AMATH 568 Homework 2

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1 Problem 1

To show that, for each fixed $\ell \geq 0$, the integral

$$H(t) = \int_0^\infty e^{-x^2-2tx}(tx)^\ell dx, \quad t \geq 0,$$

is O(1) for $t \in [0, \infty)$, first consider some special cases. If $t = 0$, then $H(t) = 0$, so $|H(t)| \leq K$ for any $K > 0$. Thus, going forward we only consider $t \in (0, \infty)$. Now, consider the case where $\ell = 0$. Then,

$$|H(t)| = \int_0^\infty e^{-x^2-2tx}dx \leq \int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2},$$

so if we take $K = \frac{\sqrt{\pi}}{2}$, we have that $|H(t)| < K * 1$ for all $t \in [0, \infty)$, meaning that $H(t)$ is O(1).
Now, consider $\ell > 0$. Then, using the change of variables $u = 2tx$,

$$|H(t)| = \int_0^\infty e^{-x^2-2tx}(tx)^\ell dx = \int_0^\infty e^{-\left(\frac{u}{2t}\right)^2-u}\left(\frac{u}{2}\right)^\ell \frac{du}{2t} \leq \frac{1}{2^{\ell+1}}\int_0^\infty e^{-u}u^{\ell-1}\frac{u}{t}e^{-\left(\frac{u}{2t}\right)^2}du.$$

Now, because we consider $t \neq 0$, let $v = \frac{u}{t}$. Then,

$$\frac{u}{t}e^{-\left(\frac{u}{2t}\right)^2} = ve^{-v^2/4} \leq \sqrt{2}e^{-1/2} = \sqrt{\frac{2}{e}} < 1,$$

because $ve^{-v^2/4}$ is maximized at $v = \sqrt{2}$ by elementary calculus. Thus,

$$|H(t)| < \frac{1}{2^{\ell+1}}\int_0^\infty e^{-u}u^{\ell-1} = \frac{\Gamma(\ell)}{2^{\ell+1}}.$$

This is constant with respect to t, so we simply take $K = \frac{\Gamma(\ell)}{2^{\ell+1}}$, and then $|H(t)| < K * 1$ for all $t \in [0, \infty)$, meaning that $H(t)$ is O(1).

2 Problem 2

We wish to apply Watson's lemma to derive an asymptotic expansion of

$$F(\lambda) = \int_0^\infty e^{-\lambda t} \frac{\sin t}{t^{3/2}} dt, \quad \lambda > 0, \quad \lambda \to \infty$$

by taking $\phi(t) = \frac{\sin t}{t^{3/2}}$ and $g(t) = \frac{\sin t}{t}$ so that $\phi(t) = t^\sigma g(t)$ where $\sigma = -1/2 > -1$. To do this, we need to show that $\phi(t)$ is absolutely integrable on $[0, \infty)$ and that $g(t)$ is infinitely differentiable in a neighborhood of $t = 0$. To see that $\phi(t)$ is absolutely integrable, write

$$\int_0^\infty |\phi(t)| dt = \underbrace{\int_0^1 \left| \frac{\sin t}{t^{3/2}} \right| dt}_{I_1} + \underbrace{\int_1^\infty \left| \frac{\sin t}{t^{3/2}} \right| dt}_{I_2}.$$

Using the fact that $|\sin(t)| = \sin(t) \leq t$ on $[0, 1]$,

$$I_1 \leq \int_0^1 \frac{t}{t^{3/2}} dt = \int_0^1 \frac{dt}{\sqrt{t}} = 2.$$

Using the fact that $|\sin t| \leq 1$,

$$I_2 \leq \int_1^\infty \frac{dt}{t^{3/2}} = 2.$$

Thus,

$$\int_0^\infty |\phi(t)| dt = I_1 + I_2 \leq 4 < \infty.$$

To see that $g(t)$ is infinitely differentiable around $t = 0$, we use the Taylor series centered at $t = 0$ for the sine function to write

$$g(t) = \frac{1}{t} \sum_{j=0}^\infty (-1)^j \frac{t^{2j+1}}{(2j+1)!} = \sum_{j=0}^\infty (-1)^j \frac{t^{2j}}{(2j+1)!} = 1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \dots.$$

