AMATH 567 Homework 5

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1 Problem 1 (2.6.1)

Let C be the unit circle centered at the origin.

1.1 Part a

Consider the function $f(z) = \sin z$ which is entire, so we can apply Cauchy's integral formula at 0 to it for any simple closed contour. Thus,

$$\sin 0 = \frac{1}{2\pi i} \oint_C \frac{\sin z}{z-0} dz.$$

This gives that

$$\oint_C \frac{\sin z}{z} dz = 2\pi i \sin 0 = 0.$$

1.2 Part b

Consider the function $f(z) = \frac{1}{4}$ which is entire and apply theorem 2.6.2 in the text (the derivatives of Cauchy's theorem) at $1/2$ for $k = 1$ to get

$$f'(\frac{1}{2}) = \frac{1!}{2\pi i} \oint_C \frac{f(z)}{(z-\frac{1}{2})^2} dz = \frac{1}{2\pi i} \oint_C \frac{1/4}{(z-\frac{1}{2})^2} dz = \frac{1}{2\pi i} \oint_C \frac{1}{(2z-1)^2} dz.$$

However $f'(z) = 0$ for any $z \in \mathbb{C}$ because f is constant, so

$$\oint_C \frac{1}{(2z-1)^2} dz = 0.$$

1.3 Part c

Now, consider the function $f(z) = \frac{1}{8}$ which is entire and apply theorem 2.6.2 in the text at $1/2$ for $k = 2$ to get

$$f''(\frac{1}{2}) = \frac{2!}{2\pi i} \oint_C \frac{f(z)}{(z-\frac{1}{2})^3} dz = \frac{1}{\pi i} \oint_C \frac{1/8}{(z-\frac{1}{2})^3} dz = \frac{1}{\pi i} \oint_C \frac{1}{(2z-1)^3} dz.$$

$f''(z) = 0$ for any $z \in \mathbb{C}$ because f is constant, so

$$\oint_C \frac{1}{(2z-1)^3} dz = 0.$$

1.4 Part d

Consider $f(z) = e^z$ which is entire, so we apply Cauchy's formula at 0 to get

$$f(0) = \frac{1!}{2\pi i} \oint_C \frac{f(z)}{z-0} dz = \frac{1}{2\pi i} \oint_C \frac{e^z}{z} dz.$$

Thus,

$$\oint_C \frac{e^z}{z} dz = 2\pi i f(0) = 2\pi i e^0 = 2\pi i.$$

1.5 Part e

Consider $f(z) = e^{z^2}$ which is once again entire, so we apply theorem 2.6.2 in the text for $k = 1$ at 0 to get that

$$f'(0) = \frac{1!}{2\pi i} \oint_C \frac{f(z)}{(z-0)^2} dz = \frac{1}{2\pi i} \oint_C \frac{e^{z^2}}{z^2} dz.$$

Note that $f'(z) = 2ze^{z^2}$, so

$$\oint_C \frac{e^{z^2}}{z^2} dz = 2\pi i f'(0) = 0.$$

Similarly, we apply the same theorem for $k = 3$ to get that

$$f''(0) = \frac{2!}{2\pi i} \oint_C \frac{f(z)}{(z-0)^3} dz = \frac{1}{\pi i} \oint_C \frac{e^{z^2}}{z^3} dz.$$

Note that $f''(z) = 2e^{z^2} + 4z^2 e^{z^2}$, so

$$\oint_C \frac{e^{z^2}}{z^3} dz = \pi i f''(0) = 2\pi i.$$

Thus, we can conclude that

$$\oint_C e^{z^2} \left(\frac{1}{z^2} - \frac{1}{z^3} \right) dz = \oint_C \frac{e^{z^2}}{z^2} dz - \oint_C \frac{e^{z^2}}{z^3} dz = 0 - 2\pi i = -2\pi i.$$

2 Problem 2 (2.6.10)

Beginning with Cauchy's integral formula and letting the contour C be a circle of unit radius centered at the origin. Parameterize as $\zeta = e^{i\theta}$ which gives $d\zeta = ie^{i\theta} d\theta$. Then,

