

# STAT 447 Assignment 2

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## Question 1 : Define a Bayesian Model

In order to perform inference on the unknown quantities, we must specify how they relate to the data; i.e., we need a probabilistic model. Assume that every Delta 7925H rocket has the same probability  $p$  of failing. For simplicity, let us assume that  $p$  is allowed to take values on an evenly space grid

$$p \in \left\{ \frac{k}{K} : k \in \{0, \dots, K\} \right\}$$

for some fixed  $k \in \mathbb{N}$ . Furthermore, we have access to a collection of numbers  $\rho_k$  such that:

$$\forall k \in \{0, \dots, K\} : \mathbb{P}\left(p = \frac{k}{K}\right) = \rho_k$$

Let  $Y_i$  denote a binary variable with  $Y_i = 1$  encoding a success, and  $Y_i = 0$  a failure. We assume that, conditionally on  $p$ , the  $Y_i$ 's are independent of each other.

We will use the following prior:

$$\rho_k \propto \frac{k}{K} \left(1 - \frac{k}{K}\right)$$

From now on, use  $K = 20$

### Part 1

What are the unknown quantities in this scenario? And what is the data?

**Solution:** In this case, we can think of two unknown quantities and one piece of known data. One unknown metric is  $Y_i$ , whether or not the rocket is going to fail. The other unknown quantity is the success probability of the rockets - this was denoted  $p$ . This is much like the “coin flip” scenario from the previous homework, where  $Y_i$  was “whether or not we flip a heads,” and the distribution of coin biases (i.e. the probability we pick a coin with bias  $k/K$ ) was  $\rho_k$ .

In this case, the data is given in the setup section, and is simply that as of Jan 2024, Delta 7925H rockets have been launched 3 times, with 0 failed launches. Further, we have  $\rho_k$ , the probability that the *true* probability  $p$  is  $k/K$  for a given discrete  $k$ .

## Part 2

Here, letting  $k = 20$ , we can think of the random Variable  $X$  as the probability that the rocket has failure rate  $k$ , for  $k \in [0, 20]$  i.e.

$$X \sim \text{unif}(\{0, 1, 2, \dots, 20\})$$

Similarly, the conditional distribution of  $Y_i$  given  $X$  can be written as:

$$Y_i \sim \text{bern}\left(\frac{X}{20}\left(1 - \frac{X}{20}\right)\right)$$

Together, these form the joint distribution  $p_{X,Y}(x,y)$  as we saw in lecture, due to the chain rule.

## Question 2 : Posterior and Point Estimates

To help you answer the following questions, we create the two vectors

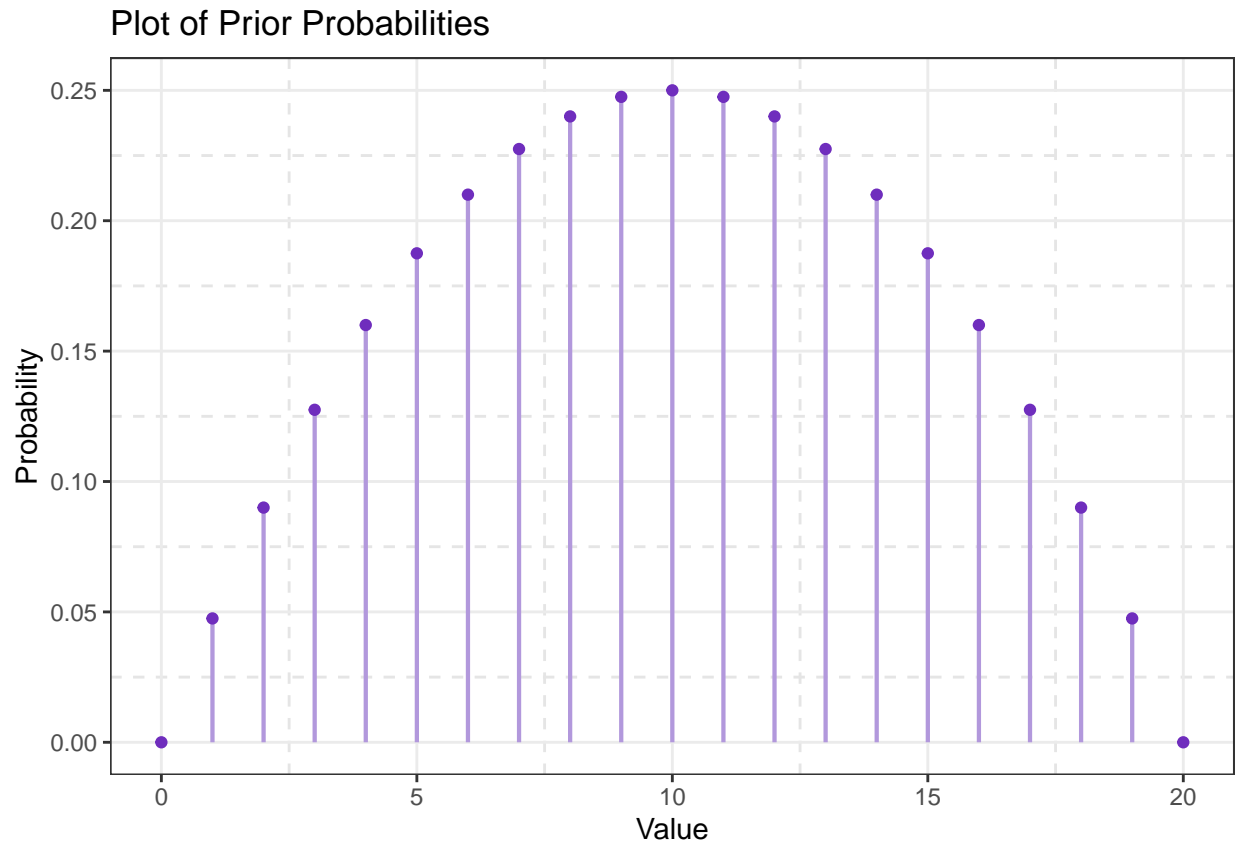
`prior_probabilities` where entry  $i$  containing the prior probability  $\rho_{i-1}$  defined in Q1 (the minus one reflects the fact that R uses indexing starting at 1), and `realizations`, a vector of possible realizations of  $p$  in the same order, namely  $(0, 1/K, 2/K \dots, 1)$ .

```
K = 20
prior_probabilities = sapply(0:K, function(k){(k / K)*(1 - k/K)})
realizations = sapply(0:K, function(k){k/K})
```

