AMATH 574 Homework 1

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January 17, 2023

1 Problem 1

1.1 Part a

Starting from (2.38)

$$\begin{aligned} \rho_t + (\rho u)_x &= 0, \\ (\rho u)_t + (\rho u^2 + P(\rho))_x &= 0, \end{aligned}$$

we wish to derive the following nonlinear equations for the pressure and velocity:

$$\begin{aligned} p_t + up_x + \rho P'(\rho)u_x &= 0, \\ u_t + (1/\rho)p_x + uu_x &= 0. \end{aligned}$$

To derive the first equation, we start we start with the conservation law for ρ_t and multiply through by $P'(\rho)$ to get that

$$P'(\rho)\rho_t + P'(\rho)\rho_x u + P'(\rho)\rho u_x = 0.$$

Now, we note that $p_x = P(\rho)_x = P'(\rho)\rho_x, p_t = P(\rho)_t = P'(\rho)\rho_t$, so this can be rewritten as

$$p_t + p_x u + \rho P'(\rho)u_x = 0,$$

the first equation. To get the second equation, we expand the conservation law for momentum and substitute $p = P(\rho)$ to get that

$$\rho_t u + \rho u_t + \rho_x u^2 + 2\rho u u_x + p_x = 0$$

which can be further simplified to

$$\rho_t u + \rho u_t + u(\rho u)_x + \rho u u_x + p_x = 0.$$

Now we plug in the conservation law for ρ_t to get that

$$0 = -(\rho u)_x u + \rho u_t + u(\rho u)_x + \rho u u_x + p_x = \rho u_t + \rho u u_x + p_x.$$

Dividing through by ρ, this yields the second equation

$$u_t + (1/\rho)p_x + uu_x = 0.$$

1.2 Part b

To write our system from part a in the form

$$q_t(x,t) + A(q(x,t))q_x(x,t) = 0,$$

we define

$$q = \begin{pmatrix} p \\ u \end{pmatrix}$$

from which we can see that

$$\begin{pmatrix} p \\ u \end{pmatrix}_t + \begin{pmatrix} u & \rho P'(\rho) \\ 1/\rho & u \end{pmatrix} \begin{pmatrix} p \\ u \end{pmatrix}_x = 0.$$

Thus, this form can be achieved with this choice of q and

$$A(q) = \begin{pmatrix} q^{(2)} & \rho P'(\rho) \\ 1/\rho & q^{(2)}. \end{pmatrix}$$

Using Mathematica, we find that $A(q)$ has eigenvalues $q^{(2)} \pm \sqrt{P'(\rho)}$. This means that $A(q)$ is real diagonalizable iff $P'(\rho) > 0$. Note that this matches the condition (2.37) assuming that $\rho > 0$.

2 Problem 2.7

Consider the p-system (2.108)

$$\begin{aligned} v_t - u_x &= 0, \\ u_t + p(v)_x &= 0. \end{aligned}$$

To determine whether this system is hyperbolic, we set

$$q = \begin{pmatrix} v \\ u \end{pmatrix}$$

which in conjunction with the chain rule allows us to write our system as

$$\begin{pmatrix} v \\ u \end{pmatrix}_t + \begin{pmatrix} 0 & -1 \\ p'(v) & 0 \end{pmatrix} \begin{pmatrix} v \\ u \end{pmatrix}_x,$$

meaning that our system is hyperbolic is hyperbolic if the matrix

$$\begin{pmatrix} 0 & -1 \\ p'(v) & 0 \end{pmatrix}$$

is diagonalizable. Using Mathematica, we find that it has eigenvalues given by $\pm\sqrt{-p'(v)}$, so our system is diagonalizable if $\sqrt{-p'(v)}$ is real for all v. Of course, this is true if $p'(v) < 0$ for all v, so it must hold that our system is hyperbolic if $p'(v) < 0$ for all v.

3 Problem 2.8

Consider isothermal flow modeled by the system (2.38) with $P(\rho) = a^2\rho$ where a is constant.

