CIRCUITS: STEP RESPONSE

INTRODUCTION

By now you have some familiarity with the three basic circuit elements: the resistor, the capacitor, and the inductor. For each of these elements there is a simple relationship between the *voltage across* the element, and the *current flowing through* the element.

When we combine these circuit elements, and connect them to a source of power such as a battery that is switched on and off or a signal generator, there will be an interesting and important time dependence to the currents and voltages in the circuit. To help you gain a qualitative understanding of the behavior of such circuits, it is helpful to review the energy properties of these elements and their power source.

1. The **signal generator** provides us with an AC (alternating current) voltage source. In other words, it puts a voltage across the circuit that varies with time. For this lab, we will be using a *square wave* input, which puts a positive voltage V_0 across the circuit for a set period of time, then a negative voltage $-V_0$ for a set time, and then repeats the process. For more information on the pulses from a signal generator, refer to Appendix I.

2. The **resistor** is a purely dissipative element. It cannot store energy; rather, it simply absorbs energy and dissipates it as heat. The voltage V dropped across a resistor with resistance R will simply follow Ohm's law: $v(t) = i(t)R$.

3. The **capacitor** can store energy in the *electric field* between its two plates, which itself is proportional to the voltage across the capacitor. When a capacitor is hooked up to a voltage source with a constant voltage V_0, current will flow and charges will build up on the capacitor plates. Once the potential difference between the plates is V_0, current will stop flowing. In essence, the capacitor always strives to have the voltage of the applied voltage source, but it takes time for the charges to build up on the plates.

Much like in the DC circuits lab for multiple resistors, the sum of the voltage across the resistor, $v_R(t)$, and the voltage across the capacitor, $v_C(t)$, in an RC circuit (refer to Figure 44) must add up to the voltage $v_{AC}(t)$ provided by the voltage source, and at every moment.[29]

$$v_{AC}(t) = v_R(t) + v_C(t) \tag{42}$$

Therefore, as the voltage is increasing across the capacitor, the voltage across the resistor will decrease (and vice versa).

4. The **inductor** can store energy in the *magnetic field*, which is proportional to the current through the inductor. Inductors are essentially coils of wire, much like the solenoids used in the **Static Magnetic Fields** lab and the **Magnetic Induction** lab. As current

[29] In all of our discussions of transient time-dependent conditions, we will use lowercase symbols, such as v, i, and q, to represent time-varying quantities.

passes through an inductor, a magnetic field is generated. Consequently, there is a magnetic flux going through the loops of the coil.

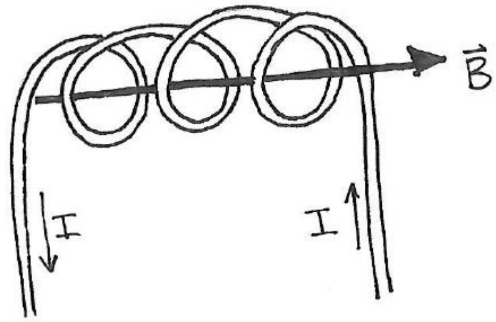

Figure 43. The magnetic field $\vec{B}$ generated by the current *I* traveling through an inductor.

If the current going through the inductor *changes*, however, the magnetic field (and hence the flux) must also change. As we have seen before, a change in magnetic flux results in an induced *emf*, which generates a magnetic field of its own that counteracts the change in the original field! As a result, it will take some time for the total current going through an inductor to change, because the induced current from the *emf* is opposing the change.

