AMATH 569 Homework 5

Cade Ballew #2120804

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1 Problem 1

Consider the Green's function for the wave equation in two-dimensions governed by

$$\begin{aligned}
&\frac{\partial^2}{\partial t^2}G - (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})G = \delta(t)\delta(x)\delta(y) \\
&G \to 0 \text{ as } r \to \infty, \text{ where } r^2 = x^2 + y^2 \\
&G = 0 \text{ for } t < 0.
\end{aligned}$$

1.1 Part a

Let $\mathcal{F}$ denote the two-dimensional Fourier transform in x and y. Letting $\mathcal{F}[G] = \iota$ and applying this to our PDE,

$$\mathcal{F}[G_{tt}] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\partial^2}{\partial t^2} Ge^{i\omega_1 x + i\omega_2 y} dxdy = \frac{\partial^2}{\partial t^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Ge^{i\omega_1 x + i\omega_2 y} dxdy = \iota_{tt}.$$

We also can find that

$$\begin{aligned}
\int_{-\infty}^{\infty} G_{xx}e^{i\omega_1 x + i\omega_2 y} dx &= e^{i\omega_2 y}\left(\left[G_x e^{i\omega_1 x}\right]_{-\infty}^{\infty} - i\omega_1 \int_{-\infty}^{\infty} G_x e^{i\omega_1 x} dx\right) \\
&= e^{i\omega_2 y}\left(\left[G_x e^{i\omega_1 x}\right]_{-\infty}^{\infty} - i\omega_1 \left[G_x e^{i\omega_1 x}\right] - \omega_1^2 \int_{-\infty}^{\infty} Ge^{i\omega_1 x} dx\right) \\
&= -\omega_1^2 \int_{-\infty}^{\infty} Ge^{i\omega_1 x + i\omega_2 y} dx
\end{aligned}$$

if we assume that $G_x \to 0$ as $x \to \pm\infty$. We can also interchange variable names to get that

$$\int_{-\infty}^{\infty} G_{yy}e^{i\omega_1 x + i\omega_2 y} dy = -\omega_2^2 \int_{-\infty}^{\infty} Ge^{i\omega_1 x + i\omega_2 y} dy$$

if we assume that $G_y \to 0$ as $x \to \pm\infty$. Then,

$$\mathcal{F}[u_{xx} + u_{yy}] = -(\omega_1^2 + \omega_2^2)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Ge^{i\omega_1 x + i\omega_2 y} dxdy = -k^2\iota$$

if we let $k = \sqrt{\omega_1^2 + \omega_2^2}$. Finally,

$$\mathcal{F}[\delta(t)\delta(x)\delta(y)] = \delta(t) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x)\delta(y) e^{i\omega_1 x + i\omega_2 y} dx dy = \delta(t).$$

Thus, our system becomes

$$\begin{aligned}
&\frac{\partial^2}{\partial t^2}\iota + k^2\iota = \delta(t) \\
&\iota \to 0 \text{ as } k \to \infty, \text{ where } k^2 = \omega_1^2 + \omega_2^2 \\
&\iota = 0 \text{ for } t < 0.
\end{aligned}$$

Using the properties of the delta function, we can instead solve the PDE

$$\frac{\partial^2}{\partial t^2}\iota + k^2\iota = 0$$

if we enforce that $\iota = 0$ at $t = 0$ for continuity and integrate across the equation to get that as $\epsilon \to 0^+$,

$$\int_0^{\epsilon} \left(\frac{\partial^2}{\partial t^2}\iota + k^2\iota \right) dt = \iota_t \Big|_{t=\epsilon} = 1 = \int_0^{\epsilon} \delta(t) dt,$$

meaning that we need $\iota_t = 1$ for $t = 0$. Plugging these in, our equation has general solution

$$\iota(k, t) = c_1 \sin(kt) + c_2 \cos(kt)$$

which becomes

$$\iota(k, t) = \frac{1}{k} \sin(kt)$$

after plugging in our initial conditions. Note that this satisfies the boundary condition. To find G, we use the 2-D inverse Fourier transform to get

$$G = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{k} \sin(kt) e^{-i(\omega_1 x + \omega_2 y)} d\omega_1 d\omega_2$$

Converting this to polar form,

$$G = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{2\pi} \int_{-\pi-\phi}^{\pi-\phi} \frac{1}{k} \sin(kt) e^{-i(kx\cos\theta + ky\sin\theta)} k d\theta dk.$$

where $\phi = \arctan(x/y)$. Using the trig identity found here, we can rewrite this as

$$G = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sin(kt) \frac{1}{2\pi} \int_{-\pi-\phi}^{\pi-\phi} e^{ikr\sin(\theta+\phi)} d\theta dk.$$

