AMATH 568 Homework 4

Cade Ballew #2120804

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1 Problem 1

Consider

$$\mathrm{AI}(x) = \frac{1}{2\pi i}\int_C e^{izx+iz^3/3}\frac{dz}{z}, \quad x \in \mathbb{R},$$

as $x \to \pm\infty$ where C is a contour in the upper-half of the complex z-plane described by

$$C = \{u(s) : s \in \mathbb{R}\},$$

for a sufficiently nice function u, oriented in the direction of increasing s where

$$\lim_{s\to\pm\infty}|u(s)| = \infty, \quad \lim_{s\to+\infty}\frac{u(s)}{|u(s)|} = e^{i\pi/6}, \quad \lim_{s\to-\infty}\frac{u(s)}{|u(s)|} = e^{i5\pi/6}.$$

We first let $x = \sigma|x|$ where $\sigma = \pm 1$ so that we can consider $x \to \pm\infty$ by considering two cases based on σ. Then,

$$\mathrm{AI}(x) = \frac{1}{2\pi i}\int_C e^{iz|x|\sigma+iz^3/3}\frac{dz}{z}.$$

Now, consider the change of variables $z = u|x|^a$ for some constant a. Then,

$$e^{iz|x|\sigma+iz^3/3} = e^{iu|x|^{1+a}\sigma+iu^3u^3|x|^{3a}/3},$$

so in order to be able to factor out all the $|x|$ terms, we need that $3a = 1 + a$, i.e. that $a = 1/2$. If we consider this change of variables $z = u|x|^{1/2}$, then

$$\mathrm{AI}(x) = \frac{1}{2\pi i}\int_C e^{iu|x|^{3/2}\sigma+iu^3|x|^{3/2}/3}\frac{|x|^{1/2}du}{u|x|^{1/2}}.$$

Renaming u to z for simplicity, we have that

$$\mathrm{AI}(x) = \frac{1}{2\pi i}\int_C e^{|x|^{3/2}(iz\sigma+iz^3/3)}\frac{dz}{z} = \frac{1}{2\pi i}\int_C e^{|x|^{3/2}h(z)}g(z)dz$$

where $h(z) = iz\sigma + iz^3/3$ and $g(z) = 1/z$. Note that $h(z)$ is entire for either value of σ and that $g(z)$ is analytic except at $z = 0$, so we can apply the method of steepest descent as $|x| \to \infty$ with potential issues only at $z = 0$.
First, we consider the case where $\sigma = 1$ which corresponds with $x \to \infty$. Then, $h(z) = iz + iz^3/3$, so $0 = h'(z) = i + iz^2$ when $z = \pm i$, so our saddle points are $z_1 = i$ and $z_2 = -i$. We also note that $h''(z) = 2iz$, so $h''(z_1) = -2 \neq 0$ and $h''(z_2) = 2 \neq 0$. Now, we investigate the real and imaginary parts of h which are given by

$$R(x,y) = -y - x^2y + \frac{y^3}{3}$$

and

$$I(x,y) = x + \frac{x^3}{3} - xy^2$$

if $z = x + iy$. We wish to find a path to which we may deform our contour C where $I(x,y)$ is constant. A reasonable choice seems to be the level curve that passes through our saddle point at $z_1 = i$, since ideally, we will not need to deform into the lower half plane to avoid the singularity at $z = 0$. We have that $h(z_1) = -2/3$, so in order to keep $I(x,y)$ constant, we need to consider where $I(x,y) = 0$. $I(x,y) = x(1 + x^2/3 - y^2)$, so this is true either where $x = 0$ or $1 + x^2/3 - y^2 = 0$ and we consider the latter. This is a hyperbola in both the upper and lower half planes, so we also restrict it to the upper half plane to match our desired angles. Thus, we consider

$$C = \{u(s) = x + iy \in \mathbb{C} \mid s \in \mathbb{R},\ y \geq 0,\ 1 + x^2/3 - y^2 = 0\}.$$

The following figure shows C drawn on a contour plot of $I(x,y)$.

