AMATH Problem set 8

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5.1. If $X=(X_t)_{t\geq 0}$ is the number of patients in the system when they arrive as a Poisson process with intensity λ and the time to treat is distributed $\text{Exp}(\mu)$, then the generator of X is given by

$$G=\begin{pmatrix} -\lambda & \lambda & & \\ \mu & -(\lambda+\mu) & \lambda & \\ & \ddots & \ddots & \ddots \end{pmatrix}$$

where the blank entries are zero. This is by the definition of a Poisson process which gives the superdiagonal, theorem 5.1.5 in Lorig which gives that gives that the leaving times are also a Poisson process, and the fact that the rows of a generator must sum to one. Theorem 5.2.7 in Lorig gives that π is an invariant distribution iff $\pi G=0$. This is equivalent to

$$\begin{cases} -\lambda\pi(0)+\mu\pi(1)=0 \\ \lambda\pi(0)-(\lambda+\mu)\pi(1)+\mu\pi(2)=0 \\ \lambda\pi(k-2)-(\lambda+\mu)\pi(k-1)+\mu\pi(k)=0, \quad k\geq 2 \end{cases}$$

Solving this, $\pi(1)=\frac{\lambda}{\mu}\pi(0)$, $\lambda\pi(0)-(\frac{\lambda^2}{\mu}+\lambda)\pi(0)+\mu\pi(2)=0 \Rightarrow \pi(2)=\left(\frac{\lambda}{\mu}\right)^2\pi(0)$.
Inductively, if we assume that $\pi(k-2)=\left(\frac{\lambda}{\mu}\right)^{k-2}\pi(0)$, $\pi(k-1)=\left(\frac{\lambda}{\mu}\right)^{k-1}\pi(0)$, $(k\geq 2)$ then $\frac{\lambda^{k-1}}{\mu^{k-2}}\pi(0)-\left(\frac{\lambda^k}{\mu^{k-1}}+\frac{\lambda^{k-1}}{\mu^{k-2}}\right)\pi(0)+\mu\pi(k)=0 \Rightarrow \pi(k)=\left(\frac{\pi}{\mu}\right)^k\pi(0)$.
Our final condition is that $1=\sum_{k=0}^{\infty}\pi(k)=\sum_{k=0}^{\infty}\pi(0)\left(\frac{\lambda}{\mu}\right)^k \Rightarrow \pi(0)=\left(\sum_{k=1}^{\infty}\left(\frac{\lambda}{\mu}\right)^k\right)^{-1}$.
This exists iff the series converges, i.e. $|\lambda/\mu|<1 \Leftrightarrow \lambda<\mu$ because $\mu,\lambda\geq 0$.
To find the actual invariant distribution, note that $\pi(0)=\frac{1}{1-\lambda/\mu}=\frac{\mu}{\mu-\lambda}=1-\frac{\mu}{\lambda}$, so $\pi(k)=\left(\frac{\lambda}{\mu}\right)^k\left(1-\frac{\mu}{\lambda}\right)\sim \text{Geo}\left(1-\frac{\lambda}{\mu}\right)$. Because this is a known distribution, we can conclude that its expectation is $\frac{1-\left(1-\frac{\lambda}{\mu}\right)}{1-\lambda/\mu}=\frac{\lambda/\mu}{1-\lambda/\mu}=\frac{\lambda}{\mu-\lambda}$.
Using Little's law, we can conclude that the total expected time a patient waits is the expected number of people in the hospital divided by the expected rate of arrival (λ because arrival is a Poisson process). Thus, the expected wait time is $\frac{\lambda}{\mu-\lambda}/\lambda=\frac{1}{\mu-\lambda}$.

5.3 If a Markov chain $X=(X_t)_{t\geq 0}$ $(S=\{0,1,2,\dots\})$ has generator

$$G=\begin{pmatrix} -\lambda & \lambda & & & \\ \mu & -(\mu+\lambda) & \lambda & & \\ & 2\mu & -(2\mu+\lambda) & \lambda & \\ & & \ddots & \ddots & \ddots \end{pmatrix},$$

then the forward Kolmogorov equation is given by

$$\frac{d}{dt}P_t(j,0)=\mu P_t(j,1)-\lambda P_t(j,0)$$

$$\frac{d}{dt}P_t(j,k)=\lambda P_t(j,k-1)-(k\mu+\lambda)P_t(j,k)+(k+1)\mu P_t(j,k+1), \quad k\geq 1.$$

when $X_0=j$ is an initial condition.

