AMATH 563 Homework 1

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1 Problem 1

To see that $C([a,b])$ equipped with the $L^2([a,b])$ norm is not a Banach space, consider the sequence of functions

$$f_n(x) = \begin{cases} 0, & a \leq x < \frac{a+b}{2}, \\ n\left(x - \frac{a+b}{2}\right), & \frac{a+b}{2} \leq x \leq \frac{a+b}{2} + \frac{1}{n}, \\ 1, & \frac{a+b}{2} + \frac{1}{n} < x \leq b. \end{cases}$$

Clearly, $f_n \in C([a,b])$ for all $n \in N$. However, this sequence converges too

$$f(x) = \begin{cases} 0, & a \leq x \leq \frac{a+b}{2}, \\ 1, & \frac{a+b}{2} \leq b, \end{cases}$$

which is clearly discontinuous, so $f \notin C([a,b])$. To see this convergence more explicitly, we compute

$$\lim_{n\to\infty} \int_a^b |f_n(x) - f(x)|^2 dx = \int_{\frac{a+b}{2}}^{\frac{a+b}{2}+\frac{1}{n}} |f_n(x) - f(x)|^2 dx = 0.$$

Thus, this space is not complete and therefore not a Banach space.

2 Problem 2

Consider normed spaces $(X_1, \|\cdot\|_1)$, $(X_2, \|\cdot\|_2)$ and their product space X with norm $\|x\| = \max\{\|x_1\|_1, \|x_2\|_2\}$. To verify that this is also a normed space, we first verify that X satisfies the axioms of a vector space, i.e. that it is closed under addition and scalar multiplication. For addition, let $(x_1, x_2), (y_1, y_2) \in X$. Then,

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2) \in X,$$

because $x_1 + y_1 \in X_1$ and $x_2 + y_2 \in X_2$ since X_1, X_2 are normed spaces and must be closed under addition by definition. Similarly, to see that X is closed under scalar multiplication, let $(x_1, x_2) \in X$ and $\alpha \in \mathbb{R}$. Then,

$$\alpha(x_1, x_2) = (\alpha x_1, \alpha x_2) \in X,$$

because $\alpha x_1 \in X_1$ and $\alpha x_2 \in X_2$ since X_1, X_2 are normed spaces and must be closed under scalar multiplication by definition. Thus, X is a vector space.

Now, we verify that $\|\cdot\|$ satisfies the axioms of a norm. Let $x = (x_1, x_2) \in X$. To see nonnegativity, note that

$$\|x\| = \max\{\|x_1\|_1, \|x_2\|_2\} \geq \|x_1\|_1 \geq 0,$$

since $\|\cdot\|_1$ is a norm and must be nonnegative by definition. To see positive definiteness, we first observe that

$$\|(0,0)\| = \max\{\|0\|_1, \|0\|_2\} = \max\{0,0\} = 0.$$

Furthermore, if $\|x\| = 0$, then $0 = \max\{\|x_1\|_1, \|x_2\|_2\}$, so $\|x_1\|_1 = \|x_2\|_2 = 0$ since these norms are nonnegative. The positive definiteness of each then implies that $x_1 = x_2 = 0$, so $x = (0,0)$. To see homogeneity, let $\alpha \in \mathbb{R}$ and observe that

$$\begin{aligned}\|\alpha x\| &= \max\{\|\alpha x_1\|_1, \|\alpha x_2\|_2\} = \max\{|\alpha|\|x_1\|_1, |\alpha|\|x_2\|_2\} \\ &= |\alpha| \max\{\|x_1\|_1, \|x_2\|_2\} = |\alpha|\|x\|,\end{aligned}$$

which follows from the homogeneity of $\|\cdot\|_1, \|\cdot\|_2$. Finally, we let $y = (y_1, y_2) \in X$, and we see the triangle inequality by taking

$$\begin{aligned}\|x+y\| &= \max\{\|x_1+y_1\|_1, \|x_2+y_2\|_2\} \leq \max\{\|x_1\|_1 + \|y_1\|_1, \|x_2\|_2 + \|y_2\|_2\} \\ &\leq \max\{\|x_1\|_1, \|x_2\|_2\} + \max\{\|y_1\|_1, \|y_2\|_2\} = \|x\| + \|y\|,\end{aligned}$$

which follows from the fact that $\|\cdot\|_1, \|\cdot\|_2$ satisfy the triangle inequality.

3 Problem 3

Let T, U be linear maps defined such that their composition TU exists, i.e. $\text{range}(U) \subset \text{domain}(T)$. Let $x, x' \in \text{domain}(U)$. Then,

$$\begin{aligned}(TU)(x+x') &= T(U(x+x')) = T(Ux + Ux') \\ &= T(Ux) + T(Ux') = (TU)x + (TU)x',\end{aligned}$$

by the linearity of T, U. Similarly, if we let α be a scalar, then

$$(TU)(\alpha x) = T(U(\alpha x)) = T(\alpha Ux) = \alpha T(Ux) = \alpha (TU)x.$$

Thus, TU is indeed a linear map.

4 Problem 4

Let $T : X \to Y$ be a linear map where the spaces X, Y have finite-dimension n. Assume that T^{-1} exists. Then part iii of the theorem on pg. 3–4 of lecture 2 gives that $\dim \text{Range}(T) = \dim \text{Dom}(T)$. By definition $\text{Dom}(T) = X$, so dim Range(T) $= n$; however, this implies that Range(T) $= Y$ since Range(T) $\subset Y$, and proper subspaces must have a strictly smaller dimension (Theorem 2.1-8 in Kreyszig).

