AMATH 568 Homework 8

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1 Problem 1

Considering the BVP

$$\begin{cases} \epsilon y''(x) + \sin(x)y'(x) + \sin(2x)y(x) = 0, \\ y(0) = \pi, \\ y(\pi) = 0, \end{cases}$$

we expect to see a boundary layer near $x = 0$, because $\sin x > 0$ arbitrarily close to $x = 0$ (and on the entire interval $(0, \pi)$). Thus, the function for our outer layer is defined by taking $\epsilon = 0$ and solving

$$\begin{cases} \sin(x)y'_{\text{out}}(x) + \sin(2x)y_{\text{out}}(x) = 0, \\ y_{\text{out}}(\pi) = 0. \end{cases}$$

Solving this with an integrating factor,

$$y_{\text{out}}(x) = C\exp\left(-\int_0^x \frac{\sin 2s}{\sin s}ds\right) = C\exp\left(-\int_0^x 2\cos s ds\right) = Ce^{-2\sin x}.$$

Plugging in our boundary condition, we get that $C = 0$, so we conclude that

$$y_{\text{out}} = 0.$$

Now, we look to find the inner expansion by considering a substitution $z = \frac{x}{\delta(\epsilon)}$ and defining $Y_{\text{in}}(z) = y(\delta z)$. Then, our differential equation becomes

$$\frac{\epsilon}{\delta^2}Y''_{\text{in}}(z) + \frac{\sin(\delta z)}{\delta}Y'_{\text{in}}(z) + \sin(2\delta z)Y_{\text{in}}(z) = 0.$$

Now, we Taylor expand the sine function around $z_0 = 0$ to get

$$\frac{\epsilon}{\delta^2}Y''_{\text{in}}(z) + \frac{1}{\delta}(\delta z + \mathrm{O}(\delta^3))Y'_{\text{in}}(z) + (2\delta z + \mathrm{O}(\delta^3))Y_{\text{in}}(z) = 0.$$

To find a dominant balance, we note that the second term will always have a lower order term than the third, so we need to balance the first and second terms. To do this, we need $\frac{\epsilon}{\delta^2} = 1$, so we take $\delta = \sqrt{\epsilon}$ and an expansion for Y_{in} is given by

$$Y_{\text{in}} \sim \sum_{n=0}^{\infty} Y_n \epsilon^{n/2}.$$

Looking at the leading order, Y_0 must satisfy the BVP

$$\begin{cases} Y_0''(z) + zY_0'(z) = 0, \\ Y_0(0) = \pi. \end{cases}$$

We can see that

$$Y_0'(z) = c_1 e^{-z^2/2}$$

satisfies this equation. Then,

$$Y_0(z) = c_1 \int e^{-z^2/2} dz + c_2 = \sqrt{\frac{\pi}{2}} c_1 \operatorname{erf}\left(\frac{z}{\sqrt{2}}\right) + c_2 = C \operatorname{erf}\left(\frac{z}{\sqrt{2}}\right) + c_2$$

where C and c_2 are constants. Noting that $\operatorname{erf} 0 = 0$, our boundary condition gives $c_2 = \pi$, so,

$$Y_{\text{in}}(z) = C \operatorname{erf}\left(\frac{z}{\sqrt{2}}\right) + \pi + \mathrm{O}(\epsilon^{1/2}).$$

Now, we find the constant C by applying the matching condition at the bottom of page 336 in the text. Namely, we need that to leading order

$$\lim_{z\to\infty, z>0} Y_{\text{in}}(z) = \lim_{x\to 0, x>0} y_{\text{out}}(x).$$

Noting that $\operatorname{erf} x \to 1$ as $x \to \infty$, this condition becomes $C + \pi = 0$, so we get that $C = -\pi$. Now, y_{out} and Y_{in} agree in the matching region

$$0 \ll \frac{x}{\sqrt{\epsilon}} \ll \epsilon^{-1/2},$$

and the matching term in this region is given by

$$\lim_{z\to\infty, z>0} Y_{\text{in}}(z) = \lim_{x\to 0, x>0} y_{\text{out}}(x) = 0.$$

Combining everything, our lowest-order uniform approximation to the BVP is

$$y(x;\epsilon) = Y_{\text{in}}(x/\sqrt{\epsilon}) + y_{\text{out}}(x) - y_{\text{match}} = -\pi \operatorname{erf}\left(\frac{x}{\sqrt{2\epsilon}}\right) + \pi.$$

2 Problem 2

2.1 Part a

Now, consider the BVP

$$\begin{cases} \epsilon y_1''(x) + x y_1'(x) = x\cos(x), \\ y_1(-1) = 2, \\ y_1(0) = c_1. \end{cases}$$

