Stat 443: Time Series and Forecasting

Assignment 4: Analysis in the Frequency Domain

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Question 1

Consider the following second-order AR process AR(2) process for $\{X_t\}_{t\in\mathbb{Z}}$, where $\{Z_t\}_{t\in\mathbb{Z}} \overset{\text{iid}}{\sim} \text{WN}(0, \sigma^2)$.

$$X_t = \frac{7}{10}X_{t-1} - \frac{1}{10}X_{t-2} + Z_t$$

We have previously shown that the autocorrelation function $\gamma(h)$ for $h \in \mathbb{Z}$ is given by:

$$\rho(h) = \frac{16}{11}\Big(\frac{1}{2}\Big)^{|h|} - \frac{5}{11}\Big(\frac{1}{5}\Big)^{|h|}, \qquad h \in \mathbb{Z}$$

Part A

Derive the normalized spectral density function $f^\star(\omega)$ for $\{X_t\}_{t\in\mathbb{Z}}$.

Solution

We begin by verifying that the Fourier Transform is well defined.

$$\begin{aligned}
\sum_{h=-\infty}^{\infty} |\rho(h)| &= \sum_{h=-\infty}^{\infty} \left|\frac{16}{11}\Big(\frac{1}{2}\Big)^{|h|} - \frac{5}{11}\Big(\frac{1}{5}\Big)^{|h|}\right| \overset{?}{<} \infty \\
\sum_{t=-\infty}^{\infty} |\rho(h)| &= \Big(\frac{16}{11} - \frac{5}{11}\Big) + 2\bigg(\frac{16}{11}\sum_{h=1}^{\infty}\Big(\frac{1}{2}\Big)^{h} - \frac{5}{11}\sum_{h=1}^{\infty}\Big(\frac{1}{5}\Big)^{h}\bigg) \\
\sum_{t=-\infty}^{\infty} |\rho(h)| &= 1 + 2\bigg(\frac{16}{11}\Big(\frac{1/2}{1-1/2}\Big) - \frac{5}{11}\Big(\frac{1/5}{1-1/5}\Big)\bigg) \\
\sum_{t=-\infty}^{\infty} |\rho(h)| &= 1 + 2\bigg(\frac{16}{11} - \frac{5}{11}\Big(\frac{1}{4}\Big)\bigg) = \boxed{\frac{81}{22} < \infty, \ \therefore \text{well-defined.}}
\end{aligned}$$

Now, we evaluate given ρ, recalling that for $\omega \in (0,1)$ and even functions, the normalized spectral density is given by:

$$f^\star(\omega) = \frac{1}{\pi}\bigg(\rho(0) + 2\sum_{h=1}^{\infty}\rho(h)\cos(\omega h)\bigg), \qquad \omega \in (0,1)$$

Where $\rho(0) = 1$.

We will evaluate the infinite sum and substitute the result into the equation above. We will re-instate coefficients A_1 and A_2 from the previous assignment during intermediate steps for simplicity. In addition, we will let $d_1 = 1/2$ and $d_2 = 1/5$, noting that the geometric series equation is usable here as $|d_1|$ and $|d_2|$ are both less than 1.

$$\sum_{h=1}^{\infty} \rho(h)\cos(\omega h) = \sum_{h=1}^{\infty} \left(\frac{16}{11}\left(\frac{1}{2}\right)^{|h|} - \frac{5}{11}\left(\frac{1}{5}\right)^{|h|} \right) \cos(\omega h)$$

$$\sum_{h=1}^{\infty} \rho(h)\cos(\omega h) = \sum_{h=1}^{\infty} \left(A_1(d_1)^{|h|} - A_2(d_2)^{|h|} \right) \cos(\omega h), \quad \text{using variable form.}$$

$$\sum_{h=1}^{\infty} \rho(h)\cos(\omega h) = \underbrace{\sum_{h=1}^{\infty} \left(A_1(d_1)^{|h|}\cos(\omega h) \right)}_{\text{Term 1}} - \underbrace{\sum_{h=1}^{\infty} \left(A_2(d_2)^{|h|}\cos(\omega h) \right)}_{\text{Term 2}}$$

We will evaluate Term 1 and Term 2 separately. We will use the following identities without proof:

$$\cos(\omega h) = \frac{1}{2}\left(e^{\mathrm{i}h\omega} + e^{-\mathrm{i}h\omega}\right), \qquad \mathrm{i} = \sqrt{-1} \tag{1}$$

$$\sum_{n=1}^{\infty} a \cdot r^n = \frac{ar}{(1-r)}, \qquad |r| < 1,\ a \in \mathbb{R} \tag{2}$$

Evaluating Term 1, noting that $|h| = h$ since the summation spans $h \in \mathbb{Z}^+$.

$$\sum_{h=1}^{\infty} \left(A_1(d_1)^{|h|}\cos(\omega h) \right) = A_1 \sum_{h=1}^{\infty} (d_1)^{|h|} \left(\frac{1}{2}\left(e^{\mathrm{i}h\omega} + e^{-\mathrm{i}h\omega}\right) \right), \qquad \text{by (1)}$$

$$\sum_{h=1}^{\infty} \left(A_1(d_1)^{|h|}\cos(\omega h) \right) = \frac{A_1}{2} \sum_{h=1}^{\infty} (d_1)^{h} \left(e^{\mathrm{i}h\omega} + e^{-\mathrm{i}h\omega}\right)$$

$$\sum_{h=1}^{\infty} \left(A_1(d_1)^{|h|}\cos(\omega h) \right) = \frac{A_1}{2} \left(\sum_{h=1}^{\infty} (d_1)^{h} e^{\mathrm{i}h\omega} + \sum_{h=1}^{\infty} (d_1)^{h} e^{-\mathrm{i}h\omega} \right)$$

