MATH 525 Homework 2

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1 Problem 1

Let X be a compact metric space and $\mathcal{F} \subset C(X)$ be equicontinuous on X. Fix $\epsilon > 0$. Then, for every $x \in X$, there exists some $\delta_x > 0$ such that for all $f \in \mathcal{F}$ and $y \in X$,

$$|f(x) - f(y)| < \frac{\epsilon}{2} \quad \text{if} \quad d(x,y) < \delta_x.$$

Consider the collection of open balls $\{\mathcal{B}_{\delta_x/2}(x) : x \in X\}$. This is an open cover of X, so it can be reduced to a finite subcover $\bigcup_{j=1}^n \mathcal{B}_{\delta_{x_j}/2}(x_j)$ of X. Let $\delta = \min_{j\in\{1,\dots,n\}} \frac{\delta_{x_j}}{2}$. Then, for any $x, y \in X$, $x \in \mathcal{B}_{\delta_{x_j}/2}(x_j)$ for some $j \in \{1, \dots, n\}$. If $d(x,y) < \delta$, then by the triangle inequality,

$$d(y, x_j) \leq d(y,x) + d(x, x_j) \leq \delta + \frac{\delta_{x_j}}{2} \leq \delta_{x_j},$$

Again applying the triangle inequality, this implies that for all $f \in \mathcal{F}$,

$$|f(x) - f(y)| \leq |f(x) - f(x_j)| + |f(x_j) - f(y)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus, $\mathcal{F}$ is uniformly equicontinuous.

2 Problem 2

Let X be a locally compact Hausdorff space and let $\mathcal{F} \subset C(X)$ be equicontinuous. Consider the closure $\overline{\mathcal{F}}$ in the topology of uniform convergence on compact sets. That is, for every $f \in \overline{\mathcal{F}}$, there exists some sequence $\{f_n\}_{n=1}^\infty$ such that $\|f_n - f\|_{u,K} \to 0$ for all compact sets $K \subset X$. Fix $\epsilon > 0$ and $x \in X$. Then, because X is locally compact, there exists some open set $V_x \ni x$ such that $\overline{V_x}$ is compact. This means that there exists some $N_f \in \mathbb{N}$ such that

$$|f_n(y) - f(y)| < \frac{\epsilon}{3},$$

for all $y \in \overline{V_x}$ and $n \geq N_f$. Furthermore, by the definition of equicontinuity, there exists some open set $W_x \ni x$, independent of f, such that for all $n \in N$,

$$|f_n(y) - f_n(x)| < \frac{\epsilon}{3} \quad \text{if} \quad y \in W_x.$$

Let $U_x = V_x \cap W_x \ni x$. Note that this set is open and, because X is Hausdorff, it contains at least one point other than x. Then, for all $y \in U_x$, by the triangle inequality,

$$|f(y) - f(x)| \leq |f(y) - f_{N_f}(y)| + |f_{N_f}(y) - f_{N_f}(x)| + |f_{N_f}(x) - f(x)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon,$$

since $x, y \in \overline{V}_x$ and $y \in W_x$. Since this construction of U_x is independent of the function f, this implies that $\overline{\mathcal{F}}$ is equicontinuous because $f \in C(X)$ since $C(X)$ is complete.

3 Problem 3

Let $\mathcal{F} \subset C_0(X)$ be compact in the uniform norm topology where X is a locally compact Hausdorff space. Then, for any $\epsilon > 0$, by total boundedness, there exist a finite number of functions $f_1, \ldots, f_n \in \mathcal{F}$ such that $\mathcal{F} \subset \bigcup_{j=1}^n \mathcal{B}_{\epsilon/2}(f_j)$ with the balls taken in the uniform norm. For all $j = 1, \ldots, n$, there exist compact sets $K_j \subset X$ such that $|f_j(x)| < \frac{\epsilon}{2}$ for all $x \in K_j^c$. Define $K = \bigcup_{j=1}^n K_j$ and note that this set is also compact. For any $f \in \mathcal{F}$, there exists some $j \in \{1, \ldots, n\}$ such that $f \in \mathcal{B}_{\epsilon/2}(f_j)$. Then, by the triangle inequality, for all $x \in K^c$,

