AMATH 569 Homework 6

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June 1, 2022

1 Problem 1

1.1 Part a

Consider the 1-dimensional heat equation for conduction in a copper rod:

$$\begin{aligned}
&\frac{\partial}{\partial t}u = \alpha^2\frac{\partial^2}{\partial x^2}u,\ 0<x<L,\ t>0\\
&u(x,t)=0 \quad \text{at } x=0 \text{ and } x=L\\
&u(x,0)=f(x),\ 0<x<L.
\end{aligned}$$

To solve this using separation of variables, we let

$$u(x,t)=\phi(x)T(t).$$

Then,

$$\frac{T'(t)}{\alpha^2T(t)}=\frac{\phi''(x)}{\phi(x)}=-\lambda^2$$

where $-\lambda^2$ must be constant as the first function depends only on t while the second depends only on x. Then,

$$T(t)=T(0)e^{-\alpha^2\lambda^2t}$$

and

$$\phi''(x)=-\lambda^2\phi(x)$$

with

$$\phi(0)=\phi(L)=0.$$

The eigenfunctions are then given by

$$\phi(x)=\phi_n(x)=\sin\left(\frac{n\pi x}{L}\right)$$

with associated eigenvalues

$$\lambda=\lambda_n=\frac{n\pi}{L}$$

for $n=1,2,\ldots$. Then,

$$u(x,t)=\sum_{n=1}^{\infty}T_n(0)e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2t}\sin\left(\frac{n\pi x}{L}\right)=\sum_{n=1}^{\infty}f_ne^{-n^2t/t_e}\sin\left(\frac{n\pi x}{L}\right)$$

where $t_e=(L/\pi\alpha)^2$ and f has sine series

$$f(x)=\sum_{n=1}^{\infty}f_n\sin\left(\frac{n\pi x}{L}\right).$$

1.2 Part b

For $t > t_e$, the exponential term in our series will be maximized when n is minimized. Since the sine term oscillates, the exponential alone will determine the dominant contribution, so the dominant mode is at $n = 1$[1]. Thus,

$$u(x,t) \approx f_1 e^{-t/t_e} \sin\left(\frac{\pi x}{L}\right).$$

This implies that as time progresses, an initial condition which may not look like a sine wave but has a sine series looks like the first term of its sine series scaled by some time-dependent term. This term has the largest wavelength and fits between the boundaries.

2 Problem 2

Consider sound waves in a box satisfying

$$\begin{cases} \text{PDE: } \frac{\partial^2}{\partial t^2} u - c^2 \nabla^2 u = 0 \text{ in } V \\ \text{BC: } u = 0 \text{ on } \partial V. \end{cases}$$

Using separation of variables,

$$u(\vec{x}, t) = T(t)\phi(\vec{x})$$

which yields that

$$\frac{T''(t)}{c^2 T(t)} = \frac{\nabla^2 \phi(\vec{x})}{\phi(\vec{x})} = -\lambda^2$$

where $-\lambda^2$ must be constant as the first function depends only on t while the second depends only on x. Regardless of our domain V, this gives that

$$T(t) = A\sin(c\lambda t) + B\cos(c\lambda t).$$

2.1 Part a

Let V be a one-dimensional box given by $0 < x < L$. Then,

$$\phi''(x) = -\lambda^2 \phi(x)$$

with $\phi(0) = \phi(L) = 0$. The eigenfunctions are then given by

$$\phi(x) = \phi_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

with associated eigenvalues

$$\lambda = \lambda_n = \frac{n\pi}{L}$$

for $n = 1, 2, \ldots$. Then,

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \sin\left(\frac{cn\pi t}{L}\right) + B_n \cos\left(\frac{cn\pi t}{L}\right)\right) \sin\left(\frac{n\pi x}{L}\right).$$

The quantized frequency of oscillation ω is then given by

$$\omega_n = c\lambda_n = \frac{cn\pi}{L}$$

for $n = 1, 2, \ldots$.

[1] Note that the other modes will decay much faster than the first.

2.2 Part b

Now, let V be a two-dimensional box given by $0 < x < L$, $0 < y < L$. Then,

$$\nabla^2 \phi(\vec{x}) = -\lambda^2 \phi(\vec{x})$$

with $\phi = 0$ at $x = 0$, $x = L$, $y = 0$, $y = L$. The eigenfunctions are then given by

$$\phi(\vec{x}) = \phi_{nm}(\vec{x}) = \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{L}\right)$$

with associated eigenvalues

$$\lambda = \lambda_{nm} = \frac{\pi\sqrt{n^2 + m^2}}{L}.$$

for $n, m = 1, 2, \ldots$. Then,

$$u(\vec{x}, t) = \sum_{n,m=1}^{\infty} \left(A_{nm} \sin\left(\frac{c\pi\sqrt{n^2 + m^2}t}{L}\right) + B_{nm} \cos\left(\frac{c\pi\sqrt{n^2 + m^2}t}{L}\right)\right) \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{L}\right).$$

The quantized frequency of oscillation ω is then given by

$$\omega_{nm} = c\lambda_{nm} = \frac{c\pi\sqrt{n^2 + m^2}}{L}$$

for $n, m = 1, 2, \ldots$.

