AMATH 573 Homework 2

Cade Ballew #2120804

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1 Problem 1

Consider the Benjamin-Ono equation

$$u_t + uu_x + \mathcal{H}u_{xx} = 0$$

where

$$\mathcal{H}f(x,t) = \frac{1}{\pi}⨍_{-\infty}^{\infty} \frac{f(z,t)}{z-x}dz.$$

We first linearize around the zero solution by taking

$$u = \epsilon v + \mathrm{O}(\epsilon^2).$$

Then, noting that the Hilbert transform is an integral and therefore linear, collecting the first order terms, we have

$$v_t + \mathcal{H}v_{xx} = 0.$$

Plugging in the ansatz $v = e^{ikx - i\omega(k)t}$, we get that

$$\mathcal{H}v_{xx} = \frac{1}{\pi}⨍_{-\infty}^{\infty} \frac{-k^2 e^{ikz - i\omega(k)t}}{z-x}dz = \frac{-k^2 e^{-i\omega(k)t}}{\pi}⨍_{-\infty}^{\infty} \frac{e^{ikz}}{z-x}dz.$$

To compute this integral, we first consider the case $k > 0$. Noting that this is very similar to example 4.3.1 in Ablowitz and Fokas, we define a semicircular contour C from $-R$ to R with a semicircular kink from $x-\epsilon$ to $x+\epsilon$ protruding inwards as in their figure 4.3.2 but shifted to be centered at $z = x$. Letting C_R denote the large semicircle, we have that

$$\lim_{R\to\infty} \int_{C_R} \frac{e^{ikz}}{z-x}dz = 0$$

by Jordan's lemma, because $k > 0$. Letting C_ϵ denote the kink, we can also use their theorem 4.3.1(b) to get that

$$\lim_{\epsilon\to 0} \int_{C_\epsilon} \frac{e^{ikz}}{z-x}dz = i(-\pi)\operatorname*{Res}_{z=x} \frac{e^{ikz}}{z-x} = -i\pi e^{ikx}.$$

Now, by Cauchy's theorem,

$$\oint_C \frac{e^{ikz}}{z-x}dz = 0,$$

and by definition,

$$⨍_{-\infty}^{\infty} \frac{e^{ikz}}{z-x}dz = \lim_{R\to\infty}\lim_{\epsilon\to 0}\left(\int_{-R}^{x-\epsilon}\frac{e^{ikz}}{z-x}dz + \int_{x+\epsilon}^{R}\frac{e^{ikz}}{z-x}dz\right),$$

so we can conclude that

$$⨍_{-\infty}^{\infty} \frac{e^{ikz}}{z-x}dz = -\left(\lim_{R\to\infty}\int_{C_R}\frac{e^{ikz}}{z-x}dz + \lim_{\epsilon\to 0}\int_{C_\epsilon}\frac{e^{ikz}}{z-x}dz\right) = i\pi e^{ikx}.$$

If we instead take $k < 0$, we need to reflect our previous contour across the real axis with the same labeling as before. We can then apply Jordan's lemma in the lower halfplane to get that

$$\lim_{R\to\infty}\int_{C_R}\frac{e^{ikz}}{z-x}dz = 0.$$

By theorem 4.3.1(b),

$$\lim_{\epsilon\to 0}\int_{C_\epsilon}\frac{e^{ikz}}{z-x}dz = i(\pi)\operatorname*{Res}_{z=x}\frac{e^{ikz}}{z-x} = i\pi e^{ikx}.$$

Integrating over the entire contour again gives 0 by Cauchy's theorem, so

$$⨍_{-\infty}^{\infty} \frac{e^{ikz}}{z-x}dz = -\left(\lim_{R\to\infty}\int_{C_R}\frac{e^{ikz}}{z-x}dz + \lim_{\epsilon\to 0}\int_{C_\epsilon}\frac{e^{ikz}}{z-x}dz\right) = -i\pi e^{ikx}.$$

