

# Stat 443: Time Series and Forecasting

## Assignment 4: Analysis in the Frequency Domain

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### Question 1

Consider the following second-order AR process AR(2) process for  $\{X_t\}_{t \in \mathbb{Z}}$ , where  $\{Z_t\}_{t \in \mathbb{Z}} \stackrel{\text{iid}}{\sim} \text{WN}(0, \sigma^2)$ .

$$X_t = \frac{7}{10}X_{t-1} - \frac{1}{10}X_{t-2} + Z_t$$

We have previously shown that the autocorrelation function  $\gamma(h)$  for  $h \in \mathbb{Z}$  is given by:

$$\rho(h) = \frac{16}{11} \left(\frac{1}{2}\right)^{|h|} - \frac{5}{11} \left(\frac{1}{5}\right)^{|h|}, \quad h \in \mathbb{Z}$$

### Part A

Derive the normalized spectral density function  $f^*(\omega)$  for  $\{X_t\}_{t \in \mathbb{Z}}$ .

#### Solution

We begin by verifying that the Fourier Transform is well defined.

$$\begin{aligned} \sum_{h=-\infty}^{\infty} |\rho(h)| &= \sum_{h=-\infty}^{\infty} \left| \frac{16}{11} \left(\frac{1}{2}\right)^{|h|} - \frac{5}{11} \left(\frac{1}{5}\right)^{|h|} \right| \stackrel{?}{<} \infty \\ \sum_{t=-\infty}^{\infty} |\rho(h)| &= \left(\frac{16}{11} - \frac{5}{11}\right) + 2 \left( \frac{16}{11} \sum_{h=1}^{\infty} \left(\frac{1}{2}\right)^h - \frac{5}{11} \sum_{h=1}^{\infty} \left(\frac{1}{5}\right)^h \right) \\ \sum_{t=-\infty}^{\infty} |\rho(h)| &= 1 + 2 \left( \frac{16}{11} \left( \frac{1/2}{1-1/2} \right) - \frac{5}{11} \left( \frac{1/5}{1-1/5} \right) \right) \\ \sum_{t=-\infty}^{\infty} |\rho(h)| &= 1 + 2 \left( \frac{16}{11} - \frac{5}{11} \left(\frac{1}{4}\right) \right) = \boxed{\frac{81}{22} < \infty, \therefore \text{well-defined.}} \end{aligned}$$

Now, we evaluate given  $\rho$ , recalling that for  $\omega \in (0, 1)$  and even functions, the normalized spectral density is given by:

$$f^*(\omega) = \frac{1}{\pi} \left( \rho(0) + 2 \sum_{h=1}^{\infty} \rho(h) \cos(\omega h) \right), \quad \omega \in (0, 1)$$

Where  $\rho(0) = 1$ .

We will evaluate the infinite sum and substitute the result into the equation above. We will re-instate coefficients  $A_1$  and  $A_2$  from the previous assignment during intermediate steps for simplicity. In addition, we will let  $d_1 = 1/2$  and  $d_2 = 1/5$ , noting that the geometric series equation is usable here as  $|d_1|$  and  $|d_2|$  are both less than 1.

$$\begin{aligned}\sum_{h=1}^{\infty} \rho(h) \cos(\omega h) &= \sum_{h=1}^{\infty} \left( \frac{16}{11} \left( \frac{1}{2} \right)^{|h|} - \frac{5}{11} \left( \frac{1}{5} \right)^{|h|} \right) \cos(\omega h) \\ \sum_{h=1}^{\infty} \rho(h) \cos(\omega h) &= \sum_{h=1}^{\infty} \left( A_1(d_1)^{|h|} - A_2(d_2)^{|h|} \right) \cos(\omega h), \quad \text{using variable form.} \\ \sum_{h=1}^{\infty} \rho(h) \cos(\omega h) &= \underbrace{\sum_{h=1}^{\infty} \left( A_1(d_1)^{|h|} \cos(\omega h) \right)}_{\text{Term 1}} - \underbrace{\sum_{h=1}^{\infty} \left( A_2(d_2)^{|h|} \cos(\omega h) \right)}_{\text{Term 2}}\end{aligned}$$

