AMATH 570 Assignment 4

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1 Problem 1 (15.4)

Using chebfun to investigate the Lebesgue constant for the set of n Chebyshev points with the endpoints removed, we get what appears to be a linear $O(n)$ relationship.

15-4-eps-converted-to.pdf

See problem15_4.m for the code that does this.

2 Problem 2

For completeness, the bound (15.5) is

$$||f - p|| \leq (\Lambda + 1)||f - p^*||$$

where f is the function we are approximating, Λ is the Lebesgue constant for a set of interpolation points, $p \in \mathcal{P}_n$ is the interpolant of f at those points, and $p^* \in \mathcal{P}_n$ is the best polynomial approximation.

2.1 Part a

Consider the function $f(x) = 1/(1 + 25x^2)$. Using chebfun, we plot both the bound and the true error for both Chebyshev and equispaced points and obtain the following:

2a-eps-converted-to.pdf

2.2 Part b

Now, consider $f(x) = 1/(1 + 500x^2)$. We again plot both the bound and the true error for both Chebyshev and equispaced points using chebfun and obtain the following:

2b-eps-converted-to.pdf

See problem2.m for the relevant MATLAB code.

3 Problem 3

3.1 Exercise 17.3

3.1.1 Part a

Using chebfun, we construct a quasimatrix A with columns corresponding to $1, x, \ldots, x^5$ on $[-1, 1]$. Using the command qr to find an orthonomal set of functions and plotting them, we get

17_1-eps-converted-to.pdf

which appears identical to the following plot of the first 6 Legendre polynomials normalized according to (17.3) which follows.

17_2-eps-converted-to.pdf

Indeed, if we plot the absolute value of the difference between the two, we observe

17_3-eps-converted-to.pdf

meaning that they are identical on the order of machine precision.

3.1.2 Part b

Now, we normalize the columns of Q such that they take value 1 at $x = 1$ and adjust the rows of R accordingly. To confirm that our modified Q and R still multiply to A, we plot the absolute value of the difference between A and $Q * R$

17_4-eps-converted-to.pdf

which shows that they are identical to machine precision.
Plotting we columns of the new quasimatrix Q, we observe

17_5-eps-converted-to.pdf

which now looks identical to the first 6 Legendre polynomials normalized in the same manner.

17_6-eps-converted-to.pdf

To confirm this, we again plot the absolute value of their difference which again shows that they are identical to machine precision.

17_7-eps-converted-to.pdf

See problem17_8.m for the code that produces the plots in this problem.

3.2 Exercise 17.8

Let $\{p_n\}$ be the family of monic polynomials associated with the inner product defined in (17.1) and let q be any monic polynomial of degree n. Then, we can write q as

$$q(x) = \sum_{k=0}^{n-1} c_k p_k(x) + p_n(x)$$

where the coefficient on p_n is 1, because both it and q are monic. We can now write

$$\begin{aligned}
(q, q) &= \int_{-1}^{1} w(x)\overline{q(x)}q(x)dx \\
&= \int_{-1}^{1} w(x) \left(\overline{\sum_{k=0}^{n-1} c_k p_k(x) + p_n(x)}\right) \left(\sum_{k=0}^{n-1} c_k p_k(x) + p_n(x)\right) dx \\
&= \int_{-1}^{1} w(x) \left(\sum_{k=0}^{n-1} \overline{c_k}\,\overline{p_k(x)} + \overline{p_n(x)}\right) \left(\sum_{k=0}^{n-1} c_k p_k(x) + p_n(x)\right) dx.
\end{aligned}$$

Now, the orthogonality of the family $\{p_n\}$ means that $\int_{-1}^{1} w(x)\overline{p_i(x)}p_j(x)dx = 0$ for $i \neq j$, so when we multiply this out and utilize the linearity of the integration operator, the cross-terms will be zero, meaning that

$$\begin{aligned}(q,q) &= \sum_{k=0}^{n-1}\int_{-1}^{1} w(x)\overline{c_k}c_k\overline{p_k(x)}p_k(x)dx + \int_{-1}^{1} w(x)\overline{p_n(x)}p_n(x)dx \\ &= \sum_{k=0}^{n-1}\overline{c_k}c_k(p_k,p_k) + (p_n,p_n) = \sum_{k=0}^{n-1}|c_k|(p_k,p_k) + (p_n,p_n) \geq (p_n,p_n),\end{aligned}$$

because $|c_k|, (p_k,p_k) \geq 0$ for any k.

4 Problem 4 (18.4)

4.1 Part a

To derive a comrade matrix for the Legendre polynomials in the manner of theorem 18.1, we wish to find a matrix whose eigenvalues are the roots of the polynomial $p(x) = \sum_{k=0}^{n} a_k P_k(x)$. We do this in the same manner as with the Chebyshev polynomials. For the interior, we use the recursive relationship $(k+1)P_{k+1}(x) - kP_{k-1}(x)$ to get that our matrix should map $P_k(x)$ to $xP_k(x) = \frac{k}{2k+1}P_{k-1}(x) + \frac{k+1}{2k+1}P_{k+1}(x)$. For the first row, our matrix should map $P_0(x)$ to $xP_0(x) = P_1(x)$. For the last row, we have the same relationship as with the Chebyshev polynomials but adjusted for our new interior coefficients. Namely,

$$P_{n-1}(x) \mapsto xP_{n-1}(x) - \frac{n-1+1}{a_n(2(n-1)+1)}(a_0P_0(x) + \ldots + a_nP_n(x)).$$

