Stat 443: Time Series and Forecasting

Assignment 4: Analysis in the Frequency Domain

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Question 1

Consider the following second-order AR process AR(2) process for $\{X_t\}_{t\in\mathbb{Z}}$, where $\{Z_t\}_{t\in\mathbb{Z}} \overset{iid}{\sim} \text{WN}(0, \sigma^2)$.

$$
X_t = \frac{7}{10}X_{t-1} - \frac{1}{10}X_{t-2} + Z_t
$$

We have previously shown that the autocorrelation function $\gamma(h)$ for $h \in \mathbb{Z}$ is given by:

$$
\rho(h) = \frac{16}{11}\left(\frac{1}{2}\right)^{|h|} - \frac{5}{11}\left(\frac{1}{5}\right)^{|h|}, \quad h \in \mathbb{Z}
$$

Part A

Derive the normalized spectral density function $f^\star(\omega)$ for $\{X_t\}_{t\in\mathbb{Z}}$.

Solution

We begin by verifying that the Fourier Transform is well defined.

$$
\sum_{h=-\infty}^{\infty} |\rho(h)| = \sum_{h=-\infty}^{\infty} \left| \frac{16}{11}\left(\frac{1}{2}\right)^{|h|} - \frac{5}{11}\left(\frac{1}{5}\right)^{|h|} \right| \overset{?}{<} \infty
$$

$$
\sum_{t=-\infty}^{\infty} |\rho(h)| = \left(\frac{16}{11} - \frac{5}{11}\right) + 2\left(\frac{16}{11}\sum_{h=1}^{\infty}\left(\frac{1}{2}\right)^h - \frac{5}{11}\sum_{h=1}^{\infty}\left(\frac{1}{5}\right)^h\right)
$$

$$
\sum_{t=-\infty}^{\infty} |\rho(h)| = 1 + 2\left(\frac{16}{11}\left(\frac{1/2}{1 - 1/2}\right) - \frac{5}{11}\left(\frac{1/5}{1 - 1/5}\right)\right)
$$

$$
\sum_{t=-\infty}^{\infty} |\rho(h)| = 1 + 2\left(\frac{16}{11} - \frac{5}{11}\left(\frac{1}{4}\right)\right) = \boxed{\frac{81}{22} < \infty, \therefore \text{ well-defined.}}
$$

Now, we evaluate given ρ, recalling that for $\omega \in (0, 1)$ and even functions, the normalized spectral density is given by:

$$
f^\star(\omega) = \frac{1}{\pi}\left(\rho(0) + 2\sum_{h=1}^{\infty}\rho(h)\cos(\omega h)\right), \quad \omega \in (0, 1)
$$

Where $\rho(0) = 1$.

We will evaluate the infinite sum and substitute the result into the equation above. We will re-instate coefficients A_1 and A_2 from the previous assignment during intermediate steps for simplicity. In addition, we will let $d_1 = 1/2$ and $d_2 = 1/5$, noting that the geometric series equation is usable here as $|d_1|$ and $|d_2|$ are both less than 1.

$$\sum_{h=1}^{\infty} \rho(h)\cos(\omega h) = \sum_{h=1}^{\infty} \left(\frac{16}{11}\left(\frac{1}{2}\right)^{|h|} - \frac{5}{11}\left(\frac{1}{5}\right)^{|h|} \right)\cos(\omega h)$$

$$\sum_{h=1}^{\infty} \rho(h)\cos(\omega h) = \sum_{h=1}^{\infty} \left(A_1(d_1)^{|h|} - A_2(d_2)^{|h|} \right)\cos(\omega h), \quad \text{using variable form.}$$

$$\sum_{h=1}^{\infty} \rho(h)\cos(\omega h) = \underbrace{\sum_{h=1}^{\infty} \left(A_1(d_1)^{|h|}\cos(\omega h) \right)}_{\text{Term 1}} - \underbrace{\sum_{h=1}^{\infty} \left(A_2(d_2)^{|h|}\cos(\omega h) \right)}_{\text{Term 2}}$$

