AMATH 569 Homework 1

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1 Problem 1

We consider the IVP

$$\begin{cases} \frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u = 0 \\ u(x,0) = f(x) \end{cases}$$

where

$$f(x) = \begin{cases} 1, & x \leq 0 \\ 1-x, & 0 < x < 1 \\ 0, & x \geq 1. \end{cases}$$

1.1 Part a

To find when then shock forms, we write the characteristic $x = \xi + tf(\xi)$ and use the condition from page 2 of lecture 2 and the bottom of page 10 in Bernard's notes that

$$t^* = -\frac{1}{c'(u_0(\xi^*))u_{0\xi}(\xi^*)}$$

where ξ^* maximizes $|c'(u_0)u_{0\xi}|$ for $c'(u_0)u_{0\xi} < 0$ for the general IVP

$$\begin{cases} \frac{\partial}{\partial t}u + c(u)\frac{\partial}{\partial x}u = 0 \\ u(x,0) = u_0(x). \end{cases}$$

In our case, $c(u) = u$ and $u_0(x) = f(x)$. We can compute

$$f'(x) = \begin{cases} 0, & x < 0,\ x > 1 \\ -1, & 0 < x < 1, \end{cases}$$

and

$$c'(u_0(\xi))u_{0\xi}(\xi) = f'(\xi).$$

Combining these, $f'(\xi) < 0$ only for $\xi \in (0,1)$ at which $f'(\xi) = -1$, so

$$t^* = -\frac{1}{-1} = 1,$$

and

$$x^* = \xi^* + t^* f(\xi^*) = \xi^* + (1 - \xi^*) = 1.$$

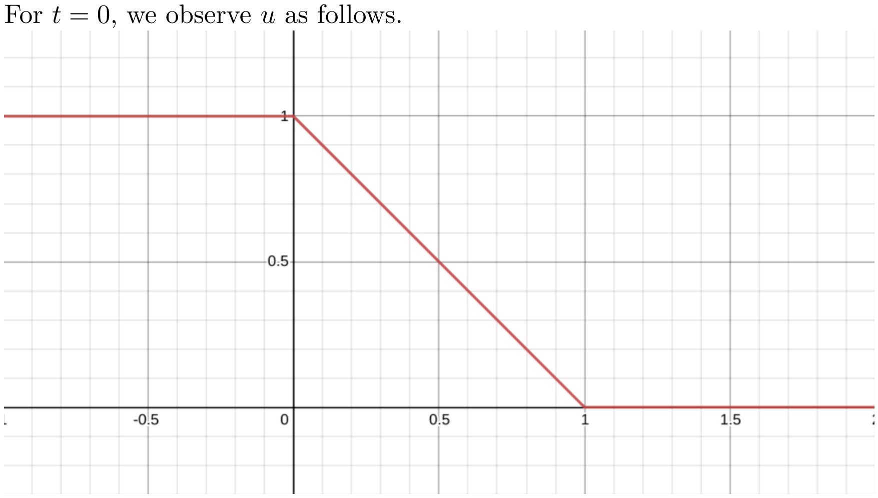
Thus, our shock first forms at time $t^* = 1$ and position $x^* = 1$. We can also see this by drawing characteristics and observing that the curves with $x \in (0, 1)$ intersect at $x^* = 1$.

1.2 Part b

The method of characteristics gives that our solution is $u(x, t) = f(\xi)$. Using our characteristic, if $\xi \leq 0$, $x = \xi + t$, so $\xi = x - t$. If $0 < \xi < 1$, $x = \xi + t(1 - \xi)$, so $\xi = \frac{x-t}{1-t}$. If $\xi \geq 1$, $x = \xi$. Thus,

