AMATH 567 Homework 3

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1 Problem 1 (2.2.4)

Consider the multi-valued function $\ln(z^\alpha)$ where $\alpha \in \mathbb{R}$. By the definition of the complex logarithm and letting $z = Re^{i \arg z}$ for any $k \in \mathbb{Z}$,

$$\ln(z^\alpha) = \ln(R^\alpha e^{i\alpha \arg z}) = \ln(R^\alpha) + i \arg(e^{i\alpha \arg z}) = \alpha \ln R + i \arg(e^{i\alpha \arg z}).$$

Note that we have multi-valued arguments. This setup also gives that

$$\alpha \ln z = \alpha(\ln R + i \arg z) = \alpha \ln R + \alpha i \arg z.$$

We wish to show that the values of these two multi-valued functions are not the same. This amounts to checking whether $\arg(e^{i\alpha \arg z}) = \alpha \arg z$ in general. If we let $\alpha = \frac{1}{2}$ and $z = 1$ such that $\arg(z) = 0$ (note that we fixed a branch to the principal argument by doing this, so $\arg(w) \in (-\pi, \pi]$ for any $w \in \mathbb{C}$), $\alpha \arg z = 0$ but $\arg(e^{i\alpha(\arg z + 2\pi k)}) = \arg(e^{ik\pi}) = \arg(\pm 1) = 0, \pi$ where $k \in \mathbb{C}$. Thus, the multvaluedness does not resolve here, so the sets of all values of these functions do not necessarily match.

2 Problem 2

Consider the function $w = z^z$ where $z \in \mathbb{C}$. We first find its real and imaginary parts by letting $z = Re^{i\theta}$ and writing

$$\begin{aligned}
z^z &= e^{z \ln z} = e^{z(\ln R + i\theta)} = e^{R(\cos\theta + i \sin\theta)(\ln R + i\theta)} = e^{R(\cos\theta \ln R - \theta \sin\theta + i(\sin\theta \ln R + \theta\cos\theta))} \\
&= e^{R\cos\theta \ln R - R\theta \sin\theta} e^{i(R \sin\theta \ln R + R\theta\cos\theta)} \\
&= e^{R\cos\theta \ln R - R\theta \sin\theta}(\cos(R\sin\theta \ln R + R\theta\cos\theta) + i \sin(R\sin\theta \ln R + R\theta\cos\theta))
\end{aligned}$$

Note that R and θ are both real by definition, so we can see that

$$\Re(w) = e^{R\cos\theta \ln R - R\theta\sin\theta} \cos(R \sin\theta \ln R + R\theta\cos\theta)$$

and

$$\Im(w) = e^{R\cos\theta \ln R - R\theta\sin\theta} \sin(R \sin\theta \ln R + R\theta\cos\theta).$$

Let $A = R\cos\theta\ln R - R\theta\sin\theta$ and $B = R\sin\theta\ln R + R\theta\cos\theta$. Then, if we let $u = \Re(w)$ and $v = \Im(w)$, $u = e^A\cos(B)$ and $v = e^A\sin(B)$. Note that $z^z = e^A e^{iB}$. We compute the partial derivatives of A and B and get

$$\begin{aligned} A_R &= (\ln R + 1)\cos\theta - \theta\sin\theta \\ A_\theta &= -R(\sin\theta\ln R + \sin\theta + \theta\cos\theta) \\ B_R &= (\ln R + 1)\sin\theta + \theta\cos\theta \\ B_\theta &= R(\cos\theta\ln R + \cos\theta - \theta\sin\theta) \end{aligned}$$

