AMATH 568 Homework 7

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1 Problem 1

Considering the ODE

$$y''(x;\lambda) + \left[\lambda\cos x - \lambda^2 \sin x\right] y(x;\lambda) = 0, \quad \lambda \to \infty,$$

we write it in the form

$$y''(x;\lambda) + f(x)y(x;\lambda) = 0, \quad \lambda \to \infty,$$

where

$$f(x) = \lambda^2 \sum_{n=0}^{\infty} f_n(x)\lambda^{-n}$$

with $f_0(x) = -\sin x$, $f_1(x) = \cos x$, and $f_j(x) = 0$ for $j \geq 2$. Then, we can apply the WKB method to find oscillatory and exponential solutions using (7.37) and (7.38) in the text.

$$y^{\pm}_{\text{osc}}(x) = \frac{1}{(-\sin x)^{1/4}} \exp\left(\pm i\lambda \int_{x_0}^{x} \sqrt{-\sin s}ds \pm i\frac{1}{2}\int_{x_0}^{x} \frac{\cos s}{\sqrt{-\sin s}}ds\right)(1+\text{o}(1))$$

$$y^{\pm}_{\text{exp}}(x) = \frac{1}{|\sin x|^{1/4}} \exp\left(\pm\lambda \int_{x_0}^{x} \sqrt{|\sin s|}ds \mp \frac{1}{2}\int_{x_0}^{x} \frac{\cos s}{\sqrt{|\sin s|}}ds\right)(1+\text{o}(1)).$$

We know that the first expansion is valid in a region $[\alpha,\beta]$ where $f_0(x) = -\sin x > 0$ and sufficiently far away from the turning points, i.e. for some region inside the interval such that $-\pi < \alpha \leq \beta < 0$ if we want to look at the region near $x = 0$ before the next sign change. Thus, we also take $x_0 \in [\alpha,\beta]$. Similarly, the second expansion is valid in some region inside the region where $f_0(x) = -\sin x < 0$ until the next turning point, i.e. in some region $[\alpha',\beta']$ inside the interval $(0,\pi)$; we also take $x_0 \in [\alpha',\beta']$.

2 Problem 2

2.1 Part a

Defining

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}, \quad n = 0, 1, 2, \ldots,$$

we first compute

$$H_n'(x) = 2x(-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} + (-1)^n e^{x^2} \frac{d^{n+1}}{dx^{n+1}} e^{-x^2} = 2xH_n(x) - H_{n+1}(x),$$

so

$$H_n''(x) = 2H_n(x) + 2xH_n'(x) - H_{n+1}'(x).$$

Then, we can reduce the differential equation to a recurrence relation as

$$\begin{aligned}
&H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 2H_n(x) + 2xH_n'(x) - H_{n+1}'(x) - 2xH_n'(x) + 2nH_n(x) \\
&= (2+2n)H_n(x) - H_{n+1}'(x) = 2(n+1)H_n(x) - 2xH_{n+1}(x) + H_{n+2}(x).
\end{aligned}$$

Now, we use the generalized Leibniz rule to compute

$$\begin{aligned}
\frac{d^{n+2}}{dx^{n+2}} e^{-x^2} &= \frac{d^{n+1}}{dx^{n+1}} \left(\frac{d}{dx} e^{-x^2} \right) = \frac{d^{n+1}}{dx^{n+1}} (-2xe^{-x^2}) \\
&= \sum_{k=0}^{n+1} \binom{n+1}{k} \frac{d^k}{dx^k}(-2x) \frac{d^{n+1-k}}{dx^{n+1-k}} e^{-x^2} = -2x \frac{d^{n+1}}{dx^{n+1}} e^{-x^2} - 2(n+1) \frac{d^n}{dx^n} e^{-x^2}.
\end{aligned}$$

From this we can use the definition of H_n to find that

$$\begin{aligned}
&2(n+1)H_n(x) - 2xH_{n+1}(x) + H_{n+2}(x) \\
&= 2(n+1)(-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} - 2x(-1)^{n+1} e^{x^2} \frac{d^{n+1}}{dx^{n+1}} e^{-x^2} + (-1)^{n+2} e^{x^2} \frac{d^{n+2}}{dx^{n+2}} e^{-x^2} \\
&= (-1)^n e^{x^2} \left(2(n+1) \frac{d^n}{dx^n} e^{-x^2} + 2x \frac{d^{n+1}}{dx^{n+1}} e^{-x^2} - 2x \frac{d^{n+1}}{dx^{n+1}} e^{-x^2} - 2(n+1) \frac{d^n}{dx^n} e^{-x^2} \right) \\
&= 0.
\end{aligned}$$