Since this Taylor series for $\sin(t)$ holds for all $t \in \mathbb{R}$ and this expression for $g(t)$ is a polynomial, $g(t)$ is clearly infinitely differentiable around $t = 0$. Furthermore, we can compute $g^{(j)}(0)$ by noting that it is precisely the constant term of the series for $g^{(j)}(t)$. This will be zero for odd j since our series contains only even powers of t. Thus, we can write

$$g^{(2j)}(0) = (-1)^j (2j)! \frac{1}{(2j+1)!} = \frac{(-1)^j}{2j+1}.$$

With this in hand, we apply Watson's lemma and use our above expression to conclude that

$$F(\lambda) \sim \sum_{n=0}^\infty \frac{g^{(n)}(0)\Gamma(\sigma+n+1)}{n!\lambda^{\sigma+n+1}} = \sum_{j=0}^\infty \frac{(-1)^j}{2j+1} \frac{\Gamma(2j+1/2)}{(2j)!\lambda^{2j+1/2}} = \sum_{j=0}^\infty \frac{(-1)^j \Gamma(2j+1/2)}{(2j+1)!\lambda^{2j+1/2}},$$

as $\lambda \to \infty$ with $\lambda > 0$ where we have reindexed $n \to 2j$.

3 Problem 3

To derive an asymptotic expansion of

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-k^2t}\hat{f}(k)dk, \quad \hat{f}(k) = \int_{-\infty}^{\infty} e^{-ikx}f(x)dx$$

for fixed x, let $\phi(k) = e^{ikx}\hat{f}(k)$, $a = -\infty$, and $b = \infty$ and consider the generalization of Watson's lemma on page 6 of part 2 of the course notes where f and $\hat{f}$ are assumed to decay rapidly enough so that $\phi(k)$ is absolutely integrable on $(-\infty, \infty)$ and $\phi(k)$ has an infinite number of continuous derivatives in a neighborhood of $k = 0$. Then, the result on page 8 of part 2 of the notes gives that

$$u(x,t) \sim \frac{1}{2\pi}\sqrt{\frac{\pi}{t}}\sum_{j=0}^{\infty}\frac{\phi^{(2j)}(0)}{t^j}\frac{1}{2^{2j}j!}$$

as $t \to \infty$, $t > 0$. Now, we compute derivatives of $\phi(k)$. By the product rule,

$$\begin{aligned}\phi'(k) &= ixe^{ikx}\int_{-\infty}^{\infty} e^{-ikx}f(x)dx + e^{ikx}\int_{-\infty}^{\infty}\frac{d}{dk}(e^{-ikx}f(x))dx \\ &= ixe^{ikx}\int_{-\infty}^{\infty} e^{-ikx}f(x)dx - e^{ikx}\int_{-\infty}^{\infty} ixe^{-ikx}f(x)dx\end{aligned}$$

Again, by the product rule,

$$\begin{aligned}\phi''(k) &= (ix)^2e^{ikx}\int_{-\infty}^{\infty} e^{-ikx}f(x)dx - ixe^{ikx}\int_{-\infty}^{\infty} ixe^{-ikx}f(x)dx \\ &- ixe^{ikx}\int_{-\infty}^{\infty} ixe^{-ikx}f(x)dx + e^{ikx}\int_{-\infty}^{\infty}(ix)^2e^{-ikx}f(x)dx \\ &= (ix)^2e^{ikx}\int_{-\infty}^{\infty} e^{-ikx}f(x)dx - 2ixe^{ikx}\int_{-\infty}^{\infty} ixe^{-ikx}f(x)dx + e^{ikx}\int_{-\infty}^{\infty}(ix)^2e^{-ikx}f(x)dx.\end{aligned}$$