Thus, in general,

$$⨍_{-\infty}^{\infty} \frac{e^{ikz}}{z-x}dz = \operatorname{sign}(k) i\pi e^{ikx},$$

and

$$\mathcal{H}v_{xx} = -\operatorname{sign}(k)k^2 e^{ikx - i\omega(k)t}.$$

Our dispersion relation can then be found by

$$-i\omega(k) - \operatorname{sign}(k) ik^2 = 0,$$

so

$$\omega(k) = -\operatorname{sign}(k)k^2$$

is the linear dispersion relationship for this equation linearized about the zero solution.

2 Problem 2

Consider the one-dimensional surface water wave problem

$$\begin{aligned}
\nabla^2\phi &= 0, && -h < z < \zeta(x,t)\\
\phi_z &= 0, && z = -h\\
\zeta_t + \phi_x\zeta_x &= \phi_z, && z = \zeta(x,t)\\
\phi_t + g\zeta + \frac{1}{2}\left(\phi_x^2 + \phi_z^2\right) &= T\frac{\zeta_{xx}}{(1+\zeta_x^2)^{3/2}}, && z = \zeta(x,t).
\end{aligned}$$

We linearize around the trivial solution by taking $\zeta = \epsilon\zeta_1 + \mathrm{O}(\epsilon^2)$ and $\phi = \epsilon\phi_1 + \mathrm{O}(\epsilon^2)$. Plugging these in, and neglecting higher order terms, we get

$$\begin{aligned}
\epsilon\phi_{1xx} + \epsilon\phi_{1zz} &= 0, && -h < z < \zeta(x,t)\\
\epsilon\phi_{1z} &= 0, && z = -h\\
\epsilon\zeta_{1t} + \epsilon^2\phi_{1x}\zeta_{1x} &= \epsilon\phi_{1z}, && z = \zeta(x,t)\\
\epsilon\phi_{1t} + \epsilon g\zeta_1 + \frac{1}{2}\left(\epsilon^2\phi_{1x}^2 + \epsilon\phi_z^2\right) &= T\frac{\epsilon\zeta_{1xx}}{(1+\epsilon^2\zeta_{1x}^2)^{3/2}}, && z = \zeta(x,t).
\end{aligned}$$

Looking at just the first order terms in ϵ, we get

$$\begin{aligned}
\phi_{1xx} + \phi_{1zz} &= 0, && -h < z < \zeta(x,t)\\
\phi_{1z} &= 0, && z = -h\\
\zeta_{1t} &= \phi_{1z}, && z = \zeta(x,t)\\
\phi_{1t} + g\zeta_1 &= T\zeta_{1xx}, && z = \zeta(x,t).
\end{aligned}$$

Now, we apply the ansatz $\zeta_1 = e^{ikx - i\omega(k)t}$, $\phi_1 = f(z)e^{ikx - i\omega(k)t}$. Plugging this in, our first two equations become

$$\begin{aligned}
-k^2 f(z) + f''(z) &= 0, && -h < z < \zeta(x,t)\\
f'(z) &= 0, && z = -h.
\end{aligned}$$

This is just an ODE with general solution

$$f(z) = c_1e^{kz} + c_2e^{-kz}.$$

Plugging in the boundary condition, $f'(z) = kc_1e^{kz} - kc_2e^{-kz}$, so we must have that $c_2 = c_1e^{-2kh}$ and

$$f(z) = c_1e^{kz} + c_1e^{-k(z+2h)} = c_1e^{-kh}(e^{k(z+h)} + e^{-k(z+h)}) = C\cosh(k(z+h))$$

where we have redefined our constant. Thus,

$$\phi_1 = C\cosh(k(z+h))e^{ikx - i\omega(k)t}.$$

Ignoring the $z = \zeta(x,t)$ condition for now, we plug our ansatz into the latter two equations to get

