MATH 525 Homework 9

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1 Problem 1

Let $T \in \mathcal{B}(X)$ and $\lambda \in \sigma_C(T)$. Then, $\lambda I - T$ cannot be bounded below. To see this, assume the contrary, i.e., there exists some $C > 0$ such that $\|(\lambda I - T)x\| \geq C\|x\|$ for all $x \in X$. Since the range of $\lambda I - T$ is dense in X, for any $x \in X$, there exists some sequence $\{x_n\}_{n=1}^{\infty} \subset X$ such that $\{x_n\} \to x$ and $x_n = (\lambda I - T)y_n$ for some $y_n \in X$. Then,

$$\lim_{n,m\to\infty} \|y_n - y_m\| \leq C^{-1} \lim_{n,m\to\infty} \|x_n - x_m\| = 0,$$

so $\{y_n\}$ is Cauchy. Because X is complete, $\{y_n\} \to y$ for some $y \in X$. Then, the continuity of $(\lambda I - T)$ implies that

$$x = \lim_{n\to\infty} (\lambda I - T)y_n = (\lambda I - T)y.$$

Thus, for all $x \in X$, there exists some $y \in X$ such that $x = (\lambda I - T)y$, so $\lambda I - T$ is surjective. By assumption, $\lambda I - T$ is injective and bounded below, so $\lambda I - T$ must be invertible which is a contradiction, meaning that $\lambda I - T$ cannot be bounded below. This implies that

$$\inf_{\|x\|=1} \|(\lambda I - T)x\| = \inf_{x\neq 0} \frac{\|(\lambda I - T)x\|}{\|x\|} = 0,$$

so there exists some sequence $\{x_n\} \subset X$ such that $\|x_n\| = 1$ for all n and

$$\lim_{n\to\infty} \|Tx_n - \lambda x_n\| = \lim_{n\to\infty} \|(\lambda I - T)x_n\| = 0,$$

as desired.

2 Problem 2

2.1 Part a

Let $\mathcal{H}$ be a Hilbert space and $S \in \mathcal{B}(\mathcal{H})$. Let $x \in \overline{S(\mathcal{H})}$. Then, there exists a sequence $\{y_n\} \subset \mathcal{H}$ such that $\{Sy_n\} \to x$. Then, for any $y \in \ker(S^*)$, by the continuity of inner products,

$$\langle x, y\rangle = \lim_{n\to\infty} \langle Sy_n, y\rangle = \lim_{n\to\infty} \langle y_n, S^*y\rangle = \lim_{n\to\infty} \langle y_n, 0\rangle = 0.$$

Thus, $x \in \ker(S^*)^{\perp}$, so $\overline{S(\mathcal{H})} \subset \ker(S^*)^{\perp}$.

Conversely, let $x \in \ker(S^*)^{\perp}$. Because $\overline{S(\mathcal{H})}$ is a closed subspace, we can uniquely decompose $x = y + z$ where $y \in \overline{S(\mathcal{H})}$ and $z \in \overline{S(\mathcal{H})}^{\perp}$. Then, for any $w \in \mathcal{H}$,

$$\langle S^*z, w\rangle = \langle z, Sw\rangle = 0,$$

so $S^*z = 0$ and $z \in \ker(S^*)$. Since $\overline{S(\mathcal{H})} \subset \ker(S^*)^{\perp}$, $y \in \ker(S^*)^{\perp}$, so

$$0 = \langle x, z\rangle = \langle y, z\rangle + \langle z, z\rangle = \langle z, z\rangle.$$

Thus, $z = 0$ and $x = y \in \overline{S(\mathcal{H})}$, so $\overline{S(\mathcal{H})} = \ker(S^*)^{\perp}$.

