Machine Learning (CE 40717)

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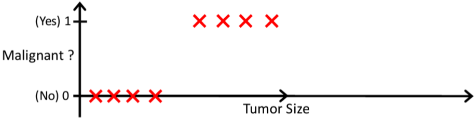
Classification problem

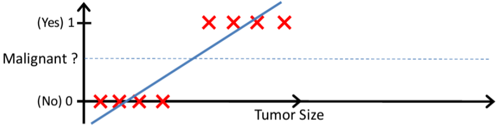
- **Classification (binary)**
  - Email: Spam / Not Spam?
  - Online Transactions: Fraudulent / Genuine?
  - Tumor: Malignant / Benign?

$$\mathbf{y} \in \{\mathbf{0}, \mathbf{1}\} \begin{cases} 0\text{: “Negative Class” (e.g., benign tumor)} \\ 1\text{: “Positive Class” (e.g., malignant tumor)} \end{cases}$$

Classification problem (cont.)

- Can we solve the problem using linear regression?



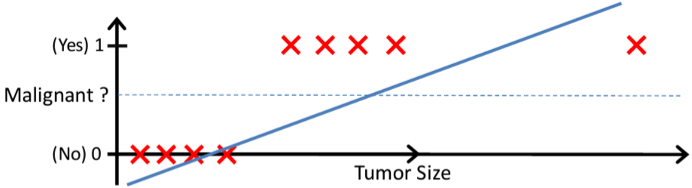


- We could fit a straight line and define a threshold at 0.5:

$$\textbf{If } h_\theta(x) \geq 0.5, \text{predict } y = 1$$
$$\textbf{If } h_\theta(x) < 0.5, \text{predict } y = 0$$

Classification problem (cont.)

- What about now? (By adding a new data point)



- Classification: $y = 0$ or $y = 1$
  - $h_\theta(x)$ can be > 1 or < 0
  - Another drawback of using linear regression for this problem
- What we need:

$$\text{Logistic regression: } 0 \leq h_\theta(x) \leq 1$$

- We also show this function with other notations: $f(x; w) = \sigma(w^T x)$

Introduction

- Suppose we have a binary classification task (so $K = 2$).
- By observing age, gender, height, weight and BMI we try to distinguish if a person is overweight or not overweight.

Age	Gender	Height (cm)	Weight (kg)	BMI	Overweight
25	Male	175	80	25.3	0
30	Female	160	60	22.5	0
		...			
35	Male	180	90	27.3	1

- We denote the features of a sample with vector x and the label with y.
- In logistic regression we try to find an $\sigma(w^T x)$ which predicts **posterior** probabilities $P(y = 1|x)$.

Introduction (cont.)

- $\sigma(w^T x)$: probability that $y = 1$ given x (parameterized by $\mathbf{w}$)

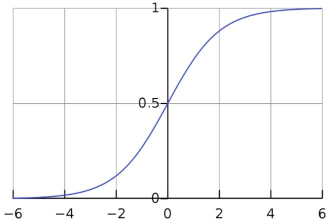
$$P(y = 1|x, \mathbf{w}) = \sigma(\mathbf{w}^T x)$$
$$P(y = 0|x, \mathbf{w}) = 1 - \sigma(\mathbf{w}^T x)$$

- We need to look for a function which gives us an output in the range [0, 1]. (like a probability).
- Let's denote this function with $\sigma(.)$ and call it the **activation function**.

Introduction (cont.)

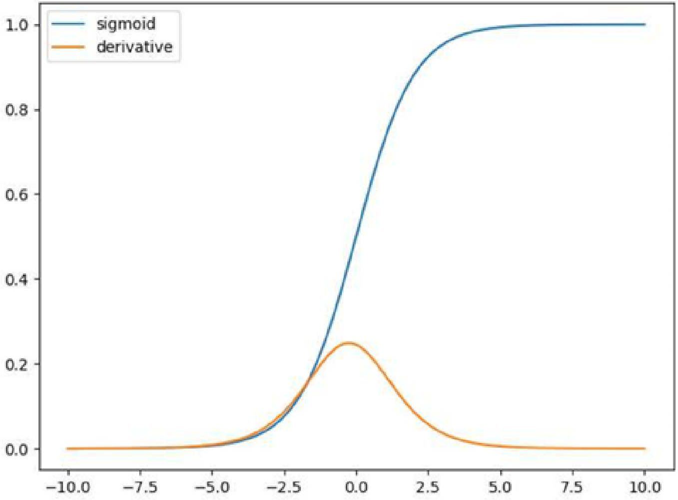
- Sigmoid (logistic) function.

$$\sigma(z) = \frac{1}{1 + e^{-z}}$$



- A good candidate for activation function.
- It gives us a number between 0 and 1 **smoothly**.
- It is also **differentiable**

Sigmoid function & its derivative



Introduction (cont.)

