

# Application of Non-Parametric Gamma-Poisson Conjugate Dirichlet Process Mixture Models

STAT 447

Caden Hewlett

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## Introduction

# TODO

## Literature Review

Before we discuss Dirichlet Processes, it is crucial to establish a groundwork in probability measure theory. We will briefly revisit the concepts of  $\sigma$ -algebra and probability measures. The relevant literature for probability measure theory are (Billingsley 2012) and (Rudin 1986). In the following, we present generalized definitions, which are discussed in rigour in these foundational measure theory works.

Let  $\mathbb{X}$  be a well-defined sample space. A  $\sigma$ -algebra  $\mathcal{F} \subseteq P(\mathbb{X})$  is a set satisfying the following:

1. The entire sample space  $\mathbb{X}$  is in  $\mathcal{F}$  and  $\emptyset$  is in  $\mathcal{F}$ . This is referred to in the literature as “non-emptiness and universality.”
2. For all sets  $A \in \mathcal{F}$ , the complement  $A^c \in \mathcal{F}$ . This property is referred to as “closure under complementation.”
3. For any countable collection of sets  $\{A_i\}_{i \in I}$ , where  $I$  is a countable index set, if  $\forall i \in I, A_i \in \mathcal{F}$  then  $\bigcup_{i \in I} A_i \in \mathcal{F}$ . This is referred to as “closure under countable unions.”

For the sake of this work, we are more interested in *probability measures*, which are built on  $\sigma$ -algebra. A probability measure  $\mu : \mathcal{F} \mapsto [0, 1]$  satisfies the following familiar axioms of probability:

1. All mappings are non-negative:  $\forall A \in \mathcal{F}, \mu(A) \geq 0$ . This is referred to as “non-negativity.”
2.  $\mu(\mathbb{X}) = 1$ , and  $\mu(\emptyset) = 0$ .
3. For any countable set  $\{A_i\} \subseteq \mathcal{F}$  where  $\forall i \neq j, A_i \cap A_j = \emptyset$ , we have that  $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$ , this is referred to as “countable additivity.”

Before discussing Dirichlet Distributions and Processes, we acknowledge that the above definitions might be abstract for those new to measure theory. For clarity, the Appendix includes examples to illustrate  $\sigma$ -algebra properties and verify a probability measure  $\mu$  on a simple finite set.

Dirichlet Process (Teh 2006)

## Methodology

### Finite-Approximation via Stick-Breaking

```
# TODO: First, Discuss finite-approximation via GEM
# TODO: Then discuss implementation via custom Mixing Distribution : https://en.wikipedia.org/wiki/Mixt
# https://dm13450.github.io/2018/02/21/Custom-Distributions-Conjugate.html
# TODO: noting that when deciding on a prior-likelihood model, the characterization
# of the system for a well-functioning mixture distribution (to be used in the DPMM)
# requires the following four characteristics, which we derive from the gamma-poisson conjugacy
# it's important to recall that while this conjugate system allows us to well-define the parameters
# for the mixing distribution as required for a Dirichlett Process sampler, the gamma distribution
# in the prior-likelihood pair is the base measure, not the prior. we can think of these as the
# parameters to the mixing process. (i.e. requires a Mixing Distribution Object `mdObject` which is
# implicitly defined by the following four features: (4 functions))
```

## Results : Data Analysis

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# TODO: Fit to Chicago Traffic model. Discuss posterior distribution curve.
```

## Conclusion and Further Work

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# TODO: Discuss Chicago Traffic Model and Implementation
```

## Appendix

*Acknowledgements:* Miscellaneous information such as knowledge on sets, power sets, subsets and countability from (Demirbas and Reznitzner 2023).

### Example of a $\sigma$ -Algebra

For additional clarity, we provide an example of a  $\sigma$ -algebra  $\mathcal{F}$  to demonstrate the properties mentioned in the literature review.

Let's consider the finitely countable and simple set  $\mathbb{X} = \{a, b, c\}$ . Directly,  $P(\mathbb{X}) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$  is the power set of  $\mathbb{X}$ . Consider  $\mathcal{F} = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}\} \subseteq P(\mathbb{X})$ .

We will apply the properties discussed in the literature review section to prove that  $\mathcal{F}$  is a  $\sigma$ -algebra.

To verify Property 1 (universality and non-emptiness), we note that we can also write  $\mathcal{F}$  as  $\{\emptyset, \{a\}, \{b, c\}, \mathbb{X}\}$ . From this definition, it is direct to see that  $\mathbb{X} \in \mathcal{F}$  and  $\emptyset \in \mathcal{F}$ , verifying universality and non-emptiness.

