AMATH 575 Problem Set 4

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1 Problem 2

Consider the "all-to-all" coupled system of pulse-coupled phase oscillators on the N-dimensional torus, with coupling strength $\epsilon > 0$

$$\dot{\theta}_i = \omega + \epsilon z(\theta_i)\frac{1}{N}\sum_{j=1}^{N} g(\theta_j) \quad \mod \quad 2\pi$$

$i = 1...N$, and let $z(\theta) = A\sin\theta + B\cos\theta$, $g(\theta) = \sum_{k=1}^{\infty} a_k \sin(k\theta) + b_k\cos(k\theta)$.

1.1 Part a

Applying the same substitution as in class, the averaged system is given by

$$\dot{\psi}_i = \epsilon\frac{1}{2\pi}\int_{\psi_i}^{2\pi+\psi_i} z(s)\sum_{j=1}^{N}\frac{1}{N}g(\psi_j - \psi_i + s)ds,$$

so we need to compute the integral

$$I = \int_{\psi_i}^{2\pi+\psi_i}(A\sin s+B\cos s)\sum_{j=1}^{N}\sum_{k=1}^{\infty}(a_k\sin(k(\psi_j-\psi_i+s))+b_k\cos(k(\psi_j-\psi_i+s)))ds.$$

Applying trig identities, the term inside the summation can be rewritten as

$$\begin{aligned}&a_k\sin(k(\psi_j-\psi_i))\cos(ks) + a_k\cos(k(\psi_j-\psi_i))\sin(ks)\\ &+ b_k\cos(k(\psi_j-\psi_i))\cos(ks) - b_k\sin(k(\psi_j-\psi_i))\sin(ks).\end{aligned}$$

Observe that this is a linear combination of $\sin(ks)$ and $\cos(ks)$, so

$$I = \sum_{j=1}^{N}\sum_{k=1}^{\infty}\int_{\psi_i}^{2\pi+\psi_i}(A\sin s + B\cos s)(C\sin(ks) + D\cos(ks))ds,$$

for some C, D independent of s. By the orthogonal of trig polynomials, this integral is zero for $k \neq 1$. Thus,

$$I = \sum_{j=1}^{N}\int_{\psi_i}^{2\pi+\psi_i}(A\sin s + B\cos s)(a_1\sin(\psi_j - \psi_i + s) + b_1\cos(\psi_j - \psi_i + s))ds.$$

We compute this integral using Mathematica and find that

$$I = \sum_{j=1}^{N} \pi \left((Ba_1 - Ab_1) \sin(\psi_j - \psi_i) + (Aa_1 + Bb_1) \cos(\psi_j - \psi_i) \right).$$

Thus, we have the averaged system

$$\dot{\psi}_i = \epsilon \frac{1}{N} \sum_{j=1}^{N} f(\psi_j - \psi_i) \quad \mod \quad 2\pi,$$

where

$$f(\psi) = \frac{1}{2} \left((Ba_1 - Ab_1) \sin\psi + (Aa_1 + Bb_1) \cos\psi \right).$$

1.2 Part b

From homework 3, we know that this averaged system is guaranteed to be a gradient system when f is an odd function. Since sine is an odd function and cosine is an even function, f will be odd iff

$$Aa_1 + Bb_1 = 0.$$

1.3 Part c

To find a general condition on our constants that guarantees that $\psi_i = c$ for all i is a fixed point for all constants c. At these points, we have that

$$\dot{\psi}_i = \epsilon \frac{1}{2N} \sum_{j=1}^{N} (Aa_1 + Bb_1) = \frac{\epsilon}{2}(Aa_1 + Bb_1),$$

so these are guaranteed to be fixed points if

$$Aa_1 + Bb_1 = 2\pi m.$$

for some $m \in \mathbb{Z}$, since we have the mod2π. Now, note that

$$\frac{\partial \dot{\psi}_i}{\partial \psi_i} = \frac{1}{2N} \sum_{j \neq i} \left(-(Ba_1 - Ab_1) \cos(\psi_j - \psi_i) + (Aa_1 + Bb_1) \sin(\psi_j - \psi_i) \right),$$

$$\frac{\partial \dot{\psi}_i}{\partial \psi_j} = \frac{1}{2N} \left((Ba_1 - Ab_1) \cos(\psi_j - \psi_i) - (Aa_1 + Bb_1) \sin(\psi_j - \psi_i) \right),$$

for $j \neq i$. Evaluating this at a fixed point $\psi_i = c$ for all i, we get that the Jacobian J has entries

$$J_{ii} = -\frac{N-1}{2N}(Ba_1 - Ab_1),$$
$$J_{ij} = \frac{1}{2N}(Ba_1 - Ab_1),$$

again for $i \neq j$. We can write this out as

$$J = \frac{1}{2N}(Ba_1 - Ab_1)\begin{pmatrix} 1-N & 1 & \cdots & 1 \\ 1 & 1-N & \cdots & 1 \\ & & & \\ 1 & 1 & \ddots & 1 \\ 1 & 1 & \cdots & 1-N \end{pmatrix}.$$