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(e^{i\theta})}{e^{i\theta} - z} ie^{i\theta} d\theta = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(e^{i\theta})e^{i\theta}}{e^{i\theta} - z} d\theta = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(\zeta)\zeta}{\zeta - z} d\theta$$

where z lies inside the circle. Now, we can observe that

$$0 = \frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{\zeta - 1/\overline{z}}d\zeta = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(\zeta)\zeta}{\zeta - 1/\overline{z}}d\theta.$$

The first equality follows because the integrand has only one singularity at $\zeta = 1/\overline{z}$ which is outside the unit circle that we are integrating over (it is equivalent to $\frac{1}{R}e^{i\theta}$ if $z = Re^{i\theta}$), so we can invoke Cauchy's theorem. The second follows because our steps in manipulating the integral to get that $\frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{\zeta - z}d\zeta = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(\zeta)\zeta}{\zeta - z}d\theta$ in the previous part did not depend on our choice of z. Now, note that $\zeta\overline{\zeta} = 1$, so

$$\frac{\zeta}{\zeta - 1/\overline{z}} = \frac{\zeta\overline{\zeta}}{\zeta\overline{\zeta} - \overline{\zeta}/\overline{z}} = \frac{1}{1 - \overline{\zeta}/\overline{z}} = \frac{\overline{z}}{\overline{z} - \overline{\zeta}}.$$

Thus,

$$0 = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(\zeta)\overline{z}}{\overline{z} - \overline{\zeta}}.$$

Now, if we subtract this from or add this to the first equation, we get that

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(\zeta)\left(\frac{\zeta}{\zeta - z} \mp \frac{\overline{z}}{\overline{z} - \overline{\zeta}}\right)d\theta = \frac{1}{2\pi}\int_0^{2\pi} f(\zeta)\left(\frac{\zeta}{\zeta - z} \pm \frac{\overline{z}}{\overline{\zeta} - \overline{z}}\right)d\theta.$$

Using the plus sign,

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(\zeta)\frac{\zeta(\overline{\zeta} - \overline{z}) + \overline{z}(\zeta - z)}{(\zeta - z)(\overline{\zeta - z})}d\theta = \frac{1}{2\pi}\int_0^{2\pi} f(\zeta)\frac{\zeta\overline{\zeta} - z\overline{z}}{|\zeta - z|^2}d\theta = \frac{1}{2\pi}\int_0^{2\pi} f(\zeta)\frac{1 - |z|^2}{|\zeta - z|^2}d\theta.$$

2.1 Part a

Now, take $z = re^{i\phi}$ and define $u(r, \phi) = \Re f$. Then,

$$\begin{aligned}
u(r, \phi) &= \Re\left(\frac{1}{2\pi}\int_0^{2\pi} f(\zeta)\frac{1 - |z|^2}{|\zeta - z|^2}d\theta\right) = \Re\left(\frac{1}{2\pi}\int_0^{2\pi} f(\zeta)\frac{1 - r^2}{|\zeta|^2 - 2\Re(\overline{\zeta}z) + r^2}d\theta\right) \\
&= \Re\left(\frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\frac{1 - r^2}{1 - 2\Re(re^{i(\phi-\theta)}) + r^2}d\theta\right) \\
&= \Re\left(\frac{1}{2\pi}\int_0^{2\pi} (u(1, \theta) + iv(1, \theta))\underbrace{\frac{1 - r^2}{1 - 2r\cos(\phi - \theta) + r^2}}_{\text{real}}d\theta\right) \\
&= \frac{1}{2\pi}\int_0^{2\pi} u(1, \theta)\frac{1 - r^2}{1 - 2r\cos(\phi - \theta) + r^2}d\theta = \frac{1}{2\pi}\int_0^{2\pi} u(\theta)\frac{1 - r^2}{1 - 2r\cos(\phi - \theta) + r^2}d\theta.
\end{aligned}$$

