AMATH 568 Homework 1

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1 Problem 1

1.1 Part a

Taking $a \in \mathbb{C}$, to show that $\dfrac{1}{z-a} = \mathrm{O}\left(\dfrac{1}{1+|z|}\right)$ as $z \to \infty$, $z \in \mathbb{C}$, we wish to find constants $K, M > 0$ such that for $|z| > M$, $\left|\frac{1}{z-a}\right| \leq K \left|\frac{1}{1+|z|}\right| = K\frac{1}{1+|z|}$. Take $M = 2|a|$ in the case where $a \neq 0$. Then, by the reverse triangle inequality, when $|z| > M$,

$$|z-a| \leq ||z| - |a|| = |z| - |a| > 2|a| - |a| = |a|.$$

Thus,

$$\left|\frac{1}{z-a}\right| < \frac{1}{|a|} = \frac{1+|z|}{|a|}\frac{1}{1+|z|}.$$

Now, note that when $|z| > M$,

$$\frac{1+|z|}{|a|} > \frac{1+M}{|a|} = \frac{1+2|a|}{|a|} = \frac{1}{|a|} + 2.$$

Thus, if we take $K \geq \frac{1}{|a|} + 2$, we have found such an M and K, so the definition is satisfied.

Now, consider the case where $a = 0$. Then, we can take $M = 1$, meaning that for $z > M$,

$$\left|\frac{1}{z-a}\right| = \frac{1}{|z|} < 1 = (1+|z|)\frac{1}{1+|z|}.$$

Also,

$$1 + |z| > 1 + M = 2,$$

so we simply need to take $K = 2$, and $\left|\frac{1}{z-a}\right| \leq K\frac{1}{1+|z|}$ for $z > M$.

1.2 Part b

Now, we wish to show that $\dfrac{1}{\text{dist}(z,[a,b])} = \text{O}\left(\dfrac{1}{1+|z|}\right)$ as $z \to \infty$, $z \in \mathbb{C}$ with $a, b \in \mathbb{R}$. First, note that if $a = b$, $\text{dist}(z,[a,b]) = |z-a|$, so this amounts to showing that $\dfrac{1}{z-a} = \text{O}\left(\dfrac{1}{1+|z|}\right)$ as $z \to \infty$, $z \in \mathbb{C}$ which was done in part a. Now, assume that $a \neq b$. Now, consider a region D defined by revolving $[a,b]$ in the complex plane. Namely, if a and b have the same sign, D is an annulus defined by $\min\{|a|,|b|\} \leq |z| \leq \max\{|a|,|b|\}$ and if they have different signs, D is a disk defined by $|z| \leq \max\{|a|,|b|\}$. Regardless, if $|z| \geq \max\{|a|,|b|\}$, $\text{dist}(z,D) = |z| - \max\{|a|,|b|\}$. Now, note that $[a,b] \subset D$, so clearly, $\text{dist}(z,[a,b]) \geq \text{dist}(z,D)$ for any $z \in \mathbb{C}$. Let $A = \max\{|a|,|b|\}$ and take $M = 2A$. Then, if $|z| > M$, by the above and the reverse triangle inequality,

$$\text{dist}(z,[a,b]) \geq \text{dist}(z,D) = |z-A| \geq ||z|-|A|| = |z| - A > 2A - A = A.$$

Thus, if $|z| > M$,

$$\left|\frac{1}{\text{dist}(z,[a,b])}\right| = \frac{1}{\text{dist}(z,[a,b])} < \frac{1}{A} = \frac{1+|z|}{A}\frac{1}{1+|z|}.$$

Now, we simply take

$$K \geq \frac{1+|z|}{A} > \frac{1+2A}{A} = \frac{1}{A} + 2 = \frac{1}{\max\{|a|,|b|\}} + 2$$

to get that

$$\left|\frac{1}{\text{dist}(z,[a,b])}\right| \leq K\frac{1}{1+|z|}$$

when $|z| > M$. Thus, $\dfrac{1}{\text{dist}(z,[a,b])} = \text{O}\left(\dfrac{1}{1+|z|}\right)$ as $z \to \infty$, $z \in \mathbb{C}$.

