AMATH 585 Homework 1

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1 Problem 1

We evaluate the second order accurate approximation

$$u''(x) \approx \frac{u(x+h)+u(x-h)-2u(x)}{h^2}$$

for $u(x) = \sin x$ and $x = \pi/6$ for $h = 10^{-1}, 10^{-2}, \ldots, 10^{-16}$ using the following MATLAB code.

```
u=@(x) sin(x);
x = pi/6;
upptrue = -sin(x); %true u''(x) value
fprintf('  h       FD Quotient      Error\n')
for k=1:16
    h = 10^-k;
    upp = (u(x+h)+u(x-h)-2*u(x))/h^2;
    err = upp-upptrue; %error term
    fprintf('%.e   %e  %e\n',h, upp, err)
end
```

The resulting output is displayed in the following table.

h	FD Approximation	Error
1e-01	-4.995835e-01	4.165278e-04
1e-02	-4.999958e-01	4.166653e-06
1e-03	-5.000000e-01	4.167450e-08
1e-04	-5.000000e-01	3.038735e-09
1e-05	-5.000012e-01	-1.151593e-06
1e-06	-4.999334e-01	6.657201e-05
1e-07	-4.996004e-01	3.996389e-04
1e-08	-1.110223e+00	-6.102230e-01
1e-09	1.110223e+02	1.115223e+02
1e-10	0.000000e+00	5.000000e-01
1e-11	0.000000e+00	5.000000e-01
1e-12	0.000000e+00	5.000000e-01
1e-13	1.110223e+10	1.110223e+10
1e-14	-1.110223e+12	-1.110223e+12
1e-15	0.000000e+00	5.000000e-01
1e-16	-1.110223e+16	-1.110223e+16

One can see that the error term appears to be second order as anticipated at first but actually increases as h decreases when $h < 10^{-4}$. As discussed in class, this is due to the limitations of finite precision arithmetic. As h gets smaller, round-off errors become more prominent and dominate our approximation error. Eventually, we run into catastrophic cancellation and obtain errors that are much larger than that of even $h = 0.1$.

2 Problem 2

Using the same FD formula for the same u and x as in problem 1, we perform two steps of Richardson extrapolation for $h = 0.2$. From class, we have that if $\phi_0(h)$ denotes our original approximation, the first step of Richardson extrapolation is given by

$$\phi_1(h) = \frac{4\phi_0(h/2) - \phi_0(h)}{3}$$

and the second step is given by

$$\phi_2(h) = \frac{16\phi_1(h/2) - \phi_1(h)}{15}.$$

The following MATLAB code computes and prints the values of ϕ_0, ϕ_1, ϕ_2 at the necessary values of h.

```
u=@(x) sin(x);
x = pi/6;
upptrue = -sin(x); %true u''(x) value
fprintf(' h       phi_0(h)       Error\n')
```

```
h = [0.2 0.1 0.05];
upp = (u(x+h)+u(x-h)-2*u(x))./h.^2; %phi_0
err = upp-upptrue;

for i=1:length(h)
    fprintf('%0.2f   %d %d\n',h(i), upp(i), err(i))
end

%phi_1
fprintf(' h       phi_1(h)        Error\n')
R1 = (4*upp(2)-upp(1))/3; %combine 0.2 and 0.1
errR1=R1-upptrue;
R2 = (4*upp(3)-upp(2))/3; %combine 0.1 and 0.05
errR2=R2-upptrue;
fprintf('%0.2f  %d  %d\n',h(1), R1, errR1)
fprintf('%0.2f  %d  %d\n',h(2), R2, errR2)

R3 = (16*R2-R1)/15; %phi_2
errR3=R3-upptrue;
fprintf(' h       phi_2(h)        Error\n')
fprintf('%0.2f  %d  %d\n',h(1), R3, errR3)
```

The resulting output is displayed in the following table.

	FD Approximation	Error
$\phi_0(0.2)$	-4.983356e-01	1.664446e-03
$\phi_0(0.1)$	-4.995835e-01	4.165278e-04
$\phi_0(0.05)$	-4.998958e-01	1.041580e-04
$\phi_1(0.2)$	-4.999994e-01	5.550598e-07
$\phi_1(0.1)$	-5.000000e-01	3.471449e-08
$\phi_2(0.2)$	-5.000000e-01	2.480538e-11

From class, we know that one step of Richardson extrapolation should give O(h^4) error and two steps should give O(h^6) error. This does appear to be roughly true of our obtained error values, as $5.55e-7/(0.2)^4 \approx 3.47e-08/(0.1)^4$ and the error when going from ϕ_1 to ϕ_2 decreases by approximately the same magnitude as when going from ϕ_0 to ϕ_1 (as they should since ϕ_0 is O(h^2)).

