MATH 525 Homework 1

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1 Problem 1

Consider the following alternative definition of a normal topology: given any two disjoint closed sets A and B, there is an open set $V \supset A$ such that $\overline{V} \cap B = \emptyset$.

To see that this is equivalent to our definition, let $(X, \mathcal{T})$ be a normal topological space. Then, given any two disjoint closed sets A and B, there exist two disjoint open sets U and V such that $A \subset V$ and $B \subset U$. This implies that

$$V \subset U^c \subset B^c.$$

Because U^c is closed and $\overline{V}$ is the smallest closed set containing V, this further implies that

$$\overline{V} \subset U^c \subset B^c,$$

meaning that $\overline{V} \cap B = \emptyset$ and the alternative definition is satisfied.

Conversely, let $(X, \mathcal{T})$ be a topological space satisfying the alternative definition. Then, given any two disjoint closed sets A and B, there is an open set $V \supset A$ such that $\overline{V} \cap B = \emptyset$. We let $U = \left(\overline{V}\right)^c$, noting that this set is open. Then, $B \subset U$ and $V \subset U^c$, so $U \cap V = \emptyset$, and $(X, \mathcal{T})$ is normal by definition.

2 Problem 2

Let $f : X \to \mathbb{R}$ be continuous. Then, for every $a \in \mathbb{R}$, $(-\infty, a)$ and (a, ∞) are both open, so their preimages $f^{-1}((-\infty, a))$ and $f^{-1}((a, \infty))$ are both open.

Conversely, consider a map $f : X \to \mathbb{R}$ such that for every $a \in \mathbb{R}$, the preimages $f^{-1}((-\infty, a))$ and $f^{-1}((a, \infty))$ are both open. Since $\mathbb{R}$ is a metric space with the standard Euclidean distance, the open balls $B(x, r) = (x - r, x + r)$ with $x \in \mathbb{R}$ and $r > 0$ form a base for $\mathbb{R}$. Thus, to show that f is continuous, it suffices to show that $f^{-1}((x - r, x + r))$ is open for all $x \in \mathbb{R}$, $r > 0$. Note that

$$(x - r, x + r) = (-\infty, x - r) \cap (x + r, \infty),$$

so

$$f^{-1}((x - r, x + r)) = f^{-1}((-\infty, x + r)) \cap f^{-1}((x - r, \infty)),$$

is open because it is the finite intersection of open sets. Thus, f is continuous.

3 Problem 3

Suppose X is compact and $f_n : X \to \mathbb{R}$ is a sequence of continuous functions on X such that $f_n(x)$ converges monotonically upwards to $f(x)$ and f is continuous. Fix $\epsilon > 0$. Define $g_n = f - f_n$ for all n and note that each g_n is continuous and the sequence $\{g_n\}$ is monotonically decreasing. For each n, define $E_n = g_n^{-1}((-\infty, \epsilon))$. By problem 2, E_n is open for all n. Because $f_n \to f$ pointwise, $X \subset \bigcup_{n=1}^{\infty} E_n$. This is an open cover of X, so it can be reduced to a finite subcover. Furthermore, because $\{g_n\}$ is monotonically decreasing, we must have that for all n, $E_n \subset E_{n+1}$. Thus, the finite subcover is equivalent to its largest member set, meaning that $X \subset E_N$ for some $N \in \mathbb{N}$. This means that for any $n \geq N$,

$$|f(x) - f_n(x)| < \epsilon,$$

for all $x \in X$. Thus, by definition, $f_n \to f$ uniformly.

4 Problem 4

Let A denote a set and A' its limit points.

4.1 Part a

Let $x \in U$ where U is an open set that does not intersect A. Then, U^c is closed and $A \subset U^c$. Since $\overline{A}$ is the smallest closed set containing A, this implies that $\overline{A} \subset U^c$, so $U \cap \overline{A} = \emptyset$, and $x \notin \overline{A}$.

Conversely, let $x \notin \overline{A}$. Then, $x \in \left(\overline{A}\right)^c$, and $\left(\overline{A}\right)^c$ is an open set that clearly does not intersect A. Thus, there exists an open set containing x that does not intersect A.

We have established that $x \notin \overline{A}$ if and only if there exists an open set containing x that does not intersect A. The contrapositive of this statement is that $x \in \overline{A}$ if and only if every open set containing x intersects A.

4.2 Part b

By part a, we have that $A' \subset \overline{A}$, as $x \in A'$ implies that every open set containing x intersects A.

Conversely, let $x \notin A \cup A'$. Then, there exists an open set U such that $U \cap A = \emptyset$ and $x \in U$. This implies that $A \subset U^c$, but U^c is closed, so $\overline{A} \subset U^c$ since $\overline{A}$ is the smallest closed set containing A. Thus, $x \notin \overline{A}$, so $\overline{A} \subset A \cup A'$ and $\overline{A} = A \cup A'$ since $A' \subset \overline{A}$.

5 Problem 5 (Folland Problem 14)

Let X and Y be topological spaces and $f : X \to Y$. If f is continuous, then for all $A \subset X$, $f^{-1}\left(\overline{f(A)}\right)$ is a closed set. Note that $A \subset f^{-1}\left(\overline{f(A)}\right)$, so $\overline{A} \subset f^{-1}\left(\overline{f(A)}\right)$ since $\overline{A}$ is the smallest closed set containing A. Thus, $f\left(\overline{A}\right) \subset \overline{f(A)}$ for all $A \subset X$.

Now, assume that $f\left(\overline{A}\right) \subset \overline{f(A)}$ for all $A \subset X$ and let $B \subset Y$. If we take $A = f^{-1}(B)$, then

$$f\left(\overline{f^{-1}(B)}\right) \subset \overline{f\left(f^{-1}(B)\right)} = \overline{B}.$$

Thus, $\overline{f^{-1}(B)} \subset f^{-1}\left(\overline{B}\right)$ for all $B \subset Y$.

Finally, assume that $\overline{f^{-1}(B)} \subset f^{-1}\left(\overline{B}\right)$ for all $B \subset Y$ and let $V \subset Y$ be closed. Taking $B = V$ yields

$$\overline{f^{-1}(V)} \subset f^{-1}\left(\overline{V}\right) = f^{-1}(V).$$

However, since $\overline{f^{-1}(V)}$ is the smallest closed set containing $f^{-1}(V)$, we must have that $f^{-1}(V)$ is closed for all closed $V \subset Y$. Thus, f is continuous, and the three statements are equivalent.