MATH 525 Homework 3

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1 Problem 1

1.1 Part a

Let $W_1 \subsetneq W_2$ be subspaces of a normed vector space X with W_1 finite-dimensional. Let $v' \in W_2 \setminus W_1$ and let $v_0 \in W_1$ be the closest element to v'. Let $c = \|v' - v_0\| > 0$. Then, $c = \inf_{w_1 \in W_1} \|v' - w\|$. The infimum is achieved because W_1 is finite-dimensional. Let $v = v' - v_0$. Then, $\|v\| = c$ and

$$\inf_{w \in W_1} \|v - w\| = \inf_{w \in W_1} \|v' - (v_0 + w)\| = \inf_{w_1 \in W_1} \|v' - w_1\| = c,$$

since $v_0 + w \in W_1$. This means that there exists some element $v \in W_2$ such that $\|v\| = c > 0$ and $\inf_{w \in W_1} \|v - w\| = c$. By rescaling $v \to \frac{v}{c}$, this means that there exists some $v \in W_2$ such that $\|v\| = 1$ and $\inf_{w \in W_1} \|v - w\| = 1$, since $\frac{w}{c} \in W_1$ if $w \in W_1$.

1.2 Part b

Let V be an infinite-dimensional normed vector space. For all $j \in \mathbb{N}$, let $V_j \subset V$ be a j-dimensional subspace such that $V_j \subset V_{j+1}$ for all j with the convention that $V_0 = \{0\}$. For all j, let $v_j \in V_{j+1}$ satisfy $\|v_j\| = 1$ and $\inf_{w \in V_j} \|v_j - w\| = 1$. Part a guarantees the existence of such a v_j for all j. Then, the sequence $\{v_n\}_{n=1}^{\infty}$ satisfies $\|v_n\| = 1$ for all n. Furthermore, if $n \neq m$ and we take $n > m$ without loss of generality, $v_m \in V_n$, so

$$\|v_n - v_m\| \geq \inf_{w \in V_n} \|v_n - w\| = 1.$$

Thus, we have produced a sequence in the set $A = \{x : \|x\| \leq 1\}$ that does not converge to an element of A, so A is not closed and therefore not compact. From Homework 2, we have that in finite dimensions, the set A is compact[1]. Thus, the set $\{x : \|x\| \leq 1\}$ is compact if and only if V is finite-dimensional.

2 Problem 2

Let M be a finite-dimensional subspace of a normed vector space $(X, \|\cdot\|)$. Let $e_1, \dots, e_n$ denote a basis for M and $e_1^*, \dots, e_n^*$ denote its corresponding dual basis. That is, if $x \in M$ is represented as $x = \sum_{k=1}^n a_k e_k$, then $e_j^*(x) = a_j$ for all $j = 1, \dots, n$. For each j, let $C_j = \|e_j^*\|_{M^*}$ and $p_j(x) = C_j \|x\|$ for any $x \in X$. Because each C_j is a positive constant, each p_j is clearly a norm on X since scaling by a positive constant does not impact any of the axioms that a norm must satisfy. Therefore, each p_j is a sublinear functional, so Hahn-Banach implies that each e_j^* can be extended to a continuous linear functional f_j on X such that $f_j(x) = e_j^*(x)$ for all $x \in M$. From this, consider the map T defined by

$$Tx = \sum_{j=1}^n e_j f_j(x), \quad x \in X.$$

If $x \in M$ is represented as $x = \sum_{k=1}^n a_k e_k$, then

$$Tx = \sum_{j=1}^n e_j f_j \left(\sum_{k=1}^n a_k e_k \right) = \sum_{j=1}^n e_j a_j = x.$$

[1]This is from Problem 5 as we showed that A is compact in the 1-norm and that all norms are equivalent in finite-dimensions.

Because T is a linear combination of continuous linear functions and elements of M, it is a continuous map from X to M such that $Tx = x$ for all $x \in M$.