Thus, the Hermite polynomials satisfy the differential equation

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0.$$

2.2 Part b

Now, let $\psi_n(x) := (2^n n! \sqrt{\pi}) e^{-\frac{x^2}{2}} H_n(x)$. Then,

$$\begin{aligned}
&\psi_n'(x) = (2^n n! \sqrt{\pi})(-x) e^{-\frac{x^2}{2}} H_n(x) + (2^n n! \sqrt{\pi}) e^{-\frac{x^2}{2}} H_n'(x) \\
&\psi_n''(x) = (2^n n! \sqrt{\pi}) \left(-e^{-\frac{x^2}{2}} H_n(x) + x^2 e^{-\frac{x^2}{2}} H_n(x) - x e^{-\frac{x^2}{2}} H_n'(x) - x e^{-\frac{x^2}{2}} H_n'(x) + e^{-\frac{x^2}{2}} H_n''(x) \right),
\end{aligned}$$

so the differential equation

$$\psi_n''(x) + (2n+1-x^2)\psi_n(x) = (2^n n!\sqrt{\pi})\big(- e^{-\frac{x^2}{2}} H_n(x) + x^2 e^{-\frac{x^2}{2}} H_n(x) - xe^{-\frac{x^2}{2}} H_n'(x)$$
$$- xe^{-\frac{x^2}{2}} H_n'(x) + e^{-\frac{x^2}{2}} H_n''(x) + (2n+1-x^2)e^{-\frac{x^2}{2}} H_n(x)\big)$$
$$= \left(2^n n!\sqrt{\pi}e^{-\frac{x^2}{2}}\right)(2nH_n(x) - 2xH_n'(x) + H_n''(x)) = 0$$

by the differential equation from part a.

2.3 Part c

From our definitions in part b, define

$$\Psi_n(x) = \psi_n(cx)$$

where c is some undetermined constant. Then,

$$\Psi_n''(x) = c^2\psi_n''(cx),$$

so our differential equation from part b gives that

$$-\frac{1}{c^2}\Psi_n''(x) - (2n+1-c^2x^2)\Psi(x) = 0$$

which we can rewrite as

$$-\frac{1}{c^4}\Psi_n''(x) + x^2\Psi_n(x) = \frac{2n+1}{c^2}\Psi_n(x).$$

Now, we set

$$\frac{1}{c^4} = \frac{\hbar^2}{2},$$

so

$$c = \left(\frac{2}{\hbar^2}\right)^{1/4}.$$

Then, our equation becomes

$$-\frac{\hbar^2}{2}\Psi_n''(x) + x^2\Psi_n(x) = \pm\sqrt{\frac{\hbar^2}{2}}(2n+1)\Psi_n(x) = \pm\sqrt{2}\hbar\left(n+\frac{1}{2}\right)\Psi_n(x).$$

We need to choose the plus sign to ensure square integrability, so the operator $\mathcal{S}_\hbar = -\frac{\hbar^2}{2}\frac{d^2}{dx^2} + x^2$ has L^2 eigenvalues $E = \sqrt{2}\hbar\left(n+\frac{1}{2}\right)$ where $n = 0, 1, 2, \ldots$. Now, we attempt to verify (7.85) in the text by first computing $\phi(E)$ as defined on page 303. Here, we have that $V(x) = x^2$ so solving $x^2 - E = 0$ gives that $x_- = -\sqrt{E}$ and $x_+ = \sqrt{E}$, and

$$\phi(E) = \sqrt{2}\int_{-\sqrt{E}}^{\sqrt{E}} \sqrt{E - x^2}dx = \sqrt{2}\frac{\pi E}{2} = \frac{\pi}{\sqrt{2}}E.$$

Thus, (7.85) gives that

$$\frac{\pi}{\sqrt{2}}E = \pi\hbar\left(n + \frac{1}{2}\right),$$

meaning that

$$E = \sqrt{2}\hbar\left(n + \frac{1}{2}\right)$$

which matches what we already derived.

3 Problem 3

Consider the Airy equation

$$y''(x) - xy(x) = 0$$

as $x \to \infty$. We perform the substitution $x = \lambda^\alpha z$ where $\lambda \to \infty, \alpha > 0$ by letting $Y(z) = y(\lambda^\alpha z)$ so that $Y''(z) = \lambda^{2\alpha} y''(\lambda z)$ and

$$y''(x) - xy(x) = \lambda^{-2\alpha}Y''(z) - \lambda^\alpha zY(z) = 0$$

which we rewrite as

$$Y''(z) - \lambda^{3\alpha} zY(z) = 0.$$

To match the form we want for WKB, we need that $3\alpha = 2$, so $\alpha = 2/3$. Then, we can apply WKB to the new equation

$$Y''(z) - f(z)Y(z) = 0$$

where

$$f(z) = \lambda^2 \sum_{n=0}^{\infty} f_n(z)\lambda^{-n}$$

with $f_0(z) = -z$ and $f_j(z) = 0$ for $j \geq 1$. Then, we look for solutions of the form $Y(z) = e^{\phi(z)}$ by letting $u(z) = \phi'(z)$ and solving the resulting Riccati equation

$$u'(z) + u(z)^2 + f(z) = 0.$$

Then, following page 9 of part 5 of the lecture notes, we look for

$$u(z) \sim \lambda \sum_{n=0}^{\infty} u_n(z)\lambda^{-n}.$$

First, we need u_0 to solve

$$0 = u_0^2(z) + f_0(z) = u_0^2(z) - z.$$

We can consider $z > 0$ since we consider $x = \lambda^{2/3}z \to \infty$ and $\lambda \to \infty$, so we simply find that

