Exploring Invariant Manifolds for Low-Cost Trajectory Design

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DRAFT – Extended Abstract
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Periodic orbits within the Circular Restricted Three-body Problem (CR3BP) are more difficult to find than periodic orbits within the Restricted Two-body Problem (R2BP). While any elliptical or circular R2BP orbit is periodic, the periodic subset of CR3BP orbits is far more selective. Periodic CR3BP orbit-finding algorithms have been known for decades, but their implementations are not easily found in free and open-source software – the few implementations which do exist in free and open-source software don't hold *generally*. Instead, publicly accessible implementations may find periodic orbits for one particular CR3BP system (e.g. Sun-Earth), or may find periodic orbits which have non-zero z-amplitudes. General solutions for periodic CR3BP orbits are incredibly useful, as CR3BP analysis is often used as an approximate starting-point in mission design. In addition, invariant manifolds about periodic CR3BP orbits can provide low-cost interplanetary transfers when compared with traditional transfer designs. This paper introduces a matured periodic CR3BP orbit-finding implementation that works *generally*. This implementation is used to highlight several Halo orbit families in familiar CR3BP systems, calculate invariant manifolds about Halo orbits, and use the invariant manifolds to calculate low-cost interplanetary transfer designs. A CR3BP patched-conic mission design will be presented. All source code is available on GitHub as part of *UnitfulAstrodynamics.jl*, an astrodynamics package written with Julia.

Contents

A TOC is *not* included in this draft to help save space for actual content. Read on for some fun orbit plots!

I. Nomenclature

$\vec{r}$	=	Spacecraft position
$\vec{v}$	=	Spacecraft velocity
Φ_m	=	Monodromy matrix
Φ	=	State transition matrix
R2BP	=	Restricted Two-body Problem
CR3BP	=	Circular Restricted Three-body Problem
$S_L \triangleq a$	=	Distance between CR3BP bodies
S_T	=	Time scale factor
$\mu^\star$	=	Normalized CR3BP mass parameter
$\vec{r}^\star$	=	Normalized spacecraft position
$\vec{v}^\star$	=	Normalized spacecraft velocity
r_i	=	Nondimensional x position of i_{th} body

II. Introduction

As space exploration targets shift from Earth's moon to Mars and beyond, low-cost trajectory designs are increasingly important [1]. One such family of low-cost trajectory designs utilizes invariant manifolds about Lagrange points [2]. Halo orbits can be estimated analytically with an expansion, as originally shown by Richardson [3] [4]. A known numerical algorithm can iterate on non-periodic initial Halo orbit conditions to produce numerically periodic Halo orbits [5]. When placed in series, these two algorithms provide a proverbial *black box* for astrodynamicists: given desired physical features (orbit amplitude, phase, etc.), the analytical algorithm can produce a Halo orbit estimate for the numerical algorithm to iterate on [2]. These algorithms were recently implemented with the Julia programming language [6] [7] [8].

This paper presents low-cost manifold-based trajectories from the Sun-Earth system, to Sun-Mars and Sun-Jupiter systems. Manifold and traditional trajectory designs will be compared, and their costs and benefits will be analyzed.

First, an overview of the Circular Restricted Three-body Problem, and periodic CR3BP orbits, is provided. Methods for calculating periodic CR3BP orbits are briefly discussed, and a Julia implementation is introduced. Test cases are presented for the analytical and numerical Halo orbit solvers [6]. These test cases will help to guarantee that the Julia Halo solvers within UnitfulAstrodynamics.jl are working correctly, and they may serve as a helpful troubleshooting resource for others who hope to implement similar algorithims [7].

Next, a short catalog of Sun-Earth, Sun-Mars, and Sun-Jupiter Halo orbits is presented using the developed Halo solvers. This catalog will provide parameters for each Halo orbit, and initial conditions and orbital periods from which to propagate.

