AMATH 573 Homework 3

Cade Ballew #2120804

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1 Problem 1

Consider the following system of one-dimensional equations

$$\begin{cases} \dfrac{\partial n}{\partial t}+\dfrac{\partial}{\partial z}(nv) &= 0 \\ \dfrac{\partial v}{\partial t}+v\dfrac{\partial v}{\partial z} &= -\dfrac{e}{m}\dfrac{\partial \phi}{\partial z} \\ \dfrac{\partial^2 \phi}{\partial z^2} &= \dfrac{e}{\varepsilon_0}\left[N_0\exp\left(\dfrac{e\phi}{\kappa T_e}\right)-n\right] \end{cases}$$

where n denotes the ion density, v is the ion velocity, e is the electron charge, m is the mass of an ion, ϕ is the electrostatic potential, ε_0 is the vacuum permittivity, N_0 is the equilibrium density of the ions, κ is Boltzmann's constant, and T_e is the electron temperature.

1.1 Part a

We wish to verify that $c_s=\sqrt{\frac{\kappa T_e}{m}}$, $\lambda_{De}=\sqrt{\frac{\varepsilon_0\kappa T_e}{N_0e^2}}$, and $\omega_{pi}=\sqrt{\frac{N_0e^2}{\varepsilon_0 m}}$ have dimensions of velocity, length, and frequency, respectively. The SI units for our variables are $\mathrm{A}\cdot\mathrm{s}$ for e, kg for m, $\frac{\mathrm{kg}\cdot\mathrm{m}^2}{\mathrm{s}^3\cdot\mathrm{A}}$ for ϕ, $\frac{\mathrm{A}^2\cdot\mathrm{s}^4}{\mathrm{m}^3\cdot\mathrm{kg}}$ for ε_0, $\frac{\mathrm{kg}}{\mathrm{m}^3}$ for N_0, $\frac{\mathrm{kg}\cdot\mathrm{m}^2}{\mathrm{s}^2\cdot\mathrm{K}}$ for κ, and K for T_e. Then, the units for c_s are given by

$$\sqrt{\frac{\mathrm{kg}\cdot\mathrm{m}^2}{\mathrm{s}^2\cdot\mathrm{K}}\mathrm{K}\frac{1}{\mathrm{kg}}}=\frac{\mathrm{m}}{\mathrm{s}}$$

which represents velocity. The units for λ_{De} are given by

$$\sqrt{\frac{\mathrm{A}^2\cdot\mathrm{s}^4}{\mathrm{m}^3\cdot\mathrm{kg}}\frac{\mathrm{kg}\cdot\mathrm{m}^2}{\mathrm{s}^2\cdot\mathrm{K}}\mathrm{K}\cdot\mathrm{m}^3\left(\frac{1}{\mathrm{A}\cdot\mathrm{s}}\right)^2}=\mathrm{m}$$

which represents length. The units for ω_{pi} are given by

$$\sqrt{\frac{1}{\mathrm{m}^3}(\mathrm{A}\cdot\mathrm{s})^2\frac{\mathrm{m}^3\cdot\mathrm{kg}}{\mathrm{A}^2\cdot\mathrm{s}^4}\frac{1}{\mathrm{kg}}}=\frac{1}{\mathrm{s}}$$

which represents frequency.

1.2 Part b

Using

$$n = N_0 n^*, \quad v = c_s v^*, \quad z = \lambda_{De} z^*, \quad t = \frac{t^*}{\omega_{pi}}, \quad \phi = \frac{\kappa T_e}{e}\phi^*,$$

we wish to nondimensionalize the system. We compute

$$\frac{\partial n}{\partial t} = \frac{\partial n}{\partial n^*}\frac{\partial n^*}{\partial t^*}\frac{\partial t^*}{\partial t} = N_0 \omega_{pi}\frac{\partial n^*}{\partial t^*},$$

and

$$\begin{aligned}\frac{\partial}{\partial z}(nv) &= \frac{\partial n}{\partial n^*}\frac{\partial n^*}{\partial z^*}\frac{\partial z^*}{\partial z} c_s v^* + N_0 n^* \frac{\partial v}{\partial v^*}\frac{\partial v^*}{\partial z^*}\frac{\partial z^*}{\partial z} \\ &= \frac{N_0 c_s}{\lambda_{De}}\frac{\partial n^*}{\partial z^*} v + \frac{N_0 c_s}{\lambda_{De}} n \frac{\partial v^*}{\partial z^*} = \frac{N_0 c_s}{\lambda_{De}}\frac{\partial}{\partial z}(n^* v^*),\end{aligned}$$

so the first equation becomes

$$\begin{aligned}0 &= N_0\sqrt{\frac{N_0 e^2}{\varepsilon_0 m}}\frac{\partial n^*}{\partial t^*} + N_0\sqrt{\frac{\kappa T_e}{m}}\sqrt{\frac{N_0 e^2}{\varepsilon_0 \kappa T_e}}\frac{\partial}{\partial z}(n^* v^*) \\ &= N_0\sqrt{\frac{N_0 e^2}{\varepsilon_0 m}}\left(\frac{\partial n^*}{\partial t^*} + \frac{\partial}{\partial z}(n^* v^*)\right).\end{aligned}$$

Of course, this reduces to

$$\frac{\partial n^*}{\partial t^*} + \frac{\partial}{\partial z}(n^* v^*) = 0.$$

Now, we compute

$$\frac{\partial v}{\partial t} = \frac{\partial v}{\partial v^*}\frac{\partial v^*}{\partial t^*}\frac{\partial t^*}{\partial t} = c_s \omega_{pi}\frac{\partial v^*}{\partial t^*},$$

$$\frac{\partial v}{\partial z} = \frac{\partial v}{\partial v^*}\frac{\partial v^*}{\partial z^*}\frac{\partial z^*}{\partial z} = \frac{c_s}{\lambda_{De}}\frac{\partial v^*}{\partial z^*},$$

and

$$\frac{\partial \phi}{\partial z} = \frac{\partial \phi}{\partial \phi^*}\frac{\partial \phi^*}{\partial z^*}\frac{\partial z^*}{\partial z} = \frac{\kappa T_e}{e\lambda_{De}}\frac{\partial \phi^*}{\partial z^*}.$$

Then, our second equation becomes

$$-\frac{e}{m}\frac{\kappa T_e}{e}\sqrt{\frac{N_0 e^2}{\varepsilon_0 \kappa T_e}}\frac{\partial \phi^*}{\partial z^*} = \sqrt{\frac{\kappa T_e}{m}}\sqrt{\frac{N_0 e^2}{\varepsilon_0 m}}\frac{\partial v^*}{\partial t^*} + \frac{\kappa T_e}{m}\sqrt{\frac{N_0 e^2}{\varepsilon_0 \kappa T_e}} v^* \frac{\partial v^*}{\partial z^*}.$$

We can simplify this to

$$\sqrt{\frac{\kappa T_e N_0 e^2}{\varepsilon_0 m^2}}\left(\frac{\partial v^*}{\partial t^*}+\frac{\partial v^*}{\partial z^*}\right)=-\sqrt{\frac{\kappa T_e N_0 e^2}{\varepsilon_0 m^2}}\frac{\partial \phi^*}{\partial z^*}$$

which further simplifies to

$$\frac{\partial v^*}{\partial t^*}+\frac{\partial v^*}{\partial z^*}=-\frac{\partial \phi^*}{\partial z^*}.$$

