Introduction: In progress...

Background: Let $\mathcal{X}$ be the observation space, Θ be the parameter space, and $\mathcal{A}$ be the action space. For simplicity, we allow all of these spaces to be discrete. The observations $x \in \mathcal{X}$ are connected to the parameter $\theta \in \Theta$ by the probability mass function $p(x|\theta)$, referred to as the data-generating process (DGP) [1]. In a discrete setting, the DGP describes the probability of an observation $x \in \mathcal{X}$ under a given parameter θ. The primary objective of statistical inference is to infer underlying properties of the DGP [2]. From the perspective of decision theory, the decision $a \in \mathcal{A}$ will be to propose a function $\alpha(x)$ to estimate the parameter θ as precisely as possible. To illustrate this concept, suppose we are flipping a fair coin and wish to recover the parameter θ corresponding to the proportion of heads, $\theta = 0.5$. Thus, the DGP $p(x|\theta)$ is a Bernoulli distribution with parameter θ. One action $a_1 \in \mathcal{A}$ is to propose the estimator $\alpha_1(x) = \frac{1}{n}\sum_{i=1}^{n} x_i$ (where n is the number of flips) whereas $a_2 \in \mathcal{A}$ is to naively propose $\alpha_2(x) = 1$ (every flip is heads). It can be shown[1] that a_1 proposes an estimator which maximizes the likelihood of the observed data under the DGP [3], whereas a_2's estimator is clearly biased, thus trivially $a_1 \succ a_2$. Unless necessarily distinct, we henceforth use estimators α and the actions a proposing them interchangeably.

To quantify the preference orderings beyond the simple heuristics mentioned in the coin-flipping case, statisticians leverage loss functions [4], which we denote $\mathcal{L}(\theta, \alpha)$. The loss function represents the error associated with proposing a "bad" evaluation of the θ (or function of θ) of interest. Thus, the best evaluation of this function is a zero loss; therefore $\mathcal{L}(\theta, \alpha) \geq 0$ [5]. From a decision-theoretic perspective, the objective of the decision-maker is to propose an estimator α which minimizes this loss. Since the true value of parameter θ is often unknown, the preference ordering of estimators is often dictated by their *expected* loss. However, exactly how we define *expected* relies upon whether one takes a frequentist or Bayesian approach.

Frequentism and Minimax: Under the frequentist paradigm, the data $x \in \mathcal{X}$ are considered random because it arises from repeated sampling via the DGP $p(x|\theta)$. Meanwhile, θ is treated as a fixed but unknown constant in the parameter space Θ. In the coin-flipping example, a frequentist would assume that the coin has a fixed (unknown) probability θ of landing heads, and each flip outcome is then governed by $p(x|\theta)$. Thus, to evaluating a proposed estimator α, the frequentist approach focuses on expected loss, akin to how Peterson [6] considers the expected utility. Specifically, we define the expected loss (EL) as the product of the probability of observing $x \in \mathcal{X}$ and the loss associated with estimating θ with $\alpha(x)$,

$$\text{EL}(\theta, \alpha) = \mathbb{E}_\theta[\mathcal{L}(\theta, \alpha)] = \sum_{x \in \mathcal{X}} \mathcal{L}\big(\theta, \alpha(x)\big) p(x|\theta) \tag{1}$$

The above is also referred to as a risk function [7]. From this definition of expected loss, we introduce the concept of "minimax" through a game-theoretic analogy of a game against Nature. In this framework, our goal is to select an estimator $\alpha \in \mathcal{A}$ that *minimizes* our expected loss. Meanwhile, Nature acts as an adversary, selecting a parameter $\theta \in \Theta$ (i.e. a "state of the world") in an attempt to *maximize* our expected loss [8]. The expected loss in

[1] Given $p(x|\theta) = \prod_{i=1}^{n} \theta^{x_i}(1-\theta)^{1-x_i}$, the log-likelihood of the n observations is $\ell(x, \theta) = \log(\theta)\sum_{i=1}^{n} x_i + \log(1-\theta)\sum_{i=1}^{n}(1-x_i)$. Maximizing wrt θ yields $\hat{\theta}_{\text{MLE}} = \frac{1}{n}\sum_{i=1}^{n} x_i = \alpha_1(x)$.

such a game is known as the "minimax risk", which we define as

$$\overline{R} = \min_{\alpha \in \mathcal{A}} \max_{\theta \in \Theta} \mathrm{EL}(\theta, \alpha) \tag{2}$$

The estimator/decision rule $\alpha \in \mathcal{A}$ that achieves the minimax risk is known as the minimax estimator. While the minimax risk $\overline{R}$ is occasionally criticized as being overly conservative [6], the ability of an estimator to be the best in the worst case scenario (which we refer to as the "minimax guarantee") is desirable for many real-world applications including management of financial portfolios [9].

Having defined the minimax risk in Equation (2) and the corresponding guarantee, we now turn to *The Bayesian Choice* [5], in which Christian Robert demonstrates that under certain "least favorable" priors, Bayesian decision theory achieves a Bayes risk that is at least as good (and often better than) this frequentist minimax bound.

Remaining Work

1. Introduce Robert's Argument and proof. (Bayesianism, Integrated Risk, proof of Integrated Risk $\leq$ Minimax Risk using weighted sum vs. set maxima)

2. Introduce Stark's Counterargument: How the prior $\pi(\theta)$ is subjective, and Robert's proof is trivial since you are "adding information" to the risk problem which was previously constrained by objectivity.

3. Introduce the Bayesian Rebuttal: Namely, the subjectivity of choice of loss function $\mathcal{L}(\dots)$ implies the frequentist construction of the problem isn't operating under such "objective constraints," so given that subjective claims need to be made on the state of Nature, a Bayesian approach gives provable optimality.

4. Conclusion and Introduction

References

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