2.3 Part c

Now, let V be a two-dimensional box given by $0 < x < L$, $0 < y < L$, $0 < z < L$. Then,

$$\nabla^2 \phi(\vec{x}) = -\lambda^2 \phi(\vec{x})$$

with $\phi = 0$ at $x = 0$, $x = L$, $y = 0$, $y = L$, $z = 0$, $z = L$. The eigenfunctions are then given by

$$\phi(\vec{x}) = \phi_{nm\ell}(\vec{x}) = \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{L}\right) \sin\left(\frac{\ell\pi z}{L}\right)$$

with associated eigenvalues

$$\lambda = \lambda_{nm\ell} = \frac{\pi\sqrt{n^2 + m^2 + \ell^2}}{L}.$$

for $n, m, \ell = 1, 2, \ldots$. Then,

$$u(\vec{x}, t) = \sum_{\substack{n=1,\\ m=1,\\ \ell=1}}^{\infty} \left(A_{nm\ell} \sin\left(\frac{c\pi\sqrt{n^2 + m^2 + \ell^2}t}{L}\right) + B_{nm\ell} \cos\left(\frac{c\pi\sqrt{n^2 + m^2 + \ell^2}t}{L}\right)\right) \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{L}\right) \sin\left(\frac{\ell\pi z}{L}\right).$$

The quantized frequency of oscillation ω is then given by

$$\omega_{nm\ell} = c\lambda_{nm\ell} = \frac{c\pi\sqrt{n^2 + m^2 + \ell^2}}{L}$$

for $n, m, \ell = 1, 2, \ldots$.

3 Problem 3

Consider the Bessel equation as an eigenvalue problem

$$(ry')' + \left(\lambda r - \frac{m^2}{r}\right) y = 0$$

for $0 < r < a$ and y bounded at $r = 0$ with $y(a) = 0$. From page 170 of KK's 403 textbook, the eigenfunctions are given by

$$y(r) = \phi_j(r) = J_m(z_{mj} r/a)$$

with associated eigenvalues

$$\lambda_j = (z_{mj}/a)^2$$

for $n = 1, 2, \ldots$ and z_{mj} is the jth root of $J_m(z)$. Now, consider eigenpairs (ϕ_k, λ_k) and (ϕ_j, λ_j). Multiplying the first associated eigenvalue problem by ϕ_j and the second by ϕ_k, we have

$$\phi_j(r\phi_k')' + \phi_j \left(\lambda_k r - \frac{m^2}{r}\right) \phi_k = 0,$$
$$\phi_k(r\phi_j')' + \phi_k \left(\lambda_j r - \frac{m^2}{r}\right) \phi_j = 0.$$

Subtracting,

$$\phi_j(r\phi_k')' - \phi_k(r\phi_j')' = (\lambda_j - \lambda_k) r \phi_k \phi_j.$$

Now, note that

$$\text{LHS} = (\phi_j(r\phi_k') - \phi_k(r\phi_j'))',$$

so we can integrate both sides to get that

$$\left[\phi_j(r)(r\phi_k'(r)) - \phi_k(r)(r\phi_j'(r))\right]_0^a = (\lambda_j - \lambda_k) \int_0^a r\phi_k(r)\phi_j(r) dr.$$

Due to the boundary conditions on the eigenfunctions, we must have that LHS $= 0$. Plugging in the known eigenpairs,

$$\begin{aligned} 0 &= \left(\frac{z_{mj}^2}{a^2} - \frac{z_{mk}^2}{a^2}\right) \int_0^a J_m(z_{mj} r/a) J_m(z_{mk} r/a) r dr \\ &= (z_{mj}^2 - z_{mk}^2) \frac{1}{a^2} \int_0^a J_m(z_{mj} r/a) J_m(z_{mk} r/a) r dr. \end{aligned}$$

Letting

$$I_{jk} = \frac{1}{a^2} \int_0^a J_m(z_{mj} r/a) J_m(z_{mk} r/a) r dr,$$

if $j \neq k$, then $z_{mj}^2 \neq z_{mk}^2$, so it must hold that $I_{jk} = 0$. If $j = k$, then

$$I_{jk} = \frac{1}{a^2} \int_0^a J_m(z_{mj} r/a)^2 r dr.$$

However, this is just some positive constant due to the squared term and the boundedness condition that the eignefunctions must satisfy.