Finally, we compute

$$\frac{\partial^2 \phi}{\partial z^2}=\frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial z}\right)=\frac{\partial z^*}{\partial z}\frac{\partial}{\partial z^*}\left(\frac{\partial \phi}{\partial z}\right)=\frac{1}{\lambda_{De}}\frac{\partial}{\partial z^*}\left(\frac{\kappa T_e}{e\lambda_{De}}\frac{\partial \phi^*}{\partial z^*}\right)=\frac{\kappa T_e}{e\lambda_{De}^2}\frac{\partial^2 \phi^*}{\partial {z^*}^2},$$

and our final equation becomes

$$\frac{\kappa T_e}{e}\frac{N_0 e^2}{\varepsilon_0 \kappa T_e}\frac{\partial^2 \phi^*}{\partial {z^*}^2}=\frac{e}{\varepsilon_0}\left(N_0 \exp\left(\frac{e}{\kappa T_e}\frac{\kappa T_e}{e}\phi^*\right)-N_0 n^*\right).$$

This simplifies to

$$\frac{N_0 e}{\varepsilon_0}\frac{\partial^2 \phi^*}{\partial {z^*}^2}=\frac{N_0 e}{\varepsilon_0}(e^{\phi^*}-n^*)$$

which further simplifies to

$$\frac{\partial^2 \phi^*}{\partial {z^*}^2}=e^{\phi^*}-n^*.$$

Thus, we arrived at the dimensionless system

$$\left\{\begin{array}{rcl}\frac{\partial n^*}{\partial t^*}+\frac{\partial}{\partial z^*}(n^* v^*) & = & 0\\ \frac{\partial v^*}{\partial t^*}+v\frac{\partial v^*}{\partial z^*} & = & -\frac{\partial \phi^*}{\partial z^*}\\ \frac{\partial^2 \phi^*}{\partial {z^*}^2} & = & e^{\phi^*}-n^*.\end{array}\right.$$

1.3 Part c

Dropping the asterisks, we search for a linear dispersion relation for our system by linearizing around the trivial solution $n=1$, $v=0$, and $\phi=0$. Namely, we set $n=1+\epsilon n_1+\mathrm{O}(\epsilon^2)$, $v=\epsilon v_1+\mathrm{O}(\epsilon^2)$, $\phi=\epsilon\phi_1+\mathrm{O}(\epsilon^2)$. Then, dropping higher order terms, our system becomes

$$\left\{\begin{array}{rcl}\epsilon n_{1t}+(1+\epsilon n_1)\epsilon v_{1z}+\epsilon n_{1t}\epsilon v_1 & = & 0\\ \epsilon v_{1t}+\epsilon v_1 \epsilon v_{1z} & = & -\epsilon\phi_{1z}\\ \epsilon\phi_{1zz} & = & (1+\epsilon\phi_1+\ldots)-(1+\epsilon n_1)\end{array}\right. .$$

Collecting the terms that are order 1 in ϵ, we get the much simpler system

$$\begin{cases} n_{1t} + v_{1z} &= 0 \\ v_{1t} &= -\phi_{1z} \\ \phi_{1zz} &= \phi_1 - n_1. \end{cases}$$

Now, to avoid solving a system of 3 equations directly, we use the general vector case of our method for finding the dispersion relation by considering

$$u = \begin{pmatrix} n \\ v \\ \phi \end{pmatrix}$$

and the ansatz $u = A(k)e^{ikz - i\omega(k)t}$. We first write our system as

$$\begin{pmatrix} n \\ v \\ 0 \end{pmatrix}_t = \begin{pmatrix} 0 & -\partial_z & 0 \\ 0 & 0 & -\partial_z \\ -1 & 0 & 1 - \partial_z^2 \end{pmatrix} \begin{pmatrix} n \\ v \\ \phi \end{pmatrix}.$$

Applying the ansatz, to find the dispersion relation, we examine

$$\begin{pmatrix} i\omega(k) & -ik & 0 \\ 0 & i\omega(k) & -ik \\ -1 & 0 & 1+k^2 \end{pmatrix} \begin{pmatrix} n \\ v \\ \phi \end{pmatrix} = 0,$$

i.e., we set

$$\det \begin{pmatrix} i\omega(k) & -ik & 0 \\ 0 & i\omega(k) & -ik \\ -1 & 0 & 1+k^2 \end{pmatrix} = 0.$$

Using a symbolic matrix determinant calculator, we find that a matrix with this sparsity pattern has determinant

$$\begin{aligned} \det \begin{pmatrix} i\omega(k) & -ik & 0 \\ 0 & i\omega(k) & -ik \\ -1 & 0 & 1+k^2 \end{pmatrix} &= (i\omega(k))(i\omega(k))(1+k^2) + (-1)(-ik)(-ik) \\ &= -\omega^2(k)(k^2+1) + k^2. \end{aligned}$$

Thus, we find that this problem has linearized dispersion relation

$$\omega^2(k) = \frac{k^2}{k^2+1}.$$

1.4 Part d

Now, we rewrite the system using the “stretched variables”

$$\xi = \epsilon^{1/2}(z-t), \quad \tau = \epsilon^{3/2}t.$$

We compute

$$\frac{\partial}{\partial t} = \frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} + \frac{\partial \tau}{\partial t}\frac{\partial}{\partial \tau} = -\epsilon^{1/2}\frac{\partial}{\partial \xi} + \epsilon^{3/2}\frac{\partial}{\partial \tau},$$

$$\frac{\partial}{\partial z} = \frac{\partial \xi}{\partial z}\frac{\partial}{\partial \xi} = \epsilon^{1/2}\frac{\partial}{\partial \xi},$$

and

$$\frac{\partial^2}{\partial z^2} = \frac{\partial}{\partial z}\frac{\partial}{\partial z} = \epsilon\frac{\partial^2}{\partial \xi^2}.$$

Then, our system becomes

$$\begin{cases} -\epsilon^{1/2}n_\xi + \epsilon^{3/2}n_\tau + \epsilon^{1/2}(nv)_\xi &= 0 \\ -\epsilon^{1/2}v_\xi + \epsilon^{3/2}v_\tau + \epsilon^{1/2}vv_\xi &= -\epsilon^{1/2}\phi_\xi \\ \epsilon\phi_{\xi\xi} &= e^\phi - n. \end{cases}$$

We can simplify this slightly to

$$\begin{cases} -n_\xi + \epsilon n_\tau + (nv)_\xi &= 0 \\ -v_\xi + \epsilon v_\tau + vv_\xi &= -\phi_\xi \\ \epsilon\phi_{\xi\xi} &= e^\phi - n. \end{cases}$$

To justify this new choice of variables, we use Mathematica to Taylor expand the dispersion relation

$$\omega_\pm(k) = \pm\frac{k}{\sqrt{k^2+1}}$$

around $k = 0$ (since we are looking for low-frequency waves) which gives

$$\omega_+(k) = k - \frac{k^3}{2} + \mathrm{O}(k^4).$$

Then, the $e^{ikz - i\omega(k)t}$ term from our ansatz becomes

$$e^{ikz - i\left(k - \frac{k^3}{2} + \mathrm{O}(k^4)\right)t} = e^{i\left(k(z-t) + \frac{k^3}{2}t\right) + \mathrm{O}(k^4)}.$$

Taking $k = \epsilon^{1/2}$, we can see that if we write this as a product of exponentials, ξ corresponds to the first term $e^{i\epsilon^{1/2}(z-t)}$, and up to a constant scaling, τ corresponds to the second term $e^{i\epsilon^{3/2}t}$.

