Stat 443: Time Series and Forecasting

Assignment 4: Analysis in the Frequency Domain

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Question 1

Consider the following second-order AR process AR(2) process for $\{X_t\}_{t\in\mathbb{Z}}$, where $\{Z_t\}_{t\in\mathbb{Z}} \overset{\text{iid}}{\sim} \text{WN}(0, \sigma^2)$.

$$X_t = \frac{7}{10}X_{t-1} - \frac{1}{10}X_{t-2} + Z_t$$

We have previously shown that the autocorrelation function $\gamma(h)$ for $h \in \mathbb{Z}$ is given by:

$$\rho(h) = \frac{16}{11}\Big(\frac{1}{2}\Big)^{|h|} - \frac{5}{11}\Big(\frac{1}{5}\Big)^{|h|}, \qquad h \in \mathbb{Z}$$

Part A

Derive the normalized spectral density function $f^\star(\omega)$ for $\{X_t\}_{t\in\mathbb{Z}}$.

Solution

We begin by verifying that the Fourier Transform is well defined.

$$\begin{aligned}
\sum_{h=-\infty}^{\infty} |\rho(h)| &= \sum_{h=-\infty}^{\infty} \left| \frac{16}{11}\Big(\frac{1}{2}\Big)^{|h|} - \frac{5}{11}\Big(\frac{1}{5}\Big)^{|h|} \right| \overset{?}{<} \infty \\
\sum_{t=-\infty}^{\infty} |\rho(h)| &= \Big(\frac{16}{11} - \frac{5}{11}\Big) + 2\bigg(\frac{16}{11}\sum_{h=1}^{\infty}\Big(\frac{1}{2}\Big)^h - \frac{5}{11}\sum_{h=1}^{\infty}\Big(\frac{1}{5}\Big)^h\bigg) \\
\sum_{t=-\infty}^{\infty} |\rho(h)| &= 1 + 2\bigg(\frac{16}{11}\Big(\frac{1/2}{1-1/2}\Big) - \frac{5}{11}\Big(\frac{1/5}{1-1/5}\Big)\bigg) \\
\sum_{t=-\infty}^{\infty} |\rho(h)| &= 1 + 2\bigg(\frac{16}{11} - \frac{5}{11}\Big(\frac{1}{4}\Big)\bigg) = \boxed{\frac{81}{22} < \infty, \ \therefore \text{well-defined.}}
\end{aligned}$$

Now, we evaluate given ρ, recalling that for $\omega \in (0, 1)$ and even functions, the normalized spectral density is given by:

$$f^\star(\omega) = \frac{1}{\pi}\bigg(\rho(0) + 2\sum_{h=1}^{\infty}\rho(h)\cos(\omega h)\bigg), \qquad \omega \in (0, 1)$$

Where, trivially, $\rho(0) = 1$.

We will evaluate the infinite sum and substitute the result into the equation above. We will re-instate coefficients A_1 and A_2 from the previous assignment during intermediate steps for simplicity.

$$\sum_{h=1}^{\infty} \rho(h)\cos(\omega h) = \sum_{h=1}^{\infty} \left(\frac{16}{11}\left(\frac{1}{2}\right)^{|h|} - \frac{5}{11}\left(\frac{1}{5}\right)^{|h|} \right) \cos(\omega h)$$

$$\sum_{h=1}^{\infty} \rho(h)\cos(\omega h) = \sum_{h=1}^{\infty} \left(A_1\left(\frac{1}{2}\right)^{|h|} - A_2\left(\frac{1}{5}\right)^{|h|} \right) \cos(\omega h)$$

$$\sum_{h=1}^{\infty} \rho(h)\cos(\omega h) = \underbrace{\sum_{h=1}^{\infty} \left(A_1\left(\frac{1}{2}\right)^{|h|} \cos(\omega h) \right)}_{\text{Term 1}} - \underbrace{\sum_{h=1}^{\infty} \left(A_2\left(\frac{1}{5}\right)^{|h|} \cos(\omega h) \right)}_{\text{Term 2}}$$

We will evaluate Term 1 and Term 2 separately. We will use the following identity without proof:

$$\cos(\omega h) = \frac{1}{2}\left(e^{\mathrm{i}h\omega} + e^{-\mathrm{i}h\omega}\right), \qquad \mathrm{i} = \sqrt{-1}$$

Evaluating Term 1, noting that $|h| = h$ since the summation spans $h \in \mathbb{Z}^+$.

$$\sum_{h=1}^{\infty} \left(A_1\left(\frac{1}{2}\right)^{|h|} \cos(\omega h) \right) = A_1 \sum_{h=1}^{\infty} \left(\frac{1}{2}\right)^{h} \left(\frac{1}{2}\left(e^{\mathrm{i}h\omega} + e^{-\mathrm{i}h\omega}\right)\right)$$

$$\sum_{h=1}^{\infty} \left(A_1\left(\frac{1}{2}\right)^{|h|} \cos(\omega h) \right) = \frac{A_1}{2} \sum_{h=1}^{\infty} \left(\frac{1}{2}\right)^{h} \left(e^{\mathrm{i}h\omega} + e^{-\mathrm{i}h\omega}\right)$$

$$\sum_{h=1}^{\infty} \left(A_1\left(\frac{1}{2}\right)^{|h|} \cos(\omega h) \right) = \frac{A_1}{2} \left(\sum_{h=1}^{\infty} \left(\frac{1}{2}\right)^{h} e^{\mathrm{i}h\omega} + \sum_{h=1}^{\infty} \left(\frac{1}{2}\right)^{h} e^{-\mathrm{i}h\omega} \right)$$

$$\sum_{h=1}^{\infty} \left(A_1\left(\frac{1}{2}\right)^{|h|} \cos(\omega h) \right) = \frac{A_1}{2} \left(\sum_{h=1}^{\infty} \left(\frac{1}{2}e^{\mathrm{i}\omega}\right)^{h} + \sum_{h=1}^{\infty} \left(\frac{1}{2}e^{-\mathrm{i}\omega}\right)^{h} \right)$$