

AMATH 568 Homework 2

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January 19, 2022

1 Problem 1

To show that, for each fixed $\ell \geq 0$, the integral

$$H(t) = \int_0^\infty e^{-x^2-2tx} (tx)^\ell dx, \quad t \geq 0,$$

is $O(1)$ for $t \in [0, \infty)$, first consider some special cases. If $t = 0$, then $H(t) = 0$, so $|H(t)| \leq K$ for any $K > 0$. Thus, going forward we only consider $t \in (0, \infty)$. Now, consider the case where $\ell = 0$. Then,

$$|H(t)| = \int_0^\infty e^{-x^2-2tx} dx \leq \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2},$$

so if we take $K = \frac{\sqrt{\pi}}{2}$, we have that $|H(t)| < K * 1$ for all $t \in [0, \infty)$, meaning that $H(t)$ is $O(1)$.

Now, consider $\ell > 0$. Then, using the change of variables $u = 2tx$,

$$|H(t)| = \int_0^\infty e^{-x^2-2tx} (tx)^\ell dx = \int_0^\infty e^{-(\frac{u}{2t})^2-u} \left(\frac{u}{2}\right)^\ell \frac{du}{2t} \leq \frac{1}{2^{\ell+1}} \int_0^\infty e^{-u} u^{\ell-1} \frac{u}{t} e^{-(\frac{u}{2t})^2} du.$$

Now, because we consider $t \neq 0$, let $v = \frac{u}{t}$. Then,

$$\frac{u}{t} e^{-(\frac{u}{2t})^2} = v e^{-v^2/4} \leq \sqrt{2} e^{-1/2} = \sqrt{\frac{2}{e}} < 1,$$

because $v e^{-v^2/4}$ is maximized at $v = \sqrt{2}$ by elementary calculus. Thus,

$$|H(t)| < \frac{1}{2^{\ell+1}} \int_0^\infty e^{-u} u^{\ell-1} du = \frac{\Gamma(\ell)}{2^{\ell+1}}.$$

This is constant with respect to t , so we simply take $K = \frac{\Gamma(\ell)}{2^{\ell+1}}$, and then $|H(t)| < K * 1$ for all $t \in [0, \infty)$, meaning that $H(t)$ is $O(1)$.

2 Problem 2

We wish to apply Watson's lemma to derive an asymptotic expansion of

$$F(\lambda) = \int_0^\infty e^{-\lambda t} \frac{\sin t}{t^{3/2}} dt, \quad \lambda > 0, \quad \lambda \rightarrow \infty$$

by taking $\phi(t) = \frac{\sin t}{t^{3/2}}$ and $g(t) = \frac{\sin t}{t}$ so that $\phi(t) = t^\sigma g(t)$ where $\sigma = -1/2 > -1$. To do this, we need to show that $\phi(t)$ is absolutely integrable on $[0, \infty)$ and that $g(t)$ is infinitely differentiable in a neighborhood of $t = 0$. To see that $\phi(t)$ is absolutely integrable, write

$$\int_0^\infty |\phi(t)| dt = \underbrace{\int_0^1 \left| \frac{\sin t}{t^{3/2}} \right| dt}_{I_1} + \underbrace{\int_1^\infty \left| \frac{\sin t}{t^{3/2}} \right| dt}_{I_2}.$$

Using the fact that $|\sin(t)| = \sin(t) \leq t$ on $[0, 1]$,

$$I_1 \leq \int_0^1 \frac{t}{t^{3/2}} dt = \int_0^1 \frac{dt}{\sqrt{t}} = 2.$$

Using the fact that $|\sin t| \leq 1$,

$$I_2 \leq \int_1^\infty \frac{dt}{t^{3/2}} = 2.$$

Thus,

$$\int_0^\infty |\phi(t)| dt = I_1 + I_2 \leq 4 < \infty.$$

To see that $g(t)$ is infinitely differentiable around $t = 0$, we use the Taylor series centered at $t = 0$ for the sine function to write

$$g(t) = \frac{1}{t} \sum_{j=0}^\infty (-1)^j \frac{t^{2j+1}}{(2j+1)!} = \sum_{j=0}^\infty (-1)^j \frac{t^{2j}}{(2j+1)!} = 1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \dots$$

Since this Taylor series for $\sin(t)$ holds for all $t \in \mathbb{R}$ and this expression for $g(t)$ is a polynomial, $g(t)$ is clearly infinitely differentiable around $t = 0$. Furthermore, we can compute $g^{(j)}(0)$ by noting that it is precisely the constant term of the series for $g^{(j)}(t)$. This will be zero for odd j since our series contains only even powers of t . Thus, we can write

$$g^{(2j)}(0) = (-1)^j (2j)! \frac{1}{(2j+1)!} = \frac{(-1)^j}{2j+1}.$$

With this in hand, we apply Watson's lemma and use our above expression to conclude that

$$F(\lambda) \sim \sum_{n=0}^\infty \frac{g^{(n)}(0) \Gamma(\sigma + n + 1)}{n! \lambda^{\sigma+n+1}} = \sum_{j=0}^\infty \frac{(-1)^j}{2j+1} \frac{\Gamma(2j+1/2)}{(2j)! \lambda^{2j+1/2}} = \sum_{j=0}^\infty \frac{(-1)^j \Gamma(2j+1/2)}{(2j+1)! \lambda^{2j+1/2}},$$

as $\lambda \rightarrow \infty$ with $\lambda > 0$ where we have reindexed $n \rightarrow 2j$.

