Math 524 Autumn 2023

Homework 1 **Due Friday, October 6**

1. Let $\mathbb{Z}_q^{\mathbb{N}}$ denote the collection of infinite sequences $a = a_1 a_2 a_3 \ldots$ with $a_j \in \mathbb{Z}_q = (0, 1, \ldots, q-1)$. Define

$$\rho(a, b) = \sum_{j=1}^{\infty} \frac{|a_j - b_j|}{q^j}.$$

(a.) Show that ρ is a metric on $\mathbb{Z}_q^{\mathbb{N}}$.

(b.) Show that $(\mathbb{Z}_q^{\mathbb{N}}, \rho)$ is a complete metric space. (Hint: start by showing that a sequence of elements $\{a^n\} \subset \mathbb{Z}_q^{\mathbb{N}}$ is Cauchy if and only if for each j the j-th digit a_j^n is constant for n large.)

(c.) Show that $(\mathbb{Z}_q^{\mathbb{N}}, \rho)$ is totally bounded (and thus compact).

2. For $n \geq 1$ define the projections $\Pi_n : \mathbb{Z}_q^{\mathbb{N}} \to \mathbb{Z}_q^n$ by the rule

$$\Pi_n(a_1 a_2 a_3 \ldots) = (a_1, a_2, a_3, \ldots, a_n)$$

(a.) Show that the collection $\mathcal{A}$ of sets of the form $\Pi_n^{-1}(E)$, for $n \in \mathbb{N}$ and $E \subseteq \mathbb{Z}_q^n$, form an algebra of sets.

(b.) Show that each set in $\mathcal{A}$ is both open and closed in $\mathbb{Z}_q^{\mathbb{N}}$ (in the above metric topology.)

(c.) Show that the σ-algebra generated by $\mathcal{A}$ is the Borel σ-algebra on $\mathbb{Z}_q^{\mathbb{N}}$.

3. Folland page 24, Problem 1.

4. Folland page 27, Problem 8 (see page 2 for the definition of $\limsup E_j$).

5. Folland page 27, Problem 10.