AMATH 573 Homework 4

Cade Ballew #2120804

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1 Problem 1

Consider the Modified Vector Derivative NLS equation

$$\boldsymbol{B}_t + (\|\boldsymbol{B}\|^2\boldsymbol{B})_x + \gamma(\boldsymbol{e}_1 \times \boldsymbol{B}_0)\,(\boldsymbol{e}_1 \cdot (\boldsymbol{B}_x \times \boldsymbol{B}_0)) + \boldsymbol{e}_1 \times \boldsymbol{B}_{xx} = 0.$$

where $\boldsymbol{B} = (0, u, v)$, $\boldsymbol{e}_1 = (1, 0, 0)$, $\boldsymbol{B}_0 = (0, B_0, 0)$, γ is a constant, and the boundary conditions are $\boldsymbol{B} \to \boldsymbol{B}_0$, $\boldsymbol{B}_x \to 0$ as $|x| \to \infty$.

1.1 Part a

We look for stationary solutions $\boldsymbol{B} = \boldsymbol{B}(z) = \boldsymbol{B}(x - Wt)$ by plugging this into our equation via Mathematica. We get that the first entry is identically zero, so we obtain the system of ODEs

$$\begin{cases} (-W + 3u^2 + v^2)u' + 2uvv' - v'' = 0 \\ 2uvu' + u^2v' - (W + B_0^2\gamma - 3v^2)v' + u'' = 0. \end{cases}$$

We can rearrange this and collect terms to get

$$\begin{cases} u'' = -(u^2v + v^3)' + (W + \gamma B_0^2)v' \\ v'' = (uv^2 + u^3)' - Wu'. \end{cases}$$

Integrating both sides of each equation, we get that

$$\begin{cases} u' = -u^2v - v^3 + (W + \gamma B_0^2)v + C_1 \\ v' = uv^2 + u^3 - Wu + C_2 \end{cases}$$

where C_1, C_2 are integration constants. Note that these are determined by the boundary conditions which we will plug in at the start of part b. In order for $H(u, v)$ to be a Hamiltonian with canonical Poisson structure, we need that

$$u' = \frac{\partial H}{\partial v}, \quad v' = -\frac{\partial H}{\partial u}.$$

This of course yields the system

$$\begin{cases} \frac{\partial H}{\partial v} = -u^2v - v^3 + (W + \gamma B_0^2)v + C_1 \\ -\frac{\partial H}{\partial u} = uv^2 + u^3 - Wu + C_2. \end{cases}$$

Integrating both sides of each, we get

$$\begin{cases} H(u,v) = -\frac{1}{2}u^2v^2 - \frac{v^4}{4} + \frac{1}{2}(W + \gamma B_0^2)v^2 + C_1v + c_1(u) \\ H(u,v) = -\frac{1}{2}u^2v^2 - \frac{u^4}{4} + \frac{W}{2}u^2 - C_2u + c_2(v). \end{cases}$$

Combining these, we get that our system has Hamiltonian given by

$$H(u,v) = -\frac{1}{2}u^2v^2 - \frac{u^4}{4} + \frac{W}{2}u^2 - C_2u - \frac{v^4}{4} + \frac{1}{2}(W + \gamma B_0^2)v^2 + C_1v,$$

meaning that our system is Hamiltonian with canonical Poisson structure.

1.2 Part b

To find the value of the Hamiltonian such that the boundary conditions are satisfied, we first need to determine C_1, C_2 by enforcing the boundary conditions. We do this via the system of equations

$$\begin{cases} u' = -u^2v - v^3 + (W + \gamma B_0^2)v + C_1 \\ v' = uv^2 + u^3 - Wu + C_2 \end{cases}$$

from before. As $|x| \to \infty$, we have that $u \to B_0$, $v \to 0$, $u' \to 0$, $v' \to 0$. Plugging this in, we get the system

$$\begin{cases} 0 = C_1 \\ 0 = B_0^3 - WB_0 + C_2 \end{cases}$$

which can be solved to get that $C_1 = 0$, $C_2 = WB_0 - B_0^3$. This means that the value of the Hamiltonian including the boundary conditions is given by

$$H(u,v) = -\frac{1}{2}u^2v^2 - \frac{u^4}{4} + \frac{W}{2}u^2 - (WB_0 - B_0^3)u - \frac{v^4}{4} + \frac{1}{2}(W + \gamma B_0^2)v^2.$$