3.1 Part a

We wish to determine the wave speeds of the linearized equations (2.50)

$$\begin{pmatrix} p \\ u \end{pmatrix}_t + \begin{pmatrix} u_0 & K_0 \\ 1/\rho_0 & u_0 \end{pmatrix} \begin{pmatrix} p \\ u \end{pmatrix}_x = 0$$

where $K_0 = \rho_0 P'(\rho_0)$. Note that this is essentially the same system we encountered in problem 1, so we know that the eigenvalues of our matrix are given by $u_0 \pm \sqrt{P'(\rho)}$. Plugging in our specific P, we get that $P'(\rho) = a^2$, so the wave speeds are given by $u_0 \pm a$.

3.2 Part b

Consider the p-system (2.107)

$$\begin{aligned} V_t - U_\xi &= 0, \\ U_t + p(V)_\xi &= 0. \end{aligned}$$

We wish to linearize this system around V_0, U_0 in the case where $p(V) = a^2/V$. To do this, we let

$$\begin{aligned} V &= V_0 + \tilde{V}(x,t), \\ U &= U_0 + \tilde{U}(x,t). \end{aligned}$$

Plugging this into our system, we find that

$$\begin{aligned} \tilde{V}_t - \tilde{U}_\xi &= 0, \\ \tilde{U}_t + \left(\frac{a}{V_0 + \tilde{V}}\right)_\xi &= 0. \end{aligned}$$

We drop product terms by noting that

$$\frac{1}{V_0 + \tilde{V}} = \frac{1/V_0}{1 - \left(-\tilde{V}/V_0\right)} = \frac{1}{V_0}\sum_{j=0}^{\infty}\left(-\frac{\tilde{V}}{V_0}\right)^j = \frac{1}{V_0} - \frac{1}{V_0^2}\tilde{V} + \frac{1}{V_0^3}\tilde{V}^2 - \dots$$

for a sufficiently small perturbation, so

$$\left(\frac{a^2}{V_0 + \tilde{V}}\right)_\xi \approx -\frac{a^2}{V_0^2}\tilde{V}_\xi,$$

and our linearized system is given by

$$\begin{pmatrix} \tilde{V} \\ \tilde{U} \end{pmatrix}_t = \begin{pmatrix} 0 & -1 \\ -a^2/V_0^2 & 0 \end{pmatrix} \begin{pmatrix} \tilde{V} \\ \tilde{U} \end{pmatrix}_\xi .$$

Using Mathematica, we find that the eigenvalues of the matrix in this system are given by $\pm a/V_0$ which are the Lagrangian wave speeds. To verify that these are what we expect in relation to the Eulerian wave speeds, we note that the relation (2.102) is roughly $\xi = (x - x_0)/V_0$ when linearizing, so $V_0 d\xi = dx$, and $V_\xi = V_0 V_x$, $U_\xi = V_0 U_x$. Thus, the system can be rewritten as

$$\begin{pmatrix} \tilde{V} \\ \tilde{U} \end{pmatrix}_t = \begin{pmatrix} 0 & -V_0 \\ -a^2/V_0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{V} \\ \tilde{U} \end{pmatrix}_x .$$

Using Mathematica, we find that the matrix in this system has eigenvalues $\pm a$ which now matches up to the initial velocity u_0 as expected.

