Math 525 Winter 2024

Homework 3 **Due Friday, January 26**

1. (a.) Let $W_1 \subsetneq W_2$ be subspaces of a normed vector space X, with W_1 finite dimensional. Show there exists $v \in W_2$ such that $\|v\| = 1$ and $\inf_{w \in W_1} \|v - w\| = 1$. (Hint: by scaling it suffices to find $\|v\| = c > 0$ with $\inf_{w \in W_1} \|v - w\| = c$. Consider $v' - v_0$ where v_0 is a closest element in W_1 to $v' \notin W_1$.)

(b.) If V is an infinite dimensional normed vector space, show there exists a sequence $\{v_n\} \subset V$ such that $\|v_n\| = 1$ for all v, and $\|v_n - v_m\| \geq 1$ whenever $n \neq m$. Conclude that in a normed vector space, the set $\{x : \|x\| \leq 1\}$ is compact iff V is finite dimensional.

2. If M is a finite dimensional subspace of a normed vector space X, show that there is a continuous projection map of X onto M; that is, a continuous map $T : X \to M$ such that $Tx = x$ for $x \in M$. (Hint: consider a dual basis to a basis for M.)

3. Let X denote the vector space of bounded sequences: $\{x_n\}_{n=1}^{\infty}$ with $x_n \in \mathbb{C}$ such that $\sup_n |x_n| < \infty$, where $c\{x_n\} + \{y_n\} = \{cx_n + y_n\}$. Show that there is a linear mapping $f : X \to \mathbb{C}$ such that:

$$|f(\{x_n\})| \leq \limsup_{n \to \infty} |x_n|, \quad f(\{x_n\}) = \lim_{n \to \infty} x_n \quad \text{if the limit exists.}$$

(It may help to think of $X = BC(\mathbb{N})$, where $\mathbb{N}$ is the set of natural numbers in the discrete topology, which makes $\mathbb{N}$ a LCH space.)

4. Folland page 155, Problem 7.

5. Folland page 160, Problem 25.