

Stat 443: Time Series and Forecasting

Assignment 4: Analysis in the Frequency Domain

Caden Hewlett

March 31, 2024

Question 1

Consider the following second-order AR process AR(2) process for $\{X_t\}_{t \in \mathbb{Z}}$, where $\{Z_t\}_{t \in \mathbb{Z}} \stackrel{\text{iid}}{\sim} \text{WN}(0, \sigma^2)$.

$$X_t = \frac{7}{10}X_{t-1} - \frac{1}{10}X_{t-2} + Z_t$$

We have previously shown that the autocorrelation function $\gamma(h)$ for $h \in \mathbb{Z}$ is given by:

$$\rho(h) = \frac{16}{11} \left(\frac{1}{2}\right)^{|h|} - \frac{5}{11} \left(\frac{1}{5}\right)^{|h|}, \quad h \in \mathbb{Z}$$

Part A

Derive the normalized spectral density function $f^*(\omega)$ for $\{X_t\}_{t \in \mathbb{Z}}$.

Solution

We begin by verifying that the Fourier Transform is well defined.

$$\begin{aligned} \sum_{h=-\infty}^{\infty} |\rho(h)| &= \sum_{h=-\infty}^{\infty} \left| \frac{16}{11} \left(\frac{1}{2}\right)^{|h|} - \frac{5}{11} \left(\frac{1}{5}\right)^{|h|} \right| \stackrel{?}{<} \infty \\ \sum_{h=-\infty}^{\infty} |\rho(h)| &= \left(\frac{16}{11} - \frac{5}{11}\right) + 2 \left(\frac{16}{11} \sum_{h=1}^{\infty} \left(\frac{1}{2}\right)^h - \frac{5}{11} \sum_{h=1}^{\infty} \left(\frac{1}{5}\right)^h \right) \\ \sum_{h=-\infty}^{\infty} |\rho(h)| &= 1 + 2 \left(\frac{16}{11} \left(\frac{1/2}{1-1/2} \right) - \frac{5}{11} \left(\frac{1/5}{1-1/5} \right) \right) \\ \sum_{h=-\infty}^{\infty} |\rho(h)| &= 1 + 2 \left(\frac{16}{11} - \frac{5}{11} \left(\frac{1}{4}\right) \right) = \boxed{\frac{81}{22} < \infty, \therefore \text{well-defined.}} \end{aligned}$$

Now, we evaluate given ρ , recalling that for $\omega \in (0, 1)$ and even functions, the normalized spectral density is given by:

$$f^*(\omega) = \frac{1}{\pi} \left(\rho(0) + 2 \sum_{h=1}^{\infty} \rho(h) \cos(\omega h) \right), \quad \omega \in (0, 1)$$

Where $\rho(0) = 1$.

We will evaluate the infinite sum and substitute the result into the equation above. We will re-instate coefficients A_1 and A_2 from the previous assignment during intermediate steps for simplicity. In addition, we will let $d_1 = 1/2$ and $d_2 = 1/5$, noting that the geometric series equation is usable here as $|d_1|$ and $|d_2|$ are both less than 1.

$$\begin{aligned}\sum_{h=1}^{\infty} \rho(h) \cos(\omega h) &= \sum_{h=1}^{\infty} \left(\frac{16}{11} \left(\frac{1}{2} \right)^{|h|} - \frac{5}{11} \left(\frac{1}{5} \right)^{|h|} \right) \cos(\omega h) \\ \sum_{h=1}^{\infty} \rho(h) \cos(\omega h) &= \sum_{h=1}^{\infty} \left(A_1(d_1)^{|h|} - A_2(d_2)^{|h|} \right) \cos(\omega h), \quad \text{using variable form.} \\ \sum_{h=1}^{\infty} \rho(h) \cos(\omega h) &= \underbrace{\sum_{h=1}^{\infty} \left(A_1(d_1)^{|h|} \cos(\omega h) \right)}_{\text{Term 1}} - \underbrace{\sum_{h=1}^{\infty} \left(A_2(d_2)^{|h|} \cos(\omega h) \right)}_{\text{Term 2}}\end{aligned}$$

We will evaluate Term 1 and Term 2 separately. We will use the following identities without proof:

$$\cos(\omega h) = \frac{1}{2} \left(e^{i h \omega} + e^{-i h \omega} \right), \quad i = \sqrt{-1} \quad (1)$$

$$\sum_{n=1}^{\infty} a \cdot r^n = \frac{ar}{(1-r)}, \quad |r| < 1, \quad a \in \mathbb{R} \quad (2)$$

Evaluating Term 1, noting that $|h| = h$ since the summation spans $h \in \mathbb{Z}^+$.

$$\begin{aligned}\sum_{h=1}^{\infty} \left(A_1(d_1)^{|h|} \cos(\omega h) \right) &= A_1 \sum_{h=1}^{\infty} (d_1)^{|h|} \left(\frac{1}{2} \left(e^{i h \omega} + e^{-i h \omega} \right) \right), \quad \text{by (1)} \\ \sum_{h=1}^{\infty} \left(A_1(d_1)^{|h|} \cos(\omega h) \right) &= \frac{A_1}{2} \sum_{h=1}^{\infty} (d_1)^h \left(e^{i h \omega} + e^{-i h \omega} \right) \\ \sum_{h=1}^{\infty} \left(A_1(d_1)^{|h|} \cos(\omega h) \right) &= \frac{A_1}{2} \left(\sum_{h=1}^{\infty} (d_1)^h e^{i h \omega} + \sum_{h=1}^{\infty} (d_1)^h e^{-i h \omega} \right) \\ \sum_{h=1}^{\infty} \left(A_1(d_1)^{|h|} \cos(\omega h) \right) &= \frac{A_1}{2} \left(\sum_{h=1}^{\infty} \left(d_1 e^{i \omega} \right)^h + \sum_{h=1}^{\infty} \left(d_1 e^{-i \omega} \right)^h \right) \\ \sum_{h=1}^{\infty} \left(A_1(d_1)^{|h|} \cos(\omega h) \right) &= \frac{A_1}{2} \left(\frac{d_1 e^{i \omega}}{1 - d_1 e^{i \omega}} + \frac{d_1 e^{-i \omega}}{1 - d_1 e^{-i \omega}} \right), \quad \text{by (2)} \\ \sum_{h=1}^{\infty} \left(A_1(d_1)^{|h|} \cos(\omega h) \right) &= \frac{A_1}{2} \left(\frac{d_1 e^{i \omega} (1 - d_1 e^{-i \omega}) + d_1 e^{-i \omega} (1 - d_1 e^{i \omega})}{(1 - d_1 e^{i \omega})(1 - d_1 e^{-i \omega})} \right) \\ \sum_{h=1}^{\infty} \left(A_1(d_1)^{|h|} \cos(\omega h) \right) &= \frac{A_1}{2} \left(\frac{d_1 (e^{i \omega} + e^{-i \omega}) - 2d_1^2}{1 - d_1 (e^{i \omega} + e^{-i \omega}) + d_1^2} \right) \\ \sum_{h=1}^{\infty} \left(A_1(d_1)^{|h|} \cos(\omega h) \right) &= \frac{A_1}{2} \left(\frac{2d_1 \cos(\omega) - 2d_1^2}{1 - 2d_1 \cos(\omega) + d_1^2} \right) \\ \sum_{h=1}^{\infty} \left(A_1(d_1)^{|h|} \cos(\omega h) \right) &= \frac{A_1(d_1 \cos(\omega) - d_1^2)}{1 - 2d_1 \cos(\omega) + d_1^2}\end{aligned}$$

