MATH 524 Homework 3

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1 Problem 1

Suppose that X and Y are metric spaces and that $f : X \to Y$ is a continuous map. To show that if $E \in \mathcal{B}_Y$, then $f^{-1}(E) \in \mathcal{B}_X$, we first show that the family of $E \subset Y$ satisfying $f^{-1}(E) \in \mathcal{B}_X$ which we denote by $\mathcal{A}$ is a σ-algebra by verifying the required axioms.

- $f^{-1}(\emptyset) = \emptyset \in \mathcal{B}_X$, so $\emptyset \in \mathcal{A}$.
- To show that $\mathcal{A}$ is closed under countable unions, let $E_j \in \mathcal{A}$ for $j \in \mathbb{N}$. Then, since the union of preimages is the preimage of the union,

$$f^{-1}\left(\bigcup_{j=1}^{\infty} E_j\right) = \bigcup_{j=1}^{\infty} f^{-1}\left(E_j\right) \in \mathcal{B}_X,$$

  since $\mathcal{B}_X$ is a σ-algebra, so $\bigcup_{j=1}^{\infty} f^{-1}(E_j) \in \mathcal{A}$.
- To show that $\mathcal{A}$ is closed under complements, let $E \in \mathcal{A}$. Then, since the complement of the preimage is the preimage of the complement,

$$f^{-1}\left(E^c\right) = \left(f^{-1}(E)\right)^c \in \mathcal{B}_X,$$

  since $\mathcal{B}_X$ is a σ-algebra, so $E^c \in \mathcal{A}$.

Thus, $\mathcal{A}$ is a σ-algebra. Under continuous mappings, open sets have open preimages, so if $A \in Y$ is open, then $f^{-1}(A)$ is also open. Thus, $f^{-1}(A) \in \mathcal{B}_X$, so $A \in \mathcal{A}$ for all open sets A. Since $\mathcal{B}_Y$ is the σ-algebra generated by all open sets in Y, it follows from the definition that $\mathcal{B}_Y \subset \mathcal{A}$. Thus, if $E \in \mathcal{B}_Y$, then $E \in \mathcal{A}$, so $f^{-1}(E) \in \mathcal{B}_X$.

2 Problem 2

In the setup of Problem 1, consider

$$\nu(E) = \mu\left(f^{-1}(E)\right),$$

where μ is a Borel measure on X. By Problem 1, we have that $\nu(E)$ is defined for all $E \in \mathcal{B}_Y$ since this implies that $f^{-1}(E) \in \mathcal{B}_X$, and Borel measures have domain $\mathcal{B}_X$. It remains to show that ν is actually a measure on $\mathcal{B}_Y$. We do this by verifying the required axioms.

- Since μ is a measure and therefore nonnegative, it follows that for any $E \in \mathcal{B}_Y$,

$$\nu(E) = \mu\left(f^{-1}(E)\right) \geq 0,$$

  so ν is nonnegative.
- By the definition of a preimage,

$$\nu(\emptyset) = \mu\left(f^{-1}(\emptyset)\right) = \mu(\emptyset) = 0,$$

  since μ is a measure.

- Let $\{E_j\}_{j=1}^{\infty}$ be a collection of disjoint sets in $\mathcal{B}_Y$. Then,

$$\nu\left(\bigsqcup_{j=1}^{\infty} E_j\right) = \mu\left(f^{-1}\left(\bigsqcup_{j=1}^{\infty} E_j\right)\right) = \mu\left(\bigsqcup_{j=1}^{\infty} f^{-1}(E_j)\right) = \sum_{j=1}^{\infty} \mu\left(f^{-1}(E_j)\right) = \sum_{j=1}^{\infty} \nu(E_j),$$

  since μ is a measure and disjointness is preserved under preimages. To see this latter fact more explicitly, let $x \in f^{-1}(E_j)$. Then, $f(x) \in E_j$, so for any $k \neq j$, $f(x) \notin E_k$, and $x \notin f^{-1}(E_k)$. Thus, $\{f^{-1}(E_j)\}_{j=1}^{\infty}$ are disjoint.

