

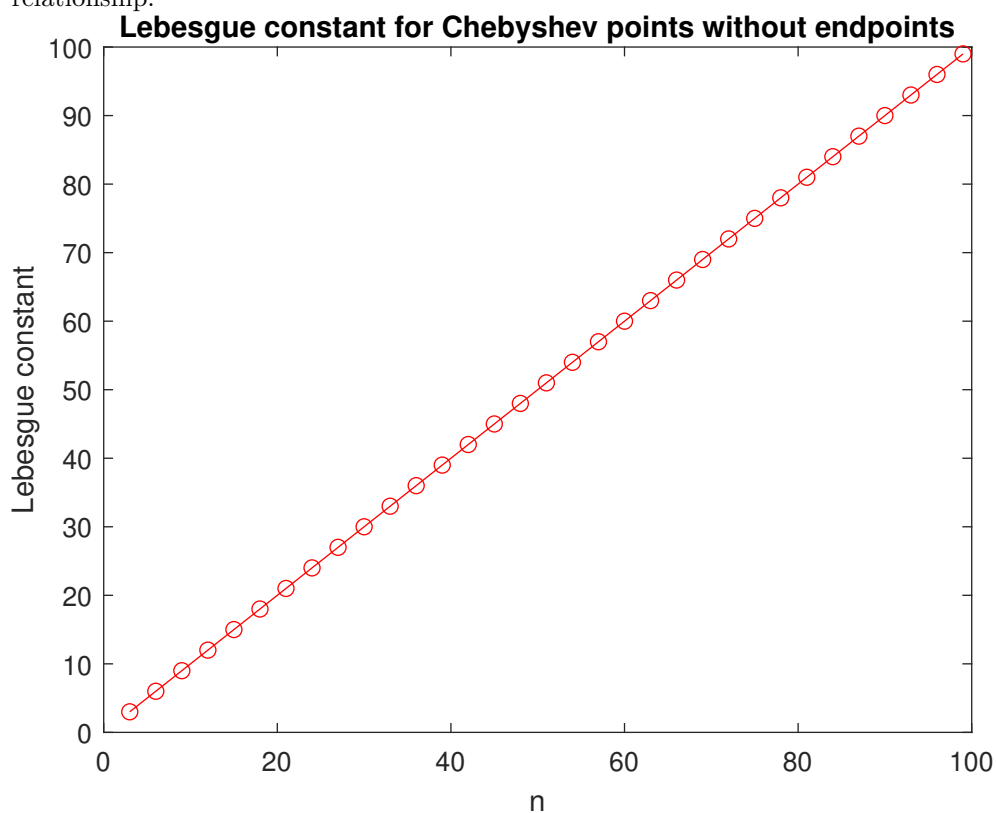
# AMATH 570 Assignment 4

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## 1 Problem 1 (15.4)

Using chebfun to investigate the Lebesgue constant for the set of  $n$  Chebyshev points with the endpoints removed, we get what appears to be a linear  $O(n)$  relationship.



See problem15.4.m for the code that does this.