$$\sum_{h=1}^{\infty} \left(A_1(d_1)^{|h|}\cos(\omega h) \right) = \frac{A_1}{2} \left(\sum_{h=1}^{\infty} \left(d_1\, e^{\mathrm{i}\omega}\right)^h + \sum_{h=1}^{\infty} \left(d_1 e^{-\mathrm{i}\omega}\right)^h \right)$$

$$\sum_{h=1}^{\infty} \left(A_1(d_1)^{|h|}\cos(\omega h) \right) = \frac{A_1}{2} \left(\frac{d_1 e^{\mathrm{i}\omega}}{1 - d_1 e^{\mathrm{i}\omega}} + \frac{d_1 e^{-\mathrm{i}\omega}}{1 - d_1 e^{-\mathrm{i}\omega}} \right), \qquad \text{by (2)}$$

$$\sum_{h=1}^{\infty} \left(A_1(d_1)^{|h|}\cos(\omega h) \right) = \frac{A_1}{2} \left(\frac{d_1 e^{\mathrm{i}\omega}(1 - d_1 e^{-\mathrm{i}\omega}) + d_1 e^{-\mathrm{i}\omega}(1 - d_1 e^{\mathrm{i}\omega})}{(1 - d_1 e^{\mathrm{i}\omega})(1 - d_1 e^{-\mathrm{i}\omega})} \right)$$

$$\sum_{h=1}^{\infty} \left(A_1(d_1)^{|h|}\cos(\omega h) \right) = \frac{A_1}{2} \left(\frac{d_1(e^{\mathrm{i}\omega} + e^{-\mathrm{i}\omega}) - 2d_1^2}{1 - d_1(e^{\mathrm{i}\omega} + e^{-\mathrm{i}\omega}) + d_1^2} \right)$$

$$\sum_{h=1}^{\infty} \left(A_1(d_1)^{|h|}\cos(\omega h) \right) = \frac{A_1}{2} \left(\frac{2d_1\cos(\omega) - 2d_1^2}{1 - 2d_1\cos(\omega) + d_1^2} \right)$$

$$\sum_{h=1}^{\infty} \left(A_1(d_1)^{|h|}\cos(\omega h) \right) = \frac{A_1(d_1\cos(\omega) - d_1^2)}{1 - 2d_1\cos(\omega) + d_1^2}$$

Similarly, if we repeat this exact same process with A_2 and d_2, noting that $|d_2| < 1$ and $A_2 \in \mathbb{R}$ also satisfy the requirements of (1) and (2), we arrive at Term 2:

$$\sum_{h=1}^{\infty} \left(A_2(d_2)^{|h|}\cos(\omega h) \right) = \frac{A_2(d_2\cos(\omega) - d_2^2)}{1 - 2d_2\cos(\omega) + d_2^2}$$

Then, we can recombine these into our original expression for the infinite sum:

$$\sum_{h=1}^{\infty} \rho(h)\cos(\omega h) = \sum_{h=1}^{\infty} \left(A_1 (d_1)^{|h|} \cos(\omega h) \right) - \sum_{h=1}^{\infty} \left(A_2 (d_2)^{|h|} \cos(\omega h) \right)$$
$$\sum_{h=1}^{\infty} \rho(h)\cos(\omega h) = \frac{A_1(d_1\cos(\omega) - d_1^2)}{1 - 2d_1\cos(\omega) + d_1^2} - \frac{A_2(d_2\cos(\omega) - d_2^2)}{1 - 2d_2\cos(\omega) + d_2^2}$$
$$\sum_{h=1}^{\infty} \rho(h)\cos(\omega h) = \left(\frac{16}{11}\right) \frac{(\frac{1}{2}\cos(\omega) - \left(\frac{1}{2}\right)^2)}{1 - 2\left(\frac{1}{2}\right)\cos(\omega) + \left(\frac{1}{2}\right)^2} - \left(\frac{5}{11}\right) \frac{(\frac{1}{5}\cos(\omega) - (\frac{1}{5})^2)}{1 - 2(\frac{1}{5})\cos(\omega) + (\frac{1}{5})^2}$$
$$\sum_{h=1}^{\infty} \rho(h)\cos(\omega h) = \left(\frac{16}{11}\right) \frac{2\cos(\omega) - 1}{5 - 4\cos(\omega)} - \left(\frac{5}{11}\right) \frac{5\cos(\omega) - 1}{2(13 - 5\cos(\omega))}$$

Then, combining the expressions we can get the final expression for the normalized spectral density:

$$f^{\star}(\omega) = \frac{1}{\pi}\left(1 + 2\sum_{h=1}^{\infty} \rho(h)\cos(\omega h)\right)$$
$$f^{\star}(\omega) = \frac{1}{\pi}\left(1 + 2\left(\left(\frac{16}{11}\right)\frac{2\cos(\omega) - 1}{5 - 4\cos(\omega)} - \left(\frac{5}{11}\right)\frac{5\cos(\omega) - 1}{2(13 - 5\cos(\omega))}\right)\right)$$
$$f^{\star}(\omega) = \boxed{\frac{1}{\pi} + \frac{32}{11\pi}\left(\frac{2\cos(\omega) - 1}{5 - 4\cos(\omega)}\right) - \frac{5}{11\pi}\left(\frac{5\cos(\omega) - 1}{13 - 5\cos(\omega)}\right)}$$

We can verify these results below:

```
w = pi/4
# define acf
rho <- function(h){  (16/11)*((1/2)^h) - (5/11)*((1/5)^h) }
# define the "infinite sum"
sum_vals = sum(sapply(1:1000, function(h){
  rho(h)*cos(w*h)
}))
# from the equation...
eqnval = (1/pi)*(rho(0) + 2*sum_vals)

# from our simplification
fw1 = 1/pi
fw2 = (32/(11*pi)) * ( (2 * cos(w) - 1) / (5 - 4*cos(w)) )
fw3 = (5/(11*pi)) * ( (5 * cos(w) - 1) / (13 - 5*cos(w)) )
# comparison
c(eqnval, fw1 + fw2 - fw3)
```

```
## [1] 0.4561755 0.4561755
```

We see that the values are identical for at least the first 10,000 lags at fixed $\omega = \pi/4$.