568hw4fig1-eps-converted-to.pdf

Now, we wish to evaluate the large z behavior for C to ensure that our contour goes to infinity in the proper direction. Substituting $z = re^{i\theta}$ into $1+x^2/3-y^2 = 0$, we have that $1 + r^2\cos^2\theta/3 - r^2\sin^2\theta = 0$ which we can rewrite as

$$\frac{3}{r^2} + \cos^2\theta - 3\sin^2\theta = 0$$

since we are considering $r \to \infty$. Now, we can let $\epsilon = 1/r$ and

$$0 = f(\theta,\epsilon) = \cos^2\theta - 3\sin^2\theta + \mathrm{O}(\epsilon^2) = -4\sin^2\theta + 1 + \mathrm{O}(\epsilon^2).$$

Solving $f(\theta, 0) = 0$, we find that $\sin\theta = \pm 1/2$, so $\theta = \pi/6, 5\pi/6, 7\pi/6, 13\pi/6$, but our contour also has the restriction that $\Im(z) \geq 0$, so we only consider $\theta_k = \pi/6, 5\pi/6$. Now, to be completely rigorous, we can note that $\frac{\partial f}{\partial \theta} \neq 0$ at these values of θ_k and apply the IFT to determine that $\theta(\epsilon) = \theta_k + \mathrm{O}(\epsilon^2) = \theta_k + \mathrm{O}(1/r^2)$ as $r \to \infty$. Thus, we observe that our contour goes to ∞ at the proper angles.
Since we know that the imaginary part of h is zero on our new contour C, we can look just at the real part on our contour. As $z \to \infty$, $z = re^{i\pi/6}$ or

$z = re^{5i\pi/6}$ as $r \to \infty$. In either case, the $iz^3/3$ term dominates $h(z)$ and is given by $ir^3 e^{i\pi/2}/3 = -r^3/3$, the real part of h looks like $-r^3/3$ as $z \to \infty$ on our contour which tends to $-\infty$ and will be decay in an exponential. The only saddle point $z_1 = i$ on our contour has $R(x_1, y_1) = -2/3$ which gives the dominant contribution. From this, define $\tilde{h}(z) = h(z) - h(z_1) = h(z) + 2/3$ and write

$$\mathrm{AI}(x) = \frac{1}{2\pi i}\int_C e^{|x|^{3/2}h(z)} g(z)dz = \frac{1}{2\pi i} e^{-2|x|^{3/2}/3} \int_C e^{|x|^{3/2}\tilde{h}(z)} g(z)dz.$$

Now, we attempt to localize our contour around our saddle point. Considering the parameterization of our contour, adopt the convention $u(s^*) = z_1$ and consider $s_- < s^* < s_+$ to be a sufficiently small neighborhood around s^* and let $u(s_-) = z_-$ and $u(s_+) = z_+$. We split our contour $C = C_1 + C_2 + C_3$ where C_1 is the contour from infinity at the left to z_-, C_2 is the contour from z_- to z_+, and C_3 is the contour from z_+ to infinity on the right. Now, we bound

$$\begin{aligned}\int_{C_1} e^{|x|^{3/2}\tilde{h}(z)} g(z)dz &= e^{|x|^{3/2}\tilde{h}(z_-)} \int_{C_1} e^{|x|^{3/2}\overbrace{(\tilde{h}(z) - \tilde{h}(z_-))}^{\text{negative}}} g(z)dz \\ &\leq e^{|x|^{3/2}\tilde{h}(z_-)} \underbrace{\int_{C_1} e^{|x|^{3/2}(\tilde{h}(z) - \tilde{h}(z_-))} |g(z)||dz|}_{\text{finite}}\end{aligned}$$

which is beyond all orders as $|x| \to \infty$. Similarly,

$$\begin{aligned}\int_{C_3} e^{|x|^{3/2}\tilde{h}(z)} g(z)dz &= e^{|x|^{3/2}\tilde{h}(z_+)} \int_{C_3} e^{|x|^{3/2}\overbrace{(\tilde{h}(z) - \tilde{h}(z_+))}^{\text{negative}}} g(z)dz \\ &\leq e^{|x|^{3/2}\tilde{h}(z_+)} \underbrace{\int_{C_3} e^{|x|^{3/2}(\tilde{h}(z) - \tilde{h}(z_+))} |g(z)||dz|}_{\text{finite}}\end{aligned}$$