1.5 Part e

Now, we expand the dependent variables as

$$\begin{cases} n &= 1 + \epsilon n_1 + \epsilon^2 n_2 + \ldots, \\ v &= \epsilon v_1 + \epsilon^2 v_2 + \ldots, \\ \phi &= \epsilon\phi_1 + \epsilon^2\phi_2 + \ldots. \end{cases}$$

Dropping higher order terms and plugging these in, the first equation becomes

$$(\epsilon n_{1\xi}+\epsilon^2 n_{2\xi})+\epsilon(\epsilon n_{1\tau})+(\epsilon n1\xi+\epsilon^2 n_{2\xi})(\epsilon v_1+\epsilon^2 v_2)+(1+\epsilon n_1+\epsilon^2 n_2)(\epsilon v_{1\xi}+\epsilon^2 v_{2\xi})=0,$$

the second equation becomes

$$-(\epsilon v_{1\xi}+\epsilon^2 v_{2\xi})+\epsilon(\epsilon v_{1\tau}+\epsilon^2 v_{2\tau})+(\epsilon v_1+\epsilon^2 v_2)(\epsilon v_{1\xi}+\epsilon^2 v_{2\xi})=-(\epsilon\phi_{1\xi}+\epsilon^2\phi_{2\xi}),$$

and the third equation becomes

$$\begin{aligned}\epsilon(\epsilon\phi_{1\xi\xi}+\epsilon^2\phi_{2\xi\xi}) &= e^{\epsilon\phi_1+\epsilon^2\phi_2}-(1+\epsilon n_1+\epsilon^2 n_2)\\ &= \big(1+(\epsilon\phi_1+\epsilon^2\phi_2)+\tfrac{1}{2}(\epsilon\phi_1+\epsilon^2\phi_2)^2\big)-(1+\epsilon n_1+\epsilon^2 n_2).\end{aligned}$$

Collecting the first order terms in ϵ, we get the system

$$\begin{cases} -n_{1\xi}+v_{1\xi} &= 0\\ -v_{1\xi} &= -\phi_{1\xi}\\ 0 &= \phi_1-n_1.\end{cases}$$

Due to the fact that all disturbances return to their equilibrium values as $\xi\to\pm\infty$, $\tau\to\infty$, we can conclude that $n_1=v_1=\phi_1$. Collecting the second order terms in ϵ, we get the system

$$\begin{cases} -n_{2\xi}+n_{1\tau}+n_{1\xi}v_1+n_1 v_{1\xi}+v_{2\xi} &= 0\\ -v_{2\xi}+v_{1\tau}+v_1 v_{1\xi} &= -\phi_{2\xi}\\ \phi_{1\xi\xi} &= \phi_2+\tfrac{1}{2}\phi_1^2-n_2.\end{cases}$$

Using our conclusion from the first order system, we reduce this to

$$\begin{cases} -n_{2\xi}+\phi_{1\tau}+2\phi_1\phi_{1\xi}+v_{2\xi} &= 0\\ -v_{2\xi}+\phi_{1\tau}+\phi_1\phi_{1\xi} &= -\phi_{2\xi}\\ \phi_{1\xi\xi} &= \phi_2+\tfrac{1}{2}\phi_1^2-n_2.\end{cases}$$

We can reduce the first two equations to

$$n_{2\xi}=\phi_{1\tau}+2\phi_1\phi_{1\xi}+\phi_{1\tau}+\phi_1\phi_{1\xi}+\phi_{2\xi}=2\phi_{1\tau}+3\phi_1\phi_{1\xi}+\phi_{2\xi}.$$

Differentiating the third gives

$$n_{2\xi}=-\phi_{1\xi\xi\xi}+\phi_{2\xi}+\phi_1\phi_{1\xi}.$$

Setting these equal yields

$$2\phi_{1\tau}+2\phi_1\phi_{1\xi}+\phi_{1\xi\xi\xi}=0$$

which, of course, is the KdV equation.

2 Problem 2

Consider the defocusing NLS equation

$$ia_t = -a_{xx} + |a|^2 a.$$

2.1 Part a

Let

$$a(x,t) = e^{i\int V dx}\rho^{1/2}.$$

where $V(x,t)$ is the phase and $\rho(x,t)$ is the amplitude. Using Mathematica, we plug this choice of a into the defocusing NLS equation and obtain the following equation

$$-\frac{1}{4\rho^{3/2}}e^{i\int V dx}\left(4\rho^2\int V_t dx + 4V^2\rho^2 + 4\rho^3 - 2i\rho\rho_t - 4i\rho^2 V_x - 4iV\rho\rho_x + \rho_x^2 - 2\rho\rho_{xx}\right) = 0.$$

Note that we Mathematica required the assumption that V be real-valued and ρ be nonnegative to obtain this, but one should expect a phase to be real and an amplitude to be nonnegative. Dividing through by the terms outside the parentheses and splitting into real and imaginary parts, we get

$$4\rho^2\int V_t dx + 4V^2\rho^2 + 4\rho^3\rho_x^2 - 2\rho\rho_{xx} = 0$$

from the real part and

$$-2\rho\rho_t - 4\rho^2 V_x - 4V\rho\rho_x = 0$$

from the imaginary part. The first equation can then be written as

$$\int V_t dx = -V^2 - \rho - \frac{\rho_x^2}{4\rho^2} + \frac{\rho_{xx}}{2\rho}.$$

Using Mathematica to differentiate both sides, we get our equation for V_t. Namely,

$$V_t = -2VV_x - \rho_x + \frac{\rho_x^3}{2\rho^3} - \frac{\rho_x\rho_{xx}}{\rho^2} + \frac{\rho_{xxx}}{2\rho}.$$

Our equation for ρ_t can be solved for directly and is given by

$$\rho_t = -2\rho V_x - 2V\rho_x,$$

so the hydrodynamic form of the NLS equation is given by

$$\begin{cases} V_t &= -2VV_x - \rho_x + \frac{\rho_x^3}{2\rho^3} - \frac{\rho_x\rho_{xx}}{\rho^2} + \frac{\rho_{xxx}}{2\rho} \\ \rho_t &= -2\rho V_x - 2V\rho_x. \end{cases}$$

