AMATH 573 Homework 6

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1 Problem 1

Consider the second integral formula for $\bar{N}$ as given in the lecture notes and in class

$$\bar{N}(x,k) = 1 + \int_x^\infty u(z)\frac{e^{2ik(x-z)} - 1}{2ik}\bar{N}(z,k)dz.$$

Using this, we wish to show that $\bar{N}(x,k)$ is analytic in the open lower-half plane, $\Im k < 0$, by showing $\bar{N}(x,k)$ and $\partial \bar{N}/\partial k$ are bounded there. To show the former, we recall the lemma from class and the lecture notes which gives that if $\Im k > 0$, then

$$\left|\frac{e^{2ikx} - 1}{2ik}\right| \leq \frac{1}{|k|}$$

for all $x \geq 0$. We solve the integral equation by iteration, using

$$\bar{N}(x,k) = 1 + \sum_{j=1}^\infty \bar{N}_j(x,k)$$

where

$$\bar{N}_j(x,k) = \int_x^\infty \frac{e^{2ik(x-z)} - 1}{2ik}u(z)\bar{N}_{j-1}(z,k)dz$$

for $j = 1, 2, \ldots$ and $\bar{N}_0(x,k) = 1$. To apply the lemma, note that $\Im(-k) > 0$, so when $z \geq x$,

$$\left|\frac{e^{2ik(x-z)} - 1}{2ik}\right| = \left|\frac{e^{2i(-k)(z-x)} - 1}{2ik}\right| \leq \frac{1}{|-k|} = \frac{1}{|k|}.$$

Then,

$$|\bar{N}_1(x,k)| = \left|\int_x^\infty \frac{e^{2ik(x-z)} - 1}{2ik}u(z)\bar{N}_0(z,k)dz\right| \leq \int_x^\infty \left|\frac{e^{2ik(x-z)} - 1}{2ik}\right||u(z)|dz$$

$$\leq \frac{1}{|k|}\int_x^\infty |u(z)|dz = \frac{1}{|k|}U(x)$$

where we define
$$U(x) = \int_x^\infty |u(z)|dz$$
and assume that $U(x)$ is defined for all x. Then,

$$|\bar{N}_2(x,k)| = \left|\int_x^\infty \frac{e^{2ik(x-z)}-1}{2ik}u(z)\bar{N}_1(z,k)dz\right| \le \int_x^\infty \left|\frac{e^{2ik(x-z)}-1}{2ik}\right||u(z)|\frac{U(z)}{|k|}dz$$

$$\le \frac{1}{|k|^2}\int_x^\infty U'(z)U(z)dz = \frac{U^2(x)}{2|k|^2}.$$

Similarly,

$$|\bar{N}_3(x,k)| = \left|\int_x^\infty \frac{e^{2ik(x-z)}-1}{2ik}u(z)\bar{N}_2(z,k)dz\right| \le \int_x^\infty \left|\frac{e^{2ik(x-z)}-1}{2ik}\right||u(z)|\frac{U^2(z)}{2|k|^2}dz$$

$$\le \frac{1}{2|k|^3}\int_x^\infty U'(z)U^2(z)dz = \frac{U^3(x)}{3!|k|^3}.$$

Using the same amount of rigor as the lecture notes, we can see that by induction,

$$|\bar{N}_j(x,k)| \le \frac{U^j(x)}{j!|k|^j}$$

for $j = 1, 2, \ldots$. Thus,

$$|\bar{N}(x,k)| \le \sum_{j=1}^\infty |\bar{N}_j(x,k)| \le \frac{U^j(x)}{j!|k|^j} = e^{U(x)/|k|},$$

meaning that if

$$\int_{-\infty}^\infty |u(x)|dx = U(-\infty) < \infty,$$

then

$$|\bar{N}(x,k)| \le e^{U(-\infty)/|k|},$$

meaning that $\bar{N}$ is bounded in the open lower halfplane. To show that its derivative is bounded, note that

$$\frac{\partial\bar{N}}{\partial k} = F(x,k) + \int_x^\infty u(z)\frac{e^{2ik(x-z)}-1}{2ik}\frac{\partial\bar{N}}{\partial k}(z,k)dz$$

where

$$F(x,k) = \int_x^\infty \frac{e^{2ik(x-z)}-1-2ik(x-z)e^{2ik(x-z)}}{2ik^2}u(z)\bar{N}(z,k)dz,$$

so $\partial\bar{N}/\partial k$ satisfies an integral equation which differs from the one for $\bar{N}$ only in the inhomogeneous term, meaning that if we can bound $F(x,k)$ uniformly in x, then

$$\left|\frac{\partial\bar{N}}{\partial k}\right| \le F_0(k)e^{U(-\infty)/|k|}$$

where $F_0(k)$ is the uniform bound on F. To do this, we first rewrite F as

$$F(x,k) = \int_x^\infty \frac{e^{2ik(x-z)} - 1}{2ik^2} u(z)\bar{N}(z,k)dz - \int_x^\infty \frac{(x-z)e^{2ik(x-z)}}{k} u(z)\bar{N}(z,k)dz.$$

