MATH 524 Homework 1

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1 Problem 1

Let $\mathbb{Z}_q^{\mathbb{N}}$ denote the collection of infinite sequences $a = a_1a_2a_3\ldots$ with $a_j \in \mathbb{Z}_q = \{0, 1, \ldots, q-1\}$ and define

$$\rho(a, b) = \sum_{j=1}^{\infty} \frac{|a_j - b_j|}{q^j}.$$

1.1 Part a

To show that ρ is a metric on $\mathbb{Z}_q^{\mathbb{N}}$, we first note that it is well-defined (i.e., the sum converges) by noting that for $a, b \in \mathbb{Z}_q^{\mathbb{N}}$ and $q > 1$,

$$\rho(a, b) = \sum_{j=1}^{\infty} \frac{|a_j - b_j|}{q^j} \le \sum_{j=1}^{\infty} \frac{q-1}{q^j} = \sum_{j=1}^{\infty} \frac{1}{q^{j-1}} - \sum_{j=1}^{\infty} \frac{1}{q^j} = 1,$$

and that $a = b = 00\ldots$ for all $a, b \in \mathbb{Z}_1^{\mathbb{N}}$, so the sum is always zero in this case as the space contains only the zero element.

We now verify the required axioms.

- If $a, b \in \mathbb{Z}_q^{\mathbb{N}}$, then $\rho(a, b) \ge 0$ because each term in the infinite sum that defines it is nonnegative since $q > 0$, and the absolute value function is nonnegative. Furthermore, we have that

$$\rho(a, a) = \sum_{j=1}^{\infty} \frac{|a_j - a_j|}{q^j} = 0,$$

  and if $\rho(a, b) = 0$, then because each term of the sum is nonnegative, they must all be zero, so $a_j = b_j$ for all $j \in \mathbb{N}$. Thus, $a = b$.

- If $a, b \in \mathbb{Z}_q^{\mathbb{N}}$, then

$$\rho(a, b) = \sum_{j=1}^{\infty} \frac{|a_j - b_j|}{q^j} = \sum_{j=1}^{\infty} \frac{|b_j - a_j|}{q^j} = \rho(b, a).$$

- To show the triangle inequality we let $a, b, c \in \mathbb{Z}_q^{\mathbb{N}}$. Then, by the triangle inequality for the absolute value,

$$\rho(a, b) = \sum_{j=1}^{\infty} \frac{|a_j - b_j|}{q^j} \le \sum_{j=1}^{\infty} \frac{|a_j - c_j|}{q^j} + \sum_{j=1}^{\infty} \frac{|c_j - b_j|}{q^j} = \rho(a, c) + \rho(c, b),$$

  where we can split the sums because they converge.

1.2 Part b

Consider a sequence of elements $\{a_n\} \subset \mathbb{Z}_q^{\mathbb{N}}$ such that a_j^n is constant in n for $n \geq N$ for each j. This sequence is trivially Cauchy. More specifically, if we fix $\epsilon > 0$, then for an $n, m \geq N$,

$$\rho(a^n, a^m) = \sum_{j=1}^{\infty} \frac{|a_j^n - a_j^m|}{q^j} = 0 < \epsilon.$$

To show that all Cauchy sequences must have this property, say that there exists a sequence that does not, i.e., there exists $\{a_n\} \subset \mathbb{Z}_q^{\mathbb{N}}$ such that for any $N \in \mathbb{N}$, there exist $n, m \geq N$ such that $a_k^n \neq a_k^m$ for some k. Then,

$$\rho(a^n, a^m) = \sum_{j=1}^{\infty} \frac{|a_j^n - a_j^m|}{q^j} \geq \frac{|a_k^n - a_k^m|}{q^k} \geq \frac{1}{q^k}.$$

This means that there is no $N \in \mathbb{N}$ such that $\rho(a^n, a^m) < \epsilon$ for $n, m \geq N$ when $\epsilon \geq \frac{1}{q^k}$, so the sequence cannot be Cauchy. Thus, sequences in $\mathbb{Z}_q^{\mathbb{N}}$ are Cauchy iff they have this property.

To see that $(\mathbb{Z}_q^{\mathbb{N}}, \rho)$ is complete, let $\{a_n\} \subset \mathbb{Z}_q^{\mathbb{N}}$ be Cauchy, and let $a_j = a_j^n$ for each j for all $n \geq N$. Then, for any $\epsilon > 0$,

$$\rho(a, a^n) = \sum_{j=1}^{\infty} \frac{|a_j^n - a_j^m|}{q^j} = 0 < \epsilon,$$

for all $n \geq N$, so $\{a_n\}$ converges to a. Clearly, $a \in \mathbb{Z}_q^{\mathbb{N}}$, so all Cauchy sequences must converge.