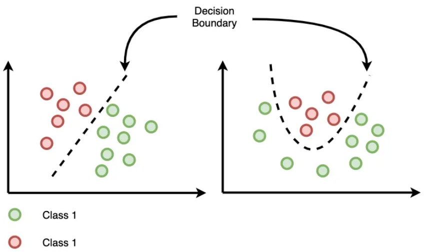
- The sigmoid function takes a number as input but we have:

$$x = [x_0 = 1, x_1, \ldots, x_d]$$
$$w = [w_0, w_1, \ldots, w_d]$$

- So we can use the **dot product** of x and w.
- We have $0 \leq \sigma(\mathbf{w}^T x) \leq 1$. which is the estimated probability of $y = 1$ on input x.
- An Example : A basketball game (Win, Lose)
  - $\sigma(\mathbf{w}^T x) = 0.7$
  - In other terms 70 percent chance of winning the game.

Decision surface

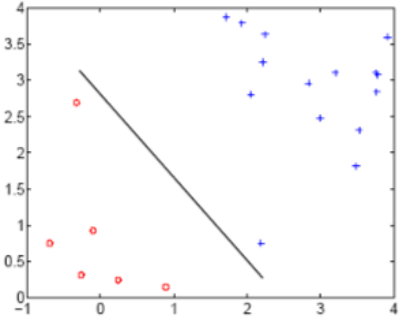
- Decision surface or decision boundary is the region of a problem space in which the output label of a classifier is ambiguous. (could be linear or non-linear)
- In binary classification it is where the probability of a sample belonging to each $y = 0$ and $y = 1$ is equal.



- Decision boundary hyperplane always has **one less dimension** than the feature space.

Decision surface (cont.)

- An example of linear decision boundaries:



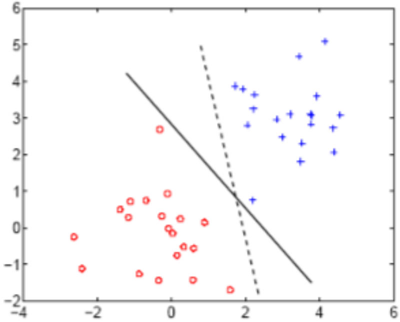


Figure adapted from Eric Xing, Machine Learning, CMU

Decision surface (cont.)

- Back to our logistic regression problem.
- Decision surface $\sigma(\mathbf{w}^T x) =$ **constant**.

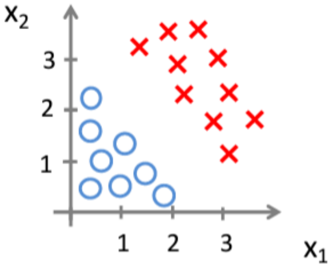
$$\sigma(\mathbf{w}^T x) = \frac{1}{1 + e^{-(\mathbf{w}^T x)}} = 0.5$$

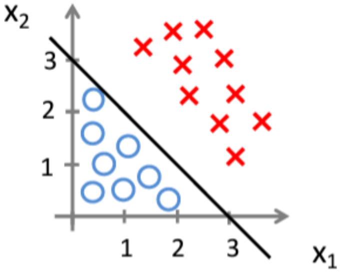
- Decision surfaces are **linear functions** of x
  - if $\sigma(\mathbf{w}^T x) \geq 0.5$ then $\hat{y} = 1$, else $\hat{y} = 0$
  - Equivalently, if $\mathbf{w}^T x + w_0 \geq 0.5$ then decide $\hat{y} = 1$, else $\hat{y} = 0$

$\hat{y}$ is the predicted label

Decision boundary example

$$\sigma(\mathbf{w}^T x) = \sigma(w_0 + w_1 x_1 + w_2 x_2)$$



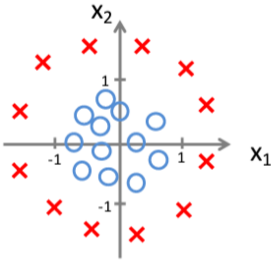


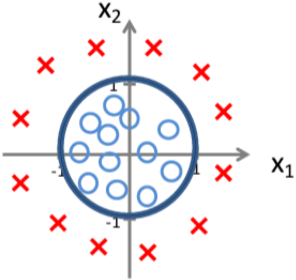
Predict $y = 1$ if $-3 + x_1 + x_2 \geq 0$

