Dynamic Trading with Market Impact Costs

A Report prepared under partial fulfillment of the course

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by

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Certificate

This is to certify that the following project work on **Dynamic Trading with Market Impact Costs** by Tapan Samangadkar and Mehul Garg was completed successfully under the guidance of Dr. Mayank Goel in the partial fulfillment of the course SPECIAL PROJECT (MATH F266) during the 4^{th} Year, 2^{nd} Semester 2017-2018 at Birla Institute of Technology and Science, K.K. Birla Goa Campus.

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Contents

Abstract

The impact of market costs in a portfolio optimization scenario affords a realistic model and a much closer estimate of actual profits when the trading is frequent and orders are of substantial size. We consider the mean-variance portfolio optimization problem in continuous time, accounting for the user's risk preference as a constraint and the aim of maximizing wealth at the end of trading horizon. We aim to find out the optimal intervals at which the portfolio should be rebalanced, while modelling the transaction costs of the trade and its impact on market prices. We also aim to find out the optimal rate of trading, by penalizing the investor for trading faster, and thus leading to more distortion in prices. We aim to employ the concept of trading towards the optimal portfolio in the future in order to increase our desirable holdings over the long term to generate better performance.

Keywords

Continuous time, mean-variance portfolio optimization, transaction costs, market impact costs, dynamic trading, Simulated Annealing.

1 INTRODUCTION

Maintaining a portfolio of investments entails significant amount of analysis on behalf of investors and asset managers to predict security returns, correctly at that, in order to profit from the market. Such active management of portfolio entails rebalancing of positions at regular intervals of time during the investment horizon, which incurs a lot of turnover costs, i.e. proportional transaction costs for the trade. Apart from this, it is important to recognize that the scale of the trades affect the market prices, making it difficult to execute it at the desired price, and thus giving the investor a rather inaccurate estimate of future returns. Also, in order to reduce the turnover of the changing positions in various securities it is necessary to account for the composition of the optimal portfolio at a future time, or in the long term, to effectively trade over a given time horizon. We try to incorporate all these factors in a continuous time mean-variance portfolio selection framework with specified investor risk preference, accounting for transaction and market impact costs, in order to derive an optimal trading strategy with respect to trading rate and rebalancing intervals.

1.1 Dynamic Portfolio Selection Problem

The Markowitz (1952, 1959) paper on mean variance portfolio selection is a cornerstone of modern quantitative finance and portfolio selection theory. Markowitz aims to minimize the variance in expected portfolio returns subject to maximization of the wealth of the investor. Besides the concept of risk being represented as the variance of security returns, Konno and Yamazaki (1991) use semi-variance and partial moment and downside risk are used by Ogryczak and Ruszcynski (1999), to construct the optimal portfolio. Mossin (1968), Samuelson (1969) and Merton (1969, 1971) investigated the optimal portfolio choice in a dynamic environment without transaction costs, extending Markowitz model to multi-period case.

However, implementing a dynamic portfolio policy entails rebalancing of positions at frequent intervals of time, resulting in high transaction costs. This results in a continuous-time mean-variance portfolio selection problem with constraints for a given expected terminal target, usually wealth maximization, subject to investor's risk appetite. Chen and Huang (2010) study the incorporation of transaction costs in the Markowitz framework. Cadenillas (2000) surveys the impact on transaction costs on dynamic rebalancing of portfolios. Dybvig (2005) considers proportional i.e. linear transactional costs with mean-variance utility.

1.2 Solutions to Dynamic Portfolio Selection

The dynamic mean-variance problem can be treated in three different ways. One of the most researched and developed strategy is what is called 'pre-commitment strategy' which optimizes the objective function at the initial time. The optimality of the function in future time is disregarded. Richardson (1989) and Bajeux-Besnainou and Portait (1998) developed a continuous-time version of the mean-variance model in this pre-committed framework. An auxiliary stochastic linear quadratic approach to the same problem is seen in Li and Ng (2000).

Zhou and Li (2000), Lim and Zhou (2002), Lim (2004), Bielecki et al. (2005), Xia (2005) provide a solution to the continuous-time mean-variance portfolio selection problem, without transaction costs. Zhou and Yin (2003), Chen et al. (2008) and Chen and Yang (2011). Dai et al. (2010) further provide a pre-committed strategy when the transaction cost is also taken into account.

The second approach aims at maximizing the current wealth of the investor at all times, dynamically evaluating the portfolio positions. The third approach recognises that the dynamic programming approach may not be applicable to since the expectations iterated in time contain an non-linear function due to the variance term. Basak and Chabakauri (2010) derive a recursive formulation for the mean-variance criteria and obtain a closed-form solution for their strategy, which can be applied to all formulations of the portfolio selection problem with stochastic control and mean-variance objective function. The latest work in this area includes Bjork and Murgoci (2010, 2014), who establish the strategy using Hamilton-Jacobi-Bellman equation applied to both discrete and continuous time models within a game-theoretic framework.