1.3 Part c

To see that $(\mathbb{Z}_q^{\mathbb{N}}, \rho)$ is totally bounded, fix $\epsilon > 0$ and take $k \in \mathbb{N}$ sufficently large such that $\epsilon > \frac{1}{q^k}$. Define

$$Z = \{a \in \mathbb{Z}_q^{\mathbb{N}} \mid 0 = a_{k+1} = a_{k+2} = \ldots\},$$

which is clearly a finite set. Given $a \in \mathbb{Z}_q^{\mathbb{N}}$, let a^* be the element of Z such that $a_j = a_j^*$ for all $j \leq k$. Then,

$$\rho(a, a^*) = \sum_{j=k+1}^{\infty} \frac{|a_j|}{q^j} = \frac{1}{q^k} \sum_{j=1}^{\infty} \frac{|a_j|}{q^j} \leq \frac{1}{q^k} \sum_{j=1}^{\infty} \frac{q-1}{q^j} = \frac{1}{q^k} < \epsilon.$$

Thus,

$$\mathbb{Z}_q^{\mathbb{N}} \subset \bigcup_{z \in Z} \mathcal{B}_\epsilon(z),$$

so $\mathbb{Z}_q^{\mathbb{N}}$ is totally bounded.

2 Problem 2

For $n \geq 1$, define the projections $\Pi_n : \mathbb{Z}_q^{\mathbb{N}} \to \mathbb{Z}_q^n$ by the rule

$$\Pi_n(a_1 a_2 a_3 \ldots) = (a_1, a_2, \ldots, a_n).$$

2.1 Part a

Let $\mathcal{A}$ be the collection of sets of the form $\Pi_n^{-1}(E)$ for $n \in \mathbb{N}$ and $E \subset \mathbb{Z}_q^{\mathbb{N}}$. To show that $\mathcal{A}$ is an algebra, we verify the required axioms.

- $\emptyset \subset \mathbb{Z}_q^n$, so $\Pi_n^{-1}(\emptyset) = \emptyset \in \mathcal{A}$.

- Let $F_1, F_2 \in \mathcal{A}$ where $F_1 = \Pi_{n_1}^{-1}(E_1)$, $F_2 = \Pi_{n_2}^{-1}(E_2)$, $E_1 \subset \mathbb{Z}_q^{n_1}$, $E_2 \subset \mathbb{Z}_q^{n_2}$. Assume WLOG that $n_1 \leq n_2$ and define
$$E_3 = \{a \in \mathbb{Z}_q^{n_2} \mid (a_1, \ldots, a_{n_1}) \in E_1\} \subset \mathbb{Z}_q^{n_2}.$$
Then,
$$F_1 \cup F_2 = \Pi_{n_2}^{-1}(E_2 \cup E_3) \in \mathcal{A},$$
so $\mathcal{A}$ is closed under finite unions.

- To show that $\mathcal{A}$ is closed under compliments, let $A \in \mathcal{A}$ such that $A = \Pi_n^{-1}(E)$, $E \subset \mathbb{Z}_q^n$. Then,
$$A^c = \{a \in \mathbb{Z}_q^{\mathbb{N}} \mid (a_1, \ldots, a_n) \in E^c\} \in \mathcal{A},$$
since $E^c = \mathbb{Z}_q^n \setminus E \subset \mathbb{Z}_q^n$.

2.2 Part b

Let $A \in \mathcal{A}$ such that $A = \Pi_n^{-1}(E)$, $E \subset \mathbb{Z}_q^n$. To see that A is open, let $a \in A$, $r < \frac{1}{q^n}$, and $b \in \mathcal{B}_r(a)$. If $a_k \neq b_k$ for some $k \leq n$, then
$$\rho(a,b) \geq \frac{a_k - b_k}{q^k} \geq \frac{1}{q^k} \geq \frac{1}{q^n} > r,$$
so it it must hold that $b \in \mathcal{B}_r(a)$ only if $a_k = b_k$ for $k \leq n$, i.e., $\Pi_n(b) \in E$, so $b \in A$. Thus, $\mathcal{B}_r(a) \subset A$, so A must be open.

To see that A is closed, note that since $\mathcal{A}$ is an algebra, $A^c \in \mathcal{A}$. Therefore, A^c is open, so A must be closed.

2.3 Part c

To prove that the σ-algebra generated by $\mathcal{A}$ is the Borel σ-algebra on $Z_q^{\mathbb{N}}$, we show that all open sets in X are contained in a countable union of elements of $\mathcal{A}$. In this case, all open sets are contained in $\mathcal{M}(\mathcal{A})$, so the Borel σ-algebra is contained in $\mathcal{M}(\mathcal{A})$ by Folland Lemma 1.1. Since all elements of $\mathcal{A}$ are open, it follows that $\mathcal{M}(\mathcal{A})$ is precisely the Borel σ-algebra in this case.

