Cade Ballew

AMATH 561 Problem Set 3 -

1. Consider a probability space (Ω, F, P) where $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $F = 2^\Omega$, and $P(\{\omega_i\}) = 1/3$ $\forall i \in \{1,2,3\}$. Of course, P is a probability measure if we let $P(\emptyset)=0$ and $P(A) = \sum_{\omega_i \in A} P(\{\omega_i\})$ (This of course gives that $P(\bigcup A_i) = \sum P(A_i)$ for disjoint A_i). Because $F = 2^\Omega$ contains all subsets of Ω, $X: \Omega \to \mathbb{R}$ is a random variable $\forall$ function X. Let $X(\omega_i) = i-2$ $\forall i \in \{1,2,3\}$. Then, $\sigma(X) = 2^\Omega$. This is because $X(\omega_1) = -1$, $X(\omega_2) = 0$, and $X(\omega_3) = 1$, so we can find Borel sets that contain any combination of these values. Now, consider the function $f(x) = |x|$. Then, for a Borel set $B \in \mathcal{B}$,

$$f(X)^{-1}(B) = \begin{cases} \emptyset \text{ if } 0,1 \notin B \\ \{\omega_1, \omega_3\} \text{ if } 0 \notin B, 1 \in B \\ \{\omega_2\} \text{ if } 0 \in B, 1 \notin B \\ \Omega \text{ if } 0,1 \in B \end{cases}$$

Note that there cannot be a Borel set B such that $f(X)^{-1}(B) = \{\omega_1\}$. Thus, $\sigma(X) \ni \{\omega_1\} \notin \sigma(f(X))$. We must have that $\sigma(f(X)) \subset \sigma(X)$ because $\sigma(X) = 2^\Omega$, so it must hold that $\sigma(f(X)) \subsetneq \sigma(X)$ but clearly, $\sigma(f(X)) \neq \{\emptyset, \Omega\}$ because $\{\omega_1, \omega_3\} \in \sigma(f(X))$.

Now, consider the function g s.t. $g(x) = 0$ $\forall x \in \mathbb{R}$. Then, for a Borel set $B \in \mathcal{B}$,

$$g(X)^{-1}(B) = \begin{cases} \emptyset \text{ if } 0 \notin B \\ \Omega \text{ if } 0 \in B \end{cases}$$

because $g(X)(\omega) = 0$ $\forall \omega \in \Omega$. Thus, $\sigma(g(X)) = \{\emptyset, \Omega\}$.

2. Consider the probability space (Ω, F, P) where $\Omega = \{\omega_1, \dots, \omega_8\}$, $F = 2^{\Omega}$, and $P(\{\omega_i\}) = 1/8$ $\forall i \in \{1, \dots, 8\}$. As in problem 1, this is a probability measure if we let $P(\emptyset) = 0$ and let $P(A) = \sum_{\omega_i \in A} P(\omega_i)$. Consider events $A, B, C \in F$ where $A = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, $B = \{\omega_3, \omega_4, \omega_5, \omega_6\}$, $C = \{\omega_1, \omega_4, \omega_7, \omega_8\}$. Then, $P(A) = P(B) = P(C) = \frac{1}{2}$, $P(A \cap B) = P(\{\omega_3, \omega_4\}) = \frac{1}{4}$, $P(A \cap C) = P(\{\omega_1, \omega_4\}) = \frac{1}{4}$, $P(A \cap B \cap C) = P(\{\omega_4\}) = \frac{1}{8}$, and $P(B \cap C) = P(\{\omega_4\}) = \frac{1}{8}$. Hence, $P(A \cap B) = P(A)P(B)$, $P(A \cap C) = P(A)P(C)$, $P(A \cap B \cap C) = P(A)P(B)P(C)$, but $P(B \cap C) \neq P(B)P(C)$. $A, B,$ and C are not independent, because our definition of independence (lecture 6 slide 12) requires that $P\left(\bigcap_{X \in I} X\right) = \prod_{X \in I} P(X)$ $\forall I \subset \{A, B, C\}$. Letting $I = \{B, C\}$, this does not hold.

