AMATH 586 Homework 2

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1 Problem 1

To derive a 3rd order RK method, we choose to make our method explicit for ease of implementation, so by page 128 of the text, our method must have at least $r = 3$ stages. We arrive at the Butcher tableau

$$\begin{array}{c|ccc} c_1 & a_{11} & a_{12} & a_{13} \\ c_2 & a_{21} & a_{22} & a_{22} \\ c_3 & a_{31} & a_{32} & a_{33} \\ \hline 1 & b_1 & b_2 & b_3 \end{array}$$

By (5.35), for consistency we need that

$$\sum_{j=1}^{3} a_{ij} = c_i, \quad i = 1, 2, 3,$$

$$\sum_{j=1}^{3} b_j = 1.$$

By (5.37), for 2nd order accuracy we need that

$$\sum_{j=1}^{3} b_j c_j = \frac{1}{2}.$$

By (5.38), for 3rd order accuracy we need that

$$\sum_{j=1}^{3} b_j c_j^2 = \frac{1}{3},$$

$$\sum_{i=1}^{3} \sum_{j=1}^{3} b_i a_{ij} c_j = \frac{1}{6}.$$

Since we want our method to be explicit, we set all a_{ij} on or above the diagonal to zero which leaves us with the Butcher tableau

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ c_2 & c_2 & 0 & 0 \\ c_3 & a_{31} & a_{32} & 0 \\ \hline 1 & b_1 & b_2 & b_3 \end{array}$$

as our first condition requires that $c_1 = 0$ and $a_{21} = c_2$ after we impose this. Now, in order to obtain a simple method, we set $a_{31} = b_2 = 0$, so our tableau becomes

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ c_2 & c_2 & 0 & 0 \\ c_3 & 0 & c_3 & 0 \\ \hline 1 & b_1 & 0 & b_3 \end{array}$$

and the constraints that we have not already imposed become

$$\begin{aligned} b_1 + b_3 &= 1 \\ b_3 c_3 &= \frac{1}{2} \\ b_3 c_3^2 &= \frac{1}{3} \\ b_3 c_2 c_3 &= \frac{1}{6}. \end{aligned}$$

Combining the 2nd and 3rd equations allows us to find that $c_3 = \frac{2}{3}$ which then gives that $b_3 = \frac{3}{4}$. Using the 1st and 4th equations, we then find that $b_1 = \frac{1}{4}$ and $c_2 = \frac{1}{3}$. Thus, our Butcher tableau is

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ 1/3 & 1/3 & 0 & 0 \\ 2/3 & 0 & 2/3 & 0 \\ \hline 1 & 1/4 & 0 & 3/4 \end{array}$$

Now, we use the general form for RK methods (5.34) to convert our tableau into the following scheme:

$$\begin{aligned} y_1 &= u^n \\ y_2 &= u^n + \frac{k}{3} f(y_1, t_n) \\ y_3 &= u^n + \frac{2k}{3} f(y_n, t_n + k/3) \\ u^{n+1} &= u^n + k\left(\frac{1}{4} f(y_1, t_n) + \frac{3}{4} f(y_3, t_n + 2k/3)\right) \end{aligned}$$

for a problem $u'(t) = f(u(t), t)$. Note that this is Heun's 3rd order method. Using Julia, we test this method on the problem

$$v'''(t) + v'(t)v(t) - \frac{\beta_1 + \beta_2 + \beta_3}{3} v'(t) = 0,$$

where $\beta_1 < \beta_2 < \beta_3$ with initial conditions

$$v(0) = \beta_3, v'(0) = 0, v''(0) = -\frac{(\beta_3 - \beta_1)(\beta_3 - \beta_2)}{6}.$$

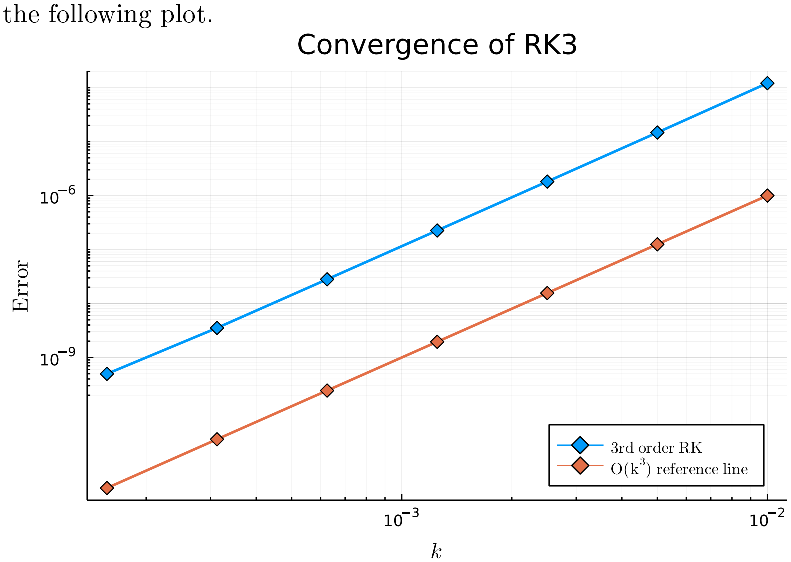
We turn it into a system

$$u_1'(t) = v'(t) = u_2(t), u_2'(t) = v''(t) = u_3(t), u_3''(t) = \frac{\beta_1 + \beta_2 + \beta_3}{3} u_2(t) - u_2(t)u_1(t)$$