We have omitted the multivaluedness of the complex logarithm (technically we should have $\theta = \arg x + 2\pi k$ for any $k \in \mathbb{Z}$ but per Piazza, we are permitted to omit this). Note that $RA_R = B_\theta$ and $-RB_R = A_\theta$.
We now wish to find the derivative of w. To ensure that we can do this, we first check that the Cauchy-Riemann equations (in polar form) are satisfied

$$\begin{aligned} u_R &= e^A(A_R\cos B - B_R\sin B) \\ v_R &= e^A(A_R\sin B + B_R\cos B) \\ u_\theta &= e^A(A_\theta\cos B - B_\theta\sin B) && = Re^A(-B_R\cos B - A_R\sin B) = -Rv_R \\ v_\theta &= e^A(A_\theta\sin B + B_\theta\cos B) && = Re^A(-B_R\sin B + A_R\cos B) = Ru_R. \end{aligned}$$

These are precisely the polar-form Cauchy-Riemann equations, so our function is indeed analytic, because the partial derivatives are clearly continuous with the exception of the origin due to the presence of $\ln R$. Thus, we can evaluate

$$\begin{aligned} w' &= e^{-i\theta}(u_R + iv_R) = e^{-i\theta}e^A(A_R\cos B - B_R\sin B + i(A_R\sin B + B_R\cos B)) \\ &= e^{-i\theta}e^A(A_R e^{iB} + iB_R(\cos B + i\sin B)) = e^{-i\theta}(A_R e^A e^{iB} + iB_R e^A e^{iB}) \\ &= e^{-i\theta}(A_R + iB_R)z^z = e^{-i\theta}z^z((\ln R + 1)\cos\theta - \theta\sin\theta + i(\ln R + 1)\sin\theta + i\theta\cos\theta) \\ &= e^{-i\theta}z^z((\ln R + 1)e^{i\theta} + i\theta e^{i\theta}) = z^z(\ln R + i\theta + 1). \end{aligned}$$

Now recall that when starting the problem, we glossed over the multivaluedness of the complex logarithm and noted that we should technically have $\theta = \arg x + 2\pi k$. If we make this substitution, then $\ln R + i\theta = \ln(Re^{i\theta}) = \ln z$. Thus, $w' = z^z(\ln z + 1)$, as desired.
Now, we wish to evaluate i^i. We could do so by using the real and imaginary parts of z^z that we found, but it is much cleaner to just use the definitions of the complex power and complex logarithm. Note that i has modulus 1 and argument $\pi/2$.

$$i^i = e^{i\ln i} = e^{i(\ln(1) + i(\pi/2 + 2\pi k)} = e^{-(\pi/2 + 2\pi k)}$$

for any $k \in \mathbb{Z}$.

3 Problem 3

Consider the multi-valued function $w(z)$ such that $w^2 = \prod_{j=1}^{n=N}(z - a_j)$ where all $a_j \in \mathbb{C}$ are distinct. The problem statement asks us to consider specific values of N, but for finding the branch points, we just consider the cases where N is even or odd. In both cases, $z = a_j$ is a branch point for all j. We can see this via a similar argument to the $N = 2$ case on page 32 of the course notes. Taking $z = a_k + \epsilon e^{i\theta}$ for some $1 \leq k \leq N$ where $\epsilon > 0$ is small,

$$w = (\epsilon e^{i\theta} \prod_{j \neq k} (a_k - a_j + \epsilon e^{i\theta}))^{1/2} \sim \sqrt{\epsilon} (\prod_{j \neq k} (a_k - a_j)^{1/2} e^{i\theta/2}),$$

ignoring terms of order ϵ. In the vicinity of $z = a_k$, this behaves like the $n = N - 1$ case. We know from the notes that the $n = 2$ case holds, so a standard induction argument on n gives the general case.

Now, we consider whether or not ∞ is a branch point which will depend on whether N is odd or even. For very large values of z, $w \sim (z^N)^{1/2}$. If $N = 2k$ is even, then $(z^N)^{1/2} = \pm z^k$. This has 2 values, but because $k \in \mathbb{Z}$, w returns to its original value as z traverses a circle of very large radius. Thus, there are two different points at ∞ for w, but the behavior near each of those is single-valued, so ∞ is not a branch point. If $N = 2k + 1$ is odd and taking $1/z = t = Re^{i\theta}$ to investigate large z, then

