Cade Ballew #2120804

AMATH 567 Homework #1 10/6/21

1. c. If we let $1+i=z=x+iy$, $x=1$, $y=1$.
$\rho=|z|=\sqrt{1^2+1^2}=\sqrt{2}$. $\theta=\arctan(\frac{1}{1})=\pi/4$.
Thus, $z=\rho e^{i\theta}=\sqrt{2}e^{i\pi/4}$

e. Letting $\frac{1}{2}-\frac{\sqrt{3}}{2}i=z=x+iy$, $x=\frac{1}{2}$, $y=\frac{-\sqrt{3}}{2}$.
$\rho=|z|=\sqrt{(\frac{1}{2})^2+(\frac{-\sqrt{3}}{2})^2}=\sqrt{\frac{1}{4}+\frac{3}{4}}=1$
$\theta=\arctan(y/x)=\arctan\left(\frac{-\sqrt{3}/2}{1/2}\right)=-\pi/3$
Thus, $z=\rho e^{i\theta}=e^{-i\pi/3}$

2. a. Let $z=\rho e^{i\theta}=e^{2+i\pi/2}$. Then, $z=e^2e^{i\pi/2}=e^2(\cos(\pi/2)+i\sin(\frac{\pi}{2}))$
(Euler's formula) $=e^2(0+i\cdot 1)=0+e^2i$.

b. $\frac{1}{1+i}=\frac{1}{1+i}\cdot\frac{1-i}{1-i}=\frac{1-i}{1^2-i^2}=\frac{1-i}{2}=\frac{1}{2}+i\left(\frac{-1}{2}\right)$

c. $(1+i)^3=1^3+3\cdot 1^2\cdot i+3\cdot 1\cdot i^2+i^3=1+3i+3i^2+i\cdot i^2$
$=1+3i+3(-1)+i(-1)=-2+2i$

d. $|3+4i|=\sqrt{3^2+4^2}=\sqrt{25}=5$

3. b. $z^4=-1=e^{i\pi}$ (as $e^{i\pi}=\cos\pi+i\sin\pi=-1+i\cdot 0=-1$)
$=e^{i(\pi+2\pi k)}$ ($\forall k\in\mathbb{N}$). Thus, $z=(z^4)^{1/4}=(e^{i(\pi+2\pi k)})^{1/4}$
$=e^{i(\frac{\pi}{4}+\frac{\pi k}{2})}$ $\forall k\in\mathbb{N}$. This has unique values at $z=e^{\frac{i\pi}{4}}, e^{i\frac{3\pi}{4}}, e^{i\frac{5\pi}{4}}, e^{i\frac{7\pi}{4}}$
(note that $e^{i\frac{9\pi}{4}}=e^{i\frac{\pi}{4}}$) which solve the equation.

4. a. Let $z=x+iy$, $w=u+iv$ with $z,w\in\mathbb{C}$. Then,
$\overline{z+w}=\overline{(x+iy)+(u+iv)}=\overline{(x+u)+i(y+v)}=(x+u)-i(y+v)$
$=(x-iy)+(u-iv)=\bar{z}+\bar{w}$.

d. Let $\mathbb{C}\ni z=x+iy$. Then, $|z|=\sqrt{x^2+y^2}\geq\sqrt{x^2}\geq x=\mathrm{Re}(z)$.
Thus, $\mathrm{Re}(z)\leq|z|$.

f. Let $z_1=x_1+iy_1$, $z_2=x_2+iy_2$ with $z,w\in\mathbb{C}$. Then,
$|z_1z_2|=|(x_1x_2-y_1y_2)+i(x_1y_2+x_2y_1)|=\sqrt{(x_1x_2-y_1y_2)^2+(x_1y_2+x_2y_1)^2}$
$=\sqrt{x_1^2x_2^2-\cancel{2x_1x_2y_1y_2}+y_1^2y_2^2+x_1^2y_2^2+\cancel{2x_1x_2y_1y_2}+x_2^2y_1^2}$
$=\sqrt{x_1^2(x_2^2+y_2^2)+y_1^2(x_2^2+y_2^2)}=\sqrt{(x_1^2+y_1^2)(x_2^2+y_2^2)}=\sqrt{x_1^2+y_1^2}\sqrt{x_2^2+y_2^2}$
$=|z_1||z_2|$.