To verify Property 2 (closure under complementation), we note that  $A^c = \mathbb{X} \setminus A$ . Hence, if we show  $A \in \mathcal{F} \implies A^c \in \mathcal{F}$  this is equivalent to showing that  $A \in \mathcal{F} \iff A^c \in \mathcal{F}$ . Since  $\mathcal{F}$  is finitely countable, we can consider a case-wise basis for verification. Firstly, we have  $A = \emptyset$ . By definition,  $A^c = \mathbb{X}$ . We see that  $\mathbb{X} \in \mathcal{F}$ . As mentioned before, this implies that the case where  $A = \mathbb{X}$  also holds. Now, we can proceed to verify the case where  $A = \{a\}$ . We note that  $\{a\}^c = \{b, c\}$ , and that  $\{b, c\} \in \mathcal{F}$ . Therefore, this case holds and thus so does the case where  $A = \{b, c\}$  by biconditionality. Hence, we can conclude that  $\mathcal{F}$  is closed under complementation.

To verify Property 3 (closure under countable unions), we first consider the concrete case where  $\{A_i\}_{i=1}^2 = \{\{a\}, \{b, c\}\}$  where  $A_1 = \{a\}$  and  $A_2 = \{b, c\}$ . We note that  $A_1, A_2 \in \mathcal{F}$ , so we would expect that  $A_1 \cup A_2 \in \mathcal{F}$ . Directly,  $A_1 \cup A_2 = \{a\} \cup \{b, c\} = \{a, b, c\} = \mathbb{X}$ , and we see that  $\mathbb{X} \in \mathcal{F}$ . For the remaining cases,  $\forall A \in \mathcal{F}, A \cup \emptyset = A$  by fundamental set properties, and directly  $A \in \mathcal{F}$  by construction. Similarly,  $\forall A \in \mathcal{F}, A \cup \mathbb{X} = \mathbb{X}$  and we know that  $\mathbb{X} \in \mathcal{F}$ . Hence, for all cases,  $\mathcal{F}$  is closed under countable unions.

From all of these properties, we can conclude that  $\mathcal{F}$  is a  $\sigma$ -algebra, as required  $\square$ .

### Example of a Probability Measure

Using the  $\sigma$ -algebra  $\mathcal{F} = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}\}$  we will define  $\mu : \mathcal{F} \mapsto [0, 1]$  and prove that  $\mu$  is a probability measure using the properties discussed in the literature review.

We will define a concrete example as follows, and show it is a probability measure on  $\mathcal{F}$ .

$$\mu(A) = \begin{cases} 2/3, & |A| = 1 \\ 1/3, & |A| = 2 \\ 1, & |A| = 3 \\ 0, & \text{otherwise} \end{cases}$$

Let's evaluate the properties discussed in the literature review to verify that  $\mu$  is in fact a probability measure.

First, we evaluate Property 1 (non-negativity.) In effect, we wish to verify that  $\forall A \in \mathcal{F}, \mu(A) \geq 0$ . We can easily evaluate the universal in a case-wise basis.

- a.  $\emptyset \in \mathcal{F}$  and  $\mu(\emptyset) = 0 \geq 0$ .
- b.  $\{a\} \in \mathcal{F}$  and  $\mu(\{a\}) = 2/3 \geq 0$ .
- c.  $\{b, c\} \in \mathcal{F}$  and  $\mu(\{b, c\}) = 1/3 \geq 0$ .
- d.  $\mathbb{X} \in \mathcal{F}$  and  $\mu(\mathbb{X}) = 1 \geq 0$ .

Hence,  $\forall A \in \mathcal{F}, \mu(A) \geq 0$ , verifying the non-negativity clause. Further, we see that  $\forall A \in \mathcal{F}, 0 \leq \mu(A) \leq 1$ .

In addition, we can utilize the evaluations above to verify Property 2. Directly, we see that  $\mu(\emptyset) = 0$  and  $\mu(\mathbb{X}) = 1$ , verifying Property 2 that a value of 1 is assigned to the sample space  $\mathbb{X}$ .

Finally, we verify Property 3 (countable additivity.) Again, because the  $\sigma$ -algebra is finitely countable, we verify all pairwise disjoint intersections on a case-wise basis.

