Optimization Problems

LING 572
Advanced Statistical Methods for NLP
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What is an optimization problem?

- The problem of finding the best solution from all feasible solutions.
- Given a function f: X➔R, find x0 in X that optimizes f(x).
- f is called
  - an objective function,
  - a loss function or cost function (minimization), or
  - a utility function or fitness function (maximization), etc.

- X is a n-variable vector:
  - discrete (possible values are countable): combinatorial optimization problem
  - continuous: e.g., constrained problems

Components of each optimization problem

- Decision variables X: describe our choices that are under our control.
  - We normally use n to represent the number of decision variables, and x_i to represent the i-th decision variable.

- Objective function f: the function we wish to optimize

- Constraints: describe the limitations that restrict our choice for decision variables.

Standard form of a continuous optimization problem

$$
\begin{aligned}
\underset{x}{\text{minimize}} \quad & f(x) \\
\text{subject to} \quad & g_i(x) \leq 0, \quad i = 1, \ldots, m \\
& h_i(x) = 0, \quad i = 1, \ldots, p
\end{aligned}
$$

re

$f(x) : \mathbb{R}^n \to \mathbb{R}$ is the **objective function** to be minimized over the n-variable vector x,

$g_i(x) \leq 0$ are called **inequality constraints**, and

$h_i(x) = 0$ are called **equality constraints**.

onvention, the standard form defines a **minimization problem**. A **maximization problem** can be treated

ating the objective function.

Common types of optimization problem

- Linear programming (LP) problems:
  - Definition: Both objective function and constraints are linear
  - The problems can be solved in polynomial time.
  - https://en.wikipedia.org/wiki/Linear_programming

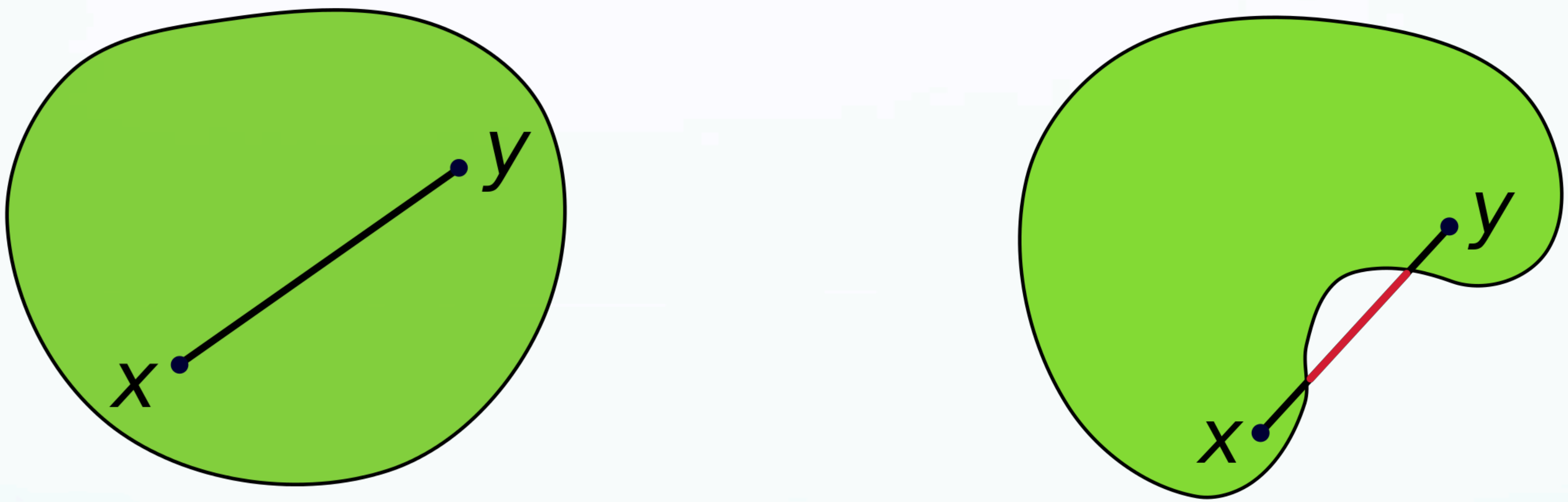
- Integer linear programming (ILP) problems:
  - Definition: LP problem in which some or all of the variables are restricted to be integers
  - Often, solving ILP problem is NP-hard.
  - https://en.wikipedia.org/wiki/Integer_programming

Common types of optimization problem (cont'd)

