

# AMATH 570 Homework 3

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## 1 Problem 1

### 1.1 Exercise 5.1

#### 1.1.1 Part a

Given interpolation points  $x_0 = -1$ ,  $x_1 = 0$ ,  $x_2 = 1/2$ , and  $x_3 = 1$ , we find the barycentric interpolation coefficients.

$$\begin{aligned}\lambda_0 &= \frac{1}{(-1)(-1 - 1/2)(-1 - 1)} = \frac{-1}{3} \\ \lambda_1 &= \frac{1}{(1)(-1/2)(-1)} = 2 \\ \lambda_2 &= \frac{1}{(1/2 + 1)(1/2)(1/2 - 1)} = \frac{-8}{3} \\ \lambda_3 &= \frac{1}{(1 + 1)(1)(1 - 1/2)} = 1\end{aligned}$$

#### 1.1.2 Part b

Now, we verify that (5.9) in ATAP gives the same result as standard polynomial interpolation via (5.3) at  $x = -1/2$ . First, compute

$$\ell(-1/2) = (-1/2 + 1)(-1/2)(-1/2 - 1/2)(-1/2 - 1) = \frac{-3}{8}.$$

Also,

$$\begin{aligned}\frac{\lambda_0}{-1/2 - x_0} &= \frac{-1/3}{1/2} = \frac{-2}{3} \\ \frac{\lambda_1}{-1/2 - x_1} &= \frac{2}{-1/2} = -4 \\ \frac{\lambda_2}{-1/2 - x_2} &= \frac{-8/3}{-1} = \frac{8}{3} \\ \frac{\lambda_3}{-1/2 - x_3} &= \frac{1}{-3/2} = \frac{-2}{3}\end{aligned}$$

Finally, we get that

$$\begin{aligned}\ell_0(-1/2) &= \frac{-3}{8} \frac{-2}{3} = \frac{1}{4} \\ \ell_1(-1/2) &= \frac{-3}{8} (-4) = \frac{3}{2} \\ \ell_2(-1/2) &= \frac{-3}{8} \frac{8}{3} = -1 \\ \ell_3(-1/2) &= \frac{-3}{8} \frac{-2}{3} = \frac{1}{4}.\end{aligned}$$

Now, we compute these same values in the normal manner

$$\begin{aligned}\ell_0(-1/2) &= \frac{(-1/2)(-1/2 - 1/2)(-1/2 - 1)}{(-1)(-1 - 1/2)(-1 - 1)} = \frac{-3/4}{-3} = \frac{1}{4} \\ \ell_1(-1/2) &= \frac{(-1/2 + 1)(-1/2 - 1/2)(-1/2 - 1)}{(1)(-1/2)(-1)} = \frac{3/4}{1/2} = \frac{3}{2} \\ \ell_2(-1/2) &= \frac{(-1/2 + 1)(-1/2)(-1/2 - 1)}{(1/2 + 1)(1/2)(1/2 - 1)} = \frac{3/8}{-3/8} = -1 \\ \ell_3(-1/2) &= \frac{(-1/2 + 1)(-1/2)(-1/2 - 1/2)}{(1 + 1)(1)(1 - 1/2)} = \frac{1/4}{1} = \frac{1}{4}.\end{aligned}$$

Clearly, these values are identical.

## 1.2 Exercise 5.2

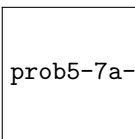
For the function  $f(x) = \cos kx$ , we use MATLAB to compute the value of various interpolants (chebfun, polyfit, and polyfitA) as well as the condition number of Vandermonde matrix used by polyfit for various values of k and obtain the results

	Chebfun	polyfit	polyfitA	condition number
k=10:	1	1.000000e+00	1.000000e+00	8.506226e+12
k=20:	1.000000e+00	1.000000e+00	1.000000e+00	6.676559e+16
k=30:	1.000000e+00	9.999932e-01	1	2.615273e+17
k=40:	1.000000e+00	9.919971e-01	1.000000e+00	6.059928e+18
k=50:	1.000000e+00	5.934477e-01	1.000000e+00	6.469850e+18
k=60:	1.000000e+00	-2.137573e-01	1.000000e+00	7.351968e+18
k=70:	1.000000e+00	-1.448931e-02	1.000000e+00	8.177436e+18
k=80:	1.000000e+00	3.892366e-02	1.000000e+00	1.005588e+19
k=90:	1.000000e+00	4.893440e-02	1.000000e+00	2.872486e+19
k=100:	1.000000e+00	2.043693e-02	1.000000e+00	7.184542e+18

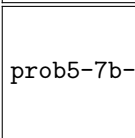
See problem5.2.m for the code that produces this table.