## 2 Problem 2

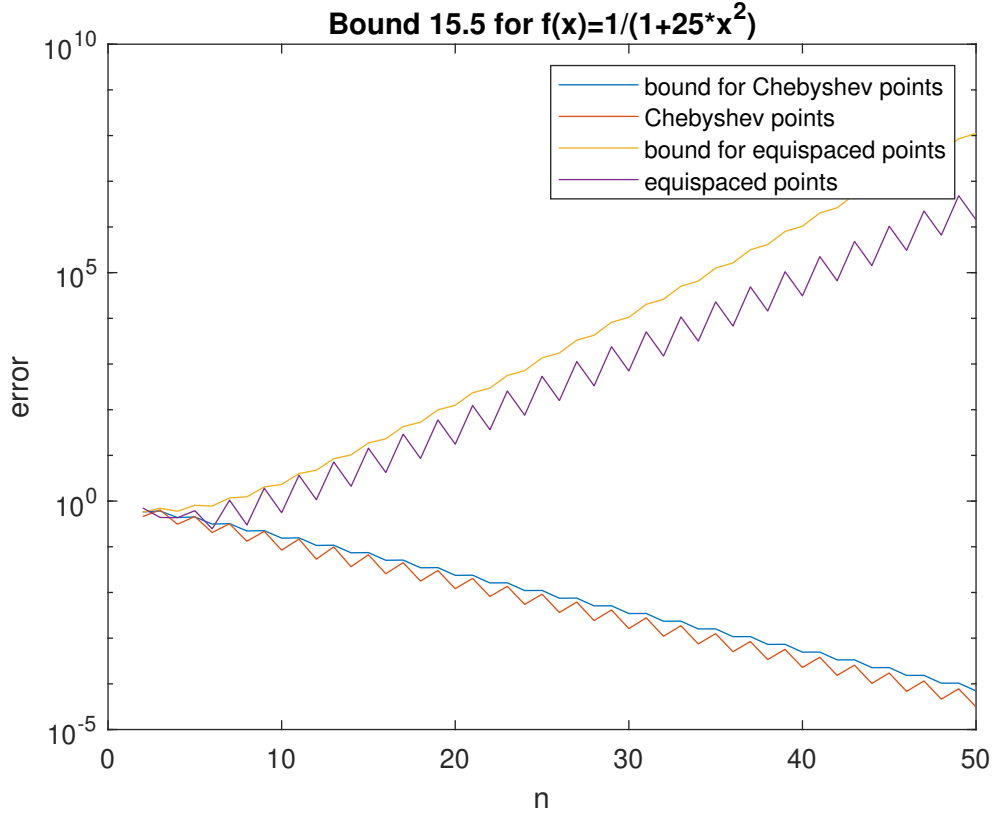
For completeness, the bound (15.5) is

$$\|f - p\| \leq (\Lambda + 1) \|f - p^*\|$$

where  $f$  is the function we are approximating,  $\Lambda$  is the Lebesgue constant for a set of interpolation points,  $p \in \mathcal{P}_n$  is the interpolant of  $f$  at those points, and  $p^* \in \mathcal{P}_n$  is the best polynomial approximation.

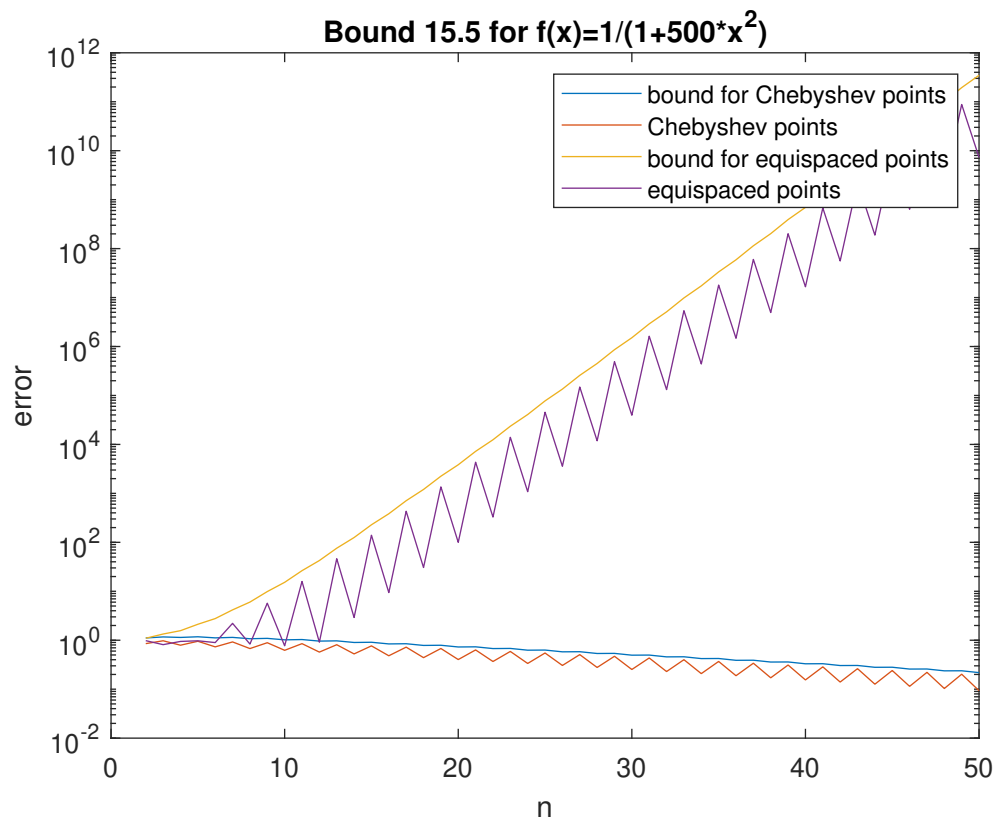
### 2.1 Part a

Consider the function  $f(x) = 1/(1 + 25x^2)$ . Using chebfun, we plot both the bound and the true error for both Chebyshev and equispaced points and obtain the following:



