Stat 443: Time Series and Forecasting

Assignment 4: Analysis in the Frequency Domain

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March 28, 2024

Question 1

Consider the following second-order AR process AR(2) process for $\{X_t\}_{t\in\mathbb{Z}}$, where $\{Z_t\}_{t\in\mathbb{Z}} \overset{\text{iid}}{\sim} \text{WN}(0, \sigma^2)$.

$$X_t = \frac{7}{10}X_{t-1} - \frac{1}{10}X_{t-2} + Z_t$$

We have previously shown that the autocorrelation function $\gamma(h)$ for $h \in \mathbb{Z}$ is given by:

$$\rho(h) = \frac{16}{11}\Big(\frac{1}{2}\Big)^{|h|} - \frac{5}{11}\Big(\frac{1}{5}\Big)^{|h|}, \qquad h \in \mathbb{Z}$$

Part A

Derive the normalized spectral density function $f^\star(\omega)$ for $\{X_t\}_{t\in\mathbb{Z}}$.

Solution

We begin by verifying that the Fourier Transform is well defined.

$$\begin{aligned}
\sum_{h=-\infty}^{\infty} |\rho(h)| &= \sum_{h=-\infty}^{\infty} \left| \frac{16}{11}\Big(\frac{1}{2}\Big)^{|h|} - \frac{5}{11}\Big(\frac{1}{5}\Big)^{|h|} \right| \overset{?}{<} \infty \\
\sum_{t=-\infty}^{\infty} |\rho(h)| &= \Big(\frac{16}{11} - \frac{5}{11}\Big) + 2\bigg(\frac{16}{11}\sum_{h=1}^{\infty}\Big(\frac{1}{2}\Big)^h - \frac{5}{11}\sum_{h=1}^{\infty}\Big(\frac{1}{5}\Big)^h\bigg) \\
\sum_{t=-\infty}^{\infty} |\rho(h)| &= 1 + 2\bigg(\frac{16}{11}\Big(\frac{1/2}{1-1/2}\Big) - \frac{5}{11}\Big(\frac{1/5}{1-1/5}\Big)\bigg) \\
\sum_{t=-\infty}^{\infty} |\rho(h)| &= 1 + 2\bigg(\frac{16}{11} - \frac{5}{11}\Big(\frac{1}{4}\Big)\bigg) = \boxed{\frac{81}{22} < \infty, \ \therefore \text{well-defined.}}
\end{aligned}$$

Now, we evaluate given ρ, recalling that for $\omega \in (0, 1)$ and even functions, the normalized spectral density is given by:

$$f^\star(\omega) = \frac{1}{\pi}\bigg(\rho(0) + 2\sum_{h=1}^{\infty}\rho(h)\cos(\omega h)\bigg), \qquad \omega \in (0, 1)$$

Where $\rho(0) = 1$.

We will evaluate the infinite sum and substitute the result into the equation above. We will re-instate coefficients A_1 and A_2 from the previous assignment during intermediate steps for simplicity. In addition, we will let $d_1 = 1/2$ and $d_2 = 1/5$, noting that the geometric series equation is usable here as $|d_1|$ and $|d_2|$ are both less than 1.

$$\sum_{h=1}^{\infty} \rho(h)\cos(\omega h) = \sum_{h=1}^{\infty} \left(\frac{16}{11} \left(\frac{1}{2}\right)^{|h|} - \frac{5}{11} \left(\frac{1}{5}\right)^{|h|} \right) \cos(\omega h)$$

$$\sum_{h=1}^{\infty} \rho(h)\cos(\omega h) = \sum_{h=1}^{\infty} \left(A_1 (d_1)^{|h|} - A_2 (d_2)^{|h|} \right) \cos(\omega h), \quad \text{using variable form.}$$

$$\sum_{h=1}^{\infty} \rho(h)\cos(\omega h) = \underbrace{\sum_{h=1}^{\infty} \left(A_1 (d_1)^{|h|} \cos(\omega h) \right)}_{\text{Term 1}} - \underbrace{\sum_{h=1}^{\infty} \left(A_2 (d_2)^{|h|} \cos(\omega h) \right)}_{\text{Term 2}}$$