2 Problem 2 (Exercise 1.8)

Suppose that μ is a continuous parameter and that for each $\mu \in [0,1]$ we have that $f(z,\mu) = \text{O}(g(z,\mu))$ as $z \to z_0$ from D. Despite the proof on page 21 of the text, it is not true that if the integrals exist in the Riemann sense for all z close enough to z_0 then

$$\int_0^1 f(z,\mu)d\mu = \text{O}\left(\int_0^1 |g(z,\mu)|d\mu\right) \quad \text{as } z \to z_0 \text{ from } D.$$

Even though both integrals can be written as Riemann sums,

$$\int_0^1 f(z,\mu)d\mu = \lim_{N\to\infty}\sum_{i=0}^{N}\Delta\mu f(z,\mu_i), \int_0^1 g(z,\mu)d\mu = \lim_{N\to\infty}\sum_{i=0}^{N}\Delta\mu g(z,\mu_i)$$

this needs to be true in the limit, requiring an infinite number of terms, so we may not be able to find the minimum of $\delta_1, \delta_2, \ldots$ and the maximum of $K_1, K_2, \ldots$ as is done in the proof. If we include the additional hypothesis that there are positive real numbers δ, K such that $|f(z,\mu)| \leq K|g(z,\mu)|$ for all $0 < |z - z_0| < \delta$ and all $\mu \in [0,1]$, then this is no longer an issue and the proof is valid with our new definitions of δ and K.

As a counterexample, consider $f(z,\mu) = \mu^{-1}e^{-z/\mu}$ and $g(z,\mu) = e^{-z/\mu}$ as $z \to 0$, $z > 0$. Clearly, for each given $\mu \in (0,1]$, $f(z,\mu) = \mathrm{O}(g(z,\mu))$, because $|\mu^{-1}e^{-z/\mu}| \leq K|e^{-z/\mu}|$ if $K \geq \mu^{-1}$. First, note that

$$\left|\int_0^1 g(z,\mu)d\mu\right| = \left|\int_0^1 e^{-z/\mu}d\mu\right| \leq \max_{\mu\in(0,1]} e^{-z/\mu} = e^{-z} \leq 1$$

since $z > 0$. However, using the substitution $u = z/\mu$ which gives that $d\mu = \frac{-\mu}{u}du$ and taking $z < 1$,

$$\begin{aligned}\int_0^1 f(z,\mu)d\mu = \int_0^1 \mu^{-1}e^{-z/\mu}d\mu &= \int_z^\infty \frac{e^{-u}}{u}du \\ &= \int_1^\infty \frac{e^{-u}}{u}du + \int_z^1 \frac{e^{-u}-1}{u}du + \int_z^1 \frac{du}{u}.\end{aligned}$$

The last integral in this expression for the integral of f is unbounded (tends to infinity) as $z \to 0$, so there is no possible way we can find a $\delta, K > 0$ such that

$$\left|\int_0^1 f(z,\mu)d\mu\right| \leq K\left|\int_0^1 g(z,\mu)d\mu\right| \leq K$$

when $z < \delta$ since we can always increase $\int_z^1 \frac{du}{u}$ by taking z to be closer to 0.

3 Problem 3 (Exercise 1.11)

Consider the function e^{-2z} for $|\arg z| < \pi/4$. This implies that $|\arg 2z| < \pi/2$, so $2z$ is in the right half-plane or, equivalently, $\Re(2z) > 0$. Now, fix $\epsilon > 0$ and observe that

$$|e^{-2z}| = |e^{-2\Re(z)}||e^{-2\Im(z)}| = |e^{-2\Re(z)}| = e^{-2\Re(z)}.$$

Because we restrict z to the sector $|\arg z| < \pi/4$, we know that $\Re(z) \to \infty$ as $z \to \infty$. We wish to show that e^{-2z} is $\mathrm{o}(1)$, so we wish to find an $M > 0$ such that $|e^{-2z}| < \epsilon$ whenever $|z| > M$. However, this is precisely the definition of a limit at infinity in the complex plane, so we need only show that $\lim_{z\to\infty, |\arg z|<\pi/4} e^{-2z} = 0$. However,

