9.4 RADIUS OF CONVERGENCE

$$\sum_{n=1}^{\infty} n^2 = 1+4+9+16+\cdots \quad \sum \text{ DIVERGES}$$

$$S = \frac{a}{1-r} = \frac{1/2}{1-1/2}$$

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots = 1 \quad \text{CONVERGES}$$

GEOMETRIC SERIES $-1 < r < 1$

$\boxed{\text{THE } n^{th} \text{ TERM TEST}}$

$$\sum_{n=1}^{\infty} a_n \text{ DIVERGES IF } \lim_{n\to\infty} a_n \neq 0$$

$$\sum_{n=1}^{\infty} \frac{n}{2n+100} \quad \underline{\text{DIVERGES}} \text{ BECAUSE } \lim_{n\to\infty} a_n = \frac{n}{2n} = \frac{1}{2}$$

$$\sum_{n=1}^{\infty} (-1)^n = -1+1-1+1\cdots \quad \underline{\text{DIVERGES}} \quad \lim_{n\to\infty} a_n \; \underline{\text{D.N.E.}}$$

$\boxed{\text{THE RATIO TEST}}$ LET $\sum a_n$ BE A POSITIVE SERIES

AND $\left| \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \right| = L$

(a) $\sum a_n$ CONVERGES IF $L < 1$

(b) $\sum a_n$ DIVERGES IF $L > 1$

(c) INCONCLUSIVE IF $L = 1$

EXAMPLE $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 3^n}{5^n}$

$$L = \left| \lim_{n\to\infty} \frac{\frac{(n+1)^2 (3)^{n+1}}{(5)^{n+1}}}{\frac{n^2 3^n}{5^n}} \right| = \rightarrow$$

$$L = \lim_{n\to\infty} \left| \frac{(n+1)^2 3^{n+1}}{5^{n+1}} \cdot \frac{5^n}{n^2 3^n} \right|$$

$$= \lim_{n\to\infty} \left| \left(\frac{n+1}{n}\right)^2 \cdot \frac{3^n \cdot 3^1}{3^n} \cdot \frac{5^n}{5^n \cdot 5^1} \right| = \left| 1 \cdot 3 \cdot \frac{1}{5} \right| = \frac{3}{5}$$

SINCE $L < 1$ $\sum_{n=1}^{\infty} \frac{n^2 3^n}{5^n}$ CONVERGES

HOMEWORK p. 495 → 1–15 ODD, 2, 4, 8, 12

9.5 TESTING CONVERGENCE

THE INTEGRAL TEST

GIVEN A SERIES $\sum_{n=1}^{\infty} a_n$ AND A FUNCTION $f(n) = a_n$

THEN $\sum_{n=1}^{\infty} a_n$ AND $\int_1^{\infty} f(n)\,dn$ PROOF p. 496–497

EITHER BOTH CONVERGE OR BOTH DIVERGE.

Ex. 1 p. 497 $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$

$$\int_1^{\infty} n^{-3/2}\,dn = \frac{n^{-1/2}}{-\frac{1}{2}} = \frac{-2}{\sqrt{n}}\Big]_1^{\infty} = \frac{-2}{\sqrt{\infty}} - \frac{-2}{\sqrt{1}} = 2$$

SINCE THE INTEGRAL CONVERGES,

THE SERIES CONVERGES.

THE p-SERIES TEST

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{CONVERGES IF } p > 1 \\ \text{DIVERGES IF } p \leq 1 \end{cases}$$

$\sum_{n=1}^{\infty} \frac{1}{n^4}$ <u>CONVERGES</u> $(p=4)$

THE LIMIT COMPARISON TEST p. 498

IF $\lim_{n\to\infty} \frac{a_n}{b_n} = C,\ 0 < C < \infty$ THEN $\sum a_n$ AND $\sum b_n$ EITHER BOTH CONVERGE OR BOTH DIVERGE.