3 Problem 3

Let X denote the vector space of bounded sequences $\{x_n\}_{n=1}^{\infty}$ with $x_n \in \mathbb{C}$ such that $\sup_n |x_n| < \infty$ where $c\{x_n\} + \{y_n\} = \{cx_n + y_n\}$. Let M denote the set of sequences $\{x_n\} \in X$ such that $\lim_{n\to\infty} x_n$ exists. To see that this is a subspace of X, let $\{x_n\}, \{y_n\} \in X$ and $c \in \mathbb{C}$. Then,

$$\lim_{n\to\infty}(c\{x_n\} + \{y_n\}) = \lim_{n\to\infty}\{cx_n + y_n\} = c\lim_{n\to\infty}\{x_n\} + \lim_{n\to\infty}\{y_n\},$$

so this limit exists and $c\{x_n\} + \{y_n\} \in M$. Based on this, define $g : M \to \mathbb{C}$ by

$$g(\{x_n\}) = \lim_{n\to\infty} x_n,$$

and $p : X \to (0, \infty)$ by

$$p(\{x_n\}) = \limsup_{n\to\infty} |x_n|.$$

To see that p is a sublinear functional on X, let $\{x_n\}, \{y_n\} \in X$ and $\lambda \geq 0$. Then,

$$p(\{x_n\} + \{y_n\}) = \limsup_{n\to\infty} |x_n + y_n| \leq \limsup_{n\to\infty} |x_n| + \limsup_{n\to\infty} |y_n| = p(\{x_n\}) + p(\{y_n\}),$$

and

$$p(\lambda\{x_n\}) = \limsup_{n\to\infty} |\lambda x_n| = \lambda \limsup_{n\to\infty} |x_n| = \lambda p(\{x_n\}),$$

so p satisfies the required axioms. Clearly, g is a bounded linear functional on M as

$$|g(\{x_n\})| = \left|\lim_{n\to\infty} x_n\right| = \lim_{n\to\infty} |x_n| \leq \sup_n |x_n| < \infty.$$

Furthermore, for any $\{x_n\} \in M$,

$$|g(\{x_n\})| = \left|\lim_{n\to\infty} x_n\right| = \lim_{n\to\infty} |x_n| \leq \limsup_{n\to\infty} |x_n| = p(\{x_n\}).$$

Thus, we can apply Hahn-Banach to get that there is a linear functional $f : X \to \mathbb{C}$ such that $|f(\{x_n\})| \leq p(\{x_n\})$ for all $\{x_n\}$ in X and $g(\{x_n\}) = f(\{x_n\})$ for all $\{x_n\} \in M$. That is, $f : X \to \mathbb{C}$ is a linear mapping such that

$$|f(\{x_n\})| \leq \limsup_{n\to\infty} |x_n|, \quad f(\{x_n\}) = \lim_{n\to\infty} x_n, \quad \text{if the limit exists.}$$

4 Problem 4

Let X be a Banach space.

4.1 Part a

Let $T \in \mathcal{L}(X)$ and $\|I - T\| < 1$. Then, the series $\sum_{n=0}^{\infty}(I - T)^n$ converges absolutely because

$$\sum_{n=0}^{\infty} \|(I - T)^n\| \leq \sum_{n=0}^{\infty} \|I - T\|^n = \frac{1}{1 - \|I - T\|} < \infty.$$

By Proposition 5.4, $\mathcal{L}(X)$ is complete, so the series $\sum_{n=0}^{\infty}(I - T)^n$ converges to some $S \in \mathcal{L}(X)$ by Theorem 5.1. Now, we compute

$$ST = \lim_{N\to\infty} \sum_{n=0}^{N}(I - T)^n T = \lim_{N\to\infty} \sum_{n=0}^{N}(I - T)^n(I - (I - T)) \lim_{N\to\infty} \left(\sum_{n=0}^{N}(I - T)^n - \sum_{n=0}^{N}(I - T)^{n+1}\right)$$
$$= \lim_{N\to\infty} \left(\sum_{n=0}^{N}(I - T)^n - \sum_{n=1}^{N+1}(I - T)^n\right) = \lim_{N\to\infty} \left(I - (I - T)^{N+1}\right) = I - \lim_{N\to\infty}(I - T)^{N+1}.$$

Now, we note that

$$\lim_{N\to\infty} \|(I-T)^{N+1}\| \le \lim_{N\to\infty} \|I-T\|^{N+1} = 0,$$

so $\lim_{N\to\infty}(I-T)^{N+1}$ must converge to the zero operator, meaning that $ST = I$. Similarly,

$$\begin{aligned} TS &= \lim_{N\to\infty} \sum_{n=0}^{N} T(I-T)^n = \lim_{N\to\infty} \sum_{n=0}^{N} (I-(I-T))(I-T)^n \lim_{N\to\infty} \left(\sum_{n=0}^{N} (I-T)^n - \sum_{n=0}^{N} (I-T)^{n+1} \right) \\ &= \lim_{N\to\infty} \left(\sum_{n=0}^{N} (I-T)^n - \sum_{n=1}^{N+1} (I-T)^n \right) = \lim_{N\to\infty} \left(I - (I-T)^{N+1} \right) = I - \lim_{N\to\infty} (I-T)^{N+1} = I. \end{aligned}$$

Thus, $S = T^{-1}$, meaning that T is bijective. Furthermore, T is invertible because $T^{-1} \in \mathcal{L}(X)$, so it is bounded.

