Math 524 Autumn 2023

Homework 3 **Due Friday, October 20**

1. Suppose that X and Y are metric spaces, and that $f : X \to Y$ is continuous map. For $E \subset Y$ recall that

$$f^{-1}(E) = \{x \in X : f(x) \in E\}$$

Show that if $E \in \mathcal{B}_Y$ then $f^{-1}(E) \in \mathcal{B}_X$. (First show that the family of E satisfying $f^{-1}(E) \in \mathcal{B}_X$ is a σ-algebra .)

2. If μ is a Borel measure on X, show that

$$\nu(E) = \mu\big(f^{-1}(E)\big)$$

is a Borel measure on Y.

3. In this problem we work with binary sequences $\mathbb{Z}_2^{\mathbb{N}}$, but the results extend to $\mathbb{Z}_q^{\mathbb{N}}$ for all q. We let $(\mathbb{Z}_2^{\mathbb{N}}, \mathcal{B}, \mu)$ be the Borel measure space where μ is the extension of the premeasure μ_0 of Homework to the Borel sets in $\mathbb{Z}_2^{\mathbb{N}}$.

(a.) Show that the map that sends $b = (b_1, b_2, \ldots)$ to

$$f(b) = \sum_{j=1}^{\infty} \frac{b_j}{2^j}$$

is continuous from $\mathbb{Z}_2^{\mathbb{N}}$ to $[0,1]$, with the metric topology on $\mathbb{Z}_2^{\mathbb{N}}$ defined in Homework 1.

(b.) Show that, if $a = (a_1, \ldots, a_n) \in \mathbb{Z}_2^n$, then the image of the set

$$A = \{b : b_j = a_j \ \text{ for } \ 1 \le j \le n\}$$

is the closed interval $\left[\frac{q}{2^n}, \frac{q+1}{2^n}\right]$, $q = \sum_{j=1}^n a_j 2^{n-j}$.

(c.) If p, q are integers with $0 \le p < q \le 2^n$, describe the set

$$f^{-1}\left(\left[\frac{p}{2^n}, \frac{q}{2^n}\right]\right) \subset \mathbb{Z}_2^{\mathbb{N}}$$

Show that it consists of a set in $\mathcal{A}$ (see Homework 1) plus 2 points in $\mathbb{Z}_2^{\mathbb{N}}$, and find its μ measure.

(d.) Show that $\mu(f^{-1}(a,b]) = b - a$ for all $0 \le a < b \le 1$, and use this to show that

$$\mu(f^{-1}(E)) = m(E) \ \text{ for all } \ E \in \mathcal{B}_{[0,1]}, \quad m = \text{Lebesgue measure.}$$

4. Folland Page 39, Problem 26.