AMATH 574 Homework 4

Cade Ballew #2120804

February 8, 2023

1 Problem 8.3

We consider the equation

$$q_t + \bar{u} q_x = aq, \quad q(x,0) = \mathring{q}(x).$$

1.1 Part a

We wish to show that the method

$$Q_j^{n+1} = Q_j^n - \frac{\bar{u}\Delta t}{\Delta x}(Q_j^n - Q_{j-1}^n) + \Delta t a Q_i^n$$

is first-order accurate when applied to this problem. We compute the LTE and Taylor expand

$$\begin{aligned}
\tau^n &= \frac{1}{\Delta t}\left(q(x_j,t_n) - \frac{\bar{u}\Delta t}{\Delta x}(q(x_j,t_n) - q(x_{j-1},t_n)) + \Delta t a q(x_j,t_n) - q(x_j,t_{n+1})\right) \\
&= aq + \frac{1}{\Delta t}\left(\left(-q_t\Delta t - \frac{1}{2}q_{tt}(\Delta t)^2 + \mathrm{O}(\Delta t^3)\right) + \frac{\bar{u}\Delta t}{\Delta x}\left(-q_x\Delta x + \frac{1}{2}q_{xx}(\Delta x)^2 + \mathrm{O}(\Delta x^3)\right)\right) \\
&= -(q_t + \bar{u}q_x - aq) + \frac{1}{2}\Delta x\bar{u}q_{xx} + \mathrm{O}(\Delta x^2) - \frac{1}{2}\Delta t q_{tt} + \mathrm{O}(\Delta t^2) \\
&= \frac{1}{2}\Delta x(1-\nu)\bar{u}q_{xx}(x_j,t_n) + \mathrm{O}(\Delta t^2).
\end{aligned}$$

where $\nu = \bar{u}\Delta t/\Delta x$ is the Courant number, so the method is indeed first-order accurate.

1.2 Part b

Now, we assume that $0 \leq \nu \leq 1$. To see that our method is 1-norm Lax-Richtmyer stable, we bound

$$\begin{aligned}\|Q^{n+1}\|_1 &= \Delta x \sum_j |Q_j^n - \nu(Q_j^n - Q_{j-1}^n) + \Delta t a Q_i^n| \\ &\leq \Delta x \left((1 - \nu + |a|\Delta t) \sum_j |Q_j^n| + \nu \sum_j |Q_{j-1}^n| \right) \\ &= (1 - \nu + |a|\Delta t)\|Q^n\|_1 + \nu \|Q^n\|_1 = (1 + |a|\Delta t)\|Q^n\|_1\end{aligned}$$

which follows from the triangle inequality and reindexing. This is precisely a bound of the form (8.23) with $\alpha = |a|$, so our method is indeed 1-norm Lax-Richtmyer stable.

1.3 Part c

To see that our method is TVB, we bound

$$\begin{aligned}\mathrm{TV}(Q^{n+1}) &= \sum_j |Q_j^{n+1} - Q_{j-1}^{n+1}| \\ &= \sum_j |(1 - \nu + a\Delta t)(Q_j^n - Q_{j-1}^n) + \nu(Q_{j-1}^n - Q_{j-2}^n)| \\ &\leq (1 - \nu + |a|\Delta t) \sum_j |Q_j^n - Q_{j-1}^n| + \nu \sum_j |Q_{j-1}^n - Q_{j-2}^n| \\ &= (1 - \nu + |a|\Delta t)\mathrm{TV}(Q^n) + \nu \mathrm{TV}(Q^n) = (1 + |a|\Delta t)\mathrm{TV}(Q^n)\end{aligned}$$

by the triangle inequality and reindexing with the same restrictions on ν (between 0 and 1) as before. This is a bound of the form (8.38), so our method is indeed TVB; however, it is only necessarily TVD if $a = 0$.

