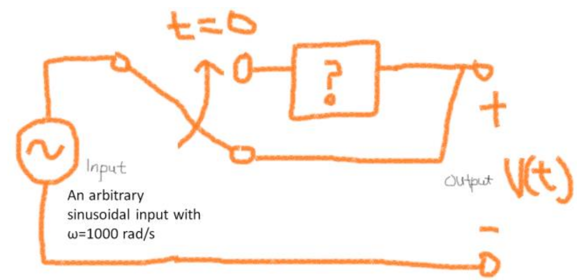
$$X' + aX = f(t)$$ →Where X=X(t) a function of time

f(t)	$X_p(t)$ guess
A constant	A constant
$A \cdot t$	B$\cdot\, t + C$
$A \cdot e^{\boldsymbol{\omega} t}$	B$\cdot\, e^{\boldsymbol{\omega} t}$
Sinusoidal with ω	Sinusoidal with ω diff amplitude and phase (φ)

<u>Where A, B, and C are constants</u>



Before switching
$v(t) = A\cos(1000t + \theta)$
at switchin $(\mathrm{t} = 0)$
$\mathrm{v}(0) = \mathrm{A}\cos\theta$



- 1st order PDE i.e: $\mathrm{Z} * \cos(1000t + \vartheta) = X * v(t) + Y * \frac{d}{dt} v(t)$
- The complete sol for v(t)

$$v(t) = \underbrace{K_1 \cos(1000t + \varphi)}_{} + \underbrace{K_2 e^{-t/\tau}}_{} [\mathrm{V}]$$

- Particul. sol
- Steady-state R.
- Dominant R. (stable) when t→∞

- Homogen. sol
- Transient R
- Becomes 0 when t→∞

7. Simple RC, RL circuits' analysis

- Two properties to be used
  1. When in steady state, t<0 ($t<t_0$) or t→∞
     - A Capacitor
       - Open circuit (i=0)
       - Use $v_c(0^-) = v_c(0^+)$ to get $v_c(0)$

       or $v_c({t_0}^-) = v_c({t_0}^+)$ to get $v_c(t_0)$
     - An inductor
       - Short circuit (v=0)
       - Use $i_L(0^-) = i_L(0^+)$ to get $i_L(0)$

       or $i_L({t_0}^-) = i_L({t_0}^+)$ to get $i_L(t_0)$

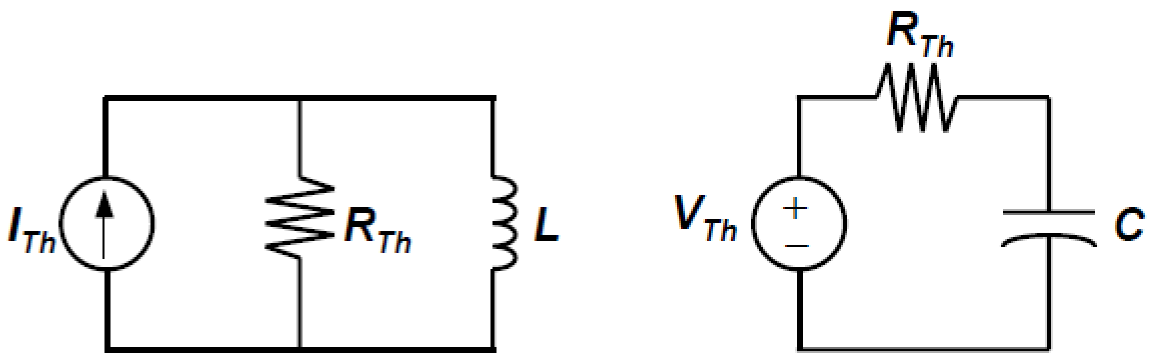
To define $i_c(0)$, it has to be differentiable

