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AMATH 561 Problem set 2 -

1. Say that X and Y are random variables on the probability space (Ω, F, P) and $A \in F$. Define $Z: \Omega \to \mathbb{R}$ s.t. $Z(\omega) = X(\omega)$ for $\omega \in A$ and $Z(\omega) = Y(\omega)$ for $\omega \in A^c$.
Let $B \subset \mathbb{R}$ be a Borel set. Then,
$Z^{-1}(B) = \{\omega \mid Z(\omega) \in B\} = \{\omega \in A \mid Z(\omega) \in B\} \cup \{\omega \in A^c \mid Z(\omega) \in B\}$
(because $A \cup A^c = \Omega$) $= \{\omega \in A \mid X(\omega) \in B\} \cup \{\omega \in A^c \mid Y(\omega) \in B\}$
(by the defition of Z) $= (X^{-1}(B) \cap A) \cup (Y^{-1}(B) \cap A^c)$.
Because σ-algebras are closed to compliments, $A^c \in F$. $X^{-1}(B), Y^{-1}(B) \in F$ by the definition of a random variable. From the lecture notes, we have that σ-algebras are closed to finite unions and intersections. Thus, $Z^{-1}(B) = (X^{-1}(B) \cap A) \cup (Y^{-1}(B) \cap A^c) \in F$, meaning that Z is a random variable by definition.

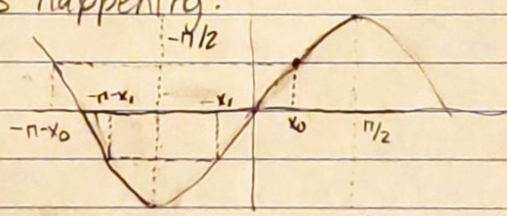
2. Let X be a continuous r.v. with distribution function F_X. Define $Y = g(X)$ s.t. g is a strictly increasing function.
a. Let F_Y denote the distribution function of Y.
Because g is strictly increasing, it must be injective, so we can write its inverse function g^{-1}. Then,
$F_Y(y) = P(Y \le y) = P(g(X) \le y) = P(X \le g^{-1}(y))$ (because g is strictly increasing) $= F_X(g^{-1}(y))$. However, this is only defined for y in the range R of g. We are allowed to assume that g is differentiable per Piazza, so we know that R doesn't have "holes." Thus, we can let $F_Y(y) = 1$ if $y \ge \sup_{x \in \mathbb{R}} g(x)$ and $F_Y(y) = 0$ if $y \le \inf_{x \in \mathbb{R}} g(x)$ and cover all cases.
b. Because X is a continuous r.v. we know that its density function f_X exists. Now,
$f_Y(y) = F_Y'(y) = \frac{d}{dy}(F_X(g^{-1}(y))) = \frac{d}{dy}(g^{-1}(y)) f_X(g^{-1}(y)) = \frac{f_X(g^{-1}(y))}{g'(y)}$. Of course, this is still only defined for $y \in R$. However, if $y \notin R$, $f_Y(y) = 0$ because F_Y is constant (either 0 or 1 depending on the side) outside the range of g.

3. Let X be a continuous r.v. with distribution function F_X.

a. Let $Y=X^2$. Then, $F_Y(y)=P(X^2\le y)=0$ if $y<0$, because $X^2\le y$ cannot hold. If $y\ge 0$,
$F_Y(y)=P(-\sqrt{y}\le X\le\sqrt{y})=F_X(\sqrt{y})-F_X(-\sqrt{y})$.
Thus, $F_Y(y)=\begin{cases} F_X(\sqrt{y})-F_X(-\sqrt{y}) & \text{if } y\ge 0\\ 0 & \text{if } y<0\end{cases}$

b. Let $Y=\sqrt{|X|}$. Then, $F_Y(y)=P(\sqrt{|X|}\le y)=0$ if $y<0$.
If $y\ge 0$, $F_Y(y)=P(|X|\le y^2)=P(-y^2\le X\le y^2)$
$=F_X(y^2)-F_X(-y^2)$. Thus,
$F_Y(y)=\begin{cases} F_X(y^2)-F_X(-y^2) & \text{if } y\ge 0\\ 0 & \text{if } y<0\end{cases}$

c. Let $Y=\sin X$. Then, $F_Y(y)=P(\sin X\le y)=\begin{cases} 0 & \text{if } y<-1\\ 1 & \text{if } y>1\end{cases}$
because $-1\le \sin X\le 1$.
If $-1\le y\le 1$, $-\frac{\pi}{2}\le \arcsin y\le \pi/2$. We draw a graph to see what's happening:



To have $\sin x_0\le y$ in this period, we must have that $-\pi-x_0\le y\le x_0$. Similarly, to have $\sin x_1\le y$, we must have that $-\pi-x_1\le y\le x_1$. Of course, we have period 2π, so to find all the y s.t $\sin x_0\le y$, $-\pi-x_0+2\pi k\le y\le x_0+2\pi k$ $\forall k\in\mathbb{Z}$. This gives that
$F_Y(y)=P(\sum_{k\in\mathbb{Z}}(2k-1)\pi-X\le\arcsin y\le 2\pi k+X)$
$=\sum_{k\in\mathbb{Z}}(F_X(\arcsin y-2\pi k)-F_X(-\arcsin y+(2k-1)\pi))$.

