MATH 524 Homework 4

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October 27, 2023

1 Problem 1

For a bounded real-valued function $f(x)$ defined on a metric space, define the upper and lower envelope functions by the rule

$$\overline{f}(x) = \lim_{r>0}\left(\sup_{d(y,x)<r} f(y)\right), \quad \underline{f}(x) = \lim_{r>0}\left(\inf_{d(y,x)<r} f(y)\right),$$

and define the oscillation function for f as

$$\omega(x) = \overline{f}(x) - \underline{f}(x).$$

1.1 Part a

To show that the function f is continuous at x if and only if $\omega(x) = 0$, first let f be continuous and fix $\epsilon > 0$. Then, there exists some $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ if $d(x, y) < \delta$. This implies that

$$\sup_{d(y,x)<\delta} f(y) \leq f(x) + \epsilon.$$

This shows that $\overline{f}(x) = f(x)$ because

$$\left|\sup_{d(y,x)<r} f(y) - f(x)\right| \leq \epsilon,$$

whenever $r \leq \delta$. Similarly,

$$\inf_{d(y,x)<\delta} f(y) \geq f(x) - \epsilon,$$

so

$$\left|\inf_{d(y,x)<r} f(y) - f(x)\right| \leq \epsilon,$$

whenever $r \leq \delta$. Thus, $\underline{f}(x) = f(x)$, and $\omega(x) = 0$.

Now, assume that $\omega(x) = 0$ for some x and fix $\epsilon > 0$. By definition,

$$\underline{f}(x) \leq f(x) \leq \overline{f}(x),$$

so $\omega(x) = 0$ implies that $\underline{f}(x) = \overline{f}(x) = f(x)$. Thus, there exist some $\delta_1, \delta_2 > 0$ such that

$$\sup_{d(y,x)<\delta_1} f(y) - f(x) < \epsilon, \quad f(x) - \inf_{d(y,x)<\delta_2} f(y) < \epsilon.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then, if $d(y, x) < \delta$,

$$|f(y) - f(x)| \leq \max\left\{\sup_{d(y,x)<\delta_1} f(y) - f(x), f(x) - \inf_{d(y,x)<\delta_2} f(y)\right\} < \epsilon.$$

Thus, f is continuous at x.

1.2 Part b

For a given $\epsilon > 0$, consider the set $A_\epsilon = \{x \mid \omega(x) < \epsilon\}$. Let $x \in A_\epsilon$ and fix $\varepsilon > 0$. Then, by definition, there exists some $2\delta > 0$ such that

$$\sup_{d(y,x)<2\delta} f(y) - \inf_{d(y,x)<2\delta} f(y) < \epsilon + \varepsilon.$$

Let $z \in \mathcal{B}_\delta(x)$. Then, if $d(y,z) < \delta$, the triangle inequality gives that

$$d(y,x) \leq d(y,z) + d(z,x) < \delta + \delta = 2\delta.$$

Thus,

$$\sup_{d(y,z)<\delta} f(y) - \inf_{d(y,z)<\delta} f(y) \leq \sup_{d(y,x)<2\delta} f(y) - \inf_{d(y,x)<2\delta} f(y) < \epsilon + \varepsilon.$$

Thus, for any $\varepsilon > 0$, there exists some $\delta > 0$ for which this is true, so we have the limit

$$\omega(z) = \lim_{r>0} \left(\sup_{d(y,z)<r} f(y) - \inf_{d(y,z)<r} f(y) \right) < \epsilon.$$

Thus, $z \in A_\epsilon$, meaning that $\mathcal{B}_\delta(x) \subset A_\epsilon$, and A_ϵ is an open set.

1.3 Part c

Now, consider the set

$$A = \bigcap_{j=1}^{\infty} A_{1/j} = \{x \mid \omega(x) = 0\}.$$

From part a, we have that A is precisely the set of x where $f(x)$ is continuous. Furthermore, part b gives that A is a countable intersection of open sets, meaning that A is also a Borel set. Thus, the set of x where $f(x)$ is continuous is a Borel set.

2 Problem 2 (Folland Problem 9)

Let $f : [0,1] \to [0,1]$ be the Cantor function and let $g(x) = f(x) + x$.