Thus, we can compute

$$\phi(0) = \hat{f}(0) = \int_{-\infty}^{\infty} f(x)dx$$

and

$$\begin{aligned}\phi''(0) &= (ix)^2\int_{-\infty}^{\infty} f(x)dx - 2ix\int_{-\infty}^{\infty} ixf(x)dx + \int_{-\infty}^{\infty}(ix)^2f(x)dx \\ &= -x^2\int_{-\infty}^{\infty} f(x)dx + 2x\int_{-\infty}^{\infty} xf(x)dx - \int_{-\infty}^{\infty} x^2f(x)dx.\end{aligned}$$

Now, we can simply plug in $j = 0, 1$ to find the first two nonzero terms of our expansion. Namely, the first term is given by

$$\frac{1}{2\pi}\sqrt{\frac{\pi}{t}}\phi(0) = \frac{1}{2\pi}\sqrt{\frac{\pi}{t}}\int_{-\infty}^{\infty} f(x)dx$$

and the second term is given by

$$\frac{1}{2\pi}\sqrt{\frac{\pi}{t}}\frac{\phi''(0)}{t}\frac{1}{2^2 1!} = \frac{1}{2\pi}\sqrt{\frac{\pi}{t}}\frac{1}{4t}\left(-x^2\int_{-\infty}^{\infty} f(x)dx + 2x\int_{-\infty}^{\infty} xf(x)dx - \int_{-\infty}^{\infty} x^2 f(x)dx\right).$$

4 Problem 4

To compute the asymptotic expansion of

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-k^4t}\hat{f}(k)dk, \quad \hat{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x)dx,$$

we derive an additional generalization of Watson's lemma based on the generalization used in problem 3. Namely, we consider

$$F(\lambda) = \int_a^b e^{-\lambda t^4}\phi(t)dt$$

where $a < 0 < b$, $\lambda > 0$, and $\phi(t)$ is absolutely integrable and has an infinite number of continuous derivatives in a neighborhood of $t = 0$.
Steps 1 and 2 of our derivation are essentially the same as their counterparts on page 7 of part 2 of the course notes. Namely, to localize, fix $\epsilon > 0$. Then,

$$\int_a^b e^{-\lambda t^4}\phi(t)dt = \int_{-\epsilon}^{\epsilon} e^{-\lambda t^4}\phi(t)dt + \mathrm{O}(\lambda^{-N})$$

as $\lambda \to \infty$, $\lambda > 0$ for all $N > 0$ since the absolute integrability of $\phi(t)$ allows us to bound the integrals on the remaining domain by a constant times $e^{-\lambda\epsilon^4}$.
Using the remainder estimate,

$$\int_{-\epsilon}^{\epsilon} e^{-\lambda t^4}\phi(t)dt = \int_{-\epsilon}^{\epsilon} e^{-\lambda t^4}\left(\sum_{j=0}^{N}\phi^{(j)}(0)\frac{t^j}{j!} - r_N(t)\right)dt$$

where $r_N(t) = \mathrm{O}(t^{N+1})$ as $t \to 0$.
To find large λ limits, we use the substitution $s = \sqrt[4]{\lambda}t$ to get

$$\int_{-\epsilon}^{\epsilon} e^{-\lambda t^4}t^\ell dt = \int_{-\sqrt[4]{\lambda}\epsilon}^{\sqrt[4]{\lambda}\epsilon} e^{-s^4}\left(\frac{s}{\sqrt[4]{\lambda}}\right)^\ell \frac{ds}{\sqrt[4]{\lambda}} = \frac{1}{\lambda^{\ell/4+1/4}}\int_{-\sqrt[4]{\lambda}\epsilon}^{\sqrt[4]{\lambda}\epsilon} e^{-s^4}s^\ell ds = \mathrm{O}(\lambda^{-\ell/4-1/4}).$$

