AMATH 567 Homework 8

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1 Problem 1 (3.6.7)

Consider

$$\pi \cot(\pi z) - \left(\frac{1}{z} + \sum_{j=-\infty}^{\infty}{}' \left(\frac{1}{z-j} + \frac{1}{j}\right)\right) = h(z).$$

1.1 Part a

To see that $h(z)$ is periodic of period 1, we show that the LHS is periodic of period 1. Clearly, $\pi \cot(\pi(z+1)) = \pi \cot(\pi z)$, because cot is π-periodic. Let $S(z)$ denote the term in parentheses. Then,

$$\begin{aligned}
S(z+1) &= \frac{1}{z+1} + \sum_{j=-\infty}^{\infty}{}' \left(\frac{1}{z+1-j} + \frac{1}{j}\right) \\
&= \frac{1}{z+1} + \left(\frac{1}{z} + 1\right) + \sum_{\substack{j=-\infty,\\ j\neq 1}}^{\infty}{}' \left(\frac{1}{z+1-j} + \frac{1}{j}\right) \\
&= \frac{1}{z} + \left(\frac{1}{z+1} + 1\right) + \sum_{\substack{j=-\infty,\\ j\neq -1}}^{\infty}{}' \left(\frac{1}{z-j} + \frac{1}{j+1}\right)
\end{aligned}$$

Now, consider the series

$$\begin{aligned}
\sum_{\substack{j=-\infty,\\ j\neq -1}}^{\infty}{}' \left(\frac{1}{j} - \frac{1}{j+1}\right) &= \sum_{j=-\infty}^{-2} \left(\frac{1}{j} - \frac{1}{j+1}\right) + \sum_{j=1}^{\infty} \left(\frac{1}{j} - \frac{1}{j+1}\right) \\
&= \lim_{N\to-\infty} \left(\frac{1}{N} - \frac{1}{-2+1}\right) + \lim_{N\to\infty} \left(\frac{1}{1} - \frac{1}{N+1}\right) \\
&= 1 + 1 = 2.
\end{aligned}$$

Note that this is uniformly convergent as a function of z, because it is constant as a function of z. Now,

$$\begin{aligned}
S(z+1) &= -2 + \frac{1}{z} + \left(\frac{1}{z+1} + 1\right) + \sum_{\substack{j=-\infty,\\ j\neq -1}}^{\infty}{}' \left(\frac{1}{z-j} + \frac{1}{j+1}\right) + \sum_{\substack{j=-\infty,\\ j\neq -1}}^{\infty}{}' \left(\frac{1}{j} - \frac{1}{j+1}\right) \\
&= \frac{1}{z} + \left(\frac{1}{z-(-1)} - 1\right) + \sum_{\substack{j=-\infty,\\ j\neq -1}}^{\infty}{}' \left(\frac{1}{z-j} + \frac{1}{j}\right) \\
&= \frac{1}{z} + \sum_{j=-\infty}^{\infty}{}' \left(\frac{1}{z-j} + \frac{1}{j}\right) = S(z)
\end{aligned}$$

where we have combined the series because they are both uniformly convergent and reinserted the $j = -1$ term. Thus, the entire LHS is periodic of period 1, so it must hold that $h(z)$ is periodic of period 1.

1.2 Part b

We first wish to show that as $y \to \pm\infty$, $\pi\cot(\pi z)$ is bounded for $x \in [0, 1]$ where $z = x + iy$. To see this, use the definition of the complex cotangent which gives that

$$\pi\cot(\pi z) = \pi i \frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}} = \pi i \frac{e^{i\pi x}e^{-\pi y} + e^{-i\pi x}e^{\pi y}}{e^{i\pi x}e^{-\pi y} - e^{-i\pi x}e^{\pi y}}.$$

Now, by the triangle inequality

$$|e^{i\pi x}e^{-\pi y} + e^{-i\pi x}e^{\pi y}| \leq |e^{i\pi x}e^{-\pi y}| + |e^{-i\pi x}e^{\pi y}| = e^{-\pi y} + e^{\pi y} = e^{-\pi|y|} + e^{\pi|y|}$$

where the last step follows by symmetry. Similarly, the reverse triangle inequality gives that

$$|e^{i\pi x}e^{-\pi y} - e^{-i\pi x}e^{\pi y}| \geq \left||e^{i\pi x}e^{-\pi y}| - |e^{-i\pi x}e^{\pi y}|\right| = |e^{-\pi y} - e^{\pi y}| = e^{\pi|y|} - e^{-\pi|y|}$$

where the last step follows from symmetry and the fact that $e^a \geq e^{-a}$ for $a \geq 0$. Then,

$$\begin{aligned}
|\pi\cot(\pi z)| = \left|\pi i \frac{e^{i\pi x}e^{-\pi y} + e^{-i\pi x}e^{\pi y}}{e^{i\pi x}e^{-\pi y} - e^{-i\pi x}e^{\pi y}}\right| &\leq \pi \frac{e^{-\pi|y|} + e^{\pi|y|}}{e^{\pi|y|} - e^{-\pi|y|}} \\
&= \pi \frac{e^{-\pi|y|} + e^{\pi|y|}}{e^{\pi|y|} - e^{-\pi|y|}} = \pi \frac{e^{-2\pi|y|} + 1}{1 - e^{-2\pi|y|}}
\end{aligned}$$

