AMATH 573 Homework 5

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1 Problem 1

We wish to show that

$$X = \begin{pmatrix} -i\zeta & q \\ \pm q^* & i\zeta \end{pmatrix}, T = \begin{pmatrix} -i\zeta^2 \mp \frac{i}{2}|q|^2 & q\zeta + \frac{i}{2}q_x \\ \pm q^*\zeta \mp \frac{i}{2}q_x^* & i\zeta^2 \pm \frac{i}{2}|q|^2 \end{pmatrix}$$

are a Lax Pair for the Nonlinear Schrödinger equations

$$iq_t = -\frac{1}{2}q_{xx} \pm |q|^2 q.$$

We do this by explicitly computing that $X_t + XT - T_x - TX = 0$ via the attached Mathematica file.

2 Problem 2

Let $\psi_n = \psi_n(t)$, $n \in \mathbb{Z}$. We consider the difference equation

$$\psi_{n+1} = X_n \psi_n,$$

and the differential equation

$$\frac{\partial \psi_n}{\partial t} = T_n \psi_n.$$

To find the compatibility condition, we differentiate and substitute the equations into each other to get

$$\psi_{n+1,t} = X_{nt}\psi_n + X_n\psi_{nt} = X_{nt}\psi_n + X_nT_n\psi_n = (X_{nt} + X_nT_n)\psi_n$$

and

$$\psi_{n+1,t} = T_{n+1}\psi_{n+1} = T_{n+1}X_n\psi_n.$$

Thus, we a compatability condition is given by

$$X_{nt} + X_nT_n = T_{n+1}X_n.$$

Now, we consider

$$X_n = \begin{pmatrix} z & q_n \\ q_n^* & 1/z \end{pmatrix}, T_n = \begin{pmatrix} iq_n q_{n-1}^* - \frac{i}{2}\left(1/z - z\right)^2 & \frac{i}{z}q_{n-1} - izq_n \\ -izq_{n-1}^* + \frac{i}{z}q_n^* & -iq_n^* q_{n-1} + \frac{i}{2}\left(1/z - z\right)^2 \end{pmatrix}.$$

We show that it is a Lax Pair for the semi-discrete equation

$$i\frac{\partial q_n}{\partial t} = q_{n+1} - 2q_n + q_{n-1} - |q_n|^2(q_{n+1} + q_{n-1})$$

by directly verifying that $X_{nt} + X_n T_n - T_{n+1}X_n = 0$ in the attached Mathematica notebook.

3 Problem 3

Consider the forward scattering problem for the KdV equation $u_t + 6uu_x + u_{xxx} = 0$ with initial condition $u(x,0) = 0$ for $x \in (-\infty, -L) \cup (L, \infty)$, and $u(x,0) = d$ for $x \in (-L, L)$, with L and d both positive. To find $a(k)$ for all time t, we first determine the function $\phi(x,k)$ for our initial data. For $x < -L$ or $x > L$, ϕ must satisfy

$$\phi_{xx} + k^2\phi = 0$$

which has general solution

$$\phi = c_1 e^{ikx} + c_2 e^{-ikx}.$$

However, by definition, we must have that $\phi(x,k) \to e^{-ikx}$ as $x \to -\infty$, so we must have that $c_1 = 0, c_2 = 1$ for $x < -L$. For $-L < x < L$, ϕ must satisfy

$$\phi_{xx} + (k^2 + d)\phi = 0$$

which has general solution

$$\phi = c_3 e^{i\sqrt{k^2+d}x} + c_4 e^{-i\sqrt{k^2+d}x}.$$

Thus, we have

$$\phi(x,k) = \begin{cases} e^{-ikx}, & x < -L, \\ c_3 e^{i\sqrt{k^2+d}x} + c_4 e^{-i\sqrt{k^2+d}x}, & -L < x < L, \\ c_1 e^{ikx} + c_2 e^{-ikx}, & x > L. \end{cases}$$