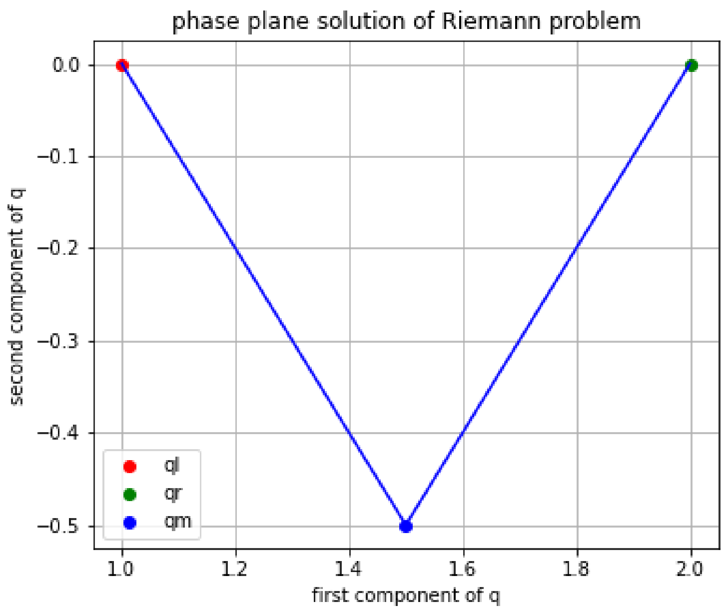
4 Problem 3.1

Using the code from problem 3.2, we provide phase plane plots for the following Riemann problems.

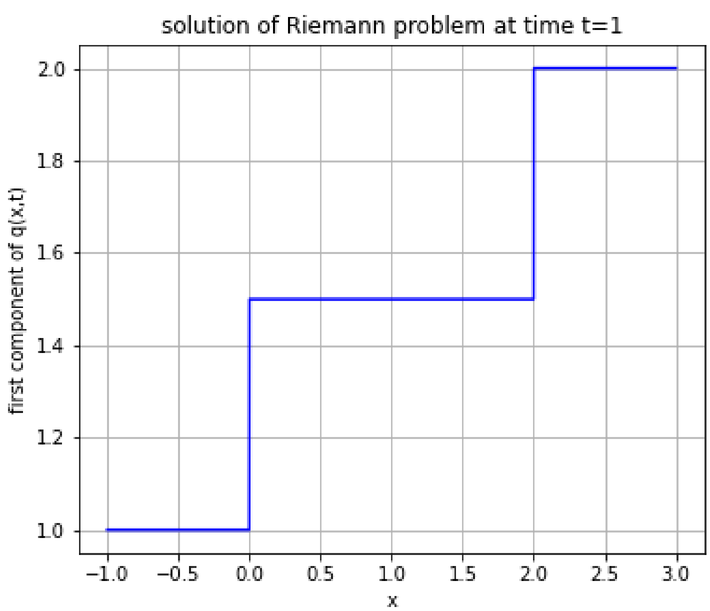
4.1 Part d

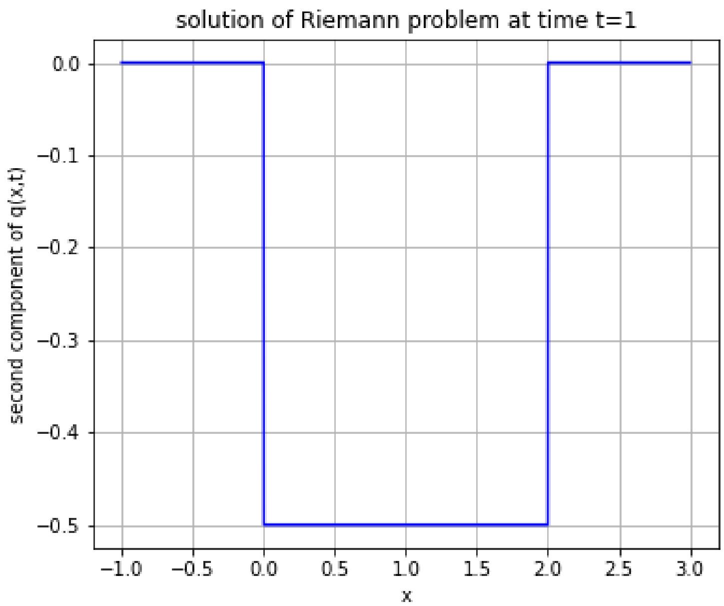
$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad q_l = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad q_r = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

yields the following phase plot



and the following solution plots at time $t = 1$.