Now, we perform the change of variables $\theta \to -(\theta + \phi)$ to get

$$\begin{aligned}
G &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \sin(kt) \frac{1}{2\pi} \int_{\pi}^{-\pi} e^{ikr\sin(-\theta)} d(-\theta) dk \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \sin(kt) \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikr\sin\theta} d\theta dk.
\end{aligned}$$

Looking up books on Bessel functions, we find that

$$G = \frac{1}{2\pi}\int_{-\infty}^{\infty} \sin(kt) J_0(kr)dk.$$

Consulting an integral table, we can then conclude that

$$G = \frac{1}{2\pi}\frac{H(t-r)}{\sqrt{t^2-r^2}}.$$

Since the derivative of a Heaviside function is a delta function, we can easily see that the additional assumptions that we imposed are satisfied by this function.

1.2 Part b

Now, we let $\mathcal{L}$ denote the Laplace transform in t and let $\tilde{G}$ the Laplace transform of G so that we may transform the PDE.

$$\begin{aligned}\mathcal{L}[G_{tt}] &= \int_0^\infty G_{tt}e^{st}dt = \left[G_t e^{st}\right]_0^\infty - s\int_0^\infty G_t e^{st}dt \\ &= \left[G_t e^{st}\right]_0^\infty - s\left[Ge^{st}\right]_0^\infty + s^2\int_0^\infty Ge^{st}dt = s^2\tilde{G}\end{aligned}$$

if we assume that $G, G_t = 0$ for $t = 0$, $t \to \infty$. We also see that

$$\mathcal{L}[\nabla^2 G] = \int_0^\infty \nabla^2 Ge^{st}dt = \nabla^2\int_0^\infty Ge^{st}dt = \nabla^2\tilde{G}$$

and

$$\mathcal{L}[\delta(t)\delta(x)\delta(y)] = \int_0^\infty \delta(t)\delta(x)\delta(y)e^{st}dt = \delta(x)\delta(y)\int_0^\infty \delta(t)e^{st}dt = \delta(x)\delta(y).$$

Thus, our system becomes

$$\begin{aligned}&s^2\tilde{G} - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\tilde{G} = \delta(x)\delta(y) \\ &\tilde{G} \to 0 \text{ as } r \to \infty, \text{ where } r^2 = x^2 + y^2 \\ &\tilde{G} = 0 \text{ for } t < 0.\end{aligned}$$

Now, we use the fact that the Laplace operator is rotationally invariant (meaning that $\tilde{G}$ is as well) to rewrite our system in polar coordinates. Note[1] that the RHS in polar coordinates is given by

$$\frac{1}{\pi r}\delta(r)$$

[1] We showed this in class.

and the Laplacian is given by

$$\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \cancel{\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}}.$$

Thus, our system becomes

$$\begin{aligned}
s^2\tilde{G} - \frac{\partial^2}{\partial r^2}\tilde{G} - \frac{1}{r}\frac{\partial}{\partial r}\tilde{G} \ &= \frac{1}{\pi r}\delta(r) \\
\tilde{G} \to 0 \text{ as } r \to \infty,\ &\text{where } r^2 = x^2 + y^2 \\
\tilde{G} = 0 \text{ for } t < 0.&
\end{aligned}$$

Multiplying through by r, our RHS is zero when $r > 0$, so we can consider the ODE

$$r\tilde{G}''(r) + \tilde{G}'(r) - s^2 r\tilde{G}(r) = 0.$$

Now, define the function $g(r) = \tilde{G}(r/s)$, so $g'(r) = \frac{1}{s}\tilde{G}'(r/s)$ and $g''(r) = \frac{1}{s^2}\tilde{G}''(r/s)$, so we have new ODE

$$\frac{r}{s}s^2 g''(r) + sg'(r) - s^2\frac{r}{s}g(r) = 0$$

which becomes

$$rg''(r) + g'(r) - rg(r) = 0$$

after dividing through by s. This is a form of the modified Bessel equation which has general solution

$$g(r) = c_1 I_0(r) + c_2 Y_0(r),$$

so we have general solution

$$\tilde{G}(r) = c_1 I_0(sr) + c_2 Y_0(sr).$$

To find the boundary conditions that we need to impose, we need to integrate both sides of the system

$$\frac{\partial^2}{\partial r^2}\tilde{G} + \frac{1}{r}\frac{\partial}{\partial r}\tilde{G} - s^2\tilde{G} \ = -\frac{1}{\pi r}\delta(r).$$