and set $c = \frac{\beta_1+\beta_2+\beta_3}{3}$

$$f(u) = \begin{bmatrix} u_2 \\ u_3 \\ u_2(c - u_1) \end{bmatrix}.$$

Solving this problem for various stepsizes, we compute the error at each by comparing to the known solution of this problem. Plotting the error for each stepsize k along with a reference line $g(k) = k^3$ on a log-log scale, we observe the following plot.



These lines appear to be parallel meaning that our method appears to be third order as anticipated. We also print a table of error values and the error reduction ratio

$$\frac{\text{Error with time step } 2k}{\text{Error with time step } k}.$$

k	error	error reduction
0.010000	1.2102e-04	
0.005000	1.4730e-05	8.2161
0.002500	1.8163e-06	8.1097
0.001250	2.2547e-07	8.0556
0.000625	2.8075e-08	8.0311
0.000313	3.5287e-09	7.9562
0.000156	4.9731e-10	7.0956

For an rth-order method we should see this be approximately 2^r, so this suggests that our method is third order as most of our values are approximately 8.
See Appenix A for the code used in this problem.

2 Problem 2

Note that a general LMM has form

$$\sum_{j=0}^{r} \alpha_j u^{n+j} = k \sum_{j=0}^{r} \beta_j f(u^{n+j}, t_{n+j})$$

and that (5.48) gives that such a method is consistent if

$$\sum_{j=0}^{r} \alpha_j = 0,$$

$$\sum_{j=0}^{r} j\alpha_j = \sum_{j=0}^{r} \beta_j.$$

2.1 Part a

The LMM

$$U^{n+3} = U^{n+1} + 2kf(U^n)$$

corresponds to $r = 3$, $\alpha_3 = 1$, $\alpha_2 = 0$, $\alpha_1 = -1$, $\alpha_0 = 0$, $\beta_3 = \beta_2 = \beta_1 = 0$, $\beta_0 = 2$. Thus,

$$\sum_{j=0}^{r} \alpha_j = 0,$$

$$\sum_{j=0}^{r} j\alpha_j = 3 - 1 = 2 = \sum_{j=0}^{r} \beta_j,$$

so it is consistent. It has characteristic polynomial

$$\rho(\zeta) = \zeta^3 - \zeta = \zeta(\zeta + 1)(\zeta - 1)$$

which has roots $\zeta = 0, \pm 1$ which satisfy the root condition, so it is also zero-stable. Thus, the method is convergent.

2.2 Part b

The LMM

$$U^{n+2} = \tfrac{1}{2}U^{n+1} + \tfrac{1}{2}U^n + 2kf(U^{n+1})$$

corresponds to $r = 2$, $\alpha_2 = 1$, $\alpha_1 = -\frac{1}{2}$, $\alpha_0 = -\frac{1}{2}$, $\beta_2 = \beta_0 = 0$, $\beta_1 = 2$, so

$$\sum_{j=0}^{r} j\alpha_j = 2 - \frac{1}{2} = \frac{3}{2} \neq 2 = \sum_{j=0}^{r} \beta_j,$$

so it is not consistent and therefore not convergent. It has characeristic polynomial

$$\rho(\zeta) = \zeta^2 - \tfrac{1}{2}\zeta - \tfrac{1}{2} = (\zeta + \tfrac{1}{2})(\zeta - 1)$$

which has roots $-\frac{1}{2}, 1$ meaning that it is zero-stable.

2.3 Part c

The LMM

$$U^{n+1} = U^n$$

corresponds to $r = 1$, $\alpha_1 = 1$, $\alpha_0 = -1$, $\beta_1 = \beta_0 = 0$, so

$$\sum_{j=0}^{r} j\alpha_j = 1 \neq 0 = \sum_{j=0}^{r} \beta_j,$$

so it is not consistent and therefore not convergent. It has characteristic polynomial

$$\rho(\zeta) = \zeta - 1$$

which has root $\zeta = 1$, so it is zero-stable.

