AMATH 567 Homework 7

Cade Ballew #2120804

November 17, 2021

1 Problem 1 (3.2.2)

1.1 Part b

From equation 3.2.25 in the text, we have that $\frac{1}{1+z^2} = \sum_{n=0}^{\infty}(-1)^n z^{2n}$ for $|z| < 1$. Thus,

$$\frac{z}{1+z^2} = z\sum_{n=0}^{\infty}(-1)^n z^{2n} = \sum_{n=0}^{\infty}(-1)^n z^{2n+1}$$

for $|z| < 1$.

1.2 Part c

Note that the derivatives of $f(x) = \cosh x$ follow a pattern, namely, for $k \geq 0$, $f^{(2k+1)}(z) = \sinh z$ and $f^{(2k)}(z) = \cosh z$. Thus, $f^{(2k+1)}(0) = 0$ and $f^{(2k)}(0) = 1$. Invoking theorem 3.2.2 in the text to write a Taylor expansion, we get that

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$$

which is valid for $|z| < \infty$, because $f(z)$ is entire.

2 Problem 2 (3.3.1)

Consider $f(z) = \frac{1}{1+z^2}$

2.1 Part a

Equation 3.2.25 in the text gives that the Taylor series for f is

$$\frac{1}{1+z^2} = \sum_{n=0}^{\infty}(-1)^n z^{2n}$$

for $|z| < 1$.

2.2 Part b

Now, note that $|z| > 1$ iff $|1/z| < 1$, so we can write

$$\frac{1}{1+z^2} = \frac{1}{z^2}\frac{1}{1+(1/z)^2} = \frac{1}{z^2}\sum_{n=0}^{\infty}(-1)^n(1/z)^{2n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{z^{2n+2}}$$

for $|z| > 1$.

3 Problem 3 (3.3.5)

Let

$$f(z) = e^{\frac{t}{2}(z-1/z)} = \sum_{n=-\infty}^{\infty} J_n(t)z^n.$$

By the definition of a Laurent series,

$$J_n(t) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z^{n+1}}dz = \frac{1}{2\pi i}\oint_C \frac{e^{\frac{t}{2}(z-1/z)}}{z^{n+1}}dz$$

for some closed contour C enclosing $z = 0$, because f is analytic except for at the origin. Take C to be the unit circle counter-clockwise and parameterize this as $z = e^{i\theta}$ which gives $dz = ie^{i\theta}d\theta$ for $\theta \in [-\pi, \pi]$. Then,

$$\begin{aligned}J_n(t) &= \frac{1}{2\pi i}\int_{-\pi}^{\pi}\frac{e^{t/2(e^{i\theta}-e^{-i\theta}})}{e^{(n+1)i\theta}}ie^{i\theta}d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{it\sin\theta}}{e^{in\theta}}d\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-i(n\theta - t\sin\theta)}d\theta\end{aligned}$$

follows from the fact that $e^{i\theta} - e^{-i\theta} = 2i\sin\theta$. Using Euler's formula to break up this integral,

$$J_n(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\underbrace{\cos(n\theta - t\sin\theta)}_{A(\theta)}d\theta - \frac{i}{2\pi}\int_{-\pi}^{\pi}\underbrace{\sin(n\theta - t\sin\theta)}_{B(\theta)}d\theta.$$

Observe that as defined above,

$$\begin{aligned}A(-\theta) &= \cos(-n\theta - t\sin(-\theta)) = \cos(-n\theta + t\sin\theta) = \cos(-(n\theta - t\sin\theta)) \\ &= \cos(n\theta - t\sin\theta) = A(\theta)\end{aligned}$$

and

$$\begin{aligned}B(-\theta) &= \sin(-n\theta - t\sin(-\theta)) = \cos(-n\theta + t\sin\theta) = \sin(-(n\theta - t\sin\theta)) \\ &= -\sin(n\theta - t\sin\theta) = -B(\theta).\end{aligned}$$

