MATH 525 Homework 5

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1 Problem 1

Let X be a locally convex vector space with seminorms $\{p_\alpha\}_{\alpha\in A}$. If the topology is equivalent to one defined by a norm p, then for all $\alpha \in A$ and some $C > 0$,

$$p_\alpha(x) \leq C_\alpha p(x).$$

Let $E = B_{1,p}$, the one-ball in the norm. Then, for all $\alpha \in A$,

$$\sup_{x\in E} p_\alpha(x) \leq \sup_{x\in E} C_\alpha p(x) \leq C_\alpha,$$

so each seminorm defining the topology is bounded on E. As the unit ball, E is open, so E is an open bounded set.

Conversely, let X contain an open bounded set E. Then, because E is open and X is locally convex, there exist some $\alpha_1, \dots, \alpha_n \in A$ and $\epsilon_1, \dots, \epsilon_n > 0$ such that

$$E \supset \bigcap_{j=1}^{n} B_{\epsilon_j,\alpha_j} = \{x : p_{\alpha_j}(x) < \epsilon_j,\ j = 1, \dots, n\}.$$

Define

$$p(x) = \sum_{j=1}^{n} p_{\alpha_j}(x), \quad \epsilon = \sum_{j=1}^{n} \epsilon_j.$$

Then, p is clearly a seminorm because nonnegativity, homogeneity, and the triangle inequality all immediately follow from its definition as the finite sum of seminorms. Furthermore,

$$E \supset \{x : p_{\alpha_j}(x) < \epsilon_j,\ j = 1, \dots, n\} \supset B_{\epsilon,p}.$$

Since E is bounded, for each $\alpha \in A$, there exists some $C_\alpha > 0$ such that $p_\alpha(x) < C_\alpha$ for all $x \in E$. In particular, this implies that on a neighborhood of the origin $F = B_{\epsilon,p}$, $p(x) < \epsilon$ and $p_\alpha(x) < C_\alpha$ for all $x \in F$. Then, for $x \in F$ such that $p(x) \neq 0$, $\frac{\epsilon}{p(x)}x \in F$, so

$$\frac{\epsilon}{p(x)} p_\alpha(x) = p_\alpha\left(\frac{\epsilon}{p(x)}x\right) < C_\alpha,$$

and $p_\alpha(x) < \frac{C_\alpha}{\epsilon}p(x)$. If instead $p(x) = 0$, then $p(cx) = 0$ for all $c \in \mathbb{C}$, so $x \in F$, but $p_\alpha(cx) = |c|p_\alpha(x) < C_\alpha$ for all $c \in \mathbb{C}$. Thus, $p_\alpha(x) = 0 < \frac{C_\alpha}{\epsilon}p(x)$, and the inequality is satisfied for all $x \in F$. Because F is absorbing, for any $x \in X$, $x = \lambda y$ for some $\lambda \geq 0$, $y \in F$, so by homeogeneity, this implies that for all $x \in X$, $p_\alpha(x) < \frac{C_\alpha}{\epsilon}p(x)$, since a factor $1/\lambda$ divides through both sides. Since $p(x) = \sum_{j=1}^{n} p_{\alpha_j}(x)$ for all $x \in X$ by construction, p generates the same topology as $\{p_\alpha\}_{\alpha\in A}$. Finally, assuming that X is Hausdorff, for every $x \neq 0$, there exists some $\alpha \in A$ for which $p_\alpha(x) > 0$. For this α,

$$0 < \frac{\epsilon}{C_\alpha} p_\alpha(x) < p(x),$$

so $p(x) \neq 0$ for all $x \neq 0$. Thus, p is nondegenerate and therefore a norm, so the locally convex topology generated by the seminorms $\{p_\alpha\}_{\alpha\in A}$ is equivalent to one generated by the norm p.

