MATH 525 Homework 4

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February 2, 2024

1 Problem 1

Let E be a barrel in a Banach space. Because E is closed, $E^o \subset E$ by definition, and $\overline{E^o}$ is the smallest closed set containing E^o, we must have that $\overline{E^o} \subset E$. Let $x \in E$ and $t \in [0, 1)$. Because E is a barrel, there exists some $r > 0$ such that $\mathcal{B}_r(0) \subset E$. Let $y \in \mathcal{B}_{(1-t)r}(tx)$. Then, y can be represented as $y = tx + (1-t)rv$ for some v with $\|v\| < 1$. This means that $rv \in \mathcal{B}_r(0) \subset E$, so $y \in E$ since E is convex. Thus, $\mathcal{B}_{(1-t)r}(tx) \subset E$, meaning that $tx \in E^o$ since E^o must contain all of the interior points of E. Now, consider the sequence of points $\{x_n\}_{n=1}^{\infty} \subset E^o$ where $x_n = \frac{n}{n+1}x$ for all n. Clearly, $\{x_n\} \to x$ and $x_n \subset E^o$ for all n. Thus, $x \in \overline{E^o}$, so we conclude that $\overline{E^o} = E$.

2 Problem 2

Let X be a normed vector space and M a vector subspace of X considered as a normed vector space itself. Let $M^\perp \subset X^*$ be the set of all $f \in X^*$ such that $M \subset \ker(f)$.

2.1 Part a

Let M be closed. If $x \in M$, then by the definition of $M^\perp$, $f(x) = 0$ for all $f \in M^\perp$. Conversely, assume that for some x, $f(x) = 0$ for all $f \in M^\perp$. If $x \notin M$, then Theorem 5.8a implies that there exists some $f \in X^*$ such that $f(x) \neq 0$ and $f(y) = 0$ for all $y \in M$. This implies that $f \in M^\perp$ but $f(x) \neq 0$ which contradicts the assumptions. Thus, $x \in M$.

2.2 Part b

To see that there is a natural equivalence $M^* \equiv X^*/M^\perp$, let $f \in M^*$ and consider the norm $p(x) = \|f\|_{M^*}\|x\|$. Since $\|f\|_{M^*}$ is a positive constant independent of X, this is clearly a norm on X. By definition, $|f(x)| \leq p(x)$ for all $x \in M$, so Hahn–Banach implies that there exists some $F \in X^*$ such that $F(x) = f(x)$ for all $x \in M$ and $|F(x)| \leq p(x)$ for all $x \in X$. We let f correspond to $F + M^\perp \in X^*/M^\perp$. Then,

$$\|F\|_{X^*} = \sup_{x \in X} \frac{|F(x)|}{\|x\|} \leq \sup_{x \in X} \|f\|_{M^*} = \|f\|_{M^*}.$$

Noting that the zero functional is in $M^\perp$,

$$\|F + M\|_{X^*/M^\perp} = \inf_{g \in M^\perp} \|F + g\|_{X^*} \leq \|F + 0\|_{X^*} = \|F\|_{X^*} \leq \|f\|_{M^*}.$$

Now, note that if $g + M^\perp = h + M^\perp \in X^*/M^\perp$, then $g - h \in M^\perp$, so $(g - h)(x) = 0$ for all $x \in M$, meaning that $g(x) = h(x)$ for all $x \in M$. Thus, given $F + M^\perp \in X^*/M^\perp$, we can uniquely define $f \in M^*$ by $f(x) = (F + M^\perp)(x)$ for all $x \in M$ where $f(x) = g(x)$ for all $g \in F + M^\perp$. Then, for all such g,

$$\|f\|_{M^*} = \sup_{x \in M} \frac{|f(x)|}{\|x\|} = \sup_{x \in M} \frac{|g(x)|}{\|x\|} \leq \sup_{x \in X} \frac{|g(x)|}{\|x\|} = \|g\|_{X^*}.$$

Since this holds for all $g \in F + M^{\perp}$, this implies that that

$$\|f\|_{M^*} \leq \inf_{g \in F+M^{\perp}} \|g\|_{X^*} = \inf_{v \in M^{\perp}} \|F + v\|_{X^*} = \|F + M^{\perp}\|_{X^*/M\perp}.$$

Thus, we have established a equivalence between $f \in M^*$ and $F + M^{\perp} \in X^*/M \perp$ such that $\|f\|_{M^*} = \|F + M^{\perp}\|_{X^*/M^{\perp}}$ for all elements of M^* and $X^*/M^{\perp}$, so $M^* \equiv X^*/M^{\perp}$.