Namely, take $\epsilon > 0$ small and set

$$\int_0^{2\pi}\int_0^{\epsilon}\left(\frac{\partial^2}{\partial r^2}\tilde{G} + \frac{1}{r}\frac{\partial}{\partial r}\tilde{G} - s^2\tilde{G}\right) r\,dr\,d\theta = -\int_0^{2\pi}\int_0^{\epsilon}\frac{1}{\pi r}\delta(r) r\,dr\,d\theta.$$

Note that the RHS is -1 by construction. The LHS becomes

$$\int_0^{2\pi}\int_0^{\epsilon}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\tilde{G}\right) dr\,d\theta - s^2\int_0^{2\pi}\int_0^{\epsilon} r\tilde{G}\,dr\,d\theta.$$

As $\epsilon \to 0$, the second term vanishes due to continuity of G. The remaining term can be written as

$$\int_0^{2\pi} \int_0^{\epsilon} \frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r}\tilde{G}\right) dr d\theta = 2\pi \left[r\tilde{G}'\right]_0^{\epsilon} = 2\pi\epsilon\tilde{G}'(\epsilon).$$

Thus, as $\epsilon \to 0$, we need that

$$\epsilon\tilde{G}'(\epsilon) = -\frac{1}{2\pi}.$$

Consulting DLMF for properties of the modified Bessel functions, we need that $c_1 = 0$ since we need that $\tilde{G} \to 0$ as $r \to \infty$ and $I_0 \to \infty$ as $r \to \infty$. Thus,

$$\tilde{G}(r) = c_2 K_0(sr).$$

Then,

$$\tilde{G}'(r) = -c_2 s K_1(sr).$$

As $\epsilon \to 0^+$,

$$K_1(s\epsilon) \sim \tfrac{1}{2}\Gamma(1)(\tfrac{1}{2}s\epsilon)^{-1} = (s\epsilon)^{-1}.$$

Thus, as $\epsilon \to 0^+$,

$$\tilde{G}'(\epsilon) \sim -c_2,$$

so our matching condition is met by taking $c_2 = \frac{1}{2\pi}$. Thus,

$$\tilde{G} = \frac{1}{2\pi} K_0(sr).$$

Finally, we consult a Laplace transform table to conclude that

$$G = \frac{1}{2\pi} \frac{H(t-r)}{\sqrt{t^2 - r^2}}.$$

From this, it is easy to see that the assumptions we posed, namely that $G, G_t = 0$ for $t = 0$, $t \to \infty$ indeed hold.

2 Problem 2

2.1 Part a

Consider the ODE

$$\frac{d^2}{dx^2}u + \left(k_0^2 + \frac{i\epsilon k_0}{c}\right) u = -\frac{\delta(x-y)}{c^2}, \quad -\infty < x < \infty$$

where $\epsilon > 0$ and y is finite and subject to the boundary condition $u \to 0$ as $x \to \pm\infty$. First, we consider the case where $x < y$ so that we may solve the homogeneous equation

$$\frac{d^2}{dx^2}u + \left(k_0^2 + \frac{i\epsilon k_0}{c}\right) u = 0, \quad -\infty < x < y$$

with boundary condition $u \to 0$ as $x \to -\infty$. We find that the roots of the characteristic polynomial are given by

$$\lambda_{1,2} = \pm\sqrt{-\left(k_0^2 + \frac{i\epsilon k_0}{c}\right)}.$$

With this being a square root in the complex plane, we need to choose a branch. We take the principal branch of the square root with branch cut $(-\infty, 0]$. Assuming that $k_0, c > 0$, let

$$\lambda_2 = \sqrt{-\left(k_0^2 + \frac{i\epsilon k_0}{c}\right)}$$

which has negative real and positive imaginary part and let $\lambda_1 = -\lambda_2$. Then,

$$u(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}.$$

To enforce our boundary condition, we note that the second term exhibits exponential growth as $x \to -\infty$, so we need $c_2 = 0$ and

$$u(x) = c_1 e^{\lambda_1 x}.$$

If we instead consider $x > y$, our problem becomes

$$\frac{d^2}{dx^2}u + \left(k_0^2 + \frac{i\epsilon k_0}{c}\right) u = 0, \quad y < x < \infty$$

with boundary condition $u \to 0$ as $x \to \infty$. Which has the same general solution as the $x < y$ case. However, we now enforce the boundary condition by $c_1 = 0$ as the first term exhibits exponential growth as $x \to \infty$. Thus,

$$u(x) = \begin{cases} c_1 e^{\lambda_1 x}, & x < y \\ c_2 e^{\lambda_2 x}, & x > y. \end{cases}$$