2.1 Part a

To show that g is a bijection from $[0,1]$ to $[0,2]$, we first let $x, y \in [0,1]$ and assume that $g(x) = g(y)$. Then, $f(x) + x = f(y) + y$, so $f(x) - f(y) = y - x$. Assume without loss of generality that $y \geq x$. Then, we have from the first paragraph on page 39 of Folland that f is nondecreasing, so $f(y) \geq f(x)$. Thus,

$$0 \geq f(x) - f(y) = y - x \geq 0,$$

so we must have that $f(x) - f(y) = y - x = 0$. Thus, $x = y$ and g is injective. From the same paragraph in Folland, f is continuous, so g is also continuous. Since $g(0) = f(0) = 0$ and $g(1) = f(1) + 1 = 2$, the intermediate value theorem implies that every value in $[0,2]$ is attained. Thus, g is surjective and therefore a bijection.

From undergrad analysis (Theorem 4.17 in Baby Rudin), we have that a continuous bijective function defined on a compact set has a continuous inverse. Since $[0,1]$ is compact, it immediately follows that $h = g^{-1}$ is continuous from $[0,2]$ to $[0,1]$.

2.2 Part b

By construction, we have that

$$C^c = \bigsqcup_{j=1}^{\infty} I_j,$$

where each I_j is an open interval and $I_j \cap I_k = \emptyset$ when $j \neq k$. Define $(a_j, b_j) = I_j$. Then, $f(x) = a_j$ for all $x \in I_j$. Thus, $g(x) = a_j + x$ for all $x \in I_j$. The continuity of g and the intermediate value theorem give that $g(I_j) = (2a_j, a_j + b_j)$. Thus,

$$m(g(C^c)) = m\left(\bigsqcup_{j=1}^{\infty} g(I_j)\right) = \sum_{j=1}^{\infty} m(g(I_j)) = \sum_{j=1}^{\infty}(b_j - a_j) = \sum_{j=1}^{\infty} m(I_j) = m(C^c),$$

since the image of a disjoint union is the disjoint union of the images because g is a bijection. Now, Theorem 1.22b in Folland gives that $m(C) = 0$, so $m(C^c) = 1$. Thus,

$$2 = m([0,2]) = m\left(g(C^c) \sqcup g(C)\right) = m\left(g(C^c)\right) + m\left(g(C)\right) = 1 + m\left(g(C)\right),$$

so $m\left(g(C)\right) = 1$.

3 Problem 3 (Folland Problem 14)

Let $f \in L^+$ and define $\lambda(E) = \int_E f \mathrm{d}\mu$ for any $E \in \mathcal{M}$. To show that λ is a measure on $\mathcal{M}$, we verify the required axioms.

- For any $E \in \mathcal{M}$,
$$\lambda(E) = \int_E f \mathrm{d}\mu = \int f \mathbb{1}_E \mathrm{d}\mu \geq 0,$$
because $f\mathbb{1}_E \in L^+$ since both f and $\mathbb{1}_E$ are nonnegative. Thus, λ is nonnegative.
- $$\lambda(\emptyset) = \int_\emptyset f \mathrm{d}\mu = \int f \mathbb{1}_\emptyset \mathrm{d}\mu = \int 0 \mathrm{d}\mu = 0,$$
since $\mathbb{1}_\emptyset$ is the zero function by definition.
- Let $\{E_j\}_{j=1}^\infty$ be a collection of disjoint elements of $\mathcal{M}$ and let $E = \bigcup_{j=1}^\infty E_j$. Then,
$$\lambda(E) = \int f \mathbb{1}_E \mathrm{d}\mu = \int f \sum_{j=1}^{\infty} \mathbb{1}_{E_j} \mathrm{d}\mu = \int \lim_{n\to\infty} f \sum_{j=1}^{n} \mathbb{1}_{E_j} \mathrm{d}\mu.$$
Now, define
$$f_n = f \sum_{j=1}^{n} \mathbb{1}_{E_j}.$$
Then, $f_n \in L^+$ and $f_n \leq f_{n+1}$ for all n, so we apply the monotone convergence theorem to get that
$$\begin{aligned}\lambda(E) &= \int \lim_{n\to\infty} f \sum_{j=1}^{n} \mathbb{1}_{E_j} \mathrm{d}\mu = \lim_{n\to\infty} \int f \sum_{j=1}^{n} \mathbb{1}_{E_j} \mathrm{d}\mu \\ &= \lim_{n\to\infty} \sum_{j=1}^{n} \int f \mathbb{1}_{E_j} \mathrm{d}\mu = \lim_{n\to\infty} \sum_{j=1}^{n} \lambda(E_j) = \sum_{j=1}^{\infty} \lambda(E_j).\end{aligned}$$
Thus, λ is a measure on $\mathcal{M}$.