We have established that ν is a measure defined on $\mathcal{B}_Y$, so ν is a Borel measure on Y.

3 Problem 3

Let $(\mathbb{Z}_2^{\mathbb{N}}, \mathcal{B}, \mu)$ be the Borel measure space where μ is the extension of the premeasure μ_0 of Homework 2 to the Borel sets in $\mathbb{Z}_2^{\mathbb{N}}$.

3.1 Part a

Consider the map f defined such that for $b = (b_1, b_2, \ldots)$,

$$f(b) = \sum_{j=1}^{\infty} \frac{b_j}{2^j}.$$

To see that f is continuous from $\mathbb{Z}_2^{\mathbb{N}}$ to $[0,1]$ with respect to the metric topology ρ defined in Homework 1, fix $\epsilon > 0$. By the triangle inequality,

$$|f(b) - f(a)| = \left|\sum_{j=1}^{\infty} \frac{b_j - a_j}{2^j}\right| \leq \sum_{j=1}^{\infty} \frac{|b_j - a_j|}{2^j} = \rho(b, a),$$

where we recall from Homework 1 that these infinite sums converge absolutely as they are bounded by 1. Thus, if $\delta = \epsilon$, then $|f(b) - f(a)| < \epsilon$ whenever $\rho(b,a) < \delta$, so f is continuous from $(\mathbb{Z}_2^{\mathbb{N}}, \rho)$ to $([0,1], |\cdot|)$.

3.2 Part b

Given $a = (a_1, \ldots, a_n) \in \mathbb{Z}_2^n$, define the set

$$A = \{b \mid b_j = a_j,\ j = 1, \ldots, n\}.$$

To show that the image $f(A)$ is the closed interval $\left[\frac{q}{2^n}, \frac{q+1}{2^n}\right]$ where $q = \sum_{j=1}^{n} a_j 2^{n-j}$, let $b \in A$. Then,

$$f(b) = \sum_{j=1}^{\infty} \frac{b_j}{2^j} = \frac{1}{2^n}\sum_{j=1}^{\infty} b_j 2^{n-j} = \frac{1}{2^n}\sum_{j=1}^{n} a_j 2^{n-j} + \frac{1}{2^n}\sum_{j=n+1}^{\infty} b_j 2^{n-j} = \frac{q}{2^n} + \frac{1}{2^n}\sum_{j=1}^{\infty} \frac{b_{j+n}}{2^j}.$$

Now, we bound

$$\sum_{j=1}^{\infty} \frac{b_{j+n}}{2^j} \leq \sum_{j=1}^{\infty} \frac{1}{2^j} = 1,$$

and

$$\sum_{j=1}^{\infty} \frac{b_{j+n}}{2^j} \geq \sum_{j=1}^{\infty} \frac{0}{2^j} = 0.$$

Thus,

$$\frac{q}{2^n} \leq f(b) \leq \frac{q}{2^n} + \frac{1}{2^n},$$

so $f(A) \subset \left[\frac{q}{2^n}, \frac{q+1}{2^n}\right]$.

Now, let $x \in \left[\frac{q}{2^n}, \frac{q+1}{2^n}\right]$, that is,

$$x = \frac{q}{2^n} + \frac{1}{2^n}c,$$

for some $c \in [0,1]$. Denote a binary expansion of c by

$$c = \sum_{j=1}^{\infty} \frac{c_j}{2^j},$$

for some $(c_1, c_2, \ldots) \in \mathbb{Z}_2^{\mathbb{N}}$. From class, we know that such an expansion exists, although it may not be unique. In the case where the expansion is not unique, there are exactly two expansions, so we simply choose one to denote as above. Then,

$$x = \frac{q}{2^n} + \frac{1}{2^n}\sum_{j=1}^{\infty}\frac{c_j}{2^j} = \frac{1}{2^n}\sum_{j=1}^{n} a_j 2^{n-j} + \sum_{j=1}^{\infty}\frac{c_j}{2^{n+j}} = \sum_{j=1}^{n}\frac{a_j}{2^j} + \sum_{j=n+1}^{\infty}\frac{c_{j-n}}{2^j}.$$