To solve for our constants, we first impose that ϕ be continuous. Imposing this at $x = \pm L$ gives the conditions

$$\begin{aligned} e^{ikL} &= c_3 e^{-i\sqrt{k^2+d}L} + c_4 e^{i\sqrt{k^2+d}L}, \\ c_1 e^{ikL} + c_2 e^{-ikL} &= c_3 e^{i\sqrt{k^2+d}L} + c_4 e^{-i\sqrt{k^2+d}L}. \end{aligned}$$

We obtain more conditions so that the constants can be solved for explicitly, we integrate over our differential equation. Namely, we let $u_0(x)$ represent our initial condition as described above and consider

$$0 = \int_{-L-\epsilon}^{-L+\epsilon} (\phi_{xx} + (k^2 + u_0)\phi)dx$$

where $\epsilon > 0$ is arbitrarily small. Then, the continuity of ϕ at $x = -L$ gives that as $\epsilon \to 0$,

$$0 = [\phi_x]_{-L-\epsilon}^{-L+\epsilon} + d\int_{-L}^{-L+\epsilon} \phi dx + k^2 \int_{-L-\epsilon}^{-L+\epsilon} \phi dx = \phi_x(-L+\epsilon) - \phi_x(-L-\epsilon),$$

implying that ϕ_x must also be continuous at $-L$. An analogous argument gives that ϕ_x must also be continuous at $x = L$. Enforcing this, we obtain the additional conditions

$$\begin{aligned} -ike^{ikL} &= i\sqrt{k^2+d}c_3 e^{-i\sqrt{k^2+d}L} - i\sqrt{k^2+d}c_4 e^{i\sqrt{k^2+d}L}, \\ ikc_1e^{ikL} - ikc_2e^{-ikL} &= i\sqrt{k^2+d}c_3 e^{i\sqrt{k^2+d}L} - i\sqrt{k^2+d}c_4 e^{-i\sqrt{k^2+d}L}. \end{aligned}$$

Now, we have 4 equations and 4 unknown constants, so we use Mathematica to solve this system and obtain c_1, c_2, c_3, c_4 which are printed in the attached notebook, meaning that ϕ is completely determined. To determine $a(k)$, we need to also know how φ behaves. However, by definition $\varphi(x,k) \to e^{ikx}$ as $x \to \infty$, and φ satisfies the same differential equation as ϕ, so we know that for $x > L$,

$$\varphi(x,k) = e^{ikx}.$$

Because the Wronskian can be evaluated at any x-value, we can just take $x > L$ and compute

$$a(k) = \frac{W(\phi,\varphi)}{2ik} = \frac{W(c_1e^{ikx} + c_2e^{-ikx}, e^{ikx})}{2ik}.$$

Using Mathematica, we can evaluate this with our correct c_1, c_2 and conclude that

$$a(k) = \frac{1}{2}e^{2ikL}\left(2\cos\left(2\sqrt{d+k^2}L\right) - \frac{i(d+2k^2)\sin\left(2\sqrt{d+k^2}L\right)}{k\sqrt{d+k^2}}\right).$$

Since $a(k)$ is constant in time, we know that this is $a(k)$ for all time t.
Since the number of solitons is equivalent to the zeros of $a(k)$, we begin to search for the zeros by plugging in $k = i\kappa$ to get

$$a(i\kappa) = \frac{1}{2}e^{-2\kappa L}\left(2\cos\left(2\sqrt{d-\kappa^2}L\right) - \frac{i(d-2\kappa^2)\sin\left(2\sqrt{d-\kappa^2}L\right)}{\kappa\sqrt{d-\kappa^2}}\right)$$

and note that $\kappa > 0$. We first consider the case where $d - \kappa^2 < 0$. Taking the principle branch of the square root, we have that $\sqrt{d - \kappa^2} = im$ for some $m > 0$. Using Mathematica aid in this substitution, we get that

$$a(i\kappa) = \frac{1}{2} e^{-2\kappa L} \left(2\cosh{(2Lm)} + \frac{(2\kappa^2 - d)\sinh{(2Lm)}}{\kappa m} \right)$$

