MATH 524 Homework 5

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1 Problem 1

Let μ and ν be regular Borel measures on $\mathbb{R}^n$ with the property that

$$\int \phi \mathrm{d}\mu = \int \phi \mathrm{d}\nu,$$

for all $\phi \in C_c(\mathbb{R}^n)$. Let $E \in \mathcal{B}_{\mathbb{R}^n}$ and assume that E is contained in some finite rectangle which implies that $\mu(E), \nu(E) < \infty$. Then, because μ is regular,

$$\mu(E) = \sup\{\mu(K) \mid K \subset E,\ K \text{ compact}\},$$

so for any $\epsilon > 0$, there exists some compact set $K_1 \subset E$ such that

$$\mu(E) \leq \mu(K_1) + \frac{\epsilon}{2}.$$

Similarly, because ν is regular,

$$\nu(E) = \inf\{\nu(U) \mid E \subset U,\ U \text{ open}\},$$

so there exists some open set U_1 such that

$$\nu(U_1) \leq \nu(E) + \frac{\epsilon}{2},$$

and $E \subset U_1$. From class (and the 12th lecture notes), we have that there exists some continuous function $f_1 : \mathbb{R}^n \to [0,1]$ such that $f_1 = 1$ on K_1 and $f_1 = 0$ on $\mathbb{R}^n \setminus U_1$. In particular, $f_1 \in C_c(\mathbb{R}^n)$ and $\mathbb{1}_{K_1} \leq f_1 \leq \mathbb{1}_{U_1}$. Then, by monotonicity and the definition of the Lebesgue integral,

$$\mu(E) \leq \mu(K_1) + \frac{\epsilon}{2} = \int \mathbb{1}_{K_1} \mathrm{d}\mu + \frac{\epsilon}{2} \leq \int f_1 \mathrm{d}\mu + \frac{\epsilon}{2} = \int f_1 \mathrm{d}\nu + \frac{\epsilon}{2} \leq \int \mathbb{1}_{U_1} \mathrm{d}\nu + \frac{\epsilon}{2} = \nu(U_1) + \frac{\epsilon}{2} \leq \nu(E) + \epsilon.$$

Now, we apply the same argument with μ and ν interchanged. That is, because ν is regular, for any $\epsilon > 0$, there exists some compact set $K_2 \subset E$ such that

$$\nu(E) \leq \nu(K_2) + \frac{\epsilon}{2}.$$

Because μ is regular, there exists some open set U_2 such that

$$\mu(U_2) \leq \mu(E) + \frac{\epsilon}{2},$$

and $E \subset U_2$. There exists some continuous function $f_2 : \mathbb{R}^n \to [0,1]$ such that $f_2 = 1$ on K_2 and $f_2 = 0$ on $\mathbb{R}^n \setminus U_2$. In particular, $f_2 \in C_c(\mathbb{R}^n)$ and $\mathbb{1}_{K_2} \leq f_2 \leq \mathbb{1}_{U_2}$. Then, by monotonicity and the definition of the Lebesgue integral,

$$\nu(E) \leq \nu(K_2) + \frac{\epsilon}{2} = \int \mathbb{1}_{K_2} \mathrm{d}\nu + \frac{\epsilon}{2} \leq \int f_2 \mathrm{d}\nu + \frac{\epsilon}{2} = \int f_2 \mathrm{d}\mu + \frac{\epsilon}{2} \leq \int \mathbb{1}_{U_2} \mathrm{d}\mu + \frac{\epsilon}{2} = \mu(U_2) + \frac{\epsilon}{2} \leq \mu(E) + \epsilon.$$

Thus, for any $\epsilon > 0$,

$$\nu(E) - \epsilon \leq \mu(E) \leq \nu(E) + \epsilon,$$

so $\mu(E) = \nu(E)$, and the measures agree on sets contained in finite rectangles.

Now, $\mathbb{R}^n$ can be written as the increasing countable union of finite rectangles, so for any $E \in \mathcal{B}_{\mathbb{R}^n}$, we can write E as

$$E = \bigcup_{j=1}^{\infty} E_j,$$

where each E_j is contained in some finite rectangle, meaning that $\mu(E_j) = \nu(E_j)$ for all $j \in \mathbb{N}$. Thus, applying continuity from below twice,

$$\mu(E) = \lim_{j\to\infty} \mu(E_j) = \lim_{j\to\infty} \nu(E_j) = \nu(E).$$

Thus, the measures agree on all Borel sets, so $\mu = \nu$.