Let $d = (a_1, \ldots, a_n, c_1, c_2, \ldots) \in A$. Then,

$$f(d) = \sum_{j=1}^{n}\frac{a_j}{2^j} + \sum_{j=n+1}^{\infty}\frac{c_{j-n}}{2^j} = x,$$

so $x \in f(A)$, and $\left[\frac{q}{2^n}, \frac{q+1}{2^n}\right] \subset f(A)$. Thus, $f(A) = \left[\frac{q}{2^n}, \frac{q+1}{2^n}\right]$.

3.3 Part c

Let p and q be integers such that $0 \le p < q \le 2^n$. Then, there exists a unique expansion of p in terms of lower powers of 2:

$$p = \sum_{j=1}^{n} a_j^0 2^{n-j}, \quad a_j^0 \in \{0,1\}.$$

Assuming for now that $p \neq 0$, let k_0 denote the last index j for which $a_j^0 = 1$. Then, we have from lecture that $\frac{p}{2^n}$ has exactly two binary expansions:

$$(a_1^0, \ldots, a_n^0, 0, 0, \ldots), \quad (a_1^0, \ldots, a_{k_0-1}^0, 0, 1, 1, \ldots).$$

Furthermore, if we assume that $p+1 < 2^n$, $p+1$ has a unique expansion

$$p+1 = \sum_{j=1}^{n} a_j^1 2^{n-j}, \quad a_j^1 \in \{0,1\},$$

and $\frac{p+1}{2^n}$ has exactly two binary expansions:

$$(a_1^1, \ldots, a_n^1, 0, 0, \ldots) = (a_1^0, \ldots, a_n^0, 0, 1, 1, \ldots).$$

Consider the set

$$B_0 = A_0 \cup \{(a_1^0, \ldots, a_{k_0-1}^0, 0, 1, 1, \ldots), (a_1^1, \ldots, a_n^1, 0, 0, \ldots)\},$$

where

$$A_0 = \{b \mid b_j = a_j^0,\ j = 1, \ldots, n\}.$$

We claim that $f^{-1}\left(\left[\frac{p}{2^n}, \frac{p+1}{2^n}\right]\right) = B_0$. To verify this, we first note that part b implies that $A_0 \subset f^{-1}\left(\left[\frac{p}{2^n}, \frac{p+1}{2^n}\right]\right)$, so we need only verify that the two remaining points are in $f^{-1}\left(\left[\frac{p}{2^n}, \frac{p+1}{2^n}\right]\right)$ to show that $B_0 \subset f^{-1}\left(\left[\frac{p}{2^n}, \frac{p+1}{2^n}\right]\right)$. Observe that

$$f\left((a_1^0, \ldots, a_{k_0-1}^0, 0, 1, 1, \ldots)\right) = \sum_{j=1}^{k_0-1}\frac{a_j^0}{2^j} + \sum_{j=k_0+1}^{\infty}\frac{1}{2^j} = \sum_{j=1}^{k_0-1}\frac{a_j^0}{2^j} + \frac{1}{2^{k_0}} = \sum_{j=1}^{k_0}\frac{a_j^0}{2^j} = \sum_{j=1}^{n}\frac{a_j^0}{2^j} = \frac{p}{2^n},$$

so $(a_1^0, \ldots, a_{k_0-1}^0, 0, 1, 1, \ldots) \in f^{-1}\left(\left[\frac{p}{2^n}, \frac{p+1}{2^n}\right]\right)$. Similarly,

$$f\left((a_1^1, \ldots, a_n^1, 0, 0, \ldots)\right) = \sum_{j=1}^{n} \frac{a_j^1}{2^j} = \frac{p+1}{2^n},$$

so $(a_1^1, \ldots, a_n^1, 0, 0, \ldots) \in f^{-1}\left(\left[\frac{p}{2^n}, \frac{p+1}{2^n}\right]\right)$. Thus, $B_0 \subset f^{-1}\left(\left[\frac{p}{2^n}, \frac{p+1}{2^n}\right]\right)$.