We will evaluate Term 1 and Term 2 separately. We will use the following identities without proof:

$$\cos(\omega h) = \frac{1}{2} \left( e^{i h \omega} + e^{-i h \omega} \right), \quad i = \sqrt{-1} \quad (1)$$

$$\sum_{n=1}^{\infty} a \cdot r^n = \frac{ar}{(1-r)}, \quad |r| < 1, \quad a \in \mathbb{R} \quad (2)$$

Evaluating Term 1, noting that  $|h| = h$  since the summation spans  $h \in \mathbb{Z}^+$ .

$$\begin{aligned}\sum_{h=1}^{\infty} \left( A_1(d_1)^{|h|} \cos(\omega h) \right) &= A_1 \sum_{h=1}^{\infty} (d_1)^{|h|} \left( \frac{1}{2} \left( e^{i h \omega} + e^{-i h \omega} \right) \right), \quad \text{by (1)} \\ \sum_{h=1}^{\infty} \left( A_1(d_1)^{|h|} \cos(\omega h) \right) &= \frac{A_1}{2} \sum_{h=1}^{\infty} (d_1)^h \left( e^{i h \omega} + e^{-i h \omega} \right) \\ \sum_{h=1}^{\infty} \left( A_1(d_1)^{|h|} \cos(\omega h) \right) &= \frac{A_1}{2} \left( \sum_{h=1}^{\infty} (d_1)^h e^{i h \omega} + \sum_{h=1}^{\infty} (d_1)^h e^{-i h \omega} \right) \\ \sum_{h=1}^{\infty} \left( A_1(d_1)^{|h|} \cos(\omega h) \right) &= \frac{A_1}{2} \left( \sum_{h=1}^{\infty} \left( d_1 e^{i \omega} \right)^h + \sum_{h=1}^{\infty} \left( d_1 e^{-i \omega} \right)^h \right) \\ \sum_{h=1}^{\infty} \left( A_1(d_1)^{|h|} \cos(\omega h) \right) &= \frac{A_1}{2} \left( \frac{d_1 e^{i \omega}}{1 - d_1 e^{i \omega}} + \frac{d_1 e^{-i \omega}}{1 - d_1 e^{-i \omega}} \right), \quad \text{by (2)} \\ \sum_{h=1}^{\infty} \left( A_1(d_1)^{|h|} \cos(\omega h) \right) &= \frac{A_1}{2} \left( \frac{d_1 e^{i \omega} (1 - d_1 e^{-i \omega}) + d_1 e^{-i \omega} (1 - d_1 e^{i \omega})}{(1 - d_1 e^{i \omega})(1 - d_1 e^{-i \omega})} \right) \\ \sum_{h=1}^{\infty} \left( A_1(d_1)^{|h|} \cos(\omega h) \right) &= \frac{A_1}{2} \left( \frac{d_1 (e^{i \omega} + e^{-i \omega}) - 2d_1^2}{1 - d_1 (e^{i \omega} + e^{-i \omega}) + d_1^2} \right) \\ \sum_{h=1}^{\infty} \left( A_1(d_1)^{|h|} \cos(\omega h) \right) &= \frac{A_1}{2} \left( \frac{2d_1 \cos(\omega) - 2d_1^2}{1 - 2d_1 \cos(\omega) + d_1^2} \right) \\ \sum_{h=1}^{\infty} \left( A_1(d_1)^{|h|} \cos(\omega h) \right) &= \frac{A_1(d_1 \cos(\omega) - d_1^2)}{1 - 2d_1 \cos(\omega) + d_1^2}\end{aligned}$$

Similarly, if we repeat this exact same process with  $A_2$  and  $d_2$ , noting that  $|d_2| < 1$  and  $A_2 \in \mathbb{R}$  also satisfy the requirements of (1) and (2), we arrive at Term 2:

$$\sum_{h=1}^{\infty} \left( A_2(d_2)^{|h|} \cos(\omega h) \right) = \frac{A_2(d_2 \cos(\omega) - d_2^2)}{1 - 2d_2 \cos(\omega) + d_2^2}$$

