MATH 524 Homework 7

Cade Ballew #2120804

December 8, 2023

1 Problem 1 (Folland Problem 23)

Define the following variant of the Hardy-Littlewood maximal function:

$$H^*f(x) = \sup\left\{\frac{1}{m(B)}\int_B |f(y)|\mathrm{d}y : B \text{ is a ball and } x \in B\right\}.$$

It is easy to see that $Hf \leq H^*f$, as any ball centered at x is a ball containing x, so clearly,

$$\sup_{r>0}\left\{\frac{1}{m(B(x,r))}\int_{B(x,r)} |f(y)|\mathrm{d}y\right\} \leq \sup\left\{\frac{1}{m(B)}\int_B |f(y)|\mathrm{d}y : B \text{ is a ball and } x \in B\right\}.$$

On the other hand, note that if for a ball B, if $x \in B$ and B has radius r, then $B \subset B(x, 2r)$. Also note that $M(B) = c_n r^n$ and $m(B(x, 2r)) = c_n(2r)^n$, so $m(B) = \frac{m(B(x,2r))}{2^n}$. Then,

$$\frac{1}{m(B)}\int_B |f(y)|\mathrm{d}y \leq \frac{1}{m(B)}\int_{B(x,2r)} |f(y)|\mathrm{d}y = 2^n\frac{1}{m(B(x,2r))}\int_{B(x,2r)} |f(y)|\mathrm{d}y.$$

Since this holds for all balls B contianing x, this demonstrates that $H^*f \leq 2^n Hf$.

2 Problem 2 (Folland Problem 26)

Let λ and μ be positive, mutually singular Borel measures on $\mathbb{R}^n$ and $\lambda + \mu$ be regular. By definition, there exist $L, M \in \mathcal{B}_{\mathbb{R}^n}$ such that $\mathbb{R}^n = L \sqcup M$ and λ is null on L and μ is null on M. To see that λ and μ are both regular, we first note that because $\lambda + \mu$ is regular, for any compact $K \in \mathcal{B}_{\mathbb{R}^n}$,

$$\lambda(K) + \mu(K) = (\lambda + \mu)(K) < \infty.$$

Because both λ and μ are measures,

$$\lambda(K) \leq (\lambda + \mu)(K) < \infty.$$

Likewise, $\mu(K) < \infty$.

To see that λ and μ satisfy outer regularity, fix $\epsilon > 0$. Then, because $\lambda + \mu$ is regular, there exists some open set U such that $E \subset U$ and

$$(\lambda + \mu)(U) \leq (\lambda + \mu)(E) + \epsilon.$$

Because $\mathbb{R}^n$ is σ-compact, it suffices to consider the case where E and U are bounded sets. That is, all quantities in this expression are finite. Expanding this,

$$\lambda(U \cap M) + \mu(U \cap L) \leq \lambda(E \cap M) + \mu(E \cap L) + \epsilon.$$

Rearranging,

$$\lambda(U \cap M) \leq \lambda(E \cap M) - \mu\left((U \setminus E) \cap L\right) + \epsilon \leq \lambda(E \cap M) + \epsilon,$$

since $E \subset U$. Because λ is null on L, this gives that

$$\lambda(U) \leq \lambda(E) + \epsilon.$$

Since we can find such a $U \supset E$ for any $\epsilon > 0$, it must follow that λ is outer regular. Since the argument is symmetric when interchanging λ and μ and L and M, μ must be outer regular as well. Thus, both measures are regular.

3 Problem 3 (Folland Problem 28)

Let $F \in NBV$ and $G(x) = |\mu_F|((-\infty, x])$. We prove that $|\mu_F| = \mu_{T_F}$ by showing that $G = T_F$ through the following steps.

3.1 Part a

By definition,

$$T_F(x) = \sup \left\{ \sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| \ : \ n \in \mathbb{N}, \ -\infty < x_0 < \ldots < x_n = x \right\}.$$

For any partition $x_0, \ldots, x_n$ as in the definition of T_F, by Proposition 3.13a,

$$\sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| = \sum_{j=1}^{n} |\mu_F((x_{j-1}, x_j])| \leq \sum_{j=1}^{n} |\mu_F|((x_{j-1}, x_j]) = |\mu_F|((x_0, x]) \leq G(x).$$

Thus, $T_f \leq G$.

