MATH 524 Homework 2

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October 13, 2023

1 Problem 1

Let $\mathcal{A}$ be the algebra on $\mathbb{Z}_q^{\mathbb{N}}$ defined last week, and define a function μ_0 on $\mathcal{A}$ by the rule

$$\mu_0(\Pi_n^{-1}(E)) = \frac{1}{q^n}\operatorname{card}(E),$$

when $E \in \mathbb{Z}_q^n$.

1.1 Part a

To see that μ_0 is well-defined on $\mathcal{A}$, we note that

$$\mu_0(\emptyset) = \mu_0(\Pi_n^{-1}(\emptyset)) = \frac{1}{q^n}\operatorname{card}(\emptyset) = 0,$$

for any $n \in \mathbb{N}$, so it is defined on the empty set. Furthermore, let $A \in \mathcal{A}$ have two different representations, i.e., $A = \Pi_{n_1}^{-1}(E_1) = \Pi_{n_2}^{-1}(E_2)$ for $E_1 \in \mathbb{Z}_q^{n_1}$, $E_2 \in \mathbb{Z}_q^{n_2}$. Assume without loss of generality that $n_1 \leq n_2$. Then, it must hold that

$$E_2 = \{a \in \mathbb{Z}_q^{n_2} \mid (a_1, \ldots, a_{n_1}) \in E_1\},$$

since all elements of $a \in A$ must satisfy $\Pi_{n_1} a \in E_1$, and restrictions on the elements $a_{n_1+1}, \ldots, a_{n_2}$ cannot be imposed if $A = \Pi_{n_1}^{-1}(E_1)$. From this, it is easy to see that

$$\operatorname{card}(E_2) = q^{n_2 - n_1}\operatorname{card}(E_1),$$

since there are $n_2 - n_1$ elements that can be chosen freely. Thus,

$$\mu_0(\Pi_{n_2}^{-1}(E_2)) = \frac{1}{q^{n_2}} q^{n_2-n_1}\operatorname{card}(E_1) = \frac{1}{q^{n_1}}\operatorname{card}(E_1) = \mu_0(\Pi_{n_1}^{-1}(E_1)).$$

We show that μ_0 is finitely additive in a similar manner. Let $\Pi_{n_1}^{-1}(E_1), \Pi_{n_2}^{-1}(E_2) \in \mathcal{A}$ be disjoint, and assume without loss of generality that $n_1 \leq n_2$. Define

$$E_3 = \{a \in \mathbb{Z}_q^{n_2} \mid (a_1, \ldots, a_{n_1}) \in E_1\},$$

and observe that $\Pi_{n_2}^{-1}(E_3) = \Pi_{n_1}^{-1}(E_1)$ and that $E_2 \cap E_3 = \emptyset$. Then,

$$\begin{aligned}
\mu_0(\Pi_{n_1}^{-1}(E_1) \sqcup \Pi_{n_2}^{-1}(E_2)) &= \mu_0(\Pi_{n_2}^{-1}(E_3) \sqcup \Pi_{n_2}^{-1}(E_2)) = \mu_0(\Pi_{n_2}^{-1}(E_3 \sqcup E_2)) = \frac{1}{q^{n_2}}\operatorname{card}(E_3 \sqcup E_2) \\
&= \frac{1}{q^{n_2}}\operatorname{card}(E_3) + \frac{1}{q^{n_2}}\operatorname{card}(E_2) = \frac{1}{q^{n_2}} q^{n_2-n_1}\operatorname{card}(E_1) + \frac{1}{q^{n_2}}\operatorname{card}(E_2) = \mu_0(\Pi_{n_1}^{-1}(E_1)) + \mu_0(\Pi_{n_2}^{-1}(E_2)).
\end{aligned}$$

1.2 Part b

To show that $(\mathbb{Z}_q^{\mathbb{N}}, \mathcal{A}, \mu_0)$ is a premeasure space, we need only show countable disjoint additivity and nonnegativity, as the other axioms have already been verified in part a or on Homework 1. Note that μ_0 is nonnegative, because cardinality is nonnegative. Let $\mathcal{A} \ni A = \bigsqcup_{j=1}^{\infty} A_j$, where each $A_j \in \mathcal{A}$ and $A_i \cap A_j = \emptyset$ for each $i \neq j$. From Homework 1, we know that $A \subset \mathbb{Z}_q^{\mathbb{N}}$ is closed and that $\mathbb{Z}_q^{\mathbb{N}}$ is compact, so A must also be compact. Furthermore, each $A_j \in \mathcal{A}$ is open, so the countable open cover defining A can be reduced to a finite subcover. Thus, by finite disjoint additivity,

$$\mu_0(A) = \mu_0\left(\bigsqcup_{j=1}^{n} A_j\right) = \sum_{j=1}^{n} \mu_0(A_j),$$

where the A_js have possibly been reindexed. However, because this union is disjoint, we must have that $A_k = \emptyset$ for $k > n$, as otherwise, it could not hold that

$$\bigsqcup_{j=1}^{\infty} A_j \subset \bigsqcup_{j=1}^{n} A_j.$$

Thus, $\mu_0(A_k) = 0$ for $k > n$, so

$$\mu_0(A) = \sum_{j=1}^{n} \mu_0(A_j) = \sum_{j=1}^{\infty} \mu_0(A_j),$$

and countable disjoint additivity holds.

