# AMATH 575 Problem Set 3

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## 1 Problem 1

Consider the 2-D map

$$\begin{cases} x_{n+1} = -x_n + x_n y_n \\ y_{n+1} = -\frac{y_n}{2} - x_n y_n - x_n^2 \end{cases}.$$

Using Mathematica, we find that the only real fixed point is given by $(x, y) = (0, 0)$. Computing the Jacobian at this fixed point,

$$Df(0,0) = \begin{pmatrix} -1 & 0 \\ 0 & -1/2 \end{pmatrix},$$

is diagonal, so we do not need to transform into eigenspace to get the center manifold and just set $y = h(x)$ and plug this in to get

$$-h(x)/2 - xh(x) - x^2 = h(-x + xh(x)).$$

To find a polynomial expansion for the center manifold up to order 3, we set $h(x) = ax + bx^2 + cx^3$. Using Mathematica to collect coefficients, we get the system of equations

$$\begin{aligned} a/2 &= 0, \\ -1 - a - a^2 - 3b/2 &= 0, \\ -b + ab + c/2 &= 0. \end{aligned}$$

Solving this with Mathematica, we get

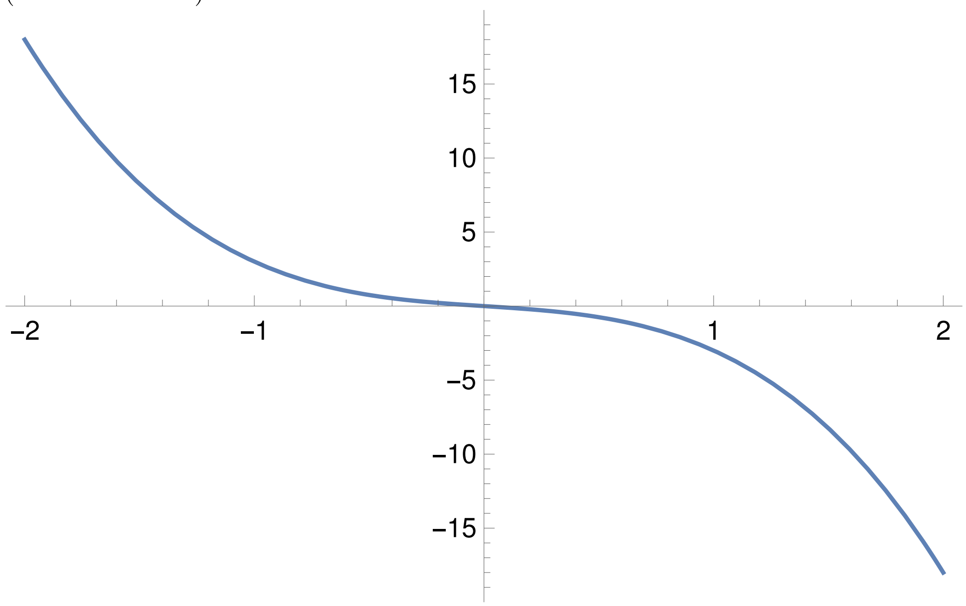
$$y = -\frac{2x^2}{3} - \frac{4x^3}{3},$$

as our center manifold. This means that the map restricted to the center manifold is given by

$$x \to -x - \frac{2x^2}{3} - \frac{4x^3}{3}.$$

Using Mathematica, we plot this map, from which we can see that the origin is attracting, and thus, the fixed point at the origin is asymptotically stable

(Theorem 18.4.2).




# 2 Problem 2

From Theorem 6.0.1 and Corollary 6.0.2, we can immediately see that in Figure 4.1.2, subfigures b, e, and f cannot occur. This is because periodic orbits must have index +1, but the sum of the indices of the fixed points inside this orbit is zero in each case.

# 3 Problem 3

## 3.1 Part a

The fact that gradient vector fields cannot have periodic orbits is a direct result of Theorem 15.0.3. From class, we have that every point of a periodic is an $\omega$ limit point, so Theorem 15.0.3 implies that every point in a periodic orbit is an equilibrium point which is obviously a contradiction.