Then, we can recombine these into our original expression for the infinite sum:

$$\begin{aligned}
\sum_{h=1}^{\infty} \rho(h) \cos(\omega h) &= \sum_{h=1}^{\infty} \left( A_1(d_1)^{|h|} \cos(\omega h) \right) - \sum_{h=1}^{\infty} \left( A_2(d_2)^{|h|} \cos(\omega h) \right) \\
\sum_{h=1}^{\infty} \rho(h) \cos(\omega h) &= \frac{A_1(d_1 \cos(\omega) - d_1^2)}{1 - 2d_1 \cos(\omega) + d_1^2} - \frac{A_2(d_2 \cos(\omega) - d_2^2)}{1 - 2d_2 \cos(\omega) + d_2^2} \\
\sum_{h=1}^{\infty} \rho(h) \cos(\omega h) &= \left( \frac{16}{11} \right) \frac{\left( \frac{1}{2} \cos(\omega) - \left( \frac{1}{2} \right)^2 \right)}{1 - 2\left( \frac{1}{2} \right) \cos(\omega) + \left( \frac{1}{2} \right)^2} - \left( \frac{5}{11} \right) \frac{\left( \frac{1}{5} \cos(\omega) - \left( \frac{1}{5} \right)^2 \right)}{1 - 2\left( \frac{1}{5} \right) \cos(\omega) + \left( \frac{1}{5} \right)^2} \\
\sum_{h=1}^{\infty} \rho(h) \cos(\omega h) &= \left( \frac{16}{11} \right) \frac{2 \cos(\omega) - 1}{5 - 4 \cos(\omega)} - \left( \frac{5}{11} \right) \frac{5 \cos(\omega) - 1}{2(13 - 5 \cos(\omega))}
\end{aligned}$$

Then, combining the expressions we can get the final expression for the normalized spectral density:

$$\begin{aligned}
f^*(\omega) &= \frac{1}{\pi} \left( 1 + 2 \sum_{h=1}^{\infty} \rho(h) \cos(\omega h) \right) \\
f^*(\omega) &= \frac{1}{\pi} \left( 1 + 2 \left( \left( \frac{16}{11} \right) \frac{2 \cos(\omega) - 1}{5 - 4 \cos(\omega)} - \left( \frac{5}{11} \right) \frac{5 \cos(\omega) - 1}{2(13 - 5 \cos(\omega))} \right) \right) \\
f^*(\omega) &= \boxed{\frac{1}{\pi} + \frac{32}{11\pi} \left( \frac{2 \cos(\omega) - 1}{5 - 4 \cos(\omega)} \right) - \frac{5}{11\pi} \left( \frac{5 \cos(\omega) - 1}{13 - 5 \cos(\omega)} \right)}
\end{aligned}$$

We can verify these results below:

```

w = pi/4
# define acf
rho <- function(h){ (16/11)*((1/2)^h) - (5/11)*((1/5)^h) }
# define the "infinite sum"
sum_vals = sum(sapply(1:1000, function(h){
  rho(h)*cos(w*h)
}))
# from the equation...
eqnval = (1/pi)*(rho(0) + 2*sum_vals)

# from our simplification
fw1 = 1/pi
fw2 = (32/(11*pi)) * ( (2 * cos(w) - 1) / (5 - 4*cos(w)) )
fw3 = (5/(11*pi)) * ( (5 * cos(w) - 1) / (13 - 5*cos(w)) )
# comparison
c(eqnval, fw1 + fw2 - fw3)

```

```
## [1] 0.4561755 0.4561755
```

We see that the values are identical for at least the first 10,000 lags at fixed  $\omega = \pi/4$ .

## Part B

Write down the power spectral density function of  $\{X_t\}_{t \in \mathbb{Z}}$ .

### Solution

We recall from the definition of normalized spectral density that

$$f^*(\omega) = \frac{f(\omega)}{\sigma_X^2}$$

Where  $\sigma_X^2$  is the variance of  $\{X_t\}_{t \in \mathbb{Z}}$ .