We will evaluate Term 1 and Term 2 separately. We will use the following identities without proof:

$$\cos(\omega h) = \frac{1}{2}\left(e^{\mathrm{i}h\omega} + e^{-\mathrm{i}h\omega}\right), \qquad \mathrm{i} = \sqrt{-1} \tag{1}$$

$$\sum_{n=1}^{\infty} a \cdot r^n = \frac{ar}{(1-r)}, \qquad |r| < 1,\ a \in \mathbb{R} \tag{2}$$

Evaluating Term 1, noting that $|h| = h$ since the summation spans $h \in \mathbb{Z}^+$.

$$\sum_{h=1}^{\infty} \left(A_1 (d_1)^{|h|} \cos(\omega h) \right) = A_1 \sum_{h=1}^{\infty} (d_1)^{|h|} \left(\frac{1}{2} \left(e^{\mathrm{i}h\omega} + e^{-\mathrm{i}h\omega} \right) \right), \qquad \text{by (1)}$$

$$\sum_{h=1}^{\infty} \left(A_1 (d_1)^{|h|} \cos(\omega h) \right) = \frac{A_1}{2} \sum_{h=1}^{\infty} (d_1)^{h} \left(e^{\mathrm{i}h\omega} + e^{-\mathrm{i}h\omega} \right)$$

$$\sum_{h=1}^{\infty} \left(A_1 (d_1)^{|h|} \cos(\omega h) \right) = \frac{A_1}{2} \left(\sum_{h=1}^{\infty} (d_1)^{h} e^{\mathrm{i}h\omega} + \sum_{h=1}^{\infty} (d_1)^{h} e^{-\mathrm{i}h\omega} \right)$$

$$\sum_{h=1}^{\infty} \left(A_1 (d_1)^{|h|} \cos(\omega h) \right) = \frac{A_1}{2} \left(\sum_{h=1}^{\infty} \left(d_1\, e^{\mathrm{i}\omega} \right)^{h} + \sum_{h=1}^{\infty} \left(d_1 e^{-\mathrm{i}\omega} \right)^{h} \right)$$

$$\sum_{h=1}^{\infty} \left(A_1 (d_1)^{|h|} \cos(\omega h) \right) = \frac{A_1}{2} \left(\frac{d_1 e^{\mathrm{i}\omega}}{1 - d_1 e^{\mathrm{i}\omega}} + \frac{d_1 e^{-\mathrm{i}\omega}}{1 - d_1 e^{-\mathrm{i}\omega}} \right), \qquad \text{by (2)}$$

$$\sum_{h=1}^{\infty} \left(A_1 (d_1)^{|h|} \cos(\omega h) \right) = \frac{A_1}{2} \left(\frac{d_1 e^{\mathrm{i}\omega}(1 - d_1 e^{-\mathrm{i}\omega}) + d_1 e^{-\mathrm{i}\omega}(1 - d_1 e^{\mathrm{i}\omega})}{(1 - d_1 e^{\mathrm{i}\omega})(1 - d_1 e^{-\mathrm{i}\omega})} \right)$$

$$\sum_{h=1}^{\infty} \left(A_1 (d_1)^{|h|} \cos(\omega h) \right) = \frac{A_1}{2} \left(\frac{d_1(e^{\mathrm{i}\omega} + e^{-\mathrm{i}\omega}) - 2d_1^2}{1 - d_1(e^{\mathrm{i}\omega} + e^{-\mathrm{i}\omega}) + d_1^2} \right)$$

$$\sum_{h=1}^{\infty} \left(A_1 (d_1)^{|h|} \cos(\omega h) \right) = \frac{A_1}{2} \left(\frac{2d_1 \cos(\omega) - 2d_1^2}{1 - 2d_1 \cos(\omega) + d_1^2} \right)$$

$$\sum_{h=1}^{\infty} \left(A_1 (d_1)^{|h|} \cos(\omega h) \right) = \frac{A_1(d_1 \cos(\omega) - d_1^2)}{1 - 2d_1 \cos(\omega) + d_1^2}$$