Now, we plug the boundary conditions into our Hamiltonian to get

$$H(B_0, 0) = -\frac{B_0^4}{4} + \frac{W}{2}B_0^2 - (WB_0 - B_0^3)B_0 = \frac{3}{4}B_0^4 - \frac{W}{2}B_0^2.$$

Setting this constant equal to $H(u,v)$ and defining $U = u/B_0$, $V = v/B_0$, and $W_0 = W/B_0^2$ lets us write

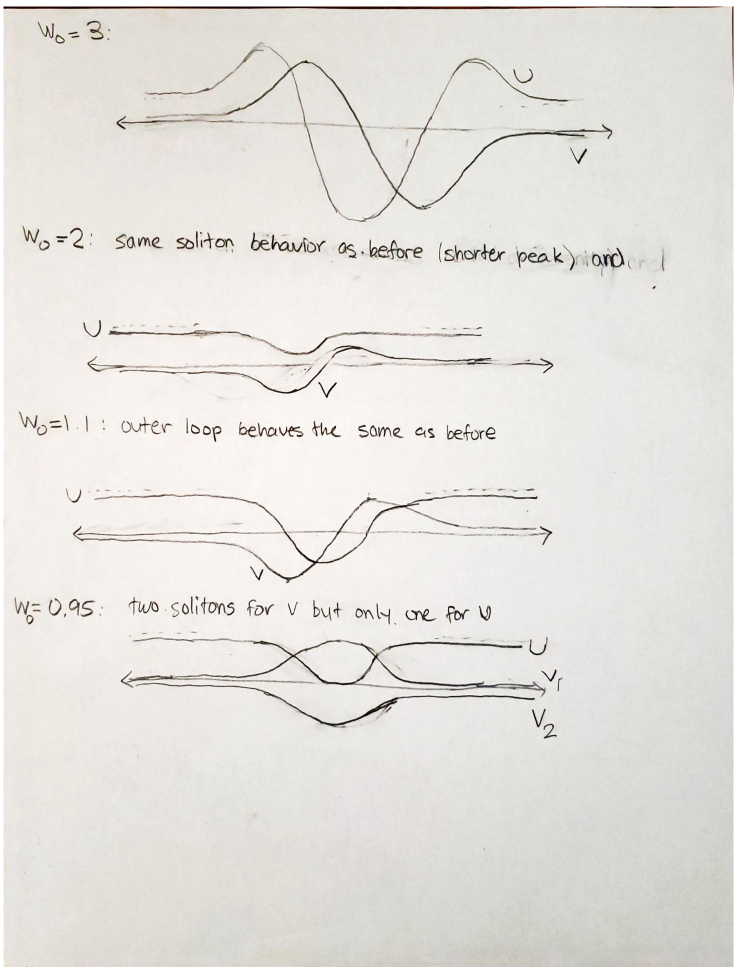
$$\frac{3}{4}B_0^4 - \frac{W_0}{2}B_0^4 = -\frac{1}{2}B_0^4U^2V^2 - B_0^4\frac{U^4}{4} + B_0^4\frac{W_0}{2}U^2 - B_0^4(W_0 - 1)U - B_0^4\frac{V^4}{4} + \frac{1}{2}B_0^4(W_0 + \gamma)V^2$$

which reduces to the curve

$$\frac{3}{4} - \frac{W_0}{2} = -\frac{1}{2}U^2V^2 - \frac{U^4}{4} + \frac{W_0}{2}U^2 - (W_0 - 1)U - \frac{V^4}{4} + \frac{1}{2}(W_0 + \gamma)V^2.$$

1.3 Part c

See Mathematica for plots of this curve with $\gamma = 1/10$ and $W_0 = 3$, $W_0 = 2$, $W_0 = 1.1$, $W_0 = 1$, $W_0 = 0.95$, $W_0 = 0.9$ and a discussion of the corresponding solitons. We include crude sketches of the solitons here.