With respect to the *voltage* across an inductor, you will see behavior that is opposite that of the capacitor. If you hook up a constant voltage source to an LR circuit (refer to Figure 47), the inductor will initially have the voltage V_0 of the source (zero current will be flowing); as time goes on, the voltage $v_L(t)$ across the inductor will decrease (as the current flowing through the inductor builds to its maximum value). In the *RL* circuit, the same rule applies to the inductor as to the capacitor: the sum of the voltage across the resistor $v_R(t)$ and the voltage across the inductor $v_L(t)$ must add up to the voltage $v_{AC}(t)$ provided by the voltage source at every moment.

$$v_{AC}(t) = v_R(t) + v_L(t) \tag{43}$$

Thus, when circuit elements are connected together and connected to a power source, it is possible for energy to be stored and released by capacitors and inductors, and for energy to be dissipated into heat by resistors.

RC CIRCUITS

We can describe the total charge Q on a capacitor as the integral of the current $i(t)$ that has flowed into the capacitor:

$$q(t) = \int i(t')dt' \tag{44}$$

In differential form, using $q(t)$ and $i(t)$ for the time-dependent charge and current, we can write:

$$\frac{dq}{dt} = i(t) \tag{45}$$

The fundamental relation between the voltage, $v_C(t)$, on a capacitor and the charge is given by $v_C(t) = q(t)/C$. If we differentiate both sides and take into account the previous equations,

$$\frac{dv_C(t)}{dt} = \frac{1}{C} i(t) \tag{46}$$

This is the fundamental relationship between the voltage across a capacitor and the current flowing into the capacitor. In other words, this means that if we graph the voltage v_C *vs.* time t, the graph of the current $i(t)$ *vs*. t will look just like the derivative dv_C/dt of the voltage (with a proportionality constant).

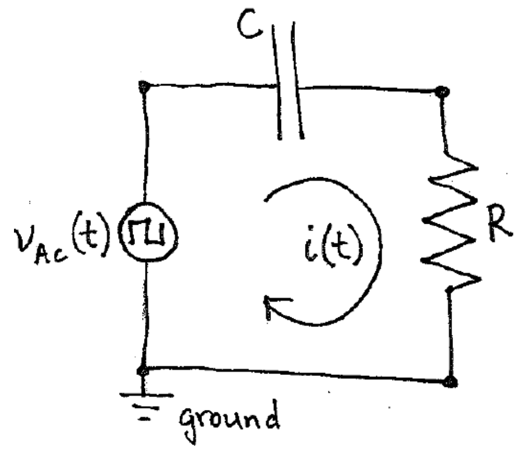

Figure 44. The *RC* circuit (note the current will be the same even if if *R* and *C* are interchanged). The arrow around $i(t)$ indicates the direction of positive current.

We are now equipped to understand the simplest circuit, the so-called *RC* circuit, as shown in Figure 44. Note that $v_{AC}(t)$ is the step voltage put forth by the signal generator: it will have a value of either V_0 or $-V_0$. This circuit will work fine, but in order to measure the voltages across various parts we will have to hook up an oscilloscope. Figure 45 shows the circuit with the oscilloscope attached, as well as how the capacitor and resistor are physically connected.

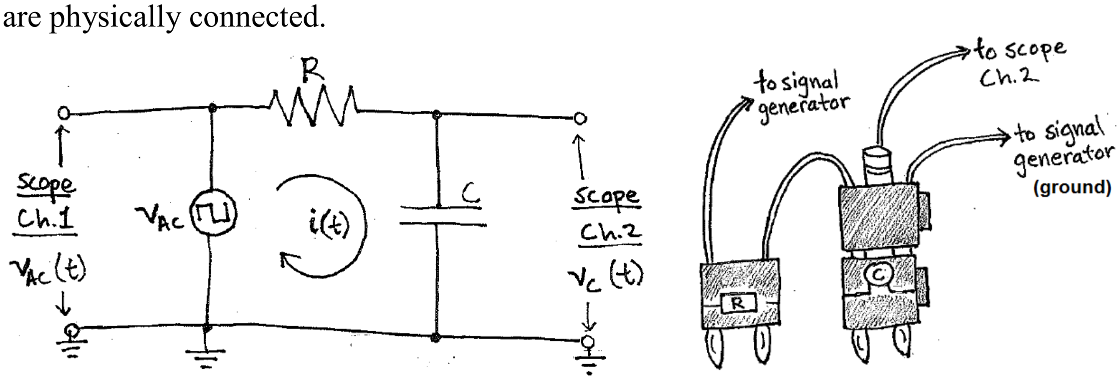

Figure 45. *Left* - The RC circuit with scope measuring $v_{AC}(t)$ across Channel 1 and $v_C(t)$ across Channel 2. *Right* - Connections to the oscilloscope. Note orientation of tabs on the right side.