As $y \to \pm\infty$, this tends to π, because $e^a \to 0$ ad $a \to -\infty$. Thus, $\pi\cot\pi z$ is bounded as $y \to \pm\infty$.

Now, if we define $S(z)$ as in part a, we can observe that

$$\begin{aligned}
S(z) &= \frac{1}{z} + \sum_{j=-\infty}^{-1}\left(\frac{1}{z-j}+\frac{1}{j}\right) + \sum_{j=1}^{\infty}\left(\frac{1}{z-j}+\frac{1}{j}\right) \\
&= \frac{1}{z} + \sum_{j=\infty}^{1}\left(\frac{1}{z+j}+\frac{1}{-j}\right) + \sum_{j=1}^{\infty}\left(\frac{1}{z-j}+\frac{1}{j}\right) \\
&= \frac{1}{z} + \sum_{j=1}^{\infty}\left(\frac{1}{z+j}-\frac{1}{j}\right) + \sum_{j=1}^{\infty}\left(\frac{1}{z-j}+\frac{1}{j}\right) = \frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{z+n}+\frac{1}{z-n}\right) \\
&= \frac{1}{z} + \sum_{n=1}^{\infty}\frac{2n}{z^2-n^2}
\end{aligned}$$

where we have negated the dummy variable in the first summation and are able to combine the series due to unifrom convergence. Then, the triangle inequality and the inequality $|z^2-n^2| \geq \frac{1}{\sqrt{2}}(y^2+n^2)$ where $z = x+iy$ and $x \in [0,1]$, $|y| \geq 2$ give that

$$\begin{aligned}
|S(z)| &\leq \frac{1}{|z|} + \sum_{n=1}^{\infty}\frac{2|z|}{|z^2-n^2|} \leq \frac{1}{|z|} + \sum_{n=1}^{\infty}\frac{2\sqrt{2}y}{|z^2-n^2|} \leq \frac{1}{|z|} + y\sum_{n=1}^{\infty}\frac{2\sqrt{2}}{(y^2+n^2)/\sqrt{2}} \\
&= \frac{1}{|z|} + 4y\sum_{n=1}^{\infty}\frac{1}{y^2+n^2}.
\end{aligned}$$

Also,

$$y\sum_{n=1}^{\infty}\frac{1}{y^2+n^2} = y\sum_{n=1}^{\infty}\frac{1}{y^2(1+n^2/y^2)} = \frac{1}{y}\sum_{n=1}^{\infty}\frac{1}{1+(n/y)^2} \leq \frac{1}{y}\int_0^{\infty}\frac{1}{1+(r/y)^2}dr$$

where the last step follows from the integral test because our integrand is strictly decreasing and these are right Riemann sums. Thus, taking the u-substitution $u = r/y$,

$$y\sum_{n=1}^{\infty}\frac{1}{y^2+n^2} = \frac{1}{y}\int_0^{\infty}\frac{1}{1+u^2}y\,du = \int_0^{\infty}\frac{1}{1+u^2}du = [\arctan u]_0^{\infty} = \pi/2.$$

Thus, the reverse triangle inequality gives that

$$|S(z)| \leq \frac{1}{|z^2|} + 4\frac{\pi}{2} \leq \frac{1}{||x|-|y||} + 2\pi = \frac{1}{|y-x|} + 2\pi$$

for y sufficiently large. Clearly, this tends to 2π as $y \to \infty$, so $S(z)$ is bounded for $0 < x < 1$ and $y \to \infty$. The same argument works for $y \to -\infty$, so we have that $S(z)$ is bounded for $0 < x < 1$.
Now, note that from page 87 of the lecture notes we have that $\pi \cot \pi z$ has simple poles at $z_j = j$ for $j \in \mathbb{Z}$ and these poles have residue 1. Thus, its only

pole on the strip $0 \leq \Re(z) < 1$ is at $z = 0$. However, this cancels with the $1/z$ term from $S(z)$ when we consider the entire LHS. Thus, $h(z)$ is bounded and analytic in this strip, and the periodicity we derived in part a gives that it is bounded and analytic everywhere. Therefore, Liouville's theorem gives that $h(z)$ must be constant.
Now, in search of extra credit, we prove the bound stated above. Letting $n \in \mathbb{N}$ and $z = x + iy$,