Part B

Write down the power spectral density function of $\{X_t\}_{t\in\mathbb{Z}}$.

Solution

We recall from the definition of normalized spectral density that

$$f^\star(\omega) = \frac{f(\omega)}{\sigma_X^2}$$

Where σ_X^2 is the variance of $\{X_t\}_{t\in\mathbb{Z}}$.

Directly, then, we can write $f(\omega)$ as:

$$f(\omega) = \sigma_X^2 f^\star(\omega) = \gamma(0) f^\star(\omega)$$

We re-establish the Yule-Walker equations, where $\alpha_1 = 7/10$ and $\alpha_2 = -1/10$

$$\begin{aligned}
\mathbb{E}(X_t X_t) &= \alpha_1 \mathbb{E}(X_t X_{t-1}) - \alpha_2 \mathbb{E}(X_t X_{t-2}) + \mathbb{E}(X_t Z_t) \\
\mathbb{E}(X_t X_{t-1}) &= \alpha_1 \mathbb{E}(X_{t-1} X_{t-1}) - \alpha_2 \mathbb{E}(X_{t-1} X_{t-2}) + \mathbb{E}(Z_t X_{t-1}) \\
\mathbb{E}(X_t X_{t-2}) &= \alpha_1 \mathbb{E}(X_{t-1} X_{t-2}) - \alpha_2 \mathbb{E}(X_{t-2} X_{t-2}) + \mathbb{E}(Z_t X_{t-2})
\end{aligned}$$

Which becomes the following system of three equations:

$$\begin{aligned}
\gamma(0) &= \alpha_1\gamma(1) + \alpha_2\gamma(2) + \sigma^2 \\
\gamma(1) &= \alpha_1\gamma(0) + \alpha_2\gamma(1) \\
\gamma(2) &= \alpha_1\gamma(1) + \alpha_2\gamma(0)
\end{aligned}$$

test area:

$$\begin{aligned}
\gamma(1) &= \alpha_1\gamma(0) + \alpha_2\gamma(1) \\
\gamma(1) &= \frac{\alpha_1}{1-\alpha_2}\gamma(0)
\end{aligned}$$

then

$$\begin{aligned}
\gamma(2) &= \alpha_1\gamma(1) + \alpha_2\gamma(0) \\
\gamma(2) &= \alpha_1\Big(\frac{\alpha_1}{1-\alpha_2}\gamma(0)\Big) + \alpha_2\gamma(0) \\
\gamma(2) &= \Big(\frac{\alpha_1^2 + \alpha_2 - \alpha_2^2}{1-\alpha_2}\Big)\gamma(0)
\end{aligned}$$

finally

$$\begin{aligned}
\gamma(0) &= \alpha_1\gamma(1) + \alpha_2\gamma(2) + \sigma^2 \\
\gamma(0) &= \alpha_1\Big(\frac{\alpha_1}{1-\alpha_2}\Big)\gamma(0) + \alpha_2\Big(\frac{\alpha_1^2 + \alpha_2 - \alpha_2^2}{1-\alpha_2}\Big)\gamma(0) + \sigma^2 \\
\gamma(0) &= \Big(\frac{\alpha_1^2 + \alpha_1^2\alpha_2 + \alpha_2^2 - \alpha_2^3}{1-\alpha_2}\Big)\gamma(0) + \sigma^2 \\
\gamma(0)\Big(1 - \frac{\alpha_1^2 + \alpha_1^2\alpha_2 + \alpha_2^2 - \alpha_2^3}{1-\alpha_2}\Big) &= \sigma^2 \\
\gamma(0)\Big(\frac{(1-\alpha_2) - \alpha_1^2 - \alpha_1^2\alpha_2 - \alpha_2^2 + \alpha_2^3}{1-\alpha_2}\Big) &= \sigma^2 \\
\gamma(0) &= \frac{\sigma^2(1-\alpha_2)}{1 - \alpha_2 - \alpha_1^2 - \alpha_1^2\alpha_2 - \alpha_2^2 + \alpha_2^3}
\end{aligned}$$

Then, we can evaluate at our given α_1 and α_2, simplifying the fraction using Python to avoid human error.

```python
from sympy import *

alpha_1, alpha_2, sigma_sq = symbols('alpha_1 alpha_2 sigma_sq')

gamma_0_expr = ( sigma_sq * (1 - alpha_2) /
    (1 - alpha_2 - alpha_1**2 - alpha_1**2 * alpha_2 - alpha_2**2 + alpha_2**3))

alpha_1_val = Rational(7, 10)
alpha_2_val = Rational(-1, 10)

gamma_0_eval = gamma_0_expr.subs({alpha_1: alpha_1_val, alpha_2: alpha_2_val})

gamma_0_simplified = nsimplify(gamma_0_eval)
print(gamma_0_simplified)
```

```
## 275*sigma_sq/162
```