Now, we find c_1, c_2 by matching across $x = y$. We first make the substitution $\lambda_2 = -\lambda_1$ and enforce $c_1 e^{\lambda_1 y} = c_2 e^{-\lambda_1 y}$, so $c_2 = c_1 e^{2\lambda_1 y}$. Integrating across our differential equation with bounds y^-, y^+ which are arbitrarily close to y from their respective sides, we get that

$$u'(y^+) - u'(y^-) = -\frac{1}{c^2}.$$

Using our function values, this amounts to the condition that

$$c_1 e^{2\lambda_1 y}(-\lambda_1) e^{-\lambda_1 y} - c_1 \lambda_1 e^{\lambda_1 y} = -\frac{1}{c^2}$$

which becomes

$$2c_1 \lambda_1 e^{\lambda_1 y} = \frac{1}{c^2},$$

so

$$c_1 = \frac{1}{2c^2\lambda_1}e^{-\lambda_1 y}, \quad c_2 = \frac{1}{2c^2\lambda_1}e^{\lambda_1 y}.$$

Thus,

$$u(x) = \begin{cases} \frac{1}{2c^2\lambda_1}e^{\lambda_1(x-y)}, & x < y \\ \frac{1}{2c^2\lambda_1}e^{\lambda_1(y-x)}, & x > y \end{cases}$$

which when written out in full is

$$u(x) = \begin{cases} \frac{-1}{2c^2\sqrt{-\left(k_0^2+\frac{i\epsilon k_0}{c}\right)}}e^{\sqrt{-\left(k_0^2+\frac{i\epsilon k_0}{c}\right)}(y-x)}, & x < y \\ \frac{-1}{2c^2\sqrt{-\left(k_0^2+\frac{i\epsilon k_0}{c}\right)}}e^{\sqrt{-\left(k_0^2+\frac{i\epsilon k_0}{c}\right)}(x-y)}, & x > y. \end{cases}$$

2.2 Part b

Solving the equation from part a with $\epsilon = 0$ subject to the Sommerfeld radiation condition, we again first consider the case where $x < y$ which leads to the homogeneous equation

$$\frac{d^2}{dx^2}u + k_0^2 u = 0, \quad -\infty < x < y$$

which has general solution

$$u(x) = c_1 e^{ik_0 x} + c_2 e^{-ik_0 x}.$$

Taking $k_0 > 0$, to impose Sommerfield's radiation condition, we need the first term to vanish, so we take $c_1 = 0$. Now, we consider $x > y$ which gives the homoegeneous equation

$$\frac{d^2}{dx^2}u + k_0^2 u = 0, \quad y < x < \infty$$

with the same general solution

$$u(x) = c_1 e^{ik_0 x} + c_2 e^{-ik_0 x}.$$

Now, we instead eliminate the second term by taking $c_2 = 0$. Thus, we have

$$u(x) = \begin{cases} c_1 e^{-ik_0 x}, & x < y \\ c_2 e^{ik_0 x}, & x > y. \end{cases}$$

We find c_1, c_2 by matching across $x = y$, taking $c_1 e^{-ik_0 y} = c_2 e^{ik_0 y}$, so $c_2 = c_1 e^{-2ik_0 y}$. Integrating across our differential equation with bounds y^-, y^+ which are arbitrarily close to y from their respective sides, we get that

$$u'(y^+) - u'(y^-) = -\frac{1}{c^2}.$$

This condition amounts to

$$(c_1 e^{-2ik_0 y})(ik_0)e^{ik_0 y} - c_1(-ik_0)e^{-ik_0 y} = -\frac{1}{c^2}$$

which yields that

$$c_1 = \frac{-1}{2c^2 ik_0} e^{ik_0 y}$$

and

$$c_2 = \frac{-1}{2c^2 ik_0} e^{-ik_0 y}.$$

Thus,

$$u(x) = \begin{cases} \frac{i}{2c^2 k_0} e^{ik_0(y-x)}, & x < y \\ \frac{i}{2c^2 k_0} e^{ik_0(x-y)}, & x > y. \end{cases}$$

This can alternatively be obtained by directly applying the formula at the bottom of page 3 of the radiation lecture which yields the same result via a simple calculation.
Now, we verify that this matches our solution from part a by taking $\epsilon \to 0^+$. Note that with our choice of branch,

$$\lim_{\epsilon \to 0^+} \sqrt{-\left(k_0^2 + \frac{i\epsilon k_0}{c}\right)} = \sqrt{-k_0^2} = ik_0,$$

so $\lambda_1 = -ik_0$ and

$$u(x) = \begin{cases} \frac{i}{2c^2 k_0} e^{ik_0(y-x)}, & x < y \\ \frac{i}{2c^2 k_0} e^{ik_0(x-y)}, & x > y. \end{cases}$$

which matches our new solution.