Practice Question 1

Let $\{X_t\}_{t\in\mathbb{Z}}$ be defined such that $X_t = Z_t + \beta_1 Z_{t-1} + \beta_2 Z_{t-2}$, where $Z_t \overset{\text{iid}}{\sim} \text{WN}(0, \sigma^2)$.

Part A

Given the model equation definitions discussed in this class, what type of process is $\{X_t\}_{t\in\mathbb{Z}}$?

It is a moving average process of order two, or simply $\boxed{\text{MA}(2)}$

Part B

Given that $\beta_1 = 0.5$ and $\beta_2 = -0.4$, is $\{X_t\}$ invertible? Is it stationary? Justify your answers.

Since β_1 and β_2 are well-defined and finite,$\{X_t\}$ is stationary by definition.

To verify invertibility, we see if the roots of the characteristic polynomial $\theta(B)$ have modulus greater than the roots of unity.

$$\begin{aligned} X_t &= Z_t + 0.5Z_{t-1} - 0.4Z_{t-2} \\ X_t &= Z_t(1 + 0.5B - 0.4B^2) \\ \text{So, } \theta(B) &= -0.4B^2 + 0.5B + 1 \end{aligned}$$

As an aside, we know by the Fundamental Theorem of Algebra that there will be exactly two roots $\vartheta_1, \vartheta_2 \in \mathbb{C}$ (not taught in this course.)

$$\begin{aligned} \vartheta &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ \vartheta &= \frac{-(-0.5) \pm \sqrt{(-0.5)^2 - 4(-0.4)(1)}}{2(-0.4)} \\ \vartheta &= \frac{0.5 \pm \sqrt{1.85}}{-0.8} \\ \text{Hence, } \vartheta_1 &\approx -1.075 \text{ and } \vartheta_2 \approx 2.325 \end{aligned}$$

Since $\|-1.075\| > 1$ and $\|2.325\| > 1$, we know that both roots have modulus greater than the roots of unity.

Hence, $\{X_t\}_{t\in\mathbb{Z}}$ is invertible as well as stationary. $\square$

Part C

Define the normalized spectral density function $f^\star(\omega)$ for $\{X_t\}_{t\in\mathbb{Z}}$ for arbitrary coefficients β_1 and β_2. Assume that the process is stationary and invertible. Show your work.

We first recall the equation for finding the power spectral density from a given autocovariance function:

$$f(\omega) = \frac{1}{\pi}\left(\gamma(0) + 2\sum_{h=1}^{\infty} \gamma(h)\cos(\omega h)\right), \text{ where } h \in \mathbb{Z} \text{ and } \omega \in (0,1) \subseteq \mathbb{R}$$

We first derive $\gamma(0)$. This is equivalent to computing the variance directly.

$$\begin{aligned}
\gamma(0) &= \text{cov}(X_t, X_t) \\
\gamma(0) &= \text{cov}(Z_t + \beta_1 Z_{t-1} + \beta_2 Z_{t-2}, Z_t + \beta_1 Z_{t-1} + \beta_2 Z_{t-2}) \\
\gamma(0) &= \text{cov}(Z_t, Z_t) + \text{cov}(\beta_1 Z_{t-1}, \beta_1 Z_{t-1}) + \text{cov}(\beta_2 Z_{t-2}, \beta_2 Z_{t-2}) + 2 \sum_{i<j;\ i,j\in[0,2]} \text{cov}(\beta_i Z_{t+i}, \beta_j Z_{t+j}) \\
\gamma(0) &= \text{var}(Z_t) + \beta_1^2 \text{var}(Z_t) + \beta_2^2 \text{var}(Z_t) + 0 \\
\gamma(0) &= \sigma^2(1 + \beta_1^2 + \beta_2^2)
\end{aligned}$$

Similarly, we find $\gamma(1)$:

$$\begin{aligned}
\gamma(1) &= \text{cov}(X_t, X_{t+1}) \\
\gamma(1) &= \text{cov}(Z_t + \beta_1 Z_{t-1} + \beta_2 Z_{t-2}, Z_{t+1} + \beta_1 Z_t + \beta_2 Z_{t-1}) \\
\gamma(1) &= \text{cov}(Z_t, \beta_1 Z_t) + \text{cov}(\beta_1 Z_{t-1}, \beta_2 Z_{t-1}) + 0 \\
\gamma(1) &= \sigma^2(\beta_1 + \beta_1\beta_2)
\end{aligned}$$

And $\gamma(2)$

$$\begin{aligned}
\gamma(2) &= \text{cov}(X_t, X_{t+2}) \\
\gamma(2) &= \text{cov}(Z_t + \beta_1 Z_{t-1} + \beta_2 Z_{t-2}, Z_{t+2} + \beta_1 Z_{t+1} + \beta_2 Z_t) \\
\gamma(2) &= \sigma^2 \beta_2
\end{aligned}$$

We can directly infer from the pattern above that $\forall h \in \mathbb{Z}$ s.t. $|h| > 2, \gamma(h) = 0$.