To show this, let $B \in \mathbb{Z}_q^{\mathbb{N}}$ be an open set, and let $a \in B$. Then, $\mathcal{B}_\epsilon(a) \subset B$ for some $\epsilon > 0$. Let $n \in \mathbb{N}$ be chosen such that $\epsilon > \frac{1}{q^n}$. Define $A \in \mathcal{A}$ by
$$A = \{b \in \mathbb{Z}_q^{\mathbb{N}} \mid \Pi_n(b) = a\}.$$
Then, for every $b \in A$,
$$\rho(a,b) = \sum_{j=n+1}^{\infty} \frac{|a_j - b_j|}{q^j} = \frac{1}{q^n} \sum_{j=1}^{\infty} \frac{|a_j - b_j|}{q^j} \leq \frac{1}{q^n} \sum_{j=1}^{\infty} \frac{q-1}{q^j} = \frac{1}{q^n} < \epsilon,$$
so $b \in \mathcal{B}_\epsilon(a)$ and $A \subset \mathcal{B}_\epsilon(a) \subset B$. Because this can be done for each $a \in B$, it must then hold that
$$B = \bigcup_{A_j \in \mathcal{A}, A_j \subset B} A_j,$$
and this union is countable because $\mathcal{A}$ is countable. Thus, the σ-algebra generated by $\mathcal{A}$ is the Borel σ-algebra on $Z_q^{\mathbb{N}}$.

3 Problem 3 (Folland problem 1)

3.1 Part a

Let $\mathcal{R} \subset \mathcal{P}(X)$ be a (σ-)ring. To see that (σ-)rings are closed under finite (countable) intersections, let $E_j \in \mathcal{R}$ for $j = 1, \ldots, n$ ($j = 1, 2, \ldots$). Then,
$$\bigcap_{j=1}^{n} E_j = E_1 \cap \left(\bigcap_{j=2}^{n} E_j \right) = E_1 \setminus \left(\bigcap_{j=2}^{n} E_j \right)^c = E_1 \setminus \left(\bigcup_{j=2}^{n} E_j \right) \in \mathcal{R}.$$

Note that the (σ-)ring case follows from simply replacing the finite unions/intersections with countably infinite ones.

3.2 Part b

Let $\mathcal{R} \subset \mathcal{P}(X)$ be a (σ-)ring. To see that $\mathcal{R}$ is an (σ-)algebra if $X \in \mathcal{R}$, we verify the required axioms.

- $\emptyset = X \setminus X \in \mathcal{R}$.
- $\mathcal{R}$ is closed under finite (countable) unions by the definition of a (σ-)ring.
- If $E \in \mathcal{R}$, then $E^c = X \setminus E \in \mathcal{R}$.

Thus, $\mathcal{R}$ is an (σ-)algebra.

On the other hand, if $\mathcal{R}$ is also an (σ-)algebra, then by the definition of an (σ-)algebra, $X \in \mathcal{R}$. Thus, $\mathcal{R}$ is an (σ-)algebra iff $X \in \mathcal{R}$.

3.3 Part c

Let $\mathcal{R} \subset \mathcal{P}(X)$ be a σ-ring and define $\mathcal{A} = \{E \subset X \mid E \in \mathcal{R} \text{ or } E^c \in \mathcal{R}\}$. To see that $\mathcal{A}$ is a σ-algebra, we verify the required axioms.

- For any $E \in \mathcal{R}$,
$$\emptyset = E \setminus E \in \mathcal{R},$$
so $\emptyset \in \mathcal{A}$.
- To see that $\mathcal{A}$ is closed under countable intersections, let $E_j \in \mathcal{A}$ for $j \in \mathbb{N}$, and let $J = \{j \in \mathbb{N} \mid E_j \in \mathcal{R}\}$. Note that $E_j^c \in \mathcal{R}$ for $j \in \mathbb{N} \setminus J$. Then,
$$\bigcap_{j=1}^{\infty} E_j = \left(\bigcap_{j \in J} E_j\right) \cap \left(\bigcap_{j \in \mathbb{N}\setminus J} E_j\right) = \left(\bigcap_{j \in J} E_j\right) \cap \left(\bigcup_{j \in \mathbb{N}\setminus J} E_j^c\right)^c = \left(\bigcap_{j \in J} E_j\right) \setminus \left(\bigcup_{j \in \mathbb{N}\setminus J} E_j^c\right) \in \mathcal{R},$$
since σ-rings are closed under countable unions and intersections and differences. Thus, $\bigcup_{j=1}^{\infty} E_j \in \mathcal{A}$.
- To see that $\mathcal{A}$ is closed under compliments, let $E \in \mathcal{A}$. If $E \in \mathcal{R}$, then $(E^c)^c \in \mathcal{R}$, so $E^c \in \mathcal{A}$. If $E^c \in \mathcal{R}$, then $E^c \in \mathcal{A}$.