Thus, $A(\theta)$ is an even function of θ and $B(\theta)$ is an odd function of θ. We can use this fact to rewrite our integral as

$$J_n(t) = \frac{1}{\pi}\int_0^{\pi}\cos(n\theta - t\sin\theta)d\theta - \frac{i}{2\pi}0 = \frac{1}{\pi}\int_0^{\pi}\cos(n\theta - t\sin\theta)d\theta.$$

4 Problem 4 (3.5.1)

4.1 Part a

Note that

$$\frac{e^{z^2}-1}{z^2} = \frac{1}{z^2}\sum_{n=1}^{\infty}\frac{z^{2n}}{n!} = \sum_{n=1}^{\infty}\frac{z^{2n-2}}{n!}.$$

e^{z^2} is entire, so the only singularity is where $z^2 = 0$, namely at $z = 0$. However, we have shown that this singularity is removable, because the above series converges to 0 at $z = 0$ and has no negative powers of z. This singularity is also isolated, because it is the only singularity.

4.2 Part b

Now, note that

$$\frac{e^{2z}-1}{z^2} = \frac{1}{z^2}\sum_{n=1}^{\infty}\frac{(2z)^n}{n!} = \sum_{n=1}^{\infty}\frac{2^n}{n!}z^{2n-2}.$$

As in part a, the only singularity is at $z = 0$, because e^{2z} is entire and it is the only zero of z^2. We have now derived a Laurent series centered at $z = 0$ for this function whose first nonzero term is at $j = -1$, meaning that $z = 0$ is a simple pole with strength 2 (take $n = 1$ to find this). Of course, it is isolated, because it is the only singularity.

4.3 Part c

When analyzing the function $e^{1/z}$, we saw that isolated simple poles in the exponent function lead to essential singularities in the combined function. Thus, the isolated simple poles of $\tan z$ are essential singularities of $e^{\tan z} = \sum_{n=0}^{\infty}\frac{\tan^n z}{n!}$. From example 3.5.3 in the text, we know that $\tan z$ has only isolated simple poles which occur at $\pi/2 + k\pi$ for any $k \in \mathbb{Z}$, meaning that $e^{\tan z}$ has essential singularities at $\pi/2 + k\pi$ for any $k \in \mathbb{Z}$. These are the only singularities, because e^z is an entire function. These singularities are isolated by the definition of an essential singularity.

4.4 Part g

The function $\begin{cases} z^2 & |z| \leq 1 \\ 1/z^2 & |z| > 1 \end{cases}$ exhibits a boundary jump discontinuity at $|z| = 1$. This is because both functions are analytic and both have continuous limits to the boundary $|z| = 1$ (we can just plug in a value at a point on the boundary to find a limit, because both functions are continuous there) but are not necessarily equal on the boundary. We can see this by plugging in $z = e^{i\theta}$ which yields the functions $e^{2i\theta}$ and $e^{-2i\theta}$, functions that are only equivalent when $\theta = \pi k$ for $k \in \mathbb{Z}$, isolated points.
Clearly, these singularities are not isolated.

4.5 Part h

Consider

$$f(z) = \sum_{n=1}^{\infty} \frac{z^{n!}}{n!}.$$

Taking a ratio test

$$\frac{|z^{(n+1)!}/(n+1)!|}{|z^{n!}/n!|} = \frac{|z^{((n+1)!-n!)}|}{n+1} = \frac{|z^{n!n}|}{n+1}$$

yields a radius of convergence $R < 1$. However, on the boundary $|z| = 1$, we find that our terms are bounded

$$\left| \frac{z^{n!}}{n!} \right| = \frac{|z|^{n!}}{n!} = \frac{1}{n!},$$

so our series is bounded by

$$\sum_{n=1}^{\infty} \frac{1}{n!} = e - 1,$$

meaning that we converge uniformly on the boundary as well by the Weierstrass M-test if we take $M_n = \frac{1}{n!}$. Thus, we can take the derivative termwise which gives that

$$f'(z) = \sum_{n=1}^{\infty} z^{n!-1} = \frac{1}{z} \sum_{n=1}^{\infty} z^{n!}.$$

Now, this diverges on a dense set of points on $|z| = 1$, because for any arc on $|z| = 1$, we can find some $N \in \mathbb{N}$ such that $z^{N!} = 1$ due to the density of the rationals in $\mathbb{R}$. However, $N!$ is a factor of $n!$ for any $n \geq N$, so $z^{n!} = 1$ for any such n. This means that $\sum\limits_{n=N}^{\infty} z^{n!}$ diverges, meaning that $f'(z)$ diverges. Thus, analytic continuation is not possible, and $|z| = 1$ forms a natural barrier which is, of course, not isolated.