3 Problem 3

Let X be a reflexive Banach space and M a closed subspace of X. Let $\hat{m} \in M^{**}$ and define $\hat{M} \in X^{**}$ by $\hat{M}(F) = \hat{m}\left(F\big|_M\right)$ for all $F \in X^{**}$. Because X is reflexive, there exists some $x \in X$ such that $\hat{M}(F) = F(x)$ for all $F \in X^*$, meaning that $\hat{m}\left(F\big|_M\right) = F(x)$ for all $F \in X^*$. Let $G \in M^{\perp}$. Then,

$$G(x) = \hat{m}\left(G\big|_M\right) = \hat{m}(0) = 0,$$

so Problem 2a implies that $x \in M$ because $G(x) = 0$ for all $G \in M^{\perp}$. Now, let $f \in M^*$. Hahn–Banach implies that there exists some $F \in X^*$ such that $F\big|_M = f$ since $p(x) = \|f\|_{M^*}\|x\|$ is a norm and $|f(x)| \leq p(x)$ for all $x \in M$. This implies that for any $f \in M^*$,

$$\hat{m}(f) = \hat{m}\left(F\big|_M\right) = \hat{M}(F) = F(x) = f(x),$$

since $x \in M$. Thus, for any $\hat{m} \in M^{**}$, we can find some $x \in M$ such that $\hat{m}(f) = f(x)$ for all $f \in M^*$. This means that the double dual map $M \to M^{**}$ is surjective, so M is also reflexive.

4 Problem 4

Let E and F be closed subspaces of a Banach space X such that $E \cap F = \{0\}$ and $X = \operatorname{span}(E \cup F)$. Define the map $\varphi : E \times F \to X$ by $\varphi(v, w) = v + w$. To see that φ is injective, assume that $\varphi(e_1, f_1) = \varphi(e_2, f_2)$. Then, $e_1 + f_1 = e_2 + f_2$, so

$$E \ni e_1 - e_2 = f_2 - f_1 \in F.$$

Since $E \cap F = \{0\}$, this means that

$$0 = e_1 - e_2 = f_2 - f_1,$$

so $(e_1, f_1) = (e_2, f_2)$.

To see that φ is surjective, let $x \in X$. Since $X = \operatorname{span}(E \cup F)$, we can write $x = e + f$ for some $e \in E$ and $f \in F$. This implies that there exists $(e, f) \in E \times F$ such that $\varphi(e, f) = e + f = x$, so φ is surjective.

To see that φ is continuous, fix $\epsilon > 0$ and $(e, f) \in E \times F$ and let $\delta = \epsilon$. Then, for any $(c, d) \in E \times F$ such that $\|(e, f) - (c, d)\|_{E \times F} < \delta$,

$$\begin{aligned}\|\varphi(e, f) - \varphi(c, d)\|_X &= \|(e + f) - (c + d)\|_X \leq \|e - c\|_X + \|f - d\|_X \\ &= \|(e - c, f - d)\|_{E \times F} = \|(e, f) - (c, d)\|_{E \times F} < \epsilon.\end{aligned}$$

Note that this uses the product space norm as defined in the course notes rather than the one in Folland. Thus, φ is continuous on $E \times F$.

Finally, we note that E and F are Banach spaces since they are closed subspaces of a Banach space. Thus, $E \times F$ is a Banach space, so the open mapping theorem implies that φ is open since it is continuous and surjective. This means that φ^{-1} is continuous, so we can conclude that φ is a homeomorphism of $E \times F$ onto X.

If E is a closed subspace of a Banach space with a closed complement F, denote by Π_E the continuous surjective projection map from $E \times F$ to E. Then, we can conclude that the composition map $\Pi_E \circ \varphi^{-1}$ is a continuous onto projection from X to E since both Π_E and φ^{-1} are. Conversely, if E has a continuous projection map Π_E from X to E, define $F = \{x \in X : \Pi_E(x) = 0\}$ and note that F is a closed subspace of X because Π_E is continuous, $F = \Pi_E^{-1}(\{0\})$, and $\{0\}$ is a closed subspace of X. Then, by construction, $E \cap F = \{0\}$. Furthermore, for any $x \in X$, $x = \Pi_E(x) + (x - \Pi_E(x))$. Then, $\Pi_E(x) \in E$ and

$$\Pi_E(x - \Pi_E(x)) = \Pi_E(x) - \Pi_E(\Pi_E(x)) = \Pi_E(x) - \Pi_E(x) = 0,$$

so $x - \Pi_E(x) \in F$ and $x \in \operatorname{span}(E \cup F)$. Thus, $X = \operatorname{span}(E \cup F)$, so E has a closed complement, namely F.

