A Note on Linearity

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Introduction

Previously, we used the Cochrane-Orcutt Procedure with the Sliding Windows Regression (SWR) model given by.

$$y_t = \sum_{i=1}^{k} \beta^{(i)}(x * \kappa^{(i)})[t] + \eta_t$$

Where η_t are the uncorrelated errors.

The mathematical justification for the Cochrane-Orcutt procedure is as follows:

$$\begin{aligned}
y_t - \varphi y_{t-1} &= \sum_{i=1}^{k} \beta^{(i)}(x * \kappa^{(i)})[t] - \varphi \sum_{i=1}^{k} \beta^{(i)}(x * \kappa^{(i)})[t-1] + \eta_t \\
&= \sum_{i=1}^{k} \beta^{(i)} \Big((x * \kappa^{(i)})[t] - \varphi (x * \kappa^{(i)})[t-1] \Big) + \eta_t \\
&= \sum_{i=1}^{k} \beta^{(i)} \Big(\sum_{s=1}^{t} (x_s - \varphi x_{s-1}) \kappa^{(i)}_{t-s} \Big) + \eta_t
\end{aligned}$$

In the SWR paper, it was dictated that the term $x_0 * \kappa^{(i)}_t$, originating from index $s = 0$, is ignored. The justification is that the kernel $\kappa^{(i)}$ is padded with zero at the end when being aadjusted to the length of $x_{[t]}$.

Thus, the model can be written as:

$$\breve{y}_t = \sum_{i=1}^{k} \beta^{(i)} \cdot \big(\breve{x} * \kappa^{(i)} \big)[t] + \eta_t, \text{ where } \breve{w}_t = w_t - \varphi w_{t-1}$$

The question is - can our DKR model be written in a similar transformation?

Dynamic Kernel Regression Procedure

In a DKR model, we have a somewhat different structure to our estimates y_t. Specifically,

$$y_t = \sum_{i=1}^{k} \big(\beta_0^{(i)} + \beta_1^{(i)} (z * \kappa^{(i)})[t] \big) \cdot (x * \kappa^{(i)})[t] + \eta_t$$

Where z is a modulating variable.

The first important thing to note about the DKR fit is that when the primary brackets containing the beta terms are expanded, we arrive at the following:

$$y_t = \underbrace{\sum_{i=1}^{k} \beta_0^{(i)} (x * \kappa^{(i)})[t]}_{\text{Identical to SWR}} + \sum_{i=1}^{k} \beta_1^{(i)} (z * \kappa^{(i)})[t](x * \kappa^{(i)})[t] + \eta_t$$

Notice that the first term in the sum is identical to the Sliding Windows Regression model. Thus, when taking the difference $y_t - \varphi y_{t-1}$, this term will not be problematic.

Thus, we focus on the second term and deal with proportionalities.

$$\begin{aligned} y_t - \varphi y_{t-1} &\propto \sum_{i=1}^{k} \beta_1^{(i)} (z * \kappa^{(i)})[t](x * \kappa^{(i)})[t] - \varphi \sum_{i=1}^{k} \beta_1^{(i)} (z * \kappa^{(i)})[t-1](x * \kappa^{(i)})[t-1] + \eta_t \\ &\propto \sum_{i=1}^{k} \beta_1^{(i)} \Big((z * \kappa^{(i)})[t](x * \kappa^{(i)})[t] - \varphi (z * \kappa^{(i)})[t-1](x * \kappa^{(i)})[t-1] \Big) \end{aligned}$$

Now, let $Z_t = (z * \kappa^{(i)})[t]$ and $X_t = (x * \kappa^{(i)})[t]$.

$$\begin{aligned} y_t - \varphi y_{t-1} &\propto \sum_{i=1}^{k} \beta_1^{(i)} (Z_t X_t - \varphi Z_{t-1} X_{t-1}) \\ &\propto \sum_{i=1}^{k} \beta_1^{(i)} (Z_t X_t \overbrace{- \varphi X_{t-1} Z_t + \varphi X_{t-1} Z_t}^{\text{Insert terms}} - \varphi Z_{t-1} X_{t-1}) \\ &\propto \sum_{i=1}^{k} \beta_1^{(i)} \big(Z_t \big(X_t - \varphi X_{t-1}\big) + \varphi X_{t-1} \big(Z_t - \varphi Z_{t-1}\big)\big) \\ &\propto \sum_{i=1}^{k} \beta_1^{(i)} \big(Z_t \big(X_t - \varphi X_{t-1}\big) + \varphi X_{t-1} \big(Z_t - \varphi Z_{t-1}\big)\big) \\ &\propto \sum_{i=1}^{k} \beta_1^{(i)} \Big((z * \kappa^{(i)})[t] \breve{x}_t + \varphi (x * \kappa^{(i)}) \breve{z}_t \Big) \end{aligned}$$

Which isn't the same simplification as in the 1D case.

As an Atomic Term

While somewhat unsatisfying, if instead we treat each product term as an inseperable ('atomic') variable w_t, i.e.

$$w_t^{(i)} = \Big((z * \kappa^{(i)})[t](x * \kappa^{(i)}) \Big) [t]$$

Then, our DKR equation becomes:

$$y_t = \sum_{i=1}^{k} \beta_0^{(i)} (x * \kappa^{(i)})[t] + \sum_{i=1}^{k} \beta_1^{(i)} w_t^{(i)} + \eta_t$$

And the difference is trivially

$$\begin{aligned} y_t - \varphi y_{t-1} &= \left(\sum_{i=1}^{k} \beta_0^{(i)} (x * \kappa^{(i)})[t] - \varphi \sum_{i=1}^{k} \beta_0^{(i)} (x * \kappa^{(i)})[t] \right) \\ &\qquad + \left(\sum_{i=1}^{k} \beta_1^{(i)} w_t^{(i)} - \varphi \sum_{i=1}^{k} \beta_1^{(i)} w_t^{(i)} \right) + \eta_t \\ &= \sum_{i=1}^{k} \beta_0^{(i)} \cdot \left(\breve{x} * \kappa^{(i)} \right)[t] + \sum_{i=1}^{k} \beta_1^{(i)} \breve{w}_t^{(i)} + \eta_t \end{aligned}$$

So there really isn't a hope of expressing everything through only $\mathbf{x}_T$ and $\mathbf{z}_T$ due to the nonlinearity, but the above is about as close as we can get. All we have to do is treat each modulating-variable term as an inseperable scalar.

However, this does allow us to recover the original form of our DKR expression just in terms of the transformation $\breve{g}_t = g_t - \varphi g_{t-1}$, since the simplification above yields

$$\breve{y}_t = \sum_{i=1}^{k} \beta_0^{(i)} \cdot \left(\breve{x} * \kappa^{(i)} \right)[t] + \sum_{i=1}^{k} \beta_1^{(i)} \breve{w}_t^{(i)} + \eta_t$$