$$\boldsymbol{i_c(t) = C\frac{d}{dt}v_c(t)}$$

– In steady state, an inductor behaves like a short circuit

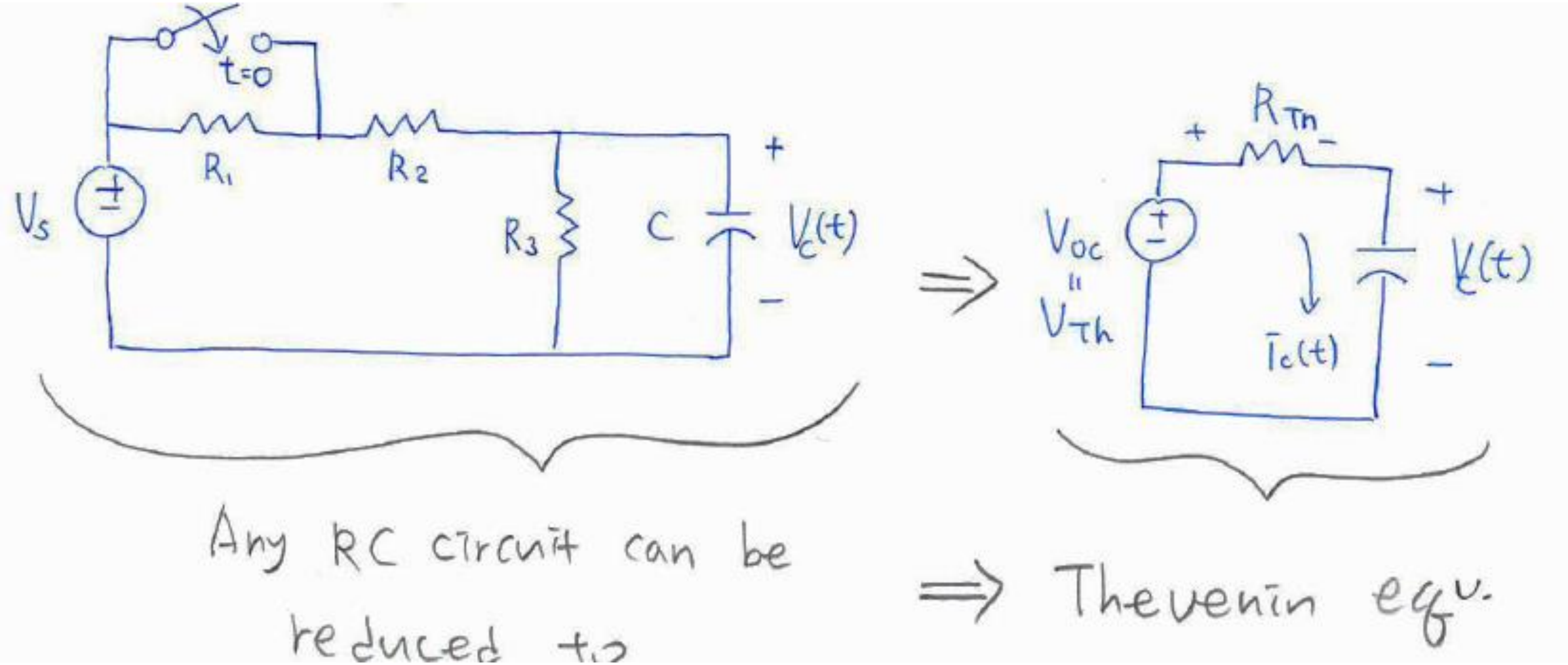
– In steady state, a capacitor behaves like an open circuit

2. When $t>0$ or $t>t_0$
    - RC circuits: Thevenin equivalent → KVL or node methods
    - RL circuits: Norton equivalent → KCL or node methods



- Then, solve for 1st order PDE
  - Usually 2 unknowns (i.e., K1 and K2)
  - Need 2 boundary conditions including an initial condition ($t=0$)
    - Usually conditions at $t=0$ and $t \rightarrow \infty$

