# AMATH 568 Homework 6

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## 1 Problem 1

Considering the boundary-value problem

$$\begin{cases} y_1''(x) + \frac{1}{16}y_1(x) = f(x), \quad x \in (-\pi, \pi), \\ y_1(-\pi) = 0, \\ y_1(\pi) = 0, \end{cases}$$

for a continuous function $f : [\pi, \pi] \to \mathbb{R}$, we apply the method of variation of parameters to solve for $y_1$. Solving the homogeneous equation

$$y_1''(x) + \frac{1}{16}y_1(x) = 0,$$

we get characteristic polynomial $r^2 + 1/16 = 0$, so $r = \pm i/4$ and a general solution is given by $2c_1 \cos(x/4) + 2c_2 \sin(x/4) = c_1 c(x) + c_2 s(x)$. Then, by (7.14) in the text, variation of parameters gives that a general solution to our original equation is given by

$$y_1(x) = s(x) \int_{x_0}^{x} \frac{c(t)f(t)}{W[c,s](t)} dt - c(x) \int_{x_0}^{x} \frac{s(t)f(t)}{W[c,s](t)} dt + c_1 c(x) + c_2 s(x).$$

First, compute the Wronskian

$$W[c,s](x) = \det \begin{pmatrix} c(x) & s(x) \\ c'(x) & s'(x) \end{pmatrix} = \det \begin{pmatrix} 2\cos\frac{x}{4} & 2\sin\frac{x}{4} \\ -\frac{1}{2}\sin\frac{x}{4} & \frac{1}{2}\cos\frac{x}{4} \end{pmatrix} = \cos^2\frac{x}{4} + \sin^2\frac{x}{4} = 1.$$

Now, we take $x_0 = -\pi$ and impose our boundary conditions.

$$0 = y_1(-\pi) = c_1 c(-\pi) + c_2 s(-\pi) = \sqrt{2}c_1 - \sqrt{2}c_2,$$

so $c_1 = c_2$. Making this substitution,

$$0 = y_1(\pi) = s(\pi) \int_{-\pi}^{\pi} c(t)f(t)dt - c(\pi) \int_{-\pi}^{\pi} s(t)f(t)dt + c_1(c(\pi) + s(\pi)),$$

$c(\pi)+s(\pi)=2\sqrt{2}$, so $s(\pi)/(c(\pi)+s(\pi))=c(\pi)/(c(\pi)+s(\pi))=1/2$ and

$$c_2=c_1=\frac{1}{2}\int_{-\pi}^{\pi}s(t)f(t)dt-\frac{1}{2}\int_{-\pi}^{\pi}c(t)f(t)dt.$$

Thus,

$$\begin{aligned}
y_1(x)=c(x)&\left(-\int_{-\pi}^{x}s(t)f(t)dt+\frac{1}{2}\int_{-\pi}^{\pi}s(t)f(t)dt-\frac{1}{2}\int_{-\pi}^{\pi}c(t)f(t)dt\right)\\
&+s(x)\left(\int_{-\pi}^{x}c(t)f(t)dt+\frac{1}{2}\int_{-\pi}^{\pi}s(t)f(t)dt-\frac{1}{2}\int_{-\pi}^{\pi}c(t)f(t)dt\right)\\
&=c(x)\left(\int_{x}^{-\pi}s(t)f(t)dt+\int_{-\pi}^{\pi}s(t)f(t)dt-\frac{1}{2}\int_{-\pi}^{\pi}s(t)f(t)dt-\frac{1}{2}\int_{-\pi}^{\pi}c(t)f(t)dt\right)\\
&+s(x)\left(\int_{-\pi}^{x}c(t)f(t)dt-\frac{1}{2}\int_{-\pi}^{\pi}[c(t)-s(t)]f(t)dt\right)\\
&=c(x)\left(\int_{x}^{\pi}s(t)f(t)dt-\frac{1}{2}\int_{-\pi}^{\pi}[s(t)+c(t)]f(t)dt\right)\\
&+s(x)\left(\int_{-\pi}^{x}c(t)f(t)dt-\frac{1}{2}\int_{-\pi}^{\pi}[c(t)-s(t)]f(t)dt\right).
\end{aligned}$$

