MATH 524 Homework 6

Cade Ballew #2120804

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1 Problem 1

Let $f, g \in L^1(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, m)$. Then, since both $|f|$ and $|g|$ are measurable, the functions $K(x,y) = |f(x-y)|$ and $G(x,y) = |g(y)|$ are Lebesgue measurable in $\mathbb{R} \times \mathbb{R}$. More explicitly, $K = |f| \circ s$ where $s(x,y) = x - y$ is continuous, so K is Lebesgue measurable. Thus, the product $|f(x-y)g(y)|$ is measurable as a function of x and y, so we apply Tonnelli's theorem and utilize the shift invariance of the Lebesgue measure to get

$$\begin{aligned}\iint |f(x-y)||g(y)|\mathrm{d}x\mathrm{d}y &= \int \left(\int |f(x-y)||g(y)|\mathrm{d}x \right) \mathrm{d}y = \int \left(\int |f(x)|\mathrm{d}x \right) |g(y)|\mathrm{d}y \\ &= \left(\int |f(x)|\mathrm{d}x \right) \left(\int |g(y)|\mathrm{d}y \right) < \infty.\end{aligned}$$

In particular, this implies that

$$\iint |f(x-y)||g(y)|\mathrm{d}x\mathrm{d}y = \int \left(\int |f(x-y)g(y)|\mathrm{d}y \right) \mathrm{d}x < \infty,$$

meaning that $\int |f(x-y)g(y)|\mathrm{d}y < \infty$ for almost every x. Thus, $f(x-y)g(y)$ is integrable in y for almost all $x \in \mathbb{R}$.

Using the substitution $z = x - y$ and the shift invariance of the Lebesgue measure, we see that

$$(f * g)(x) = \int f(x-y)g(y)\mathrm{d}y = \int f(z)g(x-z)\mathrm{d}z = (g * f)(x),$$

for all x for which these functions are defined. That is, $f * g = g * f$ almost everywhere. Finally, we use our first result to conclude that

$$\|f * g\|_1 = \int \left| \int f(x-y)g(y)\mathrm{d}x \right| \mathrm{d}y \le \iint |f(x-y)||g(y)|\mathrm{d}x\mathrm{d}y = \left(\int |f(x)|\mathrm{d}x \right) \left(\int |g(y)|\mathrm{d}y \right) = \|f\|_1 \|g\|_1.$$

2 Problem 2 (Folland Problem 11)

2.1 Part a

Consider a finite subset $\{f_j\}_{j=1}^n \subset L^1(\mu)$ and fix $\epsilon > 0$. By Corollary 3.6 in Folland, for each f_j, there exists some $\delta_j > 0$ such that $\left|\int_E f_j \mathrm{d}\mu\right| < \epsilon$ whenever $\mu(E) < \delta_j$. Let $\delta = \min_{j \in \{1,\ldots,n\}} \delta_j$. Then, for all $j = 1, \ldots, n$, $\left|\int_E f_j \mathrm{d}\mu\right| < \epsilon$ whenever $\mu(E) < \delta$. Thus, $\{f_j\}_{j=1}^n$ is uniformly integrable.

2.2 Part b

Now, let $\{f_n\} \subset L^1(\mu)$ converge to $f \in L^1(\mu)$ in the L^1 metric and fix $\epsilon > 0$. Then, there exists some $N \in \mathbb{N}$ such that $\int |f_n - f|\mathrm{d}\mu < \frac{\epsilon}{2}$ for $n \ge N$. By the reverse triangle inequality,

$$\left|\int_E f_n \mathrm{d}\mu\right| - \left|\int_E f \mathrm{d}\mu\right| \le \left|\int_E (f_n - f)\mathrm{d}\mu\right| \le \int_E |f_n - f|\mathrm{d}\mu \le \int |f_n - f|\mathrm{d}\mu < \frac{\epsilon}{2}.$$