Then, we can verify both our results and Python's simplification as follows, assuming $\sigma^2 = 1$.

```r
set.seed(3)
alpha_1 <- 7/10
alpha_2 <- -1/10
sigma_sq <- 1

gamma_0 <- (sigma_sq * (1 - alpha_2) /
        (1 - alpha_2 - alpha_1^2 - alpha_1^2 * alpha_2 - alpha_2^2 + alpha_2^3))

ar_params <- c(alpha_1, alpha_2)
simulated_data <- arima.sim(n = 5000000,
                            model = list(ar = c(7/10, -1/10)))

simulated_gamma_0 <- var(simulated_data)
# use both regular and simplified fraction version
cat("Computed Variance Using Simplified Fraction:", 275/162, "\n")
```

```
## Computed Variance Using Simplified Fraction: 1.697531
```

```r
cat("Computed Variance Using Computation:", gamma_0, "\n")
```

```
## Computed Variance Using Computation: 1.697531
```

```r
cat("Variance of Simulated AR(2) process:", simulated_gamma_0, "\n")
```

```
## Variance of Simulated AR(2) process: 1.697897
```

It seems the computation very closely approximates the truth. Hence, we conclude that:

$$f(\omega) = \gamma(0) f^{\star}(\omega) = \frac{275\sigma^2}{162}\left(\frac{1}{\pi} + \frac{32}{11\pi}\left(\frac{2\cos(\omega) - 1}{5 - 4\cos(\omega)}\right) - \frac{5}{11\pi}\left(\frac{5\cos(\omega) - 1}{13 - 5\cos(\omega)}\right)\right)$$

Part C

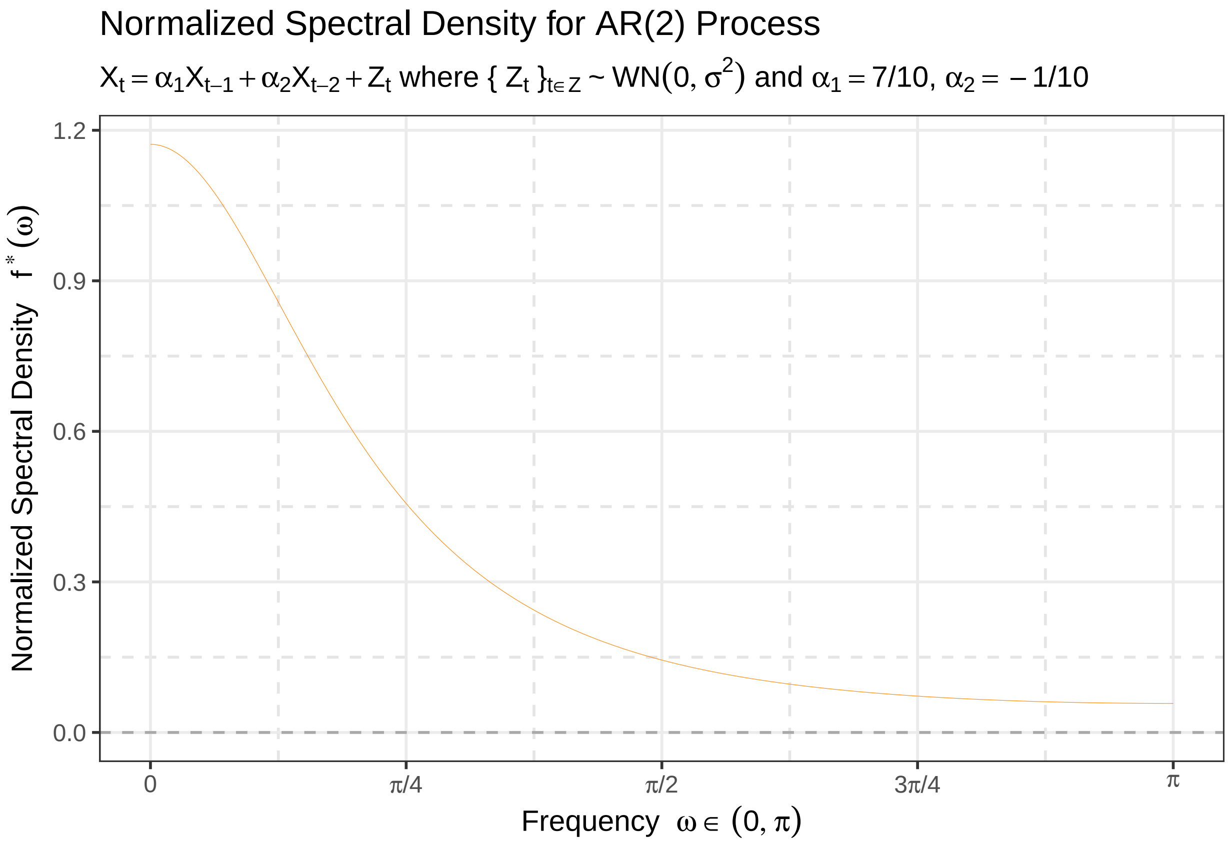
Plot the normalized spectral density and comment on its behaviour.

The normalized spectral density equation ended up being quite long, so we'll define each term one-by-one in the function below:

```r
f = function(w){
  term_1 = 1
  term_2 = (32/11)*((2*cos(w) - 1) / (5 - 4*cos(w)))
  term_3 = (5 /11)*((5*cos(w) - 1) / (13 - 5*cos(w)))
  return ( (1/pi)*(term_1 + term_2 - term_3) )
}
```