2.4 Part d

The LMM

$$U^{n+4} = U^n + \frac{4}{3}k(f(U^{n+3}) + f(U^{n+2}) + f(U^{n+1}))$$

corresponds to $r = 4$, $\alpha_4 = 1$, $\alpha_3, \alpha_2, \alpha_1 = 0$, $\alpha_0 = -1$, $\beta_4 = 0$, $\beta_3 = \beta_2 = \beta_1 = \frac{4}{3}$, $\beta_0 = 0$, so

$$\sum_{j=0}^{r} \alpha_j = 0,$$

$$\sum_{j=0}^{r} j\alpha_j = 4 = 3 \cdot \frac{4}{3} = \sum_{j=0}^{r} \beta_j,$$

meaning that it is consistent. It has characteristic polynomial

$$\rho(\zeta) = \zeta^4 - 1$$

which has roots $\zeta = \pm 1, \pm i$ which all have magnitude 1 but are not repeated, so the root condition is satisfied. Thus, it is zero-stable meaning that it is also convergent.

2.5 Part e

The LMM

$$U^{n+3} = -U^{n+2} + U^{n+1} + U^n + 2k(f(U^{n+2}) + f(U^{n+1}))$$

corresponds to $r = 3$, $\alpha_3 = \alpha_2 = 1$, $\alpha_1 = \alpha_0 = -1$, $\beta_3 = \beta_0 = 0$, $\beta_2 = \beta_1 = 2$, so

$$\sum_{j=0}^{r} \alpha_j = 0,$$

$$\sum_{j=0}^{r} j\alpha_j = 3 + 2 - 1 = 4 = \sum_{j=0}^{r} \beta_j,$$

so it is consistent. However, it has characteristic polynomial

$$\rho(\zeta) = \zeta^3 + \zeta^2 - \zeta - 1 = (\zeta - 1)(\zeta + 1)^2$$

which has roots $\zeta = \pm 1$. While these both have magnitude 1, the $\zeta = -1$ root is repeated, so the root condition is not satisfied and it is not zero-stable and therefore not convergent.

3 Problem 3

3.1 Part a

Consider the recursion

$$U^{n+2} = U^{n+1} + U^n.$$

To find a general solution, we make an ansatz $U^n = \zeta^n$ so that we can consider the charcteristic polynomial

$$\rho(\zeta) = \zeta^2 - \zeta - 1 = 0.$$

The roots of this polynomial are given by

$$\zeta_{1,2} = \frac{1 \pm \sqrt{5}}{2},$$

so our general solution is given by

$$U^n = c_1 \left(\frac{1 - \sqrt{5}}{2} \right)^n + c_2 \left(\frac{1 + \sqrt{5}}{2} \right)^n.$$

3.2 Part b

If we take starting values $U^0 = 1$, $U^1 = 1$, we can plug these into our general solution and determine c_1, c_2 by solving the linear system

$$\begin{cases} c_1 + c_2 = 1 \\ c_1 \frac{1-\sqrt{5}}{2} + c_2 \frac{1+\sqrt{5}}{2} = 1. \end{cases}$$

Substituting the first equation into the second, ,

$$c_1 \frac{1-\sqrt{5}}{2} + (1-c_1)\frac{1+\sqrt{5}}{2} = 1,$$

so

$$c_1 = \frac{1 - \frac{1+\sqrt{5}}{2}}{\frac{1-\sqrt{5}}{2} - \frac{1+\sqrt{5}}{2}} = \frac{1}{-\sqrt{5}} \frac{1-\sqrt{5}}{2} = \frac{5-\sqrt{5}}{10},$$

and

$$c_2 = 1 - \frac{5-\sqrt{5}}{10} = \frac{5+\sqrt{5}}{10}.$$

Thus,

$$U^n = \frac{5-\sqrt{5}}{10}\left(\frac{1-\sqrt{5}}{2}\right)^n + \frac{5+\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)^n,$$

so we can use Wolfram-Alpha to plug $n = 30$ into this formula and conclude that

$$U^{30} = 1346269.$$

3.3 Part c

Now, note that because $\left|\frac{1+\sqrt{5}}{2}\right| > 1$, $\left|\frac{1-\sqrt{5}}{2}\right| < 1$,

$$\lim_{n\to\infty} \frac{u^n}{u^{n-1}} = \frac{\frac{5-\sqrt{5}}{10}\cancelto{0}{\left(\frac{1-\sqrt{5}}{2}\right)^n} + \frac{5+\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)^n}{\frac{5-\sqrt{5}}{10}\cancelto{0}{\left(\frac{1-\sqrt{5}}{2}\right)^{n-1}} + \frac{5+\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)^{n-1}} = \frac{1+\sqrt{5}}{2} = \phi.$$

4 Problem 4

Consider the difference equation

$$U^{n+1} + U^{n-1} = 2xU^n, \quad n \geq 0,$$
$$U^0 = 1, \quad U^{-1} = 0$$

where $-1 \leq x \leq 1$.