Non-linear decision boundary example

$$\sigma(\mathbf{w}^T x) = \sigma(w_0 + w_1 x_1 + w_2 x_2 + w_3 x_1^2 + w_4 x_2^2)$$

We can learn more complex decision boundaries when having higher order terms





Predict $y = 1$ if $-1 + x_1^2 + x_2^2 \geq 0$

ML estimation

- We had posterior of a sample as:

$$P(y^{(i)}|x^{(i)},\mathbf{w})$$

- Logistic regression should maximize production of all these sample posteriors.
- Maximum (conditional) log likelihood:

$$\hat{\mathbf{w}} = \underset{w}{\arg\max} \ \ \log\prod_{i=1}^{n} P(y^{(i)}|x^{(i)},\mathbf{w})$$

- Note that in **binary** classification y is either 1 or 0, So we can have posterior term simplified as follows:

$$P(y^{(i)}|x^{(i)},\mathbf{w}) = \sigma(\mathbf{w}^T x^{(i)})^{y^{(i)}}(1-\sigma(\mathbf{w}^T x^{(i)}))^{(1-y^{(i)})}$$

ML estimation

- Logarithm of the posterior probability:

$$\log P(y^{(i)}|x^{(i)},\mathbf{w}) = y^{(i)}\log(\sigma(\mathbf{w}^T x^{(i)})) + (1-y^{(i)})\log(1-\sigma(\mathbf{w}^T x^{(i)}))$$

- Hence the log likelihood is as follows:

$$\log \prod_{i=1}^{n} P(y^{(i)}|x^{(i)},\mathbf{w}) = \sum_{i=1}^{n} \log P(y^{(i)}|x^{(i)},\mathbf{w})$$

$$= \sum_{i=1}^{n} [y^{(i)}\log(\sigma(\mathbf{w}^T x^{(i)})) + (1-y^{(i)})\log(1-\sigma(\mathbf{w}^T x^{(i)}))]$$

Cost function

- We should find

$$\hat{\mathbf{w}} = \arg\min_{w} \; J(w)$$

- MLE finds parameters that best describe a classification problem so cost function should be negative of log likelihood term:

$$J(w) = -\sum_{i=1}^{n} \log P(y^{(i)} | x^{(i)}, \mathbf{w})$$
$$= \sum_{i=1}^{n} -y^{(i)} \log(\sigma(\mathbf{w}^T x^{(i)})) - (1 - y^{(i)}) \log(1 - \sigma(\mathbf{w}^T x^{(i)}))$$

- No closed form solution for $\nabla_w J(w) = 0$
- However $J(w)$ is **convex**.

Cost function (cont.)

- Convexity of $J(w)$ can easily be proved:
  - We use the lemma that sum of several convex functions is still convex (you can prove it on your own).
  - Each term in the summation is differentiable (twice).
  - If you twice get derivative of (with respect to σ):

$$-y^{(i)}\log(\sigma(\mathbf{w}^T x^{(i)})) - (1-y^{(i)})\log(1-\sigma(\mathbf{w}^T x^{(i)}))$$

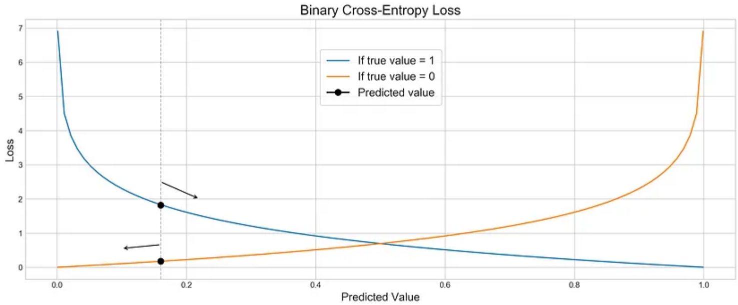
  - You get:

$$\frac{y}{\sigma^2} + \frac{1-y}{(1-\sigma)^2}$$

  - Which for both $y=0$ and $y=1$ is positive.
  - Each $\log P(y^{(i)}|x^{(i)},\mathbf{w})$ is convex, hence the summation is convex as well.

Cost function (cont.)

- Visualization of each binary cross entropy loss term:



- As you can see if the model predicted value is $\hat{y} = 0.16$ and true label is $y = 1$ then the error is high but if the true label is $y = 0$ the error would be low.