2 Problem 2

We wish to show that the canonical Poisson bracket

$$\{f, g\} = \sum_{j=1}^{N} \left(\frac{\partial f}{\partial q_j} \frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q_j} \right)$$

satisfies the Jacobi identity

$$\{\{f,g\},h\}+\{\{g,h\},f\}+\{\{h,f\},g\}=0.$$

We do this by brute-force expansion of each term. First,

$$\{\{f,g\},h\}=\left\{\sum_{j=1}^{N}\left(\frac{\partial f}{\partial q_j}\frac{\partial g}{\partial p_j}-\frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q_j}\right),h\right\}$$

$$=\sum_{k=1}^{N}\left(\frac{\partial}{\partial q_k}\left(\sum_{j=1}^{N}\left(\frac{\partial f}{\partial q_j}\frac{\partial g}{\partial p_j}-\frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q_j}\right)\right)\frac{\partial h}{\partial p_k}-\frac{\partial}{\partial p_k}\left(\sum_{j=1}^{N}\left(\frac{\partial f}{\partial q_j}\frac{\partial g}{\partial p_j}-\frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q_j}\right)\right)\frac{\partial h}{\partial q_k}\right)$$

$$=\sum_{j,k=1}^{N}\left(\frac{\partial^2 f}{\partial q_j\partial q_k}\frac{\partial g}{\partial p_j}\frac{\partial h}{\partial p_k}+\frac{\partial f}{\partial q_j}\frac{\partial^2 g}{\partial p_j\partial q_k}\frac{\partial h}{\partial p_k}-\frac{\partial^2 f}{\partial p_j\partial q_k}\frac{\partial g}{\partial q_j}\frac{\partial h}{\partial p_k}-\frac{\partial f}{\partial p_j}\frac{\partial^2 g}{\partial q_j\partial q_k}\frac{\partial h}{\partial p_k}\right)$$

$$+\sum_{j,k=1}^{N}\left(-\frac{\partial^2 f}{\partial p_k\partial q_j}\frac{\partial g}{\partial p_j}\frac{\partial h}{\partial q_k}-\frac{\partial f}{\partial q_j}\frac{\partial^2 g}{\partial p_j\partial p_k}\frac{\partial h}{\partial q_k}+\frac{\partial^2 f}{\partial p_j\partial p_k}\frac{\partial g}{\partial q_j}\frac{\partial h}{\partial q_k}+\frac{\partial f}{\partial p_j}\frac{\partial^2 g}{\partial p_k\partial q_j}\frac{\partial h}{\partial q_k}\right).$$

Similarly,

$$\{\{g,h\},f\}=\sum_{j,k=1}^{N}\left(\frac{\partial^2 g}{\partial q_j\partial q_k}\frac{\partial h}{\partial p_j}\frac{\partial f}{\partial p_k}+\frac{\partial g}{\partial q_j}\frac{\partial^2 h}{\partial p_j\partial q_k}\frac{\partial f}{\partial p_k}-\frac{\partial^2 g}{\partial p_j\partial q_k}\frac{\partial h}{\partial q_j}\frac{\partial f}{\partial p_k}-\frac{\partial g}{\partial p_j}\frac{\partial^2 h}{\partial q_j\partial q_k}\frac{\partial f}{\partial p_k}\right)$$