4.2 Part b

Now, let $T \in \mathcal{L}(X)$ be invertible and $\|S - T\| < \|T^{-1}\|^{-1}$. Then,

$$\|I - T^{-1}S\| = \|T^{-1}(T-S)\| \le \|T^{-1}\|\|S-T\| < 1,$$

so $T^{-1}S$ is invertible by part a. Let $R = (T^{-1}S)^{-1}T^{-1} \in \mathcal{L}(X)$. Then,

$$SR = S(T^{-1}S)^{-1}T^{-1} = T(T^{-1}S)(T^{-1}S)^{-1}T^{-1} = TT^{-1} = I,$$

and

$$RS = (T^{-1}S)^{-1}T^{-1}S = I,$$

so $S^{-1} = R \in \mathcal{L}(X)$ and S is bijective. S^{-1} is bounded as it is the composition of bounded linear operators. More explicitly,

$$\|S^{-1}\| \le \|(T^{-1}S)^{-1}\|\|T^{-1}\| < \infty,$$

by assumption and part a. Thus, S is invertible. This implies that the set of invertible linear operators is open in $\mathcal{L}(X)$ because given T in this set, any S satisfying $\|S-T\| < \delta_T$ is also in this set for $\delta_T = \|T^{-1}\|^{-1}$.

5 Problem 5

Let X be a Banach space and assume that X^* is separable. Let $\{f_n\}_{n=1}^{\infty} \subset X^*$ be dense and for all $n \in \mathbb{N}$, choose $x_n \in X$ such that $\|x_n\| = 1$ and $|f_n(x_n)| \ge \frac{1}{2}\|f_n\|$. Consider $M = \overline{\text{span}\{\{x_n\}_{n=1}^{\infty}\}}$. M is clearly a closed subspace of X as the closure is closed and the span satisfies the axioms of a subspace by construction. If $M \subsetneq X$, then Theorem 5.8a implies that given some $x \in X \setminus M$, there exists some $f \in X^*$ such that $f(x) = \delta > 0$, $f(y) = 0$ for all $y \in M$, and $\|f\| = 1$. Because $\{f_n\}$ is dense, for any $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that $\|f_n - f\| < \epsilon$. This implies that

$$|f_n(x_n) - f(x_n)| \le \|f_n - f\|\|x_n\| < \epsilon.$$

For ϵ sufficiently small, the reverse traingle inequality implies that

$$0 = |f(x_n)| \ge ||f_n(x_n)| - |f_n(x_n) - f(x_n)|| \ge \frac{1}{2}\|f_n\| - \epsilon,$$

so $\|f_n\| < 2\epsilon$ for any $\epsilon > 0$ sufficiently small, meaning that we must have that $\|f_n\| = 0$ and f_n must be the zero functional. This then implies that $\|f\| < \epsilon$ for all $\epsilon > 0$, so f must also be the zero functional. This is a contradiction because $f(x) > 0$, so the assumption that $M \subsetneq X$ must be false. Since $M \subset X$, this means that $M = X$. Now, note that the set of rationals is countable and dense in $\mathbb{R}$ and the set of complex numbers with rational real and imaginary part is countable and dense in $\mathbb{C}$. Denote the set corresponding to the field $\mathbb{K}$ by $\mathbb{J}$. Let N be the subset of M with coefficients in $\mathbb{J}$. Clearly, N is countable as it is the set of linear combinations of a countable number of vectors with a countable number of coefficients. Let $\sum_{j=1}^{\infty} a_j x_j \in M$ and fix $\epsilon > 0$. For each $j \in \mathbb{N}$, there exists some $b_j \in \mathbb{J}$ such that $|a_j - b_j| < \epsilon 2^{-j}$. Then, $\sum_{j=1}^{\infty} b_j x_j \in N$ and

$$\left\| \sum_{j=1}^{\infty} a_j x_j - \sum_{j=1}^{\infty} b_j x_j \right\| \leq \sum_{j=1}^{\infty} |a_j - b_j| \|x_j\| \leq \sum_{j=1}^{\infty} \epsilon 2^{-j} = \epsilon.$$

Thus, N is dense in $M = X$, so X must also be separable.