5. We first wish to show the triangle inequality

$$\left|\sum_{j=1}^{N} z_j\right| \leq \sum_{j=1}^{N} |z_j| \quad \forall z_j \in \mathbb{C},\ j \in \{1, \ldots, N\}$$ by induction on N.

Consider the base case $N=2$, i.e. $|z_1+z_2| \leq |z_1|+|z_2|$.

Take $|z_1+z_2|^2 - (|z_1|+|z_2|)^2 = (|z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + \bar{z}_1 z_2) - (|z_1|^2+|z_2|^2+2|z_1||z_2|)$
$= z_1\bar{z}_2 + \bar{z}_1 z_2 - 2|z_1||z_2| = z_1\bar{z}_2 + \overline{z_1\bar{z}_2} - 2|z_1||\bar{z}_2|$
(as $|z| = |\bar{z}|$ and $z\bar{w} = \overline{\bar{z}w}$) $= 2(\mathrm{Re}(z_1\bar{z}_2) - |z_1\bar{z}_2|) \leq 0$
($z \geq \mathrm{Re}(z)$). Thus, $|z_1+z_2| \leq |z_1|+|z_2|$. Note that this proof can be found in the course notes.

Now, assume that the inequality holds for $N=n-1$, i.e. $\left|\sum_{j=1}^{n-1} w_j\right| \leq \sum_{j=1}^{n-1} |w_j| \ \forall w_j \in \mathbb{C},\ j \in \{1, \ldots, n-1\}$. Given $z_j \in \mathbb{C}$, $j=1,\ldots,n$,

$$\left|\sum_{j=1}^{n} z_j\right| = \left|\sum_{j=1}^{n-1} z_j + z_n\right| \leq \left|\sum_{j=1}^{n-1} z_j\right| + |z_n| \text{ (by the } N=2 \text{ case)}$$

$$\leq \sum_{j=1}^{n-1} |z_j| + |z_n| \text{ (by the inductive hypothesis)} = \sum_{j=1}^{n} |z_j|.$$

Thus, the inequality holds for $N=n$, implying that the triangle inequality holds $\forall N \in \mathbb{N}$ s.t. $N \geq 2$ by induction. (Note that the $N=1$ case is trivial).

Now, we claim that equality is achieved iff $z_1, \ldots, z_N$ are colinear.

We again show this by induction on N starting with the case $N=2$. (If $N=1$, equality is clearly achieved $\forall z_1 \in \mathbb{C}$).

From above, we know that equality is achieved iff $2(\mathrm{Re}(z_1\bar{z}_2) - |z_1\bar{z}_2|) = 0$ which occurs iff $\mathrm{Re}(z_1\bar{z}_2) = |z_1\bar{z}_2|$
$= \sqrt{(\mathrm{Re}(z_1\bar{z}_2))^2 + (\mathrm{Im}(z_1\bar{z}_2))^2}$ which occurs iff $\mathrm{Im}(z_1\bar{z}_2) = 0$.
$\mathrm{Im}(z_1\bar{z}_2) = \mathrm{Im}((x_1+iy_1)(x_2-iy_2)) = -x_1y_2 + x_2y_1 = 0$
$\Leftrightarrow x_1y_2 = x_2y_1$. This implies colinearity as either $y_1/x_1 = y_2/x_2$ and $\theta_j = \arctan\left(\frac{y_j}{x_j}\right)$ so the arguments must match, or $x_1 = x_2 = 0$

Now, say the hypothesis holds for $N=n-1$, i.e. equality is achieved iff $w_1, \ldots, w_{n-1}$ are colinear. Then, in the series of inequalities

$$\left|\sum_{j=0}^{n} z_j\right| \leq \left|\sum_{k=1}^{n-1} z_k\right| + |z_n| \leq \sum_{j=1}^{n-1} |z_j| + |z_n| = \sum_{j=1}^{n} |z_j|,$$

we know that the first inequality becomes an inequality iff $\sum_{j=1}^{n-1} z_k \sim z_n$ (by the base case) and the second holds by the inductive hypothesis.