We will evaluate Term 1 and Term 2 separately. We will use the following identities without proof:

$$\cos(\omega h) = \frac{1}{2}\left(e^{ih\omega} + e^{-ih\omega} \right), \quad i = \sqrt{-1} \tag{1}$$

$$\sum_{n=1}^{\infty} a \cdot r^n = \frac{ar}{(1-r)}, \quad |r| < 1,\ a \in \mathbb{R} \tag{2}$$

Evaluating Term 1, noting that $|h| = h$ since the summation spans $h \in \mathbb{Z}^+$.

$$\sum_{h=1}^{\infty} \left(A_1(d_1)^{|h|}\cos(\omega h) \right) = A_1 \sum_{h=1}^{\infty}(d_1)^{|h|}\left(\frac{1}{2}\left(e^{ih\omega} + e^{-ih\omega} \right) \right), \quad \text{by (1)}$$

$$\sum_{h=1}^{\infty} \left(A_1(d_1)^{|h|}\cos(\omega h) \right) = \frac{A_1}{2} \sum_{h=1}^{\infty}(d_1)^{h}\left(e^{ih\omega} + e^{-ih\omega} \right)$$

$$\sum_{h=1}^{\infty} \left(A_1(d_1)^{|h|}\cos(\omega h) \right) = \frac{A_1}{2} \left(\sum_{h=1}^{\infty}(d_1)^{h}e^{ih\omega} + \sum_{h=1}^{\infty}(d_1)^{h}e^{-ih\omega} \right)$$

$$\sum_{h=1}^{\infty} \left(A_1(d_1)^{|h|}\cos(\omega h) \right) = \frac{A_1}{2} \left(\sum_{h=1}^{\infty}\left(d_1\,e^{i\omega} \right)^{h} + \sum_{h=1}^{\infty}\left(d_1 e^{-i\omega} \right)^{h} \right)$$

$$\sum_{h=1}^{\infty} \left(A_1(d_1)^{|h|}\cos(\omega h) \right) = \frac{A_1}{2} \left(\frac{d_1 e^{i\omega}}{1 - d_1 e^{i\omega}} + \frac{d_1 e^{-i\omega}}{1 - d_1 e^{-i\omega}} \right), \quad \text{by (2)}$$

$$\sum_{h=1}^{\infty} \left(A_1(d_1)^{|h|}\cos(\omega h) \right) = \frac{A_1}{2} \left(\frac{d_1 e^{i\omega}(1 - d_1 e^{-i\omega}) + d_1 e^{-i\omega}(1 - d_1 e^{i\omega})}{(1 - d_1 e^{i\omega})(1 - d_1 e^{-i\omega})} \right)$$

$$\sum_{h=1}^{\infty} \left(A_1(d_1)^{|h|}\cos(\omega h) \right) = \frac{A_1}{2} \left(\frac{d_1(e^{i\omega} + e^{-i\omega}) - 2d_1^2}{1 - d_1(e^{i\omega} + e^{-i\omega}) + d_1^2} \right)$$

$$\sum_{h=1}^{\infty} \left(A_1(d_1)^{|h|}\cos(\omega h) \right) = \frac{A_1}{2} \left(\frac{2d_1 \cos(\omega) - 2d_1^2}{1 - 2d_1 \cos(\omega) + d_1^2} \right)$$

$$\sum_{h=1}^{\infty} \left(A_1(d_1)^{|h|}\cos(\omega h) \right) = \frac{A_1(d_1 \cos(\omega) - d_1^2)}{1 - 2d_1 \cos(\omega) + d_1^2}$$

Similarly, if we repeat this exact same process with A_2 and d_2, noting that $|d_2| < 1$ and $A_2 \in \mathbb{R}$ also satisfy the requirements of (1) and (2), we arrive at Term 2:

$$\sum_{h=1}^{\infty} \left(A_2(d_2)^{|h|}\cos(\omega h) \right) = \frac{A_2(d_2 \cos(\omega) - d_2^2)}{1 - 2d_2 \cos(\omega) + d_2^2}$$