## 3.2 Part b

To see that gradient vector fields cannot have homoclinic orbits, we consider such an orbit $\Gamma$ with fixed point $\bar{x}$. By proposition (15.0.1), $\dot{V}(\bar{x}) = 0$ and $\dot{V}(x) \leq 0$ for $x \in \Gamma$. Since this curve starts and ends in a neighborhood of $\bar{x}$, continuity implies that $\dot{V}(x) = 0$ for $x \in \Gamma$. Thus,

$$0 = \dot{V}(x) = -|\nabla V(x)|^2,$$

for $x \in \Gamma$, so $\dot{x} = 0$ for all points on the homoclinic orbit which is a contradiction.

### 3.3 Part c

Finally, to see that gradient vector fields cannot have heteroclinic cycles, we consider such a cycle $\Gamma$ with fixed points $\bar{x}_1, \ldots \bar{x}_n$ in that order. Then, the fact that $\dot{V}(\bar{x}_j) = 0$ for $j = 1 \ldots, n$ and $\dot{V}(x) \leq 0$ for $x \in \Gamma$ along with continuity implies that

$$V(\bar{x}_1) \geq \ldots \geq V(\bar{x}_n).$$

However, the last connection in the cycle implies that

$$V(\bar{x}_n) \geq V(\bar{x}_1),$$

so we must have that

$$V(\bar{x}_1) = \ldots = V(\bar{x}_n),$$

so $\dot{V}(x) = 0$ for $x \in \Gamma$. Thus, the same argument as in part b implies that $\dot{x} = 0$ for all points on the heteroclinic cycle which is a contradiction.

## 4 Problem 4

Consider the "all-to-all" coupled system of phase oscillators on the N-dimensional torus, with coupling strength $\alpha > 0$:

$$\dot{\phi}_i = -\alpha/N \sum_{j=1}^{N} f(\phi_j - \phi_i) \ \ mod \ \ 2\pi$$

$i = 1...N$ where $f$ is an odd function. To see that this is a gradient system, consider

$$V = \frac{\alpha}{N} \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} G(\phi_j - \phi_i),$$

where $G$ is an anti-derivative of $f$. Then,

$$(\nabla V)_k = \frac{\alpha}{N} \left( \sum_{j=k+1}^{N} f(\phi_j - \phi_k) + \sum_{i=1}^{k-1} f(\phi_k - \phi_i) \right) = \frac{\alpha}{N} \sum_{j=1}^{N} f(\phi_j - \phi_k).$$

Thus, this is indeed a gradient system.

## 5 Problem 5

Consider the 2-D flow

$$\begin{cases} \dot{x} = y \\ \dot{y} = \mu_1 x - x^3 + \mu_2 y - x^2 y \end{cases}.$$

We use Bendixson's criterion to show that this system has no periodic orbits for any $\mu_1 \in \mathbb{R}$ and $\mu_2 < 0$. To do this, we compute

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = \mu_2 - x^2 < 0,$$

for all $x, y \in \mathbb{R}$. Thus, taking $D = \mathbb{R}^2$, Bendixson's criterion implies that there are no closed orbits in $\mathbb{R}^2$.

# 6 Problem 6

Consider the Lorenz equations

$$\begin{aligned}
\dot{x} &= \sigma(y - x), \\
\dot{y} &= \rho - y - xz, \\
\dot{z} &= -\beta z + xy.
\end{aligned}$$

We compute the divergence

$$\nabla \cdot f = -\sigma - 1 - \beta,$$

which is constant. Then, we can use (7.6.3) in Wiggins to get that the volume evolves according to

$$V(t) = \mathrm{e}^{-(\sigma+1+\beta)t} V(0),$$

which is strictly decreasing.

# 7 Problem 7

The divergence free property of a vector field does not necessarily imply that the vector field has a first integral. While one may be tempted to take it to be $I(t) = V(t)$, $V(t)$ is constant while integrals must be nonconstant. As an example of nonexistence, consider the vector field

$$\dot{x} = c,$$

where $c \in \mathbb{R}$. Then, it must hold that

$$\dot{I}(x) = \nabla I(x) f(x) = c \nabla I(x) = 0,$$

implying that $I$ must be constant and therefore not an integral.