AMATH 515 Homework 0

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1 Problem 2

1.1 Part a

For $f(x) = \sin(x_1 + x_2 + x_3 + x_4)$,

$$\nabla f(x) = \begin{bmatrix} \cos(x_1 + x_2 + x_3 + x_4) \\ \cos(x_1 + x_2 + x_3 + x_4) \\ \cos(x_1 + x_2 + x_3 + x_4) \\ \cos(x_1 + x_2 + x_3 + x_4) \end{bmatrix}$$

and

$$\nabla^2 f(x) = \begin{bmatrix} -\sin(x_1 + x_2 + x_3 + x_4) & -\sin(x_1 + x_2 + x_3 + x_4) & -\sin(x_1 + x_2 + x_3 + x_4) & -\sin(x_1 + x_2 + x_3 + x_4) \\ -\sin(x_1 + x_2 + x_3 + x_4) & -\sin(x_1 + x_2 + x_3 + x_4) & -\sin(x_1 + x_2 + x_3 + x_4) & -\sin(x_1 + x_2 + x_3 + x_4) \\ -\sin(x_1 + x_2 + x_3 + x_4) & -\sin(x_1 + x_2 + x_3 + x_4) & -\sin(x_1 + x_2 + x_3 + x_4) & -\sin(x_1 + x_2 + x_3 + x_4) \\ -\sin(x_1 + x_2 + x_3 + x_4) & -\sin(x_1 + x_2 + x_3 + x_4) & -\sin(x_1 + x_2 + x_3 + x_4) & -\sin(x_1 + x_2 + x_3 + x_4) \end{bmatrix}.$$

1.2 Part b

For $f(x) = \|x\|^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$,

$$\nabla f(x) = \begin{bmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \\ 2x_4 \end{bmatrix}$$

and

$$\nabla^2 f(x) = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

1.3 Part c

For $f(x) = \ln(x_1 x_2 x_3 x_4)$,

$$\nabla f(x) = \begin{bmatrix} \frac{1}{x_1} \\ \frac{1}{x_2} \\ \frac{1}{x_3} \\ \frac{1}{x_4} \end{bmatrix}$$

and

$$\nabla^2 f(x) = \begin{bmatrix} -\frac{1}{x_1^2} & 0 & 0 & 0 \\ 0 & -\frac{1}{x_2^2} & 0 & 0 \\ 0 & 0 & -\frac{1}{x_3^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{x_4^2} \end{bmatrix}.$$

2 Problem 3

2.1 Part a

Because the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \pi & 2 & 0 & 0 \\ 64 & -15 & 3 & 0 \\ 321 & 0 & 0 & 5 \end{bmatrix}$$

is lower triangular, its eigenvalues are simply its diagonal entries $1, 2, 3, 5$.

2.2 Part b

The matrix

$$A = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

has all linearly dependent columns, so a basis for its range space can be given by just one of them, namely

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

A vector in the nullspace of A must satisfy

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

, so satisfy $x_1 + x_2 + x_3 = 0$. Thus, it must have the form

$$\begin{bmatrix} x_1 \\ x_2 \\ -x_1 - x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix},$$

so a basis for the nullspace of A is given by

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}.$$

2.3 Part c

If A is a 10×5 matrix, and b is a vector in $\mathbb{R}^{10}$, then $A^T A$ is 5×5 and $A^T b$ is 5×1.
The system $Ax = b$ could have zero or infinitely many solutions.
The system $A^T Ax = A^T b$ may have one or infinitely many solutions.
If the columns of A are linearly independent, the system $Ax = b$ has infinitely many solutions and the system $A^T Ax = A^T b$ has one solution.