3. Consider a probability space (Ω, F, P) such that Ω is countable and $F = 2^{\Omega}$. Suppose that $\exists$ a countable collection of events $A_1, A_2, \dots \in F$ such that $P(A_i) = \frac{1}{2}$ $\forall i \in \mathbb{N}$. Fix $n \in \mathbb{N}$. For each $\omega \in \Omega$, either $\omega \in A_i$ or $\omega \in A_i^c$ $\forall i \in \{1, \dots, n\}$. For a given ω, define B_i as $B_i = A_i$ if $\omega \in A_i$ or $B_i = A_i^c$ if $\omega \notin A_i$ for $i = 1, \dots, n$. Then, the theorem on slide 13 of lecture 6 gives that $B_1, \dots, B_n$ are independent. This is because we can apply the theorem m times where m is the number of $B_i = A_i^c$, and reordering the indices as necessary. Then, by the properties of a probability measure (noting that $\{\omega\} \subset \bigcup_{i=1}^{n} B_i$) and the definition of independent events,

$$P(\{\omega\}) \leq P\left(\bigcup_{i=1}^{n} B_i\right) = \prod_{i=1}^{n} P(B_i) = \prod_{i=1}^{n} \frac{1}{2} = \frac{1}{2^n}$$

because $P(A_i^c) = 1 - P(A_i) = \frac{1}{2}$ $\forall i \in \mathbb{N}$. Take the limit as $n \to \infty$ of both sides to get that $P(\{\omega\}) \leq 0$. Then, $P(\{\omega\}) = 0$ $\forall \omega \in \Omega$, because measures are nonnegative.

Now, because this holds $\forall\, \omega \in \Omega$ and Ω is countable, the definition of a probability measure gives that

$$P(\Omega) = P\left(\bigcup_{\omega\in\Omega} \{\omega\}\right) = \sum_{\omega\in\Omega} P(\{\omega\}) = \sum_{\omega\in\Omega} 0 = 0.$$

However, this is a contradiction, because $P(\Omega)=1$ by the definition of a probability measure. Thus, we cannot find such a collection of events $A_1, A_2, \ldots \in F$.

4. a. Let $X \geq 0$ and $Y \geq 0$ be independent random variables with distribution functions F and G, respectively. Consider the function $h(x,y) = \mathbb{1}_{\{xy\leq z\}}$. Then,

$$E[h(X,Y)] = E\,\mathbb{1}_{\{xy\leq z\}} = P(XY\leq z) \text{ and}$$

$$E[h(X,Y)] = \int_0^\infty \int_0^\infty \mathbb{1}_{\{xy\leq z\}}\, dF(x)\, dG(y).$$

Note that we're integrating from 0 to infinity because $X, Y \geq 0$. Looking at the inner integral,

$$\int_0^\infty \mathbb{1}_{\{xy\leq z\}}\, dF(x) = \int_0^\infty \mathbb{1}_{\{x\leq z/y\}}\, dF(x)$$

$= P(X \leq z/y) = F(z/y)$. Thus,

$$P(XY\leq z) = \int_0^\infty F(z/y)\, dG(y).$$

Of course, this only works for $z \geq 0$, but $P(XY \leq a) = 0$ if $a<0$. Thus, if we let F_{XY} denote the distribution function of XY,

$$F_{XY}(z) = \begin{cases} \int_0^\infty F(z/y)\, dG(y) & \text{if } z\geq 0 \\ 0 & \text{if } z<0. \end{cases}$$

b. Now, say that X and Y are also continuous and have respective density functions f and g. Then, for $z \geq 0$,

$$F_{XY}(z) = \int_0^\infty F\left(\frac{z}{y}\right) dG(y) = \int_0^\infty \int_0^{z/y} f(u)\, du\, dG(y).$$

Let $u = x/y$ so $du = dx/y$. Then,

$$F_{XY}(z) = \int_0^\infty \int_0^z f\left(\frac{x}{y}\right) \frac{dx}{y}\, dG(y) = \int_0^z \int_0^\infty \frac{f(x/y)}{y}\, g(y)\, dy\, dx.$$

where we invoke Fubini's theorem in the last step. Note that we have considered the density functions to be defined for $x \geq 0$ which is why the bounds of integration start at 0. Thus, by the definition of a density function, the density function f_{xy} of XY is given by

$$f_{xy}(z) = \int_0^\infty \frac{f(z/y)}{y} g(y)\,dy.$$

for $z \geq 0$. If we wish to consider $z < 0$, then clearly $f_{xy}(z) = 0$, because $\int_{-\infty}^{x} f(u)\,du = P(X \leq x) = 0$ $\forall\, x < 0 \Rightarrow f(x') = 0$ a.s. for $x' < 0$ (of course, this depends on whether we let f, g be defined for values less than 0. We could have instead included $-\infty$ as our lower bound in the integral and used this observation to change the bound to 0. I've just chosen to treat it like this for simplicity.)

c. Consider the density function $f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$ of the exponentially distributed r.v. with parameter λ. By part b, the density function for XY where X and Y are both such an r.v., is given by

$$f_{xy}(z) = \int_0^\infty \frac{\lambda e^{-\lambda z/y}}{y} \lambda e^{-\lambda y}\,dy = \int_0^\infty \frac{\lambda^2}{y} e^{-\lambda\left(\frac{z}{y} + y\right)}\,dy$$

if $z \geq 0$. As before, $f_{xy}(z) = 0$ if $z < 0$.