2.2 Part b

To find the linear dispersion relation for the hydrodynamic form of the defocusing NLS equation, linearized around the trivial solution $V = 0$, $\rho = 1$, we set $V = \epsilon V_1 + \mathrm{O}(\epsilon^2)$ and $\rho = 1 + \epsilon\rho_1 + \mathrm{O}(\epsilon^2)$. Dropping higher order terms, our system becomes

$$\begin{cases} \epsilon V_{1t} &= -2\epsilon V_1 \epsilon V_{1x} - \epsilon\rho_{1x} + \frac{\epsilon^3\rho_{1x}^3}{2(1+\epsilon\rho_1)^3} - \frac{\epsilon\rho_{1x}\epsilon\rho_{1xx}}{(1+\epsilon\rho_1)^2} + \frac{\epsilon\rho_{1xxx}}{2(1+\epsilon\rho_1)} \\ \epsilon\rho_{1t} &= -2(1+\epsilon\rho_1)\epsilon V_{1x} - 2\epsilon V_1 \epsilon\rho_{1x}. \end{cases}$$

To collect the first order terms in ϵ, we utilize the geometric series

$$\frac{1}{1+\epsilon\rho_1} = \frac{1}{1-(-\epsilon\rho_1)} = \sum_{j=0}^{\infty}(-\epsilon\rho_1)^j = 1 - \epsilon\rho_1 + \mathrm{O}(\epsilon^2).$$

Then, we can see that first order terms in ϵ give

$$\begin{cases} V_{1t} &= -\rho_{1x} + \frac{1}{2}\rho_{1xxx} \\ \rho_{1t} &= -2V_{1x}. \end{cases}$$

As in problem 1, we use the general vector case of our method to find the dispersion relation, but since we a smaller system this time, we consider it componentwise. Namely, apply the ansatz $V = a_1(k)e^{ikx-i\omega(k)t}$, $\rho = a_2(k)e^{ikx-i\omega(k)t}$,. Then, we get the system of equations

$$\begin{cases} -a_1 i\omega &= -a_2 ik - \frac{1}{2}a_2 ik^3 \\ -a_2 i\omega &= -2a_1 ik. \end{cases}$$

Then, the second equation gives $a_1 = \frac{a_2\omega}{2k}$, so

$$\frac{-ia_2\omega^2}{2k} = -a_2 ik - \tfrac{1}{2}a_2 ik^3.$$

Solving this, we conclude that our dispersion relation is given by

$$\omega^2 = k^4 + 2k^2.$$

2.3 Part c

Now, we wish to rewrite our system using the "stretched variables"

$$\xi = \epsilon(x - \beta t), \quad \tau = \epsilon^3 t.$$

We compute

$$\frac{\partial}{\partial t} = \frac{\partial\xi}{\partial t}\frac{\partial}{\partial\xi} + \frac{\partial\tau}{\partial t}\frac{\partial}{\partial\tau} = -\epsilon\beta\frac{\partial}{\partial\xi} + \epsilon^3\frac{\partial}{\partial\tau},$$

$$\frac{\partial}{\partial x} = \frac{\partial\xi}{\partial x}\frac{\partial}{\partial\xi} = \epsilon\frac{\partial}{\partial\xi},$$

$$\frac{\partial^2}{\partial z^2} = \frac{\partial}{\partial z}\frac{\partial}{\partial z} = \epsilon^2 \frac{\partial^2}{\partial \xi^2},$$

and

$$\frac{\partial^3}{\partial z^3} = \frac{\partial}{\partial z}\frac{\partial^2}{\partial z^2} = \epsilon^3 \frac{\partial^3}{\partial \xi^3}.$$

Then, our system becomes

$$\left\{ \begin{array}{rcl} -\epsilon\beta V_\xi + \epsilon^3 V_\tau & = & -2V\epsilon V_\xi - \epsilon\rho_\xi + \frac{\epsilon^3 \rho_\xi^3}{2\rho^3} - \frac{\epsilon\rho_\xi \epsilon^2 \rho_{\xi\xi}}{\rho^2} + \frac{\epsilon^3 \rho_{\xi\xi\xi}}{2\rho} \\ -\epsilon\beta\rho_\xi + \epsilon^3 \rho_\tau & = & -2\rho\epsilon V_\xi - 2V\epsilon\rho_\xi. \end{array} \right.$$

We can divide through by ϵ to get

$$\left\{ \begin{array}{rcl} -\beta V_\xi + \epsilon^2 V_\tau & = & -2VV_\xi - \rho_\xi + \frac{\epsilon^2 \rho_\xi^3}{2\rho^3} - \frac{\epsilon^2 \rho_\xi \rho_{\xi\xi}}{\rho^2} + \frac{\epsilon^2 \rho_{\xi\xi\xi}}{2\rho} \\ -\beta\rho_\xi + \epsilon^2 \rho_\tau & = & -2\rho V_\xi - 2V\rho_\xi. \end{array} \right.$$

To justify our choice of stretched variables, we again look at the series expansion of

$$\omega_+(k) = k\sqrt{k^2+2}$$

using Mathematica. This gives

$$\omega_+(k) = \sqrt{2}k + \frac{k^3}{2\sqrt{2}} + \mathrm{O}(k^4).$$

Then, our ansatz term $e^{ikx - i\omega t}$ becomes

$$e^{ikx - i\omega(k)t} = e^{i\left((x-\sqrt{2})t + \frac{k^3}{2\sqrt{2}}t + \mathrm{O}(k^4)\right)}.$$

If we take $k = \epsilon$ and write this as a product of exponentials, we can see that the first two arguments are $(x - \sqrt{2})t$ and $\frac{k^3}{2\sqrt{2}}t$. If we take $\beta = \sqrt{2}$, these match our stetched variables up to a constant scaling on the second one. Using this value of β, our system becomes

$$\left\{ \begin{array}{rcl} -\sqrt{2}V_\xi + \epsilon^2 V_tau & = & -2VV_\xi - \rho_\xi + \frac{\epsilon^2 \rho_\xi^3}{2\rho^3} - \frac{\epsilon^2 \rho_\xi \rho_{\xi\xi}}{\rho^2} + \frac{\epsilon^2 \rho_{\xi\xi\xi}}{2\rho} \\ -\sqrt{2}\rho_\xi + \epsilon^2 \rho_\tau & = & -2\rho V_\xi - 2V\rho_\xi. \end{array} \right.$$

2.4 Part d

Now, we wish to expand the dependent variables as

$$\left\{ \begin{array}{rcl} V & = & \epsilon^2 V_1 + \epsilon^4 V_2 + \ldots, \\ \rho & = & 1 + \epsilon^2 \rho_1 + \epsilon^4 \rho_2 + \ldots. \end{array} \right.$$

Dropping higher order terms and multiplying through by ρ, our first equation becomes

$$\begin{aligned}
&-\sqrt{2}(\epsilon^2 V_{1\xi}+\epsilon^4 V_{2\xi})(1+\epsilon^2\rho_1+\epsilon^4\rho_2)^3+\epsilon^2(\epsilon^2 V_{1\tau}+\epsilon^4 V_{2\tau})(1+\epsilon^2\rho_1+\epsilon^4\rho_2)\\
&=-2(\epsilon^2 V_1+\epsilon^4 V_2)(\epsilon^2 V_{1\xi}+\epsilon^4 V_{2\xi})(1+\epsilon^2\rho_1+\epsilon^4\rho_2)^3-(1+\epsilon^2\rho_1+\epsilon^4\rho_2)^3(\epsilon^2\rho_{1\xi}+\epsilon^4\rho_{2\xi})\\
&+\tfrac{1}{2}\epsilon^2(\epsilon^2\rho_{1\xi}+\epsilon^4\rho_{2\xi})^3-\epsilon^2(\epsilon^2\rho_{1\xi}+\epsilon^4\rho_{2\xi})(\epsilon^2\rho_{1\xi\xi}+\epsilon^4\rho_{2\xi\xi})(1+\epsilon^2\rho_1+\epsilon^4\rho_2)\\
&+\tfrac{1}{2}\epsilon^2(\epsilon^2\rho_{1\xi\xi\xi}+\epsilon^4\rho_{2\xi\xi\xi})(1+\epsilon^2\rho_1+\epsilon^4\rho_2)^2.
\end{aligned}$$