Then, we plot the function for a long sequence of ω values in $(0, \pi)$:

```r
omega = seq(from = 0, to = pi, length.out = 1e4)
# define data frame for values
p1df = data.frame(omega = omega,
           f_star_omega = f(omega))
# buld plot
p1 <- ggplot(p1df, aes(x = omega, y = f_star_omega)) +
  geom_line(color = "#ff8600", linewidth = 0.1) +
  labs(
    title = "Normalized Spectral Density for AR(2) Process",
    subtitle = TeX(paste(
      "$X_t = \\alpha_1 X_{t-1} + \\alpha_2 X_{t-2} + Z_t$",
      "where",
      "$\\{\\,Z_t\\,\\}_{t \\in {Z}} \\sim WN(0, \\sigma^2)$",
      "and $\\alpha_1 =$7/10, $\\alpha_2 =-$1/10")),
    y = TeX("Normalized Spectral Density   $f^{*}(\\omega)$"),
    x = TeX("Frequency  $\\omega \\in (0, \\pi)$")
  ) + theme_bw() +
  geom_hline(yintercept = 0, lty = 'dashed', col = "darkgrey")+
  scale_x_continuous(
    breaks = c(0, pi/4, pi/2, 3*pi/4, pi),
    labels = c(TeX("0"), TeX("$\\pi$/4"), TeX("$\\pi$/2"),
               TeX("$3\\pi$/4"), TeX("$\\pi$")))+
  theme(panel.grid.minor = element_line(
    color = "grey90",
    linetype = "dashed",
    linewidth = 0.5
  ))
print(p1)
```



Comments: It appears that the normalized spectral density plot is largely dominated by low frequencies. We can tell this is the case due to the fact that the largest values of $f^\star(\omega)$ is for small $\omega \in (0, \pi)$ specifically for $\omega \in (0, \pi/4)$ we see the majority of frequencies with observed normalized density greater than 0.5. This tells us that a greater proportion of the variance inherent in the process X_t can be attributed to the lower frequencies. As we would expect of a normalized spectral density dominated by low ω values, we see that $f^\star(\omega)$ is strictly decreasing as $\omega \to \pi$.

Question 2

Given the spectral density function

$$f(\omega) = \frac{1}{\pi}\Big(62 - 70\cos(\omega) + 12\cos(2\omega)\Big), \qquad \omega \in (0, 1)$$

compute the autocovariance function $\gamma(h)$ and autocorrelation function $\rho(h)$ of the underlying stochastic process, where $h \in \mathbb{Z}$.

We will begin with computing the autocovariance function $\gamma(h)$ at lag $h = 0$, then at $h = 1$, $k = 2$ and finally $h > 2$.

Further, for notational simplicity, we let $a = 62$, $b = 70$ and $c = 12$, meaning that:

$$f(\omega) = \frac{1}{\pi}\Big(a - b\cos(\omega) + c\cos(2\omega)\Big), \qquad \omega \in (0, 1)$$

Part A: Lag 0

For $h = 0$, we have the following:

$$\gamma(0) = \int_0^\pi f(\omega)\cos(h\omega)\mathrm{d}\omega = \int_0^\pi \left[\frac{1}{\pi}\Big(a - b\cos(\omega) + c\cos(2\omega)\Big)\right]\cos(0 \times \omega)\mathrm{d}\omega$$

Let's evaluate this integral to find the autocovariance at lag zero.

$$\begin{aligned}
\gamma(0) &= \int_0^\pi \left[\frac{1}{\pi}\Big(a - b\cos(\omega) + c\cos(2\omega)\Big)\right]\cos(0)\mathrm{d}\omega \\
\gamma(0) &= \frac{a}{\pi}\int_0^\pi \mathrm{d}\omega - \frac{b}{\pi}\int_0^\pi \cos(\omega)\mathrm{d}\omega + \frac{c}{\pi}\int_0^\pi \cos(2\omega)\mathrm{d}\omega \\
\gamma(0) &= \frac{a}{\pi}\Big(\omega\,\Big|_0^\pi\Big) - \frac{b}{\pi}\Big(\sin(\omega)\,\Big|_0^\pi\Big) + \frac{c}{\pi}\Big(\frac{1}{2}\sin(2\omega)\,\Big|_0^\pi\Big) \\
\gamma(0) &= \frac{a}{\pi}(\pi - 0) - \frac{b}{\pi}\Big(\sin(\pi) - \sin(0)\Big) + \frac{c}{2\pi}\Big(\sin(2\pi) - \sin(0)\Big) \\
\gamma(0) &= a - \frac{b}{\pi}\Big(0 - 0\Big) + \frac{c}{2\pi}\Big(0 - 0\Big) \\
\gamma(0) &= a = \boxed{62}
\end{aligned}$$

Part B: Lag 1

We repeat these calculations, but now for $h = 1$. We retain $a = 62$, $b = 70$ and $c = 12$.

For $h = 1$, we have the following:

$$\gamma(1) = \int_0^\pi f(\omega)\cos(h\omega)\mathrm{d}\omega = \int_0^\pi \left[\frac{1}{\pi}\Big(a - b\cos(\omega) + c\cos(2\omega)\Big)\right]\cos(1 \times \omega)\mathrm{d}\omega$$

As before, we expand and evaluate this integral.

We will use the following trigonometric identities without proof:

