Math 524 Autumn 2023

Homework 2 **Due Friday, October 13**

1. (a.) Let $\mathcal{A}$ be the algebra on $\mathbb{Z}_q^{\mathbb{N}}$ defined last week. Define a function μ_0 on $\mathcal{A}$ by the rule

$$\mu_0(\Pi_n^{-1}(E)) = \frac{1}{q^n}\text{card}(E)$$

when $E \in \mathbb{Z}_q^n$ and card(E) is the number of elements in E. Verify that μ_0 is well defined on $\mathcal{A}$, and is finitely additive.

(b.) Show that $(\mathbb{Z}_q^{\mathbb{N}}, \mathcal{A}, \mu_0)$ is a pre-measure space. (Hint: use 1(c) and 2(b) of last week's homework.)

2. Folland Page 27, Problem 11.

3. Suppose that $(X, \mathcal{A}, \mu_0)$ is a pre-measure space, μ^* is the corresponding outer measure, $\mathcal{M}^*$ the σ-algebra of sets satisfying the Carathéodory condition. Let $\mathcal{M}$ be the σ-algebra generated by $\mathcal{A}$.

(a.) Show that if $E \subset X$ and $\mu^*(E) = 0$ then $E \in \mathcal{M}^*$, and $E \subset F$ for some $F \in \mathcal{M}$ with $\mu^*(F) = 0$.

(b.) If $E \in \mathcal{M}^*$ and $\mu^*(E) < \infty$, show there exists $A \in \mathcal{M}$ with $\mu^*(A \setminus E) = 0$.

(c.) If $E \in \mathcal{M}^*$ and $\mu^*(E) < \infty$, show there exists $B \in \mathcal{M}$ with $\mu^*(E \setminus B) = 0$.

(d.) Show that (b) and (c) hold if $E = \cup_{j=1}^{\infty} E_j$ with $\mu^*(E_j) < \infty$.

(e.) Conclude that if $(X, \mathcal{A}, \mu_0)$ is σ-finite then $(X, \mathcal{M}^*, \mu^*)$ is the completion of $(X, \mathcal{M}, \mu)$ (where μ is the restriction of μ^* to $\mathcal{M}$).

4. Folland Page 39, Problem 28.

5. Folland Page 39, Problem 30. (Hint: use Theorem 1.18. If you prefer you may assume $E \in \mathcal{B}_{\mathbb{R}}$.)