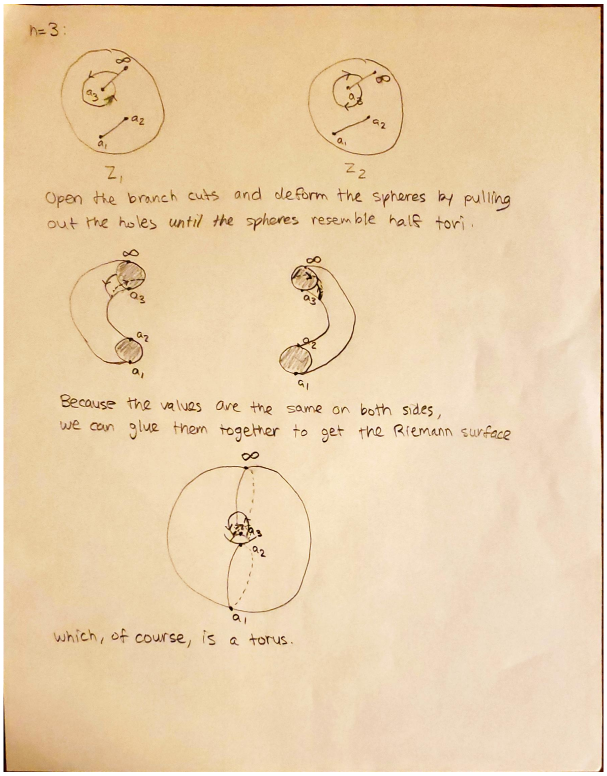
$$w_1 \sim z^{k+1/2} = (\frac{1}{Re^{i\theta}})^{k+1/2} = R^{-(k+1/2)} (e^{-i\theta})^{-(k+1/2)} = R^{-(k+1/2)} e^{-i(\theta + 2\pi m)(2k+1)/2}$$

and $w_2 = -w_1$.

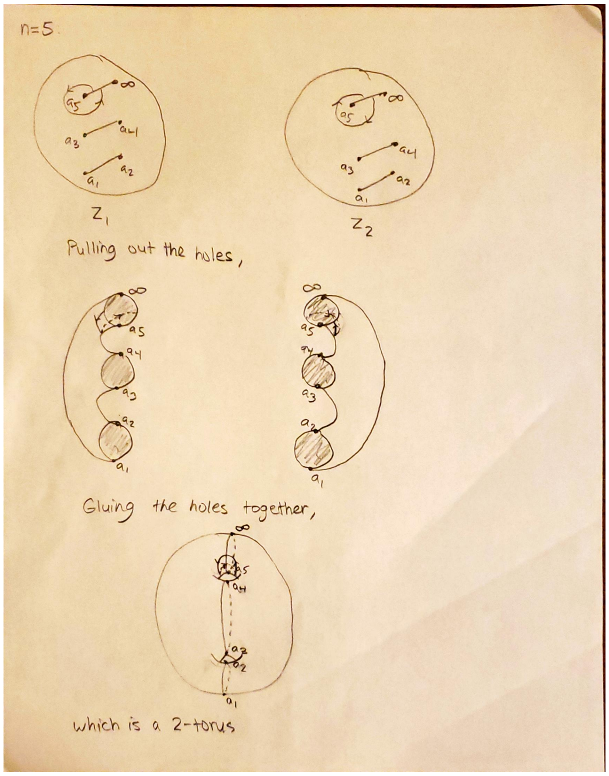
Note that this is still multi-valued, so the terms do not reconcile by the same logic as our square root example from class. Thus, ∞ is a branch point. We know that these are the only branch points, because the function w is analytic elsewhere.

Note: In office hours, Bernard stated that we need not prove that the a_is are branch points or that there are no other branch points. I have left my rough arguments in for completeness, but they may be vague as a result.