$$\begin{aligned}
|z^2 - n^2|^2 &= |(x+iy)^2 - n^2|^2 = |(x^2 + 2ixy - y^2) - n^2|^2 = |(x^2 - y^2 - n^2) + i2xy|^2 \\
&= (x^2 - y^2 - n^2)^2 + 4x^2y^2 = (x^2 - y^2)^2 - 2n^2(x^2 - y^2) + n^4 + 4x^2y^2 \\
&= (x^2 + y^2)^2 - 2n^2(x^2 - y^2) + n^4 \geq y^4 - 2n^2x^2 + 2n^2y^2 + n^4 \\
&= -2n^2x^2 + (y^2 + n^2)^2 = -2n^2x^2 + \frac{1}{2}(y^2 + n^2)^2 + \frac{1}{2}(y^2 + n^2)^2.
\end{aligned}$$

Now, we use the fact that $x \in [0, 1]$ to get that $-2n^2x^2 \geq -2n^2$ which gives

$$\begin{aligned}
|z^2 - n^2|^2 &\geq -2n^2 + \frac{1}{2}y^4 + y^2n^2 + \frac{1}{2}n^4 + \frac{1}{2}(y^2 + n^2)^2 \\
&= n^2(y^2 - 2) + \frac{1}{2}(y^4 + n^4) + \frac{1}{2}(y^2 + n^2)^2 \\
&\geq n^2(y^2 - 2) + \frac{1}{2}(y^2 + n^2)^2 \geq \frac{1}{2}(y^2 + n^2)^2
\end{aligned}$$

where the last line follows because $|y| \geq 2$. Now, we simply take the square root of both sides to get that $|z^2 - n^2| \geq \frac{1}{\sqrt{2}}(y^2 + n^2)$, the desired inequality.

1.3 Part c

Now, to see that both terms on the LHS are odd in z, note that cotangent is an odd function, so $\pi \cot(\pi z)$ is also odd. Additionally,

$$S(-z) = \frac{1}{-z} + \sum_{n=1}^{\infty} \frac{2(-z)}{(-z)^2 - n^2} = -\frac{1}{z} - \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} = -S(z).$$

Therefore the LHS is odd in z, meaning that $h(z)$ is odd. However, we know that $h(z)$ is constant, so we can write $h(z) = c$. Then, $h(z) = -h(-z)$, so $c = -c$, meaning that $c = 0$ and $h(z) = 0$.

2 Problem 2 (3.3.1)

We know that

$$\frac{\sin \pi z}{\pi z} = \prod_{j=1}^{\infty} \left(1 - \frac{z^2}{j^2}\right).$$

2.1 Part a

First, the Taylor series for sine gives that

$$\frac{\sin \pi z}{\pi z} = \frac{1}{\pi z}\left(\pi z - \frac{(\pi z)^3}{3!} + \frac{(\pi z)^5}{5!} - \dots\right) = 1 - \frac{\pi^2}{3!}z^2 + \frac{\pi^4}{5!}z^4 - \dots$$

meaning that the coefficient for the z^2 term on the RHS should be $-\frac{\pi^2}{3!}$. Looking at the partial product, we know that $\prod_{j=1}^{N}\left(1 - \frac{z^2}{j^2}\right)$ has z^2 term $\sum_{j=1}^{N} -\frac{1}{j^2}$. Since the infinite product is just the limit of the partial product as $N \to \infty$, we get that the infinite product has z^2 coefficients

$$\lim_{N\to\infty} \sum_{j=1}^{N} -\frac{1}{j^2} = -\sum_{j=1}^{\infty} \frac{1}{j^2}.$$

Equating these coefficients and dividing by -1, we get that

$$\sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{3!} = \frac{\pi^2}{6}.$$

2.2 Part b

Now, we note that the LHS has z^4 coefficient $\frac{\pi^4}{5!}$. If we again consider the RHS as a partial product from $j = 1$ to N, we get that the z^4 coefficient is $\sum_{j\neq k} \frac{1}{j^2k^2}$ where j and k are taken from 1 to N. By the same argument as in part a when we take the limit as $N \to \infty$, we get that the z^4 coefficient is the same sum but with j and k taken from 1 to infinity. Manipulating this using our result from part a,

$$\sum_{j\neq k} \frac{1}{j^2k^2} = \frac{1}{2}\sum_{j=1}^{\infty}\sum_{\substack{k=1,\\ k\neq j}}^{\infty} \frac{1}{j^2k^2} = \frac{1}{2}\sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{1}{j^2k^2} - \frac{1}{j^4}\right) = \frac{1}{2}\sum_{j=1}^{\infty}\frac{1}{j^2}\left(\sum_{k=1}^{\infty}\frac{1}{k^2} - \frac{1}{j^2}\right)$$