5 Problem 5 (3.5.3)

5.1 Part b

Example 3.5.3 shows that $f(z) = \tan z$ is meromorphic and has simple poles of strength -1 at $z = \pi/2 + k\pi$ for any $k \in \mathbb{Z}$ (Bernard said it was okay to just cite this).

5.2 Part d

Note that

$$f(z) = \frac{e^z - 1 - z}{z^4} = \frac{1}{z^4} \sum_{n=2}^{\infty} \frac{z^n}{n!} = \sum_{n=2}^{\infty} \frac{z^{n-4}}{n!}.$$

The only possible singularity is at $z = 0$ because the numerator of f is entire and it is the only root of z^4. The above Laurent series has its first nonzero term at $j = -2$, so $z = 0$ is a double pole with strength $1/2$ (look at the n=2 term to see this). Clearly, this means that f is meromorphic.

5.3 Part e

Consider
$$f(z) = \frac{1}{2\pi i} \oint_C \frac{w}{(w^2-2)(w-z)} dw$$
for $|z| < 1$ where C is the unit circle centered at the origin. The function $w/(w^2-2)$ is analytic in and on C, so we apply Cauchy's integral formula to find that
$$f(z) = \frac{z}{z^2-2} = \frac{z}{(z+\sqrt{2})(z-\sqrt{2})}$$
for $|z| < 1$. Now, we analytically extend this function to the rest of the complex plane sans its singularities. Then, we can see from the above expression for f that we have single poles at $z = \pm\sqrt{2}$ which are the only singularities, meaning that f is meromorphic. Plugging in these values to the function sans the problematic factor, we get that these poles have strength $\pm\sqrt{2}/(\pm\sqrt{2} \pm \sqrt{2}) = 1/2$.

6 Problem 6 (3.6.6)

Let Γ be given by

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}$$
for $z \neq 0, -1, -2, \ldots$ and $\gamma =$ constant.

6.1 Part a

Letting ln denote the principal complex logarithm, we take the log of both sides to get
$$-\ln(\Gamma(z)) = \ln z + \gamma z + \sum_{n=1}^{\infty} \left(\ln\left(1+\frac{z}{n}\right) - \frac{z}{n}\right).$$
Differentiating both sides,

$$-\frac{\Gamma'(z)}{\Gamma(z)} = \frac{1}{z} + \gamma + \sum_{n=1}^{\infty} \left(\frac{1/n}{1+z/n} - \frac{1}{n}\right),$$
so
$$\frac{\Gamma'(z)}{\Gamma(z)} = -\frac{1}{z} - \gamma - \sum_{n=1}^{\infty} \left(\frac{1}{n+z} - \frac{1}{n}\right).$$

Note that we can differentiate inside the summation because page 83 of the lecture notes gives that the above infinite product is uniformly convergent (if we take $z = -z$) and the proof of the M-test for infinite products in the notes gives that the corresponding infinite sum is uniformly convergent if the infinite product is uniformly convergent.

6.2 Part b

Observe that

$$\begin{aligned}\frac{\Gamma'(z+1)}{\Gamma(z+1)} - \frac{\Gamma'(z)}{\Gamma(z)} - \frac{1}{z} &= -\frac{1}{z+1} - \gamma - \sum_{n=1}^{\infty}\left(\frac{1}{n+z+1} - \frac{1}{n}\right) + \frac{1}{z} + \gamma + \sum_{n=1}^{\infty}\left(\frac{1}{n+z} - \frac{1}{n}\right) - \frac{1}{z} \\ &= -\frac{1}{z+1} - \sum_{n=1}^{\infty}\left(\frac{1}{n+z+1} - \frac{1}{n}\right) + \sum_{n=1}^{\infty}\left(\frac{1}{n+z} - \frac{1}{n}\right).\end{aligned}$$