$$+\sum_{j,k=1}^{N}\left(-\frac{\partial^2 g}{\partial p_k\partial q_j}\frac{\partial h}{\partial p_j}\frac{\partial f}{\partial q_k}-\frac{\partial g}{\partial q_j}\frac{\partial^2 h}{\partial p_j\partial p_k}\frac{\partial f}{\partial q_k}+\frac{\partial^2 g}{\partial p_j\partial p_k}\frac{\partial h}{\partial q_j}\frac{\partial f}{\partial q_k}+\frac{\partial g}{\partial p_j}\frac{\partial^2 h}{\partial p_k\partial q_j}\frac{\partial f}{\partial q_k}\right),$$

and

$$\{\{h,f\},g\}=\sum_{j,k=1}^{N}\left(\frac{\partial^2 h}{\partial q_j\partial q_k}\frac{\partial f}{\partial p_j}\frac{\partial g}{\partial p_k}+\frac{\partial h}{\partial q_j}\frac{\partial^2 f}{\partial p_j\partial q_k}\frac{\partial g}{\partial p_k}-\frac{\partial^2 h}{\partial p_j\partial q_k}\frac{\partial f}{\partial q_j}\frac{\partial g}{\partial p_k}-\frac{\partial h}{\partial p_j}\frac{\partial^2 f}{\partial q_j\partial q_k}\frac{\partial g}{\partial p_k}\right)$$

$$+\sum_{j,k=1}^{N}\left(-\frac{\partial^2 h}{\partial p_k\partial q_j}\frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q_k}-\frac{\partial h}{\partial q_j}\frac{\partial^2 f}{\partial p_j\partial p_k}\frac{\partial g}{\partial q_k}+\frac{\partial^2 h}{\partial p_j\partial p_k}\frac{\partial f}{\partial q_j}\frac{\partial g}{\partial q_k}+\frac{\partial h}{\partial p_j}\frac{\partial^2 f}{\partial p_k\partial q_j}\frac{\partial g}{\partial q_k}\right).$$

