Electro-mechanical model

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Contents

1 Introduction

The electro-mechanical system described consists of:

- a power supply circuit;
- an electromechanical device;
- a system of mechanical loads connected to the moving part.

The system has a single degree of freedom, denoted by x. The mechanical part of the system is described by the Firestone analogy:

- the *through* variable: electromagnetic current $i(t) \leftrightarrow$ mechanical force $f(t)$;
- the *across* variable: electromagnetic voltage $v(t) \leftrightarrow$ mechanical velocity $\dot{x}(t)$.

According to these conventions, the following analogies hold:

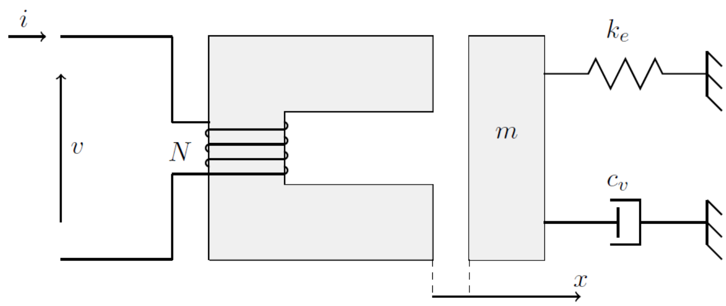


Figure 1: Electro-mechanical circuit.

The mechanical part of the system is described by the Firestone analogy:

- *through* variable:

Electromagnetic current $i(t) \leftrightarrow$ Mechanical force $f(t)$;

- *across* variable:

Electromagnetic voltage $v(t) \leftrightarrow$ Mechanical velocity $\dot{x}(t)$;

According to these conventions, the following analogies hold:

- mass represented by a capacitor:

$$i(t) = C\frac{dv}{dt} \rightarrow f(t) = m\frac{d\dot{x}}{dt} \Rightarrow C \Leftrightarrow m \tag{1}$$

  the position and velocity of the mass are absolute.

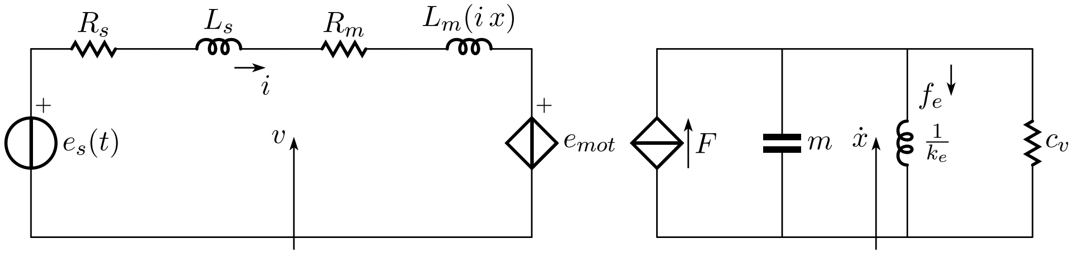
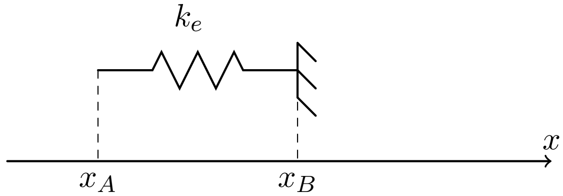


Figure 2: Circuito equivalente elettro-meccanico

- the spring is represented by an inductor:



Sign conventions: To respect the physics that dictates that the sign of a displacement Δ across the ends of a spring should be opposite to the sign of the spring's reaction force, it must be:

$$x = x_A - x_B \Rightarrow \dot{x} = \frac{d(x_A - x_B)}{dt} \tag{2}$$

in other words, x is the difference between the coordinates of the point where the force is applied (x_A) and the anchor point (x_B).