We break k into real and imaginary parts $k = k_R + ik_I$, noting that $k_I < 0$. Then, for $z \geq x \in \mathbb{R}$, we can bound

$$|e^{2ik(x-z)} - 1| \leq |e^{2ik_R(x-z)}||e^{-2k_I(x-z)}| + 1 = e^{2k_I(z-x)} + 1 \leq 2$$

which follows from the triangle inequality, the fact that $2ik_R(x-z)$ is purely imaginary, and the fact that $2k_I(z-x) \leq 0$. This logic also lets us see that under the same assumptions,

$$|e^{2ik(x-z)}| \leq e^{2k_I(z-x)} < 1.$$

Using these, we can bound

$$
\begin{aligned}
|F(x,k)| &\leq \int_x^\infty \frac{|e^{2ik(x-z)} - 1|}{2|k|^2} |u(z)||\bar{N}(z,k)|dz + \int_x^\infty \frac{|(x-z)e^{2ik(x-z)}|}{|k|} |u(z)||\bar{N}(z,k)|dz \\
&\leq \frac{1}{|k|^2} \int_x^\infty |u(z)||\bar{N}(z,k)|dz + \frac{1}{|k|} \int_x^\infty (z-x)|u(z)||\bar{N}(z,k)|dz \\
&\leq \frac{1}{|k|^2} e^{U(-\infty)/|k|} \int_x^\infty |u(z)|dz + \frac{1}{|k|} e^{U(-\infty)/|k|} \int_x^\infty (z-x)|u(z)|dz \\
&= \frac{1}{|k|^2} e^{U(-\infty)/|k|} U(x) + \frac{1}{|k|} e^{U(-\infty)/|k|} V(x) \\
&\leq \frac{1}{|k|^2} e^{U(-\infty)/|k|} U(-\infty) + \frac{1}{|k|} e^{U(-\infty)/|k|} V(-\infty)
\end{aligned}
$$

where

$$V(x) = \int_x^\infty (z-x)|u(z)|dz,$$

a monotonically decreasing function of x. This means that $\partial \bar{N}/\partial k$ is bounded in the open lower halfplane provided that

$$\int_{-\infty}^\infty |u(x)|dx < \infty, \quad \lim_{x \to -\infty} \int_x^\infty (z-x)|u(z)|dz < \infty.$$

Thus, under these conditions, we have that $\bar{N}$ is analytic in the open lower halfplane.

2 Problem 2

From homework 4, we recall the second member of the stationary KdV hierarchy from HW4

$$30u^2 u_x + 20u_x u_{xx} + 10uu_{xxx} + u_{5x} + c_1(6uu_x + u_{xxx}) + c_0 u_x = 0.$$

Integrating once, it can be rewritten as

$$(10u^3 + 10u_x^2 + 10uu_{xx} - 5u_x^2 + u_{xxxx}) + c_1(3u^2 + u_{xx}) + c_0 u + c_{-1} = 0. \quad (1)$$

This equation can be written as

$$\frac{\delta T_2}{\delta u} = 0 \quad \Longleftrightarrow \quad \frac{\delta}{\delta u}(F_2 + c_1 F_1 + c_0 F_0 + c_{-1} F_{-1}) = 0, \quad (2)$$

where F_k, $k = -1, 0, \ldots$ are the conserved quantities of the KdV equation. These conserved quantities are in involution,

$$\{F_j, F_k\} = 0 \quad \Rightarrow \{T_j, F_k\} = 0,$$

for $j, k = -1, 0, \ldots$. From homework 4, this implies

$$\frac{\delta T_2}{\delta u} \frac{d}{dx} \frac{\delta F_k}{\delta u} = \frac{dH_j}{dx}, \quad k = 0, 1. \quad (3)$$

For some functions H_0 and H_1. Thus, H_0 and H_1 are conserved quantities of (1). Note that we avoid $k = -1$ because that would produce a trivial H_{-1}.