Ordering the terms alphabetically,

$$
\begin{aligned}
&\{\{f,g\},h\}+\{\{g,h\},f\}+\{\{h,f\},g\}\\
&=\sum_{j,k=1}^{N}\left(\frac{\partial^2 f}{\partial q_j\partial q_k}\frac{\partial g}{\partial p_j}\frac{\partial h}{\partial p_k}+\frac{\partial f}{\partial q_j}\frac{\partial^2 g}{\partial p_j\partial q_k}\frac{\partial h}{\partial p_k}-\frac{\partial^2 f}{\partial p_j\partial q_k}\frac{\partial g}{\partial q_j}\frac{\partial h}{\partial p_k}-\frac{\partial f}{\partial p_j}\frac{\partial^2 g}{\partial q_j\partial q_k}\frac{\partial h}{\partial p_k}\right)\\
&+\sum_{j,k=1}^{N}\left(-\frac{\partial^2 f}{\partial p_k\partial q_j}\frac{\partial g}{\partial p_j}\frac{\partial h}{\partial q_k}-\frac{\partial f}{\partial q_j}\frac{\partial^2 g}{\partial p_j\partial p_k}\frac{\partial h}{\partial q_k}+\frac{\partial^2 f}{\partial p_j\partial p_k}\frac{\partial g}{\partial q_j}\frac{\partial h}{\partial q_k}+\frac{\partial f}{\partial p_j}\frac{\partial^2 g}{\partial p_k\partial q_j}\frac{\partial h}{\partial q_k}\right)\\
&+\sum_{j,k=1}^{N}\left(\frac{\partial f}{\partial p_k}\frac{\partial^2 g}{\partial q_j\partial q_k}\frac{\partial h}{\partial p_j}+\frac{\partial f}{\partial p_k}\frac{\partial g}{\partial q_j}\frac{\partial^2 h}{\partial p_j\partial q_k}-\frac{\partial f}{\partial p_k}\frac{\partial^2 g}{\partial p_j\partial q_k}\frac{\partial h}{\partial q_j}-\frac{\partial f}{\partial p_k}\frac{\partial g}{\partial p_j}\frac{\partial^2 h}{\partial q_j\partial q_k}\right)\\
&+\sum_{j,k=1}^{N}\left(-\frac{\partial f}{\partial q_k}\frac{\partial^2 g}{\partial p_k\partial q_j}\frac{\partial h}{\partial p_j}-\frac{\partial f}{\partial q_k}\frac{\partial g}{\partial q_j}\frac{\partial^2 h}{\partial p_j\partial p_k}+\frac{\partial f}{\partial q_k}\frac{\partial^2 g}{\partial p_j\partial p_k}\frac{\partial h}{\partial q_j}+\frac{\partial f}{\partial q_k}\frac{\partial g}{\partial p_j}\frac{\partial^2 h}{\partial p_k\partial q_j}\right)\\
&+\sum_{j,k=1}^{N}\left(\frac{\partial f}{\partial p_j}\frac{\partial g}{\partial p_k}\frac{\partial^2 h}{\partial q_j\partial q_k}+\frac{\partial^2 f}{\partial p_j\partial q_k}\frac{\partial g}{\partial p_k}\frac{\partial h}{\partial q_j}-\frac{\partial f}{\partial q_j}\frac{\partial g}{\partial p_k}\frac{\partial^2 h}{\partial p_j\partial q_k}-\frac{\partial^2 f}{\partial q_j\partial q_k}\frac{\partial g}{\partial p_k}\frac{\partial h}{\partial p_j}\right)\\
&+\sum_{j,k=1}^{N}\left(-\frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q_k}\frac{\partial^2 h}{\partial p_k\partial q_j}-\frac{\partial^2 f}{\partial p_j\partial p_k}\frac{\partial g}{\partial q_k}\frac{\partial h}{\partial q_j}+\frac{\partial f}{\partial q_j}\frac{\partial g}{\partial q_k}\frac{\partial^2 h}{\partial p_j\partial p_k}+\frac{\partial^2 f}{\partial p_k\partial q_j}\frac{\partial g}{\partial q_k}\frac{\partial h}{\partial p_j}\right).
\end{aligned}
$$

