Stat 443: Time Series and Forecasting

Assignment 4: Analysis in the Frequency Domain

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Question 1

Consider the following second-order AR process AR(2) process for $\{X_t\}_{t\in\mathbb{Z}}$, where $\{Z_t\}_{t\in\mathbb{Z}} \overset{\text{iid}}{\sim} \text{WN}(0,\sigma^2)$.

$$X_t = \frac{7}{10}X_{t-1} - \frac{1}{10}X_{t-2} + Z_t$$

We have previously shown that the autocorrelation function $\gamma(h)$ for $h \in \mathbb{Z}$ is given by:

$$\rho(h) = \frac{16}{11}\Big(\frac{1}{2}\Big)^{|h|} - \frac{5}{11}\Big(\frac{1}{5}\Big)^{|h|}, \qquad h \in \mathbb{Z}$$

Part A

Derive the normalized spectral density function $f^\star(\omega)$ for $\{X_t\}_{t\in\mathbb{Z}}$.

Solution

We begin by verifying that the Fourier Transform is well defined.

$$\begin{aligned}
\sum_{h=-\infty}^{\infty} |\rho(h)| &= \sum_{h=-\infty}^{\infty} \left| \frac{16}{11}\Big(\frac{1}{2}\Big)^{|h|} - \frac{5}{11}\Big(\frac{1}{5}\Big)^{|h|}\right| \overset{?}{<} \infty \\
\sum_{t=-\infty}^{\infty} |\rho(h)| &= \Big(\frac{16}{11} - \frac{5}{11}\Big) + 2\bigg(\frac{16}{11}\sum_{h=1}^{\infty}\Big(\frac{1}{2}\Big)^{h} - \frac{5}{11}\sum_{h=1}^{\infty}\Big(\frac{1}{5}\Big)^{h}\bigg) \\
\sum_{t=-\infty}^{\infty} |\rho(h)| &= 1 + 2\bigg(\frac{16}{11}\Big(\frac{1/2}{1-1/2}\Big) - \frac{5}{11}\Big(\frac{1/5}{1-1/5}\Big)\bigg) \\
\sum_{t=-\infty}^{\infty} |\rho(h)| &= 1 + 2\bigg(\frac{16}{11} - \frac{5}{11}\Big(\frac{1}{4}\Big)\bigg) = \boxed{\frac{81}{22} < \infty, \ \therefore \text{well-defined.}}
\end{aligned}$$

Now, we evaluate given ρ, recalling that for $\omega \in (0,1)$ and even functions, the normalized spectral density is given by:

$$f^\star(\omega) = \frac{1}{\pi}\bigg(\rho(0) + 2\sum_{h=1}^{\infty}\rho(h)\cos(\omega h)\bigg), \qquad \omega \in (0,1)$$

Where, trivially, $\rho(0) = 1$.

We will evaluate the infinite sum and substitute the result into the equation above. We will re-instate coefficients A_1 and A_2 from the previous assignment during intermediate steps for simplicity. In addition, we will let $d_1 = 1/2$ and $d_2 = 1/5$, noting that the geometric series equation is usable here as $|d_1|$ and $|d_2|$ are both less than 1.

$$\sum_{h=1}^{\infty} \rho(h)\cos(\omega h) = \sum_{h=1}^{\infty} \left(\frac{16}{11}\left(\frac{1}{2}\right)^{|h|} - \frac{5}{11}\left(\frac{1}{5}\right)^{|h|} \right) \cos(\omega h)$$

$$\sum_{h=1}^{\infty} \rho(h)\cos(\omega h) = \sum_{h=1}^{\infty} \left(A_1(d_1)^{|h|} - A_2(d_2)^{|h|} \right) \cos(\omega h), \quad \text{using variable form.}$$

$$\sum_{h=1}^{\infty} \rho(h)\cos(\omega h) = \underbrace{\sum_{h=1}^{\infty} \left(A_1(d_1)^{|h|}\cos(\omega h) \right)}_{\text{Term 1}} - \underbrace{\sum_{h=1}^{\infty} \left(A_2(d_2)^{|h|}\cos(\omega h) \right)}_{\text{Term 2}}$$

We will evaluate Term 1 and Term 2 separately. We will use the following identities without proof:

$$\cos(\omega h) = \frac{1}{2}\left(e^{\mathrm{i}h\omega} + e^{-\mathrm{i}h\omega}\right), \quad \mathrm{i} = \sqrt{-1} \tag{1}$$

$$\sum_{n=1}^{\infty} a \cdot r^n = \frac{ar}{(1-r)}, \quad |r| < 1,\ a \in \mathbb{R} \tag{2}$$

Evaluating Term 1, noting that $|h| = h$ since the summation spans $h \in \mathbb{Z}^+$.