2.1 Part a

Reusing portions of the Mathematica code from homework 4 with updated $F_{-1}, \ldots, F_2$ for this form of the KdV equation, we can easily compute the LHS of (3). We do so in the attached Mathematica notebook and integrate to get that

$$H_0 = \int u_x \left((10u^3 + 5u_x^2 + 10uu_{xx} + u_{xxxx}) + c_1(3u^2 + u_{xx}) + c_0 u + c_{-1}\right) dx$$

and

$$H_1 = \int (6uu_x + u_{xxx}) \left((10u^3 + 5u_x^2 + 10uu_{xx} + u_{xxxx}) + c_1(3u^2 + u_{xx}) + c_0 u + c_{-1}\right) dx.$$

We also compute the integrals in the attached Mathematica notebook to get more explicit formulae.

2.2 Part b

To verify explicitly that H_0 and H_1 are conserved along solutions of (1), we need only look at the integrand to see that it contains the LHS of (1). However, we also verify explicitly that the integrand is zero in the attached Mathematica notebook.

3 Problem 3

Consider the Ostrovsky equation

$$(\eta_t + \eta\eta_x + \eta_{xxx})_x = \gamma\eta,$$

and assume that as $|x| \to \infty$, η and its derivatives approach zero as fast as we need them to. From this equation,

$$\eta = \frac{1}{\gamma}(\eta_t + \eta\eta_x + \eta_{xxx})_x,$$

so

$$\int_{-\infty}^{\infty} \eta dx = \frac{1}{\gamma}\int_{-\infty}^{\infty}(\eta_t + \eta\eta_x + \eta_{xxx})_x\, dx = [\eta_t + \eta\eta_x + \eta_{xxx}]_{-\infty}^{\infty} = 0$$

by our decay assumptions. Thus, not only is $\int_{-\infty}^{\infty} \eta dx$ conserved, but its value is fixed at zero.

To show that $\int_{-\infty}^{\infty} \eta^2 dx$ is a conserved quantity, we first integrate the Ostrovsky equation to get it in evolution form

$$\eta_t = \gamma\int_{-\infty}^{x} \eta dx' - \eta\eta_x - \eta_{xxx}$$

where our lower bound is valid per our decay assumptions and chosen for convenience. Let

$$\varphi(x) = \int_{-\infty}^{x} \eta(x')dx'.$$

Now, consider

$$\frac{d}{dt}\int_{-\infty}^{\infty} \eta^2 dx = 2\int_{-\infty}^{\infty} \eta\eta_t dx = 2\gamma\int_{-\infty}^{\infty} \eta\varphi dx - 2\int_{-\infty}^{\infty} \eta^2\eta_x dx - 2\int_{-\infty}^{\infty} \eta\eta_{xxx}dx.$$

First, we evaluate

$$2\int_{-\infty}^{\infty} \eta\varphi dx = [\varphi^2(x)]_{-\infty}^{\infty} = \left(\int_{-\infty}^{\infty} \eta(x')dx'\right) - \left(\int_{-\infty}^{-\infty} \eta(x')dx'\right) = 0$$

due to the fact that $\int_{-\infty}^{\infty} \eta dx$ has value fixed at zero. Next,

$$2\int_{-\infty}^{\infty} \eta^2\eta_x dx = \frac{2}{3}[\eta^3(x)]_{-\infty}^{\infty} = 0$$

by our decay assumptions. Finally, we use integration by parts to compute

$$\int_{-\infty}^{\infty} \eta\eta_{xxx}dx = [\eta\eta_{xx}]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \eta_x\eta_{xx}dx = \tfrac{1}{2}[\eta_x^2]_{-\infty}^{\infty} = 0$$

by our decay assumptions. Thus, we can conclude that

$$\frac{d}{dt}\int_{-\infty}^{\infty}\eta^2 dx = 0,$$

meaning that $\int_{-\infty}^{\infty}\eta^2 dx$ is a conserved quantity.

Now, we wish to verify that the Ostrovsky equation is Hamiltonian with Poisson operator ∂_x and Hamiltonian

$$H = \frac{1}{2}\int_{-\infty}^{\infty}\left(\eta_x^2 - \frac{1}{3}\eta^3 - \gamma\phi^2\right)dx$$

where $\phi_x = \eta$. Namely, we wish to show that

$$\partial_x\frac{\delta\mathcal{H}}{\delta\eta} = \eta_t.$$

To do this, we need to extend the definition of the variation derivative to include a $j = -1$ term. We do so by invesigating the sufficiency proof of the Euler-Lagrange equations. Using the same notation as the course notes[1] and following the general outline, we again consider a small perturbation $\epsilon\eta$ to the path γ where η and its derivatives are zero at the endpoints of $[a, b]$. We impose the additional assumption that $\partial^{-1}\eta$ is zero at a, b. We consider

$$S[x(t) + \epsilon\eta(t)] = \int_a^b L(X + \epsilon\partial^{-1}\eta, x + \epsilon\eta, x' + \epsilon\eta')dt$$

where $X_t = x$. Taylor expanding,

$$S[x(t) + \epsilon\eta(t)] = \int_a^b L(X, x, x')dt + \epsilon\sum_{i=1}^n\int_a^b\left(\frac{\partial L}{\partial X_i}\partial^{-1}\eta_i + \frac{\partial L}{\partial x_i}\eta_i + \frac{\partial L}{\partial x_i'}\eta_i'\right) + \mathrm{O}(\epsilon^2).$$