Directly, then, we can write  $f(\omega)$  as:

$$f(\omega) = \sigma_X^2 f^*(\omega) = \gamma(0) f^*(\omega)$$

We re-establish the Yule-Walker equations, where  $\alpha_1 = 7/10$  and  $\alpha_2 = -1/10$

$$\begin{aligned}\mathbb{E}(X_t X_t) &= \alpha_1 \mathbb{E}(X_t X_{t-1}) - \alpha_2 \mathbb{E}(X_t X_{t-2}) + \mathbb{E}(X_t Z_t) \\ \mathbb{E}(X_t X_{t-1}) &= \alpha_1 \mathbb{E}(X_{t-1} X_{t-1}) - \alpha_2 \mathbb{E}(X_{t-1} X_{t-2}) + \mathbb{E}(Z_t X_{t-1}) \\ \mathbb{E}(X_t X_{t-2}) &= \alpha_1 \mathbb{E}(X_{t-1} X_{t-2}) - \alpha_2 \mathbb{E}(X_{t-2} X_{t-2}) + \mathbb{E}(Z_t X_{t-2})\end{aligned}$$

Which becomes the following system of three equations:

$$\begin{aligned}\gamma(0) &= \alpha_1 \gamma(1) + \alpha_2 \gamma(2) + \sigma^2 \\ \gamma(1) &= \alpha_1 \gamma(0) + \alpha_2 \gamma(1) \\ \gamma(2) &= \alpha_1 \gamma(1) + \alpha_2 \gamma(0)\end{aligned}$$

test area:

$$\begin{aligned}\gamma(1) &= \alpha_1 \gamma(0) + \alpha_2 \gamma(1) \\ \gamma(1) &= \frac{\alpha_1}{1 - \alpha_2} \gamma(0)\end{aligned}$$

then

$$\begin{aligned}\gamma(2) &= \alpha_1 \gamma(1) + \alpha_2 \gamma(0) \\ \gamma(2) &= \alpha_1 \left( \frac{\alpha_1}{1 - \alpha_2} \gamma(0) \right) + \alpha_2 \gamma(0) \\ \gamma(2) &= \left( \frac{\alpha_1^2 + \alpha_2 - \alpha_2^2}{1 - \alpha_2} \right) \gamma(0)\end{aligned}$$

finally

$$\begin{aligned}\gamma(0) &= \alpha_1 \gamma(1) + \alpha_2 \gamma(2) + \sigma^2 \\ \gamma(0) &= \alpha_1 \left( \frac{\alpha_1}{1 - \alpha_2} \right) \gamma(0) + \alpha_2 \left( \frac{\alpha_1^2 + \alpha_2 - \alpha_2^2}{1 - \alpha_2} \right) \gamma(0) + \sigma^2 \\ \gamma(0) &= \left( \frac{\alpha_1^2 + \alpha_1^2 \alpha_2 + \alpha_2^2 - \alpha_2^3}{1 - \alpha_2} \right) \gamma(0) + \sigma^2 \\ \gamma(0) \left( 1 - \frac{\alpha_1^2 + \alpha_1^2 \alpha_2 + \alpha_2^2 - \alpha_2^3}{1 - \alpha_2} \right) &= \sigma^2 \\ \gamma(0) \left( \frac{(1 - \alpha_2) - \alpha_1^2 - \alpha_1^2 \alpha_2 - \alpha_2^2 + \alpha_2^3}{1 - \alpha_2} \right) &= \sigma^2 \\ \gamma(0) &= \frac{\sigma^2 (1 - \alpha_2)}{1 - \alpha_2 - \alpha_1^2 - \alpha_1^2 \alpha_2 - \alpha_2^2 + \alpha_2^3}\end{aligned}$$