Now, assume that Range(T) $= Y$. Then, for any given $y \in Y$, there exists some $x \in X$ such that $T(x) = y$. Let $x_1, \ldots, x_n$ be a basis in X and write $x = \sum_{j=1}^n c_j x_j$. Then,

$$y = T\left(\sum_{j=1}^n c_j x_j\right) = \sum_{j=1}^n c_j T x_j.$$

Thus, $Tx_1, \ldots, Tx_n$ form a basis for Y since they are a length-n spanning set in an n-dimensional space. This means that

$$0 = \sum_{j=1}^n c_j T x_j = T\left(\sum_{j=1}^n c_j x_j\right)$$

is satisfied iff $c_j = 0$ for all j. Since any $x \in X$ can be written as $x = \sum_{j=1}^n c_j x_j$, this implies that $Tx = 0$ iff $x = 0$. By part i of the same theorem from the notes, this means that T^{-1} exists. Thus, T^{-1} exists iff Range(T) $= Y$.

5 Problem 5

Let T be a bounded linear operator from a normed space X onto a normed space Y and assume there is a positive constant b such that $\|Tx\| \geq b\|x\|$ for all $x \in X$. We can immediately see that T^{-1} exists due to part i of the theorem on pg. 3–4 of lecture 2 since if $Tx = 0$, then $b\|x\| \leq 0$ which can only hold when $x = 0$ by the nonnegativity of norms. Since T is onto, $T^{-1} : Y \to X$, so for any $y \in Y$, we have that for some x,

$$\|T^{-1}y\| = \|x\| \leq \frac{1}{b}\|Tx\| = \frac{1}{b}\|y\|,$$

implying that T^{-1} is bounded since $1/b > 0$.

6 Problem 6

Consider the functional $f(x) = \max_{t\in[a,b]} x(t)$ on $C([a,b])$ equipped with the sup norm. This functional is not linear, which we can see by considering $x, y \in C([a,b])$ which obtain their maxima in different places. Namely, let $x(t) = 2t$ and $y(t) = -t$. Then,

$$f(x+y) = \max_{t\in[a,b]} t = b,$$

but

$$f(x) + f(y) = \max_{t\in[a,b]} 2t + \max_{t\in[a,b]} (-t) = 2b - a,$$

so if we take $a = 0$, $b = 1$, we have that $f(x + y) \neq f(x) + f(y)$.

However, f is bounded. To see this let $x \in C([a, b])$. Then,

$$|f(x)| = \left| \max_{t \in [a,b]} x(t) \right| \leq \max_{t \in [a,b]} |x(t)| = \|x\|,$$

so the definition of boundedness is satisfied with $c = 1$.

7 Problem 7

Let X be a Banach space with dual X^*. We show that $\|\varphi\| : \varphi \mapsto \sup_{\|x\|=1} |\varphi(x)|$ is a norm on X^* by verifying the required axioms for a given $\varphi \in X^*$.

Nonnegativity is obvious since $|\varphi(x)| \geq 0$ for all x by the definition of absolute value.

For positive definiteness, first note that the zero functional clearly satisfies $\|0\| = \sup_{\|x\|=1} |0| = 0$. Conversely, if $\|\varphi\| = 0$, then we must have that $\varphi(x) = 0$ for all $\|x\| = 1$. Since φ is linear, for any nonzero x, we must have that

$$\varphi(x) = \frac{1}{\|x\|} \varphi\left(\frac{x}{\|x\|} \right) = 0.$$

Thus $\varphi(x) = 0$ for all $x \in X$, so φ must be the zero function on X.

To see homogeneity, let $\alpha \in \mathbb{R}$. Then,

$$\|\alpha\varphi\| = \sup_{\|x\|=1} |\alpha\varphi(x)| = \sup_{\|x\|=1} |\alpha||\varphi(x)| = |\alpha| \sup_{\|x\|=1} |\varphi(x)| = |\alpha|\|\varphi\|.$$

Finally, to see the triangle inequality, let $\phi, \varphi \in X^*$, and observe that

$$\begin{aligned} \|\phi + \varphi\| &= \sup_{\|x\|=1} |\phi(x) + \varphi(x)| \leq \sup_{\|x\|=1} \{|\phi(x)| + |\varphi(x)|\} \\ &\leq \sup_{\|x\|=1} |\phi(x)| + \sup_{\|x\|=1} |\varphi(x)| = \|\phi\| + \|\varphi\|. \end{aligned}$$

8 Problem 8

To prove the Schwartz inequality on inner product spaces, let $x, y \in X$ where X is a (real) inner product space. For $y \neq 0$ and $\alpha \in \mathbb{R}$, we have that

$$0 \leq \|x - \alpha y\|^2 = \|x\|^2 - 2\alpha\langle x, y\rangle + \alpha^2\|y\|^2.$$

If we let $\alpha = \langle x, y\rangle / \|y\|^2$, then

$$0 \leq \|x\|^2 - \frac{\langle x, y\rangle^2}{\|y\|^2},$$

which can be rearranged to get the Schwartz inequality

$$\langle x, y\rangle \leq \|x\|\|y\|,$$

for $y \neq 0$. If $y = 0$, the inequality is trivially true as both sides are zero. Thus, the Schwartz inequality holds for all $x, y \in X$.

By the positive definiteness of norms and the work above, the Schwartz inequality holds with equality iff $y = 0$ or $x = \alpha y$. This is true iff x and y are linearly dependent by definition.