2.2 Part b

Let $\lambda \in \sigma(T) \setminus \sigma_P(T)$. We first note that for any $x, y \in \mathcal{H}$,

$$\langle x, (\lambda I - T)^* y\rangle = \langle (\lambda I - T)x, y\rangle = \lambda\langle x, y\rangle - \langle Tx, y\rangle = \langle x, \overline{\lambda} y\rangle + \langle x, T^* y\rangle = \langle x, (\overline{\lambda} I - T^*)y\rangle,$$

so $(\lambda I - T)^* = \overline{\lambda} I - T^*$. If $\overline{\lambda} \in \sigma_P(T^*)$, then $\ker(\overline{\lambda} I - T^*) \neq \{0\}$. This implies that $\ker\left((\lambda I - T)^*\right)^\perp \neq \mathcal{H}$, so by part a, $\overline{(\lambda I - T)(\mathcal{H})} \neq \mathcal{H}$, and the range of $\lambda I - T$ is not dense, meaning that $\lambda \in \sigma_R(T)$ since $\lambda \in \sigma(T) \backslash \sigma_\rho(T)$. Conversely, if $\overline{\lambda} \notin \sigma_P(T^*)$, then $\ker(\overline{\lambda} I - T^*) = \{0\}$. This means that $\ker\left((\lambda I - T)^*\right)^\perp = \mathcal{H}$, so by part a, $\overline{(\lambda I - T)(\mathcal{H})} = \mathcal{H}$. Thus, the range of $\lambda I - T$ is dense in $\mathcal{H}$, so $\lambda \notin \sigma_R(T)$. Thus, $\lambda \in \sigma_R(T)$ if and only if $\overline{\lambda} \in \sigma_P(T^*)$.

3 Problem 3

Let $\mathcal{H} = \ell^2(\mathbb{N})$ and define the left and right shift operators:

$$S_L\left(x_1, x_2, \ldots\right) = \left(x_2, x_3, \ldots\right), \quad S_R\left(x_1, x_2, \ldots\right) = \left(0, x_1, x_2, \ldots\right).$$

3.1 Part a

For any $x, y \in \ell^2(\mathbb{N})$,

$$\langle x, S_R y\rangle = \sum_{j=2}^{\infty} x_j y_{j-1} = \sum_{j=1}^{\infty} x_{j+1} y_j = \langle S_L x, y\rangle,$$

so $S_L^* = S_R$.

3.2 Part b

For any $z \in \mathbb{C}$, consider the series expansion for $z \neq 0$

$$(zI - S_L)^{-1} = \sum_{k=0}^{\infty} z^{-k-1} S_L^k.$$

Then, for any $x \in \mathcal{H}$,

$$\left((zI - S_L)^{-1} x\right)_j = \sum_{k=0}^{\infty} z^{-k-1} x_{k+j}.$$

If x is any finite sequence, this series has only a finite number of nonzero terms, so it must converge. Furthermore, $(zI - S_L)^{-1}x$ can only have a finite number of nonzero terms, so $(zI - S_L)^{-1}x \in \ell^2(\mathbb{N})$ is defined, and x is in the range of $zI - S_L$. Since finite sequences are dense in $\ell^2(\mathbb{N})$, the range of $zI - S_L$ is dense in $\mathcal{H}$ for all $z \in \mathbb{C}$, meaning that $\sigma_R(S_L) = \emptyset$. Now, let $x \in \ker(zI - S_L)$. Then,

$$0 = \left((zI - S_L)x\right)_j = z x_j - x_{j+1},$$

so $x_{j+1} = z x_j$ for all $j \in \mathbb{N}$ and $x = x_1(1, z, z^2, \ldots)$. Then,

$$\|x\|^2 = \sum_{j=1}^{\infty} |x_1|^2 |z|^{2j}.$$

This sum is finite if and only if $|z| < 1$, so $\ker(zI - S_L) \neq \{0\}$ if and only if $|z| < 1$, meaning that $\{z : |z| < 1\} \subset \sigma_P(S_L)$. Since $\sigma(S_L)$ is closed, $\overline{\{z : |z| < 1\}} = \{z : |z| \leq 1\} \subset \sigma(S_L)$. Now, we observe that for any $x \in \mathcal{H}$,

$$\|S_L x\|^2 = \sum_{j=1}^{\infty} |x_{j+1}|^2 = \sum_{j=2}^{\infty} |x_j|^2 \leq \|x\|^2,$$

so $\|S_L\| \leq 1$; however, for any $x \in \mathcal{H}$ such that $x_1 = 0$, $\|x\| = \|S_L x\|$, so $\|S_L\| = 1$. This means that $zI - T$ is invertible for all $z \in \mathbb{C}$ such that $|z| > 1$, so we must have that $\{z : |z| \leq 1\} = \sigma(S_L)$. Since $\ker(zI - S_L) \neq \{0\}$ only if $|z| < 1$ and there is no residual spectrum, we conclude that $\sigma_P(S_L) = \{z : |z| < 1\}$ and $\sigma_C(S_L) = \{z : |z| = 1\}$.

