MATH 525 Homework 6

Cade Ballew #2120804

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1 Problem 1 (Folland Problem 55)

Let $\mathcal{H}$ be a Hilbert space.

1.1 Part a

Let $x, y \in \mathcal{H}$. Then,

$$\|x+y\|^2 - \|x-y\|^2 = (\|x\|^2 + 2\operatorname{Re}\langle x, y\rangle + \|y\|^2) - (\|x\|^2 - 2\operatorname{Re}\langle x, y\rangle + \|y\|^2) = 4\operatorname{Re}\langle x, y\rangle.$$

Similarly,

$$\begin{aligned}\|x+\mathrm{i}y\|^2 - \|x-\mathrm{i}y\|^2 &= (\|x\|^2 + 2\operatorname{Re}\langle x, \mathrm{i}y\rangle + \|\mathrm{i}y\|^2) - (\|x\|^2 - 2\operatorname{Re}\langle x, \mathrm{i}y\rangle + \|\mathrm{i}y\|^2) \\ &= 4\operatorname{Re}\langle x, \mathrm{i}y\rangle = 4\operatorname{Re}\left(-\mathrm{i}\langle x, y\rangle\right) = 4\operatorname{Im}\langle x, y\rangle.\end{aligned}$$

Thus,

$$\frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + \mathrm{i}\|x+\mathrm{i}y\|^2 - \mathrm{i}\|x-\mathrm{i}y\|^2\right) = \operatorname{Re}\langle x, y\rangle + \mathrm{i}\operatorname{Im}\langle x, y\rangle = \langle x, y\rangle,$$

and the polarization identity is satisfied.

1.2 Part b

Let $\mathcal{H}'$ be another Hilbert space. If a linear map $U : \mathcal{H} \to \mathcal{H}'$ is unitary, then it is by definition invertible and therefore surjective. Since U preserves inner products, for any $x \in \mathcal{H}$,

$$\|Ux\|_{\mathcal{H}'} = \sqrt{\langle Ux, Ux\rangle_{\mathcal{H}'}} = \sqrt{\langle x, x\rangle_{\mathcal{H}}}\|x\|_{\mathcal{H}},$$

so U must be isometric.

Conversely, if U is isometric and surjective, then for any $x, y \in \mathcal{H}$,

$$\begin{aligned}\langle Ux, Uy\rangle_{\mathcal{H}'} &= \frac{1}{4}\left(\|Ux+Uy\|^2_{\mathcal{H}'} - \|Ux-Uy\|^2_{\mathcal{H}'} + \mathrm{i}\|Ux+\mathrm{i}Uy\|^2_{\mathcal{H}'} - \mathrm{i}\|Ux-\mathrm{i}Uy\|^2_{\mathcal{H}'}\right) \\ &= \frac{1}{4}\left(\|U(x+y)\|^2_{\mathcal{H}'} - \|U(x-y)\|^2_{\mathcal{H}'} + \mathrm{i}\|U(x+\mathrm{i}y)\|^2_{\mathcal{H}'} - \mathrm{i}\|U(x-\mathrm{i}y)\|^2_{\mathcal{H}'}\right) \\ &= \frac{1}{4}\left(\|x+y\|^2_{\mathcal{H}} - \|x-y\|^2_{\mathcal{H}} + \mathrm{i}\|x+\mathrm{i}y\|^2_{\mathcal{H}} - \mathrm{i}\|x-\mathrm{i}y\|^2_{\mathcal{H}}\right) = \langle x, y\rangle_{\mathcal{H}},\end{aligned}$$

so U preserves inner products. Furthermore, U injective as for any $x, y \in \mathcal{H}$ such that $Ux = Uy$,

$$0 = \|Ux - Uy\|_{\mathcal{H}'} = \|U(x-y)\|_{\mathcal{H}'} = \|x-y\|\|Ux-Uy\|_{\mathcal{H}},$$

so $x - y = 0$ and $x = y$. Finally, both U and U^{-1} are bounded since $\|Ux\|_{\mathcal{H}'} = \|x\|_{\mathcal{H}}$ for all $x \in \mathcal{H}$, so $\|Ux\|_{\mathcal{H}'} \le C_1\|x\|_{\mathcal{H}}$ and $\|Ux\|_{\mathcal{H}'} \ge C_2\|x\|_{\mathcal{H}}$ for $C_1 = C_2 = 1$. Thus, U is invertible, so it is also a unitary map.