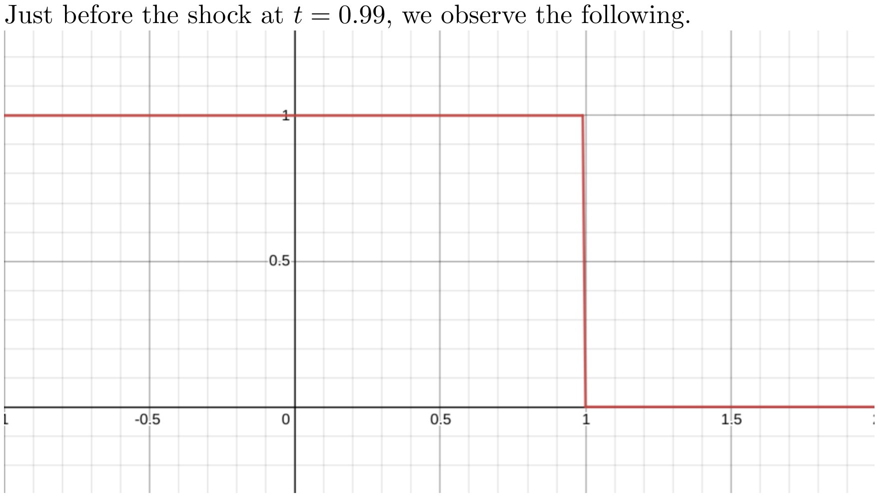
$$\begin{aligned} u(x,t) &= \begin{cases} f(x-t), & x - t \leq 0 \\ f\left(\frac{x-t}{1-t}\right), & 0 < \frac{x-t}{1-t} < 1 \\ f(x), & x \geq 1 \end{cases} = \begin{cases} 1, & x - t \leq 0 \\ 1 - \frac{x-t}{1-t}, & 0 < \frac{x-t}{1-t} < 1 \\ 0, & x \geq 1 \end{cases} \\ &= \begin{cases} 1, & x \leq t \\ \frac{1-x}{1-t}, & t < x < 1 \\ 0, & x \geq 1. \end{cases} \end{aligned}$$

Note that this is consistent with our observation that the shock occurs at time $t^* = 1$ and position $x^* = 1$. Using Desmos, we plot this solution for various t. For $t = 0$, we observe u as follows.



For $t = 0.5$, we see

Just before the shock at $t = 0.99$, we observe the following.



1.3 Part c

Note that

$$u\frac{\partial}{\partial x}u = \frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right).$$

Then, the Rankine-Hugoniot condition gives that the shock speed is

$$\dot{X}(t) = \frac{q_2 - q_1}{u_2 - u_1} = \frac{\frac{1}{2}u_2^2 - \frac{1}{2}u_1^2}{u_2 - u_1} = \frac{1}{2}(u_1 + u_2) = \frac{1}{2}(1+0) = \frac{1}{2}.$$

1.4 Part d

Using the shock speed, we can integrate to find that the position of the shock is

$$X(t) = \frac{1}{2}t + C.$$

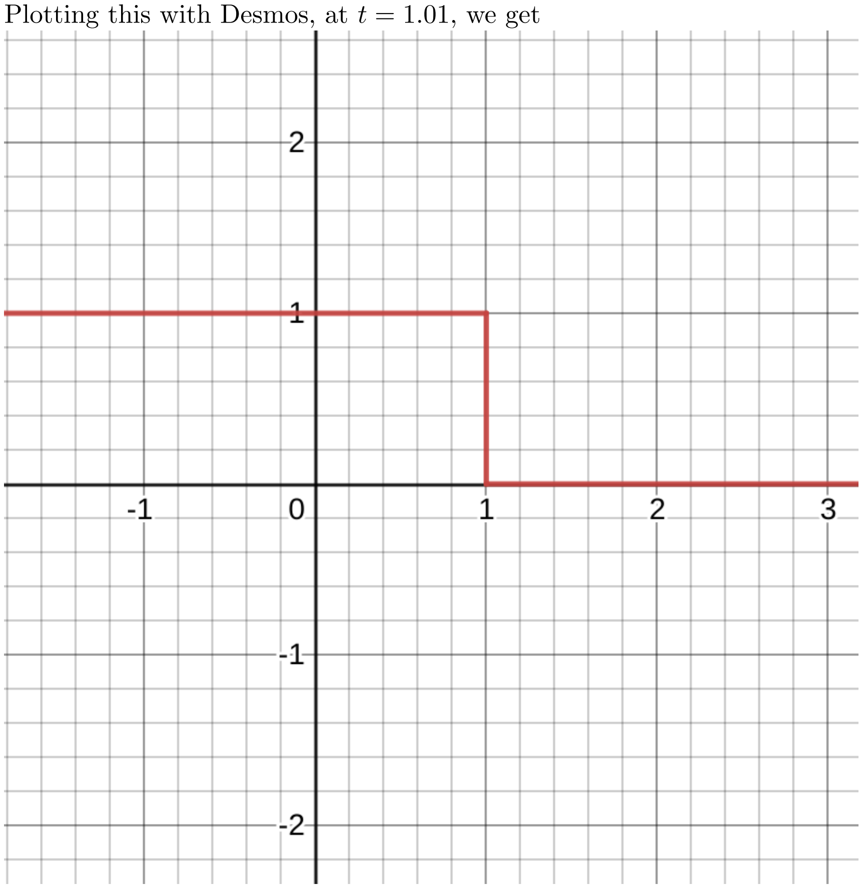
To determine C, we note that we have already found that our shock has location 1 at time 1, so $X(1) = 1$, meaning that

