AMATH 561 Problem Set 7

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1. a. If we let X_n be the largest number rolled up to the nth roll of a 6-sided die, it is a Markov chain, because whether or not the maximum changes (and what the new maximum is) depends only on the value of the new roll relative to the previous maximum. The one-step transition matrix is given by

$$P=\begin{pmatrix} 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 2/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 3/6 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 0 & 4/6 & 1/6 & 1/6 \\ 0 & 0 & 0 & 0 & 5/6 & 1/6 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where our state space is $S=\{1,\dots,6\}$.

b. If we let X_n be the number of sixes rolled in the first n rolls, it is a Markov chain, because $X_n=X_{n-1}+1$ if we roll a 6 on the nth roll and $X_n=X_{n-1}$ if not, so we depend only on the previous state. The one-step transition matrix is

$$P=\begin{pmatrix} 5/6 & 1/6 & 0 & & \\ & 0 & 5/6 & 1/6 & 0 \\ & & & \ddots & \ddots \\ & & & & \ddots \end{pmatrix}$$

where the blank entries are zero and our state space is $S=\{0,1,2,\dots\}$.

c. If X_n is the time since the last 6 was rolled at time n, it is a Markov chain, because $X_n=X_{n-1}+1$ if we don't roll a 6 on the nth roll and $X_n=0$ otherwise, so we only depend on the previous state. The one-step transition matrix is

$$P=\begin{pmatrix} 1/6 & 5/6 & & \\ 1/6 & 0 & 5/6 & \\ \vdots & & & \ddots \\ \vdots & & & \end{pmatrix}$$

where the blank entries are zero and our state space is $S=\{0,1,2,\dots\}$

d. If X_n is the time until the next 6 is rolled at time n, it is a Markov chain, because $X_n=X_{n-1}-1$ unless $X_{n-1}=0$ in which case X_n is k with the probability of needing k rolls to achieve the next 6. In either case, it only depends on the previous state. The one-step transition matrix is $\left(\left(\frac{5}{6}\right)^k\frac{1}{6}\right)$

$$P=\begin{pmatrix} \frac{1}{6} & \left(\frac{5}{6}\right)\frac{1}{6} & \left(\frac{5}{6}\right)^2\frac{1}{6} & \cdots \\ 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ & & & \ddots \end{pmatrix}$$

where the blank entries are zero and the state space is $S=\{0,1,2,\dots\}$.

2. Let $Y_n = X_{2n}$.

a. If X is a simple random walk that increases by one with probability p and decreases by one with probability q, in two steps, X can either increase by one twice, increase by 1 once and decrease by 1 once (no net change), or decrease by one twice. These occur with probability p^2, $2pq$, and q^2, respectively. Thus, the transition matrix P for Y has entries

$$p(i,j) = \begin{cases} p^2, & j = i-2 \\ 2pq, & j = i \\ q^2, & j = i+2 \\ 0, & \text{otherwise} \end{cases} \quad \text{for all } i,j \in S.$$

b. Now, let X be a branching process with generating function G. Then, in the case where $X_0 = i$, $G_{X_2}(s) = G_{X_1}(G(s)) = G_{X_0}(G(G(s))) = (G(G(s)))^i$ by theorem 3.2.1 in Long and definition $G_{X_0}(s) = E[s^{X_0}] = E[s^i] = s^i$.
We also know that $G_{X_2} = \sum_{k=0}^{\infty} p_k s^k$ where $p_k = P(X_2 = k \mid X_0 = i)$ are given by the corresponding Taylor coefficients for G_{X_2}. Thus,

$$p(i,j) = P(Y_n = j \mid Y_{n-1} = i) = P(X_{2n} = j \mid X_{2n-2} = i) = P(X_2 = j \mid X_0 = i)$$
$$= \frac{1}{j!}\frac{d^j}{ds^j}\left((G(G(s)))^i\right)\Big|_{s=0} \quad \text{for all } i,j \in S.$$