$$-i\omega(k) = Ck\sinh(k(z+h))$$
$$-iC\cosh(k(z+h))\omega(k) + g = -Tk^2.$$

Then,

$$C = \frac{-i\omega(k)}{k\sinh(k(z+h))},$$

so

$$-i\frac{-i\omega(k)}{k\sinh(k(z+h))}\cosh(k(z+h))\omega(k) + g = -Tk^2.$$

This yields

$$\frac{-\coth(k(z+h))\omega^2(k)}{k} = -Tk^2 - g,$$

and

$$\omega^2(k) = k(g+Tk^2)\tanh(k(z+h)).$$

Now, we enforce $z = \zeta(x,t) = \epsilon\zeta_1 + \mathrm{O}(\epsilon^2)$ which gives

$$\omega^2(k) = k(g+Tk^2)\tanh(k(\epsilon\zeta_1 + h + \mathrm{O}(\epsilon^2))).$$

To leading order in ϵ, this simply yields

$$\omega^2(k) = k(g+Tk^2)\tanh(kh),$$

our linear dispersion relationship.

3 Problem 3

Take $T = 0$. Then, the dispersion relationship from our previous problem is $\omega^2 = gk\tanh(kh)$. If $|k|h \ll 1$, then Taylor expanding gives that $\tanh(kh) \approx kh$, so $\omega^2 = ghk^2$ and $\omega_\pm = \pm\sqrt{gh}|k|$. Then, the group velocity is given by

$$c_g = \frac{d\omega}{dk} = \pm\,\mathrm{sign}(k)\sqrt{gh}.$$

If instead $|k|h \gg 1$,

$$\tanh(kh) = \frac{e^{kh} + e^{-kh}}{e^{kh} - e^{-kh}} \sim \mathrm{sign}(k).$$

Then, $\omega^2 = g|k|$, and $\omega_\pm = \pm\sqrt{g|k|}$, so the group velocity is given by

$$c_g = \frac{d\omega}{dk} = \pm\frac{\mathrm{sign}(k)}{2}\sqrt{\frac{g}{|k|}}.$$

4 Problem 4

4.1 Part a

Consider the Whitham equation

$$u_t + uu_x + \int_{-\infty}^{\infty} K(x-y)u_y(y,t)dy = 0,$$

where

$$K(x) = \frac{1}{2\pi}\fint_{-\infty}^{\infty} c(k)e^{ikx}dk,$$

and $c(k)$ is the positive phase speed for the water-wave problem: $c(k) = \sqrt{g\tanh(kh)/k}$. To compute the linear dispersion relation, we linearize around zero for simplicity, taking

$$u = \epsilon v + \mathrm{O}(\epsilon^2).$$

Then, because integrals are linear operators, by collecting first order terms we get

$$v_t + \int_{-\infty}^{\infty} K(x-y)v_y(y,t)dy = 0.$$

Now, consider the ansatz $v = e^{ikx-i\omega(k)t}$. We can then compute

$$\begin{aligned} I &:= \int_{-\infty}^{\infty} K(x-y)v_y(y,t)dy = \int_{-\infty}^{\infty} \frac{1}{2\pi}\fint_{-\infty}^{\infty} c(k')e^{ik'(x-y)}dk' ike^{iky-i\omega(k)t}dy \\ &= \frac{ik}{2\pi}e^{-i\omega(k)t}\int_{-\infty}^{\infty}\fint_{-\infty}^{\infty} c(k')e^{ik'x+iy(k-k')}dk'dy \\ &= \frac{ik}{2\pi}e^{ikx-i\omega(k)t}\int_{-\infty}^{\infty}\fint_{-\infty}^{\infty} c(k')e^{i(k-k')(y-x)}dk'dy. \end{aligned}$$