- RC analysis

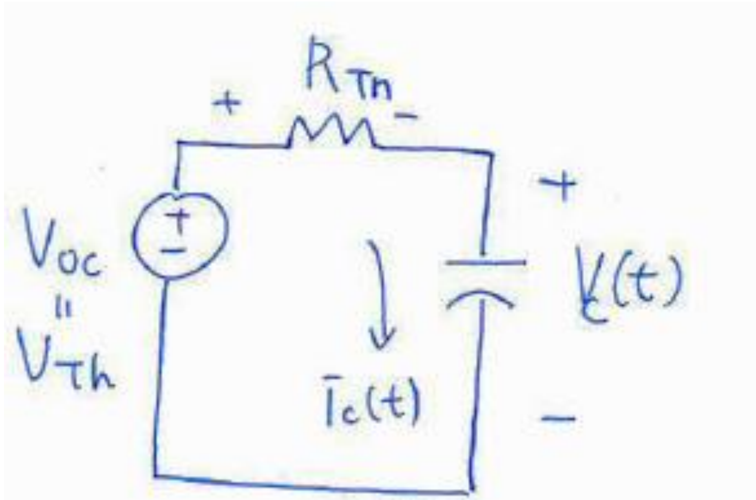


When $t = 0^+$ the switch is closed

$$V_{Th} = V_{oc} = \frac{R_3}{R_2 + R_3} V_s$$

$$R_{Th} = R_2 || R_3 = \frac{R_2 \cdot R_3}{R_2 + R_3}$$

- RC analysis (cont..)
  - KVL on Thevenin



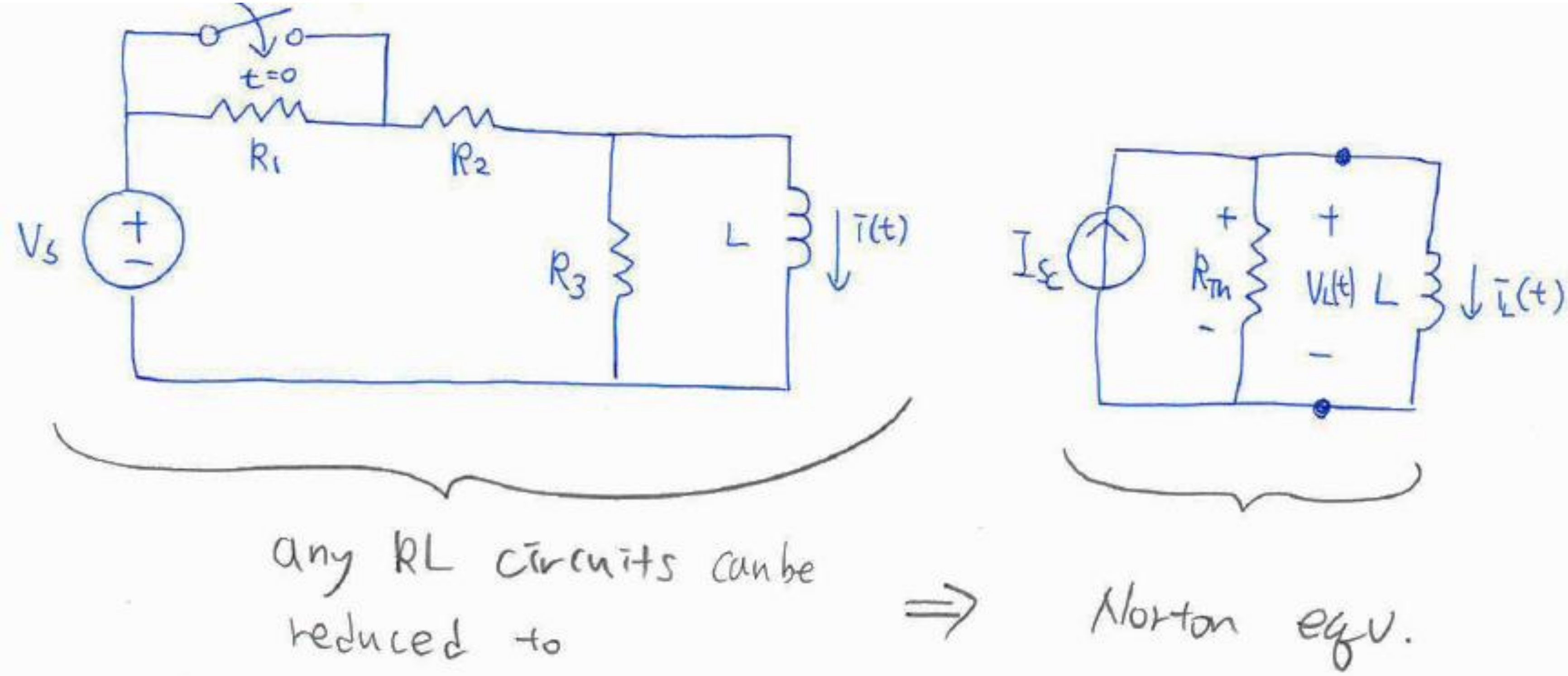
$$(\text{Gain}) = (\text{Drop})$$

$$V_{oc} = R_{Th}\, I_c(t) + V_c(t)$$

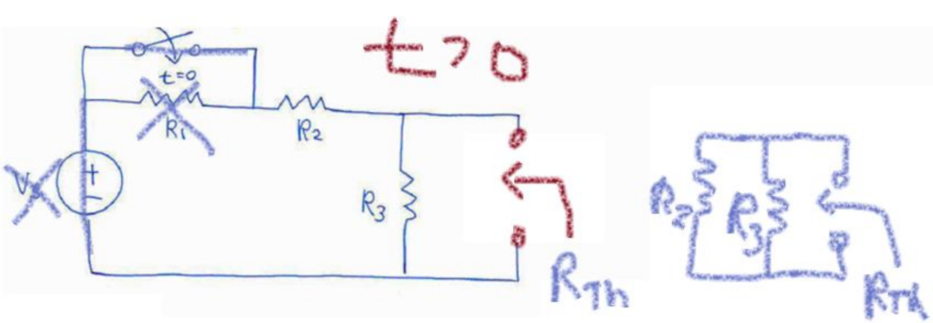