Figure adopted from https://towardsdatascience.com/logistic-regression-from-scratch-69db4f587e17

Gradient descent

- Remember from previous slides:

$$J(w) = \sum_{i=1}^{n} -y^{(i)} \log(\sigma(\mathbf{w}^T x^{(i)})) - (1 - y^{(i)}) \log(1 - \sigma(\mathbf{w}^T x^{(i)}))$$

- Update rule for **gradient descent**:

$$w^{t+1} = w^t - \eta \nabla_w J(w^t)$$

- With $J(w)$ definition for logistic regression we get:

$$\nabla_w J(w) = \sum_{i=1}^{n} (\sigma(\mathbf{w}^T x^{(i)}) - y^{(i)}) x^{(i)}$$

Gradient descent

- Compare the gradient of logistic regression with the gradient of SSE in linear regression :

$$\nabla_w J(w) = \sum_{i=1}^{n} (\sigma(\mathbf{w}^T x^{(i)}) - y^{(i)}) x^{(i)}$$

$$\nabla_w J(w) = \sum_{i=1}^{n} (\mathbf{w}^T x^{(i)} - y^{(i)}) x^{(i)}$$

Loss function

- Loss function is a single overall measure of loss incurred for taking our decisions (over entire dataset).
- We have:

$$Loss(y, \sigma(\mathbf{w}^T x)) = -y \times \log(\sigma(\mathbf{w}^T x)) - (1-y) \times \log(1-\sigma(\mathbf{w}^T x))$$

- Since in binary classification either $y = 1$ or $y = 0$ we have:

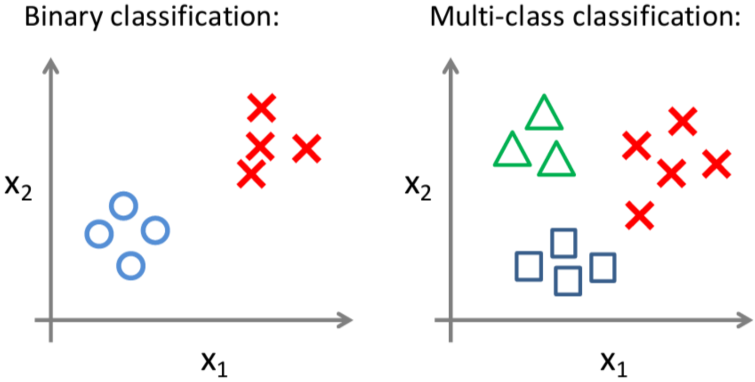
$$Loss(y, \sigma(\mathbf{w}^T x)) = \begin{cases} -\log(\sigma(\mathbf{w}^T x)) & \textbf{if } y = 1 \\ -\log(1-\sigma(\mathbf{w}^T x)) & \textbf{if } y = 0 \end{cases}$$

- How is it related to zero-one loss? ($\hat{y}$ is the predicted label and y is the ture label)

$$Loss(y, \hat{y}) = \begin{cases} 1 & \textbf{if } y \neq \hat{y} \\ 0 & \textbf{if } y = \hat{y} \end{cases}$$

Multi-class logistic regression

- Now consider a problem where we have K classes and every sample only belongs to one class (for simplicity).



Multi-class logistic regression (cont.)

- For each class k, $\sigma_k(x;\mathbf{W})$ predicts the probability of $y = k$.
  - i.e., $P(y = k|x, \mathbf{W})$
- For each data point x_0, $\sum_{k=1}^{K} P(y = k|x_0, \mathbf{W})$ must be 1
  - W denotes a matrix of w_i's in which each w_i is a weight vector dedicated for class label i.
- On a new input x, to make a prediction, we pick the class that maximizes $\sigma_k(x;\mathbf{W})$:

$$\alpha(x) = \underset{k=1,\dots,K}{\arg\max} \ \sigma_k(x;\mathbf{W})$$

if $\sigma_k(x;\mathbf{W}) > \sigma_j(x;\mathbf{W}) \ \forall j \neq k$ **then decide** C_k

Multi-class logistic regression (cont.)

- $K > 2$ and $y \in \{1, 2, \ldots, K\}$

$$\sigma_k(x, \mathbf{W}) = P(y = k|x) = \frac{\exp\left(w_k^T x\right)}{\sum_{j=1}^{K} \exp\left(w_j^T x\right)}$$

