4.6 RELATED RATES

AN EQUATION CAN BE DIFFERENTIATED WITH RESPECT TO ANY VARIABLE

$$Y = X^2 \qquad \frac{dY}{dX} = 2X$$

$$Y = X^2 \qquad \frac{dY}{dt} = 2X\frac{dX}{dt}$$

EXAMPLE

HOW FAST DOES THE RADIUS OF A SPHERE CHANGE WHEN THE VOLUME CHANGES 10 cm³/sec?

$$V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2\frac{dr}{dt}$$

OR $\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt} \Rightarrow 10 = 4\pi r^2\frac{dr}{dt}$

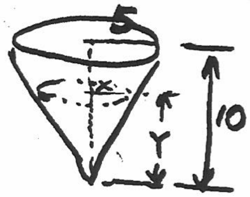
$$\frac{dr}{dt} = \frac{10}{4\pi r^2}$$

a) AT $r = 1$ cm

$$\frac{dr}{dt} = \frac{10}{4\pi \cdot 1^2} \approx .8 \text{ cm/sec}$$

b) AT $r = 10$ cm $\quad \frac{dr}{dt} = \frac{10}{4\pi \cdot 10^2} \approx .008 \text{ cm/sec}$

p.235 EXAMPLE 4 (SEE PICTURE p.236)



$\frac{dV}{dt} = 9\frac{ft^3}{min}$ $Y = 6ft$

FIND $\frac{dY}{dt}$. $\frac{X}{Y} = \frac{5}{10}$

$V = \frac{1}{3}\pi r^2 h$ $V = \frac{1}{3}\pi X^2 Y$ $X = \frac{Y}{2}$

$V = \frac{1}{3}\pi\left(\frac{Y}{2}\right)^2 Y$ $V = \frac{\pi}{12}Y^3$ $\frac{dV}{dt} = \frac{\pi}{12}3Y^2\frac{dY}{dt}$

$9 = \frac{\pi}{12}3(6)^2\frac{dY}{dt}$ $\frac{dY}{dt} = \frac{9 \cdot 12}{\pi \cdot 3 \cdot 36} = \frac{1}{\pi}$

$$\boxed{\frac{dY}{dt} = .32\ \frac{ft}{min}}$$

EXAMPLE ⟶ $S = \sqrt{x^2 + y^2}$

a) HOW ARE $\frac{ds}{dt}$, $\frac{dx}{dt}$, AND $\frac{dy}{dt}$ RELATED?

$$\frac{ds}{dt} = \frac{1}{2}(X^2+Y^2)^{-1/2}\left(2X\frac{dx}{dt} + 2Y\frac{dy}{dt}\right)$$

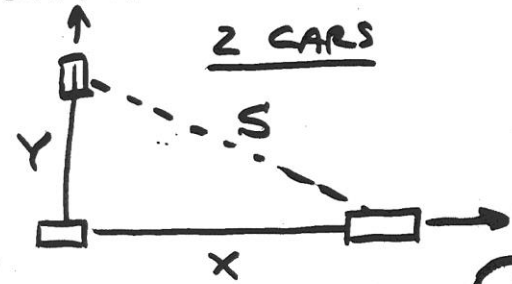
$$\text{OR } \frac{ds}{dt} = \frac{X\frac{dx}{dt} + Y\frac{dy}{dt}}{\sqrt{X^2+Y^2}} = \boxed{\frac{X\frac{dx}{dt} + Y\frac{dy}{dt}}{S} = \frac{ds}{dt}}$$

HOMEWORK p.237 EX. 1-7, 16-20, 32

4.6 CONTINUED

EXAMPLE

2 CARS



GIVEN INFORMATION

$X=4 \quad Y=3$

$\frac{dx}{dt}=40\text{ mph} \quad \frac{dy}{dt}=30\text{ m/h}$

$(x=40t) \quad (Y=30t)$

HOW FAST IS THE DISTANCE CHANGING?

$$\frac{d}{dt}\left(s^2=x^2+y^2\right)$$

$$2s\frac{ds}{dt}=2x\frac{dx}{dt}+2y\frac{dy}{dt} \qquad \frac{ds}{dt}=\frac{1}{s}\left(x\frac{dx}{dt}+y\frac{dy}{dt}\right)$$

OR $$\boxed{\frac{ds}{dt}=\frac{1}{\sqrt{x^2+y^2}}\left(x\frac{dx}{dt}+y\frac{dy}{dt}\right)}$$