4.1 Part a

Since $-1 \leq x \leq 1$ and the cosine function takes on all values between -1 and 1, we know that $x = \cos\theta$ for some $\theta \in \mathbb{R}$. By definition,

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2},$$

so our difference equation becomes

$$\begin{aligned} U^{n+1} + U^{n-1} &= (e^{i\theta} + e^{-i\theta})U^n, \quad n \geq 0, \\ U^0 = 1, \quad U^{-1} &= 0, \quad \theta \in \mathbb{R}. \end{aligned}$$

4.2 Part b

If we define $V^n = U^n - e^{i\theta}U^{n-1}$, we can use our recurrence to find that

$$\begin{aligned} V^n &= U^n - e^{i\theta}U^{n-1} = U^n - e^{i\theta}(-U^{n+1} + (e^{i\theta} + e^{-i\theta})U^n) \\ &= e^{i\theta}U^{n+1} - e^{2i\theta}U^n = e^{i\theta}(U^{n+1} + e^{i\theta}U^n) = e^{i\theta}V^{n+1}, \end{aligned}$$

so $V^{n+1} = e^{-i\theta}V^n$. Note that our initial conditions give that $V^0 = 1$, so we now have the recurrence

$$\begin{aligned} V^{n+1} &= e^{-i\theta}V^n, \quad n \geq 0, \\ V^0 &= 1, \quad \theta \in \mathbb{R}. \end{aligned}$$

Now, this recurrence has characteristic polynomial

$$\rho(\zeta) = \zeta - e^{-i\theta}$$

which has root $\zeta = e^{-i\theta}$, so we have general solution

$$V^n = c(e^{-i\theta})^n.$$

Plugging in $n = 0$, we find that $c = 1$, so

$$V^n = e^{-in\theta}.$$

4.3 Part c

Having solved for V^n, we now have the nonlinear recurrence relation

$$\begin{aligned} &U^n = e^{i\theta}U^{n-1} + e^{-in\theta}, \quad n \geq 1, \\ &U^0 = 1, \quad \theta \in \mathbb{R}. \end{aligned}$$

To solve this, note that a general nonlinear recurrence $u^n = \alpha u^{n-1} + h_n$ has solution

$$\begin{aligned} u^1 &= \alpha u^0 + h_1 \\ u^2 &= \alpha^2 u^0 + \alpha h_1 + h_2 \\ u^3 &= \alpha^3 u^0 + \alpha^2 h_1 + \alpha h_2 + h_3 \\ &\vdots \\ u^n &= \alpha^n u^0 + \sum_{j=1}^{n} \alpha^{n-j} h_j. \end{aligned}$$

Here, we have $u^0 = U^0 = 1$, $\alpha = e^{i\theta}$, $h_n = e^{-in\theta}$, so we have that for $n \geq 1$,

$$U^n = e^{in\theta} + \sum_{j=1}^{n} e^{i(n-j)\theta} e^{-ij\theta} = e^{in\theta} + \sum_{j=1}^{n} e^{i(n-2j)\theta} = \sum_{j=0}^{n} e^{i(n-2j)\theta}.$$

If n is odd,

$$U^n = \sum_{j=0}^{(n-1)/2} e^{i(n-2j)\theta} + \sum_{j=(n+1)/2}^{n} e^{i(n-2j)\theta}.$$

Making the change of index $j \to n-j$ in the second sum,

$$\begin{aligned} U^n &= \sum_{j=0}^{(n-1)/2} e^{i(n-2j)\theta} + \sum_{j=(n-1)/2}^{0} e^{i(n-2(n-j))\theta} = \sum_{j=0}^{(n-1)/2} \left(e^{i(n-2j)\theta} + e^{-i(n-2j)\theta}\right) \\ &= 2 \sum_{j=0}^{(n-1)/2} \cos(n-2j)\theta. \end{aligned}$$

If we again reindex, $j \to \frac{n-1-2j}{2}$,

$$U^n = 2 \sum_{j=0}^{(n-1)/2} \cos(2j+1)\theta.$$

If n is even,

$$U^n = \sum_{j=0}^{n/2-1} e^{i(n-2j)\theta} + e^{i(n-n)\theta} + \sum_{j=n/2+1}^{n} e^{i(n-2j)\theta}.$$

Reindexing $j \to n-j$ in the second sum,

$$\begin{aligned} U^n &= 1 + \sum_{j=0}^{n/2-1} e^{i(n-2j)\theta} + \sum_{j=n/2-1}^{0} e^{i(n-2(n-j))\theta} = 1 + \sum_{j=0}^{n/2-1} \left(e^{i(n-2j)\theta} + e^{-i(n-2j)\theta}\right) \\ &= 1 + 2 \sum_{j=0}^{n/2-1} \cos(n-2j)\theta. \end{aligned}$$