Now, we have found above that both of these series are uniformly convergent, so we can combine them which gives

$$\begin{aligned}\frac{\Gamma'(z+1)}{\Gamma(z+1)} - \frac{\Gamma'(z)}{\Gamma(z)} - \frac{1}{z} &= -\frac{1}{z+1} + \sum_{n=1}^{\infty}\left(\frac{1}{n+z} - \frac{1}{n+z+1}\right) \\ &= -\frac{1}{z+1} + \lim_{N\to\infty}\sum_{n=1}^{N}\left(\frac{1}{n+z} - \frac{1}{n+z+1}\right) \\ &= -\frac{1}{z+1} + \lim_{N\to\infty}\left(\frac{1}{1+z} - \frac{1}{N+z+1}\right) \\ &= -\frac{1}{z+1} + \frac{1}{1+z} = 0\end{aligned}$$

because our series is a telescoping sum.
Taking the antiderivative of both sides, we get that $\ln(\Gamma(z+1)) - \ln(\Gamma(z)) - \ln(z) + c_1 = 0$, so

$$\ln\frac{\Gamma(z+1)}{z\Gamma(z)} = -c_1$$

which gives that $\Gamma(z+1) = Cz\Gamma(z)$ after exponentiating both sides and renaming our constant.

6.3 Part c

From the definition of Γ, we have that

$$\lim_{z\to 0} z\Gamma(z) = \lim_{z\to 0}\frac{1}{e^{\gamma z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n}} = \frac{1}{1\prod_{n=1}^{\infty}1*1} = 1.$$

Therefore, $C = \Gamma(1)$ by part b. Note that there is some subtlety here, because we are switching the order of the limits (one limit comes from the infinite product) in order to plug in $z = 0$ directly, but this is okay because we have seen that the infinite product is uniformly convergent in part a.

6.4 Part d

Plugging in $z = 1$ to our definition for Γ and using part c, we get that

$$1 = \frac{1}{\Gamma(1)} = e^{\gamma} \prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right) e^{-1/n},$$

so

$$e^{-\gamma} = \prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right) e^{-1/n}.$$

6.5 Part e

By definition,

$$\begin{aligned} e^{-\gamma} &= \prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right) e^{-1/n} = \lim_{n\to\infty} \frac{2}{1}\frac{3}{2}\frac{4}{3}\cdots\frac{n+1}{n} e^{\sum_{n=1}^{\infty} -1/n} \\ &= \lim_{n\to\infty} (n+1) e^{-S(n)} \end{aligned}$$

where $S(n) = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$. Exponentiating both sides,

$$-\gamma = \lim_{n\to\infty} (\ln(n+1) - S(n))$$

which gives that

$$\gamma = \lim_{n\to\infty} \left(\sum_{k=1}^{\infty} \frac{1}{k} - \ln(n+1) \right).$$

7 Problem 7

Define the Weierstrass $\wp$-function as

$$\wp(z) = \frac{1}{z^2} + \sum_{j,k=-\infty}^{\infty}{}' \left(\frac{1}{(z - j\omega_1 - k\omega_2)^2} - \frac{1}{(j\omega_1 + k\omega_2)^2} \right),$$

where ω_1 is a positive real number, and ω_2 is on the positive imaginary axis.

7.1 Part a

We first show that $\wp(z + \omega_1) = \wp(z)$.

$$\begin{aligned} \wp(z+\omega_1) &= \frac{1}{(z+\omega_1)^2} + \sum_{j,k=-\infty}^{\infty}{}' \left(\frac{1}{(z-(j-1)\omega_1 - k\omega_2)^2} - \frac{1}{(j\omega_1 + k\omega_2)^2} \right) \\ &= \frac{1}{(z+\omega_1)^2} + \left(\frac{1}{z^2} - \frac{1}{\omega_1^2}\right) + \sum_{j,k=-\infty,(j,k)\neq(1,0)}^{\infty}{}' \left(\frac{1}{(z-(j-1)\omega_1 - k\omega_2)^2} - \frac{1}{(j\omega_1 + k\omega_2)^2} \right) \\ &= \frac{1}{(z+\omega_1)^2} + \left(\frac{1}{z^2} - \frac{1}{\omega_1^2}\right) + \sum_{j,k=-\infty,(j,k)\neq(-1,0)}^{\infty}{}' \left(\frac{1}{(z - j\omega_1 - k\omega_2)^2} - \frac{1}{((j+1)\omega_1 + k\omega_2)^2} \right) \end{aligned}$$