2 Problem 8.5

To prove Harten's theorem, we bound

$$\begin{aligned}\mathrm{TV}(Q^{n+1}) &= \sum_j |Q_{j+1}^{n+1} - Q_j^{n+1}| \\ &= \sum_j \left|(1 - C_j^n - D_j^n)(Q_{j+1}^n - Q_j^n) + D_{j+1}^n(Q_{j+2}^n - Q_{j+1}^n) + C_{j-1}^n(Q_j^n - Q_{j-1}^n)\right| \\ &\leq (1 - C_j^n - D_j^n) \sum_j |Q_{j+1}^n - Q_j^n| + \sum_j D_{j+1}^n |Q_{j+2}^n - Q_{j+1}^n| + \sum_j C_{j-1}^n |Q_j^n - Q_{j-1}^n| \\ &= (1 - C_j^n - D_j^n)\mathrm{TV}(Q^n) + D_j^n \mathrm{TV}(Q^n) + C_j^n \mathrm{TV}(Q^n) = \mathrm{TV}(Q^n)\end{aligned}$$

by the triangle inequality and reindexing. Thus, such a method satisfying the property that these coefficients be nonnegative is in fact TVD.

3 Problem 8.6

We wish to show that the method (4.64) is 1-norm stable when $1 \leq \nu \leq 2$ where $\nu = \bar{u}\Delta t/\Delta x$ is the Courant number. We bound

$$\begin{aligned}
\|Q^{n+1}\|_1 &= \Delta x \sum_j \left|(2-\nu)Q^n_{j-1} - (\nu-1)Q^n_{j-2}\right| \\
&\leq \Delta x \left((2-\nu)\sum_j |Q^n_{j-1}| + (\nu-1)\sum_j |Q^n_{j-2}| \right) \\
&= (2-\nu)\|Q^n\|_1 + (\nu-1)\|Q^n\|_1 = \|Q^n\|_1
\end{aligned}$$

by the triangle inequality and reindexing along with the fact that $2-\nu, \nu-1 \geq 0$ by our assumption. Thus, this method is indeed 1-norm stable with our choice of ν.

4 Problem 11.1

Assume that we are solving the scalar conservation law $q_t + f(q)_x = 0$ with smooth $q(x,0)$. Then, differentiating (11.11) gives that

$$q_x = \xi_x q_x(\xi, 0).$$

Since our initial condition is smooth, q_x becomes infinite when ξ_x becomes infinite. Differentiating (11.12),

$$1 = \xi_x + \xi_x q_x(\xi,0) f''(q(\xi,0))t,$$

so

$$\xi_x = \frac{1}{1 + q_x(\xi,0) f''(q(\xi,0))t},$$

so ξ_x becomes infinite when

$$t = \frac{-1}{f''(q(\xi,0))q_x(\xi,0)}.$$

Since we assume $t \geq 0$, this only occurs if the denominator is negative for some ξ. If this holds, the we wish to find the smallest time for which it occurs, so

$$T_b = \min_\xi \left\{ \frac{-1}{f''(q(\xi,0))q_x(\xi,0)} \right\} = \frac{-1}{\min_x \left\{ f''(q(x,0))q_x(x,0) \right\}}$$

where we have changed our dummy variable.

5 Problem 11.3

Consider (11.21)

$$s = \frac{f(q_r) - f(q_l)}{q_r - q_l}$$

for a general smooth scalar flux function f. If we Taylor expand the first term in the numerator around q_l and the second around q_r, we get that

$$\begin{aligned} s = {} & \frac{1}{q_r - q_l}\bigg(f(q_l) + f'(q_l)(q_r - q_l) + \frac{1}{2}f''(q_l)(q_r - q_l)^2 - f(q_r) + f'(q_r)(q_r - q_l) \\ & - \frac{1}{2}f''(q_r)(q_r - q_l)^2 + \mathrm{O}(|q_r - q_l|^3)\bigg) \\ & = -s + (f'(q_l) + f'(q_r)) + \frac{1}{2}(f''(q_l) - f''(q_r))(q_r - q_l) + \mathrm{O}(|q_r - q_l|^2). \end{aligned}$$

Now, we Taylor expand

$$f''(q_l) = f''(q_r) + \mathrm{O}(|q_r - q_l|),$$

which we plug in to get that

$$s = -s + (f'(q_l) + f'(q_r)) + \mathrm{O}(|q_r - q_l|^2).$$

Solving this for s yields that

$$s = \frac{1}{2}(f'(q_l) + f'(q_r)) + \mathrm{O}(|q_r - q_l|^2).$$

6 Problem 11.5

Consider Burgers' equation with initial data

$$\mathring{u}(x) = \begin{cases} 2, & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

