MATH 525 Homework 8

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1 Problem 1 (Folland Problem 2)

Let μ be a Radon measure on X.

1.1 Part a

Let N be the union of all open $U \subset X$ such that $\mu(U) = 0$. Then, N is the union of open sets, so it is also open. Let $K \subset N$ be compact. Then, all open $U \subset X$ such that $\mu(U) = 0$ form an open cover of K, so this can be reduced to a finite subcover $K \subset \bigcup_{j=1}^{n} U_j$ such that $\mu(U_j) = 0$ for all j. Thus,

$$\mu(K) \leq \sum_{j=1}^{n} \mu(U_j) = 0.$$

By inner regularity for open sets, $\mu(N)$ is the supremum over $\mu(K)$ for all compact $K \subset U$, but $\mu(K) = 0$ for all such K, so $\mu(N) = 0$ as well.

1.2 Part b

Let $x \in \operatorname{supp}(\mu)$ and let $f \in C_c(X, [0, 1])$ such that $f(x) > 0$ be given. Let $f(x) = a$ and let $0 < \epsilon < a$. Then, since f is continuous, $U = f^{-1}\left((a - \epsilon, 1)\right)$ is open, so $\mu(U) > 0$ since $x \in U$. Thus,

$$\int f \mathrm{d}\mu \geq \int_U f \mathrm{d}\mu > (a - \epsilon)\mu(U) > 0.$$

Conversely, assume that $\int f \mathrm{d}\mu > 0$ for all $f \in C_c(X, [0, 1])$ such that $f(x) > 0$ for some $x \in X$. Let U be any open set containing x. $\{x\}$ is compact, so there is some $f \in C_c(X, [0, 1])$ such that $\mathbb{1}_{\{x\}} \leq f \prec U$. In particular, $f(x) > 0$ and $f \prec U$. By assumption,

$$I(f) = \int f \mathrm{d}\mu > 0,$$

so by the Riesz representation theorem,

$$\mu(U) = \sup\{I(f) : f \prec U\} > 0.$$

Since this holds for all open U containing x, we conclude that $x \in \operatorname{supp}(\mu)$ if and only if $\int f \mathrm{d}\mu > 0$ for all $f \in C_c(X, [0, 1])$ such that $f(x) > 0$.

2 Problem 2 (Folland Problem 8)

Let μ be a Radon measure on X, $\phi \in L^1(\mu)$, and $\phi \geq 0$. By Exercise 2.14 from last quarter, ν defined by $\nu(E) = \int_E \phi \mathrm{d}\mu$ is a measure, so we need only show that it is Radon. ν is finite as for any $E \in \mathcal{B}_X$,

$$\nu(E) = \int_E \phi \mathrm{d}\mu \leq \int \phi \mathrm{d}\mu < \infty,$$

so in particular, ν is finite on compact sets. To see that ν is inner regular, fix $E \in \mathcal{B}_X$ and define $F_n = \phi^{-1}\left(\left(\frac{1}{n}, \infty\right)\right)$, $E_n = E \cap F_n$ for all $n \in \mathbb{N}$. Then, $E \setminus \phi^{-1}(\{0\}) = \bigcup_{n=1}^{\infty} E_n$, so continuity from below implies that $\nu\left(E \setminus \phi^{-1}(\{0\})\right) = \lim_{n\to\infty} \nu(E_n)$. Furthermore,

$$\nu(E) = \int_E \phi \mathrm{d}\mu = \int_{E\setminus\phi^{-1}(\{0\})} \phi \mathrm{d}\mu = \nu\left(E \setminus \phi^{-1}(\{0\})\right) = \lim_{n\to\infty} \nu(E_n).$$

Now, we observe that for each n,

$$\mu(E_n) < n \int_{E_n} \phi \mathrm{d}\mu \leq n \int \phi \mathrm{d}\mu < \infty,$$

so μ is finite on each E_n. By Corollary 3.6, there exists some $\delta_n > 0$ such that $\nu(E) < \frac{1}{n}$ whenever $\mu(E) < \delta_n$. By Proposition 7.5, for all $n \in \mathbb{N}$, there exists some compact $K_n \subset E_n$ such that $\mu(E_n \setminus K_n) < \delta_n$, meaning that $\nu(E_n \setminus K_n) < \frac{1}{n}$, and since ν is finite, $\nu(E_n) < \frac{1}{n} + \nu(K_n)$. Thus,

$$\nu(E) = \lim_{n\to\infty} \left(\frac{1}{n} + \nu(K_n)\right) = \lim_{n\to\infty} \nu(K_n).$$

