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AMATH 561 Problem Set 4 -

1\. a. $\sigma(X) = \sigma(\{a,b\},\{c,d\}) = \{\emptyset, \Omega, \{a,b\}, \{c,d\}\}$ because $X(a)=X(b)=1$ means that $a,b \in X^{-1}(B)\ \forall\ B \in \mathcal{B}$ containing 1 and $X(c)=X(d)=-1 \Rightarrow c,d \in X^{-1}(B)\ \forall\ B \in \mathcal{B}$ containing -1.

b. Partition Ω into $\Omega_1 = \{a,b\}$ and $\Omega_2 = \{c,d\}$ and note that $E[Y|X] = E[Y|\sigma(X)]$. Then, on Ω_1,

$$E[Y|X] = \frac{E[Y;\Omega_1]}{P(\Omega_1)} = \frac{\frac{1}{6}\cdot 1 + \frac{1}{3}(-1)}{\frac{1}{6}+\frac{1}{3}} = \frac{-1}{3}$$

and on Ω_2,

$$E[Y|X] = \frac{E[Y;\Omega_2]}{P(\Omega_2)} = \frac{\frac{1}{4}\cdot 1 + \frac{1}{4}(-1)}{\frac{1}{4}+\frac{1}{4}} = 0.$$

Thus, $E[Y|X](\omega) = \begin{cases} -1/3 & \text{if } \omega = a,b \\ 0 & \text{if } \omega = c,d. \end{cases}$

c. Using the linearity of conditional expectation,
$E[Z|X] = E[X+Y|X] = E[X|X] + E[Y|X] = X + E[Y|X]$
(because X is $\sigma(X)$-measurable). Thus,
$E[Z|X] = 1 + \left(\frac{-1}{3}\right) = \frac{2}{3}$ on Ω_1 and
$E[Z|X] = -1 + 0 = -1$ on Ω_2. Thus,

$$E[Z|X](\omega) = \begin{cases} 2/3 & \text{if } \omega = a,b \\ -1 & \text{if } \omega = c,d \end{cases}$$

2\. a. Let our probability space be represented by $(\Omega, \mathcal{F}_0, P)$. The definition of conditional expectation gives that
$\int_A X\,dP = \int_A E[X|\mathcal{F}]\,dP$ for any $A \in \mathcal{F} \subset \mathcal{F}_0$ where $\mathcal{F}$ is a sigma algebra.
However, we know that $\Omega \in \mathcal{F}$ if $\mathcal{F}$ is a σ-algebra, so $\int_\Omega X\,dP = \int_\Omega E[X|\mathcal{F}]\,dP$ if we take $A = \Omega$.
This is precisely the definition of the expectation of a random value, namely,
$E[X] = \int_\Omega X\,dP$ and $E[E[X|\mathcal{F}]] = \int_\Omega E[X|\mathcal{F}]\,dP$.
Thus, $E[X] = E[E[X|\mathcal{F}]]$.

b. Now, consider a σ-algebra $G \subset F$ and let $E[X^2] < \infty$. Also, assume that $E|X\,E[X|F]| < \infty$ and $E|X\,E[X|G]| < \infty$ (Ivana said this is fine to assume, but it can be shown from the assumptions we already have.) Then, $E[(X - E[X|F])^2] + E[(E[X|F] - E[X|G])^2]$
$= E[X^2] - 2E[X\,E[X|F]] + E[(E[X|F])^2] + E[(E[X|F])^2] - 2E[E[X|F]\,E[X|G]]$
$+ E[(E[X|G])^2]$ (by the linearity of expectation). Now, note that $E[X|F]$ is F-measurable by definition and that $E[X|G]$ is G-measurable by definition (and also F-measurable because $G \subset F$), so we invoke the theorem on slide 7 of lecture 10 to get that this expression is equal to (using the assumptions above) $E[X^2] - 2E[X\,E[X|F]] + 2E[E[X\,E[X|F]\,|\,F]]$
$-2E[E[X\,E[X|G]\,|\,F]] + E[(E[X|G])^2] = E[X^2] - 2E[X\,E[X|F]]$
$+ 2E[X\,E[X|F]] - 2E[X\,E[X|G]] + E[(E[X|G])^2]$
(by part a) $= E[X^2 - 2X\,E[X|F] + (E[X|G])^2]$ (by the linearity of expectation) $= E[(X - E[X|G])^2]$.

3. Consider a probability space (Ω, F_0, P) and let $F \subset F_0$ be a σ-algebra. Defining $var(X|F) = E[X^2|F] - (E[X|F])^2$,
$E[var(X|F)] + var(E[X|F]) = E[E[X^2|F] - (E[X|F])^2] + E[(E[X|F])^2] - (E[E[X|F]])^2$
(by the definition of variance) $= E[E[X^2|F]] - E[(E[X|F])^2] + E[(E[X|F])^2]$
$-(E[E[X|F]])^2$ (by the linearity of expectation)
$= E[X^2] - (E[X])^2$ (by problem 2 part a) $= var(X)$ (by the definition of variance).

4. Given $Y_1, Y_2, \ldots$ iid r.v.s with mean μ and variance σ^2, independent from N, a positive integer-valued r.v. with $E[N^2] < \infty$, and $X = Y_1 + \ldots + Y_N$, to evaluate $var(X)$, we first consider two lemmas.

Lemma 1 - If X and Y are independent r.v.s, then $var(X|Y) = var(X)$.

proof - $var(X|Y) = E[X^2|Y] - (E[X|Y])^2 = E[X^2] - (E[X])^2 = var(X)$, because X and X^2 are independent from Y.

Lemma 2 - If X, Y, and Z are independent random variables (not pairwise), then $var(X+Y|Z) = var(X|Z) + var(Y|Z) = var(X) + var(Y)$.

proof - $var(X+Y|Z) = E[(X+Y)^2|Z] - (E[X+Y|Z])^2$
$= E[X^2 + 2XY + Y^2|Z] - (E[X|Z] + E[Y|Z])^2$ (linearity of expectation)

$= E[X^2|Z] + 2E[XY|Z] + E[Y^2|Z] - (E[X|Z])^2 - 2E[X|Z]E[Y|Z] - (E[Y|Z])^2$
(by the linearity of expectation) $= var(X|Z) + var(Y|Z) + 2E[XY] - 2E[X]E[Y]$
(because XY, X, and Y are independent from Z)
$= var(X|Z) + var(Y|Z) + 2E[X]E[Y] - 2E[X]E[Y]$ (X and Y independent)
$= var(X|Z) + var(Y|Z) = var X + var Y$ by lemma 1.
Now, we use these lemmas to compute
$var(X) = E[var(X|N)] + var(E[X|N])$ (by problem 3)
$= E[var(Y_1 + \ldots + Y_N | N)] + var(E[Y_1 + \ldots + Y_N | N])$ (by definition of X)
$= E[var(Y_1|N) + \ldots + var(Y_N|N)] + var(E[Y_1|N] + \ldots + E[Y_N|N])$
(by lemma 2 applied $N-1$ times and the linearity of conditional expectation) $= E[var(Y_1) + \ldots + var(Y_N)] + var(E[Y_1] + \ldots + E[Y_N])$
(by lemma 1 and the fact that Y_i and N are independent $\forall i$)
$= E[N\sigma^2] + var(N\mu) = \sigma^2 E[N] + \mu^2 var(N)$ by the properties of expectation and variance given that μ and σ are constants.