It is easy to see that this matrix has a zero eigenvalue with eigenvector

$$\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

and an eigenvalue of $\frac{1}{2}(Ab_1 - Ba_1)$ with $N-1$ linear independent eigenvectors

$$\begin{pmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -1 \end{pmatrix}.$$

Thus, if S, U, C denote the stable, unstable, and center manifolds, respectively, if $Ab_1 - Ba_1 < 0$, then $\dim(S) = N-1, \dim(U) = 0, \dim(C) = 1$, if $Ab_1 - Ba_1 > 0$, then $\dim(S) = 0, \dim(U) = N-1, \dim(C) = 1$, and if $Ab_1 - Ba_1 = 0$, then $\dim(S) = 0, \dim(U) = 0, \dim(C) = N$.

2 Problem 3

Consider a two-dimensional flow with linear part

$$J = \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix}$$

where $\lambda \neq 0$ is an arbitrary real parameter. Following Bernard's notes, we apply (11.26) to the basis for H_2 (11.25). Using Mathematica, we get

$$\begin{aligned}
L_J^{(2)} \begin{pmatrix} x^2 \\ 0 \end{pmatrix} &= \begin{pmatrix} -x^2 \\ 0 \end{pmatrix}, \\
L_J^{(2)} \begin{pmatrix} xy \\ 0 \end{pmatrix} &= \begin{pmatrix} -\lambda xy \\ 0 \end{pmatrix}, \\
L_J^{(2)} \begin{pmatrix} y^2 \\ 0 \end{pmatrix} &= \begin{pmatrix} (1-2\lambda)y^2 \\ 0 \end{pmatrix}, \\
L_J^{(2)} \begin{pmatrix} 0 \\ x^2 \end{pmatrix} &= \begin{pmatrix} 0 \\ (-2+\lambda)x^2 \end{pmatrix}, \\
L_J^{(2)} \begin{pmatrix} 0 \\ xy \end{pmatrix} &= \begin{pmatrix} 0 \\ -xy \end{pmatrix}, \\
L_J^{(2)} \begin{pmatrix} 0 \\ y^2 \end{pmatrix} &= \begin{pmatrix} 0 \\ -\lambda y^2 \end{pmatrix}.
\end{aligned}$$

In the case $\lambda \neq 1/2, 2$, we can see that range $\left(L_J^{(2)}\right) = H_2$. Thus, there are no quadratic terms in this case, so up to quadratic terms

$$\begin{cases} \dot{x} = y, \\ \dot{y} = \lambda x. \end{cases}$$

In the case $\lambda = 1/2$, we have that

$$\begin{pmatrix} y^2 \\ 0 \end{pmatrix},$$

is not in the range of $L_J^{(2)}$. Thus, the normal form up to quadratic terms is given by

$$\begin{cases} \dot{x} = y + ay^2, \\ \dot{y} = \lambda x, \end{cases}$$

for some nonzero constant a. For the case $\lambda = 2$, we have that

$$\begin{pmatrix} 0 \\ x^2 \end{pmatrix},$$

is not in the range of $L_J^{(2)}$. Thus, the normal form up to quadratic terms is given by

$$\begin{cases} \dot{x} = y, \\ \dot{y} = \lambda x + ax^2, \end{cases}$$

for some nonzero constant a.