Now, let $x \in \left(\frac{p}{2^n}, \frac{p+1}{2^n}\right)$. Then,

$$x = \frac{p}{2^n} + \frac{c}{2^n} = \frac{p+c}{2^n},$$

for some $c \in (0,1)$. From class we know that c has a binary expansion which we denote

$$c = \sum_{j=1}^{n} \frac{c_j}{2^j}.$$

Then, the unique binary expansion for x is given by

$$x = \sum_{j=1}^{n} a_j^0 2^{n-j} + \frac{1}{2^n} \sum_{j=1}^{\infty} \frac{c_j}{2^j} = \sum_{j=1}^{n} a_j^0 2^{n-j} + \sum_{j=n+1}^{\infty} \frac{c_{j-n}}{2^j},$$

so $f\left((a_1^0, \ldots, a_n^0, c_{n+1}, c_{n+2}, \ldots)\right) = x$. Note that $(a_1^0, \ldots, a_n^0, c_{n+1}, c_{n+2}, \ldots) \in A_0 \subset B$. If $x = \frac{p}{2^n}$, then we have already established that $f\left((a_1^0, \ldots, a_n^0, 0, 0, \ldots)\right) = f\left((a_1^0, \ldots, a_{k_0-1}^0, 0, 1, 1, \ldots)\right) = x$, $(a_1^0, \ldots, a_n^0, 0, 0, \ldots) \in A_0 \subset B$, $(a_1^0, \ldots, a_{k_0-1}^0, 0, 1, 1, \ldots) \in B$, and these are the only two points that map to x. Similarly, if $x = \frac{p+1}{2^n}$, then $f\left((a_1^1, \ldots, a_{k_1-1}^1, 1, 0, 0, \ldots)\right) = f\left((a_1^0, \ldots, a_n^0, 0, 1, 1, \ldots)\right) = x$, $(a_1^0, \ldots, a_n^0, 0, 1, 1, \ldots) \in A_0 \subset B$, $(a_1^1, \ldots, a_n^1, 0, 0, \ldots) \in B$, and these are the only two points that map to x. Thus, $f^{-1}\left(\left[\frac{p}{2^n}, \frac{p+1}{2^n}\right]\right) \subset B_0$, so $f^{-1}\left(\left[\frac{p}{2^n}, \frac{p+1}{2^n}\right]\right) = B_0$

Now, observe that by taking $p \leftarrow p + k$ for $\ell = 0, \ldots q - p - 1$, we get that $f^{-1}\left(\left[\frac{p+\ell}{2^n}, \frac{p+\ell+1}{2^n}\right]\right) = B_\ell$, where

$$B_\ell = A_\ell \cup \{(a_1^\ell, \ldots, a_{k_\ell-1}^\ell, 0, 1, 1, \ldots), (a_1^{\ell+1}, \ldots, a_n^{\ell+1}, 0, 0, \ldots)\},$$

with

$$A_\ell = \{b \mid b_j = a_j^\ell,\ j = 1, \ldots, n\},$$

and k_ℓ denotes the last index j for which $a_j^\ell = 1$ in the expansion

$$p + \ell = \sum_{j=1}^{n} a_j^\ell 2^{n-j}, \quad a_j^\ell \in \{0,1\}.$$

Then,

$$f^{-1}\left(\left[\frac{p}{2^n}, \frac{q}{2^n}\right]\right) = f^{-1}\left(\bigcup_{\ell=0}^{q-p-1} \left[\frac{p+\ell}{2^n}, \frac{p+\ell+1}{2^n}\right]\right) = \bigcup_{\ell=0}^{q-p-1} B_\ell.$$

