The Duck Problem:
a statistical adventure

Cade E. Brown
Greg T. Croisdale

A Long Time Ago

Advanced Placement Statistics L4
Mark J. Waxmonsky
L&N STEM Academy

Figure 1: Duck

1 Introduction

1.1 Statement

There are 10 fraternity brothers (Figure 2) at a shooting gallery at the state fair. Each brother is a perfect shot. Ten ducks (Figure 1) appear simultaneously, and each shooter[1] picks one of the ten ducks at random and hits his target.

Figure 2: Fraternity Brother

1.2 Inquiries

We are attempting to provide an approximate answer to each of the following problems:

1.2.1 Average ducks shot per round

This asks on average, how many ducks are shot per round. For example, if all 10 fraternity brothers shot and only hit 5 distinct ducks, the number of ducks shot would be 5, even though all 10 fired at ducks. This is necessary due to the fact that two or more

[1] Throughout this paper, 'shooter' is analogous to 'fraternity brother' and may be used interchangeably

fraternity brothers may shoot the same duck, and only one[2] is added to the amount shot in one round. In this case, the total number of ducks shot is 5.

1.2.2 Probability that at least half the ducks are hit

This asks the probability that at least half the ducks are shot in a trial as described in 1.2.1.

For example, having 10 shots fired, and 6 ducks hit we would say that 5 or more ducks have been hit.

1.2.3 Modified version of 1.2.1 and 1.2.2 in which there are 20 hunters

Given the stipulations in 1.2.1 and 1.2.2, how does the solution to each change if there are now 20 hunters and only 10 ducks?

1.3 Assumptions

In order to make an accurate experiment, some assumptions must be made:

1.3.1 Duck

The ducks at the state fair are of the *Lophonetta Specularioides* species. For the purposes of this experiment, we will assume that they are adequately fed and lack any supernatural abilities. The ten ducks in the original experiment are named:

1. Frank
2. Frank Jr.
3. Dr. Frank
4. Mr. Frank
5. Mrs. Frank
6. Frank the I, Esq.
7. Frank the III: Back in Action
8. Frank IV: A New Hope
9. Joe
10. Also Frank

Their little duck families will mourn their inevitable demise[3].

[2] We only count distinct ducks hit by 1 or more shot, rather than shots total

[3] All other ducks are named $Frank_{n+1}$

1.3.2 Fraternity

The Fraternity in question is $\alpha\kappa\kappa\pi$, or, "impossibly good duck hunters". We will assume they all have a "B -" class average and are 22 years of age. Their names are as follows:

1. Neville "Heyguys" Nestle
2. Darren "Wassup'" Dollard
3. Preston "Didya" Pratico
4. Greg "Catch" Gelinas
5. Broderick "The" Beiler
6. Cruz "Game" Counter
7. Mario "Last" Mckain
8. Earle "Night" Eyman
9. Harland "?" Hipple
10. Tedcruz "is" Zodiackiller

We also assume that each shot by a fraternity brother is completely independent of all other shots. i.e. no duck is more or less likely to be shot due to an earlier shot.

2 Methods

In order to approximate the solutions to inquiries outlined in 1.2, we will employ programs[4] in a format which can be ran and verified online.

These methods run a series of tests which each run the original stipulation[5]. By changing variable *num_ducks* from 10 to 20, 1.2.3 can be simulated.

[4]For the Python language, see 3.1 in 3

[5]Found in 1.1 and 1.2

3 Programs

3.1 Python

```
# used to randomly pick targets
import random

# how many trials to test?
trials = 1000

# how many hunters?
num_hunters = 10

# how many ducks
num_ducks = 10

# to keep track
total_hit = 0
half_hit = 0

# do the trials
for i in range(0, trials):
    # set is a data type which does not allow duplicates
    # i.e. if two hunters hit the same one, only one is added
    targets = set()

    # get a target for each hunter
    for hunter in range(0, num_hunters):
        targets.add(random.randint(0, num_ducks))

    # how many ducks were hit
    ducks_hit = len(targets)
    total_hit += ducks_hit

    # if we hit more than half, add it to the count
    if (float(ducks_hit)/num_ducks) > 0.5:
        half_hit += 1

# ducks hit / trials
average_ducks_hit = float(total_hit) / trials

# times we hit more than half / trials
prob_half_hit = float(half_hit) / trials
```

4 Results

In a simulation with 2^{24} trials (as described in 1.1), the values for 1.2.1, 1.2.2, and 1.2.3 are as follows:

4.1 1.2.1 results

6.51332 average ducks hit per round, assuming 10 shooters The theoretical average for this is $10(1 - .9^{10})$, for which the justification is:

The probability that a single ducks survives a single shot is $a = 9/10$, so the chance they survive all 10 is a^{10} (since these are independent events) Therefore, the probability that a single duck dies is $1 - a^{10}$ (the negation), and multiplied by the number of ducks is $10(1 - .9^{10})$

4.2 1.2.2 results

$0.85360 = 85.36\%$ of the time, more than half of the ducks were shot, assuming 10 shooters.

