Problem 4.2 | Ask to calculate the plasma frequency with the ion motions included.

We are going to follow ~~th~~ section 4.3 in the 2nd edition and apply perturbation theory to the equations of motion.

Perturbation is a ~~two~~ three step procedure:

(1) perturb an equilibrium; stable or unstable.

$\cancel{f = \epsilon^{(0)} f^{(0)} + \epsilon^{(1)} f^{(1)} + \epsilon^{(2)}}$

$f = \epsilon f^{(0)} + \epsilon f^{(1)} + \epsilon^2 f^{(2)} + \ldots$

$\epsilon \ll 1.$

or

$f = f^{(0)} + f^{(1)} + f^{(2)} + \ldots$

where ϵ is absorbed into the terms.

(3) Assume a form for the perturbations

$\sim f = f^{(0)} e^{i(kx - \omega t)}$

(2) linearize: remove all terms of order 2+.

We will also assume: * similiar to Dr. Mordijk

(1) there is no magnetic field

(2) there are no thermal motions ($kT = 0$) * Cold plasma *

(3) The motions occure only in the x-direction.

(4) The text assumes the ~~ions~~ ions are not moving.

Similar to the text we have the electron ~~EoM~~ equation of motion (EoM), & Poisson's equation

$$m_e n_e \left[\frac{\partial \vec{v}_e}{\partial t} + (\vec{v}_e \cdot \vec{\nabla}) \vec{v}_e \right] = -e n_e \vec{E}$$

$$\frac{\partial n_e}{\partial t} + \vec{\nabla} \cdot (n_e \vec{v}_e) = 0$$

Electron EoM

$$\epsilon_0 \vec{\nabla} \cdot \vec{E} = e(n_i - n_e)$$ – Poisson's equation

Since we are asked to consider the motion of the ions we have to include the ion EoM.

$$m_i n_i \left[\frac{\partial \vec{v}_i}{\partial t} + (\vec{v}_i \cdot \vec{\nabla}) \vec{v}_i \right] = e n_i \vec{E}$$

$$\frac{\partial n_i}{\partial t} + \vec{\nabla} \cdot (n_i \vec{v}_i) = 0$$

- The first step is to perturb the dynamical variables:

$$n_{i,e} = n^{(0)}_{i,e} + n^{(1)}_{i,e} \rightarrow (\#) - \text{to organize the terms.}$$

$n_{i,e}$: i → ion, e → electron

$n^{(1)}_{i,e}$ → first-order perturbation

$n^{(0)}_{i,e}$ → equilibrium, $n^{(0)}_i \simeq n^{(0)}_e \simeq n^{(0)}$ - quasineutral

$$\dot{n}^{(0)}_{i,e} = n'^{(0)}_{i,e} = 0$$

$$\frac{\partial n^{(0)}_{i,e}}{\partial t} = \frac{\partial n^{(0)}_{i,e}}{\partial x} = 0$$

$$\vec{v}_{i,e} = \underbrace{\vec{v}^{(0)}_{i,e}}_{\vec{v}_i = \vec{v}_e = \vec{0}} + \vec{v}^{(1)}_{i,e}$$

$$\vec{E} = \vec{E}^{(0)} + \vec{E}^{(1)}$$

→ $\cancel{\vec{E}^{(0)} = \vec{\nabla} \cdot \frac{e}{\epsilon_0} (n^{(0)}_i - n^{(0)}_e)}$

$\cancel{\vec{\nabla} \cdot \vec{E}^{(0)}}$

$\cdot\ \vec{E}^{(0)}$ b/c $n^{(0)}_i - n^{(0)}_e \simeq 0$ quasineutral.

- Then substitute the perturbed dynamical variables into the EoMs and ignore terms with order 2 & higher.