3.2 Part b

Let E be an interval with endpoints at $a, b \in \overline{\mathbb{R}}$ with $a < b$. Then, noting that $x_0 = a, x_1 = b$ is a partition of E,

$$\begin{aligned} |\mu_F(E)| = |F(b) - F(a)| &\leq \sup \left\{ \sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| \ : \ n \in \mathbb{N}, \ a = x_0 < \ldots < x_n = b \right\} \\ &= T_F(b) - T_F(a) = \mu_{T_F}(E). \end{aligned}$$

A trivial subtlety here is that limits are needed if $a, b = \pm\infty$. Since this inequality holds for all intervals, it must also hold for all Borel sets since all quantities involved are finite.

3.3 Part c

By Exercise 21, and part b, we have that for any $E \in \mathcal{B}_{\mathbb{R}}$,

$$\begin{aligned} |\mu_F|(E) &= \sup \left\{ \sum_{1}^{n} |\mu_F(E_j)| : n \in \mathbb{N}, E_1, \ldots, E_n \text{ disjoint}, \ E = \bigsqcup_{1}^{n} E_j \right\} \\ &\leq \sup \left\{ \sum_{1}^{n} |\mu_{T_F}(E_j)| : n \in \mathbb{N}, E_1, \ldots, E_n \text{ disjoint}, \ E = \bigsqcup_{1}^{n} E_j \right\} = \mu_{T_F}(E). \end{aligned}$$

Thus, $|\mu_F| \leq \mu_{T_F}$, and hence $G \leq T_F$.

4 Problem 4 (Folland Problem 31)

Let $F(x) = x^2 \sin(x^{-1})$ and $G(x) = x^2 \sin(x^{-2})$ for $x \neq 0$ and $F(0) = G(0) = 0$.

4.1 Part a

We see that $F(x)$ and $G(x)$ are differentiable for $x \neq 0$ by computing

$$F'(x) = 2x \sin(x^{-1}) - \cos(x^{-1}), \quad G'(x) = 2x \sin(x^{-2}) - \frac{2}{x} \cos(x^{-2}),$$

which are clearly defined for $x \neq 0$. We also compute

$$F'(0) = \lim_{h \to 0} \frac{F(h) - F(0)}{h} = h \sin(h^{-1}) = 0,$$

by the squeeze theorem since $-1 \leq \sin(h^{-1}) \leq 1$. Similarly,

$$G'(0) = \lim_{h \to 0} \frac{G(h) - G(0)}{h} = h \sin(h^{-2}) = 0.$$

Thus, F and G are both differentiable everywhere.

4.2 Part b

To see that F is of bounded variation, note that for $x \in [-1, 1]$,

$$|F'(x)| \leq 2|x| \left|\sin(x^{-1})\right| + \left|\cos(x^{-1})\right| \leq 2 + 1 = 3.$$

Thus, by Example 3.25c, since F is differentiable on $\mathbb{R}$ and F' is bounded on $[-1, 1]$, $F \in BV([-1, 1])$.

On the other hand, consider the partition

$$x_0 = -1,\ x_j = -\sqrt{\frac{2}{j\pi}},\ x_n = 1,$$

defined for $j = 1, \ldots, n-1$. For simplicity, let $n \geq 3$. Then,

$$\sum_{j=1}^{n} |G(x_j) - G(x_{j-1})| \geq \sum_{j=2}^{n-1} \left| -\frac{2}{j\pi} \sin\left(\frac{j\pi}{2}\right) + \frac{2}{(j-1)\pi} \left(\frac{(j-1)\pi}{2}\right) \right| \geq \sum_{j=2}^{n-1} \frac{2}{j\pi}.$$

Since

$$\lim_{n \to \infty} \sum_{j=2}^{n-1} \frac{2}{j\pi} = \infty,$$

by definition, $T_G(1) - T_G(-1) = \infty$ and $G \notin BV([-1, 1])$.

5 Problem 5 (Folland Problem 33)

Let F be increasing on $\mathbb{R}$. Define

$$\hat{F}(x) = \begin{cases} F(x), & x \leq b, \\ F(b), & x > b, \end{cases}$$

and let $G(x) = \hat{F}(x+)$, noting that G is increasing and right-continuous. By Theorem 3.23, $\hat{F}$ and G are differentiable almost everywhere and $\hat{F}' = G'$ almost everywhere. This implies that on the interval $(a, b]$, F and G are differentiable almost everywhere and $F' = G'$ almost everywhere. Furthermore, Theorem 3.22 gives that $\mu_G = G'\mathrm{d}m$. Thus, by construction,

$$F(b) - F(a) \geq G(b) - G(a) = \mu_G\left((a, b]\right) = \int_{(a,b]} G'\mathrm{d}m = \int_a^b G'(t)\mathrm{d}t = \int_a^b F'(t)\mathrm{d}t,$$

as desired.