2 Problem 2 (Folland Problem 20)

Let $f_n, g_n, f, g \in L^1$, $f_n \to f$ and $g_n \to g$ almost everywhere, $|f_n| \le g_n$, and $\int g_n \to \int g$. First, as a lemma, we claim that

$$\inf_k \{a_k + b_k\} \le \sup_k \{a_k\} + \inf_k \{b_k\},$$

and therefore

$$\liminf_n (a_n + b_n) \le \limsup_n a_n + \liminf_n b_n.$$

To see this, we first note that the infimum of the sums is at least as large as the sum of the infimums. Thus,

$$\inf_k \{a_k\} = \inf_k \{(a_k + b_k) - b_k\} \ge \inf_k \{a_k + b_k\} + \inf_k \{-b_k\} = \inf_k \{a_k + b_k\} - \sup_k \{b_k\},$$

and the result follows.

Now, following Folland's proof of the dominated convergence theorem, we note that by taking real and imaginary parts, it suffices to assume that f_n and f are real-valued. Thus, $g_n + f_n$ and $g_n - f_n$ are nonnegative almost everywhere. Applying Fatou's lemma to the sequence $\{g_n + f_n\}$, and utilizing our lemma,

$$\int g + \int f = \int (g + f) \le \liminf_n \int (g_n + f_n) \le \limsup_n \int g_n + \liminf_n \int f_n = \int g + \liminf_n \int f_n.$$

Similarly, applying Fatou's lemma to the sequence $\{g_n - f_n\}$,

$$\int g - \int f = \int (g - f) \le \liminf_n \int (g_n - f_n) \le \limsup_n \int g_n - \limsup_n \int f_n = \int g - \limsup_n \int f_n.$$

Thus,

$$\liminf_n \int f_n \ge f \ge \limsup_n \int f_n,$$

so it follows that

$$\int f = \lim_{n\to\infty} f_n.$$

3 Problem 3 (Folland Problem 21)

Suppose $f_n, f \in L^1$ and $f_n \to f$ almost everywhere. First, assume that $\int |f_n - f| \to 0$. By the triangle inequality,

$$|f_n| \le |f_n - f| + |f|.$$

Taking limits,

$$\lim_{n\to\infty} \int |f_n| \le \lim_{n\to\infty} \left(\int |f_n - f| + \int |f| \right) = \int |f|.$$

Furthermore, $|f_n| \in L^+$, so Fatou's lemma implies that

$$\int |f| \leq \liminf_{n\to\infty} \int |f_n| \leq \lim_{n\to\infty} \int |f_n|.$$

Thus,

$$\lim_{n\to\infty} \int |f_n| = \int |f|.$$

Now, assume that $\int |f_n| \to \int |f|$. Then, by the triangle inequality,

$$|f_n - f| \leq |f_n| + |f|.$$

Additionally,

$$\lim_{n\to\infty} \int (|f_n| + |f|) = \lim_{n\to\infty} \int |f_n| + \int |f| = \int 2|f|.$$

Now, we can apply the result of Exercise 20 by denoting

$$\hat{f}_n := |f_n - f|, \quad \hat{f} := 0, \quad \hat{g}_n := |f_n| + |f|, \quad \hat{g} := 2|f|.$$

These are all L^1 functions that satisfy $\hat{f}_n \to \hat{f}$ and $\hat{g}_n \to \hat{g}$ almost everywhere, $|\hat{f}_n| \leq \hat{g}_n$, and $\int \hat{g}_n \to \int \hat{g}$. Thus, Exercise 20 gives that $\int \hat{f}_n \to \hat{f}$, which in our notation means

$$\lim_{n\to\infty} \int |f_n - f| = \int 0 = 0.$$

4 Problem 4 (Folland Problem 33)

Assume that $f_n \geq 0$ and $f_n \to f$ in measure. We can always find a subsequence that converges to the lim inf, so applying this to the sequence $\int f_n$, there exists some subsequence $\{f_{n_j}\}$ such that