2 Problem 2 (Folland Problem 56)

Let E be a subset of a Hilbert space $\mathcal{H}$. Consider $\left(E^{\perp}\right)^{\perp}$. Since all orthogonal complements are closed subspaces, this set is a closed subspace of $\mathcal{H}$. Let $x \in E$. Then, for any $y \in E^{\perp}$, by definition $\langle x, y\rangle = 0$, so $x \in \left(E^{\perp}\right)^{\perp}$ and $E \subset \left(E^{\perp}\right)^{\perp}$.

Now, as a lemma, we note that for any two subsets A, B of $\mathcal{H}$ such that $A \subset B$, $B^{\perp} \subset A^{\perp}$. Indeed, if $x \in B^{\perp}$, then $\langle x, y\rangle = 0$ for all $y \in B$. This implies that $\langle x, y\rangle = 0$ for all $y \in A$ since $A \subset B$, so $x \in A^{\perp}$ as well.

Let F be any closed subspace such that $E \subset F$. Then, by the lemma, $F^{\perp} \subset E^{\perp}$ and $\left(E^{\perp}\right)^{\perp} \subset \left(F^{\perp}\right)^{\perp}$. Let $x \in \left(F^{\perp}\right)^{\perp}$. Because F is a closed subspace of $\mathcal{H}$, $\mathcal{H} = F \oplus F^{\perp}$, x can be expressed uniquely as $x = y + z$ with $y \in F$ and $z \in F^{\perp}$. By definition, we must also have that $0 = \langle x, z\rangle$ and $0 = \langle y, z\rangle$. Thus,

$$0 = \langle y + z, z\rangle = \langle y, z\rangle + \langle z, z\rangle = \|z\|^2,$$

so $z = 0$ and $x = y \in F$. This means that $\left(E^{\perp}\right)^{\perp} \subset \left(F^{\perp}\right)^{\perp} \subset F$, so $\left(E^{\perp}\right)^{\perp}$ is contained in any closed subspace containing E. Thus, it must be the smallest closed subspace of $\mathcal{H}$ containing E.

3 Problem 3 (Folland Problem 7)

Let $f \in L^p \cap L^\infty$ for some $p < \infty$ such that $f \in L^q$ for all $q > p$. First, note that if $f = 0$ almost everywhere, then $\|f\|_\infty = \|f\|_q = 0$, so $\|f\|_\infty = \lim_{q\to\infty} \|f\|q$ holds trivially. Assume that this is not the case, i.e., that $\|f\|_\infty > 0$. By setting $r = \infty$, Proposition 6.10 in Folland gives that $\|f\|_q \leq \|f\|_p^{\frac{p}{q}} \|f\|_\infty^{1-\frac{p}{q}}$. Taking limits,

$$\limsup_{q\to\infty} \|f\|_q \leq \limsup_{q\to\infty} \|f\|_p^{\frac{p}{q}} \|f\|_\infty^{1-\frac{p}{q}} = \|f\|_\infty.$$

For the other direction, fix $\epsilon > 0$ such that $\epsilon < \|f\|_\infty$ and let $E = \{x : |f(x)| > \|f\|_\infty - \epsilon\}$. By construction, $(\|f\|_\infty - \epsilon)\mathbb{1}_E \leq |f|$. Thus,

$$(\|f\|_\infty - \epsilon)^q \mu(E) \leq \int_E |f|^q \mathrm{d}\mu \leq \int |f|^q \mathrm{d}\mu = \|f\|^q < \infty,$$

so $\mu(E) < \infty$ as well. Taking the qth root and limits,

$$\|f\|_\infty - \epsilon = \liminf_{q\to\infty} (\|f\|_\infty - \epsilon)\mu(E)^{\frac{1}{q}} \leq \liminf_{q\to\infty} \|f\|_q.$$