Then, we can recombine these into our original expression for the infinite sum:

$$\sum_{h=1}^{\infty} \rho(h)\cos(\omega h) = \sum_{h=1}^{\infty}\left(A_1(d_1)^{|h|}\cos(\omega h)\right) - \sum_{h=1}^{\infty}\left(A_2(d_2)^{|h|}\cos(\omega h)\right)$$

$$\sum_{h=1}^{\infty} \rho(h)\cos(\omega h) = \frac{A_1(d_1\cos(\omega) - d_1^2)}{1 - 2d_1\cos(\omega) + d_1^2} - \frac{A_2(d_2\cos(\omega) - d_2^2)}{1 - 2d_2\cos(\omega) + d_2^2}$$

$$\sum_{h=1}^{\infty} \rho(h)\cos(\omega h) = \left(\frac{16}{11}\right)\frac{\left(\frac{1}{2}\cos(\omega) - \left(\frac{1}{2}\right)^2\right)}{1 - 2\left(\frac{1}{2}\right)\cos(\omega) + \left(\frac{1}{2}\right)^2} - \left(\frac{5}{11}\right)\frac{\left(\frac{1}{5}\cos(\omega) - \left(\frac{1}{5}\right)^2\right)}{1 - 2\left(\frac{1}{5}\right)\cos(\omega) + \left(\frac{1}{5}\right)^2}$$

$$\sum_{h=1}^{\infty} \rho(h)\cos(\omega h) = \left(\frac{16}{11}\right)\frac{2\cos(\omega) - 1}{5 - 4\cos(\omega)} - \left(\frac{5}{11}\right)\frac{5\cos(\omega) - 1}{2(13 - 5\cos(\omega))}$$

Then, combining the expressions we can get the final expression for the normalized spectral density:

$$f^{\star}(\omega) = \frac{1}{\pi}\left(1 + 2\sum_{h=1}^{\infty}\rho(h)\cos(\omega h)\right)$$

$$f^{\star}(\omega) = \frac{1}{\pi}\left(1 + 2\left(\left(\frac{16}{11}\right)\frac{2\cos(\omega) - 1}{5 - 4\cos(\omega)} - \left(\frac{5}{11}\right)\frac{5\cos(\omega) - 1}{2(13 - 5\cos(\omega))}\right)\right)$$

$$\boxed{f^{\star}(\omega) = \frac{1}{\pi} + \frac{32}{11\pi}\left(\frac{2\cos(\omega) - 1}{5 - 4\cos(\omega)}\right) - \frac{5}{11\pi}\left(\frac{5\cos(\omega) - 1}{13 - 5\cos(\omega)}\right)}$$

We can verify these results below:

```r
w = pi/4
# define acf
rho <- function(h){  (16/11)*((1/2)^h) - (5/11)*((1/5)^h) }
# define the "infinite sum"
sum_vals = sum(sapply(1:1000, function(h){
  rho(h)*cos(w*h)
}))
# from the equation...
eqnval = (1/pi)*(rho(0) + 2*sum_vals)

# from our simplification
fw1 = 1/pi
fw2 = (32/(11*pi)) * ( (2 * cos(w) - 1) / (5 - 4*cos(w)) )
fw3 = (5/(11*pi)) * ( (5 * cos(w) - 1) / (13 - 5*cos(w)) )
# comparison
c(eqnval, fw1 + fw2 - fw3)
```

```
## [1] 0.4561755 0.4561755
```

We see that the values are identical for at least the first $10,000$ lags at fixed $\omega = \pi/4$.

Part B

Write down the power spectral density function of $\{X_t\}_{t \in \mathbb{Z}}$.

Solution

We recall from the definition of normalized spectral density that

$$f^\star(\omega) = \frac{f(\omega)}{\sigma_X^2}$$

Where σ_X^2 is the variance of $\{X_t\}_{t \in \mathbb{Z}}$.

