AMATH 569 Homework 2

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1 Problem 1

Consider the PDE

$$\begin{aligned} &\nabla^2 u = 0, \quad y > 0, \ -\infty < x < \infty \\ &u(x,0) = f(x), \quad -\infty < x < \infty \end{aligned}$$

and assume that f is of compact support and $u(x,t) \to 0$ as $x \to \pm\infty$. Let $\mathcal{F}$ denote the Fourier transform applied in the x domain and let

$$U(x,y) = \mathcal{F}[u(x,y)].$$

Then,

$$\begin{aligned} \mathcal{F}[u_{xx}] &= \int_{-\infty}^{\infty} u_{xx} e^{i\omega x} dx = \left[u_x e^{i\omega x}\right]_{-\infty}^{\infty} - i\omega \int_{-\infty}^{\infty} u_x e^{i\omega x} dx \\ &= \left[u_x e^{i\omega x}\right]_{-\infty}^{\infty} - i\omega \cancel{\left[u e^{i\omega x}\right]_{-\infty}^{\infty}} - \omega^2 \int_{-\infty}^{\infty} u e^{i\omega x} dx \\ &= -\omega^2 \int_{-\infty}^{\infty} u e^{i\omega x} dx = -\omega^2 U \end{aligned}$$

if we make the additional assumption that $u_x(x,t) \to 0$ as $x \to \pm\infty$. We also have that

$$\mathcal{F}[u_{yy}] = \int_{-\infty}^{\infty} u_{yy} e^{i\omega x} dx = \frac{\partial^2}{\partial y^2} \int_{-\infty}^{\infty} u e^{i\omega x} dx = U_{yy}.$$

Thus, Fourier transforming our equation gives new equation

$$-\omega^2 U(x,y) + U_{yy}(x,y) = 0.$$

This is an ODE that can easily be solved to have general solution

$$U(x,y) = c_1(\omega) e^{|\omega| y} + c_2(\omega) e^{-|\omega| y}.$$

To find a particular solution, note that we now have initial condition

$$U(\omega,0) = \mathcal{F}[u(x,0)] = F(\omega)$$

if we set $F(\omega) = \mathcal{F}[f(x)]$. We now also make the assumption that $u(x, y) \to 0$ as $y \to \infty$ which implies that $U(\omega, y) \to 0$ as $y \to \infty$. The latter requires that $c_1(x) = 0$ which means that former then gives that $c_2(\omega) = F(\omega)$. Thus,

$$U(x, y) = F(\omega)e^{-|\omega|y}.$$

Our solution u is then obtained by applying the inverse Fourier transform

$$u(x, y) = \mathcal{F}^{-1}[U(\omega, y)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{-|\omega|y}e^{-i\omega x}d\omega.$$

If we wish to obtain a simplified solution, we need to apply the convolution theorem which says that

$$\mathcal{F}^{-1}[gh] = \mathcal{F}^{-1}[g] * \mathcal{F}^{-1}[h]$$

for arbitrary functions g, h. Thus,

$$u(x, y) = \mathcal{F}^{-1}[F(\omega)] * \mathcal{F}^{-1}[e^{-|\omega y|}] = f(x) * \mathcal{F}^{-1}[e^{-|\omega y|}].$$

We find that

$$\begin{aligned}\mathcal{F}^{-1}[e^{-|\omega y|}] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-|\omega|y}e^{-i\omega x}d\omega = \frac{1}{2\pi}\left(\int_{-\infty}^{0} e^{\omega(y-ix)}d\omega + \frac{1}{2\pi}\int_{0}^{\infty} e^{-\omega(y+ix)}d\omega\right)\\ &= \frac{1}{2\pi}\left(\frac{1}{y-ix} + \frac{1}{y+ix}\right) = \frac{1}{\pi}\frac{y}{x^2+y^2}.\end{aligned}$$

This gives that

$$u(x, y) = f(x) * \frac{1}{\pi}\frac{y}{x^2+y^2} = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\tau)\frac{y}{(x-\tau)^2+y^2}d\tau.$$

Now, we verify that the assumptions we made earlier are in fact valid. Noting that f has compact support, our integral essentially has finite bounds in practice, so we can push limits inside the integral. As $y \to \infty$, $\frac{y}{(x-\tau)^2+y^2} \to 0$, so $u(x, y) \to 0$ as $y \to \infty$. We can also compute

$$u_x(x, y) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\tau)\frac{-2y(x-\tau)}{((x-\tau)^2+y^2)^2}d\tau.$$

Clearly, our assumption that $u_x(x, t) \to 0$ as $x \to \pm\infty$ hold here as well as the integrand tends to zero.