Similarly, if we repeat this exact same process with A_2 and d_2 , noting that $|d_2| < 1$ and $A_2 \in \mathbb{R}$ also satisfy the requirements of (1) and (2), we arrive at Term 2:

$$\sum_{h=1}^{\infty} \left(A_2(d_2)^{|h|} \cos(\omega h) \right) = \frac{A_2(d_2 \cos(\omega) - d_2^2)}{1 - 2d_2 \cos(\omega) + d_2^2}$$

Then, we can recombine these into our original expression for the infinite sum:

$$\begin{aligned}
\sum_{h=1}^{\infty} \rho(h) \cos(\omega h) &= \sum_{h=1}^{\infty} \left(A_1(d_1)^{|h|} \cos(\omega h) \right) - \sum_{h=1}^{\infty} \left(A_2(d_2)^{|h|} \cos(\omega h) \right) \\
\sum_{h=1}^{\infty} \rho(h) \cos(\omega h) &= \frac{A_1(d_1 \cos(\omega) - d_1^2)}{1 - 2d_1 \cos(\omega) + d_1^2} - \frac{A_2(d_2 \cos(\omega) - d_2^2)}{1 - 2d_2 \cos(\omega) + d_2^2} \\
\sum_{h=1}^{\infty} \rho(h) \cos(\omega h) &= \left(\frac{16}{11} \right) \frac{\left(\frac{1}{2} \cos(\omega) - \left(\frac{1}{2} \right)^2 \right)}{1 - 2\left(\frac{1}{2} \right) \cos(\omega) + \left(\frac{1}{2} \right)^2} - \left(\frac{5}{11} \right) \frac{\left(\frac{1}{5} \cos(\omega) - \left(\frac{1}{5} \right)^2 \right)}{1 - 2\left(\frac{1}{5} \right) \cos(\omega) + \left(\frac{1}{5} \right)^2} \\
\sum_{h=1}^{\infty} \rho(h) \cos(\omega h) &= \left(\frac{16}{11} \right) \frac{2 \cos(\omega) - 1}{5 - 4 \cos(\omega)} - \left(\frac{5}{11} \right) \frac{5 \cos(\omega) - 1}{2(13 - 5 \cos(\omega))}
\end{aligned}$$

Then, combining the expressions we can get the final expression for the normalized spectral density:

$$\begin{aligned}
f^*(\omega) &= \frac{1}{\pi} \left(1 + 2 \sum_{h=1}^{\infty} \rho(h) \cos(\omega h) \right) \\
f^*(\omega) &= \frac{1}{\pi} \left(1 + 2 \left(\left(\frac{16}{11} \right) \frac{2 \cos(\omega) - 1}{5 - 4 \cos(\omega)} - \left(\frac{5}{11} \right) \frac{5 \cos(\omega) - 1}{2(13 - 5 \cos(\omega))} \right) \right) \\
f^*(\omega) &= \boxed{\frac{1}{\pi} + \frac{32}{11\pi} \left(\frac{2 \cos(\omega) - 1}{5 - 4 \cos(\omega)} \right) - \frac{5}{11\pi} \left(\frac{5 \cos(\omega) - 1}{13 - 5 \cos(\omega)} \right)}
\end{aligned}$$

We can verify these results below:

```

w = pi/4
# define acf
rho <- function(h){ (16/11)*((1/2)^h) - (5/11)*((1/5)^h) }
# define the "infinite sum"
sum_vals = sum(sapply(1:1000, function(h){
  rho(h)*cos(w*h)
}))
# from the equation...
eqnval = (1/pi)*(rho(0) + 2*sum_vals)

# from our simplification
fw1 = 1/pi
fw2 = (32/(11*pi)) * ( (2 * cos(w) - 1) / (5 - 4*cos(w)) )
fw3 = (5/(11*pi)) * ( (5 * cos(w) - 1) / (13 - 5*cos(w)) )
# comparison
c(eqnval, fw1 + fw2 - fw3)

```

```
## [1] 0.4561755 0.4561755
```

We see that the values are identical for at least the first 10,000 lags at fixed $\omega = \pi/4$.

Part B

Write down the power spectral density function of $\{X_t\}_{t \in \mathbb{Z}}$.

Solution

We recall from the definition of normalized spectral density that

$$f^*(\omega) = \frac{f(\omega)}{\sigma_X^2}$$

Where σ_X^2 is the variance of $\{X_t\}_{t \in \mathbb{Z}}$.

Directly, then, we can write $f(\omega)$ as:

$$f(\omega) = \sigma_X^2 f^*(\omega) = \gamma(0) f^*(\omega)$$

We re-establish the Yule-Walker equations, where $\alpha_1 = 7/10$ and $\alpha_2 = -1/10$

$$\begin{aligned}\mathbb{E}(X_t X_t) &= \alpha_1 \mathbb{E}(X_t X_{t-1}) - \alpha_2 \mathbb{E}(X_t X_{t-2}) + \mathbb{E}(X_t Z_t) \\ \mathbb{E}(X_t X_{t-1}) &= \alpha_1 \mathbb{E}(X_{t-1} X_{t-1}) - \alpha_2 \mathbb{E}(X_{t-1} X_{t-2}) + \mathbb{E}(Z_t X_{t-1}) \\ \mathbb{E}(X_t X_{t-2}) &= \alpha_1 \mathbb{E}(X_{t-1} X_{t-2}) - \alpha_2 \mathbb{E}(X_{t-2} X_{t-2}) + \mathbb{E}(Z_t X_{t-2})\end{aligned}$$

Which becomes the following system of three equations:

$$\begin{aligned}\gamma(0) &= \alpha_1 \gamma(1) + \alpha_2 \gamma(2) + \sigma^2 \\ \gamma(1) &= \alpha_1 \gamma(0) + \alpha_2 \gamma(1) \\ \gamma(2) &= \alpha_1 \gamma(1) + \alpha_2 \gamma(0)\end{aligned}$$

test area:

$$\begin{aligned}\gamma(1) &= \alpha_1 \gamma(0) + \alpha_2 \gamma(1) \\ \gamma(1) &= \frac{\alpha_1}{1 - \alpha_2} \gamma(0)\end{aligned}$$

then

$$\begin{aligned}\gamma(2) &= \alpha_1 \gamma(1) + \alpha_2 \gamma(0) \\ \gamma(2) &= \alpha_1 \left(\frac{\alpha_1}{1 - \alpha_2} \gamma(0) \right) + \alpha_2 \gamma(0) \\ \gamma(2) &= \left(\frac{\alpha_1^2 + \alpha_2 - \alpha_2^2}{1 - \alpha_2} \right) \gamma(0)\end{aligned}$$

finally

$$\begin{aligned}\gamma(0) &= \alpha_1 \gamma(1) + \alpha_2 \gamma(2) + \sigma^2 \\ \gamma(0) &= \alpha_1 \left(\frac{\alpha_1}{1 - \alpha_2} \right) \gamma(0) + \alpha_2 \left(\frac{\alpha_1^2 + \alpha_2 - \alpha_2^2}{1 - \alpha_2} \right) \gamma(0) + \sigma^2 \\ \gamma(0) &= \left(\frac{\alpha_1^2 + \alpha_1^2 \alpha_2 + \alpha_2^2 - \alpha_2^3}{1 - \alpha_2} \right) \gamma(0) + \sigma^2 \\ \gamma(0) \left(1 - \frac{\alpha_1^2 + \alpha_1^2 \alpha_2 + \alpha_2^2 - \alpha_2^3}{1 - \alpha_2} \right) &= \sigma^2 \\ \gamma(0) \left(\frac{(1 - \alpha_2) - \alpha_1^2 - \alpha_1^2 \alpha_2 - \alpha_2^2 + \alpha_2^3}{1 - \alpha_2} \right) &= \sigma^2 \\ \gamma(0) &= \frac{\sigma^2 (1 - \alpha_2)}{1 - \alpha_2 - \alpha_1^2 - \alpha_1^2 \alpha_2 - \alpha_2^2 + \alpha_2^3}\end{aligned}$$

