Math 525 Winter 2024

Homework 9 **Due Friday, March 8**

1. Show that if $T \in \mathcal{B}(X)$, and $\lambda \in \sigma_C(T)$, there is a sequence $x_n \in X$ with $\|x_n\| = 1$ and $\lim_{n\to\infty} \|Tx_n - \lambda x_n\| = 0$.

2. (a.) If $\mathcal{H}$ is a Hilbert space, and $S \in \mathcal{B}(\mathcal{H})$, show that

$$\overline{S(\mathcal{H})} = \ker(S^*)^\perp$$

(b.) If $\lambda \in \sigma(T) \setminus \sigma_P(T)$, then $\lambda \in \sigma_R(T)$ iff $\bar{\lambda} \in \sigma_P(T^*)$.

3. Consider $\mathcal{H} = \ell^2(\mathbb{N})$, and define left and right shift operators:

$$S_L(x_1, x_2, \cdots) = (x_2, x_3, \cdots), \qquad S_R(x_1, x_2, \cdots) = (0, x_1, x_2, \cdots).$$

(a.) Show that $S_L^* = S_R$.

(b.) Show that $\sigma_P(S_L) = \{z : |z| < 1\}$, and $\sigma_C(S_L) = \{z : |z| = 1\}$. [Hint: show that for all $z \in \mathbb{C}$ the range of $zI - S_L$ contains all finite sequences, so there is no residual spectrum, by considering the formal expansion of $(zI - S_L)^{-1}$ if $z \neq 0$.]

(c.) Show that $\sigma_R(S_R) = \{z : |z| < 1\}$, and $\sigma_C(S_R) = \{z : |z| = 1\}$. [Hint: Use problem 2 above.]

4. Suppose that X, Y are Banach spaces with duals X^*, Y^*. Show that if $T^* : Y^* \to X^*$ is compact, then T is compact.

One way to do this is to use compactness of T^{**}, and the norm preserving embedding $X \to X^{**}$ that identifies $x \in X$ with $\hat{x} \in X^{**}$, where $\hat{x}(v) = v(x)$. To proceed, first show that $T^{**}(\hat{x}) = \widehat{Tx}$.

5. Consider a continuous function $h : [0, 1] \to \mathbb{C}$, and the multiplication operator on $L^2([0, 1], m)$ defined by $(T_h f)(x) = h(x)f(x)$. Show that $\sigma(T_h) = \text{range}(h)$, and that $\sigma_R(T_h) = \emptyset$. Under what conditions on f is a given $\lambda \in \text{range}(h)$ an eigenvalue of T_h?