2 Problem 2

Let M be a vector subspace of a normed vector space X. If $M = X$, then it is closed in any topology, so it is trivially closed in the norm topology if and only if it is weakly closed. Assume that $M \subsetneq X$ and let M be closed in the norm topology. Then, for any $x \in M^c$, Theorem 5.8a gives that there exists some $f \in X^*$ such that $f\big|_M = 0$ and $f(x) = \delta > 0$. Consider the open ball with respect to the seminorm $|f|$ given by

$$B_{\delta,f}(x) = \{y \in X : |f(x-y)| < \delta\} = \{y \in X : 0 < f(y) < 2\delta\}.$$

Since $f\big|_M = 0$, we have that $B_{\delta,f}(x) \subset M^c$. Since we can find such an open ball in some seminorm for any $x \in M^c$, M^c must be open in the seminorm topology, meaning that M is weakly closed.

Conversely, let M be weakly closed and let $x \in \overline{M}$ with $\overline{M}$ defined in the norm topology. That is, there exists some sequence $\{x_n\}_{n=1}^\infty$ such that $\lim_{n\to\infty} \|x_n - x\| = 0$. Then, for any $f \in X^*$,

$$\lim_{n\to\infty} |f(x_n) - f(x)| \leq \lim_{n\to\infty} \|f\|_{X^*}\|x_n - x\| = 0,$$

since $\|f\|_{X^*}$ is finite. Thus, $\{x_n\}_{n=1}^\infty \to x$ in the weak topology, so $x \in M$ since M is closed under this topology. This means that in the norm topology, $M = \overline{M}$, so M is closed in the norm topology as well.

3 Problem 3

Let X be a normed space and $f_j \in X^*$ converge weak* to f. That is, $f_j(x) \to f(x)$ for all $x \in X$. Then, for all $x \in X$,

$$|f(x)| = \lim_{j\to\infty} |f_j(x)| = \liminf_{j\to\infty} |f_j(x)| \leq \liminf_{j\to\infty} \|f_j\|_{X^*}\|x\| = \|x\| \liminf_{j\to\infty} \|f_j\|_{X^*}.$$

Thus,

$$\|f\|_{X^*} = \sup_{x\in X} \frac{|f(x)|}{\|x\|} \leq \liminf_{j\to\infty} \|f_j\|_{X^*}.$$

As an example for which this inequality is strict, let $X = \ell^1(\mathbb{N})$ and for any $x = \{x_j\}_{j\in\mathbb{N}} \in X$, define the functionals

$$f_n(x) = \sum_{j=n+1}^\infty x_j.$$

Because we have the correspondence $\ell^1(\mathbb{N})^* \equiv \ell^\infty(\mathbb{N})$, each f_n corresponds to a sequence in $\ell^\infty(\mathbb{N})$ defined by

$$(f_n)_j = f_n(e_j) = \begin{cases} 1, & j \geq n+1, \\ 0, & j < n+1. \end{cases}$$

Define f to be the zero functional on $X = \ell^1(\mathbb{N})$. Then, for any $x \in X$,

$$\lim_{n\to\infty} |f_n(x) - f(x)| \leq \lim_{n\to\infty} \sum_{j=n+1}^\infty |x_j| = 0,$$

so $\{f_n\}$ converges weak* to f. However, $\|f\|_{X^*} = 0$, but for any $n \in \mathbb{N}$,

$$\|f_n\|_{X^*} = \sup_{j\in\mathbb{N}} (f_n)_j = 1,$$

so $\liminf_{j\to\infty} \|f_j\|_{X^*} = 1$, and the inequality is sharp.

4 Problem 4 (Folland Problem 38)

Let X and Y be Banach spaces and $\{T_n\} \subset \mathcal{L}(X,Y)$ such that $\lim_{n\to\infty} T_n x$ exists for every $x \in X$. Define T by $Tx = \lim_{n\to\infty} T_n x$. To show that $T \in \mathcal{L}(X,Y)$, we first verify that T is linear.