Now that we know what our branch points are, we can consider specific values of N, starting with $n = 3$. In this case, we have branch points at a_1, a_2, a_3, ∞, meaning that we can draw two branch cuts arbitrarily (such that all branch points are part of exactly one branch cut). Take the two spheres z_1 and z_2 with these cuts and manipulate as follows (please forgive my poor quality drawings):



In the case $n = 4$, we have the same number of branch points, so everything is the same except $z = a_4$ replaces ∞ as a branch point.
For the case $n = 5$, we have branch points at $a_1, a_2, a_3, a_4, a_5, \infty$, meaning that we can draw 3 branch cuts and do the following:



In general, we will have N branch points if N is even and $N+1$ branch points if N is odd, meaning that the Riemann surface will be a k-torus for an even $N = 2k$ and an odd $N = 2k-1$.

4 Problem 4 (2.2.7)

Consider the function $\Omega(z) = k\ln(z - z_0)$ where $k \in \mathbb{R}$ and $z_0 \in \mathbb{C}$ are constants. The velocity potential ϕ and the stream function ψ are just the real and imaginary parts of this function respectively, so $\Omega(z) = k(\ln|z - z_0| + i\arg(z - z_0))$ gives that $\phi(z) = k\ln|z - z_0|$ and $\psi(z) = k\arg(z - z_0)$.
From page 41 of the text, we have that velocity is given by

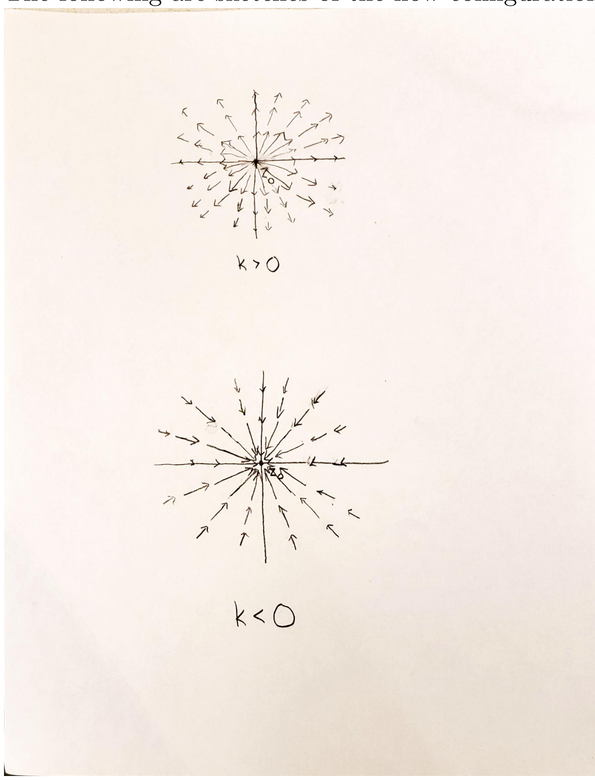
$$\overline{\Omega}'(z) = \overline{\frac{k}{z - z_0}} = \frac{k}{\overline{z - z_0}} = \frac{k(z - z_0)}{|z - z_0|^2}.$$

We wish to show that this is purely radial relative to $z = z_0$, so consider a shifted polar coordinate system such that $z - z_0 = Re^{i\theta}$. Then,

$$\overline{\Omega}'(z) = \frac{kRe^{i\theta}}{R^2} = \frac{k}{R}e^{i\theta}.$$

Written in vector form, velocity is $(\frac{k}{R}\cos\theta, \frac{k}{R}\sin\theta)$; this is just a normal vector pointing radially outward from the origin (defined relative to z_0) scaled by $\frac{k}{R}$

(as the collection of these vectors is a circle for fixed R). Thus, velocity is purely radial relative to $z = z_0$ (given by $V_r = \frac{k}{R}$) .
The following are sketches of the flow configuration for different values of k:



Let $M = \oint_C V_r ds$ where V_r is the radial velocity and ds is the increment of arclength in the direction tangent to the circle C enclosing $z = z_0$. Continuing with our modified coordinate system, let $C = Re^{i\theta}$. Then,

$$M = \oint_C V_r ds = \int_0^{2\pi} (\frac{k}{R})(Rd\theta) = k \int_0^{2\pi} d\theta = 2\pi k.$$