Since this holds for all $\epsilon > 0$ sufficiently small, we have that

$$\limsup_{q\to\infty} \|f\|_q \leq \|f\|_\infty \leq \liminf_{q\to\infty} \|f\|_q,$$

so we conclude that

$$\lim_{q\to\infty} \|f\|_q = \|f\|_\infty$$

4 Problem 4 (Folland Problem 13)

Consider the space $L^p(\mathbb{R}^n, m)$ for $1 \leq p < \infty$. By Proposition 6.7 in Folland, the set of simple functions F of the form $\sum_{j=1}^n a_j \mathbb{1}_{E_j}$ where $m(E_j) < \infty$ for all j is dense in $L^p(\mathbb{R}^n, m)$, so to show that $L^p(\mathbb{R}^n, m)$ is separable, it suffices to show that this set is separable. Consider the set of simple functions G of the form $\sum_{j=1}^n b_j \mathbb{1}_{F_j}$ where b_j is rational and F_j is a finite union of rectangles whose sides are intervals with rational coordinates. G is clearly countable as it is the collection of finite sums of products of countable sets. Let $f = \sum_{j=1}^n a_j \mathbb{1}_{E_j} \in F$ be given and fix $\epsilon > 0$. By the construction of simple functions, we can assume without loss of generality that $a_j, m(E_j) \neq 0$ for any j. By the density of the rationals in $\mathbb{R}$, for each j, we can find some $b_j \in \mathbb{Q}$ such that $|a_j - b_j| < \frac{\epsilon}{3nm(E_j)^{1/p}}$. By Theorem 2.40c in Folland, there is some collection of disjoint rectangles whose sides are intervals E_j' such that $m(E_j \triangle E_j') < \left(\frac{\epsilon}{3n|b_j|}\right)^p$. In the case $b_j = 0$, E_j' can be chosen arbitrarily. Finally, the density rationals in $\mathbb{R}$ implies that there is some finite union of rectangles whose sides are intervals with rational coordinates F_j such that $m(E'_j \triangle F_j) < \left(\frac{\epsilon}{3n|b_j|}\right)^p$, where, as before, F_j can be chosen arbitrarily if $b_j = 0$. Define $g = \sum_{j=1}^n b_j \mathbb{1}_{F_j} \in G$. Then,

$$\begin{aligned}
\|f-g\|_p &\leq \left\| \sum_{j=1}^n (a_j - b_j) \mathbb{1}_{E_j} \right\|_p + \left\| \sum_{j=1}^n b_j \left(\mathbb{1}_{E_j} - \mathbb{1}_{E'_j} \right) \right\|_p + \left\| \sum_{j=1}^n b_j \left(\mathbb{1}_{E'_j} - \mathbb{1}_{F_j} \right) \right\|_p \\
&\leq \sum_{j=1}^n |a_j - b_j| m(E_j)^{1/p} + \sum_{j=1}^n |b_j| m(E_j \triangle E'_j)^{1/p} + \sum_{j=1}^n |b_j| m(E'_j \triangle F_j)^{1/p} \\
&\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}$$

Thus, G is dense in F, so $L^p(\mathbb{R}^n, m)$ is separable.

Now, consider the space $L^\infty(\mathbb{R}^n, m)$ and the set F of functions of the form $f_r = \mathbb{1}_{\mathcal{B}_r(0)}$ for some $r > 0$. Then, for any $f_r, f_{r'} \in F$ with $r \neq s$, $\|f_r - f_{r'}\|_\infty \geq 1$ since $\|f_r(x) - f_{r'}(x)\|$ for any $x \in \mathcal{B}_r(0) \triangle \mathcal{B}_{r'}(0)$. This means that the set

$$\bigcup_{r>0} \mathcal{B}_{1/2}(f_r),$$

is an uncountable collection of disjoint open balls each containing at least one element of $L^\infty(\mathbb{R}^n, m)$. Therefore, any countable subset of $L^\infty(\mathbb{R}^n, m)$ cannot be dense as there must be some open ball in this collection that does not contain an element of the countable set. Thus, $L^\infty(\mathbb{R}^n, m)$ is not separable.