$$u_0(z) = \pm z^{1/2}.$$

We then use our hierarchy of equations to compute

$$u_1(z) = \frac{-1}{2u_0(z)}(u_0'(z) + f_1(z)) = \frac{-1}{\pm 2z^{1/2}} \frac{\pm 1}{2z^{1/2}} = -\frac{1}{4z}$$

and

$$u_2(z) = \frac{-1}{2u_0(z)}(u_1'(z) + u_1(z)^2 + f_2(z)) = \frac{-1}{\pm 2z^{1/2}} \left(\frac{1}{4z^2} + \frac{1}{16z^2} \right) = \mp \frac{5}{32z^{5/2}},$$

so

$$u(z) \sim \pm z^{1/2}\lambda - \frac{1}{4z} \mp \frac{5}{32z^{5/2}\lambda} + \mathrm{O}(\lambda^{-2})$$

From this, we can compute

$$\begin{aligned}
\phi(z) &= \lambda \sum_{n=0}^{\infty} \int_{z_0}^{z} u_n(s) ds \lambda^{-n} = \pm\lambda \int_{z_0}^{z} s^{1/2} ds - \int_{z_0}^{z} \frac{ds}{4s} \mp \lambda^{-1} \int_{z_0}^{z} \frac{5}{32s^{5/2}} ds + \mathrm{O}(\lambda^{-2}) \\
&= \pm\lambda \left(\frac{2}{3} z^{3/2} - \frac{2}{3} z_0^{3/2} \right) - \left(\frac{1}{4} \log z - \frac{1}{4} \log z_0 \right) \mp \lambda^{-1} \left(-\frac{5}{48z^{3/2}} + \frac{5}{48z_0^{3/2}} \right) + \mathrm{O}(\lambda^{-2}) \\
&= \pm\lambda \frac{2}{3} z^{3/2} - \frac{1}{4} \log z \pm \lambda^{-1} \frac{5}{48z^{3/2}} + C_1\lambda + C_0 + C_{-1}\lambda^{-1} + \mathrm{O}(\lambda^{-2})
\end{aligned}$$

where C_1, C_0, C_{-1} are constants depending on our choice of z_0. Thus, we have that

$$Y(z) = \exp\left(\pm \frac{2}{3} (\lambda^{2/3} z)^{3/2} - \frac{1}{4} \log z \pm \frac{5}{48(\lambda^{2/3} z)^{3/2}} + C_1\lambda + C_0 + C_{-1}\lambda^{-1} + \mathrm{O}(\lambda^{-2}) \right).$$

Undoing our substitution, we find

$$\begin{aligned}
y(x) &= \exp\left(\pm \frac{2}{3} x^{3/2} - \frac{1}{4} \log \lambda^{-2/3} x \pm \frac{5}{48x^{3/2}} + C_1\lambda + C_0 + C_{-1}\lambda^{-1} + \mathrm{O}(x^{-3}) \right) \\
&= C x^{-1/4} \exp\left(\pm \frac{2}{3} x^{3/2} \right) \exp\left(\pm \frac{5}{48x^{3/2}} + \mathrm{O}(x^{-3}) \right)
\end{aligned}$$

where we have grouped C_0 and all terms depending on λ into a single constant C also depending on λ. Taylor expanding,

$$y(x) = C x^{-1/4} \exp\left(\pm \frac{2}{3} x^{3/2} \right) \sum_{k=0}^{\infty} \frac{\left(\pm \frac{5}{48x^{3/2}} + \mathrm{O}(x^{-3}) \right)^k}{k!}.$$

Taking the 0th and 1st terms of this power series,

$$y(x) \sim C x^{-1/4} \exp\left(\pm \frac{2}{3} x^{3/2} \right) \left(1 \pm \frac{5}{48x^{3/2}} + \mathrm{O}(x^{-3}) \right).$$

To match this with the expansion on DLMF, we need to take the minus sign and let $C = \frac{1}{2\sqrt{\pi}}$ (which amounts to choosing z_0 such that this holds). Then,

$$y(x) \sim \frac{1}{2\sqrt{\pi} x^{1/4}} e^{-\frac{2}{3} x^{3/2}} \left(1 - \frac{5}{48x^{3/2}} + \mathrm{O}(x^{-3}) \right).$$

To verify that this is the same as DLMF, we compute

$$u_1 = \frac{15}{216} u_0 = \frac{5}{72}$$

so

$$\mathrm{Ai}(x) \sim \frac{e^{-\frac{2}{3}x^{3/2}}}{2\sqrt{\pi} x^{1/4}} \sum_{k=0}^{\infty} (-1)^k \frac{u_k}{\left(\frac{2}{3}x^{3/2}\right)^k} = x^{-1/4} \exp\left(\pm \frac{2}{3} x^{3/2}\right) \left(1 \pm \frac{5}{48 x^{3/2}} + \mathrm{O}(x^{-3})\right)$$

which is precisely what we obtained above.