Note that the exponential and cosh are positive when their arguments are real and that sinh is positive when its argument is positive. This is the case here, so $a(i\kappa)$ is guaranteed to be positive if

$$\frac{2\kappa^2 - d}{\kappa m} \geq 0.$$

However, this follows directly from our assumptions that $\kappa, m > 0$ and $d - \kappa^2 < 0$, since

$$2\kappa^2 - d > \kappa^2 - d > 0.$$

Thus, we have shown that $a(i\kappa) > 0$ for all κ such that $d - \kappa^2 < 0$, meaning that we cannot have any zeros for such values of κ, and it suffices to consider $d - \kappa^2 \geq 0$. We define a scaling $s = 2\sqrt{d - \kappa^2} L$ which is now valid because it is guaranteed to be real (and nonnegative). Using Mathematica to make this substitution, we get

$$e^{-\sqrt{4dL^2 - s^2}} \left(\cos s + \frac{2dL^2 - s^2}{s\sqrt{4dL^2 - s^2}} \sin s \right).$$

Letting $p = 2dL^2$, we set this expression equal to zero. One way to rewrite this new equation is as

$$\cot s = \frac{s^2 - p}{s\sqrt{2p - s^2}}.$$

We plot both sides of this equation as functions of s for a range of values of p using the Manipulate function in Mathematica and look for points of intersection, noting that we need only consider $s \geq 0$, $p > 0$. In doing so, we observe that the RHS appears to be monotonically increasing with vertical asymptotes at $s = 0, \sqrt{2p}$. We also know that the cotangent is monotonically decreasing on each of its periods which have vertical asymptotes at $s = j\pi, (j+1)\pi$, $j \in \mathbb{N}$, so we expect to have exactly one intersection on each period of $\cot s$ for which the RHS is defined. Thus, we conjecture that the number of zeros (and therefore the number of solitons) is given by $\lceil \sqrt{2p}/\pi \rceil = \lceil 2L\sqrt{d}/\pi \rceil$ when we assume that $d > 0$.

If we instead wish to consider $d < 0$, we actually encounter a case which we have already considered. Namely, because $\kappa > 0$, $d - \kappa^2 < 0$ will always hold if $d < 0$. We have shown that this case cannot produce any solitons ($a(k)$ always has no zeros), so if $d < 0$, we will not obtain any solitons. Of course, this matches what we'd expect from the KdV equation.

Finally, we return to our original expression but make the substitution $2dL = \alpha$ and compute

$$\lim_{L\to 0} a(k) = 1 - \frac{i\alpha}{2k}$$

via Mathematica. One can rewrite this as

$$\frac{\alpha + 2ik}{2ik}$$

which is precisely the $a(k)$ obtained from the delta function potential with scaling α. This makes sense to some extent, because the support of our initial condition is collapsing to a single point in this limit.

4 Problem 5

Consider Liouville's equation

$$u_{xy} = e^u,$$

and the transformation

$$\begin{aligned} v_x &= -u_x + \sqrt{2}e^{(u-v)/2}, \\ v_y &= u_y - \sqrt{2}e^{(u+v)/2}, \end{aligned}$$

where $u(x,y)$ satisfies Liouville's equation.

4.1 Part a

Taking the y derivative of the first term and plugging in the second, we get that

$$\begin{aligned} v_{xy} &= -u_{xy} + \frac{u_y - v_y}{\sqrt{2}} e^{(u-v)/2} = -u_{xy} + \frac{u_y - (u_y - \sqrt{2}e^{(u+v)/2})}{\sqrt{2}} e^{(u-v)/2} \\ &= -u_{xy} + e^u = 0. \end{aligned}$$