$$|f(x)| \leq |f(x) - f_j(x)| + |f_j(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since the compact set K does not depend on the specific j, this implies that for all $\epsilon > 0$, there is a compact set K such that for all $f \in \mathcal{F}$, $|f(x)| < \epsilon$ on K^c. To see that $\mathcal{F}$ is bounded fix $\epsilon > 0$ and let K be the associated compact set that we just found. Then, $\mathcal{F}$ is pointwise bounded on K^c because $|f(x)| < \epsilon$ for all $f \in \mathcal{F}$ and $x \in K^c$. By Arzelà–Ascoli, $\mathcal{F}$ is pointwise bounded on K since it is a compact set. Thus, $\mathcal{F}$ is pointwise bounded on all of X. To see that $\mathcal{F}$ is equicontinuous, fix $\epsilon > 0$ and let K be a compact set such that $|f(x)| < \frac{\epsilon}{2}$ for all $x \in K^c$ and $f \in \mathcal{F}$. Then, by Arzelà–Ascoli, $\mathcal{F}$ is equicontinuous on K since it is compact. Fix $x \in K^c$. Then, $U = K^c$ is an open set containing x such that for any $y \in U$,

$$|f(y) - f(x)| \leq |f(y)| + |f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus, $\mathcal{F}$ is equicontinuous on K^c, so it is equicontinuous on all of X.

Conversely, let $\mathcal{F} \subset C_0(X)$ with the same assumptions on X be closed, pointwise bounded, equicontinuous, and satisfy the property that for each $\epsilon > 0$, there is a compact set K such that for all $f \in \mathcal{F}$, $|f(x)| \leq \epsilon$ on K^c. To show that $\mathcal{F}$ is sequentially compact, let $\{f_n\}_{n=1}^\infty \subset \mathcal{F}$ be given. For all $j \in \mathbb{N}$, let K_j be a compact set such that for all $f \in \mathcal{F}$ and $x \in K_j^c$, $|f(x)| \leq \frac{1}{j}$. Because each K_j is compact, Arzelà–Ascoli implies that $\mathcal{F} \subset C(K_j)$ is compact.[1] Denote by $\{f_{n,1}\}_{n=1}^\infty$ a subsequence of $\{f_n\}_{n=1}^\infty$ that is uniformly Cauchy on K_1. Noting that $K_j \subset K_{j+1}$ for all j and proceeding inductively, there exists a subsequence $\{f_{n,j+1}\}_{n=1}^\infty$ of $\{f_{n,j}\}_{n=1}^\infty$ that is uniformly Cauchy on K_{j+1} for all j. By a standard diagonalization argument, letting $g_j = f_{j,j}$, there exists a subsequence $\{g_n\}_{n=1}^\infty$ of $\{f_n\}_{n=1}^\infty$ that is uniformly Cauchy on K_j for all $j \in \mathbb{N}$. Fix $\epsilon > 0$ and choose $m \in \mathbb{N}$ such that $\frac{2}{m} < \epsilon$. Then, if $x \in K_m^c$, for all $j, k \in \mathbb{N}$,

$$|g_j(x) - g_k(x)| \leq |g_j(x)| + |g_k(x)| \leq \frac{1}{m} + \frac{1}{m} = \frac{2}{m} < \epsilon.$$

If $x \in K_m$, then because $\{g_n\}_{n=1}^\infty$ is uniformly Cauchy, there exists some $N \in \mathbb{N}$ such that $|g_j(x) - g_k(x)| < \epsilon$ if $j, k \geq N$. Thus, there exists some $N \in \mathbb{N}$ such that for all $x \in X$, $|g_j(x) - g_k(x)| < \epsilon$ if $j, k \geq N$, so $\{g_n\}_{n=1}^\infty$ is uniformly Cauchy on all of X. Since $BC(X)$ is complete and $C_0(X) \subset BC(X)$ is closed, this implies that $\{g_n\}_{n=1}^\infty$ converges. Thus, every sequence in $\mathcal{F}$ has a convergent subsequence, so $\mathcal{F}$ is sequentially compact. Since $C_0(X)$ with the topology of uniform convergence is a metric space, this implies that $\mathcal{F}$ is compact.