3 Problem 3

To derive an asymptotic expansion of

$$u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx - k^2 t} \hat{f}(k) dk, \quad \hat{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x) dx$$

for fixed x , let $\phi(k) = e^{ikx} \hat{f}(k)$, $a = -\infty$, and $b = \infty$ and consider the generalization of Watson's lemma on page 6 of part 2 of the course notes where f and \hat{f} are assumed to decay rapidly enough so that $\phi(k)$ is absolutely integrable on $(-\infty, \infty)$ and $\phi(k)$ has an infinite number of continuous derivatives in a neighborhood of $k = 0$. Then, the result on page 8 of part 2 of the notes gives that

$$u(x, t) \sim \frac{1}{2\pi} \sqrt{\frac{\pi}{t}} \sum_{j=0}^{\infty} \frac{\phi^{(2j)}(0)}{t^j} \frac{1}{2^{2j} j!}$$

as $t \rightarrow \infty$, $t > 0$. Now, we compute derivatives of $\phi(k)$. By the product rule,

$$\begin{aligned} \phi'(k) &= ix e^{ikx} \int_{-\infty}^{\infty} e^{-ikx} f(x) dx + e^{ikx} \int_{-\infty}^{\infty} \frac{d}{dk} (e^{-ikx} f(x)) dx \\ &= ix e^{ikx} \int_{-\infty}^{\infty} e^{-ikx} f(x) dx - e^{ikx} \int_{-\infty}^{\infty} ix e^{-ikx} f(x) dx \end{aligned}$$

Again, by the product rule,

$$\begin{aligned} \phi''(k) &= (ix)^2 e^{ikx} \int_{-\infty}^{\infty} e^{-ikx} f(x) dx - ix e^{ikx} \int_{-\infty}^{\infty} ix e^{-ikx} f(x) dx \\ &\quad - ix e^{ikx} \int_{-\infty}^{\infty} ix e^{-ikx} f(x) dx + e^{ikx} \int_{-\infty}^{\infty} (ix)^2 e^{-ikx} f(x) dx \\ &= (ix)^2 e^{ikx} \int_{-\infty}^{\infty} e^{-ikx} f(x) dx - 2ix e^{ikx} \int_{-\infty}^{\infty} ix e^{-ikx} f(x) dx + e^{ikx} \int_{-\infty}^{\infty} (ix)^2 e^{-ikx} f(x) dx. \end{aligned}$$

Thus, we can compute

$$\phi(0) = \hat{f}(0) = \int_{-\infty}^{\infty} f(x) dx$$

and

$$\begin{aligned} \phi''(0) &= (ix)^2 \int_{-\infty}^{\infty} f(x) dx - 2ix \int_{-\infty}^{\infty} ix f(x) dx + \int_{-\infty}^{\infty} (ix)^2 f(x) dx \\ &= -x^2 \int_{-\infty}^{\infty} f(x) dx + 2x \int_{-\infty}^{\infty} x f(x) dx - \int_{-\infty}^{\infty} x^2 f(x) dx. \end{aligned}$$

Now, we can simply plug in $j = 0, 1$ to find the first two nonzero terms of our expansion. Namely, the first term is given by

$$\frac{1}{2\pi} \sqrt{\frac{\pi}{t}} \phi(0) = \frac{1}{2\pi} \sqrt{\frac{\pi}{t}} \int_{-\infty}^{\infty} f(x) dx$$

and the second term is given by

$$\frac{1}{2\pi} \sqrt{\frac{\pi}{t}} \frac{\phi''(0)}{t} \frac{1}{2^2 1!} = \frac{1}{2\pi} \sqrt{\frac{\pi}{t}} \frac{1}{4t} \left(-x^2 \int_{-\infty}^{\infty} f(x) dx + 2x \int_{-\infty}^{\infty} x f(x) dx - \int_{-\infty}^{\infty} x^2 f(x) dx \right).$$

4 Problem 4

To compute the asymptotic expansion of

$$u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx - k^4 t} \hat{f}(k) dk, \quad \hat{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x) dx,$$

we derive an additional generalization of Watson's lemma based on the generalization used in problem 3. Namely, we consider

$$F(\lambda) = \int_a^b e^{-\lambda t^4} \phi(t) dt$$

where $a < 0 < b$, $\lambda > 0$, and $\phi(t)$ is absolutely integrable and has an infinite number of continuous derivatives in a neighborhood of $t = 0$.

Steps 1 and 2 of our derivation are essentially the same as their counterparts on page 7 of part 2 of the course notes.