MATH 525 Homework 7

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1 Problem 1

Let $1 \leq p \leq q < \infty$ and (X, μ), (Y, ν) be σ-finite. Then, $1 \leq \frac{q}{p} < \infty$, so applying Minkowski's integral inequality to $|f|^p$ with index $\frac{q}{p}$ gives that

$$\left(\int_Y \left(\int_X |f(x,y)|^p \mathrm{d}\mu(x)\right)^{\frac{q}{p}} \mathrm{d}\nu(y)\right)^{\frac{p}{q}} \leq \int_X \left(\int_Y |f(x,y)|^q \mathrm{d}\nu(y)\right)^{\frac{p}{q}} \mathrm{d}\mu(x).$$

Taking the pth root of both sides,

$$\left(\int_Y \left(\int_X |f(x,y)|^p \mathrm{d}\mu(x)\right)^{\frac{q}{p}} \mathrm{d}\nu(y)\right)^{\frac{1}{q}} \leq \left(\int_X \left(\int_Y |f(x,y)|^q \mathrm{d}\nu(y)\right)^{\frac{p}{q}} \mathrm{d}\mu(x)\right)^{\frac{1}{p}},$$

as desired.

To see that this inequality can fail if $q < p$, consider $f(x,y) = \cos(2\pi(x-y)) + 1$ on the unit square with $q = 1$, $p = 2$. Then, since f is nonnegative, the inequality is given by

$$\int_0^1 \left(\int_0^1 (\cos(2\pi(x-y)) + 1)^2 \mathrm{d}x\right)^{1/2} \mathrm{d}y \leq \left(\int_0^1 \left(\int_0^1 (\cos(2\pi(x-y)) + 1)\mathrm{d}y\right)^2 \mathrm{d}x\right)^{1/2}.$$

We have that

$$\int_0^1 (\cos(2\pi(x-y)) + 1)^2 \mathrm{d}x = \frac{3}{2},$$

so

$$\int_0^1 \left(\int_0^1 (\cos(2\pi(x-y)) + 1)^2 \mathrm{d}x\right)^{1/2} \mathrm{d}y = \sqrt{\frac{3}{2}}.$$

On the other hand,

$$\int_0^1 (\cos(2\pi(x-y)) + 1)\mathrm{d}y = 1,$$

so

$$\left(\int_0^1 \left(\int_0^1 (\cos(2\pi(x-y)) + 1)\mathrm{d}y\right)^2 \mathrm{d}x\right)^{1/2} = 1,$$

and the inequality fails.

2 Problem 2 (Folland Problem 21)

Let $1 < p < \infty$ and assume that $f_n \to f$ weakly in $l^p(A)$. Then, for each $a \in A$, $\mathbb{1}_{\{a\}} \in l^q(A)$ where q is the dual index to p corresponds to an element $\phi_a \in l^p(A)^*$ defined such that

$$\phi_a(g) = \int g \mathbb{1}_{\{a\}} = \sum_{a' \in A} g(a') \mathbb{1}_{\{a\}}(a') = g(a).$$

Thus, weak convergence applied to each ϕ_a gives that $f_n(a) \to f(a)$ for all $a \in A$. That is, $f_n \to f$ pointwise. Since $l^p(A)$ is reflexive, let $\hat{f}_n$ denote the double dual element corresponding to each f_n. Then, for each $\phi \in l^p(A)^*$, $\lim_{n\to\infty} \phi(f_n)$ converges to $\phi(f)$, so

$$\sup_n |\hat{f}_n(\phi)| = \sup_n |\phi(f_n)| < \infty.$$

Since this holds for all $\phi \in l^p(A)^*$, the uniform boundedness principle implies that

$$\sup_n \|f_n\|_p = \sup_n \|\hat{f}_n\|_{l^p(A)^{**}} < \infty,$$

as desired.

Conversely, assume that $\sup_n \|f_n\|_p = M < \infty$ and $f_n \to f$ pointwise. First, by Fatou's lemma and the continuity of exponentiation, we have that

$$\|f\|_p^p = \int |f|^p \leq \liminf_{n\to\infty} |f_n|^p = \|f_n\|_p^p,$$

so $f \in l^p(A)$ and $\|f\|_p \leq M$. Now, fix $\epsilon > 0$ and let $\phi \in l^p(A)^*$ be given where g denotes its corresponding function in $l^q(A)$. Since

$$\|g\|_g^q = \sum_{a\in A} |g(a)|^q < \infty,$$

there must be some countable subset $B \subset A$ such that $g(a) = 0$ for $a \in A \setminus B$. Denote the elements of b by $b_1, b_2, \ldots$. Then, $\sum_{j=1}^\infty |g(x_j)|^q < \infty$, so there must be some $J \in \mathbb{N}$ such that

$$\sum_{j=J+1}^\infty |g(x_j)|^q < \left(\frac{\epsilon}{4M}\right)^q.$$

Since $f_n \to f$ pointwise, for each $j = 1, \ldots J$, we can find some N_j such that for all $n \geq N_j$,