<u>Electron:</u>

$$m_e(n^{(0)}_0 + \cancel{n^{(1)}_e}) \left[\frac{\partial}{\partial t}(\vec{v}^{(1)}_e) + (\cancel{\vec{v}^{(1)}_i \cdot \vec{\nabla}}) \vec{v}^{(1)}_i \right] = -e(n^{(0)}_0 + \cancel{n^{(1)}_e}) \vec{E}^{(1)}$$

2nd-order 2nd-order 2nd-order

$$\cancel{m n^{(0)} \frac{\partial \vec{v}^{(1)}_e}{\partial t}} \quad \boxed{m \cancel{n^{(0)}} \, \dot{\vec{v}}^{(1)}_e = -e \cancel{n^{(0)}} \vec{E}^{(1)}}$$

$$\cancel{\frac{\partial n^{(1)}}{\partial t}} \quad \dot{n}^{(1)}_{e} + \vec{\nabla}\cdot\left(n^{(0)}\vec{V}^{(1)}_{e} + \cancel{2^{nd}\text{-order}}^{\,0}\right) = 0$$

$$\boxed{\dot{n}^{(1)}_{e} + n^{(0)}\, V'^{(1)}_{e} = 0}$$

$\vec{\nabla}\cdot\vec{v}^{(1)}_{e}$ – keep this form

~~The ion~~

ion:

$$\boxed{m_i \cancel{n^{(0)}}\, \dot{\vec{V}}^{(1)}_{i} = +e\cancel{n^{(0)}}\,\vec{E}^{(1)}}$$

$$\boxed{\dot{n}^{(1)}_{i} + n^{(0)}\, \vec{V}'^{(1)}_{i} = 0}$$

$\vec{\nabla}\cdot\vec{v}^{(1)}_{i}$ – ...

Poisson's Equation

$$\boxed{\epsilon_0 \vec{\nabla}\cdot\vec{E}^{(1)} = e\left(n^{(1)}_{i} - n^{(1)}_{e}\right)}$$

Next we assume the perturbations ~~follows~~ a are sinusoidal.

$$n^{(1)}_{i,e} = n^{(1)}_{i,e}\, e^{i(kx-\omega t)}$$

non-fluctuating term

fluctuating term

$$\vec{V}^{(1)}_{i,e} = V^{(1)}_{i,e}\, e^{i(kx-\omega t)}\,\hat{x}$$

$$\vec{E}^{(1)} = E^{(1)}\, e^{i(kx-\omega t)}\,\hat{x}$$

* Normally fluctuating terms are marked by a "~", $\tilde{n}_{i,e}$. I got lazy and did not add the "~".

$$f = A\, e^{i(kx-\omega t)}$$

$$\boxed{*\ \dot{f} = -i\omega f \qquad *\ f' = ikf}$$

* The sinusoidal assumption makes dealing with the derivatives easy.

Electron: EoM

$$\begin{cases} m_e \cancel{n^{(0)}} \dot{\vec{V}}_e^{(1)} = -e\cancel{n^{(0)}} \vec{E}^{(1)} & \text{(electron)} \\ m_i \cancel{n^{(0)}} \dot{\vec{V}}_i^{(1)} = e\cancel{n^{(0)}} \vec{E}^{(1)} & \text{(ion)} \end{cases}$$

$$\begin{cases} m_e \cancel{n^{(0)}} [-i\omega v_e \cancel{e^{i(kx-\omega t)}}] = -e\cancel{n^{(0)}} E^{(1)} \cancel{e^{i(kx-\omega t)}} \\ m_i \cancel{n^{(0)}} [-i\omega v_i \cancel{e^{i(kx-\omega t)}}] = e\cancel{n^{(0)}} E^{(1)} \cancel{e^{i(kx-\omega t)}} \end{cases}$$

$$\boxed{V_{i,e}^{(1)} = \frac{q_{i,e}}{m_{i,e}\omega} i \, E^{(1)}}$$

What does "i" imply? $\pi/2$ - phase shift

Recall: SHM

$x(t) = A\cos(\omega t)$

$v(t) = -\omega A \sin(\omega t)$

$a(t) = -\omega^2 A \cos(\omega t)$

~~$\sin(\omega t)$~~ $\cos(\omega t + \pi/2) = \sin(\omega t)$

$\therefore$

$v(t) = -\omega A \cos(\omega t + \pi/2)$

$\downarrow$

$\pi/2$ - phase shift to $a(t)$.