Our second equation becomes

$$\begin{aligned}
&-\sqrt{2}(\epsilon^2\rho_{1\xi}+\epsilon^4\rho_{2\xi})+\epsilon^2(\epsilon^2\rho_{1\tau}+\epsilon^4\rho_{2\tau})\\
&=-2(1+\epsilon^2\rho_1+\epsilon^4\rho_2)(\epsilon^2 V_{1\xi}+\epsilon^4 V_{2\xi})-2(\epsilon^2 V_1+\epsilon^4 V_2)(\epsilon^2\rho_{1\xi}+\epsilon^4\rho_{2\xi}).
\end{aligned}$$

Collecting second order terms in ϵ, we get the system

$$\begin{cases} -\sqrt{2}V_{1\xi} &= -\rho_{1\xi}\\ -\sqrt{2}\rho_{1\xi} &= -2V_{1\xi}. \end{cases}$$

This yields $\rho_{1\xi}=\sqrt{2}V_{1\xi}$ which combined with the fact that all disturbances return to their equilibrium values as $\xi\to\pm\infty$, $\tau\to\infty$, yields that $\rho_1=\sqrt{2}V_1$. Collecting fourth order terms in ϵ, we get the system

$$\begin{cases} -\sqrt{2}V_{2\xi}+V_{1\tau} &= -2V_1V_{1\xi}-\rho_{2\xi}+\frac{1}{2}\rho_{1\xi\xi\xi}\\ -\sqrt{2}\rho_{2\xi}+\rho_{1\tau} &= -2\rho_1V_{1\xi}-2V_{2\xi}-2V_1\rho_{1\xi}. \end{cases}$$

Substituting $\rho_1=\sqrt{2}V_1$ yields

$$\begin{cases} -\sqrt{2}V_{2\xi}+V_{1\tau} &= -2V_1V_{1\xi}-\rho_{2\xi}+\frac{1}{\sqrt{2}}V_{1\xi\xi\xi}\\ -\sqrt{2}\rho_{2\xi}+\sqrt{2}V_{1\tau} &= -4\sqrt{2}V_1V_{1\xi}-2V_{2\xi}. \end{cases}$$

Solving each for $\rho_{2\xi}$, we get the equation

$$\sqrt{2}V_{2\xi}-V_{1\tau}-2V_1V_{1\xi}+\frac{1}{\sqrt{2}}V_{1\xi\xi\xi}=V_{1\tau}+4V_1V_{1\xi}+\sqrt{2}V_{2\xi}.$$

Simplifying, we again get the KdV equation

$$2V_{1\tau}+6V_1V_{1\xi}-\frac{1}{\sqrt{2}}V_{1\xi\xi\xi}=0.$$

3 Problem 3

Now, consider the previous problem, but with the focusing NLS equation

$$ia_t=-a_{xx}-|a|^2a.$$

To see why the method presented in the previous problem does not allow one to describe the dynamics of long-wave solutions of the focusing NLS equation using the KdV equation, we follow the same steps as before noting the effect of the sign change. Plugging the same $a(x,t)$ into the equation via Mathematica, we now get

$$-\frac{1}{4\rho^{3/2}}e^{i\int V dx}\left(4\rho^2\int V_t dx + 4V^2\rho^2 - 4\rho^3 - 2i\rho\rho_t - 4i\rho^2 V_x - 4iV\rho\rho_x + \rho_x^2 - 2\rho\rho_{xx}\right) = 0.$$

Note that only the sign on the $4\rho^3$ has changed. This changes the hydrodynamic form of the NLS equation to

$$\begin{cases} V_t &= -2VV_x + \rho_x + \frac{\rho_x^3}{2\rho^3} - \frac{\rho_x\rho_{xx}}{\rho^2} + \frac{\rho_{xxx}}{2\rho} \\ \rho_t &= -2\rho V_x - 2V\rho_x. \end{cases}$$

Linearizing in the same manner as before, the first order terms in ϵ now give

$$\begin{cases} V_{1t} &= \rho_{1x} + \frac{1}{2}\rho_{1xxx} \\ \rho_{1t} &= -2V_{1x}. \end{cases}$$

Plugging in the same ansatz, the sign change causes our dispersion relation to become

$$\omega^2(k) = k^4 - 2k^2,$$

so

$$\omega_\pm(k) = \pm k\sqrt{k^2-2}.$$

This is problematic, because when $k^2 < 2$, $\omega(k)$ cannot be real. This shows up if one attempts to series expand around zero. Mathematica now gives that the series is given by

$$\omega_+(z) = -i\sqrt{2}k + \frac{i}{2\sqrt{2}}k^3 + \mathrm{O}(k^4).$$

Applying this to our ansatz in the same way as before gives that we need $\beta = -i\sqrt{2}$. However, this would mean that our stretched variable ξ would be nonreal which makes this an invalid scaling.

4 Problem 4

Consider the defocusing mKdV equation

$$4u_t = -6u^2u_x + u_{xxx}.$$

4.1 Part a

We first examine the traveling-wave solutions via the potential energy method. Namely, we first set

$$u(x,t) = U(x-vt) = U(z)$$

where $z = x - vt$ and v is constant. Substituting this in gives

$$-4vU' = -6U^2U' + U'''.$$

Integrating yields

$$-4vU = -2U^3 + U''' + \alpha$$

where α is an integration constant. Multiplying both sides by U' and integrating again, we get

$$\tfrac{1}{2}U'^2 + V(U; v, \alpha) = \beta$$

where

$$V(U; v, \alpha) = -\tfrac{1}{2}U^4 + 2vU^2 + \alpha U$$

and β is another integration constant. Note that V is quartic in U and $V \to -\infty$ as $U \to \pm\infty$, so when plotted, it looks like Figure 5.1 in the notes but upside down. We include two plots of $V(U)$ in the Mathematica notebook for different values of v, α. Let U_1 and U_2 denote the locations of the local maxima named such that $\beta_s = V(U_1) \leq V(U_2) = \beta_t$. As a first case, consider $\beta > \beta_t$. Clearly, we have no real-valued solutions. If $\beta = \beta_t$, we see only a double real root at U_2, so our solution takes infinite time to get to U_2 from either $U \in (-\infty, U_2)$ or $U \in (U_2, \infty)$. If $\beta_s < \beta < \beta_t$, we have two simple roots which we label $U_{\min} < U_{\max}$. We have two classes of solutions which reach either $U_{\min}$ or $U_{\max}$ in finite time depending on if $U \in (-\infty, U_2)$ or $U \in (U_2, \infty)$. If $\beta = \beta_s$, we have two simple roots and one double root. Using the same labeling for the simple roots, we have three solution classes. One takes an infinite time to reach U_1 from $U \in (-\infty, U_1)$, another reaches $U_{\min}$ in finite time from $U \in (U_1, U_{\min})$, and one reaches $U_{\max}$ in finite time from $U \in (U_{\max}, \infty)$. Finally, if $0 < \beta < \beta_s$, we have four simple roots which we label $U_3 < U_4 < U_5 < U_6$. we have periodic solutions in the gaps, e.g., $U \in (U_4, U_5)$, which reach their endpoints in finite time as well as solitary solutions on $U \in (-\infty, U_3)$ and $U \in (U_6, \infty)$ which also reach their endpoints in finite time. As a special case, when $\alpha = 0$ and $\beta = \beta_s = \beta_t$, our solution takes an infinite amount of time to get to both U_1 and U_2 (a shock).