$$\lim_{j\to\infty} \int f_{n_j} = \liminf_n \int f_n.$$

This subsequence also converges to f in measure. To see this, for any $\epsilon > 0$, consider the sequence $\{a_n\}$ where

$$a_n = \mu(\{x \mid |f_n(x) - f(x)| \geq \epsilon\}).$$

Then, $\{a_n\}$ converges to 0, so any subsequence of $\{a_n\}$ also converges to zero. Namely,

$$\lim_{j\to\infty} \mu(\{x \mid |f_{n_j}(x) - f(x)| \geq \epsilon\}) = 0,$$

so $\{f_{n_j}\}$ converges to f in measure. This implies that $\{f_{n_j}\}$ is Cauchy in measure, so Theorem 2.30 in Folland implies that there exists a subsequence $\{f_{n_{j_k}}\}$ that converges to f almost everywhere. Applying Fatou's lemma to this sequence

$$\int f \leq \liminf_k \int f_{n_{j_k}} = \lim_{j\to\infty} \int f_{n_j} = \liminf_n \int f_n,$$

where the first equality follows from the fact that $\int f_{n_{j_k}}$ is a subsequence of a convergent sequence, so its lim inf must be the limit of the whole sequence.

5 Problem 5 (Folland Problem 46)

Let $X = Y = [0,1]$, $\mathcal{M}, \mathcal{N} = \mathcal{B}_{[0,1]}$, μ be the Lebesgue measure, ν the counting measure, and $D = \{(x,x) \mid x \in [0,1]\}$. Then, observe that

$$D_x = \{y \in Y \mid (x,y) \in D\} = \{x\},$$

and

$$D^y = \{x \in X \mid (x,y) \in D\} = \{y\}.$$

Then,

$$\iint \mathbb{1}_D \mathrm{d}\mu \mathrm{d}\nu = \int_Y \mu(D^y)\mathrm{d}\nu = \int_Y 0 \mathrm{d}\nu = 0,$$

while

$$\iint \mathbb{1}_D \mathrm{d}\nu \mathrm{d}\mu = \int_X \nu(D_x)\mathrm{d}\mu = \int_X \mathrm{d}\mu = \mu(X) = 1.$$

Finally, by definition,

$$(\mu \times \nu)(D) = \inf\left\{\sum_{j=1}^{\infty} \mu(A_j)\nu(B_j),\ D \subset \bigsqcup_{j=1}^{\infty}(A_j \times B_j),\ A_j, B_j \text{ are disjoint rectangles}\right\}.$$

Thus, for any $\epsilon > 0$, there exists a set of disjoint rectangles $A_j \times B_j$, $j \in \mathbb{N}$ such that $D \subset \bigsqcup_{j=1}^{\infty}(A_j \times B_j)$ and

$$(\mu \times \nu)(D) \geq \sum_{j=1}^{\infty} \mu(A_j)\nu(B_j) + \epsilon.$$

Furthermore, we can assume without loss of generality that $A_j = B_j$ for all $j \in \mathbb{N}$. This is because if there exists some $a \in A_j$ such that $a \notin B_j$, a can be removed from A_j while retaining $D \subset \bigsqcup_{j=1}^{\infty}(A_j \times B_j)$ and not increasing $\mu(A_j)$ by monotonicity. Applying the same argument to some $b \in B_j$ such that $b \notin A_j$ yields the desired result.

Then, we must have that $[0,1] \subset \bigcup_{j=1}^{\infty} A_j$ as otherwise we could find an element of D not included in the rectangles. Monotonicity implies that

$$1 = \mu([0,1]) \leq \mu\left(\bigcup_{j=1}^{\infty} A_j\right) \leq \sum_{j=1}^{\infty} \mu(A_j),$$

which implies that there must exist some index $k \in \mathbb{N}$ such that $\mu(A_k) > 0$. This implies that A_k is uncountable, since all countable sets have Lebesgue measure zero. Thus, $B_k = A_k$ is also uncountable, so $\nu(B_k) = \infty$. This means that $\mu(A_k)\nu(B_k) = \infty$, so

$$(\mu \times \nu)(D) \geq \sum_{j=1}^{\infty} \mu(A_j)\nu(B_j) + \epsilon = \infty.$$

Thus, $(\mu \times \nu)(D) = \infty$.