is also beyond all orders as $|x| \to \infty$. Thus, we can localize our contour to C_2 at the cost of a BAO error.
Now that we have localized, we consider the behavior in a neighborhood around our saddle point by applying a change of variables with the IFT. Namely, we consider

$$\frac{\tilde{h}(z_1 + s\phi)}{s^2} + 1 = 0.$$

Now, we can simply apply the formula on page 108 of the text to get that

$$\int_{C_2} e^{|x|^{3/2}h(z)} g(z)dz = |x|^{3/2 - 1/2} e^{-2|x|^{3/2}/3} \left(e^{i\theta(z_1)} \sqrt{\frac{2\pi}{|h''(z_1)|}} g(z_1) + \mathrm{O}(|x|^{-3/2}) \right)$$

where $\theta(z_1)$ is the angle at which C leaves z_1. We can see from our change of variables that this is given by the angle of $\phi(0)$, so we compute

$$\phi(0) = \pm\sqrt{\frac{-2}{h''(z_1)}} = \pm 1.$$

Thus, we enter/leave at angles of $\theta = 0, \pi$, and $\theta(z_1) = 0$ with our convention. With this, we can write

$$\begin{aligned} \mathrm{AI}(x) &= \frac{1}{2\pi i}|x|^{-3/4}e^{-2|x|^{3/2}/3}\left(\sqrt{\pi}\frac{1}{i} + \mathrm{O}(|x|^{-3/2})\right) \\ &= -\frac{1}{2\sqrt{\pi}}e^{-2|x|^{3/2}/3}(|x|^{-3/4} + \mathrm{O}(|x|^{-9/4})) + \mathrm{BAO} \end{aligned}$$

as $x \to \infty$.

Now, consider the case where $x \to -\infty$, i.e. that $\sigma = -1$ so that $h(z) = -iz + iz^3/3$ and $0 = h'(z) = -i + iz^2$ when $z = \pm 1$; take our saddle points to be $z_1 = -1$, $z_2 = 1$. Also note that $h''(z) = 2iz$, so $h''(z_1) = -2i \neq 0$ and $h''(z_2) = 2i \neq 0$. Now, we investigate the real and imaginary parts of h which are

$$R(x,y) = y - x^2y + \frac{y^3}{3}$$

and

$$I(x,y) = -x + \frac{x^3}{3} - xy^2.$$

To go about deforming our contour, we look to choose a new contour that passes through our saddle points with constant imaginary part. At $z = 1$, $h(z) = -2i/3$ and at $z = -1$, $h(z) = 2i/3$. Thus, we consider

$$C_1 = \{u(s) = x + iy \in \mathbb{C} \mid s \in \mathbb{R},\ x \leq 0,\ -x + \frac{x^3}{3} - xy^2 = 2/3\}.$$

This set actually includes two separate curves, so we will abuse notation and adopt the convention that C_1 only refers to the curve for which y decreases as x increases. The following figure shows C_1 drawn on a contour plot of $I(x,y)$.

568hw4fig2-eps-converted-to.pdf

We also consider

$$C_2 = \{u(s) = x + iy \in \mathbb{C} \mid s \in \mathbb{R},\ x \geq 0,\ -x + \frac{x^3}{3} - xy^2 = -2/3\}$$

where we again abuse notation and say that this curve is increasing in y when it is increasing in x. The following figure shows C_2 drawn on a contour plot of $I(x,y)$.

568hw4fig3-eps-converted-to.pdf

Now, we compute

$$\operatorname*{Res}_{z=0} e^{|x|^{3/2}h(z)}g(z) = \operatorname*{Res}_{z=0} \frac{e^{|x|^{3/2}(-iz+iz^3/3)}}{z} = \lim_{z\to 0} e^{|x|^{3/2}(-iz+iz^3/3)} = 1.$$