To perform phase plane analysis, we consider

$$U'' = -\frac{\partial V}{\partial U} = 2U^3 - 4vU - \alpha.$$

Letting $u_1 = U$, $u_2 = U'$, we have the system

$$\begin{cases} u_1' &= u_2, \\ u_2' &= 2u_1^3 - 4vu_1 - \alpha. \end{cases}$$

Our critical points are given by

$$\frac{\partial V}{\partial U} = 0 = -2U^3 + 4vU + \alpha.$$

This cubic has discriminant

$$\Delta = 512v^3 - 108\alpha^2,$$

so we consider cases based on whether Δ is positive, negative, or zero. See the attached Mathematica notebook for phase plane plots. Note that we observe homoclinic connection when $\Delta > 0$ which becomes heteroclinic when we also take $\alpha = 0$.

4.2 Part b

We wish to find the explicit form of the profiles corresponding to heteroclinic connection, so we take $\alpha = 0$. Then, we have

$$V(U) = -\tfrac{1}{2}U^4 + 2vU^2, \quad V'(U) = -2U^3 + 4vU.$$

Setting $V'(U) = 0$ yields

$$-2U(U^2 - 2vU) = 0,$$

so we have critical points at $U = 0, \pm\sqrt{2v}$. Note that this requires $v > 0$, but that requirement corresponds to $\Delta > 0$ which is what enabled us to find the heteroclinic orbit in the first place. Then, as $x \to \pm\infty$, $U \to \pm\sqrt{2v}$, so

$$\beta = \lim_{x\to\infty} = \left(\tfrac{1}{2}U'^2 - \tfrac{1}{2}U^4 + 2vU^2\right) = -\tfrac{1}{2}(2v)^2 + 2v(2v) = 2v^2$$

since we assume that derivatives go to zero at infinity. Then, we can write

$$U' = \pm\sqrt{2(\beta - V(U))},$$

so

$$\pm z = \int_{U_0}^{U} \frac{du}{\sqrt{2(2v^2 + \frac{1}{2}U^4 - 2vU^2)}} = \mp\frac{1}{\sqrt{2v}}\left(\operatorname{arctanh}\left(\frac{U}{\sqrt{2v}}\right) + C\right)$$

where we have used Mathematica to compute the integral and replaced the term induced by U_0 with a constant C. Solving for U,

$$U = \mp\sqrt{2v}\tanh(\sqrt{2v}z + C).$$

Letting $x_0 = C$, we can conclude

$$u(x,t) = U(x - vt) = \pm\sqrt{2v}\tanh(\sqrt{2v}(x - vt) + x_0).$$

5 Problem 5

Consider the DNLS equation

$$b_t + \alpha\left(b|b|^2\right)_x - ib_{xx} = 0.$$

where $b(x,t)$ is a complex-valued function.

5.1 Part a

Consider a polar decomposition

$$b(x,t) = B(x,t)e^{i\theta(x,t)},$$

where B and θ are real-valued functions. Using Mathematica, we plug this ansatz into the DNLS equation, noting that due to real-valuedness, $|b|^2 = B^2$, so we can make that replacement in our Mathematica code. This yields the equation

$$e^{i\theta}(B_t + 3\alpha B^2 B_x + i\alpha B^3\theta_x + 2B_x\theta_x - iB_{xx} + B(i\theta_t + i\theta_x^2 + \theta_{xx})) = 0.$$

Dividing by the exponential and separating real and imaginary parts, we get the system of equations

$$B_t + 3\alpha B^2 B_x + 2B_x\theta_x + B\theta_{xx} = 0,$$
$$\alpha B^3\theta_x - B_{xx} + B\theta_t + B\theta_x^2 = 0.$$

We can simplify this by noting that $\frac{1}{B}(B^2\theta_x)_x = 2B_x\theta_x + B\theta_{xx}$ and dividing the second equation by B to get the system

$$B_t + 3\alpha B^2 B_x + \frac{1}{B}(B^2\theta_x)_x = 0,$$
$$\theta_t + \alpha B^2\theta_x + \theta_x^2 - \frac{1}{B}B_{xx} = 0.$$

5.2 Part b

Assuming a traveling-wave envelope, $B(x,t) = R(z)$, with $z = x - vt$ and constant v, we consider an ansatz $\theta(x,t) = \Phi(z) - \Omega t$, with constant Ω. To show that this ansatz is consistent with our equations, we plug it into them. Using the form of the equations from part a before we simplified and noting that

$$\partial_x = \partial_z, \quad \partial_t = -v\partial_z,$$

we get the system

$$-vR_z + 3\alpha R^2 R_z + 2R_z\Phi_z + R\Phi_{zz} = 0,$$
$$\alpha R^3\Phi_z - R_{zz} + R(-v\Phi_z - \Omega) + R\Phi_z^2 = 0.$$

Note that this system does not contain x, t outside of the variable z, so this ansatz is in fact consistent with our equations.

5.3 Part c

Now, we assume $B(x,t) = R(z)$, with $z = x - vt$ and constant v and $\theta(x,t) = \Phi(z) - \Omega t$. Plugging this into the equation associated with the real part, we get that

$$-vR' + 3\alpha R^2 R' + \frac{1}{R}(R^2\Phi')_x = 0.$$

Letting $s = \alpha R^2/2$, we can write this as

$$\frac{2s}{\alpha}\Phi' = \int (vRR' - 3\alpha R^3 R')dx = \frac{v}{2}R^2 - \frac{3\alpha}{4}R^4 + C_1$$

where C_1 is an integration constant since $\partial_x = \partial_z$. Letting $C = \alpha C_1$, we get that

$$\Phi' = \frac{C + vs - 3s^2}{2s}.$$

5.4 Part d

Plugging in our ansatz $B(x,t) = R(z)$, with $z = x - vt$ and constant v and $\theta(x,t) = \Phi(z) - \Omega t$ into the second equation, we get that

$$\alpha R^2 \Phi' - \frac{R''}{R} - v\Phi' - \Omega + \Phi'^2 = 0.$$

We let

$$\begin{aligned}
F(s) &= \alpha R^2 \Phi' - v\Phi' - \Omega + \Phi'^2 \\
&= 2s\frac{C + vs - 3s^2}{2s} - v\frac{C + vs - 3s^2}{2s} - \Omega + \left(\frac{C + vs - 3s^2}{2s}\right)^2 \\
&= \frac{C^2}{4s^2} - \left(\frac{C}{2} + \frac{v^2}{4} + \Omega\right) + vs - \frac{3s^2}{4}
\end{aligned}$$

where we have expanded terms in Mathematica. Note that $s' = \alpha R R'$. Multiplying through by s', we get that

$$\alpha R' R'' - F(s)s' = 0.$$

Integrating both sides, we get that

$$0 = \alpha R'^2 - \int F(s)s'dz = \frac{s'^2}{\alpha R^2} - \int F(s)s'dz = \frac{s'^2}{2s} - \int F(s)s'dz = 0.$$