The most tractable framework, in this regard is provided by Garleanu and Pedersen (2012) who model price changes to accommodate transaction costs and market impact costs, and give closed form expressions for the optimal dynamic portfolio choices. They suggest the idea of trading partially towards the optimal portfolio in the future, which is somewhat between the first and third approaches to dynamic portfolio selection, discussed above. Their strategy mimics a trader constantly placing orders, as per available market liquidity, thus reducing market impact costs of the trades, to follow the optimal portfolio at the current time, as well as the optimal portfolio in the future of the trading horizon.

1.3 Proposed model

We suggest some changes to this mean-variance framework proposed above. We evaluate the strategy for a fixed trading horizon rather than infinite time used by Garleanu. We suggest the inclusion of a risk-free asset in the investor's portfolio which is used to compensate for the transaction costs incurred during the rebalancing of portfolio at regular intervals. We liquidate the risk-free assets to increase our positions in the risky assets which promise better returns (derived from return predictors). The price impact costs are modelled into the objective function to penalize fast trading and not implicitly as a part of price dynamics. Also the previous model assumes a constant risk-averseness factor of the investor. We suggest variable risk-averseness proportional to the investor wealth at the time, so that increases in investment up to time t may benefit the investor in the future by selecting high return and more risky investment options. This factor incorporates a factor of behavioral finance related to investor risk preferences.

2 Mathematical Framework

2.1 General Framework

- We consider a market with multiple securities being traded and our portfolio consisting of risky assets, being traded over a finite time horizon [0,T].
- We consider a continuous time mean-variance portfolio optimization/selection problem in which transaction costs are incurred to rebalance a portfolio (say x') into a more efficient portfolio in the long term (say x).
- The portfolio is updated periodically based on the information from a set of return predictors, which are allotted weights as per their signal decay time, a major feature in Garleanu (2013).
- In addition to the transaction costs, the continuous trading of securities in bulk amounts causes distortion in market prices, hereby termed as market impact costs, which will be modelled as a part of the objective function in order to penalize fast trading.
- The risk aversion parameter of the investor, in the utility function, is modelled as a function of current investor wealth, thereby implying more the accumulation of wealth in previous rebalancings, the greater proportion of investment in the risky assets.
- Our optimal strategy focusses on determining the optimal trading rate, keeping in mind not only to trade towards the current optimal portfolio (aim portfolio) but also towards the optimal portfolio in the future (target portfolio).

Our framework is based on that proposed by Garleanu and Pedersen (2012), herein GP.

2.2 Mathematical Model

We consider a universe of N risky assets with a $N \times 1$ vector x_t denoting the positions in the assets at any time t. We consider the investment in the multi-period setting, where the investor is maximizing the expected terminal wealth at the end of the trading horizon [0,T], by choosing the positions in each of the risky assets of the universe, rebalanced at regular intervals as per an objective function. We consider three types of trading costs: proportional transaction costs, market impact costs and quadratic transaction costs. We assume price changes in excess of the return rate are independently distributed with mean μ and covariance matrix Σ. We consider finite time horizon T.
The investor decision, in our framework can be written as:

$$\max_{\{x_t\}_{t=1}^T} \sum_{t=1}^T \left[(1-\rho)^t(x_t^\top \mu - \tfrac{\gamma}{2} x_t^\top \Sigma x_t) \text{ - } (1-\rho)^{t-1}\kappa \|\Lambda^{\frac{1}{p}}(x_t - x_{t-1})\|_p^p\right]$$

where ρ is discount factor, γ is the risk-aversion parameter of the investor, and κ is the transaction cost parameter. $\Lambda \in R^{N\times N}$ is the transaction cost matrix and $\|s\|_p$ is the p-norm of the vector s; $\|s\|_p^p = \sum_{i=1}^N |s_i|^p$. The transaction cost matrix Λ captures the distortion of asset prices caused due to interaction between the securities of the universe and is a multi-dimensional representation of Kyle's Lambda.

In the consequent section we will explain the derivation of the objective function accounting for 3 costs: **proportional**, **market impact** and **quadratic** transaction costs.

2.3 Transaction Costs

2.3.1 Proportional costs

Here we consider the proportional transaction costs that arise when investor makes small trades that hardly affect the price of the underlying asset. Essentially for this case, we have $p = 1$, in our general equation given above. These proportional costs effectively model the costs arising from bid-ask spread and the fixed brokerage charges associated with these trades.