Thus, we can write

$$C = \begin{pmatrix} 0 & 1 & & & & \\ 1/3 & 0 & 2/3 & & & \\ & 2/5 & 0 & 3/5 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \frac{n-2}{2n-3} & 0 & \frac{n-1}{2n-3} \\ & & & & \frac{n-1}{2n-1} & 0 \end{pmatrix} - \frac{n}{a_n(2n-1)}\begin{pmatrix} & & & & \\ & & & & \\ & & & & \\ & & & & \\ a_0 & a_1 & a_2 & \ldots & a_{n-1} \end{pmatrix}.$$

Note that the kth Legendre polynomial corresponds to the $(k+1)$st row of C, so we need to shift the relations above accordingly to explicitly write what $C_{i,j}$ is (see part b for code that does this or problem 5 for it written out explicitly). Then, our theorem is as follows:
The roots of the polynomial

$$p(x) = \sum_{k=0}^{n} a_k P_k(x), \quad a_n \neq 0,$$

are the eigenvalues of the matrix C as defined above.

4.2 Part b

To verify that the comrade matrix for $p = P_0 + \ldots + P_5$ has the same eigenvalues are p does roots, we note that $a_i = 1$ for $i = 0, \ldots, 5$ and use MATLAB to construct C for this case which outputs

```
The eigenvalues of the comrade matrix are
 -1.000000000000000 + 0.000000000000000i
  0.634846841685047 + 0.225134736423368i
  0.634846841685047 - 0.225134736423368i
 -0.412624619462826 + 0.273188868980396i
 -0.412624619462826 - 0.273188868980396i
```

and

```
The roots of our polynomial are
 -0.412624619462826 - 0.273188868980397i
 -0.412624619462826 + 0.273188868980397i
  0.634846841685048 - 0.225134736423369i
  0.634846841685048 + 0.225134736423369i
 -1.000000000000001 + 0.000000000000000i
```

which are given in different order, but match to machine precision as one would expect.
See problem18_4.m for the code that does this.

5 Problem 5 (19.7)

5.1 Part a

Knowing that the nodes $\{x_j\}$ of the $(n+1)$-point Gauss quadrature are the zeroes of the Legendre polynomial P_{n+1}, we wish to find a comrade matrix for which these are its eigenvalues. However, in problem 4, we derived such a comrade matrix for linear combinations of Legendre polynomials. Now, we need an $(n+1)$ by $(n+1)$ matrix, but we have that $a_0 = \ldots = a_n = 0$ and $a_{n+1} = 1$, so the second matrix is the zero matrix and

$$A = \begin{pmatrix} 0 & 1 & & & & \\ 1/3 & 0 & 2/3 & & & \\ & 2/5 & 0 & 3/5 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \frac{n-1}{2n-1} & 0 & \frac{n}{2n-1} \\ & & & & \frac{n}{2n+1} & 0 \end{pmatrix}$$

We can reindex our formula from problem 4 to explicitly find the entries of this matrix. Namely, for $k \in \{1, \dots, n\}$,

$$\begin{cases} A_{k,k+1} = \frac{k}{2k-1}, \\ A_{k,k-1} = \frac{k-1}{2k-1}, \\ A_{k,j} = 0, \quad \text{otherwise.} \end{cases}$$

Additionally, we have that $A_{1,2} = 1$, $A_{n,n+1} = \frac{n}{2n+1}$ and 0 in all other entries.

5.2 Part b

To find a diagonal matrix $D = \text{diag}(d_0, \dots, d_n)$ such that $B = DAD^{-1}$ is real symmetric, we consider that for $k > 1$,

$$B_{k,k+1} = d_{k-1} A_{k,k+1} / d_k$$

and for $k < n + 1$

$$B_{k+1,k} = d_k A_{k+1,k} / d_{k-1}.$$

Because we need B to be symmetric, we can equate these in the interior $2 \leq k \leq n$. Thus,

$$\frac{d_{k-1}}{d_k} \frac{k}{2k-1} = \frac{d_k}{d_{k-1}} \frac{k}{2k+1}.$$

This yields a quadratic equation

$$\frac{d_k}{d_{k-1}} = \frac{2k+1}{2k-1}$$

for which we can consider only the positive solution, because we require that $d_j > 0$ for $j \geq 1$. Thus,

$$d_k = d_{k-1} \sqrt{\frac{2k+1}{2k-1}}$$

for $2 \leq k \leq n$. Since we require that $d_0 = 1$, this recursion actually cancels due to the fact that $2k - 1 = 2(k-1) + 1$ and simply yields that $d_k = \sqrt{2k+1}$. Now, using our work above, we find that B has entries

$$B_{k,k+1} = \frac{k}{2k-1} \sqrt{\frac{2k-1}{2k+1}} = \frac{k}{\sqrt{(2k-1)(2k+1)}}$$

and of course, $B_{k+1,k} = B_{k,k+1}$ due to symmetry. More explicitly we can reindex to find that

$$B_{k,k-1} = \frac{k-1}{\sqrt{(2k-3)(2k-1)}}.$$

Of course, the first equation hold for $k = 1, \dots, n$, the second holds for $k = 2, \dots, n+1$, and all other entries are zero.