Continuity -

$$\begin{cases} \dot{n}_e^{(1)} + n^{(0)} \vec{\nabla}\cdot\vec{V}_e^{(1)} = 0 \\ \dot{n}_i^{(1)} + n^{(0)} \vec{\nabla}\cdot\vec{V}_i^{(1)} = 0 \end{cases}$$

$$\begin{cases} -\cancel{i}\omega n_e^{(1)} + \cancel{i}k n^{(0)} v_e^{(1)} = 0 \\ -\cancel{i}\omega n_i^{(1)} + \cancel{i}k n^{(0)} v_i^{(1)} = 0 \end{cases}$$

$$\boxed{n_{i,e}^{(1)} = \frac{k n^{(0)}}{\omega} V_{i,e}^{(1)}}$$

- Poisson's -

$$\epsilon_0 \vec{\nabla}\cdot\vec{E}^{(1)} = e(n_i^{(1)} - n_e^{(1)})$$

fluctuating $\tilde{n}_{i,e}^{(1)} = \underbrace{n_{i,e}^{(1)}}_{\text{non-fluctuating}} e^{i(kx-\omega t)}$

$$\boxed{ik\epsilon_0 E^{(1)} = e(n_i^{(1)} - n_e^{(1)})}$$

* We can do two things obtain $E^{(1)}$ as a function $V_i^{(1)} - V_e^{(1)}$ or obtain another equation for $E^{(1)}$ as a function of $n_i^{(1)} - n_e^{(1)}$.

Let use the EoMs to obtain $v_i^{(1)} - v_e^{(1)}$,

$$v_i^{(1)} - v_e^{(1)} = \frac{1}{\omega}\left(\frac{e}{m_i} + \frac{e}{m_e}\right) \frac{i E^{(1)}}{\cancel{(\omega)}} \quad (\mathrm{I})$$

Next ~~let~~ use the continuity equations to obtain $v_i^{(1)} - v_e^{(1)}$

$$v_i^{(1)} - v_e^{(1)} = \frac{\omega}{k n^{(0)}}\left(n_i^{(1)} - n_e^{(1)}\right) \quad (\mathrm{II})$$

Poisson's equation to obtain $iE^{(1)}$,

$$iE^{(1)} = \frac{e}{k\epsilon_0}\left(n_i^{(1)} - n_e^{(1)}\right) \quad (\mathrm{III})$$

Substitute equations (III) + (II) into (I),

$$\left[\frac{\omega}{k n^{(0)}}\left(n_i^{(1)} - n_e^{(1)}\right)\right] = \frac{1}{\omega}\left(\frac{e}{m_i} + \frac{e}{m_e}\right)\left[\frac{e}{k\epsilon_0}\left(n_i^{(1)} - n_e^{(1)}\right)\right]$$

$$\frac{\omega}{k n^{(0)}}\left(n_i^{(1)} - n_e^{(1)}\right) = \frac{1}{\omega k}\left(\frac{e^2}{m_i\epsilon_0} + \frac{e^2}{m_e\epsilon_0}\right)\left(n_i^{(1)} - n_e^{(1)}\right)$$

* $n_i^{(1)} - n_e^{(1)} \neq 0$ Or there is no perturbation,

$\therefore$ the coefficients must equate.

~~The coeff~~ c

$$\frac{\omega}{\cancel{k} n^{(0)}} = \frac{1}{\omega \cancel{k}}\left(\frac{e^2}{m_i\epsilon_0} + \frac{e^2}{m_e\epsilon_0}\right)$$

$$\omega_p^2 = \left(\frac{e^2 n^{(0)}}{m_i\epsilon_0} + \frac{e^2 n^{(0)}}{m_e\epsilon_0}\right); \quad \omega_{p_{i,e}} = \frac{q_{i,e}^2 n^{(0)}}{m_i\,\epsilon_0}$$

$$\omega_p^2 = \left(\omega_{pe}^2 + \omega_{pi}^2\right) \rightarrow \omega_p^2 = \omega_{pe}^2\left(1 + \left(\omega_{pi}/\omega_{pe}\right)^2\right)$$

Z - atomic # of ion, I have ignored the term and assumed the ion is ~~a~~ hydrogen ion.

$$\left(\frac{\omega_{pi}}{\omega_{pe}}\right)^2 = Z\frac{m_e}{m_i} \ll 1$$

$\therefore$

$$\boxed{\omega_p^2 \simeq \omega_{pe}^2}$$

* We can ignore the ion contribution b/c of the electron-proton ~~ratio~~. mass ratio.*