5 Problem 5

Let X be a normed vector space over $\mathbb{C}$ and let $f \in X^*$ with $\|f\|_{X^*} = 1$.

5.1 Part a

Define the map $\varphi : X/\ker(f) \to \mathbb{C}$ by $\varphi(y) = f(y)$ for any $y \in x + \ker(f)$ and $x \in X$. This is well-defined because if $x + \ker(f) = y + \ker(f)$, then $x - y \in \ker(f)$, so $f(x-y) = 0$ and, because f is linear, $f(x) = f(y)$. Thus, we can define the map by $\varphi(x + \ker(f)) = f(x)$.

To see that is map is onto, note that because $\|f\|_{X^*} = 1$, there exists some $\tilde{x} \in X$ such that $f(\tilde{x}) = c \neq 0$ since f is not the zero functional. Given some $a \in \mathbb{C}$, let $x = \frac{a}{c}\tilde{x}$. Then,

$$f(x) = \frac{a}{c} f(\tilde{x}) = a.$$

Thus, φ is onto.

To see that φ is norm-preserving, we first observe that for any $v \in \ker(f)$,

$$|f(x)| \leq |f(x) - f(v)| + |f(v)| = |f(x-v)| \leq \|x - v\|.$$

Thus,

$$\|\varphi(x + \ker(f))\| = |f(x)| \leq \inf_{v \in \ker(f)} \|x - v\| = \|x + \ker(f)\|_{X/\ker(f)}.$$

To show this inequality in the opposite direction, we first note that φ is injective because if $\varphi(x + \ker(f)) = \varphi(y + \ker(f))$, then $f(x) = f(y)$, so $f(x - y) = 0$ and $x - y \in \ker(f)$, so x and y are in the same equivalence class and $x + \ker(f) = y + \ker(f)$. This means that φ is an isomorphism, so $X/\ker(f)$ is one-dimensional since $\mathbb{C}$ is. Now, because $\|f\|_{X^*} = 1$, for any $\epsilon > 0$, there exists some $y \in X$ such that $\|y\| \leq |f(y)| + \epsilon$. Then,

$$\|y + \ker(f)\|_{X/\ker(f)} = \inf_{v \in \ker(f)} \|y - v\| \leq \|y\| \leq |f(y)| + \epsilon.$$

Because $X/\ker(f)$ is one-dimensional, for any $x \in X$ nonzero, there exists some $\lambda \in \mathbb{C}$ such that $x + \ker(f) = \lambda(y + \ker(f))$. This implies that $x - \lambda y \in \ker(f)$, so $f(x) = f(\lambda y)$ and $f\left(\frac{x}{\lambda}\right) = f(y)$. Thus,

$$\begin{aligned}
\|x + \ker(f)\|_{X/\ker(f)} &= |\lambda| \|y + \ker(f)\|_{X/\ker(f)} \leq |\lambda||f(y)| + |\lambda|\epsilon = |\lambda| \left| f\left(\frac{x}{\lambda}\right)\right| + |\lambda|\epsilon \\
&= |f(x)| + |\lambda|\epsilon = \|\varphi(x + \ker(f))\| + |\lambda|\epsilon.
\end{aligned}$$

By rescaling ϵ, this implies that for any nonzero $x \in X$ and $\epsilon > 0$, $\|x + \ker(f)\|_{X/\ker(f)} \leq \|\varphi(x + \ker(f))\| + \epsilon$. Furthermore, this is trivially true for $x = 0$, so we can conclude that for all $x \in X$, $\|x + \ker(f)\|_{X/\ker(f)} \leq \|\varphi(x + \ker(f))\|$. Thus, φ is norm-preserving.

5.2 Part b

From class, we have that if X is reflexive and $f \in X^*$ with $\|f\|_{X^*} = 1$, then there is an $x \in X$ with $\|x\| = 1$ such that

$$f(x) = \|f\|_{X^*} = 1.$$

By part a, there is a norm-preserving onto map φ from $X/\ker(f)$ to $\mathbb{C}$ such that $\varphi(y + \ker(f)) = f(y)$. Thus, by definition,

$$1 = |f(x)| = \|x + \ker(f)\|_{X + \ker(f)} = \inf_{v \in \ker(f)} \|x - v\|,$$

so we have established that there is an $x \in X$ with $\|x\| = 1$ such that $\inf_{v \in \ker(f)} \|x - v\| = 1$.