2.2 Part b

Using the minus sign in the formula for $f(z)$,

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(\zeta)\frac{\zeta(\overline{\zeta}-\overline{z})-\overline{z}(\zeta-z)}{(\zeta-z)(\overline{\zeta-z})}d\theta = \frac{1}{2\pi}\int_0^{2\pi} f(\zeta)\frac{1-2\zeta\overline{z}+|z|^2}{|\zeta-z|^2}d\theta.$$

Again taking $z = re^{i\phi}$,

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(\zeta)\frac{1-2e^{i\theta}re^{-i\phi}+r^2}{1-2r\cos(\phi-\theta)+r^2}d\theta = \frac{1}{2\pi}\int_0^{2\pi} f(\zeta)\frac{1+r^2-2re^{i(\theta-\phi)}}{1-2r\cos(\phi-\theta)+r^2}d\theta.$$

Taking the imaginary part of this,

$$\begin{aligned}
v(r,\phi) &= \Im\left(\frac{1}{2\pi}\int_0^{2\pi} f(\zeta)\frac{1+r^2-2re^{i(\theta-\phi)}}{1-2r\cos(\phi-\theta)+r^2}d\theta\right) \\
&= \Im\left(\frac{1}{2\pi}\int_0^{2\pi}(u(1,\theta)+iv(1,\theta))\left(\frac{1+r^2-2r\cos(\theta-\phi)}{1-2r\cos(\phi-\theta)+r^2} - i\frac{2r\sin(\theta-\phi)}{1-2r\cos(\phi-\theta)+r^2}\right)d\theta\right) \\
&= -\frac{1}{2\pi}\int_0^{2\pi} u(\theta)\frac{2r\sin(\theta-\phi)}{1-2r\cos(\phi-\theta)+r^2}d\theta + \frac{1}{2\pi}\int_0^{2\pi} v(\theta)\frac{1+r^2-2r\cos(\theta-\phi)}{1-2r\cos(\phi-\theta)+r^2}d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} v(\theta)d\theta + \frac{1}{\pi}\int_0^{2\pi} u(\theta)\frac{r\sin(\phi-\theta)}{1-2r\cos(\phi-\theta)+r^2}d\theta \\
&= v(r=0) + \frac{1}{\pi}\int_0^{2\pi} u(\theta)\frac{r\sin(\phi-\theta)}{1-2r\cos(\phi-\theta)+r^2}d\theta
\end{aligned}$$

where the penultimate step follows because sine is an odd function, and the last step from the first equation we derived in this problem taken at $z = 0$. Note that the first term is constant with respect to r and ϕ.

2.3 Part c

Consider

$$\begin{aligned}
\Im\left(\frac{\zeta+z}{\zeta-z}\right) &= \Im\left(\frac{(\zeta+z)(\overline{\zeta-z})}{(\zeta-z)(\overline{\zeta-z})}\right) = \Im\left(\frac{1-r^2+2i\Im(z\overline{\zeta})}{|\zeta-z|^2}\right) \\
&= \Im\left(\frac{1-r^2+2i\Im(re^{i(\phi-\theta)})}{1-2r\cos(\phi-\theta)+r^2}\right) = \Im\left(\frac{1-r^2+2ir\sin(\phi-\theta)}{1-2r\cos(\phi-\theta)+r^2}\right) \\
&= \Im\left(\frac{1-r^2}{1-2r\cos(\phi-\theta)+r^2} + i\frac{2r\sin(\phi-\theta)}{1-2r\cos(\phi-\theta)+r^2}\right) \\
&= \frac{2r\sin(\phi-\theta)}{1-2r\cos(\phi-\theta)+r^2}.
\end{aligned}$$

Note that we need not worry about dividing by 0, because we cannot have that $\overline{z} = \overline{\zeta}$, since z inside the circle, but ζ is on it, so $\phi \neq \theta$. Thus, the result from part b can be expressed as

$$v(r,\phi) = v(0) + \frac{1}{2\pi}\Im\left(\int_0^{2\pi} u(\theta)\frac{\zeta+z}{\zeta-z}d\theta\right).$$