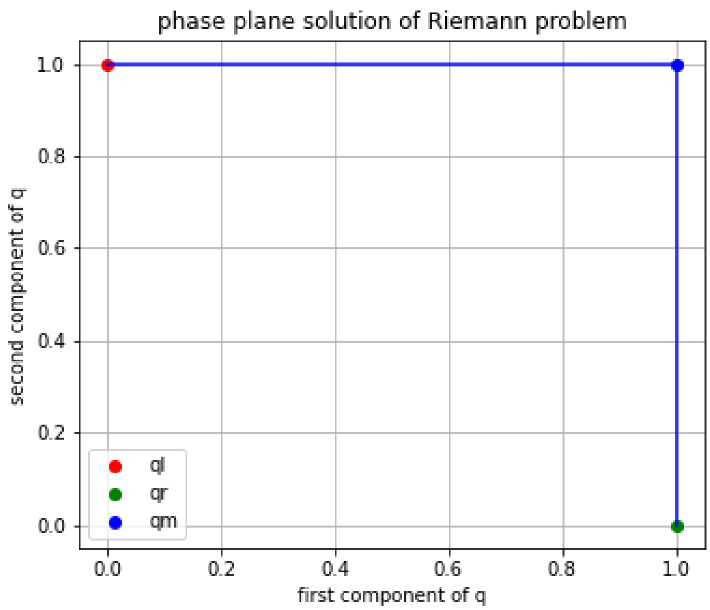




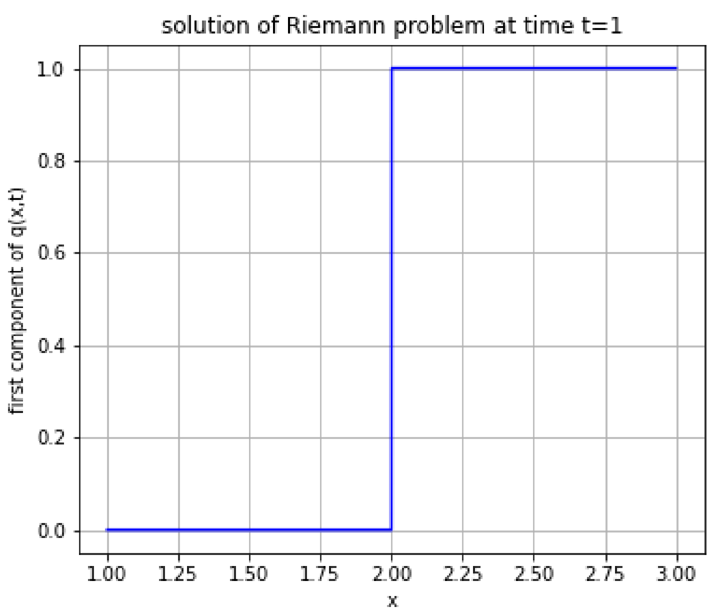
4.2 Part e

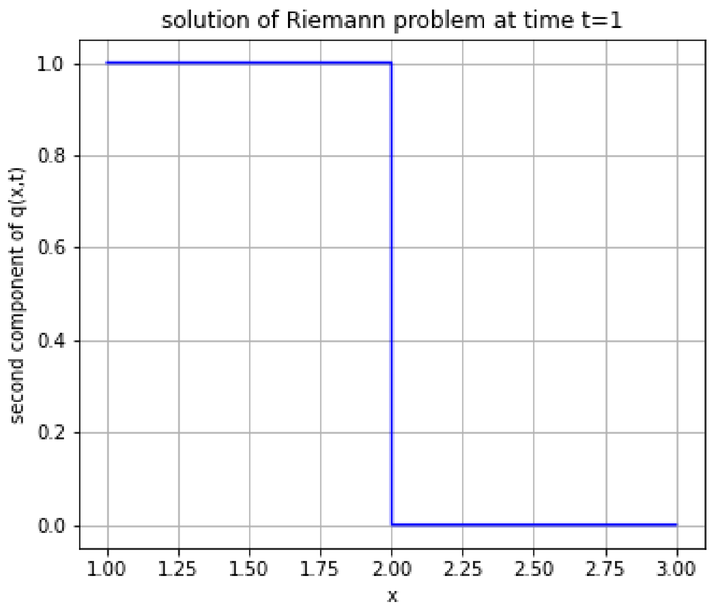
$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad q_l = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad q_r = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

yields the following phase plot



and the following solution plots at time $t = 1$.