EX. 3 p. 499

a) $\sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2}$ BEHAVES LIKE $\frac{2n}{n^2}$ AS $n \to \infty$

$$\sum_{n=1}^{\infty} \frac{2n}{n^2} = 2\sum_{n=1}^{\infty} \frac{1}{n^1} \quad p=1 \ \underline{\text{DIVERGENT}}$$

SINCE $\sum_{n=1}^{\infty} \frac{2n}{n^2}$ DIVERGES, $\boxed{\sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2} \ \underline{\text{DIVERGES}}}$

---

b) $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ BEHAVES LIKE $\sum_{n=1}^{\infty} \frac{1}{2^n}$ AS $n \to \infty$

$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ GEOMETRIC $S = \frac{a}{1-r} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$ <u>CONVERGE</u>

SINCE $\sum_{n=1}^{\infty} \frac{1}{2^n}$ CONVERGES, $\boxed{\sum_{n=1}^{\infty} \frac{1}{2^n - 1} \ \underline{\text{CONVERGES}}}$

HOMEWORK p. 506-507 → 1-11 <u>ALL</u>

ALTERNATING SERIES TEST p.501

IF A SERIES ALTERNATES SIGNS $(+,-,+,-\dots)$ CONTINUALLY DECREASES AND THE n^{th} TERM APPROACHES ZERO, THE SERIES CONVERGES.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$$

CONVERGES BY ALTERNATING SERIES TEST.

IF THE ABSOLUTE VALUE OF AN ALTERNATING SERIES CONVERGES THEN THE SERIES IS SAID TO CONVERGE ABSOLUTELY

IF AN ALTERNATING SERIES CONVERGES BUT THE ABSOLUTE VALUE OF IT DIVERGES, THE SERIES IS SAID TO CONVERGE CONDITIONALLY

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n^1} \quad p=1 \text{ DIVERGES}$$

SO $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ CONVERGES CONDITIONALLY

IF A SERIES DOES NOT ALTERNATE, IT EITHER CONVERES ABSOLUTELY OR DIVERGES.

$\boxed{\text{HWORK p. 507} \rightarrow \text{17-26 ALL}}$

9.4 POWER SERIES RADIOS OF CONVERGENCE

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots$$ SUM = 1 CONVERGES

$$\sum_{n=1}^{\infty} 2^n = 2+4+8+16+\cdots$$ SUM = ∞ DIVERGES

$$\sum_{n=1}^{\infty}(-1)^n = -1+1+-1+1+\cdots$$ SUM DOESN'T EXIST DIVERGES

$\sum_{n=1}^{\infty} x^n$ FOR WHAT VALUES OF X DOES THIS POWER SERIES CONVERGE?

REMEMBER THEOREM 9 p.491 (9.4)

$$L = \rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$ 1) CONVERGES ABS. $(\rho<1)$ 2) DIVERGES $\rho>1$

$L = \rho = 1$ INCONCLUSIVE

FOR POWER SERIES

$$\rho = \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|$$ u_n = POWER SERIES IN X. (n^{th} TERM)

1) CONVERGES ABS. $(\rho<1)$ 2) DIVERGES $(\rho>1)$

AT $\rho=1$ TEST THE "ENDPOINTS".

$$\sum_{n=1}^{\infty} x^n \quad L=\rho=\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x^{n+1}}{x^n}\right| = \left|\frac{x^n\cdot x^1}{x^n}\right| = |x|$$

$\rho<1$ CONVERGES $|x|<1$ CONVERGES

INTERVAL OF CONVERGENCE: CONVERGES $-1<x<1$

$|x|=1$ ENDPOINTS (1 OR −1)

$x=1$ $\sum_{n=1}^{\infty} x^n = \sum_{n=1}^{\infty} 1^n = 1+1+1+1+\cdots$ DIVERGES

$x=-1$ $\sum_{n=1}^{\infty} x^n = \sum_{n=1}^{\infty} (-1)^n = -1+1-1+1+\cdots$ DIVERGES

RADIUS OF CONVERGENCE = 1

EXAMPLE $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n\cdot 3^n}$

FIND THE INTERVAL OF CONVERGENCE.