To see this, first note that if ℓ is odd,

$$\int_{-\sqrt[4]{\lambda}\epsilon}^{\sqrt[4]{\lambda}\epsilon} e^{-s^4}s^\ell ds = 0$$

by the symmetry of e^{-s^4}. If ℓ is even, the same symmetry gives that

$$\int_{-\sqrt[4]{\lambda}\epsilon}^{\sqrt[4]{\lambda}\epsilon} e^{-s^4}s^\ell ds = \int_{-\infty}^{\infty} e^{-s^4}s^\ell ds - 2\int_{\sqrt[4]{\lambda}\epsilon}^{\infty} e^{-s^4}s^\ell ds.$$

Then, using the change of variables $s = \sqrt[4]{\lambda}\epsilon + x$,

$$\begin{aligned}
\int_{\sqrt[4]{\lambda}\epsilon}^{\infty} e^{-s^4} s^\ell ds &= \int_0^\infty e^{-\lambda\epsilon^4 - 4\sqrt[4]{\lambda^3}\epsilon^3 x - 6\sqrt{\lambda}\epsilon^2 x^2 - 4\sqrt[4]{\lambda}\epsilon x^3 - x^4} (\sqrt[4]{\lambda}\epsilon + x)^\ell dx \\
&\leq e^{-\lambda\epsilon^4} \int_0^\infty e^{-x^4/2} (\underbrace{\sqrt[4]{\lambda}\epsilon e^{-\frac{x^4}{2\ell}}}_{\leq \sqrt[4]{\lambda}\epsilon} + \underbrace{x e^{-\frac{x^4}{2\ell}}}_{\leq C})^\ell dx \\
&\leq e^{-\lambda\epsilon^4} (\sqrt[4]{\lambda}\epsilon + C)^\ell \int_0^\infty e^{-x^4/2} dx = \mathrm{O}(\lambda^{-N})
\end{aligned}$$

for all $N > 0$ as $\lambda \to \infty$ when $\lambda > 0$. Additionally, when ℓ is even, using the substitution $t = s^4$,

$$\int_{-\infty}^{\infty} e^{-s^4} s^\ell ds = 2\int_0^\infty e^{-s^4} s^\ell ds = 2\int_0^\infty e^{-t} t^{\ell/4} \frac{dt}{4t^{3/4}} = \frac{1}{2}\Gamma(\frac{\ell}{4} + \frac{1}{4}).$$

Now, we can assemble everything to conclude that

$$F(\lambda) \sim \sum_{j=0}^{\infty} \frac{\phi^{(2j)}(0)}{(2j)!\lambda^{j/2+1/4}} \frac{\Gamma(j/2 + 1/4)}{2}$$

where we have reindexed $\ell \to 2j$ to account for only the even terms being nonzero.
Now, taking $\phi(k) = e^{ikx}\hat{f}(k)$, $a = -\infty$, and $b = \infty$ as in problem 3 and again assuming that f and $\hat{f}$ decay rapidly so that $\phi(k)$ is absolutely integrable on $(-\infty, \infty)$ and $\phi(k)$ has an infinite number of continuous derivatives in a neighborhood of $k = 0$, we have that

$$u(x,t) \sim \frac{1}{2\pi} \sum_{j=0}^{\infty} \frac{\phi^{(2j)}(0)}{(2j)!t^{j/2+1/4}} \frac{\Gamma(j/2 + 1/4)}{2}$$

as $t \to \infty$, $t > 0$. Since we have computed the required derivatives in problem 3, we plug in $j = 0, 1$ and conclude that the first term is given by

$$\frac{1}{2\pi} \frac{\Gamma(1/4)}{2t^{1/4}} \int_{-\infty}^{\infty} f(x) dx$$

and the second term is given by

$$\frac{1}{2\pi} \frac{\Gamma(3/4)}{4t^{3/4}} \left(-x^2 \int_{-\infty}^{\infty} f(x)dx + 2x \int_{-\infty}^{\infty} x f(x) dx - \int_{-\infty}^{\infty} x^2 f(x) dx \right).$$

This expansion differs from that in problem 3 in that its leading term has t raised to $-1/4$ instead of $-1/2$. Additionally, each successive term has the power of its t term decreased by $1/2$ instead of 1, meaning that it will take more terms to get an approximation of a similar order if we are considering large t.