Directly, then, we can write $f(\omega)$ as:

$$f(\omega) = \sigma_X^2 f^\star(\omega) = \gamma(0) f^\star(\omega)$$

We re-establish the Yule-Walker equations, where $\alpha_1 = 7/10$ and $\alpha_2 = -1/10$

$$\mathbb{E}(X_t X_t) = \alpha_1 \mathbb{E}(X_t X_{t-1}) - \alpha_2 \mathbb{E}(X_t X_{t-2}) + \mathbb{E}(X_t Z_t)$$
$$\mathbb{E}(X_t X_{t-1}) = \alpha_1 \mathbb{E}(X_{t-1} X_{t-1}) - \alpha_2 \mathbb{E}(X_{t-1} X_{t-2}) + \mathbb{E}(Z_t X_{t-1})$$
$$\mathbb{E}(X_t X_{t-2}) = \alpha_1 \mathbb{E}(X_{t-1} X_{t-2}) - \alpha_2 \mathbb{E}(X_{t-2} X_{t-2}) + \mathbb{E}(Z_t X_{t-2})$$

Which becomes the following system of three equations:

$$\gamma(0) = \alpha_1 \gamma(1) + \alpha_2 \gamma(2) + \sigma^2$$
$$\gamma(1) = \alpha_1 \gamma(0) + \alpha_2 \gamma(1)$$
$$\gamma(2) = \alpha_1 \gamma(1) + \alpha_2 \gamma(0)$$

test area:
$$\gamma(1) = \alpha_1 \gamma(0) + \alpha_2 \gamma(1)$$
$$\gamma(1) = \frac{\alpha_1}{1 - \alpha_2} \gamma(0)$$

then
$$\gamma(2) = \alpha_1 \gamma(1) + \alpha_2 \gamma(0)$$
$$\gamma(2) = \alpha_1 \left(\frac{\alpha_1}{1 - \alpha_2} \gamma(0) \right) + \alpha_2 \gamma(0)$$
$$\gamma(2) = \left(\frac{\alpha_1^2 + \alpha_2 - \alpha_2^2}{1 - \alpha_2} \right) \gamma(0)$$

finally
$$\gamma(0) = \alpha_1 \gamma(1) + \alpha_2 \gamma(2) + \sigma^2$$
$$\gamma(0) = \alpha_1 \left(\frac{\alpha_1}{1 - \alpha_2} \right) \gamma(0) + \alpha_2 \left(\frac{\alpha_1^2 + \alpha_2 - \alpha_2^2}{1 - \alpha_2} \right) \gamma(0) + \sigma^2$$
$$\gamma(0) = \left(\frac{\alpha_1^2 + \alpha_1^2 \alpha_2 + \alpha_2^2 - \alpha_2^3}{1 - \alpha_2} \right) \gamma(0) + \sigma^2$$
$$\gamma(0) \left(1 - \frac{\alpha_1^2 + \alpha_1^2 \alpha_2 + \alpha_2^2 - \alpha_2^3}{1 - \alpha_2} \right) = \sigma^2$$
$$\gamma(0) \left(\frac{(1 - \alpha_2) - \alpha_1^2 - \alpha_1^2 \alpha_2 - \alpha_2^2 + \alpha_2^3}{1 - \alpha_2} \right) = \sigma^2$$
$$\gamma(0) = \frac{\sigma^2 (1 - \alpha_2)}{1 - \alpha_2 - \alpha_1^2 - \alpha_1^2 \alpha_2 - \alpha_2^2 + \alpha_2^3}$$

Then, we can evaluate at our given α_1 and α_2, simplifying the fraction using Python to avoid human error.

```python
from sympy import *

alpha_1, alpha_2, sigma_sq = symbols('alpha_1 alpha_2 sigma_sq')

gamma_0_expr = ( sigma_sq * (1 - alpha_2) /
    (1 - alpha_2 - alpha_1**2 - alpha_1**2 * alpha_2 - alpha_2**2 + alpha_2**3))

alpha_1_val = Rational(7, 10)
alpha_2_val = Rational(-1, 10)

gamma_0_eval = gamma_0_expr.subs({alpha_1: alpha_1_val, alpha_2: alpha_2_val})

gamma_0_simplified = nsimplify(gamma_0_eval)
print(gamma_0_simplified)
```

```
## 275*sigma_sq/162
```