Applying the residue theorem to deform our contour,

$$\frac{1}{2\pi i}\int_{-C} e^{|x|^{3/2}h(z)}g(z)dz + \frac{1}{2\pi i}\int_{C_1} e^{|x|^{3/2}h(z)}g(z)dz + \frac{1}{2\pi i}\int_{C_2} e^{|x|^{3/2}h(z)}g(z)dz = \operatorname*{Res}_{z=0} e^{|x|^{3/2}h(z)}g(z),$$

so

$$\frac{1}{2\pi i}\int_{C} e^{|x|^{3/2}h(z)}g(z)dz = \frac{1}{2\pi i}\int_{C_1} e^{|x|^{3/2}h(z)}g(z)dz + \frac{1}{2\pi i}\int_{C_2} e^{|x|^{3/2}h(z)}g(z)dz - 1.$$

We do need to confirm that this is a valid deformation by showing that C_1 and C_2 go off to infinity at the correct angles. To do this, we substitute $z = re^{i\theta}$ into $-x + x^3/3 - xy^2 = 2/3$, we have that $-r\cos\theta + r^3\cos^3\theta/3 - r^3\cos\theta\sin^2\theta = 2/3$ which we can rewrite as

$$-\frac{2}{r^3} + \cos\theta\left(-\frac{3}{r^2} + \cos^2\theta - 3\sin^2\theta\right) = 0$$

since we are considering $r \to \infty$. Now, we can let $\epsilon = 1/r$ and

$$0 = f(\theta, \epsilon) = \cos\theta(\cos^2\theta - 3\sin^2\theta) + \mathrm{O}(\epsilon^2) = \cos\theta(-4\sin^2\theta + 1) + \mathrm{O}(\epsilon^2).$$

Solving $f(\theta, 0) = 0$, we find that either $\sin\theta = \pm 1/2$, so $\theta = \pi/6, 5\pi/6, 7\pi/6, 13\pi/6$ or $\cos\theta = 0$, so $\theta = \pi/2, 3\pi/2$. Our contour also has the restriction that $\Re(z) \le 0$, so we only consider $\theta_k = \pi/2, 5\pi/6, 7\pi/6, 3\pi/2$. Now, to be completely rigorous, we can note that $\frac{\partial f}{\partial \theta} \neq 0$ at these values of θ_k and apply the IFT to determine that $\theta(\epsilon) = \theta_k + \mathrm{O}(\epsilon^2) = \theta_k + \mathrm{O}(1/r^2)$ as $r \to \infty$. If we plot the two curves associated with this region, we can see that one goes to ∞ above the real axis and along the imaginary axis in the lower half plane. This is the curve that we have chosen, so clearly it goes to zero at $\theta_k = 5\pi/6, 3\pi/2$ as desired. Similarly, if we consider $-x + x^3/3 - xy^2 = -2/3$, we essentially have the same $f(\theta, \epsilon)$ since the only term that changes is absorbed into the $\mathrm{O}(\epsilon^2)$ term. However, we now consider $\Re(z) \ge 0$, so we instead have $\theta_k = \pi/6, \pi/2, 3\pi/2, 13\pi/6$. Again, we pick the curve that goes to ∞ along the imaginary axis in the lower half plane and above the real axis. Thus, our deformation is valid.

First consider the contour C_1. As $z \to \infty$, $z = re^{3i\pi/2}$ or $z = re^{5i\pi/6}$ as $r \to \infty$. In either case, the $iz^3/3$ term dominates $h(z)$ and is given by $ir^3e^{i\pi/2}/3 = -r^3/3$, the real part of h looks like $-r^3/3$ as $z \to \infty$ on our contour which goes to $-\infty$ as $r \to \infty$, giving decay when in an exponential. Similarly, we can see that this is true on the contour C_2 as well, because as $z \to \infty$, $z = re^{3i\pi/2}$ or $z = re^{i\pi/6}$ as $r \to \infty$. In either case, the $iz^3/3$ term dominates $h(z)$ and is given by $ir^3e^{i\pi/2}/3 = -r^3/3$, the real part of which also looks like $-r^3/3$. Now, we compute $h(z_1) = 2i/3$ and $h(z_2) = -2i/3$ and define $h_1(z) = h(z) - h(z_1) = h(z) - 2i/3$ and $h_2(z) = h(z) - h(z_2) = h(z) + 2i/3$. We can see two equal "dominant" contributions, so we will seek to localize around each saddle point. Now, we write

$$\mathrm{AI}(x) = \frac{1}{2\pi i} e^{2i|x|^{3/2}/3} \int_{C_1} e^{|x|^{3/2} h_1(z)} g(z) dz + \frac{1}{2\pi i} e^{-2i|x|^{3/2}/3} \int_{C_2} e^{|x|^{3/2} h_2(z)} g(z) dz - 1.$$