Multiplying the kth equation by s^k and summing,

$$\sum_{k=0}^{\infty} s^k \frac{d}{dt} p_t(j,k) = \sum_{k=1}^{\infty} \lambda s^k p_t(j,k-1) - \sum_{k=0}^{\infty} (k\mu+\lambda) s^k p_t(j,k) + \sum_{k=0}^{\infty} (k+1)\mu s^k p_t(j,k+1)$$

$$= \sum_{k=0}^{\infty} \lambda s^{k+1} p_t(j,k) - s\mu \sum_{k=0}^{\infty} k s^{k-1} p_t(j,k) - \lambda \sum_{k=0}^{\infty} s^k p_t(j,k) + \mu \sum_{k=0}^{\infty} \frac{d}{ds} s^{k+1} p_t(j,k+1)$$

$$= \lambda(s-1) \sum_{k=0}^{\infty} s^k p_t(j,k) - s\mu \sum_{k=0}^{\infty} \frac{d}{ds} s^k p_t(j,k) + \mu \sum_{k=0}^{\infty} \frac{d}{ds} s^k p_t(j,k)$$

(Note that the $k=0$ term is zero, so we continue to index from 0 despite shifting $k+1 \to k$)

$= \lambda(s-1) G_{X_t}(s) - \mu(s-1) \frac{\partial}{\partial s} G_{X_t}(s)$ by definition if we move the derivatives outside the sums. This also gives that LHS $= \frac{\partial}{\partial t} G_{X_t}(s)$, so we have the PDE $\frac{\partial}{\partial t} G_{X_t}(s) = \lambda(s-1) G_{X_t}(s) - \mu(s-1) \frac{\partial}{\partial s} G_{X_t}(s)$, $G_{X_0} = s^j$

Solving this with Mathematica, we get that

$$G_{X_t}(s) = (1+e^{-\mu t}(s-1))^j \exp\left(\frac{\lambda}{\mu}(1-e^{-\mu t})(s-1)\right).$$

As $t \to \infty$, $G_{X_t}(s) \to 1^j \exp\left(\frac{\lambda}{\mu}(s-1)\right) = e^{\frac{\lambda}{\mu}(s-1)}$ because $e^{-\mu t} \to 0$ as $t \to \infty$ if $\mu > 0$. We recognize this as the generating function of a Poisson r.v. (we can also see this as $e^{\frac{\lambda}{\mu}(s-1)} = e^{-\lambda/\mu} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\lambda}{\mu}\right)^k s^k \Rightarrow p_k = \left(\frac{\lambda}{\mu}\right)^k$), so $X_t \sim \text{Pois}\left(\frac{\lambda}{\mu}\right)$ as $t \to \infty$.

5.4. This Poisson process N_t has generator

$$G = \begin{pmatrix} -\lambda(t) & \lambda(t) & & \\ 0 & -\lambda(t) & \lambda(t) & \\ & & \ddots & \ddots \end{pmatrix},$$

so the forward kolmogorov equation is given by

$$\begin{cases} \frac{d}{dt} p_t(i,0) = -\lambda(t) p_t(i,0) \\ \frac{d}{dt} p_t(i,j) = \lambda(t) p_t(i,j-1) - \lambda(t) p_t(i,j), \ j \geq 1 \end{cases}$$

and the backward equation is given by $\frac{d}{dt} p_t(i,j) = -\lambda(t) p_t(i,j) + \lambda(t) p_t(i+1,j) \ \forall j$.

Solving the forward equation by multiplying the jth equation by s^j and summing,

$$\sum_{j=0}^{\infty} s^j \frac{\partial}{\partial t} p_t(i,j) = \sum_{j=0}^{\infty} s^j p_t(i,j-1)\lambda(t) - \sum_{j=0}^{\infty} s^j p_t(i,j)\lambda(t)$$

$$= \lambda(t) \sum_{j=0}^{\infty} s^{j+1} p_t(i,j) - \sum_{j=0}^{\infty} s^j p_t(i,j)\lambda(t) = \lambda(t)(s-1) \sum_{j=0}^{\infty} s^j p_t(i,j)$$

Thus, $\frac{\partial}{\partial t} G_{N_t}(s) = \lambda(t)(s-1) G_{N_t}(s)$ by definition.

