AMATH 561 Problem Set 6

Cade Ballew

1. Let $X \sim \text{Binomial}(n, U)$ where $U \sim \text{Uniform}(0,1)$. Then, by the law of total expectation

$$G_X(s) = E[s^X] = E\left[E[s^X | U=p]\right] = E\left[G_{X|p}(s)\right]$$
$$= E\left[\sum_{k=0}^{n} \binom{n}{x} p^x (1-p)^{n-x} s^x\right] = E[(ps+(1-p))^n]$$

where we have used the PGF of a binomial distribution as given on slide 3 from lecture 15. Then,

$$G_X(s) = \int_0^1 (ps+(1-p))^n \, dp = \left[\frac{(ps+(1-p))^{n+1}}{(s-1)(n+1)}\right] = \frac{s^{n+1}-1}{(s-1)(n+1)} = \frac{1-s^{n+1}}{(n+1)(1-s)}$$
$$= \frac{1}{n+1} \sum_{k=0}^{n} s^k.$$

By definition, $P(X=k) = p_k = \frac{1}{n+1} \quad \forall\, k \in \{0, \dots, n\}$.

2. Let $Z_0 = 1$ and $Z_{n+1} = \sum_{i=1}^{Z_n} \xi_i^{n+1} + Y_{n+1}$ where (ξ_i^{n+1}) are iid with distribution ξ, (Y_n) are iid with distribution Y, and (ξ_i^{n+1}) and (Y_{n+1}) are independent. Then, $\sum_{i=1}^{Z_n} \xi_i^{n+1}$ and Y_{n+1} are independent, so theorem 2 from lecture 15 gives that

$$G_{Z_{n+1}}(s) = G_{\sum_{i=1}^{Z_n} \xi_i^{n+1} + Y_{n+1}}(s) = G_{\sum_{i=1}^{Z_n} \xi_i^{n+1}}(s)\, G_{Y_{n+1}}(s) = G_{\sum_{i=1}^{Z_n} \xi_i^{n+1}}(s)\, G_Y(s).$$

Then, Z_n is independent of $(\xi_i^{n+1})_{i \geq 1}$, so theorem 2 from lecture 14 gives

$$G_{Z_{n+1}}(s) = G_{Z_n}(G_\xi(s))\, G_Y(s).$$

Using this, $G_{Z_2}(s) = G_{Z_1}(G_\xi(s)) G_Y(s)$ and $G_{Z_1}(s) = G_{Z_0}(G_\xi(s)) G_Y(s)$. $G_{Z_0}(s) = E[s^{Z_0}] = E[s] = s$, so $G_{Z_1}(s) = G_\xi(s) G_Y(s)$ and

$$G_{Z_2}(s) = G_\xi(G_\xi(s))\, G_Y(G_\xi(s))\, G_Y(s).$$

3. a. From slide 9 from lecture 15, we have that

$$E[e^{itX}] = \int_0^\infty e^{itx}\lambda e^{-\lambda x} = \lim_{b\to\infty}\int_0^b \lambda e^{(it-\lambda)x}dx$$

$$= \lim_{b\to\infty}\left[\frac{\lambda}{it-\lambda}e^{(it-\lambda)x}\right]_0^b = \lim_{b\to\infty}\left(\frac{\lambda}{it-\lambda}e^{itb}e^{-\lambda b} - \frac{\lambda}{it-\lambda}\right)$$

$$= 0 - \frac{\lambda}{it-\lambda} = \frac{\lambda}{\lambda - it}$$ because $|e^{itb}| = 1 \ \forall\, b \in \mathbb{R}$ and $e^{-\lambda b} \to 0$ as $b\to\infty$ if $\lambda > 0$.

b. By definition,

$$\phi_f(t) = E[e^{itX}] = \int_{-\infty}^\infty e^{itx}dP = \int_{-\infty}^\infty e^{itx}\frac{1}{2}e^{-|x|}dx$$

$$= \frac{1}{2}\left(\lim_{a\to-\infty}\int_a^0 e^{x(it+1)}dx + \lim_{b\to\infty}\int_0^b e^{x(it-1)}dx\right)$$

$$= \frac{1}{2}\left(\lim_{a\to-\infty}\left[\frac{e^{x(it+1)}}{it+1}\right]_a^0 + \lim_{b\to\infty}\left[\frac{e^{x(it-1)}}{it-1}\right]_0^b\right)$$

$$= \frac{1}{2}\left(\lim_{a\to-\infty}\left(\frac{1}{it+1} - \frac{e^{ait}e^a}{it+1}\right) + \lim_{b\to\infty}\left(\frac{e^{bit}e^{-b}}{it-1} - \frac{1}{it-1}\right)\right)$$

$$= \frac{1}{2}\left(\frac{1}{it+1} - \frac{1}{it-1}\right) = \frac{1}{2}\,\frac{-2}{(it)^2-1} = \frac{-1}{-t^2-1} = \frac{1}{t^2+1}$$ by the same reasoning as above.

($e^a, e^{-b} \to 0$ as $a\to-\infty$, $b\to\infty$ and $|e^{ait}| = |e^{bit}| = 1$.)

4. We know that Geo(p) represents the number of coin flips with a probability p of heads needed to get the first head, so we can let

$N = X_1 + \dots + X_k$ where $(X_j)_{j=1}^k$ are iid and $X_i \sim \text{Geo}(p)$. Slide 9 from lecture 16 gives that $\phi_{X_j}(t) = \dfrac{pe^{it}}{1-(1-p)e^{it}}$ $\forall\, j$. Using the properties of characteristic functions,

$$\phi_{2Np}(t) = \phi_N(2pt) = \phi_{\sum_{j=1}^k X_j}(2pt) = (\phi(2pt))^k = \left(\frac{pe^{it}}{1-(1-p)e^{it}}\right)^k.$$

Using Wolfram-Alpha,

$$\lim_{p\to 0}\left(\frac{pe^{it}}{1-(1-p)e^{it}}\right)^k = \left(\frac{1}{1-2it}\right)^k.$$ Thus, $\lim_{p\to 0}\phi_{2Np}(t) = (1-2it)^{-k}$

$= \phi_{\Gamma(k,1/2)}(t)$ where $\phi_{\Gamma(k,1/2)}(t)$ is the characteristic function of $\Gamma(k, 1/2)$ as given on Piazza. This is continuous at $t=0$, so the continuity theorem gives that $F_{2Np} \to F_{\Gamma(k,1/2)}$ as $p \to 0$.