### 2.2 Part b

Now, consider  $f(x) = 1/(1 + 500x^2)$ . We again plot both the bound and the true error for both Chebyshev and equispaced points using chebfun and obtain the following:



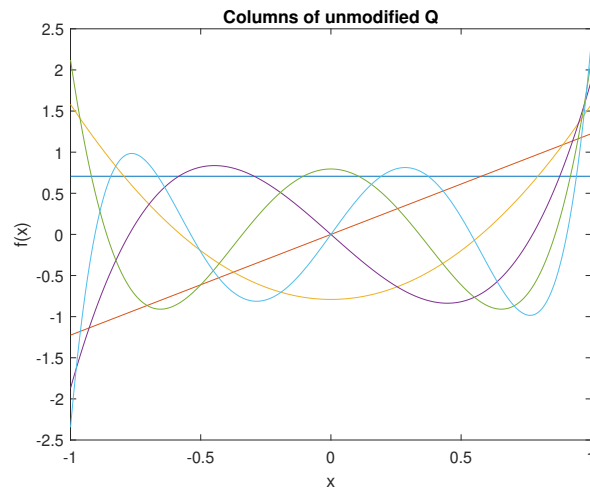
See problem2.m for the relevant MATLAB code.

### 3 Problem 3

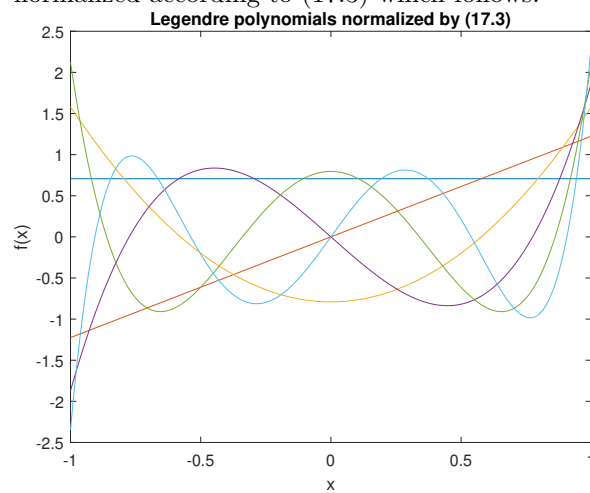
#### 3.1 Exercise 17.3

##### 3.1.1 Part a

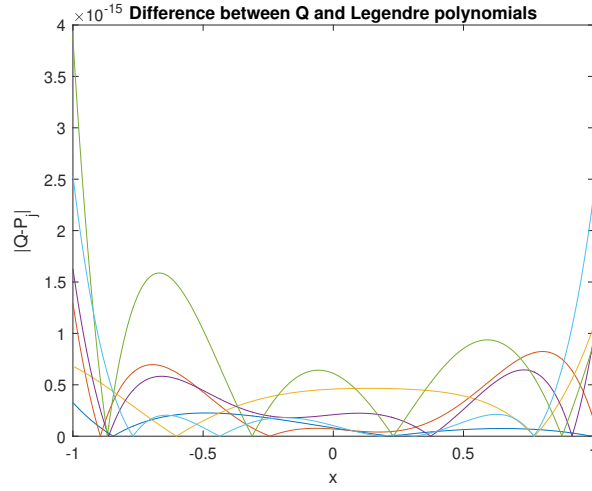
Using `chebfun`, we construct a quasimatrix  $A$  with columns corresponding to  $1, x, \dots, x^5$  on  $[-1, 1]$ . Using the command `qr` to find an orthonormal set of functions and plotting them, we get



which appears identical to the following plot of the first 6 Legendre polynomials normalized according to (17.3) which follows.