## Part 1

Plot the prior PMF. Do you think this is a reasonable prior?

```
plot_data <- data.frame(
  value = 0:K,
  probability = prior_probabilities
)
# plot prior
p <- ggplot(plot_data, aes(x = value, y = 0, yend = probability)) +
  geom_segment(aes(xend = value), color = "#b298dc", linewidth = 0.75) +
  geom_point(aes(x = value, y = probability), color = "#6f2dbd") +
  labs(x = "Value", y = "Probability") +
  ggtitle("Plot of Prior Probabilities") + theme_bw() +
  theme(panel.grid.minor = element_line(
    color = "grey90",
    linetype = "dashed",
    linewidth = 0.5
  ))
print(p)
```



I'm not sure that this prior is exactly reasonable for this specific situation, but it's not bad. My main issue with it is the lower probability placed on higher outcomes. For example,  $\rho_{20}$ , i.e.  $P(p = 20/K = 1)$  is assigned the same (zero) probability as  $\rho_0$ , the probability of the rocket being guaranteed to fail. This symmetry in the fail probability likelihoods doesn't exactly square with our existing data - we know that we've had three successful launches already, so it may not be correct to assume that there isn't a "heavier tail" for greater values of  $k$ . Further, as was discussed in lecture, it's best not to assume any of the values in the prior are exactly zero.

However, there is nuance to this problem (as in all real-world scenarios.) This prior distribution is quite conservative - i.e., it isn't placing too much potentially risky weight on the historical successes, which is *also* a sensible choice given the cost of these rockets.

## Part 2

Let  $\pi_k = \mathbb{P}(p = k/K | Y_{1:3} = (1, 1, 1))$  denote the posterior probabilities, for  $k \in \{0, 1, 2, \dots, K\}$ . Create a vector `posterior_probabilities` where entry  $i$  is  $\pi_{i-1}$ . Plot the posterior PMF.

We can start by computing  $P(Y_{1:3} = \vec{1} | p = \frac{k}{20})$  using the `realizations` vector from earlier. This probability decomposes as follows (letting  $k/20 = p_{\text{obs}}$  for simplicity):

$$P(Y_{1:3} = \vec{1} | p = p_{\text{obs}}) = P(Y_1 = 1 | p = p_{\text{obs}})P(Y_2 = 1 | Y_1 = 1, p = p_{\text{obs}})P(Y_3 = 1 | Y_2 = 1, Y_1 = 1, p = p_{\text{obs}})$$

Which can simplify nicely into:

$$P(Y_{1:3} = \vec{1} | p = p_{\text{obs}}) = P(Y_1 = 1 | p = p_{\text{obs}})P(Y_2 = 1 | p_{\text{obs}})P(Y_3 = 1 | p = p_{\text{obs}})$$

Then, assuming the rocket launches are *iid*  $\text{bern}(p_{\text{obs}})$  as discussed previously, with success probability  $p_{\text{obs}}$ , this can be simplified as:

$$P(Y_{1:3} = \vec{1} \mid p = p_{\text{obs}}) = (p_{\text{obs}})(p_{\text{obs}})(p_{\text{obs}}) = (p_{\text{obs}})^3$$

This is really just a long-winded way of saying we simply take the cube of our `realizations` vector to get the likelihood.

```
likelihood = (realizations)^3
```

Then,  $\gamma(k/20)$  for  $k \in \{0, 1, \dots, 20\}$  is simply the product of the prior probabilities with the likelihood.