## 1.3 Exercise 5.7

Using a greedy algorithm to determine the interpolation points for the function  $f(x) = |x|$ , we obtain the following error plots



prob5-7a-eps-converted-to.pdf

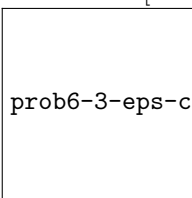


prob5-7b-eps-converted-to.pdf

where the first includes the error plots for all interpolants used throughout the algorithm, and the second is just the final error plot for  $n = 25$ . Looking at the spacing of the grid, the chosen points look quite similar to the Chebyshev points, but as one can check using `chebfun`, are not the Chebyshev points exactly. See `problem5_7.m` for the code that does this.

## 2 Problem 2 (Exercise 6.3)

We compute the Chebyshev interpolant of  $f(x) = \sin(1/x) \sin(1/\sin(1/x))$  on the interval  $[0.07, 0.4]$  for various values of  $n$  and plot the errors.



prob6-3-eps-converted-to.pdf

We can see that the convergence appears to be roughly  $O(n^{-1/3})$  which implies that in order to get accuracy around  $10^{-16}$ , we need  $n$  to roughly be  $10^{48}$ . See `problem6_3.m` for the code that does this.

## 3 Problem 3 (Exercise 7.1)

### 3.1 Part a

Using `chebfun` to compute the total variation of  $f(x) = \sin(100x)/(1+x^2)$  on  $[-1, 1]$ , we get that it is roughly 99.836. See `problem7_1.m` for the code that does this.

### 3.2 Part b

To see that the total variation of  $f(x) = \sin(Mx)/(1+x^2)$  on  $[-1, 1]$  is asymptotic to  $M$  as  $M \rightarrow \infty$ , first write out the definition of total variation

$$\int_{-1}^1 \left| \left( \frac{\sin(Mx)}{1+x^2} \right)' \right| dx = \int_{-1}^1 \left| \frac{(1+x^2)M \cos(Mx) - 2x \sin(Mx)}{(1+x^2)^2} \right| dx$$

. Now, consider that

$$\int_{-1}^1 \left| \frac{2x \sin(Mx)}{1+x^2} \right| dx \leq \int_{-1}^1 \left| \frac{2x}{1+x^2} \right| dx = 1$$

because  $|\sin(Mx)| \leq 1$ . Thus, this term becomes negligible as  $M \rightarrow \infty$ , so we can consider only

$$M \int_{-1}^1 \left| \frac{\cos(Mx)}{1+x^2} \right| dx$$

The key observation here is that  $|\cos(Mx)|$  rapidly oscillates as  $M \rightarrow \infty$ , so its average over any region will tend to its average over a period. Over a given period, it will have average

$$\frac{1}{\pi/M} \int_{-\pi/2M}^{\pi/2M} |\cos(Mx)| dx = \frac{M}{\pi} \left[ \frac{\sin(Mx)}{M} \right]_{-\pi/2M}^{\pi/2M} = \frac{2}{\pi}.$$

From this, we can infer that the total variation as  $M \rightarrow \infty$  looks like

$$M \int_{-1}^1 \frac{2/\pi}{1+x^2} dx = \frac{2M}{\pi} \int_{-1}^1 \frac{dx}{1+x^2} = \frac{2M}{\pi} \frac{\pi}{2} = M.$$

## 4 Problem 4

### 4.1 Exercise 8.1

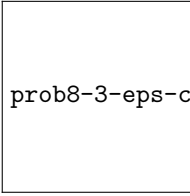
Consider a Bernstein ellipse  $E_\rho$  where  $\rho > 1$ . We know that  $E_\rho$  is the circle  $z = \rho e^{i\theta}$  under the Joukowski map, so to find the rightmost endpoint of the ellipse, we need to find the  $\theta$  that maximizes the real part of  $\frac{1}{2}(\rho e^{i\theta} + \frac{1}{\rho e^{i\theta}})$ . This is precisely  $\theta = 0$ , because that is the value of  $\theta$  (ignoring periodicity) that maximizes the real part of  $\rho e^{i\theta}$ . Thus, the rightmost endpoint is given by  $\frac{1}{2}(\rho + \frac{1}{\rho})$ . Similarly, the uppermost endpoint of the ellipse is given by the point that maximizes the imaginary part which is  $\theta = \pi/2$ , so the uppermost endpoint is given by  $\frac{i}{2}(\rho - \frac{1}{\rho})$ . Of course, the ellipse is centered at the origin, so clearly the length of the semimajor axis is  $\frac{1}{2}(\rho + \frac{1}{\rho})$  and the length of the semiminor axis is  $\frac{1}{2}(\rho - \frac{1}{\rho})$ , so their sum is  $\rho$ .