Then, we can evaluate at our given α_1 and α_2 , simplifying the fraction using Python to avoid human error.

```

from sympy import *

alpha_1, alpha_2, sigma_sq = symbols('alpha_1 alpha_2 sigma_sq')

gamma_0_expr = ( sigma_sq * (1 - alpha_2) /
    (1 - alpha_2 - alpha_1**2 - alpha_1**2 * alpha_2 - alpha_2**2 + alpha_2**3))

alpha_1_val = Rational(7, 10)
alpha_2_val = Rational(-1, 10)

gamma_0_eval = gamma_0_expr.subs({alpha_1: alpha_1_val, alpha_2: alpha_2_val})

gamma_0_simplified = nsimplify(gamma_0_eval)
print(gamma_0_simplified)

```

```
## 275*sigma_sq/162
```

Then, we can verify both our results and Python's simplification as follows, assuming $\sigma^2 = 1$.

```

set.seed(3)
alpha_1 <- 7/10
alpha_2 <- -1/10
sigma_sq <- 1

gamma_0 <- (sigma_sq * (1 - alpha_2) /
    (1 - alpha_2 - alpha_1^2 - alpha_1^2 * alpha_2 - alpha_2^2 + alpha_2^3))

ar_params <- c(alpha_1, alpha_2)
simulated_data <- arima.sim(n = 500000,
    model = list(ar = c(7/10, -1/10)))

simulated_gamma_0 <- var(simulated_data)
# use both regular and simplified fraction version
cat("Computed Variance Using Simplified Fraction:", 275/162, "\n")

```

```
## Computed Variance Using Simplified Fraction: 1.697531
```

```
cat("Computed Variance Using Computation:", gamma_0, "\n")
```

```
## Computed Variance Using Computation: 1.697531
```

```
cat("Variance of Simulated AR(2) process:", simulated_gamma_0, "\n")
```

```
## Variance of Simulated AR(2) process: 1.697897
```

It seems the computation very closely approximates the truth. Hence, we conclude that:

$$f(\omega) = \gamma(0)f^*(\omega) = \frac{275\sigma^2}{162} \left(\frac{1}{\pi} + \frac{32}{11\pi} \left(\frac{2\cos(\omega) - 1}{5 - 4\cos(\omega)} \right) - \frac{5}{11\pi} \left(\frac{5\cos(\omega) - 1}{13 - 5\cos(\omega)} \right) \right)$$

Part C

Plot the normalized spectral density and comment on its behaviour.

The normalized spectral density equation ended up being quite long, so we'll define each term one-by-one in the function below:

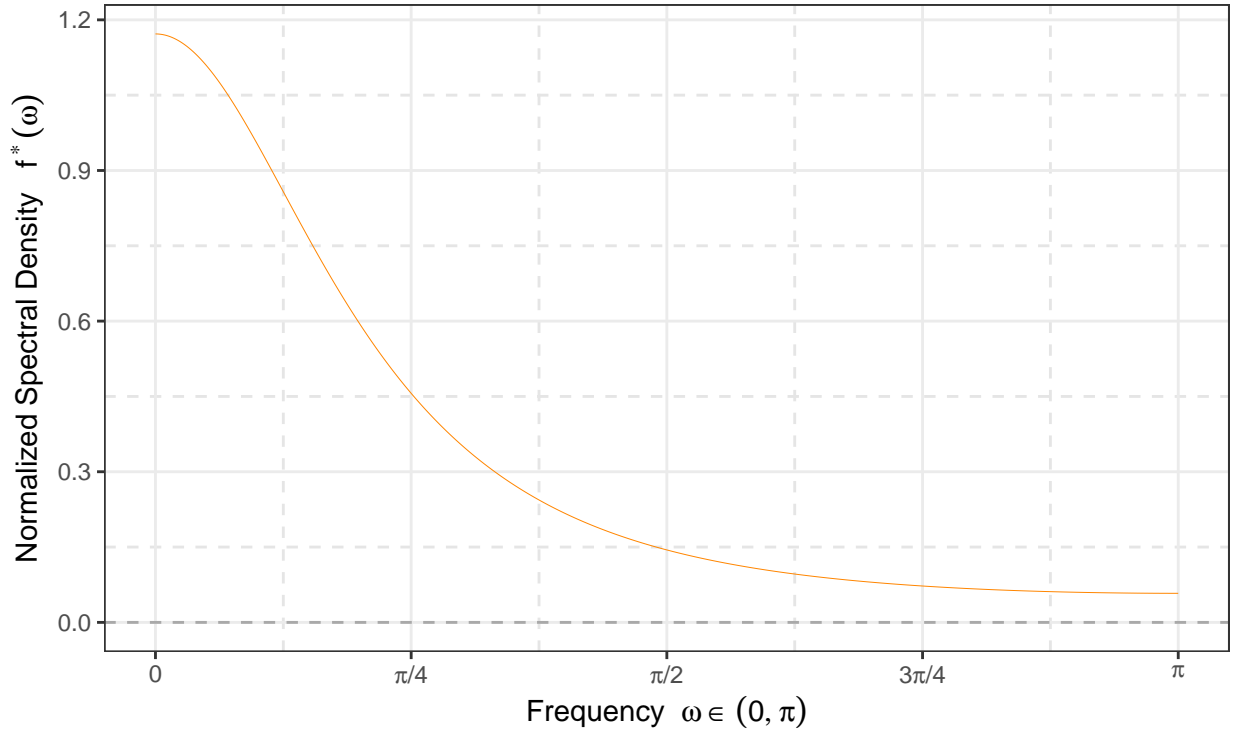
```
f = function(w){
  term_1 = 1
  term_2 = (32/11)*((2*cos(w) - 1) / (5 - 4*cos(w)))
  term_3 = (5 / 11)*((5*cos(w) - 1) / (13 - 5*cos(w)))
  return ( (1/pi)*(term_1 + term_2 - term_3) )
}
```

Then, we plot the function for a long sequence of ω values in $(0, \pi)$:

```
omega = seq(from = 0, to = pi, length.out = 1e4)
# define data frame for values
p1df = data.frame(omega = omega,
  f_star_omega = f(omega))
# build plot
p1 <- ggplot(p1df, aes(x = omega, y = f_star_omega)) +
  geom_line(color = "#ff8600", linewidth = 0.1) +
  labs(
    title = "Normalized Spectral Density for AR(2) Process",
    subtitle = TeX(paste(
      "$X_t = \alpha_1 X_{t-1} + \alpha_2 X_{t-2} + Z_t$",
      "where",
      "$\{Z_t\}_{t \in \mathbb{Z}} \sim \text{WN}(0, \sigma^2)$",
      "and $\alpha_1 = 7/10, \alpha_2 = -1/10$")),
    y = TeX("Normalized Spectral Density $f^*(\omega)$"),
    x = TeX("Frequency $\omega \in (0, \pi)$")) +
  theme_bw() +
  geom_hline(yintercept = 0, lty = 'dashed', col = "darkgrey") +
  scale_x_continuous(
    breaks = c(0, pi/4, pi/2, 3*pi/4, pi),
    labels = c(TeX("0"), TeX("$\pi/4$"), TeX("$\pi/2$"),
      TeX("$3\pi/4$"), TeX("$\pi$")))+
  theme(panel.grid.minor = element_line(
    color = "grey90",
    linetype = "dashed",
    linewidth = 0.5
  ))
print(p1)
```

Normalized Spectral Density for AR(2) Process

$X_t = \alpha_1 X_{t-1} + \alpha_2 X_{t-2} + Z_t$ where $\{Z_t\}_{t \in \mathbb{Z}} \sim \text{WN}(0, \sigma^2)$ and $\alpha_1 = 7/10$, $\alpha_2 = -1/10$



Comments: It appears that the normalized spectral density plot is largely dominated by low frequencies. We can tell this is the case due to the fact that the largest values of $f^*(\omega)$ is for small $\omega \in (0, \pi)$ specifically for $\omega \in (0, \pi/4)$ we see the majority of frequencies with observed normalized density greater than 0.5. This tells us that a greater proportion of the variance inherent in the process X_t can be attributed to the lower frequencies. As we would expect of a normalized spectral density dominated by low ω values, we see that $f^*(\omega)$ is strictly decreasing as $\omega \rightarrow \pi$.