$$v(t) = L\frac{di}{dt} \to i(t) = \frac{1}{L}\int_{-\infty}^{t} v(t')dt' \to f(t) = k_e\left(x(0) + \int_0^t \dot{x}dt'\right) \Rightarrow \frac{1}{L} \Leftrightarrow k_e \tag{3}$$

- viscous friction is represented by a resistor:

$$v(t) = Ri(t) \to f(t) = c_v\dot{x} \Rightarrow R \Leftrightarrow c_v \tag{4}$$

1.1 Electromechanical Conversion Equations

In general, the study of an electromechanical system requires the calculation of:

- $\lambda(i, x)$: the magnetic flux linked to the winding as a function of position x and supply current i. In static conditions, i.e., in the absence of parasitic currents, this characteristic can be calculated through a series of magnetostatic field analyses at different values of i and x. The characteristic may be nonlinear with respect to x due to geometry, and nonlinear with respect to i in the presence of ferromagnetic materials.

- $F(i, x)$: the force exerted on the moving element by the magnetic field. Again, in static conditions, the force characteristic can be calculated through magnetostatic field analysis. In the presence of reluctance forces, the characteristic becomes quadratic with respect to i, while in the presence of permanent magnets, it has a linear dependence on the current. However, saturation of the ferromagnetic material can also modify the force characteristic.

Once the characteristic $\lambda(i, x)$ is determined, the electromagnetic induction terms related to its temporal variation can be evaluated. In the electromechanical case, the term $\frac{d\lambda}{dt}$ depends on two effects:

- *Transformer effect*: modification of λ due to temporal variations of the supply quantities, assuming the system to be static from the mechanical point of view.
- *Motion effect*: modification of λ due to variations in the position of the magnetic system under constant supply conditions.

Specifically, the equation for the inductive terms becomes:

$$\lambda = \lambda(i, x) \rightarrow e_{ind} = \frac{\partial \lambda}{\partial i}\frac{di}{dt} - \frac{\partial \lambda}{\partial x}\frac{dx}{dt} = e_{trasf} + e_{mot} \tag{5}$$

The term e_{trasf} is related to the temporal variation of the current and is represented by the inductive term $L_m(i, x)$. The sign depends on the user's convention for L_m. The term e_{mot} is associated with the movement of the structure and is represented by the controlled voltage generator. Again, the sign depends on the convention adopted for the generator term.

From the *electromechanical circuit* perspective, the controlled generator e_{mot} reflects the effect of velocity $\dot{x}$ on the electrical side. On the other hand, the force action is represented by the controlled current generator with a characteristic given by $F = F(i, x)$. For computational purposes, the force characteristic can be linearized using a first-order Taylor series expansion as:

$$\begin{aligned} F = F(i, x) \simeq F(i_0, x_0) + \left.\tfrac{\partial F(i,x)}{\partial i}\right|_{(i_0,x_0)} (i - i_0) = \\ = \left(F(i_0, x_0) - \left.\tfrac{\partial F(i,x)}{\partial i}\right|_{(i_0,x_0)} i_0\right) + \left.\tfrac{\partial F(i,x)}{\partial i}\right|_{(i_0,x_0)} i = \\ = F_0 + k_F i \end{aligned} \tag{6}$$

1.2 Analytical solution of the cyrcuit

C-shaped magnetic circuit with an armature, and the permeability of the ferromagnetic material, $\mu_{Fe} \rightarrow \infty$. The reluctance of the magnetic circuit is given by:

$$\mathcal{R}(x) = \frac{2x}{\mu_0 S} \tag{7}$$

The magnetic flux is defined as:

$$\Phi = \frac{\mathcal{F}}{\mathcal{R}(x)} = \frac{Ni}{\mathcal{R}(x)} \tag{8}$$

The magnetic flux linkage is given by:

$$\lambda = \lambda(i, x) = N\Phi = \frac{N^2}{\mathcal{R}(x)} i = \frac{N^2 \mu_0 S}{2}\frac{i}{x} = \frac{G}{2}\frac{i}{x} \tag{9}$$

where $G = N^2 \mu_0 S$ depends on the characteristics of the circuit.
The transformative term is:

$$e_{trasf} = \left.\frac{\partial \lambda}{\partial i}\right|_{x=const} \frac{di}{dt} = \frac{G}{2x}\frac{di}{dt} = L_m(x)\frac{di}{dt} \tag{10}$$

In this case, due to the linearity of the material, the inductance L_m does not depend on the current i.

