3.6 THE CHAIN RULE

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$$

PROOF $\quad \dfrac{dy}{du} \cdot \dfrac{du}{dx} = \dfrac{dy\,\cancel{du}}{\cancel{du}\,dx} = \dfrac{dy}{dx}$

EXAMPLE p.146 #2

$y = \sin(7-5x) \qquad y' = ?$

$y = \sin u \qquad u = 7-5x$

$\dfrac{dy}{du} = \cos u \qquad \dfrac{du}{dx} = -5$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \cos u\,(-5)$$

$$= -5\cos(7-5x)$$

EXAMPLE

$Y = \sqrt{x} + \csc(1-2x) \qquad u = 1-2x$

$Y = x^{1/2} + \csc u \qquad \dfrac{du}{dx} = -2$

$Y' = \frac{1}{2}x^{-1/2} + \dfrac{d\csc u}{du}\,\dfrac{du}{dx} = \dfrac{1}{2\sqrt{x}} + -\csc u \cot u\,(-2)$

B p 146 → 1,3,4,5

$$\boxed{Y' = \frac{1}{2\sqrt{x}} + 2\csc(1-2x)\cot(1-2x)}$$

EXAMPLE

$$Y = X\sec(3-8X)$$

$$Y = u \cdot V$$

$u = X$

$\frac{du}{dx} = 1$

$V = \sec(3-8X)$

$$Y' = u\frac{dv}{dx} + V\frac{du}{dx}$$

$$\frac{dv}{dx} = \sec(3-8x)\tan(3-8x)(-8)$$

$$Y' = X\sec(3-8x)\tan(3-8x)(-8) + \sec(3-8x)(1)$$

$$\boxed{Y' = -8x\sec(3-8x)\tan(3-8x) + \sec(3-8x)}$$

EXAMPLE p. 146 #8

$$Y = \sec(\tan x)$$

$$Y = \sec u$$

$u = \tan x$

$\frac{du}{dx} = \sec^2 x$

$$Y' = \frac{dy}{du}\cdot\frac{du}{dx} = \sec u \tan u \sec^2 x$$

$$\boxed{Y' = \sec(\tan x)\cdot\tan(\tan x)\sec^2 x}$$

EXAMPLE

$$Y = \sin^{-3} x + \sqrt{\tan x}$$

$$Y = (\sin x)^{-3} + (\tan x)^{1/2}$$

$$Y = u^{-3} + V^{1/2}$$

$$Y' = -3u^{-4}\frac{du}{dx} + \frac{1}{2}V^{-1/2}\frac{dv}{dx}$$

$$= -3(\sin x)^{-4}\cos x + \frac{1}{2}(\tan x)^{-1/2}\sec^2 x$$

$$= \frac{-3\cos x}{\sin^4 x} + \frac{\sec^2 x}{2\sqrt{\tan x}}$$

3.6 CONTINUED

REMEMBER $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$

EXAMPLE

$Y = \left(\frac{x}{2} - 1\right)^{-10}$ $\qquad u = \frac{1}{2}x - 1$

$Y = u^{-10}$ $\qquad \dfrac{du}{dx} = \dfrac{1}{2}$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = -10u^{-11}\left(\frac{1}{2}\right)$$

$$\frac{dy}{dx} = -5\left(\frac{x}{2} - 1\right)^{-11} \qquad \boxed{\frac{dy}{dx} = \frac{-5}{\left(\frac{x}{2} - 1\right)^{11}}}$$

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p.226 TABLE 3.1 CHAIN RULE FORMULAS

$\dfrac{d}{dx}u^n = nu^{n-1}\dfrac{du}{dx}$	
$\dfrac{d}{dx}\sin u = \cos u \dfrac{du}{dx}$	$\dfrac{d}{dx}\sec u = \sec u \tan u \dfrac{du}{dx}$
$\dfrac{d}{dx}\cos u = -\sin u \dfrac{du}{dx}$	$\dfrac{d}{dx}\cot u = -\csc^2 u \dfrac{du}{dx}$
$\dfrac{d}{dx}\tan u = \sec^2 u \dfrac{du}{dx}$	$\dfrac{d}{dx}\csc u = -\csc u \cot u \dfrac{du}{dx}$

HOMEWORK p. 146 → 1-20 OMIT 2,5,6, 8, 15, 16

EXAMPLE p.147 #28 FIND r'

$$r = \overset{u}{2\theta}\,\overset{v}{\sqrt{\sec\theta}}$$

$u = 2\theta$ $\quad \dfrac{du}{d\theta} = 2$

$V = (\sec\theta)^{1/2}$ $\quad \dfrac{dv}{d\theta} = \dfrac{1}{2}(\sec\theta)^{-1/2}\sec\theta\tan\theta$

$$\frac{dr}{d\theta} = u\frac{dv}{d\theta} + V\frac{du}{d\theta} \qquad \frac{dv}{d\theta} = \frac{1}{2}\sqrt{\sec\theta}\,\tan\theta$$

$$\frac{dr}{d\theta} = 2\theta\,\frac{1}{2}\sqrt{\sec\theta}\,\tan\theta + \sqrt{\sec\theta}\;2$$

$$\boxed{\frac{dr}{d\theta} = \theta\sqrt{\sec\theta}\,\tan\theta + 2\sqrt{\sec\theta}}$$