Question 2

Given the spectral density function

$$f(\omega) = \frac{1}{\pi} (62 - 70 \cos(\omega) + 12 \cos(2\omega)), \quad \omega \in (0, 1)$$

compute the autocovariance function $\gamma(h)$ and autocorrelation function $\rho(h)$ of the underlying stochastic process, where $h \in \mathbb{Z}$.

We will begin with computing the autocovariance function $\gamma(h)$ at lag $h = 0$, then at $h = 1$, $k = 2$ and finally $h > 2$.

Further, for notational simplicity, we let $a = 62$, $b = 70$ and $c = 12$, meaning that:

$$f(\omega) = \frac{1}{\pi} (a - b \cos(\omega) + c \cos(2\omega)), \quad \omega \in (0, 1)$$

Part A: Lag 0

For $h = 0$, we have the following:

$$\gamma(0) = \int_0^\pi f(\omega) \cos(h\omega) d\omega = \int_0^\pi \left[\frac{1}{\pi} (a - b \cos(\omega) + c \cos(2\omega)) \right] \cos(0 \times \omega) d\omega$$

Let's evaluate this integral to find the autocovariance at lag zero.

$$\begin{aligned} \gamma(0) &= \int_0^\pi \left[\frac{1}{\pi} (a - b \cos(\omega) + c \cos(2\omega)) \right] \cos(0) d\omega \\ \gamma(0) &= \frac{a}{\pi} \int_0^\pi d\omega - \frac{b}{\pi} \int_0^\pi \cos(\omega) d\omega + \frac{c}{\pi} \int_0^\pi \cos(2\omega) d\omega \\ \gamma(0) &= \frac{a}{\pi} \left(\omega \Big|_0^\pi \right) - \frac{b}{\pi} \left(\sin(\omega) \Big|_0^\pi \right) + \frac{c}{\pi} \left(\frac{1}{2} \sin(2\omega) \Big|_0^\pi \right) \\ \gamma(0) &= \frac{a}{\pi} (\pi - 0) - \frac{b}{\pi} (\sin(\pi) - \sin(0)) + \frac{c}{2\pi} (\sin(2\pi) - \sin(0)) \\ \gamma(0) &= a - \frac{b}{\pi} (0 - 0) + \frac{c}{2\pi} (0 - 0) \\ \gamma(0) &= a = \boxed{62} \end{aligned}$$

Part B: Lag 1

We repeat these calculations, but now for $h = 1$. We retain $a = 62$, $b = 70$ and $c = 12$.

For $h = 1$, we have the following:

$$\gamma(1) = \int_0^\pi f(\omega) \cos(h\omega) d\omega = \int_0^\pi \left[\frac{1}{\pi} (a - b \cos(\omega) + c \cos(2\omega)) \right] \cos(1 \times \omega) d\omega$$

As before, we expand and evaluate this integral.

We will use the following trigonometric identities without proof:

$$\cos^2(\theta) = 1 - \sin^2(\theta) \tag{3}$$

$$\sin^2(\theta) = \frac{1}{2} (1 - \cos(2\theta)) \tag{4}$$

$$\begin{aligned}
\gamma(1) &= \int_0^\pi \left[\frac{1}{\pi} \left(a - b \cos(\omega) + c \cos(2\omega) \right) \right] \cos(\omega) d\omega \\
\gamma(1) &= \frac{a}{\pi} \int_0^\pi \cos(\omega) d\omega - \frac{b}{\pi} \int_0^\pi \cos(\omega) \cos(\omega) d\omega + \frac{c}{\pi} \int_0^\pi \cos(2\omega) \cos(\omega) d\omega \\
\gamma(1) &= \frac{a}{\pi} \left(\sin(\omega) \Big|_0^\pi \right) - \frac{b}{\pi} \int_0^\pi \cos^2(\omega) d\omega + \frac{c}{\pi} \int_0^\pi \cos(\omega) \cos(2\omega) d\omega \\
\gamma(1) &= 0 - \frac{b}{\pi} \int_0^\pi (1 - \sin^2(\omega)) d\omega + \frac{c}{\pi} \int_0^\pi \cos(\omega) \cos(2\omega) d\omega \quad \text{applying (3)} \\
\gamma(1) &= -\frac{b}{\pi} \int_0^\pi d\omega + \frac{b}{\pi} \int_0^\pi \sin^2(\omega) d\omega + \frac{c}{\pi} \int_0^\pi \cos(\omega) \cos(2\omega) d\omega \\
\gamma(1) &= -b + \frac{b}{\pi} \int_0^\pi \frac{1}{2} (1 - \cos(2\theta)) d\omega + \frac{c}{\pi} \int_0^\pi \cos(\omega) \cos(2\omega) d\omega \quad \text{applying (4)} \\
\gamma(1) &= -b + \frac{b}{\pi} \int_0^\pi \frac{1}{2} d\omega - \frac{b}{2\pi} \underbrace{\int_0^\pi \cos(2\theta) d\omega}_{\text{shown to be 0 in Part A}} + \frac{c}{\pi} \int_0^\pi \cos(\omega) \cos(2\omega) d\omega \\
\gamma(1) &= -b + \frac{b}{\pi} \int_0^\pi \frac{1}{2} d\omega + \frac{c}{\pi} \int_0^\pi \cos(\omega) \cos(2\omega) d\omega \\
\gamma(1) &= -b + \frac{b}{2} + \frac{c}{\pi} \int_0^\pi \cos(\omega) \cos(2\omega) d\omega
\end{aligned}$$

We apply a manipulation of equation (4) to evaluate the final integral:

$$\begin{aligned}
\gamma(1) &= -\frac{b}{2} + \frac{c}{\pi} \int_0^\pi \cos(\omega) \cos(2\omega) d\omega \\
\gamma(1) &= -\frac{b}{2} + \frac{c}{\pi} \int_0^\pi \cos(\omega) (1 - 2\sin^2(\omega)) d\omega \\
\gamma(1) &= -\frac{b}{2} + \frac{c}{\pi} \int_0^\pi \cos(\omega) d\omega - \frac{2c}{\pi} \int_0^\pi \cos(\omega) \sin^2(\omega) d\omega \\
\gamma(1) &= -\frac{b}{2} + \frac{c}{\pi} \left(\sin(\omega) \Big|_0^\pi \right) - \frac{2c}{\pi} \underbrace{\int_0^\pi \cos(\omega) \sin^2(\omega) d\omega}_{\text{let } q = \sin(\omega)} \\
\gamma(1) &= -\frac{b}{2} + 0 - \frac{2c}{\pi} \int_0^\pi \cos(\omega) q^2 \left(\frac{1}{\cos(\omega)} \right) dq \\
\gamma(1) &= -\frac{b}{2} - \frac{2c}{3\pi} \left(q^3 \Big|_{\omega=0}^\pi \right) \\
\gamma(1) &= -\frac{b}{2} - \frac{2c}{3\pi} \left(\sin^3(\pi) - \sin^3(0) \right) \\
\gamma(1) &= -\frac{b}{2} = -\frac{70}{2} = \boxed{-35}
\end{aligned}$$

Part C: Lag 2

We repeat these calculations, but now for $h = 2$. We retain $a = 62$, $b = 70$ and $c = 12$.