$$= R_{Th}\left(C \cdot \frac{d}{dt} V_c(t)\right) + V_c(t)$$

$$= \underbrace{R_{Th}\,(C}_{\tau} \cdot \frac{d}{dt} V_c(t)) + 1\, V_c(t)$$

- RL analysis



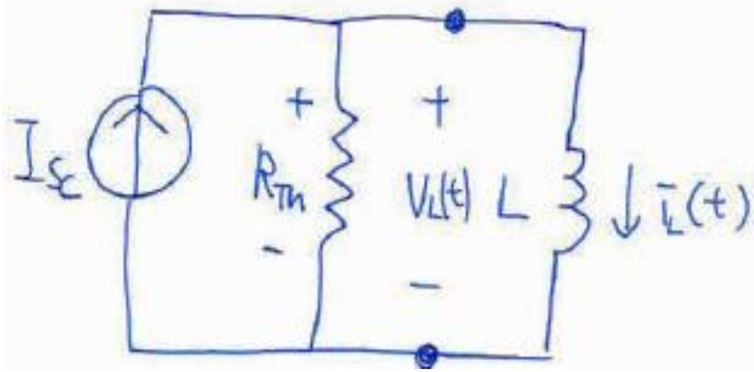
When $t = 0^+$ the switch is closed



$$I_{SC} = \frac{V_{OC}}{R_{Th}}$$

$$R_{Th} = R_2||R_3 = \frac{R_2 \cdot R_3}{R_2 + R_3}$$

- RL analysis (cont..)
  - KCL on Norton



In = Out.