Motional term:

$$\lambda = \lambda(i, x) = \frac{G}{2}\frac{i}{x} \Rightarrow \left.\frac{\partial \lambda}{\partial x}\right|_{i=cost} = -\frac{Gi}{2x^2} \tag{11}$$

Table 1: Reference values for magnetic circuit

Variable	Value	Unit
Gap:	0.0200	m
Section S:	0.0010	m2
Turns N:	1000	
Current I:	3.0000	A
Force F:	-7.0686	N
Resistance Rs:	1.8833	ohm
Voltage Vs:	5.6498	V

$$e_{mot} = \left.\frac{\partial \lambda}{\partial x}\right|_{i=cost} \frac{dx}{dt} = -\frac{Gi}{2x^2}\dot{x} \tag{12}$$

The force can be calculated based on the surface force density across the air gap:

$$f = \frac{1}{2}\frac{B_0^2}{\mu_0}, \qquad [f] = \text{ N/m}^2 \tag{13}$$

where B_0 is the component of magnetic induction normal to the air gap. Expressing B_0 as $B_0 = \frac{\Phi}{S}$, we obtain:

$$F = -\frac{\Phi^2}{\mu_0 S} \tag{14}$$

By expressing the flux Φ in terms of current and reluctance:

$$F(i,x) = -\frac{1}{4}\left(\frac{(Ni)^2\mu_0 S}{x^2}\right) = -\frac{1}{4}G\left(\frac{i}{x}\right)^2 \tag{15}$$

Referring to equation (6), we have:

$$F_0 = \left(F(i_0,x_0) - \left.\frac{\partial F(i,x)}{\partial i}\right|_{(i_0,x_0)} i_0\right) = -\frac{1}{4}G\frac{i_0^2}{x_0^2} - \left(-\frac{1}{2}G\frac{i_0}{x_0^2}\right)i_0 = \frac{1}{4}G\frac{i_0^2}{x_0^2} \tag{16}$$

$$k_f = \left.\frac{\partial F(i,x)}{\partial i}\right|_{(i_0,x_0)} = -\frac{1}{2}G\frac{i_0}{x_0^2} \tag{17}$$

In a time integration cycle of the differential equations, both terms F_0 and k_f need to be recalculated at each time instant.

1.2.1 Numerical example

Considering that the variable x can vary between $x = 10^{-4} \div 4\cdot 10^{-2}$ m, the previous magnetic circuit can assume the specific values for $x = 2\cdot 10^{-2}$ m as shown in Table 1.

Given the force exerted by the electromagnet, we can consider a spring that provides zero force for $x = 2\cdot 10^{-2}$ m and has an elastic constant $k_e = 2000$ N/m.

The mass of the moving element can be assumed to give rise to free oscillations with a frequency of $f = 10$ Hz, thus:

$$\omega^2 = \frac{k_e}{m} = (2\pi 10)^2 \Rightarrow m = \frac{2000}{31.4^2} \simeq 2 \text{ kg} \tag{18}$$

1.3 State-Space Formulation

From the circuit in Figure 2, we can write the following equations. For the electrical side:

$$LKT: \quad e_s(t) - R_s i - L_s \frac{di}{dt} - R_m i - L_m \frac{di}{dt} - e_{mot} = 0 \tag{19}$$

$$(L_s + L_m)\frac{di}{dt} + R_t i + \left(-\frac{\partial \lambda}{\partial x}\dot{x}\right) = e_s(t) \tag{20}$$

where $R_t = R_s + R_m$.