# 2 Problem 2

Considering the nonlinear boundary-value problem

$$\begin{cases}y''(x;\epsilon)+\left(\frac{\sigma}{2}\right)^2y(x;\epsilon)+\epsilon\sin y(x;\epsilon)=0, & x\in(-\pi,\pi),\quad 0<\epsilon\ll 1,\\ y(-\pi)=1,\\ y(\pi)=0,\end{cases}$$

we first take $\sigma=1/2$ and $y(x;\epsilon)=\sum_{n=0}^{\infty}\epsilon^n y_n(x)$. Note that this is a regularly perturbed problem, because it has a solution when we take $\epsilon=0$ which we can see when solving for $y_0(x)$. Plugging in our expansion for $y$ and matching powers of $\epsilon$, $y_0$ must satisfy the BVP

$$\begin{cases}y_0''(x)+\frac{1}{16}y_0(x)=0, & x\in(-\pi,\pi),\\ y_0(-\pi)=1,\\ y_0(\pi)=0.\end{cases}$$

We know from problem 1 that a general solution to this problem is given by

$$y_0=c_1c(x)+c_2s(x).$$

Plugging in the boundary conditions, we get the system

$$\begin{aligned}1=y_0(-\pi)&=\sqrt{2}(c_1-c_2)\\ 0=y_0(\pi)&=\sqrt{2}(c_1+c_2)\end{aligned}$$

which gives that $c_1 = \frac{1}{2\sqrt{2}}$, $c_2 = -\frac{1}{2\sqrt{2}}$, so

$$y_0(x) = \frac{1}{2\sqrt{2}}c(x) - \frac{1}{2\sqrt{2}}s(x).$$

Now, looking at the terms with $\epsilon^1$, we see that $y_1$ must satisfy the BVP

$$\begin{cases} y_1''(x) + \frac{1}{16}y_1(x) + \sin(y_0(x)) = 0, & x \in (-\pi, \pi), \\ y_1(-\pi) = 0, \\ y_1(\pi) = 0, \end{cases}$$

but this is precisely the BVP that we solved in problem 1 if we take

$$f(x) = -\sin(y_0(x)) = -\sin\left(\frac{1}{2\sqrt{2}}c(x) - \frac{1}{2\sqrt{2}}s(x)\right).$$

Thus, we can use our prior work to conclude that

$$\begin{aligned} y_1(x) = &-c(x)\left(\int_x^\pi s(t)\sin\left(\frac{1}{2\sqrt{2}}c(t) - \frac{1}{2\sqrt{2}}s(t)\right)dt - \frac{1}{2}\int_{-\pi}^{\pi}[s(t)+c(t)]\sin\left(\frac{1}{2\sqrt{2}}c(t) - \frac{1}{2\sqrt{2}}s(t)\right)dt\right) \\ &- s(x)\left(\int_{-\pi}^{x} c(t)\sin\left(\frac{1}{2\sqrt{2}}c(t) - \frac{1}{2\sqrt{2}}s(t)\right)dt - \frac{1}{2}\int_{-\pi}^{\pi}[c(t)-s(t)]\sin\left(\frac{1}{2\sqrt{2}}c(t) - \frac{1}{2\sqrt{2}}s(t)\right)dt\right). \end{aligned}$$

If we instead consider $\sigma = 2$, our problem is singularly perturbed, because it is no longer possible to find a function that satisfies the boundary conditions when $\epsilon = 0$. More explicitly, the BVP

$$\begin{cases} y_0''(x) + y_0(x) = 0, & x \in (-\pi, \pi), \\ y_0(-\pi) = 1, \\ y_0(\pi) = 0 \end{cases}$$

has characteristic polynomial $r^2 + 1$ with roots $r = \pm i$ which yields a general solution of

$$y_0(x) = c_1 \cos x + c_2 \sin x.$$

Imposing the first boundary condition requires $1 = y(-\pi) = -c_1$, but imposing the second requires $0 = y(\pi) = c_1$. Clearly, these cannot both hold simultaneously.