Similarly, if we repeat this exact same process with A_2 and d_2, noting that $|d_2| < 1$ and $A_2 \in \mathbb{R}$ also satisfy the requirements of (1) and (2), we arrive at Term 2:

$$\sum_{h=1}^{\infty} \left(A_2 (d_2)^{|h|} \cos(\omega h) \right) = \frac{A_2(d_2 \cos(\omega) - d_2^2)}{1 - 2d_2 \cos(\omega) + d_2^2}$$

Then, we can recombine these into our original expression for the infinite sum:

$$\sum_{h=1}^{\infty}\rho(h)\cos(\omega h) = \sum_{h=1}^{\infty}\Big(A_1(d_1)^{|h|}\cos(\omega h)\Big) - \sum_{h=1}^{\infty}\Big(A_2(d_2)^{|h|}\cos(\omega h)\Big)$$
$$\sum_{h=1}^{\infty}\rho(h)\cos(\omega h) = \frac{A_1(d_1\cos(\omega) - d_1^2)}{1-2d_1\cos(\omega)+d_1^2} - \frac{A_2(d_2\cos(\omega) - d_2^2)}{1-2d_2\cos(\omega)+d_2^2}$$
$$\sum_{h=1}^{\infty}\rho(h)\cos(\omega h) = \left(\frac{16}{11}\right)\frac{(\frac{1}{2}\cos(\omega) - \left(\frac{1}{2}\right)^2)}{1-2\left(\frac{1}{2}\right)\cos(\omega)+\left(\frac{1}{2}\right)^2} - \left(\frac{5}{11}\right)\frac{(\frac{1}{5}\cos(\omega) - (\frac{1}{5})^2)}{1-2(\frac{1}{5})\cos(\omega)+(\frac{1}{5})^2}$$
$$\sum_{h=1}^{\infty}\rho(h)\cos(\omega h) = \left(\frac{16}{11}\right)\frac{2\cos(\omega)-1}{5-4\cos(\omega)} - \left(\frac{5}{11}\right)\frac{5\cos(\omega)-1}{2(13-5\cos(\omega))}$$

Then, combining the expressions we can get the final expression for the normalized spectral density:

$$f^{\star}(\omega) = \frac{1}{\pi}\left(1+2\sum_{h=1}^{\infty}\rho(h)\cos(\omega h)\right)$$
$$f^{\star}(\omega) = \frac{1}{\pi}\left(1+2\left(\left(\frac{16}{11}\right)\frac{2\cos(\omega)-1}{5-4\cos(\omega)} - \left(\frac{5}{11}\right)\frac{5\cos(\omega)-1}{2(13-5\cos(\omega))}\right)\right)$$
$$f^{\star}(\omega) = \boxed{\frac{1}{\pi} + \frac{32}{11\pi}\left(\frac{2\cos(\omega)-1}{5-4\cos(\omega)}\right) - \frac{5}{11\pi}\left(\frac{5\cos(\omega)-1}{13-5\cos(\omega)}\right)}$$

We can verify these results below:

```
w = pi/4
# define acf
rho <- function(h){  (16/11)*((1/2)^h) - (5/11)*((1/5)^h) }
# define the "infinite sum"
sum_vals = sum(sapply(1:1000, function(h){
  rho(h)*cos(w*h)
}))
# from the equation...
eqnval = (1/pi)*(rho(0) + 2*sum_vals)

# from our simplification
fw1 = 1/pi
fw2 = (32/(11*pi)) * ( (2 * cos(w) - 1) / (5 - 4*cos(w)) )
fw3 = (5/(11*pi)) * ( (5 * cos(w) - 1) / (13 - 5*cos(w)) )
# comparison
c(eqnval, fw1 + fw2 - fw3)
```

```
## [1] 0.4561755 0.4561755
```

We see that the values are identical for at least the first $10,000$ lags at fixed $\omega = \pi/4$.

Part B

Write down the power spectral density function of $\{X_t\}_{t\in\mathbb{Z}}$.