$$|f_n(b_j) - f(b_j)| < \frac{\epsilon}{2J|g(b_j)|},$$

when $g(b_j) \neq 0$. If $g(b_j) = 0$, it suffices to choose $N_j = 1$. Let $N = \max_{j=1,\ldots,J} N_j$. Then, for all $n \geq N$, by Hölder's inequality on $B \setminus \{b_1, \ldots, b_j\}$,

$$\begin{aligned}
|\phi(f_n) - \phi(f)| &= \sum_{j=1}^\infty |f_n(b_j) - f(b_j)||g(b_j)| = \sum_{j=1}^J |f_n(b_j) - f(b_j)||g(b_j)| + \sum_{j=J+1}^\infty |f_n(b_j) - f(b_j)||g(b_j)| \\
&< \frac{\epsilon}{2} + \left(\sum_{j=J+1}^\infty |f_n(b_j) - f(b_j)|^p\right)^{1/p} \left(\sum_{j=J+1}^\infty |g(b_j)|^q\right)^{1/q} < \frac{\epsilon}{2} + \frac{\epsilon}{4M}\left(\sum_{a\in A} |f_n(a) - f(a)|^p\right)^{1/p} \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{4M}\|f_n - f\|_p \leq \frac{\epsilon}{2} + \frac{\epsilon}{4M}(\|f_n\|_p + \|f\|_p) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Thus, $f_n \to f$ weakly in $l^p(A)$.

3 Problem 3 (Folland Problem 31)

Let $1 \leq p_j \leq \infty$, $\sum_{j=1}^n p_j^{-1} = r^{-1} \leq 1$, and $f_j \in L^{p_j}$ for $j = 1, \ldots, n$. First consider the case $n = 2$. If $1 < p_1, p_2 < \infty$, then by Hölder's inequality applied to $|f_1|^r |f_2|^r$ with indices p_1/r and p_2/r,

$$\|f_1 f_2\|_r^r = \int |f_1 f_2|^r \leq \left(\int (|f_1|^r)^{p_1/r}\right)^{r/p_1} \left(\int (|f_2|^r)^{p_2/r}\right)^{r/p_2} = \|f_1\|_{p_1}^r \|f_2\|_{p_2}^r.$$

Note that this quantity is finite since $f_1 \in L^{p_1}$ and $f_2 \in L^{p_2}$, so $f_1 f_2 \in L^r$. Furthermore, taking the rth root of each side gives that $\|f_1 f_2\|_r \leq \|f_1\|_{p_1} \|f_2\|_{p_2}$. If either p_1 or p_2 is 1, then the other must be infinity and $r = 1$, so the result is equivalent to Hölder's inequality with indices 1 and infinity. If instead we assume without loss of generality that $p_1 = \infty$, then $r = p_2$ and

$$\|f_1 f_2\|_r^r = \int |f_1 f_2|^r \leq \|f_1\|_\infty^r \int |f_2|^r = \|f_1\|_\infty^r \|f_2\|_r^r,$$

and the result again follows by taking the rth root of each side.

To apply induction, we assume that the result holds for $n = k$. That is, $\prod_{j=1}^k f_j \in L^{r'}$ and $\| \prod_{j=1}^k f_j\|_{r'} \leq \prod_{j=1}^k \|f_j\|_{p_j}$, where $r'^{-1} = \sum_{j=1}^k p_j^{-1}$. To show that the result holds for $n = k+1$, we again let $1 < p_{k+1}, r' < \infty$ and apply Hölder's inequality with indices p_{k+1}/r and r'/r. Then,

$$\left\|\prod_{j=1}^{k+1} f_j\right\|_r^r \leq \left(\int (|f_{k+1}|^r)^{p_{k+1}/r}\right)^{r/p_{k+1}} \left(\int \left(\left|\prod_{j=1}^{k+1} f_j\right|^r\right)^{r'/r}\right)^{r/r'} = \|f_{k+1}\|_{p_{k+1}}^r \left\|\prod_{j=1}^{k} f_j\right\|_{r'}^r.$$

This quantity is finite by the inductive hypothesis, so $\prod_{j=1}^{k+1} f_j \in L^r$, and taking rth roots gives that

$$\left\|\prod_{j=1}^{k+1} f_j\right\|_r \leq \|f_{k+1}\|_{p_{k+1}} \left\|\prod_{j=1}^{k} f_j\right\|_{r'} \leq \|f_{k+1}\|_{p_{k+1}} \prod_{j=1}^{k} \|f_j\|_{p_j} = \prod_{j=1}^{k+1} \|f_j\|_{p_j}.$$

The case where either p_{k+1} or r' is 1 or infinity follows by the same argument as before, as this inductive step simply amounts to applying the $n = 2$ case to f_{k+1} and $\prod_{j=1}^k f_j$. Thus, the inductive step holds for all $1 \leq p_{k+1}, r' \leq \infty$, so by induction, we have that $\prod_{j=1}^k f_j \in L^{r'}$ and $\| \prod_{j=1}^n f_j\|_{r'} \leq \prod_{j=1}^n \|f_j\|_{p_j}$ for all $n \geq 2$.

