8.3 IMPROPER INTEGRALS

(INTEGRALS INVOLVING INFINITY OR UNDEFINED)

EXAMPLE

$$\int_0^1 \sqrt{\frac{1+x}{1-x}}\,dx$$

NOTE: AT $x=1$ $\sqrt{\frac{1+1}{1-1}}$ APPROACHES ∞

RATIONALIZE NUMERATOR

$$\int_0^1 \frac{\sqrt{1+x}}{\sqrt{1-x}}\frac{\sqrt{1+x}}{\sqrt{1+x}}\,dx = \int_0^1 \frac{1+x}{\sqrt{1-x^2}}\,dx$$

AB OMIT $3^*, 12^*, 23$

BC OMIT 3, 12

$u=1-x^2$

$du=-2x\,dx$

$$= \int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx + -\frac{1}{2}\int_0^1 \frac{-2x}{\sqrt{1-x^2}}\,dx$$

$$= \sin^{-1}x\Big]_0^1 + -\frac{1}{2}\int_0^1 (1-x^2)^{-1/2}(-2x\,dx)$$

$$= \sin^{-1}x\Big]_0^1 - \frac{1}{2}\int u^{-1/2}\,du = \sin^{-1}x\Big]_0^1 - \frac{1}{2}\frac{u^{1/2}}{\frac{1}{2}}$$

$$= \sin^{-1}x\Big]_0^1 - \sqrt{1-x^2}\Big]_0^1$$

(FINITE SUM) * CONVERGENT

$$= \sin^{-1}1 - \sin^{-1}0 - \left(\sqrt{1-1^2} - \sqrt{1-0^2}\right) = \frac{\pi}{2}+1$$

EXAMPLE 1 p. 434

(NO FINITE SUM)

$$\int_1^\infty \frac{1}{x}\,dx = \ln x\Big]_1^\infty = \ln\infty - \ln 1$$

$$= \ln\infty = \underline{\underline{\infty}}$$

DIVERGENT

EXAMPLE

$$\int_0^1 \frac{1}{x}\,dx = \ln x\Big]_0^1 = \ln 1 - \ln 0$$

DIVERGENT

$$= 0-(-\infty) = \underline{\underline{\infty}}$$

HWORK p. 442 EX. 1-3 b, c ONLY, 7, 9-12, 15-17 ← $u=x^{1/2}$, 21 ← $u=\tan^{-1}x$, 23 INT BY PARTS

8.3 STILL TESTS FOR
(BC) CONVERGENCE OR DIVERGENCE.

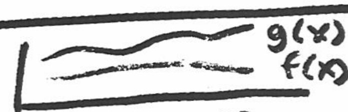
THE INTEGRAL TEST

IF AN INTEGRAL HAS A FINITE RESULT, THEN THE INTEGRAL CONVERGES.

IF AN INTEGRAL HAS NO FINITE RESULT, THEN THE INTEGRAL DIVERGES

$$\int_1^\infty x^{-2}\,dx = \frac{x^{-1}}{-1}\Big]_1^\infty = \frac{-1}{\infty} - \frac{-1}{1} = 1 \quad \text{CONVERGE}$$

$$\int_1^\infty x^{-1}\,dx = \ln|x|\Big]_1^\infty = \infty - 0 = \infty \quad \text{DIVERGE}$$



F $0 \le f(x) \le g(x)$, THEN

1) IF $\int_a^\infty g(x)\,dx$ CONVERGES, SO DOES $\int_a^\infty f(x)\,dx$

2) IF $\int_a^\infty f(x)\,dx$ DIVERGES, SO DOES $\int_a^\infty g(x)\,dx$

THE COMPARISON TEST

EXAMPLE: DOES $\int_0^{\pi/2} \frac{dx}{\sqrt{x}+\cos x}$ CONVERGE OR DIVERGE

$f(x) = \frac{1}{\sqrt{x}+\cos x}$ $\quad g(x) = \frac{1}{\sqrt{x}}$ $\quad 0 \le f(x) \le g(x)$

SINCE $\int g(x)\,dx$ CONVERGES

$$\int_0^{\pi/2} \frac{1}{\sqrt{x}}\,dx = \frac{x^{1/2}}{\frac{1}{2}}\Big]_0^{\pi/2} = 2\sqrt{\pi/2}$$