Then, we can evaluate at our given  $\alpha_1$  and  $\alpha_2$ , simplifying the fraction above using Python to avoid human error.

```

from sympy import *

alpha_1, alpha_2, sigma_sq = symbols('alpha_1 alpha_2 sigma_sq')

gamma_0_expr = ( sigma_sq * (1 - alpha_2) /
    (1 - alpha_2 - alpha_1**2 - alpha_1**2 * alpha_2 - alpha_2**2 + alpha_2**3))

alpha_1_val = Rational(7, 10)
alpha_2_val = Rational(-1, 10)

gamma_0_eval = gamma_0_expr.subs({alpha_1: alpha_1_val, alpha_2: alpha_2_val})

gamma_0_simplified = nsimplify(gamma_0_eval)
print(gamma_0_simplified)

```

```
## 275*sigma_sq/162
```

Then, we can verify both our results and Python's simplification as follows, assuming  $\sigma^2 = 1$ .

```

set.seed(3)
alpha_1 <- 7/10
alpha_2 <- -1/10
sigma_sq <- 1

gamma_0 <- (sigma_sq * (1 - alpha_2) /
    (1 - alpha_2 - alpha_1^2 - alpha_1^2 * alpha_2 - alpha_2^2 + alpha_2^3))

ar_params <- c(alpha_1, alpha_2)
simulated_data <- arima.sim(n = 500000,
    model = list(ar = c(7/10, -1/10)))

simulated_gamma_0 <- var(simulated_data)
# use both regular and simplified fraction version
cat("Computed Variance Using Simplified Fraction:", 275/162, "\n")

```

```
## Computed Variance Using Simplified Fraction: 1.697531
```

```
cat("Computed Variance Using Computation:", gamma_0, "\n")
```

```
## Computed Variance Using Computation: 1.697531
```

```
cat("Variance of Simulated AR(2) process:", simulated_gamma_0, "\n")
```

```
## Variance of Simulated AR(2) process: 1.697897
```

It seems the computation very closely approximates the truth. Hence, we conclude that:

$$f(\omega) = \gamma(0)f^*(\omega) = \frac{275\sigma^2}{162} \left( \frac{1}{\pi} + \frac{32}{11\pi} \left( \frac{2\cos(\omega) - 1}{5 - 4\cos(\omega)} \right) - \frac{5}{11\pi} \left( \frac{5\cos(\omega) - 1}{13 - 5\cos(\omega)} \right) \right)$$

## Part C

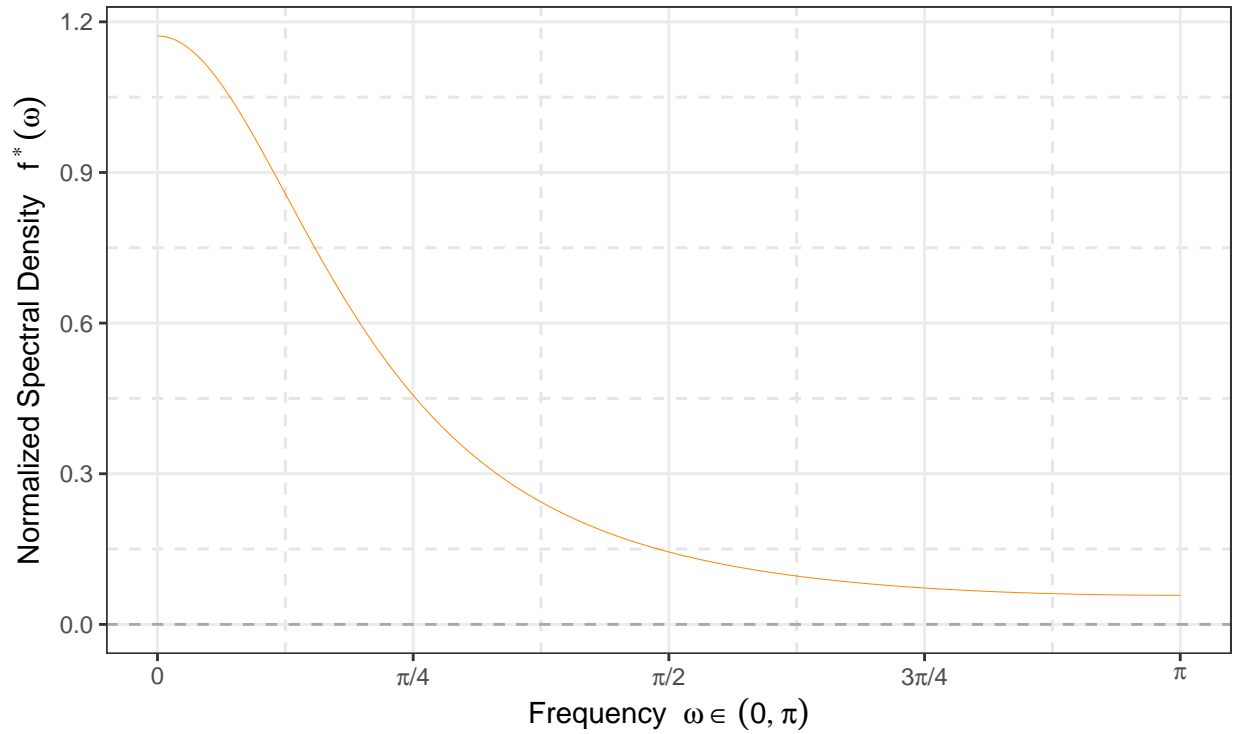
Plot the normalized spectral density and comment on its behaviour.