$$X(t) = \frac{1}{2}t + \frac{1}{2} = \frac{t+1}{2}.$$

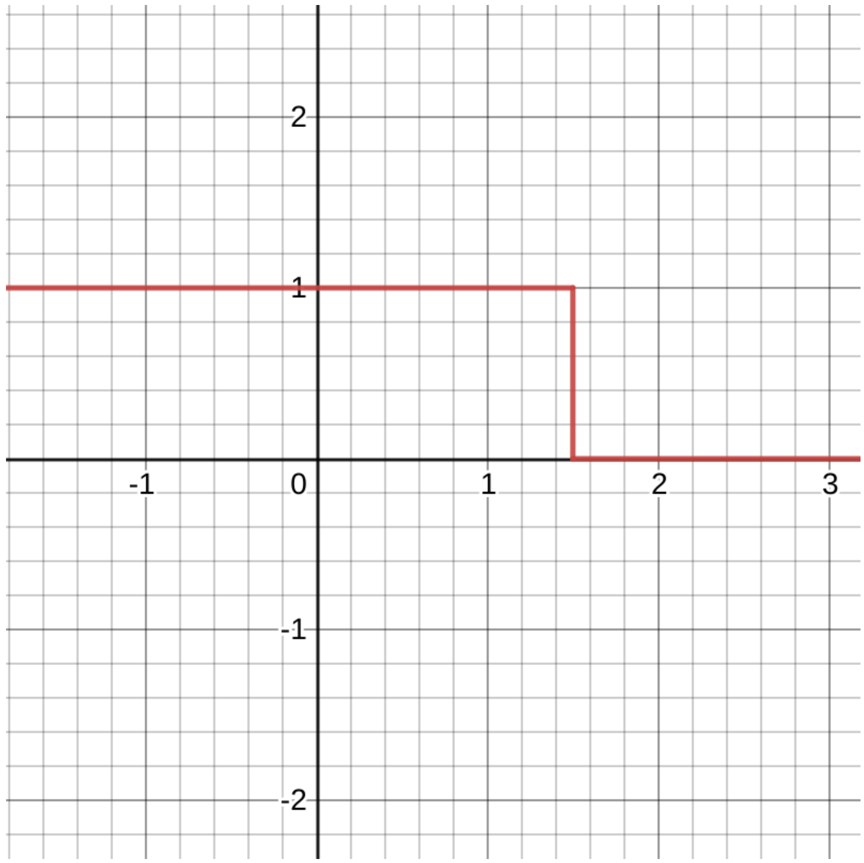
Knowing this, we expect that after the shock

$$u(x,t) = \begin{cases} 1, & x < X(t) \\ 0, & x > X(t) \end{cases} = \begin{cases} 1, & x < \frac{t+1}{2} \\ 0, & x > \frac{t+1}{2}. \end{cases}$$

Plotting this with Desmos, at $t = 1.01$, we get



and at $t = 2$, we get



2 Problem 2

Now, we consider the IVP

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u = \epsilon\frac{\partial^2}{\partial x^2}u, \quad -\infty < x < \infty,\ t > 0,$$

Subject to the initial condition

$$u(x,0) = u_0(x) = \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases}$$

where $0 < \epsilon < 1$. To find the outer solution, we set $\epsilon \to 0$ and keep all the terms O(1). Namely, we solve the IVP

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u = 0, \quad -\infty < x < \infty,\ t > 0,$$

with the same initial condition. Since this does not have a value at $x = 0$, we set $u_0(0) = c \in [-1, 1]$. Using the method of characteristics,

$$x = \xi + tu_0(\xi).$$

If $\xi < 0$, $x = \xi - t$, so $\xi = x + t$. If $\xi > 0$, $x = \xi + t$, so $\xi = x - t$. If $\xi = 0$, $x = ct$, so $c = x/t$. Note that this implies that $-1 \leq \frac{x}{t} \leq 1$. Thus, the outer solution is given by

$$u^0(x,t) = \begin{cases} u_0(x+t), & x+t < 0 \\ u_0(0), & -1 \leq \frac{x}{t} \leq 1 \\ u_0(x-t), & x-t > 0 \end{cases} = \begin{cases} -1, & x < -t \\ \frac{x}{t}, & -t \leq x \leq t \\ 1, & x > t. \end{cases}$$

3 Problem 3

Now, we instead consider the IVP

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u = \epsilon\frac{\partial^2}{\partial x^2}u, \quad -\infty < x < \infty,\ t > 0,$$

Subject to the initial condition

$$u(x,0) = u_0(x) = \begin{cases} 1 & x < 0 \\ -1 & x > 0 \end{cases}$$

where $0 < \epsilon < 1$.