- Quadratic programming (QP):
  - Definition: The objective function is quadratic, and the constraints are linear
  - Solving QP problems is simple under certain conditions
  - https://en.wikipedia.org/wiki/Quadratic_programming

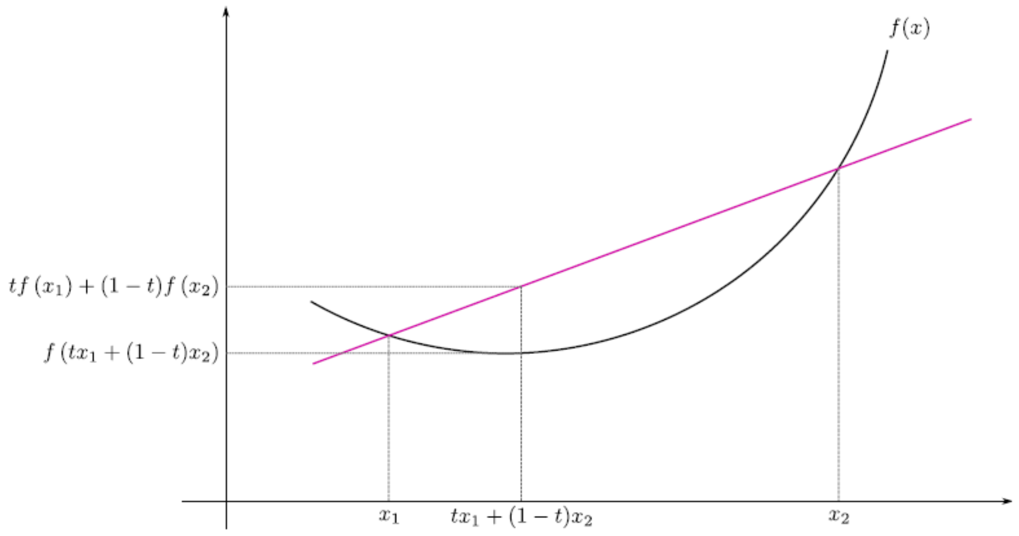
- Convex optimization:
  - Definition: f(x) is a convex function, and X is a convex set.
  - Property: if a local minimum exists, then it is a global minimum.
  - https://en.wikipedia.org/wiki/Convex_optimization

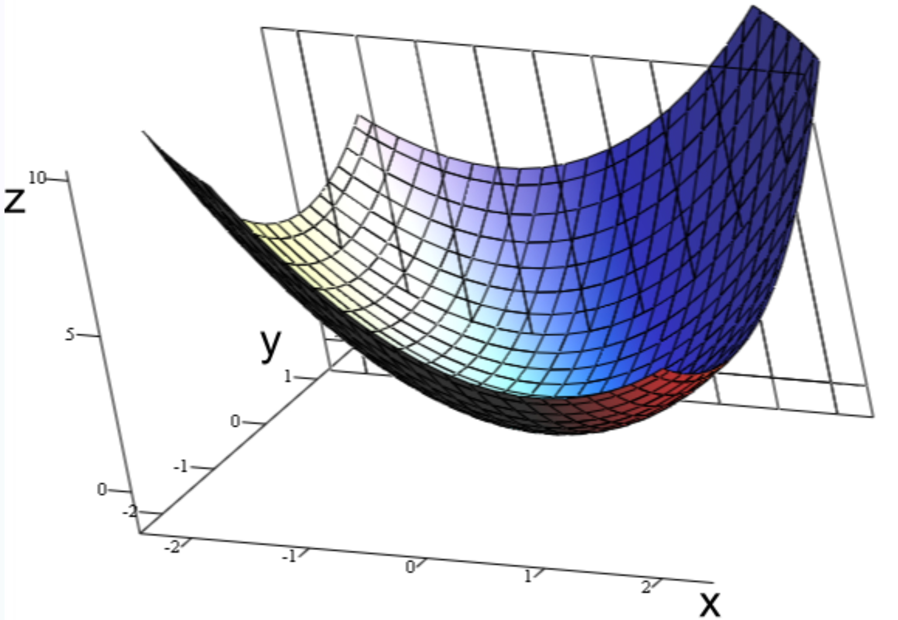
Convex set



A set C is said to be **convex** if, for all x and y in C and all t in the interval $(0, 1)$, the point $(1 - t)x + ty$ also belongs to C

Convex function





Let X be a convex set in a real vector space and let $f: X \to \mathbf{R}$ be a function.

