Proof. Let us fix $\mathbf{x} \in R$.

We proceed by induction. Let us show that if

$$F^{\ell} \in \mathrm{Robust}_{\prod_{\lambda \leq \ell} \alpha^{\lambda}}(\mathrm{R}, \mathrm{r}) \ ,$$

then

$$F^{\ell+1} \in \mathrm{Robust}_{\prod_{\lambda \leq \ell+1} \alpha^{\lambda}}(\mathrm{R}, \mathrm{r}) \ .$$

Indeed, let us fix ε s.t. $\|\varepsilon\| < r$, then:

$$\begin{aligned} & \|F^{\ell+1}(\mathbf{x}+\varepsilon) - F^{\ell+1}(\mathbf{x})\| \\ = \ & \|f^{\ell+1}(F^{\ell}(\mathbf{x}+\varepsilon)) - f^{\ell+1}(F^{\ell}(\mathbf{x}))\| \ . \end{aligned}$$

Note that as $F^{\ell} \in \mathrm{Robust}_{\prod_{\lambda \leq \ell} \alpha^{\lambda}}(\mathrm{R}, \mathrm{r})$, it holds that:

$$\|F^{\ell}(\mathbf{x}+\varepsilon) - F^{\ell}(\mathbf{x})\| \leq \prod_{\lambda \leq \ell} \left(\alpha^{\lambda}\right) \|\varepsilon\| \ .$$

So we can write $F^{\ell}(\mathbf{x}+\varepsilon) = F^{\ell}(\mathbf{x}) + \varepsilon'$, where $\|\varepsilon'\| \leq \prod_{\lambda \leq \ell} \left(\alpha^{\ell}\right) \|\varepsilon\| \leq r \prod_{\lambda \leq \ell} \left(\alpha^{\ell}\right)$.

So, we obtain:

$$\begin{aligned} & \|F^{\ell+1}(\mathbf{x}+\varepsilon) - F^{\ell+1}(\mathbf{x})\| \\ = \ & \|f^{\ell+1}(F^{\ell}(\mathbf{x}) + \varepsilon') - f^{\ell+1}(F^{\ell}(\mathbf{x}))\| \\ \leq \ & \alpha^{\ell+1} \|\varepsilon'\| \leq \prod_{\lambda \leq \ell+1} \left(\alpha^{\lambda}\right) \|\varepsilon\| \ . \end{aligned}$$

□