$$\cos^2(\theta) = 1 - \sin^2(\theta) \tag{3}$$

$$\sin^2(\theta) = \frac{1}{2}\big(1 - \cos(2\theta)\big) \tag{4}$$

$$
\begin{aligned}
\gamma(1) &= \int_0^\pi \left[\frac{1}{\pi}\Big(a - b\cos(\omega) + c\cos(2\omega)\Big)\right]\cos(\omega)\mathrm{d}\omega \\
\gamma(1) &= \frac{a}{\pi}\int_0^\pi \cos(\omega)\mathrm{d}\omega - \frac{b}{\pi}\int_0^\pi \cos(\omega)\cos(\omega)\mathrm{d}\omega + \frac{c}{\pi}\int_0^\pi \cos(2\omega)\cos(\omega)\mathrm{d}\omega \\
\gamma(1) &= \frac{a}{\pi}\Big(\sin(\omega)\Big|_0^\pi\Big) - \frac{b}{\pi}\int_0^\pi \cos^2(\omega)\mathrm{d}\omega + \frac{c}{\pi}\int_0^\pi \cos(\omega)\cos(2\omega)\mathrm{d}\omega \\
\gamma(1) &= 0 - \frac{b}{\pi}\int_0^\pi \big(1 - \sin^2(\omega)\big)\mathrm{d}\omega + \frac{c}{\pi}\int_0^\pi \cos(\omega)\cos(2\omega)\mathrm{d}\omega \qquad \text{applying (3)} \\
\gamma(1) &= -\frac{b}{\pi}\int_0^\pi \mathrm{d}\omega + \frac{b}{\pi}\int_0^\pi \sin^2(\omega)\mathrm{d}\omega + \frac{c}{\pi}\int_0^\pi \cos(\omega)\cos(2\omega)\mathrm{d}\omega \\
\gamma(1) &= -b + \frac{b}{\pi}\int_0^\pi \frac{1}{2}\big(1 - \cos(2\theta)\big)\mathrm{d}\omega + \frac{c}{\pi}\int_0^\pi \cos(\omega)\cos(2\omega)\mathrm{d}\omega \qquad \text{applying (4)} \\
\gamma(1) &= -b + \frac{b}{\pi}\int_0^\pi \frac{1}{2}\mathrm{d}\omega - \frac{b}{2\pi}\underbrace{\int_0^\pi \cos(2\theta)\mathrm{d}\omega}_{\text{shown to be 0 in Part A}} + \frac{c}{\pi}\int_0^\pi \cos(\omega)\cos(2\omega)\mathrm{d}\omega \\
\gamma(1) &= -b + \frac{b}{\pi}\int_0^\pi \frac{1}{2}\mathrm{d}\omega + \frac{c}{\pi}\int_0^\pi \cos(\omega)\cos(2\omega)\mathrm{d}\omega \\
\gamma(1) &= -b + \frac{b}{2} + \frac{c}{\pi}\int_0^\pi \cos(\omega)\cos(2\omega)\mathrm{d}\omega
\end{aligned}
$$

We apply a manipulation of equation (4) to evaluate the final integral:

$$
\begin{aligned}
\gamma(1) &= -\frac{b}{2} + \frac{c}{\pi}\int_0^\pi \cos(\omega)\cos(2\omega)\mathrm{d}\omega \\
\gamma(1) &= -\frac{b}{2} + \frac{c}{\pi}\int_0^\pi \cos(\omega)\Big(1 - 2\sin^2(\omega)\Big)\mathrm{d}\omega \\
\gamma(1) &= -\frac{b}{2} + \frac{c}{\pi}\int_0^\pi \cos(\omega)\mathrm{d}\omega - \frac{2c}{\pi}\int_0^\pi \cos(\omega)\sin^2(\omega)\mathrm{d}\omega \\
\gamma(1) &= -\frac{b}{2} + \frac{c}{\pi}\Big(\sin(\omega)\Big|_0^\pi\Big) - \underbrace{\frac{2c}{\pi}\int_0^\pi \cos(\omega)\sin^2(\omega)\mathrm{d}\omega}_{\text{let } q = \sin(\omega)} \\
\gamma(1) &= -\frac{b}{2} + 0 - \frac{2c}{\pi}\int_0^\pi \cos(\omega)q^2\Big(\frac{1}{\cos(\omega)}\Big)\mathrm{d}q \\
\gamma(1) &= -\frac{b}{2} - \frac{2c}{3\pi}\left(q^3\Big|_{\omega=0}^\pi\right) \\
\gamma(1) &= -\frac{b}{2} - \frac{2c}{3\pi}\Big(\sin^3(\pi) - \sin^3(0)\Big) \\
\gamma(1) &= -\frac{b}{2} = -\frac{70}{2} = \boxed{-35}
\end{aligned}
$$

Part C: Lag 2

We repeat these calculations, but now for $h = 2$. We retain $a = 62$, $b = 70$ and $c = 12$.

For $h = 2$, we have the following:

$$\gamma(2) = \int_0^\pi f(\omega)\cos(h\omega)\mathrm{d}\omega = \int_0^\pi \left[\frac{1}{\pi}\Big(a - b\cos(\omega) + c\cos(2\omega)\Big)\right]\cos(2\times\omega)\mathrm{d}\omega$$

As before, we expand $f(\omega)$ and evaluate this integral.

$$\begin{aligned}
\gamma(2) &= \int_0^\pi \left[\frac{1}{\pi}\Big(a - b\cos(\omega) + c\cos(2\omega)\Big)\right]\cos(2\omega)\mathrm{d}\omega \\
\gamma(2) &= \frac{a}{\pi}\int_0^\pi \cos(2\omega)\mathrm{d}\omega - \frac{b}{\pi}\underbrace{\int_0^\pi \cos(\omega)\cos(2\omega)\mathrm{d}\omega}_{\text{Shown in Part B to be zero.}} + \frac{c}{\pi}\int_0^\pi \cos(2\omega)\cos(2\omega)\mathrm{d}\omega \\
\gamma(2) &= \frac{a}{2\pi}\Big(\sin(2\omega)\Big|_0^\pi\Big) - \frac{b}{\pi}(0) + \frac{c}{\pi}\int_0^\pi \cos^2(2\omega)\mathrm{d}\omega, \qquad \text{applying (3)} \\
\gamma(2) &= \frac{a}{2\pi}\Big(\sin(2\pi) - \sin(0)\Big) + \frac{c}{\pi}\int_0^\pi \Big(1 - \sin^2(2\omega)\Big)\mathrm{d}\omega \\
\gamma(2) &= \frac{a}{2\pi}(0) + \frac{c}{\pi}\int_0^\pi 1\mathrm{d}\omega - \frac{c}{\pi}\int_0^\pi \sin^2(2\omega)\mathrm{d}\omega \\
\gamma(2) &= \frac{c}{\pi}(\pi) - \frac{c}{\pi}\int_0^\pi \frac{1}{2}\Big(1 - \cos(2\cdot 2\omega)\Big)\mathrm{d}\omega, \qquad \text{applying (4)} \\
\gamma(2) &= c - \frac{c}{2\pi}\int_0^\pi 1\mathrm{d}\omega + \frac{c}{2\pi}\int_0^\pi \cos(4\omega)\mathrm{d}\omega \\
\gamma(2) &= c - \frac{c}{2\pi}(\pi) + \frac{c}{2\pi}\Big(\frac{1}{4}\sin(4\omega)\Big|_0^\pi\Big) \\
\gamma(2) &= c - \frac{c}{2} + \frac{c}{8\pi}\Big(\sin(4\pi) - \sin(0)\Big) \\
\gamma(2) &= \frac{c}{2} = \frac{12}{2} = \boxed{6}
\end{aligned}$$