Since $E_n \subset E$ for all n, we have constucted a sequence of compact sets $\{K_n\}$ such that $K_n \subset E$ for all $n \in \mathbb{N}$ and $\nu(E) = \lim_{n\to\infty} \nu(K_n)$. Thus, for any $E \in \mathcal{B}_X$,

$$\nu(E) = \sup\{\nu(E) : K \subset E,\ K \text{ compact}\},$$

so ν is inner regular on all Borel sets. To show outer regularity, fix $\epsilon > 0$ and $E \in \mathcal{B}_X$. Then, ν is inner regular on E^c and finite, so there exists some compact $K \subset E^c$ such that $\nu(E^c \setminus K) < \epsilon$. The set $U = K^c$ is open, $E \subset U$, and

$$\nu(U \setminus E) = \nu(E^c \cap U) = \nu(E^c \setminus K) < \epsilon.$$

Thus, for any $\epsilon > 0$ we can find some $U \supset E$ such that $\nu(E) > \nu(U) - \epsilon$, so

$$\nu(E) = \inf\{\nu(E) : U \supset E,\ U \text{ open}\},$$

meaning that ν is outer regular and therefore Borel.

3 Problem 3 (Folland Problem 9)

Let μ be a Radon measure on X, $\phi \in C(X, (0, \infty))$, $\nu(E) = \int_E \phi \mathrm{d}\mu$, and ν' be the Radon measure associated to the functional $I(f) = \int f\phi \mathrm{d}\mu$ on $C_c(X)$.

3.1 Part a

Let $U \subset X$ be open. By the Riesz representation theorem,

$$\nu'(U) = \sup\left\{\int f\phi \mathrm{d}\mu : f \in C_c(X),\ f \prec U\right\}.$$

By Theorem 7.13 applied to $\phi \mathbb{1}_U$,

$$\nu(U) = \int_U \phi \mathrm{d}\mu = \sup\left\{\int g \mathrm{d}\mu : g \in C_c(X),\ 0 \leq g \leq \phi \mathbb{1}_U\right\}.$$

This theorem requires that $\phi\mathbb{1}_U$ be lower semicontinuous, but this is easy to verify: for any $a \in \mathbb{R}$, $\{x : (\phi\mathbb{1}_U)(x) > a\} = U \cap \{x : \phi(x) > a\}$ is open because U is open and ϕ is continuous. To see that $\nu'(U) = \nu(U)$, let $f \in C_c(X)$ and $f \prec U$. Then, $0 \leq f \leq \mathbb{1}_U$, so if we let $g = f\phi$, $0 \leq g \leq \phi\mathbb{1}_U$ and $\int f\phi \mathrm{d}\mu = \int g \mathrm{d}\mu$. Since f is continuous and compactly supported and ϕ is continuous, $g \in C_c(X)$. Thus, every element in the set defining $\nu'(U)$ can be written as an element in the set defining $\nu(U)$. Conversely, let $g \in C_c(X)$ and $0 \leq g \leq \phi\mathbb{1}_U$. Since g is continuous and compactly supported and ϕ is continuous and positive, $f = \frac{g}{\phi}$ satisfies $0 \leq f \leq \mathbb{1}_U$ and $f \in C_c(x)$. Then, $f \prec U$ and $\int f\phi \mathrm{d}\mu = \int g \mathrm{d}\mu$, so every element in the set defining $\nu(U)$ can be written as an element in the set defining $\nu'(U)$. Since we take the supremum of both sets, we conclude that $\nu'(U) = \nu(U)$ for all open $U \subset X$.

