Gamma AR Processes with Variable Shape

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Introduction

Following the construction of Sim (1990), we consider a model autoregressive gamma model of the following form:

$$X_n = \alpha * X_{n-1} + \varepsilon_n, \quad \text{where} \quad \alpha * X_{n-1} = \sum_{i=0}^{N(X)} W_i$$

In the above,

- $\varepsilon_n \sim \text{Gamma}(\nu, \alpha)$ with (unintuitively) **shape** ν and **scale** α.
- Each W_i are *iid* exponential random variables with rate α
- For a fixed value of x, the $N(x)$ is a Poisson distributed random variable with rate paraemeter $\lambda = p\alpha$
- Where p indicates the magnitude of the autocorrelation in the process

Allowing Temporal Dependence in Shape

We now consider the case where the shape parameter ν is unique for each X_n; as such, it is temporally dependent and potentially a function of other random variables, i.e. $\nu \to \nu_n$, for $n > 0$.

Derivations 2.1.1, 2.1.2, 2.2

We begin by noting that the Laplace transform (LT) of $\alpha * X$ conditioned on a fixed $X = x$ (derivations 2.1.1 and 2.1.2) remain unchanged, as both W_i and $N(X)$ are independent of ν_t.

From this independence, the cumulative process $\alpha * X_{n-1}$ remains independent of ε_t. Thus the LT of X_n given $X_{n-1} = x$ can be written as a slight adjustment ot the previously-derived Equation 2.2., i.e.

$$\mathbb{E}[\exp(-sX_n) \mid X_{n-1} = x] = \left(\frac{\alpha}{s+\alpha}\right)^{\nu_t} \exp\left(\frac{-\lambda x s}{\alpha + s}\right)$$

As before, allowing $\Phi_{X_n}(s)$ to be the Laplace transform of the PDF of X_n (i.e. $\Phi_{X_n}(s) = \mathbb{E}[\exp(-sX_n)]$), yields

$$\Phi_{X_n}(s) = \left(\frac{\alpha}{s+\alpha}\right)^{\nu_n} \Phi_{X_{n-1}}\left(\frac{\alpha p s}{\alpha + s}\right)$$

Derivation 2.2.2: The Trouble Point

Previously, we solved the recursive relationship for $\Phi_{X_n}(s)$ to prove that $\Phi_{X_n}(s) \to (1+\theta^{-1})^{-\nu}$ as $n \to \infty$.

In order to do this, we recognized and simiplified a telescoping series that allowed us to write:

$$\Phi_{X_n}(s) = \Big(\frac{\alpha}{\alpha + s\sum_{k=0}^{n-1} p^k}\Big)^{\nu} \Phi_{X_0}\Big(\frac{\alpha p^n s}{\alpha + s\sum_{k=0}^{n-1} p^k}\Big)$$

In my proof of this, I wrote up a generalized $n = 2$ case and noticed the emerging pattern. However, in my rough work I also derived the $n = 3$ instance to identify the telescoping properties. Specifically, letting C_2 be the coefficients of $\Phi_{X_2}(s)$ I had that

$$\begin{aligned}
\Phi_{X_3}(s) &= C_2 \cdot \Phi_{X_1}\Big(\frac{\alpha p^2 s}{\alpha + p(1-s)}\Big) \\
&= C_2 \cdot \left(\frac{\alpha}{\alpha + \dfrac{\alpha p^2 s}{\alpha + p(1-s)}} \cdot \Phi_{X_0}(\dots)\right)^{\nu} \\
&= C_2 \Big(\frac{\alpha + p(1+s)}{\alpha + p^2 s + ps + p}\Big)^{\nu} \\
&= \Big(\frac{\alpha}{\alpha+s}\Big)^{\nu}\Big(\frac{\alpha+s}{\alpha+p(1+s)}\Big)^{\nu}\Big(\frac{\alpha+p(1+s)}{\alpha+p^2 s+ps+p}\Big)^{\nu}\Phi_0(\dots) \qquad (\star)
\end{aligned}$$

The telescoping pattern in $(\star)$ allows us to simplify the pattern for general n to a geometric sum in which $\Phi_0(\dots)$ becomes negligible as $n \to \infty$. However, if we allow for different values of ν for each n, the series cannot simplify in this fashion.

Instead, it becomes some strange non-simplifying product sequence.

$$\begin{aligned}
\Phi_{X_3} &= \Big(\frac{\alpha}{\alpha+s}\Big)^{\nu_1}\Big(\frac{\alpha+s}{\alpha+p(1+s)}\Big)^{\nu_2}\Big(\frac{\alpha+p(1+s)}{\alpha+p^2 s+ps+p}\Big)^{\nu_3}\Phi_0(\dots) \\
&= \alpha^{\nu_1}(\alpha+s)^{\nu_2-\nu_1}\big(\alpha+p(1+s)\big)^{\nu_3-\nu_2}\big(\alpha+p^2+ps+p\big)^{-\nu_3}
\end{aligned}$$

So then the general form would be something like this (omitting the $\Phi_0(\dots)$ contribution for simplicity):

$$\Phi_{X_n}(s) \propto \alpha^{\nu_1} P_n^{-\nu_n} \prod_{k=1}^{n-1} P_k^{\nu_k - \nu_{k-1}}, \text{ where } P_k = \alpha + p + s\sum_{m=1}^{k-1} p^m$$

Since $\nu_k - \nu_{k-1} \neq 0$ by assumption of time-varying shape terms, the product does not reduce to a gamma distribution form (although it may converge), and hence we cannot write $\Phi_{X_n}(s)$ as a Gamma distribution as $n \to \infty$.

