Math 525 | Winter 2024
Homework 4 | **Due Friday, February 2**

1. Prove that a barrel in a Banach space is the closure of its interior.

2. Let X be a normed vector space, and M a vector subspace of X considered as a normed vector space itself. Let $M^\perp \subset X^*$ be the set of all $f \in X^*$ such that $M \subset \ker(f)$; that is, $f \in M^\perp$ iff $f|_M = 0$. It is easy to see that $M^\perp$ is closed in X^*.

(a.) If M is closed, show that $x \in M$ iff $f(x) = 0$ for all $f \in M^\perp$.

(b.) Show that there is a natural equivalence $M^* \equiv X^*/M^\perp$ which agrees in the norm.

3. Show that if X is a reflexive Banach space, and M is a closed subspace of X, then M is reflexive.

4. Assume that E, F are closed subspaces of a Banach space X such that $E \cap F = \{0\}$, and $X = \mathrm{span}(E \cup F)$. (Such spaces are called closed complements of each other.) Show that the map $(v, w) \to v + w$ is a homeomorphism of $E \times F$ onto X. Conclude that for a closed subspace E of a Banach space there is a continuous projection map $X \to E$ iff E has a closed complement.

5. (a.) Let X be a normed vector space over $\mathbb{C}$, and let $f \in X^*$ with $\|f\|_{X^*} = 1$. Show that the map $f : X \to \mathbb{C}$ induces a norm preserving map of the quotient space $X/\ker(f)$ onto $\mathbb{C}$.

(b.) If X is reflexive, show that there is $x \in X$ with $\|x\| = 1$ such that $\inf_{v \in \ker(f)} \|x - v\| = 1$.