Grouping terms,

$$
\begin{aligned}
&\{\{f,g\},h\}+\{\{g,h\},f\}+\{\{h,f\},g\}\\
&=\left(\sum_{j,k=1}^{N}\frac{\partial^2 f}{\partial q_j\partial q_k}\frac{\partial g}{\partial p_j}\frac{\partial h}{\partial p_k}-\sum_{j,k=1}^{N}\frac{\partial^2 f}{\partial q_j\partial q_k}\frac{\partial g}{\partial p_k}\frac{\partial h}{\partial p_j}\right)+\left(\sum_{j,k=1}^{N}\frac{\partial f}{\partial q_j}\frac{\partial^2 g}{\partial p_j\partial q_k}\frac{\partial h}{\partial p_k}-\sum_{j,k=1}^{N}\frac{\partial f}{\partial q_k}\frac{\partial^2 g}{\partial p_k\partial q_j}\frac{\partial h}{\partial p_j}\right)\\
&+\left(\sum_{j,k=1}^{N}\frac{\partial^2 f}{\partial p_k\partial q_j}\frac{\partial g}{\partial q_k}\frac{\partial h}{\partial p_j}-\sum_{j,k=1}^{N}\frac{\partial^2 f}{\partial p_j\partial q_k}\frac{\partial g}{\partial q_j}\frac{\partial h}{\partial p_k}\right)+\left(\sum_{j,k=1}^{N}\frac{\partial f}{\partial p_k}\frac{\partial^2 g}{\partial q_j\partial q_k}\frac{\partial h}{\partial p_j}-\sum_{j,k=1}^{N}\frac{\partial f}{\partial p_j}\frac{\partial^2 g}{\partial q_j\partial q_k}\frac{\partial h}{\partial p_k}\right)\\
&+\left(\sum_{j,k=1}^{N}\frac{\partial^2 f}{\partial p_j\partial q_k}\frac{\partial g}{\partial p_k}\frac{\partial h}{\partial q_j}-\sum_{j,k=1}^{N}\frac{\partial^2 f}{\partial p_k\partial q_j}\frac{\partial g}{\partial p_j}\frac{\partial h}{\partial q_k}\right)+\left(\sum_{j,k=1}^{N}\frac{\partial f}{\partial q_k}\frac{\partial^2 g}{\partial p_j\partial p_k}\frac{\partial h}{\partial q_j}-\sum_{j,k=1}^{N}\frac{\partial f}{\partial q_j}\frac{\partial^2 g}{\partial p_j\partial p_k}\frac{\partial h}{\partial q_k}\right)\\
&+\left(\sum_{j,k=1}^{N}\frac{\partial^2 f}{\partial p_j\partial p_k}\frac{\partial g}{\partial q_j}\frac{\partial h}{\partial q_k}-\sum_{j,k=1}^{N}\frac{\partial^2 f}{\partial p_j\partial p_k}\frac{\partial g}{\partial q_k}\frac{\partial h}{\partial q_j}\right)+\left(\sum_{j,k=1}^{N}\frac{\partial f}{\partial p_j}\frac{\partial^2 g}{\partial p_k\partial q_j}\frac{\partial h}{\partial q_k}-\sum_{j,k=1}^{N}\frac{\partial f}{\partial p_k}\frac{\partial^2 g}{\partial p_j\partial q_k}\frac{\partial h}{\partial q_j}\right)\\
&+\left(\sum_{j,k=1}^{N}\frac{\partial f}{\partial p_k}\frac{\partial g}{\partial q_j}\frac{\partial^2 h}{\partial p_j\partial q_k}-\sum_{j,k=1}^{N}\frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q_k}\frac{\partial^2 h}{\partial p_k\partial q_j}\right)+\left(\sum_{j,k=1}^{N}\frac{\partial f}{\partial p_j}\frac{\partial g}{\partial p_k}\frac{\partial^2 h}{\partial q_j\partial q_k}-\sum_{j,k=1}^{N}\frac{\partial f}{\partial p_k}\frac{\partial g}{\partial p_j}\frac{\partial^2 h}{\partial q_j\partial q_k}\right)\\
&+\left(\sum_{j,k=1}^{N}\frac{\partial f}{\partial q_j}\frac{\partial g}{\partial q_k}\frac{\partial^2 h}{\partial p_j\partial p_k}-\sum_{j,k=1}^{N}\frac{\partial f}{\partial q_k}\frac{\partial g}{\partial q_j}\frac{\partial^2 h}{\partial p_j\partial p_k}\right)+\left(\sum_{j,k=1}^{N}\frac{\partial f}{\partial q_k}\frac{\partial g}{\partial p_j}\frac{\partial^2 h}{\partial p_k\partial q_j}-\sum_{j,k=1}^{N}\frac{\partial f}{\partial q_j}\frac{\partial g}{\partial p_k}\frac{\partial^2 h}{\partial p_j\partial q_k}\right).
\end{aligned}
$$

However, each group of terms in parentheses is identically zero. If one switches the dummy index on the second summation in each group $j \to k$, $k \to j$, the first and second sums in each group are identical, meaning that the sums cancel through subtraction. Thus,

$$\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0$$

as desired.