- f is called convex if:

$$\forall x_1, x_2 \in X, \forall t \in [0,1]: \qquad f(tx_1 + (1-t)x_2) \le tf(x_1) + (1-t)f(x_2).$$

- f is called strictly convex if:

$$\forall x_1 \neq x_2 \in X, \forall t \in (0,1): \qquad f(tx_1 + (1-t)x_2) < tf(x_1) + (1-t)f(x_2).$$

Terms

- A solution is the assignment of values to all the decision variables
- A solution is called feasible if it satisfies all the constraints.
- The set of all the feasible solutions forms a feasible region.
- A feasible solution is called optimal if f(x) attains the optimal value at the solution.

- If a problem has no feasible solution, the problem itself is called infeasible.

- If the value of the objective function can be infinitely large, the problem is called unbounded.

Linear programming

Linear Programming

- The linear programming method was first developed by Leonid Kantorovich in late 1930s.
- Main applications: diet problem, supply problem
- A primary method for solving LP is the simplex method.
- LP problems can be solved in polynomial time.

An example

Suppose that a farmer has a piece of farm land, say L km^2, to be planted with either wheat or barley or some combina
of the two. The farmer has a limited amount of fertilizer, F kilograms, and pesticide, P kilograms. Every square kilom
of wheat requires F_1 kilograms of fertilizer and P_1 kilograms of pesticide, while every square kilometer of barley requ
F_2 kilograms of fertilizer and P_2 kilograms of pesticide. Let S_1 be the selling price of wheat per square kilometer, and S
the selling price of barley. If we denote the area of land planted with wheat and barley by x_1 and x_2 respectively, then pr
can be maximized by choosing optimal values for x_1 and x_2. This problem can be expressed with the following li
programming problem in the standard form:

Maximize:	$S_1 \cdot x_1 + S_2 \cdot x_2$	(maximize the revenue—revenue is the "objective function")
Subject to:	$x_1 + x_2 \leq L$	(limit on total area)
	$F_1 \cdot x_1 + F_2 \cdot x_2 \leq F$	(limit on fertilizer)
	$P_1 \cdot x_1 + P_2 \cdot x_2 \leq P$	(limit on pesticide)
	$x_1 \geq 0, x_2 \geq 0$	(cannot plant a negative area).

Property of LP

- The feasible region is convex

- If the feasible region is non-empty and bounded, then
  - optimal solutions exist, and
  - there is an optimal solution that is a corner point

  ➔ We only need to check the corner points

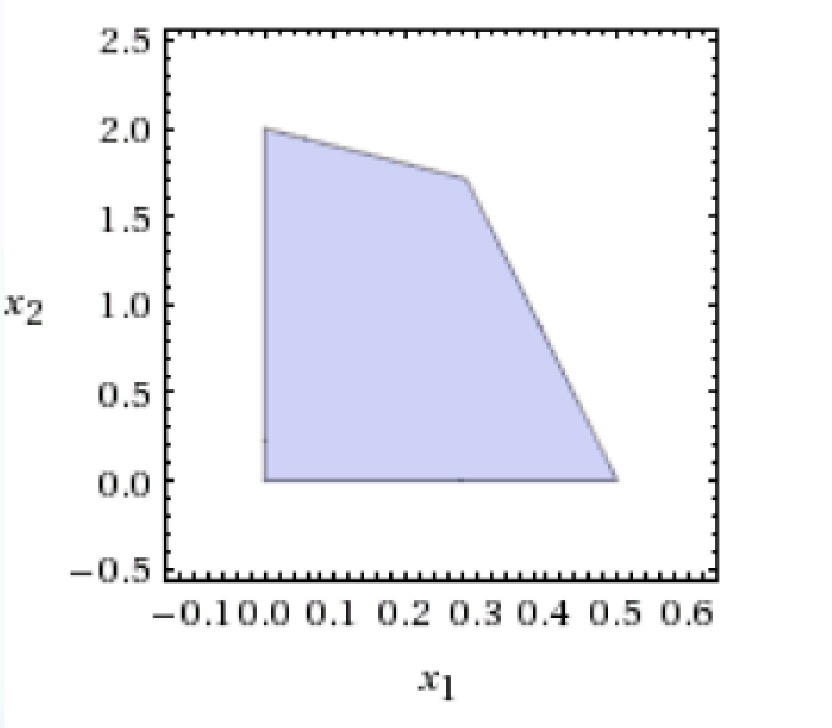
- The most well-known method is called the simplex method.