3. Let X be a Markov chain with state space S and absorbing state k and assume that $j \to k$ for all $j \in S$. To show that all states $j \neq k$ are transient, we need to show that $P(X_n = j \text{ for some } n>0 \mid X_0 = j) = 1$ cannot hold $\forall\, j \neq k$: However,
$P(X_n = j \text{ for some } n>0 \mid X_0 = j) + P(X_n \neq j\ \forall\, n>0 \mid X_0 = j) = 1$, so it suffices to show that $P(X_n \neq j\ \forall\, n>0 \mid X_0 = j) > 0$. Because $j \to k$, we know that $\exists$ some $n>0$ s.t. $p_n(j,k) > 0$. Let N be the smallest such n. Then, $X_n \neq j\ \forall\, n \leq N$ because if this doesn't hold, then $p_N(j,k) = p_n(j,j)\, p_{N-n}(j,k)$
$\Rightarrow p_{N-n}(i,k) = \frac{p_N(j,k)}{p_n(j,j)} > 0$ (Note that $p_n(j,j) = 0$ implies that $p_N(j,k) = 0$, a contradiction.), which contradicts the assumption that N is the shortest path because $N-n < N$. Thus, we can now conclude that
$P(X_n \neq j\ \forall n \mid X_0 = j) \geq P(X_N = k \mid X_0 = j) = p_N(j,k) > 0$, meaning that all states $j \neq k$ must be transient.

4. Assume that two distinct states i, j satisfy $p = P(\tau_j < \tau_i | X_0 = i) = P(\tau_i < \tau_j | X_0 = j)$. Note that $1-p = P(\tau_j > \tau_i | X_0 = i) = P(\tau_i > \tau_j | X_0 = j)$ and let V denote the number of visits to j prior to revisiting i. Then,

$$E[V | X_0 = i] = \sum_{n=0}^{\infty} n P(V=n | X_0 = i) = \sum_{n=0}^{\infty} n P(\tau_j < \tau_i | X_0 = i)\left(P(\tau_j < \tau_i | X_0 = j)\right)^{n-1} P(\tau_i < \tau_j | X_0 = j)$$

$$= \sum_{n=0}^{\infty} n p (1-p)^{n-1} p = \sum_{n=0}^{\infty} n p^2 (1-p)^{n-1} = -p^2 \sum_{n=0}^{\infty} \frac{d}{dp}(1-p)^n = -p^2 \frac{d}{dp}\left(\sum_{n=0}^{\infty}(1-p)^n\right)$$

$$= -p^2 \frac{d}{dp}\left(\frac{1}{1-(1-p)}\right) = -p^2 \frac{d}{dp}(p^{-1}) = -p^2(-p^{-2}) = 1.$$

There are a couple things to note in these calculations. First, we are able to start our series at $n=0$ despite the $n-1$ exponent because we multiply by n, so the term is zero. Second, uniform convergence of this series which enables us to compute the sum and move the derivative outside it requires $p>0$. However, in the case where $p=0$, $E[V | X_0 = i] = 0$.

5. Given a transition matrix

$$P = \begin{pmatrix} 1-2p & 2p & 0 \\ p & 1-2p & p \\ 0 & 2p & 1-2p \end{pmatrix},$$

we use a matrix diagonalization calculator to factor it as

$$P = \underbrace{\begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix}}_{U^{-1}} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-2p & 0 \\ 0 & 0 & 1-4p \end{pmatrix}}_{\Lambda} \underbrace{\begin{pmatrix} 1/4 & 1/2 & 1/4 \\ -1/2 & 0 & 1/2 \\ 1/4 & -1/2 & 1/4 \end{pmatrix}}_{U}$$

Thus, $$P^n = U^{-1}\Lambda^n U = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & (1-2p)^n & 0 \\ 0 & 0 & (1-4p)^n \end{pmatrix} \begin{pmatrix} 1/4 & 1/2 & 1/4 \\ -1/2 & 0 & 1/2 \\ 1/4 & -1/2 & 1/4 \end{pmatrix}.$$

An invariant distribution π which satisfies $\pi = \pi P$ is by definition a left eigenvector corresponding to eigenvalue 1. From our diagonalization, we can see that $\pi = (1/4 \;\; 1/2 \;\; 1/4)$ is such an eigenvector. By definition, $\bar{\tau}_i = 1/\pi(i) \;\forall i \in S$, so we find that $\bar{\tau}_1 = 4$, $\bar{\tau}_2 = 2$, $\bar{\tau}_3 = 4$.