2 Problem 2

Now, we seek to solve the problem

$$\begin{aligned}&\frac{\partial}{\partial t}u = \frac{\partial^2}{\partial x^2}u, \quad 0 < t,\ 0 < x < \infty\\ &u(x, 0) = 0, \quad 0 < x < \infty\\ &u(0, t) = T_0, \quad t > 0.\end{aligned}$$

2.1 Part a

Consider the similarity transformation

$$\eta = \frac{x}{t^\alpha}.$$

Then,

$$\frac{\partial u}{\partial t} = \frac{\partial \eta}{\partial t}\frac{\partial u}{\partial \eta} = \frac{-\alpha x}{t^{\alpha+1}}\frac{\partial u}{\partial \eta} = -\frac{\alpha}{t}\eta\frac{\partial u}{\partial \eta},$$

$$\frac{\partial u}{\partial x} = \frac{\partial \eta}{\partial x}\frac{\partial u}{\partial \eta} = \frac{1}{t^\alpha}\frac{\partial u}{\partial \eta},$$

and

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{1}{t^\alpha}\frac{\partial u}{\partial \eta}\right) = \frac{1}{t^\alpha}\left(\frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \eta}\right)\right) = \frac{1}{t^\alpha}\left(\frac{1}{t^\alpha}\frac{\partial^2 u}{\partial \eta^2}\right) = \frac{1}{t^{2\alpha}}\frac{\partial^2 u}{\partial \eta^2}.$$

Thus, our differential equation becomes

$$-\frac{\alpha}{t}\eta\frac{\partial u}{\partial \eta} = \frac{1}{t^{2\alpha}}\frac{\partial^2 u}{\partial \eta^2},$$

so we can eliminate t by taking $\alpha = \frac{1}{2}$. Adjusting our boundary/initial conditions for the change of variables $\eta = x/\sqrt{t}$, we now have

$$\begin{aligned} -\frac{1}{2}\eta\frac{\partial u}{\partial \eta} &= \frac{\partial^2 u}{\partial \eta^2} \\ u(\infty) &= 0, \\ u(0) &= T_0. \end{aligned}$$

To solve this ODE, we set $v = u'$ so that we have

$$\frac{v'}{v} = -\frac{1}{2}$$

which has solution

$$v(\eta) = Ce^{-\eta^2/4}$$

after integrating. Then,

$$u(\eta) = \int_0^\eta Ce^{-(\eta/2)^2}d\eta = c_1 \operatorname{erf}\left(\frac{\eta}{2}\right) + c_2.$$

Noting that $\operatorname{erf}(0) = 0$, $\operatorname{erf}(\infty) = 1$, our second condition gives that $c_2 = T_0$ which in combination with our second condition gives that $c_1 = -T_0$. Thus,

$$u(\eta) = -T_0 \operatorname{erf}\left(\frac{\eta}{2}\right) + T_0 = T_0 \operatorname{erfc}\left(\frac{\eta}{2}\right).$$

Undoing our change of variables, we conclude that

$$u(x,t) = T_0 \operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right).$$

2.2 Part b

Letting $\mathcal{L}$ denote the Laplace transform in t and using a tilde to denote a function that has been transformed in this way, we now compute

$$\mathcal{L}[u_t(x,t)] = \int_0^\infty u_t e^{-st} dt = \left[ue^{-st}\right]_0^\infty + s\int_0^\infty ue^{-st} dt = s\tilde{u}(x,s)$$

if we impose the additional assumption that $u(x,t)$ vanish as $t \to \infty$. We also have that

$$\mathcal{L}[u_{xx}(x,t)] = \int_0^\infty u_{xx} e^{-st} dt = \frac{\partial^2}{\partial x^2}\int_0^\infty ue^{-st} dt = \tilde{u}_{xx}(x,s).$$

Thus, our PDE becomes

$$s\tilde{u} = \tilde{u}_{xx},$$

an ODE. Since we assume that $s > 0$, this has general solution

$$\tilde{u}(x,s) = c_1(s)e^{\sqrt{s}x} + c_2(s)e^{-\sqrt{s}x}.$$

Now, we find one boundary condition by computing

$$\tilde{u}(0,s) = \int_0^\infty u(0,t)e^{-st} dt = \int_0^\infty T_0 e^{-st} dt = \frac{T_0}{s}.$$

We also impose the assumption that $u(x,t) \to 0$ as $x \to \infty$ which implies that $\tilde{u}(x,s) \to 0$ as $x \to \infty$ and allows us to conclude that $c_1(s) = 0$ which then enables us to find that $c_2(s) = \frac{T_0}{s}$, so

$$\tilde{u}(x,s) = \frac{T_0}{s} e^{-\sqrt{s}x}.$$

Now, we consult Wolfram-Alpha to compute the inverse transform

$$u(x,t) = \mathcal{L}^{-1}[\tilde{u}(x,s)] = T_0 \operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right)$$

which is the same solution we found in part a. From this solution, we notice that our assumption that $u(x,t) \to 0$ as $t \to \infty$ clearly holds since $\frac{x}{2\sqrt{t}} \to 0$ as $t \to \infty$ and $\operatorname{erfc} 0 = 0$. Also, $\operatorname{erfc} z \to 0$ as $z \to \infty$, so our assumption that $u(x,t) \to 0$ as $x \to \infty$ also clearly holds.