Part D: Lags Greater Than 2

Before we evaluate lags $h > 2 \subset \mathbb{Z}$, we will establish the following lemma:

Lemma 1: Let $m \in \mathbb{Z}, n \in \mathbb{Z}^+$ We will show that:

$$\forall (m,n) \in \mathbb{Z}^+ \text{ s.t. } m \neq n, \int_0^\pi \cos(m\omega)\cos(n\omega)\,\mathrm{d}\omega = 0$$

Proof: In order to show this equality holds, we will use the following product-to-sum identity of cosines without proof. Let $\theta, \vartheta \in \mathbb{R}$.

$$\cos(\theta)\cos(\vartheta) = \frac{1}{2}\cos(\theta+\vartheta) + \frac{1}{2}\cos(\theta-\vartheta) \tag{2.1}$$

We will use (2.1) to evaluate the integral.

$$\int_0^\pi \cos(m\omega)\cos(n\omega)\,\mathrm{d}\omega = \int_0^\pi \left(\frac{1}{2}\cos(m\omega+n\omega)+\frac{1}{2}\cos(m\omega-n\omega)\right)\mathrm{d}\omega$$
$$\int_0^\pi \cos(m\omega)\cos(n\omega)\,\mathrm{d}\omega = \frac{1}{2}\int_0^\pi \underbrace{\cos\left((m+n)\omega\right)}_{\text{Let } q=(m+n)\omega}\mathrm{d}\omega + \frac{1}{2}\int_0^\pi \underbrace{\cos\left((m-n)\omega\right)}_{\text{Let } \nu=(m-n)\omega}\mathrm{d}\omega$$
$$\int_0^\pi \cos(m\omega)\cos(n\omega)\,\mathrm{d}\omega = \frac{1}{2(m+n)}\int_0^\pi \cos(q)\,\mathrm{d}q + \frac{1}{2(m-n)}\int_0^\pi \cos(\nu)\,\mathrm{d}\nu, \qquad \text{requires } m\neq n$$
$$\int_0^\pi \cos(m\omega)\cos(n\omega)\,\mathrm{d}\omega = \frac{1}{2(m+n)}\left(\sin(q)\Big|_{\omega=0}^{\pi}\right) + \frac{1}{2(m-n)}\left(\sin(\nu)\Big|_{\omega=0}^{\pi}\right)$$
$$\int_0^\pi \cos(m\omega)\cos(n\omega)\,\mathrm{d}\omega = \frac{1}{2(m+n)}\left(\sin(m\omega+n\omega)\Big|_0^\pi\right) + \frac{1}{2(m-n)}\left(\sin(m\omega-n\omega)\Big|_0^\pi\right)$$
$$\int_0^\pi \cos(m\omega)\cos(n\omega)\,\mathrm{d}\omega = \frac{1}{2(m+n)}\Big(\sin\left((m+n)\pi\right)-\sin(0)\Big) + \frac{1}{2(m-n)}\Big(\sin\left((m-n)\pi\right)-\sin(0)\Big)\Big)$$
$$\int_0^\pi \cos(m\omega)\cos(n\omega)\,\mathrm{d}\omega = \frac{1}{2(m+n)}\sin\left((m+n)\pi\right) + \frac{1}{2(m-n)}\sin\left((m-n)\pi\right)$$

We note that $m\in\mathbb{Z}$ and $n\in\mathbb{Z}$. Hence, by properties of integers, we know that $m+n\in\mathbb{Z}$ and $m-n\in\mathbb{Z}$.

We can hence note that:
$$(m\in\mathbb{Z}\wedge n\in\mathbb{Z}) \implies \exists \ell\in\mathbb{Z} \text{ s.t. } \ell = m+n$$

Since $(m,n)\in\mathbb{Z}^+$ and $m\neq n$, we know that $m+n>0$, $\therefore \ell>0$.

Similarly, we have that:
$$(m\in\mathbb{Z}\wedge n\in\mathbb{Z}) \implies \exists \varrho\in\mathbb{Z} \text{ s.t. } \varrho = m-n$$

Further, since $(m,n)\in\mathbb{Z}^+$ and $m\neq n$, we know that $\varrho\neq 0$. We also note by properties of sine that $\forall k\in\mathbb{Z}, \sin(k\cdot\pi)=0$.