3 Problem 3 (2.6.2 in lecture notes)

Let $f(z) = R(x, y)\exp(i\theta(x, y))$ where R and θ are real-valued functions of x and y. If $f(z)$ is analytic, then it satisfies the Cauchy-Riemann equations, so using the fact that $f(x, y) = \underbrace{R(x, y)\cos(\theta(x, y))}_{u} + i\,\underbrace{R(x, y)\sin(\theta(x, y))}_{v}$, we can write

$$u_x = R_x\cos(\theta(x, y)) - \theta_x R(x, y)\sin(\theta(x, y)) = R_y\sin(\theta(x, y)) + \theta_y R(x, y)\cos(\theta(x, y)) = v_y$$

and

$$v_x = R_x\sin(\theta(x, y)) + \theta_x R(x, y)\cos(\theta(x, y)) = -R_y\cos(\theta(x, y)) + \theta_y R(x, y)\sin(\theta(x, y)) = -u_y.$$

To simplify these, multiply the first equation by $\cos(\theta(x, y))$ and the second by $\sin(\theta(x, y))$ and add which gives that

$$R_x = R_x(\cos^2(\theta(x, y)) + \sin^2(\theta(x, y))) = \theta_y(\cos^2(R(x, y))(\theta(x, y)) + \sin^2(\theta(x, y))) = \theta_y R(x, y).$$

Multiplying the first equation by $-\sin(\theta(x, y))$ and the second by $\cos(\theta(x, y))$ and adding gives that

$$\theta_x R(x, y) = \theta_x R(x, y)(\cos^2(\theta(x, y)) + \sin^2(\theta(x, y))) = -R_y(\cos^2(\theta(x, y)) + \sin^2(\theta(x, y))) = -R_y.$$

Now that the x and y derivatives of R and θ are connected, we consider the case where $R(x, y) = R$ is a constant. Then, $R_x, R_y = 0$ so our equations become

$$\theta_y R = 0$$
$$\theta_x R = 0.$$

If $R = 0$, then these equations hold for all $\theta(x, y)$, but in that case, $f(z) = 0$ is constant. If we assume $R \neq 0$, then we have $\theta_x, \theta_y = 0$, so $\theta(x, y) = \theta$ is constant with respect to x and y. Thus, $f(z) = R\exp i\theta$ is constant.
If $\theta(x, y) = \theta$ is constant, $\theta_x, \theta_y = 0$, so the equations become

$$R_x = 0$$
$$-R_y = 0.$$

Thus, $R(x, y) = R$ is constant with respect to x and y, meaning that $f(z) = R\exp i\theta$ is constant.

4 Problem 4 (2.6.11 in lecture notes)

Consider the Legendre polynomials defined by

$$P_n(z) = \frac{1}{2^n n!}\frac{d^n}{dz^n}(z^2 - 1)^n$$

for $n \in \mathbb{N}$. Because $(z^2-1)^n$ is entire for any $n \in \mathbb{N}$, we can apply theorem 2.6.2 to compute the derivative above by taking $f(z) = (z^2-1)^n$ and C to be any simple closed contour encircling z. Namely,

$$\frac{d^n}{dz^n}(z^2-1)^n = \frac{n!}{2\pi i}\oint_C \frac{(t^2-1)^n}{(t-z)^{n+1}}dt.$$

Thus,

$$P_n(z) = \frac{1}{2^n n!}\frac{n!}{2\pi i}\oint_C \frac{(t^2-1)^n}{(t-z)^{n+1}}dt = \frac{1}{2\pi i}\oint_C \frac{(t^2-1)^n}{2^n(t-z)^{n+1}}dt$$

for any $n \in \mathbb{N}$ and any simple closed contour C encircling z.