Thus, $\begin{cases} 0 & \text{if } y<-1\\ 1 & \text{if } y>1\\ \sum_{k\in\mathbb{Z}}(F_X(\arcsin y-2\pi k)-F_X(-\arcsin y+(2k-1)\pi)) & \text{if } y\in[-1,1].\end{cases}$

d Let $Y = F_X(X)$. Then, $F_Y(y) = P(F_X(X) \le y) = \begin{cases} 1 \text{ if } y > 1 \\ 0 \text{ if } y < 0 \end{cases}$ because $0 \le F_X(x) \le 1$ by definition.

Let $0 \le y \le 1$. $F_Y(y) = P(\{\omega \mid P(X \le X(\omega)) \le y\})$
$= P(\{\omega \mid X(\omega) \le \sup\{x \mid F_X(x) \le y\}\})$
$= F_X(\sup\{x \mid F_X(x) \le y\})$. This is identically y due to the fact that F_X is right-continuous.

Thus,

$$F_Y(y) = \begin{cases} 0 & \text{if } y < 0 \\ y & \text{if } 0 \le y \le 1 \\ 1 & \text{if } y > 0 \end{cases}$$

4. a. Let $x \in \mathbb{Q}$. It is a standard argument in measure theory that $(0, r) = \bigcup_{n=1}^{\infty} (0, r - \frac{1}{n}]$ $\forall\, 0 < r \le 1$. To see this, let $x \in (0, r)$. Then, choose some $n' \in \mathbb{N}$ s.t. $\frac{1}{n'} \ge r - x$. We can do this because $x < r$ and $1/n'$ can be made arbitrarily small. Then, $x \ge r - \frac{1}{n'} \Rightarrow x \in (0, r - \frac{1}{n'}] \Rightarrow x \in \bigcup_{n=1}^{\infty} (0, r - \frac{1}{n'}]$ $\Rightarrow (0, r) \subset \bigcup_{n=1}^{\infty} (0, r - \frac{1}{n}]$. Clearly, $\bigcup_{n=1}^{\infty} (0, r - \frac{1}{n}] \subset (0, r)$, because $(0, r - \frac{1}{n}] \subset (0, r)$ $\forall\, n \in \mathbb{N}$. Thus, $(0, r) \in \mathcal{B}[0,1]$ $\forall\, 0 < r \le 1$, because the Borel σ-algebra is closed under countable unions. Also, note that $\{0\} \in \mathcal{B}[0,1]$ because $\{0\} = (0,1]^c \in \mathcal{B}[0,1]$ and $\{1\} \in \mathcal{B}[0,1]$ because $\{1\} = [0,1)^c = ((0,1) \cup \{0\})^c \in \mathcal{B}[0,1]$. Now, let $x \in \mathbb{Q} \cap (0,1)$. Then, $\{x\} = (\{0\} \cup (0,x) \cup (x,1])^c \in \mathcal{B}[0,1]$. because the Borel σ-algebra is closed under countable unions and compliments. Thus, $\mathbb{Q} \cap [0,1] = \bigcup_{k \in \mathbb{Q} \cap [0,1]} \{k\} \in \mathcal{B}[0,1]$, because the rationals are countable, meaning that this is a countable union. So yes, the set of rational numbers in $[0,1]$ is a Borel set.

b. Let $X : [0,1] \to \mathbb{R}$ be defined s.t. $X(\omega) = 0$ for $\omega \in \mathbb{Q} \cap [0,1]$ and $X(\omega) = 1$ for $\omega \in \mathbb{Q} \setminus [0,1]$ on the probablity space $([0,1], \mathcal{B}[0,1], P)$ where P is the Lebesgue measure. We have that

$$X^{-1}(B) = \{\omega \mid X(\omega) \in B\} = \begin{cases} \mathbb{Q} \cap [0,1] & \text{if } 0 \in B,\ 1 \notin B \\ [0,1] \cap \mathbb{Q}^c & \text{if } 0 \notin B,\ 1 \in B \\ [0,1] & \text{if } 0, 1 \in B \\ \emptyset & \text{if } 0, 1 \notin B \end{cases}$$

for any Borel set $B \subset [0,1]$.

Clearly, all four possible preimages are elements of $\mathcal{B}[0,1]$ (we showed that $\mathbb{Q} \in \mathcal{B}[0,1]$ in part a), so X is indeed a random variable. As we showed in class, $\mathbb{Q} \cap [0,1]$ has a Lebesgue measure of 0 because it is countable. Hence, $P(X=0)=0$, $P(X=1)=1$, meaning that $F(x) = \begin{cases} 0 & \text{if } x<0 \\ 1 & \text{if } x \geq 0 \end{cases}$

and $E[X] = 0 \cdot 0 + 1 \cdot 1 = 1$. This distribution function is precisely our point mass example from class which means that X is discrete and does not have a distribution function as a result.