4.2 Part b

To find a general solution for $v(x,y)$, we simply integrate this relation twice to get that

$$v(x,y) = a(x) + b(y).$$

4.3 Part c

Plugging this into our Bäcklund transformation, we get the system of equations

$$\begin{cases} a'(x) = -u_x + \sqrt{2}e^{(u-a(x)-b(y))/2} \\ b'(y) = u_y - \sqrt{2}e^{(u+a(x)+b(y))/2}. \end{cases}$$

To get an integrating factor, we can rewrite this system as

$$\begin{cases} e^{-(u+a(x))/2}(u+a(x))_x = \sqrt{2}e^{-b(y)/2}e^{-a(x)} \\ e^{-(u-b(y))/2}(u-b(y))_y = \sqrt{2}e^{a(x)/2}e^{b(y)}. \end{cases}$$

Integrating both sides of each,

$$\begin{cases} -2e^{-(u+a(x))/2} = \sqrt{2}e^{-b(y)/2}\int e^{-a(x)}dx \\ -2e^{-(u-b(y))/2} = \sqrt{2}e^{a(x)/2}\int e^{b(y)}dy. \end{cases}$$

Taking logarithms and adding in integration constants,

$$\begin{cases} -(u+a(x))/2 = \log\left(-\frac{1}{\sqrt{2}}e^{-b(y)/2}\left(\int e^{-a(x)}dx + c_1(y)\right)\right) \\ -(u-b(y))/2 = \log\left(-\frac{1}{\sqrt{2}}e^{a(x)/2}\left(\int e^{b(y)}dy + c_2(x)\right)\right). \end{cases}$$

We can finally solve for u in each which gives

$$\begin{cases} u = -a(x) + \log 2 + b(y) - 2\log\left(-\int e^{-a(x)}dx - c_1(y)\right) \\ u = b(y) + \log 2 - a(x) - 2\log\left(-\int e^{b(y)}dy - c_2(x)\right). \end{cases}$$

Combining these to reconcile our integration constants, we conclude that

$$u(x,y) = -a(x) + \log 2 + b(y) - 2\log\left(-\int e^{-a(x)}dx - \int e^{b(y)}dy\right).$$

5 Problem 6

Consider the sine-Gordon equation

$$u_{xt} = \sin u.$$

5.1 Part a

Consider the transformation

$$\begin{aligned} v_x &= u_x + 2\sin\frac{u+v}{2}, \\ v_t &= -u_t - 2\sin\frac{u-v}{2}. \end{aligned}$$

To see that this is an auto-Bäcklund transformation for the sine-Gordon equation, we assume that u satisfies the sine-Gordon equation and differentiate the first equation in t to get that

$$v_{xt} = u_{xt} + (u_t + v_t)\cos\frac{u+v}{2} = \sin u + (u_t + v_t)\cos\frac{u+v}{2}.$$

Plugging in the second equation and applying a product-to-sum identity, we get that

$$\begin{aligned}
v_{xt} &= \sin u - 2\sin\frac{u-v}{2}\cos\frac{u+v}{2} \\
&= \sin u - \left(\sin\left(\frac{u-v}{2}+\frac{u+v}{2}\right) - \sin\left(\frac{u+v}{2}-\frac{u-v}{2}\right)\right) \\
&= \sin u - (\sin u - \sin v) = \sin v,
\end{aligned}$$

meaning that v also satisfies the sine-Gordon equation.

5.2 Part b

Let $u(x,t) = 0$, the simplest solution of the sine-Gordon equation. Substituting this in, the auto-Bäcklund transformation yields the system

$$\begin{cases} v_x = 2\sin\frac{v}{2} \\ v_t = -2\sin\frac{-v}{2} = 2\sin\frac{v}{2}. \end{cases}$$

From this, we can write

$$dx = \frac{dv}{2\sin\frac{v}{2}} = dt$$

and integrate. Doing this via Mathematica and including integration constants, we get that

$$x + c_1(t) = \log\left(\tan\frac{v}{4}\right) = t + c_2(x).$$

Reconciling c_1 and c_2 and solving for v, we conclude that

$$v(x,t) = 4\arctan\left(e^{x+t}\right).$$