Variable Scale

Similarly to earlier, should we allow the scale to vary with time, i.e. $\alpha \to \alpha_t$, there are a few notable changes.

Firstly, for a fixed $X_n = x$ we still have:

$$\{\mathcal{L}^*_{\alpha_n * X}\}(s) = \exp(-\frac{p\alpha_n xs}{\alpha_n + s})$$

However, this is now dependent on α_n, implying that

$$\begin{aligned}
\mathbb{E}[\exp(-sX_n) \mid X_{n-1} = x] &= \mathbb{E}[\exp(-s\varepsilon_n)]\mathbb{E}\big[-(\alpha * X_{n-1}) \mid X_{n-1} = x\big] \\
&= \Big(\frac{\alpha_n}{s+\alpha_n}\Big)^{\nu} \exp\Big(\frac{-p\alpha_{n-1} xs}{\alpha_{n-1}+s}\Big)
\end{aligned}$$

Thus, the LT of the unconditional PDF of X_n (Derivation 2.2.2) faces a similar problem to before, since

$$\Phi_{X_n}(s) = \left(\frac{\alpha_n}{s+\alpha_n}\right)^{\nu} \Phi_{X_{n-1}}\left(\frac{p\alpha_{n-1}s}{\alpha_{n-1}+s}\right)$$

Which means that upon solving recursively (for, say, $n = 2$) we have:

$$\begin{aligned}
\Phi_{X_2}(s) &= \left(\frac{\alpha_2}{s+\alpha_2}\right)^{\nu} \Phi_{X_1}\left(\frac{p\alpha_1 s}{\alpha_1+s}\right) \\
&= \left(\frac{\alpha_2}{s+\alpha_2}\right)^{\nu} \left[\left(\frac{\alpha_1}{\frac{p\alpha_1 s}{\alpha_1+s}+\alpha_1}\right)^{\nu} \Phi_{X_0}\left(\frac{\alpha_0 p \frac{p\alpha_1 s}{\alpha_1+s}}{\alpha_0+\frac{p\alpha_1 s}{\alpha_1+s}}\right)\right] \\
&= \left(\frac{\alpha_2}{s+\alpha_2}\right)^{\nu} \left(\frac{\alpha_1+s}{\alpha_1+s+ps}\right)^{\nu} \Phi_{X_0}\left(\frac{\alpha_0\alpha_1 p^2 s}{\alpha_0\alpha_1+\alpha_0 s+p\alpha_1 s}\right)
\end{aligned}$$

Notice that the third term in the product $\Phi_{X_0}(\dots)$ doesn't simplify as nicely as before. Because $\alpha_1 \neq \alpha_0$, we cannot factor out the α term as we did before.

This means that in the $n \geq 3$ cases we lose the telescoping property.

$$\Phi_{X_3}(s) = \left(\frac{\alpha_3}{s+\alpha_3}\right)^{\nu} \left(\frac{\alpha_2+s}{\alpha_2+s+ps}\right)^{\nu} \left(\frac{\alpha_1\alpha_2+\alpha_2 p^2 s+\alpha_1 s+p\alpha_2}{\alpha_1\alpha_2+\alpha_1 s+p\alpha_2 s}\right)^{\nu} \Phi_{X_0}(\dots)$$

Thus, this form doesn't converge to a Gamma distribution, either.

Which Factors Still Hold?

If we allow ν to vary with time, the LT of the PDF of the conditional density of X_n remains

$$\{\mathcal{L}^{\star}_{X_n|X_{n-1}}\}(s) = \mathbb{E}[\exp(-sX_n) \mid X_{n-1} = x] = \left(\frac{\alpha}{s+\alpha}\right)^{\nu_n} \exp\left(\frac{-p\alpha x s}{\alpha+s}\right)$$

From this, we could theoretically compute the inverse Laplace transform and construct a log-likelihood function

$$\ell(\boldsymbol{\theta}; x_0, x_1, \dots x_n) \propto \sum_{t=1}^{n} \log(\{\mathcal{L}^{\star}_{X_n|X_{n-1}}(s)\}^{-1}) = \sum_{t=1}^{n} f_{X_t|X_{t-1}}(x_t \mid x_{t-1})$$

However, in doing this we could not admit that our X_t are Gamma-distributed. We could allow X_t to be our DKR process, and consider this

Conclusion

Since neither adjustment converges X_n to a Gamma distribution, the subsequent derivations and properties will not hold either. Further, we cannot use it for our DKR model by the lack of Gamma-distributed X_n, which we had intended to equate our model to one of the coefficients with.