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AMATH 561 Problem Set 5 -

1. Let X and $Y_0, Y_1, \ldots$ be r.v.s on a probability space (Ω, F, P) with $E|X| < \infty$ and define $F_n = \sigma(Y_0, \ldots, Y_n)$ and $X_n = E[X \mid F_n]$. We show that X_n is a martingale w.r.t. the filtration F_n by verifying the required axioms.

i. $E|X_n| = E[|E[X \mid F_n]|] \leq E[E[|X| \mid F_n]] = E|X| < \infty$.

This follows from the theorem on slide 4 from lecture 10 by taking $\phi(x) = |x|$ and the law of total expectation.

ii. $X_n = E[X \mid F_n] \in F_n$ by the definition of conditional expectation.

iii. $E[X_{n+1} \mid F] = E[E[X \mid F_n] \mid F_{n+1}] = E[X \mid F_n] = X_n$ because $F_{n+1} \subset F_n$ and the result that the "smaller σ-algebra wins".

$\therefore X_n$ is a martingale.

2. Let $X_0, X_1, \ldots$ be iid Bernoulli r.v.s w/ parameter p and define $S_n = \sum_{i=1}^{n} X_i$ with $S_0 = 0$, $F_n = \sigma(X_0, \ldots, X_n)$,

$$Z_n = \left(\frac{1-p}{p}\right)^{2S_n - n} \quad n = 0, 1, 2, \ldots$$

We show that Z_n is a martingale w.r.t. F_n by verifying the required axioms.

i. Note that $|2S_n - n| \leq n$, so $E|Z_n| < \infty$ must hold because Z_n is a constant raised to a bounded power.

ii. We know that $S_n \in F_n$ and Z_n is a continuous function of S_n, so it must also hold that $Z_n \in F_n$.

iii. $E[Z_{n+1} \mid F_n] = E\left[\left(\frac{1-p}{p}\right)^{2S_{n+1} - (n+1)} \middle| F_n\right] = E\left[\left(\frac{1-p}{p}\right)^{2X_{n+1} - 1}\left(\frac{1-p}{p}\right)^{2S_n - n} \middle| F_n\right]$

$= E\left[\left(\frac{1-p}{p}\right)^{2X_{n+1} - 1}\right] E[Z_n \mid F_n] = \left(\left(\frac{1-p}{p}\right) \cdot p + \left(\frac{1-p}{p}\right)^{-1}(1-p)\right) Z_n$

$= ((1-p) + p) Z_n = Z_n$ because $Z_n \in F_n$ and $\left(\frac{1-p}{p}\right)^{2X_{n+1} - 1}$ is independent from F_n, because the X_is are iid.

$\therefore Z_n$ is a martingale w.r.t. F_n.

3. Given a sequence of r.v.s ξ_i s.t. $X_n = \xi_0 + \dots + \xi_n$ determine a martingale $(n \geq 0)$, we first observe that by definition $E[X_{n+1} | F_n] = X_n$. Taking the expectation of both sides, $E[E[X_{n+1} | F_n]] = E[X_n] \Rightarrow E[X_{n+1}] = E[X_n]$ by the law of total expectation. However, the linearity of expectation implies that $E[\xi_0] + \dots + E[\xi_n] + E[\xi_{n+1}] = E[\xi_0] + \dots + E[\xi_n]$, so $E[\xi_{n+1}] = 0 \ \forall n \geq 0$, meaning that $E[\xi_i] = 0 \ \forall i \geq 1$. Now, noting that $X_{i+1} = X_i + \xi_{i+1}$, $X_i = E[X_{i+1} | F_i] = E[X_i | F_i] + E[\xi_{i+1} | F_i] = X_i + E[\xi_{i+1} | F_i]$ by the linearity of conditional expectation and the fact that $X_i \in F_i$. Thus, $E[\xi_{i+1} | F_i] = 0 \ \forall i \geq 0 \Rightarrow E[\xi_i | F_{i-1}] = 0 \ \forall i \geq 1$.

Now, assume WLOG that $i > j$ (strict inequality because $i \neq j$). Then, by the law of total expectation and the fact that $\xi_j = X_j - X_{j-1}$ (if $j=0$, $\xi_j = X_j$ so consider the following while taking $X_{j-1} = 0$ which doesn't affect anything)
$E[\xi_i \xi_j] = E[\xi_i X_j] - E[\xi_i X_{j-1}] = E[E[\xi_i X_j | F_{i-1}]] - E[E[\xi_i X_{j-1} | F_{i-1}]]$. Now, $X_{j-1}, X_j \in F_{i-1}$ because $F_{j-1} \subset F_j \subset F_{i-1}$ (note that $j = i-1$ may hold). Thus, $E[\xi_i \xi_j] = E[X_j E[\xi_i | F_{i-1}]] - E[X_{j-1} E[\xi_i | F_{i-1}]]$. Now, note that $i \geq 1$ because we took $i > j$ WLOG, so $E[\xi_i | F_{i-1}] = 0$ and $E[\xi_i \xi_j] = 0$. We also have that $E[\xi_i] = 0$, so $0 = E[\xi_i \xi_j] = E[\xi_i] E[\xi_j]$, meaning that ξ_i and ξ_j are mutually uncorrelated.

4. For the branching process with $p_0 = 1/8$, $p_1 = 3/8$, $p_2 = 3/8$, $p_3 = 1/8$ and $Z_0 = 1$, $\mu = \frac{3}{8} \cdot 1 + \frac{3}{8} \cdot 2 + \frac{1}{8} \cdot 3 = \frac{3}{2} > 1$, so theorem 4.3.12 in Durrett gives that the extinction probability ρ is the unique solution in $[0,1)$ of the equation $\frac{1}{8} + \frac{3}{8}s + \frac{3}{8}s^2 + \frac{1}{8}s^3 = s$ $\Rightarrow \frac{1}{8}(s^3 + 3s^2 - 5s + 1) = 0$. Note that 1 is a root so we can factor $\frac{1}{8}(s-1)(s^2+4s-1) = 0$. The quadratic has roots $-2 \pm \sqrt{5}$, so $s = 1, -2 \pm \sqrt{5}$. $-2+\sqrt{5} \in [0,1)$, so $\rho = -2 + \sqrt{5}$.

The case where families have 2 children has $p_0 = \frac{1}{4}$, $p_1 = \frac{1}{2}$, $p_2 = \frac{1}{4}$ and $Z_0 = 1$. Here, $\mu = \frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 2 = 1$ and $p_1 < \frac{1}{2}$, so theorem 4.3.11 in Durrett gives that $Z_n = 0 \ \forall$ sufficiently large n, so $\rho = 1$.