$$\sum_{h=1}^{\infty} \left(A_1(d_1)^{|h|}\cos(\omega h) \right) = A_1 \sum_{h=1}^{\infty} (d_1)^{|h|} \left(\frac{1}{2}\left(e^{\mathrm{i}h\omega} + e^{-\mathrm{i}h\omega}\right) \right), \quad \text{by (1)}$$

$$\sum_{h=1}^{\infty} \left(A_1(d_1)^{|h|}\cos(\omega h) \right) = \frac{A_1}{2} \sum_{h=1}^{\infty} (d_1)^{h} \left(e^{\mathrm{i}h\omega} + e^{-\mathrm{i}h\omega}\right)$$

$$\sum_{h=1}^{\infty} \left(A_1(d_1)^{|h|}\cos(\omega h) \right) = \frac{A_1}{2} \left(\sum_{h=1}^{\infty} (d_1)^{h} e^{\mathrm{i}h\omega} + \sum_{h=1}^{\infty} (d_1)^{h} e^{-\mathrm{i}h\omega} \right)$$

$$\sum_{h=1}^{\infty} \left(A_1(d_1)^{|h|}\cos(\omega h) \right) = \frac{A_1}{2} \left(\sum_{h=1}^{\infty} \left(d_1\, e^{\mathrm{i}\omega}\right)^h + \sum_{h=1}^{\infty} \left(d_1 e^{-\mathrm{i}\omega}\right)^h \right)$$

$$\sum_{h=1}^{\infty} \left(A_1(d_1)^{|h|}\cos(\omega h) \right) = \frac{A_1}{2} \left(\frac{d_1 e^{\mathrm{i}\omega}}{1 - d_1 e^{\mathrm{i}\omega}} + \frac{d_1 e^{-\mathrm{i}\omega}}{1 - d_1 e^{-\mathrm{i}\omega}} \right), \quad \text{by (2)}$$

$$\sum_{h=1}^{\infty} \left(A_1(d_1)^{|h|}\cos(\omega h) \right) = \frac{A_1}{2} \left(\frac{d_1 e^{\mathrm{i}\omega}(1 - d_1 e^{-\mathrm{i}\omega}) + d_1 e^{-\mathrm{i}\omega}(1 - d_1 e^{\mathrm{i}\omega})}{(1 - d_1 e^{\mathrm{i}\omega})(1 - d_1 e^{-\mathrm{i}\omega})} \right)$$

$$\sum_{h=1}^{\infty} \left(A_1(d_1)^{|h|}\cos(\omega h) \right) = \frac{A_1}{2} \left(\frac{d_1(e^{\mathrm{i}\omega} + e^{-\mathrm{i}\omega}) - 2d_1^2}{1 - d_1(e^{\mathrm{i}\omega} + e^{-\mathrm{i}\omega}) + d_1^2} \right)$$

$$\sum_{h=1}^{\infty} \left(A_1(d_1)^{|h|}\cos(\omega h) \right) = \frac{A_1}{2} \left(\frac{2d_1\cos(\omega) - 2d_1^2}{1 - 2d_1\cos(\omega) + d_1^2} \right)$$

$$\sum_{h=1}^{\infty} \left(A_1(d_1)^{|h|}\cos(\omega h) \right) = \frac{A_1(d_1\cos(\omega) - d_1^2)}{1 - 2d_1\cos(\omega) + d_1^2}$$

Similarly, if we repeat this exact same process with A_2 and d_2, noting that $|d_2| < 1$ and $A_2 \in \mathbb{R}$ also satisfy the requirements of (1) and (2), we arrive at Term 2:

$$\sum_{h=1}^{\infty} \left(A_2(d_2)^{|h|}\cos(\omega h) \right) = \frac{A_2(d_2\cos(\omega) - d_2^2)}{1 - 2d_2\cos(\omega) + d_2^2}$$

Then, we can recombine these into our original expression for the infinite sum:

$$\sum_{h=1}^{\infty} \rho(h)\cos(\omega h) = \sum_{h=1}^{\infty}\left(A_1(d_1)^{|h|}\cos(\omega h)\right) - \sum_{h=1}^{\infty}\left(A_2(d_2)^{|h|}\cos(\omega h)\right)$$
$$\sum_{h=1}^{\infty} \rho(h)\cos(\omega h) = \frac{A_1(d_1\cos(\omega) - d_1^2)}{1-2d_1\cos(\omega)+d_1^2} - \frac{A_2(d_2\cos(\omega) - d_2^2)}{1-2d_2\cos(\omega)+d_2^2}$$
$$\sum_{h=1}^{\infty} \rho(h)\cos(\omega h) = \left(\frac{16}{11}\right)\frac{(\frac{1}{2}\cos(\omega) - \left(\frac{1}{2}\right)^2)}{1-2\left(\frac{1}{2}\right)\cos(\omega)+\left(\frac{1}{2}\right)^2} - \left(\right)\frac{(d_2\cos(\omega) - d_2^2)}{1-2d_2\cos(\omega)+d_2^2}$$

```
from sympy import *
from pytexit import py2tex


w = symbols('w')
# numerator and denominator
numerator = 16 * ( (1/2)*cos(w) - (1/2)**(2) )
denominator = 11 * (1 - 2*(1/2)*cos(w) + (1/2)**2)
# simplify first term
term_1 = nsimplify(numerator / denominator)
# pytex_obj = py2tex(str(term_1))
together(term_1)
```

```
## 16*(2*cos(w) - 1)/(11*(5 - 4*cos(w)))
```