4.3 1.2.3 results

These are answers to modified 1.2.1 and 1.2.2 in that they contain 20 fraternity brothers and 10 ducks instead of 10 fraternity brothers and 10 ducks.

4.3.1 Modified 1.2.1

8.78425 average ducks hit per round, assuming 20 shooters.

The theoretical average for this is $1 - .9^{20}$

Similar to 1.2.1 results, the closed form for this is simple to deduce: The chance that a single duck survives a single shot is $a = 9/10$, and the chance that the duck survives all 20 shots is a^{20}. Thus, the chance it has been shot at least once is $1 - a^{20}$

4.3.2 Modified 1.2.2

$0.99977 = 99.977\%$ of the time, more than half of the ducks were shot, assuming 20 shooters.

5 Analysis

5.1 1.2.1

The calculated average was 6.51322, therefore our simulation had a percent error of 0.001603 %. Our simulation was extremely accurate to theoretical values.

5.2 1.2.2

Our calculated value, 0.85360, was somewhat expected. Since the average from 1.2.1 was $> 10/2$, we would expect that on a normal (or approximately normal) distribution, more than half the time more half the ducks were shot.

We have not found an easy closed form for this, so we cannot measure percent error.

5.3 1.2.3

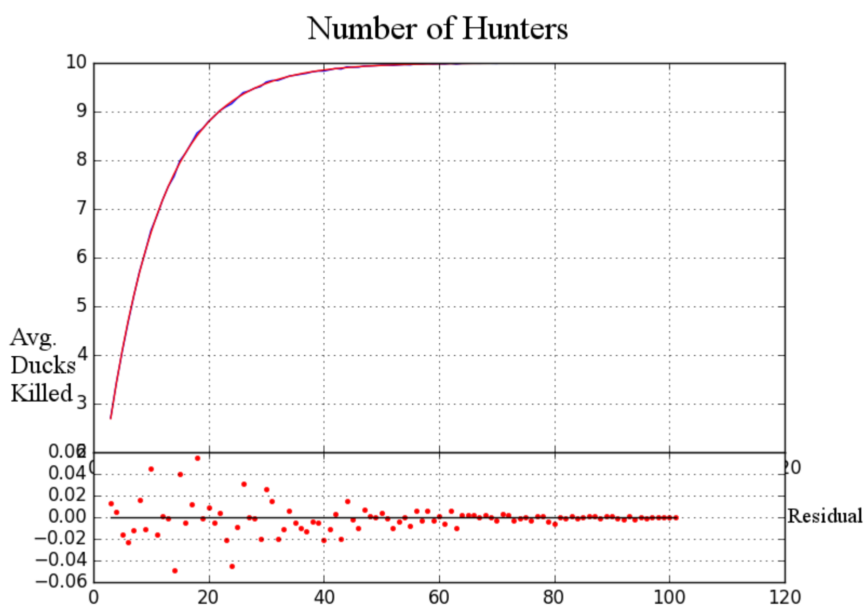

Figure 3: Average ducks shot per round with x hunters, where x is the horizontal axis. The blue line is the simulated data points, and the red line is the theoretical model of $10(1-(.9)^h)$. The subplot is the residuals at each point.

More generally, let $d, h > 0$, where d is the number of ducks, and h is the number of fraternity brothers, call this expected value $f(d, h)$

The expected value of unique ducks shot per round is: $f(d, h) = d * (1 - (\frac{d-1}{d})^h)$

Proof:

The probability that any one duck is not shot by a shooter during one shot is $\frac{h-1}{h}$, so the probability that they are not shot all h times is $\frac{h-1}{h}^h$, assuming random independent events[6].

[6]See 1.3

Thus, the probability that they are shot at least one time is $1-(\frac{h-1}{h})^h$. This is done over d ducks, so the predicted amount of ducks surviving per round is $d(1-(\frac{h-1}{h})^h)$

Using the definition of e, Euler's Number, the following limit can be shown: $\lim_{x\to\infty} f(x,x) = x\frac{e-1}{e}$

5.3.1 Modified 1.2.1

Our calculated average, 8.78425 had a percent error of 0.000188 %. This was even more accurate than 1.2.1.

We assume this to, on average, be higher than the normal 1.2.1 due to a simple heuristic:

If we have 20 shooters, it must be at least as large as with 10 shooters because, 20 shooters is equivalent to having 10 shooters take their shot, and then have 10 more shots be fired without replacing the ducks. Because no ducks can come back to life (see 1.3), the number is strictly increasing or staying the same.

5.3.2 Modified 1.2.2

Our calculated value, 0.99977 is to be expected as larger than the analog experiment with 10 shooters of 0.85360. Although we do not have a closed form, this form makes heuristic sense.

Similarly to modified 1.2.1, modified 1.2.2 is expected to be larger or the same as 1.2.1 due to the same heuristic. If we have 10 shooters take their shot, then have 10 more take one shot each, the number of ducks shot can only increase or stay the same, and thus the probability that more than half of the ducks would be shot can only increase or stay the same.