Hence, the Power Spectral Density expression becomes:

$$\begin{aligned}
f(\omega) &= \frac{1}{\pi}\left(\gamma(0) + 2\sum_{h=1}^{\infty} \gamma(h), \cos(\omega h)\right) \\
f(\omega) &= \frac{1}{\pi}\left(\gamma(0) + 2\gamma(1)\cos(\omega) + 2\gamma(2)\cos(2\omega) + \sum_{h=3}^{\infty} \gamma(h)\cos(\omega h)\right) \\
f(\omega) &= \frac{1}{\pi}\left(\sigma^2(1 + \beta_1^2 + \beta_2^2) + 2\sigma^2(\beta_1 + \beta_1\beta_2)\cos(\omega) + 2\sigma^2\beta_2\cos(2\omega) + 0\right) \\
f(\omega) &= \frac{\sigma^2}{\pi}\left(1 + \beta_1^2 + \beta_2^2 + 2\beta_1\cos(\omega) + 2\beta_1\beta_2\cos(\omega) + 2\beta_2\cos(2\omega)\right)
\end{aligned}$$

Then, finally, the *normalized* spectral density $f^\star(\omega)$ is found via the relation:

$$f^\star(\omega) = \frac{f(\omega)}{\sigma_X^2} = \frac{f(\omega)}{\gamma(0)} = \frac{1 + \beta_1^2 + \beta_2^2 + 2\beta_1\cos(\omega) + 2\beta_1\beta_2\cos(\omega) + 2\beta_2\cos(2\omega)}{\pi(1 + \beta_1^2 + \beta_2^2)} \square$$

Practice Question 2

Part A

For well-defined coefficients, an AR process is always invertible.

To check stationarity, we use the roots of the characteristic polynomial:

$$\text{Roots} = \frac{1.2 \pm 0.4\mathrm{i}}{0.8}$$

These have modulus greater than the roots of unity, therefore the process is starionary and invertible.

Part B

Use the Yule-Walker Equations (I was too tired to typeset all of my work).

For $h \in [-2, 2] \subseteq \mathbb{Z}$, we have:

$$\rho(h) = \begin{cases} \rho(-h), & h < 0 \\ 1 & h = 0 \\ 5/8, & h = 1 \\ 7/20, & h = 2 \end{cases}$$

Part C

To find $\hat{x}_3(2)$, we do the following, recalling $x_1 = 5$, $x_2 = 3$ and $x_3 = 4$.

$$\begin{aligned}
\hat{x}_3(2) &= \mathbb{E}(X_{3+2} \mid X_3, X_2, X_1) \\
\hat{x}_3(2) &= \mathbb{E}(1.2X_{5-1} - 0.4X_{5-2} + Z_5 \mid X_3, X_2, X_1) \\
\hat{x}_3(2) &= 1.2\,\underbrace{\mathbb{E}(X_4 \mid X_3, X_2, X_1)}_{\text{1 -step ahead forecast}} - 0.4x_3 \\
\hat{x}_3(2) &= 1.2\hat{x}_3(1) - 0.4x_3 \\
\hat{x}_3(2) &= 1.2\Big(\mathbb{E}(1.2X_{4-1} - 0.4X_{4-2} + Z_4 \mid X_{1:3})\Big) - 0.4x_3 \\
\hat{x}_3(2) &= 1.2(1.2x_3 - 0.4x_2) - 0.4x_3 \\
\hat{x}_3(2) &= 1.2\big(1.2(4) - 0.4(3)\big) - 0.4(4) = \boxed{2.72}
\end{aligned}$$

Practice Question 3

Consider the following bivariate time series $(\{X_t, Y_t\})_{t\in\mathbb{Z}}$, assumed to be stationary for well-defined $\alpha, \beta \in \mathbb{R}$ and $Z_t \overset{\text{iid}}{\sim} \text{WN}(0, \sigma^2)$.

$$X_t = Z_t + \alpha Z_{t-1}, \text{ and } Y_t = Z_t + \beta Z_{t-1}$$

Part A

Compute the cross-covariance function $\gamma_{XY}(h)$ for this bivariate series.

$$\gamma_{XY}(h) = \begin{cases} \alpha\sigma^2, & h = -1 \\ \sigma^2 + \alpha\beta\sigma^2, & h = 0 \\ \beta\sigma^2, & h = 1 \\ 0, & \text{otherwise} \end{cases}$$

Part B

For an stationary bivariate time series $(\{X_t, Y_t\})_{t\in\mathbb{Z}}$, show that $\gamma_{XY}(-h) = \gamma_{YX}(h)$.

$$\begin{aligned}
\gamma_{XY}(-h) &= \text{cov}(X_t, Y_{t-h}) \\
&\quad \text{Let } s = t - h, \ \therefore t = s + h, \text{ where } t, s, h \in \mathbb{Z} \\
\gamma_{XY}(-h) &= \text{cov}(X_{s+h}, Y_{(s+h)-h}) \\
\gamma_{XY}(-h) &= \text{cov}(X_{s+h}, Y_s) \\
\gamma_{XY}(-h) &= \text{cov}(Y_s, X_{s+h}) \\
\gamma_{XY}(-h) &= \gamma_{YX}(h) \quad \square
\end{aligned}$$