We expect a boundary layer near 0, because $x < 0$ on the interval $(-1, 0)$. Thus, the function for our outer layer is given by taking $\epsilon = 0$ and solving

$$\begin{cases} xy_{\text{out}}'(x) = x\cos(x), \\ y_{\text{out}}(-1) = 2. \end{cases}$$

This gives that $y_{\text{out}}'(x) = \cos x$, so

$$y_{\text{out}}(x) = \sin x + C.$$

Incorporating our boundary condition, $2 = \sin(-1) + C$, so $C = 2 - \sin(-1) = 2 + \sin 1$. Thus,

$$y_{\text{out}}(x) = \sin x + 2 + \sin 1.$$

Now, we look to find the inner expansion by considering a substitution $z = \frac{x}{\delta(\epsilon)}$ and defining $Y_{\text{in}}(z) = y(\delta z)$. Then, our differential equation becomes

$$\frac{\epsilon}{\delta^2} Y_{\text{in}}''(z) + \frac{\delta z}{\delta} Y_{\text{in}}'(z) = \delta z \cos \delta z.$$

Taylor expanding the RHS at $z_0 = 0$,

$$\frac{\epsilon}{\delta^2} Y_{\text{in}}''(z) + zY_{\text{in}}'(z) = \delta z(\delta z + \mathrm{O}(\delta^3)).$$

To find a dominant balance, we need to balance the first two terms, meaning that $\frac{\epsilon}{\delta^2} = 1$, so $\delta = \sqrt{\epsilon}$. Then,

$$Y_{\text{in}}''(z) + zY_{\text{in}}'(z) = \epsilon z^2 + \mathrm{O}(\epsilon^2),$$

so an expansion for Y_{in} is given by

$$Y_{\text{in}} \sim \sum_{n=0}^{\infty} Y_n \epsilon^{n/2}.$$

To leading order, we must then have that

$$\begin{cases} Y_0''(z) + zY_0'(z) = 0, \\ Y_0(0) = c_1. \end{cases}$$

From problem 1, we know that a general solution is given by

$$Y_0(z) = C \operatorname{erf}\left(\frac{z}{\sqrt{2}}\right) + C',$$

and plugging in our boundary condition gives that $C' = c_1$, so

$$Y_0(z) = C \operatorname{erf}\left(\frac{z}{\sqrt{2}}\right) + c_1.$$

Now, we find the constant C by applying the matching condition at the bottom of page 336 in the text (noting that our interval is negative instead of positive). Namely, we need that to leading order

$$\lim_{z\to-\infty, z<0} Y_{\text{in}}(z) = \lim_{x\to 0, x<0} y_{\text{out}}(x).$$

Noting that $\operatorname{erf} x \to -1$ as $x \to -\infty$, this condition becomes $-C + c_1 = 2 + \sin 1$, so $C = -(2 + \sin 1 - c_1)$. Thus,

$$Y_{\text{in}} \sim -(2 + \sin 1 - c_1) \operatorname{erf}\left(\frac{z}{\sqrt{2}}\right) + c_1 + \mathrm{O}(\epsilon^{1/2}).$$

Now, y_{out} and Y_{in} agree in the matching region

$$0 \ll \frac{x}{\sqrt{\epsilon}} \ll \epsilon^{-1/2},$$

and the matching term in this region is given by

$$y_{\text{match}} = \lim_{z\to-\infty, z<0} Y_{\text{in}}(z) = \lim_{x\to 0, x<0} y_{\text{out}}(x) = 2 + \sin 1.$$

Combining everything, our lowest-order uniform approximation to the BVP is

$$\begin{aligned}
y(x;\epsilon) &= Y_{\text{in}}(x/\sqrt{\epsilon}) + y_{\text{out}}(x) - y_{\text{match}} \\
&= \sin x + 2 + \sin 1 - (2 + \sin 1 - c_1) \operatorname{erf}\left(\frac{x}{\sqrt{2\epsilon}}\right) + c_1 - (2 + \sin 1) \\
&= \sin x - (2 + \sin 1 - c_1) \operatorname{erf}\left(\frac{x}{\sqrt{2\epsilon}}\right) + c_1.
\end{aligned}$$

2.2 Part b

Now, consider the BVP

$$\begin{cases} \epsilon y_2''(x) + x y_2'(x) = x\cos(x), \\ y_2(1) = 2, \\ y_2(0) = c_1. \end{cases}$$