For $h = 2$, we have the following:

$$\gamma(2) = \int_0^\pi f(\omega) \cos(h\omega) d\omega = \int_0^\pi \left[\frac{1}{\pi} (a - b \cos(\omega) + c \cos(2\omega)) \right] \cos(2 \times \omega) d\omega$$

As before, we expand $f(\omega)$ and evaluate this integral.

$$\begin{aligned} \gamma(2) &= \int_0^\pi \left[\frac{1}{\pi} (a - b \cos(\omega) + c \cos(2\omega)) \right] \cos(2\omega) d\omega \\ \gamma(2) &= \frac{a}{\pi} \int_0^\pi \cos(2\omega) d\omega - \frac{b}{\pi} \underbrace{\int_0^\pi \cos(\omega) \cos(2\omega) d\omega}_{\text{Shown in Part B to be zero.}} + \frac{c}{\pi} \int_0^\pi \cos(2\omega) \cos(2\omega) d\omega \\ \gamma(2) &= \frac{a}{2\pi} \left(\sin(2\omega) \Big|_0^\pi \right) - \frac{b}{\pi} (0) + \frac{c}{\pi} \int_0^\pi \cos^2(2\omega) d\omega, \quad \text{applying (3)} \\ \gamma(2) &= \frac{a}{2\pi} \left(\sin(2\pi) - \sin(0) \right) + \frac{c}{\pi} \int_0^\pi \left(1 - \sin^2(2\omega) \right) d\omega \\ \gamma(2) &= \frac{a}{2\pi} (0) + \frac{c}{\pi} \int_0^\pi 1 d\omega - \frac{c}{\pi} \int_0^\pi \sin^2(2\omega) d\omega \\ \gamma(2) &= \frac{c}{\pi} (\pi) - \frac{c}{\pi} \int_0^\pi \frac{1}{2} (1 - \cos(2 \cdot 2\omega)) d\omega, \quad \text{applying (4)} \\ \gamma(2) &= c - \frac{c}{2\pi} \int_0^\pi 1 d\omega + \frac{c}{2\pi} \int_0^\pi \cos(4\omega) d\omega \\ \gamma(2) &= c - \frac{c}{2\pi} (\pi) + \frac{c}{2\pi} \left(\frac{1}{4} \sin(4\omega) \Big|_0^\pi \right) \\ \gamma(2) &= c - \frac{c}{2} + \frac{c}{8\pi} \left(\sin(4\pi) - \sin(0) \right) \\ \gamma(2) &= \frac{c}{2} = \frac{12}{2} = \boxed{6} \end{aligned}$$

Part D: Lags Greater Than 2

Before we evaluate lags $h > 2 \in \mathbb{Z}$, we will establish the following lemma:

Lemma 1: Let $m \in \mathbb{Z}, n \in \mathbb{Z}^+$ We will show that:

$$\forall (m, n) \in \mathbb{Z}^+ \text{ s.t. } m \neq n, \int_0^\pi \cos(m\omega) \cos(n\omega) d\omega = 0$$

Proof: In order to show this equality holds, we will use the following product-to-sum identity of cosines without proof. Let $\theta, \vartheta \in \mathbb{R}$.

$$\cos(\theta) \cos(\vartheta) = \frac{1}{2} \cos(\theta + \vartheta) + \frac{1}{2} \cos(\theta - \vartheta) \quad (2.1)$$

We will use (2.1) to evaluate the integral.

$$\begin{aligned}
\int_0^\pi \cos(m\omega) \cos(n\omega) d\omega &= \int_0^\pi \left(\frac{1}{2} \cos(m\omega + n\omega) + \frac{1}{2} \cos(m\omega - n\omega) \right) d\omega \\
\int_0^\pi \cos(m\omega) \cos(n\omega) d\omega &= \frac{1}{2} \int_0^\pi \underbrace{\cos((m+n)\omega)}_{\text{Let } q=(m+n)\omega} d\omega + \frac{1}{2} \int_0^\pi \underbrace{\cos((m-n)\omega)}_{\text{Let } \nu=(m-n)\omega} d\omega \\
\int_0^\pi \cos(m\omega) \cos(n\omega) d\omega &= \frac{1}{2(m+n)} \int_0^\pi \cos(q) dq + \frac{1}{2(m-n)} \int_0^\pi \cos(\nu) d\nu, \quad \text{requires } m \neq n \\
\int_0^\pi \cos(m\omega) \cos(n\omega) d\omega &= \frac{1}{2(m+n)} \left(\sin(q) \Big|_{\omega=0}^\pi \right) + \frac{1}{2(m-n)} \left(\sin(\nu) \Big|_{\omega=0}^\pi \right) \\
\int_0^\pi \cos(m\omega) \cos(n\omega) d\omega &= \frac{1}{2(m+n)} \left(\sin(m\omega + n\omega) \Big|_0^\pi \right) + \frac{1}{2(m-n)} \left(\sin(m\omega - n\omega) \Big|_0^\pi \right) \\
\int_0^\pi \cos(m\omega) \cos(n\omega) d\omega &= \frac{1}{2(m+n)} \left(\sin((m+n)\pi) - \sin(0) \right) + \frac{1}{2(m-n)} \left(\sin((m-n)\pi) - \sin(0) \right) \\
\int_0^\pi \cos(m\omega) \cos(n\omega) d\omega &= \frac{1}{2(m+n)} \sin((m+n)\pi) + \frac{1}{2(m-n)} \sin((m-n)\pi)
\end{aligned}$$

We note that $m \in \mathbb{Z}$ and $n \in \mathbb{Z}$. Hence, by properties of integers, we know that $m+n \in \mathbb{Z}$ and $m-n \in \mathbb{Z}$.

We can hence note that:

$$(m \in \mathbb{Z} \wedge n \in \mathbb{Z}) \implies \exists \ell \in \mathbb{Z} \text{ s.t. } \ell = m+n$$

Since $(m, n) \in \mathbb{Z}^+$ and $m \neq n$, we know that $m+n > 0$, $\therefore \ell > 0$.

Similarly, we have that:

$$(m \in \mathbb{Z} \wedge n \in \mathbb{Z}) \implies \exists \varrho \in \mathbb{Z} \text{ s.t. } \varrho = m-n$$

Further, since $(m, n) \in \mathbb{Z}^+$ and $m \neq n$, we know that $\varrho \neq 0$. We also note by properties of sine that $\forall k \in \mathbb{Z}, \sin(k \cdot \pi) = 0$.

Finally, we can rewrite our simplified integral in terms of ℓ and ϱ and solve.

$$\begin{aligned}
\int_0^\pi \cos(m\omega) \cos(n\omega) d\omega &= \frac{1}{2\ell} \sin(\ell\pi) + \frac{1}{2\varrho} \sin(\varrho\pi) \\
\int_0^\pi \cos(m\omega) \cos(n\omega) d\omega &= \frac{1}{2\ell}(0) + \frac{1}{2\varrho}(0), \quad \text{since } \forall k \in \mathbb{Z}, \sin(k \cdot \pi) = 0 \\
\int_0^\pi \cos(m\omega) \cos(n\omega) d\omega &= (0) + (0), \quad \text{since } \ell \neq 0 \text{ and } \varrho \neq 0 \\
\int_0^\pi \cos(m\omega) \cos(n\omega) d\omega &= 0
\end{aligned}$$

Hence, we conclude that

$$\forall (m, n) \in \mathbb{Z}^+ \text{ s.t. } m \neq n, \int_0^\pi \cos(m\omega) \cos(n\omega) d\omega = 0, \quad \text{as required. } \square$$