Using integration by parts, we compute

$$\int_a^b\frac{\partial L}{\partial X_i}\partial^{-1}\eta_i = \left[\partial^{-1}\frac{\partial L}{\partial X_i}\partial^{-1}\eta_i\right]_a^b - \int_a^b\eta_i\partial^{-1}\frac{\partial L}{\partial X_i}dt = -\int_a^b\eta_i\partial^{-1}\frac{\partial L}{\partial X_i}dt$$

by our decay assumption. The remaining terms behave identically to the notes, so we find that

$$\frac{dS}{d\epsilon}[x + \epsilon\eta]\bigg|_{\epsilon=0} = \sum_{i=1}^n\int_a^b\eta_i\left(-\partial_t^{-1}\frac{\partial L}{\partial X_i} + \frac{\partial L}{\partial x_i} - \partial_t\frac{\partial L}{\partial x_i'}\right)dt,$$

meaning that the variational derivative is given by

$$-\partial_t^{-1}\frac{\partial L}{\partial X_i} + \frac{\partial L}{\partial x_i} - \partial_t\frac{\partial L}{\partial x_i'}$$

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[1]This is unfortunate because it uses a different η from the one in this problem.

as the proof of necessity follows in the same manner as the notes. Applying this to our problem[2],

$$\partial_x \frac{\delta \mathcal{H}}{\delta \eta} = \partial_x \left(-\partial_x^{-1} \frac{\partial \mathcal{H}}{\partial \phi} + \frac{\partial \mathcal{H}}{\partial \eta} - \partial_x \frac{\partial \mathcal{H}}{\partial \eta_x} \right) = -\left(\frac{\partial \mathcal{H}}{\partial \phi} - \partial_x \frac{\partial \mathcal{H}}{\partial \phi_x} + \partial_x^2 \frac{\partial \mathcal{H}}{\partial \phi_{xx}} \right) = -\frac{\delta \mathcal{H}}{\delta \phi}.$$

Using Mathematica, we compute

$$-\frac{\delta \mathcal{H}}{\delta \phi} = \gamma \phi - \phi_x \phi_{xx} - \phi_{xxxx} = \gamma \phi - \eta \eta_x - \eta_{xxx} = \eta_t,$$

meaning that this equation is in fact Hamiltonian with the given Poisson operator and Hamiltonian.

4 Problem 4

We wish to discuss the integrability of

$$u_t = u^p u_x + u_{xxx}$$

where p is a positive integer using the Painlevé test. To do this, we look for solutions of the form

$$u(x,t) = \frac{1}{(x - x_0(t))^\rho} \sum_{n=0}^\infty \alpha_n(t)(x - x_0(t))^n = \sum_{n=0}^\infty \alpha_n (x - x_0)^{n-\rho}.$$

Looking at leading order terms,

$$u \sim \alpha_0 (x - x_0)^{-\rho}$$
$$u^p \sim \alpha_0^p (x - x_0)^{-p\rho}$$
$$u_x \sim -\rho \alpha_0 (x - x_0)^{-\rho-1}$$
$$u_{xx} \sim \rho(\rho+1)\alpha_0 (x - x_0)^{-\rho-2}$$
$$u_{xxx} \sim -\rho(\rho+1)(\rho+2)\alpha_0 (x - x_0)^{-\rho-3}$$
$$u_t \sim \alpha_0'(x - x_0)^{-\rho} + \rho \alpha_0 x_0'(x - x_0)^{-\rho-1}.$$

Substituting these into our PDE,

$$\alpha_0'(x - x_0)^{-\rho} + \rho \alpha_0 x_0'(x - x_0)^{-\rho-1} + \rho \alpha_0^{p+1}(x - x_0)^{-(p+1)\rho-1}$$
$$+ \rho(\rho+1)(\rho+2)\alpha_0 (x - x_0)^{-\rho-3} = 0.$$

Looking at only the most singular terms and applying the principle of dominant balance, we simply need to set

$$-(p+1)\rho - 1 = -\rho - 3$$

which yields that $\rho = \frac{2}{p}$. This means that our function is meromorphic when $2/p$ is integer, so we expect by the Painlevé test that our PDE is integrable when $2/p$ an is integer. Since we restrict p to the positive integers, our PDE is only integrable when $p = 1, 2$.

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[2] with the unfortunate flip in notation