AMATH 515 Homework 4

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1 Problem 1

Let $\alpha \in \mathbb{R}$. Then,

$$
\begin{aligned}
&\|\alpha x+(1-\alpha)y\|^2+\alpha(1-\alpha)\|x-y\|^2\\
&=\|\alpha x\|^2+2\langle \alpha x,(1-\alpha)y\rangle+\|(1-\alpha)y\|^2+\alpha(1-\alpha)\left(\|x\|^2-2\langle x,y\rangle+\|y\|^2\right)\\
&=\alpha^2\|x\|^2+\cancel{2\alpha(1-\alpha)\langle x,y\rangle}+(1-\alpha)^2\|y\|^2+\alpha(1-\alpha)\|x\|^2-\cancel{2\alpha(1-\alpha)\langle x,y\rangle}+\alpha(1-\alpha)\|y\|^2\\
&=\alpha(\alpha+(1-\alpha))\|x\|^2+(1-\alpha)((1-\alpha)+\alpha)\|y\|^2=\alpha\|x\|^2+(1-\alpha)\|y\|^2.
\end{aligned}
$$

2 Problem 2

Let

$$T_\lambda=(1-\lambda)I+\lambda T$$

where T is nonexpansive.

2.1 Part a

To see that T_λ and T have the same fixed points, consider an x such that $x=Tx$ (a fixed point of T). Then,

$$T_\lambda x=((1-\lambda)I+\lambda T)x=(1-\lambda)x+\lambda Tx=(1-\lambda)x+\lambda x=x,$$

so x is a fixed point of T_λ. If x is instead a fixed point of T_λ, then $T_\lambda x=x$, so

$$x=T_\lambda x=((1-\lambda)I+\lambda T)x=(1-\lambda)x+\lambda Tx.$$

Then,

$$\lambda Tx=x-(1-\lambda)x=\lambda x,$$

so $Tx=x$, meaning that x is also a fixed point of T. Thus, T and T_λ have the same fixed points.

2.2 Part b

Let $\overline{z}$ be a fixed point of T. Then, $\overline{z} = T\overline{z}$, so

$$\|T_\lambda z - \overline{z}\|^2 = \|(1-\lambda)z + \lambda Tz - \lambda\overline{z} - (1-\lambda)\overline{z}\|^2 = \|\lambda(Tz - T\bar{z}) + (1-\lambda)(z-\bar{z})\|^2.$$

Now, we apply problem 1 and use the fact that T is nonexpansive to get that

$$\begin{aligned}
\|T_\lambda z - \overline{z}\|^2 &= \lambda\|Tz - T\bar{z}\|^2 + (1-\lambda)\|z-\bar{z}\|^2 - \lambda(1-\lambda)\|(Tz - T\bar{z}) - (z-\bar{z})\|^2 \\
&\leq \lambda\|z-\bar{z}\|^2 + (1-\lambda)\|z-\bar{z}\|^2 - \lambda(1-\lambda)\|Tz - \underbrace{T\bar{z}}_{=\bar{z}} - z + \bar{z}\|^2 \\
&= \|z-\bar{z}\|^2 - \lambda(1-\lambda)\|Tz - z\|^2 = \|z-\bar{z}\|^2 - \lambda(1-\lambda)\|z - Tz\|^2.
\end{aligned}$$

3 Problem 3

3.1 Part a

Let T be a firmly nonexpansive operator. Then,

$$\|Tx - Ty\|^2 + \|(I-T)x - (I-T)y\|^2 \leq \|x-y\|^2.$$

From this,

$$\begin{aligned}
\text{LHS} &= \|Tx - Ty\|^2 + \|(x-y) - (Tx - Ty)\|^2 \\
&= \|Tx - Ty\|^2 + \|x-y\|^2 - 2\langle x-y, Tx-Ty\rangle + \|Tx-Ty\|^2 \\
&= 2\|Tx - Ty\|^2 + \|x-y\|^2 - 2\langle x-y, Tx-Ty\rangle.
\end{aligned}$$

Thus, by definition, T is firmly nonexpansive iff

$$2\|Tx - Ty\|^2 + \|x-y\|^2 - 2\langle x-y, Tx-Ty\rangle \leq \|x-y\|^2.$$

Of course, we can just move terms around to see that this is true iff,

$$\langle x-y, Tx-Ty\rangle \geq \|Tx-Ty\|^2,$$

so T is firmly nonexpansive iff this holds.

3.2 Part b

Using the result of part a, we can write the norm as an inner product to get that T is firmly nonexpansive iff

$$\langle x-y, Tx-Ty\rangle - \langle Tx-Ty, Tx-Ty\rangle \geq 0.$$

Now, note that

$$\begin{aligned}
&\langle x-y, Tx-Ty\rangle - \langle Tx-Ty, Tx-Ty\rangle \\
&= \langle (x-y) - (Tx-Ty), Tx-Ty\rangle = \langle (I-T)x - (I-T)y, Tx-Ty\rangle.
\end{aligned}$$

Thus, T is firmly nonexpansive iff

$$\langle Tx-Ty, (I-T)x - (I-T)y\rangle \geq 0.$$

3.3 Part c

Now, suppose that $S = 2T - I$. Let

$$\mu = \|Tx - Ty\|^2 + \|(I-T)x - (I-T)y\|^2 - \|x-y\|^2$$

and let

$$\nu = \|Sx - Sy\|^2 - \|x-y\|^2.$$

Then,

$$\begin{aligned}
\nu &= \|(2T-I)x - (2T-I)y\|^2 - \|x-y\|^2 \\
&= \|(T-I)x - (T-I)y + (Tx - Ty)\|^2 + \|x-y\|^2 - 2\|x-y\|^2 \\
&= \|(T-I)x - (T-I)y + (Tx - Ty)\|^2 \\
&+ \|(T-I)x - (T-I)y - (Tx-Ty)\|^2 - 2\|x-y\|^2,
\end{aligned}$$

so

$$\begin{aligned}
\frac{1}{4}\nu = \|\frac{1}{2}((T-I)x - (T-I)y) + \frac{1}{2}(Tx - Ty)\|^2 \\
+ \frac{1}{4}\|(T-I)x - (T-I)y - (Tx-Ty)\|^2 - \frac{1}{2}\|x-y\|^2.
\end{aligned}$$

Applying problem 1 with $\alpha = \frac{1}{2}$ to the first two terms, we get that

$$\frac{1}{4}\nu = \frac{1}{2}\|(T-I)x - (T-I)y\|^2 + \frac{1}{2}\|Tx - Ty\|^2 - \frac{1}{2}\|x-y\|^2,$$

so

$$\nu = 2\|(T-I)x - (T-I)y\|^2 + 2\|Tx - Ty\|^2 - 2\|x-y\|^2 = 2\mu.$$

We know from the definition that T is firmly nonexpansive iff $\mu \leq 0$ which we now know is true iff $\nu \leq 0$. Of course, this is true iff S is nonexpansive.