For the mechanical side, considering a shaft that contains all the C elements and the co-shaft equation for the L components, we obtain the following equations:

$$LKC: \quad F(t) - m\frac{d\dot{x}}{dt} - f_e - \frac{1}{c_v}\dot{x} = 0 \tag{21}$$

Using the expression defined in eq. (6), we can rewrite it as:

$$F_0 + k_f i - m\frac{d\dot{x}}{dt} - f_e - \frac{1}{c_v}\dot{x} = 0 \tag{22}$$

$$k_f i - m\frac{d\dot{x}}{dt} - f_e - \frac{1}{c_v}\dot{x} = -F_0 \tag{23}$$

For the co-shaft side:

$$\frac{1}{k_e}\frac{df_e}{dt} - \dot{x} = 0 \tag{24}$$

The above equations can be rewritten as:

$$(L_s + L_m(i,x))\frac{di}{dt} + R_t i + \left(\frac{\partial \lambda}{\partial x}(i,x)\right)\dot{x} = e_s(t) \tag{25}$$

$$m\frac{d\dot{x}}{dt} - k_f(i,x)i + \frac{1}{c_v}\dot{x} - f_e = F_0(i,x) \tag{26}$$

Note that the minus sign in front of the term f_e depends on the convention regarding the polarity of the spring force described in (2).

$$\frac{1}{k_e}\frac{df_e}{dt} + \dot{x} = 0 \tag{27}$$

Assuming the state vector as:

$$\mathbf{X} = \left\{ \begin{array}{c} i \\ \dot{x} \\ f_e \end{array} \right\} \tag{28}$$

the state equations can be written as:

$$[\mathbf{A}]\frac{d\mathbf{X}}{dt} + [\mathbf{B}]\,\mathbf{X} = \mathbf{u} \tag{29}$$

where:

$$[\mathbf{A}] = \begin{bmatrix} L_s + L_m(i,x) & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \frac{1}{k_e} \end{bmatrix} \tag{30}$$

$$[\mathbf{B}] = \begin{bmatrix} R_t & \frac{\partial \lambda}{\partial x}(i,x) & 0 \\ -k_f(i,x) & \frac{1}{c_v} & -1 \\ 0 & +1 & 0 \end{bmatrix} \tag{31}$$

$$[\mathbf{u}] = \begin{Bmatrix} e_s \\ F_0(i,x) \\ 0 \end{Bmatrix} \tag{32}$$

$$\mathbf{A}\frac{d\mathbf{x}}{dt} + \mathbf{B}\mathbf{x} = \mathbf{u} \tag{33}$$

The system is discretized under the following assumptions:

- The time axis is divided into intervals of equal length Δt, and we indicate $\mathbf{x}k = \mathbf{x}(t_k) = \mathbf{x}(k \cdot \Delta t)$.
- The Crank-Nicolson scheme is used.
- The nonlinear coefficients are computed at instant $tk - 1$.

We obtain:

$$[\mathbf{A}_{k-1}] = \begin{bmatrix} L_s + L_m(i_{k-1}, x_{k-1}) & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \frac{1}{k_e} \end{bmatrix} \tag{34}$$

$$[\mathbf{B}_{k-1}] = \begin{bmatrix} R_t & \frac{\partial \lambda}{\partial x}(i_{k-1}, x_{k-1}) & 0 \\ -k_f(i_{k-1}, x_{k-1}) & \frac{1}{c_v} & -1 \\ 0 & +1 & 0 \end{bmatrix} \tag{35}$$

$$\left(\frac{\mathbf{A}_{k-1}}{\Delta t} + \frac{\mathbf{B}_{k-1}}{2}\right)\mathbf{x}_k = \left(\frac{\mathbf{A}_{k-1}}{\Delta t} - \frac{\mathbf{B}_{k-1}}{2}\right)\mathbf{x}_{k-1} + \frac{1}{2}\left(\mathbf{u}_k + \mathbf{u}_{k-1}\right) \tag{36}$$

$$\mathbf{M}_1\mathbf{x}_k = \mathbf{M}_2\mathbf{x}_{k-1} + \frac{1}{2}\left(\mathbf{u}_k + \mathbf{u}_{k-1}\right) \tag{37}$$

where $\mathbf{M}_1 = \left(\frac{\mathbf{A}_{k-1}}{\Delta t} + \frac{\mathbf{B}_{k-1}}{2}\right)$ e $\mathbf{M}_2 = \left(\frac{\mathbf{A}_{k-1}}{\Delta t} - \frac{\mathbf{B}_{k-1}}{2}\right)$.