Thus, $\mathcal{A}$ is a σ-algebra.

3.4 Part d

Let $\mathcal{R} \subset \mathcal{P}(X)$ be a σ-ring and define $\mathcal{A} = \{E \subset X \mid E \cap F \in \mathcal{R} \text{ for all } F \in \mathcal{R}\}$. To see that $\mathcal{A}$ is a σ-algebra, we verify the required axioms.

- For any $E \in \mathcal{R}$,
$$\emptyset = E \setminus E \in \mathcal{R},$$
so $\emptyset \in \mathcal{A}$.
- To see that $\mathcal{A}$ is closed under countable intersections, let $E_j \in \mathcal{A}$ for $j \in \mathbb{N}$. Then, for any $F \in \mathcal{R}$,
$$\bigcup_{j=1}^{\infty} (E_j \cap F) = \left(\bigcup_{j=1}^{\infty} E_j\right) \cap F \in \mathcal{R},$$
so $\bigcup_{j=1}^{\infty} E_j \in \mathcal{A}$.
- To see that $\mathcal{A}$ is closed under compliments, let $E \in \mathcal{A}$. Then, for any $F \in \mathcal{R}$,
$$\mathcal{R} \ni F \setminus (E \cap F) = F \cap (E \cap F)^c = F \cap (E^c \cup F^c) = E^c \cap F,$$
so $E^c \in \mathcal{A}$.

Thus, $\mathcal{A}$ is a σ-algebra.

4 Problem 4 (Folland problem 8)

Let $(X, \mathcal{M}, \mu)$ be a measure space and $\{E_j\}_{j=1}^{\infty} \subset \mathcal{M}$. Then, noting that $\bigcap_{n=k}^{\infty} E_n \subset \bigcap_{n=k+1}^{\infty} E_n$, continuity from below implies that

$$\mu(\liminf E_j) = \mu\left(\bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty} E_n\right) = \lim_{k\to\infty} \mu\left(\bigcap_{n=k}^{\infty} E_n\right).$$

Noting that $\bigcap_{n=k}^{\infty} E_n \subset E_{n^*}$ for any $n^* \geq k$, monotonicity implies that

$$\mu(\liminf E_j) = \lim_{k\to\infty} \mu\left(\bigcap_{n=k}^{\infty} E_n\right) \leq \lim_{k\to\infty} \inf_{n\geq k} \mu(E_n) = \liminf \mu(E_j).$$

Similarly, if $\mu\left(\bigcup_{j=1}^{\infty} E_j\right) < \infty$, continuity from above implies that

$$\mu(\limsup E_j) = \mu\left(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} E_n\right) = \lim_{k\to\infty} \mu\left(\bigcup_{n=k}^{\infty} E_n\right).$$

since $\bigcup_{n=k+1}^{\infty} E_n \subset \bigcup_{n=k}^{\infty} E_n$. Noting that $E_{n^*} \subset \bigcup_{n=k}^{\infty} E_n$ for any $n^* \geq k$, monotonicity implies that

$$\mu(\limsup E_j) = \lim_{k\to\infty} \mu\left(\bigcup_{n=k}^{\infty} E_n\right) \leq \lim_{k\to\infty} \sup_{n\geq k} \mu(E_n) = \limsup \mu(E_j).$$

5 Problem 5 (Folland problem 10)

Let $(X, \mathcal{M}, \mu)$ be a measure space and fix $E \in \mathcal{M}$. Define $\mu_E(A) = \mu(A \cap E)$. To see that μ_E is a measure, we verify the required axioms.

- $$\mu_E(\emptyset) = \mu(\emptyset \cap E) = \mu(\emptyset) = 0.$$
- Let $E_1, E_2, \ldots \in \mathcal{M}$ be disjoint. Then,

$$\mu_E\left(\bigsqcup_{j=1}^{\infty} E_j\right) = \mu\left(\left(\bigsqcup_{j=1}^{\infty} E_j\right) \cap E\right) = \mu\left(\bigsqcup_{j=1}^{\infty} (E_j \cap E)\right) = \sum_{j=1}^{\infty} \mu(E_j \cap E) = \sum_{j=1}^{\infty} \mu_E(E_j).$$

Thus, μ_E is a measure.