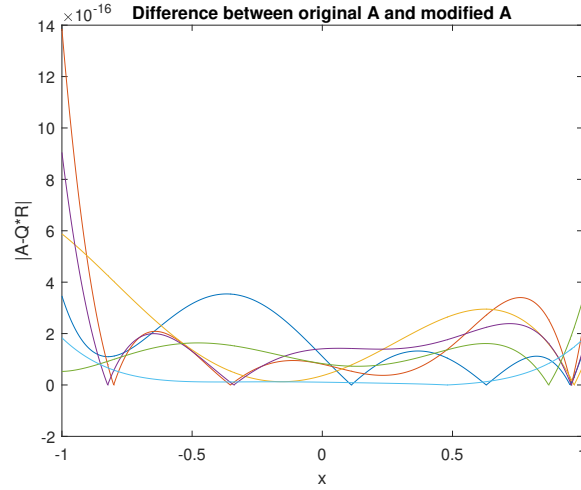
Indeed, if we plot the absolute value of the difference between the two, we observe



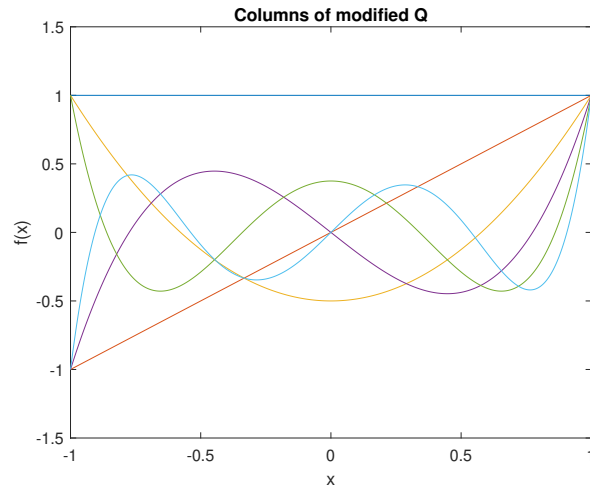
meaning that they are identical on the order of machine precision.

### 3.1.2 Part b

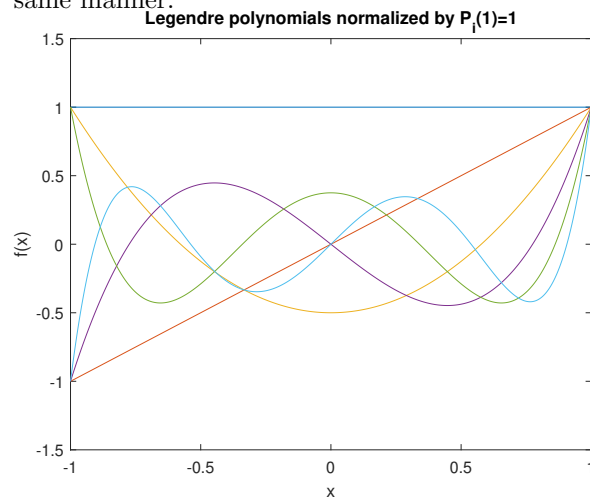
Now, we normalize the columns of  $Q$  such that they take value 1 at  $x = 1$  and adjust the rows of  $R$  accordingly. To confirm that our modified  $Q$  and  $R$  still multiply to  $A$ , we plot the absolute value of the difference between  $A$  and  $Q * R$



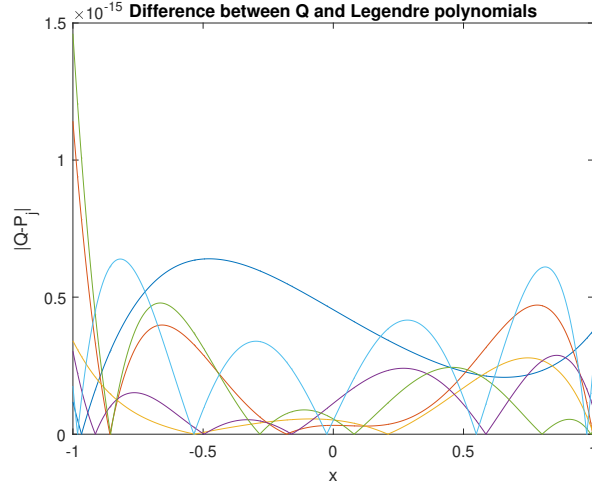
which shows that they are identical to machine precision. Plotting we columns of the new quasimatrix  $Q$ , we observe



which now looks identical to the first 6 Legendre polynomials normalized in the same manner.