Application of the Lemma

Now, we can show that $\forall h > 2 \subseteq \mathbb{Z}^+$ that:

$$\gamma(h) = \int_0^\pi f(\omega) \cos(h\omega) d\omega = 0$$

Proof: We begin by expanding $f(\omega)$ with our previous variable assignments:

$$\begin{aligned}\int_0^\pi f(\omega) \cos(h\omega) d\omega &= \int_0^\pi \left[\frac{1}{\pi} (a - b \cos(\omega) + c \cos(2\omega)) \right] \cos(h\omega) d\omega \\ \int_0^\pi f(\omega) \cos(h\omega) d\omega &= \frac{a}{\pi} \int_0^\pi \cos(h\omega) d\omega - \frac{b}{\pi} \int_0^\pi \cos(\omega) \cos(h\omega) d\omega + \frac{c}{\pi} \int_0^\pi \cos(2\omega) \cos(h\omega) d\omega \\ \int_0^\pi f(\omega) \cos(h\omega) d\omega &= \frac{a}{\pi} \int_0^\pi \cos(0 \cdot \omega) \cos(h \cdot \omega) d\omega - \frac{b}{\pi} \int_0^\pi \cos(1 \cdot \omega) \cos(h \cdot \omega) d\omega + \frac{c}{\pi} \int_0^\pi \cos(2 \cdot \omega) \cos(h \cdot \omega) d\omega\end{aligned}$$

Now, from the expansion on the third line, we can draw an interesting parallel to the lemma established previously. Let $M = \{0, 1, 2\}$ and $N = \{n \in \mathbb{Z} : n > 2\}$.

We see by construction of these sets $M \cap N = \emptyset$. Importantly,

$$M \cap N = \emptyset \implies \forall n \in N, \forall m \in M, m \neq n$$

Now, if we let $M = \{m_1, m_2, m_3\} = \{0, 1, 2\}$ and $n \in N := \{h \in \mathbb{Z} : h > 2\}$, we can rewrite our integral in terms of m and n and simplify using our lemma.

$$\begin{aligned}\int_0^\pi f(\omega) \cos(h\omega) d\omega &= \frac{a}{\pi} \int_0^\pi \underbrace{\cos(m_1 \cdot \omega) \cos(n \cdot \omega)}_{\forall n \in N, m_1 \neq n} d\omega - \frac{b}{\pi} \int_0^\pi \underbrace{\cos(m_2 \cdot \omega) \cos(n \cdot \omega)}_{\forall n \in N, m_2 \neq n} d\omega + \frac{c}{\pi} \int_0^\pi \underbrace{\cos(m_3 \cdot \omega) \cos(n \cdot \omega)}_{\forall n \in N, m_3 \neq n} d\omega \\ \int_0^\pi f(\omega) \cos(h\omega) d\omega &= \frac{a}{\pi}(0) - \frac{b}{\pi}(0) + \frac{c}{\pi}(0), \text{ by set construction and Lemma 1.} \\ \int_0^\pi f(\omega) \cos(h\omega) d\omega &= \boxed{0}\end{aligned}$$

So, we can conclude that:

$$\forall h > 2, \gamma(h) = \int_0^\pi f(\omega) \cos(h\omega) d\omega = 0, \text{ as required. } \square$$

Conclusions

Hence, bringing all the previous parts together, we can conclude that for $h \in \mathbb{Z}$:

$$\gamma(h) = \begin{cases} \gamma(-h), & h < 0 \\ 62, & h = 0 \\ -35, & h = 1 \\ 6, & h = 2 \\ 0, & \text{otherwise} \end{cases}$$

Similarly, we find $\rho(h)$ for $h \in \mathbb{Z}$ by dividing $\gamma(h)$ by $\sigma_X^2 = \gamma(0) = 62$.

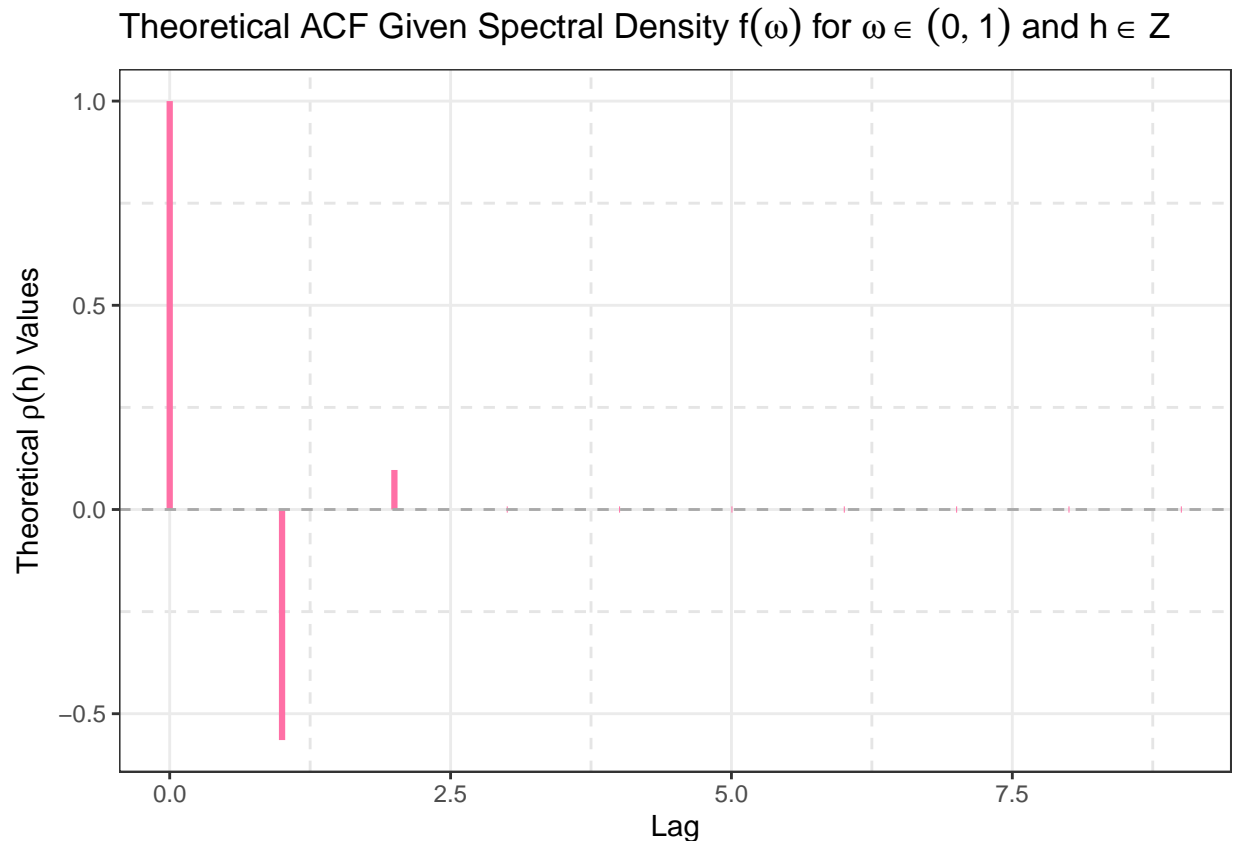
$$\rho(h) = \begin{cases} \rho(-h), & h < 0 \\ 1, & h = 0 \\ -35/62, & h = 1 \\ 3/31, & h = 2 \\ 0, & \text{otherwise} \end{cases}$$

We can also provide a plot of the theoretical ACF to get a more easily interpretable idea of how the autocorrelation of X_t changes for lags $h \in \mathbb{Z}$.

```

p2DF = data.frame(
  h = 0:9,
  rh = c(1, -35/62, 3/31, rep(0, times = 7))
)
p2 <- ggplot(p2DF, aes(x = h, y = rh)) +
  geom_segment(aes(xend = h, yend = 0),
    color = "#FF70A6",
    linewidth = 1.1) +
  geom_hline(yintercept = 0,
    linetype = "dashed",
    color = "darkgray") +
  labs(x = "Lag", y = TeX("Theoretical  $\rho(h)$  Values"),
    title = TeX(paste("Theoretical ACF Given Spectral Density  $f(\omega)$ ",
      "for  $\omega \in (0, 1)$  and  $h \in \mathbb{Z}$ ")) ) +
  theme_bw() +
  theme(panel.grid.minor = element_line(color = "grey90",
    linetype = "dashed", linewidth = 0.5))
print(p2)

```



Question 3

The data file `accel_watch.csv` contains three axes of accelerometer data for a test subject that was walking at a steady pace while wearing a biologging watch. The accelerometer data was recorded in meters per second squared and measured every 0.05 seconds (i.e., at a rate of 20 observations per second)

Part A: Data Preprocessing and Investigation

Part A.1.