maximize $z = x_1 - 3x_2$

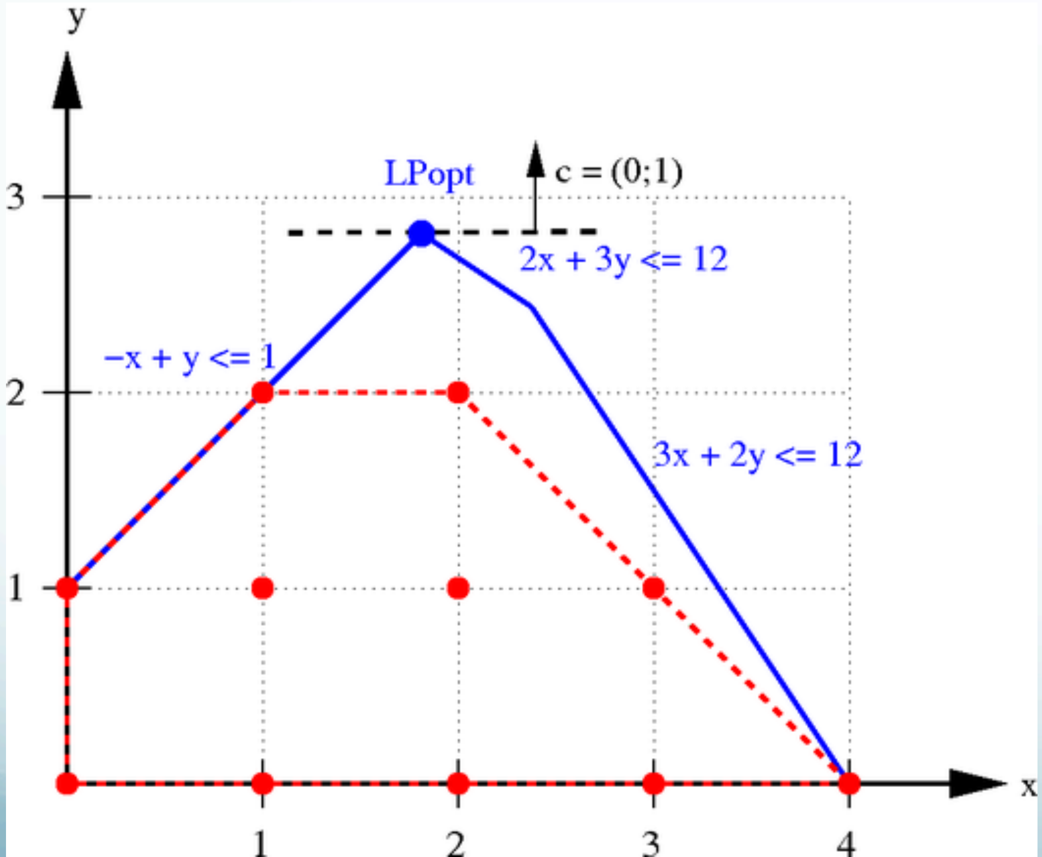
subject to $x_1 + x_2 \leq 2$

$8x_1 + x_2 \leq 4$

$x_1, x_2 \geq 0$



Feasible region and simplex method



Simplex method:

- Start with a feasible solution, move to another on
  to increase f(x)

Integer linear programming

Integer programming

- IP is an active research area and there are still many unsolved problems.
- IP is more difficult to solve than LP.
- Methods:
  - Branch and Bound
  - Use LP relaxation

Tournament	Assigned Number (i)	Chance of bidding (c)[1]	Cost (m)[2]	Time needed (t)[3]	Date[4]
Greenhill	1	0.4	$500	3	9/20-9/21
Yale	2	0.3	$700	4	9/19-9/21
Valley	3	0.5	$500	3	9/27-9/29
Cypress Bay*	4	0.4	$600	4	10/10-10/12
Presentation	5	0.5	$350	3	10/11-10/13
Bronx Science	6	0.3	$600	3	10/17-10/19
St. Mark's	7	0.5	$500	3	10/18-10/20
Meadows	8	0.6	$450	4	10/31-11/2
Apple Valley*	9	0.3	$300	2	11/7-11/8
Glenbrooks*	10	0.5	$400	3	11/22-11/24
Alta*	11	0.4	$500	4	12/4-12/6
Blake	12	0.5	$450	4	12/19-12/21
College Prep	13	0.4	$350	3	12/20-12/21
Harvard Westlake	14	0.5	$400	3	1/3-1/5
Sunvitational	15	0.3	$600	3	1/11-1/12
Lexington	16	0.4	$500	3	1/17-1/18
Emory	17	0.5	$600	4	1/23-1/25
Stanford	18	0.6	$350	4	2/7-2/9
Berkeley	19	0.4	$350	3	2/14-2/16
Harvard	20	0.5	$500	4	2/14-2/16