Visualizations for stable and unstable manifolds near each chosen Halo orbit are shown. Points along the Halo orbits' stable manifolds are found, and transfers between LEO and these stable manifolds are defined. These manifolds are

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then used to generate interplanetary transfers. The manifold-based transfers will be compared and contrasted against traditional transfer designs (e.g. Hohmann, Lambert). The following analysis metrics are addressed: total mission Δv, total mission duration, and (ideal) number of maneuvers.

All code written for this project is included in the open-source Julia package titled *UnitfulAstrodynamics.jl* [7]. A Pluto (Jupyter-like) notebook with code examples, and an informal project summary is provided [9] [6].

III. Technical Overview

A. CR3BP Overview

The Circular Restricted Three-body Problem is a simplified dynamical model of an infinitesimally small spacecraft traveling near two finite-mass celestial bodies. All masses are described as point masses. The system's barycenter is the center of mass of the two celestial bodies, and both celestial bodies are constrained to travel in a circle about an inertial frame placed at the system barycenter. CR3BP spacecraft dynamics are typically described in the *Synodic* frame, which is placed at the barycenter of the system, and rotates about its Z axis such that each celestial body is fixed on the Synodic X axis. CR3BP spacecraft dynamics are also typically described with normalized units, as shown in equations (1) (2) (3) (4). Normalized equations of motion for a spacecraft within the CR3BP are shown in equations (5) (6) (7) (8) (9). Note that $\mu_2 < \mu_1$.

Relevant CR3BP definitions, calculations, and equations of motion will be presented here.

B. Periodic CR3BP Orbits

Planar periodic CR3BP orbits about the first or second Lagrange point (L1 or L2) are known as Lypunov orbits, while periodic CR3BP orbits about L1 or L2 with non-zero Z components are known as Halo orbits [4]. Periodic CR3BP orbits are desirable for many reasons, including eclipse avoidance [10]. Of course, orbits within the CR3BP are **not** guaranteed to be periodic. An estimated analytical periodic orbit solution can be found with a third-order expansion, as originally shown by Dr. Richardson [3] [4] [2]. Dr. Howell developed an iterative algorithm find a numerically periodic CR3BP orbit [5] [4] [2]. The analytical and iterative periodic orbit-finding algorithms are described in more detail in the remainder of this section.

1. Analytical Solution

Approximate analytical solutions exist for periodic orbits within the Circular Restricted Three-body Problem. CR3BP dynamics can be written as a polynomial expansion [4]. Dr. Richardson used a third-order expansion to develop an analytical solution for periodic CR3BP orbits about L1 or L2; the solution involves changing coordinates to be centered at the desired Lagrange point and normalized to the distance between the lagrange point and the less-massive celestial body [4] [3] [2]. This analytical solution is described in far more detail by Rund and Koon et al [2] [4]. Here, the reader should know two important points about this analytical method: each periodic orbit can be completely described by its Z amplitude (as a result of a X and Z amplitude constraint), and because each analytical solution is only an *approximation*, propagating any analytical result does *not* result in a numerically periodic orbit [4]. The latter point is shown by Fig.1.

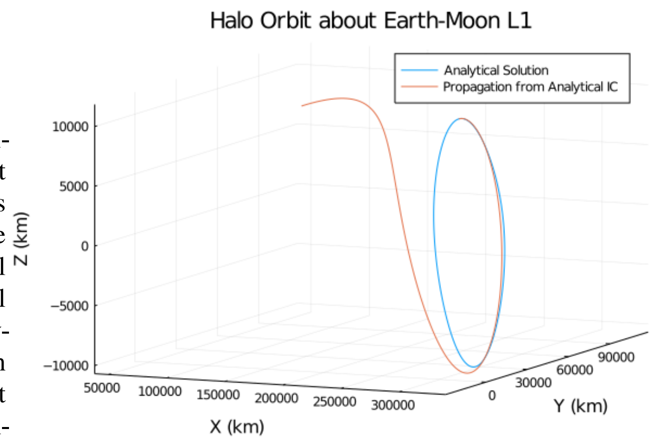

Fig. 1 Numerically propagated analytical solution

2. Numerical Solution

A description of the iterative Halo orbit-finding algorithm will be provided here. The description will be at a somewhat high level, similar to the analytical solution description above. Still, the ***algorithm*** *package will be used to outline the numerical algorithm, and a Julia implementation will be briefly presented. The plots below will be shown in the final paper: the first shows how the numerical algorithm iterates on a periodic orbit candidate produced by the analytical solution, and the second shows a family of periodic CR3BP orbits that were produced by the Julia implementation.*