3.2 Part b

Let $E \in \mathcal{B}_X$ and fix $\epsilon > 0$. If $\nu(E) = \infty$, then $\nu(U) = \infty$ for any open set $U \supset E$, so outer regularity is trivially satisfied for such sets, and we can assume that $\nu(E)$ is finite. Noting that $X = \bigcup_{k\in\mathbb{Z}} V_k$ where $V_k = \{x : 2^k < \phi(x) < 2^{k+2}\}$ is open, for each k, define $F_k = E \cap V_k$ and $E_k = F_k \setminus F_{k-1}$. Then, $E = \bigcup_{k\in\mathbb{Z}} E_k$. Since $E_k \subset V_k$,

$$\mu(E_k) < 2^{-k} \int_{E_k} \phi \mathrm{d}\mu = 2^{-k}\nu(E_k),$$

so $\mu(E_k) < \infty$. Then, because μ is outer regular on all Borel sets, there exists some open set $U_k \supset E_k$ such that $\mu(U_k \setminus E_k) < \frac{\epsilon 2^{-|k|-k-2}}{3}$. We can assume without loss of generality that $U_k \subset V_k$ by redefining $U_k \to U_k \cap V_k$ since V_k is open. Then, $U_k \setminus E_k \subset V_k$, so

$$\nu(U_k \setminus E_k) = \int_{U_k \setminus E_k} \phi \mathrm{d}\mu < 2^{k+2}\mu(U_k \setminus E_k) < \frac{\epsilon 2^{-|k|}}{3}.$$

Define $U = \bigcup_{k\in\mathbb{Z}} U_k$. Then,

$$\nu(E) = \sum_{k\in\mathbb{Z}} \nu(E_k) > \sum_{k\in\mathbb{Z}} \nu(U_k) - \sum_{k\in\mathbb{Z}} \frac{\epsilon 2^{-|k|}}{3} \geq \nu(U) - \epsilon.$$

Thus, for any $\epsilon > 0$, there exists some open $U \supset E$ such that $\nu(E) > \nu(U) - \epsilon$, so

$$\nu(E) = \inf\left\{\nu(E) : U \supset E,\ U \text{ open}\right\},$$

meaning that ν is outer regular.

3.3 Part c

For any $E \in \mathcal{B}_X$, since ν and ν' agree on open sets and both are outer regular,

$$\nu(E) = \inf\left\{\nu(E) : U \supset E,\ U \text{ open}\right\} = \inf\left\{\nu'(E) : U \supset E,\ U \text{ open}\right\} = \nu'(E),$$

so $\nu = \nu'$. Since ν' is Radon, ν is as well.

4 Problem 4 (Folland Problem 18)

Let μ be a σ-finite Radon measure on X and $\nu \in M(X)$ where $\nu = \nu_1 + \nu_2$ is the Lebesgue decomposition of ν with respect to μ. That is, $\nu_1 \perp \mu$ and $\nu_2 \ll \mu$, meaning that $\mathrm{d}\nu_2 = f\mathrm{d}\mu$ for some $f \in L^1(\mu)$. We decompose f into positive and negative real and imaginary parts by $f = f_R^+ - f_R^- + \mathrm{i}f_I^+ - \mathrm{i}f_I^-$ where $f_R^+, f_R^-, f_I^+, f_I^- \in L^1(\mu)$ are all nonnegative. Then, by Problem 2, the measures defined by $\mathrm{d}\nu_R^+ = f_R^+\mathrm{d}\mu$, $\mathrm{d}\nu_R^- = f_R^-\mathrm{d}\mu$, $\mathrm{d}\nu_I^+ = f_I^+\mathrm{d}\mu$, $\mathrm{d}\nu_I^- = f_I^-\mathrm{d}\mu$ are all Radon, i.e., $\nu_R^+, \nu_R^-, \nu_I^+, \nu_I^- \in M(X)$. By Proposition 7.16, $M(X)$ is a vector space, so $\nu_2 = \nu_R^+ - \nu_R^- + \mathrm{i}\nu_I^+ - \mathrm{i}\nu_I^- \in M(X)$, meaning that ν_2 is Radon. Since $\nu \in M(X)$, this also implies that $\nu_1 = \nu - \nu_2 \in M(X)$. Thus, ν_1 and ν_2 are both Radon, as desired.