We expect a boundary layer near 0, because $x > 0$ on the interval $(0,1)$. Thus, the function for our outer layer is given by taking $\epsilon = 0$ and solving

$$\begin{cases} x y_{\text{out}}'(x) = x\cos(x), \\ y_{\text{out}}(1) = 2. \end{cases}$$

This gives that $y_{\text{out}}'(x) = \cos x$, so

$$y_{\text{out}}(x) = \sin x + C.$$

Incorporating our boundary condition, $2 = \sin 1 + C$, so $C = 2 - \sin 1$. Thus,

$$y_{\text{out}}(x) = \sin x + 2 - \sin 1.$$

Now, we look to find the inner expansion by considering a substitution $z = \frac{x}{\delta(\epsilon)}$ and defining $Y_{\text{in}}(z) = y(\delta z)$. Then, our differential equation becomes

$$\frac{\epsilon}{\delta^2} Y''_{\text{in}}(z) + \frac{\delta z}{\delta} Y'_{\text{in}}(z) = \delta z \cos \delta z.$$

Taylor expanding the RHS at $z_0 = 0$,

$$\frac{\epsilon}{\delta^2} Y''_{\text{in}}(z) + z Y'_{\text{in}}(z) = \delta z(\delta z + \mathrm{O}(\delta^3)).$$

To find a dominant balance, we need to balance the first two terms, meaning that $\frac{\epsilon}{\delta^2} = 1$, so $\delta = \sqrt{\epsilon}$. Then,

$$Y''_{\text{in}}(z) + z Y'_{\text{in}}(z) = \epsilon z^2 + \mathrm{O}(\epsilon^2),$$

so an expansion for Y_{in} is given by

$$Y_{\text{in}} \sim \sum_{n=0}^{\infty} Y_n \epsilon^{n/2}.$$

To leading order, we must then have that

$$\begin{cases} Y_0''(z) + z Y_0'(z) = 0, \\ Y_0(0) = c_2. \end{cases}$$

From problem 1, we know that a general solution is given by

$$Y_0(z) = C \operatorname{erf}\left(\frac{z}{\sqrt{2}}\right) + C',$$

and plugging in our boundary condition gives that $C' = c_1$, so

$$Y_0(z) = C \operatorname{erf}\left(\frac{z}{\sqrt{2}}\right) + c_2.$$

Now, we find the constant C by applying the matching condition at the bottom of page 336 in the text. Namely, we need that to leading order

$$\lim_{z \to \infty, z > 0} Y_{\text{in}}(z) = \lim_{x \to 0, x > 0} y_{\text{out}}(x).$$

Noting that $\operatorname{erf} x \to 1$ as $x \to \infty$, this condition becomes $C + c_2 = 2 - \sin 1$, so $C = 2 - \sin 1 - c_2$. Thus, to leading order

$$Y_{\text{in}} \sim (2 - \sin 1 - c_2) \operatorname{erf}\left(\frac{z}{\sqrt{2}}\right) + c_2.$$

Now, y_{out} and Y_{in} agree in the matching region

$$0 \ll \frac{x}{\sqrt{\epsilon}} \ll \epsilon^{-1/2},$$

and the matching term in this region is given by

$$y_{\text{match}} = \lim_{z\to\infty, z>0} Y_{\text{in}}(z) = \lim_{x\to 0, x>0} y_{\text{out}}(x) = 2 - \sin 1.$$

Combining everything, our lowest-order uniform approximation to the BVP is

$$\begin{aligned}
y_2(x;\epsilon) &= Y_{\text{in}}(x/\sqrt{\epsilon}) + y_{\text{out}}(x) - y_{\text{match}} \\
&= \sin x + 2 - \sin 1 + (2 - \sin 1 - c_2)\operatorname{erf}\left(\frac{x}{\sqrt{2\epsilon}}\right) + c_2 - (2 - \sin 1) \\
&= \sin x + (2 - \sin 1 - c_2)\operatorname{erf}\left(\frac{x}{\sqrt{2\epsilon}}\right) + c_2.
\end{aligned}$$

2.3 Part c

Now, we wish to solve the BVP

$$\begin{cases} \epsilon y''(x) + xy'(x) = x\cos(x), \\ y(1) = 2, \\ y(-1) = 2. \end{cases}$$