$$\lim_{z\to\infty, |\arg z|<\pi/4} |e^{-2z}| = \lim_{z\to\infty, |\arg z|<\pi/4} |e^{-2\Re(z)}| = 0$$

because $\Re(z) \to \infty$ here, so $\lim_{z\to\infty, |\arg z|<\pi/4} e^{-2z} = 0$ is indeed true. Thus, e^{-2z} is o(1), meaning that as $z \to \infty$ in this sector

$$2\cosh(z) = e^z + e^{-z} = e^z(1 + e^{-2z}) = e^z(1 + \mathrm{o}(1)),$$

so $2\cosh(z)$ is approximated by e^z in the sector $|\arg z| < \pi/4$ in the sense of small relative error. This is also true in the sense of absolute error, because the sector is contained in the right half-plane, $\Re(z) > 0$, meaning that

$$\begin{aligned}\lim_{z\to\infty, |\arg z|<\pi/4} |e^{-z}| &= \lim_{z\to\infty, |\arg z|<\pi/4} |e^{-\Re(z)}||e^{-\Im(z)}| \\ &= \lim_{z\to\infty, |\arg z|<\pi/4} |e^{-\Re(z)}| = \lim_{z\to\infty, |\arg z|<\pi/4} e^{-\Re(z)} = 0,\end{aligned}$$

so e^{-z} is o(1) as $z \to \infty$ in the sector $|\arg z| < \pi/4$, and $f(z) = e^z + \mathrm{o}(1)$ as $z \to \infty$ in the sector $|\arg z| < \pi/4$.
Now, consider the function e^{2z} in the sector $|\arg(-z)| < \pi/4$. Then, $|\arg(-2z)| < \pi/2$, so $-2z$ is in the right half-plane or, equivalently, $\Re(2z) < 0$. Now observe that

$$|e^{2z}| = |e^{2\Re(z)}||e^{2\Im(z)}| = |e^{2\Re(z)}| = e^{2\Re(z)}.$$

Because we restrict z to the sector $|\arg(-z)| < \pi/4$, we know that $\Re(z) \to -\infty$ as $z \to \infty$. Thus,

$$\lim_{z\to\infty, |\arg(-z)|<\pi/4} |e^{2z}| = \lim_{z\to\infty, |\arg(-z)|<\pi/4} |e^{2\Re(z)}| = 0$$

because $\Re(z) \to -\infty$ here, so $\lim_{z\to\infty, |\arg(-z)|<\pi/4} e^{2z} = 0$, meaning that e^{2z} is o(1), and as $z \to \infty$ in this sector,

$$2\cosh(z) = e^z + e^{-z} = e^{-z}(1 + e^{2z}) = e^{-z}(1 + \mathrm{o}(1)),$$

so $2\cosh(z)$ is approximated by e^{-z} in the sector $|\arg(-z)| < \pi/4$ in the sense of small relative error. Again, this is also true in the sense of absolute error, because the sector is contained in the left half plane, $\Re(z) < 0$, meaning that

$$\begin{aligned}\lim_{z\to\infty, |\arg(-z)|<\pi/4} |e^{z}| &= \lim_{z\to\infty, |\arg(-z)|<\pi/4} |e^{\Re(z)}||e^{\Im(z)}| \\ &= \lim_{z\to\infty, |\arg(-z)|<\pi/4} |e^{\Re(z)}| = \lim_{z\to\infty, |\arg(-z)|<\pi/4} e^{\Re(z)} = 0,\end{aligned}$$

so e^z is o(1) as $z \to \infty$ in the sector $|\arg(-z)| < \pi/4$, and $f(z) = e^{-z} + \mathrm{o}(1)$ as $z \to \infty$ in the sector $|\arg(-z)| < \pi/4$.