However, because $z_1 \sim \cdots \sim z_{n-1}$, $\sum_{j=1}^{n-1} z_k \sim z_n$ iff $z_n \sim z_1 \sim \cdots \sim z_{n-1}$ (colinearity is equivalent to matching arguments, and arguments are retained in summation), thus the triangle inequality becomes an equality iff $z_1 \sim \cdots \sim z_n$ and the $N=n$ case holds. Thus, by induction, the triangle inequality holds with equality iff $z_1 \sim \cdots \sim z_N$ $\forall N \in \mathbb{N}$ s.t. $N \geq 2$.

6. $\forall z_1, z_2 \in \mathbb{C}$, $E(z_1)E(z_2) = \sum_{n=0}^{\infty} \frac{z_1^n}{n!} \sum_{m=0}^{\infty} \frac{z_2^m}{m!}$. By taking $a_n = \frac{z_1^n}{n!}$, $b_n = \frac{z_2^m}{m!}$, the Cauchy product allows us to write

$$E(z_1)E(z_2) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{z_1^k}{k!} \frac{z_2^{n-k}}{(n-k)!}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} z_1^k z_2^{n-k} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} z_1^k z_2^{n-k}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} (z_1+z_2)^n \text{ (binomial theorem)} = E(z_1+z_2).$$

Now, consider an arbitrary power series $F(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z_1^n$ (we can include the factorial term because Borel's theorem gives that any power series is the Taylor series of some C^∞ function and Taylor series include this factorial term) taken to be centered at 0 WLOG (we can always shift it) with coefficients $a_n \in \mathbb{C}$. $\forall n \in \mathbb{N}$. Then, the Cauchy product gives that $\forall z_1, z_2 \in \mathbb{C}$.

$$F(z_1)F(z_2) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z_1^n \sum_{m=0}^{\infty} \frac{a_m}{m!} z_2^m = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{a_k}{k!} z_1^k \frac{a_{n-k}}{(n-k)!} z_2^{n-k}$$

By the binomial theorem,

$$F(z_1+z_2) = \sum_{n=0}^{\infty} \frac{a_n}{n!} (z_1+z_2)^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{a_n}{\cancel{n!}} \cdot \frac{\cancel{n!}}{k!(n-k)!} z_1^k z_2^{n-k}$$

Thus, in order for $F(z_1)F(z_2) = F(z_1+z_2)$ $\forall z_1, z_2 \in \mathbb{C}$, we need that $\frac{a_n}{k!(n-k)!} = \frac{a_k a_{n-k}}{k!(n-k)!}$ $\forall n, k \in \mathbb{N}$ s.t. $n-k \geq 0$

which happens iff $a_n = a_k a_{n-k}$ $\forall n, k \in \mathbb{N}$ s.t. $n-k \geq 0$.

First consider $n=0, k=0$. This gives that $a_0 = a_0 a_0 = a_0^2$, so $a_0 = 0, 1$. In the case where $a_0 = 0$, we can take $k=0$ to find that $a_n = a_0 a_n = 0 \ \forall \ n \in \mathbb{N}$ which of course implies that $F(z)$ is the zero power series. Now consider the case where $a_0 = 1$. We claim that this implies that $a_N = a_1^N \ \forall \ N \in \mathbb{N}$ s.t. $N \geq 1$. This is trivially true for $N=1$. We show this by induction on N for $N \geq 2$. Starting with the base case $N=2$, we take $n=2, k=1$ to get $a_2 = a_1 a_1 = a_1^2$, meaning that the base case holds. Assuming that the $N=m-1$ case holds $(a_{m-1} = a_1^{m-1})$, we take $n=m, k=1$ to get that $a_m = a_1 a_{m-1} = a_1 a_1^{m-1} = a_1^m$. Thus, we have shown by induction that $a_N = a_1^N \ \forall \ N \in \mathbb{N}$ with $N \geq 2$. This implies that our power series must be of the form