ecision variables: $x_i = \begin{cases} 1 & \text{if decided to compete at the } i^{\text{th}} \text{ tournament} \\ 0 & \text{otherwise} \end{cases}$

pjective function: $z = \sum_{i=1}^{n} c_i x_i$

nstraints:

$$\sum_{i=1}^{n} m_i x_i \leq M$$

$$\sum_{i=1}^{n} t_i x_i \leq T$$

$$x_1 + x_2, x_4 + x_5, x_6 + x_7, x_{12} + x_{13}, x_{19} + x_{20} \leq 1$$

$$\text{maximize } z = \sum_{i=1}^{n} c_i x_i$$

$$\text{subject to} \sum_{i=1}^{n} m_i x_i \leq M$$

$$\sum_{i=1}^{n} t_i x_i \leq T$$

$$0 \leq x_i \leq 1 \text{ for every } i$$

$$x_i \text{ is an integer for every } i$$

$$x_1 + x_2 \leq 1$$

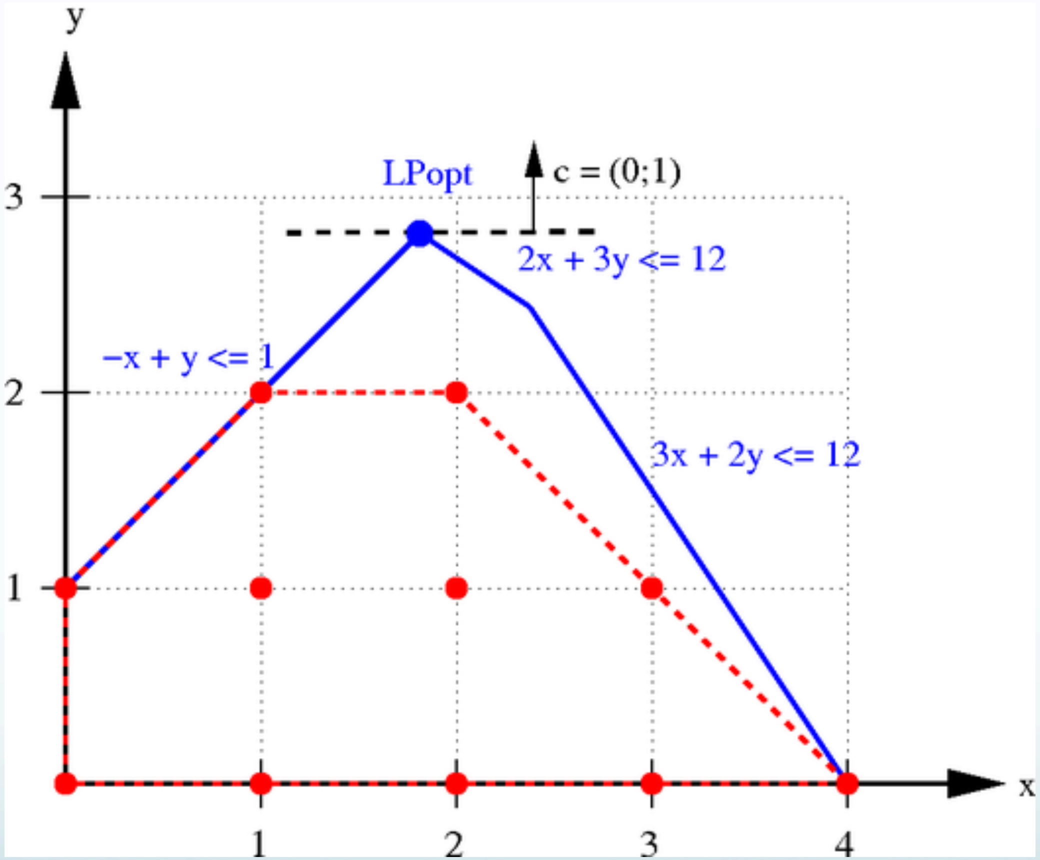
$$x_4 + x_5 \leq 1$$

$$x_6 + x_7 \leq 1$$

$$x_{12} + x_{13} \leq 1$$

$$x_{19} + x_{20} \leq 1$$

LP vs. ILP



Summary

- Optimization problems have many real-life applications.
- Common types: LP, IP, ILP, QP, Convex optimization problem
- LP is easy to solve; the most well-known method is the simplex method.
- IP is hard to resolve.
- QP and Convex optimization are used the most in our field.