Now, by Corollary 3.6 in Folland, since $f \in L^1(\mu)$, there exists some $\hat{\delta} > 0$ such that $\left|\int_E f \mathrm{d}\mu\right| < \frac{\epsilon}{2}$ if $\mu(E) < \hat{\delta}$. Therefore, $\left|\int_E f_n \mathrm{d}\mu\right| < \epsilon$ if $\mu(E) < \hat{\delta}$ and $n \geq N$. As in part a, for each f_n with $n < N$, there exists some $\delta_n > 0$ such that $\left|\int_E f_n \mathrm{d}\mu\right| < \epsilon$ whenever $\mu(E) < \delta_n$. Now, let $\delta = \left\{\min_{n \in \{1,\ldots,N-1\}} \delta_n, \hat{\delta}\right\}$. Then, clearly, $\left|\int_E f_n \mathrm{d}\mu\right| < \epsilon$ whenever $\mu(E) < \delta$ for all $n \in \mathbb{N}$. Thus, $\{f_n\}$ is uniformly integrable.

3 Problem 3 (Folland Problem 17)

Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space, $\mathcal{N} \subset \mathcal{M}$ a sub-σ-algebra, $\nu = \mu | \mathcal{N}$, and $f \in L^1(\mu)$. Define the signed measure $\hat{\lambda}$ on $(X, \mathcal{M})$ by

$$\hat{\lambda}(E) = \int_E f \mathrm{d}\mu, \quad E \in \mathcal{M},$$

and let $\lambda = \hat{\lambda} | \mathcal{N}$. Then, $\lambda \ll \nu$ because for any $E \in \mathcal{N}$ such that $\nu(E) = 0$, $\mu(E) = 0$, and $\lambda(E) = \int_E f \mathrm{d}\mu = 0$. Thus, the Lebesgue–Radon–Nikodym theorem (Theorem 3.8 in Folland) gives that there exists some $g = \frac{\mathrm{d}\lambda}{\mathrm{d}\nu}$ that is extended ν-integrable such that $\mathrm{d}\lambda = g \mathrm{d}\nu$. Furthermore, if g' is another such function, then $g' = g$ ν-almost everywhere. This means that for any $E \in \mathcal{N}$,

$$\int_E f \mathrm{d}\mu = \lambda(E) = \int_E \mathrm{d}\lambda = \int_E g \mathrm{d}\nu.$$

Since $f \in L^1(\mu)$, the leftmost integral is always finite, so the rightmost integral is as well, meaning that $g \in L^1(\nu)$.

4 Problem 4 (Folland Problem 21)

Let ν be a complex measure on $(X, \mathcal{M})$. For $E \in \mathcal{M}$, define

$$\mu_1(E) = \sup\left\{\sum_1^n |\nu(E_j)| : n \in \mathbb{N}, E_1, \ldots, E_n \text{ disjoint}, E = \bigsqcup_1^n E_j\right\},$$
$$\mu_2(E) = \sup\left\{\sum_1^\infty |\nu(E_j)| : E_1, E_2, \ldots \text{ disjoint}, E = \bigsqcup_1^\infty E_j\right\},$$
$$\mu_3(E) = \sup\left\{\left|\int_E f d\nu\right| : |f| \leq 1\right\}.$$

For any $E \in \mathcal{M}$, we trivially have that $\mu_1(E) \leq \mu_2(E)$, since

$$\left\{\sum_1^n |\nu(E_j)| : n \in \mathbb{N}, E_1, \ldots, E_n \text{ disjoint}, E = \bigsqcup_1^n E_j\right\} \subset \left\{\sum_1^\infty |\nu(E_j)| : E_1, E_2, \ldots \text{ disjoint}, E = \bigsqcup_1^\infty E_j\right\},$$

clearly holds. If $E = \bigsqcup_1^\infty E_j$, then proposition 3.13a gives that

$$\sum_{j=1}^\infty |\nu(E_j)| \leq \sum_{j=1}^\infty |\nu|(E_j) = |\nu|(E),$$

so we must have that $\mu_2(E) \leq |\nu|(E)$ for all $E \in \mathcal{M}$. Following the first hint, for any $E \in \mathcal{M}$,

$$\mu_3(E) \geq \left|\int_E \overline{\frac{\mathrm{d}\nu}{\mathrm{d}|\nu|}} \mathrm{d}\nu\right|.$$