Solution

We recall from the definition of normalized spectral density that

$$f^\star(\omega) = \frac{f(\omega)}{\sigma_X^2}$$

Where σ_X^2 is the variance of $\{X_t\}_{t\in\mathbb{Z}}$.

Directly, then, we can write $f(\omega)$ as:

$$f(\omega) = \sigma_X^2 f^\star(\omega) = \gamma(0) f^\star(\omega)$$

We re-establish the Yule-Walker equations, where $\alpha_1 = 7/10$ and $\alpha_2 = -1/10$

$$\begin{aligned}
\mathbb{E}(X_tX_t) &= \alpha_1\mathbb{E}(X_tX_{t-1}) - \alpha_2\mathbb{E}(X_tX_{t-2}) + \mathbb{E}(X_tZ_t)\\
\mathbb{E}(X_tX_{t-1}) &= \alpha_1\mathbb{E}(X_{t-1}X_{t-1}) - \alpha_2\mathbb{E}(X_{t-1}X_{t-2}) + \mathbb{E}(Z_tX_{t-1})\\
\mathbb{E}(X_tX_{t-2}) &= \alpha_1\mathbb{E}(X_{t-1}X_{t-2}) - \alpha_2\mathbb{E}(X_{t-2}X_{t-2}) + \mathbb{E}(Z_tX_{t-2})
\end{aligned}$$

Which becomes the following system of three equations:

$$\begin{aligned}
\gamma(0) &= \alpha_1\gamma(1) + \alpha_2\gamma(2) + \sigma^2\\
\gamma(1) &= \alpha_1\gamma(0) + \alpha_2\gamma(1)\\
\gamma(2) &= \alpha_1\gamma(1) + \alpha_2\gamma(0)
\end{aligned}$$

test area:

$$\begin{aligned}
\gamma(1) &= \alpha_1\gamma(0) + \alpha_2\gamma(1)\\
\gamma(1) &= \frac{\alpha_1}{1-\alpha_2}\gamma(0)
\end{aligned}$$

then

$$\begin{aligned}
\gamma(2) &= \alpha_1\gamma(1) + \alpha_2\gamma(0)\\
\gamma(2) &= \alpha_1\Big(\frac{\alpha_1}{1-\alpha_2}\gamma(0)\Big) + \alpha_2\gamma(0)\\
\gamma(2) &= \Big(\frac{\alpha_1^2+\alpha_2-\alpha_2^2}{1-\alpha_2}\Big)\gamma(0)
\end{aligned}$$

finally

$$\begin{aligned}
\gamma(0) &= \alpha_1\gamma(1) + \alpha_2\gamma(2) + \sigma^2\\
\gamma(0) &= \alpha_1\Big(\frac{\alpha_1}{1-\alpha_2}\Big)\gamma(0) + \alpha_2\Big(\frac{\alpha_1^2+\alpha_2-\alpha_2^2}{1-\alpha_2}\Big)\gamma(0) + \sigma^2\\
\gamma(0) &= \Big(\frac{\alpha_1^2+\alpha_1^2\alpha_2+\alpha_2^2-\alpha_2^3}{1-\alpha_2}\Big)\gamma(0) + \sigma^2\\
\gamma(0)&\Big(1-\frac{\alpha_1^2+\alpha_1^2\alpha_2+\alpha_2^2-\alpha_2^3}{1-\alpha_2}\Big) = \sigma^2\\
\gamma(0)&\Big(\frac{(1-\alpha_2)-\alpha_1^2-\alpha_1^2\alpha_2-\alpha_2^2+\alpha_2^3}{1-\alpha_2}\Big) = \sigma^2\\
\gamma(0) &= \frac{\sigma^2(1-\alpha_2)}{1-\alpha_2-\alpha_1^2-\alpha_1^2\alpha_2-\alpha_2^2+\alpha_2^3}
\end{aligned}$$