Finally, we can rewrite our simplified integral in terms of ℓ and ϱ and solve.

$$\int_0^\pi \cos(m\omega)\cos(n\omega)\,\mathrm{d}\omega = \frac{1}{2\ell}\sin(\ell\pi)+\frac{1}{2\varrho}\sin(\varrho\pi)$$
$$\int_0^\pi \cos(m\omega)\cos(n\omega)\,\mathrm{d}\omega = \frac{1}{2\ell}(0)+\frac{1}{2\varrho}(0), \quad \text{since } \forall k\in\mathbb{Z}, \sin(k\cdot\pi)=0$$
$$\int_0^\pi \cos(m\omega)\cos(n\omega)\,\mathrm{d}\omega = (0)+(0), \qquad \text{since } \ell\neq 0 \text{ and } \varrho\neq 0$$
$$\int_0^\pi \cos(m\omega)\cos(n\omega)\,\mathrm{d}\omega = 0$$

Hence, we conclude that

$$\forall (m,n)\in\mathbb{Z}^+ \text{ s.t. } m\neq n,\ \int_0^\pi \cos(m\omega)\cos(n\omega)\,\mathrm{d}\omega = 0, \quad \text{as required. } \square$$

Application of the Lemma

Now, we can show that $\forall h>2\subseteq\mathbb{Z}^+$ that:

$$\gamma(h) = \int_0^\pi f(\omega)\cos(h\omega)\mathrm{d}\omega = 0$$

Proof: We begin by expanding $f(\omega)$ with our previous variable assignments:

$$\int_0^\pi f(\omega)\cos(h\omega)\mathrm{d}\omega = \int_0^\pi \left[\frac{1}{\pi}\Big(a - b\cos(\omega) + c\cos(2\omega)\Big)\right]\cos(h\omega)\mathrm{d}\omega$$
$$\int_0^\pi f(\omega)\cos(h\omega)\mathrm{d}\omega = \frac{a}{\pi}\int_0^\pi \cos(h\omega)\mathrm{d}\omega - \frac{b}{\pi}\int_0^\pi \cos(\omega)\cos(h\omega)\mathrm{d}\omega + \frac{c}{\pi}\int_0^\pi \cos(2\omega)\cos(h\omega)\mathrm{d}\omega$$
$$\int_0^\pi f(\omega)\cos(h\omega)\mathrm{d}\omega = \frac{a}{\pi}\int_0^\pi \cos(0\cdot\omega)\cos(h\cdot\omega)\mathrm{d}\omega - \frac{b}{\pi}\int_0^\pi \cos(1\cdot\omega)\cos(h\cdot\omega)\mathrm{d}\omega + \frac{c}{\pi}\int_0^\pi \cos(2\cdot\omega)\cos(h\cdot\omega)\mathrm{d}\omega$$

Now, from the expansion on the third line, we can draw an interesting parallel to the lemma established previously. Let $M = \{0, 1, 2\}$ and $N = \{n \in \mathbb{Z} : n > 2\}$.

We see by construction of these sets $M \cap N = \emptyset$. Importantly,

$$M \cap N = \emptyset \implies \forall n \in N, \forall m \in M,\ m \neq n$$

Now, if we let $M = \{m_1, m_2, m_3\} = \{0, 1, 2\}$ and $n \in N := \{h \in \mathbb{Z} : h > 2\}$, we can rewrite our integral in terms of m and n and simplify using our lemma.

$$\int_0^\pi f(\omega)\cos(h\omega)\mathrm{d}\omega = \frac{a}{\pi}\int_0^\pi \underbrace{\cos(m_1\cdot\omega)\cos(n\cdot\omega)}_{\forall n\in N,\, m_1\neq n}\mathrm{d}\omega - \frac{b}{\pi}\int_0^\pi \underbrace{\cos(m_2\cdot\omega)\cos(n\cdot\omega)}_{\forall n\in N,\, m_2\neq n}\mathrm{d}\omega + \frac{c}{\pi}\int_0^\pi \underbrace{\cos(m_2\cdot\omega)\cos(n\cdot\omega)}_{\forall n\in N,\, m_3\neq n}\mathrm{d}\omega$$
$$\int_0^\pi f(\omega)\cos(h\omega)\mathrm{d}\omega = \frac{a}{\pi}(0) - \frac{b}{\pi}(0) + \frac{c}{\pi}(0),\ \text{by set construction and Lemma 1.}$$
$$\int_0^\pi f(\omega)\cos(h\omega)\mathrm{d}\omega = \boxed{0}$$

So, we can conclude that:

$$\forall h > 2, \gamma(h) = \int_0^\pi f(\omega)\cos(h\omega)\mathrm{d}\omega = 0,\ \text{as required.} \quad \square$$

Conclusions

Hence, bringing all the previous parts together, we can conclude that for $h \in \mathbb{Z}$:

$$\gamma(h) = \begin{cases} \gamma(-h), & h < 0 \\ 62, & h = 0 \\ -35, & h = 1 \\ 6, & h = 2 \\ 0, & \text{otherwise} \end{cases}$$

Similarly, we find $\rho(h)$ for $h \in \mathbb{Z}$ by dividing $\gamma(h)$ by $\sigma_X^2 = \gamma(0) = 62$.

$$\rho(h) = \begin{cases} \rho(-h), & h < 0 \\ 1, & h = 0 \\ -35/62, & h = 1 \\ 3/31, & h = 2 \\ 0, & \text{otherwise} \end{cases}$$

We can also provide a plot of the theoretical ACF to get a more easily interpretable idea of how the autocorrelation of X_t changes for lags $h \in \mathbb{Z}$.

```r
p2DF = data.frame(
  h = 0:9,
  rh = c(1, -35/62, 3/31, rep(0, times = 7))
)
p2 <- ggplot(p2DF, aes(x = h, y = rh)) +
  geom_segment(aes(xend = h, yend = 0),
               color = "#FF70A6",
               linewidth = 1.1) +
  geom_hline(yintercept = 0,
             linetype = "dashed",
             color = "darkgray") +
  labs(x = "Lag",  y = TeX("Theoretical $\\rho(h)$ Values"),
       title =  TeX(paste("Theoretical ACF Given Spectral Density $f(\\omega)$",
       "for $\\omega \\in (0, 1)$ and $h \\in Z$")) )+
  theme_bw() +
  theme(panel.grid.minor = element_line(color = "grey90",
                                        linetype = "dashed", linewidth = 0.5))
print(p2)
```