In this circuit, Channel 1 of the oscilloscope is directly connected to the signal generator and monitors the input voltage $v_{AC}(t)$, while Channel 2 monitors the voltage $v_C(t)$, across the capacitor.

Notice that one terminal of the signal generator is at ground potential, as well as one terminal of each of the two channels of the oscilloscope. These connections are hard-wired, and cannot be changed. Thus, this oscilloscope can only measure voltages with respect to ground potential. An unfortunate consequence of this property of the oscilloscope is that we cannot measure directly the voltage across the resistor unless we interchange the capacitor and resistor in the circuit (we will do this in Experiment 2).

Important note: You will need an adaptor to convert from the coaxial connector on the oscilloscope to two banana sockets; and from the coaxial connector on the signal generator to two banana sockets. The adaptor has a tab on one edge, associated with the black banana jack. This means that the *black, tabbed* banana jack is connected to the grounded side of the coaxial connector. In order for your circuit to work properly, you must ensure the tabs are lined up on all three adaptors.

In the following experiments, we will first examine qualitatively the voltage across a capacitor in an *RC* circuit. Next, we will examine quantitatively the voltage across the resistor in the *RC* circuit, and calculate the time constant τ of the circuit.

Experiment 1 setup:

1. Set up the circuit of Figure 45, ensuring that the tabbed sides of the adaptors and the black banana jack on the signal generator are all lined up. On the oscilloscope, **CH 1** will connect directly to the signal generator, and **CH 2** will connect to the adaptor on the capacitor.
2. Set your signal generator to square wave mode, with an amplitude V_0 of about 2 volts. The amplitude is adjusted with the AMPL knob on the signal generator. Set the frequency to about 6 kHz using the FREQUENCY knob.
3. Go to the **CH 1** and **CH 2** menus on the oscilloscope and set the vertical sensitivity (**VOLTS/DIV)** to about 1 V/division. Set the oscilloscope sweep (**SEC/DIV**) to about 25 μs/div; this will show one full cycle of the input square wave on the scope (you may have to adjust this).

Experiment 1

1. Draw the square waveform from the signal generator and the resulting waveform from the capacitor below it. What is the maximum value of the voltage $v_C(t)$ across the capacitor plates? Is this what you would expect?
2. Keeping in mind Equation(43), draw what you would expect $v_R(t)$ to look like over the course of the step function.
3. Try adjusting the frequency on the signal generator and observing what happens to the voltage across the capacitor. What happens at higher frequencies (higher than 15 kHz)? Explain. Does anything change for lower frequencies?

We will now examine quantitatively the voltage across the resistor, $v_R(t)$. The corresponding circuit and connections is shown in
Figure 46.