- a. First, we consider the set of pairwise disjoint sets  $\{\emptyset, A\}$  for  $A \in \mathcal{F}$ . We see that  $\forall A \in \mathcal{F}, \emptyset \cap A = \emptyset$  and  $\emptyset \cup A = A$ . Using,  $\mu(\emptyset) = 0$ , we observe that  $\mu\left(\bigcup_i A_i\right) = \mu(A) = \mu(A) + \mu(\emptyset) = \sum_{i=1} \mu(A_i)$ , as required.
- b. Similarly, we consider  $\mathbb{X} \in \mathcal{F}$ , noting that  $\forall A \in \mathcal{F}, \mathbb{X} \cap A = A$  is non-disjoint, so the case holds vacuously by falsity of the antecedent. A similar argument can be applied for  $\{A, A\}, A \in \mathcal{F}$  which is evidently a non-disjoint pair.
- c. The other pairwise disjoint set in  $\mathcal{F}$  is  $\{A_i\} = \{\{a\}, \{b, c\}\}$ , since  $\{a\} \cap \{b, c\} = \emptyset$ . Hence, this pair should be countably additive. We can see directly that  $\bigcup_i A_i = \{a\} \cup \{b, c\} = \mathbb{X}$ . Hence,  $\mu\left(\bigcup_i A_i\right) = \mu(\mathbb{X}) = 1$ , so we anticipate  $\sum_i \mu(A_i) = 1$ . To verify, we see that  $\sum_i \mu(A_i) = \mu(\{a\}) + \mu(\{b, c\}) = 2/3 + 1/3 = 1$ , so countable additivity holds.

From all of these properties, we can conclude that  $\mu$  is a probability measure on  $\sigma$ -algebra  $\mathcal{F}$ , as required  $\square$ .

$$\begin{aligned}
& \{\pi_k\}_{k=1}^K \sim \text{GEM}(\alpha), \text{ stick-breaking process} \\
& \{\theta_k\}_{k=1}^K \sim \mathbb{G}_0, \text{ base measure} \\
& z_i \sim \text{categorical}(\{1, 2, \dots, K\}, \{\pi_k\}), \text{ cluster assignment } i \in [1, N] \\
& y_i \mid z_i, \{\theta_k\}_{k=1}^K \sim F(x_i; \theta_{z_i}), \text{ likelihood given cluster} \\
& \text{Infinite Dirichlet Process Mixture Model} \\
& \{\pi_k\}_{k=1}^\infty \sim \text{GEM}(\alpha) \\
& \{\langle \alpha_k, \beta_k \rangle\}_{k=1}^\infty \sim \mathbb{G}_0 \\
& y_i \mid \{\pi_k\}, \{\langle \alpha_k, \beta_k \rangle\} \sim \sum_{k=1}^\infty \pi_k \text{beta}(y_i \mid \langle \alpha_k, \beta_k \rangle)
\end{aligned}$$

Where the MCMC implementation is facilitated by the `dirichletprocess` package, created by (Markwick 2023).

Finite-approximated infinite mixture DPMM, similar

$$\begin{aligned}
& \{\pi_k\}_{k=1}^K \sim \text{GEM}(\alpha), \text{ stick-breaking process} \\
& \{\theta_k\}_{k=1}^K \sim \mathbb{G}_0, \text{ base measure} \\
& z_i \sim \text{categorical}(\{1, 2, \dots, K\}, \{\pi_k\}), \text{ cluster assignment } i \in [1, N] \\
& y_i \mid z_i, \{\theta_k\}_{k=1}^K \sim F(x_i; \theta_{z_i}), \text{ likelihood given cluster} \\
& \{\pi_k\}_{k=1}^\infty \sim \text{GEM}(\alpha), \text{ stick-breaking process} \\
& \{\theta_k\}_{k=1}^\infty \sim \mathbb{G}_0, \text{ base measure,} \\
& F = \prod_{i=1}^N \left( \sum_{k=1}^\infty \pi_k N(x_i \mid \theta_k) \right), \text{ likelihood from normal kernel}
\end{aligned}$$

## Sources (Jones 2021)

Billingsley, Patrick. 2012. *Probability and Measure, Anniversary Edition*. Wiley.

Demirbas, Seckin, and Andrew Rechnitzer. 2023. “An Introduction to Mathematical Proof : MATH 220.” Free web and pdf textbook. <https://personal.math.ubc.ca/~PLP/>.

Jones, Andy. 2021. “Dirichlet Process Mixture Models.” Online. <https://andrewcharlesjones.github.io/journal/dpmm.html>.

Markwick, Dean. 2023. *Dirichletprocess: An ‘r’ Package for Dirichlet Process Mixture Models*. <https://dm13450.github.io/dirichletprocess/>.

Rudin, Walter. 1986. *Real and Complex Analysis*. 3rd ed. McGraw-Hill.

Teh, Yee Whye. 2006. “Dirichlet Process.” Course Notes for Gatsby Computational Neuroscience Unit Tutorial. <https://mlg.eng.cam.ac.uk/zoubin/tut06/ywt.pdf>.