Then, we can evaluate at our given α_1 and α_2, simplifying the fraction above using Python to avoid human error.

```python
from sympy import *

alpha_1, alpha_2, sigma_sq = symbols('alpha_1 alpha_2 sigma_sq')

gamma_0_expr = ( sigma_sq * (1 - alpha_2) /
    (1 - alpha_2 - alpha_1**2 - alpha_1**2 * alpha_2 - alpha_2**2 + alpha_2**3))

alpha_1_val = Rational(7, 10)
alpha_2_val = Rational(-1, 10)

gamma_0_eval = gamma_0_expr.subs({alpha_1: alpha_1_val, alpha_2: alpha_2_val})

gamma_0_simplified = nsimplify(gamma_0_eval)
print(gamma_0_simplified)
```

```
## 275*sigma_sq/162
```

Then, we can verify both our results and Python's simplification as follows, assuming $\sigma^2 = 1$.

```r
set.seed(3)
alpha_1 <- 7/10
alpha_2 <- -1/10
sigma_sq <- 1

gamma_0 <- (sigma_sq * (1 - alpha_2) /
        (1 - alpha_2 - alpha_1^2 - alpha_1^2 * alpha_2 - alpha_2^2 + alpha_2^3))

ar_params <- c(alpha_1, alpha_2)
simulated_data <- arima.sim(n = 5000000,
                            model = list(ar = c(7/10, -1/10)))

simulated_gamma_0 <- var(simulated_data)
# use both regular and simplified fraction version
cat("Computed Variance Using Simplified Fraction:", 275/162, "\n")
```

```
## Computed Variance Using Simplified Fraction: 1.697531
```

```r
cat("Computed Variance Using Computation:", gamma_0, "\n")
```

```
## Computed Variance Using Computation: 1.697531
```

```r
cat("Variance of Simulated ARIMA process:", simulated_gamma_0, "\n")
```

```
## Variance of Simulated ARIMA process: 1.697897
```

It seems the computation very closely approximates the truth. Hence, we conclude that:

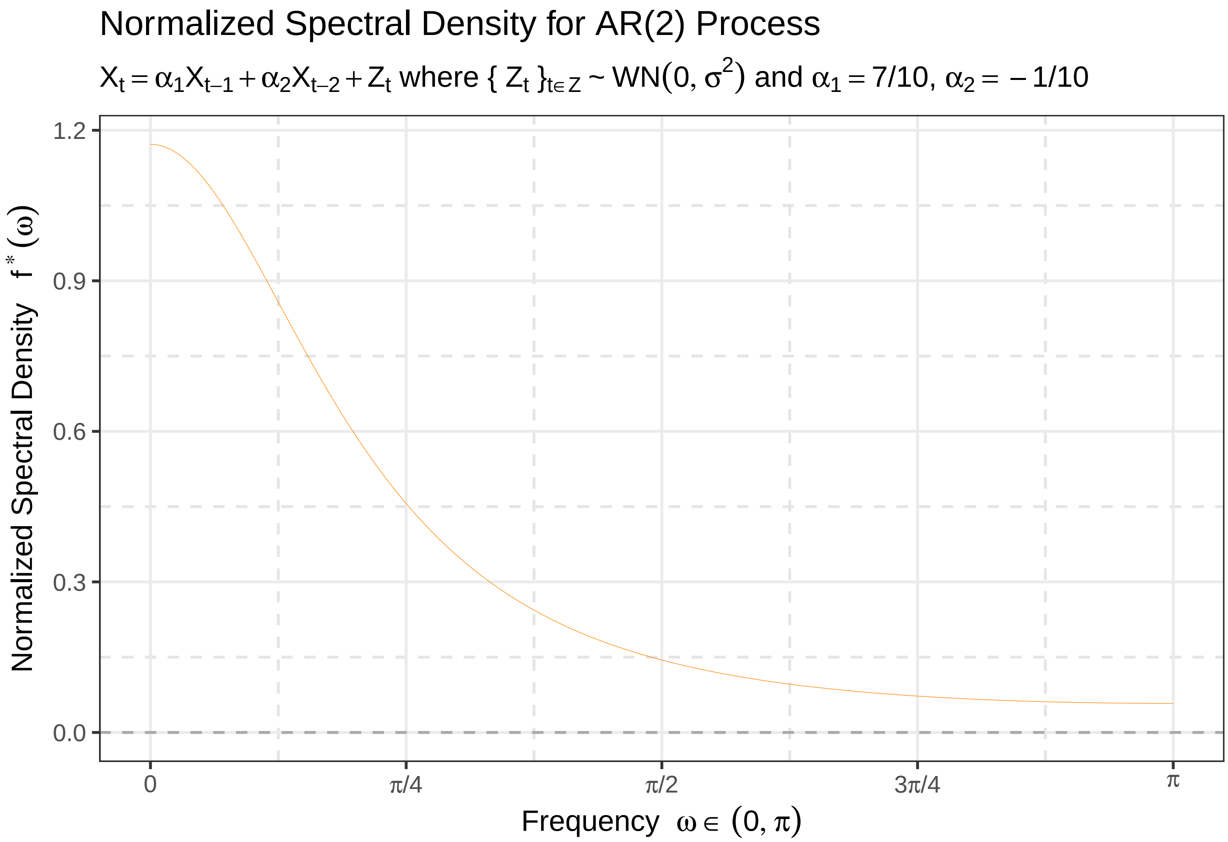
$$f(\omega) = \gamma(0) f^{\star}(\omega) = \frac{275\sigma^2}{162}\left(\frac{1}{\pi} + \frac{32}{11\pi}\left(\frac{2\cos(\omega) - 1}{5 - 4\cos(\omega)}\right) - \frac{5}{11\pi}\left(\frac{5\cos(\omega) - 1}{13 - 5\cos(\omega)}\right)\right)$$