# 3 Problem 3

## 3.1 Part a

Now, define an integral operator $\mathcal{L}$ such that $\mathcal{L}f = y_1$ where $y_1$ is as found in problem 1. Then, by the triangle inequality

$$\begin{aligned}
\|\mathcal{L}f\|_\infty &= \max_{-\pi\le x\le\pi}\left|c(x)\left(\int_x^\pi s(t)f(t)dt - \frac{1}{2}\int_{-\pi}^\pi [s(t)+c(t)]f(t)dt\right)\right. \\
&\left. + s(x)\left(\int_{-\pi}^x c(t)f(t)dt - \frac{1}{2}\int_{-\pi}^\pi [c(t)-s(t)]f(t)dt\right)\right| \\
&\le \max_{-\pi\le x\le\pi}\left\{|c(x)|\left(\int_x^\pi |s(t)||f(t)|dt + \frac{1}{2}\int_{-\pi}^\pi |s(t)+c(t)||f(t)|dt\right)\right. \\
&\left. + |s(x)|\left(\int_{-\pi}^x |c(t)||f(t)|dt + \frac{1}{2}\int_{-\pi}^\pi |c(t)-s(t)||f(t)|dt\right)\right\}.
\end{aligned}$$

Now, note that $|c(x)|, |s(x)| \le 2$ for all $x \in \mathbb{R}$, so $|c(x) - s(x)|, |c(x) + s(x)| \le 4$ by the triangle inequality as well. Then,

$$\begin{aligned}
\|\mathcal{L}f\|_\infty &\le \max_{-\pi\le x\le\pi}\left\{2\left(\int_x^\pi 2|f(t)|dt + \frac{1}{2}\int_{-\pi}^\pi 4|f(t)|dt\right)\right. \\
&\left. + 2\left(\int_{-\pi}^x 2|f(t)|dt + \frac{1}{2}\int_{-\pi}^\pi 4|f(t)|dt\right)\right\} \\
&= \max_{-\pi\le x\le\pi}\left\{4\int_x^\pi |f(t)|dt + 8\int_{-\pi}^\pi |f(t)|dt + 4\int_{-\pi}^x |f(t)|dt\right\} \\
&= \max_{-\pi\le x\le\pi}\left\{12\int_{-\pi}^\pi |f(t)|dt\right\} \le \max_{-\pi\le x\le\pi}\left\{12\int_{-\pi}^\pi \max_{-\pi\le x\le\pi}|f(x)|dt\right\} \\
&= \max_{-\pi\le x\le\pi}\left\{12\max_{-\pi\le x\le\pi}|f(x)|\int_{-\pi}^\pi dt\right\} = \max_{-\pi\le x\le\pi}\left\{24\pi\max_{-\pi\le x\le\pi}|f(x)|\right\} \\
&= 24\pi\max_{-\pi\le x\le\pi}|f(x)| = 24\pi\|f\|_\infty.
\end{aligned}$$

Thus, $\|\mathcal{L}f\|_\infty \le C\|f\|_\infty$ where $C \ge 24\pi > 0$.

## 3.2 Part b

Now, consider $Y(x;\epsilon) = y(x;\epsilon) - y_0(x)$ where $y, y_0$ satisfy their respective differential equations as in problem 2. Then,

$$\begin{aligned}
&Y''(x;\epsilon) + \left(\frac{\sigma}{2}\right)^2 Y(x;\epsilon) + \epsilon\sin\left(Y(x;\epsilon) + y_0(x)\right) \\
&= y''(x;\epsilon) - y_0(x) + \left(\frac{\sigma}{2}\right)^2 (y(x;\epsilon) - y_0(x)) + \epsilon\sin y(x;\epsilon) \\
&= \left(y''(x;\epsilon) + \left(\frac{\sigma}{2}\right)^2 y(x;\epsilon) + \epsilon\sin y(x;\epsilon)\right) - \left(y_0''(x) + \left(\frac{\sigma}{2}\right)^2 y_0(x)\right) = 0.
\end{aligned}$$

Similarly,

$$Y(-\pi) = y(-\pi) - y_0(-\pi) = 1 - 1 = 0$$

and

$$Y(\pi) = y(\pi) - y_0(\pi) = 0 - 0 = 0,$$

so $Y$ must satisfy the BVP

$$\begin{cases} Y''(x;\epsilon) + \left(\frac{\sigma}{2}\right)^2 Y(x;\epsilon) + \epsilon \sin\left(Y(x;\epsilon) + y_0(x)\right) = 0, \quad x \in (-\pi, \pi), \\ Y(-\pi) = 0, \\ Y(\pi) = 0. \end{cases}$$

Now, observe that this is the BVP from problem 1 taken with $f(x) = -\epsilon \sin\left(Y(x;\epsilon) + y_0(x)\right)$. Thus, the result of problem 1 tells us that if a solution to this BVP exists, it is given by

$$Y = -\epsilon \mathcal{L} \sin(Y + y_0).$$

Also, if such a function exists, it clearly solves the BVP. Thus, a solution to this BVP exists iff a solution to the above fixed point problem exists.