Again reindexing $j \to n/2 - j$,

$$U^n = 1 + 2\sum_{j=1}^{n/2} \cos 2j\theta.$$

To find the x such that U^n is maximized, we again need to consider the even and odd cases separately. In the case where n is odd, each term in the series is of the form $\cos k\theta$ where k is odd. The cosine function has maximum value 1 which it obtains at arguments of the form $2k\pi$ where $k \in \mathbb{Z}$. In the case where n is odd, if we have that $(2j+1)\theta = 2k\pi$ for $j = 0, \ldots, (n-1)/2$, then U^n must be maximized. In order for this to hold at $j = 0$, we need that $\theta = 2k\pi$ for $k \in \mathbb{Z}$. If θ has this form, then clearly $(2j+1)\theta = 2k'\pi$ for some $k' \in \mathbb{Z}$, so each term of the sum is maximized. Thus, U^n is maximized at $\theta = 2k\pi$ for $k \in \mathbb{Z}$. This gives that $x = \cos 2k\pi = 1$, so U^n is maximized at $x = 1$.
If n is even, we have that if $2j\theta = 2k\pi$ for $j = 1, \ldots, n/2$, then U^n must be maximized since its individual components are maximized. In order for this to hold at $j = 1$, we need that $\theta = k\pi$ where $k \in \mathbb{Z}$. If θ has this form, then clearly $2j\theta = 2k'\pi$ for some $k' \in \mathbb{Z}$, so each term of the sum is maximized. Thus, U^n is maximized at $\theta = k\pi$ for $k \in Z$. This gives that $x = \cos k\pi = \pm 1$, so U^n is maximized at $x = \pm 1$.

5 Problem 5

Consider the RK4 method of example 5.13 in the text applied to the problem $u'(t) = \lambda u(t)$. We find that if $z = k\lambda$,

$$\begin{aligned}
y_1 &= u^n \\
y_2 &= u^n + \frac{1}{2}k\lambda y_1 = \left(1 + \frac{z}{2}\right) u^n \\
y_3 &= u^n + \frac{1}{2}k\lambda y_2 = \left(1 + \frac{z}{2} + \frac{z^2}{4}\right) u^n \\
y_4 &= u^n + k\lambda y_3 = \left(1 + z + \frac{z^2}{2} + \frac{z^3}{4}\right) u^n \\
u^{n+1} &= u^n + \frac{k\lambda}{6}\left(y_1 + 2y_2 + 2y_3 + y_4\right) = \left(1 + \frac{z}{6}\left(6 + 3z + z^2 + \frac{z^3}{4}\right)\right) u^n \\
&= \left(1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24}\right) u^n.
\end{aligned}$$

This implies that

$$R(z) = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!}$$

which clearly matches the power series of e^z through the fourth order.

6 Problem 6

Applying the simplification of TR-BDF2 given in (8.6) to the problem $u'(t) = \lambda u(t)$, we first have that

$$u^* = u^n + \frac{k}{4}(\lambda u^n + \lambda u^*),$$

so we can find that

$$u^* = \frac{4+z}{4-z}$$

where $z = k\lambda$. We then have that

$$u^{n+1} = \frac{1}{3}(4u^* - u^n + k\lambda u^{n+1}).$$

Using our value of u^*,

$$3u^{n+1} = \left(4\frac{4+z}{4-z} - 1\right) u^n + zu^{n+1},$$

so

$$(3-z)u^{n+1} = \frac{12+5z}{4-z}u^n,$$

and

$$u^{n+1} = \frac{12+5z}{(3-z)(4-z)}u^n,$$

so we have

$$R(z) = \frac{12+5z}{(3-z)(4-z)}.$$

Now, we expand

$$\frac{1}{3-z} = \frac{1}{3}\frac{1}{1-\frac{z}{3}} = \frac{1}{3}\sum_{j=0}^{\infty}\left(\frac{z}{3}\right)^j = \sum_{j=0}^{\infty}\frac{z^j}{3^{j+1}}.$$

Similarly,

$$\frac{1}{4-z} = \sum_{j=0}^{\infty}\frac{z^j}{4^{j+1}}.$$

Now, using the Cauchy product,

$$\frac{1}{(3-z)(4-z)} = \sum_{j=0}^{\infty}\frac{z^j}{3^{j+1}}\sum_{k=0}^{\infty}\frac{z^k}{4^{k+1}} = \sum_{j=0}^{\infty}\sum_{\ell=0}^{j}\frac{z^\ell}{3^{\ell+1}}\frac{z^{j-\ell}}{4^{j-\ell+1}} = \sum_{j=0}^{\infty}\sum_{\ell=0}^{j}\frac{z^j}{3^{\ell+1}4^{j-\ell+1}}.$$

Thus,

$$\begin{aligned}
R(z) = (12+5z)\sum_{j=0}^{\infty}\sum_{\ell=0}^{j}\frac{z^j}{3^{\ell+1}4^{j-\ell+1}} &= \sum_{j=0}^{\infty}\sum_{\ell=0}^{j}\left(\frac{12z^j}{3^{\ell+1}4^{j-\ell+1}} + \frac{5z^{j+1}}{3^{\ell+1}4^{j-\ell+1}}\right) \\
&= 1 + \left(\left(\frac{z}{3}+\frac{z}{4}\right)+\frac{5z}{12}\right) + \left(\left(\frac{z^2}{16}+\frac{z^2}{12}+\frac{z^2}{9}\right)+\left(\frac{5z^2}{36}+\frac{5z^2}{48}\right)\right) + \dots \\
&= 1 + z + \frac{z^2}{2} + \dots
\end{aligned}$$

which clearly matches the power series of e^z through the second order.