Then, we can verify both our results and Python's simplification as follows, assuming $\sigma^2 = 1$.

```r
set.seed(3)
alpha_1 <- 7/10
alpha_2 <- -1/10
sigma_sq <- 1

gamma_0 <- (sigma_sq * (1 - alpha_2) /
        (1 - alpha_2 - alpha_1^2 - alpha_1^2 * alpha_2 - alpha_2^2 + alpha_2^3))

ar_params <- c(alpha_1, alpha_2)
simulated_data <- arima.sim(n = 5000000,
                            model = list(ar = c(7/10, -1/10)))

simulated_gamma_0 <- var(simulated_data)
# use both regular and simplified fraction version
cat("Computed Variance Using Simplified Fraction:", 275/162, "\n")
```

```
## Computed Variance Using Simplified Fraction: 1.697531
```

```r
cat("Computed Variance Using Computation:", gamma_0, "\n")
```

```
## Computed Variance Using Computation: 1.697531
```

```r
cat("Variance of Simulated AR(2) process:", simulated_gamma_0, "\n")
```

```
## Variance of Simulated AR(2) process: 1.697897
```

It seems the computation very closely approximates the truth. Hence, we conclude that:

$$f(\omega) = \gamma(0)f^{\star}(\omega) = \frac{275\sigma^2}{162}\left(\frac{1}{\pi} + \frac{32}{11\pi}\left(\frac{2\cos(\omega) - 1}{5 - 4\cos(\omega)}\right) - \frac{5}{11\pi}\left(\frac{5\cos(\omega) - 1}{13 - 5\cos(\omega)}\right)\right)$$

Part C

Plot the normalized spectral density and comment on its behaviour.

The normalized spectral density equation ended up being quite long, so we'll define each term one-by-one in the function below:

```
f = function(w){
  term_1 = 1
  term_2 = (32/11)*((2*cos(w) - 1) / (5 - 4*cos(w)))
  term_3 = (5 /11)*((5*cos(w) - 1) / (13 - 5*cos(w)))
  return ( (1/pi)*(term_1 + term_2 - term_3) )
}
```

Then, we plot the function for a long sequence of ω values in $(0, \pi)$:

```
omega = seq(from = 0, to = pi, length.out = 1e4)
# define data frame for values
p1df = data.frame(omega = omega,
          f_star_omega = f(omega))
# buld plot
p1 <- ggplot(p1df, aes(x = omega, y = f_star_omega)) +
  geom_line(color = "#ff8600", linewidth = 0.1) +
  labs(
    title = "Normalized Spectral Density for AR(2) Process",
    subtitle = TeX(paste(
      "$X_t = \\alpha_1 X_{t-1} + \\alpha_2 X_{t-2} + Z_t$",
      "where",
      "$\\{\\,Z_t\\,\\}_{t \\in {Z}} \\sim WN(0, \\sigma^2)$",
      "and $\\alpha_1 =$7/10, $\\alpha_2 =-$1/10")),
    y = TeX("Normalized Spectral Density   $f^{*}(\\omega)$"),
    x = TeX("Frequency  $\\omega \\in (0, \\pi)$")
  ) + theme_bw() +
  geom_hline(yintercept = 0, lty = 'dashed', col = "darkgrey")+
  scale_x_continuous(
    breaks = c(0, pi/4, pi/2, 3*pi/4, pi),
    labels = c(TeX("0"), TeX("$\\pi$/4"), TeX("$\\pi$/2"),
              TeX("$3\\pi$/4"), TeX("$\\pi$")))+
  theme(panel.grid.minor = element_line(
    color = "grey90",
    linetype = "dashed",
    linewidth = 0.5
  ))
print(p1)
```