To confirm this, we again plot the absolute value of their difference which again shows that they are identical to machine precision.



See problem17.8.m for the code that produces the plots in this problem.

### 3.2 Exercise 17.8

Let  $\{p_n\}$  be the family of monic polynomials associated with the inner product defined in (17.1) and let  $q$  be any monic polynomial of degree  $n$ . Then, we can write  $q$  as

$$q(x) = \sum_{k=0}^{n-1} c_k p_k(x) + p_n(x)$$

where the coefficient on  $p_n$  is 1, because both it and  $q$  are monic. We can now write

$$\begin{aligned} (q, q) &= \int_{-1}^1 w(x) \overline{q(x)} q(x) dx \\ &= \int_{-1}^1 w(x) \left( \sum_{k=0}^{n-1} c_k p_k(x) + p_n(x) \right) \left( \sum_{k=0}^{n-1} \overline{c_k} \overline{p_k(x)} + \overline{p_n(x)} \right) dx \\ &= \int_{-1}^1 w(x) \left( \sum_{k=0}^{n-1} \overline{c_k} \overline{p_k(x)} + \overline{p_n(x)} \right) \left( \sum_{k=0}^{n-1} c_k p_k(x) + p_n(x) \right) dx. \end{aligned}$$

Now, the orthogonality of the family  $\{p_n\}$  means that  $\int_{-1}^1 w(x) \overline{p_i(x)} p_j(x) dx = 0$  for  $i \neq j$ , so when we multiply this out and utilize the linearity of the integration operator, the cross-terms will be zero, meaning that

$$\begin{aligned} (q, q) &= \sum_{k=0}^{n-1} \int_{-1}^1 w(x) \overline{c_k} \overline{p_k(x)} p_k(x) dx + \int_{-1}^1 w(x) \overline{p_n(x)} p_n(x) dx \\ &= \sum_{k=0}^{n-1} \overline{c_k} c_k (p_k, p_k) + (p_n, p_n) = \sum_{k=0}^{n-1} |c_k|^2 (p_k, p_k) + (p_n, p_n) \geq (p_n, p_n), \end{aligned}$$

because  $|c_k|, (p_k, p_k) \geq 0$  for any  $k$ .

## 4 Problem 4 (18.4)

### 4.1 Part a

To derive a comrade matrix for the Legendre polynomials in the manner of theorem 18.1, we wish to find a matrix whose eigenvalues are the roots of the polynomial  $p(x) = \sum_{k=0}^n a_k P_k(x)$ . We do this in the same manner as with the Chebyshev polynomials. For the interior, we use the recursive relationship  $(k+1)P_{k+1}(x) - kP_{k-1}(x)$  to get that our matrix should map  $P_k(x)$  to  $xP_k(x) = \frac{k}{2k+1}P_{k-1}(x) + \frac{k+1}{2k+1}P_{k+1}(x)$ . For the first row, our matrix should map  $P_0(x)$  to  $xP_0(x) = P_1(x)$ . For the last row, we have the same relationship as with the Chebyshev polynomials but adjusted for our new interior coefficients. Namely,

$$P_{n-1}(x) \mapsto xP_{n-1}(x) - \frac{n-1+1}{a_n(2(n-1)+1)}(a_0P_0(x) + \dots + a_nP_n(x)).$$

Thus, we can write

$$C = \begin{pmatrix} 0 & 1 & & & & \\ 1/3 & 0 & 2/3 & & & \\ & 2/5 & 0 & 3/5 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \frac{n-2}{2n-3} & 0 & \frac{n-1}{2n-3} \\ & & & & \frac{n-1}{2n-1} & 0 \end{pmatrix} - \frac{n}{a_n(2n-1)} \begin{pmatrix} & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ a_0 & a_1 & a_2 & \dots & a_{n-1} \end{pmatrix}.$$

Note that the  $k$ th Legendre polynomial corresponds to the  $(k+1)$ st row of  $C$ , so we need to shift the relations above accordingly to explicitly write what  $C_{i,j}$  is (see part b for code that does this or problem 5 for it written out explicitly). Then, our theorem is as follows:  
The roots of the polynomial

$$p(x) = \sum_{k=0}^n a_k P_k(x), \quad a_n \neq 0,$$

are the eigenvalues of the matrix  $C$  as defined above.