The normalized spectral density ended up being quite long, so we'll define each term one-by-one in the function below:

```
f = function(w){
  term_1 = 1
  term_2 = (32/11)*((2*cos(w) - 1) / (5 - 4*cos(w)))
  term_3 = (5 /11)*((5*cos(w) - 1) / (13 - 5*cos(w)))
  return ( (1/pi)*(term_1 + term_2 - term_3) )
}
omega = seq(from = 0, to = pi, length.out = 1e4)
# define data frame for values
p1df = data.frame(omega = omega,
  f_star_omega = f(omega))
# build plot
p1 <- ggplot(p1df, aes(x = omega, y = f_star_omega)) +
  geom_line(color = "#ff8600", linewidth = 0.1) +
  labs(
    title = "Normalized Spectral Density for AR(2) Process",
    subtitle = TeX(paste(
      "$X_t = \\alpha_1 X_{t-1} + \\alpha_2 X_{t-2} + Z_t$",
      "where",
      "$\\{\\{Z_t\\}\\}_{t \\in \\{Z\\}} \\sim WN(0, \\sigma^2)$",
      "and $\\alpha_1 = 7/10, \\alpha_2 = -1/10$")),
    y = TeX("Normalized Spectral Density $f^*(\\omega)$"),
    x = TeX("Frequency $\\omega \\in (0, \\pi)$"))
  ) + theme_bw() +
  geom_hline(yintercept = 0, lty = 'dashed', col = "darkgrey")+
  scale_x_continuous(
    breaks = c(0, pi/4, pi/2, 3*pi/4, pi),
    labels = c(TeX("0"), TeX("$\\pi/4$"), TeX("$\\pi/2$"),
      TeX("$3\\pi/4$"), TeX("$\\pi$")))+
  theme(panel.grid.minor = element_line(
    color = "grey90",
    linetype = "dashed",
    linewidth = 0.5
  ))
print(p1)
```

## Normalized Spectral Density for AR(2) Process

$X_t = \alpha_1 X_{t-1} + \alpha_2 X_{t-2} + Z_t$  where  $\{Z_t\}_{t \in \mathbb{Z}} \sim \text{WN}(0, \sigma^2)$  and  $\alpha_1 = 7/10$ ,  $\alpha_2 = -1/10$



**Comments:** It appears that the normalized spectral density plot is largely dominated by low frequencies. We can tell this is the case due to the fact that the largest values of  $f^*(\omega)$  is for small  $\omega \in (0, \pi)$  specifically for  $\omega \in (0, \pi/4)$  we see the majority of frequencies with observed normalized density greater than 0.5. This tells us that a greater proportion of the variance inherent in the process  $X_t$  can be attributed to the lower frequencies. As we would expect of a normalized spectral density dominated by low  $\omega$  values, we see that  $f^*(\omega)$  is strictly decreasing as  $\omega \rightarrow \pi$ .

## Question 2