3 Problem 3

To derive the error term for the approximation

$$u'(x) \approx \frac{1}{2h}[-3u(x) + 4u(x+h) - u(x+2h)],$$

we first write out the Taylor series

$$
\begin{aligned}
u(x+h) &= u(x) + hu'(x) + \frac{h^2}{2!}u''(x) + \frac{h^3}{3!}u'''(x) + \mathrm{O}(h^4) \\
&= u(x) + hu'(x) + \frac{h^2}{2}u''(x) + \frac{h^3}{6}u'''(x) + \mathrm{O}(h^4)
\end{aligned}
$$

$$
\begin{aligned}
u(x+2h) &= u(x) + 2hu'(x) + \frac{(2h)^2}{2!}u''(x) + \frac{(2h)^3}{3!}u'''(x) + \mathrm{O}(h^4) \\
&= u(x) + 2hu'(x) + 2h^2u''(x) + \frac{4h^3}{3}u'''(x) + \mathrm{O}(h^4)
\end{aligned}
$$

From this, we can compute

$$
\begin{aligned}
&\frac{1}{2h}[-3u(x) + 4u(x+h) - u(x+2h)] \\
&= \frac{1}{2h}\left((-3+4-1)u(x) + (4h-2h)u'(x) + \left(4\frac{h^2}{2} - 2h^2\right)u''(x) + \left(4\frac{h^3}{6} - \frac{4h^3}{3}\right)u'''(x) + \mathrm{O}(h^4)\right) \\
&= \frac{1}{2h}\left(2hu'(x) - \frac{2h^3}{3}u'''(x) + \mathrm{O}(h^4)\right) = u'(x) - \frac{h^2}{3}u'''(x) + \mathrm{O}(h^3).
\end{aligned}
$$

Thus, the error term of the approximation is given by

$$
u'(x) - \frac{h^2}{3}u'''(x) + \mathrm{O}(h^3) - u'(x) = -\frac{h^2}{3}u'''(x) + \mathrm{O}(h^3).
$$

4 Problem 4

Re-purposing the Taylor series written out in problem 3, we can compute

$$
\begin{aligned}
&Au(x) + Bu(x+h) + Cu(x+2h) \\
&= (A+B+C)u(x) + (B+2C)hu'(x) + \left(\frac{B}{2} + 2C\right)h^2u''(x) + \left(\frac{B}{6} + \frac{4C}{3}\right)h^3u'''(x) + \dots.
\end{aligned}
$$

In order to achieve maximal order of accuracy when using this as an approximation to $u''(x)$, we require that

$$
\begin{aligned}
&A + B + C = 0 \\
&B + 2C = 0 \\
&\frac{B}{2} + 2C = \frac{1}{h^2}
\end{aligned}
$$

Of course, we would obtain higher order accuracy if we could also set $\frac{B}{6} + \frac{4C}{3} = 0$, but we already have three equations and three unknowns, so we cannot do this in general.
Solving this system of equations, we get that $A = \frac{1}{h^2}$, $B = -\frac{2}{h^2}$, $C = \frac{1}{h^2}$. To

make the above statement about this being the maximal order of accuracy more clear, note that $\left(\frac{B}{6}+\frac{4C}{3}\right)h^3 = h$, so the term $\left(\frac{B}{6}+\frac{4C}{3}\right)h^3 u'''(x)$ is only zero when $u'''(x)=0$ and not in general. Now, we plug in these values of $A, B,$ and C to conclude that

$$Au(x) + Bu(x+h) + Cu(x+2h) = u''(x) + hu'''(x) + \ldots = u''(x) + \mathrm{O}(h),$$

so this approximation is of order h accuracy.

5 Problem 5

Using the centered difference formulae

$$u''(x) \approx \frac{u_{j+1}+u_{j-1}-2u_j}{h^2}$$

and

$$u'(x) \approx \frac{u_{j+1}-u_{j-1}}{2h},$$

the solution of the BVP

$$u'' + 2xu' - x^2 u = x^2, \quad u(0) = 1, \quad u(1) = 0$$

can be approximated by

$$\frac{u_{j+1}+u_{j-1}-2u_j}{h^2} + 2x_j\frac{u_{j+1}-u_{j-1}}{2h} - x_j^2 u_j = x_j^2$$

for all interior gridpoints x_j. If we take $h = 1/4$, our grid becomes $x_0 = 0,\ x_1 = 1/4,\ x_2 = 1/2,\ x_3 = 3/4,\ x_4 = 1$. Note that to impose the boundary conditions, we take $u_0 = 1,\ u_1 = 0$. Now, we write out this equation explicitly for $j = 1, 2, 3$.

$$\begin{aligned}
&\frac{u_2+1-2u_1}{(1/4)^2} + x_1\frac{u_2-1}{1/4} - (1/4)^2 u_1 = (1/4)^2,\\
&\frac{u_3+u_1-2u_2}{(1/4)^2} + (1/2)\frac{u_3-u_1}{1/4} - (1/2)^2 u_2 = (1/2)^2,\\
&\frac{0+u_2-2u_3}{(1/4)^2} + (3/4)\frac{0-u_2}{h} - (3/4)^2 u_3 = (3/4)^2.
\end{aligned}$$

These can be simplified the equations

$$\begin{aligned}
&(-32-1/16)u_1 + (16+1)u_2 = 1/16 + 1 - 16,\\
&(16-2)u_1 + (-32-1/4)u_2 + (16+2)u_3 = 1/4,\\
&(16-3)u_2 + (-32-9/16)u_3 = 9/16,
\end{aligned}$$

which can be further simplified to

$$\begin{aligned}
&-513/16u_1 + 17u_2 = -239/16,\\
&14u_1 - 129/4u_2 + 18u_3 = 1/4,\\
&13u_2 - 521/16u_3 = 9/16.
\end{aligned}$$

Thus can be rewritten as the matrix equation

$$\begin{pmatrix} -513/16 & 17 & 0 \\ 14 & -129/4 & 18 \\ 0 & 13 & -521/16 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} -239/16 \\ 1/4 \\ 9/16 \end{pmatrix}.$$