### 4.2 Part b

To verify that the comrade matrix for  $p = P_0 + \dots + P_5$  has the same eigenvalues as  $p$  does roots, we note that  $a_i = 1$  for  $i = 0, \dots, 5$  and use MATLAB to construct  $C$  for this case which outputs

The eigenvalues of the comrade matrix are  
-1.0000000000000000 + 0.0000000000000000i



```

0.634846841685047 + 0.225134736423368i
0.634846841685047 - 0.225134736423368i
-0.412624619462826 + 0.273188868980396i
-0.412624619462826 - 0.273188868980396i

```

and

The roots of our polynomial are

```

-0.412624619462826 - 0.273188868980397i
-0.412624619462826 + 0.273188868980397i
0.634846841685048 - 0.225134736423369i
0.634846841685048 + 0.225134736423369i
-1.000000000000001 + 0.000000000000001i

```

which are given in different order, but match to machine precision as one would expect.

See problem18.4.m for the code that does this.

## 5 Problem 5 (19.7)

### 5.1 Part a

Knowing that the nodes  $\{x_j\}$  of the  $(n+1)$ -point Gauss quadrature are the zeroes of the Legendre polynomial  $P_{n+1}$ , we wish to find a comrade matrix for which these are its eigenvalues. However, in problem 4, we derived such a comrade matrix for linear combinations of Legendre polynomials. Now, we need an  $(n+1)$  by  $(n+1)$  matrix, but we have that  $a_0 = \dots = a_n = 0$  and  $a_{n+1} = 1$ , so the second matrix is the zero matrix and

$$A = \begin{pmatrix} 0 & 1 & & & & \\ 1/3 & 0 & 2/3 & & & \\ & 2/5 & 0 & 3/5 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \frac{n-1}{2n-1} & 0 & \frac{n}{2n-1} \\ & & & & \frac{n}{2n+1} & 0 \end{pmatrix}$$

We can reindex our formula from problem 4 to explicitly find the entries of this matrix. Namely, for  $k \in \{1, \dots, n\}$ ,

$$\begin{cases} A_{k,k+1} = \frac{k}{2k-1}, \\ A_{k,k-1} = \frac{k-1}{2k-1}, \\ A_{k,j} = 0, \quad \text{otherwise.} \end{cases}$$

Additionally, we have that  $A_{1,2} = 1$ ,  $A_{n,n+1} = \frac{n}{2n+1}$  and 0 in all other entries.

## 5.2 Part b

To find a diagonal matrix  $D = \text{diag}(d_0, \dots, d_n)$  such that  $B = DAD^{-1}$  is real symmetric, we consider that for  $k > 1$ ,

$$B_{k,k+1} = d_{k-1}A_{k,k+1}/d_k$$

and for  $k < n + 1$

$$B_{k+1,k} = d_kA_{k+1,k}/d_{k-1}.$$

Because we need  $B$  to be symmetric, we can equate these in the interior  $2 \leq k \leq n$ . Thus,

$$\frac{d_{k-1}}{d_k} \frac{k}{2k-1} = \frac{d_k}{d_{k-1}} \frac{k}{2k+1}.$$

This yields a quadratic equation

$$\frac{d_k}{d_{k-1}} = \frac{2k+1}{2k-1}$$

for which we can consider only the positive solution, because we require that  $d_j > 0$  for  $j \geq 1$ . Thus,

$$d_k = d_{k-1} \sqrt{\frac{2k+1}{2k-1}}$$

for  $2 \leq k \leq n$ . Since we require that  $d_0 = 1$ , this recursion actually cancels due to the fact that  $2k-1 = 2(k-1)+1$  and simply yields that  $d_k = \sqrt{2k+1}$ . Now, using our work above, we find that  $B$  has entries

$$B_{k,k+1} = \frac{k}{2k-1} \sqrt{\frac{2k-1}{2k+1}} = \frac{k}{\sqrt{(2k-1)(2k+1)}}$$

and of course,  $B_{k+1,k} = B_{k,k+1}$  due to symmetry. More explicitly we can reindex to find that

$$B_{k,k-1} = \frac{k-1}{\sqrt{(2k-3)(2k-1)}}.$$

Of course, the first equation hold for  $k = 1, \dots, n$ , the second holds for  $k = 2, \dots, n+1$ , and all other entries are zero.