Normalized Spectral Density for AR(2) Process

$X_t = \alpha_1 X_{t-1} + \alpha_2 X_{t-2} + Z_t$ where $\{ Z_t \}_{t \in Z} \sim WN(0, \sigma^2)$ and $\alpha_1 = 7/10$, $\alpha_2 = -1/10$
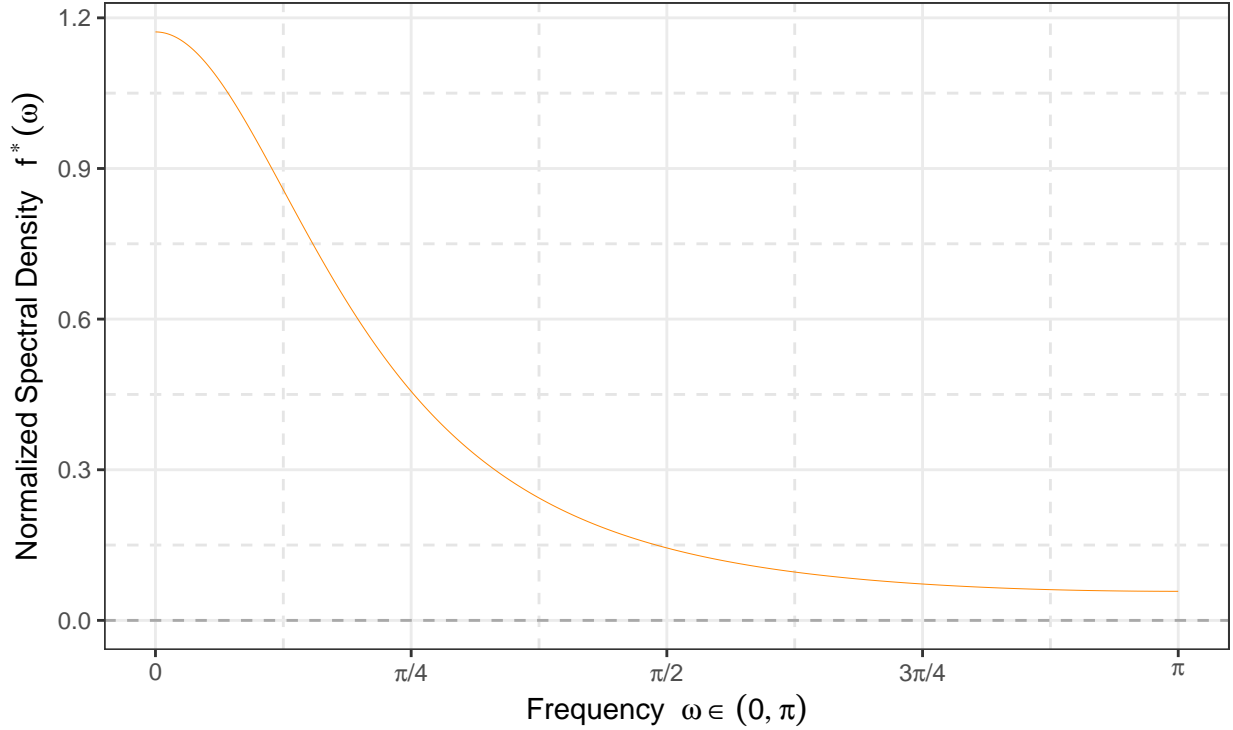
Comments: It appears that the normalized spectral density plot is largely dominated by low frequencies. We can tell this is the case due to the fact that the largest values of $f^\star(\omega)$ is for small $\omega \in (0, \pi)$ specifically for $\omega \in (0, \pi/4)$ we see the majority of frequencies with observed normalized density greater than 0.5. This tells us that a greater proportion of the variance inherent in the process X_t can be attributed to the lower frequencies. As we would expect of a normalized spectral density dominated by low ω values, we see that $f^\star(\omega)$ is strictly decreasing as $\omega \to \pi$.

Question 2

Given the spectral density function

$$f(\omega) = \frac{1}{\pi}\Big(62 - 70\cos(\omega) + 12\cos(2w)\Big), \quad \omega \in (0, 1)$$

compute the autocovariance function $\gamma(h)$ and autocorrelation function $\rho(h)$ of the underlying stochastic process, where $h \in \mathbb{Z}$.

We will begin with computing the autocovariance function $\gamma(h)$ at lag $h = 0$, then at $h = 1$, $k = 2$ and finally $k > 2$.

Part D: Lags Greater than 2

Before we evaluate lags $h > 2 \subset \mathbb{Z}$, we will establish the following lemma:

Lemma 1: Let $m \in \mathbb{Z}, n \in \mathbb{Z}^+$ We will show that:

$$\forall (m,n) \in \mathbb{Z}^+ \text{ s.t. } m \neq n, \int_0^\pi \cos(m\omega)\cos(n\omega)\,d\omega = 0$$

Proof: In order to show this equality holds, we will use the following product-to-sum identity of cosines without proof. Let $\theta, \vartheta \in \mathbb{R}$.

$$\cos(\theta)\cos(\vartheta) = \frac{1}{2}\cos(\theta + \vartheta) + \frac{1}{2}\cos(\theta - \vartheta) \tag{2.1}$$