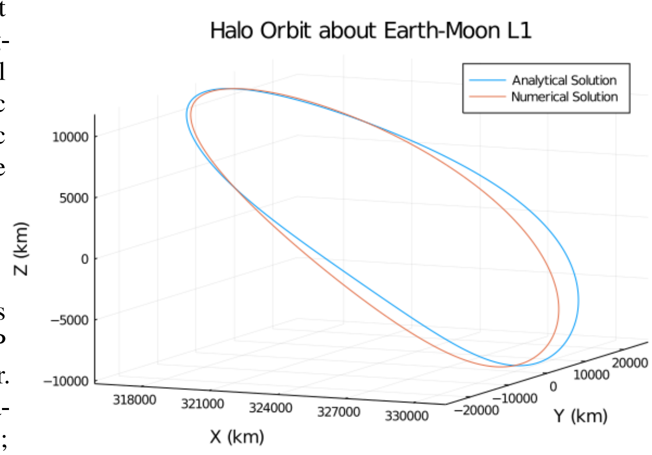

Fig. 2 Analytical and Iterative Numerical solutions

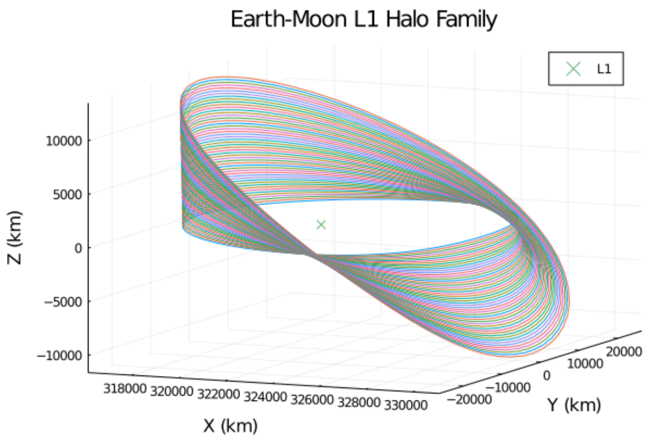

Fig. 3 Famly of Halo orbits about Earth-Moon L1

C. Invariant Manifolds

The mathematics behind invariant manifolds will be presented. An invariant manifold-finding algorithm will be presented, as shown by Rund and Koon et al [2] [4]. Stable and unstable invariant manifolds near Halo orbits will be shown, as produced by a Julia implementation of the previously mentioned algorithm.

D. Manifold-based Transfers

An algorithm for using invariant manifolds to design interplanetary transfers will be presented, as summarized by Rund [2]. A Julia implementation will (hopefully!) be presented here as well.

IV. Results

A. Numerically Periodic CR3BP Orbits

A large table (1-2 pages) of autogenerated periodic orbits about Lagrange points of familiar CR3BP systems will be provided here. I'll write some code to find several families of Halo orbits about the Sun-Earth, Earth-Moon, Sun-Mars, and Sun-Jupiter systems. This will hopefully be a helpful reference for future astrodynamics students.

B. Transfer Design Comparisons

A table (half a page) of Hohmann, Lambert, and manifold-based transfer designs from Earth to Mars and Jupiter will be presented. The results will be discussed, including cost and time duration comparisons between transfer designs. If there is time, a GMAT-like tool will also be presented. This tool will allow for user inputs, such as "start at Earth, transfer to Jupiter, end at 7000 km Z amplitude orbit about Io."

V. Conclusion

VI. References

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VII. Appendix

Will present all source code necessary to replicate this project, including copy-pasted UnitfulAstrodynamics.jl code that is available on GitHub. Any MATLAB implementations will also be provided here, as MATLAB is a far more common tool used by astrodynamics students.