Stat 443: Time Series and Forecasting

Assignment 2: Time Series Models

Caden Hewlett

February 04, 2024

Questions

1. Stationary Second-Order AR Process

Consider the stationary second-order AR process:

$$X_t = \frac{7}{10}X_{t-1} - \frac{1}{10}X_{t-2} + Z_t$$

Where $\{Z_t\}_{t \in \mathbb{Z}} \sim \mathrm{WN}(0, \sigma^2)$.

a) Yule-Walker Equations

Derive the Yule-Walker equations and find the autocorrelation functions of $X_{t\ t \in \mathbb{Z}}$.

b) Simulation and Plot

Assume $\sigma^2 = 1$. Using `set.seed(443)` to set the simulation seed and `arima.sim()` function simulate 1000 observations from the AR(2) process defined above. Plot its sample ACF for the first 15 lags along with the theoretical ACF obtained in part (a). Compare the sample and theoretical ACFs.

```
# R code block for simulation and plotting
set.seed(443)
# Further simulation and plotting code goes here
```

2. ARMA(2,1) Process

Consider the ARMA(2,1) process:

$$X_t = -0.5X_{t-2} + Z_t + 0.5Z_{t-1}$$

Where $\{Z_t\}_{t \in \mathbb{Z}} \sim \mathrm{WN}(0, \sigma^2)$.

a) Stationarity and Invertibility

Check whether the process is stationary and invertible. Justify your answers.

We will begin by converting $\{X_t\}$ into a function of the characteristic polynomials $\varphi(B)$ and $\theta(B)$. We let B denote the backshift operator.

By rearrangement,

$$X_t = -0.5X_{t-2} + Z_t + 0.5Z_{t-1}$$
$$X_t + 0.5X_{t-2} = Z_t + 0.5Z_{t-1}$$
$$X_t(B^0 + 0.5B^2) = Z_t(B^0 + 0.5B^1)$$
$$X_t(1 + 0.5B^2) = Z_t(1 + 0.5B)$$

Therefore, we can write our ARMA(2,1) process as:

$$\text{ARMA}(2,1): \varphi(B)X_t = \theta(B)Z_t$$

$$\text{Where} \begin{cases} \varphi(B) = 1 + 0.5B^2 \\ \theta(B) = 1 + 0.5B \end{cases}$$

Now, we have everything to need to verify invertibility and stationarity.

Invertibility: We know that $\{X_t\}$ is invertible.

Proof: Let $\tilde{\theta}_i \in \mathbb{C}$ be the i-th root of $\theta(B)$. In order for the ARMA process to be invertible, it must hold that all roots of $\theta(B)$ are greater than the roots of unity. In other words, we wish to show that:

$$\forall i \in [1, q], \|\tilde{\theta}_i\| > 1$$

Hence, we must find the roots of $\theta(B)$ and their magnitude. By the Fundamental Theorem of Algebra, we know that there will exist exactly one root $\tilde{\theta}_1 \in \mathbb{C}$, as $\theta(B)$ has order 1.

Directly, we see:

$$1 + 0.5B = 0$$
$$2 + B = 0$$
$$B = -2$$

$$\text{Hence, } \tilde{\theta}_1 = -2, \text{ and } \|\tilde{\theta}_1\| = 2 > 1 \therefore \{X_t\} \text{ is invertible. } \square$$

Thus, we have proved $\{X_t\}$ is invertible, as required.

Stationarity We know that $\{X_t\}$ is stationary.

Proof: Let $\tilde{\varphi}_j \in \mathbb{C}$ be the j-th root of $\varphi(B)$. In order for the ARMA process to be stationary, it must hold that all roots of $\varphi(B)$ are greater than the roots of unity. In other words, we wish to show that:

$$\forall j \in [1, p], \|\tilde{\varphi}_i\| > 1$$

Hence, we must find the roots of $\varphi(B)$ and their magnitude. By the Fundamental Theorem of Algebra, we know that there will exist exactly two roots $\{\tilde{\varphi}_1, \tilde{\varphi}_2\} \in \mathbb{C}$, as $\varphi(B)$ has order 2.

We can find them directly.

$$1 + 0.5B^2 = 0$$
$$2 + B^2 = 0$$
$$B = \pm\sqrt{-2}$$

Hence,
$$\{\tilde\varphi_1, \tilde\varphi_2\} = \{i\sqrt{2}, -i\sqrt{2}\}$$

Further, we know that
$$\|\tilde\varphi_1\| = \|\tilde\varphi_2\| = \sqrt{2} > 1, \therefore \{X_t\} \text{ is stationary. } \square$$

Thus, we have proved $\{X_t\}$ is stationary, as required.