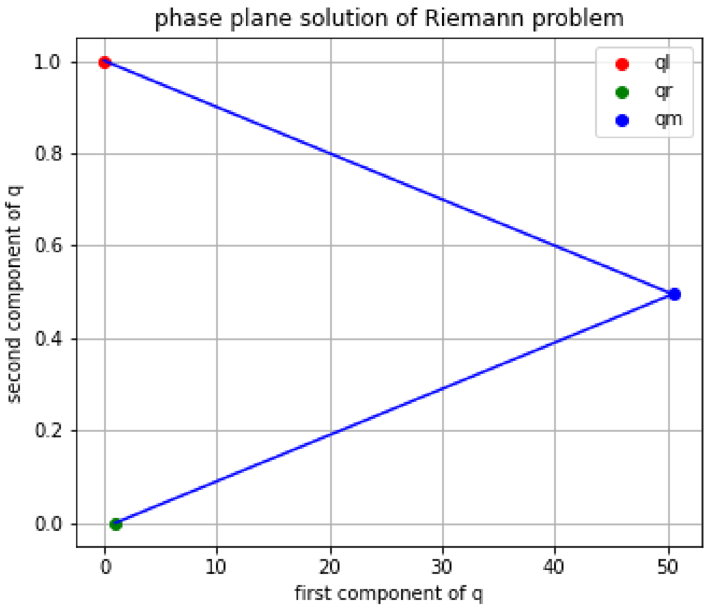




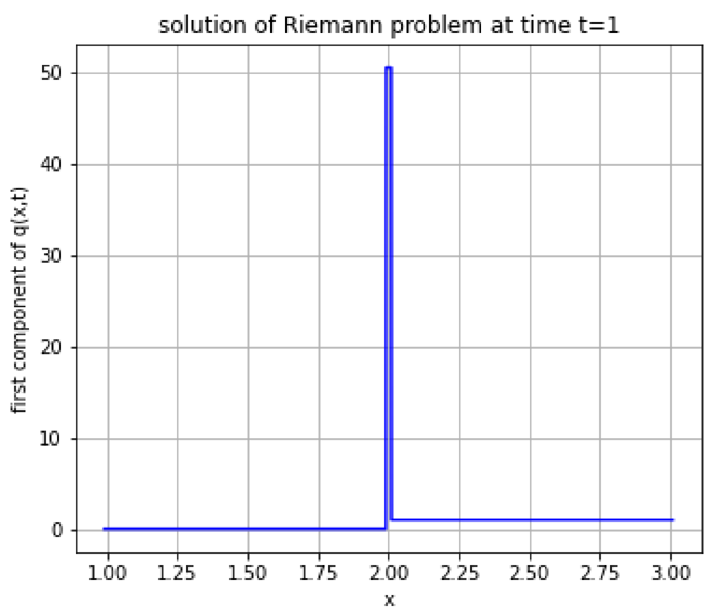
4.3 Part f

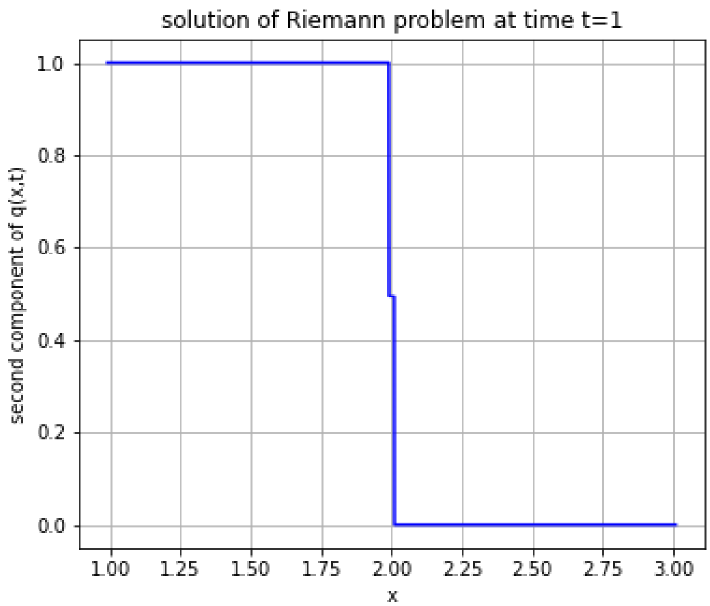
$$A = \begin{pmatrix} 2 & 1 \\ 10^{-4} & 2 \end{pmatrix}, \quad q_l = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad q_r = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

yields the following phase plot



and the following solution plots at time $t = 1$.