The investor decision, only considering proprtional transaction costs is given by:

$$\max_{\{x_t\}_{t=1}^T} \sum_{t=1}^T \left[(1-\rho)^t(x_t^\top \mu - \tfrac{\gamma}{2} x_t^\top \Sigma x_t) \text{ - } (1-\rho)^{t-1}\kappa\|(x_t - x_{t-1})\|_1\right]$$

where $\kappa\|(x_t - x_{t-1})\|_1$ is the transaction cost associated with trading taking place in time interval [t-1,t] and the change in position from x_{t-1} to x_t.
The rebalancing region is given by:

$$\frac{\|\sum_{s=t}^T (1-\rho)^{s-t}\Lambda^{-\frac{1}{p}}\Sigma(x_s - x^*)\|_q}{p\|\Lambda^{\frac{1}{p}}(x_t - x_{t-1})\|_p^{p-1}} \leq \frac{\kappa}{(1-\rho)\gamma}$$

where q is such that $\frac{1}{p} + \frac{1}{q} = 1$.

We can make the following comments about the behaviour of solution space with respect to the control parameters seen in the above equation.

- The no-trade region expands as the proportional transaction parameter κ increases.
- The no-trade region expands as the discount factor parameter ρ increases.
- The no-trade region shrinks as the investment horizon T increases.
- The no-trade region shrinks as the risk-aversion parameter γ increases.

These observations will be useful while selecting the feasible solution during each iteration of our optimization process.

2.3.2 Market Impact Costs

Market Impact costs arise when the investor makes large trades, distorting market prices. Prevalent theory on market impact costs suggest they are linear to the amount traded thus giving rise to quadratic transaction costs, however recent estimates of market impact costs, taking limit order book-size and market volume into consideration, suggests a square-root function better approximates the market impact. We assume a general case, where $p \in (1,2)$. The distortion in prices is captured by our multi-dimensional Kyle's lambda matrix Λ.

The investor decision, now updated to include market impact costs, is given by:

$$\max_{\{x_t\}_{t=1}^T} \sum_{t=1}^T \left[(1-\rho)^t(x_t^\top \mu - \frac{\gamma}{2} x_t^\top \Sigma x_t) \text{ - } (1-\rho)^{t-1}\kappa \|\Lambda^{\frac{1}{p}}(x_t - x_{t-1})\|_p^p\right]$$

And the rebalancing region is: $\frac{\| \sum_{s=t}^T (1-\rho)^{s-t} \Lambda^{-\frac{1}{p}} \Sigma^{\frac{1}{q}} (x_s - x^*)\|_q}{p\|\Lambda^{\frac{1}{p}}(x_t - x_{t-1})\|_p^{p-1}} \leq \frac{\kappa}{(1-\rho)\gamma}$

The behaviour of the solution space, essentially remains the same with respect to the control parameters, even after inclusion of market impact costs, but we gain the following insight into the future optimal positions:

- The rebalancing region for the t-th period contains the rebalancing region for every subsequent period.
- Every rebalancing region contains the Markowitz portfolio,
- The rebalancing region converges to the Markowitz portfolio in the limit when the investment horizon goes to infinity.

Clearly, the behaviour of solution space is perfectly fitting with our proposed optimal strategy of trading only partially towards the target portfolio, so as to minimize turnover over the trading horizon. The above conclusions are consistent with our mathematical framework and reconfirm the viability of our strategy.

2.3.3 Quadratic Transaction Costs

For the particular case of $p = 2$, i.e., in presence of quadratic transaction costs, the investor decision is given by:

$$\max_{\{x_t\}_{t=1}^T} \sum_{t=1}^T \left[(1-\rho)^t(x_t^\top \mu - \frac{\gamma}{2} x_t^\top \Sigma x_t) \text{ - } (1-\rho)^{t-1}\kappa \|\Lambda^{\frac{1}{2}}(x_t - x_{t-1})\|_2^2\right]$$

We can conclude the following rules for establishing an optimal rebalancing strategy:

- The optimal portfolio for each stage is a combination of the Markowitz strategy (the target portfolio), the previous period portfolio, and the next period portfolio.

- For the case with quadratic transaction costs, the optimal portfolio policy is to trade at every period along a straight line that converges to the Markowitz portfolio.

- For the case with market impact costs, the investor trades, at every period, towards the boundary of the corresponding rebalancing region. The resulting path of the portfolio is not a straight line, reconfirming our strategy of trading only partially towards the optimal portfolio.

- It can also be appreciated that the investor trades more aggressively at the first periods compared to the final periods, the strategy ensuring trade-off between market impact costs and discounted utility.