In conclusion, we have shown that the ARMA(2, 1) process is both stationary and invertible.

b) Pure AR Process

Write the above ARMA(2,1) process as a pure AR process.

In order to write the model as a pure AR process, we will express it in the following form:
$$Z_t = \left(\frac{\varphi(B)}{\theta(B)}\right)X_t = \pi(B)X_t$$

Where $\theta(B)$ and $\varphi(B)$ are as defined in the previous question. Let us find an expression for $\pi(B)$:
$$\pi(B) = \frac{\varphi(B)}{\theta(B)} = \frac{(1 + 0.5B^2)}{(1 + 0.5B)} = \frac{(1 + 0.5B^2)}{\left(1 - (-0.5B)\right)} = (1 + 0.5B^2)\sum_{n=1}^{\infty}(-0.5B)^n$$

Where the last expression in the equality follows from the definition of a geometric series, under the assumption that $|-0.5B| < 1$. We expand this process, letting $\delta = 0.5$ for readability.

$$\pi(B) = (1 + \delta B^2)\sum_{n=1}^{\infty}(-\delta B)^n$$
$$\pi(B) = (1 + \delta B^2)\left[1 + (-\delta B) + (-\delta B)^2 + (-\delta B)^3 + \ldots\right]$$
$$\pi(B) = \left[1 + (-\delta B) + \left(\delta B^2 + (-\delta B)^2\right) + \left((\delta B^2)(-\delta B) + (-\delta B)^3\right) + \ldots\right]$$

Now, we can begin to attempt to investigate each $\pi_i \in \pi(B)$ where i indicates the power of B in the expression.

$$\pi_0 = 1$$
$$\pi_1 = (-\delta B)$$
$$\pi_2 = \delta B^2 + (-\delta B)^2$$
$$\pi_3 = (\delta B^2)(-\delta B) + (-\delta B)^3$$
$$\pi_4 = (\delta B^2)(-\delta B)^2 + (-\delta B)^4$$
$$\vdots$$

Investigating these terms, we begin to notice a pattern:

$$\pi_0 = (0 + (-\delta)^0)B^0$$
$$\pi_1 = (0 + (-\delta)^1)B^1$$
$$\pi_2 = (\delta + (-\delta)^2)B^2$$
$$\pi_3 = (\delta(-\delta) + (-\delta)^3)B^3$$
$$\pi_4 = (\delta(-\delta)^2 + (-\delta)^4)B^4$$
$$\vdots$$
$$\pi_j = (\delta(-\delta)^{j-2} + (-\delta)^j)B^j$$

With this in mind, we can denote the following term to define all elements of $\pi(B)$ for $j \in \mathbb{N}$ where, still using $\delta = 0.5$, each π_j term is defined by the rule stated (using indicators to account for the fact that π_0 and π_1 follow a slightly different general pattern.)

$$\pi(B) = \sum_{j=0}^{\infty} B^j \pi_j, \quad \text{where } \pi_j = \left(\mathbb{1}[j \geq 2]\delta(-\delta)^{j-2} \right) + (-\delta)^j$$

Now that we have a clear definition for $\pi(B)$, we can conclude by writing:

$$\pi(B)X_t = Z_t$$

Thus, we've converted the ARMA process to a pure infinite-order AR process with a new characteristic polynomial $\pi(B)$ as required.

c) Pure MA Process

Write the above ARMA(2,1) process as a pure MA process.

d) Autocorrelation Function

Find the ACF of $X_{t\,t\in\mathbb{Z}}$.

3. AR(2) Process

Consider the AR(2) process given by:

$$2aX_t + \frac{3}{6a}X_{t-1} - \frac{1}{6a}X_{t-2} = Z_t, \quad a \in \mathbb{R}$$

a) Stationarity Conditions

Under which conditions on constant a is the process $X_{t\,t\in\mathbb{Z}}$ stationary?

b) Autocorrelation Function

Assuming that a satisfies conditions found in part (a), find the ACF of $X_{t\,t\in\mathbb{Z}}$.

4. SARIMA Model

Show that SARIMA(2,1,1)$\times$(0,1,1)$_{12}$ can be written as an ARMA(p,q) process and specify values of p and q.