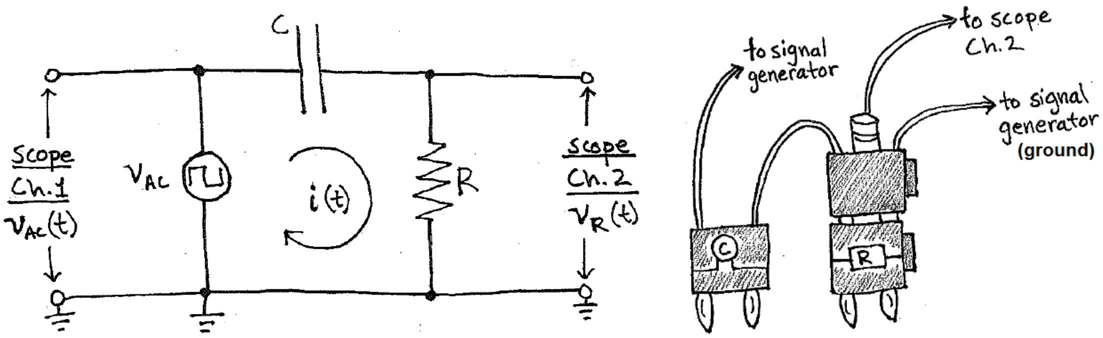

Figure 46. *Left* - The RC circuit with scope measuring $v_{AC}(t)$ across Channel 1 and $v_R(t)$ across Channel 2. *Right* - Connections for measuring $v_R(t)$ with the scope. Note orientation of tabs on the right side.

Channel 1 of the oscilloscope monitors the input voltage $v_{AC}(t)$, and Channel 2 monitors the voltage across the resistor $v_R(t) = Ri(t)$. Since the current flowing into the capacitor is the same as the current flowing into the resistor, Channel 2 monitors a quantity proportional to the circuit current. [30]

Experiment 2 setup:

1. Prior to hooking up your circuit, use the capacitance meter provided to measure the capacitance C of your capacitor, and the multimeter provided to measure the resistor R of your sensing resistor. These values will prove useful shortly.
2. Set up the circuit of Figure 46, ensuring that the tabbed sides of the adaptors and the black banana jack on the signal generator are all lined up. On the oscilloscope, **CH 1** will connect directly to the signal generator, and **CH 2** will connect to the adaptor on the resistor.
3. Set your signal generator to square wave mode, with an amplitude V_0 of about 2 volts and a frequency of about 6 kHz.
4. On the **CH 1** and **CH 2** menus, set the vertical sensitivity (**VOLTS/DIV)** to about 1 V/division. Set the oscilloscope sweep (**SEC/DIV**) to about 25 μs/div, adjusting as needed.
5. To adjust the relative positions of the waveforms on the scope, use the POSITION knobs.

If you have set up the circuit properly, you will notice that the voltage across the sensing resistor jumps by an amount exactly equal[31] to the amplitude V_0 of the square wave, and then decays according to the formula

$$v_R(t) = V_0 \exp(-t/\tau) \tag{47}$$

[30] You may be wondering if we need to worry about currents flowing to the oscilloscope channel, which would invalidate the assumption that the current through the capacitor is the same as the current through the resistor. For the purposes of this experiment, this effect is completely negligible, because the resistance of the oscilloscope is nearly infinite in comparison to the resistance R. The oscilloscope acts like an ideal voltmeter; *i.e.* an open circuit.

[31] There is a small transient effect at the top of the resistor voltage which is due to the signal generator not generating a *perfect* square wave.

Using Ohm's law *($v = iR$),* we see that the current through the *RC* circuit must have similar behavior, given by

$$i(t) = I_0 \exp(-t/\tau) \tag{48}$$

The equation for the voltage across the capacitor that you saw in Experiment 1 can thus be written as

$$v_C(t) = V_0(1-\exp(-t/\tau)) \tag{49}$$

In all of the above, the rate of exponential decay is $1/\tau$, where τ is called the *time constant* of the circuit. For an *RC* circuit, the time constant is expected to be $\tau = RC$. With the above equation, we are able to predict our circuit's *theoretical* time constant using the values you found for *R* and *C.* In order to verify our τ *experimentally*, it is convenient to relate τ to the time $t_{1/2}$ that it takes for the voltage across the sensing resistor to decay to ½ of its value:[32]

$$v_R(t_{1/2}) = V_0/2 \tag{50}$$

$$\frac{V_0}{2} = V_0 \exp(-t_{1/2}/\tau) \tag{51}$$

From the above equation, it is easy to prove that:

$$\tau = t_{1/2}/\ln(2) = 1.443 t_{1/2} \tag{52}$$

You will go through this proof in the Prelab.