4 Problem 4 (Folland Problem 63)

Let $K \in C([0,1] \times [0,1])$ and for any $f \in C([0,1])$, define

$$Tf(x) = \int_0^1 K(x,y) f(y) \mathrm{d}y.$$

To see that $Tf \in C([0,1])$, fix $\epsilon > 0$ and $x \in [0,1]$. Let $M = \max_{y \in [0,1]} |f(y)|$. We know that M is finite because $[0,1] \subset \mathbb{R}$ is compact and f is continuous. Note that K is uniformly continuous because $[0,1] \times [0,1]$ is compact. Thus, there exists some $\delta > 0$ such that $|K(x_1, x_2) - K(y_1, y_2)| < \frac{\epsilon}{M}$ if $\|(x_1, x_2) - (y_1, y_2)\| < \delta$. This implies that if $|x - y| < \delta$, then

$$|Tf(x) - Tf(y)| \leq \int_0^1 |K(x,z) - K(y,z)||f(z)| \mathrm{d}z < \frac{\epsilon}{M} M = \epsilon,$$

[1] This is a slight abuse of notation. When we say $\mathcal{F} \subset C(K_j)$, we're really considering $\mathcal{F}$ to be its composite functions restricted to K_j.

since

$$\|(x,z)-(y,z)\| = |x-y| < \delta,$$

for all $z \in [0,1]$. Thus, $Tf \in C([0,1])$.

Let $\mathcal{F} = \{Tf : \|f\|_u \le 1\}$. To see that $\mathcal{F}$ is equicontinuous, fix $\epsilon > 0$, $x \in [0,1]$, and $Tf \in \mathcal{F}$. Then, as before, there exists some $\delta > 0$ such that $|K(x_1,x_2) - K(y_1,y_2)| < \epsilon$ if $\|(x_1,x_2)-(y_1,y_2)\| < \delta$. This implies that if $|x-y| < \delta$, then

$$|Tf(x) - Tf(y)| \le \int_0^1 |K(x,z) - K(y,z)||f(z)|\mathrm{d}z < \epsilon,$$

since $\|f\|_u \le 1$. Since the choice of δ is independent of f, this implies that $\mathcal{F}$ is equicontinuous. To see that $\mathcal{F}$ is pointwise bounded, let $M = \max_{(x,y)\in[0,1]\times[0,1]} |K(x,y)|$. We know that M is finite because $[0,1]\times[0,1]$ is compact and K is continuous. Then, for any $Tf \in \mathcal{F}$ and $x \in [0,1]$,

$$|Tf(x)| \le \int_0^1 |K(x,y)||f(y)|\mathrm{d}y \le M.$$

Thus, $\mathcal{F}$ is pointwise bounded, so $\mathcal{F}$ is precompact in $C([0,1])$ by Arzelà–Ascoli since $[0,1]$ is a compact Hausdorff space.

5 Problem 5 (Folland Problem 6)

Let X be a finite-dimensional vector space with a basis given by $e_1, \ldots, e_n$. Define $\|\sum_{j=1}^n a_j e_j\|_1 = \sum_{j=1}^n |a_j|$.

5.1 Part a

We show that $\|\cdot\|_1$ is a norm on X by verifying the required axioms. In the following, let $x, y \in X$ be represented in the basis as $x = \sum_{j=1}^n a_j e_j$, $y = \sum_{j=1}^n b_j e_j$.

- By the triangle inequality on $\mathbb{K}$,

$$\|x+y\|_1 = \left\|\sum_{j=1}^n (a_j + b_j)e_j\right\|_1 = \sum_{j=1}^n |a_j + b_j| \le \sum_{j=1}^n |a_j| + \sum_{j=1}^n |b_j| = \|x\|_1 + \|y\|_1,$$

  so $\|\cdot\|_1$ satisfies the triangle inequality.

