Stat 443: Time Series and Forecasting

Assignment 2: Time Series Models

Caden Hewlett

February 12, 2024

Questions

1. Stationary Second-Order AR Process

Consider the stationary second-order AR process:

$$X_t = \frac{7}{10}X_{t-1} - \frac{1}{10}X_{t-2} + Z_t$$

Where $\{Z_t\}_{t\in\mathbb{Z}} \sim \text{WN}(0, \sigma^2)$.

a) Yule-Walker Equations

Derive the Yule-Walker equations and find the autocorrelation functions of $X_{tt\in\mathbb{Z}}$.

Let us begin by deriving the Yule-Walker Equations.

Step 1: The first step is to multiply each side of the equation by X_{t-i} for $i \in [1, p]$.

$$X_tX_{t-1} = \frac{7}{10}X_{t-1}X_{t-1} - \frac{1}{10}X_{t-1}X_{t-2} + Z_tX_{t-1}$$
$$X_tX_{t-2} = \frac{7}{10}X_{t-1}X_{t-2} - \frac{1}{10}X_{t-2}X_{t-2} + Z_tX_{t-2}$$

Step 2: Then we take the expectation of these equations, using linearity of expectation directly.

$$\mathbb{E}(X_tX_{t-1}) = \frac{7}{10}\mathbb{E}(X_{t-1}X_{t-1}) - \frac{1}{10}\mathbb{E}(X_{t-1}X_{t-2}) + \mathbb{E}(Z_tX_{t-1})$$
$$\mathbb{E}(X_tX_{t-2}) = \frac{7}{10}\mathbb{E}(X_{t-1}X_{t-2}) - \frac{1}{10}\mathbb{E}(X_{t-2}X_{t-2}) + \mathbb{E}(Z_tX_{t-2})$$

This simplifies to...

$$\gamma(1) = \frac{7}{10}\gamma(0) - \frac{1}{10}\gamma(1)$$
$$\gamma(2) = \frac{7}{10}\gamma(1) - \frac{1}{10}\gamma(0)$$

Step 3: Divide everything by $\gamma(0)$ to get correlation values.

$$\rho(1) = \frac{7}{10} - \frac{1}{10}\rho(1)$$
$$\rho(2) = \frac{7}{10}\rho(1) - \frac{1}{10}$$

Then, we can solve directly to find:

$$\rho(1) = \frac{7}{11}, \text{ and } \rho(2) = \frac{19}{55}$$

However, we're not done! We still need to find the general autocorrelation function for all h values, i.e. $\rho(h)$.

To do this, we must reconsider the original AR model, which I'll write as:

$$\alpha_0 X_t = \alpha_1 X_{t-1} + \alpha_2 X_{t-2} + Z_t$$
$$\text{Where } \alpha_0 = 1, \alpha_1 = \frac{7}{10} \text{ and } \alpha_2 = -\frac{1}{10}$$

With this definition of α_i for $i \in [0, 2]$ we can write the polynomial:

$$D^p - \alpha_1 D^{p-1} - \alpha_2 D^{p-2} = D^2 - \frac{7}{10}D + \frac{1}{10}$$

Then, we solve the roots of this polynomial:

$$\begin{aligned}
0 &= D^2 - \frac{7}{10}D + \frac{1}{10} \\
0 &= 10D^2 - 7D + 1 \\
0 &= (5D - 1)(2D - 1) \\
&\text{Hence, } d_1 = \frac{1}{5} \text{ and } d_2 = \frac{1}{2}
\end{aligned}$$

We then substitue these into the General Equation for $\rho(h)$.

$$\rho(h) = A_1 d_1^{|h|} + A_2 d_2^{|h|} \to \rho(h) = A_1\left(\frac{1}{5}\right)^{|h|} + A_2\left(\frac{1}{2}\right)^{|h|}$$

We'll use our known solutions of $\rho(0) = 1$ and $\rho(1) = 7/11$ to find A_1 and A_2.

$$\begin{aligned}
\rho(0) &= A_1 + A_2 = 1 \text{ hence } A_1 = 1 - A_2 \\
\rho(1) &= \frac{A_1}{5} + \frac{A_2}{2} = \frac{7}{11}
\end{aligned}$$

Substituting $A_1 = 1 - A_2$ into the second equation,

$$\begin{aligned}
\frac{1}{5}(1 - A_2) + \frac{1}{2}(A_2) &= \frac{7}{11} \\
\frac{3}{10}A_2 &= \frac{24}{55} \\
\text{Hence } A_2 &= \frac{16}{11}, \text{ and } A_1 = -\frac{5}{11}
\end{aligned}$$