Note that $(a_1^{\ell+1}, \ldots, a_n^{\ell+1}, 0, 0, \ldots) \in A_{\ell+1}$ and that

$$(a_1^\ell, \ldots, a_{k_\ell-1}^\ell, 0, 1, 1, \ldots) = (a_1^{\ell-1}, \ldots, a_n^{\ell-1}, 0, 1, 1, \ldots),$$

since these are both binary expansions of $\frac{p+\ell}{2^j}$ distinct from $(a_1^\ell, \ldots, a_{k_\ell}^\ell, 0, 0, \ldots)$ and we know from class that there are exactly two such expansions, so $(a_1^\ell, \ldots, a_{k_\ell-1}^\ell, 0, 1, 1, \ldots) \in A_{\ell-1}$. Thus,

$$f^{-1}\left(\left[\frac{p}{2^n}, \frac{q}{2^n}\right]\right) = \left(\bigcup_{\ell=0}^{q-p-1} A_\ell\right) \cup \{(a_1^0, \ldots, a_{k_0-1}^0, 0, 1, 1, \ldots), (a_1^{q-p}, \ldots, a_n^{q-p}, 0, 0, \ldots)\}.$$

Note that $\bigcup_{\ell=0}^{q-p-1} A_\ell \in \mathcal{A}$ as defined on Homework 1 since $A_\ell = \Pi_n^{-1}(E_\ell)$ for $E_\ell = \{(a_0^\ell, \ldots, a_n^\ell)\}$.

Circling back to the cases $p = 0$ and $q = 2^n$, the first and last additional points drop out in each case, respectively. This follows from the fact that 0 and 1 have unique binary expansions, and the definitions of these points are not valid since these expansions require an infinite number of zeroes or ones, respectively.

To see that the measure of a single point $a \in \mathbb{Z}_2^{\mathbb{N}}$ is zero, define

$$D_j = \Pi_j^{-1}\left(\{(a_1, \ldots, a_j)\}\right), \quad j \in \mathbb{N}.$$

Then, $\mu(D_1) = \frac{1}{2} < \infty$ and $D_1 \supset D_2 \supset \ldots$, so by continuity from above,

$$\mu(\{a\}) = \mu\left(\bigcup_{j=1}^{\infty} D_j\right) = \lim_{j\to\infty} \mu(D_j) = \lim_{j\to\infty} \frac{1}{2^j}\,\mathrm{card}(\{(a_1, \ldots, a_j)\}) = 0.$$

Thus, in all cases

$$\mu\left(f^{-1}\left(\left[\frac{p}{2^n}, \frac{q}{2^n}\right]\right)\right) = \mu\left(\bigcup_{\ell=0}^{q-p-1} A_\ell\right) = \mu\left(\Pi_n^{-1}\left(\bigcup_{\ell=0}^{q-p-1} E_\ell\right)\right) = \frac{1}{2^n}\,\mathrm{card}\left(\bigcup_{\ell=0}^{q-p-1} E_\ell\right) = \frac{q-p}{2^n},$$

since each E_ℓ is distinct due to the fact that each integer has a different binary expansion in powers of 2.