We note that Proposition 3.9a generalizes to complex measures as mentioned in Section 3.3, so by this and Proposition 3.13b,

$$\mu_3(E) \geq \left|\int_E \overline{\frac{\mathrm{d}\nu}{\mathrm{d}|\nu|}} \frac{\mathrm{d}\nu}{\mathrm{d}|\nu|} \mathrm{d}|\nu|\right| = \left|\int_E \left|\frac{\mathrm{d}\nu}{\mathrm{d}|\nu|}\right|^2 \mathrm{d}|\nu|\right| = \left|\int_E \mathrm{d}|\nu|\right| = ||\nu|(E)| = |\nu|(E).$$

Thus, for all $E \in \mathcal{M}$, $\mu_1(E) \leq \mu_2(E) \leq |\nu|(E) \leq \mu_3(E)$. To show that these are equalities, fix $\epsilon > 0$ and let $|f| \leq 1$ as in the definition of μ_3. Then, by Theorem 2.26 in Folland, there exists some simple function ϕ such that

$$\left| \int_E f d\nu \right| \leq \left| \int_E \phi d\nu \right| + \epsilon.$$

Let the representation of ϕ be given by $\phi = \sum_{j=1}^n c_j \nu(F_j)$ for $F_1, \ldots, F_n$ disjoint and $X = \bigsqcup_{j=1}^n F_j$. Then, by the triangle inequality,

$$\left| \int_E f d\nu \right| \leq \left| \sum_{j=1}^n c_j \nu(F_j \cap E) \right| + \epsilon \leq \sum_{j=1}^n |c_j||\nu(F_j \cap E)| + \epsilon.$$

Now, define $E_j = F_j \cap E$ and note that because $|f| \leq 1$, $|c_j| \leq 1$ for all $j = 1, \ldots, n$. Thus,

$$\left| \int_E f d\nu \right| \leq \sum_{j=1}^n |\nu(E_j)| + \epsilon.$$

Furthermore, $E_1, \ldots, E_n$ are disjoint and $E = \bigsqcup_{j=1}^n E_j$. Since this holds for all $\epsilon > 0$ and $|f| \leq 1$, we must have that $\mu_3(E) \leq \mu_1(E)$ for all $E \in \mathcal{M}$. Thus, $\mu_1(E) = \mu_2(E) = \mu_3(E) = |\nu|(E)$ for all $E \in \mathcal{M}$.

5 Problem 5

Consider the algebra $\mathcal{A}_n \subset \mathcal{B}_{(0,1]}$ on $(0, 1]$ generated by sets of the form $E_k = \left(\frac{k}{2^n}, \frac{k+1}{2^n}\right]$ with $0 \leq k \leq 2^n - 1$ and let $f \in L^1\left((0, 1], \mathcal{B}_{(0,1]}, m\right)$ be given. Let g be the conditional expectation of f on $\mathcal{A}_n$ as defined in Problem 3. Then, since g is $\mathcal{A}_n$-measurable, it must be constant on each E_k. If it weren't, then there would exist some a such that $A_a = \{x \in X : g(x) < a\}$ would contain a nontrivial subset of E_k. Because $\mathcal{A}_n$ contains only unions of the disjoint sets E_k, a nontrivial subset cannot be $\mathcal{A}_n$-measurable. Let $\hat{m}$ denote the restriction of m to $\mathcal{A}_n$ and c_k denote the value of g on E_k. Then, it must hold that for all E_k,

$$\int_{E_k} f \mathrm{d}m = \int_{E_k} g \mathrm{d}\hat{m} = c_k \hat{m}(E_k) = c_k m(E_k) = \frac{c_k}{2^n}.$$

Thus, for $x \in E_k$,

$$g(x) = c_k = 2^n \int_{E_k} f \mathrm{d}m = 2^n \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} f(x) \mathrm{d}x.$$

Since the sets E_k cover $(0, 1]$ disjointly, this defines g on all of $(0, 1]$. Since any $E \in \mathcal{A}_n$ can be written as the disjoint union of the sets E_k, it immediately follows by additivity that

$$\int_E f \mathrm{d}m = \int_E g \mathrm{d}\hat{m},$$

for all $E \in \mathcal{A}_n$, so g as defined above must be the conditional expectation of f on $\mathcal{A}_n$ that is unique m-almost everywhere.