- Normalized exponential (Aka **Softmax**)
- if $w_k^T x \gg w_j^T x$ for all $j \neq k$ then $P(C_k|x) \approx 1$ and $P(C_j|x) \approx 0$
- Note : remember from Bayes theorem:

$$P(C_k|x) = \frac{P(x|C_k)P(C_k)}{\sum_{j=1}^{K} P(x|C_j)P(C_j)}$$

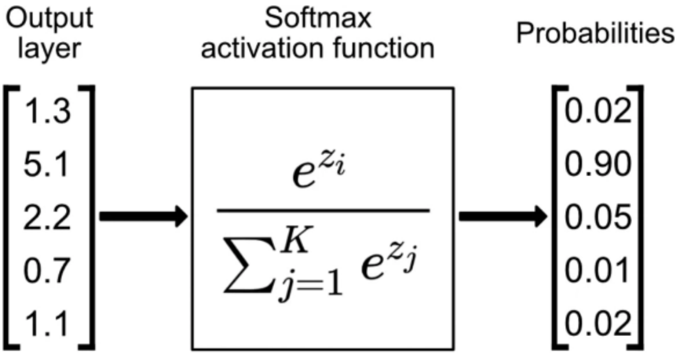
Multi-class logistic regression (cont.)

- Softmax function **smoothly** highlights the maximum probability and is differentiable.
- Compare it with max(.) function which is strict and non-differentiable
- Softmax can also handle negative values because we are using exponential function
- And it gives us probability for each class since:

$$\sum_{k=1}^{K} \frac{\exp(w_k^T x)}{\sum_{j=1}^{K} \exp(w_j^T x)} = 1$$

Multi-class logistic regression (cont.)

- An example of applying softmax (note that $z_i = w^T x_i$):



Multi-class logistic regression (cont.)

- Again we set $J(W)$ as negative of log likelihood.
- We need $\hat{W} = \underset{W}{\arg\min}\; J(W)$

$$
\begin{aligned}
J(W) &= -\log \prod_{i=1}^{n} P(y^{(i)} | x^{(i)}, \mathbf{W}) \\
&= -\log \prod_{i=1}^{n} \prod_{k=1}^{K} \sigma_k(x^{(i)}; \mathbf{W})^{y_k^{(i)}} \\
&= -\sum_{i=1}^{n} \sum_{k=1}^{K} y_k^{(i)} \log(\sigma_k(x^{(i)}; \mathbf{W}))
\end{aligned}
$$

- If **i-th** sample belongs to class k then $y_k^{(i)}$ is 1 else 0.
- Again no closed-from solution for $\hat{W}$

Multi-class logistic regression (cont.)

- From previous slides we have:

$$J(W) = -\sum_{i=1}^{n}\sum_{k=1}^{K} y_k^{(i)} \log(\sigma_k(x^{(i)};\mathbf{W}))$$

- In which:

$$W = [w_1, w_2, \dots, w_K], \quad Y = \begin{pmatrix} y^{(1)} \\ y^{(2)} \\ \vdots \\ y^{(n)} \end{pmatrix} = \begin{pmatrix} y_1^{(1)} & \cdots & y_K^{(1)} \\ y_1^{(2)} & \cdots & y_K^{(2)} \\ \vdots & \ddots & \vdots \\ y_1^{(n)} & \cdots & y_K^{(n)} \end{pmatrix}$$

- y is a vector of length K (1-of-K encoding)
  - For example $y = [0, 0, 1, 0]^T$ when the target class is C_3.

Multi-class logistic regression (cont.)

- Update rule for gradient descent:

$$w_j^{t+1} = w_j^t - \eta \nabla_W J(W^t)$$

$$\nabla_{w_j} J(W) = \sum_{i=1}^{n} (\sigma_j(x^{(i)}; \mathbf{W}) - y_j^{(i)}) x^{(i)}$$

- w_j^t denotes the weight vector for class j (since in multi-class LR, each class has its own weight vector) in the t-th iteration

Logistic regression (LR) summary

- LR is a **linear** classifier
- LR optimization problem is obtained by **maximum likelihood**
- No closed-form solution for its optimization problem
  - But convex cost function and global optimum can be found by gradient ascent

1 Introduction

2 Logistic Regression

3 Summary

4 Extra reading
- Probabilistic view in classification
- Probabilistic classifiers

5 References

Probabilistic view in classification problem

- In a classification problem:
  - Each **feature** is a **random variable** (e.g. a person's height)
  - The **class label** is also considered a **random variable** (e.g. a person could be overweight or not)
- We observe the feature values for a random sample and intend to find its class label
  - Evidence: Feature vector x
  - Objective: Class label

Definitions

- Posterior probability : The probability of a class label C_k given a sample x

$$P(C_k|x)$$

- Likelihood or class conditional probability : PDF of feature vector x for samples of class C_k

$$P(x|C_k)$$

- Prior probability : Probability of the label be C_k

$$P(C_k)$$

- $P(x)$: PDF of feature vector x
    - From total probability theorem:

$$P(x) = \sum_{k=1}^{K} P(x|C_k)P(C_k)$$