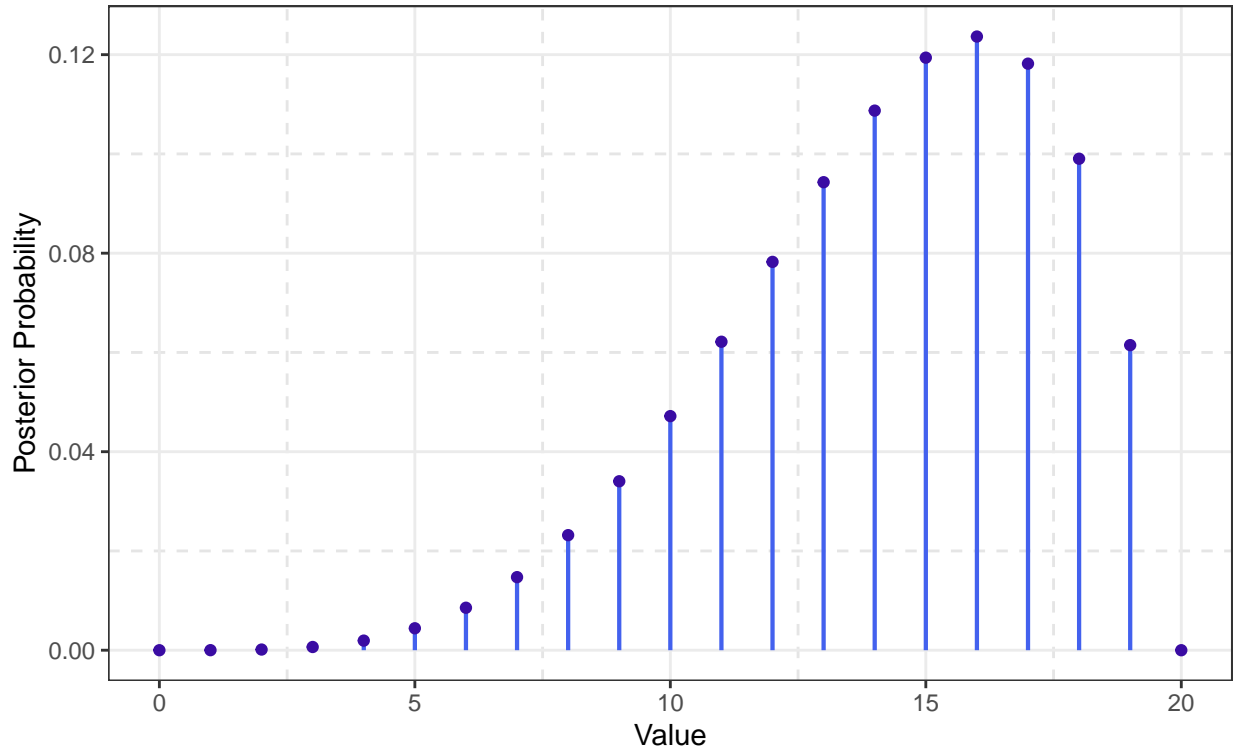
```
gamma = prior_probabilities*likelihood
```

Finally, we divide by the normalizing constant  $Z$  so the probabilities sum to 1. This gives  $\pi_k$ , which we calculate and plot below.

```
pi = gamma/sum(gamma)

plot_data_2 <- data.frame(
  value = 0:K,
  probability = pi
)
# plot posterior
p2 <- ggplot(plot_data_2, aes(x = value, y = 0, yend = probability)) +
  geom_segment(aes(xend = value), color = "#4361ee", linewidth = 0.75) +
  geom_point(aes(x = value, y = probability), color = "#3a0ca3") +
  labs(x = "Value", y = "Posterior Probability",
       title = "Plot of Posterior Probabilities",
       subtitle = "Given the Three Successful Rocket Launches") + theme_bw() +
  theme(panel.grid.minor = element_line(
    color = "grey90",
    linetype = "dashed",
    linewidth = 0.5
  ))
print(p2)
```

Plot of Posterior Probabilities  
Given the Three Successful Rocket Launches



### Part 3

What is the posterior mode?

By definition, we know that:

$$\text{mode}(\pi(x)) = \underset{x \in X}{\operatorname{argmax}} [\pi(x)]$$

For our prior distribution, this is found to be:

```
(0:K)[which.max(pi)]
```

```
## [1] 16
```

Corresponding to  $k = 16$ , or  $p = 16/20$ .

### Part 4

Write a function that compute the posterior mean of  $p$ . (Hint: you should obtain  $\mathbb{E}[p|Y_{1:3} = (1, 1, 1)] \approx 0.7$ .)

We know that the posterior mean  $\mathbb{E}[p|Y_{1:3} = (1, 1, 1)]$  can be found by calculating

$$\sum_{\{x \text{ s.t. } \pi(x) > 0\}} x \pi(x)$$

Thankfully, with our `realizations` and `posterior` distributions, we already have most of the tools at our disposal to make this calculation.