where we reindexed $j \to j+1$. Now, consider the series

$$\sum_{j,k=-\infty,(j,k)\neq(-1,0)}^{\infty}{}' \left(\frac{1}{((j+1)\omega_1 + k\omega_2)^2} - \frac{1}{(j\omega_1 + k\omega_2)^2} \right)$$
$$= \sum_{k=-\infty}^{\infty}{}' \left(\sum_{j=-\infty}^{0} \left(\frac{1}{((j+1)\omega_1 + k\omega_2)^2} - \frac{1}{(j\omega_1 + k\omega_2)^2} \right) + \sum_{j=1}^{\infty} \left(\frac{1}{((j+1)\omega_1 + k\omega_2)^2} - \frac{1}{(j\omega_1 + k\omega_2)^2} \right) \right)$$
$$+ \sum_{j=-\infty}^{-2} \left(\frac{1}{((j+1)\omega_1)^2} - \frac{1}{(j\omega_1)^2} \right) + \sum_{j=1}^{\infty} \left(\frac{1}{((j+1)\omega_1)^2} - \frac{1}{(j\omega_1)^2} \right)$$
$$= \sum_{k=-\infty}^{\infty}{}' \left(\lim_{N\to-\infty} \left(\frac{1}{(\omega_1 + k\omega_2)^2} - \frac{1}{(N\omega_1 + k\omega_2)^2} \right) + \lim_{N\to\infty} \left(\frac{1}{((N+1)\omega_1 + k\omega_2)^2} - \frac{1}{(\omega_1 + k\omega_2)^2} \right) \right)$$
$$+ \lim_{N\to-\infty} \left(\frac{1}{(-\omega_1)^2} - \frac{1}{(N\omega_1)^2} \right) + \lim_{N\to\infty} \left(\frac{1}{((N+1)\omega_1)^2} - \frac{1}{(\omega_1)^2} \right)$$
$$= \sum_{k=-\infty}^{\infty}{}' \left(\frac{1}{(\omega_1 + k\omega_2)^2} - \frac{1}{(\omega_1 + k\omega_2)^2} \right) + \frac{1}{\omega_1} - \frac{1}{\omega_1} = 0.$$

Because this is zero, we can add it to our equation for $\wp(z+\omega_1)$ which gives

$$\wp(z+\omega_1) = \frac{1}{(z+\omega_1)^2} + \left(\frac{1}{z^2} - \frac{1}{\omega_1^2} \right) + \sum_{j,k=-\infty,(j,k)\neq(-1,0)}^{\infty}{}' \left(\frac{1}{(z - j\omega_1 - k\omega_2)^2} - \frac{1}{((j+1)\omega_1 + k\omega_2)^2} \right)$$
$$+ \sum_{j,k=-\infty,(j,k)\neq(-1,0)}^{\infty}{}' \left(\frac{1}{((j+1)\omega_1 + k\omega_2)^2} - \frac{1}{(j\omega_1 + k\omega_2)^2} \right).$$

We know that the first series is uniformly convergent, because it is just a shifted version of the Weierstrass $\wp$ function defined at $z+\omega_1$ which we know is uniformly convergent. The second series is also uniformly convergent with respect to z, because it is constant with respect to z. Thus, we can combine the series and get that