$$= \frac{1}{2}\sum_{j=1}^{\infty}\frac{1}{j^2}\left(\frac{\pi^2}{6} - \frac{1}{j^2}\right) = \frac{1}{2}\frac{\pi^2}{6}\sum_{j=1}^{\infty}\frac{1}{j^2} - \frac{1}{2}\sum_{j=1}^{\infty}\frac{1}{j^4} = \frac{\pi^4}{72} - \frac{1}{2}\sum_{j=1}^{\infty}\frac{1}{j^4}$$

where we have split the sums because these are p-series which are absolutely convergent. Thus,

$$\sum_{j=1}^{\infty}\frac{1}{j^4} = 2\left(\frac{\pi^4}{72} - \frac{\pi^4}{5!}\right) = 2\left(\frac{\pi^2}{72} - \frac{\pi^4}{120}\right) = \frac{\pi^4}{90}.$$

3 Problem 3 (4.1.2)

3.1 Part a

Let

$$f(z) = \frac{z^2+1}{z^2-a^2}$$

where $a^2 < 1$.

3.1.1 Part i

Assuming $a \neq 0$, the residue theorem gives that

$$\frac{1}{2\pi i}\oint_C f(z)dz = \operatorname*{Res}_{z=-a}\frac{z^2+1}{(z-a)(z+a)} + \operatorname*{Res}_{z=a}\frac{z^2+1}{(z-a)(z+a)} = \frac{a^2+1}{-2a} + \frac{a^2+1}{2a} = 0.$$

In the case where $a = 0$,

$$\frac{1}{2\pi i}\oint_C f(z)dz = \operatorname*{Res}_{z=0}\frac{z^2+1}{z^2} = \operatorname*{Res}_{z=0}\left(1+\frac{1}{z^2}\right) = 0,$$

the same solution.

3.1.2 Part ii

Enclosing the singular points outside C, the residue theorem gives that

$$\frac{1}{2\pi i}\oint_C f(z)dz = \operatorname*{Res}_{z=\infty} f(z) = \operatorname*{Res}_{t=0}\frac{1}{t^2}\frac{1/t^2+1}{1/t^2-a^2} = 0.$$

This is because the function that we are taking the residue of is even as a function of t, meaning that its Laurent series cannot have coefficients of 0 for all its odd terms because this would require that $c_j = -c_j$ for all odd $j \in \mathbb{Z}$. Thus, $c_{-1} = 0$.
Clearly, we have obtained the same result with both methods.

3.2 Part b

Let

$$f(z) = \frac{z^2+1}{z^3}.$$

3.2.1 Part i

By the residue theorem,

$$\frac{1}{2\pi i}\oint_C f(z)dz = \operatorname*{Res}_{z=0}\frac{z^2+1}{z^3} = \operatorname*{Res}_{z=0}\left(\frac{1}{z^3}+\frac{1}{z}\right) = 1.$$

3.2.2 Part ii

Enclosing the singular points outside C, the residue theorem gives that

$$\frac{1}{2\pi i}\oint_C f(z)dz = \operatorname*{Res}_{z=\infty} f(z) = \operatorname*{Res}_{t=0}\frac{1/t^2+1}{1/t^3}\frac{1}{t^2} = \operatorname*{Res}_{t=0}\frac{1/t^2+1}{1/t} = \operatorname*{Res}_{t=0}\left(\frac{1}{t}+t\right) = 1.$$

Clearly, we have obtained the same result with both methods.

3.3 Part c

Let $f(z) = z^2 e^{-1/z}$.

3.3.1 Part i

By the residue theorem,

$$\frac{1}{2\pi i}\oint_C f(z)dz = \operatorname*{Res}_{z=0} z^2 e^{-1/z} = \operatorname*{Res}_{z=0} z^2 \sum_{j=0}^{\infty} \frac{(-1/z)^j}{j!} = \operatorname*{Res}_{z=0} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} z^{2-j} = -\frac{1}{6}.$$

3.3.2 Part ii

Enclosing the singular points outside C, the residue theorem gives that

$$\begin{aligned}\frac{1}{2\pi i}\oint_C f(z)dz &= \operatorname*{Res}_{z=\infty} f(z) = \operatorname*{Res}_{z=0} \frac{1}{t^2}\frac{1}{t^2}e^{-t} = \operatorname*{Res}_{z=0} \frac{1}{t^4}\sum_{j=0}^{\infty}\frac{(-t)^j}{j!} \\ &= \operatorname*{Res}_{z=0} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} t^{j-4} = -\frac{1}{6}.\end{aligned}$$

Clearly, we have obtained the same result with both methods.