Noting the chain rule, we can compute the integral in Mathematica as

$$\int F(s)s'dz = -\frac{C^2}{4s} - \left(\frac{C}{2} + \frac{v^2}{4} + \Omega\right)s + \frac{vs^2}{2} - \frac{s^3}{4} + C_1.$$

where C_1 is an integration constant. We can then get that

$$s'^2 + \frac{C^2}{2} + 2C_1 s + \left(C + \frac{v^2}{2} + 2\Omega\right)s^2 - vs^3 + \frac{s^4}{2} = 0$$

which we can write as

$$s'^2 + V(s) = E$$

where

$$V(s) = 2C_1 s + \left(C + \frac{v^2}{2} + 2\Omega\right)s^2 - vs^3 + \frac{s^4}{2}$$

and

$$E = -\frac{C^2}{2}.$$

6 Problem 6

Consider example 5.2 in the notes. To check that $y = x^2/t$ and $t^{1/2}q$ are both scaling invariant, we note that the NLS equation has scaling symmetry

$$x = \frac{\hat{x}}{a}, \quad t = \frac{\hat{t}}{a^2}, \quad q = a\hat{q},$$

so we simply see that

$$y = \frac{x^2}{t} = \frac{\hat{x}^2/a^2}{\hat{t}/a^2} = \frac{\hat{x}^2}{\hat{t}}$$

and

$$t^{1/2}q = \frac{\hat{t}^{1/2}}{a}a\hat{q} = \hat{t}^{1/2}\hat{q}.$$

Now we find the ordinary differential equation satisfied by $G(y)$, for similarity solutions of the form $q(x,t) = t^{-1/2}G(y)$ by computing

$$q_t = t^{-1/2}G'(y)\left(-\frac{x^2}{t^2}\right) - \tfrac{1}{2}t^{-3/2}G(y) = -x^2t^{-5/2}G'(y) - \tfrac{1}{2}t^{-3/2}G(y),$$

$$q_x = t^{-1/2}G'(y)\frac{2x}{t} = 2xt^{-3/2}G'(y),$$

and

$$q_{xx} = 2t^{-3/2}G'(y) + 4x^2t^{-5/2}G''(y).$$

Plugging this into the NLS equation,

$$i\left(-x^2t^{-5/2}G'(y) - \tfrac{1}{2}t^{-3/2}G(y)\right) = -\left(2t^{-3/2}G'(y) + 4x^2t^{-5/2}G''(y)\right) + \sigma t^{-3/2}|G(y)|^2|G(y)|.$$

Dividing through by $t^{-3/2}$,

$$i\left(-x^2t^{-1}G'(y) - \tfrac{1}{2}G(y)\right) = -\left(2G'(y) + 4x^2t^{-1}G''(y)\right) + \sigma|G(y)|^2|G(y)|$$

which simplifies to the ODE

$$i\left(-yG'(y) - \tfrac{1}{2}G(y)\right) = -2G'(y) - 4yG''(y) + \sigma|G(y)|^2|G(y)|.$$

To see that this result is in the same similarity solutions as the example, we note that $z = \sqrt{y}$, so we want to apply this change of variables, letting $\tilde{G}(z) = G(y)$. Then,

$$\tilde{G}'(z) = (G(z^2))' = 2zG'(z^2),$$

so

$$G'(z^2) = \frac{1}{2z}\tilde{G}'(z),$$

and

$$\tilde{G}''(z) = (G(z^2))'' = (2zG'(z^2))' = 2\frac{dz}{dy}G'(z^2) + 4z^2G''(z^2) = \frac{1}{z}G'(z^2) + 4z^2G''(z^2).$$

This gives us that

$$G''(z^2) = -\frac{1}{4z^3}\tilde{G}'(z) + \frac{1}{4z^2}\tilde{G}''(z).$$

Then, our ODE becomes

$$\begin{aligned} i\left(-z^2\frac{1}{2z}\tilde{G}'(z) - \tfrac{1}{2}\tilde{G}(z)\right) &= -2\frac{1}{2z}\tilde{G}'(z) - 4z^2\left(-\frac{1}{4z^3}\tilde{G}'(z) + \frac{1}{4z^2}\tilde{G}''(z)\right) \\ &\quad + \sigma|\tilde{G}(z)|^2|\tilde{G}(z)|. \end{aligned}$$

This reduces to

$$i\left(-\tfrac{1}{2}z\tilde{G}'(z) - \tfrac{1}{2}\tilde{G}(z)\right) = -\tilde{G}''(z) + \sigma|\tilde{G}(z)|^2|\tilde{G}(z)|$$

which is precisely the ODE that F in the notes satisfies, so clearly these are the same similarity solutions.

7 Problem 7

Consider the Toda lattice

$$\begin{aligned} \frac{da_n}{dt} &= a_n(b_{n+1} - b_n), \\ \frac{db_n}{dt} &= 2(a_n^2 - a_{n-1}^2) \end{aligned}$$

where a_n, b_n, $n \in \mathbb{Z}$, are functions of t.

7.1 Part a

To find a scaling symmetry, let $a_n = \alpha A_n$, $b_n = \beta B_n$, $t = \gamma\tau$. Then,

$$\frac{d}{dt} = \frac{\partial\tau}{\partial t} = \frac{1}{\gamma}\frac{d}{d\tau},$$

so the lattice becomes

$$\begin{aligned} \frac{1}{\gamma}\frac{d(\alpha A_n)}{d\tau} &= \alpha A_n(\beta B_{n+1} - \beta B_n), \\ \frac{1}{\gamma}\frac{d(\beta B_n)}{dt} &= 2(\alpha^2 A_n^2 - \alpha^2 A_{n-1}^2). \end{aligned}$$

In order for the system to be the same, we need that

$$\frac{\alpha}{\gamma\alpha\beta} = \frac{\beta}{\gamma\alpha^2} = 1.$$

This requires $\gamma = \frac{1}{\beta}$ which in turn requires $\alpha^2 = \beta^2$. For simplicity, we choose to require that $\alpha = \beta$. Then, our scaling symmetry is given by

$$a_n = \alpha A_n, \quad b_n = \alpha B_n, \quad t = \frac{\tau}{\alpha}.$$

7.2 Part b

As a similarity ansatz, take

$$x_n = ta_n \quad y_n = tb_n.$$

where x_n and y_n are constants. Then,

$$\frac{da_n}{dt} = \frac{d}{dt}\frac{x_n}{t} = -\frac{x_n}{t^2}, \quad \frac{db_n}{dt} = -\frac{y_n}{t^2}.$$