$$\begin{aligned}\wp(z+\omega_1) &= \frac{1}{(z+\omega_1)^2} + \left(\frac{1}{z^2} - \frac{1}{\omega_1^2} \right) + \sum_{j,k=-\infty,(j,k)\neq(-1,0)}^{\infty}{}' \left(\frac{1}{(z - j\omega_1 - k\omega_2)^2} - \frac{1}{(j\omega_1 + k\omega_2)^2} \right) \\ &= \frac{1}{z^2} + \left(\frac{1}{(z+\omega_1)^2} - \frac{1}{\omega_1^2} \right) + \sum_{j,k=-\infty,(j,k)\neq(-1,0)}^{\infty}{}' \left(\frac{1}{(z - j\omega_1 - k\omega_2)^2} - \frac{1}{(j\omega_1 + k\omega_2)^2} \right) \\ &= \frac{1}{z^2} + \sum_{j,k=-\infty}^{\infty}{}' \left(\frac{1}{(z - j\omega_1 - k\omega_2)^2} - \frac{1}{(j\omega_1 + k\omega_2)^2} \right) = \wp(z)\end{aligned}$$

noting that $\frac{1}{(z+\omega_1)^2} - \frac{1}{\omega_1^2}$ is the $(j,k) = (-1,0)$ term of this series.
Now, we argue that showing this is sufficient for showing that $\wp(z + M\omega_1 + N\omega_2) = \wp(z)$ for any $M, N \in \mathbb{Z}$. We first use an induction argument on M,

noting that if we plug in $z+\omega_1$ to our relation, we get that $\wp(z+\omega_1) = \wp(z+2\omega_1)$, so $\wp(z) = \wp(z+2\omega_1)$. Similarly, $\wp(z+(M-1)\omega_1) = \wp(z+M\omega_1)$, so if we take this to be our inductive step, we get that $\wp(z) = \wp(z+M\omega_1)$, meaning that this relation holds for all $M \in \mathbb{N}$. Similarly, if we plug in $z-\omega_1$, we get that $\wp(z-\omega_1) = \wp(z-\omega_1+\omega_1) = \wp(z)$. Then, we can use the same inductive argument in the negative direction by noting that $\wp(z-2\omega_1) = \wp(z-2\omega_1+\omega_1) = \wp(z-\omega_1)$ and that $\wp(z-M\omega_1) = \wp(z-M\omega_1+\omega_1) = \wp(z-(M-1)\omega_1)$. Thus, we have that $\wp(z+M\omega_1) = \wp(z)$ for all $M \in \mathbb{Z}$. To get the ω_2 direction, we simply note that $\wp$ is symmetric with respect to ω_1 and ω_2, so the manipulations we made also imply that $\wp(z+\omega_2) = \wp(z)$. The induction arguments we already made then give that $\wp(z+N\omega_2) = \wp(z)$ for any $N \in \mathbb{Z}$. Then, we plug $z+M\omega_1$ into this equation to get that $\wp(z+M\omega_1+N\omega_2) = \wp(z+M\omega_1) = \wp(z)$ for all $M, N \in \mathbb{Z}$.

7.2 Part b

To show that $\wp$ is an even function, consider

$$\begin{aligned}\wp(-z) &= \frac{1}{(-z)^2} + \sum_{j,k=-\infty}^{\infty}{}' \left(\frac{1}{(-z-j\omega_1-k\omega_2)^2} - \frac{1}{(j\omega_1+k\omega_2)^2}\right) \\ &= \frac{1}{z^2} + \sum_{j,k=-\infty}^{\infty}{}' \left(\frac{1}{(z+j\omega_1+k\omega_2)^2} - \frac{1}{(j\omega_1+k\omega_2)^2}\right).\end{aligned}$$

Now, reindex $j \to -j$, $k \to -k$

$$\begin{aligned}\wp(-z) &= \frac{1}{z^2} + \sum_{j,k=\infty}^{-\infty}{}' \left(\frac{1}{(z-j\omega_1-k\omega_2)^2} - \frac{1}{(-j\omega_1-k\omega_2)^2}\right) \\ &= \frac{1}{z^2} + \sum_{j,k=-\infty}^{\infty}{}' \left(\frac{1}{(z-j\omega_1-k\omega_2)^2} - \frac{1}{(j\omega_1+k\omega_2)^2}\right) = \wp(z)\end{aligned}$$

where we have simply changed the order of summation. Thus, $\wp(z)$ is an even function.