- Let $\lambda \in \mathbb{K}$. Then,

$$\|\lambda x\|_1 = \left\|\sum_{j=1}^n (\lambda a_j)e_j\right\|_1 = \sum_{j=1}^n |\lambda a_j| = |\lambda| \sum_{j=1}^n |a_j| = |\lambda|\|x\|_1,$$

  so $\|\cdot\|_1$ is homogeneous.

- If $\|x\| = 0$, then

$$\sum_{j=1}^n |a_j| = 0.$$

  Since the absolute value function is nonnegative, this implies that $a_1, \ldots, a_n = 0$, meaning that $x = 0$.

Thus, $\|\cdot\|_1$ is a norm on X.

5.2 Part b

Consider the map f defined by $(a_1, \ldots, a_n) \mapsto \sum_{j=1}^n a_j e_j$ from $\mathbb{K}^n$ with the Euclidean topology to X with the topology defined by $\|\cdot\|_1$. To see that this is continuous, fix $\epsilon > 0$ and $(a_1, \ldots, a_n) \in \mathbb{K}^n$. Let $\delta = \frac{\epsilon}{n}$. Then, if $\|(a_1, \ldots, a_n) - (b_1, \ldots, b_n)\| < \delta$,

$$\|f(a_1, \ldots, a_n) - f(b_1, \ldots, b_n)\|_1 = \sum_{j=1}^n |b_j - a_j| \leq n \max_{j \in \{1, \ldots, n\}} |b_j - a_j| \leq n \sqrt{\sum_{j=1}^n (b_j - a_j)^2} < n \frac{\epsilon}{n} = \epsilon.$$

Thus, f is continuous at all $(a_1, \ldots, a_n) \in \mathbb{K}^n$.

5.3 Part c

Consider the set $A = \left\{(a_1, \ldots, a_n) \in \mathbb{K}^n : \sum_{j=1}^n |a_j| = 1\right\}$. To see that this is compact in the Euclidean topology, we need only show that it is closed and bounded. The function $g(a_1, \ldots, a_n) = \sum_{j=1}^n |a_j|$ is a continuous function from $\mathbb{K}^n$ to $\mathbb{R}$. The set $\{1\} \subset \mathbb{R}$ is closed, and $g^{-1}(\{1\}) = A$, so A must also be closed. Also, for any $(a_1, \ldots, a_n) \in A$,

$$\|(a_1, \ldots, a_n)\| = \sqrt{\sum_{j=1}^n a_j^2} \leq \sqrt{n \max_{j \in \{1, \ldots, n\}} a_j^2} = \sqrt{n} \max_{j \in \{1, \ldots, n\}} |a_j| \leq \sqrt{n} \sum_{j=1}^n |a_j| = \sqrt{n},$$

so A is also bounded. Thus, A is compact. Letting f be the continuous map from part b, the set $f(A) = \{x \in X : \|x\|_1 = 1\}$ is also compact.

5.4 Part d

Let $\|\cdot\|$ denote an arbitrary norm on X. Let $C_2 = \max_{j \in \{1, \ldots, n\}} \|e_j\|$. Then, for any $x = \sum_{j=1}^n a_j e_j \in X$,

$$\|x\| \leq \sum_{j=1}^n |a_j| \|e_j\| \leq C_2 \sum_{j=1}^n |a_j| = C_2 \|x\|_1.$$

Additionally, let $C_1 = \min_{\|y\|_1 = 1} \|y\|$. This is defined because $\{x \in X : \|x\|_1 = 1\}$ is compact by part c and norms are continuous. Then,

$$\|x\| = \|x\|_1 \left\| \frac{x}{\|x\|_1} \right\| \geq C_1 \|x\|_1.$$

Thus, we have found constants $C_1, C_2 > 0$ such that

$$C_1 \|x\|_1 \leq \|x\| \leq C_2 \|x\|_1,$$

for all $x \in X$. Thus, any norms are equivalent to the 1-norm, meaning that all norms are equivalent in finite dimensions.