5 Problem 5 (2.6.12 in lecture notes)

5.1 Part a

Parameterizing the unit circle $C(0,1)$ as $z = e^{it}$, $dz = ie^{it}dt$, we get that

$$\begin{aligned}\frac{1}{2\pi i}\oint_{C(0,1)}\left(z+\frac{1}{z}\right)^n\frac{dz}{z} &= \frac{1}{2\pi i}\int_0^{2\pi}(e^{it}+e^{-it})^n\frac{ie^{it}dt}{e^{it}}\\ &= \frac{1}{2\pi}\int_0^{2\pi}(2\cos t)^n\,dt = \frac{2^n}{2\pi}\int_0^{2\pi}\cos^n t dt\end{aligned}$$

by the definition of the complex cosine.

5.2 Part b

Using the binomial formula on the LHS of part a,

$$\begin{aligned}\frac{1}{2\pi i}\oint_{C(0,1)}\left(z+\frac{1}{z}\right)^n\frac{dz}{z} &= \frac{1}{2\pi i}\oint_{C(0,1)}\sum_{\ell=0}^{n}\binom{n}{\ell}z^{n-\ell}\frac{1}{z^\ell}\frac{dz}{z}\\ &= \frac{1}{2\pi i}\oint_{C(0,1)}\sum_{\ell=0}^{n}\binom{n}{\ell}z^{n-2\ell-1}dz\\ &= \frac{1}{2\pi i}\sum_{\ell=0}^{n}\binom{n}{\ell}\oint_{C(0,1)}z^{n-2\ell-1}dz.\end{aligned}$$

From this and part a, we find that for $k \in \mathbb{N}$,

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\cos^{2k} t dt &= \frac{1}{2^{2k}}\frac{1}{2\pi i}\oint_{C(0,1)}\left(z+\frac{1}{z}\right)^{2k}\frac{dz}{z}\\ &= \frac{1}{2^{2k}}\frac{1}{2\pi i}\sum_{\ell=0}^{2k}\binom{2k}{\ell}\oint_{C(0,1)}z^{2k-2\ell-1}dz\end{aligned}$$

From pages 44 and 45 of the lecture notes, because our contour encircles the origin and $2k-2\ell-1$ is an integer for $\ell \in \{1,\dots,2k+1\}$, we have that

$$\oint_{C(0,1)} z^{2k-2\ell-1}dz = \begin{cases} 0, & 2k-2\ell-1 \neq -1 \\ 2\pi i, & 2k-2\ell-1 = -1 \end{cases}.$$

Thus, $\oint_{C(0,1)} z^{2k-2\ell-1}dz$ will only be nonzero for $\ell = k$, meaning that we can write

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} \cos^{2k} t dt &= \frac{1}{2^{2k}}\frac{1}{2\pi i}\binom{2k}{k}\oint_{C(0,1)} z^{-1}dz = \frac{1}{2^{2k}}\frac{1}{2\pi i}\binom{2k}{k}2\pi i \\ &= \frac{1}{2^{2k}}\frac{(2k)!}{k!k!} = \frac{(2k)!}{2^{2k}(k!)^2}. \end{aligned}$$

Similarly, for $k \in \mathbb{N}$,

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} \cos^{2k+1} t dt &= \frac{1}{2^{2k+1}}\frac{1}{2\pi i}\oint_{C(0,1)} \left(z+\frac{1}{z}\right)^{2k+1}\frac{dz}{z} \\ &= \frac{1}{2^{2k+1}}\frac{1}{2\pi i}\sum_{\ell=0}^{2k+1}\binom{2k+1}{\ell}\oint_{C(0,1)} z^{2k+1-2\ell-1}dz \end{aligned}$$

Once again, we have that

$$\oint_{C(0,1)} z^{2k-2\ell-1}dz = \begin{cases} 0, & 2k+1-2\ell-1 \neq -1 \\ 2\pi i, & 2k+1-2\ell-1 = -1 \end{cases}.$$

by the same reasoning as before. However, $2k+1-2\ell-1 = -1$ cannot hold for integers k and ℓ, because solving this yields $\ell = k+\frac{1}{2}$. Thus, $\oint_{C(0,1)} z^{2k-2\ell-1}dz = 0$ for all $\ell \in \{1,\dots,2k+1\}$, meaning that

$$\frac{1}{2\pi}\int_0^{2\pi} \cos^{2k+1} t dt = 0.$$