### 4.2 Exercise 8.3

Consider  $f(x) = \exp(-x^2)$ . Because it is entire, we can apply Theorem 8.2 for any positive  $\rho$ , but must choose  $M$  to be the maximum value that  $f$  takes on  $E_\rho$ . For this choice of  $f$ , this occurs on the top (or bottom) of the ellipse, so we take

$$M = e^{-(i\frac{\rho-\rho^{-1}}{2})^2} = e^{(\frac{\rho-\rho^{-1}}{2})^2}.$$

Plotting error against the bound given in the theorem for various values of  $\rho$ , we get



For high values of  $\rho$ , this bound is pretty tight, but the bound does not fit the data very well for low values of  $\rho$ .

## 5 Problem 5

Assume  $0 < m < M$  and let  $\kappa = M/m$ . Consider the  $k$ th scaled and shifted Chebyshev polynomial:

$$T_k \left( \frac{2x - M - m}{M - m} \right) / T_k \left( \frac{-M - m}{M - m} \right).$$

The numerator is shifted so that the interval  $[m, M]$  maps linearly to  $[-1, 1]$ . We know that a Chebyshev polynomial  $T_k$  satisfies  $-1 \leq T_k(x) \leq 1$  for  $x \in [-1, 1]$ , so we can bound  $\left| T_k \left( \frac{2x - M - m}{M - m} \right) \right| \leq 1$  for  $x \in [m, M]$ . Looking at the denominator,

$$T_k \left( \frac{-M - m}{M - m} \right) = T_k \left( -\frac{\frac{M}{m} + 1}{\frac{M}{m} - 1} \right) = T_k \left( -\frac{\kappa + 1}{\kappa - 1} \right).$$

Per the hint, we now look to find a  $z$  such that  $\frac{1}{2}(z + z^{-1}) = -\frac{\kappa + 1}{\kappa - 1}$ . This yields a quadratic equation  $z^2 + 2\frac{\kappa + 1}{\kappa - 1}z + 1 = 0$  which has solution

$$z = -\frac{\kappa + 1}{\kappa - 1} \pm \sqrt{\left( \frac{\kappa + 1}{\kappa - 1} \right)^2 - 1} = -\frac{\kappa + 1}{\kappa - 1} \pm \frac{2\sqrt{\kappa}}{\kappa - 1} = -\frac{(\sqrt{\kappa} \pm 1)^2}{(\sqrt{\kappa} + 1)(\sqrt{\kappa} - 1)} = -\frac{\sqrt{\kappa} \pm 1}{\sqrt{\kappa} \mp 1}$$

Note that for one choice of  $z$ ,  $z^{-1}$  yields the other, so we get that

$$\begin{aligned} T_k \left| \left( \frac{-M - m}{M - m} \right) \right| &= \left| T_k \left( -\frac{1}{2} \left( \frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1} + \frac{\sqrt{\kappa} + 1}{\sqrt{\kappa} - 1} \right) \right) \right| \\ &= \left| T_k \left( \frac{1}{2} \left( \frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1} + \frac{\sqrt{\kappa} + 1}{\sqrt{\kappa} - 1} \right) \right) \right| \\ &= \frac{1}{2} \left( \left( \frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1} \right)^k + \left( \frac{\sqrt{\kappa} + 1}{\sqrt{\kappa} - 1} \right)^k \right) \end{aligned}$$

by the second part of the hint. Combined with the numerator, this gives that the absolute value polynomial is bounded by

$$\left| T_k \left( \frac{2x - M - m}{M - m} \right) / T_k \left( \frac{-M - m}{M - m} \right) \right| \leq 2 \left( \left( \frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1} \right)^k + \left( \frac{\sqrt{\kappa} + 1}{\sqrt{\kappa} - 1} \right)^k \right)^{-1}$$

for  $x \in [m, M]$ .