Read the data into R.

```
df = read.csv("accel_watch.csv")
```

Part A.2.

Create a vector that contains the magnitude of acceleration at each time index.

We will use the following to define the column:

$$\|\vec{a}\| = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

```
df$Mag = sqrt(df$Ax^2 + df$Ay^2 + df$Az^2)
kable( t( head(df$Mag) ), caption = "Preview of  $\|\vec{a}\|$  Vector")
```

Table 1: Preview of $\|\vec{a}\|$ Vector

17.88641	15.61458	12.2083	12.02874	11.79417	12.63943
----------	----------	---------	----------	----------	----------

Part A.3.

Coerce the vector into a time series object.

We are told that the accelerometer data was recorded in meters per second squared and measured every 0.05 seconds (i.e., at a rate of 20 observations per second.) Hence, we set `frequency = 20`, as `?ts` informs us that the `frequency` parameter is “the number of observations per unit of time.”

```
accel_ts = ts(df$Mag, frequency = 20)
```

Part A.4.

Plot the resulting time series and its sample acf. (Make sure to properly label the axes and provide titles for the plots.)

Time Series Plot

```
# coerce time series into a plottable object
p3DF = fortify.zoo(accel_ts)
# build plot
p3 <- ggplot(p3DF, aes(x = Index, y = accel_ts)) +
  geom_line(color = "#70D6FF", linewidth = 0.8) +
  labs(
    title = TeX(paste("Acceleration Magnitude",
      "$\\| \\textbf{a} \\|$ of Subject over Time, in $(m / s^2)$")),
    subtitle = "Test Subject Walking at Steady Pace, Wearing a Biologging watch.",
    y = TeX("Acceleration Magnitude $\\| \\textbf{a} \\|$ in $(m / s^2)$"),
    x = TeX("Elapsed Time (seconds)"))
```

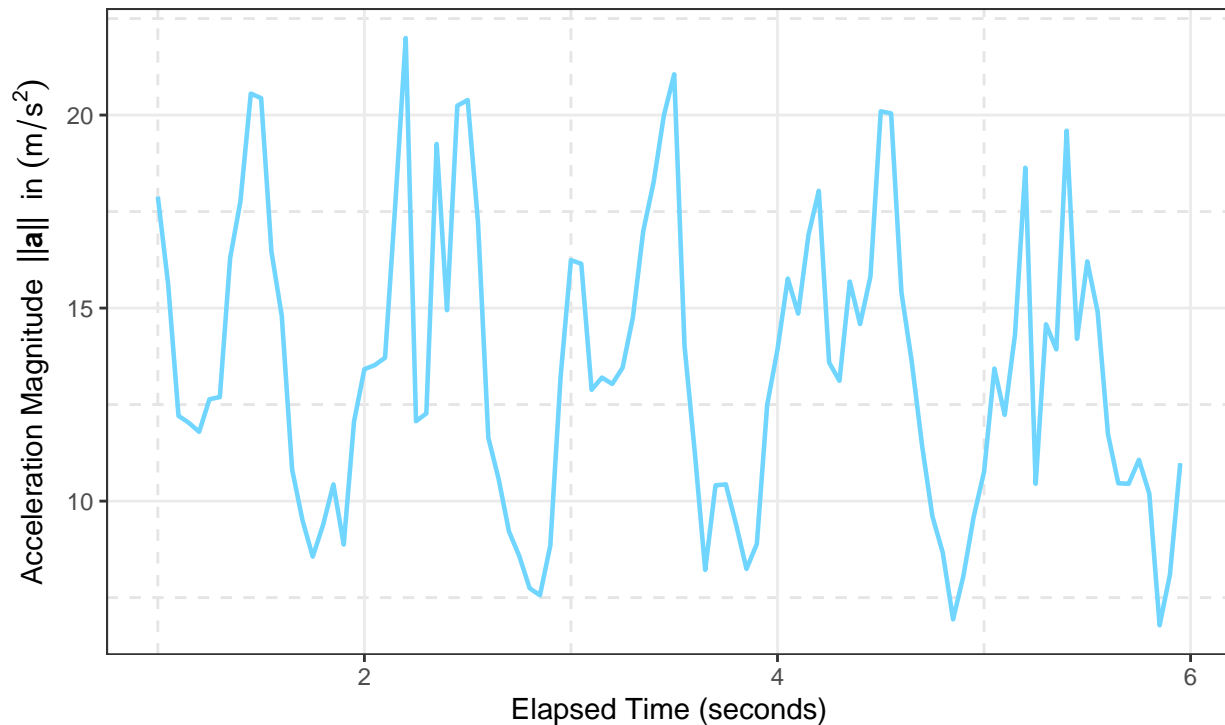
```

) + theme_bw() +
theme(panel.grid.minor = element_line(
  color = "grey90",
  linetype = "dashed",
  linewidth = 0.5
))
# show results
print(p3)

```

Acceleration Magnitude $\|a\|$ of Subject over Time, in (m/s^2)

Test Subject Walking at Steady Pace, Wearing a Biologging watch.



Sample ACF

```

# get sample acf values
p5DF = data.frame(
  h = 0:36,
  rh = acf(accel_ts, plot = FALSE, lag.max = 36)$acf
)
# find n for WN bound calculation
n = length(accel_ts)
# create plot
p5 <- ggplot(p5DF, aes(x = h, y = rh)) +
  geom_hline(yintercept = 2/sqrt(n),
    linetype = "dashed",
    col = "#59054d") +
  geom_hline(yintercept = -2/sqrt(n),
    linetype = "dashed",
    col = "#59054d") +