3.3 Part c

First, let $|z| < 1$. Then, $\overline{z} \in \sigma_P(S_L)$, so $\ker(\overline{z}I - S_L) = \ker((zI - S_R)^*) \neq 0$. By Problem 2a, this implies that $zI - S_R$ is not dense in $\mathcal{H}$, so $zI - S_R$ is not invertible and $z \in \sigma_R(S_R)$. This means that $\{z : |z| < 1\} \subset \sigma(S_R)$, so $\overline{\{z : |z| < 1\}} = \{z : |z| \leq 1\} \subset \sigma(S_R)$. For any $x \in \ell^2(\mathbb{N})$, $\|S_R x\| = \|x\|$, so $\|S_R\| = 1$, meaning that $\sigma(S_R) \subset \{z : |z| \leq 1\}$, so $\sigma(S_R) = \{z : |z| \leq 1\}$. Now, we note that $\sigma_P(S_R) = \emptyset$ because for any $x \in \ker(zI - S_R)$, for all $j \in \mathbb{N}$,

$$0 = ((zI - S_R)x)_j = \begin{cases} zx_j - x_{j-1}, & j \geq 2, \\ zx_1, & j = 1, \end{cases}$$

so $x_j = 0$ for all j, meaning that $\ker(zI - S_R) = \{0\}$ for all $z \in \mathbb{C}$. Then, Problem 2b gives that if $z \in \sigma(S_R)$, then $z \in \sigma_R(S_R)$ if and only if $\overline{z} \in \sigma_P(S_L)$. If $|z| = 1$, then $\overline{z} \notin \sigma_R(S_L)$, so $z \notin \sigma_R(S_R)$. Since S_R has no eigenvalues, we must have that $z \in \sigma_C(S_R)$, so $\{z : |z| = 1\} \subset \sigma_C(S_R)$. We already have that $\{z : |z| < 1\} \subset \sigma_R(S_R)$, so we have classified the entire spectrum and can conclude that $\{z : |z| < 1\} = \sigma_R(S_R)$ and $\{z : |z| = 1\} = \sigma_C(S_R)$.

4 Problem 4

Let X and Y be Banach spaces with respective duals X^* and Y^* and assume that $T^* : Y^* \to X^*$ is compact. Then, $T^{**} : X^{**} \to Y^{**}$ is compact. For any $y \in X^*$ and $x \in X$,

$$(T^{**}\hat{x})(y) = \hat{x}(T^* y) = T^* y(x) = y(Tx) = \widehat{Tx}(y),$$

so $T^{**}\hat{x} = \widehat{Tx}$ for all $x \in X$. Let $\{x_n\} \subset X$ be a bounded sequence. Then, since the embedding $X \to X^{**}$ is norm-preserving, $\{\widehat{x_n}\} \subset X^{**}$ is also a bounded sequence. The compactness of T^{**} then implies that the sequence $\{\widehat{Tx_n}\} = \{T^{**}\widehat{x_n}\} \subset Y$ has a convergent subsequence. Denote this subsequence by $\{\widehat{Tx_{n_j}}\}$. This subsequence is Cauchy, so

$$0 = \lim_{j,k\to\infty} \left\|\widehat{Tx_{n_j}} - \widehat{Tx_{n_k}}\right\| = \lim_{j,k\to\infty} \left\|T(\widehat{x_{n_j} - x_{n_k}})\right\| = \lim_{j,k\to\infty} \left\|T(x_{n_j} - x_{n_k})\right\| = \lim_{j,k\to\infty} \left\|Tx_{n_j} - Tx_{n_k}\right\|,$$

meaning that $\{Tx_{n_j}\} \subset Y$ is also Cauchy. Since Y is complete, $\{Tx_{n_j}\}$ is convergent, meaning that we have found a convergent subsequence of $\{Tx_n\} \subset Y$. Thus, T is also compact.