$$\mathbf{x}_k = \mathbf{M}_1^{-1}\left(\mathbf{M}_2\mathbf{x}_{k-1} + \frac{1}{2}\left(\mathbf{u}_k + \mathbf{u}_{k-1}\right)\right) \tag{38}$$

2 Simulink resolution

The problem has been developed in Simulink. The initial values are defined in Matlab and then the Simulink program is opened:

Listing 1: Variables definition.

```
mu0 =  4*pi*10^(-7); %[H/m] Vacuum magnetic permeability
S = 0.001;           %[m^2] Section of the ferromagnet
x0 = 0.02;           %[m]   Initial displacement
N = 1000;            %[-]   Number of windings
k = 2000;            %[N/m] Elastic constant
c = 0.2;             %[Ns/m] Mechanical damper
m = 2;               %[kg]  Mass
R_m = 1.8833;        %[Ohm] Resistance of the magnetic circuit
R_s = 0;             %[Ohm] Resistance of the power supply
L_s = 0;             %[H]   Inductance of the power supply
V_s = 5.6498;        %[V]   Fem of the power supply
open('test_Sim.slx')
```

The overall Simulink program is displayed in Figure 3. The resulution of the problem is divided in three subsystems:

- the magnetic one, where the quantities given by the magnetic circuit are calculated,
- the electrical one, where the electric circuit is solved,
- the mechanical one, where the mechanical system is solved.

All of them are dependent on to each other since the the input of one subsystem is the output of another subsystem.

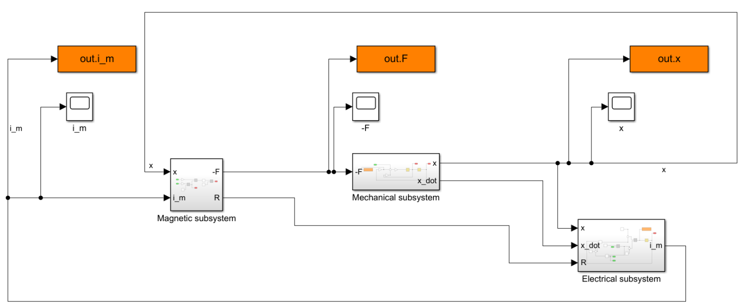


Figure 3: Simulink resolution.

2.1 Magnetic subsystem

In the magnetic subsection (Figure 4), the reluctance of the magnetic circuit and the attractive magnetic force created by the current flowing in the coils:

$$R(x) = \frac{2x}{\mu_0 S} \tag{39}$$

$$F = -\frac{\Phi^2}{\mu_0 S} \tag{40}$$

where

$$\Phi = \frac{Ni}{R(x)} \tag{41}$$

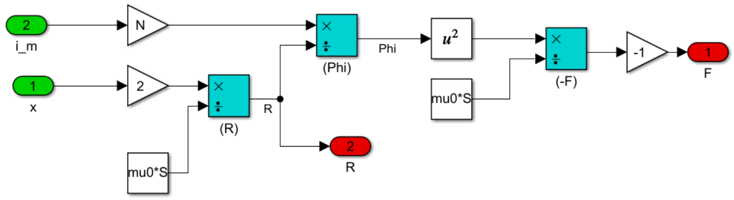


Figure 4: Magnetic subsystem, Simulink resolution.