$$\rho = \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{\left(x^{n+1}\right)/(n+1)\cdot 3^{n+1}}{x^n/n\cdot 3^n}\right|$$

$$= \left|\frac{x^{n+1}}{(n+1)\,3^{n+1}}\cdot\frac{n\cdot 3^n}{x^n}\right| = \left|\frac{x^{n+1}}{x^n}\cdot\frac{n}{n+1}\cdot\frac{3^n}{3^{n+1}}\right|$$

RADIUS OF CONVERGENCE = 3

$$= \left|x\cdot 1\cdot\frac{1}{3}\right| = \left|\frac{x}{3}\right| \qquad \left|\frac{x}{3}\right|<1 \qquad |x|<3$$

CONVERGES

$\left|\frac{x}{3}\right| = 1$ ENDPOINTS $x=3$ OR -3

p-series $p=1$

$$x=3 \quad \sum_{n=1}^{\infty}\frac{3^n}{n\cdot 3^n} = \sum_{n=1}^{\infty}\frac{1}{n^1} = \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots \text{ DIVERGES}$$

Th.1 p.675

$$x=-3 \quad \sum_{n=1}^{\infty}\frac{(-3)^n}{n\cdot 3^n} = -\frac{1}{1}+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}+\cdots \text{ CONVERGES}$$

ALT. SERIES TEST

INTERVAL OF CONVERGENCE

So $\displaystyle\sum_{n=1}^{\infty}\frac{x^n}{n\cdot 3^n}$ CONVERGES $-3\le x<3$

EXAMPLE $\displaystyle\sum_{n=1}^{\infty} n^n x^n$ FIND INT. OF CONVERGENCE

$$\rho = \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{(n+1)^{n+1}x^{n+1}}{n^n x^n}\right| = \left|\frac{(n+1)^n(n+1)^1}{n^n}\,\frac{x^n\cdot x^1}{x^n}\right|$$

$$= |\infty\cdot x| \qquad \rho<1 \quad \text{AT } x=0 \text{ ONLY CONVERGES}$$

$x=0$

$$\sum_{n=1}^{\infty} n^n x^n = \sum_{n=1}^{\infty} n^n 0^n = 1^1\cdot 0^1 + 2^2\cdot 0^2 + 3^3\cdot 0^3 + \cdots = \underline{\underline{0}}$$

p.495 # 36 FIND THE INTERVAL OF CONVERGENCE

150

$$\sum_{n=0}^{\infty} \frac{(x+1)^{2n}}{9^n}$$

$$\rho = L = \lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \left| \frac{\frac{(x+1)^{2(n+1)}}{9^{n+1}}}{\frac{(x+1)^{2n}}{9^n}} \right|$$

DIVERGES AT BOTH ENDPOINTS

$$= \left| \frac{(x+1)^{2n+2}}{9^{n+1}} \cdot \frac{9^n}{(x+1)^{2n}} \right| = \left| \frac{(x+1)^2}{9} \right| \to \left| \frac{(x+1)^2}{9} \right| < 1$$

$$(x+1)^2 < 9 \to -3 < x+1 < 3 \to -4 < x < 2$$

REWRITE Σ AS A FUNCTION OF X $-4 < x < 2$

$$S = \frac{a}{1-r} = \sum_{n=0}^{\infty} 1 \cdot \left[\frac{(x+1)^2}{9}\right]^n = \sum_{n=1}^{\infty} 1 \cdot \left[\left(\frac{x+1}{3}\right)^2\right]^{n-1}$$

$a = 1 \quad r = \left(\frac{x+1}{3}\right)^2$

$$S = \frac{1}{1 - \frac{(x+1)^2}{9}} = \frac{9}{9-(x+1)^2} = \frac{9}{9-(x^2+2x+1)}$$

$$\boxed{S = \frac{9}{8-x^2-2x}}$$

RADIUS & INTERVAL

HWORK p. 495 → 21-39 ODD
p. 507 → 29-35 a ONLY

REVIEW FOR TEST → p.466 → 2, 9, 29, 43
p.478 → 6, 17, 27 p.486 → 4, 14, 32
p.495 → 1, 3, 23, 26 p.507 → 17, 19, 31a, 35a