5 Problem 5

Let $h : [0,1] \to \mathbb{C}$ be a continuous function and define the multiplication operator T_h on $L^2([0,1], m)$ by $(T_h f)(x) = h(x)f(x)$. First, let $\lambda \notin \text{range}(h)$. Denoting $S_\lambda = \lambda I - T_h$, let $S_\lambda f = 0$. Then, $(\lambda - h(x))f(x) = 0$ for almost all x, meaning that $f(x) = 0$ for all x since $\lambda \neq h(x)$ for any x. This means that $f = 0$ in the L^2-sense, so S_λ is injective. Similarly, for any $f \in L^2$, since $\lambda \neq h(x)$ for any x, $g(x) = \frac{f(x)}{\lambda - h(x)}$ is in L^2, and $S_\lambda g = f$, so S_λ is surjective. Corollary 5.11 in Folland then implies that S_λ is invertible, so $\sigma(T_h) \subset \text{range}(h)$.

To see that range(h) is precisely the spectrum, let $\lambda \in \text{range}(h)$. Let $y \in [0,1]$ be a point such that $h(y) = \lambda$. Since h is continuous, there exists some positive decreasing sequence $\{\delta_n\}$ such that $|h(x) - h(y)| < \frac{1}{n}$ whenever $|x - y| < \delta_n$. Define the functions $f_n = \frac{1}{2\delta_n}\mathbb{1}_{(x-\delta_n, x+\delta_n)}$ and note that these have norm 1. Then,

$$\|S_\lambda f_n\|_{L^2}^2 = \int \left|(h(x) - h(y))\frac{1}{2\delta_n}\mathbb{1}_{(x-\delta_n, x+\delta_n)}(x)\right|^2 \mathrm{d}x \leq \frac{1}{n^2}.$$

Thus, $\lim_{n\to\infty} \|S_\lambda f_n\|_{L^2} = 0$, so

$$\inf_{\|f\|=1} \|S_\lambda f\| = 0,$$

meaning that S_λ and $\lambda \in \sigma(T_h)$.

To find the eigenvalues of T_h, let $\lambda \in \text{range}(h)$ and set $S_\lambda f = 0$. Then, $(\lambda - h(x))f(x) = 0$ for almost all x. This means that $h(x) = \lambda$ at all x for which $f(x) = 0$. For f to be nonzero in the L^2-sense, there must be some set A such that $m(A) > 0$ and $f \neq 0$ on A. Thus, S_λ has a nontrivial kernel if and only if $h(x) = \lambda$ on a set of positive measure. That is, λ is an eigenvalue if and only if there is a set A such that $m(A) > 0$ and $h(x) = \lambda$ for all $x \in A$.

Using this, we may show that $\sigma_R(T_h) = \emptyset$. Since L^2 is a Hilbert space, we find the adjoint of T_h by noting that for any $f, g \in L^2$,

$$\langle T_h f, g\rangle = \int h(x)f(x)\overline{g(x)}\mathrm{d}x = \int f(x)\overline{\overline{h(x)}g(x)}\mathrm{d}x = \langle f, T_{\overline{h}}g\rangle,$$

so $T_h^* = T_{\overline{h}}$. Let $\lambda \in \sigma(T_h) \setminus \sigma_P(T_h)$. Then, by Problem 2, $\lambda \in \sigma_R(T_h)$ if and only if $\overline{\lambda} \in \sigma_P(T_{\overline{h}})$. This occurs if and only if $\overline{h(x)} = \overline{\lambda}$ for all x in some set of positive measure. Conjugating, this is true if and only if $h(x) = \lambda$ for all x in some set of positive measure, i.e., $\lambda \in \sigma_P(T_h)$. However, we assumed that $\lambda \notin \sigma_P(T_h)$, so $\lambda \notin \sigma_R(T_h)$ as well. Thus, $\sigma_R(T_h) = \emptyset$.