Experiment 2

1. Draw the square waveform from the signal generator and the resulting waveform from the resistor below it. Is this what you would expect given Equation(47)? Does this match your prediction from Part 2 of Experiment 1?
2. Go to the CURSOR menu on the oscilloscope. Here you can set the "Type" of cursors to either Time (horizontal axis) or Voltage (vertical axis) and control their locations using the POSITION knobs (refer to the appendix on the oscilloscope if needed). Use the cursors to measure the time $t_{1/2}$ that it takes for the v_R signal to decay to 50% of its value.
3. Convert $t_{1/2}$ to the *RC* time constant (this is your *experimental* τ) and compare with the *theoretical* time constant using the *R* and *C* measurements you took earlier.
4. How would the time constant change if you replaced *R* with a much smaller resistor, say 100 Ω? Explain how this would affect the rate of charge building up on the capacitor plates.
5. How would the time constant change if you replaced *C* with a much larger capacitor? Explain.

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[32] In nuclear physics, $t_{1/2}$ is known as the *half-life* of a nucleus.

RL CIRCUITS

We now turn to the transient response of a similar circuit, the *RL* circuit, shown in Figure 47. In the following experiments, we will focus primarily on the voltage across the sensing resistor, $v_R(t)$.

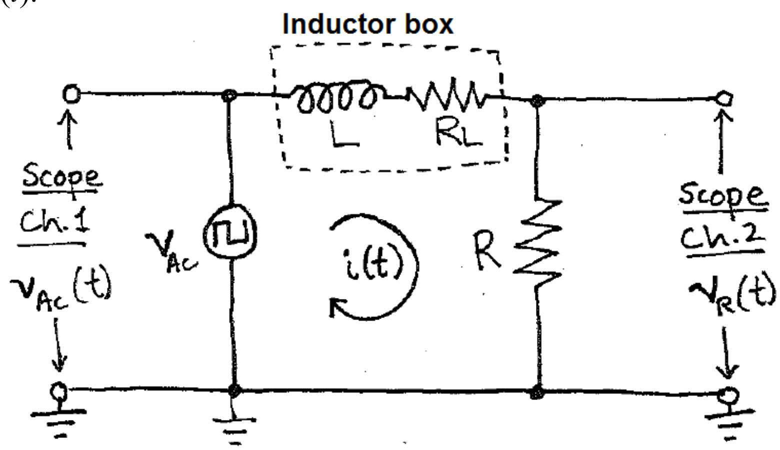

Figure 47. The *RL* circuit with scope measuring $v_{AC}(t)$ across Channel 1 and $v_R(t)$ across Channel 2

Note that the inductor box is drawn as a combination of both an inductor with inductance L (measured in milliHenries, mH) and a resistor R_L. This is because all real inductors have some inherent resistance – they are essentially coils of wire, and the wire has a non-negligible resistance. As such, when doing calculations with inductors as elements in a real circuit, we must always take into account R_L. For simplicity's sake, our theoretical discussion in the next few paragraphs will concern a "perfect" inductor ($R_L = 0$).

If we expand on the equations for induced voltage and magnetic field from the **Magnetic Inductance** lab (the curious reader is encouraged to derive the following relation for themselves), we find that the fundamental relation between the current and the voltage in an inductor, v_L, is as follows:

$$v_L(t) = L\frac{di(t)}{dt}, \text{ or } \frac{di(t)}{dt} = \frac{1}{L}v_L(t) \tag{53}$$

This equation tells us that the current through a perfect inductor will increase uniformly with time if a constant voltage is impressed across it. Conversely, if there is no voltage across a perfect inductor, the current will remain zero or constant; *i.e.* if you short-circuit a perfect, current-carrying inductor, the current will keep on flowing forever.[33]

Just after the voltage of the signal switches from 0 to some positive voltage V_0, since the initial current is zero the current will start to rise slowly from 0 at a uniform rate $di/dt = V_0/L$. The initial voltage amplitude across the inductor will be $V_L = V_0$, so the voltage amplitude across R (in an *RL* circuit) will initially be very small.