7 Problem 7

Consider the time-dependent matrices

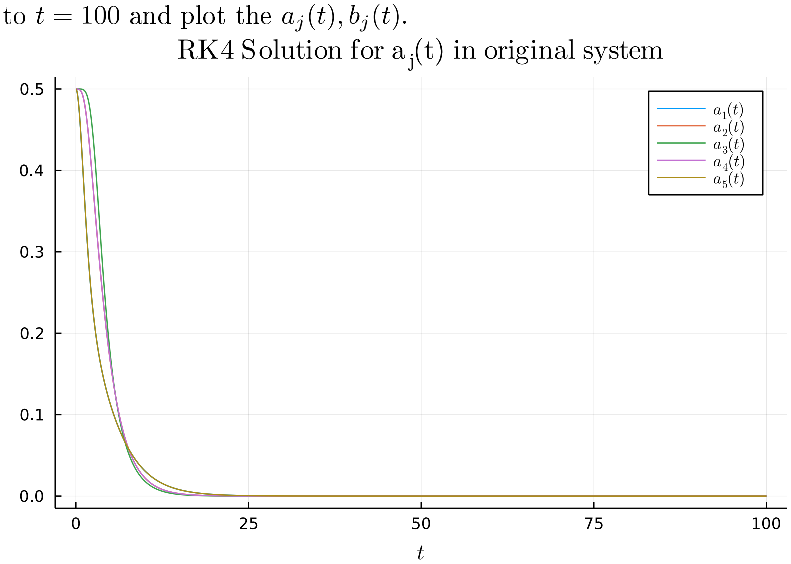
$$T_N(t) = \begin{bmatrix} b_1(t) & a_1(t) & & & \\ a_1(t) & b_2(t) & a_2(t) & & \\ & a_2(t) & b_3(t) & \ddots & \\ & & \ddots & \ddots & a_{N-1}(t) \\ & & & a_{N-1}(t) & b_N(t) \end{bmatrix},$$

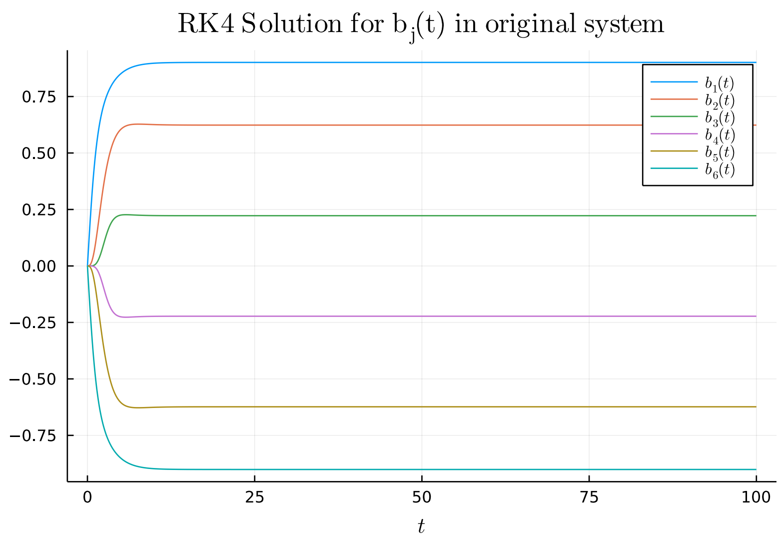
$$S_N(t) = \begin{bmatrix} 0 & a_1(t) & & & \\ -a_1(t) & 0 & a_2(t) & & \\ & -a_2(t) & 0 & \ddots & \\ & & \ddots & \ddots & a_{N-1}(t) \\ & & & -a_{N-1}(t) & 0. \end{bmatrix}$$

