Brief Explanation of Integration Schemes

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Explicit Integration

Let's use a simple example to frame our investigation of explicit integration. Take a look at Equation 1

$$\dot{x} = f(x, t) \tag{1}$$

If we discretize t as we would in any numerical scheme $t = t_0, t_0 + \Delta t, t_0 + 2\Delta t, \ldots, t_0 + n\Delta t$ we can transform Equation 1 into the discrete time equivalent shown in Equation 2.

$$\dot{x}_n = f(x_n, t_n) \tag{2}$$

Now lets approximate the derivative at a given step n as follows.

$$\dot{x}_n = \frac{(x_{n+1} - x_n)}{\Delta t} \tag{3}$$

Now substituting Equation 3 into Equation 1 we have the following.

$$\begin{aligned}\frac{(x_{n+1} - x_n)}{\Delta t} &= f(t_n, x_n) \\ x_{n+1} &= x_n + \Delta t f(t_n, x_n) \qquad (4)\end{aligned}$$

Using this formulation we can **explicitly** write down the solution for each consecutive step in time given initial conditions. (i.e. $x_0 = c$)

$$\begin{aligned} x_1 &= \Delta t f(t_0, c) + c \\ x_2 &= \Delta t f(t_1, x_1) + x_1 \\ &\vdots \end{aligned}$$

The particular method is known as Euler's method which is the simplest explicit integration scheme. It allows for the calculation of x_{n+1} in terms of x_n in a straightforward manner at each time step, but is sensitive to stability issues (like all explicit methods) which will be discussed next.

Stability of Explicit Integration

Let's attempt to solve the equation below using Euler's method.

$$\dot{x} = \lambda x(t) \tag{5}$$

If we assume $x(0) = c$ we know Equation 5 has the exact solution as follows

$$x(t) = ce^{\lambda t} \tag{6}$$

We also know that Equation 5 is only bounded (stable) when $\Re(\lambda) \leq 0$. Let's keep this in mind as we investigate the stability of the numerical method. Discretize Equation 5 and plug into Euler's formula.

$$\begin{aligned} x_{n+1} &= \lambda \Delta t x_n + x_n \\ &= (1 + \lambda \Delta t) x_n \\ &= (1 + \lambda \Delta t)^2 x_{n-1} \\ &\vdots \\ &= (1 + \lambda \Delta t)^{n+1} x_0 \end{aligned} \tag{7}$$

Equation 7 shows us that as n increases the only way x_{n+1} will not grow indefinitely is for the following to hold true.

$$|1 + \lambda \Delta t| \leq 1 \tag{8}$$

Solving Equation 8 shows us that we must choose at time step that satisfies Equation 9 for the solution to be stable.

$$\Delta t \leq \frac{2}{|\lambda|} \tag{9}$$

Therefore, even though the calculation of x_{n+1} at each step is simple in an explicit scheme the time step must be small enough to ensure stability.

Implicit Integration

We will now use a Backward Euler method to demonstrate an implicit integration scheme. We will only make a slight change to how we approximate the derivative.

$$\dot{x}_n = \frac{(x_n - x_{n-1})}{\Delta t} \tag{10}$$

Substitute Equation 10 into Equation 2

$$\begin{aligned} \frac{(x_n - x_{n-1})}{\Delta t} &= f(x_n, t_n) \\ x_n &= \Delta t f(x_n, t_n) + x_{n-1} \end{aligned}$$

If we reindex n, we have the equation.

$$x_{n+1} = \Delta t f(x_{n+1}, t_{n+1}) + x_n \tag{11}$$

Since x_{n+1} appears on both sides of the Equation 11 it is said to be **implicit** in x_{n+1}. This sometimes requires unique solution techniques to solve for x_{n+1} at each time step. So each time step is computationally more expensive than an explicit method, but implicit methods have advantages in stability.

Stability of Implicit Integration

Discretizing Equation 5 and plugging into the Backward Euler equation we have the following.

$$\begin{aligned}
x_{n+1} &= \lambda \Delta t x_{n+1} + x_n \\
x_{n+1} - \lambda \Delta t x_{n+1} &= x_n \\
(1 - \lambda \Delta t) x_{n+1} &= x_n \\
x_{n+1} &= \frac{x_n}{1 - \lambda \Delta t} \\
&= \frac{x_{n-1}}{(1 - \lambda \Delta t)^2} \\
\\
&= \frac{x_0}{(1 - \lambda \Delta t)^{n+1}}
\end{aligned} \tag{12}$$

Equation 12 is stable as long as the following inequality holds true.

$$|1 - \lambda \Delta t| \geq 1 \tag{13}$$

Because of the stability criterion on λ that requires $\Re(\lambda) \leq 0$, Equation 13 is always true. This is said to be unconditionally stable. Herein lies the advantages of implicit schemes.