Now, performing the change of variables $y - x \to y$ inside the integral and flipping the order of integration,

$$\begin{aligned} I &= \frac{ik}{2\pi}e^{ikx-i\omega(k)t}\int_{-\infty}^{\infty}\fint_{-\infty}^{\infty} c(k')e^{i(k-k')y}dk'dy \\ &= ike^{ikx-i\omega(k)t}\fint_{-\infty}^{\infty}\left(\frac{1}{2\pi}c(k')\int_{-\infty}^{\infty} e^{i(k-k')y}dy\right)dk' \\ &= ike^{ikx-i\omega(k)t}\fint_{-\infty}^{\infty} c(k')\delta(k-k')dk' = ike^{ikx-i\omega(k)t}c(k) \end{aligned}$$

by the exponential representation of the Dirac delta function and integrating the Dirac delta function. Thus, we can get the linear dispersion relation by taking

$$-i\omega(k) + ikc(k) = 0$$

which gives

$$\omega(k) = kc(k).$$

4.2 Part b

Consider the KdV equation

$$u_t + vu_x + uu_x + \gamma u_{xxx} = 0.$$

To compute its linear dispersion relation, we linearize around zero for simplicity, taking

$$u = \epsilon u_1 + \mathrm{O}(\epsilon^2).$$

Collecting first order terms, we get that

$$u_{1t} + vu_{1x} + \gamma u_{1xxx} = 0.$$

Applying our usual ansatz $u_1 = e^{ikx - i\omega(k)t}$, we get

$$-i\omega(k) + vik - \gamma ik^3 = 0$$

which gives a linear dispersion relation of

$$\omega(k) = k(v - \gamma k^2).$$

For this to be an approximation of the linear dispersion relation of the Whitham equation as $k \to 0$, we need to choose v and γ to match the coefficients in the series expansion for $c(k)$ centered at zero. Using Mathematica to compute this expansion, we get that

$$c(k) = \sqrt{gh} - \frac{h^2\sqrt{gh}}{6}k^2 + \mathrm{O}(k^3).$$

Thus, taking $v = \sqrt{gh}$, $\gamma = \frac{h^2\sqrt{gh}}{6}$ gives that the dispersion relation is an approximation for long waves.

5 Problem 5

Consider the linear free Schrödinger equation

$$i\psi_t + \psi_{xx} = 0, \qquad -\infty < x < \infty, \quad t > 0, \quad \psi \to 0 \text{ as } |x| \to \infty,$$

with $\psi(x, 0) = \psi_0(x)$ such that $\int_{-\infty}^{\infty} |\psi_0|^2 dx < \infty$.

5.1 Part a

To solve this problem, we first need to find the dispersion relation. Since the equation is already linear, we can plug in our usual ansatz $\psi = e^{ikx - i\omega(k)t}$ which gives

$$i(-i\omega(k)) - k^2 = 0,$$

so the dispersion relation is $\omega(k) = k^2$. Then, the solution is given by

$$\psi(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} a(k)e^{ikx - ik^2 t}dk$$

where

$$a(k) = \int_{-\infty}^{\infty} \psi_0(x)e^{-ikx}dx.$$

5.2 Part b

To apply the method of stationary phase, we set

$$\phi(k) = k\frac{x}{t} - \omega(k) = k\frac{x}{t} - k^2.$$

We compute $\phi'(k) = \frac{x}{t} - 2k$ and $\phi''(k) = -2$. We find our stationary points by setting $\phi'(k) = 0$ which gives one stationary point at $k_0 = \frac{x}{2t}$. We can then use the result (2.7) in the lecture notes to get

$$\begin{aligned}\psi(x,t) &\approx \frac{a(k_0)}{\sqrt{2\pi t|\phi''(k_0)|}} \exp\left(i\phi(k_0)t + \frac{i\pi \operatorname{sign}(\phi''(k_0))}{4}\right) \\ &= \frac{\exp\left(\frac{ix^2}{4t} - \frac{i\pi}{4}\right)}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} \psi_0(x) e^{-\frac{ix^2}{2t}}\, dx.\end{aligned}$$

5.3 Parts c and d

See the attached Mathematica notebook for plots of the true solution when $\psi_0(x) = e^{-x^2}$ compared against the stationary phase approximation and an approximation computed by Mathematica's NIntegrate function on the lines $x/t = 1$ and $x/t = 2$. In general, we observe that stationary phase performs well as $x, t \to \infty$ but not for x, t close to zero, but the numerical integrator performs well for small x, t but not as $x, t \to \infty$.