With initial conditions $b_j(0) = 0$, $j = 1, 2, \dots, N$, $a_j(0) = 1/2$, $j = 1, 2, \dots, N-1$ and $N = 6$. In Julia, we use RK4 to solve

$$T_N'(t) = S_N(t)T_N(t) - T_N(t)S_N(t),$$

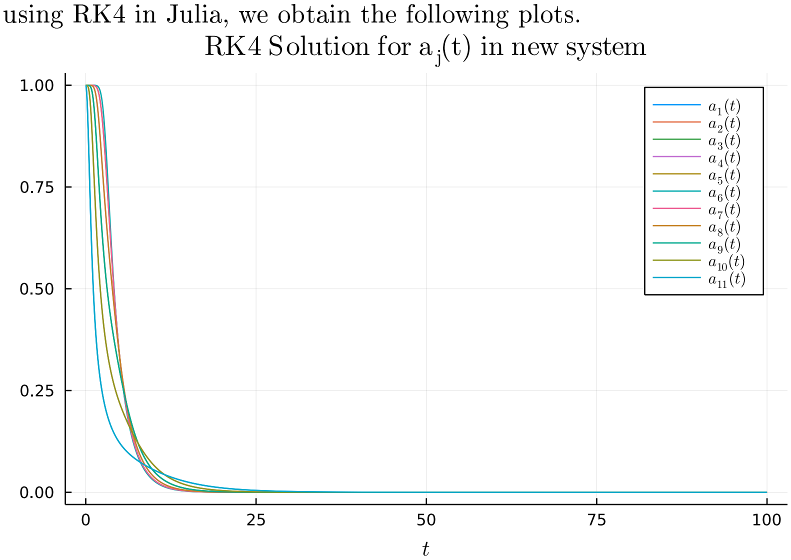
to $t = 100$ and plot the $a_j(t), b_j(t)$.

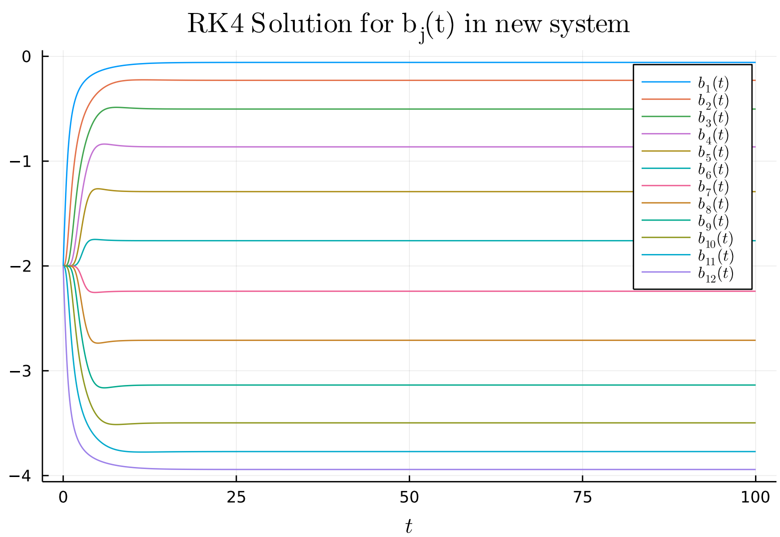




It appears that the off-diagonal entries of $T_N(t)$ tend to zero and the diagonal entries tend to the eigenvalues of $T_N(0)$. This is because the matrices constitute a Lax pair which implies that the eigenvalues of $T_N(t)$ are independent of t, so if the off-diagonal entries tend to zero, the diagonal entries must tend to the eigenvalues.

Now, consider $b_j(0) = -2$ and $a_j(0) = 1$, $j = 1, 2, \ldots, N$ and $N = 12$. Again using RK4 in Julia, we obtain the following plots.





Again, it appears that the off-diagonal entries of $T_N(t)$ tend to zero and the diagonal entries tend to the eigenvalues of $T_N(0)$.
See Appendix A for the Julia code used to solve this problem.

8 Appendix A

The following is the code used for problem 1.

```
using Plots, LaTeXStrings, Elliptic.Jacobi, Printf

β₁ = 0.
β₂ = 1.
β₃ = 10.
c = (β₁ + β₂ + β₃)/3
v = t -> β₂ + (β₃ - β₂)*cn(sqrt((β₃-β₁)/12)*t, (β₃-β₂)/(β₃-β₁) )^2

f = u -> [u[2], u[3], u[2]*(c - u[1])]
Df = u -> [0. 1. 0.; 0. 0. 1.; -u[2] c-u[1] 0.]
u₀ = [β₃,0.,-1.0/6*(β₃-β₁)*(β₃-β₂)]

h = 0.0001
[v(0), (v(h)-v(-h))/(2h), (v(h)-2v(0)+v(-h))/(h^2)]

T = 10.# Final time.

k = .02 #stepsize
p = 7
data = zeros(p)
ks = zeros(p)
for i = 1:p
    global k = k/2
    n = convert(Int64,ceil(T/k))
    println("Number of time steps = ", n)
    U = zeros(3,n+1) # To save the solution values
    U[:,1] = u₀
    global t = zeros(n+1,1)
    t[1] = 0.
```

```
    max_iter = 10
    # implementation of Heun's method
    for i = 2:n+1
        t[i] = t[i-1] + k
        Y1 = U[:,i-1]
        f1 = f(Y1)
        Y2 = U[:,i-1] + (k/3)*f1
        f2 = f(Y2)
        Y3 = U[:,i-1] + (2k/3)*f2
        f3 = f(Y3)
        U[:,i] = U[:,i-1] + (k/4)*(f1+3*f3)
    end
    data[i] = abs(U[1,end] - v(t[end]))
    ks[i] = k
end
p1 = plot(ks,data,lw=2,ms=5,marker=:d, minorgrid = true, xaxis=:log, yaxis= :log,
    label=L"\mathrm{3rd~order~RK}", legend = :bottomright)
plot!(ks,ks.^3,lw=2,ms=5,marker=:d, minorgrid = true,
     label=L"\mathrm{O(k^3)~reference~line}")
xlabel!(latexstring("k"))
ylabel!(L"\mathrm{Error}")
title!("Convergence of RK3")
savefig(p1, "p1.pdf")
display(p1)

#print table
@printf("%s        | %s      | %s   \n","k","error","error reduction")
@printf("%f | %0.4e |       \n",ks[1],data[1])
for j=2:7
    @printf("%f | %0.4e |      %0.4f  \n",ks[j],data[j],data[j-1]/data[j])
end
```