2 Problem 2 (Folland problem 11)

To show that a finitely additive measure μ is a measure iff it is continuous from below, we need only show the "if" statement, as the other direction is trivially true by Theorem 1.8c in Folland. To do this, let μ be a finitely additive measure that is continuous from below; to show that μ is a measure, we need only verify that it is countably additive. Let $\{E_j\}_{j=1}^{\infty} \subset \mathcal{M}$ be disjoint. Define $F_k = \bigsqcup_{j=1}^{k} E_j$ for all $k \in \mathbb{N}$. Then, $F_1 \subset F_2 \subset \ldots$, so we apply continuity from below and finite additivity to get

$$\mu\left(\bigsqcup_{j=1}^{\infty} E_j\right) = \mu\left(\bigsqcup_{k=1}^{\infty} F_k\right) = \lim_{k\to\infty} \mu(F_k) = \lim_{k\to\infty} \mu\left(\bigsqcup_{j=1}^{k} E_j\right) = \lim_{k\to\infty} \sum_{j=1}^{k} \mu(E_j) = \sum_{j=1}^{\infty} \mu(E_j),$$

so μ is countably additive and therefore a measure.

Similarly, to show that if $\mu(X) < \infty$, a finitely additive measure μ is a measure iff it is continuous from above, we need only show the "if" statement, as the other direction is trivially true by Theorem 1.8d in Folland (since montonicity implies that $\mu(E) < \infty$ for all $E \subset X$). To do this, let μ be a finitely additive measure that is continuous from above and satisfies $\mu(X) < \infty$. Let $\{E_j\}_{j=1}^{\infty} \subset \mathcal{M}$ be disjoint and define for each $k \in \mathbb{N}$,

$$F_k = \left(\bigsqcup_{j=1}^{k} E_j\right)^c = \bigcap_{j=1}^{k} E_j^c.$$

Then, $F_1 \supset F_2 \supset \ldots$ and $\mu(F_1) = \mu(X) - \mu(E_1) < \infty$, so we apply continuity from above and finite additivity to get

$$\mu\left(\bigsqcup_{j=1}^{\infty} E_j\right) = \mu(X) - \mu\left(\left(\bigsqcup_{j=1}^{\infty} E_j\right)^c\right) = \mu(X) - \mu\left(\bigcap_{k=1}^{\infty} F_k\right) = \mu(X) - \lim_{k\to\infty} \mu(F_k)$$

$$= \mu(X) - \lim_{k\to\infty} \mu\left(X \setminus \left(\bigsqcup_{j=1}^{k} E_j\right)\right) = \mu(X) - \lim_{k\to\infty}\left(\mu(X) - \mu\left(\bigsqcup_{j=1}^{k} E_j\right)\right) = \lim_{k\to\infty} \mu\left(\bigsqcup_{j=1}^{k} E_j\right)$$

$$= \lim_{k\to\infty} \sum_{j=1}^{k} E_j = \sum_{j=1}^{\infty} E_j,$$

so μ is countably additive and therefore a measure.

3 Problem 3

Suppose that $(X, \mathcal{A}, \mu_0)$ is a premeasure space, μ^* is the corresponding outer measure, and $\mathcal{M}^*$ is the σ-algebra of sets satisfying the Carathéodory condition. Let $\mathcal{M}$ be the σ-algebra generated by $\mathcal{A}$.

3.1 Part a

Let $E \subset X$ and $\mu^*(E) = 0$. Then, for any $B \subset X$, by monotonicity,

$$0 = \mu^*(E) \geq \mu^*(E \cap B),$$

so $\mu^*(E \cap B) = 0$, and

$$\mu^*(B) \geq \mu^*(B \cap E^c) = \mu^*(E \cap B) + \mu^*(B \cap E^c).$$

Thus, E satisfies the Carathéodory condition, and $E \in \mathcal{M}^*$.