Thus, we can write the general acf function at lag h as:

$$\rho(h) = \frac{16}{11}\left(\frac{1}{2}\right)^{|h|} - \frac{5}{11}\left(\frac{1}{5}\right)^{|h|}$$

b) Simulation and Plot

Assume $\sigma^2 = 1$. Using `set.seed(443)` to set the simulation seed and `arima.sim()` function simulate 1000 observations from the AR(2) process defined above. Plot its sample ACF for the first 15 lags along with the theoretical ACF obtained in part (a). Compare the sample and theoretical ACFs.

```r
# R code block for simulation and plotting
set.seed(443)
# Further simulation and plotting code goes here
```

2. ARMA(2,1) Process

Consider the ARMA(2,1) process:

$$X_t = -0.5X_{t-2} + Z_t + 0.5Z_{t-1}$$

Where $\{Z_t\}_{t\in\mathbb{Z}} \sim \mathrm{WN}(0, \sigma^2)$.

a) Stationarity and Invertibility

Check whether the process is stationary and invertible. Justify your answers.

We will begin by converting $\{X_t\}$ into a function of the characteristic polynomials $\varphi(B)$ and $\theta(B)$. We let B denote the backshift operator.

By rearrangement,

$$\begin{aligned} X_t &= -0.5X_{t-2} + Z_t + 0.5Z_{t-1} \\ X_t + 0.5X_{t-2} &= Z_t + 0.5Z_{t-1} \\ X_t(B^0 + 0.5B^2) &= Z_t(B^0 + 0.5B^1) \\ X_t(1 + 0.5B^2) &= Z_t(1 + 0.5B) \end{aligned}$$

Therefore, we can write our ARMA(2,1) process as:

$$\mathrm{ARMA}(2,1) : \varphi(B)X_t = \theta(B)Z_t$$
$$\text{Where } \begin{cases} \varphi(B) = 1 + 0.5B^2 \\ \theta(B) = 1 + 0.5B \end{cases}$$

Now, we have everything to need to verify invertibility and stationarity.

Invertibility: We know that $\{X_t\}$ is invertible.

Proof: Let $\tilde{\theta}_i \in \mathbb{C}$ be the i-th root of $\theta(B)$. In order for the ARMA process to be invertible, it must hold that all roots of $\theta(B)$ are greater than the roots of unity. In other words, we wish to show that:

$$\forall i \in [1, q],\ \|\tilde{\theta}_i\| > 1$$

Hence, we must find the roots of $\theta(B)$ and their magnitude. By the Fundamental Theorem of Algebra, we know that there will exist exactly one root $\tilde{\theta}_1 \in \mathbb{C}$, as $\theta(B)$ has order 1.

Directly, we see:

$$\begin{aligned} 1 + 0.5B &= 0 \\ 2 + B &= 0 \\ B &= -2 \end{aligned}$$

$$\text{Hence, } \tilde{\theta}_1 = -2, \text{ and } \|\tilde{\theta}_1\| = 2 > 1 \ \therefore \{X_t\} \text{ is invertible. } \square$$

Thus, we have proved $\{X_t\}$ is invertible, as required.

Stationarity We know that $\{X_t\}$ is stationary.

Proof: Let $\tilde{\varphi}_j \in \mathbb{C}$ be the j-th root of $\varphi(B)$. In order for the ARMA process to be stationary, it must hold that all roots of $\varphi(B)$ are greater than the roots of unity. In other words, we wish to show that:

$$\forall j \in [1, p],\ \|\tilde{\varphi}_i\| > 1$$

Hence, we must find the roots of $\varphi(B)$ and their magnitude. By the Fundamental Theorem of Algebra, we know that there will exist exactly two roots $\{\tilde{\varphi}_1, \tilde{\varphi}_2\} \in \mathbb{C}$, as $\varphi(B)$ has order 2.

We can find them directly.

$$\begin{aligned}1 + 0.5B^2 &= 0\\ 2 + B^2 &= 0\\ B &= \pm\sqrt{-2}\end{aligned}$$

Hence,

$$\{\tilde{\varphi}_1, \tilde{\varphi}_2\} = \{i\sqrt{2}, -i\sqrt{2}\}$$

Further, we know that

$$\|\tilde{\varphi}_1\| = \|\tilde{\varphi}_2\| = \sqrt{2} > 1,\ \therefore \{X_t\} \text{ is stationary. } \square$$

Thus, we have proved $\{X_t\}$ is stationary, as required.