- If $x, y \in X$, then

$$T(x+y) = \lim_{n\to\infty} T_n(x+y) = \lim_{n\to\infty} (T_n x + T_n y) = \lim_{n\to\infty} T_n x + \lim_{n\to\infty} T_n y = Tx + Ty.$$

- If $x \in X$ and $\lambda \in \mathbb{C}$, then

$$T(\lambda x) = \lim_{n\to\infty} T_n(\lambda x) = \lim_{n\to\infty} \lambda T_n x = \lambda \lim_{n\to\infty} T_n x = \lambda T_n x.$$

To see that T is bounded, we note that because $\lim_{n\to\infty} T_n x$ exists for every $x \in X$, we must have that

$$\sup_{n\in\mathbb{N}} \|T_n x\| < \infty,$$

for all $x \in X$. Thus, an application of the uniform boundedness principle yields that $\sup_{n\in\mathbb{N}} \|T_n\| = C < \infty$ for some constant C. Then, for any $x \in X$,

$$\|Tx\| = \lim_{n\to\infty} \|T_n x\| \leq \lim_{n\to\infty} \|T_n\|\|x\| \leq \sup_{n\in\mathbb{N}} \|T_n\|\|x\| = C\|x\|.$$

Thus, $\|T\| \leq C$, so T is bounded and $T \in \mathcal{L}(X,Y)$.

5 Problem 5 (Folland Problem 48)

Let X be a Banach space.

5.1 Part a

Consider the norm-closed unit ball $B = \{x \in X : \|x\| \leq 1\}$. To see that B is also weakly closed, let $\{x_n\}_{n=1}^{\infty} \subset B$ such that $f(x_n) \to f(x)$ for all $f \in X^*$. To show that B is weakly closed, we need to show that $x \in B$. For any $f \in X^*$,

$$|\hat{x}(f)| = |f(x)| = \lim_{n\to\infty} |f(x_n)| \leq \lim_{n\to\infty} \|f\|\|x_n\| = \|f\|,$$

so

$$\|\hat{x}\| = \sup_{f\in X^*} \frac{|\hat{x}(f)|}{\|f\|} \leq 1.$$

By Theorem 5.8d, the map $x \mapsto \hat{x}$ is norm-preserving, so $\|x\| = \|\hat{x}\| \leq 1$, so $x \in B$, and B is weakly closed.

5.2 Part b

Let $E \subset X$ be bounded in the norm topology. Then $E \subset \overline{B(r,0)}$ for some $r > 0$, so $r^{-1}E \subset B$ using the notation of part a. Since B is closed in the weak topology, $\overline{r^{-1}E} \subset B$ with the closure taken in the weak topology. Since this is just a dilation, $r^{-1}\overline{E} \subset B$, and $\overline{E} \subset rB$. Thus, for any $\epsilon > 0$, $\overline{E} \subset B(r+\epsilon, 0)$, so the weak closure of E is bounded in the norm topology.

5.3 Part c

Let $F \subset X^*$ be bounded in the norm topology. Then, as before, $F \subset B(r,0)$ for some $r > 0$, so $r^{-1}E \subset B^*$ where $B^* = \{f \in X^* : \|f\| \leq 1\}$. By Alaoglu's theorem, B^* is compact in the weak* topology and therefore closed. Thus, by the same argument as in part b, $\overline{r^{-1}F} \subset B^*$, $r^{-1}\overline{F} \subset B^*$, $\overline{F} \subset rB^*$, and for any $\epsilon > 0$, $\overline{F} \subset B(r+\epsilon, 0)$, so the weak* closure of F is bounded in the norm topology.

5.4 Part d

Let $\{f_n\}_{n=1}^{\infty}$ be a weak* Cauchy sequence in X^*. That is, for a given $x \in X$,

$$\lim_{m,n\to\infty} |f_n(x) - f_m(x)| = 0.$$

This means that for any given $x \in X$, $\{f_n(x)\}_{n=1}^{\infty}$ is a Cauchy sequence in the usual metric in $\mathbb{C}$. Since $\mathbb{C}$ is complete, $\{f_n(x)\}_{n=1}^{\infty}$ is a convergent sequence. Define the functional f by

$$f(x) = \lim_{n\to\infty} f_n(x),$$

for all $x \in X$. Because $\{f_n(x)\}_{n=1}^{\infty}$ is convergent, $\lim_{n\to\infty} f_n(x)$ exists for all $x \in X$. X and $\mathbb{C}$ are both Banach spaces, so Problem 4 implies that $f \in \mathcal{L}(X, \mathbb{C})$, i.e., $f \in X^*$. Since $f_n(x) \to f(x)$ for all $x \in X$, $\{f_n\}_{n=1}^{\infty}$ converges to f in the weak topology. Thus, any weak* Cauchy sequence in X^* converges.