3 Problem 3

We wish to show that the Sine-Gordon equation

$$u_{tt} - u_{xx} + \sin(u) = 0$$

is Hamiltonian with canonical Poisson structure and Hamiltonian

$$H = \int \left(\frac{1}{2}p^2 + \frac{1}{2}q_x^2 + 1 - \cos(q) \right) dx,$$

where $q = u$, and $p = u_t$. To do this, we simply need to show that

$$\frac{\partial q}{\partial t} = \frac{\delta H}{\delta p}, \quad \frac{\partial p}{\partial t} = -\frac{\delta H}{\delta q}.$$

We compute the variational derivatives

$$\begin{aligned} \frac{\delta H}{\delta p} &= \frac{\partial \mathcal{H}}{\partial u_t} = u_t, \\ \frac{\delta H}{\delta q} &= \frac{\partial \mathcal{H}}{\partial u} - \partial_x \frac{\partial \mathcal{H}}{\partial u_x} = \sin u - \partial_x(u_x) = \sin u - u_{xx}. \end{aligned}$$

Now, note that

$$\begin{aligned} \frac{\partial q}{\partial t} &= u_t = \frac{\delta H}{\delta p}, \\ \frac{\partial p}{\partial t} &= u_{tt} = u_{xx} - \sin u = -\frac{\delta H}{\delta q}, \end{aligned}$$

so the Sine-Gordon equation is indeed Hamiltonian with canonical Poisson structure and the given Hamiltonian.

4 Problem 4

We wish to check that the conserved quantities $F_{-1} = \int u dx$, $F_0 = \int \frac{1}{2}u^2 dx$, $F_1 = \int \left(\frac{1}{6}u^3 - \frac{1}{2}u_x^2\right) dx$, $F_2 = \int \left(\frac{1}{24}u^4 - \frac{1}{2}uu_x^2 + \frac{3}{10}u_{xx}^2\right) dx$ are mutually in involution with respect to the Poisson bracket defined by the Poisson structure given by ∂_x. By the antisymmetry of the Poisson bracket, it suffices to check that

$$\{F_{-1}, F_0\} = \{F_{-1}, F_1\} = \{F_{-1}, F_2\} = \{F_0, F_1\} = \{F_0, F_2\} = \{F_1, F_2\} = 0.$$

In the attached Mathematica notebook, we compute each of these and get that this is indeed true.

5 Problem 5

To find the fourth conserved quantity of the KdV equation, we assemble the nontrivial weight 8 terms

$$F_2 = \int (a_1 u^4 + a_2 u u_x^2 + a_3 u_{xx}^2) dx.$$

Using Mathematica, we compute

$$\frac{\delta f_{2t}}{\delta x} = 2((12a_1+a_2)u_x^3 - 2(3a_2+5a_3)u_{xx}u_{xxx} + u_x(3(12a_1+a_2)uu_{xx} - (3a_2+5a_3)u_{xxxx})).$$

Since we need this to be zero for all x, t, we get the linear system

$$\begin{cases} 12a_1 + a_2 = 0 \\ 3a_2 + 5a_3 = 0. \end{cases}$$

If we let $a_1 = \frac{1}{24}$, we get that $a_2 = -\frac{1}{2}$, $a_3 = \frac{3}{10}$, meaning that

$$F_2 = \int \left(\frac{1}{24}u^4 - \frac{1}{2}uu_x^2 + \frac{3}{10}u_{xx}^2 \right) dx.$$

Of course, this matches the given F_2 from problem 4 as expected.

6 Problem 6

Consider a recursion operator $R = B_1 B_0^{-1}$, for the KdV equation with $B_0 = \partial_x$ and $B_1 = \partial_{xxx} + \frac{1}{3}(u\partial_x + \partial_x u)$. Using Mathematica, compute that for a given function v,

$$Rv = \frac{2}{3}uv + \frac{1}{3}u_x \int v dx + v_{xx}.$$

Again using Mathematica, we compute the first KdV flow

$$u_{t_1} = Ru_x = uu_x + u_{xxx}.$$

Applying R to this result, we get the second KdV flow up to a rescaling on t_2

$$u_{t_2} = \frac{5}{6}u^2 u_x + \frac{10}{3}u_x u_{xx} + \frac{5}{3}uu_{xxx} + u_{xxxxx}.$$