Now, let $\phi \in L^+$ be simple, that is

$$\phi = \sum_{j=1}^{n} a_j \mathbb{1}_{E_j},$$

for some constants a_j and sets $E_j \in \mathcal{M}$. Then,

$$\int \phi \mathrm{d}\lambda = \sum_{j=1}^{n} a_j \lambda(E_j) = \sum_{j=1}^{n} a_j \int_{E_j} f \mathrm{d}\mu = \sum_{j=1}^{n} a_j \int f \mathbb{1}_{E_j} \mathrm{d}\mu = \int f \sum_{j=1}^{n} a_j \mathbb{1}_{E_j} \mathrm{d}\mu = \int f\phi \mathrm{d}\mu.$$

Now, let $g \in L^+$. Then, by Theorem 2.10a in Folland, there exists a sequence of simple functions $\{\phi_n\}_{n=1}^{\infty}$ such that $0 \leq \phi_1 \leq \phi_2 \leq \ldots \leq g$, and $\phi_n \to g$ pointwise. Since f is nonnegative, we can multiply this through by f to get that $0 \leq f\phi_1 \leq f\phi_2 \leq \ldots \leq fg$, and $f\phi_n \to fg$ pointwise. Then, applying the monotone convergence theorem twice and using the above equality for simple functions,

$$\int g \mathrm{d}\lambda = \lim_{n\to\infty} \int \phi_n \mathrm{d}\lambda = \lim_{n\to\infty} \int f\phi_n \mathrm{d}\mu = \int fg \mathrm{d}\mu.$$

4 Problem 4 (Folland Problem 15)

Let $\{f_n\}_{n=1}^{\infty} \subset L^+$ such that f_n decreases pointwise to f and $\int f_1 \mathrm{d}\mu < \infty$. Define the functions $g_n = f_1 - f_n$ for all $n \in \mathbb{N}$. Then,

$$\lim_{n\to\infty} g_n = f_1 - \lim_{n\to\infty} f_n = f_1 - f,$$

pointwise. Furthermore, $g_n \leq g_{n+1}$ and $g_n \in L^+$ for all $n \in \mathbb{N}$. Thus, we can apply the monotone convergence theorem to get that

$$\int (f_1 - f)\mathrm{d}\mu = \lim_{n\to\infty} \int (f_1 - f_n)\mathrm{d}\mu. \tag{1}$$

Now, as a lemma, we note that if $g, h, g-h \in L^+$, then by Theorem 2.15,

$$\int g\mathrm{d}\mu = \int (h + (g-h))\mathrm{d}\mu = \int h\mathrm{d}\mu + \int (g-h)\mathrm{d}\mu.$$

Thus,

$$\int (g-h)\mathrm{d}\mu = \int g\mathrm{d}\mu - \int h\mathrm{d}\mu.$$

Applying this to both sides of (1),

$$\int f_1\mathrm{d}\mu - \int f\mathrm{d}\mu = \lim_{n\to\infty} \int f_1\mathrm{d}\mu - \lim_{n\to\infty} \int f\mathrm{d}\mu = \int f_1\mathrm{d}\mu - \lim_{n\to\infty} \int f_n\mathrm{d}\mu.$$

Since $\int f_1\mathrm{d}\mu < \infty$, we can subtract this from both sides and conclude that

$$\int f\mathrm{d}\mu = \lim_{n\to\infty} \int f_n\mathrm{d}\mu.$$

5 Problem 5 (Folland Problem 16)

Let $f \in L^+$ with $\int f\mathrm{d}\mu < \infty$ and fix $\epsilon > 0$. Then, by the definition of the Lebesgue integral, there exists some ϕ simple such that

$$\int f\mathrm{d}\mu < \int \phi\mathrm{d}\mu + \epsilon,$$

and $\phi \leq f$. Let ϕ be given by $\phi = \sum_{j=1}^{n} a_j \mathbb{1}_{E_j}$ for some constants a_j and sets $E_j \in \mathcal{M}$, and define $E = \bigcup_{j=1}^{n} E_j$. We first observe that ϕ is nonzero only on E, so $\phi\mathbb{1}_E = \phi$. Then,

$$\int_E f\mathrm{d}\mu = \int f\mathbb{1}_E\mathrm{d}\mu \geq \int \phi\mathbb{1}_E\mathrm{d}\mu = \int \phi\mathrm{d}\mu > \int f\mathrm{d}\mu - \epsilon.$$

Now, $\int \phi\mathrm{d}\mu < \infty$ because $\phi \leq f$. Since

$$\mu(E) \leq \sum_{j=1}^{n} a_j \mu(E_j) = \int \phi\mathrm{d}\mu,$$

we have that $\mu(E) < \infty$ as well.