WHEN $X=4$, $Y=3$ $$\frac{ds}{dt}=\frac{1}{\sqrt{4^2+3^2}}\left(4\cdot 40+3\cdot 30\right)=\boxed{50\text{ m/h}}$$

ALSO SEE EX 3. p. 234 (NEG. VELOCITY)

HOMEWORK
p. 238-240
13, 22, 31, 34, 35

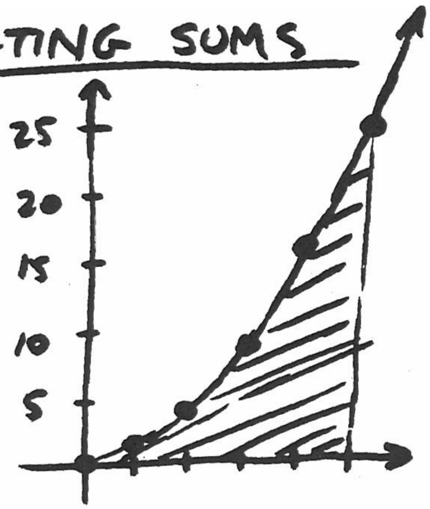
CONSIDER THIS PART OF THE REVIEW TOO.

REVIEW p. 242-245 → 1, 5, 7, 9, 12, 17, 27, 36, 37, 39, 55, 57, 58, 62

5.1 ESTIMATING SUMS

$Y = X^2$

X	Y
0	0
1	1
2	4
3	9
4	16
5	25



$Y = X^2$

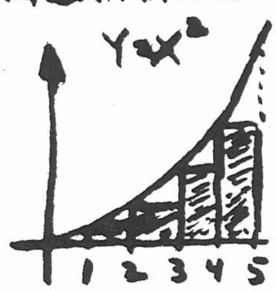
X	Y
.5	.25
1.5	2.25
2.5	6.25
3.5	12.25
4.5	20.25

WHAT IS THE AREA OF THE SHADED REGION?

L = LEFT   R = RIGHT   M = MIDPOINT

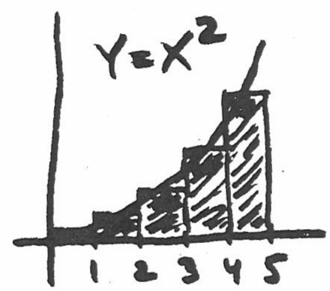
5 LEFT POINTS   5 RIGHT POINTS   5 MIDPOINTS

APPROXIMATING AREA UNDER CURVE



$LRAM_5(X^2)$

$RRAM_5(X^2)$



$MRAM_5(X^2)$

LEFT RECTANGLE APPROXIMATION METHOD
WITH 5 DIVISIONS

TO APPROXIMATE, ADD THE RECTANGLES.

SIGMA → (Σ)

AREA = Σ RECTANGLES

$= \Sigma$ HEIGHT $\cdot$ WIDTH $= \Sigma f(x) \cdot \Delta X$

GRAPH $Y1 = X^2$

TABLE TBLSET TblStart=0 ΔTbl=.5

$LRAM_5(x^2)$ FROM 0 TO 5

$= 0\cdot 1 + 1\cdot 1 + 4\cdot 1 + 9\cdot 1 + 16\cdot 1 = 30$ LOW

$RRAM_5(x^2) = 1\cdot 1 + 4\cdot 1 + 9\cdot 1 + 16\cdot 1 + 25\cdot 1 = 55$ HIGH

$MRAM_5(x^2) = .25(1) + 2.25(1) + 6.25(1) + 12.25(1) + 20.25(1) = \underline{\underline{41.25}}$

IT TURNS OUT, THE EXACT AREA UNDER THE CURVE IS THE DIFFERENCE OF THE ANTIDERIVATIVES EVALUATE AT THE ENDPOINTS.

ANTIDERIVATIVE OF X^2 IS $\frac{X^3}{3} + C$

$$\frac{5^3}{3} + C - \left(\frac{0^3}{3} + C\right) = \frac{125}{3} = 41\tfrac{2}{3}$$

$\boxed{\text{HOMEWORK p. 254-257} \quad 1, 2, 10, 11, 12, 22ab, 24ab}$

#1 HINT $\Delta X = \frac{2}{4} = .5$

X	Y
0	✓
.5	✓
1	✓
1.5	✓
2	X

L RAM