4 Problem 4

By definition, for $z \in \mathbb{C}$,

$$\mathrm{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-s^2} ds = \frac{2}{\sqrt{\pi}} \left(\int_z^0 e^{-s^2} ds + \int_0^\infty e^{-s^2} ds \right).$$

Additionally, using the change of variables $s \to -s$,

$$\begin{aligned}
\operatorname{erfc}(-z) &= \frac{2}{\sqrt{\pi}} \int_{-z}^{\infty} e^{-s^2} ds = \frac{2}{\sqrt{\pi}} \left(\int_{-z}^{0} e^{-s^2} ds + \int_{0}^{\infty} e^{-s^2} ds \right) \\
&= \frac{2}{\sqrt{\pi}} \left(\int_{z}^{0} e^{-s^2} (-ds) + \int_{0}^{-\infty} e^{-s^2} (-ds) \right) \\
&= \frac{2}{\sqrt{\pi}} \left(\int_{0}^{z} e^{-s^2} ds + \int_{-\infty}^{0} e^{-s^2} ds \right).
\end{aligned}$$

Thus,

$$\begin{aligned}
\operatorname{erfc}(z) + \operatorname{erfc}(-z) &= \frac{2}{\sqrt{\pi}} \left(\int_{z}^{0} e^{-s^2} ds + \int_{0}^{\infty} e^{-s^2} ds + \int_{0}^{z} e^{-s^2} ds + \int_{-\infty}^{0} e^{-s^2} ds \right) \\
&= \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-s^2} ds = \frac{2}{\sqrt{\pi}} \sqrt{\pi} = 2.
\end{aligned}$$

From class, we have that when $\Re(z) \geq 0$ and as $|z| \to \infty$,

$$\operatorname{erfc}(z) \sim \frac{e^{-z^2}}{\sqrt{\pi}} \sum_{m=0}^{\infty} (-1)^m \frac{\left(\frac{1}{2}\right)_m}{z^{2m+1}},$$

so the above identity gives that

$$\operatorname{erfc}(-z) \sim 2 - \frac{e^{-z^2}}{\sqrt{\pi}} \sum_{m=0}^{\infty} (-1)^m \frac{\left(\frac{1}{2}\right)_m}{z^{2m+1}}$$

when $\Re(z) \geq 0$ and as $|z| \to \infty$. To find an expression for $\operatorname{erfc}(z)$ when $\Re(z) < 0$, we simply swap z and $-z$ to get that

$$\operatorname{erfc}(z) \sim 2 - \frac{e^{-(-z)^2}}{\sqrt{\pi}} \sum_{m=0}^{\infty} (-1)^m \frac{\left(\frac{1}{2}\right)_m}{(-z)^{2m+1}} = 2 + \frac{e^{-z^2}}{\sqrt{\pi}} \sum_{m=0}^{\infty} (-1)^m \frac{\left(\frac{1}{2}\right)_m}{z^{2m+1}}$$

for $\Re(z) < 0$ as $|z| \to \infty$.

5 Problem 5

Consider the function defined on $\mathbb{R}$

$$f(z) = \begin{cases} e^{-1/z}, & z > 0 \\ 0, & z \leq 0 \end{cases}$$

at the point $z_0 = 0$. The derivative of $e^{-1/z}$ is $z^{-2} e^{-1/z}$, so in general, the product rule tells us that the nth derivative will be some linear combination of products $z^{-k} e^{-1/z}$, where $k \in \mathbb{Z}$. However, from the right, $e^{-1/z}$ tends to zero faster than any polynomial, so

$$\lim_{z \to 0+} \left(e^{-1/z} \right)^{(k)} = 0$$

for all k. Thus, our function is infinitely differentiable at 0 since this matches the derivative from the left at 0. However, this implies that its Taylor series centered at $z_0 = 0$ is identically 0, but $f(z)$ takes nonzero values arbitraily close to $z_0 = 0$ when $z > 0$, as $e^{-1/z} \neq 0$ for any $z \in \mathbb{R}$.