$$F(z) = 1 + \frac{c}{1!} z + \frac{c^2}{2!} z^2 + \dots + \frac{c^n}{n!} z^n + \dots \quad \text{for some } c \in \mathbb{C}$$

or $F(z) = 0$ in order for $F(z_1)F(z_2) = F(z_1 + z_2)$ to hold $\forall \ z_1, z_2 \in \mathbb{C}$. We also have that $F(z) = E(cz)$, meaning that we can find other power series with this property, but they must be a multiplicatively scaled form of $E(z)$ or identically 0.

7. We start with the given equation $x^3 + ax^2 + bx + c = 0$. Letting $x = y - a/3$, we get new equation

$$0 = (y - a/3)^3 + a(y - a/3)^2 + b(y - a/3) + c$$
$$= (y^3 - \cancel{ay^2} + \frac{a^2}{3}y - \frac{a^3}{27}) + (ay^2 - \frac{2a^2}{3}y + a^3/9) + (by - \frac{ab}{3}) + c$$
$$= y^3 + \frac{3b - a^2}{3} y + \frac{2a^3 - 9ab + 27c}{27}.$$

Thus, $y^3 + py + q = 0$ for

$$p = \frac{3b - a^2}{3}, \quad q = \frac{2a^3 - 9ab + 27c}{27}.$$

Now let $y = u + v$. Then, we have $0 = (u+v)^3 + p(u+v) + q$ $= u^3 + 3u^2v + 3uv^2 + v^3 + p(u+v) + q = u^3 + v^3 + 3uv(u+v) + p(u+v) + q$.
Thus, $u^3 + v^3 + (3uv + p)(u+v) + q = 0$.
Now, impose $3uv + p = 0 \Rightarrow u^3v^3 = -p/27$. Then, $u^3 + v^3 = -q$.

We can now substitute $v^3 = -q - u^3$ into $u^3v^3 = -p^3/27$ to get $u^3(-q-u^3) = -p^3/27 \Leftrightarrow u^6 + qu^3 - p^3/27 = 0$. Because u and v are defined symmetrically, we also have that $v^6 + qv^3 - p^3/27 = 0$, i.e. u^3 and v^3 both satisfy the quadratic equation $d^2 + qd - p^3/27 = 0$. Solving this with the quadratic formula, we get $u^3 = \frac{-q \pm \sqrt{q^2 + \frac{4p^3}{27}}}{2}$. We know that v^3 takes on the same values because it satisfies the same quadratic equation, but we can also find which value of v^3 corresponds to a given value of u^3 because $u^3v^3 = -p^3/27$. If $u^3 = 0$, then $v^3 = 0$ because it satisfies the same quadratic. If $u^3 \neq 0$, then

$$v^3 = \frac{-p^3}{27u^3} = \frac{-2p^3}{27\left(-q \pm \sqrt{q^2 + \frac{4p^3}{27}}\right)} = \frac{-2p^3\left(-q \mp \sqrt{q^2 + \frac{4p^3}{27}}\right)}{27\left(q^2 - \left(q^2 + \frac{4p^3}{27}\right)\right)}$$

$$= \frac{-2p^3\left(-q \mp \sqrt{q^2 + \frac{4p^3}{27}}\right)}{-4p^3} = \frac{-q \mp \sqrt{q^2 + \frac{4p^3}{27}}}{2}.$$

Thus, we can have either

$$(u^3, v^3) = \left(\frac{-q + \sqrt{q^2 + \frac{4p^3}{27}}}{2}, \frac{-q - \sqrt{q^2 + \frac{4p^3}{27}}}{2}\right) \text{ or } (u^3, v^3) = \left(\frac{-q - \sqrt{q^2 + \frac{4p^3}{27}}}{2}, \frac{-q + \sqrt{q^2 + \frac{4p^3}{27}}}{2}\right).$$