Furthermore, by the definition of the corresponding outer measure, there exist some $A_j^n \in \mathcal{A}$, $j \in \mathbb{N}$ such that $E \subset \bigcup_{j=1}^\infty A_j^n$ and

$$\sum_{j=1}^\infty \mu_0(A_j^n) < \mu^*(E) + \frac{1}{n} = \frac{1}{n}.$$

Define $F^n = \bigcup_{j=1}^\infty A_j^n$ and $F = \bigcap_{n=1}^\infty F^n \in \mathcal{M}$ by the definition of a σ-algebra. Note that $E \subset F$. Assume without loss of generality that $F^1 \supset F^2 \supset \ldots$[1], and note that $\mu_0(F_1) < \infty$. Then, continuity from above for premeasures implies that

$$\mu_0(F) = \lim_{n\to\infty} \mu_0(F^n) \leq \lim_{n\to\infty} \frac{1}{n} = 0.$$

Then, by the definition of the corresponding outer measure, it follows that $\mu^*(F) = 0$, so there exists some $F \in \mathcal{M}$ such that $E \subset F$ and $\mu^*(F) = 0$.

3.2 Part b

Let $E \in \mathcal{M}^*$ and $\mu^*(E) = 0$. Following a similar argument to part a, by the definition of the corresponding outer measure, there exist some $A_j^n \in \mathcal{A}$, $j \in \mathbb{N}$ such that $E \subset \bigcup_{j=1}^\infty A_j^n$ and

$$\sum_{j=1}^\infty \mu_0(A_j^n) < \mu^*(E) + \frac{1}{n}.$$

Define $A^n = \bigcup_{j=1}^\infty A_j^n$ and $A = \bigcap_{n=1}^\infty A^n \in \mathcal{M}$ by the definition of a σ-algebra. Assume without loss of generality that $F^1 \supset F^2 \supset \ldots$, and note that $\mu_0(F_1) < \infty$. Then, continuity from above for premeasures implies that

$$\mu^*(A) = \mu_0(A) = \lim_{n\to\infty} \mu_0(A^n) \leq \lim_{n\to\infty} \left(\mu^*(E) + \frac{1}{n}\right) = \mu^*(E).$$

Then, since E satisfies the Carathéodory condition and $\mu^*(E) < \infty$, it immediately follows that $\mu^*(A \backslash E) = 0$.

3.3 Part c

Let $E \in \mathcal{M}^*$ and $\mu^*(E) = 0$. By part b, there exists $A \in \mathcal{M}$ such that $\mu^*(A \setminus E) = 0$. Noting that $\mathcal{M} \subset \mathcal{M}^*$ and applying part a, there exists $B \in \mathcal{M}^*$ such that $\mu^*(B) = 0$ and $A \setminus E \subset B$. Applying the Carathéodory condition,

$$0 = \mu^*(B) = \mu^*(E \cap B) + \mu^*(E \cap B^c) = \mu^*(E \setminus B).$$

[1] If this does not hold, one may redefine $F^n \leftarrow \bigcap_{j=1}^n F^n \in \mathcal{M}$ which has smaller premeasure by homogeneity.

3.4 Part d

Because σ-algebras are closed under countable unions, it immediately follows that parts b and c also hold if $E = \bigcup_{j=1}^{\infty} E_j$ where $E_j \in \mathcal{M}^*$ and $\mu^*(E_j) < \infty$.

3.5 Part e

Let $(X, \mathcal{A}, \mu_0)$ be σ-finite and let $(X, \overline{\mathcal{M}}, \overline{\mu})$ denote the completion of $(X, \mathcal{M}, \mu)$ where μ is the restriction of μ^* to $\mathcal{M}$. Let $E \in \mathcal{M}^*$. By σ-finiteness, $E = \bigcup_{j=1}^{\infty} E_j$ where $E_j \in \mathcal{M}^*$ and $\mu^*(E_j) < \infty$, so we apply part d to get that there exists $B \in \mathcal{M}$ such that $\mu^*(E \setminus B) = 0$. Furthermore, part a implies that there exists $F \in \mathcal{M}$ such that $\mu(F) = 0$ and $E \setminus B \subset F$. Thus, by the definition of the completion,

$$E = E \cup (E \setminus B) \in \overline{\mathcal{M}}.$$

Now, let $E \cup F \in \overline{\mathcal{M}}$. By definition, $E \in \mathcal{M} \subset \mathcal{M}^*$, and there exists $N \in \mathcal{M}$ such that $\mu^*(N) = \mu(N) = 0$ and $F \subset N$. By the monotonicity of outer measures, $\mu^*(F) = \mu^*(N) = 0$. Part a implies that $F \in \mathcal{M}^*$, so $E \cup F \in \mathcal{M}^*$ since σ-algebras are closed under finite unions. Thus, $\mathcal{M}^* = \overline{\mathcal{M}}$.

By Theorem 1.14 in Folland, the extension of μ of μ_0 to $\mathcal{M}$ is unique. Futhermore, Theorem 1.9 in Folland gives that the extension $\overline{\mu}$ of μ to $\overline{\mathcal{M}}$ is also unique. Since we established that $\mathcal{M}^* = \overline{\mathcal{M}}$, it then follows that $(X, \mathcal{M}^*, \mu^*)$ is the completion of $(X, \mathcal{M}, \mu)$.

