Final Project

STAT 447

2024-04-10

Introduction and Review:

Before we discuss Dirichlet Processes, it is crucial to establish a groundwork in probability measure theory. We will briefly revisit the concepts of σ-algebra, probability measures and Dirichlet Distributions. In the following, we present slightly more generalized definitions than can be found in formal measure theory works such as (Billingsley 2012).

Let $\mathbb{X}$ be a well-defined sample space. A σ-algebra $\mathcal{F} \subseteq P(\mathbb{X})$ is a set satisfying the following:

1. The entire sample space $\mathbb{X}$ is in $\mathcal{F}$.
2. For all sets $A \in \mathcal{F}$, the complement $A^c \in \mathcal{F}$. This property is referred to as "closure under complementation."
3. For any countable collection of sets $\{A_i\}_{i\in I}$, where I is a countable index set, if $\forall i \in I, A_i \in \mathcal{F}$ then $\bigcup_{i\in I} A_i \in \mathcal{F}$. This is referred to as "closure under countable unions." It should be noted that by properties (1) and (2) that $\mathbb{X}^c = \emptyset \in \mathcal{F}$. This extended definition of (1) is referred to as "non-emptiness and universality." More detail on these properties can be found in foundational works such as (Rudin 1986).

For the sake of this work, we are more interested in *probability measures*, which are built on σ-algebra. A probability μ measure $\mu : \mathcal{F} \mapsto [0, 1]$ satisfies the following familiar axioms of probability:

1. All mappings are non-negative: $\forall A \in \mathcal{F},\ \mu(A) \geq 0$. THis is referred to as "non-negativity."
2. $\mu(\mathbb{X}) = 1$, and $\mu(\emptyset) = 0$.
3. For any countable set $\{A_i\} \subseteq \mathcal{F}$ where $\forall i \neq j, A_i \cap A_j = \emptyset$, we have that $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$, this is referred to as "countable additivity."

(Teh 2006)

Appendix

Acknowledegements: Miscellaneous information such as knowledge on sets, power sets, subsets and countability from (Demirbas and Rechnitzer 2023).

Example of a σ-Algebra

For additional clarity, we provide an example of a σ-algebra $\mathcal{F}$ to demonstrate the properties mentioned in the literature review.

Let $\mathbb{X} = \{a, b, c\}$. Directly, $P(\mathbb{X}) = \big\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\big\}$ is the power set of $\mathbb{X}$. Consider $\mathcal{F} = \big\{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}\big\} \subseteq P(\mathbb{X})$.

We will apply the properties discussed in the literature review section to prove that $\mathcal{F}$ is a σ-algebra.

To verify Property 1 (universality and non-emptiness), we note that we can also write $\mathcal{F}$ as $\big\{\emptyset, \{a\}, \{b, c\}, \mathbb{X}\big\}$. From this definition, it is direct to see that $\mathbb{X} \in \mathcal{F}$ and $\emptyset \in \mathcal{F}$, verifying universality and non-emptiness.

To verify Property 2 (closure under complementation), we note that ${A^c}^c = A$. Hence, if we show $A \in \mathcal{F} \implies A^c \in \mathcal{F}$ this is equivalent to showing that $A \in \mathcal{F} \iff A^c \in \mathcal{F}$. Since $\mathcal{F}$ is finitely countable, we can consider a case-wise basis for verification. Firstly, we have $A = \emptyset$. By definition, $A^c = \mathbb{X}$. We see that $\mathbb{X} \in \mathcal{F}$. As mentioned before, this implies that the case where $A = \mathbb{X}$ also holds. Now, we can proceed to verify the case where $A = \{a\}$. We note that $\{a\}^c = \{b, c\}$, and that $\{b, c\} \in \mathcal{F}$. Therefore, this case holds and thus so does the case where $A = \{b, c\}$ by biconditionality. Hence, we can conclude that $\mathcal{F}$ is closed under complementation.

To verify Property (closure under countable unions), we first consider the concrete case where $\{A_i\}_{i=1}^2 = \big\{\{a\}, \{b, c\}\big\}$ where $A_1 = \{a\}$ and $A_2 = \{b, c\}$. We note that $A_1, A_2 \in \mathcal{F}$, so we would expect that $A_1 \cup A_2 \in \mathcal{F}$. Directly, $A_1 \cup A_2 = \{a\} \cup \{b, c\} = \{a, b, c\} = \mathbb{X}$, and we see that $\mathbb{X} \in \mathcal{F}$. For the remaining cases, $\forall A \in \mathcal{F}, A \cup \emptyset = A$ by fundamental set properties, and directly $A \in \mathcal{F}$ by construction. Similarly, $\forall A \in \mathcal{F}, A \cup \mathbb{X} = \mathbb{X}$ and we know that $\mathbb{X} \in \mathcal{F}$. Hence, for all cases, $\mathcal{F}$ is closed under countable unions.

From all of these properties, we can conclude that $\mathcal{F}$ is a σ-algebra, as required $\square$.

Sources

Billingsley, Patrick. 2012. *Probability and Measure, Anniversary Edition.* Wiley.

Demirbas, Seckin, and Andrew Rechnitzer. 2023. "An Introduction to Mathematical Proof : MATH 220." Free web and pdf textbook. https://personal.math.ubc.ca/~PLP/.

Rudin, Walter. 1986. *Real and Complex Analysis.* 3rd ed. McGraw-Hill.

Teh, Yee Whye. 2006. "Dirichlet Process." Course Notes for Gatsby Computational Neuroscience Unit Tutorial. https://mlg.eng.cam.ac.uk/zoubin/tut06/ywt.pdf.