Cade
Ballew

AMATH 561 Problem Set 1 -

1. a. In a given string of coin flips, let H represent heads and T represent tails. Also assume that for any given toss, the probability of heads is p and the probability of tails is 1-p. Then, we can write the probability space (Ω, F, P) as follows:
$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$
and take $F = 2^{\Omega}$ to be the power set of Ω.
Define the function $q: \Omega \to \mathbb{R}$ s.t.
$q(HHH) = p^3$, $q(HHT) = q(HTH) = q(THH) = p^2(1-p)$,
$q(HTT) = q(THT) = q(TTH) = p(1-p)^2$, $q(TTT) = (1-p)^3$.
Then, $\forall A \in F$, $P(A) = \Sigma_{\omega \in A} q(\omega)$ defines our probability measure P. Clearly, $P(A) \geq P(\emptyset) = 0 \ \forall A \in F$ and $\forall$ sequence of disjoint sets $A_i \in F$ (which must be finite because F is finite), $P(\bigcup_i A_i) = \Sigma_i P(A_i)$ (because we just decompose into events) and $P(\Omega) = p^3 + 3p^2(1-p) + 3p(1-p)^2 + (1-p)^3 = (p + (1-p))^3 = 1$. Thus, this is indeed a probability measure.

b. Now, in a given string of balls drawn, let B represent blue and R represent red. Then, we can write the probability space (Ω, F, P) as $\Omega = \{RR, RB, BR, BB\}$, $F = 2^{\Omega}$ and define $p: \Omega \to \mathbb{R}$ s.t. $p(RR) = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$, $p(RB) = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$, $p(BR) = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$, $p(BB) = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$ and let $P(A) = \Sigma_{\omega \in A} p(\omega)$ $\forall A \in F$ define our probability measure P. As before, this is clearly a measure and $P(\Omega) = \frac{1}{6} + \frac{1}{3} + \frac{1}{3} + \frac{1}{6} = 1$
$\Rightarrow$ P is a probability measure.

2. Consider a probability measure P on $\Omega = \mathbb{Z}$ with the σ-algebra $F = 2^{\mathbb{Z}}$ that satisfies the translation-invariance property. Let $B = \{0\} \in 2^{\mathbb{Z}}$. Then, by the translation invariance property, $P(\{0\}) = P(B) = P(B+n) = P(\{n\}) \ \forall n \in \mathbb{Z}$. Of course, $\Omega = \mathbb{Z} = \bigcup_{i \in \mathbb{Z}} \{i\}$. If we let $A_i = \{i\} \ \forall i \in \mathbb{Z}$, then the set of A_is are disjoint. Also note that $\mathbb{Z}$ is countable, so $\bigcup_{i \in \mathbb{Z}} A_i$ is a disjoint, countable union. Thus, in order for P to be a probability measure, it must hold

that $1 = P(\Omega) = P(\bigcup_{i\in\mathbb{Z}} A_i) = \sum_{i\in\mathbb{Z}} P(A_i) = \sum_{i\in\mathbb{Z}} P(B) = P(B)\sum_{i\in\mathbb{Z}} 1$. This sum diverges unless $P(B)=0$, but if $P(B)=0$, then $P(\Omega)=0$. Thus $P(\Omega)\neq 1$, meaning that P cannot be a probability measure which in turn implies that no such probability measure exists.

3. Now, consider a probability measure P on the set $\Omega = \mathbb{R}$ with σ-algebra $F = \mathcal{B}(\mathbb{R})$ with the translation-invariance property. We apply a very similar method as the previous problem by letting $B=(0,1]$. Then, $\forall\, n\in\mathbb{Z}$, $P(B) = P(B+n) = P((n, n+1])$. Let $A_i = (i, i+1]$ $\forall\, i\in\mathbb{Z}$. Then, $\Omega = \mathbb{R} = \bigcup_{i\in\mathbb{Z}} A_i$. Note that the A_is are all disjoint. As before, this is a countable union, so it must hold that $1 = P(\Omega) = P(\bigcup_{i\in\mathbb{Z}} A_i) = \sum_{i\in\mathbb{Z}} P(A_i) = \sum_{i\in\mathbb{Z}} P(B) = P(B)\sum_{i\in\mathbb{Z}} 1$. As before, this is not possible, so P cannot be a probability measure, meaning that no such probability measure exists.

4. Consider (Ω, F, P) s.t. $\Omega = \mathbb{R}$, F is the set of all subsets $A \subseteq \mathbb{R}$ s.t. A is countable or A^c is countable, and $P(A)=0$ if A is countable and $P(A)=1$ if A^c is countable. We first show that F is a σ-algebra (as Ω is clearly a set) by verifying the required axioms:

i. If $A\in F$ is countable, then $A^c \in F$ because $(A^c)^c = A$ is countable. If $A\in F$ is uncountable, then $A^c\in F$ because A^c is countable. Thus, F is closed to compliments.

ii. Let $A = \bigcup_{i=1}^{\infty} A_i$ be a countable union of sets $A_i \in F$ $\forall\, i\in\mathbb{N}$. If A_i is countable $\forall i$, then $A\in F$ because the countable union of countable sets is countable.
Now, say that not all A_is are countable, i.e. $\exists\, k\in\mathbb{N}$ s.t. A_k is uncountable. $A^c = \bigcap_{i=1}^{\infty} A_i^c$ (de Morgan's law) $= \bigcap_{i\neq k} A_i^c \cap \{A_k^c\} \subset A_k^c$. However, A_k^c must be countable because A_k is uncountable, so A^c must be countable $\Rightarrow A\in F$. Thus, F is closed to countable unions and is therefore a σ-algebra

Now, we wish to show that P satisfies the axioms of a probability measure.

i. $\emptyset$ is countable, so $P(A) \geq P(\emptyset) = 0 \ \forall A \in F$.

ii. Consider $F \ni A = \bigcup_{i=1}^{\infty} A_i$ where the A_is $\in F$ are disjoint.
First, say assume that A_i is countable $\forall i$. Then, A is also countable as a countable union of countable sets,
so $\sum_{i=1}^{\infty} P(A_i) = 0 = P(A) = P(\bigcup_{i=1}^{\infty} A_i)$.
Now, assume that $\exists$ some $k \in \mathbb{N}$ s.t. A_k is uncountable.
Because all A_is are disjoint, $A_i \subset A_k^c \ \forall i \neq k$.
A_k is uncountable, so A_k^c must be countable
$\Rightarrow A_i$ is countable $\forall i \neq k$. Note that $A_k \subset A$, so A must be uncountable. Then,
$\sum_{i=1}^{\infty} P(A_i) = P(A_k) + \sum_{i \neq k}^{\infty} P(A_i) = 1 + 0 = 1 = P(A) = P(\bigcup_{i=1}^{\infty} A_i)$.
Thus, $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$ in all cases.

iii. $\Omega = \mathbb{R}$ is uncountable, so $P(\Omega) = 1$.

Thus, P satisfies all the axioms and is indeed a probability measure, meaning that (Ω, F, P) is a probability space.