The reluctance depends on the displacement x, while the force is a function of both the current flowing into the wire i and the the displacement x. This two variables vary in time and are calculated from the electrical and mechanical subsystems.

2.2 Electrical subsystem

In the electrical subsection (Figure 5), the current flowing in the wire is computed by solving the electric circuit equation:

$$e_s - (L_s + L_m)\frac{di}{dt} - (R_m + R_s)i - e_{mot} = 0 \tag{42}$$

where

$$e_{mot} = -\frac{N^2 \mu_0 S i}{2x^2}\dot{x} = -\frac{L_m i}{x}\dot{ix} \tag{43}$$

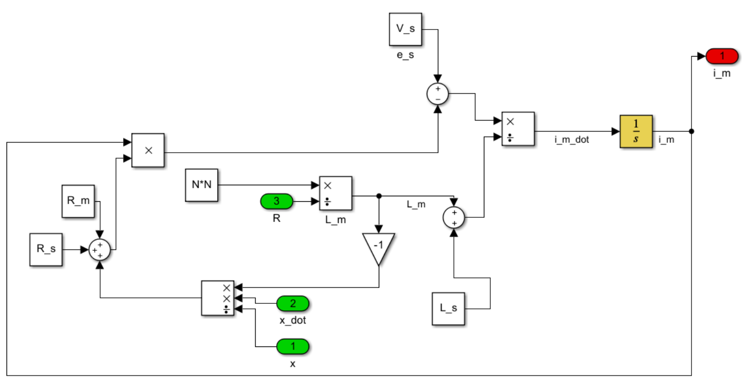


Figure 5: Electric subsystem, Simulink resolution.

2.3 Mechanical subsystem

In the mechanical subsection (Figure 5), the velocity ($\dot{x}$) and the displacement (x) are evaluated by solving the mechanical system equation:

$$F = m\ddot{x} + \frac{1}{c}\dot{x} + F_{spring} \tag{44}$$

where

$$F_{spring} = -k(x - x_0) \tag{45}$$

Indeed, given the definition of x depicted in Figure 1 (versus towards right) and the boundary condition that the spring resting length is at $x_0 = 0.02m$, the spring force contribution is in opposition to the shift Δx. When x increases (towards right), the spring is in compression and the force is negative with respect the x reference system.

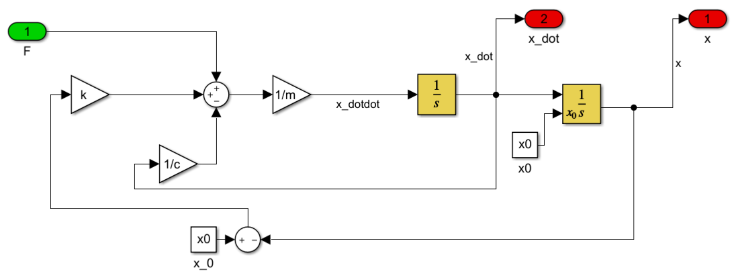
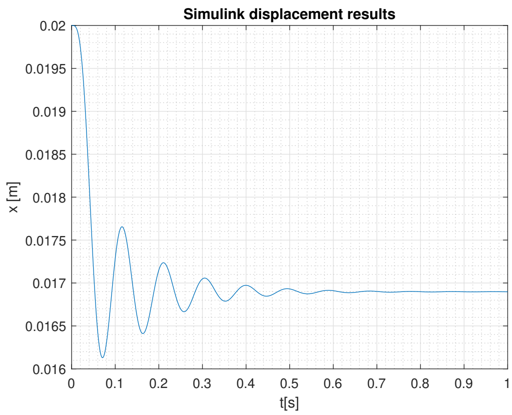


Figure 6: Mechanical subsystem, Simulink resolution.

3 Results

Given the constant f.e.m. of the power supply (Vs), there will be a transitory where the mass will move back and forth till reaching the equilibrium. Since the stiffness of the spring is low, there is also a transitory in the current, before reaching the steady-state value. As visible in Figure 7, the results obtained in Python, with the use of the state-space formulation, and in Simulink are exactly the same. The equilibrium is reached with $x = 0.0169m$ and $i = 5.31A$.