This produces a rarefaction wave initially at $x = 0$ and a shock initially at $x = 1$. Letting T_c denote the time at which the shock catches up, we first solve the equation for $t < T_c$. In the rarefaction wave, we have a similarity solution, $u(x,t) = \tilde{u}(x,t)$. (11.27) tells us that $\tilde{u}(x,t) = x/t$ which we note occurs for $0 < x/t < 2$. To fit the shock, Rankine-Hugoniot gives that it moves with speed

$$s = \frac{1}{2}(0 + 2) = 1.$$

Thus, our solution for $t < T_c$ is given by

$$u(x,t) = \begin{cases} 0, & x < 0, \\ x/t, & 0 < x < 2t, \\ 2, & 2t < x < t + 1, \\ 0, & x > t + 1. \end{cases}$$

From this, we infer that $T_c = 1$ which is where this solution fails to hold.

6.1 Part a

Let $x_s(t)$ denote the shock location at time t. Rankine-Hugoniot now gives that

$$x_s'(t) = \frac{1}{2}(u_l + u_r) = \frac{x_s(t)}{2t}.$$

This is a separable ODE with general solution

$$x_s(t) = c_1\sqrt{t}.$$

Plugging in the location $x = 2$ at time $t = T_c = 1$ gives that $c_1 = 2$, so

$$x_s(t) = 2\sqrt{t}.$$

6.2 Part b

We can instead obtain x_s by noting that the exact solution is triangular with base x_s and height x_s/t. Noting that our solution has initial area 2 which is conserved, we get that

$$\frac{1}{2}\frac{x_s}{t}x_s = 2$$

which simplifies to

$$x_s(t) = 2\sqrt{t}.$$

7 Problem 11.8

Consider the scalar conservation law $u_t + (e^u)_x = 0$.

7.1 Part a

Let the initial data be given by

$$\mathring{u}(x) = \begin{cases} 1, & x < 0, \\ 0, & x > 0. \end{cases}$$

Here, a shock forms immediately, so we use Rankine-Hugoniot to compute its speed as

$$s = \frac{e^0 - e^1}{0 - 1} = e - 1.$$

Thus, our solution is given by

$$u(x,t) = \begin{cases} 1, & x < (e-1)t, \\ 0, & x > (e-1)t. \end{cases}$$

7.2 Part b

Let the initial data be given by

$$\mathring{u}(x) = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases}$$

Here, we instead get a rarefaction wave. (11.27) tells us that

$$e^{\tilde{u}(x,t)} = \frac{x}{t},$$

so following (11.28), the solution is given by

$$u(x,t) = \begin{cases} 0, & x < t, \\ \log(x/t), & t < x < et, \\ 1, & x > et. \end{cases}$$

7.3 Part c

Let the initial data be given by

$$\mathring{u}(x) = \begin{cases} 2, & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Now, we have a rarefaction wave followed by a shock which collide at some time $t = T_c$. We can solve this for $t < T_c$ by piecing together our solutions from parts a and b modified for a jump of 2. Namely, our solution is given by

$$u(x,t) = \begin{cases} 0, & x < t, \\ \log(x/t), & t < x < e^2 t, \\ 2, & e^2 t < x < 1 + (e^2 - 1)t/2, \\ 0, & x > 1 + (e^2 - 1)t/2 \end{cases}$$

since the shock speed is now $s = (e^2 - 1)/2$. Thus, this breaks at $T_c = 2/(e^2 + 1)$. To solve this for $t > T_c$, we let $x_s(t)$ denote the shock location. Rankine-Hugoniot then gives that

$$x_s' = \frac{1 - x_s/t}{0 - \log(x_s/t)},$$

so the shock location at time t is obtained by solving the ODE

$$x_s'(t) = \frac{x_s(t)}{t} \log \frac{x_s(t)}{t},$$
$$x_s(2/(e^2 + 1)) = \frac{2e^2}{e^2 + 1}.$$

8 Coding problem

See the attached Jupyter notebook.