Non-Normal Time Series Estimation

Replication Study: Implementation

2025-06-19

Introduction

TODO

Proof of 2.7: Conditional Density

We now use the fact that each X_n has $\Phi_{X_n}(s) = \big(1+\theta s\big)^{-\nu}$ (the so-called 'equilibrium version' used in 2.3.1) which defines a Gamma distribution with shape ν and scale θ^{-1} (equivalently, rate θ) with corresponding marginal probability density function:

$$f_{X_n}(x_n) = \frac{1}{\Gamma(\nu)\theta^{\nu}} x^{\nu-1} e^{-x/\theta}$$

As well as bivariate joint density (2.6.1)

$$f_{X_{n+j},X_n}(x,y) = \frac{1}{\Gamma(\nu)\theta^{\nu+1}(1-p^j)} \Big(\frac{xy}{p^j}\Big)^{(\nu-1)/2} \exp\Big(\frac{-(x+y)}{\theta(1-p^j)}\Big) I_{\nu-1}\Big(\frac{2(p^j xy)^{1/2}}{\theta(1-p^j)}\Big)$$

Consequently, the conditional density $f_{X_{t+j}|X_t}(x \mid y)$ can be computed as follows, letting the modified Bessel function component (rightmost term in the above) be written as C_{xy} for simplicity.

$$\begin{aligned}
f_{X_{t+j}|X_t}(x \mid y) &= \frac{f_{X_{t+j},X_t}(x,y)}{f_{X_t}(y)} \\
&= \frac{\frac{1}{\Gamma(\nu)\theta^{\nu+1}(1-p^j)} \Big(\frac{xy}{p^j}\Big)^{(\nu-1)/2} \exp\Big(\frac{-(x+y)}{\theta(1-p^j)}\Big) C_{xy}}{\frac{1}{\Gamma(\nu)\theta^{\nu}} y^{\nu-1} \exp\big(-\frac{y}{\theta}\big)} \\
&= \frac{\theta^{\nu-(\nu+1)}}{(1-p^j)} \Big(\frac{xy}{p^j}\Big)^{(\nu-1)/2} y^{-(\nu-1)} \exp\Big(\frac{-(x+y)}{\theta(1-p^j)} - \big(-\frac{y}{\theta}\big)\Big) C_{xy} \\
&= \frac{1}{\theta(1-p^j)} \Big(\frac{x}{p^j y}\Big)^{(\nu-1)/2} \exp\Big(-\frac{x+p^j y}{\theta(1-p^j)}\Big) I_{\nu-1}\Big(\frac{2(p^j xy)^{1/2}}{\theta(1-p^j)}\Big)
\end{aligned}$$

Recalling that $\theta = 1/(\varphi(1-p))$, we can write the above as:

$$f_{X_{t+j}|X_t}(x \mid y) = \frac{\varphi(1-p)}{(1-p^j)} \Big(\frac{x}{p^j y}\Big)^{(\nu-1)/2} \exp\Big(-\varphi(1-p)\frac{x+p^j y}{(1-p^j)}\Big) I_{\nu-1}\Big(\varphi(1-p)\frac{2(p^j xy)^{1/2}}{(1-p^j)}\Big)$$

Finally, letting $\zeta = \varphi(1-p)/(1-p^j)$, we have the derivation provided in the labeled (2.7) of the Second Paper.

$$\boxed{f_{X_{t+j}|X_t}(x \mid y) = \zeta\Big(\frac{x}{p^j y}\Big)^{(\nu-1)/2} \exp\big(-\zeta(x+p^j y)\big) I_{\nu-1}\big(2\zeta(p^j xy)^{1/2}\big)}$$

Where $I_r(x)$ is the modified Bessel function of the first kind and order r, given by (source):

$$I_r(x) = \sum_{m=0}^{\infty} \frac{1}{m!\,\Gamma(m+\alpha+1)} \Big(\frac{x}{2}\Big)^{2m+a}$$

We will use the boxed equation above in the estimation of the model.