First for C_1, considering the parameterization of our contour, adopt the convention $u(s^*) = z_1$ and consider $s_- < s^* < s_+$ be a sufficiently small neighborhood around s^* and let $u(s_-) = z_-$ and $u(s_+) = z_+$. We split our contour $C_1 = C_{1a} + C_{1b} + C_{1c}$ where C_{1a} is the contour from infinity at the left to z_-, C_{1b} is the contour from z_- to z_+, and C_{1c} is the contour from z_+ to infinity on the right. Now, we bound

$$\begin{aligned}\int_{C_{1a}} e^{|x|^{3/2} h_1(z)} g(z) dz &= e^{|x|^{3/2} h_1(z_-)} \int_{C_{1a}} e^{|x|^{3/2} \overbrace{(h_1(z) - h_1(z_-))}^{\text{negative}}} g(z) dz \\ &\leq e^{|x|^{3/2} h_1(z_-)} \underbrace{\int_{C_{1a}} e^{|x|^{3/2}(h_1(z) - h_1(z_-))} |g(z)||dz|}_{\text{finite}}\end{aligned}$$

which is beyond all orders as $|x| \to \infty$. Similarly,

$$\begin{aligned}\int_{C_{1c}} e^{|x|^{3/2} h_1(z)} g(z) dz &= e^{|x|^{3/2} h_1(z_+)} \int_{C_{1c}} e^{|x|^{3/2} \overbrace{(h_1(z) - h_1(z_+))}^{\text{negative}}} g(z) dz \\ &\leq e^{|x|^{3/2} h_1(z_+)} \underbrace{\int_{C_{1c}} e^{|x|^{3/2}(h_1(z) - h_1(z_+))} |g(z)||dz|}_{\text{finite}}\end{aligned}$$

is also beyond all orders as $|x| \to \infty$. Thus, we can localize C_1 to C_{1b} at the cost of a BAO error. We can make the same argument for C_2. Namely, if we reuse the convention (these variables have the same names but are obviously different from the C_1 case) $u(s^*) = z_1$ and consider $s_- < s^* < s_+$ be a sufficiently small neighborhood around s^* and let $u(s_-) = z_-$ and $u(s_+) = z_+$. We split our contour $C_2 = C_{2a} + C_{2b} + C_{2c}$ where C_{2a} is the contour from infinity at the left to z_-, C_{2b} is the contour from z_- to z_+, and C_{2c} is the contour from z_+ to infinity on the right. Now, we bound

$$\begin{aligned}\int_{C_{2a}} e^{|x|^{3/2} h_2(z)} g(z) dz &= e^{|x|^{3/2} h_2(z_-)} \int_{C_{2a}} e^{|x|^{3/2} \overbrace{(h_2(z) - h_2(z_-))}^{\text{negative}}} g(z) dz \\ &\leq e^{|x|^{3/2} h_2(z_-)} \underbrace{\int_{C_{2a}} e^{|x|^{3/2}(h_2(z) - h_2(z_-))} |g(z)||dz|}_{\text{finite}}\end{aligned}$$

which is beyond all orders as $|x| \to \infty$. Similarly,

$$\begin{aligned}\int_{C_{2c}} e^{|x|^{3/2} h_2(z)} g(z) dz &= e^{|x|^{3/2} h_2(z_+)} \int_{C_{2c}} e^{|x|^{3/2} \overbrace{(h_2(z) - h_2(z_+))}^{\text{negative}}} g(z) dz \\ &\leq e^{|x|^{3/2} h_2(z_+)} \underbrace{\int_{C_{2c}} e^{|x|^{3/2}(h_2(z) - h_2(z_+))} |g(z)||dz|}_{\text{finite}}\end{aligned}$$

is also beyond all orders as $|x| \to \infty$. Thus, we can localize C_2 to C_{2b} at the cost of a BAO error.
Now that we have localized, let us first consider the behavior in a neighborhood around z_1 by applying a change of variables with the IFT. Namely, we consider