4 Problem 4 (Folland problem 28)

Let F be increasing and right continuous, and let μ_F be the associated measure. Then, for any $a < b$,

- Note that $\{a\} = \bigcap_{j=1}^{\infty} \left(a - \frac{1}{j}, a\right]$, that $\mu_F\left((a-1, a]\right) < \infty$, and that these intervals decrease in size, each contained in the previous. Then, continuity from above for premeasures gives that

$$\mu_F(\{a\}) = \lim_{j\to\infty} \mu_F\left(\left(a - \frac{1}{j}, a\right]\right) = \lim_{j\to\infty}\left(F(a) - F\left(a - \frac{1}{j}\right)\right) = F(a) - F(a-).$$

- Now, note that $[a, b) = \left(\bigcap_{j=1}^{\infty} \left(a - \frac{1}{j}, a\right]\right) \sqcup \left(\bigcup_{j=1}^{\infty} \left(a, b - \frac{1}{j}\right]\right)$ and that the intervals in the latter union increase in size, each containing the previous. Then, continuity from below for premeasures gives that

$$\mu_F\left(\bigcup_{j=1}^{\infty}\left(a, b - \frac{1}{j}\right]\right) = \lim_{j\to\infty} \mu_F\left(\left(a, b - \frac{1}{j}\right]\right) = \lim_{j\to\infty}\left(F\left(b - \frac{1}{j}\right) - F(a)\right) = F(b-) - F(a).$$

  Finite disjoint addivity for premeasures then gives that

$$\mu_F\left([a, b)\right) = F(a) - F(a-) + F(b-) - F(a) = F(b-) - F(a-).$$

- Note that $[a, b] = \bigcap_{j=1}^{\infty} \left(a - \frac{1}{j}, b\right]$, that $\mu_F((a-1), b) < \infty$, and that these intervals decrease in size, each contained in the previous.. Then, continuity from above for premeasures gives that

$$\mu_F([a, b]) = \lim_{j\to\infty} \mu_F\left(\left(a - \frac{1}{j}, b\right]\right) = \lim_{j\to\infty}\left(F(b) - F\left(a - \frac{1}{j}\right)\right) = F(b) - F(a-).$$

- Finally, note that $(a, b) = \bigcup_{j=1}^{\infty} \left(a, b - \frac{1}{j}\right]$ and that the intervals increase in size, each containing the previous.. It follows directly from the previous work for the $[a, b)$ case that

$$\mu_F((a, b)) = \mu_F\left(\bigcup_{j=1}^{\infty}\left(a, b - \frac{1}{j}\right]\right) = F(b-) - F(a).$$

5 Problem 5 (Folland problem 30)

Let $E \in \mathcal{L}$ and $m(E) > 0$. To show that for any $\alpha < 1$, there is an open interval I such that $m(E \cap I) > \alpha m(I)$, we assume the contrary, i.e., that for all open intervals I, $m(E \cap I) \leq \alpha m(I)$ for all $\alpha < 1$. By Lemma 1.17, there exist open intervals I_j such that $E \subset \bigcup_{j=1}^{\infty} I_j$ and

$$\sum_{j=1}^{\infty} m(I_j) \leq m(E) + \epsilon,$$

for any $\epsilon > 0$. Then, by our assumptions,

$$\sum_{j=1}^{\infty} m(E \cap I_j) \leq \sum_{j=1}^{\infty} \alpha m(I_j) \leq \alpha m(E) + \alpha\epsilon.$$

On the other hand, since $E \subset \bigcup_{j=1}^{\infty} I_j$ and by subadditivity,

$$m(E) = m\left(E \cap \left(\bigcup_{j=1}^{\infty} I_j\right)\right) = m\left(\bigcup_{j=1}^{\infty} (E \cap I_j)\right) \leq \sum_{j=1}^{\infty} (E \cap I_j).$$

Thus,

$$m(E) \leq \alpha m(E) + \alpha\epsilon.$$

This yields the desired contradiction if we take

$$\epsilon \geq \frac{1-\alpha}{\alpha} m(E),$$

for any $0 < \alpha < 1$, provided that $m(E) < \infty$.

If $m(E) = \infty$, let $E_j = E \cap (j, j+1]$. Then, $E = \bigcup_{j=-\infty}^{\infty} E_j$, and $m(E_j) \leq m((j, j+1]) = 1 < \infty$ for all $j \in \mathbb{Z}$. Let I be the open interval corresponding to some E_j such that $m(E_j \cap I) \leq \alpha m(I)$ for all $\alpha < 1$. Then, by monotonicity,

$$m(E \cap I) \geq m(E_j \cap I) > \alpha m(I),$$

for all $\alpha < 1$.