5 Problem 3.2

See the attached Jupyter notebook for code which solve 2×2 Riemann problems and produces their phase and solution plots.

6 Problem 3.3

6.1 Part a

We wish to solve the Riemann problem with

$$A = \begin{pmatrix} 0 & 0 & 4 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad q_l = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad q_r = \begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix}.$$

Using Mathematica, we find that A has eigenvalues $\lambda^1 = -2, \lambda^2 = 1, \lambda^3 = 2$ with corresponding eigenvectors

$$r^1 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}, \quad r^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad r^3 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix},$$

so we set

$$R = \begin{pmatrix} -2 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

and solve the linear system

$$R\alpha = q_r - q_l = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$$

via Mathematica to get

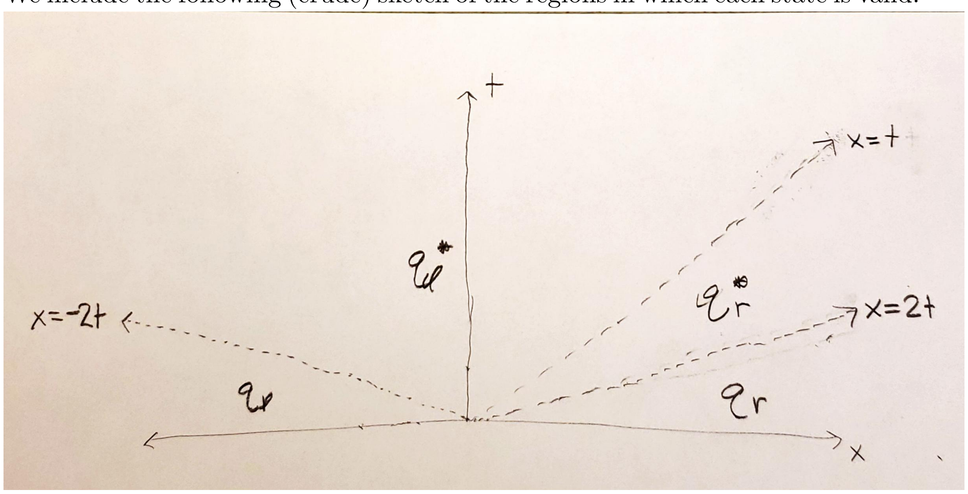
$$\alpha = \begin{pmatrix} 1/2 \\ 3 \\ 1/2 \end{pmatrix}.$$

From this, we can compute q at our two middle states as follows.

$$q_l^* = q_l + \alpha^1 r^1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 1/2 \end{pmatrix},$$

$$q_r^* = q_r - \alpha^3 r^3 = \begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ 1/2 \end{pmatrix}.$$

We include the following (crude) sketch of the regions in which each state is valid.



6.2 Part b

We wish to solve the Riemann problem with

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad q_l = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad q_r = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}.$$

Using Mathematica, we find that A has eigenvalues $\lambda^1 = 1, \lambda^2 = 2, \lambda^3 = 3$ with corresponding eigenvectors

$$r^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad r^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad r^3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

so we set

$$R = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and solve the linear system

$$R\alpha = q_r - q_l = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

via Mathematica to get

$$\alpha = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}.$$

From this, we can compute q at our two middle states as follows.

$$q_l^* = q_l + \alpha^1 r^1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - 0 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

$$q_r^* = q_r - \alpha^3 r^3 = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} - 2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}.$$

We include the following sketch of the regions in which each state is valid. Note that $q_l = q_l^*$ so we really only have three states.