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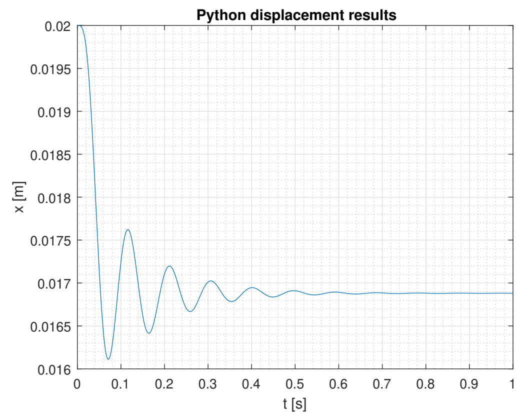

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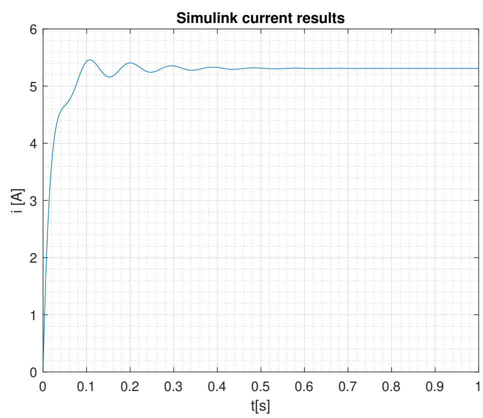

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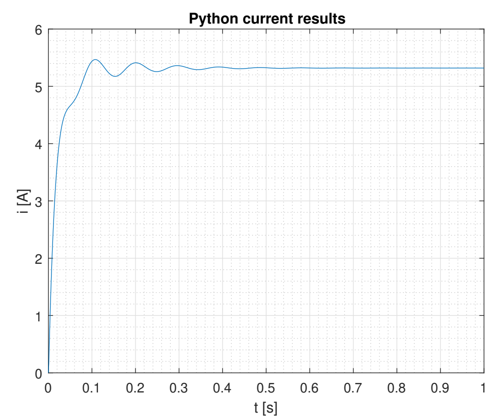

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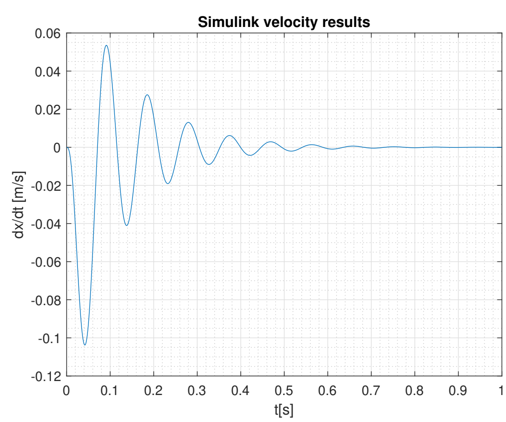
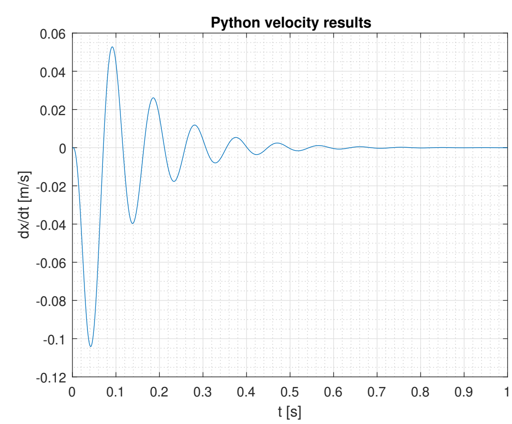


(e)



(f)

Figure 7: Displacement, current and velocity resulting from the simulations in Simulink (left) and Python (right).