3 Problem 4

To determine the Takens–Bogdanov normal form to third order, we again look at

$$L_J^{(3)} = Jh_3 - Dh_3 J \begin{pmatrix} x \\ y \end{pmatrix}.$$

Now, consider

$$H_3 = \operatorname{span}\left\{ \begin{pmatrix} x^3 \\ 0 \end{pmatrix}, \begin{pmatrix} x^2 y \\ 0 \end{pmatrix}, \begin{pmatrix} xy^2 \\ 0 \end{pmatrix}, \begin{pmatrix} y^3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x^3 \end{pmatrix}, \begin{pmatrix} 0 \\ x^2 y \end{pmatrix}, \begin{pmatrix} 0 \\ xy^2 \end{pmatrix}, \begin{pmatrix} 0 \\ y^3 \end{pmatrix} \right\}.$$

Using Mathematica, we find

$$\begin{aligned}
L_J^{(2)} \begin{pmatrix} x^3 \\ 0 \end{pmatrix} &= \begin{pmatrix} -3x^2 y \\ 0 \end{pmatrix}, \\
L_J^{(2)} \begin{pmatrix} x^2 y \\ 0 \end{pmatrix} &= \begin{pmatrix} -2xy^2 \\ 0 \end{pmatrix}, \\
L_J^{(2)} \begin{pmatrix} xy^2 \\ 0 \end{pmatrix} &= \begin{pmatrix} -y^3 \\ 0 \end{pmatrix}, \\
L_J^{(2)} \begin{pmatrix} y^3 \\ 0 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\
L_J^{(2)} \begin{pmatrix} 0 \\ x^3 \end{pmatrix} &= \begin{pmatrix} x^3 \\ -3x^2 y \end{pmatrix}, \\
L_J^{(2)} \begin{pmatrix} 0 \\ x^2 y \end{pmatrix} &= \begin{pmatrix} x^2 y \\ -2xy^2 \end{pmatrix}, \\
L_J^{(2)} \begin{pmatrix} 0 \\ xy^2 \end{pmatrix} &= \begin{pmatrix} xy^2 \\ -y^3 \end{pmatrix}, \\
L_J^{(2)} \begin{pmatrix} 0 \\ y^3 \end{pmatrix} &= \begin{pmatrix} y^3 \\ 0 \end{pmatrix}.
\end{aligned}$$

By inspection, we see that this is a 6-dimensional set, and that

$$\begin{pmatrix} x^3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x^3 \end{pmatrix},$$

are not contained in it. Thus, the normal form up to third order terms is given by

$$\begin{cases} \dot{x} = y + a_1 x^2 + b_1 x^3, \\ \dot{y} = a_2 x^2 + b_2 x^3, \end{cases}$$

for nonzero constants a_1, a_2, b_1, b_2.

4 Problem 5

Recall from homework 1 that the Lorenz equations

$$\begin{cases} x' = 10(-x+y) \\ y' = rx - y - xz \\ z' = -\frac{8}{3}z + xy \end{cases},$$

have a fixed point at the origin that is stable for $r < 1$ and unstable for $r > 1$ and two additional fixed points at $\left(-2\sqrt{\frac{2}{3}(r-1)}, -2\sqrt{\frac{2}{3}(r-1)}, r-1\right)$, $\left(2\sqrt{\frac{2}{3}(r-1)}, 2\sqrt{\frac{2}{3}(r-1)}, r-1\right)$ when $r > 1$ that are stable for $r < \frac{470}{19}$. This implies that we have a pitchfork bifurcation at $r = 1$ since we go from 1 fixed point to 3 and undergo a change in stability.

5 Problem 6

Consider the one-parameter family of one-dimensional maps,

$$x \mapsto x^2 + c,$$

where c is a real-valued parameter.

5.1 Part 1

Using Mathematica, we find that this map has fixed points at

$$x = \frac{1}{2} \pm \frac{1}{2}\sqrt{1-4c}$$

if $c < 1/4$. if $c = 1/4$, we have one fixed point $\bar{x} = 1/2$, and if $c > 1/4$, we have no fixed points. The Jacobian of our map at these fixed points is given by

$$1 \pm \sqrt{1-4c},$$

is greater than 1 in absolute value when $c < 1/4$, so both fixed points are unstable in this case. Because of this, there is a bifurcation at $c = 1/4$.

5.2 Part 2

We consider $c = -3/4$ and let $p_- = -1/2$ be the smaller fixed point. Then,

$$f'_{-3/4}(p_-) = -1.$$

I convinced myself that as c descends through $-3/4$, we see the emergence of an (attracting) 2-cycle.

5.3 Part 3

We solve for the period 2 points by using Mathematica to find the fixed points of the map $f_c^2(\cdot)$ where $c \leq 1/4$. This gives two fixed points at

$$x = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - 4c},$$

for all $c \leq 1/4$ and two additional fixed points at

$$x = -\frac{1}{2} \pm \frac{1}{2}\sqrt{-3 - 4c},$$

for $c < -3/4$.