### 3.3 Part c

Considering $V, Y$, real valued and continuous on $[-\pi, \pi]$, note that the operator $\mathcal{L}$ is linear, because all integrals in $\mathcal{L}$ involve $f$ and integrals are known to be linear operators. Thus,

$$\begin{aligned} &\|\epsilon \mathcal{L} \sin(Y + y_0) - \epsilon \mathcal{L} \sin(V + y_0)\|_\infty = \|\mathcal{L}(\epsilon(\sin(Y + y_0) - \sin(V + y_0)))\|_\infty \\ &\leq C\|\epsilon(\sin(Y + y_0) - \sin(V + y_0))\|_\infty = C \max_{-\pi \leq x \leq \pi} |\epsilon(\sin(Y + y_0) - \sin(V + y_0))(x)| \\ &= \epsilon C \max_{-\pi \leq x \leq \pi} \left|\sin(Y(x) + y_0(x)) - \sin(V(x) + y_0(x))\right|. \end{aligned}$$

Now, we apply the hint that the MVT implies that $\sin(x) - \sin(y) = \cos(\xi)(x - y)$ for $\xi$ between $x$ and $y$ for $x = \sin(Y(x') + y_0(x'))$ and $y = \sin(V(x') + y_0(x'))$ where

$$x' = \arg \max_{-\pi \leq x \leq \pi} \left|\sin(Y(x) + y_0(x)) - \sin(V(x) + y_0(x))\right|.$$

Then,

$$\sin(Y(x') + y_0(x')) - \sin(V(x') + y_0(x')) = \cos\xi(Y(x') - V(x')),$$

for some $\xi$ between $Y(x') + y_0(x')$ and $V(x') + y_0(x')$, so

$$\begin{aligned} &\max_{-\pi \leq x \leq \pi} \left|\sin(Y(x) + y_0(x)) - \sin(V(x) + y_0(x))\right| \\ &= |\sin(Y(x') + y_0(x')) - \sin(V(x') + y_0(x'))| \\ &= |\cos\xi(Y(x') - V(x'))| \leq \max_{-\pi \leq x \leq \pi} |\cos\xi(Y(x) - V(x))|. \end{aligned}$$

Thus,

$$\|\epsilon\mathcal{L}\sin(Y+y_0)-\epsilon\mathcal{L}\sin(V+y_0)\|_\infty \leq \epsilon C \max_{-\pi\leq x\leq\pi}|\cos\xi(Y(x)-V(x))|$$
$$= \epsilon C \underbrace{|\cos\xi|}_{\leq 1} \max_{-\pi\leq x\leq\pi}|Y(x)-V(x)| \leq \epsilon C\|V-Y\|_\infty.$$

Thus, if we take $0<\epsilon<\frac{1}{C}=\frac{1}{24\pi}$ and $L=\epsilon C$, then

$$\|\epsilon\mathcal{L}\sin(Y+y_0)-\epsilon\mathcal{L}\sin(V+y_0)\|_\infty \leq L\|V-Y\|_\infty, \quad 0<L<1.$$

Now, we can invoke theorem 6.2 in the text by taking $\mathcal{B}$ to be the space of real-valued functions on the interval $[-\pi,\pi]$ taken with the infinity norm. This space is known to be complete, so $\mathcal{B}$ is a Banach space. We take $X$ to be the subset of bounded functions in $\mathcal{B}$ and consider $T_\epsilon : X\to X$ such that $T_\epsilon(Y)=-\epsilon\mathcal{L}\sin(Y+y_0)$. Taking $\rho=L$, then there exists a unique $f\in X$ such that $f=T_\epsilon(f)$, i.e. a solution to the problem $Y=-\epsilon\mathcal{L}\sin(Y+y_0)$ both exists and is unique. Therefore, we know that a solution our original differential equation both exists and is unique.