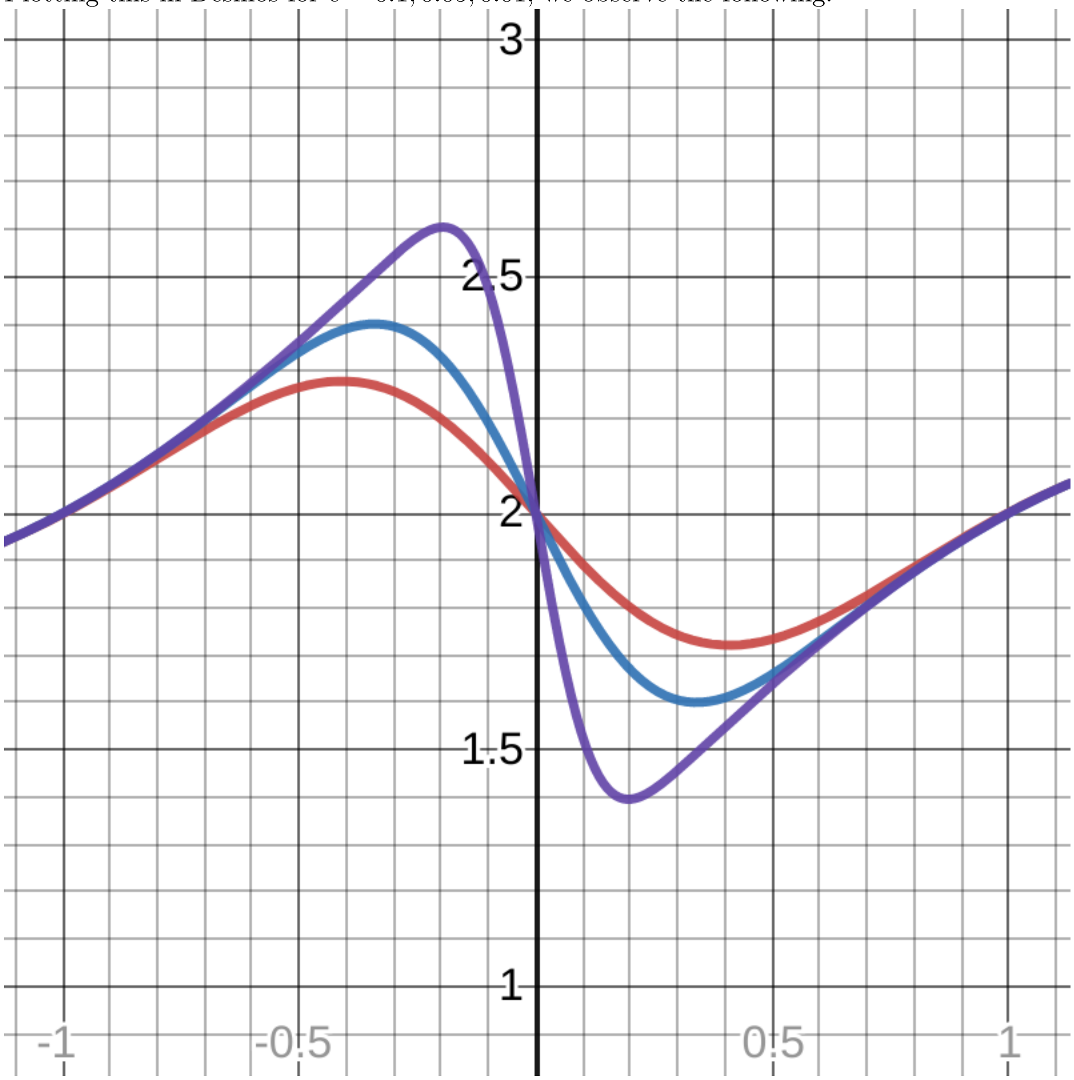
by combining our solutions from parts a and b. To do this, we set $c_1 = c_2$ and enforce that our functions are identical on our new extended domain. Combining these,

$$\sin x - (2 + \sin 1 - c_1)\operatorname{erf}\left(\frac{x}{\sqrt{2\epsilon}}\right) + c_1 = \sin x + (2 - \sin 1 - c_1)\operatorname{erf}\left(\frac{x}{\sqrt{2\epsilon}}\right) + c_1$$

which gives that $-2 - \sin 1 + c_1 = 2 - \sin 1 - c_1$, so $c_1 = c_2 = 2$. Thus, we can plug this in for y_1 and y_2 to find that a uniform approximation of the combined BVP is given by

$$y(x;\epsilon) = \sin x - \sin 1 \operatorname{erf}\left(\frac{x}{\sqrt{2\epsilon}}\right) + 2.$$

Plotting this in Desmos for $\epsilon = 0.1, 0.05, 0.01$, we observe the following.



where the red plot is $\epsilon = 0.1$, the blue plot is $\epsilon = 0.05$, and the purple plot is $\epsilon = 0.01$.