3.1 Part a

To find the outer solution, we again set $\epsilon \to 0$ and keep all the terms O(1). Namely, we solve the IVP

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u = 0, \quad -\infty < x < \infty,\ t > 0,$$

with the same initial condition. Using the method of characteristics,

$$x = \xi + tu_0(\xi).$$

If $\xi < 0$, $x = \xi - t$, so $\xi = x + t$. If $\xi > 0$, $x = \xi - t$, so $\xi = x + t$. Thus, the outer solution is given by

$$u^0(x,t) = \begin{cases} u_0(x-t), & x-t < 0 \\ u_0(x+t), & x+t > 0 \end{cases} = \begin{cases} 1, & x < t \\ -1, & x > -t. \end{cases}$$

From this solution, it is clear that we have a shock when the two piecewise components of our solutions overlap, i.e., our characteristics cross when $-t < x < t$. Thus, our solution is valid in the region

$$\{(x,t) : |x| > t\}.$$

One can also see this by plotting the characteristics and observing that this region is where they cross.

3.2 Part b

Let $U = \dot{X}(t)$ be the shock speed. We let

$$\hat{x} = x - \int_0^t U dt.$$

Then,

$$u_t = \hat{x}_t u_{\hat{x}} = -U u_{\hat{x}},$$

and

$$u_x = \hat{x}_x u_{\hat{x}} = u_{\hat{x}},$$

so

$$u_{xx} = (u_{\hat{x}})_x = \hat{x}_x u_{\hat{x}\hat{x}} = u_{\hat{x}\hat{x}}.$$

Then, our equation becomes

$$(u - U)u_{\hat{x}} = \epsilon u_{\hat{x}\hat{x}}.$$

If we scale $\bar{x} = \frac{\hat{x}}{\delta}$, then

$$(u - U)u_{\bar{x}} \frac{1}{\delta} = \frac{\epsilon}{\delta^2} u_{\bar{x}\bar{x}}$$

which can be rewritten as

$$(u - U)u_{\bar{x}} = \frac{\epsilon}{\delta} u_{\bar{x}\bar{x}},$$

so we balance by taking $\delta = \epsilon$ and get that the inner equation is given by

$$(u - U)u_{\bar{x}} = u_{\bar{x}\bar{x}}.$$

3.3 Part c

We solve this equation by first integrating with respect to $\bar{x}$ to get that

$$\frac{1}{2}u^2 - Uu + C_1 = u_{\bar{x}}.$$

To match the outer solution, we need that $u \to u^{0-}$ as $\bar{x} \to -\infty$ and $u \to u^{0+}$ as $\bar{x} \to \infty$. Noting that $u_{\bar{x}} \to 0$ as $\bar{x} \to \pm\infty$ and that $u^{0\pm} = \mp 1$, we can write equations

$$\begin{cases} \frac{1}{2}1^2 - U \cdot 1 + C_1 = 0 \\ \frac{1}{2}(-1)^2 - U \cdot (-1) + C_1 = 0 \end{cases}$$

which become the linear system

$$\begin{cases} -U + C_1 = -\frac{1}{2} \\ U + C_1 = -\frac{1}{2} \end{cases}$$

which has solution[1] $U = 0$ and $C_1 = -\frac{1}{2}$. Now, we have differential equation

$$\frac{1}{2}u^2 - \frac{1}{2} = u_{\bar{x}}$$

which is separable. We write

$$d\bar{x} = \frac{2du}{(u-1)(u+1)} = \left(\frac{1}{u-1} - \frac{1}{u+1}\right) du.$$

Then,

$$\frac{u-1}{u+1} = C_2 e^{\bar{x}},$$

so

$$u = \frac{1 + C_2 e^{\bar{x}}}{1 - C_2 e^{\bar{x}}}.$$

Note that $\hat{x} = x - \int_0^t U dt = x$, so $\bar{x} = x/\epsilon$, and

$$u(x,t) = \frac{1 + C_2 e^{x/\epsilon}}{1 - C_2 e^{x/\epsilon}}.$$

Now, we find the constant C_2 by enforcing that our solution be skew-symmetric as our outer solution is. In order for this to be possible for a continuous function, we need that

$$u(0,t) = 0.$$

Enforcing this,

$$\frac{1 + C_2}{1 - C_2} = 0,$$

so $C_2 = -1$, and

$$u(x,t) = \frac{1 - e^{x/\epsilon}}{1 + e^{x/\epsilon}} = -\tanh\left(\frac{x}{2\epsilon}\right).$$

[1]Note that U matches the Rankine-Hugoniot condition that we found previously $\dot{X}(t) = \frac{1}{2}(u^{0+} + u^{0-}) = 0$.