### 3.4 Part d

Now that we know that there is in fact a solution such that $Y=-\epsilon\mathcal{L}\sin(Y+y_0)$, then by part a,

$$\|Y\|_\infty = \|-\epsilon\mathcal{L}\sin(Y+y_0)\|_\infty = \epsilon\|\mathcal{L}\sin(Y+y_0)\|_\infty \leq C\epsilon\|\sin(Y+y_0)\|_\infty$$
$$= C\epsilon \max_{-\pi\leq x\leq\pi}\underbrace{|\sin(Y(x)+y_0(x))|}_{\leq 1} \leq C\epsilon.$$

Thus, $\|Y\|_\infty = \mathrm{O}(\epsilon)$ as $\epsilon\to 0$ by definition.

## 4 Problem 4

Now consider $W(x;\epsilon)=y(x;\epsilon)-y_0(x)-\epsilon y_1(x)$. Then, using the differential equations that $y$, $y_0$, $y_1$ must satisfy,

$$W''(x;\epsilon)+\left(\frac{\sigma}{2}\right)^2 W(x;\epsilon)+\epsilon\sin\left(W(x;\epsilon)+y_0(x)+\epsilon y_1(x)\right)-\epsilon\sin(y_0(x))$$
$$= y''(x;\epsilon)-y_0''(x)-\epsilon y_1''(x)+\left(\frac{\sigma}{2}\right)^2(y(x;\epsilon)-y_0(x)-\epsilon y_1(x))+\epsilon\sin y(x;\epsilon)-\epsilon\sin(y_0(x))$$
$$= \left(y''(x;\epsilon)+\left(\frac{\sigma}{2}\right)^2 y(x;\epsilon)+\epsilon\sin y(x;\epsilon)\right)-\left(y_0''(x)+\left(\frac{\sigma}{2}\right)^2 y_0(x)\right)-\epsilon\left(y_1''(x)+\left(\frac{\sigma}{2}\right)^2 y_1(x)+\sin(y_0(x))\right)$$
$$= 0.$$

Also, we have boundary conditions

$$W(-\pi)=y(\pi)-y_0(-\pi)-\epsilon y_1(-\pi)=1-1-0=0$$
$$W(\pi)=y(\pi)-y_0(\pi)-\epsilon y_1(\pi)=0-0-0=0.$$

Thus, $W$ satisfies the BVP

$$\begin{cases} W''(x;\epsilon) + \left(\frac{\sigma}{2}\right)^2 W(x;\epsilon) + \epsilon \sin\left(W(x;\epsilon) + y_0(x) + \epsilon y_1(x)\right) - \epsilon \sin(y_0(x)) = 0, \quad x \in (-\pi,\pi), \\ Y(-\pi) = 0, \\ Y(\pi) = 0. \end{cases}$$

which is equivalent to the fixed point problem

$$W = -\epsilon \mathcal{L}(\sin(W + y_0 + \epsilon y_1) - \sin(y_0)).$$

Since we know that $Y$ and therefore $W$ exists, we can use the linearity of $W$ (which is clear since it's an integral operator), the result of part a, and the Taylor expansion of $\sin(y(x))$ (as on page 2 of part 5 of the lecture notes) to find

$$\begin{aligned} \|W\|_\infty &= \| -\epsilon \mathcal{L}(\sin(W + y_0 + \epsilon y_1) - \sin(y_0))\|_\infty \le \epsilon C \|\sin(W + y_0 + \epsilon y_1) - \sin(y_0)\|_\infty \\ &= \epsilon C \|\sin(y) - \sin(y_0)\|_\infty = \epsilon C \|\sin(y_0) + \epsilon \cos(y_0) y_1 + \mathrm{O}(\epsilon^2) - \sin(y_0)\|_\infty \\ &= \epsilon^2 C \|\cos(y_0) y_1 + \mathrm{O}(\epsilon)\|_\infty \end{aligned}$$

Since our functions are defined on a bounded interval and continuous, we know that they must be bounded on that interval. In particular, this means that

$$\begin{aligned} \|W\|_\infty &\le \epsilon^2 C \max_{-\pi \le x \le \pi} |\cos(y_0(x)) y_1(x) + \mathrm{O}(\epsilon)| = \epsilon^2 C \max_{-\pi \le x \le \pi} \underbrace{|\cos(y_0(x))|}_{\le 1} |y_1(x)| + \mathrm{O}(\epsilon^3) \\ &\le \epsilon^2 C M + \mathrm{O}(\epsilon^3) \end{aligned}$$

for some $M > 0$. Clearly, this means that $\|W\|_\infty = \mathrm{O}(\epsilon^2)$ as $\epsilon \to 0$.