Applying R again to this result, we get the third KdV equation up to a scaling on t_3

$$u_{t_3} = \frac{35}{54}u^3 u_x + \frac{35}{18}u_x^3 + \frac{35}{18}u^2 u_{xxx} + \frac{35}{3}u_{xx}u_{xxx} + 7u_x u_{xxxx} + \frac{7}{9}u(10u_x u_{xx} + 3u_{xxxxx}) + u_{xxxxxxx}.$$

7 Problem 7

Consider the function $U(x) = 2\partial_x^2 \ln\left(1 + e^{kx+\alpha}\right)$. We show that for a suitable k, $U(x)$ is a solution of the first member of the stationary KdV hierarchy

$$6uu_x + u_{xxx} + c_0 u_x = 0.$$

by plugging it in via Mathematica and solving for c_0 in terms of k. This yields that our function is indeed a solution if $c_0 = -k^2$. Now, we let $u(x, t_1, t_2, t_3, \ldots) = U(x)|_{\alpha=\alpha(t_1,t_2,t_3,\ldots)}$ and determine the dependence of α on t_1, t_2 and t_3 such that $u(x, t_1, t_2, t_3, \ldots)$ is simultaneously a solution of the first, second and third KdV equations

$$u_{t_1} = 6uu_x + u_{xxx},$$
$$u_{t_2} = 30u^2 u_x + 20u_x u_{xx} + 10uu_{xxx} + u_{5x},$$
$$u_{t_3} = 140u^3 u_x + 70u_x^3 + 280uu_x u_{xx} + 70u_{xx}u_{xxx} + 70u^2 u_{xxx} + 42u_x u_{xxxx} + 14uu_{5x} + u_{7x}.$$

by plugging it into each via Mathematica and solving for α. This yields that

$$\alpha_{t_1} = k^3 \quad \alpha_{t_2} = k^5 \quad \alpha_{t_3} = k^7.$$

Based on this, we guess that

$$\alpha = \sum_{j=1}^{\infty} k^{2j+1} t_j$$

yields a one-soliton solution that solves the entire KdV hierarchy.

8 Problem 8

Consider the function

$$U(x) = 2\partial_x^2 \ln\left(1 + e^{k_1 x+\alpha} + e^{k2_x+\beta} + \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 e^{k_1 x + k_2 x + \alpha + \beta}\right).$$

We show that for a suitable k_1, k_2, $U(x)$ is a solution of the second member of the stationary KdV hierarchy

$$30u^2 u_x + 20u_x u_{xx} + 10uu_{xxx} + u_{5x} + c_1(6uu_x + u_{xxx}) + c_0 u_x = 0.$$

by plugging it into Mathematica and solving for c_0, c_1 in terms of k_1, k_2. This yields that our function is indeed a solution if

$$c_0 = k_1^2 k_2^2, \quad c_1 = -k_1^2 - k_2^2.$$

Now, we let $u(x, t_1, t_2, t_3, \ldots) = U(x)|_{\alpha=\alpha(t_1,t_2,t_3,\ldots),\beta=\beta(t_1,t_2,t_3,\ldots)}$ and determine the dependence of α and β on t_1, t_2 and t_3 such that $u(x, t_1, t_2, t_3, \ldots)$ is simultaneously a solution of the first, second and third KdV equations by it into each via Mathematica and solving for α, β. This yields that

$$\alpha_{t_1} = k_1^3 \quad \alpha_{t_2} = k_1^5 \quad \alpha_{t_3} = k_1^7$$

and

$$\beta_{t_1} = k_2^3 \quad \beta_{t_2} = k_2^5 \quad \beta_{t_3} = k_2^7.$$

Based on this, we guess that

$$\alpha = \sum_{j=1}^{\infty} k_1^{2j+1} t_j, \quad \beta = \sum_{j=1}^{\infty} k_2^{2j+1} t_j$$

yields a two-soliton solution that solves the entire KdV hierarchy.