In conclusion, we have shown that the ARMA(2, 1) process is both stationary and invertible.

b) Pure AR Process:

Write the above ARMA(2,1) process as a pure AR process.

In order to write the model as a pure AR process, we will express it in the following form:

$$Z_t = \left(\frac{\varphi(B)}{\theta(B)}\right) X_t = \pi(B) X_t$$

Where $\theta(B)$ and $\varphi(B)$ are as defined in the previous question. Let us find an expression for $\pi(B)$:

$$\pi(B) = \frac{\varphi(B)}{\theta(B)} = \frac{(1 + 0.5B^2)}{(1 + 0.5B)} = \frac{(1 + 0.5B^2)}{\big(1 - (-0.5B)\big)} = (1 + 0.5B^2) \sum_{n=0}^{\infty} (-0.5B)^n$$

Where the last expression in the equality follows from the definition of a geometric series, under the assumption that $|-0.5B| < 1$. We expand this process, letting $\delta = 0.5$ for readability.

$$\begin{aligned}\pi(B) &= (1 + \delta B^2) \sum_{n=1}^{\infty} (-\delta B)^n\\ \pi(B) &= (1 + \delta B^2)\Big[1 + (-\delta B) + (-\delta B)^2 + (-\delta B)^3 + \dots\Big]\\ \pi(B) &= \Big[1 + (-\delta B) + \big(\delta B^2 + (-\delta B)^2\big) + \big((\delta B^2)(-\delta B) + (-\delta B)^3\big) + \dots\Big]\end{aligned}$$

Now, we can begin to attempt to investigate each $\pi_i \in \pi(B)$ where i indicates the power of B in the expression.

$$\begin{aligned}\pi_0 &= 1\\ \pi_1 &= (-\delta B)\\ \pi_2 &= \delta B^2 + (-\delta B)^2\\ \pi_3 &= (\delta B^2)(-\delta B) + (-\delta B)^3\\ \pi_4 &= (\delta B^2)(-\delta B)^2 + (-\delta B)^4\\ &\vdots\end{aligned}$$

Investigating these terms, we begin to notice a pattern:

$$\begin{aligned}
\pi_0 &= (0 + (-\delta)^0)B^0 \\
\pi_1 &= (0 + (-\delta)^1)B^1 \\
\pi_2 &= (\delta + (-\delta)^2)B^2 \\
\pi_3 &= (\delta(-\delta) + (-\delta)^3)B^3 \\
\pi_4 &= (\delta(-\delta)^2 + (-\delta)^4)B^4 \\
&\vdots \\
\pi_j &= (\delta(-\delta)^{j-2} + (-\delta)^j)B^j
\end{aligned}$$

With this in mind, we can denote the following term to define all elements of $\pi(B)$ for $j \in \mathbb{N}$ where, still using $\delta = 0.5$, each π_j term is defined by the rule stated (using indicators to account for the fact that π_0 and π_1 follow a slightly different general pattern.)

$$\pi(B) = 1 - \sum_{j=0}^{\infty} B^j \pi_j, \text{ where } \{\pi_j\} = \big(\mathbb{1}[j \geq 2]\delta(-\delta)^{j-2}\big) + (-\delta)^j$$

Now that we have a clear definition for $\pi(B)$, we can conclude by writing:

$$\pi(B)X_t = Z_t$$

Thus, we've converted the ARMA process to a pure infinite-order AR process with a new characteristic polynomial $\pi(B)$ as required.

c) Pure MA Process (Pure AR?)

Write the above ARMA(2,1) process as a pure MA process.

In order to write the model as a pure MA process, we will express it in the following form:

$$X_t = \left(\frac{\theta(B)}{\varphi(B)}\right)Z_t = \Psi(B)Z_t$$

Where $\theta(B)$ and $\varphi(B)$ are as defined in the previous question.Let us find an expression for $\Psi(B)$:

$$\Psi(B) = \frac{\theta(B)}{\varphi(B)} = \frac{(1 + 0.5B)}{(1 + 0.5B^2)} = \frac{(1 + 0.5B)}{\big(1 - (-0.5B^2)\big)} = (1 + 0.5B)\sum_{n=0}^{\infty}(-0.5B^2)^n$$

Where the last expression in the equality follows from the definition of a geometric series, under the assumption that $|-0.5B^2| < 1$. Let's expand this product's infinite sum a bit, again letting $\delta = 0.5$.