```

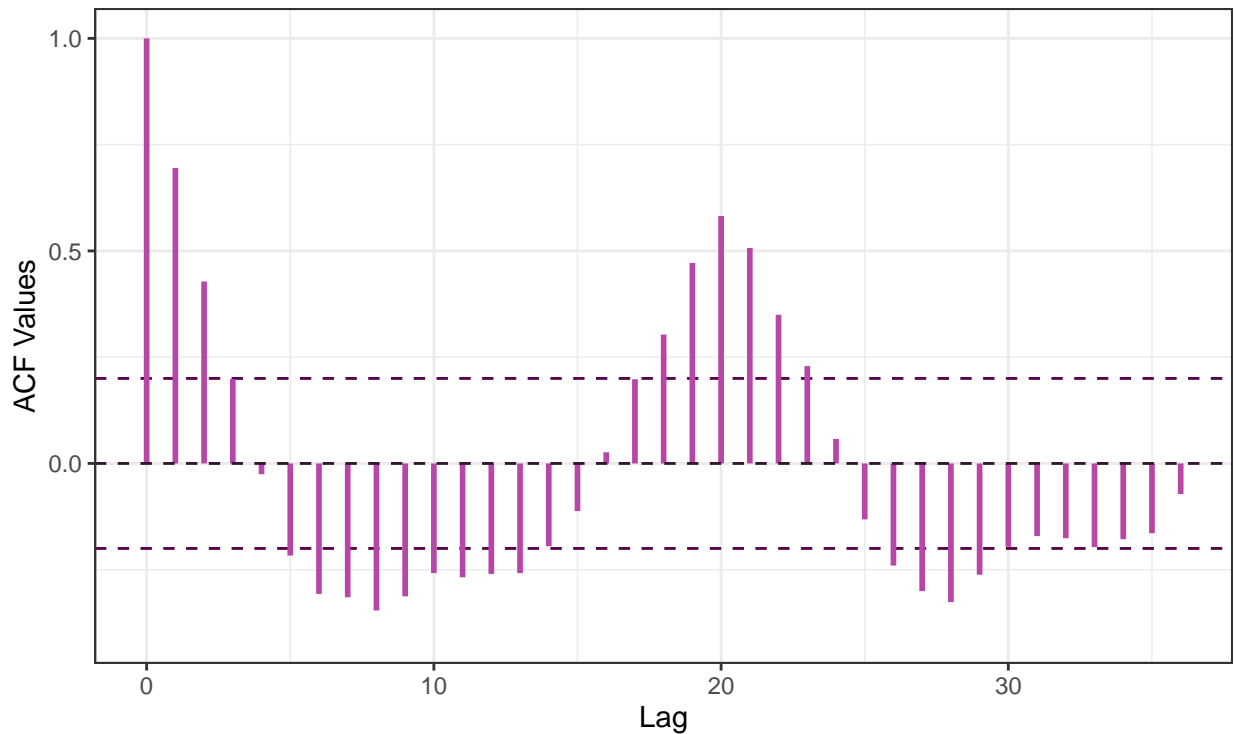
```

ylim(-0.4, 1) +
geom_segment(aes(xend = h, yend = 0),
             color = "#b846a7",
             linewidth = 1) +
geom_hline(yintercept = 0,
           linetype = "dashed",
           color = "#33172f") +
labs(x = "Lag", y = "ACF Values",
     title = TeX(paste("Correlogram of Acceleration Magnitude",
                        "$\\|a\\|$ of Subject")),
     subtitle = paste("Test Subject Walking at Steady Pace for appx.",
                       "Six Seconds, Wearing a Biologging Watch")) +
theme_bw()
print(p5)

```

Correlogram of Acceleration Magnitude $\|a\|$ of Subject

Test Subject Walking at Steady Pace for appx. Six Seconds, Wearing a Biologging Watch



Part A.5.

Comment on what you observe.

Part B

Part B.1.

Plot the raw periodogram for the series.

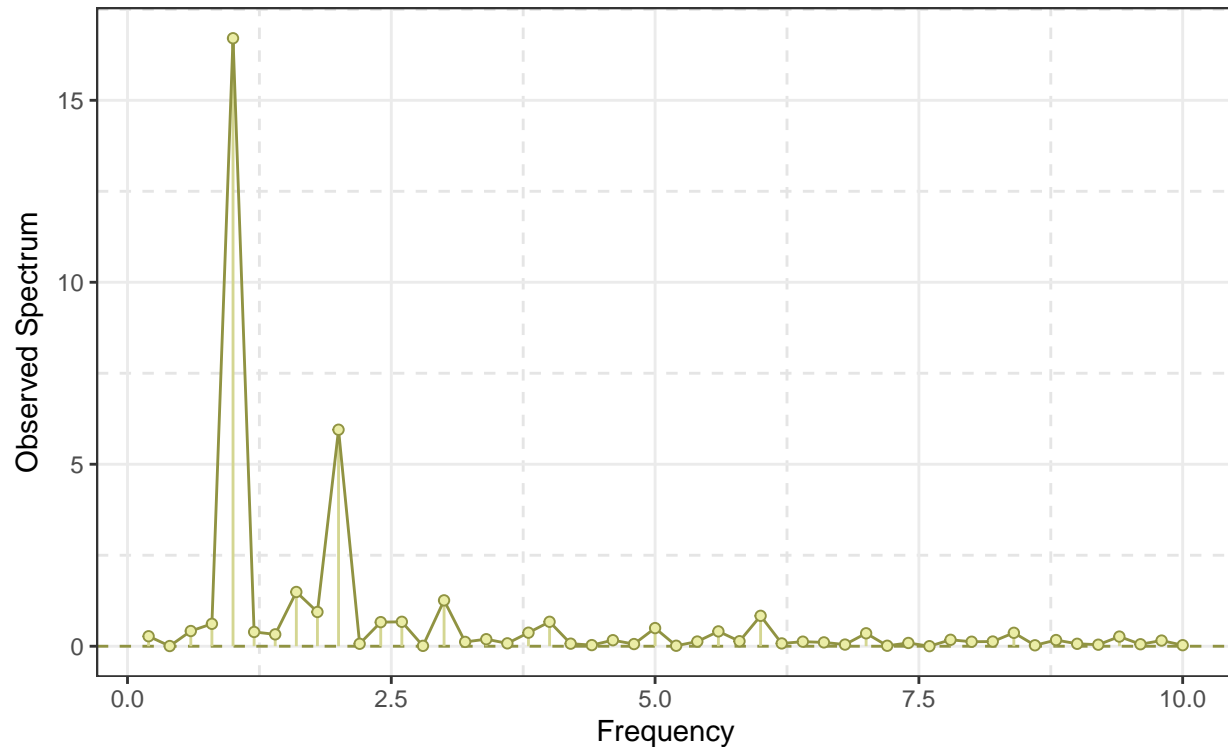

```

# compute raw periodogram
raw_pdg = spec.pgram(accel_ts, log = "no", plot = F)
# coerce to data frame
p6DF = data.frame(
  Frequency = raw_pdg$freq,
  Spectrum = raw_pdg$spec
)
# plot
p6 <- ggplot(p6DF, aes(x = Frequency, y = Spectrum)) +
  geom_hline(yintercept = 0, linetype = "dashed", color = "#909342") +
  geom_segment(aes(
    x = Frequency, xend = Frequency,
    y = 0, yend = Spectrum,
  ), color = "#d4d692") +
  geom_line(color = "#909342", linewidth = 0.5) +
  geom_point(color = "#909342", fill = "#ebeda6", pch = 21) +
  labs(
    title = "Raw Periodogram of Acceleration Magnitude Time Series",
    subtitle = paste(
      "Test Subject Walking at Steady Pace,",
      "Wearing a Biologging Watch for appx. 6s."
    ),
    y = TeX("Observed Spectrum"),
    x = TeX("Frequency")
  ) +
  theme_bw() +
  theme(panel.grid.minor = element_line(
    color = "grey90",
    linetype = "dashed",
    linewidth = 0.5
  ))
print(p6)

```

Raw Periodogram of Acceleration Magnitude Time Series

Test Subject Walking at Steady Pace, Wearing a Biologging Watch for appx. 6s.



Comments: Dominated by low frequencies, two peaks in spectrum magnitude at (...) and (...), tails off into noise as $f \uparrow$, for higher frequencies. Thereby we know a (... larger proportion of the total variation in the series is attributed to the lower frequencies in the series (?))

Part B.2.

```
LOC = which.max(raw_pdg$spec)
```

```
# all relevant data for this maximum
```

```
print(paste("Spectrum at Max:", raw_pdg$spec[LOC]))
```

```
## [1] "Spectrum at Max: 16.7043024601403"
```

```
print(paste("Angular Frequency at Max:", 2*pi*raw_pdg$freq[LOC]))
```

```
## [1] "Angular Frequency at Max: 6.28318530717959"
```

```
print(paste("Acceleration at Max:", accel_ts[LOC]))
```

```
## [1] "Acceleration at Max: 11.7941735976911"
```

```
print(paste("Elapsed Time at Max:", time(accel_ts)[LOC]))
```

```
## [1] "Elapsed Time at Max: 1.2"
```

Angular Frequency: $\omega = (2\pi)/T$ where T is period, and $\omega = 2\pi f$ where f is frequency.