We will use (2.1) to evaluate the integral.

$$\int_0^\pi \cos(m\omega)\cos(n\omega)\,d\omega = \int_0^\pi \left(\frac{1}{2}\cos(m\omega + n\omega) + \frac{1}{2}\cos(m\omega - n\omega) \right) d\omega$$

$$\int_0^\pi \cos(m\omega)\cos(n\omega)\,d\omega = \frac{1}{2}\int_0^\pi \underbrace{\cos\left((m+n)\omega\right)}_{\text{Let } q=(m+n)\omega}\,d\omega + \frac{1}{2}\int_0^\pi \underbrace{\cos\left((m-n)\omega\right)}_{\text{Let } \nu=(m-n)\omega}\,d\omega$$

$$\int_0^\pi \cos(m\omega)\cos(n\omega)\,d\omega = \frac{1}{2(m+n)}\int_0^\pi \cos(q)\,dq + \frac{1}{2(m-n)}\int_0^\pi \cos(\nu)\,d\nu, \qquad \text{requires } m \neq n$$

$$\int_0^\pi \cos(m\omega)\cos(n\omega)\,d\omega = \frac{1}{2(m+n)}\left(\sin(q)\Big|_{\omega=0}^\pi \right) + \frac{1}{2(m-n)}\left(\sin(\nu)\Big|_{\omega=0}^\pi \right)$$

$$\int_0^\pi \cos(m\omega)\cos(n\omega)\,d\omega = \frac{1}{2(m+n)}\left(\sin(m\omega + n\omega)\Big|_0^\pi \right) + \frac{1}{2(m-n)}\left(\sin(m\omega - n\omega)\Big|_0^\pi \right)$$

$$\int_0^\pi \cos(m\omega)\cos(n\omega)\,d\omega = \frac{1}{2(m+n)}\left(\sin\left((m+n)\pi\right) - \sin(0) \right) + \frac{1}{2(m-n)}\left(\sin\left((m-n)\pi\right) - \sin(0) \right)$$

$$\int_0^\pi \cos(m\omega)\cos(n\omega)\,d\omega = \frac{1}{2(m+n)}\sin\left((m+n)\pi\right) + \frac{1}{2(m-n)}\sin\left((m-n)\pi\right)$$

We note that $m \in \mathbb{Z}$ and $n \in \mathbb{Z}$. Hence, by properties of integers, we know that $m + n \in \mathbb{Z}$ and $m - n \in \mathbb{Z}$.

We can hence note that:
$$(m \in \mathbb{Z} \wedge n \in \mathbb{Z}) \implies \exists \ell \in \mathbb{Z} \text{ s.t. } \ell = m + n$$

Since $(m,n) \in \mathbb{Z}^+$ and $m \neq n$, we know that $m + n > 0$, $\therefore \ell > 0$.

Similarly, we have that:
$$(m \in \mathbb{Z} \wedge n \in \mathbb{Z}) \implies \exists \varrho \in \mathbb{Z} \text{ s.t. } \varrho = m - n$$

Further, since $(m,n) \in \mathbb{Z}^+$ and $m \neq n$, we know that $\varrho \neq 0$. We also note by properties of sine that $\forall k \in \mathbb{Z}, \sin(k \cdot \pi) = 0$.

Finally, we can rewrite our simplified integral in terms of ℓ and ϱ and solve.

$$\int_0^\pi \cos(m\omega)\cos(n\omega)\,d\omega = \frac{1}{2\ell}\sin(\ell\pi) + \frac{1}{2\varrho}\sin(\varrho\pi)$$

$$\int_0^\pi \cos(m\omega)\cos(n\omega)\,d\omega = \frac{1}{2\ell}(0) + \frac{1}{2\varrho}(0), \quad \text{since } \forall k \in \mathbb{Z}, \sin(k \cdot \pi) = 0$$

$$\int_0^\pi \cos(m\omega)\cos(n\omega)\,d\omega = (0) + (0), \qquad \text{since } \ell \neq 0 \text{ and } \varrho \neq 0$$

$$\int_0^\pi \cos(m\omega)\cos(n\omega)\,d\omega = 0$$

Hence, we conclude that

$$\forall (m,n) \in \mathbb{Z}^+ \text{ s.t. } m \neq n, \int_0^\pi \cos(m\omega)\cos(n\omega)\,d\omega = 0, \quad \text{as required. } \square$$