Then, the lattice becomes

$$\begin{aligned} -\frac{x_n}{t^2} &= \frac{x_n}{t}\left(\frac{y_{n+1}}{t} - \frac{y_n}{t}\right), \\ -\frac{y_n}{t^2} &= 2\left(\frac{x_n^2}{t^2} - \frac{x_{n-1}^2}{t^2}\right). \end{aligned}$$

This reduces to

$$\begin{aligned} -1 &= y_{n+1} - y_n, \\ -y_n &= 2\left(x_n^2 - x_{n-1}^2\right). \end{aligned}$$

If $n \geq 0$, the first equation can be solved inductively to get that

$$y_n = y_0 - n$$

which when plugged into the second equation gives

$$n - y_0 = 2x_n^2 - 2x_{n-1}^2,$$

so

$$x_n^2 = \frac{n - y_0}{2} + x_{n-1}^2 = x_0^2 + \tfrac{1}{2}\sum_{j=0}^{n} j - n\frac{y_0}{2} = x_0^2 + \frac{n(n+1)}{4} - \frac{ny_0}{2}.$$

In order for x_n to be real for all $n \geq 0$, we need that

$$x_0^2 + \frac{n(n+1)}{4} - \frac{ny_0}{2} \geq 0$$

for all n. To get a sufficient condition that does not depend on n, we note that $\frac{n(n+1)}{4} - \frac{ny_0}{2}$ is convex in n, so we can set its deriviative to zero to find a minimizer. This gives that $n^* = y_0 - \frac{1}{2}$ which allows us to bound

$$x_0^2 + \frac{n(n+1)}{4} - \frac{ny_0}{2} \geq x_0^2 - \frac{4y_0^2 - 4y_0 + 1}{16},$$

so we can get a bound if

$$x_0^2 \geq \left(\frac{2y_0 - 1}{4}\right)^2.$$

Note that this may not be tight since $y_0 - \frac{1}{2}$ is not necessarily a nonnegative integer. In this case, we could in principle instead consider $n^* = \max\{0, \text{round}(y_0 - \frac{1}{2})\}$.
If instead $n \leq 0$, we can inductively find that

$$y_n = y_0 + n$$

which instead gives that

$$x_n^2 = x_{n+1}^2 - \frac{y_0 + (n+1)}{2} = x_0^2 - \frac{ny_0}{2} + \frac{1}{2}\sum_{j=0}^{n}(j+1) = x_0^2 - \frac{(n+1)(n+2)}{2} - \frac{ny_0}{2},$$

so we need that

$$x_0^2 \geq \frac{(n+1)(n+2)}{2} + \frac{ny_0}{2}$$

for all $n \leq 0$. Again optimizing this over n, we get $n^* = -\frac{y_0+3}{2}$ which gives the requirement that

$$x_0^2 \geq -\frac{y_0^2 + 6y_0 + 1}{8}.$$

Again, this bound is not necessarily tight, and one should really consider $n^* = \min\{0, \text{round}\left(-\frac{y_0+3}{2}\right)\}$. These two bounds could in principle be compared and combined to get a tight bound, but regardless, the two together are sufficient to ensure that x_n, y_n are real for all $n \in \mathbb{Z}$.

8 Problem 8

Consider the equation

$$u_t = 30u^2 u_x + 20u_x u_{xx} + 10uu_{xxx} + u_{5x}.$$

To find its scaling symmetry, we set

$$x = \frac{\hat{x}}{a}, \quad t = \frac{\hat{t}}{b}, \quad u = c\hat{u}.$$

Plugging these in, our equation becomes

$$bc\hat{u}_{\hat{t}} = 30ac^3\hat{u}_{\hat{x}}^2 + 20a^3c^2\hat{u}_{\hat{x}}\hat{u}_{\hat{x}\hat{x}} + 10a^3c^2\hat{u}\hat{u}_{\hat{x}\hat{x}\hat{x}} + a^5c\hat{u}_{5\hat{x}}.$$

Dividing through by bc, in order to have the same equation, we need that

$$\frac{ac^2}{b} = \frac{a^3c}{b} + \frac{a^5}{b} = 1.$$

We get that $b = a^5$ which when plugged into either the first or second equation gives that $c = a^2$. Thus, a scaling symmetry for this equation exists and is given by the form

$$x = \frac{\hat{x}}{a}, \quad t = \frac{\hat{t}}{a^5}, \quad u = a^2\hat{u}.$$

9 Problem 9

Consider a Modified KdV equation

$$u_t - 6u^2 u_x + u_{xxx} = 0.$$

9.1 Part a

To find its scaling symmetry, we set

$$x = \frac{\hat{x}}{a}, \quad t = \frac{\hat{t}}{b}, \quad u = c\hat{u}.$$

Plugging these in, our equation becomes

$$bc\hat{u}_{\hat{t}} - 6ac^3\hat{u}^2\hat{u}_{\hat{x}} + a^3 c\hat{u}_{\hat{x}\hat{x}\hat{x}}.$$

Dividing through by bc, in order to have the same equation, we need that

$$\frac{ac^2}{b} = \frac{a^3}{b} = 0.$$

We get that $b = a^3$ which then gives that $c^2 = a^2$ which we simplify to $c = a$. Thus, the scaling symmetry is given by

$$x = \frac{\hat{x}}{a}, \quad t = \frac{\hat{t}}{a^3}, \quad u = a\hat{u}.$$

9.2 Part b

Based on this scaling symmetery, a similarity ansatz is given by

$$y = t^{-1/3}x, \quad F = t^{1/3}u$$

as both quantities are clearly scale invariant with our scaling symmetry.

9.3 Part c

To see that this ansatz is compatible with $u = (3t)^{-1/3}w(z)$ with $z = x/(3t)^{1/3}$, simply take $z = 3^{-1/3}y$, $w = 3^{-1/3}F$. Then, both z, w are clearly scale invariant since we only multiplied by a constant in both.

9.4 Part d

With z, w defined in this way, we compute

$$u_t = -(3t)^{-1/3}w'(z)x(3t)^{-4/3} - (3t)^{-4/3}w(z) = -(3t)^{-5/3}xw'(z) - (3t)^{-4/3}w(z),$$

$$u_x = (3t)^{-1/3}(3t)^{-1/3}w'(z) = (3t)^{-2/3}w'(z),$$

$$u_{xxx} = (3t)^{-4/3} w'''(z).$$

Then, the equation becomes

$$-(3t)^{-5/3} x w'(z) - (3t)^{-4/3} w(z) - 6(3t)^{-2/3} w^2(z)(3t)^{-2/3} w'(z) + (3t)^{-4/3} w'''(z) = 0.$$

Dividing through by by $(3t)^{-4/3}$, we get that

$$-(3t)^{-1/3} x w'(z) - w(z) - 6w^2(z)w'(z) + w'''(z) = 0$$

which is just

$$w'''(z) = zw'(z) + w(z) + 6w^2(z)w'(z).$$

Note that

$$zw'(z) + w(z) + 6w^2(z)w'(z) = (2w^3(z) + zw(z))_z,$$

so integrating both sides of our equation yields

$$w''(z) = 2w^3(z) + zw(z) + \alpha$$

where α is an integration constant; this is precisely the second Painlevé equation.