However, because our expression for y (and therefore x) allows u and v to be interchanged, we can consider the former case WLOG as either case will produce the same result for y. Now, we can simply take $u_1 = \sqrt[3]{\frac{-q + \sqrt{q^2 + \frac{4p^3}{27}}}{2}}$ (Assume that this expression is the principal cube root, i.e. the real root if u^3 is real. If u^3 is complex, we can let $u^3 = \rho e^{i\theta}$ and take $u_1 = \sqrt[3]{\rho}\, e^{i\theta/3}$ or just choose a cube root that we find.) We can then use the principle of roots of unity to let $u_2 = u_1 e^{i2\pi/3}$, $u_3 = u_1 e^{i4\pi/3}$ be the remaining roots of this cubic. We know that $uv = -p/3$, so we know that $v_j = \frac{-p}{3u_j}$ for $j = 1, 2, 3$ (excluding the case where $u_j = 0 \Rightarrow v_j = 0$ as stated above).

Thus, we get that $x = y - \frac{a}{3}$ where $y = u+v$

where $u = \underbrace{\sqrt[3]{\frac{-q+\sqrt{q^2+\frac{4p^3}{27}}}{2}}}_{u_1}, \underbrace{u_1 e^{i\frac{2\pi}{3}}}_{u_2}, \underbrace{u_1 e^{i\frac{4\pi}{3}}}_{u_3}$

and $v = \frac{-p}{3u_1}, \frac{-p}{3u_2}, \frac{-p}{3u_3}$ where $p = \frac{3b-a^3}{3}$, $q = \frac{2a^3-9ab+27c}{27}$.

This would appear on the surface to give 9 values for x, but the fact that values of v correspond one-to-one with values of u ($uv = -p/3$) means that we have 3 possible values of x, exactly what one would expect from a cubic.

Now, consider the equation $x^3-2x^2+x-12=0$. Then, $p = \frac{3-4}{3} = -1/3$, $q = \frac{-16+18-324}{27} = \frac{-322}{27}$. Then,

$$u_1 = \sqrt[3]{\frac{\frac{322}{27}+\sqrt{\left(\frac{322}{27}\right)^2 - \frac{4}{729}}}{2}} = \frac{7+3\sqrt{5}}{6}$$ according to Wolfram-Alpha.

$$v_1 = \frac{-p}{3u} = \frac{6}{7+3\sqrt{5}} \cdot \frac{1}{9} = \frac{2}{3} \cdot \frac{7-3\sqrt{5}}{4} = \frac{7-3\sqrt{5}}{6}.$$

Thus, $y = u_1+v_1 = 7/3 \Rightarrow x = 7/3 + 2/3 = 3$ is a solution. Checking this, $27-2\cdot 9+3-12=0$ ✓. Now, we can factor $0=(x-3)(x^2+x+4)$. By the quadratic formula $\frac{-1\pm\sqrt{1-16}}{2}$ are solutions. Thus, the roots are $x=3, \frac{-1+i\sqrt{15}}{2}, \frac{-1-i\sqrt{15}}{2}$.

Now, consider Bombelli's equation $x^3-15x-4=0$.

$p=-15$, $q=-4$, so $u^3 = \frac{4+\sqrt{16+\frac{4\cdot 15^3}{27}}}{2} = 2+11i$ by Wolfram.

$(2+i)^3 = 2+11i$, so take $u_1 = 2+i$. Then, $v = \frac{15}{3(2+i)} = 5 \cdot \frac{2-i}{5} = 2-i$

Thus, $y = u_1+v_1 = 4 \Rightarrow x = 4 - 0/3 = 4$ is a solution. Checking this, $4^3-15\cdot 4-4=0$ ✓. Now, we factor $0=(x-4)(x^2+4x+1)$. By the quadratic formula, $-2\pm\sqrt{4-1}$ are solutions. Thus, the roots are $x=4, -2+\sqrt{3}, -2-\sqrt{3}$.