7.3 Part c

To find a Laurent series for $\wp(z)$, first note that if we consider some w that is constant with respect to z,

$$\begin{aligned}\frac{1}{(z-w)^2} &= \frac{d}{dz}\left(\frac{-1}{z-w}\right) = \frac{d}{dz}\left(\frac{1}{w}\frac{1}{1-z/w}\right) = \frac{d}{dz}\left(\frac{1}{w}\sum_{k=0}^{\infty}\left(\frac{z}{w}\right)^k\right) \\ &= \frac{1}{w}\sum_{k=0}^{\infty}\frac{d}{dz}\left(\left(\frac{z}{w}\right)^k\right) = \frac{1}{w}\sum_{k=1}^{\infty}\frac{k}{w}\left(\frac{z}{w}\right)^{k-1} = \frac{1}{w^2}\sum_{k=1}^{\infty}k\left(\frac{z}{w}\right)^{k-1}\end{aligned}$$

Note that the $k = 0$ term drops out, because it is a constant, so its derivative is 0. Reindexing,

$$\frac{1}{(z-w)^2} = \frac{1}{w^2}\sum_{k=0}^{\infty}(k+1)\left(\frac{z}{w}\right)^k = \frac{1}{w^2} + \frac{1}{w^2}\sum_{k=1}^{\infty}(k+1)\left(\frac{z}{w}\right)^k.$$

Thus,

$$\frac{1}{(z-w)^2} - \frac{1}{w^2} = \frac{1}{w^2}\sum_{k=1}^{\infty}(k+1)\left(\frac{z}{w}\right)^k.$$

Now, we take a step back ensure that our steps were legitimate. In order for our geometric series to be valid, we need that $|z/w| < 1$. If this holds, we also have that our series is uniformly convergent, so we are able to differentiate termwise. Now, take $w = j\omega_1 + k\omega_2$. Then, for $|z/w| < 1$, we need that $|z| < \inf_{j,k\in\mathbb{Z}} |j\omega_1 + k\omega_2|$ in order for the above series to be valid. Then,

$$\frac{1}{(z-j\omega_1-k\omega_2)^2} - \frac{1}{(j\omega_1+k\omega_2)^2} = \frac{1}{(j\omega_1+k\omega_2)^2}\sum_{n=1}^{\infty}(n+1)\left(\frac{z}{j\omega_1+k\omega_2}\right)^n$$

which gives that

$$\begin{aligned}
\wp(z) &= \frac{1}{z^2} + \sum_{j,k=-\infty}^{\infty}{}' \left(\frac{1}{(j\omega_1+k\omega_2)^2}\sum_{n=1}^{\infty}(n+1)\left(\frac{z}{j\omega_1+k\omega_2}\right)^n\right) \\
&= \frac{1}{z^2} + \sum_{n=1}^{\infty}\sum_{j,k=-\infty}^{\infty}{}' \frac{n+1}{(j\omega_1+k\omega_2)^2}\left(\frac{z}{j\omega_1+k\omega_2}\right)^n \\
&= \frac{1}{z^2} + \sum_{n=1}^{\infty}\sum_{j,k=-\infty}^{\infty}{}' \frac{n+1}{(j\omega_1+k\omega_2)^{n+2}} z^n
\end{aligned}$$

where we are able to switch the sums, because both the geometric series and $\wp(z)$ are uniformly convergent for $|z| < \inf_{j,k\in\mathbb{Z}} |j\omega_1 + k\omega_2|$. Thus, if we let

$$\alpha_n = \sum_{j,k=-\infty}^{\infty}{}' \frac{n+1}{(j\omega_1+k\omega_2)^{n+2}},$$

then $\wp(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty}\alpha_n z^n$. From part b, we know that $\wp(z)$ is even, so it must hold that $\alpha_{2k} = 0$ for $k \in \mathbb{N}$. It is also clear that this must be true from the fact that $j\omega_1 + k\omega_2$ is raised to an even power in the formula for α_{2k}, meaning that the terms are even in both j and k, so summing over both from $-\infty$ to ∞ yields 0. Thus, we can write the Laurent series in the form

$$\wp(z) = \frac{1}{z^2} + \alpha_0 + \alpha_2 z^2 + \alpha_4 z^4 + \dots$$

where $\alpha_0 = 0$ because we do not have a constant term. Because our series are uniformly convergent, we can differentiate termwise to get that

