Chaotic Functions

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Logistic Map

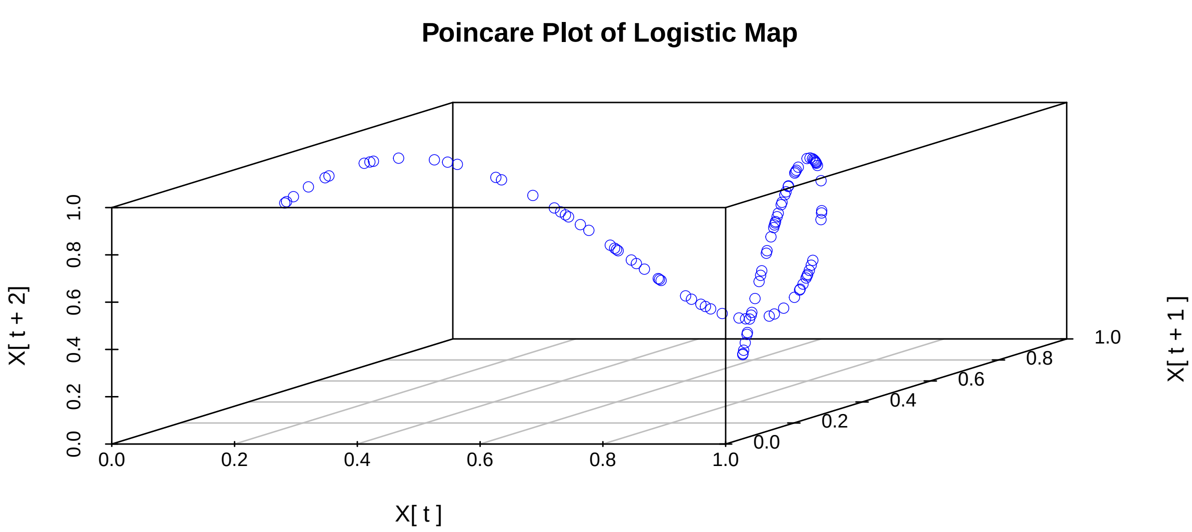
The Logistic Map depends on the parameter r, which controls the behaviour of the system and ranges between 0 and 4.

For $r \in [0, 1)$ the population will eventually die out, i.e. $x_n \to 0$. For $r \in [1, 3)$ the system stabilizes to a fixed point dependent on r, and for $r \in [3, \approx 3.57]$ the system exhibits period-doubling bifuractions between two points. Finally, for $r \in [3.57, 4]$ the system is chaotic and the values of x_n are highly dependent on the initial condition x_0.

The general evolution of a logistic map is given by:

$$x_{n+1} = r \cdot x_n \cdot (1 - x_n), \text{ where } r \in [0, 4] \text{ and } x_0 \in \mathbb{R}$$

Below is a plot of a logistic map, with $x_0 = 0.15$ and $r = 3.9$.



Henon Map

A Henon Map is a two-dimensional discrete time chaotic system that can produce strange attractors. It was developed in 1976 as a simplified model of the Poincaré section of the Lorenz system.

It is defined by state variables x_n and y_n alongside real-valued coefficients a and b, in the system below.