3 Simulated Annealing

3.1 Introduction

Simulated annealing (SA) is a probabilistic technique for approximating the global optimum of a given function. Specifically, it is a metaheuristic to approximate global optimization in a large search space. It is often used when the search space is discrete. For problems where finding an approximate global optimum is more important than finding a precise local optimum in a fixed amount of time, simulated annealing may be preferable to alternatives such as gradient descent.Detailed discussions of simulated annealing can be found in van Laarhoven and Aarts (1988), or in the survey by Pirlot (1992). This approximation of the long term rather than optimization of discrete times, is crucial to our model objective of following the target. This basic concept of this algorithm makes it ideal for implementing our proposed model. Below, we give a general overview of the technique.

3.2 The Algorithm

Simulated annealing is a generic name for a class of optimization heuristics that perform a stochastic neighborhood search of the solution space. The major advantage of SA over classical local search methods is its ability to avoid getting trapped in local minima while searching for a global minimum. The underlying idea of the heuristic arises from an analogy with certain thermodynamical processes viz. cooling of a melted solid.

For a generic problem of the form: min $F(x)$ s.t. x $\in X$,
the basic principle of the SA heuristic can be described as follows. Starting from a current solution x, another solution y is generated by taking a stochastic step in some neighborhood of x. If this new proposal improves the value of the objective function, then y replaces x as the new current solution. Otherwise, the new solution y is accepted with a probability that decreases with the magnitude of the deterioration and in the course of iterations. (Notice the difference with classical descent approaches, where only improving moves are allowed and the algorithm may end up quickly in a local optimum).

The precise steps of the Simulated Annealing Algorithm are:

- Choose an initial solution $x^{(0)}$ and compute the value of the objective function $F(x^{(0)}$. Initialize the best available solution, denoted by $(x^*$,$F^*)$ as : $(x^*$,$F^*) \leftarrow (x^{(0)}, F(x^{(0)})$.
- Until a stopping criterion is fulfilled and for n starting from 0, do:
  - Draw a solution x at random in the neighborhood $V(x^{(n)}$ of $x^{(n)}$.
  - If $F(x) \leq F(x^{(n)})$ then $x^(n+1) \leftarrow x$, and if $F(x) \leq F^*$, then $(x^*$,$F^*) \leftarrow (x, F(x))$.
  - If $F(x) > F(x^{(n)})$ then draw a number p at random from [0,1] and if $p \leq p(n, x, x^{(n)})$ then $x^{(n+1)} \leftarrow x$, else $x^{(n+1)} \leftarrow x^{(n)}$.

The function $p(n, x, x^{(n)})$ is often taken to be a Boltzmann function inspired from thermodynamics models:

$p(n, x, x^{(n)}) = \exp\big(- \frac{1}{T_n} \Delta F_n \big)$

where $\Delta F_n = F(x) - F(x^{(n)})$ and T_n is the temperature at step n, that is a nonincreasing function of the iteration counter n. In so-called geometric cooling schedules, the temperature is kept unchanged during each successive stage, where a stage consists of a constant number L of consecutive iterations. After each stage, the temperature is multiplied by a constant factor $\alpha \in (0,1)$.

3.3 SA for Portfolio Selection

The concepts involved in SA technique offers wide flexibility in the requirements of objective function like convexity, derivability, not to mention the unrestricted solution space. This helps to apply SA to a variety of optimization problems, yielding excellent practical solutions to hard problems. Most of the original applications of simulated annealing have been made to problems of a combinatorial nature, where the notions of *step* or *neighbor* usually find a natural interpretation. We will try to extend this approach to our framework for continuous maximization problem. We require a thorough understanding of the extensions required in the Simulated Annealing technique to suit our framework, which involves a mix of discrete and continuous variables, as well as variable constraints, to solve the mixed integer non-linear optimization problem.

In order to tailor the SA technique for Portfolio Selection (herein PS), we define two basic notions of **solution** and **neighborhood**.

- We *encode* a solution of PS as an n-dimensional vector x, with x_i representing the holding of asset i in the portfolio. The quality of the solution is evaluated by the variance of the returns of the portfolio, consistent with the mean-variance framework of Markowitz which is the bedrock of our model.
- There are two approaches to selecting a solution from the solution space of the PS for each iteration of the SA algorithm :
  - The first assumes that the neighborhood of current feasible solution must consist only of feasible solutions, and iterates through these to reach an optimal solution.
  - The second approach, considers infeasible solutions, by adding a penalty term to the objective function for each violated constraint. The penalty should depend on the importance of the constraint, and the actual violation for the same.

The specific modifications required to implement this SA approach to our framework requires modelling of constraints to define a solution space, a definite strategy to *step* to a new solution in each iteration, as well as defining the penalty terms required to adjust the objective functions.

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