The following is the code used from problem 7.

```
using Plots, LaTeXStrings, LinearAlgebra, SparseArrays, Arpack, Printf

# 1st case
N = 6
T = spdiagm(-1=>0.5*ones(N-1),1=>0.5*ones(N-1))
initeig = eigs(T)

t = 100. # Final time.
k = 0.01 # Step size

# function to compute ST-TS
function rhsf(T)
    S = spdiagm(-1=>-diag(T,1),1=>diag(T,1))
    S*T-T*S
end

# implement RK4
n = convert(Int64,t/k) # Number of time steps, converted to Int64
tvec = k*(0:n)
A = zeros(N-1,n+1)
B = zeros(N,n+1)
A[:,1] = diag(T,1)
B[:,1] = diag(T)
for i = 2:n+1
    local Y1 = T
    f1 = rhsf(Y1)
    Y2 = T + (k/2)*f1
    f2 = rhsf(Y2)
    Y3 = T + (k/2)*f2
    f3 = rhsf(Y3)
    Y4 = T + k*f3
```

```
    f4 = rhsf(Y4)
    global T += (k/6)*(f1+2*f2+2*f3+f4)
    A[:,i] = diag(T,1)
    B[:,i] = diag(T)
end

p1 = plot(
    title=latexstring("\\mathrm{RK4~Solution~for~b_j(t)~in~original~system}"))
xlabel!(latexstring("t"))
for j = 1:N
    plot!(tvec,B[j,:],label=latexstring(@sprintf("b_%i(t)",j)))
end
savefig(p1, "p7i.pdf")
display(p1)

p1a = plot(
    title=latexstring("\\mathrm{RK4~Solution~for~a_j(t)~in~original~system}"))
xlabel!(latexstring("t"))
for j = 1:N-1
    plot!(tvec,A[j,:],label=latexstring(@sprintf("a_%i(t)",j)))
end
savefig(p1a, "p7ii.pdf")
display(p1a)

# 2nd case
N = 12
T = spdiagm(0=>-2*ones(N),-1=>ones(N-1),1=>ones(N-1))
initeig2 = eigs(T)

# implement RK4
n = convert(Int64,t/k) # Number of time steps, converted to Int64
tvec = k*(0:n)
A2 = zeros(N-1,n+1)
B2 = zeros(N,n+1)
A2[:,1] = diag(T,1)
B2[:,1] = diag(T)
for i = 2:n+1
    local Y1 = T
    f1 = rhsf(Y1)
    Y2 = T + (k/2)*f1
    f2 = rhsf(Y2)
    Y3 = T + (k/2)*f2
    f3 = rhsf(Y3)
    Y4 = T + k*f3
    f4 = rhsf(Y4)
    global T += (k/6)*(f1+2*f2+2*f3+f4)
    A2[:,i] = diag(T,1)
    B2[:,i] = diag(T)
end

p2 = plot(
    title=latexstring("\\mathrm{RK4~Solution~for~b_j(t)~in~new~system}"))
xlabel!(latexstring("t"))
for j = 1:N
    plot!(tvec,B2[j,:],label=latexstring(@sprintf("b_{%i}(t)",j)))
end
savefig(p2, "p7iii.pdf")
display(p2)

p2a = plot(
    title=latexstring("\\mathrm{RK4~Solution~for~a_j(t)~in~new~system}"))
xlabel!(latexstring("t"))
for j = 1:N-1
    plot!(tvec,A2[j,:],label=latexstring(@sprintf("a_{%i}(t)",j)))
end
savefig(p2a, "p7iv.pdf")
display(p2a)
```