[33] Amazingly, there is such a thing as a perfect conductor, known as a *superconductor*. Rings of current have been observed to flow for years in such circuits.

It won't keep increasing steadily forever, because once a significant current starts to flow, some of the voltage V_0 appears across R. The current will eventually reach a plateau, whereupon all of the voltage appears across the resistor, and $V_R = V_0$.

We can describe the behavior of the voltage across the resistor as

$$v_R(t) = V_0(1 - \exp(-t/\tau)) \tag{54}$$

If we now take into account the inherent inductor resistance, the time constant τ will depend on L, R, and R_L.

$$\tau = L/(R + R_L) \tag{55}$$

As with the *RC* circuit, τ can be calculated by measuring the time it takes for V_R to reach half of its maximum value ($t_{1/2}$). Equation (55) still holds, though with a different τ.

Since we do not have a perfect inductor in our lab, part of the voltage V_0 will always drop across R_L. As our oscilloscope is only measuring across the sensing resistor R and *cannot* measure across R_L, the maximum value of $v_R(t)$ will always be slightly less than the voltage V_0 of the step function. As such, we use V_{max} to denote the maximum value of $v_R(t)$.

$$\begin{gathered} v_R(t) = V_{\max}((1 - \exp(-t/\tau)) \\ \text{Where } V_{\max} = \frac{R}{R + R_L} \end{gathered} \tag{56}$$

In the following experiment, we will examine $v_R(t)$ and find the time constant τ for an *RL* circuit.

Experiment 3

1. Set up the circuit of Figure 47, with Figure 48 as a reference for the connections. Use the same parameters as for the previous circuit.
2. Draw the square waveform from the signal generator and the waveform from the sensing resistor below it. Is this what you would expect from Equation (56)?
3. If you overlay both waveforms on the scope such that the zero of each waveform is the same (the small arrows on the left hand side of the scope screen indicate the zeroes), is there a discrepancy between V_0 and V_{max}? Approximately what percentage of the total voltage V_0 is the discrepancy? If you changed the sensing resistor R to a much smaller resistor (say 100 Ω) would you expect the discrepancy to become larger or smaller? Explain.
4. Using the CURSOR menu on the oscilloscope, measure $t_{1/2}$ and use it to calculate the experimental τ. Compare with the theoretical τ for this *RL* circuit, using the actual values of the components.

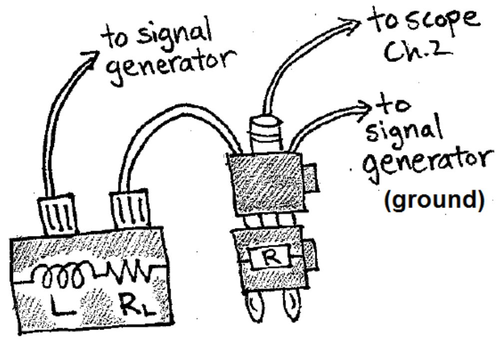

Figure 48. Connections for measuring $v_R(t)$ in an RL circuit.

APPENDIX I: TYPES OF PULSES

An AC voltage source (a.k.a. a *signal generator* or *function generator*) allows us to vary the voltage across a circuit from positive to negative values. The current in a circuit will also vary as the voltage changes – hence the term "**A**lternating **C**urrent" – but the precise way in which it varies will depend on the circuit elements (resistors, capacitors, and inductors). The signal generator we are using has three modes: square wave, triangular wave, and sinusoidal wave.