$$\frac{h_1(z_1 + s\phi)}{s^2} + 1 = 0.$$

Now, we apply the formula on page 108 of the text to get that

$$\int_{C_{1b}} e^{|x|^{3/2} h(z)} g(z) dz = |x|^{3/2 - 1/2} \left(e^{i\theta(z_1)} \sqrt{\frac{2\pi}{|h''(z_1)|}} g(z_1) + \mathrm{O}(|x|^{-3/2}) \right)$$

with $\theta(z_1)$ defined in the same way as before. We compute

$$\phi(0) = \pm \sqrt{\frac{-2}{h''(z_1)}} = \pm\sqrt{-i} = \pm e^{-i\pi/4}.$$

Thus, we enter/leave at angles of $\theta = -\pi/4, 3\pi/4$, so with our convention, $\theta(z_1) = -\pi/4$. Now, we can plug in

$$\begin{aligned}\frac{1}{2\pi i} e^{2i|x|^{3/2}/3} \int_{C_{1b}} e^{|x|^{3/2} h(z)} g(z) dz &= \frac{1}{2\pi i} |x|^{-3/4} e^{2i|x|^{3/2}/3} \left(e^{-i\pi/4} \sqrt{\frac{2\pi}{2}} \frac{1}{-1} + \mathrm{O}(|x|^{-3/2}) \right) \\ &= -\frac{1}{2i\sqrt{\pi}} e^{2i|x|^{3/2}/3 - i\pi/4} (|x|^{-3/4} + \mathrm{O}(|x|^{-9/4})).\end{aligned}$$

Now, we apply a different change of variables with the IFT to consider the behavior in a neighborhood around z_2. Namely, we consider

$$\frac{h_2(z_2 + s\phi)}{s^2} + 1 = 0.$$

Now, we apply the formula on page 108 of the text to get that

$$\int_{C_{2b}} e^{|x|^{3/2} h(z)} g(z) dz = |x|^{3/2 - 1/2} \left(e^{i\theta(z_2)} \sqrt{\frac{2\pi}{|h''(z_2)|}} g(z_2) + \mathrm{O}(|x|^{-3/2}) \right).$$

We compute

$$\phi(0) = \pm \sqrt{\frac{-2}{h''(z_1)}} = \pm\sqrt{i} = \pm e^{i\pi/4},$$

so we enter/leave at angles of $\theta = -3i\pi/4, i\pi/4$. With our convention, we get that $\theta(z_2) = \pi/4$. Plugging everything in,

$$\begin{aligned}\frac{1}{2\pi i}e^{-2i|x|^{3/2}/3}\int_{C_{2b}} e^{|x|^{3/2}h(z)}g(z)dz &= \frac{1}{2\pi i}|x|^{-3/4}e^{-2i|x|^{3/2}/3}\left(e^{i\pi/4}\sqrt{\frac{2\pi}{2}}\frac{1}{1} + \mathrm{O}(|x|^{-3/2})\right)\\ &= \frac{1}{2i\sqrt{\pi}}e^{-2i|x|^{3/2}/3+i\pi/4}(|x|^{-3/4} + \mathrm{O}(|x|^{-9/4})).\end{aligned}$$

Now, we can finally assemble our full expansion

$$\begin{aligned}&\mathrm{AI}(x) = -1 - \frac{1}{2i\sqrt{\pi}}e^{2i|x|^{3/2}/3-i\pi/4}(|x|^{-3/4} + \mathrm{O}(|x|^{-9/4}))\\ &+ \frac{1}{2i\sqrt{\pi}}e^{-2i|x|^{3/2}/3+i\pi/4}(|x|^{-3/4} + \mathrm{O}(|x|^{-9/4})) + \mathrm{BAO}\\ &= -1 + \left(-\frac{1}{2i\sqrt{\pi}}e^{2i|x|^{3/2}/3-i\pi/4} + \frac{1}{2i\sqrt{\pi}}e^{-2i|x|^{3/2}/3+i\pi/4}\right)(|x|^{-3/4} + \mathrm{O}(|x|^{-9/4})) + \mathrm{BAO}\end{aligned}$$

as $x \to -\infty$.