$$\wp'(z) = -\frac{2}{z^3} + \beta_1 z + \beta_3 z^3 + \dots$$

where

$$\beta_{n-1} = \sum_{j,k=-\infty}^{\infty}{}' \frac{(n+1)n}{(j\omega_1 + k\omega_2)^{n+2}}$$

follows from taking the derivative of $\alpha_n z^n$. Reindexing, we get that

$$\beta_n = \sum_{j,k=-\infty}^{\infty}{}' \frac{(n+2)(n+1)}{(j\omega_1 + k\omega_2)^{n+3}} = (n+1)\alpha_{n+1}.$$

Note that $\beta_{2k+1} = 0$ for $k \in \mathbb{N}$ follows from the relation to α_{2k+2}.

7.4 Part d

Note that $\alpha_0 = 0$, so we have that

$$\wp(z) = \frac{1}{z^2} + \alpha_2 z^2 + \alpha_4 z^4 + \dots,$$

and

$$\wp'(z) = -\frac{2}{z^3} + \beta_1 z + \beta_3 z^3 + \dots.$$

Let us write out the terms of the differential equation, stopping at the z^1 term

$$(\wp')^2 = \left(\frac{-2}{z^3}\right)^2 + 2\frac{-2}{z^3}\beta_1 z + 2\frac{-2}{z^3}\beta_3 z^3 + \dots = \frac{4}{z^6} - \frac{4\beta_1}{z^2} - 4\beta_3 + \dots$$

$$\wp^3 = \left(\frac{1}{z^2}\right)^3 + 3\left(\frac{1}{z^2}\right)^2 \alpha_2 z^2 + 3\left(\frac{1}{z^2}\right)^2 \alpha_4 z^4 + \dots = \frac{1}{z^6} + \frac{3\alpha_2}{z^2} + 3\alpha_4 + \dots$$

$$\wp^2 = \left(\frac{1}{z^2}\right)^2 + 2\frac{1}{z^2}\alpha_2 z^2 + \dots = \frac{1}{z^2} + 2\alpha_2 + \dots$$

$$\wp = \frac{1}{z^2} + \dots.$$

Now, define $d(z) = (\wp')^2 - a\wp^3 - b\wp^2 - c\wp$, so

$$d = \frac{4-a}{z^6} - \frac{b}{z^4} + \frac{-4\beta_1 - 3a\alpha_2 - c}{z^2} - (4\beta_3 + 3a\alpha_4 + 2b\alpha_2) + \dots.$$

Clearly, d is biperiodic, because $\wp$ is biperiodic. If we take $a = 4$, $b = 0$, $c = -4\beta_1 - 3a\alpha_2 = -4\beta_1 - 12\alpha_2 = -8\alpha_2 - 12\alpha_2 = -20\alpha_2$, then $d(z)$ does not have a singularity at $z = 0$, because the coefficients on the negative power terms of the Laurent series centered at $z = 0$ are zero. Of course, this is only valid in our radius of convergence, but the biperiodicity of $\wp(z)$ implies that we can write this same expansion at $z = M\omega_1 + N\omega_2$ for any $M, N \in \mathbb{Z}$. This is precisely where the singularities of $\wp$ lie, so if d has singularities at these points, they are removable. Thus, d is entire as a function of z. In class, we showed that an entire biperiodic function is bounded (This follows from the fact that its image on a parallelogram, a compact subset, is its image on the whole complex plane). Thus, $d(z)$ is an entire function on the complex plane that is bounded, so Liouville's theorem gives that it must be constant as a function of z. Thus, all higher order terms must be zero, meaning that $d = -(4\beta_3 + 3a\alpha_4 + 2b\alpha_2) = -4\beta_3 - 12\alpha_4 = -16\alpha_4 - 12\alpha_4 = -28\alpha_4$, so $\wp(z)$ satisfies the differential equation

$$(\wp')^2 = a\wp^3 + b\wp^2 + c\wp + d,$$

where we take $a = 4$, $b = 0$, $c = -20\alpha_2$, $d = -28\alpha_4$.