$\boxed{\int f(x)\,dx \text{ CONVERGES}}$

COMPARISON TEST CONTINUED

EXAMPLE: DOES $\int_1^\infty \frac{dx}{x-\sqrt[3]{x}}$

$f(x)=\frac{1}{x}$ $\quad g(x)=\frac{1}{x-\sqrt[3]{x}}$ $\quad 0\le f(x)\le g(x)$

$\int_1^\infty \frac{1}{x}dx = \ln|x|\Big]_1^\infty = \infty$

SINCE $\int f(x)\,dx$ DIVERGES, $\boxed{\int g(x)dx \text{ DIVERGES}}$

LIMIT COMPARISON TEST

(MY VERSION) MORE POWERFUL THAN COMPARISON

$p(x), q(x)$ CONTINUOUS $[a,\infty)$

$A=\int_a^\infty \frac{p(x)}{q(x)}dx$ CONVERGE OR DIVERGE

$B=\int_a^\infty \frac{\text{HIGHEST ORDER TERM}[p(x)]}{\text{HIGHEST ORDER TERM}[q(x)]}dx$

THEIR VERSION p. 439

A & B BOTH CONVERGE OR BOTH DIVERGE.

EXAMPLE 8 p. 439 $A=\int_1^\infty \frac{1\,dx}{1+x^2}$

$B=\int_1^\infty \frac{1\,dx}{x^2} = \frac{x^{-1}}{-1}\Big]_1^\infty = \frac{-1}{\infty}-\frac{-1}{1}=1$ CONVERGES

MY EXAMPLE $A=\int_1^\infty \frac{2x+3}{e^{x^2}+3x}dx$ $\quad B=\int_1^\infty \frac{2x}{e^{x^2}}dx$

$B=-\int_1^\infty e^{-x^2}(-2x)dx = -e^{-x^2}\Big]_1^\infty = -(e^{-\infty}-e^{-1})=\frac{1}{e}$

SINCE B CONVERGES, SO DOES A.

$\boxed{\text{HOMEWORK p. 442} \to 28, 30-35, 37-42, 45}$ (BC)

$u=\tan^{-1}y$

DISCONTINUITY IN THE INTERVAL

EXAMPLE 4 p. 437 BREAK-UP

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} \qquad \text{DISCONTINUITY AT } x=1$$

$$\int_0^1 (x-1)^{-2/3}dx + \int_1^3 (x-1)^{-2/3}dx \qquad u = x-1$$

$$\frac{(x-1)^{1/3}}{1/3}\Big]_0^1 + \frac{(x-1)^{1/3}}{1/3}\Big]_1^3 = 3\sqrt[3]{x-1}\Big]_0^1 + 3\sqrt[3]{x-1}\Big]_1^3$$

$$= 3\sqrt[3]{1-1} - 3\sqrt[3]{0-1} + 3\sqrt[3]{3-1} - 3\sqrt[3]{1-1}$$

$$= 0 - 3(-1) + 3\sqrt[3]{2} - 0 \qquad 3 + 3\sqrt[3]{2}$$

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_a^{\infty} f(x)dx$$

EXAMPLE 2 p. 434

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_0^{\infty} \frac{dx}{1+x^2}$$

$$= \tan^{-1}x\Big]_{-\infty}^{0} + \tan^{-1}x\Big]_0^{\infty} = \left(0 - \frac{-\pi}{2}\right) + \left(\frac{\pi}{2} - 0\right) = \pi$$

CONTINUOUS $[a, \infty)$

NOTE: $\int_a^{\infty} \frac{dx}{x^p}$ CONVERGES $p > 1$, DIVERGES $p \leq 1$ $\qquad \frac{\sin\theta}{\cos\theta} \leftarrow u$

AB & BC

HOMEWORK p. 442 → 4-6, 8, 22, 25*, 27, 29, 36, 43*, 44*

* HARD

$\boxed{\#5 \quad u = \frac{1}{x}}$ → $\cot\theta = \frac{\cos\theta}{\sin\theta} \leftarrow u$ $\qquad \int_1^0 \frac{1}{\sqrt{x}}dx + \int_0^4 \frac{1}{\sqrt{x}}dx$