Application of the Lemma

Now, we can show that $\forall h > 2 \subseteq \mathbb{Z}^+$ that:

$$\gamma(h) = \int_0^\pi f(\omega)\cos(h\omega)\mathrm{d}\omega = 0$$

Proof: We begin by expanding $f(\omega)$ with our previous variable assignments:

$$\int_0^\pi f(\omega)\cos(h\omega)\mathrm{d}\omega = \int_0^\pi \left[\frac{1}{\pi}\Big(a - b\cos(\omega) + c\cos(2w)\Big)\right]\cos(h\omega)\mathrm{d}\omega$$

$$\int_0^\pi f(\omega)\cos(h\omega)\mathrm{d}\omega = \frac{a}{\pi}\int_0^\pi \cos(h\omega)\mathrm{d}\omega - \frac{b}{\pi}\int_0^\pi \cos(\omega)\cos(h\omega)\mathrm{d}\omega + \frac{c}{\pi}\int_0^\pi \cos(2\omega)\cos(h\omega)\mathrm{d}\omega$$

$$\int_0^\pi f(\omega)\cos(h\omega)\mathrm{d}\omega = \frac{a}{\pi}\int_0^\pi \cos(0\cdot\omega)\cos(h\cdot\omega)\mathrm{d}\omega - \frac{b}{\pi}\int_0^\pi \cos(1\cdot\omega)\cos(h\cdot\omega)\mathrm{d}\omega + \frac{c}{\pi}\int_0^\pi \cos(2\cdot\omega)\cos(h\cdot\omega)\mathrm{d}\omega$$

Now, from the expansion on the third line, we can draw an interesting parallel to the lemma established previously. Let $M = \{0, 1, 2\}$ and $N = \{n \in \mathbb{Z} : n > 2\}$.

We see by construction of these sets $M \cap N = \emptyset$. Importantly,

$$M \cap N = \emptyset \implies \forall n \in N, \forall m \in M,\ m \neq n$$

Now, if we let $M = \{m_1, m_2, m_3\} = \{0, 1, 2\}$ and $n \in N := \{h \in \mathbb{Z} : h > 2\}$, we can rewrite our integral in terms of m and n and simplify using our lemma.

$$\int_0^\pi f(\omega)\cos(h\omega)\mathrm{d}\omega = \frac{a}{\pi}\int_0^\pi \underbrace{\cos(m_1\cdot\omega)\cos(n\cdot\omega)}_{\forall n \in N,\, m_1 \neq n}\mathrm{d}\omega - \frac{b}{\pi}\int_0^\pi \underbrace{\cos(m_2\cdot\omega)\cos(n\cdot\omega)}_{\forall n \in N,\, m_2 \neq n}\mathrm{d}\omega + \frac{c}{\pi}\int_0^\pi \underbrace{\cos(m_2\cdot\omega)\cos(n\cdot\omega)}_{\forall n \in N,\, m_3 \neq n}\mathrm{d}\omega$$

$$\int_0^\pi f(\omega)\cos(h\omega)\mathrm{d}\omega = \frac{a}{\pi}(0) - \frac{b}{\pi}(0) + \frac{c}{\pi}(0),\ \text{by set construction and Lemma 1.}$$

$$\int_0^\pi f(\omega)\cos(h\omega)\mathrm{d}\omega = \boxed{0}$$

So, we can conclude that:

$$\forall h > 2, \gamma(h) = \int_0^\pi f(\omega)\cos(h\omega)\mathrm{d}\omega = 0,\ \text{as required.}\ \square$$

Autocovariance Function

Hence, bringing all the previous parts together, we can conclude that for $h \in \mathbb{Z}$:

$$\gamma(h) = \begin{cases} \gamma(-h), & h < 0 \\ \dots, & h = 0 \\ \dots, & h = 1 \\ \dots, & h = 2 \\ 0 & \text{otherwise} \end{cases}$$

Autocorrelation Function