$$I_{sc} = \frac{V_L(t)}{R_{Th}} + i_L(t) \Rightarrow \frac{1}{R_{Th}}\left(L \cdot \frac{d}{dt} i_L(t)\right) + i_L(t)$$

$$\Rightarrow \frac{1}{R_{Th}}\left(L \frac{d}{dt} i_L(t)\right) + 1\, i_L(t)$$

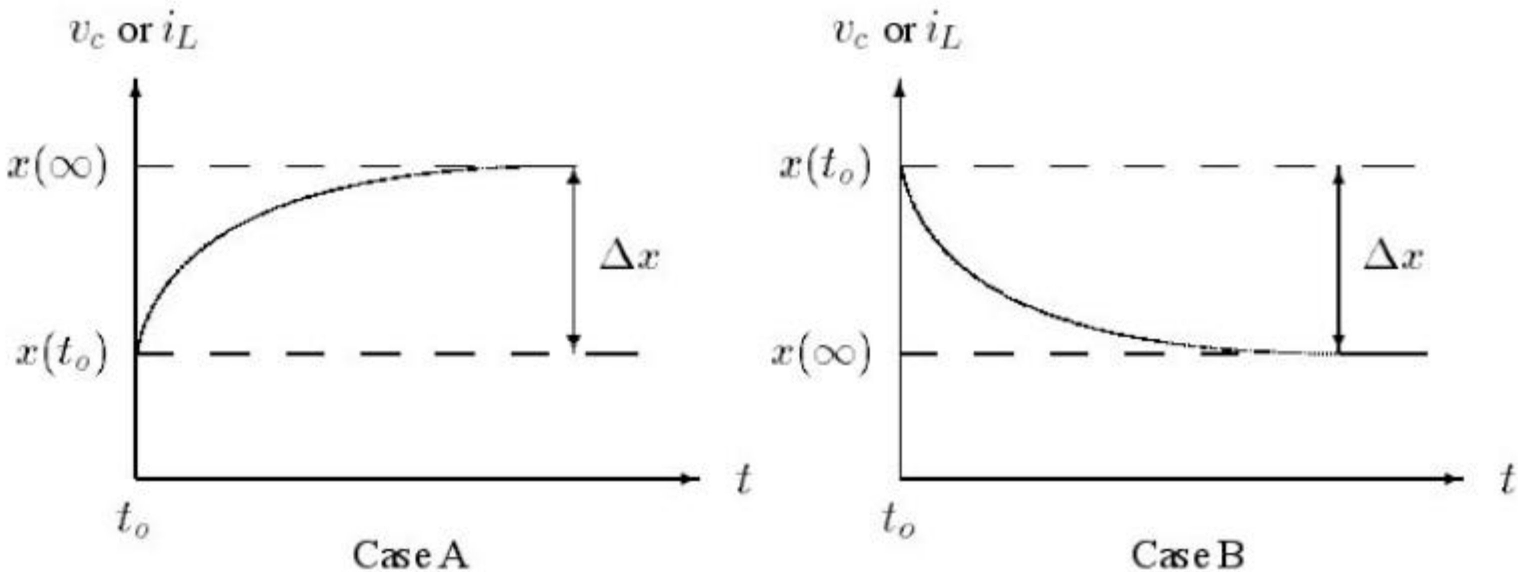
τ

- Based on previous RC and RL analysis, we can generalize the 1st order PDE form when each circuit is reduced to Thevenin or Norton equivalent

$$V_{oc} = \underbrace{R_{Th} (C}_{\tau} \frac{d}{dt} v_c(t)) + 1 v_c(t) \qquad I_{sc} = \underbrace{\frac{1}{R_{Th}} (L}_{\tau} \frac{d}{dt} i_L(t)) + 1 i_L(t)$$

$$\tau \frac{d}{dt} v_{or} i(t) + v_{or} i(t) = V_{oc} \; or \; I_{sc}$$

More in general …



Case A: $$x(t) = x(t_o) + \Delta x \left(1 - e^{-(t-t_o)/\tau}\right)$$

Case B: $$x(t) = x(\infty) + \Delta x \, e^{-(t-t_o)/\tau}$$

- Previous examples: ?