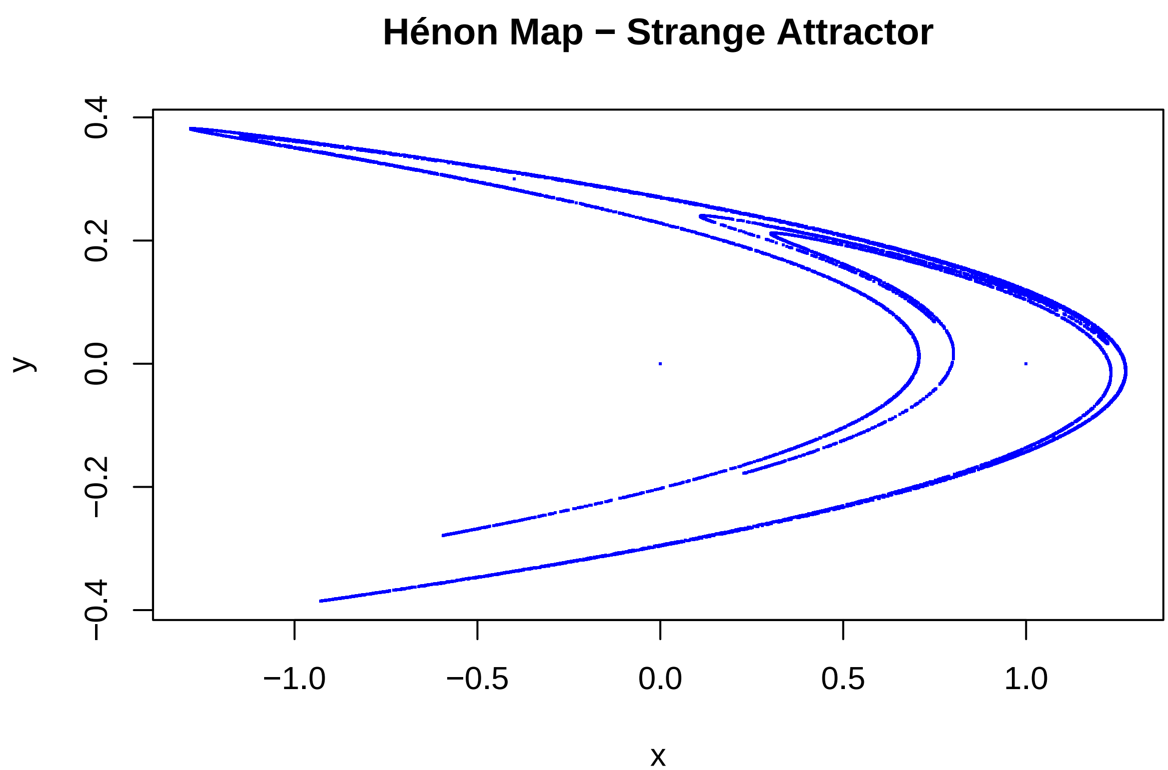
$$x_{n+1} = 1 - a \cdot x_n^2 + y_n$$
$$y_{n+1} = b \cdot x_n$$

Like all of these systems, the Henon Map evolves iteratively, i.e. with (x_0, y_0) you can compute (x_1, y_1), etc. Critically, the squaring of x_n in line one introduces nonlinearity in the evolution that tends to chaotic behaviour and strong dependence on the initial conditions.

For the standard initialization $a = 1.4$ and $b = 0.3$, the system is chaotic and does not settle into a cycle but instead a **strange attractor**.

In dynamic systems, an attractor is a set of states that the system tends towards over time. This can be a fixed point attractor $S = \{(x^\star, y^\star)\}$. Alternatively, it can be a limit **cycle attractor**, wherein the system settles into a periodic orbit dictated by some function, i.e. $S = \{\big(x(t), y(t)\big) \mid t \in T\}$ where T is the period (i.e. past discrete time T the cycle repeats.) The system can also form a **torus attractor** which is quasi-periodic and forms a donut-like shape without ever exactly repeating. Finally, a **strange attractor** tends to a fractal structure that is non-periodic. The system never follows a periodic or repeating cycle, insead following some complex pattern appearing random (but is actually determined by the initial conditions.)

Below is a plot of a Henon Map forming a Strange Attractor in $\mathbb{R}^2$, with $x_0 = y_0 = 0$.



Lorenz System

The Lorenz System is a set of three non-linear differential equations introduced by Edward Lorenz in 1963.

The System is defined by the following ODEs.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \varsigma(y - x)$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = x(\rho - z) - y$$
$$\frac{\mathrm{d}z}{\mathrm{d}t} = xy - \beta z$$

Where x, y, z are the variables that evolve over time, and ς, ρ and β define the behaviour.

The first coefficient, ς, is known as the Prandtl number. It is related to the viscosity of the fluid. The second coefficeint, ρ, is the Rayleigh number. It is related to the temperature difference controlling the convection. Finally β is the geometric factor of the series.

The first equation controls how x evolves over time. In a physical sense, it represents the rate of change of the convective velocity, where ς controls the sensitivity of the system to the difference between y and x. The second equation describes the change in y over time, by combination of convection through x and cooling through y. Finally, the third equation governs the evolution of z, which is related to the heat transfer (xy) and the dissipation βz.

Below is a plot of the system with $\varsigma = 10$, $\rho = 28$, $\beta = 8/3$ and $x_0 = y_0 = z_0 = 1$.

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## Warning: package 'deSolve' was built under R version 4.4.1
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