$$\begin{aligned}
\Psi(B) &= (1 + \delta B)\sum_{n=0}^{\infty}(-\delta B^2)^n \\
\Psi(B) &= (1 + \delta B)\big(1 + (-\delta B^2) + \delta^2 B^4 + (-\delta^3 B^6) + \dots\big) \\
\Psi(B) &= \big(1 + \delta B - \delta B^2 - \delta^2 B^3 + \delta^2 B^4 - \delta^3 B^5 - \delta^3 B^6 + \dots\big)
\end{aligned}$$

Investigating these in terms of the powers of B, we begin to see a pattern. It's slightly more complicated than the last one, because of the squared term in the infinite sum.

$$
\begin{aligned}
&\psi_0 = 1 = (-1)^0(-\delta)^0 B^0 \\
&\psi_1 = \delta B = (-1)^1(-\delta)^1 B^1 \\
&\psi_2 = -\delta B^2 = (-1)^2(-\delta)^1 B^2 \\
&\psi_3 = -\delta^2 B^3 = (-1)^3(-\delta)^2 B^3 \\
&\psi_4 = \delta^2 B^4 = (-1)^4(-\delta)^2 B^4 \\
&\psi_5 = \delta^3 B^5 = (-1)^5(-\delta)^3 B^5 \\
&\psi_6 = -\delta^3 B^6 = (-1)^6(-\delta)^3 B^6 \\
&\quad\vdots \\
&\psi_j \overset{?}{=\ldots=} (-1)^j(-\delta)^{\lceil j/2 \rceil} B^j
\end{aligned}
$$

The series for ψ_i seems to have a "lagging" component in the δ terms. It could even seem as if there's a different series for odd and even indices. However, we can encapsulate the series with the ceiling function, as shown above.

We could alternatively express the series as a piecewise function, if we don't wish to use the ceiling function.

$$
\psi_j = \begin{cases} (-\delta)^{j/2} B^j, & j \equiv 0 \text{mod}(2) \\ (-\delta)^{(j+1)/2} B^j, & j \equiv 1 \text{mod}(2) \end{cases}
$$

In either case, we can write $\Psi(B)$ in its entirety. Letting $\delta = 0.5$, we write:

$$
\Psi(B) = \sum_{i=0}^{\infty} \psi_i B^i, \text{ where } \{\psi_j\} = (-1)^j(-\delta)^{\lceil j/2 \rceil} B^j
$$

By this definition, we conclude that:

$$
X_t = \Psi(B) Z_t, \text{ for } \Psi(B) = \sum_{i=0}^{\infty} \psi_i B^i
$$

d) Autocorrelation Function

Find the ACF of $\{X_t\}_{t \in \mathbb{Z}}$.

(how..? Is my MA equation wrong?)

3. AR(2) Process

Consider the AR(2) process given by:

$$
2aX_t + \frac{3}{6a} X_{t-1} - \frac{1}{6a} X_{t-2} = Z_t, \quad a \in \mathbb{R}
$$

a) Stationarity Conditions

Under which conditions on constant a is the process $\{X_t\}_{t \in \mathbb{Z}}$ stationary?

Let's adjust the expression in terms of the backshift operator B.

$$\Big(2aB^0 + \frac{3}{6a}B - \frac{1}{6a}B^2\Big)X_t = Z_t$$

From this, we can conclude that the characteristic polynomial $\varphi(B)$ can be written as:

$$\varphi(B) = 2a + \frac{3}{6a}B - \frac{1}{6a}B^2$$

Again, let $\tilde{\varphi}_j \in \mathbb{C}$ be the j-th root of $\varphi(B)$. In order for the ARM process to be stationary, it must hold that all roots of $\varphi(B)$ are greater than the roots of unity. In other words, we wish to show that:

$$\forall j \in [1,p],\ \|\tilde{\varphi}_i\| > 1$$

By the Fundamental Theorem of Algebra, there are two roots $\{\tilde{\varphi}_1, \tilde{\varphi}_2\}$ that we can find for this characteristic polynomial. We should note from the beginning that in order for the polynomial to be well defined, $a \neq 0$.

Let us solve:

- check roots of $(2a + (3B)/6A - (Bˆ2)/6a)

b) Autocorrelation Function

Assuming that a satisfies conditions found in part (a), find the ACF of $\{X_t\}_{t\in\mathbb{Z}}$.