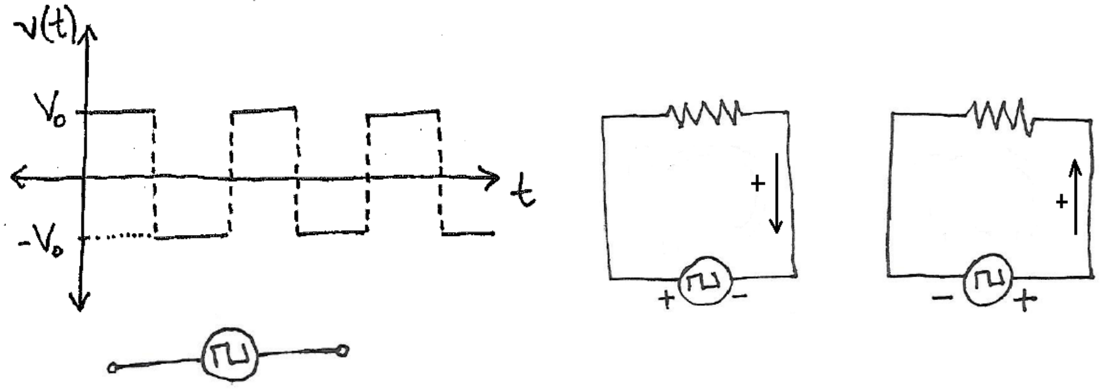

Figure 49. *Left* - Square wave pulse and corresponding symbol. *Right* - The current will alternate direction in a simple one-resistor circuit pulsed with a square wave.

In the square wave mode (Figure 49), the voltage alternates discontinuously between V_0 and $-V_0$. The current (motion of the positive charges) in a simple circuit with one resistor will thus alternate between the clockwise and counter-clockwise directions.

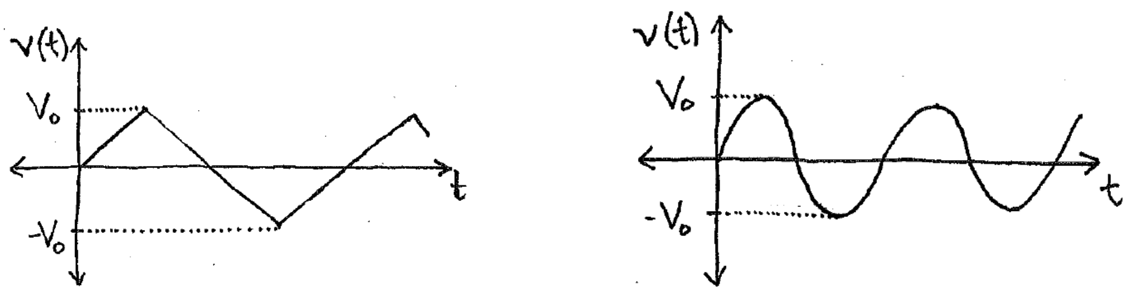

Figure 50. (a) Triangle wave pulse and symbol. **(b)** Sinusoidal wave pulse and symbol.

In the triangle wave mode shown in Figure 50a, the voltage linearly increases to V_0, then linearly decreases to $-V_0$. In the sinusoidal wave mode shown in Figure 50b, the voltage varies smoothly between V_0 and $-V_0$.

PRE-LABORATORY

Assume we have an *RC* circuit and an *RL* circuit. The *RC* circuit has a capacitor C = 10 nF and a sensing resistor of R = 1,200 Ω. The *RL* circuit has a sensing resistor R = 1,200 Ω and an inductor with L = 15 mH and R_L = 130 Ω. The input voltage in both cases is a square wave.

1. For the *RC* circuit, what is the value of the time constant τ? How about for the *RL* circuit?

2. For the *RC* circuit and the *RL* circuit, assume that the period of the source square wave is much larger than the time constant for each. Make a sketch of $v_R(t)$ as a function of t for each of the circuits

3. Starting from the equation for voltage, Equation (51) , show that $\tau = t_{1/2} / \ln(2) = 1.443\ t_{1/2}$.