Part C

Plot the normalized spectral density and comment on its behaviour.

The normalized spectral density ended up being quite long, so we'll define each term one-by-one in the function below:

```
f = function(w){
  term_1 = 1
  term_2 = (32/11)*((2*cos(w) - 1) / (5 - 4*cos(w)))
  term_3 = (5 /11)*((5*cos(w) - 1) / (13 - 5*cos(w)))
  return ( (1/pi)*(term_1 + term_2 - term_3) )
}
omega = seq(from = 0, to = pi, length.out = 1e4)
# define data frame for values
p1df = data.frame(omega = omega,
           f_star_omega = f(omega))
# buld plot
p1 <- ggplot(p1df, aes(x = omega, y = f_star_omega)) +
  geom_line(color = "#ff8600", linewidth = 0.1) +
  labs(
    title = "Normalized Spectral Density for AR(2) Process",
    subtitle = TeX(paste(
      "$X_t = \\alpha_1 X_{t-1} + \\alpha_2 X_{t-2} + Z_t$",
      "where",
      "$\\{\\,Z_t\\,\\}_{t \\in {Z}} \\sim WN(0, \\sigma^2)$",
      "and $\\alpha_1 =$7/10, $\\alpha_2 =-$1/10")),
    y = TeX("Normalized Spectral Density   $f^{*}(\\omega)$"),
    x = TeX("Frequency  $\\omega \\in (0, \\pi)$")
  ) + theme_bw() +
  geom_hline(yintercept = 0, lty = 'dashed', col = "darkgrey")+
  scale_x_continuous(
    breaks = c(0, pi/4, pi/2, 3*pi/4, pi),
    labels = c(TeX("0"), TeX("$\\pi$/4"), TeX("$\\pi$/2"),
               TeX("$3\\pi$/4"), TeX("$\\pi$")))+
  theme(panel.grid.minor = element_line(
    color = "grey90",
    linetype = "dashed",
    linewidth = 0.5
  ))
print(p1)
```



Comments: It appears that the normalized spectral density plot is largely dominated by low frequencies. We can tell this is the case due to the fact that the largest values of $f^\star(\omega)$ is for small $\omega \in (0, \pi)$ specifically for $\omega \in (0, \pi/4)$ we see the majority of frequencies with observed normalized density greater than 0.5. This tells us that a greater proportion of the variance inherent in the process X_t can be attributed to the lower frequencies. As we would expect of a normalized spectral density dominated by low ω values, we see that $f^\star(\omega)$ is strictly decreasing as $\omega \to \pi$.

Question 2