3.4 Part d

Now, consider $0 \le a < b \le 1$. Then, $[a, b] = \bigcap_{j=1}^{\infty} F_j$ where

$$F_j = \left[\frac{\lfloor 2^j a\rfloor}{2^j}, \frac{\lceil 2^j b\rceil}{2^j}\right].$$

Note that $0 \le \lfloor 2^j a\rfloor < \lceil 2^j b\rceil \le 2^n$, so part c can be applied to F_j. Observe that $F_1 \supset F_2 \supset \ldots$ and $F_1 \subset [0, 1]$, so $\mu(F_1) < \infty$ since

$$\mu\left(f^{-1}([0,1])\right) = \mu\left(f^{-1}\left(\left[\frac{0}{2^n}, \frac{2^n}{2^n}\right]\right)\right) = \frac{2^n}{2^n} = 1,$$

for any $n \in \mathbb{N}$. We know from Problem 2 that $\mu(f^{-1}(\cdot))$ defines a Borel measure on $[0, 1]$, so continuity from above implies that

$$\mu\left(f^{-1}([a,b])\right) = \mu\left(f^{-1}\left(\bigcap_{j=1}^{\infty} F_j\right)\right) = \lim_{j\to\infty} \mu\left(f^{-1}\left(\left[\frac{\lfloor 2^j a\rfloor}{2^j}, \frac{\lceil 2^j b\rceil}{2^j}\right]\right)\right) = \lim_{j\to\infty} \frac{\lceil 2^j b\rceil - \lfloor 2^j a\rfloor}{2^j} = b - a.$$

Now, we note that $f^{-1}(\{a\})$ contains either one or two points since any number in $[0, 1]$ has either one or two binary expansions; however, since we have already noted that individual points have measure zero,

$$\mu\left(f^{-1}((a,b])\right) = \mu\left(f^{-1}([a,b])\right) - \mu\left(f^{-1}(\{a\})\right) = b - a.$$

We remark that this is also true when $a = b$, since $(a, a] = \{a\}$, and we established that $\mu\left(f^{-1}(\{a\})\right) = 0$. Now, Theorem 1.16 in Folland gives that there is a unique Borel measure μ_F corresponding to $F(x) = x$ on $[0, 1]$ such that $\mu_F((a, b]) = F(b) - F(a) = b - a$ for all $0 \le a \le b \le 1$. By Problem 2, $\mu(f^{-1}(\cdot))$ is a Borel measure with this property. By definition the Lebesgue measure m is also a Borel measure with this property. Thus, it must hold that

$$\mu\left(f^{-1}(E)\right) = m(E),$$

for all $E \in \mathcal{B}_{[0,1]}$.

4 Problem 4 (Folland Problem 26)

Proposition 1.20 states that if $E \in \mathcal{M}_\mu$ and $\mu(E) < \infty$, then for every $\epsilon > 0$ there is a set A that is a finite union of open intervals such that $\mu(E \triangle A) < \epsilon$. To prove this, fix $\epsilon > 0$. Then, by Lemma 1.17, there exists a collection of open intervals $\{I_j\}_{j=1}^\infty$ such that $E \subset \bigcup_{j=1}^\infty I_j$ and

$$\sum_{j=1}^\infty \mu(I_j) < \mu(E) + \frac{\epsilon}{2}.$$

Since $\mu(E) < \infty$, this sum converges, so there exists some $N \in \mathbb{N}$ such that

$$\sum_{j=N+1}^\infty \mu(I_j) < \frac{\epsilon}{2}.$$

Let $A = \bigcup_{j=1}^N I_j$. Then, by monotonicity and subadditivity

$$\mu(E \setminus A) = \mu\left(\bigcup_{j=1}^\infty I_j \setminus \bigcup_{j=1}^N I_j\right) = \mu\left(\bigcup_{j=N+1}^\infty I_j\right) \leq \sum_{j=N+1}^\infty \mu(I_j) < \frac{\epsilon}{2}.$$

Furthermore, again using monotonicity and subadditivity,

$$\mu(A \setminus E) \leq \mu\left(\bigcup_{j=1}^\infty I_j \setminus E\right) \leq \mu\left(\bigcup_{j=1}^\infty I_j\right) - \mu(E) < \frac{\epsilon}{2}.$$

Finally, by disjoint additivity,

$$\mu(E \triangle A) = \mu(E \setminus A) + \mu(A \setminus E) < \epsilon,$$

as desired.