4 Problem 4 (Folland Problem 32)

Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces, $K \in L^2(\mu \times \nu)$, and $f \in L^2(\nu)$. Then, by Fubini–Tonelli,

$$\int \left(\int |K(x,y)|^2 \mathrm{d}\nu(y)\right) \mathrm{d}\mu(x) = \int \int |K(x,y)|^2 \mathrm{d}(\mu \times \nu)(x,y) < \infty,$$

so in particular, the inner integral is finite for almost every $x \in X$. Then, Hölder's inequality gives that for almost all $x \in X$,

$$|Tf(x)| \leq \int |K(x,y) f(y)| \mathrm{d}\nu(y) \leq \left(\int |K(x,y)|^2 \mathrm{d}\nu(y)\right)^{1/2} \|f\|_2 < \infty,$$

so $Tf(x)$ converges absolutely for almost every $x \in X$. By Minkowski's inequality,

$$\begin{aligned}\|Tf\|_2 = \left(\int |Tf(x)|^2 \,\mathrm{d}\mu(x)\right)^{1/2} &\leq \left(\int \left(\int |K(x,y) f(y)| \mathrm{d}\nu(y)\right)^2 \mathrm{d}\mu(x)\right)^{1/2} \\ &\leq \int \left(\int |K(x,y) f(y)|^2 \mathrm{d}\mu(x)\right)^{1/2} \mathrm{d}\nu(y) = \int \left(\int |K(x,y)|^2 \mathrm{d}\mu(x)\right)^{1/2} |f(y)| \mathrm{d}\nu(y).\end{aligned}$$

Applying Hölder's inequality,

$$\|Tf\|_2 \leq \|f\|_2 \left(\int |K(x,y)|^2 \mathrm{d}\mu(x) \mathrm{d}\nu(y)\right)^{1/2} = \|f\|_2 \|K\|_2,$$

by Fubini–Tonelli, as desired.

5 Problem 5 (Folland Problem 36)

Let f be weak L^p and $\mu(\{x : f(x) \neq 0\}) < \infty$. Let $M = [f]_p$ and $L = \mu(\{x : f(x) \neq 0\})$. Then, by definition, $\lambda_f(\alpha) \leq L$ for all $\alpha \in (0, \infty)$. This inequality also holds at $\alpha = 0$ if λ_f is extended to be defined there. Furthermore, for all $\alpha \in (0, \infty)$, $\alpha^p \lambda_f(\alpha) \leq M$. Let $\epsilon > 0$. Then, by Proposition 6.24, for $q < p$,

$$\begin{aligned}\|f\|_q^q &= \int |f|^q \mathrm{d}\mu = q \int_0^\infty \alpha^{q-1} \lambda_f(\alpha) \mathrm{d}\alpha = q \int_0^\epsilon \alpha^{q-1} \lambda_f(\alpha) \mathrm{d}\alpha + q \int_\epsilon^\infty \alpha^{q-1} \lambda_f(\alpha) \mathrm{d}\alpha \\ &\leq qL \int_0^\epsilon \alpha^{q-1} \mathrm{d}\alpha + qM \int_\epsilon^\infty \alpha^{q-p-1} \mathrm{d}\alpha = L \left[\alpha^q\right]_0^\epsilon + M \frac{q}{q-p} \left[\alpha^{q-p}\right]_\epsilon^\infty = L\epsilon^q - M \frac{q}{q-p} \epsilon^{q-p}.\end{aligned}$$

This quantity is finite for any ϵ, so $\|f\|_q < \infty$ and $f \in L^q$ for all $q < p$.

Now, let f be in both weak L^p and L^∞ and fix $q > p$. As before, let $M = [f]_p$, so $\alpha^p \lambda_f(\alpha) \leq M$ for all $\alpha \in (0, \infty)$. Because $f \in L^\infty$,

$$\|f\|_\infty = \inf\{\alpha > 0 : \lambda_f(\alpha) = 0\} < \infty,$$

so there exists some $a > 0$ such that $\lambda_f(\alpha) = 0$ for all $\alpha > a$. Then, by Proposition 6.24,

$$\begin{aligned}\|f\|_q^q &= \int |f|^q \mathrm{d}\mu = q \int_0^\infty \alpha^{q-1} \lambda_f(\alpha) \mathrm{d}\alpha = q \int_0^a \alpha^{q-1} \lambda_f(\alpha) \mathrm{d}\alpha \leq qM \int_0^a \alpha^{q-p-1} \mathrm{d}\alpha \\ &= \frac{q}{q-p} M \left[\alpha^{q-p}\right]_0^a = \frac{q}{q-p} M a^{q-p}.\end{aligned}$$

This quantity is finite, so $\|f\|_q < \infty$ and $f \in L^q$ for all $q > p$.