MADE UP EXAMPLE FIND $\dfrac{ds}{dt}$

$$s = \underset{u}{t}\cdot\underset{v}{\sin(\pi - 2t)} \qquad \frac{du}{dt} = 1$$

$$s' = uv' + vu' \qquad \frac{dv}{dt} = \cos(\pi - 2t)(-2)$$

$$= t\cos(\pi - 2t)(-2) + \sin(\pi - 2t)(1)$$

$$\boxed{s' = -2t\cos(\pi - 2t) + \sin(\pi - 2t)}$$

HOMEWORK

p. 146-147 → 21-27 <u>ALL</u>

MORE ON CHAIN RULE FIND Y'

35

$$Y = \left(\frac{\sin x}{2x+1}\right)^3 \qquad u = \frac{\sin x}{2x+1} \begin{matrix} \leftarrow u \\ \leftarrow v \end{matrix}$$

$$Y = u^3 \qquad \frac{du}{dx} = \frac{vu' - uv'}{v^2}$$

$$Y' = 3u^2 \cdot \frac{du}{dx} \qquad \frac{du}{dx} = \frac{(2x+1)\cos x - \sin x\,(2)}{(2x+1)^2}$$

$$\boxed{Y' = 3\left(\frac{\sin x}{2x+1}\right)^2 \cdot \left[\frac{(2x+1)\cos x - 2\sin x}{(2x+1)^2}\right]}$$

ANOTHER EXAMPLE FIND Y''

$$Y = \sec x \qquad u = \sec x$$

$$Y' = \underset{u}{\sec x}\ \underset{v}{\tan x} \qquad u' = \sec x \tan x$$

$$v = \tan x$$

$$v' = \sec^2 x$$

$$Y'' = uv' + vu'$$

$$\boxed{Y'' = \sec x(\sec^2 x) + \tan x(\sec x \tan x)}$$

HOMEWORK p. 146-147
→ $\underline{\underline{5}}$, 6, 15, 16, 29-32

EXAMPLE p.147 #36

$f(u) = u + \frac{1}{\cos^2 u}$ $\quad u = \pi x = g(x)$ $\quad$ FIND $(f \circ g)'$ AT $x = 1/4$

$$f \circ g = f(g(x)) = \pi x + \frac{1}{\cos^2(\pi x)}$$

$$f(u) = u + (\cos u)^{-2} \qquad \boxed{\frac{d}{dx} f(u) = f'(u) \frac{du}{dx}}$$

$$(f \circ g)' = f'(u) = \frac{du}{dx} + -2(\cos u)^{-3}(-\sin u)\frac{du}{dx}$$

$$f'(x) = \pi + 2(\cos \pi x)^{-3} \sin \pi x \; \pi$$

$$f'(x) = \pi + \frac{2\pi \sin \pi x}{(\cos \pi x)^3}$$

EVALUATE AT $x = 1/4$

$$f'(1/4) = \pi + \frac{2\pi \sin(\pi(\frac{1}{4}))}{(\cos(\pi \frac{1}{4}))^3} = \pi + \frac{2\pi \cdot \frac{\sqrt{2}}{2}}{\left(\frac{\sqrt{2}}{2}\right)^3} = 5\pi$$

EXAMPLE p.148 → 57

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
0	1	1	5	$1/3$
1	3	-4	$-1/3$	$-8/3$

EVALUATE THE DERIVATIVES OF THE GIVEN EXPRESSION

#57 CONTINUED

a) $Y = 5f(x) - g(x)$ $\quad x = 1$

$Y' = 5f'(x) - g'(x)$ (LOOK AT CHART)

$Y' = 5(-1/3) - (-8/3) = \frac{-5}{3} + \frac{8}{3} = \frac{3}{3} = \underline{\underline{1}}$

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b) $Y = \underbrace{f(x)}\,\underbrace{g^3(x)}$ $\quad x = 0$

$Y = u \cdot v \longrightarrow u^3$

$Y' = u\frac{dv}{dx} + v\frac{du}{dx} = f(x) \cdot 3g^2(x)g'(x) + g^3(x)f'(x)$

(LOOK AT CHART) $= 1 \cdot 3 \cdot 1^2 \cdot \frac{1}{3} + 1^3 \cdot 5$

$x = 0$

$= 1 + 5 = \underline{\underline{6}}$

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c) $Y = \frac{f(x)}{g(x)+1}$ $\begin{matrix} \leftarrow u \\ \leftarrow v \end{matrix}$ $\quad x = 1$

$Y' = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{(g(x)+1)f'(x) - f(x)g'(x)}{(g(x)+1)^2}$

LOOK AT CHART $\rightarrow$ $= \frac{(-4+1)(-1/3) - 3(-8/3)}{(-4+1)^2} = \frac{1+8}{9} = \underline{\underline{1}}$

$x = 1$

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d) $Y = f(g(x))$ $\quad \underline{x = 0}$ $\quad$ USE CHART

$Y' = f'(g(x)) \cdot g'(x) = f'(1) \cdot g'(0) = -\frac{1}{3} \cdot \frac{1}{3} = -\frac{1}{9}$

HOMEWORK p. 147 $\longrightarrow$ 33-35, 37, 51, 52, 54-56

#51 HINT $\frac{ds}{dt} = \frac{ds}{d\theta} \cdot \frac{d\theta}{dt}$