For the backward equation, multiply the jth equation by s^j to get

$$\sum_{j=0}^{\infty} s^j \frac{\partial}{\partial t} p_t(i,j) = -\sum_{j=0}^{\infty} \lambda(t) s^j p_t(i,j) + \sum_{j=0}^{\infty} \lambda(t) s^j p_t(i+1,j)$$

$$= -\lambda(t) \sum_{j=0}^{\infty} s^j p_t(i,j) + \lambda(t) \sum_{j=1}^{\infty} s^j p_t(i,j-1) = -\lambda(t) \sum_{j=0}^{\infty} s^j p_t(i,j) + \lambda(t) \sum_{j=0}^{\infty} s^{j+1} p_t(i,j)$$

$= \lambda(t)(s-1) \sum_{j=0}^{\infty} s^j p_t(i,j)$ where we have used that the probability of a jump of size one does not depend on the current state. Thus, $\frac{\partial}{\partial t} G_{N_t}(s) = \lambda(t)(s-1) G_{N_t}(s)$ by definition and the two equations are the same.

Solving these, we separate and integrate

$$\int \frac{dG_{N_t}(s)}{G_{N_t}(s)} = \int \lambda(t)(s-1)\,dt$$

which gives $\ln|G_{N_t}(s)| + C = (s-1)\int \lambda(t)\,dt$, so $G_{N_t}(s) = Ae^{(s-1)\int \lambda(t)dt}$ for some constant A. Now, consider $G_{N_0}(s) = s^0 = 1$, $\lambda(t) = \frac{c}{1+t}$,

so $G_{N_t}(s) = Ae^{(s-1)\int \frac{c}{1+t}dt} = Ae^{(s-1)c\ln(1+t)}$ (note $t \geq 0$ here)

$= A(t+1)^{c(s-1)}$. Plugging in the initial condition, $A = 1$. Thus,

$G_{N_t}(s) = (t+1)^{c(s-1)}$. To find $E[\tau_1 | N_0 = 0]$, consider that the CDF of τ_1 is $P(\tau_1 \leq t) = 1 - p_t(0,0) = 1 - G_{N_t}(0) = 1 - (t+1)^{-c}$, so its PDF is $c(t+1)^{-c-1}$.

Thus, $E[\tau_1 | N_0 = 0] = \int_0^\infty t c(t+1)^{-c-1}\,dt = \int_0^\infty \frac{ct}{(t+1)^{c+1}}\,dt < \infty$ iff $c > 1$ because the integrand looks like $\frac{A}{t^c}$ as $t \to \infty$.

5.5 By the law of iterated expectations,

$$E[s^{N_t}] = E[E[s^{N_t} | \Lambda = \lambda]] = E[G_{N_t}(s)] = E[e^{(s-1)\lambda t}]$$

$$= pe^{(s-1)\lambda_1 t} + (1-p)e^{(s-1)\lambda_2 t} =: G_{N_t}(s)$$

Now, using the results of lecture 24 slide 11,

$$E[N_t] = G_{N_t}'(1) = \lambda_1 t pe^{(s-1)\lambda_1 t} + \lambda_2 t(1-p)e^{(s-1)\lambda_2 t}\Big|_{s=1} = (p\lambda_1 + (1-p)\lambda_2)t.$$

$$\mathrm{Var}(N_t) = G_{N_t}''(1) + G_{N_t}'(1) - (G_{N_t}'(1))^2 = \lambda_1^2 t^2 pe^{(s-1)\lambda_1 t} + \lambda_2^2 t^2(1-p)e^{(s-1)\lambda_2 t}\Big|_{s=1}$$

$$+ E[N_t] - (E[N_t])^2 = (p\lambda_1^2 + (1-p)\lambda_2^2)t^2 + (p\lambda_1 + (1-p)\lambda_2)t - (p\lambda_1 + (1-p)\lambda_2)^2 t^2$$

$$= (p\lambda_1 + (1-p)\lambda_2)t - 2p(1-p)\lambda_1\lambda_2.$$