6 Problem 6

6.1 Part a

We wish to verify that the discrete analogue of the Fourier transform

$$\psi_n(t) = \frac{1}{2\pi i} \oint_{|z|=1} \hat{\psi}(z,t) z^{n-1} dz,$$

and

$$\hat{\psi}(z,t) = \sum_{m=-\infty}^{\infty} \psi_m(t) z^{-m}$$

are inverses. We do this by first plugging the latter into the former and interchanging the integral and summation which gives

$$\begin{aligned}\frac{1}{2\pi i} \oint_{|z|=1} \left(\sum_{m=-\infty}^{\infty} \psi_m(t) z^{-m}\right) z^{n-1} dz &= \sum_{m=-\infty}^{\infty} \phi_m(t) \frac{1}{2\pi i} \oint_{|z|=1} z^{n-m-1} dz \\ &= \psi_m(t).\end{aligned}$$

Note that this follows from the residue theorem which gives that

$$\frac{1}{2\pi i}\oint_{|z|=1} z^j dz = \begin{cases} 1, & j=-1, \\ 0, & j=0. \end{cases}$$

Now, we plug the former into the latter and perform the change of variables $m \to -m$ to get

$$\begin{aligned}\sum_{m=-\infty}^{\infty} \frac{1}{2\pi i}\oint_{|z|=1} \hat{\psi}(z',t) z'^{m-1} dz' z^{-m} &= \sum_{m=-\infty}^{\infty}\left(\frac{1}{2\pi i}\oint_{|z|=1}\frac{\hat{\psi}(z',t)}{z'^{m+1}}dz'\right) z^m \\ &= \hat{\psi}(z,t),\end{aligned}$$

because this is precisely the definition of a Laurent series for $\hat{\psi}$.

6.2 Part b

Consider the discrete linear Schrödinger equation:

$$i\frac{d\psi_n}{dt} + \frac{1}{h^2}(\psi_{n+1} - 2\psi_n + \psi_{n-1}) = 0,$$

where h is a real constant, n is any integer, $t > 0$, $\psi_n \to 0$ as $|n| \to \infty$, and $\psi_n(0) = \psi_{n,0}$ is given. To find the dispersion relation, we consider the anstaz $\psi_n = z^n e^{-i\omega(z)t}$. Plugging this in gives

$$i(-i\omega(z))z^n e^{-i\omega(z)t} + \frac{1}{h^2}(z^{n+1}e^{-i\omega(z)t} - 2z^n e^{-i\omega(z)t} + z^{n-1}e^{-i\omega(z)t}) = 0$$

which simplifies to

$$z\omega(z) + \frac{1}{h^2}(z^2 - 2z + 1) = 0.$$

This gives the dispersion relation

$$\omega(z) = -\frac{(z-1)^2}{zh^2}.$$

To compare this with the dispersion relation from the fully continuous problem, we note that standard anstaz can be obtained from the one we used by setting $z^n = e^{ikx}$, so $z = e^{ikx/n}$. We then note that if our spatial grid has spacing h, points are given by $x = hn$, so we set $z = e^{ikh}$ to acquire the dispersion relation

$$-\frac{(e^{ikh}-1)^2}{e^{ikh}h^2}.$$

Now, we use Mathematica to compute the limit as $h \to 0$. Namely, we get that

$$\lim_{h\to 0} -\frac{(e^{ikh}-1)^2}{e^{ikh}h^2} = k^2$$

which is precisely the dispersion relation ffrom the continuous problem.