1.) yule walker

2.) find d_i

3.) find A_i

4.) done

4. SARIMA Model

Show that SARIMA$(\underset{p}{2}, \underset{d}{1}, \underset{q}{1}) \times (\underset{P}{0}, \underset{D}{1}, \underset{Q}{1})_{\underset{s}{12}}$ can be written as an ARMA(p,q) process and specify values of p and q.

Let us begin by finding the p-value for the AR process.

For a SARIMA$(\underset{p}{2}, \underset{d}{1}, \underset{q}{1}) \times (\underset{P}{0}, \underset{D}{1}, \underset{Q}{1})_{\underset{s}{12}}$

$$\text{LHS} = \varphi(B)\Phi(B^s)W_t, \text{ where } W_t = \nabla^d \nabla_s^D X_t$$

We wish to represent it in ARMA form, which has a single polynomial on each side. In other words, let the following be our "desired" ARMA process:

$$\varphi^{\oplus}(B)X_t = \theta^{\oplus}(B)Z_t$$

Where $\varphi^{\oplus}(B)$ and $\theta^{\oplus}(B)$ are the characteristic polynomials of the ARMA model. Let's begin find p, the order of $\theta^{\oplus}(B)$, by expanding the LHS of the SARIMA model.

In order to do this, we must first expand W_t.

$$
\begin{aligned}
W_t &= \nabla^d \nabla_s^D X_t \\
W_t &= \nabla^1 \nabla_{12}^1 X_t \\
W_t &= \nabla_{12}^1 (X_t - X_{t-1}) \\
W_t &= \nabla_{12}^1 (X_t) - \nabla_{12}^1 (X_{t-1}) \\
W_t &= (X_t - X_{t-12}) - (X_{t-1} - X_{t-12-1}) \\
W_t &= X_t - X_{t-1} - X_{t-12} + X_{t-13}
\end{aligned}
$$

Now, we expand $\phi(B)$, understanding that $\Phi(B) = 1$ since $P = 0$.

$$\phi(B) = (1 - \alpha_1 B - \alpha_2 B^2)$$

Then we put this together as:

$$\varphi^{\oplus}(B) X_t = (1 - \alpha_1 B - \alpha_2 B^2)(X_t - X_{t-1} - X_{t-12} + X_{t-13})$$

We can expand this briefly (but not entirely) to find p for $\varphi^{\oplus}(B)$.

$$
\begin{aligned}
\varphi^{\oplus}(B) X_t &= (1 - \alpha_1 B - \alpha_2 B^2)(X_t - X_{t-1} - X_{t-12} + X_{t-13}) \\
\varphi^{\oplus}(B) X_t &= X_t - \alpha_1 B X_t - \alpha_2 B^2 X_t - \cdots - \alpha_1 B X_{t-13} - \alpha_2 B^2 X_{t-13} \\
\varphi^{\oplus}(B) X_t &= X_t - \alpha_1 X_{t-1} - \alpha_2 X_{t-2} - \cdots - \alpha_1 X_{t-14} - \alpha_2 X_{t-15}
\end{aligned}
$$

Hence, from the final term of $\varphi^{\oplus}(B) X_t$, we can conclude that $p = 15$.

We can similarly determine the order q of the right-hand side of the equation.

$$\text{RHS} = \theta(B) \Theta(B^s) Z_t$$

In this situation, both q and Q are nonzero. We'll expand each term independently, then consider their product.

$$\theta(B) = (1 + \beta B), \quad \Theta(B^s) = (1 + \tilde{\beta} B^{12})$$

Hence, we can expand the AR piece of the model to be:

$$
\begin{aligned}
\theta^{\oplus}(B) &= (1 + \beta B)(1 + \tilde{\beta} B^{12}) Z_t \\
\theta^{\oplus}(B) &= (1 + \beta B)(1 + \tilde{\beta} B^{12}) Z_t \\
\theta^{\oplus}(B) &= (1 + \beta B + \tilde{\beta} B^{12} + \beta \tilde{\beta} B B^{12}) \\
\theta^{\oplus}(B) &= Z_t + \beta Z_{t-1} + \tilde{\beta} Z_{t-12} + \beta \tilde{\beta} Z_{t-13}
\end{aligned}
$$

Hence, from the final term of $\theta^{\oplus}(B) Z_t$, we can conclude that $q = 13$.

Therefore, we can conclude that the SARIMA model can decompose into an ARMA$(p = 15, q = 13)$ model under this decomposition. $\square$